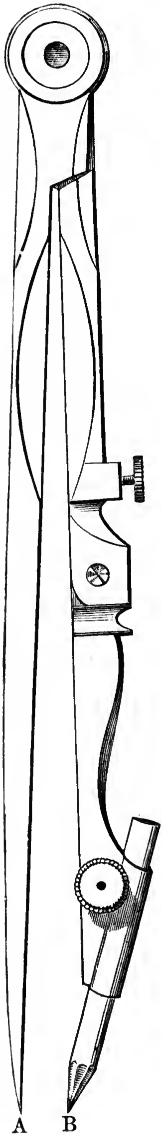

Fig. 1

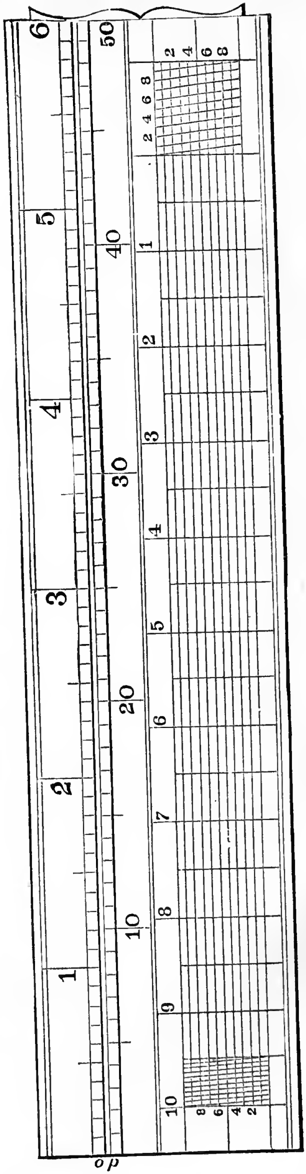

Fig. 2.

ELEMENTARY

GEOMETRY.

NEW EDITION.

INCLUDING PLANE, SOLID, AND SPHERICAL GEOMETRY, WITH PRACTICAL EXERCISES.

BY

EDWARD OLNEY,

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF MICHIGAN.

SHELDON AND COMPANY,

NEW YORK AND CHICAGO.

1883.

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PREFACE

THE first edition of OLNEY'S SPECIAL OR ELEMENTARY GEOMETRY was issued nearly twelve years ago. It contained many new features. The book has gone into use in every State in the Union, and has been tested by practical teachers in all grades of schools. This long and varied test has been watched with care by the author, and it is with the greatest pleasure that he has found that the general features of the book have been well-nigh universally approved.

To make the book still more acceptable to the teachers and schools of our country, and to keep it abreast with the real advancement in science and methods of teaching, as well as to make it a worthy exponent of the best style of the printer's art, are some of the reasons which have led to the preparation of this edition.

1. The division into Chapters and Sections, instead of Books, has been retained, as affording better means of classifying the subject-matter, and also as conforming to the usage of modern times in other literary and scientific treatises.

2. Part First of the old edition has been omitted, and the definitions and illustrations necessary to the integrity of the subject have been incorporated with the body of the work. This has been done solely in deference to the general sentiment of the teachers of our country. The author can but feel that this sentiment is wrong. That the best way to present the subject of Geometry is to present some of its leading notions and practical facts with their uses in drawing and in common life, before attempting to reason upon them, appears to him quite clear. It is in accord with one of the settled maxims of teaching which

requires "facts before reasoning," and then it is in harmony with the historic development of the science, and with the order of mental development in the individual. Moreover, since this method was presented to the American public in this treatise, the author has received books on exactly the same plan, which are in general use in Germany, and also "A Syllabus of Plane Geometry, prepared by the Association for the improvement of Geometrical teaching" in England, in which this principle is recognized by recommending quite an extended course in Geometrical constructions before entering upon the logical treatment of the subject. The author hopes to revise his Part First, and present it as a little treatise adapted to our Grammar or lower schools; as he can but think these subjects much more interesting and useful to pupils of this grade than much of the matter usually brought before them, especially the more advanced portions of arithmetic, and as he is confident that they are the proper preparation for the intelligent study of logical geometry.

3. The same general analysis of the subject is adhered to as in the first edition. All must acknowledge it a reproach to the oldest and most perfect of the sciences that, hitherto, no systematic classification of its subject-matter has been reached. That the ordinary arrangement found in our Geometries is not based upon a scientific analysis of the subject, and a systematic classification of topics will be evident to any one who attempts to give the subject-title of almost any so-called Book. A glance at the table of contents of this volume will show that the analysis of the subject-matter is simple and strictly philosophical. There are two lines of inquiry in geometry, viz., concerning position (from which form results) and magnitude. The concepts of Plane Geometry are the point, straight line, angle, and circle. Now, the measurement of magnitude is either direct or indirect. The direct measurement and comparison of magnitudes is a simple arithmetical operation, and is presented, as regards straight lines, in Section 4. The direct measurement of other magnitudes is effected in a similar manner, but is unimportant from a scientific point of view. The indirect measurement of magnitude, as when we find the third side of a triangle from the other two and their included angle, the circum-

ference or area of a circle from the radius, etc., is a somewhat remote application of more elementary principles. There is then left, as the natural first object of inquiry, the relative position of two (and hence of all) straight lines. Here we have philosophically the first inquiry of logical geometry. This inquiry divides into the three inquiries concerning perpendicular, oblique and parallel lines. In a similar manner the topics of the succeeding sections unfold themselves from the principles stated.

4. This analysis and classification of the subject-matter requires that a somewhat larger number of propositions be demonstrated from fundamental principles, that did the old method, of proving first any proposition you could, and then any other, and so on ; but who will consider this a defect ? On the other hand, it gives almost absolute unity of method of demonstration in the propositions of any one section.

5. The freedom with which revolution is used as a method of demonstration, will be observed upon a cursory reading. Of course it is assumed that the old repugnance to the introduction of the notions of time and motion into geometry is outgrown. Indeed, the old geometers could not get on without the superposition of magnitudes, and this idea involves motion. Now, revolution is but a systematic method of effecting superposition, which is well-nigh the only geometrical method of proving the equality of magnitudes.

6. The author has long desired to introduce the idea of sameness of direction in treating parallels; but could not accept what seemed to him the vague methods of writers who have made the attempt. If we cannot define the notion of direction, we certainly should have some method of estimating and measuring it before it can be made a proper subject of geometrical inquiry. This the author thinks he has secured, by giving the necessary precision to certain very common and simple notions.

7. As to the introduction of the infinitesimal method into mathematics (and if introduced at all, why not in the elements where it will do most service?), the author is confident that no one thing would do more to simplify, and hence to advance, elementary mathematical study, than the general and hearty acceptance of this method. No writer has succeeded in getting

on far, even in pure mathematics, without openly or covertly introducing the notion, and its directness, simplicity, if not absolute necessity, in the applied mathematics make its introduction into the elements exceedingly desirable. Nevertheless, the author has given alternative demonstrations, either in the body of the text or in the appendix, so that those who prefer can omit the demonstrations involving the infinitesimal conception.

8. Thanks to the spirit of the times, no geometry can now receive favor which does not give opportunity for the application of principles and for independent investigation. As in the former edition, so in this, large attention has been given to this just demand of the times. As a help to independent thinking, after the student has been fairly introduced to the methods, and had time to imbibe somewhat of the spirit of geometrical reasoning, the references to the antecedent principles on which statements in the demonstrations are based, are sometimes omitted, and their place supplied by interrogation marks.

9. In the earlier part of the work, the demonstrations are divided, according to the suggestion originally given by De Morgan, into short paragraphs, each of which presents but a single step. So, also, in this part, care has been taken to make separate paragraphs of the statement of premises and the conclusion, and to put the former in different type from the body of the demonstration. But, in the latter part of the work, this somewhat stiff and mechanical arrangement gives place to the freer and more elegant forms with which the student will need to be familiar in his subsequent reading.

10. In the preparation of the work the author has availed himself of the suggestions of a large number of the best practical teachers in all parts of onr country. His chief advisers have been Professor Benjamin F. Clarke, of Brown University, R. I., and Professor H. N. Chute, of the Ann Arbor High School, Mich. To Professor Clarke he is indebted for valuable suggestions on the whole of Chapter II., and especially on triedrals. Indeed, whatever merit there may be in the general method of treatment of triedrals, is due more to him than to the writer. His ability as a mathematician, and his knowledge of what is

practical in methods of presentation, gained by long experience in teaching the subject, appear on well-nigh every page of the latter part of the work. Professor Chute, the able and accomplished teacher of geometry in the Ann Arbor High School, has given me the free use of his careful and scholarly thought, and long and successful experience as a teacher, by several readings of the proofs, and by the use of the advance sheets of the entire work in his classes. His logical acumen, practical skill, and generous contribution of whatever he has found most valuable in matter or method, have been of the highest service. The same general acknowledgments are due to other authors as were made in the earlier edition. To the taste and skill of the stereotypers, and the lavish expenditure of patience and money of the Publishers, the author is indebted for the elegant and beautiful dress in which the book appears.

EDWARD OLNEY.

UNIVERSITY OF MICHIGAN,
ANN ARBOR, *September 1, 1883.*

N.B.—Part III. of the old edition will still be published for use in such schools as wish to push the study of geometry still further than it is carried in the ordinary treatises, and especially into the methods of what is called the *Modern Geometry.* The topics embraced in that part are Exercises in Geometrical Invention, including advanced theorems in Special or Elementary Geometry, Problems in the same, and Applications of Algebra to Geometry; and also an Introduction to Modern Geometry, including the elements of the subjects of Loci, Symmetry, Maxima and Minima, Isoperimetry, Transversals, Harmonic Proportion, Pencils and Ratio, Poles and Polars, Radical Axes and Centres of Similitude in respect to Circles.

The author's Trigonometry can also be had, bound separately or in connection with the other parts of the Geometry, the same as formerly.

E. O.

SUGGESTIONS TO TEACHERS.

1. Fix firmly in mind the fundamental definitions of the science, *in exact language,* and illustrate them so fully that the terms cannot be used in the hearing of the pupil, or by him, without bringing before his mind, *without conscious effort,* the geometrical conception.

2. By numerous and varied applications of the fundamental principles of plane geometry to the most familiar and homely things in common life, divest the pupil's mind of the impression that he is studying "higher mathematics" (as he is not), and beget in him the habit of seeing the applications and illustrations of these principles everywhere about him.

3. By means of much experience in the elements of geometrical drawing, train the taste to enjoy, the eye to perceive, and the hand to execute, geometrical forms, and by so doing fix indelibly in the mind the "working facts" of geometry.

4. Have all definitions, theorems, corollaries, &c., memorized with perfect exactitude, and repeated till they can be given without effort. Demonstrations should not be memorized by the pupil; and considerable latitude may be allowed in the use of language, provided the argument is brought out clearly. But errors in grammar, and inelegancies in style, should be carefully guarded against. One of the chief benefits to be derived from class-room drill in mathematics is the ability to think clearly and logically, and to express the thought in concise, perspicuous, and elegant language.

5. The teacher should never give a theorem or corollary in proper form, but by some such half-questions as the following, suggest the topic:

The relation between the hypotenuse and the sides of a right-angled triangle?

The relative position of two circles when the distance between the centres is less than the sum and greater than the difference of the radii?

The sum of the angles of a triangle?

The relation between the angles and the sides of a triangle? etc.

In this manner the teacher should always designate the proposition *without stating it.* The statement is one of the most important things for the pupil to learn, and have at perfect command, and hence should not be given him by the teacher.

6. The construction of the figure is a necessary part of the demonstration, and no assistance should be given the pupil, nor aids allowed.

7. All figures in plane geometry should, upon first going over the subject, be constructed by the pupils with strict accuracy, on correct geometrical principles, using ruler and string; and this should be persisted in until it can be done with ease. *In reviews,* free-hand drawing of figures may be allowed, and is even desirable.

8. The ordinary notation by letters should be used.

9. All the exercises in the book should be worked with care in the study, and in the class, and be carefully explained by the pupil; and as many additional, impromptu exercises as may be found necessary in order to render the pupil familiar with the practical import of the propositions.

10. Little, if any, original demonstration of theorems not in the book should be required of the pupil upon first going over plane geometry. In review, more or less of such work may be required.

11. Great pains should be taken that original demonstrations be given in good, workmanlike *form.* For this purpose, they should be written out with care by the pupil. Indeed, it is an excellent occasional exercise, to have demonstrations written out in full in class.

12. In review, much attention should be given to synopses of demonstrations. They are the main reliance for fixing in memory the line of argument by which a proposition is demonstrated.

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INTRODUCTION.

PRELIMINARY NOTIONS AND DEFINITIONS.

SECTION I.

GENERAL DEFINITIONS.*

1. A **Proposition** is a statement of something to be considered or done.

ILLUSTRATION.—Thus, the common statement, "Life is short," is a proposition; so, also, we make, or state a proposition, when we say, "Let us seek earnestly after truth."—"The product of the divisor and quotient, plus the remainder, equals the dividend," and the requirement, "To reduce a fraction to its lowest terms," are examples of Arithmetical propositions.

2. Propositions are distinguished as *Axioms*, *Theorems*, *Lemmas*, *Corollaries*, *Postulates*, and *Problems*.

---

* The terms here defined are such as are used in the science in consequence of its logical character, hence they are sometimes called logico-mathematical terms. The science of the Pure Mathematics may be considered as a department of practical logic.

3. An **Axiom** is a proposition which states a principle that is so elementary, and so evidently true as to require no proof.

ILLUSTRATION.—Thus, "A part of a thing is less than the whole of it," "Equimultiples of equals are equal," are examples of axioms. If any one does not admit the truth of axioms, when he understands the terms used, we say that his mind is not sound, and that we cannot reason with him.

4. A **Theorem** is a proposition which states a real or supposed fact, whose truth or falsity we are to determine by reasoning.

ILLUSTRATION.—"If the same quantity be added to both numerator and denominator of a proper fraction, the value of the fraction will be increased," is a *Theorem.* It is a statement the truth or falsity of which we are to determine by a course of reasoning.

5. A **Demonstration** is the course of reasoning by means of which the truth or falsity of a theorem is made to appear. The term is also applied to a logical statement of the reasons for the processes of a rule.

A solution tells *how* a thing is done: a demonstration tells *why* it is so done. A demonstration is often called *proof.*

6. A **Lemma** is a theorem demonstrated for the purpose of using it in the demonstration of another theorem.

ILLUSTRATION.—Thus, in order to demonstrate the rule for finding the greatest common divisor of two or more numbers, it may be best first to prove that "A divisor of two numbers is a divisor of their sum, and also of their difference." This theorem, when proved for such a purpose, is called a *Lemma.*

The term *Lemma* is not much used, and is not very important, since most theorems, once proved, become in turn auxiliary to the proof of others, and hence might be called lemmas.

7. A **Corollary** is a subordinate theorem which is suggested, or the truth of which is made evident, in the course of the demonstration of a more general theorem, or which is a direct inference from a proposition, or a definition.

ILLUSTRATION.—Thus, by the discussion of the ordinary process of performing subtraction in Arithmetic, the following *Corollary* might be suggested: "Subtraction may also be performed by addition, as we can readily observe what number must be added to the subtrahend to produce the minuend."

8. A **Postulate** is a proposition which states that something can be done, and which is so evidently true as to require no process of reasoning to show that it is possible to be done. We may or may not know how to perform the operation.

ILLUSTRATION.—Quantities of the same kind can be added together.

9. A **Problem** is a proposition to do some specified thing, and is stated with reference to developing the method of doing it.

ILLUSTRATION.—A problem is often stated as an incomplete sentence, as, "To reduce fractions to forms having a common denominator."—This incomplete statement means that "We propose to show how to reduce fractions to forms having a common denominator." Again, the problem "To construct a square," means that "We propose to draw a figure which is called a square, and to tell how it is done."

10. A **Rule** is a formal statement of the method of solving a general problem, and is designed for practical application in solving special examples of the same class.

11. A **Solution** is the process of performing a problem or an example.

A solution should usually be accompanied by a demonstration of the process.

12. A **Scholium** is a remark made at the close of a discussion, and designed to call attention to some particular feature or features of it.

ILLUSTRATION.—Thus, after having discussed the subject of multiplication and division in Arithmetic, the remark that "Division is the converse of multiplication," is a scholium.

13. An **Hypothesis** is a supposition made in the statement of a proposition, or in the course of a demonstration.

The **Data** are the things given or granted in a proposition. The **Conclusion** is the thing to be proved.

The *data* of a proposition and the *hypotheses* are the same thing.

SECTION II.

THE GEOMETRICAL CONCEPTS.*

POINTS.

14. A **Point** is a place without size. Points are designated by letters.

Illustration.—If we wish to designate any particular point (place) on the paper, we put a letter by it, and sometimes a dot in it. Thus, in Fig. 1, the ends of the line, which are points, are designated as "point A," "point D;" or, simply, as A and D. The points marked in the line are designated as "point B," "point C," or as B and C. F and E are two points above the line.

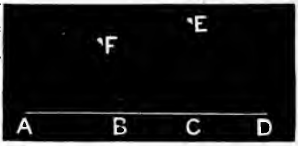


Fig. 1.

LINES.

15. A **Line** is the path of a point in motion.

Lines are represented upon paper by marks made with a pen or pencil, the point of the pen or pencil representing the moving point.

A line is designated by naming the letters written at its extremities, or somewhere upon it.

Illustration.—In each case in Fig. 2, conceive a point to start from A and move along the path indicated by the mark to B. The path thus traced is a line. *Since a point has no size, a line has no breadth*, though

* A concept is a thing thought about;—a thought-object. Thus, in Arithmetic, number is the concept; in Botany, plants; in Geometry, as will appear in this section, points, lines, surfaces, and solids. These may also be said to constitute the *subject-matter* of the science.

the marks by which we represent lines have some breadth. The first and third lines in the figure are each designated as "the line **AB**." The second line is considered as traced by a point starting from **A** and coming

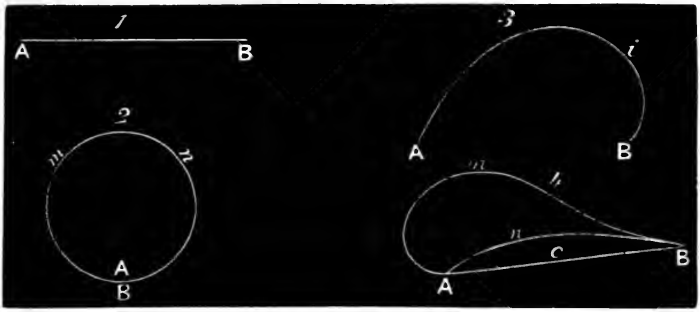


Fig. 2.

around to **A** again, so that **B** and **A** coincide. This line may be designated as the line **A**mn**A**, or **A**mn**B**. In the fourth case, there are three lines represented, which are designated, respectively, as **A**m**B**, **A**n**B**, and **A**c**B**; or, the last, as **AB**.

16. Lines are of **Two Kinds,** *Straight* and *Curved.* A straight line is also called a *Right Line.* A curved line is often called simply a *Curve.*

17. A **Straight Line** is a line traced by a point which moves constantly in the same direction. (See **46**, *a*.)

The word "line" used alone generally signifies a straight line.

18. A **Curved Line** is a line traced by a point which constantly changes its direction of motion.

Illustration.—Thus, in (1), Fig. 2, if the line **AB** is conceived as traced by a point moving from **A** to **B**, it is evident that this point moves in the same direction throughout its course; hence **AB** is a straight line. If a body, as a stone, is let fall, it moves constantly toward the centre of the earth; hence its path represents a straight line. If a weight is suspended by a string, the string represents a straight line.

Considering the line represented by **A**i**B**, (3), Fig. 2, as the path of a point moving from **A** to **B**, we see that the direction of motion is constantly changing.

Sometimes a path like that represented in Fig. 3 is called, though improperly, a *Broken Line*. It is not *a* line at all; that is, not *one* line: it is a combination of straight lines.

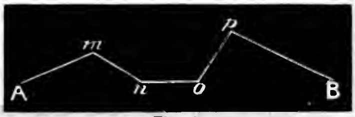


Fig. 3.

SURFACES.

19. A **Surface** is the path of a line in motion.

20. Surfaces are of **Two Kinds,** *Plane* and *Curved.*

21. A **Plane Surface,** or simply a *Plane,* is a surface such that a straight line passing through any two of its points lies wholly in the surface. Such a surface may always be conceived as the path of a straight line in motion.

ILLUSTRATION.—Let **AB**, Fig. 4, be supposed to move to the right, so that its extremities **A** and **B** move at the same rate and in the same direction, **A** tracing the line **AD**, and **B** the line **BC**. The path of the line, the figure **ABCD**, is a surface. This page is a surface, and may be conceived as the path of a line sliding like a ruler from top to bottom of it, or from one side to the other. Such a path will have length and breadth, being in the latter respect unlike a line, which has only length.

Fig. 4.

22. A **Curved Surface** is a surface in which, if various lines are drawn through any point, some or all of them will be curved.

ILLUSTRATION.—Suppose a fine wire bent into the form of the curve **A***m***B**, Fig. 5, and its ends **A** and **B** stuck into a rod **XY**. Now, taking the rod **XY** in the fingers and rolling it, it is evident that the path of the line represented by the wire **A***m***B** will be the surface of a ball (sphere).

Again, suppose the rod **XY** placed on the surface of this paper so that the wire **A***m***B** shall stand straight up from the paper, just as it would if we could take hold of the curve at *m* and raise it right up, letting **XY** lie as it does in the figure. Now slide the rod straight up or down the page, making both ends move at the same rate. The path of

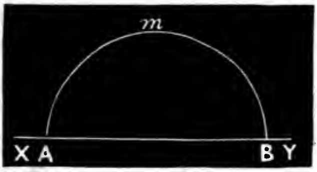


Fig. 5.

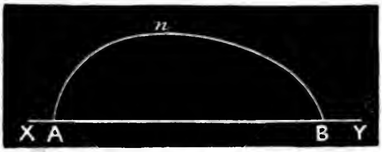


Fig. 6.

Fig. 7.

A*m***B** will be like the surface of a half-round rod (a semi-cylinder). Thus we see how surfaces, plane and curved, may be conceived as the paths of lines in motion.

Ex. 1. If the curve **A***n***B**, Fig. 6, be conceived as revolved about the line **XY**, the surface of what object will its path be like?

Ex. 2. If the figure **OMNP**, Fig. 7, be conceived as revolved about **OP**, what kind of a path will **MN** trace? What kind of paths will **PN** and **OM** trace?

Ans. One path will be like the surface of a joint of stove-pipe, *i. e.*, a cylindrical surface; and one will be like a flat wheel, *i. e.*, a circle.

Ex. 3. If you fasten one end of a cord at a point in the ceiling and hang a ball on the other end, and then make the ball swing around in a circle, what kind of a surface will the string describe?

Ex. 4. If on the surface of a stove-pipe, you were to draw various lines through the same point, might any of them be straight? Could *all* of them be straight? What kind of a surface is this, therefore?

Ex. 5. Can you draw a straight line on the surface of a ball? On the surface of an egg? What kind of surfaces are these?

Ex. 6. When the carpenter wishes to make the surface of a board perfectly flat, he takes a ruler whose edge is a straight line, and lays this straight edge on the surface in all directions, watching closely to see if it touches at all points in all positions. Which of our definitions is he illustrating by his practice?

Ex. 7. How can you conceive a straight line to move so that it shall not generate a surface?

OF THE CIRCLE.

23. A **Circle** is a plane surface bounded by a curved line all points in which are equally distant from a point within.

24. The **Circumference** of a circle is the curved line all points in which are equally distant from a point within.

25. The **Centre** of a circle is the point within, which is equally distant from all points in the circumference.

26. An **Arc** is a part of a circumference.

27. A **Radius** is a straight line drawn from the centre to any point in the circumference of a circle.

By reason of **(24)** all radii of the same circle are equal.

28. A **Diameter** of a circle is a straight line passing through the centre and limited by the circumference.

A diameter is equal to the sum of two radii; hence, all diameters of the same circle are equal.

Illustration.—A circle may be conceived as the path of a line, like **OB**, Fig. 8, one end of which, **O**, remains at the same point, while the other end, **B**, moves around it in the plane of the paper. **OB** is the *radius*, and the path described by the point **B** is the *circumference*. **AB** is a *diameter*. In Fig. 9, the curved line **ABCDA** is the *circumference*, **O** is the *centre*, and the surface within the circumference is the *circle*. Any part of

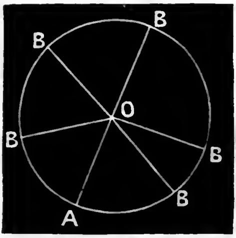


Fig. 8.

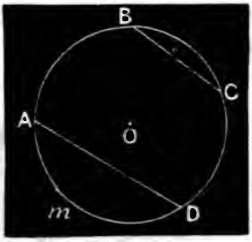


Fig. 9.

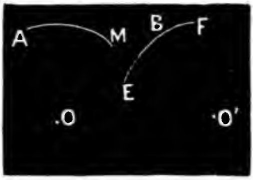


Fig. 10.

a circumference, as **AB**, or any one of the curved lines **BB**, Fig. 8, is an *arc*. So also **AM** and **EF**, Fig. 10, are arcs. **EF** is an arc drawn from **O′** as a centre, with the radius **O′B**.

29. A **Chord** is a straight line joining any two points in a circumference, as **BC** or **AD**, Fig. 9. The portion of the circle included between the chord and its arc, as **A*m*D**, is a **Segment.**

30. A **Tangent** to a circle is a straight line which touches the circumference, but does not intersect it, how far soever the line be produced.

Two circles which touch each other in but one point are said to be tangent to each other. A straight-line tangent is called a **Rectilinear Tangent.**

31. A **Secant** is a straight line which intersects the circumference.

ANGLES.

32. A **Plane Angle,** or simply an **Angle,** is the opening between two lines which meet each other.

The point in which the lines meet is called the **Vertex,** and the lines are called the **Sides.**

An angle is designated by placing a letter at its vertex, and one by each of its sides. In reading, we name the letter at the vertex when there is but one vertex at the point, and the three letters when there are two or more vertices at the same point. In the latter case, the letter at the vertex is put between the other two.

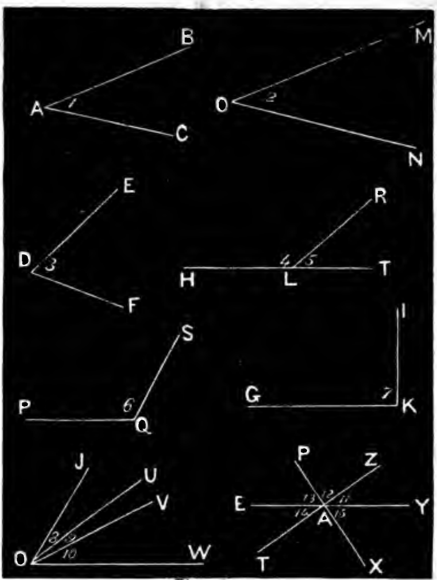


Fig. 11.

Illustration.—In common language an angle is called a *corner*. The opening between the two lines **AB** and **AC**, in which the figure 1 stands, is called the angle **A**; or, if we choose, we may call it the angle **BAC**. At **L** there are two vertices, so that were we to say the angle **L**, one would not know whether we meant the angle (corner) in which 4 stands, or that in which 5 stands. To avoid this ambiguity, we say the angle **HLR** for the former, and **RLT** for the latter. The angle **ZAY** is the corner in which 11 stands; that is, the opening between the two lines **AY** and **AZ**. In designating an angle by three letters, it is immaterial which letter stands first, so that the one at the vertex is put between the other two. Thus, **PQS** and **SQP** are both designations of the angle in which 6 stands. An angle is also frequently designated by putting a letter or figure in it and near the vertex.

33. The **Size of an Angle** depends upon the rapidity with which its sides separate, and not upon their length.

Illustration.—The angles **BAC** and **MON**, Fig. 11, are equal, since the sides separate at the same rate, although the sides of the latter are more prolonged than those of the former. The sides **DF** and **DE** separate faster than **AB** and **AC**, hence the angle **EDF** is greater than the angle **BAC**.

34. **Adjacent Angles** are angles so situated as to have a common vertex and one common side lying between them.

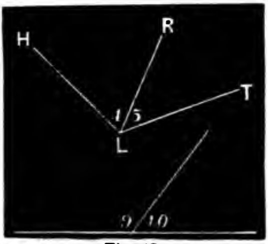


Fig. 12.

ILLUSTRATION.—In Fig. 12, angles 4 and 5 are *adjacent*, since they have the common vertex **L**, and the common side **LR**. Angles 9 and 10 are also adjacent.

35. Angles are distinguished as *Right Angles* and *Oblique Angles.* Oblique angles are either *Acute* or *Obtuse.*

36. A **Right Angle** is an angle included between two straight lines which meet each other in such a manner as to make the adjacent angles equal.

37. An **Acute Angle** is an angle which is less than a right angle, *i. e.*, one whose sides separate less rapidly than those of a right angle.

38. An **Obtuse Angle** is an angle which is greater than a right angle, *i. e.*, one whose sides separate more rapidly than those of a right angle.

39. A **Straight Angle** is an angle whose sides extend in opposite directions, and hence form one and the same straight line.

ILLUSTRATIONS.—In common language, a right angle is called a *square corner*, and an acute angle a *sharp corner*.

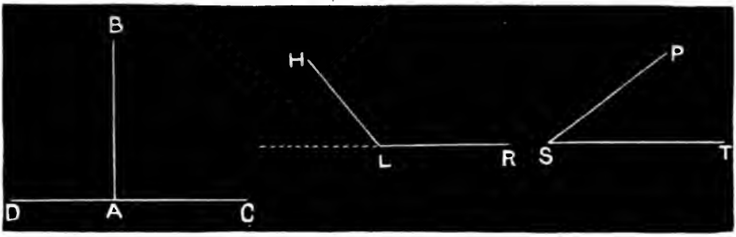


Fig. 13.

Angles **BAD** and **BAC**, Fig. 13, are *right angles*, **PST** is an *acute angle*, and **HLR** is an *obtuse angle*.

If **HL** were turned to the left until it fell in the dotted line, the angle **HLR** would increase, and when **HL** fell in the dotted line, the angle would become what is called a *straight angle*.

40. The **Sum of Two Angles** is the angle included between their non-coincident sides, when the two angles are so placed as to be adjacent angles, and their sides lie in the same plane.

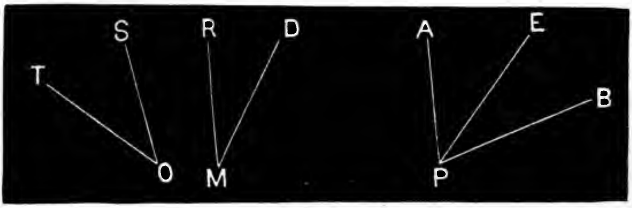


Fig. 14.

Illustration.—Let **O** and **M** be any two angles. Make **EPB** = **M**, and **APE** = **O**, thus placing the two angles **O** and **M** so that they become adjacent angles (**34**). Then is **APB** the sum of **O** and **M**, and we write,

$$O + M = APB, \quad \text{or} \quad APE + EPB = APB.$$

That is, the sum of the angles **O** and **M**, or **APE** and **EPB**, is **APB**.

41. The **Difference between Two Angles** is the angle included by their non-coincident sides, when the angles are so placed as to have a common vertex and side, the second side of the less angle lying *between* the sides of the greater.

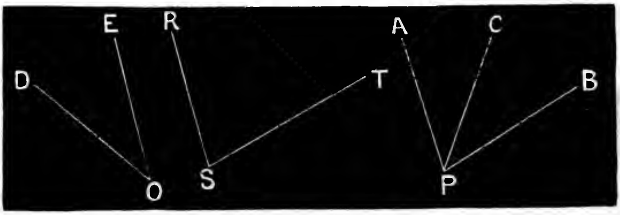


Fig. 15.

Illustration.—To find the difference between the two angles **O** and **S**, we place the vertices **O** and **S** at a common point, as at **P**, making **APB** = **RST**, and **APC** = **DOE**. Then is **CPB** the difference between **RST** and **DOE**; that is,

$$RST - DOE = CPB.$$

$$\text{So also} \quad APB - APC = CPB,$$

$$\text{and} \quad APB - CPB = APC.$$

42. Corollary 1.—(*a*) *The sum of two right angles,*

(**b**) Or, *the sum of the two adjacent angles formed by one straight line meeting another,*

(**c**) Or, *the sum of all the consecutive angles included by several lines lying on the same side of a given line and meeting it in a common point, is a straight angle.*

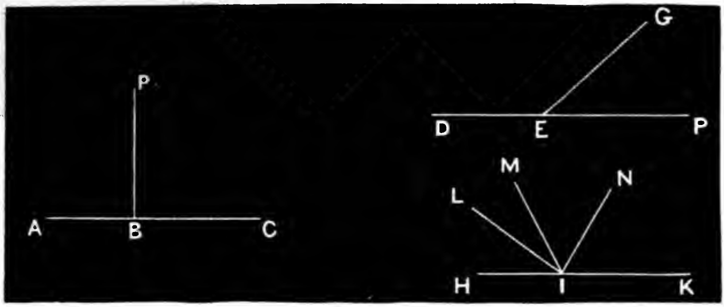
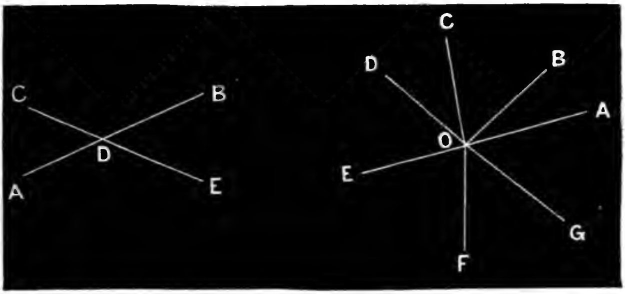


Fig. 16.

Thus, **ABP** + **PBC**, or **DEG** + **GEP**, or **HIL** + **LIM** + **MIN** + **NIK**, is a straight angle.



43. Corollary 2.—*The sum of the four angles formed by two intersecting lines, or the sum of all the consecutive angles formed by any number of lines meeting in a common point is two straight angles, or four right angles.*

Thus, the sum of the four angles **ADC**, **CDB**, **BDE**, and **EDA** is four right angles, as also is the sum of **AOB**, **BOC**, **COD**, **DOE**, **EOF**, **FOG**, and **GOA**.

44. A **Solid** is a limited portion of space.

ILLUSTRATION.—Suppose you have a block of wood like that represented in Fig. 17. Hold it still in your fingers a moment, and fix your mind upon it. Now take the block away and think of the space (place) where it was. This *space* is an example of what we call a *Solid* in Geometry. In fact, the solids of Geometry are not solids at all, in the common sense of the word solid; they are only *places of certain shapes.*

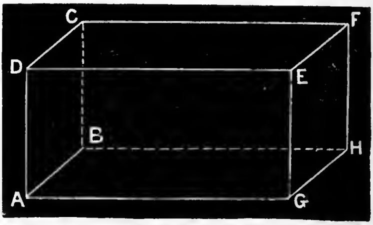


Fig. 17.

Again, hold your ball still a moment in your fingers, then let it drop, and think of the place it filled when you had it in your fingers. It is this *place*, shaped just like your ball, that we think about and talk about as a *solid* in Geometry.

GENERATION OF LINES, SURFACES, AND ANGLES.

45. When one geometrical concept is conceived to move so that its path is some other concept, the former is said to *generate* the latter, and the latter is called the *locus* of the former.

The **Locus of a Point** is the line (either straight or curved) generated by the motion of the point according to some given law.

In the same manner, a surface is conceived as the locus of a line moving in some determinate manner.

46. A **Line** is generated by a moving point **(15–18)**. Hence, the locus of a point is a line.

(*a*) The same straight line may be conceived as generated by a point moving in either of two opposite directions, or part of it may be conceived as generated by a point moving in one direction, and part by a point moving in the opposite direction. Thus, **FA**, Fig. 18, may be conceived as generated by a point moving from **F** to **A**, or from **A** to **F**; or the part **OA** may be conceived as generated by a point moving from **O** to **A**, and the part **OF** by a point moving in the opposite direction, *i. e.*, from **O** to **F**.

47. A **Surface** is generated by a moving line (**19–22**). Hence, the locus of a line is a surface.

48. An **Angle** is generated by the revolution of a straight line about one of its extremities, the line lying all the time in the same plane.

ILLUSTRATION.—The angle **BOA**, Fig. 18, may be considered as generated by the revolution of the line **BO** from the position **AO** to its present position. The angle **COB** may be considered as generated by the revolution of **CO** from the position **BO** to its present position, etc.

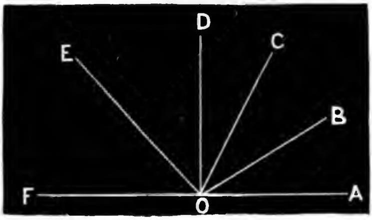


Fig. 18.

49. A **Right Angle** is generated by one-fourth of an entire revolution, an **Acute Angle** by less than one-fourth of an entire revolution, and an **Obtuse Angle** by more than one-fourth. A **Straight Angle** is generated by one-half of a revolution.

50. A **Solid** may be conceived as generated by the motion of a plane, and hence may be defined as the path of a plane in motion.

ILLUSTRATION.—Thus the solid, Fig. 17, may be conceived as generated by the movement of the plane **ABCD** from its present position to the position **GHFE**.

51. A **Sphere** may be conceived as generated by the revolution of a semi-circle about its diameter. (See illustration at the bottom of page 18.)

QUERIES.

1. If the surface **OMNP**, Fig. 19, is conceived as revolved around **OP**, what is the path through which it moves?

CAUTION.—The student should distinguish between the *surface* generated by the *line* **MN**, and the *solid* generated by the *surface* **OMNP**.

Fig. 19.

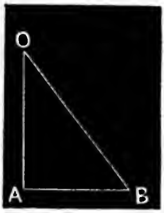


Fig. 20.

2. If the surface represented by **CAB**, Fig. 20, is conceived as revolved about its side **CA**, what kind of a solid is its path?

3. As you fill a vessel with water, what is the solid traced by the surface of the water?

Ans. The same as the space within the vessel.

4. If a circle is conceived as lying horizontally, and then moved directly up, what will be the solid described, *i. e.*, its path? Do not confound the surface described with the solid. What describes the surface? What the solid?

EXTENSION AND FORM.

52. **Extension** means a stretching, or reaching out. Hence, a *Point* has no extension. It has only position (place).

A *Line* stretches or reaches out, but only in length, as it has no width. Hence, a line is said to have *One Dimension*, viz., length.

A *Surface* extends not only in length, but also in breadth; and hence has *Two Dimensions*, viz., length and breadth.

A *Solid* has *Three Dimensions,* viz., length, breadth, and thickness.

Illustration.—Suppose we think of a point as capable of stretching out (extending) in one direction. It would become a line. Now suppose the line to stretch out (extend) in another direction—to widen. It would become a surface. Finally, suppose the surface capable of thickening, that is, extending in another direction. It would become a solid.

53. The *Limits* (extremities) of a line are points.
The *Limits* (boundaries) of a surface are lines.
The *Limits* (boundaries) of a solid are surfaces.

54. Magnitude (size) is the result of extension. Lines, surfaces, and solids are the geometrical magnitudes. A point is not a magnitude, since it has no size. The magnitude of a line is its length; of a surface, its area; of a solid, its volume.

55. Figure or **Form** (shape) is the result of position of points. The form of a line (as straight or curved) depends upon the relative position of the points in the line. The form of a surface (as plane or curved) depends upon the relative position of the points in it. The form of a solid depends upon the relative position of the points in its surface.

QUERIES.

1. Suppose a line to begin to contract in length, and continue the operation till it can contract no longer, what does it become? That is, what is the minor limit of a line?

2. If a surface contracts in one dimension, as width, till it reaches its limit, what does it become?

3. If a solid contracts to its limit in one dimension, what does it pass into? If in two dimensions? If in three dimensions?

4. What kind of a surface is that, every point in which is equally distant from a given point?

56. Geometry is that science which treats of *magnitude* and *form* as the result of extension and position.

The **Geometrical Concepts** are points, lines, surfaces (including plane and spherical angles), and solids (including solid angles).*

The **Object** of the science is the measurement and comparison of these concepts.

Plane Geometry treats of figures all of whose parts are confined to one plane. *Solid Geometry*, called also *Geometry of Space*, and *Geometry of Three Dimensions*, treats of figures whose parts lie in different planes. The division of this treatise into two chapters is founded upon this distinction.

SECTION III.

AXIOMS AND POSTULATES.

57. There are very many axioms; but, as they are truths which the mind grants on the mere statement, it is not needful to enumerate them all. We give a few of the more important, with some illustrative remarks.

58. All demonstration is based upon definitions, axioms, or previously demonstrated propositions.

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* A plane angle may be conceived as a portion of a plane, and hence as itself a surface, and thus capable of increase or diminution like the other magnitudes. The angle thus considered becomes a sort of infinity determined relatively by the rate of separation of the lines. It is thus analogous to an infinite series the law of which is determined by a few of its first terms. See definitions 32, 33, and 48, with their illustrations.

59. Axiom I.—*A straight line is the shortest line between two points.*

Illustration.—If a cord is *stretched* across the table, it marks a straight line. In this way the carpenter marks a straight line. Having rubbed a cord, called a chalk-line, with chalk, he *stretches it tightly* from one point to another on the surface upon which he wishes to mark the line, and then raising the middle of the cord, lets it snap upon the surface. So the gardener makes the edges of his paths straight by *stretching* a cord along them. These operations depend upon the principle that when the line between the points is the shortest possible, it is straight.

---

60. Axiom II.—*Two points in a straight line determine its position.*

Illustration.—If the farmer wants a straight fence built, he sets two stakes to mark its ends. From these its entire course becomes known. This is the principle upon which aligning (or sighting) depends. Two points in the required line being given, by looking from one in the direction of the other, we look along a straight line, and are thus able to locate other points in the line. If the points **A** and **B** are marked, by putting the eye at **A** and looking steadily towards **B**, we can tell whether **D** and **E** are in the same straight line with **A** and **B**, or not. So we can observe that **C′** and **C″** are *not* in the line; but that **C** is. This process of discovering other points in a line with two given points is called aligning, or sighting. In this way a row of trees is made straight, or a line of stakes set. It is the principle upon which the surveyor runs his lines, and the hunter aims his gun. In the latter case, the two sights are the given points, and the mark, or game, is a third point, which the marksman wishes to have in the same straight line as the sights.

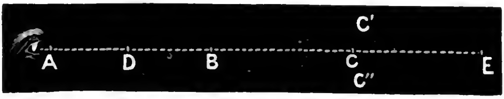

Fig. 21.

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61. Axiom III.—*Between the same two points there is one straight line, and only one.*

Illustration.—Let any two letters on this page represent the situation of two points; we readily see that there is one, and only one, straight path between them. Again, let a corner of the desk represent one point and a corner of the ceiling of the room represent another point; we perceive at once that, if a point is conceived to pass in a straight line from one to the other, it will always trace the same path. In short, as soon as two points are mentioned, we think of the distance between them as a single straight line,—for example, the centre of the earth and the centre of the sun.

Once more, conceive **A** and **B**, Fig. 21, to be two points in the path of a point moving from **A** in the direction of **B**. Now *all* the points in the same direction from **A** that **B** is, are in this path; and any point out of this line, as **C′** or **C″**, is in a different direction from **A**.

In this manner we draw a straight line on paper by laying the straight edge of a ruler on two points through which we wish the line to pass, and passing a pen or pencil along this edge.

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62. **To Intersect** is to cross; and a crossing is called an *Intersection.*

63. Corollary.—*Two straight lines can intersect in but one point;* for, if they had two points common, they would coincide and not intersect.

Ex. 1. A railroad is to be run from the town **A** to town **B**. If it is made straight, through what points will it pass? Can it pass through any points not in the same direction from **A** that **B** is?

Ex. 2. If I live on the south side of a straight railroad, and my friend on the north side, but five miles farther east and two miles farther north, and the road from my house to his is straight, how many times does it cross the railroad?

Ex. 3. Can you always draw a straight line which shall cut a curve (whatever curve it may be) in two points at least? Try it.

---

64. **Axiom IV.**—*The whole is greater than any of its parts.*

65. **Axiom V.**—*The whole is equal to the sum of all its parts.*

66. **Axiom VI.**—*Things which are equal to the same thing are equal to each other.*

67. **Axiom VII.**—*If equals be increased or diminished equally, the results will be equal.*

68. **Axiom VIII.**—*If unequals be increased or diminished equally, the greater will give the greater result, i. e., the inequality will exist in the same sense.*

POSTULATES.

69. Postulates, like axioms, are very numerous, and it would be useless to attempt to enumerate them all. We give a few simply as specimens.

70. **Postulate I.**—*A line can be produced to any length.*

71. **Postulate II.**—*From any point a straight line can be drawn to any other point.*

72. **Postulate III.**—*Geometrical magnitudes can be added, subtracted, multiplied, or divided.*

73. **Postulate IV.**—*A geometrical figure can be conceived as moved at pleasure, without changing its size or the relation of its parts (shape).*

74. **Postulate V.**—*Any number of lines can be drawn making equal angles with a given line.*

75. **Postulate VI.**—*With any point as a centre, a circumference can be drawn with any radius.*

SECTION IV.

MEASUREMENT OF RIGHT LINES.

76. The **Measure** of a line is another line which is contained in it an exact number of times.

77. A **Common Measure** of two or more lines is a line which measures each of them.

78. **Commensurable Lines** are lines which have a finite common measure.

79. The **Sum of Two Lines** is the line formed by uniting them so that one shall be the prolongation of the other.

80. The **Difference between Two Lines** is the line which remains after the length of the less has been taken from the greater.

81. **Problem.**—*To measure a straight line with the dividers and scale.*

Solution.—Let **AB**, Fig. 22, be the line to be measured. Take the dividers, Fig. 2 (frontispiece), and placing the sharp point A firmly upon the end **A** of the line **AB**, open the dividers till the other point B (the pencil point) just reaches the other end of the line, **B**. Then letting

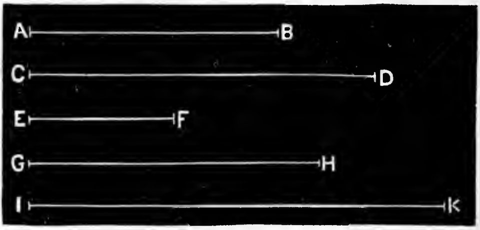

Fig. 22.

the dividers remain open just this amount, place the point A on the lower end of the left-hand scale, as at *o*, Fig. 1 (frontispiece), and notice where the point **B** reaches. In this case, it reaches 3 spaces beyond the figure 1. Now, as this scale is inches and *tenths* of inches,* the line **AB** is 1.3 inches long.

Ex. 1. What is the length of **CD**? *Ans.* .15 of a foot.
Ex. 2. What is the length of **EF**? *Ans.* .75 of an inch.
Ex. 3. What is the length of **GH**? *Ans.* $1\frac{1}{2}$ inches.
Ex. 4. What is the length of **IK**? *Ans.* .18 of a foot.
Ex. 5. Draw a line 3 inches long.
Ex. 6. Draw a line 2.15 inches long.
Ex. 7. Draw a line 1.25 inches long.
Ex. 8. Draw a line .85 of an inch long.

[NOTE.—Suppose a fine elastic cord were attached by each of its ends to the points A and B of the dividers; when they were opened so as to reach from **C** to **D**, Fig. 22, the cord would represent the line **CD**. Now applying the dividers to the scale is the same as laying this cord on the scale. Without the cord, we can imagine the distance between the points of the dividers to be a line of the same length as **CD**.]

Ex. 9. Find in the same way as above the length and width of this page. Also the distance from one corner (angle) to the opposite one (the diagonal).

---

82. Problem.—*To find the sum of two lines.*

SOLUTION.—To find the sum of **AB** and **CD**, I† first draw the indefinite line **EX**. With the dividers I obtain the length of **AB**, by placing one point on **A** and extending the other to **B**. This length I now lay off on the indefinite line **EX**, by putting one point of the dividers at **E** and with the other marking the point **F**. **EF** is thus made equal to **AB**. In the same man-

Fig. 23.

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* The next scale to the right is divided into 10ths and 100ths of a foot. Thus, from *p* to 10 is 1 tenth of a foot, and the smaller divisions are hundredths.

† These elementary solutions are sometimes put in the singular, as the more simple style.

ner, taking the length of **CD** with the dividers, I lay it off from **F** on the line **FX**. Thus I obtain

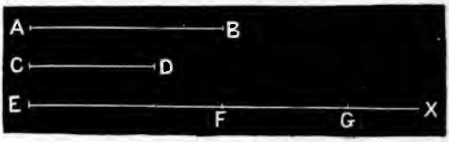

Fig. 23.

$$\begin{aligned} \mathbf{EG} &= \mathbf{EF} + \mathbf{FG} \\ &= \mathbf{AB} + \mathbf{CD}. \end{aligned}$$

Hence, the sum of **AB** and **CD** is **EG**.

Ex. 1. Find the sum of **AB** and **EF**, Fig. 22.

Ex. 2. Find the sum of **EF**, **CD**, and **GH**, Fig. 22.

Ex. 3. Make a line twice as long as **CD**, Fig. 22. Three times as long.

---

83. Problem.—*To find the difference of two lines.*

Solution.—To find the difference of **AB** and **CD**, I take the length of the less line **AB** with the dividers; and placing one point of the dividers at one extremity of **CD**, as **C**, make **CE** = **AB**. Then is **ED** the difference of **AB** and **CD**, since

Fig. 24.

$$\mathbf{ED} = \mathbf{CD} - \mathbf{CE} = \mathbf{CD} - \mathbf{AB}.$$

Ex. 1. Find the difference of **IK** and **EF**, Fig. 22.

Ex. 2. Find the difference of **GH** and **CD**, Fig. 22.

Ex. 3. Find how much longer **IK**, Fig. 22, is than the sum of **EF**, Fig. 22, and **CD**, Fig. 23.

Ex. 4. Find the difference of the sum of **AB** and **GH**, and the sum of **CD** and **EF**, Fig. 22.

---

84. Problem.—*To find the ratio of two commensurable lines.*

Solution.—Let **AB** and **CD** (Fig. 25) be the two lines whose ratio we seek.

Apply the shorter (**AB**) to the longer (**CD**) as many times as the latter will contain the former. If **AB** is contained an integral number of times (say 3, or m) in **CD**, then **AB** is a common measure of **AB** and **CD**, and we have $\frac{\mathbf{AB}}{\mathbf{CD}} = \frac{1}{3}$, or m.

But if the shorter is not contained in the longer an integral number of times, apply it as many times as it is contained, and note the remainder; thus, AB is contained in CD *once*, with a remainder aD.

Now apply this remainder, aD, to AB as many times as AB will contain it, which, in this case, is *once* with a remainder bB.

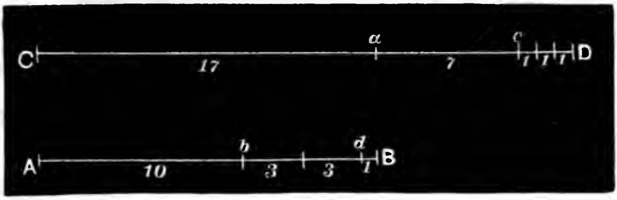

Fig. 25.

Again, apply this remainder, bB, to aD, the former remainder. In this case, it is contained *once* with a remainder cD.

Again, apply cD to bB. It is contained *twice*, with a remainder dB.

Finally, applying dB to cD, we find it contained 3 times, without any remainder.

Hence, dB is the common measure of AB and CD.

Calling dB the unit of measure, 1, we have,

$$\begin{aligned}
d\mathrm{B} &= 1;\\
c\mathrm{D} &= 3d\mathrm{B} = 3;\\
bd &= 2c\mathrm{D} = 6;\\
ac &= b\mathrm{B} = bd + d\mathrm{B} = 7;\\
a\mathrm{D} &= ac + c\mathrm{D} = 10;\\
\mathrm{AB} &= \mathrm{A}b + b\mathrm{B} = a\mathrm{D} + ac = 17;\\
\mathrm{CD} &= \mathrm{C}a + a\mathrm{D} = \mathrm{AB} + a\mathrm{D} = 27.
\end{aligned}$$

Hence the lines AB and CD are to each other as the numbers 17 and 27; AB is $\frac{17}{27}$ of CD; or, expressed in the form of a proportion,

$$\frac{\mathrm{AB}}{\mathrm{CD}} = \frac{17^*}{27}.$$

[Note.—This process will be seen to be the same as that developed in Arithmetic and Algebra for finding the greatest or highest Common Measure of two numbers. See Practical Arithmetic, p. 362, and Complete Algebra, (137).]

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* This method will not always obtain the exact ratio, both because of the imperfection of the measurement, and because some lines are incommensurable by any finite unit, as will appear hereafter.

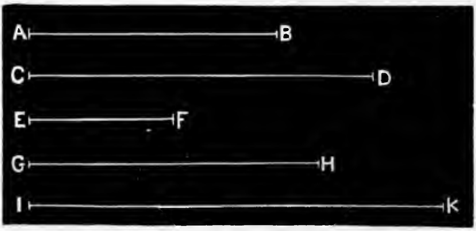

Fig. 26.

Ex. 1. Find, as above, the approximate ratio of **AB** to **CD**.

Ratio, $\frac{13}{18}$.

Ex. 2. Find, as above, the approximate ratio of **CD** and **IK**.

Ratio, $\frac{5}{6}$.

Ex. 3. Find, as above, the approximate ratio of **EF** to **GH**.

Ratio, $\frac{1}{2}$.

Ex. 4. Find, as above, the approximate ratio of **EF** to **CD**.

Ratio, $\frac{5}{12}$.

CONTINUOUS VARIATION.

85. A magnitude is said to vary **continuously** when in passing from one value to another it passes through all intermediate values.

Illustration.—Let the line **EF**, Fig. 26, be produced by placing a pencil at **F** and tracing the line to the right, until it becomes equal to **IK**. **EF** has thus been made to be successively of all intermediate lengths between its present length and the length of **IK**; *i. e.*, it has varied *continuously*.

In like manner, an angle may be conceived to vary continuously from one magnitude to another. Thus, in Fig. 27, the angle **CPB** may be made greater or less by revolving **CP** about **P**. By such a revolution of **CP** the angle **CPB** may be conceived to vary, or grow, *continuously* till it becomes **C′PB**.

CHAPTER I.

PLANE GEOMETRY.

SECTION I.

OF PERPENDICULAR STRAIGHT LINES.

86. A **Perpendicular** to a given line is a line which makes a right angle (**36**) with the given line.

87. An **Oblique** line is a line which makes an oblique angle with a given line.

PROPOSITION I.

88. Theorem.—*At any point in a straight line, one perpendicular can be erected to the line, and only one, which shall lie on the same side of the line.*

DEMONSTRATION.

Let AB represent any line, and P be any point therein.

We are to prove that, on the same side of **AB**, there can be one, and only one, perpendicular erected to **AB** at **P**.

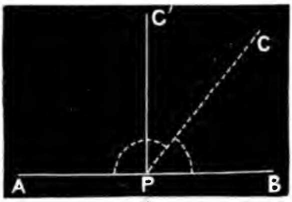

Fig. 27.

From **P** draw any oblique line, as **PC**, forming with **AB** the two angles **CPB** and **CPA**.

Now, while the extremity **P**, of **PC**, remains at **P**, conceive the line **PC** to revolve so as to increase the less of the two angles, as **CPB**, continuously. Since the sum of **CPB** and **CPA** remains constant, **CPA** will diminish continuously.

Hence, for a certain position of **CP**, as **C′P**, these angles will become equal. In this position, the line is perpendicular to **AB** (**36**, **86**). Therefore, there can be one perpendicular, **C′P**, erected to **AB** at **P**.

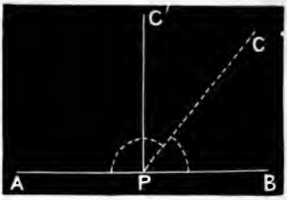

Fig. 27.

Again, if the line **C′P** revolve from the position in which the angles are equal, one angle will increase and the other diminish; hence there is *only one* position of the line on this side of **AB** in which the adjacent angles are equal.

Therefore there can be only one perpendicular erected to **AB** at **P**, which shall lie on the same side of **AB**. Q. E. D.

89. Corollary 1.—*On the other side of the line a second perpendicular, and only one, can be erected from the same point in the line.*

90. Corollary 2.—*If one straight line meets another so as to make the angle on one side of it a right angle, the angle on the other side is also a right angle.*

PROPOSITION II.

91. Theorem.—*If two straight lines intersect so as to make one of the four angles formed a right angle, the other three are right angles, and the lines are mutually perpendicular to each other.*

Demonstration.

Let CD intersect AB, making CEB a right angle.

We are to prove that **CEA**, **AED**, and **DEB** are also right angles, and that **CD** is perpendicular to **AB**, and **AB** to **CD**.

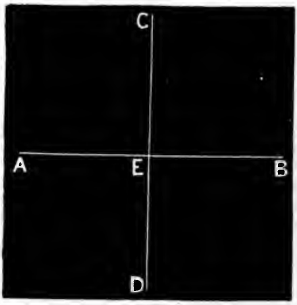

Fig. 28.

By (**90**), since **CEB** is a right angle, **CEA** is also a right angle.

In like manner, as **BE** meets **CD**, making **CEB** a right angle, **BED** is a right angle, by (**90**).

Again, since **DEB** is right, **DE** meets **AB**, making one angle right; hence the other, **AED**, is right also (**90**). Q. E. D.

Finally, since **CD** meets **AB**, making **AEC** a right angle, **CD** is perpendicular to **AB** (86); and, since **AB** meets **CD**, making **AEC** a right angle, **AB** is perpendicular to **CD**. Q. E. D.

PROPOSITION III.

92. Theorem.—*When two straight lines intersect at right angles, if the portion of the plane of the lines on one side of either line be conceived to revolve on that line as an axis until it coincides with the portion of the plane on the other side, the parts of the second line will coincide.**

DEMONSTRATION.

Let the two lines AB and CD intersect at right angles at E; and let the portion of the plane of the lines on the side of CD on which B lies be conceived to revolve on the line CD as an axis, until it falls in the portion of the plane on the other side of CD.†

We are to prove that **EB** will fall in and coincide with **EA.**

The point **E** being in **CD**, does not change position in the revolution; and, as **EB** remains perpendicular to **CD**, it must coincide with **EA** after the revolution, or there would be two perpendiculars to **CD** on the same side and from the same point, **E**, which is impossible (88).

Hence, **EB** coincides with **EA**. Q. E. D.

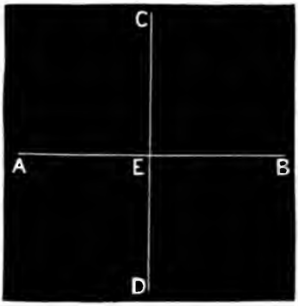

Fig. 29.

PROPOSITION IV.

93. Theorem.—*From any point without a straight line, one perpendicular can be let fall upon that line, and only one.*

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* This has nothing to do with the *lengths* of **EB** and **EA**; indeed, lines are generally supposed indefinite in length, unless limited by the data.

† This revolution may be illustrated by conceiving the paper folded in the line **CD** until **EB** is brought into **EA**.

DEMONSTRATION.

Let AB be any line and P any point without the line.

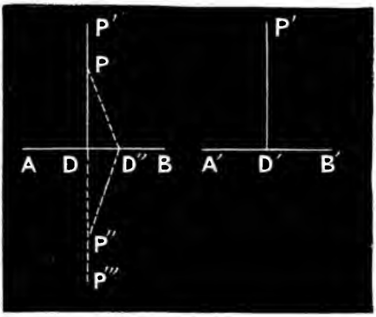

Fig. 30.

We are to prove that one perpendicular, and only one, can be let fall from **P** upon **AB**.

Let **A′B′** be an auxiliary line; and at any point in it, as **D′**, let a perpendicular **P′D′** be erected (**88**).

Now place **A′B′**, bearing **P′D′** with it, in **AB**, and move it to the right or left till **P′D′** passes through **P**, and when in this position let **D** be the point in **AB** in which **D′** falls.

Connect **P** and **D**.

Then, since angle **PDB** coincides with the right angle **P′D′B′**, **PDB** is a right angle, and **PD** is a perpendicular from **P** to the line **AB** (**86**). Q. E. D.

We are now to prove that **PD** is the only perpendicular from **P** to the line **AB**.

Suppose that there can be another, and let it be **PD″**.

Produce **PD** to **P‴**, and take **DP″**=**DP**, and draw **P″D″**.

Now let the portion of the plane above **AB** be revolved upon **AB** as an axis until it falls in the plane on the opposite side of **AB** from its first position. Then will **DP′** fall in **DP‴** (**92**), and since **DP″** is by construction equal to **DP**, **P** will fall in **P″**.

Then, since **PDB** is a right angle **BDP″** is also a right angle, and **PP″** is a straight line (**42**, *a*).

For a like reason **PD″ P″** is a straight line, and we have two straight lines from **P** to **P″**, which is impossible.

Hence there can be but one perpendicular, as **PD**, from **P** upon **AB**. Q. E. D.

PROPOSITION V.

94. Theorem.—*From a point without a straight line, the perpendicular is the shortest distance to the line.*

DEMONSTRATION.

Let AB be any straight line, P any point without it, PD a perpendicular, and PC any oblique line.

We are to prove that **PD** is shorter than any oblique line, as **PC**, 1st. Since the shortest distance from **P** to *any* point in the line **AB** is a straight line (**59**), we are to examine only straight lines.

2d. Produce **PD**, making **DP′** = **PD**, and draw **P′C**.

Now let the portion of the plane of the lines above **AB** be revolved upon **AB** as an axis until it coincides with the portion below **AB**.

Since **PP′** and **AB** intersect at right angles, **PD** will fall in **DP′** (**92**); and, since **PD**=**DP′**, **P** will fall in **P′**, and **PC**=**P′C**, since they coincide when applied.

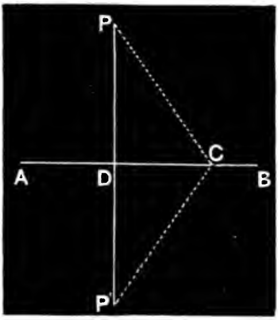

Fig. 31.

Finally, **PP′** being a straight line, is shorter than **PCP′** which is a broken line, since a straight line is the shortest distance between two points (**59**).

Now **PD**, the half of **PP′**, is less than **PC**, the half of the broken line **PCP**.

Therefore, the perpendicular, **PD**, is the shortest distance from **P** to the line **AB**. Q. E. D.

95. The **Distance** between two points is the straight line which joins them, and the **Distance** from a point to a line is the perpendicular from the point to the line.

PROPOSITION VI.

96. Theorem.—*If a perpendicular is erected at the middle point of a straight line,*

1st. *Any point in the perpendicular is equally distant from the extremities of the line.*

2d. *Any point without the perpendicular is nearer the extremity of the line on its own side of the perpendicular.*

Demonstration.

Let PD be a perpendicular to AB at its middle point, D, O any point in this perpendicular, and O′ any point without the perpendicular.

Draw **OA**, **OB**, **O′A**, and **O′B**.

We are to prove, 1st, that **OA**=**OB**; and 2d, that **O′B** < **O′A**.

1st. Revolve **ODB** on **PD** as an axis,

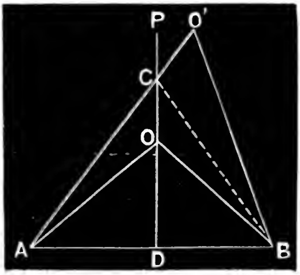

Fig. 32.

till **B** falls in the plane on the opposite side of **PD**.

Then, since **PD** is perpendicular to **AB**, **DB** will fall in **DA** (**92**). And since **DB** = **DA** by hypothesis, **B** will fall in **A**, and **OB** will coincide with **OA** (**61**).

Hence **OA** = **OB**. Q. E. D.

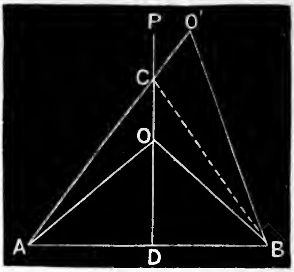

Fig. 32.

2d. **O′** being on the opposite side of **PD** from **A**, **O′A** will cut **PD** at some point, as **C**.

Draw **CB**.

Now, since **C** is a point in the perpendicular, **CA** = **CB** by the former part of the demonstration.

And, since **O′B** is a straight line and **O′C** + **CB** is a broken line,

$$O'B < O'C + CB \ (59).$$

Whence, substituting **CA** for its equal **CB**, we have

$$O'B < O'C + CA,$$

or $$O'B < O'A. \quad \text{Q. E. D.}$$

97. Corollary.—Conversely, *The locus of a point equidistant from the extremities of a given line is a perpendicular to that line at its middle point,* since any point in such perpendicular is equidistant from the extremities of the line, and any point not in the perpendicular is unequally distant from the extremities.

PROPOSITION VII.

98. Theorem.—*If each of two points in one line is equally distant from the extremities of another line, the former line is perpendicular to the latter at its middle point.*

Demonstration.

Every point equally distant from the extremities of a straight line lies in a perpendicular to that line at its middle point, by (**97**). But two points determine the position of a straight line. Hence, two points, each equally distant from the extremities of a straight line, determine the position of the perpendicular at the middle point of the line. Q. E. D.

PROPOSITION VIII.

99. Problem.—*To erect a perpendicular to a given line at a given point in the line.*

SOLUTION.

Let XY be the given line, and A the given point.

We are to erect a perpendicular to **XY**, at **A**.

From **A** lay off on each side equal distances, as **AC** = **AB**.

From **C** and **B** as centres, with a radius sufficiently great to cause the arcs to intersect at some point without **XY**, describe arcs intersecting at **O**.

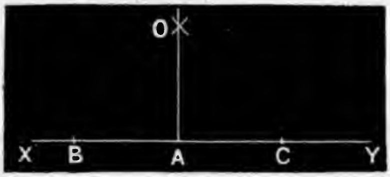

Fig. 33.

Pass a line through **O** and **A**, and it will be the perpendicular sought.

DEMONSTRATION OF SOLUTION.

Since **OA** has two points, **O** and **A**, each equally distant from **B** and **C**, **OA** is a perpendicular to **BC** at **A**, its middle point (98).

But **BC** coincides with **XY**; hence **OA** is perpendicular to **XY** at **A**.

100. DEFINITION.—**To Bisect** anything is to divide it into two equal parts.

PROPOSITION IX.

101. Problem.—*To bisect a given line.*

SOLUTION.

Let AB be the given line.

We are to bisect it, that is, to divide it into two equal parts.

From the extremities **A** and **B** as centres, with any radius sufficiently great to cause the arcs to intersect without the line **AB**, describe arcs intersecting in two points, as *m* and *n*.

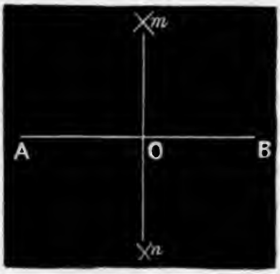

Fig. 34.

Pass a line through m and n, intersecting **AB** at **O**.

Then is **O** the middle point of **AB**, and **AO** = **OB**.

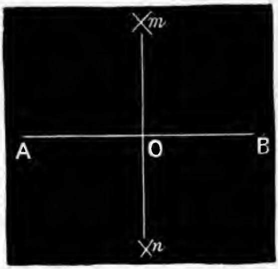

Fig. 34.

DEMONSTRATION OF SOLUTION.

Since the line mn has two points, m and n, each equally distant from **A** and **B**, it is perpendicular to **AB** at its middle point (**98**).

PROPOSITION X.

102. Problem.—*From a point without a given line, to let fall a perpendicular upon the line.*

SOLUTION.

Let XY be the given line, and O the point without the line.

We are to let fall a perpendicular from **O** to **XY**.

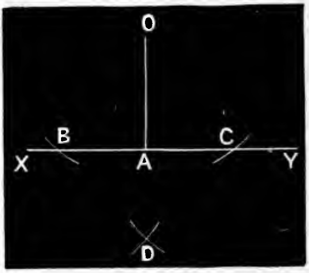

Fig. 35.

From **O** as a centre, with a radius sufficiently great to cause the arcs to intersect, describe an arc cutting **XY** in two points, as **B** and **C**.

From **B** and **C** as centres, with a radius sufficiently great to cause the arcs to intersect without **XY**, describe arcs intersecting at some point, as **D**.

Pass a line through **O** and **D**, meeting **XY** in **A**. Then is **OA** the perpendicular sought.

DEMONSTRATION OF SOLUTION.

OA being produced through **D** has two points, **O** and **D**, each equally distant from **B** and **C**, and hence is perpendicular to **BC**, which coincides with **XY**. Hence, **OA** passes through **O** and is perpendicular to **XY**.

QUERIES.

103. 1. In the solution of Proposition IX, is it necessary that the arcs which intersect at n should be struck with the same radius as those which intersect at m? Is it necessary that the two intersections be on different sides of **AB** ?

2. In the solution of Proposition X, is it necessary that the intersection **D** should fall on the opposite side of **XY** from **O**? Why is it necessary to take the radius with which these arcs are struck greater than half of **BC**?

EXERCISES.

104. 1. A mason wishes to build a wall from **O** (Fig. 36), in the wall **AB**, "straight across" (perpendicular) to the wall **CD**, which is 8 feet from **AB**. He has only his 10-foot pole, which is subdivided into feet and inches, with which to find the point in the opposite wall at which the cross wall must join. How shall he find it? What principle is involved?

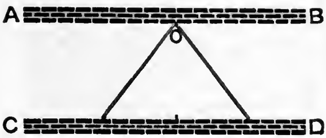

Fig. 36.

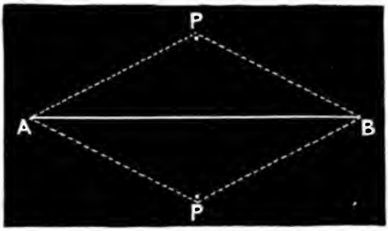

Fig. 37.

2. Wishing to erect a line perpendicular to **AB** (Fig. 37) at its centre, I take a cord or chain somewhat longer than **AB**, and fastening its ends at **A** and **B**, take hold of the middle of the cord or chain and carry it as far from **AB** as I can, first on one side and then on the other, sticking pins at the most remote points, as at **P** and **P′**. These points determine the perpendicular sought. What is the principle involved?

3. Bisect a line by making marks on only one side of it.

4. With a measuring-tape as an instrument, how would you erect on the shore a perpendicular to the straight bank of a lake, at a given point in the bank?

SECTION II.

OF OBLIQUE STRAIGHT LINES.

105. The **Supplement** of an angle is the angle which remains after it has been taken from a straight angle, or two right angles.

106. **Supplemental Angles** are, therefore, two angles whose sum is a straight angle, or two right angles (**42**, ***b***).

107. **Vertical,** or **Opposite Angles** are the non-adjacent angles formed by the intersection of two straight lines.

PROPOSITION I.

108. **Theorem.**—*Vertical, or opposite angles are equal.*

DEMONSTRATION.

Let AB and CE intersect at D.

We are to prove that **ADC** = **BDE**, and **CDB** = **ADE**.

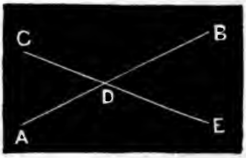

Fig. 38.

ADC + **CDB** = a straight angle (**42**, ***b***); and for the same reason **CDB** + **BDE** = a straight angle.

Hence, **ADC** + **CDB** = **CDB** + **BDE**; and subtracting **CDB** from each member, we have **ADC** = **BDE**.

In like manner, **CDB** + **BDE** = **BDE** + **ADE**; whence, **CDB** = **ADE**.

Q. E. D.

PROPOSITION II.

109. Theorem.—*If two supplemental angles are so placed as to be adjacent to each other, the two sides not common fall in the same straight line.*

DEMONSTRATION.

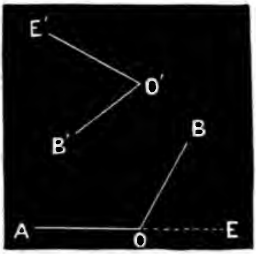

Fig. 39.

Let AOB and B′O′E′ be two supplemental angles, and let B′O′E′ be placed so as to be adjacent to AOB, *i. e.*, as BOE.

We are to prove that **AE** is a straight line.

Before considering **B′O′E′** as placed adjacent to **AOB**, produce **AO** to **E**, forming **AE**.

By (42, *b*), **AOB** + **BOE** = a straight angle, *i. e.*, two right angles, whence **BOE** is the supplement of **AOB**.

Now, as by hypothesis **B′O′E′** is the supplement of **AOB**, **B′O′E′** = **BOE**.

Place **B′O′E′** adjacent to **AOB**, **O′** in **O**, and **O′B′** in **OB**. Then will **O′E′** fall in **OE**.

Therefore the two sides not common, *i. e.*, **AO** and **O′E′**, fall in the same straight line **AE**. Q. E. D.

PROPOSITION III.

110. Theorem.—*If from a point without a line a perpendicular is drawn to the line, and oblique lines are drawn from the same point, meeting the line at equal distances from the foot of the perpendicular,*

1st. *The oblique lines are equal to each other.*

2d. *The angles which the oblique lines form with the perpendicular are equal to each other.*

3d. *The angles formed by the oblique lines with the first line are equal to each other.*

DEMONSTRATION.

Let AB (Fig. 40) be any line, P any point without it, PD a perpendicular, and PC and PE oblique lines meeting AB at C and E, so that CD = DE.

We are to prove, 1st, that **PC** = **PE**; 2d, that **CPD** = **DPE**; and 3d, that **PCD** = **PED**.

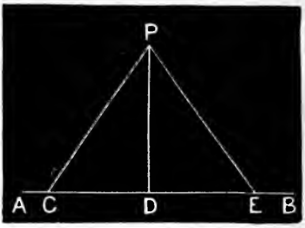

Fig. 40

Revolve **PDE** on **PD** as an axis, until **E** falls in the plane on the other side of **PD**.

Now, since **AB** is perpendicular to **PD**, **DB** will fall in **DA** (**92**). And since **DE** = **DC** by hypothesis, **E** will fall in **C**. Hence the two figures **PDE** and **PDC** coincide, and we have, 1st, **PC** = **PE**; 2d, **CPD** = **DPE**; and 3d, **PCD** = **PED**. Q. E. D.

QUERY.—How would the equality of **PC** and **PE** follow from (**96**)?

PROPOSITION IV.

111. Theorem.—*If from a point without a line a perpendicular is drawn to the line, and from the same point two oblique lines are drawn, making equal angles with the perpendicular and meeting the first line,*

1st. *The oblique lines are equal to each other.*

2d. *The oblique lines cut off equal distances from the foot of the perpendicular.*

3d. *The oblique lines make equal angles with the first line.**

DEMONSTRATION.

Let AB be a straight line, P any point without it, and PD a perpendicular to AB; and let PE and PC be drawn, making CPD = EPD.

We are to prove, 1st, that **PC** = **PE**; 2d, that **DE** = **DC**; and 3d, that **PED** = **PCD**.

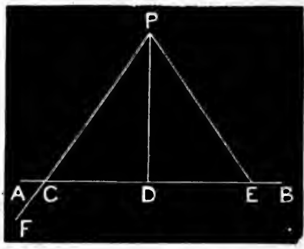

Fig. 41.

Revolve **PDE** upon **PD** as an axis, until **E** falls in the plane on the opposite side of **PD**.

Then, since **EPD** = **CPD** by hypothesis, **PE** will fall in **PF**, and the point **E** will be found somewhere in **PF**.

---

* This proposition is the *converse* of the last. The significance of this statement will be more fully developed farther on (**128**).

Again, **DE** will fall in **DA** (92), and **E** will fall somewhere in **DA**.

Now as **E** falls at the same time in **DA** and **PF**, it must fall at their intersection **C**, and the figures **PDE** and **PDC** must coincide; whence we have,

1st, **PC** = **PE**; 2d, **DE** = **DC**; and 3d, **PCD** = **PED**. Q. E. D.

PROPOSITION V.

112. **Theorem.**—*If from a point without a line a perpendicular is drawn to the line, and from the same point two oblique lines are drawn making equal angles with the first line,*

1st. *The oblique lines cut off equal distances from the foot of the perpendicular.*

2d. *The oblique lines are equal to each other.*

3d. *The oblique lines make equal angles with the perpendicular.*

Demonstration.

Let P be any point without the line AB, and PD a perpendicular from P upon AB, and let PE and PC be drawn making the angle DEP = angle DCP.

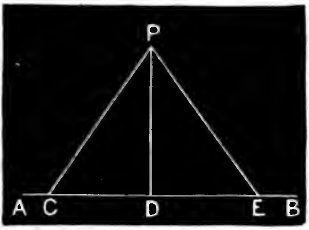

Fig. 42.

We are to prove, 1st, that **DE** = **DC**; 2d, that **PE** = **PC**; and 3d, that angle **DPE** = angle **DPC**.

Conceive a perpendicular erected at the middle point of **CE**, and let it intersect **CP**, or **CP** produced, in some point as **X**. Conceive **X** joined with **E**.

By (110, 3d.) **XED**=**XCD**, (*i. e.*, **PCD**). But by hypothesis **PED** = **PCD**. Hence, **XE** falls in **PE**, and **PD** is the perpendicular to **CE** at its middle point.

Therefore, **DE** = **DC**; and by (110) **PE** = **PC**, and **DPE** = **DPC**. Q. E. D.

PROPOSITION VI.

113. Theorem.—*If from a point without a line a perpendicular is let fall on the line, and from the same point two oblique lines are drawn, the oblique line which cuts off the greater distance from the foot of the perpendicular is the greater.*

Demonstration.

Let AB be any straight line, P any point without it, and PC and PF two oblique lines of which PF cuts off the greater distance from the foot of the perpendicular PD; that is, DF > DC.

We are to prove that **PF > PC.**

If the two oblique lines do not lie on the same side of the perpendicular, as in the case of **PF** and **PE**, take **DC = DE**, and on the side in which **PF** lies, draw **PC**. Then **PC** will be equal to **PE**, by (**110**, 1st). Hence, if we show the proposition true when both oblique lines lie on the same side of the perpendicular, it will be true in general.

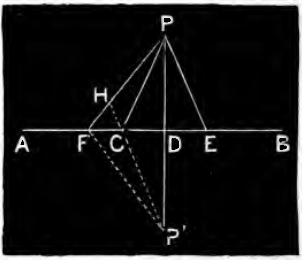

Fig. 43.

Produce **PD**, making **DP′ = PD**, and draw **P′F** and **P′C**, producing the latter until it meets **PF** in **H**.

Revolve the figure **FPD** upon **AB** as an axis, until it falls in the plane on the opposite side of **AB**.

Since **PP′** is perpendicular to **AB**, **PD** will fall in **P′D**; and, since **PD = P′D**, **P** will fall at **P′**. Then **P′C = PC** and **P′F = PF**.

Now the broken line **PCP′** < than the broken line **PHP′**, since the straight line **PC** < the broken line **PHC**.

For a like reason, the broken line **PHP′** < **PFP′**, since **HP′** < **HFP′**.

Hence **PCP′** < **PFP′**, and **PC** (the half of **PCP′**) < **PF** (the half of **PFP′**). Q. E. D.

114. Corollary.—*From a given point without a line, there can be two, and only two, equal oblique lines drawn to the line, and these will lie on opposite sides of the perpendicular drawn from the given point to the given line.*

PROPOSITION VII.

115. Theorem.—*If two equal oblique lines are drawn from the same point in a perpendicular to a given line, they cut off equal distances on that line from the foot of the perpendicular.*

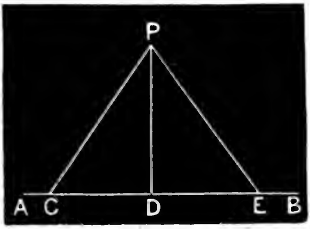

Fig. 44.

Let PD be perpendicular to AB, and PE = PC.

We are to prove that DE = DC.

If DE were greater than DC, PE would be greater than PC, and if DE were less than DC, PE would be less than PC (113); but both of these conclusions are contrary to the hypothesis PE = PC.

Hence, as DE can neither be greater nor less than DC it must be equal to DC. Q. E. D.

EXERCISES.

116. 1. Having an angle given, how can you construct its supplement? Draw on the blackboard any angle, and then construct its supplement. What is the supplement of a right angle?

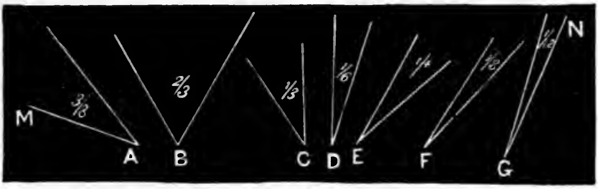

Fig. 45.

2. The several angles in Fig. 45 are such parts of a right angle as are indicated by the fractions placed in them. If these angles are added together by bringing the vertices together and causing the adjacent sides of the angles to coincide, how will the two sides not common lie? Why?

3. If two times A, B (Fig. 45), two times D, three times E, three times C, three times G, and two times F, are added in order, how will AM and GN lie with reference to each other? Why?

Ans. They will coincide.

4. If you place the vertices of any two equal angles together so that two of the sides shall extend in opposite directions and form one and the same straight line, the other two sides lying on opposite sides thereof, how will the latter sides lie? By what principle?

5. If two lines intersect, show that the line which bisects one of the angles will, if produced, bisect the opposite angle.

6. If one line meet another, show that the two lines bisecting these supplemental angles are perpendicular to each other.

7. If two lines intersect, show that two lines bisecting the two pairs of opposite angles are perpendicular to each other.

SECTION III.

OF PARALLELS.

117. The **Direction** of a straight line is defined or determined by the plane in which it lies and the angle which it makes with some fixed line, this angle being generated (**48**) from the fixed line around in the same direction,* in the same argument.

118. The assumed fixed line is called the **Direction Line,** and the angle which the line makes with the direction line is called the **Direction Angle.**

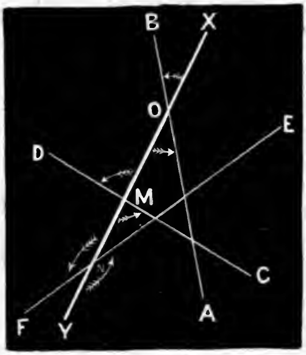

Fig. 46.

ILLUSTRATION.—Thus the directions of the several lines **AB**, **CD**, and **EF** may be defined by referring them to some assumed fixed line, as **XY**.

The direction of **AB** is defined by saying that its direction angle is **YOA**, or its equal **XOB**, this angle being conceived as generated *from* the direction line, as indicated by the arrows.

So also the direction of **CD** is defined by the angle **YMC**, or its equal **XMD**; and the direction of **EF** is in like manner defined by **YNE**, or **XNF**.

119. With reference to its generation, the same line may be conceived as having either of two opposite directions, or various parts of it may be conceived as having opposite directions.

ILLUSTRATION.—Thus, the line **AB** (Fig. 47) may be considered as generated by a point moving from **A** to **B**, whence its direction would

---

* Revolution around a fixed point is often designated as *from left to right*, or *from right to left*. To comprehend these terms, one may conceive himself in the centre of motion, and facing the moving point. Thus all the motions represented by arrows in Fig. 46 will be seen to be *from right to left.*

be from A towards B; or, it may be considered as generated by a point moving from B to A, whence its direction would be from B towards A.

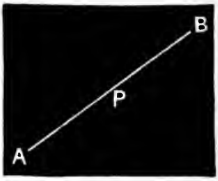

Fig. 47.

In like manner, part of the line, as PB, may be considered as having the direction from P towards B, while the other part is conceived as having the opposite direction, *i. e.*, from P towards A.

120. Lines have the **Same Direction** when they lie in the same plane and make equal direction angles with the same line.

Any line may be assumed at pleasure as the direction line, provided that in comparing the directions of different lines they all be referred to the same direction line.

121. Parallel Lines are lines which have the same or opposite directions.

122. A **Transversal** is a line cutting a system of lines.

123. When two lines are cut by a transversal, the angles formed are named as follows:

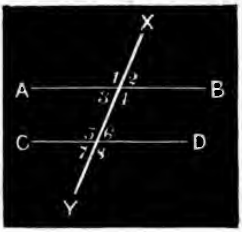

Fig. 48.

Exterior Angles are those without the two lines, as 1, 2, 7, and 8.

Interior Angles are those within the two lines, as 3, 4, 5, and 6.

Alternate Exterior Angles are those without the two lines and on different sides of the transversal, but not adjacent, as 2 and 7, 1 and 8.

Alternate Interior Angles are those within the two lines and on different sides of the transversal, but not adjacent, as 3 and 6, 4 and 5.

Corresponding Angles are one without and one within the two lines, and on the same side of the transversal, but not adjacent, as 2 and 6, 4 and 8, 1 and 5, 3 and 7.

PROPOSITION I.

124. Theorem.—*Through a given point one line can be drawn parallel to a given line, and but one.*

DEMONSTRATION.

Let AB be the given line, and P the given point.

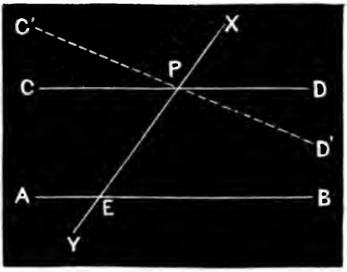

Fig. 49.

We are to prove that one line can be drawn through **P** parallel to **AB**, and but one.

Through **P** draw **XY** as the direction line, intersecting **AB** in **E**.

Also through **P** pass a line **C′D′**, making **XPD′** greater than **XEB**.

Then revolving **C′D′** about **P** as a centre, **XPD′** may be made to diminish continuously, and in some position, as **CD**, **XPD** will equal **XEB**. In this position, **CD** is parallel to **AB** (**120, 121**).

Hence there can be one line drawn through **P** parallel to **AB**. Q. E. D.

Again, there can be but one; since, if **CD** be revolved in either direction about **P**, the angle **XPD** will become unequal to **XEB**, and hence the line **CD** will not be parallel to **AB**. Q. E. D.

PROPOSITION II.

125. Theorem.—*If a transversal cuts two parallels,*

1st. *Any two corresponding angles are equal.*

2d. *Any two alternate interior, or any two alternate exterior angles are equal.*

3d. *The sum of any two interior angles on the same side of the transversal, or the sum of any two exterior angles on the same side, is two right angles, or a straight angle.*

DEMONSTRATION.

Let AB and CD (Fig. 50) be any two parallels, and EF any transversal.

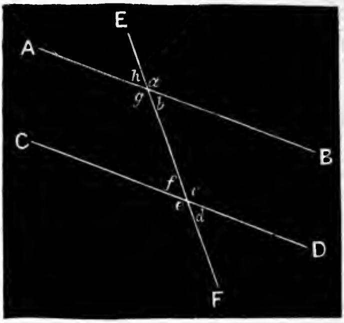

Fig. 50.

We are to prove, 1st. Of the corresponding angles, $b = d$, $a = c$, $e = g$, and $f = h$.

2d. Of the alternate interior angles, $b = f$, and $c = g$; of the alternate exterior angles, $d = h$, and $a = e$.

3d. Of the interior angles on the same side of the transversal,

$b + c = 2$ right angles,

and $g + f = 2$ right angles;

of the exterior angles on the same side,

$a + d = 2$ right angles, and $e + h = 2$ right angles.

Let **EF** be taken as the direction line, the direction angles being estimated from right to left (**120**, **121**, and foot-note, p. 55). Then,

1st. Of the corresponding angles, $b = d$, these being the direction angles, and **AB** and **CD** being parallel.

$a = c$, since they are supplements of the equal angles, b and d; and $e = g$, for the same reason.

Also, $f = h$, since they are opposite angles to the equal angles, b and d.

2d. Of the alternate interior angles, $b = f$, since $f = d$ (**108**); $c = g$, since they are supplements of b and d.

Of the alternate exterior angles, $d = h$, since $h = b$ (**108**); and $e = a$, since they are supplements of b and d.

3d. Of the interior angles on the same side,

$b + c = 2$ right angles (or a straight angle),

since $d + c = 2$ right angles (or a straight angle), (**42**, ***b***), and $b = d$;

and $g + f = 2$ right angles,

since $g + b =$ a straight angle, and $b = f$.

Of the exterior angles on the same side,

$a + d = 2$ right angles,

since $a + b =$ a straight angle, and $b = d$;

also $e + h = 2$ right angles.

since $g + h =$ a straight angle, and $e = g$. Q. E. D.

PROPOSITION III.

126. Theorem.—Conversely to Proposition II, *When two lines are cut by a transversal, the two lines are parallel,*

1st. *If any two corresponding angles are equal.*

2d. *If any two alternate interior, or any two alternate exterior angles are equal.*

3d. *If the sum of any two interior angles on the same side, or the sum of any two exterior angles on the same side is two right angles.*

Demonstration.

Let AB and CD be two lines cut by the transversal EF, making any pair of corresponding angles equal, as $b = d$, $a = c$, $g = e$, $h = f$; or any two alternate interior angles, or any two alternate exterior angles equal, as $b = f$, $g = c$, $a = e$, or $h = d$; or the sum of any two interior angles on the same side, or of any two exterior angles on the same side, equal to 2 right angles, as $b + c$, $g + f$, $a + d$, $h + e$, equal to 2 right angles.

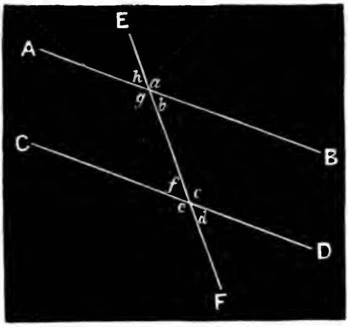

Fig. 51.

We are to prove that **AB** and **CD** are parallel.

Let **EF** be the direction line, and b and d the direction angles. If, then, these are granted or proved equal, the lines are parallel (**121**).

Now, 1st. Of the corresponding angles, if $b = d$, **AB** and **CD** are parallel by definition; but, if $a = c$, $b = d$, since b and d are supplements of a and c; or, if $g = e$, $b = d$, since b and d are supplements of g and e; or, if $h = f$, $b = d$, since $b = h$ and $d = f$ (**108**). Hence, in every case, $b = d$, and **AB** and **CD** are parallel.

2d. Of alternate interior angles, if $b = f$, $b = d$, since $f = d$; or, if $g = c$, $b = d$, since b and d are supplements of g and c. Hence, in either case, $b = d$, and **AB** and **CD** are parallel.

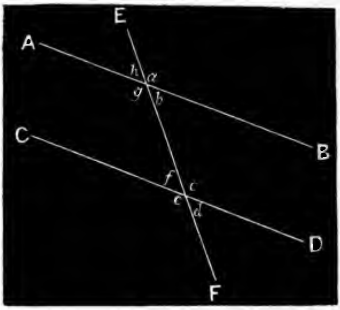

Fig. 51.

Of alternate exterior angles, if $h = d$, $b = d$, since $b = h$ (**108**); or, if $a = e$, $b = d$, since b and d are supplements of a and e. Hence, in either case, $b = d$ and, **AB** and **CD** are parallel.

3d. Of interior angles on the same side, if $b + c = 2$ right angles, $b = d$, since $d + c = 2$ right angles (**42**); or, if $g + f = 2$ right angles, $b = d$, since

$$g + f + b + c = 4 \text{ right angles};$$

hence, $b + c = 2$ right angles, and as $d + c = 2$ right angles, $d = b$. Hence, in either case, $b = d$, and **AB** and **CD** are parallel.

Of exterior angles on the same side, if $a + d = 2$ right angles, $b = d$, since $a + b = 2$ right angles; or, if $h + e = 2$ right angles, $b = d$, since $h + e + a + d = 4$ right angles; hence, $a + d = 2$ right angles, and, as $a + b = 2$ right angles, $d = b$. Hence, in either case, $b = d$, and **AB** and **CD** are parallel. Q. E. D.

127. Corollary.—*Two lines which are perpendicular to a third are parallel to each other.*

For, in such a case, all the eight angles formed are equal; hence, any of the conditions of the proposition are met.

128. Scholium.—The last two propositions are the converse of each other; *i.e.*, the hypotheses and conclusions are exchanged. Thus, in Prop. II, the hypothesis is that the two lines are parallel, and the conclusion is certain relations between the angles; while in Prop. III the hypotheses are certain relations among the angles, and the conclusion is that the lines are parallel.

The learner may think that, if a proposition is true, its converse is necessarily true; and hence, that when a proposition has been proved, its converse may be assumed as also proved. Now this is by no means the case. Although in a great number of mathematical propositions, it happens that the proposition and its converse *are* both true, we never assume one from having proved the other; and we shall occasionally find a proposition whose converse is not true.

PROPOSITION IV.

129. Theorem.—*When two straight lines are cut by a transversal, if the sum of the two interior angles on either side is less than two right angles, the two lines will meet on this side of the transversal, if sufficiently extended.*

DEMONSTRATION.

Let AB and CD be two lines cut by the transversal XY, making BEP + EPD < 2 right angles.

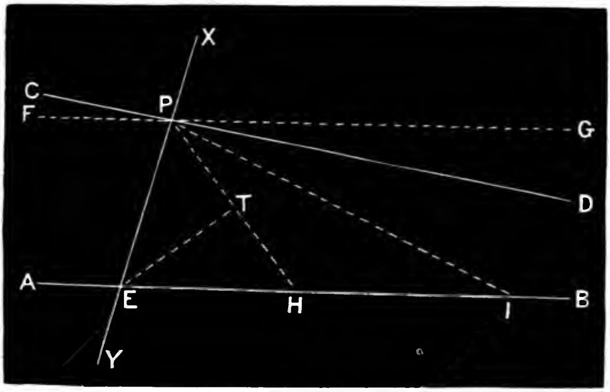

Fig. 52.

We are to prove that **AB** and **CD** will meet on the side of **XY** on which these angles lie.

Through **P** draw **FG** parallel to **AB**.

Take **EH** = **EP** and draw **PH**, and also **ET** perpendicular to **PH**. By (115), **TH** = **TP**, whence **EHT** = **EPT** (110).

But **EHT** = **GPH** (125). Hence **GPH** = $\frac{1}{2}$**GPE**.

Again, take **HI** = **PH** and draw **PI**, and it may be shown in the same manner that **GPI** = $\frac{1}{2}$**GPH** = $\frac{1}{4}$**GPE**.

In this manner we may continue to draw oblique lines through **P** cutting **AB** further and further from **E**, and may thus diminish at pleasure the angle included by the oblique line and **PG**. Hence this angle may be made less than **GPD**, the difference between **DPE** and the supplement of **PEB**, when the oblique line will fall between **PD** and **PG**. Call this line **PR**. Now as **PR** and **PE** cut **AB**, and **PD** lies between them, it must cut **AB** between **E** and **R**. Q. E. D.

130. Corollary 1.—*If a transversal cuts one of two parallels, it cuts the other also.*

131. Corollary 2.—*Non-parallel straight lines meet, if sufficiently produced.*

132. Corollary 3.—*Two straight lines in the same plane which do not meet, however far they are extended, are parallel.*

For, let **AB** and **CD** be two such lines, and **P** any point in **CD**. Now all lines through **P** which are not parallel to **AB** meet **AB** (**131**). Hence as there can be one parallel to **AB** through **P** (**124**), it is the line which does not meet **AB**.

PROPOSITION V.

133. Theorem.—*A line which is perpendicular to one of two parallels is perpendicular to the other also.*

Demonstration.

Let AB and CD be two parallels, and let EF be perpendicular to AB.

We are to prove that **EF** is also perpendicular to **CD**.

Since **EF** is a transversal cutting **AB** and **CD**, angle **EOB** = angle **EMD** (**125**, 1).

Now **EOB** is a right angle by hypothesis (**86**),

Hence **EMD** is a right angle, and **EF** is perpendicular to **CD**.

Q. E. D.

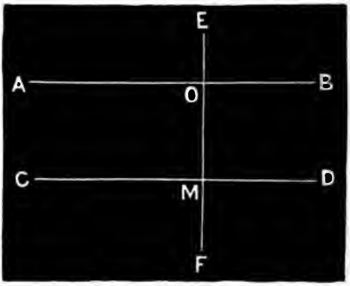

Fig. 53.

134. Corollary.—*The shortest distance between two parallels is the perpendicular which joins them.*

For, **OM** being a perpendicular from **O** to **CD**, is shorter than any other line from **O** to **CD** (**94**).

135. The **Distance** between two parallels is the perpendicular which joins them.

PROPOSITION VI.

136. Theorem.—*Two parallels are everywhere equally distant from each other, and hence never meet.*

Demonstration.

Let E and F be any two points in the line CD, and EG and FH perpendiculars measuring the distances between the parallels CD and AB at these points.

We are to prove **EG** = **FH**.

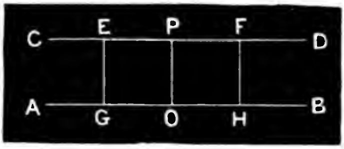

Fig. 54.

Let **P** be the middle point between **E** and **F**, and **PO** a perpendicular at this point.

Revolve the portion of the figure on the right of **PO**, upon **PO** as an axis, until it falls upon the plane of the paper at the left.

Then, since **FPO** and **EPO** are right angles, **PD** will fall in **PC**; and, as **PF** = **PE**, **F** will fall on **E**. As **F** and **E** are right angles, **FH** will take the direction **EG**, and **H** will lie in **EG** or **EG** produced. Also, as **POH** and **POG** are right angles, **OB** will fall in **OA**, and **H**, falling at the same time in **EG** and **OA**, is at their intersection **G**.

Hence, **FH** coincides with and is equal to **EG**. Q. E. D.

Hence, also, **CD** cannot meet **AB**, since the distance from any point in **CD** to **AB** is **EG**. Q. E. D.

PROPOSITION VII.

137. Theorem.—Conversely to Proposition VI., *If two points in one straight line are equally distant from a second straight line, and on the same side of it, the lines are parallel to each other.*

Demonstration.

Let AB and CD (Fig. 55) be two lines having the points P and S in CD equally distant from AB, and on the same side of it.

We are to prove that **CD** and **AB** are parallel.

From **P** and **S** draw **PE** and **SF** perpendicular to **AB**. Then is **PE** = **SF**, by hypothesis.

Through **O**, the middle point of **PE**, draw **GH** parallel to **AB**.

Since **PE** is parallel to **SF**, **GH** cuts **SF** in some point as **I** (130).

By (136), **OE** = **IF**; and since **SF** = **PE** and **OE** is $\frac{1}{2}$**PE**, **IF** is $\frac{1}{2}$**SF**, that is, **IF** = **IS**.

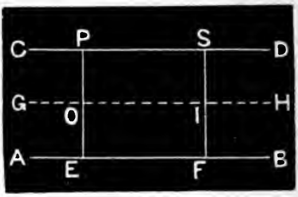

Fig. 55.

Now, as **PE** and **SF** are perpendicular to **GH** (133), if we revolve the figure **OAEFBI** on **GH**, **E** will fall in **P**, and **F** in **S** (92), and **AB** will have two points in common with **CD**, and hence will coincide with it.

Hence, **DPO** = **BEO**, and as the latter is a right angle by construction, **AB** and **CD** are perpendicular to **PE**, and hence parallel (127). Q. E. D.

PROPOSITION VIII.

138. Theorem.—*A pair of parallel transversals intercept equal portions of two parallels.*

DEMONSTRATION.

Let ST and RL be two parallel transversals, cutting the two parallels AB and CD.

We are to prove that **GE** = **HF**.

From **E** and **F** let fall the perpendiculars **EM** and **FK**. Then

EM = **FK** (136).

Now apply the figure **GEM** to **HFK**, placing **EM** in its equal **FK**. Since **M** and **K** are right angles, **MG** will fall in **KH**.

With the figures in this position, **FH** and **EG** are lines drawn from the same point in the perpendicular to **ST** and making equal angles with it (125), and are hence equal (112). Q. E. D.

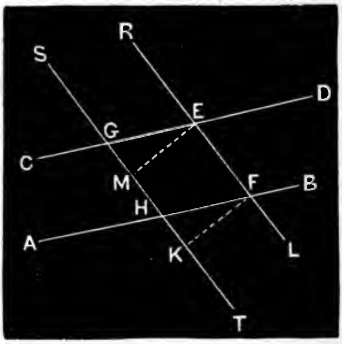

Fig. 56.

PROPOSITION IX.

139. Theorem.—*Two straight lines which are parallel to a third are parallel to each other.*

DEMONSTRATION.

Let AB and CD be each parallel to EF.

We are to prove that **AB** and **CD** are parallel to each other.

Draw **HI** perpendicular to **EF**; then will it be perpendicular to **CD** (**133**).

For a like reason, **HI** is perpendicular to **AB**.

Hence, **CD** and **AB** are both perpendicular to **HI**, and consequently parallel (**127**). Q. E. D.

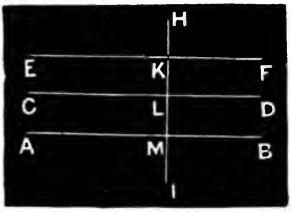

Fig. 57.

PROPOSITION X.

140. Theorem.—*If to each of two parallels perpendiculars are drawn, then are the perpendiculars parallel.*

DEMONSTRATION.

Let A and B be parallel lines, P be perpendicular to A, and Q to B.

We are to prove that **P** and **Q** are parallel to each other.

Q, which is perpendicular to **B**, one of the two parallels, is perpendicular to **A**, the other parallel, also (**133**).

Hence, **P** and **Q** are both perpendicular to **A**, and hence are parallel (**127**). Q. E. D.

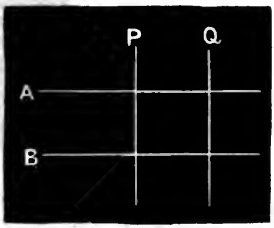

Fig. 58.

PROPOSITION XI.

141. Theorem.—*If to each of two non-parallel lines a perpendicular is drawn, the perpendiculars are non-parallel.*

DEMONSTRATION.

Let A and B be non-parallel, and P a perpendicular to A, and Q to B.

We are to prove that **P** and **Q** are non-parallel.

If **P** and **Q** were parallel, then, by the preceding proposition, **A** and **B** would be parallel, which is contrary to the hypothesis. Hence, **P** and **Q** are non-parallel. Q. E. D.

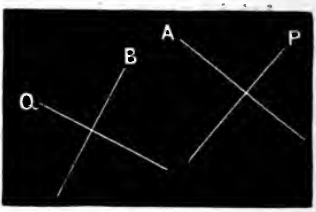

Fig. 59.

PROPOSITION XII.

142. Problem.—*Through a given point to draw a parallel to a given line.*

SOLUTION.

Let AB be the given line, and P the given point.

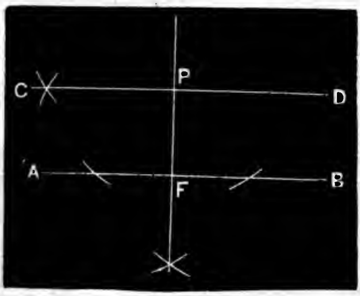

Fig. 60.

We are to draw through **P** a parallel to **AB**.

Let fall **PF**, a perpendicular from **P** to **AB** (102).

At **P** erect **CD**, a perpendicular to **PF** (99).

Then is **CD** parallel to **AB** (127). [Pupil give proof.]

EXERCISES.

143. 1. How can a farmer tell whether the opposite sides of his farm are parallel?

2. If we wish to cross over from one of two parallel roads to the other, is it of any use to travel farther in the hope that the distance across will be less? Why?

3. If a straight line intersects two parallel lines, how many angles are formed? How many angles of the same size? May they *all* be of the same size? When? When will they not be all of the same size?

4. Are the two opposite walls of a building which are carried up by the plumb line exactly parallel? Why?

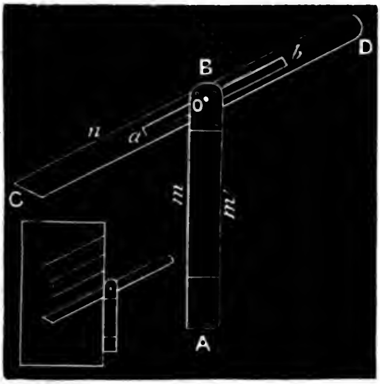

Fig. 61.

5. A *bevel* (Fig. 61) is an instrument much used by carpenters, and consists of a main limb **AB**, in which a tongue **CD** is placed, so as to open and shut like the blade of a knife. This tongue turns on the pivot **O**, which is a screw, and can be tightened so as to hold the tongue firmly at any angle with the limb. The tongue can also be adjusted so as to allow a greater or less portion to extend on a given side, as **CB**, of the limb. Now, suppose the tongue fixed in position, as represented in the figure, and the side m of the limb to be placed against the straight edge of a board, and slid up and down, while lines are drawn along the side n of the tongue. What will be the relative position of these lines? Upon what proposition does their relative position depend? How can the carpenter adjust the bevel to a right angle upon the principle in Prop. I, Sec. I? At what angle is the bevel set, when, drawing two lines from the same point in the edge of the board, one with one edge m of the bevel against the edge of the board, and the other with the other edge m', these lines are at right angles to each other?

SECTION IV.

OF THE RELATIVE POSITIONS OF STRAIGHT LINES AND CIRCUMFERENCES.

PROPOSITION I.

144. Theorem.—*Any diameter divides a circle, and also its circumference, into two equal parts.*

DEMONSTRATION.

Let AB be the diameter of the circle AmBn.

We are to prove that arc **AmB** = arc **AnB**, and that segment **AmB** = segment **AnB**.

Revolve **AnB** upon **AB** as an axis, until it falls in the plane on the opposite side of **AB**.

Then, since every point in **AnB** is at the same distance from the centre **C** as every point of **AmB** (24), the arc **AnB** falls in **AmB**, and both arcs and segments coincide; whence, arc **AnB** = arc **AmB**, and segment **AnB** = segment **AmB**. Q. E. D.

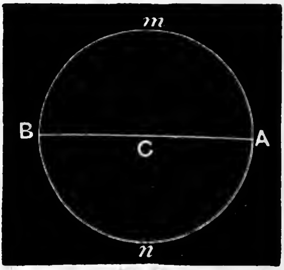

Fig. 62.

PROPOSITION II.

145. Theorem.—*The diameter of a circle is greater than any other chord of the same circle.*

DEMONSTRATION.

Let AB (Fig. 63) be a chord meeting the circumference in A and B, and not passing through the centre O; and let AC be the diameter.

We are to prove that **AB** is less than any diameter, as **AC** (**28**).

Now as **AB** is not a diameter, it does not pass through **O**, or lie in **AC**. Hence **B** is a different point from **C**.

Draw **OB**.

Now **AB** being a straight line, is less than **AO + OB**, which is a broken line (**59**); hence, as **AO + OB = AC**, **AB < AC**. Q. E. D.

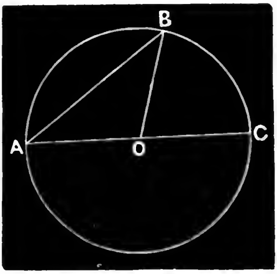

Fig. 63.

146. An arc is said to be **Subtended** by the chord which joins its extremities, and the arc is said to subtend the angle included by the radii drawn to its extremities.

PROPOSITION III.

147. Theorem.—*A radius which is perpendicular to a chord bisects the chord, the subtended arc, and the subtended angle.**

DEMONSTRATION.

Let AB be a chord subtending the arc AB, which arc subtends the angle AOB. Let the radius EO be perpendicular to AB, cutting it in D.

We are to prove that **DA = DB**, arc **AE** ≐ arc **EB**, and angle **AOE** = angle **BOE**.

Produce **EO**, forming the diameter **EC**.

Revolve the semicircle **EBC** on **EC** as an axis, till it falls in the plane on the other side of **EC**.

The semicircles will coincide (**144**), and since **AB** is perpendicular to **EO**, **DB** will fall in **DA**.

Moreover, as **OA = OB**, and there cannot

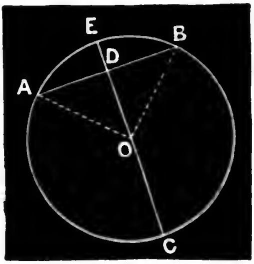

Fig. 64.

* Such statements in Plane Geometry are generally limited to the consideration of arcs less than a semi circumference, yet all the propositions in this section, except Prop. VIII, are equally true whatever the arcs may be.

be two equal oblique lines from a point to a line on the same side of a perpendicular (**114**), **OB** falls in **OA**, and **B** falls in **A**.

Hence, **DB** coincides with **DA**, **EB** with **EA**. and angle **BOE** with angle **AOE**, and we have **DA** = **DB**, arc **AE** = arc **EB**, and angle **AOE** = angle **BOE**. Q. E. D.

PROPOSITION IV.

148. Theorem.—Conversely to Proposition III, *A radius which bisects an arc bisects the chord which subtends the arc, is perpendicular to the chord, and also bisects the subtended angle.*

DEMONSTRATION.

Let arc AB be bisected by the radius OE at E. Let the straight line AB be the chord of this arc, and AOB the subtended angle.

We are to prove that **OE** bisects the chord **AB** and is perpendicular to it, and also bisects the angle **AOB**.

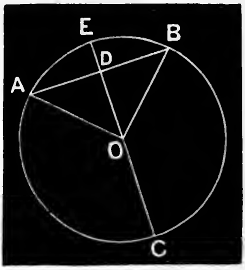

Fig. 64.

Produce **EO**, forming the diameter **EC**.

Revolve the semi-circumference **EBC** upon **EC** as an axis, till it falls in the plane at the left of **EC**.

Then will semi-circumference **EBC** coincide with **EAC**, and since arc **BE** = arc **AE** by hypothesis, **B** will fall in **A**, and **BD** = **AD**.

Hence, the line **OE** has two points, **O** and **D**, each equally distant from **A** and **B**, and is therefore perpendicular to **AB** (**98**).

Furthermore, angle **BOD** coincides with **AOD**, and **BOD** = **AOD**. Q. E. D.

PROPOSITION V.

149. Theorem.—Conversely to Propositions III and IV, *A radius which bisects the angle included by two other radii bisects the arc subtending the angle, and the chord of the arc, and is perpendicular to the chord.*

[Let the student give the demonstration.]

PROPOSITION VI.

150. **Theorem.**—*In the same circle, or in equal circles, equal chords are equally distant from the centre.*

DEMONSTRATION.

Let EF and GH be equal chords in the same circle or in equal circles; and OL and ON be the perpendiculars from the centre O upon the chords, and thus be the distances of the chords from the centre (95).

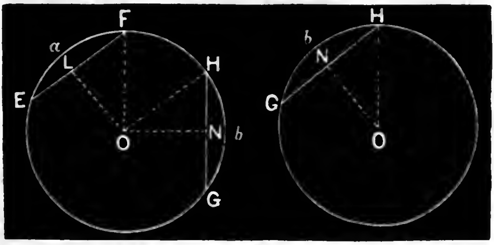

Fig. 65.

We are to prove OL = ON.

Since OL and ON are perpendiculars from the centre upon the equal chords EF and GH, HN = FL (147).

Now apply the figure HNO to FLO, placing HN in its equal FL. Then will NO coincide with LO (88).

In this position, HO and FO are equal lines drawn from the same point in the perpendicular FL to the line LO. Hence, LO = NO (115). Q. E. D.

[Let the student state and prove other converses to Propositions III. IV and VI.]

PROPOSITION VII.

151. **Theorem.**—*In the same circle, or in equal circles, if two arcs are equal, the chords which subtend them are equal; and, conversely, if two chords are equal, the subtended arcs are equal.*

DEMONSTRATION.

Let AmB and CnD be equal arcs in the same circle or in equal circles.

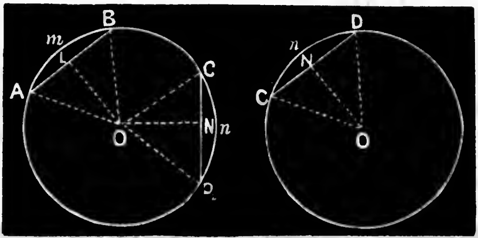

Fig. 66.

We are to prove, first, that the chords **AB** and **CD** are equal.

Apply the figure **C*n*DO** to **A*m*BO**, placing the radius **CO** in its equal **AO**, and let the arc **CD** extend in the direction of arc **AB**.

Then, since each point in arc **CD** is at the same distance from the centre as each point in arc **AB**, arc **CD** falls in arc **AB**, and since arc **CD** = arc **AB** by hypothesis, **D** falls in **B**.

Hence, chord **AB** = chord **CD** (61). Q. E. D.

Conversely, **if chord AB = chord CD, arc AB = arc CD.**

Draw the perpendiculars **OL** and **ON**. Then, since the chords are equal, **OL** = **ON** (150).

Now apply the figure **C*n*DO** to **A*m*BO**, placing **ON** in its equal **OL**. Since **CD** is perpendicular to **ON**, and **AB** to **OL**, **CD** will fall in **AB**; and, since the chords are equal by hypothesis, and are bisected at **N** and **L** (147), **D** falls in **B** and **C** in **A**.

Hence, arc **C*n*D** coincides with arc **A*m*B**, and arc **C*n*D** = arc **A*m*B**. Q. E. D.

PROPOSITION VIII.

152. Theorem.—*In the same circle, or in equal circles, if two arcs are unequal, the less arc has the less chord; and, conversely, if two chords are unequal, the less chord subtends the less arc.*

DEMONSTRATION.

In the same circle, or in equal circles, let arc A*m*B < arc C*n*D

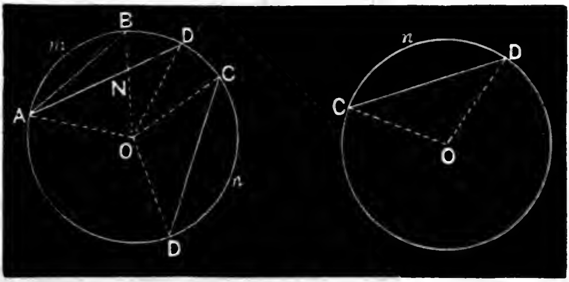

Fig. 67.

We are to prove, first, that chord **AB** < chord **CD**.

Draw **OA**, **OB**, **OD**, and **OC**.

Apply the figure **C*n*DO** to **A*m*BO**, placing **OC** in its equal **OA**, and the arc *n* in the arc *m*.

Since arc **C*n*D** > arc **A*m*B**, **D** will fall beyond **B**, as at **D′**. Draw **OD′**. **AD′** will evidently cut **OB**. Let **N** be the point of intersection.

Now $$\mathbf{AB} < \mathbf{AN} + \mathbf{NB} \ (59),$$

and $$\mathbf{BO} = \mathbf{D'O} < \mathbf{ND'} + \mathbf{ON} \ (59).$$

Adding, $$\mathbf{AB} + \mathbf{BO} < \mathbf{AN} + \mathbf{ND'} + \mathbf{ON} + \mathbf{NB},$$

or $$\mathbf{AB} + \mathbf{BO} < \mathbf{AD'} + \mathbf{BO}.$$

Subtracting **BO** from each member, we have

$$\mathbf{AB} < \mathbf{AD'}. \quad \text{Q. E. D.}$$

Conversely, if chord **AB** < chord **CD**.

We are to prove that arc **A*m*B** is less than arc **C*n*D**.

For, if arc **A*m*B** = arc **C*n*D**, chord **AB** = chord **CD** (151). And, if arc **A*m*B** > arc **C*n*D**, chord **AB** > chord **CD**, by the former part of this demonstration. But both of these conclusions are contrary to the hypothesis.

Hence, as arc **A*m*B** can neither be equal to nor greater than arc **C*n*D**, it must be less. Q. E. D.

PROPOSITION IX.

153. Theorem.—*In the same circle, or in equal circles, of two unequal chords, the less is at the greater distance from the centre; and, conversely, of two chords which are unequally distant from the centre, that which is at the greater distance is the less.*

DEMONSTRATION.

In the same circle, or in equal circles, let chord CE < chord AB, and OD and OD′ their respective distances from the centre.

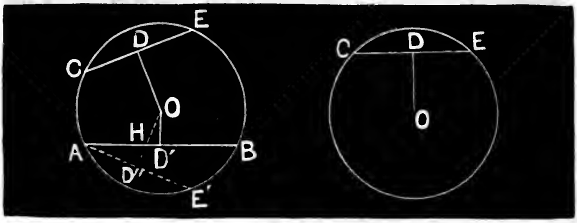

Fig 68.

We are to prove, first, that **OD** > **OD′**.

From **A**, one extremity of the greater chord, lay off towards **B**, **AE′** = **CE**. Since **AE′** < **AB**, arc **AE′** < arc **AB** (**152**), and **E′** falls somewhere on the arc **AB** between **A** and **B**.

Draw **OD″** perpendicular to **AE′**, and **OD″** = **OD**, since the equal chords are equally distant from the centre (**150**).

Now **OD″** is a different line from **OD′**, since **OD″** produced would bisect arc **AE′**, and **OD′** would bisect arc **AB**. Hence, as **OD′** is perpendicular to **AB**, **OD″** must be oblique (**93**).

Again, **OD″** cuts the line **AB** in some point as **H**, since the chord **AE′** lies on the opposite side of **AB** from the centre **O**.

Hence, **OH** > **OD′** (**94**), and much more is **OH** + **HD″** (= **OD″**) > **OD′**. Q. E. D.

Conversely, let **OD** > **OD′**.

We are to prove that **CE** < **AB**.

If **CE** = **AB**, **OD** = **OD′** (**150**), and if **CE** > **AB**, **OD** < **OD′**, both of which conclusions are contrary to the hypothesis **OD** > **OD′**.

Hence, as **CE** can neither be equal to nor greater than **AB**, it must be less. Q. E. D.

PROPOSITION X.

154. Theorem.—*A straight line which intersects a circumference in one point intersects it also in a second point, and can intersect it in but two points.*

DEMONSTRATION.

Let LM (Fig. 69) intersect the circumference in A.

We are to prove that it intersects in another point, as **B**, and in only these two points.

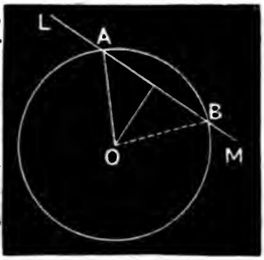

Fig. 69.

Since **LM** intersects the circumference in **A**, it passes within it, and hence has points nearer the centre **O** than **A**. **OA** is, therefore, an oblique line, and not the perpendicular from **O** upon **LM** (**94**).

Now two equal oblique lines can be drawn to a line from a point without (**114**). Let **OB** be the other oblique line equal to **OA**. But as **OA** is a radius, **OB** = **OA** must also be a radius, and **B** is in the circumference. Q. E. D.

Again, **LM** cannot have another point common with the circumference, since if it had there could be more than two equal straight lines drawn from **O** to **LM**, which is impossible. Q. E. D.

155. COROLLARY.—*Any line which is oblique to a radius at its extremity is a secant line,* since any such line has points nearer the centre than the extremity of the radius, and hence passes within the circumference.

PROPOSITION XI.

156. Theorem.—*A straight line which is perpendicular to a radius at its outer extremity is tangent to the circumference; and, conversely, a tangent to a circumference is perpendicular to a radius drawn to the point of contact.*

DEMONSTRATION.

A line perpendicular to a radius at its extremity touches the circumference because the extremity of the radius is in the circumference.

Moreover, it does not intersect the circumference, since, if it did, it would have points nearer the centre than the extremity of the radius; but these it cannot have, as the perpendicular is the shortest distance from a point to a line. Hence, as a line which is perpendicular to a radius at its extremity touches the circumference but does not intersect it, it is a tangent (**30**). Q. E. D.

Conversely, as a tangent to a circumference does not pass within, the point of contact is the nearest point to the centre, and hence is the foot of a perpendicular from the centre. Q. E. D.

157. Corollary.—*A perpendicular from the centre of a circle to a tangent meets the tangent in the point of tangency* (**93**).

PROPOSITION XII.

158. Theorem.—*The arcs of a circumference intercepted by two parallels are equal.*

Demonstration.

There may be three cases, 1st. When one parallel is a tangent and the other a secant, as AB and CD;

2d. When both parallels are secants, as CD and EF; and

3d. When both parallels are tangents, as AB and GH.

In the first case we are to prove **MI** = **MK**; in the second, **IL** = **KR**; and in the third, **M**m**N** = **M**n**N**.

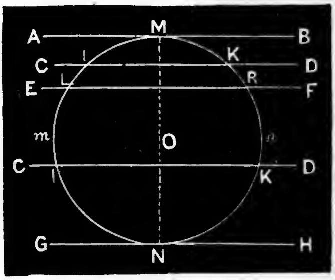

Fig. 70.

Through **O** draw **MN** perpendicular to one of the parallels, in any case, and it will be perpendicular to the other also (**133**); and as a perpendicular from the centre upon a tangent meets the tangent at the point of tangency (**157**), **M** and **N** are points of tangency, and **MN** is a diameter.

Now, since the parallels are perpendicular to **MN**, and the chords **IK** and **LR** are bisected by it, if we fold the right-hand portion of the figure on **MN** as an axis until it falls in the plane on the left of **MN**, **K** will fall in **I**, and **R** in **L**.

Hence, **MI** = **MK**, **IL** = **KR**, and **M**m**N** = **M**n**N**. Q. E. D.

PROPOSITION XIII.

159. Problem.—*To bisect a given arc.*

Solution.

Let ACB be the given arc.

We are to bisect it; that is, find its middle point.

Draw the chord **AB** joining the extremities of the arc; and bisect this chord by the perpendicular **OO′** (**101**). Then will **OO′** bisect the arc, as at **C**.

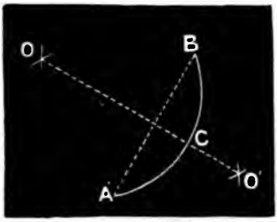

Fig. 71.

Demonstration of Solution.

OO′ being a perpendicular to the chord **AB** at its middle point, any point in it is equally distant from the extremities. Hence chord **BC** = chord **AC**, and arc **BC** = arc **AC** (**151**). Q. E. D.

PROPOSITION XIV.

160. Problem.—*To find the centre of a circle whose circumference is known, or of any arc of it.*

Solution.

Let ACB be an arc of a circumference.

We are to find the centre of the circle.

Draw any two chords of the arc, as **AC** and **BC**, not parallel, and bisect each by a perpendicular. Then will the intersection of these perpendiculars, as **O**, be the centre of the circle.

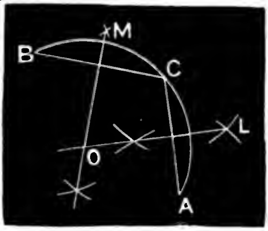

Fig. 72.

Demonstration of Solution.

OL being perpendicular to the chord **AC** at its centre, passes through the centre of the circle, since if the centre were out of **OL** it would be unequally distant from **A** and **C** (**96**). And for a similar reason, **OM**

being perpendicular to the chord **BC** at its centre, passes through the centre of the circle.

Hence, as the centre of the circle lies at the same time in **LO** and **MO**, it is their intersection **O**. Q. E. D.

PROPOSITION XV.

161. Problem.—*To pass a circumference through three given points not in the same straight line.*

Solution.

Let A, B, and C be the three given points not in the same straight line.

Join **AB** and **BC**.

Bisect **AB** by the perpendicular **MN** (**101**), and **BC** by the perpendicular **RS**.

With **O**, the intersection of **MN** and **RS**, as a centre, and any one of the distances **OA**, **OB**, **OC**, say **OA**, as a radius, describe a circumference.

Then will this circumference pass through the three points **A**, **B**, and **C**.

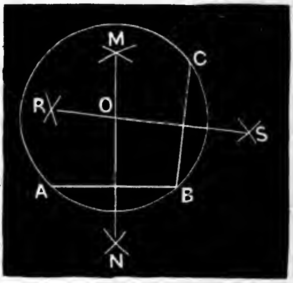

Fig. 73.

Demonstration of Solution.

Since **AB** and **BC** are non-parallel by hypothesis, **MN** and **RS** are non-parallel (**141**), and hence meet in some point, as **O** (**131**).

Now as every point in **MN** is equally distant from the extremities of **AB** (**96**), **OA** = **OB**.

In like manner, every point in **RS** is equally distant from **B** and **C**. Hence, **OB** = **OC**.

Hence, **OA** = **OB** = **OC**, and a circumference struck from **O** as a centre, with a radius **OA**, will pass through **A**, **B**, and **C**. Q. E. D.

PROPOSITION XVI.

162. Problem.—*To draw a tangent to a circle at a given point in its circumference.*

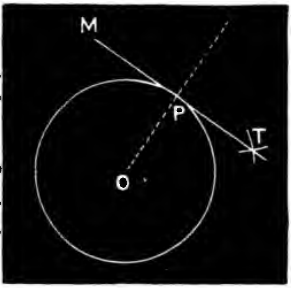

Fig. 74.

SOLUTION.

Let it be required to draw a tangent to the circle whose centre is O, at the point P in its circumference.

Draw the radius **OP**, and produce it to any convenient distance beyond the circle. Through **P** draw **MT** perpendicular to **OP**. Then is **MT** a tangent to the circle at **P**.

DEMONSTRATION OF SOLUTION.

MT being a perpendicular to the radius at its extremity, is a tangent to the circle by **(156)**. Q. E. D.

EXERCISES.

163. 1. Draw a circle and divide it into two equal parts. What proposition is involved?

2. Given a point in a circumference, to find where a semicircumference reckoned from this point terminates. What proposition is involved?

3. In a circle whose radius is 11 there are drawn two chords, one at 6 from the centre, and one at 4. Which chord is the greater? By what proposition?

4. In a certain circle there are two chords, each 15 inches in length. What are their relative distances from the centre? Quote the principle.

5. There is a circular plat of ground whose diameter is 20 rods. A straight path in passing runs within 7 rods of the centre. What is the position of the path with reference to the plat? What is the position of a straight path whose nearest point is 10 rods from the centre? One whose nearest point is 11 rods from the centre?

6. Pass a line through a given point, and parallel to a given line, by the principles contained in **(151)**, **(147)**, **(148)**, and **(127)**.

SECTION V.

OF THE RELATIVE POSITIONS OF CIRCUMFERENCES.

AXIOMS.

164. Two circles may occupy any one of five positions with reference to each other:

1st. One circle may be wholly exterior to the other.

2d. One circle may be tangent to the other externally, the circles being exterior to each other.

3d. One circumference may intersect the other.

4th. One circle may be tangent to the other internally.

5th. One circle may be wholly interior to the other.

PROPOSITION I.

165. Theorem.—*When one circle is wholly exterior to another, the distance between their centres is greater than the sum of their radii.*

DEMONSTRATION.

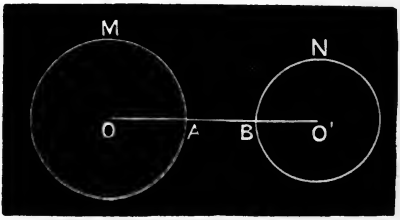

Fig. 75.

Let M and N be two circles whose centres are O and O′, and whose radii are OA = R**, and O′B =** r**, respectively; and let N be wholly exterior to M.**

We are to prove that **OO′** $> R + r$.

Draw **OO′**, and let it intersect circumference **M** in **A**, and **N** in **B**.

Since **N** is wholly exterior to **M**, **OB** > **OA**.

Adding **BO′** to each member of this inequality, we have

$$OB + BO' > OA + BO',$$

or

$$OO' > R + r,$$

since $OB + BO' = OO'$, $OA = R$, and $O'B = r$. Q. E. D.

PROPOSITION II.

166. Theorem.—*When two circles are tangent to each other externally,*

1st. *The distance between their centres is the sum of their radii.*

2d. *They have a common rectilinear tangent at their point of tangency.*

3d. *The point of tangency is in the straight line joining their centres.*

DEMONSTRATION.

Let M and N be two circles tangent to each other externally; let O and O′ be their respective centres, *R* and *r* their radii, D the point of tangency, and TR a tangent to M at D.

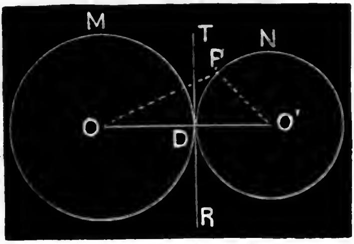

Fig. 76.

We are to prove, 1st. That $OO' = R + r$; 2d. That **TR** is tangent to **N**; 3d. That **D** is in **OO′**.

1st. Draw the radii $OD = R$, and $O'D = r$.

If we show that $OD + O'D = R + r$ is the shortest path from **O** to **O′**, we show that it is a straight line (**59**), and hence is the distance from **O** to **O′** (**95**).

Consider any other path from **O** to **O′**, crossing circumference **N** in some other point than **D**, say in **P**.

Now the shortest path from **O** to **P** is the straight line **OP** (**59**); and the shortest path from **P** to **O′** is the straight line **PO′**. Hence the shortest path from **O** to **O′** passing through **P** is $OP + PO'$.

But **OP** > R (?)*, and **PO′** = r, whence **OP** + **PO′** > $R + r$.

Hence, as **P** is the point where any other path from **O** to **O′** crosses circumference **N**, **OD** + **D′O** = $R + r$ is the distance from **O** to **O′**. Q. E. D.

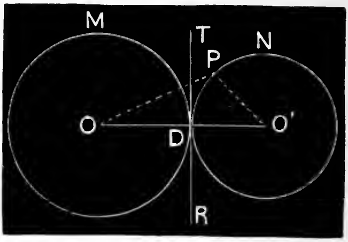

Fig. 76.

2d. As **TR** is tangent to **M** at **D**, by hypothesis, and as **ODO′** has been shown to be a straight line, **TR** is perpendicular to **DO′** (?) and hence tangent to **N** (156). Q. E. D.

3d. As **D** is the point of tangency, and **ODO′** is **OO′**, **D** is in **OO′**. Q. E. D.

PROPOSITION III.

167. Theorem.—*Two circumferences which intersect in one point intersect also in a second point, and hence have a common chord.*

Demonstration.

Let M and N be two circumferences intersecting in P.

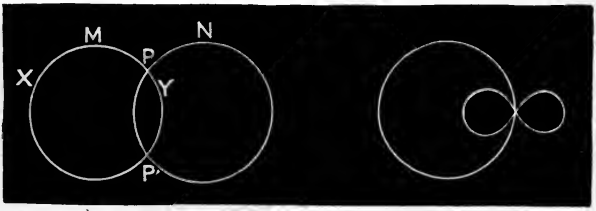

Fig. 77.

We are to prove that they intersect in another point, as **P′**, and hence have a common chord **PP′**.

As **M** intersects **N**, it has points both without and within **N**.

Now consider the circumference **M** as generated by a point moving from left to right, and let **Y** be a point within **N**. The generating point,

---

* Hereafter, minor references to principles on which a statement depends will be omitted, and the interrogation mark substituted. This indicates that the student is to give the principle. In this case. **P** is without **M** since by hypothesis **N** is external to **M**.

in passing from **Y**, a point within **N**, to **X**, any point in the circumference **M** without **N**, must cross circumference **N** at some point, as **P′**, since this is a closed curve.

Moreover, this second point, **P′**, is a different point from **P**, since a circumference of a circle does not cut itself, or become tangent to itself.

Hence, if circumference **M** cuts circumference **N** in **P**, it cuts it also in a second point, as **P′**. Q. E. D.

Finally, since **P** and **P′** are common to both circumferences, the circles **M** and **N** have a common chord **PP′**. Q. E. D.

PROPOSITION IV.

168. Theorem.—*When two circumferences intersect,*

1st. *The line joining their centres is perpendicular to their common chord at its middle point.*

2d. *The distance between their centres is less than the sum of their radii and greater than their difference.*

Demonstration.

Let M and N be two circumferences intersecting at P and P′; let O and O′ be their centres, and R and r their radii respectively, R being equal to or greater than r.

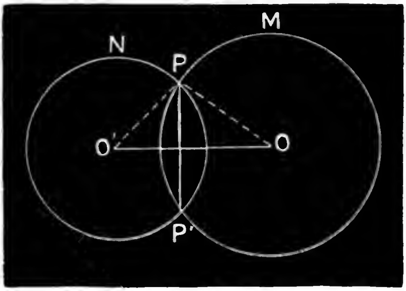

Fig. 78.

We are to prove, 1st. That **OO′** is perpendicular to **PP′** at its middle point; and 2d. That **OO′** $< R + r$, and **OO′** $> R - r$.

Draw **OP** and **O′P**.

1st. Since **O** is equally distant from **P** and **P′**, and **O′** is also equally distant from **P** and **P′** (?), **OO′** is perpendicular to **PP′** at its middle point (98). Q. E. D.

2d. As **P** is not in **OO′**, $OO' < OP + PO'$ (?), or $OO' < R + r$.

Again, $$OO' + O'P > OP,$$
or $$OO' + r > R;$$
whence, subtracting r from each member,
$$OO' > R - r. \quad \text{Q. E. D.}$$

PROPOSITION V.

169. Theorem.—*When the less of two circles is tangent to the other internally,*

1st. *They have a common rectilinear tangent at the point of tangency.*

2d. *Their centres and the point of tangency lie in the same straight line.*

3d. *The distance between the centres is equal to the difference of their radii.*

DEMONSTRATION.

Let M and N be two circles whose centres are O and O′ respectively, N being less than M and tangent to it internally; let R and r be their radii, and D the point of tangency.

We are to prove, 1st. That they have a common rectilinear tangent at **D**; 2d. That **O**, **O′**, and **D** are in the same straight line; and 3d. That **OO′** $= R - r$.

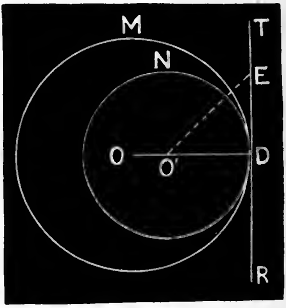

Fig. 79.

1st. Draw **TR** tangent to **M** at **D**. Draw also **O′D**, and any other line from **O′** to **TR**, as **O′E**.

Now, since **E** is without the circle **M** (?), and **M** is without **N** (?), **O′E** > **O′D**, and **O′D** is perpendicular to **TR** (94).

Hence, **TR** is tangent to **N** (156), and is therefore a common tangent. Q. E. D.

2d. Since both **OD** and **O′D** are perpendicular to **TR** at **D** (?), **OD** and **O′D** coincide (88), and **O** and **O′** lie in the same straight line with **D**. Q. E. D.

3d. Since **OO′** and **D** are in the same straight line, and **O′** is between **O** and **D**, **OO′** = **OD** − **O′D**; that is, **OO′** $= R - r$. Q. E. D.

PROPOSITION VI.

170. Theorem.—*When the less of two circles is wholly interior to the other, the distance between the centres is less than the difference of their radii.*

Demonstration.

Let M and N be two circles whose centres are O and O′, and whose radii are R and r respectively, and let N be wholly within M.

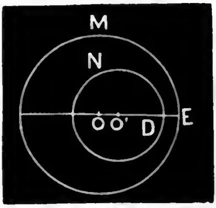

Fig. 80.

We are to prove that $OO' < R - r$.

Produce OO′ till it meets both circumferences on the same side of O that O′ is; and let the intersections with N and M respectively be D and E.

Then, as O, O′, D, and E lie in order in the same straight line,

$$OD < OE;$$

and subtracting O′D from each, and noticing that $OD - O'D = OO'$, that $OE = R$, and $O'D = r$, we have

$$OO' < R - r. \quad \text{Q. E. D.}$$

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171. General Scholium.—The converse of each of Props. I, II, IV, V, and VI is also true. Thus, if the distance between the centres is greater than the sum of the radii, the circles are wholly exterior the one to the other; since if they occupied any one of the other four possible positions, the distance between the centres would be equal to the sum of the radii, less than their sum, equal to their difference, or less than their difference; any one of which conclusions would be contrary to the hypothesis.

In like manner, the converse of any one of the five propositions may be proved.

This method of proof is called The Reductio ad Absurdum, and consists in showing that any conclusion other than the one stated would lead to an absurdity.

PROPOSITION VII.

172. Theorem.—*All the circumferences which can be passed through three points not in the same straight line coincide, and are one and the same.*

DEMONSTRATION.

Let A, B, and C be three points not in the same straight line.

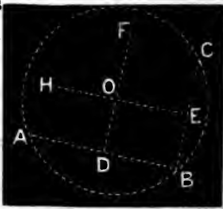

Fig. 81.

We are to prove that all the circumferences which can be passed through them coincide, and are one and the same circumference.

By (161) a circumference can be passed through **A**, **B**, and **C**.

Now every point equally distant from **A** and **B** lies in **FD**, a perpendicular to **AB** at its middle point (?). And, in like manner, every point equally distant from **B** and **C** is in **HE**, a perpendicular to **BC** at its middle point.

But the two straight lines **FD** and **HE** can intersect in only one point.

Hence all circumferences which can pass through **A**, **B**, and **C** have their centre in **O**, and their radius **OA**, and therefore they constitute one and the same circumference. Q. E. D.

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173. Cor. 1.—*Through any three points not in the same straight line a circumference can be passed, and but one.*

174. Definition.—A circle is said to be determined when the position of its centre and the length of its radius are known.

175. Cor. 2.—*Three points not in the same straight line determine a circle.*

176. Cor. 3.—*Two circumferences can intersect in only two points.*

For, if they have three points common, they coincide, and form one and the same circumference.

EXERCISES.

177. 1. The centres of two circles whose radii are 10 and 7, are at 4 from each other. What is the relative position of the circumferences? What if the distance between the centres is 17? What if 20? What if 2? What if 0? What if 3?

2. Given two circles **O** and **O′** (Fig. 82), to draw two others, one of which shall be tangent to these externally, and to the other of which the two given circles shall be tangent internally. Give *all* the principles involved in the construction. Give other methods.

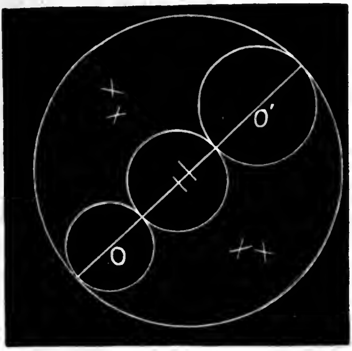

Fig. 82.

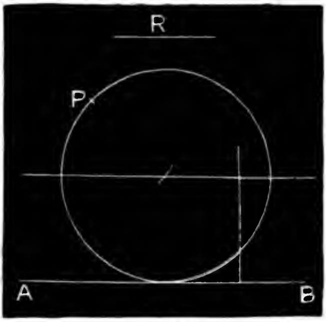

Fig. 83.

3. Given two circles whose radii are 6 and 10, and the distance between their centres 20. To draw a third circle whose radius shall be 8, and which shall be tangent to the two given circles. Can a third circle whose radius is 2 be drawn tangent to the two given circles? How will it be situated? Can one be drawn tangent to the given circles, whose radius shall be 1? Why?

4. With a given radius, draw a circumference (Fig. 83) which shall pass through a given point and be tangent to a given line.

SECTION VI.

OF THE MEASUREMENT OF ANGLES.

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178. Two angles are **Commensurable** when there is a common finite angle which measures each. When they have no such common measure, they are **Incommensurable.**

179. An **Angle at the Centre** is an angle included between two radii.

180. An **Inscribed Angle** is an angle whose vertex is in a circumference, and whose sides are chords of that circumference.

181. Angles are said to be measured by arcs, according to the principles developed in the following propositions.

PROPOSITION I.

182. Theorem.—*In the same circle, or in equal circles, two angles at the centre are in the same ratio as the arcs intercepted between their sides.*

DEMONSTRATION.

There are three cases :

CASE I.

When the angles are equal.

Let angle AOB = angle DOE (Fig. 84) in the same circle or in equal circles.

We are to prove that

$$\frac{AOB}{DOE} = \frac{\text{arc } AB}{\text{arc } DE}.*$$

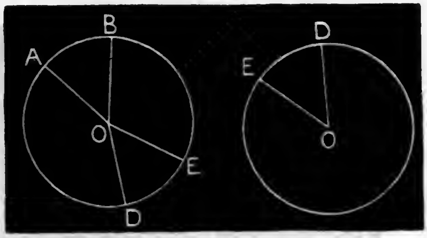

Fig. 84.

Apply the angle **DOE** to the angle **AOB**, placing the radius **OD** in its equal **OA**. By reason of the equality of the angles **DOE** and **AOB**, **OE** will fall in **OB**, and **E** in **B** (?).

Hence **DE** coincides with **AB**, and

$$\frac{\text{arc } AB}{\text{arc } DE} = 1.$$

But, by hypothesis, $$\frac{AOB}{DOE} = 1.$$

Hence, $$\frac{AOB}{DOE} = \frac{\text{arc } AB}{\text{arc } DE} \quad (66).$$ Q. E. D.

CASE II.

When the angles are commensurable.

Let AOB and DOE be two commensurable angles at the centre in the same circle, or in equal circles.

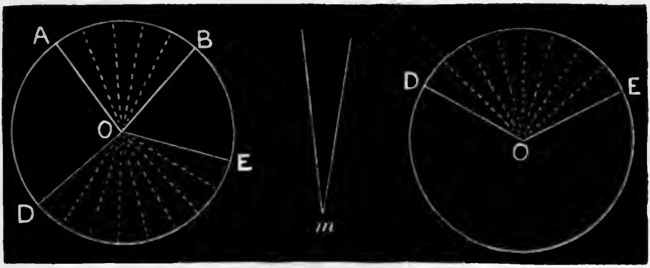

Fig. 85.

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* This method of writing a proportion is adopted in this book as the more elegant, and as it appears to be coming into exclusive use. The above is the same as

$$AOB : DOE :: \text{arc } AB : \text{arc } DE$$

and is to be read in the same manner.

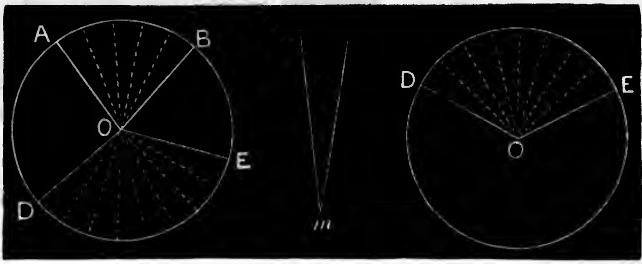

Fig. 85.

We are to prove that $\frac{\text{AOB}}{\text{DOE}} = \frac{\text{arc AB}}{\text{arc DE}}$.

As the angles are commensurable by hypothesis, let m be their common measure, and let it be contained 5 times in **AOB** and 8 times in **DOE**. so that

$$\frac{\text{AOB}}{\text{DOE}} = \frac{5}{8}.$$

Conceive the angle **AOB** divided into 5 partial angles, each equal to m, and the angle **DOE** divided into 8 such partial angles.

Now as these partial angles are equal, their intercepted arcs are equal (?), and as **AB** contains 5 of them, and **DE** 8,

$$\frac{\text{arc AB}}{\text{arc DE}} = \frac{5}{8}.$$

Hence, $$\frac{\text{AOB}}{\text{DOE}} = \frac{\text{arc AB}}{\text{arc DE}} \text{ (?)}. \quad \text{Q. E. D.}$$

CASE III.

When the angles are incommensurable.

Let AOB and DOE (Fig. 86) be two incommensurable angles at the centre, in the same circle, or in equal circles.

We are to prove that $\frac{\text{AOB}}{\text{DOE}} = \frac{\text{arc AB}}{\text{arc DE}}$.

If the ratio $\frac{\text{AOB}}{\text{DOE}}$ is not equal to the ratio $\frac{\text{arc AB}}{\text{arc DE}}$, let it be *greater;* and let

$$\frac{\text{AOB}}{\text{DOE}} = \frac{\text{arc AB}}{\text{arc DL}},$$

in which **DL** is less than **DE**.

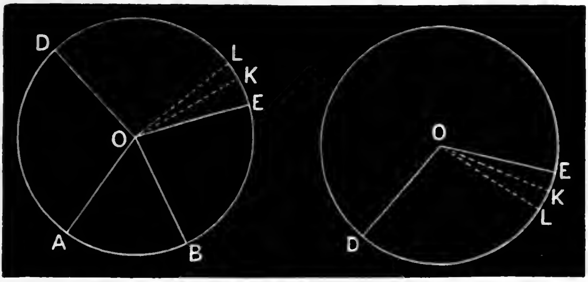

Fig. 86.

Draw **OL**, and divide **AOB** into equal parts, each less than **LOE**. Apply this measure to **DOE**, beginning at **DO**. At least one line of division will fall between **OL** and **OE**. Let this be **OK**.

Now **AOB** and **DOK** are commensurable; hence, by Case II,

$$\frac{\mathbf{AOB}}{\mathbf{DOK}} = \frac{\text{arc } \mathbf{AB}}{\text{arc } \mathbf{DK}};$$

but by hypothesis

$$\frac{\mathbf{AOB}}{\mathbf{DOE}} = \frac{\text{arc } \mathbf{AB}}{\text{arc } \mathbf{DL}}.$$

Dividing $\frac{\mathbf{AOB}}{\mathbf{DOK}}$ by $\frac{\mathbf{AOB}}{\mathbf{DOE}}$, and $\frac{\text{arc } \mathbf{AB}}{\text{arc } \mathbf{DK}}$ by $\frac{\text{arc } \mathbf{AB}}{\text{arc } \mathbf{DL}}$, we have

$$\frac{\mathbf{DOE}}{\mathbf{DOK}} = \frac{\text{arc } \mathbf{DL}}{\text{arc } \mathbf{DK}}.$$

But this conclusion is absurd, since

$$\frac{\mathbf{DOE}}{\mathbf{DOK}} > 1, \quad \text{and} \quad \frac{\text{arc } \mathbf{DL}}{\text{arc } \mathbf{DK}} < 1.$$

Thus we show that the ratio $\frac{\mathbf{AOB}}{\mathbf{DOE}}$ cannot be greater than the ratio $\frac{\text{arc } \mathbf{AB}}{\text{arc } \mathbf{DE}}$; and in a similar manner we may show that $\frac{\mathbf{AOB}}{\mathbf{DOE}}$ cannot be less than $\frac{\text{arc } \mathbf{AB}}{\text{arc } \mathbf{DE}}$.

Hence, as $\frac{\mathbf{AOB}}{\mathbf{DOE}}$ is neither greater nor less than $\frac{\text{arc } \mathbf{AB}}{\text{arc } \mathbf{DE}}$, it is equal to $\frac{\text{arc } \mathbf{AB}}{\text{arc } \mathbf{DE}}$, and we have $\frac{\mathbf{AOB}}{\mathbf{DOE}} = \frac{\text{arc } \mathbf{AB}}{\text{arc } \mathbf{DE}}$. Q. E. D.

[For other methods of demonstrating this important theorem, see Appendix.]

188. Out of the truth developed in this proposition grows the method of representing angles by degrees, minutes, and seconds, as given in Trigonometry (Part IV, **3–6**). It will be observed, that in all cases, if arcs be struck *with the same radius*, from the vertices of angles as centres, the angles bear the same ratio to each other as the arcs intercepted by their sides. *Hence the arc is said to measure the angle.* Though this language is convenient, it is not quite natural; for we naturally measure a quantity by another of *like kind.* Thus, *distance* (length) we measure by *distance*, as when we say a line is 10 inches long. The line is *length;* and its measure, an inch, is *length* also. So, likewise, we say the area of a field is 4 acres: the quantity measured is *a surface;* and the measure, an acre, is *a surface* also. Yet, notwithstanding the artificiality of the method of measuring angles by arcs, instead of directly by angles, it is not only convenient but universally used; and the student should know just what is meant by it.

189. A **Degree** is $\frac{1}{360}$ part of the circumference of a circle; a **Minute** is $\frac{1}{60}$ of a degree, and a **Second** is $\frac{1}{60}$ of a minute. This is the primary signification of these terms. But as any angle at the centre sustains the same ratio to any other angle at the centre as do their subtended arcs, we speak of an angle as an angle of so many degrees, minutes, and seconds. Thus, an angle of 45 degrees (written 45°) means an angle at the centre 45 times as large as one which subtends $\frac{1}{360}$ of the circumference, or half as large as one which subtends 90° of the circumference.

This idea, as well as the notation °, ′, ″, for degrees, minutes, and seconds, has already been made familiar in Arithmetic.

190. As the vertex of any angle may be conceived as the centre of a circle, the intercepted arc of whose circumference measures the angle, we speak of all angles in the same manner as of angles at the centre. Thus, a right angle is called an angle of 90°, one-half a right angle is an angle of 45°, a straight angle is an angle of 180°, and the sum of four right angles, being measured by the entire circumference, is an angle of 360°, etc.

PROPOSITION II.

191. Theorem.—*An inscribed angle is measured by half the arc intercepted between its sides.*

Demonstration.

Let APB be an angle inscribed in a circle whose centre is O.

We are to prove that the angle **APB** is measured by one-half the arc **AB**.

There are three cases: 1st. When the centre is in one side; 2d. When the centre is within the angle; and 3d. When it is without.

CASE I.

***When the centre,* O, *is in one side, as* PB.**

Draw the diameter **DC** parallel to **AP**.

By reason of the parallels **AP** and **CD**,

arc **AC** = arc **PD** (158);

and, since **COB** = **POD** (?),

arc **CB** = arc **PD** (?).

Hence, arc **AC** = arc **CB**,

and arc **CB** = $\frac{1}{2}$ arc **AB**.

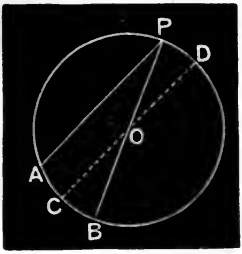

Fig. 87.

Again, since the parallels **AP** and **DC** are cut by the transversal **PB**, the angles **APB** and **COB** are equal (125).

But **COB** is measured by arc **CB** (?). Hence, **APB** is measured by arc **CB** = $\frac{1}{2}$ arc **AB**. Q. E. D.

CASE II.

When the centre is within the angle.

Draw the diameter **PC**.

Now by Case I, **APC** is measured by $\frac{1}{2}$ arc **AC**, and **CPB** by $\frac{1}{2}$ arc **CB**. Hence the sum of these angles, or **APB**, is measured by $\frac{1}{2}$ arc **AC** + $\frac{1}{2}$ arc **CB**, or $\frac{1}{2}$ arc **AB**. Q. E. D.

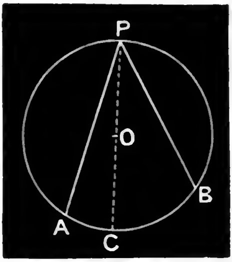

Fig. 88.

CASE III.

When the centre is without the angle.

Draw the diameter **PC**.

By Case I, **APC** is measured by $\frac{1}{2}$ arc **AC**, and **BPC** by $\frac{1}{2}$ arc **BC**. Hence, **APB**, which is **APC** — **BPC**, is measured by

$\frac{1}{2}$ arc **AC** — $\frac{1}{2}$ arc **BC**

or $\frac{1}{2}$ arc **AB**. Q. E. D.

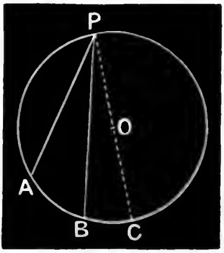

Fig. 89.

192. Corollary.—*In the same circle or in equal circles, all angles inscribed in the same segment or in equal segments intercept equal arcs, and are consequently equal. If the segment is less than a semicircle, the angles are obtuse; if a semicircle, right; if greater than a semicircle, acute.*

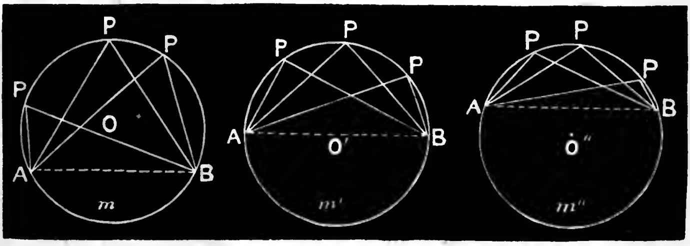

Fig. 90.

Illustration.—In each separate figure the angles **P** are equal to each other, for they are each measured by half the same arc.

In **O**, each angle **P** is acute, being measured by $\frac{1}{2}m$, which is less than a quarter of a circumference.

In **O′**, each angle **P** is a right angle, being measured by $\frac{1}{2}m'$, which is a quadrant (quarter of a circumference).

In **O″**, each angle **P** is obtuse, being measured by $\frac{1}{2}m''$, which is greater than a quadrant.

PROPOSITION III.

193. Theorem.—*Any angle formed by two chords intersecting in a circle is measured by one-half the sum of the arcs intercepted between its sides and the sides of its vertical, or opposite, angle.*

Demonstration.

Let the chords AB and CD (Fig. 91) intersect in P.

We are to prove that angle **APD** (= angle **CPB** ?) is measured by

$$\tfrac{1}{2}\,(\text{arc } \mathbf{AD} + \text{arc } \mathbf{CB});$$

and that angle **BPD** (= angle **CPA** ?) is measured by

$$\tfrac{1}{2}\,(\text{arc } \mathbf{BD} + \text{arc } \mathbf{CA}).$$

Draw **CE** parallel to **AB**.

Arc **AE** = arc **CB** (?); whence, arc **ED** = arc **AD** + arc **CB**.

Now the inscribed angle **ECD** is measured by $\frac{1}{2}$ arc **ED** = $\frac{1}{2}$ (arc **AD** + arc **CB**).

But **ECD** = **APD** (?); hence, **APD** (= **CPB**) is measured by $\frac{1}{2}$ (arc **AD** + arc **CB**). Q. E. D.

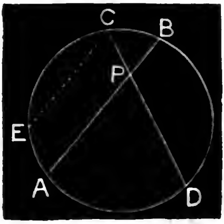

Fig. 91.

Finally, that **APC**, or its equal **BPD**, is measured by $\frac{1}{2}$ (**AC** + **BD**), appears from the fact that the sum of the four angles about **P** being equal to four right angles, is measured by a whole circumference (**190**).

But **APD** + **CPB** is measured by **AD** + **CB**; whence **APC** + **BPD**, or 2**APC**, is measured by the whole circumference minus (**AD** + **CB**); that is, by **AC** + **BD**. Hence **APC** is measured by $\frac{1}{2}$ (**AC** + **BD**). Q. E. D.

194. Scholium.—The case of the angle included between two chords passes into that of the inscribed angle in the preceding proposition, by conceiving **AB** to move parallel to its present position until **P** arrives at **C** and **BA** coincides with **CE**. The angle **APD** is all the time measured by half the sum of the intercepted arcs; but, when **P** has reached **C**, **CB** becomes 0, and **APD** becomes an inscribed angle measured by half its intercepted arc.

In a similar manner we may pass to the case of an angle at the centre, by supposing **P** to move toward the centre. All the time **APD** is measured by $\frac{1}{2}$(**AD** + **CB**); but, when **P** reaches the centre, **AD** = **CB**, and $\frac{1}{2}$ (**AD** + **CB**) = $\frac{1}{2}$ (2**AD**) = **AD**; *i. e.*, an angle at the centre is measured by its intercepted arc.

PROPOSITION IV.

195. Theorem.—*An angle included between two secants meeting without the circle is measured by one-half the difference of the intercepted arcs.*

Demonstration.

Let APB (Fig. 92) be an angle included between the secants PA and PB; and let the intersections with the circumference be C and D.

We are to prove that **APB** is measured by $\frac{1}{2}$ (arc **AB** — arc **CD**).

Draw **CE** parallel to **PB**.

Now arc **CD** = arc **EB** (?). Hence, arc **AE** = arc **AB** — arc **CD**.

Again, **ACE** = **APB** (?).

But **ACE** is measured by $\frac{1}{2}$ arc **AE** (?). Hence **APB** is measured by

$\frac{1}{2}$ arc **AE** = $\frac{1}{2}$ (arc **AB** — arc **CD**). Q. E. D.

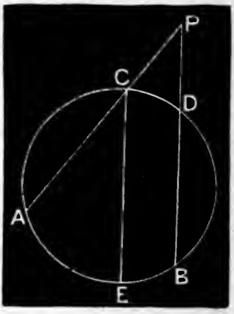

Fig. 92.

196. SCHOLIUM.—This case passes into that of an inscribed angle, by conceiving **P** to move toward **C**, thus diminishing the arc **CD**. When **P** reaches **C**, the angle becomes inscribed; and, as **CD** is then 0, $\frac{1}{2}$ (**AB** — **CD**) = $\frac{1}{2}$ **AB**. Also, by conceiving **P** to continue to move along **PA**, **CD** will reappear *on the other side of* **PA**, hence will change its sign,* and $\frac{1}{2}$ (**AE** — **CD**) will become $\frac{1}{2}$ (**AE** + **CD**), as it should, since the angle is then formed by two chords intersecting within the circumference.

PROPOSITION V.

197. Theorem.—*All equal angles whose sides intercept a given line, and whose vertices lie on the same side of that line, are inscribed in the same segment of which the intercepted line is the chord.*

DEMONSTRATION.

Let APB, AP′B, AP″B, etc., be any number of equal angles whose sides intercept the given line AB.

We are to prove that the vertices **P**, **P′**, **P″**, etc., all lie in the same arc of which **AB** is the chord.

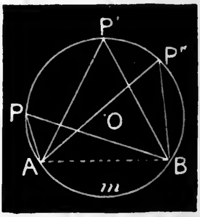

Fig. 93.

Through one of the vertices, as **P**, and **A** and **B** describe a circumference.

Now the angle **APB** is measured by $\frac{1}{2}$ the arc **A***m***B**, and as the other angles are equal to this, they must have the same measure.

---

* In accordance with the law of positive and negative quantities as used in mathematics, whenever a continuously varying quantity is conceived as diminishing till it reaches 0, and then as reappearing by the same law of change, it must change its sign.

But suppose any one of them, as **P′**, had its vertex within the segment. It would then be an angle included between two chords drawn from **A** and **B**, and hence would be measured by $\frac{1}{2}$**A**m**B** *plus* some arc (**193**).

If, on the other hand, the vertex **P′** was without the segment, the angle would be an angle included between two secants, and would be measured by $\frac{1}{2}$**A**m**B** *less* some arc (**195**).

Hence, as **P′** can lie neither without nor within the arc **APB**, it lies *in* it. Q. E. D.

198. Corollary.—*All right angles whose sides intercept a given line are inscribed in a semicircle whose diameter is the given line.*

PROPOSITION VI.

199. Theorem.—*An angle included between a tangent and a chord drawn from the point of tangency is measured by one-half the intercepted arc.*

Demonstration.

Let TPA be an angle included between the tangent TM and the chord PA.

We are to prove that **TPA** is measured by $\frac{1}{2}$ arc **P**n**A**.

Through **A** draw the chord **AD** parallel to **TM**.

Then is **PAD** = **TPA** (?).

Now **PAD** is measured by $\frac{1}{2}$**P**m**D** (?).

Whence **TPA** is measured by $\frac{1}{2}$**P**m**D**. But **P**m**D** equals **P**n**A** (?).

Hence **TPA** is measured by $\frac{1}{2}$**P**n**A**. Q. E. D.

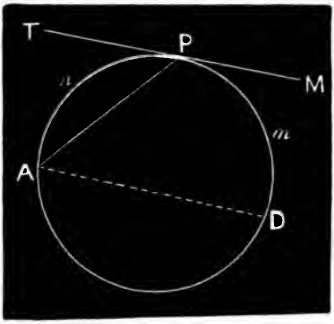

Fig. 94.

Exercise.—Show that **APM** is measured by $\frac{1}{2}$ arc **A**m**P**.

Also, observe how the case of two secants (**195**), passes into this.

PROPOSITION VII.

200. Theorem.—*An angle included between two tangents is measured by one-half the difference of the intercepted arcs.*

Demonstration.

Let **APB** be an angle included between the two tangents **PA** and **PB**, tangent at **C** and **D**.

We are to prove that **APB** is measured by

$\frac{1}{2}$ (arc **C***m***D** – arc **C***n***D**).

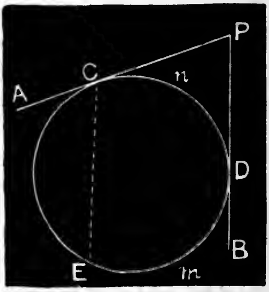

Fig. 95.

Draw the chord **CE** parallel to **PB**.

Now arc **C***n***D** = arc **E***m***D** (?).

Whence arc **CE** = arc **C***m***D** – arc **C***n***D**.

Again, **ACE** = **APB** (?).

But **ACE** is measured by $\frac{1}{2}$ arc **CE** = $\frac{1}{2}$ (arc **C***m***D** – arc **C***n***D**). Hence **APB** is measured by $\frac{1}{2}$ (arc **C***m***D** – arc **C***n***D**). Q. E. D.

201. Scholium.—The case of two secants (195) becomes this by supposing the secants to move parallel to their first position till they both become tangents.

PROPOSITION VIII.

202. Theorem. — *An angle included between a secant and a tangent is measured by one-half the difference of the intercepted arcs.*

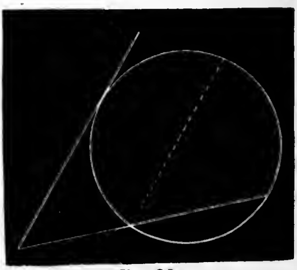
Fig. 96.

[Let the student write out the demonstration in form.]

PROPOSITION IX.

203. Problem.—*From a given point in a given line to draw a line which shall make with the given line a given angle.*

SOLUTION.

Let A be the given point in the given line AB, and O the given angle.

We are to draw from **A** a line which shall make with **AB** an angle equal to **O**.

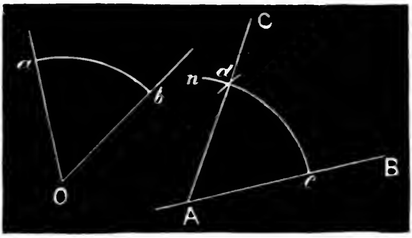

Fig. 97.

From **O** as a centre, with any convenient radius, describe an arc, as *ab*, measuring the angle **O**.

From **A** as a centre, with the same radius, describe an arc *cn* cutting **AB** and extending on that side of **AB** on which the angle is to lie. Let this arc intersect **AB** in *c*.

From *c* as a centre, with a radius equal to the chord *ab*, describe an arc cutting *cn*, as at *d*.

From **A** draw a line through *d*, as **AC**.

Then will **CAB** be the angle required.

DEMONSTRATION OF SOLUTION.

Arc *ab* measures angle **O** (?).

Arc *cd* = arc *ab* (?).

Hence, angle **CAB** = angle **O** (?).

PROPOSITION X.

204. Problem.—*Through a given point to draw a parallel to a given line.*

SOLUTION.

Let P (Fig. 98) be the given point, and AB the given line.

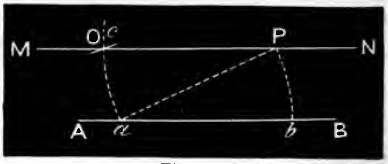

Fig. 98.

We are to draw a line through **P** which shall be parallel to **AB**.

From **P** as a centre, with any radius sufficiently great, strike an arc cutting **AB**, as at *a*, and extending on the same side of **AB** that the parallel is to lie. Let the arc be *ac*.

From *a* as a centre, with the same radius, pass an arc through **P**, cutting **AB** in some point, as *b*.

With the chord *b***P** as a radius and *a* as a centre, strike an arc cutting *ac*, as in **O**.

Draw a line through **O** and **P**, and it will be the parallel required.

Demonstration of Solution.

The arcs **O***a* and **P***b* are arcs of circles with equal radii, and have equal chords, and are hence equal arcs (?).

The angles **OP***a* and **P***ab* are equal, since they are measured by the equal arcs **O***a* and **P***b* (?).

Hence the transversal **P***a* cuts the two lines **MN** and **AB**, making the alternate angles **MP***a* and **B***a***P** equal. Wherefore **MN** is parallel to **AB**, and as it passes through the given point **P**, it is the parallel required. Q. E. D.

PROPOSITION XI.

205. Problem.—*From a point without a circle to draw a tangent to the circle.*

Solution.

Let O be the centre and OT the radius of the given circle, and P the given point.

We are to draw from **P** a tangent to the circle.

Join **P** with the centre **O** by a straight line.

On the line **OP** describe a circle intersecting the given circle in **T** and **T′**.

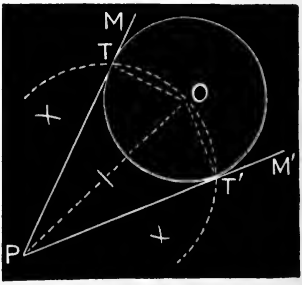

Fig. 99.

Through the points **P** and **T**, **P** and **T′** draw the straight lines **PM** and **PM′**. These will be the required tangents.

Demonstration of Solution.

Drawing **OT**, the angle **OTP** is a right angle, since it is inscribed in a semicircle (192).

Hence **PM** is a tangent to the circle, as it is a perpendicular to a radius at its extremity, and as it passes through **P** it fulfills the conditions of the problem.

In like manner, **PM′** is seen to be a tangent passing through **P**, and the problem has two solutions. Q. E. D.

206. Corollary.—*Through any point without a circle two tangents may be drawn to the circle.*

PROPOSITION XII.

207. Problem.—*On a given line to construct a segment which shall contain a given inscribed angle.*

Solution.

Let AB be the given line and O the given angle.

We are to construct a segment on **AB** which shall contain the **O** as an inscribed angle.

At one extremity of **AB**, as **B**, construct an angle **ABC** equal to **O**, and on the side of **AB** opposite to that on which the segment is to lie.

Erect a perpendicular to **CB** at **B**, and one to **AB** at its middle point **E**. Let **F** be the intersection of these perpendiculars.

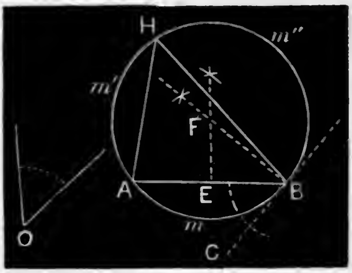

Fig. 100.

With **FB** (or **FA**) as a radius, describe a circle. Then will **A***m′m″***B** be the segment required; and any angle inscribed in this segment, as **AHB**, will be equal to **O**.

Demonstration of Solution.

Since **CB** and **AB** are non-parallel lines, perpendiculars erected to them will meet in some point as **F** (141, 131).

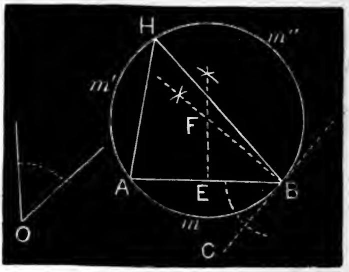

Fig. 100.

F being a point in the perpendicular to **AB** at its middle point **FA** = **FB** (**96**), and a circle struck with **FB** as a radius and **F** as a centre will pass through **A**. Moreover **CB** will be a tangent to this circle, since it is perpendicular to a radius at its extremity (**156**).

Now **O** = **ABC** by construction, and **ABC** being an angle included between a tangent and a chord, is measured by half the intercepted arc **A**m**B** (?).

But any angle inscribed in the segment **A**$m'm''$**B** is measured by $\frac{1}{2}$ arc **A**m**B** (?), and hence equals **ABC** = **O**. Q. E. D.

PROPOSITION XIII.

208. Problem.—*To bisect a given angle.*

Solution.

Let BOA be the given angle.

We are to draw a line dividing **BOA** into two equal angles.

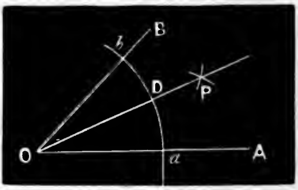

Fig. 101.

With any convenient radius and **O** as a centre, describe an arc cutting the sides **OB** and **OA** at b and a.

From a and b as centres, with equal radii, strike arcs cutting in some point, as **P**.

Through **O** and **P** draw a straight line.

Then is the angle **BOA** bisected by **OP**, and **BOP** = **POA**.

Demonstration of Solution.

OP being perpendicular to the chord of arc ab (?) bisects the arc (**147**). Hence arc b**D** = arc a**D**.

But $\dfrac{\text{angle BOP}}{\text{angle POA}} = \dfrac{\text{arc } b\text{D}}{\text{arc } a\text{D}}$. Therefore, **BOP** = **POA**. Q. E. D.

EXERCISES.

209. 1. To find a point in a plane having given its distances from two known points.

When are there two solutions?

When but one solution?

When no solution?

2. In Fig. 102 there are 4 pairs of equal angles. Which are they, and why?

Show that **COB** = **ABD** + **CDB**.

Show that **DOB** = **ABC** + **DAB**.

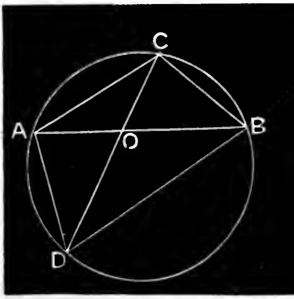

Fig. 102.

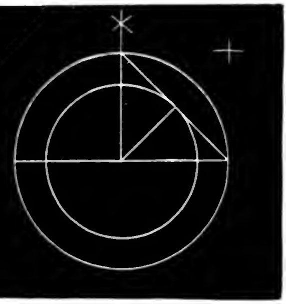

Fig. 103.

210. Concentric Circles are circles which have a common centre.

3. Draw two concentric circles (Fig. 103), such that the chords of the outer circle which are tangent to the inner shall be equal to the diameter of the inner.

4. From a point out of a given straight line to draw a line making a given angle with the first line.

5. Prove that if two circles are concentric, any chord of the outer which is tangent to the inner is bisected at the point of contact.

6. Prove that if **D** and **B** (Fig. 104) are right angles, **A** and **C** are supplementary.

7. Prove that if, in the adjoining figure, the opposite sides **AB** and **DC**, and **AD** and **BC** be produced till they meet, the lines which bisect the included angles will be perpendicular to each other.

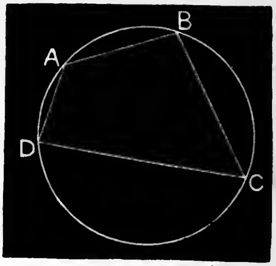

Fig. 104.

8. Draw a triangle, and then draw a circle about it so that all its angles shall be inscribed; *i. e.*, circumscribe a circle about a triangle. (See **161.**)

SECTION VII.

OF THE ANGLES OF POLYGONS, AND THE RELATION BETWEEN THE ANGLES AND SIDES.

OF TRIANGLES.

211. A **Plane Triangle,** or simply a *Triangle,* is a plane figure bounded by three straight lines.

212. With respect to their sides, triangles are distinguished as *Scalene, Isosceles,* and *Equilateral.*

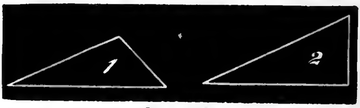

Fig. 105.

A **Scalene Triangle** is a triangle which has no two sides equal, as (1) or (2).

An **Isosceles Triangle** is a triangle which has two of its sides equal to each other, as (3).

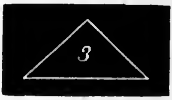

Fig. 106.

An **Equilateral Triangle** is a triangle which has all three of its sides equal each to each, as (4).

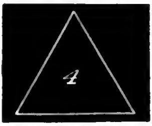

Fig 107.

213. With respect to their angles, triangles are distinguished as *acute* angled, *right* angled, and *obtuse* angled.

An **Acute Angled Triangle** is a triangle all of whose angles are acute, as (4).

A **Right Angled Triangle** is a triangle one of whose angles is right, as (2).

An **Obtuse Angled Triangle** is a triangle one of whose angles is obtuse, as (1).

214. A circle **Circumscribes** a figure when all the angles of the latter are inscribed.

PROPOSITION I.

215. **Theorem.**—*The sum of the three angles of a triangle is two right angles.*

DEMONSTRATION.

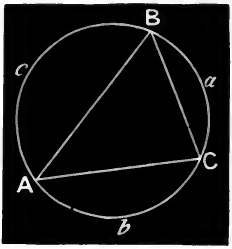

Fig. 108.

Let ABC be any triangle.

We are to prove that

$$A + B + C = 2 \text{ right angles.}$$

Circumscribe a circle about the triangle (**161**).

Then the angle A is measured by ½ the arc BaC (?), the angle B by ½ the arc CbA, and the angle C by ½ the arc AcB.

Hence the sum of the three angles, or A + B + C, is measured by ½ the sum of BaC + CbA + AcB, or ½ the circumference.

But a semi-circumference is the measure of two right angles (**190**). Hence A + B + C = 2 right angles. Q. E. D.

216. Corollary 1.—*A triangle can have only one right angle, or one obtuse angle.* Why?

217. Corollary 2.—*Two angles of a triangle, or their sum, being given, the third may be found by subtracting this sum from two right angles,* i. e., *any angle is the supplement of the sum of the other two.*

218. Corollary 3.—*The sum of the two acute angles of a right-angled triangle is equal to one right angle;* i. e., *they are complements of each other.*

219. Corollary 4.—*If the angles of a triangle are equal each to each, any one is one-third of two right angles, or two-thirds of one right angle.*

PROPOSITION II.

220. Theorem.—*The sides of a triangle sustain the same* GENERAL *relation to each other as their opposite angles; that is, the greatest side is opposite the greatest angle, the second greatest side opposite the second greatest angle, and the least side opposite the least angle.*

Demonstration.

Let ABC be any triangle having the angle C greater than B, and B greater than A.

We are to prove that **AB** opposite **C** is the greatest side, **AC** opposite **B** the next greatest, and **BC** opposite **A** the least.

Circumscribe a circle about the triangle (**161**).

If the triangle is acute-angled, the arc measuring any angle is less than a quarter of a circumference (**191**).

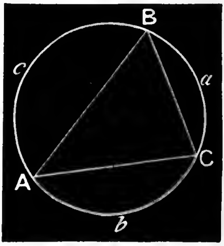

Fig. 109.

Now the angle **C** being greater than **B**, the arc *c* is greater than arc *b* (?). Hence, the chord **AB** is greater than the chord **AC**.

In like manner, the angle **B** being greater than the angle **A**, the arc b is greater than arc a (?). Hence the chord **AC** is greater than the chord **BC**.

If the triangle has one right angle, as **C**, Fig. 110, this angle is measured by $\frac{1}{2}$ the semi-circumference **A**c**B**, and inscribed in the semi-circumference **ACB**. Hence the order of magnitude of the arcs is still $c > b > a$ (?), and of the sides **AB** > **AC** > **BC**.

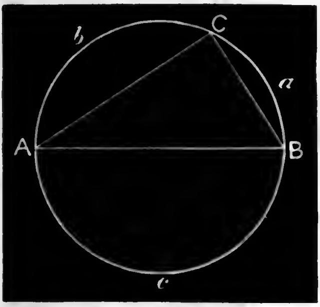

Fig. 110.

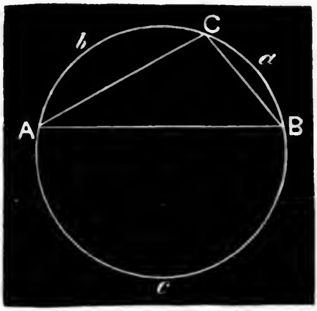

Fig. 111.

If any angle of the triangle, as **C**, is obtuse, Fig. 111, this angle is inscribed in a segment less than a semicircle (**192**), whence this arc **ACB** is less than a semi-circumference, and greater than either a or b, as it is their sum.

Hence the chord **AB** is greater than either **AC** or **BC** (?).

Thus we have shown that in all cases, the order of magnitude of the angles being **C** > **B** > **A**, the order of magnitude of the sides is

AB > **AC** > **BC**. Q. E. D.

221. Corollary 1.—Conversely, *The order of the magnitudes of the sides being* **AB** > **AC** > **BC**, *the order of the magnitudes of the angles is* **C** > **B** > **A**.

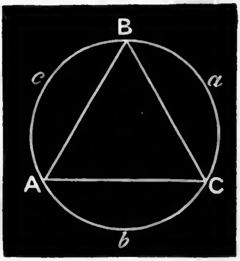

Fig. 112.

[Let the student give the demonstration in form.]

222. Corollary 2.—*An equiangular triangle is also equilateral; and, conversely, an equilateral triangle is equiangular.*

Thus, if **A** = **B** = **C**, arc a = arc b = arc c, and, consequently, chord **BC** = chord **AC** = chord **AB**. Conversely, if the chords are equal, the arcs are, and hence the angles subtended by these arcs.

223. COROLLARY 3.—*In an isosceles triangle the angles opposite the equal sides are equal; and, conversely, if two angles of a triangle are equal, the sides opposite are equal, and the triangle is isosceles.*

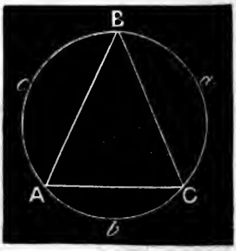

Fig. 113.

Thus, if **AB** = **BC**, arc a = arc c; and hence, angle **A**, measured by $\frac{1}{2}a$, = angle **C**, measured by $\frac{1}{2}c$.

Conversely, if **A** = **C**, arc a = arc c; and hence chord **BC** = chord **AB**.

224. SCHOLIUM.—It should be observed that the proposition gives only the *general* relation between the angles and sides of a triangle. It is not meant that the sides are *in the same ratio* as their opposite angles: this is not true. Thus, in Fig. 114, angle **C** is twice as great as angle **A**; but side c is not *twice* as great as side a, although it *is greater*. Trigonometry discovers the *exact* relation which exists between the sides and angles.

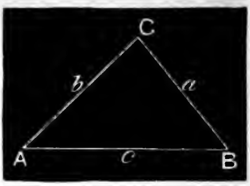

Fig. 114.

PROPOSITION III.

225. Theorem.—*If from any point within a triangle lines are drawn to the extremities of any side, the included angle is greater than the angle of the triangle opposite this side.*

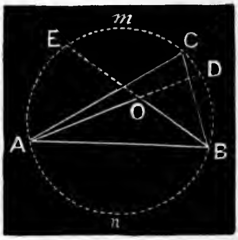

Fig. 115.

DEMONSTRATION.

Let ACB be any triangle, O any point within, and OB and OA lines drawn from this point to the extremities of AB.

We are to prove that angle **AOB** > angle **ACB**.

Circumscribe a circle about the triangle (**161**), and produce **AO** and **BO** till they meet the circumference.

Now **ACB** is measured by $\frac{1}{2}$**A***n***B** (**191**); but **AOB** is measured by $\frac{1}{2}$(**A***n***B** + **E***m***D**) (**193**). Hence, **AOB** > **ACB**. Q. E. D.

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226. An **Exterior Angle** of a triangle is an angle formed by any side with its adjacent side produced, as **CBD**, Fig. 116.

PROPOSITION IV.

227. Theorem.—*Any exterior angle of a triangle is equal to the sum of the two interior non-adjacent angles.*

DEMONSTRATION.

Let ABC be a triangle, and CBD be an exterior angle.

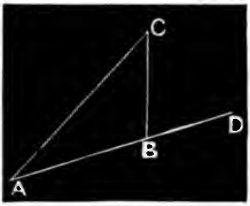

Fig. 116.

We are to prove that **CBD** = **A** + **C**.

ABC + **CBD** = a straight angle (?).
But **ABC** + **A** + **C** = a straight angle (?).
Hence, **ABC** + **CBD** = **ABC** + **A** + **C** (?).
Hence, subtracting **ABC** from each member,

CBD = **A** + **C**. Q. E. D.

228. COROLLARY.—*Either angle of a triangle not adjacent to a specified exterior angle, is equal to the difference between this exterior angle and the other non-adjacent angle.*

Thus, since **CBD** = **A** + **C**,

by transposition, **CBD** − **A** = **C**,

and **CBD** − **C** = **A**.

OF QUADRILATERALS.

229. A **Quadrilateral** is a plane surface inclosed by *four* right lines.

230. There are three *Classes* of quadrilaterals, viz., *Trapeziums, Trapezoids,* and *Parallelograms.*

231. A **Trapezium** is a quadrilateral which has no two of its sides parallel to each other.

232. A **Trapezoid** is a quadrilateral which has but two of its sides parallel to each other.

233. A **Parallelogram** is a quadrilateral which has its opposite sides parallel.

234. A **Rectangle** is a parallelogram whose angles are right angles.

235. A **Square** is an equilateral rectangle.

236. A **Rhombus** is an equilateral parallelogram whose angles are oblique.

237. A **Rhomboid** is an oblique-angled parallelogram two of whose sides are greater than the other two.

Ill.—The figures in the margin are all quadrilaterals. A is a trapezium. (Why?) B is a trapezoid. (Why?) C, D, E, and F are parallelograms. (Why?) D and E are rectangles,

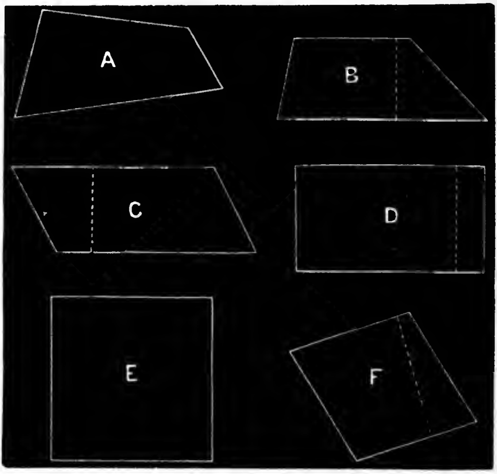

Fig. 117.

although **D** is the form usually referred to by the term rectangle. So **C** is the form usually referred to when a parallelogram is spoken of, without saying what kind of a parallelogram. **C** is also a rhomboid. (Why?) **E** is a square. (Why?) **F** is a rhombus. (Why?)

238. A **Diagonal** is a line joining the vertices of two non-consecutive angles of a figure.

239. The **Altitude** of a parallelogram is a perpendicular between its opposite sides; of a trapezoid, it is a perpendicular between its parallel sides; of a triangle, it is the perpendicular from any vertex to the side opposite or to that side produced.

240. The **Bases** of a parallelogram, or of a trapezoid, are the sides between which the altitude is conceived as taken; of a triangle, the base is the side to which the altitude is perpendicular.

PROPOSITION V.

241. Theorem.—*The sum of the angles of a quadrilateral is four right angles.**

DEMONSTRATION.

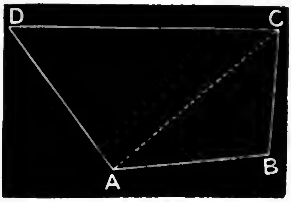

Fig. 118.

Let ABCD be any quadrilateral.

We are to prove that

DAB + **B** + **BCD** + **D** = 4 right angles.

Draw either diagonal, as **AC**.

The diagonal divides the quadrilateral into two triangles, and the sum of the angles of the two triangles is the same as the sum of the angles of the quadrilateral, since

BCA + **ACD** = **BCD**,

and **BAC** + **CAD** = **DAB**.

But the sum of the angles of the triangles is four right angles (?). Hence the sum of the angles of the quadrilateral is four right angles. Q. E. D.

---

* See (261).

PROPOSITION VI.

242. Theorem.—*The opposite angles of any quadrilateral which can be inscribed in a circle are supplementary.*

DEMONSTRATION.

Let ABCD be any inscribed quadrilateral.

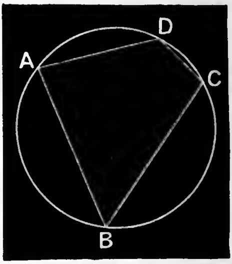

Fig. 119.

We are to prove that

$$\mathbf{A} + \mathbf{C} = 2 \text{ right angles,}$$

and also that $\mathbf{D} + \mathbf{B} = 2$ right angles.

A is measured by $\frac{1}{2}$ the arc **DCB**, and **C** by $\frac{1}{2}$ the arc **BAD**.

Hence, **A+C** is measured by $\frac{1}{2}$(**DCB+BAD**), that is, by a semi-circumference, and is therefore 2 right angles (**190**). Q. E. D.

In like manner, **B + D** is measured by $\frac{1}{2}$(**ADC + CBA**), and hence is 2 right angles. Q. E. D.

PROPOSITION VII.

243. Theorem.—*The adjacent angles of a parallelogram are supplemental, and the opposite angles are equal to each other.*

DEMONSTRATION.

Let ABCD be any parallelogram.

Fig. 120.

We are to prove, 1st. That **A + B**, or **B + C**, or **C + D**, or **D + A** is 2 right angles; and 2d. That **A = C** and **D = B**.

1st. Since, by definition (**233**), **AD** is parallel to **BC**, and the transversal **AB** cuts them, the sum of the two interior angles on the same side, that is, **A+B**, is 2 right angles (**125**).

In like manner, **B+C** is two right angles, since they are the interior angles on the same side of the transversal **BC** which cuts the parallels **AB** and **DC**.

In the same way, **C**+**D**, or **D**+**A** may be shown equal to 2 right angles.

Hence the sum of any two adjacent angles of a parallelogram is 2 right angles. Q. E. D.

2d. **A** + **B** = **B** + **C**, since each sum is 2 right angles, by the preceding part of this demonstration.

Hence, subtracting **B** from each member, we have **A** = **C**.

In a similar manner, we may show that **B** = **D**.

Hence, either two opposite angles are equal to each other. Q. E. D.

244. Corollary 1.—*The two angles of a trapezoid adjacent to either one of the two sides not parallel are supplemental.*

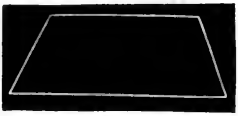

Fig. 121.

[Let the student show why.]

245. Corollary 2.—*If one angle of a parallelogram is right, the others are also, and the figure is a rectangle.*

PROPOSITION VIII.

246. Theorem.—Conversely to the last, *If three consecutive angles of a quadrilateral are such that the first and the second, and the second and the third are supplemental, or if the opposite angles are equal, the figure is a parallelogram.*

Demonstration.

Let ABCD be a quadrilateral having D and A, and A and B supplemental, or having A = C and D = B.

We are to prove that, in either case, the figure is a parallelogram.

Fig. 122.

1st. If we have **D** and **A**, and **A** and **B** supplemental.

Since the transversal **AD** cuts the lines **AB** and **DC**, making **A** + **D** = 2 right angles, the lines **AB** and **DC** are parallel (**126**).

Again, for a like reason, since **A** + **B** = 2 right angles, **AD** and **BC** are parallel.

Hence the opposite sides of the quadrilateral are parallel, and the figure is a parallelogram (**233**). Q. E. D.

Fig. 122.

2d. If **A** = **C**, and **D** = **B**, adding, we have

$$\mathbf{A} + \mathbf{D} = \mathbf{C} + \mathbf{B}.$$

But $\mathbf{A} + \mathbf{D} + \mathbf{C} + \mathbf{B} = 4$ right angles.

Hence, substituting, we have

$$\mathbf{A} + \mathbf{D} + \mathbf{A} + \mathbf{D} = 4 \text{ right angles } (?),$$

or $2(\mathbf{A} + \mathbf{D}) = 4$ right angles,

or $\mathbf{A} + \mathbf{D} = \mathbf{A} + \mathbf{B}\ (?) = 2$ right angles,

and the figure is a parallelogram by the former part of the demonstration. Q. E. D.

PROPOSITION IX.

247. Theorem.—*If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*

DEMONSTRATION.

Let ABCD, in (*a*), be a quadrilateral having the sides AB and DC equal and parallel.

We are to prove that **AD** and **BC** are parallel, and hence that the figure is a parallelogram.

Draw the diagonal **AC**.

Then, by reason of the parallels **AB** and **DC**, the angles **BAC** and **DCA** are equal (?)

Conceive the quadrilateral divided in this diagonal into two triangles, as in (*b*).

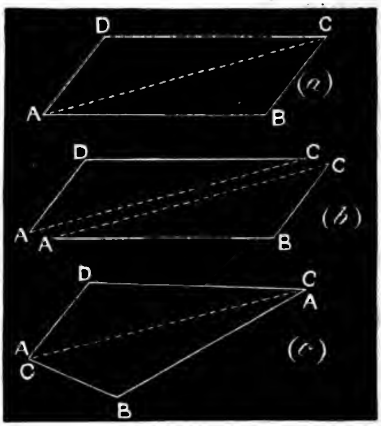

Fig. 123.

Reverse the triangle **ACB** and place it as in (*c*). Since **AC** of the triangle **ADC** = **CA** of the triangle **ABC**, **CA** may be placed in **AC**, as in (*c*).

Now revolve the triangle **CBA** on **CA** as an axis. Since, as we have shown, the angle **BAC** = angle **DCA**, **BA** will take the direction **CD**, and being equal to it, by hypothesis, **B** will fall in **D**, and the angle **BCA** coincides with and is equal to **DAC**.

But in (*a*) the angles **BCA** and **DAC** are alternate interior angles made by the transversal **AC** cutting **AD** and **BC**. Hence **AD** and **BC** are parallel, and as **AB** and **DC** are parallel by hypothesis, the quadrilateral is a parallelogram (233). Q. E. D.

PROPOSITION X.

248. Theorem.—*If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

Demonstration.

Let ABCD, (*a*), be a quadrilateral, having AD = BC and AB = DC.

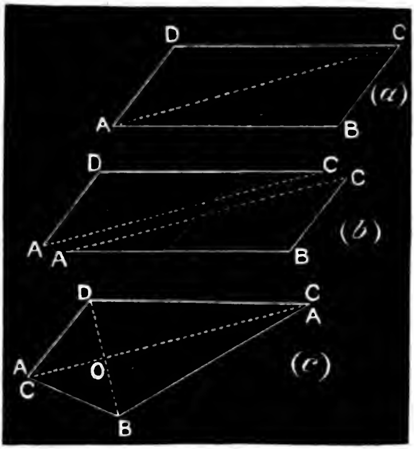

Fig. 124.

We are to prove that **ABCD** is a parallelogram, *i. e.*, that **AB** is parallel to **DC**, and **AD** to **BC**.

Draw the diagonal **AC**, and conceive the quadrilateral divided in this diagonal into two triangles, as in (*b*).

Reverse the triangle **ABC**, and place it as in (*c*). Since **AC** of the triangle **ADC** equals **CA** of the triangle **ABC**, **CA** may be placed in **AC**, as in (*c*).

Draw **DB**, intersecting **CA** (or **CA** produced), in **O**.

As **CD** = **AB**, and **AD** = **CB**, by hypothesis, the line **AC** has two points each equally distant from the extremities of **DB**, and **AC** and **DB** are perpendicular to each other (98). Moreover, since **AB** and **CD** are equal oblique lines drawn from the same point in the perpendicular to the line **DB**, angle **BAC** = angle **DCA** (98, 110, 2d).

Now in (*a*), as angles **BAC** and **DCA** are the alternate interior angles made by the transversal with the lines **AB** and **DC**, the latter are parallel, and as they are equal by hypothesis, the quadrilateral is a parallelogram by the last proposition. Q. E. D.

249. Corollary.—*A diagonal of a parallelogram divides it into two equal triangles.*

PROPOSITION XI.

250. **Theorem.**—Conversely to the last, *The opposite sides of a parallelogram are equal.*

Demonstration.

Let ABCD be a parallelogram.

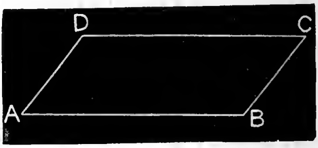

Fig. 125.

We are to prove that **AD** = **BC**, and **AB** = **DC**.

AD and **BC** being parallel transversals cutting the parallels **AB** and **DC**, their intercepted portions, which are the opposite sides of the parallelogram, are equal by (138).

For a like reason, **AB** = **DC**.

Hence, **AD** = **BC** and **AB** = **DC**. Q. E. D.

PROPOSITION XII.

251. **Theorem.**—*The diagonals of a parallelogram bisect each other.*

Demonstration.

Let ABCD be a parallelogram whose diagonals AC and DB intersect in Q.

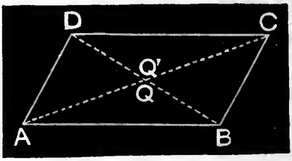

Fig. 126.

We are to prove that **AQ** = **QC**, and **DQ** = **QB**.

Angle **QDC** = angle **QBA** (?), angle **QCD** = angle **QAB** (?), and **DC** = **AB** (?).

For distinctness, let **Q′** represent the vertex at **Q** of the triangle **DQC**.

Now apply the triangle **AQB** to **DQ′C**, placing the side **BA** in its equal **DC**, with the extremity **B** in **D**, and **A** in **C**, and the vertex **Q** on the same side of **DC** that the vertex **Q′** is, and the triangles will coincide. For, since angle **QAB** = angle **Q′CD**, **AQ** will take the direction **CQ′**, and the vertex **Q** will fall somewhere in the line **CQ′**. In like manner, by reason of the equality of angles **QBA** and **Q′DC**, the vertex **Q** will fall in **DQ′**. Hence the vertex **Q** of the triangle **AQB** falling at the same time in **CQ′** and **DQ′**, falls at their intersection.

Hence, as these triangles coincide, **AQ** = **Q′C**, and **DQ′** = **QB**; that is, **AQ** = **QC**, and **DQ** = **QB**. Q. E. D.

PROPOSITION XIII.

252. Theorem.—*The diagonals of a rhombus bisect each other at right angles.*

DEMONSTRATION.

Let ABCD be a rhombus, and AC and DB its diagonals intersecting at Q.

We are to prove that **DB** and **AC** are perpendicular to each other.

Since **AB** = **AD**, and **CD** = **CB** (?), the line **AC** has two points, **A** and **C**, each equally distant from the extremities of **DB**. Hence **AC** is a perpendicular to **DB** at its middle point **Q** (**98**). Q. E. D.

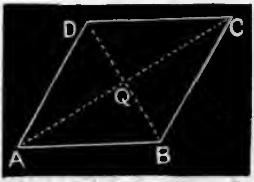

Fig. 127.

In like manner, **DB** may be shown to be perpendicular to **AC** at its middle point. Q. E. D.

253. COROLLARY.—*The diagonals of a rhombus bisect its angles.*

For, revolve **ABC** upon **AC** as an axis, and it will coincide with **ADC**. Hence angles **A** and **C** are bisected. In like manner revolve **DAB** upon **DB**, and it will coincide with **DCB**. Hence, **D** and **B** are bisected.

PROPOSITION XIV.

254. Theorem.—*The diagonals of a rectangle are equal.*

DEMONSTRATION.

Let AC and DB be the diagonals of the rectangle ABCD.

We are to prove that **AC** = **DB**.

Upon **AC** as a diameter describe a circle.

Since **ADC** and **ABC** are right angles whose sides intercept **AC**, they are inscribed in the circumference of which **AC** is a diameter (**198**).

Again, since **DCB** is a right angle and is inscribed, **DB** is a diameter (?).

Hence **AC** and **DB**, being diameters of the same circle, are equal. Q. E. D.

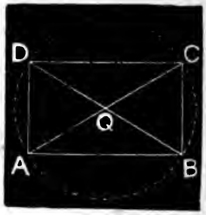

Fig. 128.

255. COROLLARY.—Conversely, *If the diagonals of a parallelogram are equal, the figure is a rectangle.*

By (**251**) the parallelogram is circumscriptible; whence, by (**192**) the angles are right angles.

OF POLYGONS OF MORE THAN FOUR SIDES.

256. A **Polygon** is a portion of a plane bounded by straight lines.

The word polygon means many-angled; so that with strict propriety we might limit the definition to plane figures with five or more sides. This limitation in the use of the word is frequently made.

257. A polygon of three sides is a *triangle;* of four, a *quadrilateral;* of five, a *pentagon;* of six, a *hexagon;* of seven, a *heptagon;* of eight, an *octagon;* of nine, a *nonagon;* of ten, a *decagon;* of twelve, a *dodecagon.*

258. The **Perimeter** of a polygon is the distance around it, or the sum of the bounding lines.

259. A **Salient Angle** of a polygon is one whose sides, when produced, can only extend *without* the polygon.

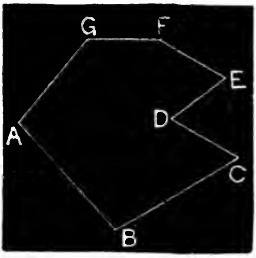

Fig. 129.

260. A **Re-entrant Angle** of a polygon is one whose sides, when produced, can extend *within* the polygon.

ILLUSTRATION.—In the polygon ABCDEFG, all the angles are salient except D, which is re-entrant.

261. A **Convex Polygon** is a polygon which has only salient angles. A polygon is always supposed to be convex, unless the contrary is stated.

262. A **Concave** or **Re-entrant Polygon** is a polygon with at least one re-entrant angle.

263. An **Equilateral Polygon** is a polygon whose sides are equal, each to each; and an **Equiangular Polygon** is a polygon whose angles are equal, each to each.

PROPOSITION XV.

264. Theorem.—*The sum of the interior angles of a polygon is equal to twice as many right angles as the polygon has sides, less four right angles.*

DEMONSTRATION.

Let n be the number of sides of any polygon.

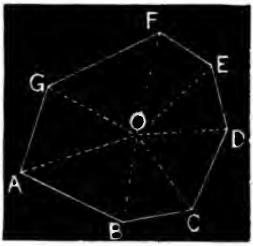

Fig. 130.

We are to prove that the sum of its angles is n times 2 right angles less 4 right angles.

From any point within, as O, draw lines to the vertices of the angles. As many triangles will then be formed as the polygon has sides, that is, n.

The sum of the angles of the triangles is n times 2 right angles.

But this sum exceeds the sum of the angles of the polygon by the sum of the angles around the common vertex O, that is, by 4 right angles.

Hence the sum of the angles of the polygon is

n times 2 right angles less 4 right angles. Q. E. D.

265. Scholium 1.—The sum of the angles of a pentagon is

5 times 2 right angles − 4 right angles, or 6 right angles.

The sum of the angles of a hexagon is 8 right angles; of a heptagon, 10; of an octagon, 12, etc.

266. Scholium 2.—This proposition is equally applicable to triangles and to quadrilaterals. Thus, the sum of the angles of a triangle is

3 times 2 right angles — 4 right angles = 2 right angles.

So also the sum of the angles of a quadrilateral is

4 times 2 right angles — 4 right angles, or 4 right angles.

267. Scholium 3.—To find the value of an angle of an equiangular polygon, divide the sum of all the angles by the number of angles.

PROPOSITION XVI.

268. Theorem.—*If one of the sides of a polygon is produced (and only one) at each vertex, the sum of the exterior angles thus formed is four right angles.*

Demonstration.

Let n be the number of the sides of any polygon, and one side be produced at each vertex.

We are to prove that the sum of the exterior angles thus formed, as $a + b + c + d$, etc., is 4 right angles.

At each of the n vertices there are two angles, an interior and an exterior one, whose sum, as $A + a$, is 2 right angles. Hence the sum of all the exterior and interior angles is

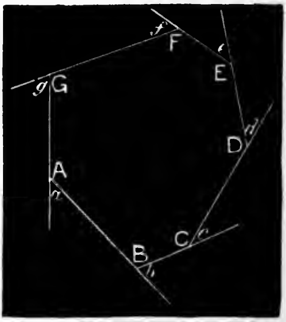

Fig. 131.

n times 2 right angles.

Now, from this sum subtracting the sum of the exterior angles, the remainder is the sum of the interior angles.

But, by the preceding proposition, 4 right angles subtracted from n times 2 right angles leaves the sum of the interior angles.

Therefore the sum of the exterior angles is 4 right angles. Q. E. D.

OF REGULAR POLYGONS.

269. A **Regular Polygon** is a polygon which is both equilateral and equiangular (**263**).

270. An **Inscribed Polygon** is a polygon whose angles are all inscribed in the same circumference.

271. A **Circumscribed Polygon** is a polygon whose sides are all tangent to the same circle. The circumference is said to be inscribed in the polygon.

PROPOSITION XVII.

272. Theorem.—*The angles of an inscribed equilateral polygon are equal; and the polygon is regular.*

DEMONSTRATION.

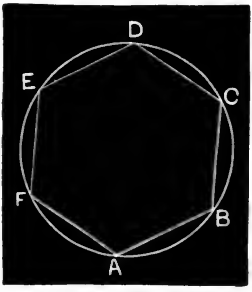

Fig. 132.

Let **ABCDEF** be an inscribed polygon, having **AB** = **BC** = **CD**, etc.

We are to prove that angle **ABC** = angle **BCD** = angle **CDE**, etc.

The sides of the polygon being equal chords, subtend equal arcs (**151**).

Now any angle of the polygon is measured by $\frac{1}{2}$ the difference between the circumference and the sum of two of these equal arcs, as angle **ABC** measured by $\frac{1}{2}$ (circumference − arc **ABC**) = $\frac{1}{2}$ arc **AFEDC**.

Hence all the angles are equal, and the polygon is regular (**269**). Q. E. D.

PROPOSITION XVIII.

273. Theorem.—*A circumference may be circumscribed about any regular polygon.*

DEMONSTRATION.

Let ABCDEF be a regular polygon.

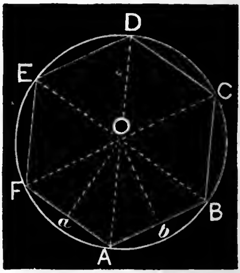

Fig. 133.

We are to prove that a circumference can be circumscribed about it.

Bisecting any two consecutive sides, as **FA** and **AB**, by perpendiculars, as **O**a and **O**b, pass a circumference through the vertices **F**, **A**, and **B** (**161**).

We will now show that this circumference passes through all the other vertices.

Revolve the quadrilateral **FO**b**A** upon **O**b as an axis until it falls in the plane of **CO**b**B**, b**A** will fall in its equal b**B** (?); and since angle **A** = angle **B**, and side **AF** = side **BC**, **F** will fall in **C**.

Thus it appears that the circumference described from **O**, and passing through **F**, **A**, and **B**, also passes through **C**.

In a similar manner it can be shown that the same circumference passes through all the vertices, and hence is circumscribed. Q. E. D.

PROPOSITION XIX.

274. Theorem.—*A circumference may be inscribed in any regular polygon.*

DEMONSTRATION.

Let ABCDEF be a regular polygon.

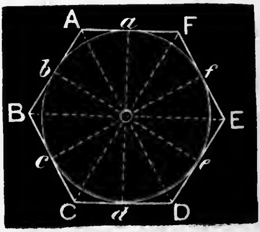

Fig 134.

We are to show that a circumference may be inscribed in it.

Let **O** be the centre of the circumscribed circumference (**273**); then the sides of the polygon are equal chords of this circle, and consequently equally distant from the centre (**150**).

Now draw the perpendiculars **O**a, **O**b, **O**c, **O**d, etc. These perpendiculars are all equal, and a circumference struck from **O** as a centre, with any one of them, as **O**a, as a radius, will pass through b, c, d, etc.

Moreover, the sides **AB**, **BC**, **CD**, etc., being perpendicular to the radii **O**a, **O**b, etc., are tangents to this circumference, which is therefore an inscribed circumference (**271**). Q. E. D.

275. COROLLARY.—*The centres of the inscribed and circumscribed circles coincide.*

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276. The **Centre** of a regular polygon is the common centre of its inscribed and circumscribed circles.

277. An **Angle at the Centre** of a regular polygon is the angle included by two lines drawn from the centre to the extremities of a side, as **FOA**, **AOB** (Fig. 133).

278. The **Apothem** of a regular polygon is the distance from the centre to any side, and is the radius of the inscribed circle.

PROPOSITION XX.

279. Theorem.—*The angles at the centre of a regular polygon are equal each to each; and any one is equal to four right angles divided by the number of sides of the polygon.*

DEMONSTRATION.

Let P be a polygon of n sides.

We are to prove, 1st. That the angles at the centre are equal; and 2d. That any one of them is $\frac{4 \text{ right angles}}{n}$.

1st. Each angle at the centre intercepts one of the equal sides of the polygon. But these sides are chords of equal arcs (?). Hence the several angles at the centre have equal measures, and are therefore equal. Q. E. D.

2d. The sum of all the angles at the centre is 4 right angles (?), and as they are equal and n in number, any one is

$$\frac{4 \text{ right angles}}{n}. \quad \text{Q. E. D.}$$

PROPOSITION XXI.

280. Theorem.—*Any side of a regular inscribed hexagon is equal to the radius.*

DEMONSTRATION.

Let ABCDEF be a regular hexagon inscribed in a circle whose radius is R.

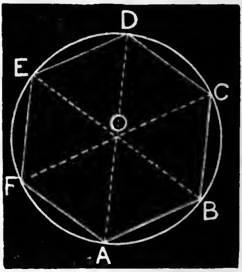

Fig. 135.

We are to prove that any one of the equal sides, as **AB**, equals R.

Let **O** be the centre of the polygon, and draw **OA**, **OB**, etc.

Now in the triangle **AOB**, angle **O** is $\frac{1}{6}$ of 4 right angles, or $\frac{1}{3}$ of 2 right angles (?).

Whence the sum of the angles **OAB** and **OBA** is $\frac{2}{3}$ of 2 right angles (?).

But the triangle **AOB** is isosceles, **OA** and **OB** being radii of the same circle. Hence, each one of the angles at the base is $\frac{1}{2}$ of $\frac{2}{3}$ of 2 right angles, or $\frac{1}{3}$ of 2 right angles. Therefore the triangle **AOB** is equiangular and consequently equilateral (222), and **AB** $=$ **OA** $= R$. Q. E. D.

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281. A *Broken Line* is said to be *Convex* when a straight line cannot be drawn which shall cut it in more than two points.

PROPOSITION XXII.

282. Theorem.—*A convex broken line is less than any broken line which envelops it and has the same extremities, the former lying between the latter and a straight line joining its extremities.*

DEMONSTRATION.

Let AbcdB be a broken line enveloped by the broken line ACDEFB, and having the same extremities A and B.

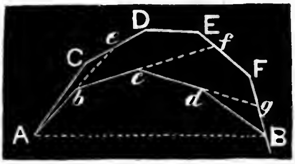

Fig. 136.

We are to prove that

$$AbcdB < ACDEFB.$$

Produce the parts of $AbcdB$ till they meet the enveloping line, as Ab to e, bc to f, and cd to g.

Now,
$$Ab + be < ACe \text{ (?)},$$
$$bc + cf < be + eDEf \text{ (?)},$$
$$cd + dg < cf + fFg,$$
$$dB < dg + gB.$$

Hence, adding, and subtracting common terms,

$$Ab + bc + cd + dB < ACe + eDEf + fFg + gB,$$

or
$$AbcdB < ACDEFB. \quad \text{Q. E. D.}$$

PROPOSITION XXIII.

283. Problem.—*To inscribe a circle in a given triangle.*

SOLUTION.

Let ABC be a triangle.

We are to inscribe a circle.

Bisect any two angles, as **A** and **B** (208).

From the intersection of the bisectors, as **O**, let fall a perpendicular, as **OD**.

Then is **O** the centre of the inscribed circle, and **OD** its radius.

Hence a circle described with **O** as a centre and **OD** as a radius will be inscribed.

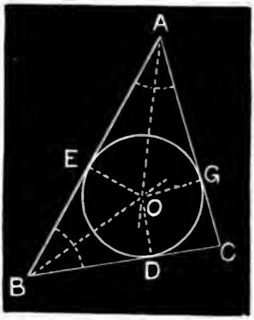

Fig. 137.

DEMONSTRATION OF SOLUTION.

From **O** let fall the perpendiculars **OD**, **OE**, and **OG** on the sides.

Now the triangle **AOE** = **AOG** (?), **BEO** = **BOD** (?).

Hence **OD** = **OE** = **OG**, and the circumference struck from **O** as a centre with a radius **OD**, passes through **E** and **G**.

Moreover, **AC**, **AB**, and **BC** are perpendicular to the radii **OG**, **OE**, and **OD** respectively, and hence are tangents to the circle.

Therefore the circle is inscribed in the triangle. Q. E. D.

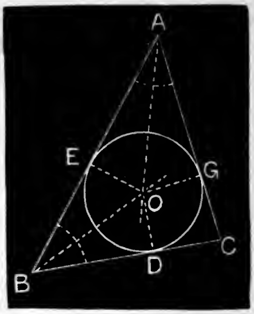

Fig. 137.

PROPOSITION XXIV.

284. Problem.—*In a given circle to inscribe a square, and hence a regular octagon, and then a regular polygon of 16 sides, etc.*

[Let the pupil give the solution and demonstration.]

PROPOSITION XXV.

285. Problem.—*In a given circle to inscribe a regular hexagon, and hence an equilateral triangle and a dodecagon.*

[Let the pupil give the solution.]

PROPOSITION XXVI.

286. Problem.—*To circumscribe a square about a given circle.*

[Let the pupil give the solution.]

PROPOSITION XXVII.

287. Problem.—*To circumscribe an equilateral triangle about a circle.*

[Let the pupil give the solution.]

PROPOSITION XXVIII.

288. Problem.—*To circumscribe a regular hexagon about a given circle.*

[Let the pupil give the solution.]

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289. Query.—Given any regular inscribed polygon, how is the regular circumscribed polygon of the same number of sides constructed?

EXERCISES.

290. 1. Given two angles of a triangle, to find the third.

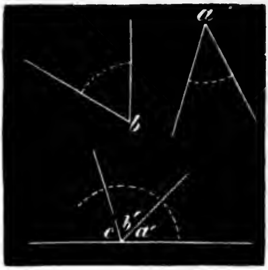

Fig. 138.

Suggestions.—The student should draw two angles on the blackboard, as *a* and *b*, and then proceed to find the third. The figure will suggest the method. The third angle is *c*.

The solution is effected also by constructing the two given angles at the extremities of any line, and producing the sides till they meet (?).

2. What part of a right angle is one of the angles of an equilateral triangle? From this fact, how can you obtain an angle equal to $\frac{1}{3}$ of a right angle?

3. Two angles of a triangle are respectively $\frac{2}{3}$ and $\frac{1}{3}$ of a right angle. What is the third angle?

4. The angles of a triangle are respectively $\frac{2}{3}$, $\frac{1}{2}$, and $\frac{5}{6}$ of a right angle. Which is the greatest side? Which the least? Can you tell the ratio of the sides?

5. What is the value of one of the equal angles of an isosceles triangle whose third angle is $\frac{1}{3}$ of a right angle?

6. Two consecutive angles of a quadrilateral are respectively $\frac{4}{3}$ and $\frac{2}{3}$ of a right angle, and the other two angles are mutually equal to each other. What is the form of the quadrilateral? What the value of each of the two latter angles?

7. One of the angles of a parallelogram is $\frac{5}{3}$ of a right angle. What are the values of the other angles?

8. The two opposite angles of a quadrilateral are respectively $\frac{2}{3}$ and $\frac{4}{3}$ of a right angle. Can a circumference be circumscribed? If so, do it.

9. Two of the opposite sides of a quadrilateral are parallel, and each is 15 in length. What is the figure? Do these facts determine the angles?

10. Two of the opposite sides of a quadrilateral are 12 each, and the other two 7 each. What do these facts determine with reference to the form of the figure?

11. What is the value of an angle of a regular dodecagon?

12. What is the sum of the angles of a nonagon? What is the value of one angle of a regular nonagon? Of one exterior angle?

13. What is the regular polygon, one of whose angles is $1\frac{3}{7}$ right angles?

14. What is the regular polygon, one of whose exterior angles is $\frac{2}{3}$ of a right angle?

15. Can you cover a plane surface with equilateral triangles without overlapping them or leaving vacant spaces? With quadrilaterals? Of what form? With pentagons? Why? With hexagons? Why? What insect puts the latter fact to practical use? Can you cover a plane surface thus with regular polygons of more than 6 sides? Why?

THEOREMS FOR ORIGINAL INVESTIGATION.

[It is quite desirable that students have exercise, early in their course, in the original demonstration of theorems. Those which are given in this and the following lists are not such as are essential to the integrity of an elementary course, and pupils may be encouraged to demonstrate more or less of them, as their time and ability will allow. But all should do some such work—it is the true test of mathematical ability and attainment.]

291. 1. **Theorem.**—*The least chord that can be drawn through a point within a circle is the chord which is perpendicular to a diameter passing through the same point.*

2. **Theorem.**—*The shortest distance from a point without a circle to the circumference is measured in a line which passes through the centre.*

3. **Theorem.**—*The sum of the angles formed by producing the alternate sides of any pentagon is two right angles.*

4. **Theorem.**—*Prove that the sum of the angles of a triangle is two right angles, by producing two of the sides about an angle, and through the vertex of this angle drawing a line parallel to the third side.*

Prove the same by producing one side of the triangle, and drawing a line through the vertex of the exterior angle parallel to the non-adjacent side.

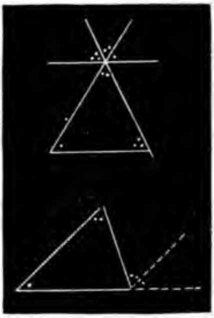

Fig. 139.

5. **Theorem.**—*If* AB *is any chord,* AC *a tangent at* A, *and* CDE *a line parallel to* AB *and cutting the circumference in* D *and* E, *the triangles* ACD, CAE, *and* ADB *are mutually equiangular.*

6. **Theorem.**—*If from any point in the base of an*

isosceles triangle lines are drawn parallel to the equal sides, a parallelogram is formed whose perimeter is equal to the sum of the equal sides.

SECTION VIII

OF EQUALITY.

292. Equality signifies likeness in every respect.

293. The equality of magnitudes is usually shown by applying one to the other, and observing that the two coincide.

OF ANGLES.

PROPOSITION I.

294. Theorem.—*Two angles whose corresponding sides are parallel, and extend in the same or in opposite directions from their vertices, are equal.*

DEMONSTRATION.

First. **In (a) and (a'), let B and E be two angles having BA parallel to ED and extending in the same direction from the vertices, and also BC parallel to EF and extending in the same direction from the vertices.**

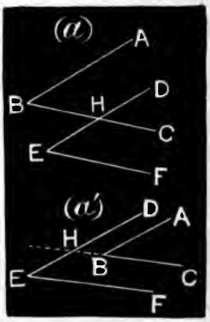

Fig. 140.

We are to prove that angles **B** and **E** are equal.

Produce (if necessary) either two non-parallel sides, as **BC** and **ED**, till they intersect, as in **H**.

$$ABC = DHC\ (?),$$

and $$DHC = DEF\ (?).$$

Therefore, $$ABC = DEF. \quad \text{Q. E. D.}$$

Second. **In (*b*) and (*b*′), let B′ and E′ have B′A′ parallel to E′F′, but extending in an opposite direction from the vertices; and in like manner B′C′ parallel to, but extending in an opposite direction from E′D′.**

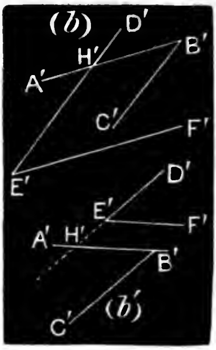

Fig. 141.

We are to prove that **B′** and **E′** are equal.

Produce (if necessary) either two non-parallel sides, as **A′B′** and **E′D′**, till they meet in some point, as **H′**.

D′H′B′ = D′E′F′ (?),

and **D′H′B′ = A′B′C′** (?).

Therefore **D′E′F′ = A′B′C′** (?). Q. E. D.

PROPOSITION II.

295. Theorem.—*Two angles having their corresponding sides parallel, while two extend in the same direction, and the other two in opposite directions from the vertices, are supplemental.*

Demonstration.

Let ABC and DEF be two angles whose corresponding sides BA and EF are parallel and extend in the same direction from B and E, while BC and ED extend in opposite directions from the vertices.

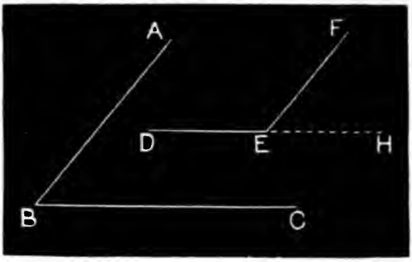

Fig. 142.

We are to prove that **ABC** and **DEF** are supplemental.

Produce one of the two sides having opposite directions, as **DE** to **H**, in the same direction from the vertex that **BC** extends.

Now **DEF** is supplemental to **FEH** (?), and **FEH** is equal to **ABC** (?)

Therefore, **DEF** and **ABC** are supplemental. Q. E. D.

PROPOSITION III.

296. Theorem.—*If the sides of one angle are perpendicular respectively to the sides of another, the angles are either equal or supplemental.*

DEMONSTRATION.

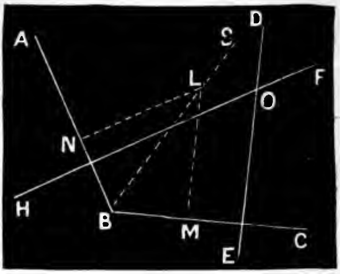

Fig. 143.

Let ABC be any angle and DE and FH be two lines drawn through any point O, DE being perpendicular to BC and FH to AB.

We are to prove that of the four angles **FOD**, **DOH**, etc., two are equal to **ABC**, and two are supplemental.

Draw **BS** bisecting **ABC**, and from any point in this bisector, as **L**, draw **LM** and **LN**, respectively parallel to **DE** and **FH**.

Now, in the quadrilateral **LNBM**, the sum of the four angles is four right angles (266); and, as **LNB** and **LMB** are right angles (?), **NLM** and **NBM** (or **ABC**) are supplemental.

But **NLM** = **FOD** (?) = **HOE** (?).

Therefore two of the four angles **FOD**, **DOH**, etc., namely, **FOD** and **HOE**, are supplemental to **ABC**. Q. E. D.

Finally, **FOE** and **DOH** are supplements of **FOD** and **HOE** (?) and hence equal to **ABC**. Q. E. D.

297. SCHOLIUM.—To determine whether the angles are equal, or whether they are supplemental, we may consider one angle as moved (if necessary) till its vertex falls in the bisector, its sides remaining parallel to their first position. Then, if both sides of one angle extend towards, or both extend from the sides of the other, the angles are supplemental, otherwise they are equal.

OF TRIANGLES.

PROPOSITION IV.

298. Theorem.—*Two triangles which have two sides and the included angle of one equal to two sides and the included angle of the other, each to each, are equal.*

DEMONSTRATION.

Let ABC and DEF be two triangles, having AC = DF, AB = DE, and angle A = angle D.

We are to prove that the triangles are equal.

Place the triangle **ABC** in the position (*b*), the side **AB** in its equal **DE**, and the angle **A** adjacent to its equal angle **D**.

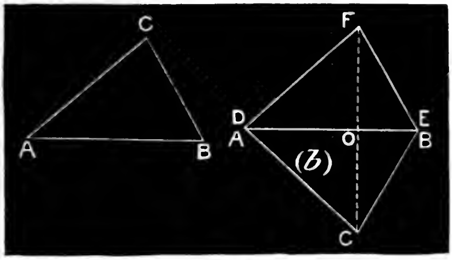

Fig. 144.

Then revolving **ABC** upon **DE**, until it falls in the plane on the opposite side of **DE**, since angle **A** = angle **D**, **AC** will take the direction **DF**; and as **AC** = **DF**, **C** will fall at **F**. Hence **BC** will fall in **EF**, and the triangles will coincide. Therefore the two triangles are equal. Q. E. D.

299. SCHOLIUM 1.—We may also make the application of **ABC** to **DEF** directly. The method here given is used for the purpose of uniformity in this and the following. We may observe that in this, as in the other cases, **DB** is perpendicular to **FC**, and bisects it at **O**.

300. SCHOLIUM 2.—This proposition signifies that the two triangles *are equal in all respects*, *i. e.*, that the two remaining sides are equal, as **CB** = **FE**; that angle **C** = angle **F**, angle **B** = angle **E**, and that the areas are equal.

PROPOSITION V.

301. Theorem.—*Two triangles which have two angles and the included side of the one equal to two angles and the included side of the other, each to each, are equal.*

DEMONSTRATION.

Let ABC and DEF be two triangles, having angle A = angle D, angle B = angle E, and side AB = side DE.

We are to prove that the triangles are equal.

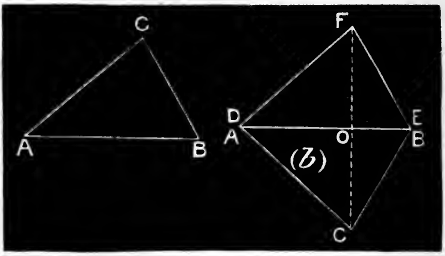

Fig. 145.

Place **ABC** in the position (*b*), the side **AB** in its equal **DE**, the angle **A** adjacent to its equal angle **D**, and **B** adjacent to its equal angle **E**.

Then revolving **ABC** upon **DE** till it falls in the plane on the same side as **DFE**, since angle **A** = angle **D**, **AC** will take the direction **DF**, and **C** will fall somewhere in **DF**, or **DF** produced.

Also, since angle **B** = angle **E**, **BC** will take the direction **EF**, and **C** will fall somewhere in **EF**, or **EF** produced.

Hence, as **C** falls at the same time in **DF** and **EF**, it falls at their intersection **F**. Therefore the two triangles coincide, and are consequently equal. Q. E. D.

302. Corollary.—*If one triangle has a side, its opposite angle, and one adjacent angle, equal to the corresponding parts in another triangle, the triangles are equal.*

For the third angles are equal to each other, since each is the supplement of the sum of the given angles. Whence the case is included in the proposition.

303. Scholium.—A triangle may have a side and one adjacent angle equal to a side and an adjacent angle in another, and the second adjacent angle of the first equal to the angle *opposite* the equal side in the second, and the triangles *not* be equal. Thus, in the figure, **AB** = **C′A′**, **A** = **A′**,

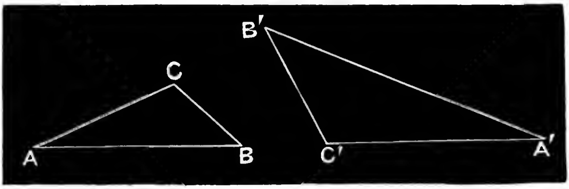

Fig. 146.

and **B** = **B′**; but the triangles are evidently not equal. [Such triangles are, however, *similar*, as will be shown hereafter.]

PROPOSITION VI.

304. Theorem.—*Two triangles which have two sides and an angle opposite one of these sides, in the one, equal to the corresponding parts in the other, are equal, if of these two sides the one opposite the given angle is equal to or greater than the one adjacent.*

DEMONSTRATION.

In the triangles ABC and DEF, let AC = DF, CB = FE, A = D, and CB (= FE) $\geqq$ AC (= DF).

We are to prove that the triangles are equal.

Apply the triangle **ABC** to **DEF**, placing **AC** in its equal **DF**, the point **A** falling at **D**, and **C** at **F**.

Since **A = D**, **AB** will take the direction **DE**.

Let fall the perpendicular **FH** upon **DE**, or **DE** produced.

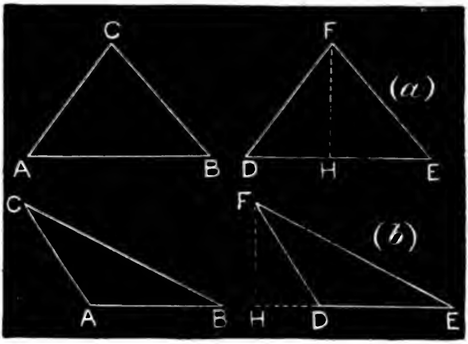

Fig. 147.

Now, **CB** being $\geqq$ **DF**, cannot fall between it and the perpendicular, but must fall in **FD** or beyond both (?).

But **CB** cannot fall in **FD**, since it is a different line from **CA**.

Again, as **CB = FE**, and both lie on the same side of **FH**, they must coincide (**114**).

Hence, the two triangles coincide, and are consequently equal. Q. E. D.

PROPOSITION VII.

305. Theorem.—*Two triangles which have the three sides of the one equal to the three sides of the other, each to each, are equal.*

DEMONSTRATION.

Let ABC and DEF be two triangles, in which AB = DE, AC = DF, and BC = EF.

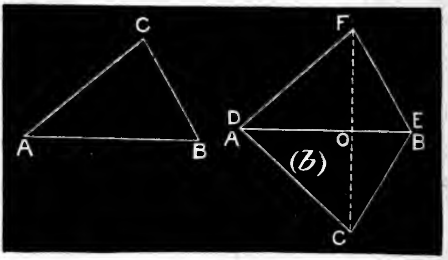

Fig. 148.

We are to prove that the triangles are equal.

Place the triangle **ABC** in the position (*b*), with the longest side, **AB**, in its equal, **DE**, so that the other equal sides shall be adjacent, as **AC** adjacent to **DF**, and **BC** to **EF**. Draw **FC** cutting **DE** in **O**.

Now, since **AC** = **DF**, and **BC** = **EF**, **DE** is perpendicular to **FC** at its middle point (?).

Hence, revolving **ABC** upon **DE**, it will coincide with **DEF** when brought into the plane of the latter, since **OC** will fall in **OF** (?) and is equal to it.

Therefore the two triangles coincide, and hence are equal. Q. E. D.

306. COROLLARY.—*In two equal triangles, the equal angles lie opposite the equal sides.*

PROPOSITION VIII.

307. Theorem.—*If two triangles have two sides of the one respectively equal to two sides of the other, and the included angles unequal, the third sides are unequal, and the greater third side belongs to the triangle having the greater included angle.*

DEMONSTRATION.

Let ABC and DEF be two triangles having AC = DF, CB = FE, and C > F.

We are to prove that **AB** > **DE**.

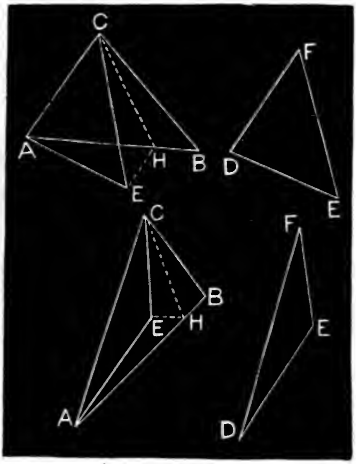

Fig. 149.

Make the angle **ACE** = **DFE**, take **CE** = **FE**, and draw **AE**. Then is the triangle **ACE** = **DFE**, and **AE** = **DE**.

Bisect **ECB** with **CH**.

Now since angle **DFE** = **ACE** < **ACB** by hypothesis, **CE** falls between **CA** and **CB**, and **CH** will meet **AB** in some point, as **H**.

Draw **HE**.

The triangles **HCB** and **HCE** have two sides and the included angle of the one, equal to the corresponding parts of the other, whence **HE** = **HB** (?).

Now $AH + HE > AE$

but $AH + HE = AH + HB = AB$.

Therefore, $AB > AE$, or $AB > DE$. Q. E. D.

308. Corollary.—Conversely, *If two sides of one triangle are respectively equal to two sides of another, and the third sides are unequal, the angle opposite this third side is the greater in the triangle which has the greater third side.*

That is, if **AC** = **DF**, **CB** = **FE**, and **AB** > **DE**, angle **C** > angle **F**. For, if **C** = **F**, the triangles would be equal, and **AB** = **DE** (298); and, if **C** were less than **F**, **AB** would be less than **DE**, by the proposition. But both these conclusions are contrary to the hypothesis. Hence, as **C** cannot be equal to **F**, or less than **F**, it must be greater.

PROPOSITION IX.

309. Theorem.—*Two right-angled triangles which have the hypotenuse and one side of the one equal to the hypotenuse and one side of the other, each to each, are equal.*

Demonstration.

In the two triangles ABC and DEF, right-angled at B and E, let AC = DF, and BC = EF.

We are to prove that the triangles are equal.

Place **FE** in its equal **CB**, with **FD** on the same side of **CB** that **AC** is.

Then, since two equal oblique lines cannot be drawn from **C** to **AB** on the same side of **CB**, **FD** will coincide with **CA**, and **DE** with **AB** (?)

Hence the two triangles are equal, as they coincide throughout when applied (**292**, **293**). Q. E. D.

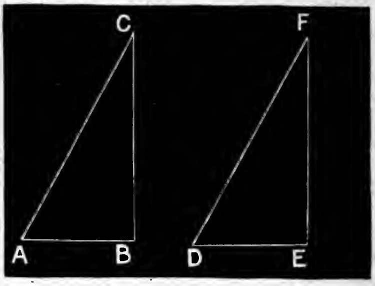

Fig. 150.

PROPOSITION X.

310. Theorem. —*Two right-angled triangles having any side and one acute angle of the one equal to the corresponding parts of the other are equal.*

Demonstration.

One acute angle in one triangle being equal to one in the other, the other acute angles are equal, since they are complements of the same angles (**218**). The case then falls under (**301**).

EXERCISES.

Exercise 1. Given the sides of a triangle, as 15, 8, and 5, to construct the triangle.

Ex. 2. Given two sides of a triangle, $a = 20$, $b = 8$, and the angle **B** opposite the side b equal $\frac{1}{3}$ of a right angle, to construct the triangle.

Ex. 3. Same as in the preceding example, except $b = 12$. Same, except that $b = 25$.

Ex. 4. Construct a triangle with angle **A** $= \frac{2}{3}$ of a right angle, angle **B** $= \frac{1}{2}$ of a right angle, and side a opposite angle **A**, 15.

Ex. 5. Construct an isosceles triangle whose vertical angle is 30°.

Ex. 6. Construct a right-angled triangle whose hypotenuse is 12 and one of whose acute angles is 60°.

Ex. 7. Construct an equilateral triangle, and let fall a perpendicular from one vertex upon the opposite side. How is this angle divided? How many degrees measure the angle between the perpendicular and one side?

THE DETERMINATION OF POLYGONS.

311. A triangle, or any polygon, is said to be **Determined** when a sufficient number of parts are known to enable us to construct the figure, or to find the unknown parts. If two different figures can be constructed, the case is said to be **Ambiguous.**

312. Since, in such a case, if several polygons were to be constructed with the same given parts all would be equal, the conditions which *determine* a polygon are, in general, the same as those which insure equality (**292**). Hence, having shown that certain given parts determine a polygon, we may assert that two polygons having these parts respectively equal are equal, except in the ambiguous cases.

PROPOSITION XI.

313. Theorem.—*A triangle is determined in the following cases:*

I. *When two sides and the included angle are known.*

II. *When two angles and the included side are known.*

III. *When the three sides are known.*

IV. *When two sides and an angle opposite one of them are known.*

(*a.*) *If the known angle is right or obtuse.*

(*b.*) *If the known angle is acute and the known side opposite it is equal to the perpendicular upon the unknown side; or equal to or greater than the other known side.*

(*c.*) *But, if the known angle is acute and the known side opposite it is intermediate in length between the other known side and the perpendicular upon the unknown side, the case is* AMBIGUOUS, i. e., *there are two triangles possible.*

DEMONSTRATION.

The demonstration of this proposition is effected in the solution of the following problems.

---

314. Problem. — *Given two sides and the included angle, to construct a triangle.*

SOLUTION.

Let A and B be the given (or known) sides, and O the given angle.

We are to construct a triangle having an angle equal to **O** included between sides equal to **A** and **B**.

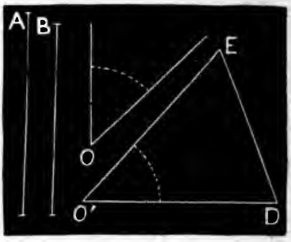

Fig. 151.

Draw any line, as **O′D**, equal to either of the given sides, as **A**.

Lay off at either extremity of **O′D**, as at **O′**, an angle equal to **O** (**203**), and make **O′E** equal to **B**, and draw **ED**.

Then will **EO′D** be the triangle required.

For, if two triangles (or any number) be constructed with the same sides and included angle, they will all be equal to each other (**298**).

---

315. Problem.—*Given two angles and the included side, to construct a triangle.*

SOLUTION.

Let M and N be the two given angles, and A the given side.

We are to construct a triangle having a side equal to **A** and included between the vertices of two angles equal respectively to **M** and **N**.

Draw **DE** equal to **A**. At one extremity, as **D**, make angle **FDE** = **M**, and at **E** make **FED** = **N**.

Then is **DEF** the triangle required (?).

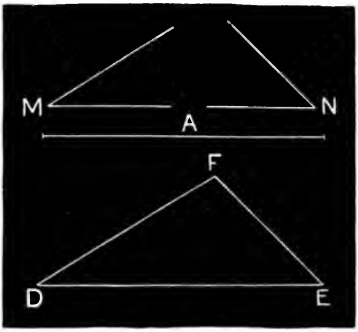

Fig. 152.

QUERY.—What is the limit of the sum of the given angles?

---

316. Problem.—*Given three sides, to construct a triangle.*

SOLUTION.

Let A, B, and C be the three given sides.

We are to construct a triangle which shall have its three sides respectively equal to **A**, **B**, and **C**.

Draw **DE** = **A**.

With **D** as a centre and a radius equal to **B**, strike an arc intersecting an arc struck from **E** as a centre, with a radius **C**.

The triangle **DEF** is the triangle sought (?).

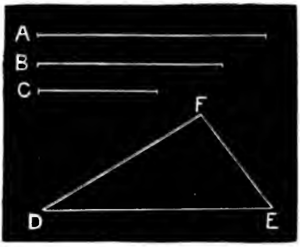

Fig. 153.

317. SCHOLIUM.—If any one of the three proposed sides is greater than the sum or less than the difference of the other two, a triangle is impossible (?).

---

318. Problem.—*To construct a triangle, having given two sides and the angle opposite one of them.*

SOLUTION.

There are three cases.

CASE (*a*).

When the given angle is right or obtuse.

Let O be the angle, and A and B the sides, the angle O to be opposite the side A.

Construct angle **NDM** = **O** (**203**), and take **FD** = **B**.

From **F** as a centre, with **A** as a radius, strike an arc cutting **DM** in **E**, and draw **FE**.

Then is **FDE** the triangle sought.

For it has **FD** = **B**, **FE** = **A** (since **FE** is a radius of a circle struck with **A** as a radius), and angle **FDE**, opposite **FE**, equal to **O**.

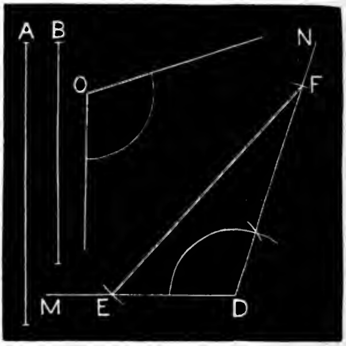

Fig. 154.

If the given angle were right, the construction would be the same.

CASE (*b*).

When the given angle is acute, and, 1st, the side opposite equal to the perpendicular upon the unknown side, and, 2d, when the side opposite is equal to or greater than the other given side.

1st. **Let A and B be the given sides and O the given angle opposite B.**

Proceed exactly as in the preceding case, but when the arc is struck from **F** as a centre with a radius equal to **B**, instead of intersecting **DM** it will be tangent to it, since **B** = **FE** is the perpendicular, and a line which is perpendicular to a radius at its extremity is tangent to the arc (**156**).

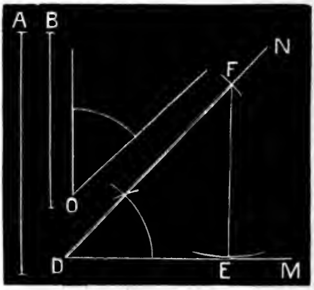

Fig. 155.

2d. If the side opposite the given angle is equal to the other given side, the arc struck from* F *with it as a radius will cut

DM ***at an equal distance with*** **FD** ***from the foot of the perpendicular (?), and the triangle formed will be isosceles (?).***

If the side opposite is greater than the other given side, it will cut **MD** ***but once (?) and there will be but one triangle.***

CASE (*c*).

When the given angle is acute, and the given side opposite it is intermediate in length between the other given side and the perpendicular to the unknown side.

Let A and B be the given sides and O the angle opposite B, B being intermediate in length between A and the perpendicular FH on the unknown side.

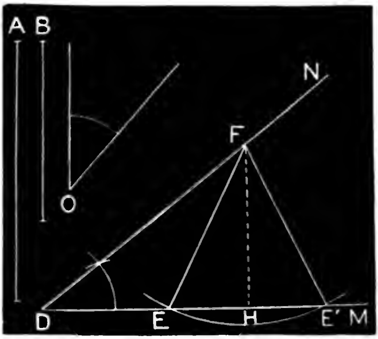

Fig. 156.

Proceed as in the two preceding cases, but instead of tangency we get *two* intersections of **DM** by the arc struck from **F** with radius **B**, as **E** and **E′**, since two equal oblique lines can be drawn from **F** to **DM** (**114**), and **B** being less than **FD** = **A**, **FE** will lie between **FD** and **FH**, and **FE′** beyond **FH** (**113**).

Thus we have two triangles, **DEF** and **DE′F**, each of which fulfills the required conditions.

319. Scholium.—In order that the triangle should be possible, the side opposite the given angle must be equal to or greater than the perpendicular upon the unknown side.

OF QUADRILATERALS.

The subject of the conditions which determine a quadrilateral or other polygon is quite an important and practical subject, especially in surveying, and we treat the problem of the equality of polygons of more than three sides in this way. (See **312**.)

PROPOSITION XII.

320. Theorem.—*A quadrilateral is determined when there are given in their order:*

I. *The four sides and either diagonal.*

II. *The four sides and one angle.*

III. 1st. *Three sides and two included angles.*

2d. *When the two angles are not both included between the known sides, the case may be ambiguous.*

IV. *Three angles and two sides, the unknown sides being non-parallel.*

Demonstration.

CASE I.

Let *a*, *b*, *c*, *d* (Fig. 157), be the sides in order, and *e* the diagonal joining the vertex of the angle between *a* and *d* with the vertex between *b* and *c*.

With **LO** = *a*, **OM** = *b*, **MN** = *c*, **NL** = *d*, and **LM** = *e*, construct, by (**316**), the triangles **LOM** and **MNL**, on **LM** as a common side.

Then is **LOMN** the quadrilateral sought.

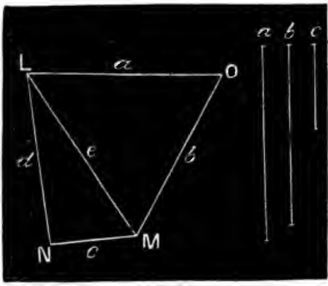

Fig. 157.

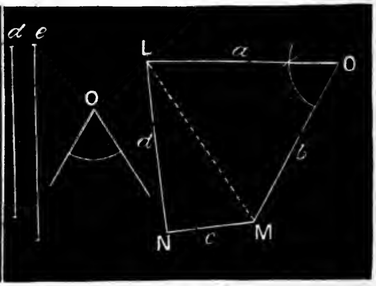

Fig. 158.

CASE II.

Let *a*, *b*, *c*, and *d* (Fig. 158), be the given sides in order, and O the angle included between *a* and *b*.

With the same notation as before, construct the triangle **LOM** by (**314**), and then **LMN** by (**316**), and the quadrilateral is constructed, *i. e.*, all the parts are found.

CASE III.

Let a, b, and c (Fig. 159) be the given sides in order.

1st. Let both the given angles* O *and* M *be included between the given sides,* O *being included by by a and b, and* M *by b and c.

Construct an angle **LOM** $=$ **O**, and take **OL** $= a$ and **OM** $= b$.

Now lay off the angle **OMN** $=$ **M**, and taking **MN** $= c$, draw **LN**.

Then is **LOMN** the quadrilateral sought.

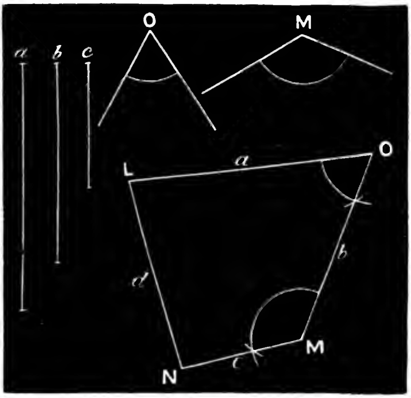

Fig. 159.

2d. AMBIGUOUS CASES. — ***If three sides and two angles of a quadrilateral are given, and both the given angles are not included between given sides, the case may be*** AMBIGUOUS.

There may be three cases: 1st. When the two given angles are consecutive, and one only is included between given sides; 2d. When the given angles are consecutive, and the included side is unknown; 3d. When the given angles are opposite.

Fig. 160 shows how an ambiguous solution may arise under Case 1. The given parts are a, b, c, and angles L and **O**.

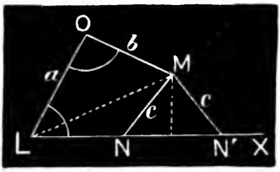

Fig. 160.

Fig. 161 shows how such solutions may arise under cases 2 and 3.

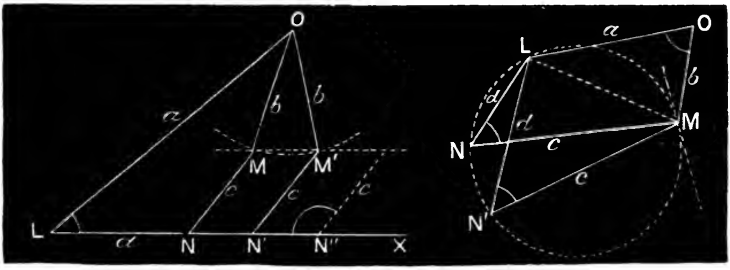

Fig. 161.

CASE IV.

1st. Let the three given angles be* O, M, *and* N, *and, first, let a and b be two consecutive given sides.

Since the sum of the angles of a quadrilateral is 4 right angles, and **O**, **M**, and **N** are given, the fourth, **L**, can be found (**241**).

[Let the student make the construction.]

2d. Let* O, M, *and* N *be the given angles, and a and c the given non-consecutive sides, d and b being non-parallel, i. e., the angles* L *and* O *not being supplemental.

Find the fourth angle by subtracting the sum of the three given angles from 4 right angles. Whence all the angles are known.

Lay off side a and at its extremities make **LOX** = **O**, and **OLY** = **L**.

Then draw any line, **A**m, making the angle m = **M**.

Take m**A** = c, and through **A** draw **AS** parallel to **OX**. Let this intersect **LY** in **N**. Through **N** draw **NM** parallel to **A**m.

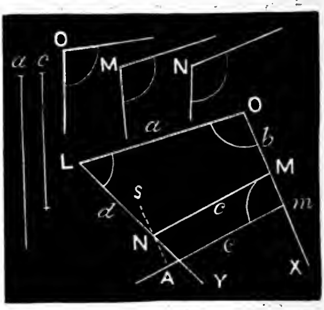

Fig. 162.

Then is **NM** = the given side c (?), and **OMN** = the given angle **M** (?), and **LNM** = the given angle **N** (?).

Hence **LOMN** is the required quadrilateral.

321. Scholium.—With a given set of parts, as above, the possibility of constructing a quadrilateral can be determined on the same principle as the possibility of a triangle.

1. In Case I, if the diagonal is less than the sum and greater than the difference of the sides of either of the two triangles into which it divides the quadrilateral, the quadrilateral is possible, but not otherwise.

2. In Case II, the two given sides and their included angle always make a triangle possible; whence the possibility of the quadrilateral will be determined by the relation of the other two sides to the third side of this triangle, as (Fig. 158) when $c + d >$ **LM**, and $c - d <$ **LM**, the quadrilateral is possible, but not otherwise.

3. In Case III, the 1st problem is always possible. The student will be able to determine when the several cases in the 2d are possible by inspecting Figs. 160 and 161.

4. In Case IV, the first problem is always possible when the sum of

the given angles is less than 4 right angles. In the second problem, if the unknown sides are parallel, the problem is *indeterminate*, *i. e.*, there may be any number of solutions, if any.

NOTE.—In problems of this class, it is usually understood that the given parts are such as to allow the construction; *i.e.*, that they are parts of a possible polygon.

322. COROLLARY 1.—*A parallelogram is determined when two sides and their included angle are given.*

Since the opposite sides of a parallelogram are equal (**250**), all the sides are known when two are given, and the case falls under Case II of the proposition.

323. COROLLARY 2.—*Two rectangles having equal bases and equal altitudes are equal.*

EXERCISE 1. Construct a quadrilateral three of whose consecutive sides are 20, 12, and 15, and the angle included between 20 and the unknown side $\frac{2}{3}$ of a right angle, and that between 15 and the unknown side $\frac{1}{2}$ a right angle.

Ex. 2. Construct a quadrilateral three of whose sides shall be 5, 4.2, and 4, and in which the angle between the unknown side and the side 5 shall be $\frac{1}{3}$ of a right angle, and that between the unknown side and side 4, $1\frac{1}{2}$ right angles. How many solutions are there? How many solutions if the second side is made 1.2, and the third 2? How many if the second side is made 1, and the third 1.5?

OF POLYGONS.

PROPOSITION XIII.

324. Theorem.—*A polygon is determined when two consecutive sides, the diagonals from the vertex of their included angle, and the consecutive angles included between these lines are given.*

[Let the student show how the construction is made, and thus demonstrate the proposition.]

PROPOSITION XIV.

325. Theorem.—*A polygon is determined by means of its sides and angles, when there are given in order:*

I. *All the parts except two angles and their included side.*

II. *All the parts except three angles.*

III. *All the parts except two non-parallel sides.*

CONSTRUCTIONS.

CASE I.

Beginning at one extremity of the unknown side, and constructing the given sides and angles in order till all are constructed, and joining the extremities of the broken line thus drawn, the polygon will be constructed.

CASE II.

1st. When the three angles are consecutive.

Suppose the polygon to be **ABCDEFG**, and the unknown angles **A**, **G**, and **F**. Commencing with side **AB**, lay off the given sides and angles in order till the unknown angle **F** is reached. Then from **F** as a centre, with a radius equal to the known side **FG**, strike an arc intersecting an arc struck from **A** as a centre with the side **AG** as a radius. This intersection determines the remaining vertex of the polygon.

QUERY.—When does this case become impossible?

2d. When two of the unknown angles are consecutive and the third is separated from both the others.

Let **A**, **B**, and **F** be the unknown angles. The two partial polygons **AIHGF** and **BCDEF** can be constructed, and thus the sides **AF** and **BF** will become known, as also the angles **AFG**, **IAF**, **BFE**, and **FBC**.

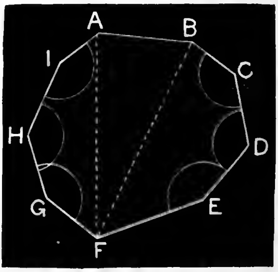

Fig. 163.

Then constructing the triangle **ABF**, whose three sides are now known, the angles **AFB**, **ABF**, and **FAB** become known. Hence all the parts of the polygon are found, for

the angle **GFE** = **AFG** + **AFB** + **BFE**, etc.

3d. When no two of the three unknown angles are consecutive.

Let **A**, **C**, and **F** be the unknown angles.

Constructing the broken lines **ABC**, **CDEF**, and **FGHIA** separately, and apart from the position where the polygon is to be constructed, the diagonals which form the sides of the triangle **ACF** can be determined by joining the extremities **A** and **C**, **C** and **F**, and **F** and **A**.

This triangle can then be constructed in the position desired, and the broken lines constructed on its sides, as in the figure.

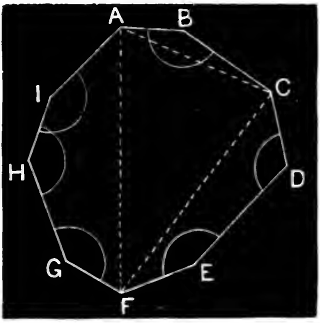

Fig. 164.

CASE III.

Under this case we have two problems:

1st. When the two unknown sides are consecutive.

2d. When the two unknown sides are separated.

[The student will be able to effect the construction. The first is similar to that of Case II, 1st problem. The second is effected by obtaining a quadrilateral similarly to the construction in Case II, 3d problem.

326. In case the unknown parts are two parallel sides, as *a* and *b*, it is evident that these may be varied in length at pleasure without changing the value of the other parts.

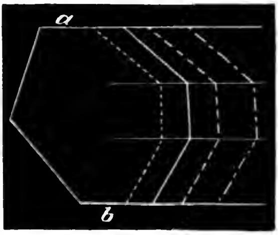

Fig. 165.

327. It will be a profitable exercise for the student to reduce the determination of polygons to that of quadrilaterals, and both to that of triangles.

PROPOSITIONS FOR ORIGINAL SOLUTION AND DEMONSTRATION.

328. 1. **Theorem.**—*The sum of the exterior angles of a polygon is four right angles.*

Prove by drawing lines from a point and parallel to the sides of the polygon.

2. **Theorem.**—*The sum of the angles of a polygon is twice as many right angles as the polygon has sides, less four right angles.*

Having proved the preceding, base the proof of this upon that.

3. **Theorem.**—*If the sum of two opposite sides of a quadrilateral is equal to the sum of the other two opposite sides, show that a circle can be inscribed in the quadrilateral.*

4. **Theorem.**—*If from a point without a circle two tangents are drawn, and also a chord joining the points of tangency, the angle included between a radius drawn to either point of tangency and the chord is half the angle included between the tangents.*

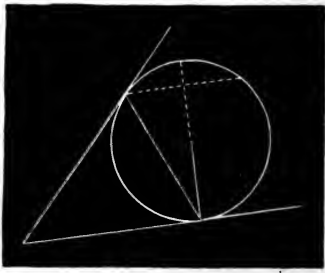

Fig. 166.

5. **Theorem.**—*In an isosceles triangle the line drawn from the vertex to the middle of the base bisects the triangle and also the angle at the vertex.*

6. **Problem.**—*With a given radius draw a circle tangent to the sides of a given angle.*

7. **Problem.**—*Through a given point within a given angle draw a line which shall make equal angles with the sides.*

8. **Problem.**—*To draw a circumference through two given points and having its centre in a given line; or, to find in a given line a point equally distant from two points out of that line.*

9. **Theorem.**—*If from the extremities of a diameter perpendiculars are let fall on any secant, the parts intercepted between the feet of these perpendiculars and the circumference are equal.*

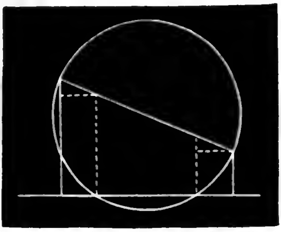

Fig. 167.

10. **Problem.**—*To trisect a right angle.*

Suggestion.—What is the value of an angle of an equilateral triangle?

SECTION IX.

OF EQUIVALENCY AND AREA.

329. Equivalent Figures are such as are equal in magnitude.

330. The **Area** of a surface is the number of times it contains some other surface taken as a unit of measure; or it is the ratio of one surface to another assumed as a standard of measure.

PROPOSITION I.

331. Theorem.—*Parallelograms having equal bases and equal altitudes are equivalent.*

DEMONSTRATION.

Let ABCD and EFGH be two parallelograms having equal bases, BC and FG, and equal altitudes.

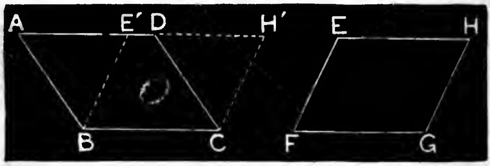

Fig. 168.

We are to prove that the parallelograms are equivalent.

Apply **EFGH** to **ABCD**, placing **FG** in its equal **BC**; and, since the altitudes are equal, the upper base **EH** will fall in **AD** or **AD** produced, as **E′H′**.

Now, the two triangles **AE′B** and **DH′C** are equal, since they have two sides and the included angle of the one equal to two sides and the included angle of the other; viz., **AB** = **DC**, being opposite sides of a parallelogram; and for a like reason **BE′** = **CH′**. Also, angle **ABE′** = angle **DCH′**, by reason of the parallelism of their sides (**294**).

These triangles being equal,

the quadrilateral **ABCH′** — the triangle **AE′B** = **ABCH′** — **DH′C**.

But **ABCH′** — **AE′B** = **E′BCH′** = **EFGH**;

and **ABCH′** — **DH′C** = **ABCD**.

Hence, **ABCD** = **EFGH**. Q. E. D.

332. COROLLARY.—*Any parallelogram is equivalent to a rectangle having the same base and altitude.*

PROPOSITION II.

333. Theorem.—*A triangle is equivalent to one-half of any parallelogram having an equal base and an equal altitude with the triangle.*

DEMONSTRATION.

Let ABC (Fig. 169) be a triangle.

We are to prove that **ABC** is equivalent to one-half a parallelogram having an equal base and an equal altitude with the triangle.

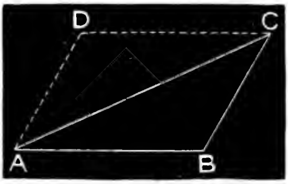

Fig. 169.

Consider **AB** as the base of the triangle, and complete the parallelogram **ABCD** by drawing **AD** parallel to **BC**, and **DC** to **AB**.

Now **ABCD** has the same base, **AB**, as the triangle, and the same altitude, since the altitude of each is the perpendicular distance between the parallels **DC** and **AB**.

But **ABC** is half of **ABCD** (**249**), and as any other parallelogram having an equal base and altitude with **ABCD** is equivalent to **ABCD** (**331**), **ABC** is equivalent to one-half of *any* parallelogram having an equal base and altitude with **ABC**. Q. E. D.

334. Corollary 1.—*A triangle is equivalent to one-half of a rectangle having an equal base and an equal altitude with the triangle.*

335. Corollary 2.—*Triangles of equal bases and equal altitudes are equivalent,* for they are halves of equivalent parallelograms.

PROPOSITION III.

336. Theorem.—*The square described on n times a line is n^2 times the square described on the line, n being any integer.*

Demonstration.

Let u be any line and AB a line n times as long, n being any integer.

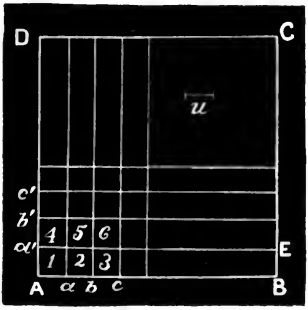

Fig. 170

We are to prove that the square described on **AB** is n^2 times the square on **A**a.

Construct on **AB** the square **ABCD**.

Since u is a measure (**76**) of **AB**, by hypothesis, divide **AB** into n equal parts by applying u, and at the points of division a, b, c, etc., draw parallels to **AD**.

In like manner divide **AD**, and draw through the points of division a', b', c', etc., parallels to **AB**.

Then are the surfaces 1, 2, 3, 4, 5, 6,

etc., squares, since their opposite sides are parallel (**139**) and equal (**138**), and their angles are right angles (**125**).

Now of these squares there are n in each of the rectangles a'**B**, b'**E**, etc. (?), and as there are n divisions in **AD**, there are n rectangles.

Hence there are n times n, or n^2 squares in **ABCD**. Q. E. D.

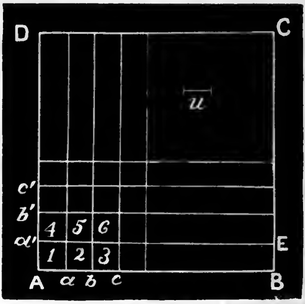

Fig. 170.

337. Corollary.—*The square described on twice a line is four times the square described on the line; that on 3 times a line is 9 times the square on the line, etc.*

PROPOSITION IV.

338. Theorem.—*A trapezoid is equivalent to two triangles having for their bases the upper and lower bases of the trapezoid, and for their common altitude the altitude of the trapezoid.*

By constructing any trapezoid, and drawing either diagonal, the student can show the truth of this theorem.

PROPOSITION V.

339. Problem.—*To reduce any polygon to an equivalent triangle.*

Solution.

Let ABCDEF (Fig. 171) be a polygon.

We are to reduce it to an equivalent triangle.

Draw any diagonal, as **EC**, between two alternate vertices, and through the intermediate vertex, **D**, draw **DH** parallel to **EC** and meeting **BC** produced in **H**. Then draw **EH**.

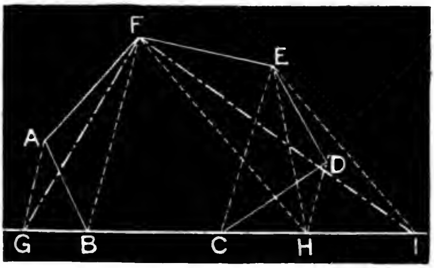

Fig. 171.

In like manner, draw **FH**, and through **E** draw **EI** parallel thereto, meeting **BH** produced in **I**. Then draw **FI**.

Again, draw the diagonal **FB**, and through **A** draw **AG** parallel thereto, meeting **BC** produced in **G**. Then draw **FG**.

Now **FGI** is equivalent to **ABCDEF**.

Demonstration of Solution.

Consider the polygon **ABCDEF** as diminished by **ECD** and then increased by **ECH**. Since these triangles have the same base **EC**, and the same altitude (as their vertices lie in **DH** parallel to **EC**, and parallels are everywhere equidistant), the triangles are equivalent (**335**). Hence, **ABHEF** is equivalent to **ABCDEF** (?).

In like manner **ABIF** is equivalent to **ABHEF**, and **FGI** to **ABIF**.

Hence **FGI** is equivalent to **ABCDEF**. Q. E. D.

AREA.

340. An **Infinitesimal** is a quantity conceived under such a law as to be less than any assignable quantity.

Illustration.—Consider a line of any finite length, as one foot. Conceive this line bisected, and one-half taken. Again conceive this half bisected, and one-half of it taken. By this process it is evident that the line may be reduced to a line less than any assignable line. Moreover, if the process be considered as repeated infinitely, the result is an infinitesimal.

This is the familiar conception of the last term of a decreasing infinite progression, the last term of which is called zero.

341. Principle I.—*In comparison with finite quantities, an infinitesimal is zero.*

Thus, suppose $$\frac{m}{n} = a,$$

m, n, and a being finite quantities. Let i represent an infinitesimal; then

$$\frac{m \pm i}{n}, \text{ or } \frac{m}{n \pm i}, \text{ or } \frac{m \pm i}{n \pm i},$$

is to be considered as still equal to a, for to consider it to differ from a by any amount we might name, would be to assign *some* value to i.

342. Principle II.—*Any two geometrical magnitudes of the same kind are to be conceived as commensurable by an infinitesimal unit.*

By the process for obtaining the common measure of two lines (**84**), the remainder may be made (in conception) less than any assignable quantity, and hence in comparison with the lines should be considered zero.

The same conception may be applied to any geometrical magnitudes.

PROPOSITION VI.

343. Theorem.– *Rectangles are to each other* as the products of their respective bases and altitudes.*

First Demonstration.

Lemma.—***Two rectangles of equal altitudes are to each other as their bases.***

Let ABCD and *abcd* be two rectangles having their altitudes AD and *ad* equal.

Suppose rectangle **ABCD** generated by the movement of **AD** from **AD** to **BC**, it remaining all the time parallel to its first position, and suppose *abcd* generated in like manner by the movement of *ad*.

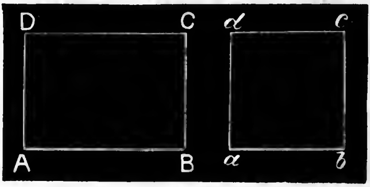

Fig. 172.

Let these equal generatrices **AD** and *ad* move with uniform and equal velocities; then it is evident that the surfaces generated will be as the distances **AB** and *ab*.

That is, $$\frac{\text{ABCD}}{abcd} = \frac{\text{AB}}{ab}.$$

* This is a common elliptical form, meaning that surfaces, or areas, are to each other.

Now let M and N be any two rectangles, the base of M being AB and the altitude BC, and the base of N BE and its altitude BG.

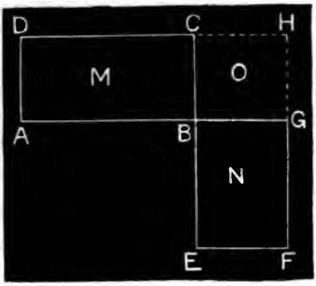

Fig. 173.

We are to prove that

$$\frac{\mathsf{M}}{\mathsf{N}} = \frac{\mathsf{AB} \times \mathsf{BC}}{\mathsf{BE} \times \mathsf{BG}}.$$

Place the rectangles so that the angles **ABC** and **GBE** shall be opposite, *i. e.*, so that **AG** and **CE** shall be straight lines (**109**).

Complete the rectangle **CBGH**, and call it **O**.

Since **M** and **O** have equal altitudes,

$$\frac{\mathsf{M}}{\mathsf{O}} = \frac{\mathsf{AB}}{\mathsf{BG}}. \qquad (1)$$

In like manner, since **N** and **O** have equal altitudes,

$$\frac{\mathsf{N}}{\mathsf{O}} = \frac{\mathsf{BE}}{\mathsf{BC}}. \qquad (2)$$

Dividing the members of (1) by the corresponding members of (2), we have

$$\frac{\mathsf{M}}{\mathsf{N}} = \frac{\mathsf{AB} \times \mathsf{BC}}{\mathsf{BE} \times \mathsf{BG}}. \quad \text{Q. E. D.}$$

Second Demonstration.

Let ABCD and EFGH be any two rectangles.

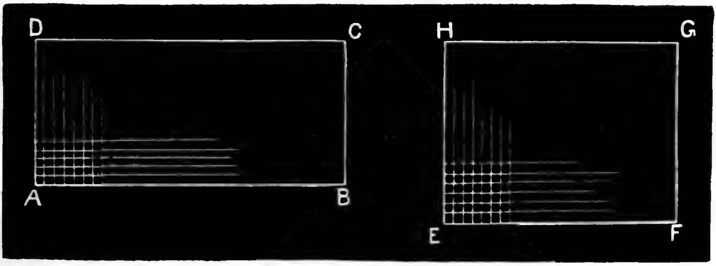

Fig. 174.

We are to prove that $\frac{\mathsf{ABCD}}{\mathsf{EFGH}} = \frac{\mathsf{AB} \times \mathsf{AD}}{\mathsf{EF} \times \mathsf{EH}}.$

The bases and altitudes of the two rectangles are at least to be considered as commensurable by an infinitesimal unit (**342**).

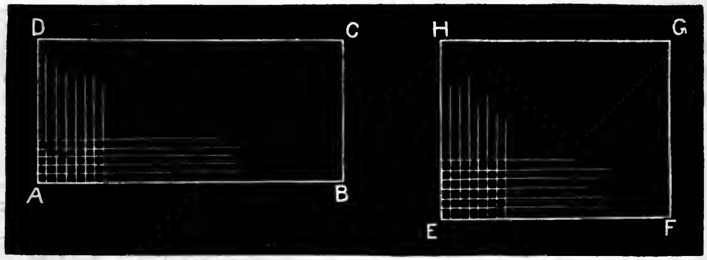

Fig. 174.

Let i be the common measure of **AB**, **AD**, **EF**, and **EH**, and suppose it contained in **AB** m times, in **AD** n times, in **EF** p times, and in **EH** q times.

Whence, $m = \frac{AB}{i}$, $n = \frac{AD}{i}$, $p = \frac{EF}{i}$, and $q = \frac{EH}{i}$.

Now conceive the rectangles divided into squares by drawing through the points of division of the bases and altitudes parallels to the altitudes and bases, as in (**336**), whence the rectangles will be divided into equal squares.

Of these equal squares, **ABCD** contains $m \times n$, and **EFGH** $p \times q$.

Therefore $$\frac{ABCD}{EFGH} = \frac{m \times n}{p \times q} = \frac{\frac{AB}{i} \times \frac{AD}{i}}{\frac{EF}{i} \times \frac{EH}{i}} = \frac{AB \times AD}{EF \times EH}.$$ Q. E. D.

PROPOSITION VII.

344. Theorem.—*The area of a rectangle is equal to the product of its base and altitude.*

DEMONSTRATION.

Let ABCD be a rectangle.

We are to prove that its area is **AB** × **AD**.

Let the square u be the proposed unit of measure, whose side is 1.

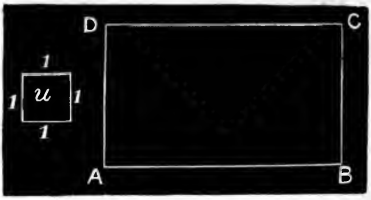

Fig. 175.

By (**343**), $\frac{\mathsf{ABCD}}{u} = \frac{\mathsf{AB} \times \mathsf{AD}}{1 \times 1} = \mathsf{AB} \times \mathsf{AD}$.

Hence, by (**330**), area $\mathsf{ABCD} = \mathsf{AB} \times \mathsf{AD}$. Q. E. D.

345. Corollary 1.—*The area of a square is equal to the second power of one of its sides,* as in this case the base and altitude are equal.

346. Corollary 2.—*The area of any parallelogram is equal to the product of its base and altitude;* for any parallelogram is equivalent to a rectangle of the same base and altitude (**332**).

347. Corollary 3.—*The area of a triangle is equal to one-half the product of its base and altitude;* for a triangle is one-half of a parallelogram of the same base and altitude (**333**).

348. Corollary 4.—*Parallelograms or triangles of equal bases are to each other as their altitudes; of equal altitudes, as their bases; and in general they are to each other as the product of their bases by their altitudes.*

349. Scholium.—The arithmetical signification of the theorem, *The area of a rectangle is equal to the product of its base and altitude,* is this:

Let the base be b and the altitude a; then we have, by the proposition,

$$\text{area} = ab.$$

Now, in order that ab may represent a surface, one of the factors must be conceived as a surface and the other as a number. Thus, we may conceive b to represent b superficial units, *i. e.*, the rectangle having the base of the rectangle for its base and being 1 linear unit in altitude.

The entire rectangle is, then, a times the rectangle which contains b superficial units, or ab superficial units.

In the expression

$$\text{area } \mathsf{ABCD} = \mathsf{AB} \times \mathsf{AD},$$

AB and **AD** may be given a similar interpretation.

PROPOSITION VIII.

350. Theorem.—*The area of a trapezoid is equal to the product of its altitude into one-half the sum of its parallel sides, or, what is the same thing, the product of its altitude into a line joining the middle points of its inclined sides.*

Demonstration.

Let ABCD be a trapezoid, whose parallel sides are AB and DC, and whose altitude is IK.

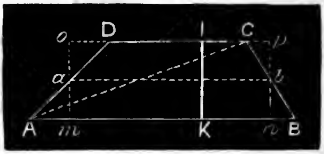

Fig. 176.

We are to prove, 1st, that

$$\text{area } \mathbf{ABCD} = \mathbf{IK} \times \frac{\mathbf{AB} + \mathbf{CD}}{2},$$

and, 2d, that area $\mathbf{ABCD} = \mathbf{IK} \times ab$,

ab being a line joining the middle points of **AD** and **BC**.

Draw either diagonal, as **AC**. The trapezoid is thus divided into two triangles, whose areas are together equal to one-half the product of their common altitude (the altitude of the trapezoid) into their bases **DC** and **AB**, or this altitude into $\frac{1}{2}(\mathbf{AB} + \mathbf{DC})$. Q. E. D.

At a and b draw the perpendiculars om and pn, meeting **DC**, produced, if necessary.

Now the triangles $ao\mathbf{D}$ and $\mathbf{A}am$ are equal, since

$$\mathbf{A}a = a\mathbf{D},$$

$$\text{angle } o = \text{angle } m,$$

both being right, and angle $oa\mathbf{D} = \mathbf{A}am$, being opposite. Whence

$$\mathbf{A}m = o\mathbf{D}.$$

In like manner, we may show that

$$\mathbf{C}p = n\mathbf{B}.$$

Hence, $ab = \frac{1}{2}(op + mn)$ (?) $= \frac{1}{2}(\mathbf{AB} + \mathbf{DC})$; and area **ABCD**, which equals $\frac{1}{2}(\mathbf{AB} + \mathbf{DC}) \times \mathbf{IK}, = ab \times \mathbf{IK}$. Q. E. D.

PROPOSITION IX.

351. Theorem.—*The area of a regular polygon is equal to one-half the product of its apothem into its perimeter.*

DEMONSTRATION.

Let ABCDEFG be a regular polygon, whose perimeter is AB + BC + CD + DE + EF + FG + GA, and whose apothem is O*a*.

We are to prove that

$$\text{area } ABCDEFG = \tfrac{1}{2}Oa(AB+BC+CD+DE+EF+FG+GA).$$

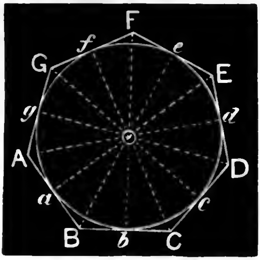

Fig. 177.

Draw the inscribed circle, the radii O*a*, O*b*, etc., to the points of tangency, and the radii of the circumscribed circle OA, OB, etc. (273, 274).

The polygon is thus divided into as many equal triangles as it has sides.

Now, the apothem (or radius of the inscribed circle) is the common altitude of these triangles, and their bases make up the perimeter of the polygon.

Hence, the area $= \tfrac{1}{2}Oa(AB+BC+CD+DE+EF+FG+GA)$. Q. E. D.

352. COROLLARY.—*The area of any polygon in which a circle can be inscribed is equal to one-half the product of the radius of the inscribed circle into the perimeter.*

The student should draw a figure and observe the fact. It is especially worthy of note in the case of a triangle. See Fig. 137.

PROPOSITION X.

353. Lemma.—*If any polygon is circumscribed about a circle and a second polygon is formed by drawing tangents to the arcs intercepted between the consecutive points of tangency, thus forming a polygon of double the number of sides, the perimeter of the second polygon is less than that of the first.*

DEMONSTRATION.

Let ABCDE be any circumscribed polygon, whose consecutive sides are tangent at K, F, G, etc., and let a second polygon be formed by drawing tangents at *f*, *g*, etc.

We are to prove that the perimeter $ab+bc+cd$, etc., is less than the perimeter $\mathsf{EA}+\mathsf{AB}+$etc.

Observing the portions of the perimeters from **K** to **F**, for the first polygon we have

$$\mathsf{KA}+\mathsf{AF}=\mathsf{K}a+(a\mathsf{A}+\mathsf{A}b)+b\mathsf{F},$$

and for the second

$$\mathsf{K}a+ab+b\mathsf{F}.$$

But $ab<a\mathsf{A}+\mathsf{A}b$ (?).

Hence,

$$\mathsf{K}a+ab+b\mathsf{F}<\mathsf{KA}+\mathsf{AF}.$$

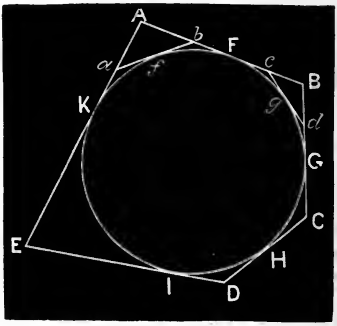

Fig. 178.

Now, as a similar reduction will take place at each vertex, the entire perimeter of the second polygon will be less than that of the first. Q. E. D.

354. The **Limit** of a varying quantity is a fixed quantity which it approaches by such a law as to be capable of being made to differ from it by less than any assignable quantity.

Such a varying quantity is often spoken of as reaching its limit after an infinite number of steps of approach.

355. COROLLARY.—*As the number of the sides of a circumscribed regular polygon is increased the perimeter is diminished, and approaches the circumference of the circle as its limit,* since the circle is the limit of such a polygon.

PROPOSITION XI.

356. Theorem.—*The area of a circle is equal to one-half the product of its radius into its circumference.*

DEMONSTRATION.

Let O*a* (Fig. 179) be the radius of the circle.

We are to prove that the area of the circle is $\frac{1}{2}Oa \times$ *the circumference.*

Circumscribe any regular polygon.

Now the area of this polygon is one-half the product of its apothem and perimeter.

Conceive the number of sides of the polygon indefinitely increased, the polygon still continuing to be circumscribed and regular.

The apothem continues to be the radius of the circle, and the perimeter approaches the circumference.

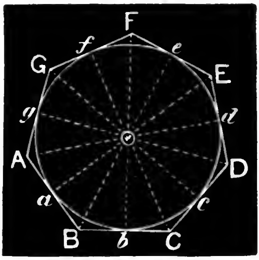

Fig. 179.

When, therefore, the number of sides of the polygon becomes infinite, it is to be considered as coinciding with the circle, and its perimeter with the circumference **(355)**.

Hence the area of the circle is equal to one-half the product of its radius into its circumference. Q. E. D.

357. A **Sector** is a part of a circle included between two radii and their intercepted arc.

358. Corollary 1.—*The area of a sector is equal to one-half the product of the radius into the arc of the sector.*

359. Corollary 2.—*The area of a sector is to the area of the circle as the arc of the sector is to the circumference, or as the angle of the sector is to four right angles.*

EXERCISES.

360. 1. What is the area in acres of a triangle whose base is 75 rods and altitude 110 rods?

2. What is the area of a right-angled triangle whose sides about the right angle are 126 feet and 72 feet?

3. If two lines are drawn from the vertex of a triangle to the base, dividing the base into parts which are to each other as 2, 3, and 5, how is the triangle divided? How does a line drawn

from an angle to the middle of the opposite side divide a triangle?

4. What is the area of the largest triangle which can be inscribed in a circle whose radius is 12, the diameter being one side?

5. What is the area of a cross section of a ditch which is 6 feet wide at the bottom, 9 feet at the top, and 3 feet deep?

6. If one of the angles at the base of an isosceles triangle is double the angle at the vertex, how many degrees in each?

SECTION X.

OF SIMILARITY.

361. The primary notion of similarity is *likeness of form.* Two figures are said to be similar which have the same shape, although they may differ in magnitude. A more scientific definition is as follows:

362. Similar Figures are such as have their angles respectively equal, and their homologous sides proportional.

363. Homologous Sides of similar figures are those which are included between equal angles in the respective figures.

364. *In similar triangles, the homologous sides are those opposite the equal angles.*

The student should be careful, at the outset, to mark the fact that *similarity involves two things,* EQUALITY OF ANGLES and PROPORTIONALITY OF SIDES. It will appear that, in the case of triangles, if *one* of these facts exists, the other exists also; but this is not so in other polygons.

365. Two figures are said to be **Mutually** equiangular when each angle in one has an equal angle in the other, and **Mutually** equilateral when each side in the one has an equal side in the other.

PROPOSITION I.

366. **Theorem.**—*Triangles which are mutually equiangular are similar.*

DEMONSTRATION.

Let ABC and DEF be two mutually equiangular triangles, in which A = D, B = E, C = F.

We are to prove that the sides opposite these equal angles are proportional, and thus that the triangles possess both the requisites of similarity, viz., equality of angles and proportionality of sides.

Lay off on **CA** **CD′** = **FD**, and on **CB** **CE′** = **FE**, and draw **D′E′**.

Triangle **CD′E′** equals triangle **FDE** (?).

Draw **AE′** and **BD′**.

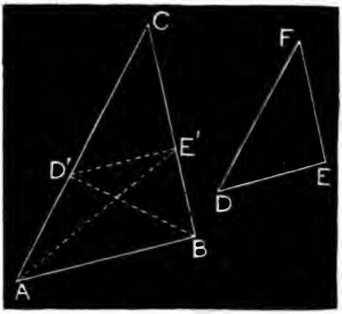

Fig. 180.

Since angle **CE′D′** = **CBA**, **D′E′** is parallel to **AB** (?), and as the triangles **D′E′B** and **D′E′A** have a common base **D′E′** and the same altitudes, their vertices being in a line parallel to their base, they are equivalent (**335**).

Now the triangles **CD′E′** and **D′E′A**, having a common altitude, are to each other as their bases (**348**).

Hence, $$\frac{CD'E'}{D'E'A} = \frac{CD'}{D'A}.$$

For like reason, $$\frac{CD'E'}{D'E'B} = \frac{CE'}{E'B}.$$

Whence, as $$D'E'B = D'E'A,$$

$$\frac{CD'}{D'A} = \frac{CE'}{E'B}.$$

By composition, $$\frac{CD'}{CD' + D'A} = \frac{CE'}{CE' + E'B},$$

or $$\frac{CD'}{CA} = \frac{CE'}{CB},$$

or $$\frac{FD}{CA} = \frac{FE}{CB}.$$

In a similar manner, by laying off **ED** and **EF** in **BA** and **BC** respectively, we can show that

$$\frac{FE}{CB} = \frac{ED}{BA}.$$

Hence, $$\frac{FD}{CA} = \frac{FE}{CB} = \frac{ED}{BA}. \quad \text{Q. E. D.}$$

367. Corollary 1.—*If two triangles have two angles of one respectively equal to two angles of the other, the triangles are similar* (?).

368. Corollary 2.—*A transversal parallel to any side of a triangle divides the other sides proportionally, and the sides are in the ratio of either two corresponding segments.*

For in the demonstration we have **D′E′** parallel to **AB**, and

$$\frac{CD'}{D'A} = \frac{CE'}{E'B},$$

or $$\frac{CD'}{CE'} = \frac{D'A}{E'B}.$$

And also $$\frac{CD'}{CA} = \frac{CE'}{CB},$$

or, by alternation, $$\frac{CA}{CB} = \frac{CD'}{CE'} = \frac{D'A}{E'B}.$$

PROPOSITION II.

369. Theorem.—*If any two transversals cut a series of parallels, their intercepted segments are proportional.*

Demonstration.

1st. Let **OA** and **O′B′** (Fig. 181) be any two parallel transversals cutting the series of parallels ab, cd, ef, gh, etc.

We are to prove that $\frac{ac}{bd} = \frac{ce}{df} = \frac{eg}{fh}$, etc.

Now $\frac{ac}{bd} = 1, \quad \frac{ce}{df} = 1, \quad \frac{eg}{fh} = 1,$ etc. (?)

Hence,

$$\frac{ac}{bd} = \frac{ce}{df} = \frac{eg}{fh}, \text{ etc.}$$

Q. E. D.

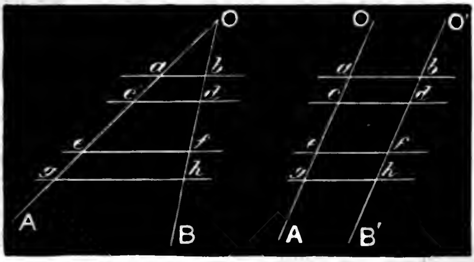

Fig. 181.

2d. **Let OA and OB be any non-parallel transversals cutting *ab*, *cd*, *ef*, *gh*, etc.**

We are to prove

$$\frac{ac}{bd} = \frac{ce}{df} = \frac{eg}{fh}, \text{ etc.}$$

Since **OA** and **OB** are non-parallel, they meet in some point, as **O**.

Then, by (**368**), we have $\frac{\mathbf{O}c}{\mathbf{O}d} = \frac{ac}{bd}$

and $$\frac{\mathbf{O}c}{\mathbf{O}d} = \frac{ce}{df}.$$

Whence, by equality of ratios, we have

$$\frac{ac}{bd} = \frac{ce}{df}.$$

Similarly, we may show that $\frac{ce}{df} = \frac{eg}{fh}$, etc.

Hence, also, by alternation, and by equality of ratios,

$$\frac{ac}{ce} = \frac{bd}{df}, \quad \frac{ac}{bd} = \frac{eg}{fh}, \quad \text{and} \quad \frac{ac}{eg} = \frac{bd}{fh}, \text{ etc.} \quad \text{Q. E. D.}$$

PROPOSITION III.

370. Theorem.—Conversely to Prop. I, *If two triangles have their corresponding sides proportional, they are similar.*

DEMONSTRATION.

Let ABC and DEF have $\frac{\mathbf{AC}}{\mathbf{DF}} = \frac{\mathbf{CB}}{\mathbf{FE}} = \frac{\mathbf{BA}}{\mathbf{ED}}$.

We are to prove that **ABC** is similar to **DEF**.

As one of the characteristics of similarity, viz., proportionality of sides, exists by hypothesis, we have only to prove the other, *i. e.*, that

$$A = D, \ C = F, \text{ and } B = E.$$

Make $CD' = FD$, and draw $D'E'$ parallel to **AB**.

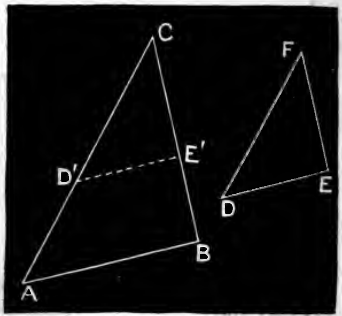

Fig. 182.

Then, by (**368**),

$$\frac{CA}{CD'} = \frac{CB}{CE'};$$

and since by construction $\qquad CD' = FD,$

and by hypothesis $\qquad \frac{CA}{FD} = \frac{CB}{FE},$

$$\therefore CE' = FE.$$

Again the triangles **D′E′C** and **ABC** are mutually equiangular, since **C** is common, angle $CD'E' = CAB$ (?), and angle $CE'D' = CBA$ (?).

Whence $\qquad \frac{CA}{CD'} = \frac{AB}{D'E'}.$

But by hypothesis and construction

$$\frac{CA}{CD'} = \frac{CA}{DF} = \frac{AB}{DE}.$$

Hence $D'E' = DE$, and the triangles **CD′E′** and **DEF** are equal (?).

Therefore **ABC** and **D′E′C** are similar; and as $D'E'C = DEF$, **ABC** and **DEF** are similar. Q. E. D.

371. Scholium.—As we now know that if two triangles are mutually equiangular, they are similar; or, if they have their corresponding sides proportional, they are similar, it will be sufficient hereafter, in any given case, to prove *either one* of these facts, in order to establish the similarity of two triangles. For, either fact being proved, the other follows as a consequence.

PROPOSITION IV.

372. Theorem.—*Two triangles which have the sides of the one respectively parallel or perpendicular to the sides of the other, are similar.*

DEMONSTRATION.

Let ABC and A′B′C′ be two triangles whose sides are respectively parallel or perpendicular to each other.

We are to prove that the triangles are similar.

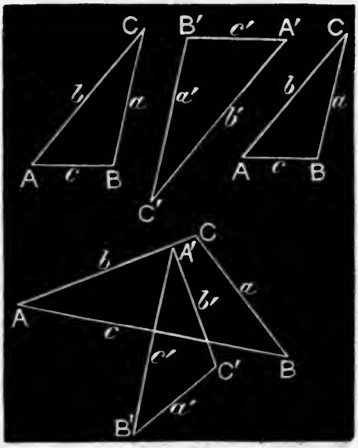

Fig. 183.

Any angle in one triangle is either equal or supplemental to the angle in the other which is included between the sides which are parallel or perpendicular to its own sides. Thus, **A** either equals **A′**, or **A** + **A′** = 2 right angles (**294**, **295**, **296**).

Now, if the corresponding angles are all supplemental, that is, if

A + **A′** = 2 right angles,
B + **B′** = 2 right angles,
and **C** + **C′** = 2 right angles,

the sum of the angles of the two triangles is 6 right angles, which is impossible.

Again, if one angle in one triangle equals the corresponding angle in the other, as **A** = **A′**, and the other angles are supplemental, the sum is 4 right angles plus twice the equal angle, which is impossible. Hence, two of the angles of one triangle must be equal respectively to two angles of the other. Therefore the triangles are similar (**367**). Q. E. D.

PROPOSITION V.

373. Theorem.—*Two triangles having an angle in one equal to an angle in the other, and the sides about the equal angles proportional, are similar.*

DEMONSTRATION.

Let ABC and DEF have the angles C and F equal, and $\frac{AC}{DF} = \frac{CB}{FE}$.

We are to prove that **ABC** and **DEF** are similar.

Make **CD′** equal to **FD**, and draw **D′E′** parallel to **AB**. Then is

angle **CD′E′** = angle **CAB**,

whence the triangles are similar (**367**), and by (**368**),

$$\frac{AC}{D'C\ (= DF)} = \frac{CB}{CE'}.$$

But, by hypothesis,

$$\frac{AC}{DF} = \frac{CB}{FE}.$$

Whence $CE' = FE$.

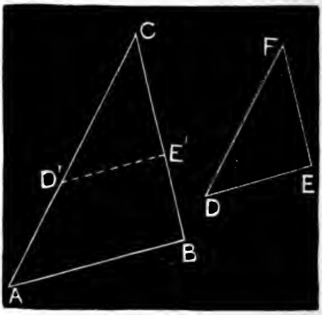

Fig. 184.

Hence the triangle **CD′E′** is equal to the triangle **FDE**. Now, **CD′E′** and **ABC** are mutually equiangular. Hence **DFE** and **ABC** are mutually equiangular and consequently similar. Q. E. D.

PROPOSITION VI.

374. Theorem.—*In any right-angled triangle, if a line is drawn from the vertex of the right angle perpendicular to the hypotenuse:*

1st. *The perpendicular divides the triangle into two triangles, which are similar to the given triangle, and consequently similar to each other.*

2d. *Either side about the right angle is a mean proportional between the whole hypotenuse and the adjacent segment.*

3d. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

DEMONSTRATION.

Let ACB be a triangle right-angled at C, and CD a perpendicular upon the hypotenuse AB ; then

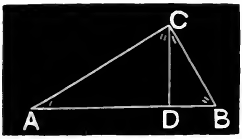

Fig. 185.

1st. The triangles **ACD** and **ACB** have the angle **A** common, and a right angle in each; hence they are similar (367). For a like reason, **CDB** and **ACB** are similar. Finally, as **ACD** and **CDB** are both similar to **ACB**, they are similar to each other. Q. E. D.

2d. By reason of the similarity of **ACD** and **ACB**, we have

$$\frac{AD}{AC} = \frac{AC}{AB};$$

and from **CDB** and **ACB**, we have $\frac{DB}{CB} = \frac{CB}{AB}$. Q. E. D.

3d. By reason of the similarity of **ACD** and **CDB**, we have

$$\frac{AD}{CD} = \frac{CD}{DB}. \quad \text{Q. E. D.}$$

QUERIES.—To which triangle does the first **CD** belong? To which the second? Why is **CD** made the consequent of **AD**? Why, in the second ratio, are **CD** and **DB** to be compared?

375. COROLLARY.—*If a perpendicular is let fall from any point in a circumference upon a diameter, this perpendicular is a mean proportional between the segments of the diameter.*

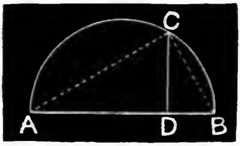

Fig. 186.

Let **CD** be such perpendicular, and draw **AC** and **CB**. Then, since **ACB** is a right angle (192), we have, by Case 3d, the proportion

$$\frac{AD}{CD} = \frac{CD}{DB}, \quad \text{or} \quad \overline{CD}^2 = AD \times DB.$$

PROPOSITION VII.

376. **Theorem.**—*The square described on the hypotenuse of a right-angled triangle is equivalent to the sum of the squares described on the other two sides.*

FIRST DEMONSTRATION.

Let ACB (Fig. 187) be any right-angled triangle.

We are to prove that $\overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2$.

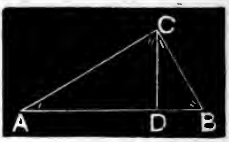

Fig. 187.

For, let fall the perpendicular **CD**, and by (374, 2d) we have

$$\frac{AD}{AC} = \frac{AC}{AB}, \quad \text{and} \quad \frac{DB}{CB} = \frac{CB}{AB};$$

$$\therefore \quad AD \times AB = \overline{AC}^2;$$

and $$DB \times AB = \overline{CB}^2.$$

Adding, we have $$AB\,(AD + DB) = \overline{AC}^2 + \overline{CB}^2,$$

or $$AB \times AB = \overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2. \quad \text{Q. E. D.}$$

Second Demonstration.

Let ABC be any right-angled triangle, right-angled at B.

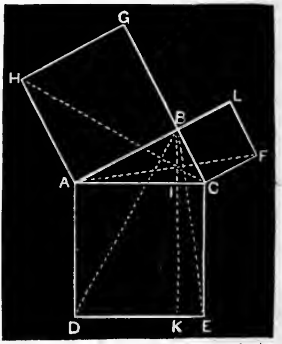

Fig. 188.

Describe the squares **AE**, **AG**, and **CL** on the hypotenuse and the other sides respectively. From the right angle let fall upon **DE** the perpendicular **BK** intersecting **AC** in **I**, and draw the diagonals **BE**, **DB**, **HC**, and **AF**.

Now the triangles **BAD** and **HAC** are equal, having two sides and the included angle of one equal to two sides and the included angle of the other; viz., **BA** = **HA**, being sides of the same square, and for a like reason **AD** = **AC**; and the angle **HAC** = **BAD**, since each is made up of a right angle and the angle **BAC**.

Since **ABG** and **ABC** are right angles, **BG** is the prolongation of **BC**, and the triangle **HAC** has the same base, **HA**, and the same altitude, **AB**, as the square **AG**. Hence the triangle **HAC** is half the square **AG**.

Moreover, the triangle **BAD** has the same base, **AD**, as the rectangle **AK**, and the same altitude as **AI**. Hence,

$$\text{triangle } BAD = \tfrac{1}{2}ADKI.$$

Therefore, as the rectangle **ADKI** and the square **AG** are twice the equal triangles **BAD** and **HAC** respectively, they are equivalent.

In like manner, the square **CL** may be shown to be equivalent to the rectangle **CK**.

Whence we have $$ADKI = ABGH,$$

and $$IKEC = BCFL;$$

and adding, $$ADEC = ABGH + BCFL. \quad \text{Q. E. D.}$$

377. Corollary 1.—*The hypotenuse of a right-angled triangle equals the square root of the sum of the squares of the other two sides.*

Also, either side about the right angle equals the square root of the square of the hypotenuse minus the square of the other side.

378. Corollary 2.—*The diagonal of a square is $\sqrt{2}$ times the side.*

For, let S be the side. Drawing the diagonal, we have a right-angled triangle of which the diagonal is the hypotenuse, and the sides about the right angle are each S. Hence, by the proposition,

$$(\text{diag.})^2 = S^2 + S^2 = 2S^2,$$

or

$$\text{diag.} = S\sqrt{2}.$$

379. Scholium.—Proposition VI with its corollary, and Prop. VII, which is a direct result of Prop. VI, are perhaps the most fruitful in direct practical results of any in Geometry. Prop. VII is called the Pythagorean Proposition, its original demonstration being attributed to Pythagoras.

PROPOSITION VIII.

380. Theorem.—*Regular polygons of the same number of sides are similar figures.*

Demonstration.

Let P and P′ be two regular polygons of the same number of sides, *a, b, c, d,* etc., being the sides of the former, and *a′, b′, c′, d′,* etc., the sides of the latter.

Now, by the definition of regular polygons, the sides a, b, c, d, etc., are equal each to each, and also a', b', c', d', etc. Hence, we have

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'}, \text{ etc.}$$

Again, the angles are equal, since n being the number of angles of each polygon, each angle is equal to

$$\frac{n \times 2 \text{ right angles} - 4 \text{ right angles}}{n} \text{ (267).}$$

Hence the polygons are mutually equiangular, and have their corresponding sides proportional; that is, they are similar. Q. E. D.

PROPOSITION IX.

381. Theorem.—*The corresponding diagonals of regular polygons of the same number of sides are in the same ratio as the sides of the polygons.*

[Let the student give the demonstration.]

PROPOSITION X.

382. Theorem.—*The radii of the circumscribed, and also of the inscribed circles, of regular polygons of the same number of sides, are in the same ratio as the sides of the polygons.*

Demonstration.

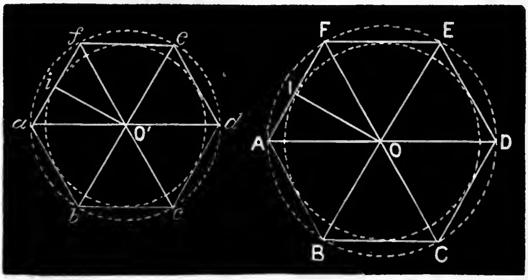

Fig. 189.

Let ABCDEF and *abcdef* be two regular polygons of the same number of sides, and R and r be the radii of their circumscribed circles, and R' and r' of their inscribed.

We are to prove that $\frac{AF}{af}\left(=\frac{FE}{fe}, \text{etc.}\right) = \frac{R}{r} = \frac{R'}{r'}$.

Let **O** and **O′** be the centres of the polygons, and draw **OA**, **OF**, **O′**a, and **O′**f, and also the apothems **OI** and **O′**i.

$$\mathsf{OA} = R, \quad \text{and} \quad \mathsf{O'}a = r \ (?);$$

also $$\mathsf{OI} = R', \quad \text{and} \quad \mathsf{O'}i = r' \ (?).$$

Now the triangles **AFO** and af**O′** are equiangular (?), and hence similar.

Therefore, $$\frac{\mathsf{AF}}{af}\left(=\frac{\mathsf{FE}}{fe}, \text{etc.}\right) = \frac{\mathsf{OA}}{\mathsf{O'}a} = \frac{R}{r}. \quad \text{Q. E. D.}$$

Again, the triangles **AIO** and ai**O′** are mutually equiangular (?), and hence similar.

Therefore, $$\frac{\mathsf{AI}}{ai} = \frac{\mathsf{OI}}{\mathsf{O'}i};$$

whence, doubling the terms of the first ratio, we have

$$\frac{\mathsf{AF}}{af}\left(=\frac{\mathsf{FE}}{fe}, \text{etc.}\right) = \frac{\mathsf{OI}}{\mathsf{O'}i} = \frac{R'}{r'}. \quad \text{Q. E. D.}$$

---

383. Homologous Altitudes in similar triangles are perpendiculars let fall from the vertices of equal angles upon the sides opposite.

384. Homologous Diagonals in similar polygons are diagonals joining the vertices of corresponding equal angles.

PROPOSITION XI.

385. Theorem.—*Homologous altitudes in similar triangles have the same ratio as the homologous sides.*

[Let the student give the demonstration.]

PROPOSITION XII.

386. Theorem.—*The bisectors of equal angles of similar triangles are to each other as the homologous sides of the triangles, hence as the homologous perpendiculars.*

[Let the student give the demonstration.]

PROPOSITION XIII.

387. Theorem.—*Homologous diagonals in similar polygons have the same ratio as the homologous sides.*

DEMONSTRATION.

Let ABCDEFG and *abcdefg* be two similar polygons, having angle A = angle *a*, B = *b*, C = *c*, etc.

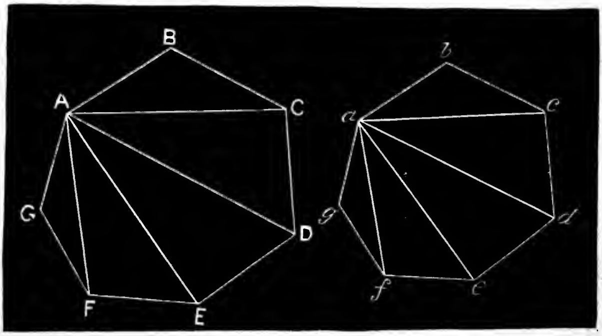

Fig. 190.

We are to prove that

$$\frac{\mathsf{AC}}{ac},\text{ or }\frac{\mathsf{AD}}{ad},\text{ etc.} = \frac{\mathsf{AB}}{ab},$$

the ratio $\frac{\mathsf{AB}}{ab}$ being the ratio of any two homologous sides of the polygons.

The triangles **ABC** and *abc* are similar (?), and hence

$$\frac{\mathsf{AC}}{ac} = \frac{\mathsf{AB}}{ab}.$$

Also, since triangle **ABC** is similar to *abc*,

angle **BCA** = angle *bca*,

and subtracting these respectively from the equal angles (?) **BCD** and *bcd*, we have

angle **ACD** = angle *acd*.

Hence the two triangles **ACD** and *acd* have an angle in each equal and the including sides proportional (?), and are consequently similar.

Therefore $$\frac{\mathsf{AD}}{ad} = \frac{\mathsf{AC}}{ac} = \frac{\mathsf{AB}}{ab}.$$

In like manner, any homologous diagonals may be shown to have the ratio $\frac{AB}{ab}$, which is the ratio of any two homologous sides. Q. E. D.

388. Corollary 1.—*Any two similar polygons are divided by their homologous diagonals into an equal number of similar triangles similarly placed.*

389. Corollary 2.—Conversely, *Two polygons which can be divided by diagonals into the same number of mutually similar triangles, similarly placed, are similar.*

PROPOSITION XIV.

390. Theorem.—*Circles are similar figures.*

Demonstration.

Let Oa and OA be the radii of any two circles.

Place the circles so that they shall be concentric, as in the figure. Inscribe the regular hexagons, as *abcdef*, **ABCDEF**.

Conceive the arcs **AB**, **BC**, etc., of the outer circumference bisected, and the regular dodecagon inscribed, and also the corresponding regular dodecagon in the inner circumference.

These are similar figures by (**380**).

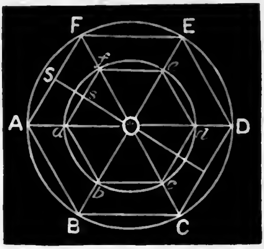

Fig. 191.

Now, as the process of bisecting the arcs of the exterior circumference can be conceived as indefinitely repeated, and the corresponding *regular* polygons as inscribed in each circle, the circles may be considered as regular polygons of the same number of sides, and hence similar. Q. E. D.

391. Corollary.—*Sectors which correspond to equal angles at the centre are similar figures.*

Since a radius is perpendicular to the circumference of its circle, such sectors are mutually equiangular; and by the proposition it is evident that the arcs are to each other as the radii.

$$i.\ e.,\ \frac{\text{arc } fe}{\text{arc FE}} = \frac{\text{O}f}{\text{OF}}.$$

Scholium.—The circle is said to be the *limit* of the inscribed polygon, and the circumference the *limit* of the perimeter. By this is meant that as the number of the sides of the inscribed polygon is increased it approaches nearer and nearer to equality with the circle. The apothem approaches equality with the radius, and hence has the radius for its limit.

PROPOSITION XV.

392. Problem.—*To divide a given line into parts which shall be proportional to several given lines.*

SOLUTION.*

Let it be required to divide OP into parts proportional to the lines A, B, C, and D.

Draw **ON** making any convenient angle with **OP**, and on it lay off **A**, **B**, **C**, and **D**, in succession, terminating at **M**.

Join **M** with the extremity **P**, and draw parallels to **MP** through the other points of division.

Then by reason of the parallels we shall have

$$\mathsf{A} : \mathsf{B} : \mathsf{C} : \mathsf{D} :: a : b : c : d \ (369).$$

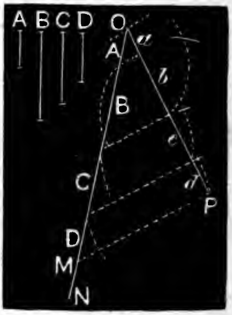

Fig. 192.

393. The notation $\mathsf{A} : \mathsf{B} : \mathsf{C} : \mathsf{D} :: a : b : c : d$ is of such frequent occurrence in mathematical writing that we feel constrained to retain it. It means that the successive ratios

$$\frac{\mathsf{A}}{\mathsf{B}}, \ \frac{\mathsf{B}}{\mathsf{C}}, \ \frac{\mathsf{C}}{\mathsf{D}},$$

are equal to the successive ratios

$$\frac{a}{b}, \ \frac{b}{c}, \ \frac{c}{d}.$$

We may read the expression thus: "The successive ratios **A** to **B**, **B** to **C**, **C** to **D** = the successive ratios a to b. b to c, c to d." *It does not mean that the ratio* **A** to **B** = **B** to **C**, etc.

---

* Hereafter we shall change somewhat the style of our demonstrations, from the elementary form hitherto used to the more common and free form used by writers generally. In the "Solution" of a problem we shall hereafter usually include the "Demonstration of the Solution."

PROPOSITION XVI.

394. Problem.—*To find a fourth proportional to three given lines.*

SOLUTION.

Let it be required to find D, a fourth proportional to the lines A, B, and C, so that we shall have $\frac{A}{B} = \frac{C}{D}$.

From some point O, draw two indefinite lines OX, OY. Lay off on OX, $Oa = A$, and $Oc = B$. Also, on OY lay off $Ob = C$, and draw ab. Through c draw cd parallel to ab. Then is Od the fourth proportional, D, which was sought.

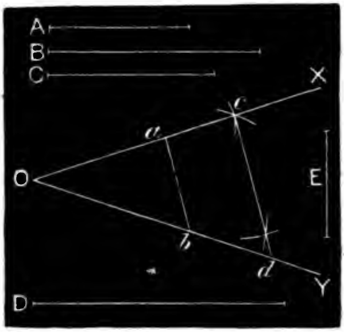

Fig. 193.

For, since ab and cd are parallel, we have, by (368),

$$\frac{Oa \text{ (or A)}}{Oc \text{ (or B)}} = \frac{Ob \text{ (or C)}}{Od \text{ (or D)}}.$$

Hence D is the fourth proportional sought.

395. SCHOLIUM.—In speaking of the fourth proportional to three given lines, it is necessary that the order in which the three are to occur be specified. This order is usually understood to be that in which the lines are named. Thus, a fourth proportional to A, B, and C, is D, as found above. But a fourth proportional to B, A, and C is quite a different line from D.

PROPOSITION XVII.

396. Problem.—*To find a third proportional to two given lines.*

SOLUTION.

Let A and B be the two given lines.

We are to find a third proportional, x, such that

$$\frac{\mathsf{A}}{\mathsf{B}} = \frac{\mathsf{B}}{x}.$$

The usual solution is the same as the last, **C** being equal to **B**. [Let the student execute it.]

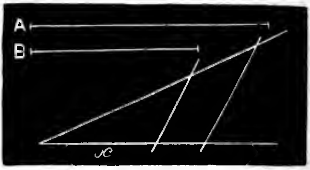

Fig. 194.

ANOTHER SOLUTION.

Let A and B be the two lines.

Draw an indefinite line **AM**, and take **AD** = **A**.

At **D** erect a perpendicular **BD** and make it equal to **B**.

Join **A** and **B**, and bisect it by the perpendicular **ON**.

NO will intersect **AM**; since, as **A** is less than a right angle (?), the sum of the two angles **ONA** and **OAN** is less than two right angles (**129**).

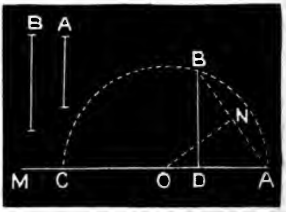

Fig. 195.

From **O** as a centre, with **OA** as a radius, describe a semi-circumference. It will pass through **B** (?).

Now $$\frac{\mathsf{AD}\ (\text{or } \mathsf{A})}{\mathsf{BD}\ (\text{or } \mathsf{B})} = \frac{\mathsf{BD}\ (\text{or } \mathsf{B})}{\mathsf{DC}\ (\text{or } x)}\ (?).$$

Hence, **CD** = x, the required third proportional.

PROPOSITION XVIII.

397. Problem.—*To find a mean proportional between two given lines.*

SOLUTION.

Let it be required to find a mean proportional, x, between M and N, so that

$$\frac{\mathsf{M}}{x} = \frac{x}{\mathsf{N}}$$

or $$x = \sqrt{\mathsf{M} \times \mathsf{N}}.$$

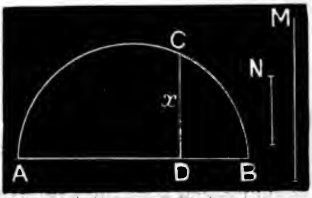

Fig. 196.

Draw an indefinite line, and on it lay off **AD** = **M**, and **DB** = **N**. On **AB** as a diameter draw a semi-circumference, and erect **DC** perpendicular to **AB**. Then **CD** = x, the mean proportional required.

[Let the student give the proof.]

PROPOSITION XIX.

398. Problem.—*To construct a square equivalent to a given triangle.*

Find a mean proportional between the altitude and half the base. On this construct a square.

[Let the student execute the problem and demonstrate it.]

EXERCISES.

399. 1. Draw any line, and divide it into 3, 5, 8, or 10 equal parts.

2. Draw any line and divide it into parts which shall be to each other as 2, 3, and 5.

3. Construct the square root of 7, 11, 2.

Fig. 197 will suggest the construction of $\sqrt{7}$.

Fig. 197.

4. The diameter of a circle is 20. What is the perpendicular distance to the circumference from a point in the diameter 15 from one extremity? What are the distances from the point where this perpendicular meets the circumference to the extremities of the diameter?

5. The sides of one triangle are 7, 9, and 11. The side of a second similar triangle, homologous with side 9, is $4\frac{1}{2}$. What are the other sides of the latter?

6. **DE** being parallel to **BC**, prove that the triangles **DOE** and **BOC** are similar, and hence that

$$\frac{OD}{OC} = \frac{OE}{OB}$$

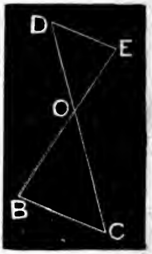

Fig. 198.

Are the following proportions true?

$$\frac{OD}{OC} = \frac{OE}{OB}, \quad \frac{OD}{DE} = \frac{OC}{BC},$$

$$\frac{OD}{OE} = \frac{OC}{BC}, \quad \frac{OB}{BC} = \frac{OE}{DE}.$$

7. Draw any triangle or polygon, and then construct a similar one whose homologous sides shall be $\frac{2}{3}$ as long.

8. Show that if **ABCDEF** is a regular polygon, **A***bcdef* is also regular, *bc*, *cd*, etc., being parallel to **BC**, **CD**, etc. Show that any two similar polygons may be placed in similar relative positions, and hence show that the corresponding diagonals are in the same ratio as the homologous sides.

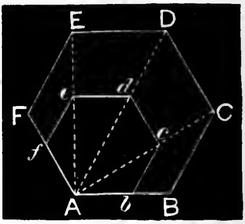

Fig. 199.

PROPOSITIONS FOR ORIGINAL INVESTIGATION.

400. 1. *If two straight lines join the alternate ends of two parallels, the line joining their centres is half the difference of the parallels.*

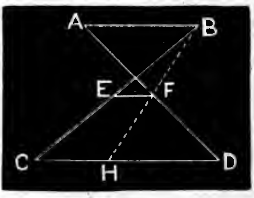

Fig. 200.

We are to prove that

$$EF = \tfrac{1}{2}(CD - AB).$$

$$\tfrac{1}{2}CH = EF = \tfrac{1}{2}(CD - AB).$$

2. *To construct a square equivalent to a given polygon.*

First reduce the polygon to a triangle (**339**). Then construct an equivalent square (**398**).

3. *The area of a regular inscribed dodecagon is three times the square on the radius.*

4. *If the sides of a quadrilateral be divided into m equal parts, and the n^{th} points of division, reckoning from two opposite vertices, be joined so as to form a quadrilateral, the quadrilateral will be a parallelogram.*

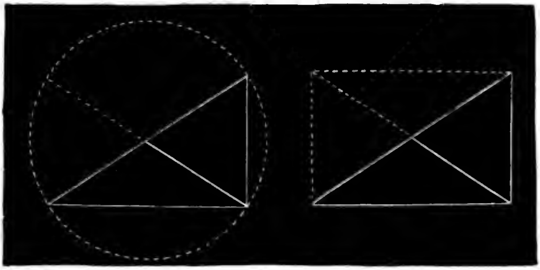

Fig 201.

5. *The line drawn from the vertex of the right angle of a right-angled triangle to the middle of the hypotenuse is half the hypotenuse.*

Prove from either figure.

6. *In any triangle the rectangle of two sides is equivalent to the rectangle of the perpendicular let fall from their included angle upon the third side, into the diameter of the circumscribed circle.*

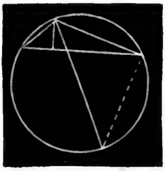

Fig. 202.

This proposition is an immediate consequence of the similarity of two triangles in the figure.

SECTION XI.

APPLICATIONS OF THE DOCTRINE OF SIMILARITY TO THE DEVELOPMENT OF GEOMETRICAL PROPERTIES OF FIGURES.

401. The doctrine of similarity, as presented in the preceding section, is the chief reliance for the development of the geometrical properties of figures. This section will be devoted to the investigation of a few of the more elementary properties of plane figures, which we are able to discover by means of this doctrine.

OF THE RELATIONS

OF THE SEGMENTS OF TWO LINES INTERSECTING EACH OTHER, AND INTERSECTED BY A CIRCUMFERENCE.

PROPOSITION I.

402. Theorem.—*If two chords intersect each other in a circle, their segments are reciprocally proportional; whence the product of the segments of one chord equals the product of the segments of the other.*

DEMONSTRATION.

Let the chords AC and BD (Fig. 203) intersect at O.

We are to prove that $$\frac{OB}{OA} = \frac{OC}{OD},$$

whence $$OB \times OD = OA \times OC.$$

Draw **AD** and **BC**.

The two triangles **AOD** and **BOC** are similar (?).

Hence, $$\frac{OB}{OA} = \frac{OC}{OD};$$

whence $$OB \times OD = OA \times OC. \quad \text{Q. E. D.}$$

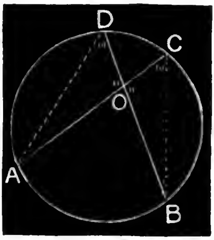

Fig. 203.

QUERIES.—Why is **OB** compared with **OA**? Why **OC** with **OD**? Would **AO : CO :: BO : DO** be true? Would **AO : DO :: BO : CO**? What is the force of the word "reciprocally," as used in the proposition?

PROPOSITION II.

403. Theorem.—*If from a point without a circle, two secants are drawn terminating in the concave arc, the whole secants are reciprocally proportional to their external segments; whence the product of one secant into its external segment equals the product of the other into its external segment.*

DEMONSTRATION.

Let OA and OB be two secants intersecting the circumference in D and C respectively.

We are to prove

$$\frac{OB}{OA} = \frac{OD}{OC};$$

whence, $$OB \times OC = OA \times OD.$$

Draw **AC** and **BD**.

The two triangles **AOC** and **BOD** are similar (?).

Hence, $$\frac{OB}{OA} = \frac{OD}{OC};$$

whence, $$OB \times OC = OA \times OD. \quad \text{Q. E. D.}$$

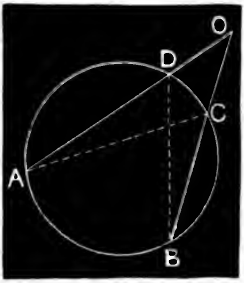

Fig. 204.

QUERIES.—Same as under preceding demonstration.

PROPOSITION III.

404. Theorem.—*If from a point without a circle a tangent is drawn, and a secant terminating in the concave arc, the tangent is a mean proportional between the whole secant and its external segment; whence the square of the tangent equals the product of the secant into its external segment.*

DEMONSTRATION.

Let OA be a tangent and OB a secant intersecting the circumference in C.

We are to prove that

$$\frac{\mathbf{OB}}{\mathbf{OA}} = \frac{\mathbf{OA}}{\mathbf{OC}};$$

whence, $\mathbf{OB} \times \mathbf{OC} = \overline{\mathbf{OA}}^2.$

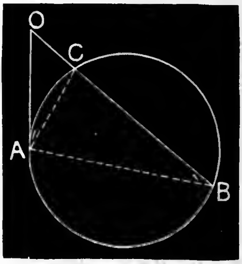

Fig. 205.

Draw **AC** and **AB**.

The two triangles **AOB** and **AOC** are similar, since angle **O** is common, and angle **OAC** = angle **B** (?).

Hence, $$\frac{\mathbf{OB}}{\mathbf{OA}} = \frac{\mathbf{OA}}{\mathbf{OC}};$$

whence, $\mathbf{OB} \times \mathbf{OC} = \overline{\mathbf{OA}}^2.$ Q. E. D.

OF THE BISECTOR OF AN ANGLE OF A TRIANGLE.

PROPOSITION IV.

405. Theorem.—*A line which bisects any angle of a triangle divides the opposite side into segments proportional to the adjacent sides.*

DEMONSTRATION.

In the triangle ABC (Fig. 206) let CD bisect the angle ACB.

Then is $\frac{AD}{DB} = \frac{AC}{CB}$.*

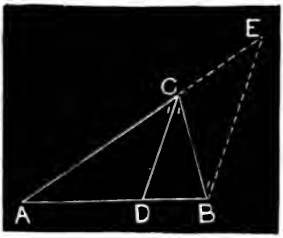

Fig. 206.

Draw **BE** parallel to **CD**, and produce it till it meets **AC** produced in **E**.

By reason of the parallels **CD** and **EB**,

angle **ACD** = **AEB**,

and **DCB** = **CBE**.

But, by hypothesis, **ACD** = **DCB**.

Therefore, **AEB** (or **CEB**) = **CBE**,

and **CE** = **CB** (?).

Hence, finally, $\frac{AD}{DB} = \frac{AC}{CE\ (= CB)}$ (368). Q. E. D.

PROPOSITION V.

406. Theorem.—*If a line is drawn from any vertex of a triangle bisecting the exterior angle and intersecting the opposite side produced, the distances from the other vertices to this intersection are proportional to the adjacent sides.*

Demonstration.

Let CD bisect the exterior angle BCF of the triangle ACB.

Then is $\frac{AD}{BD} = \frac{AC}{CB}$.

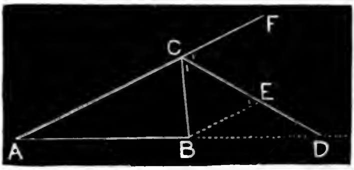

Fig. 207.

For, draw **BE** parallel to **AC**. By reason of these parallels,

angle **FCE** = **CEB**,

and **BCE** = **FCE**, by hypothesis.

Hence, **CEB** = **BCE**,

and **CB** = **BE**.

Also, by reason of the similar triangles **ACD** and **BED**,

$$\frac{AD}{BD} = \frac{AC}{BE\ (\text{or } CB)}.\quad \text{Q. E. D.}$$

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* See note at the bottom of p. 178.

PROPOSITION VI.

407. Theorem.—*If a line is drawn bisecting any angle of a triangle and intersecting the opposite side, the product of the sides about the bisected angle equals the product of the segments of the third side, plus the square of the bisector.*

DEMONSTRATION.

In the triangle ACB, let CD bisect the angle ACB.

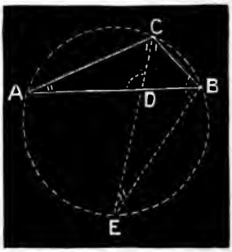

Fig. 208.

Then $AC \times CB = AD \times DB + \overline{CD}^2$.

For, circumscribe the circle about the triangle, produce the bisector till it meets the circumference at E, and draw EB. The triangles ADC and CBE are similar, since angle $ACD = ECB$, by hypothesis, and $A = E$, because each is measured by $\frac{1}{2}$ arc CB.

Therefore, $$\frac{AC}{CE} = \frac{CD}{CB};$$

whence, $$AC \times CB = CE \times CD = (DE + CD)\, CD$$
$$= DE \times CD + \overline{CD}^2.$$

For $DE \times CD$, substituting its equivalent $AD \times DB$ (402), we have

$$AC \times CB = AD \times DB + \overline{CD}^2. \quad \text{Q. E. D.}$$

AREAS OF SIMILAR FIGURES.

PROPOSITION VII.

408. Theorem.—*The areas of similar triangles are to each other as the squares described on their homologous sides.*

DEMONSTRATION.

Let ABC and EFG be two similar triangles, the homologous sides being AB and EF, BC and FG, and AC and EG.

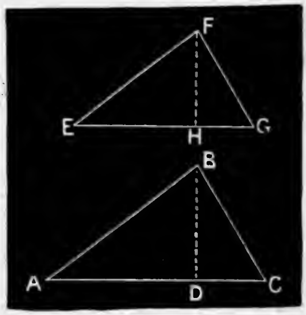

Fig. 209.

Then is

$$\frac{\text{area } ABC}{\text{area } EFG} = \frac{\overline{AC}^2}{\overline{EG}^2} = \frac{\overline{AB}^2}{\overline{EF}^2} = \frac{\overline{BC}^2}{\overline{FG}^2}.$$

From the greatest* angle in each triangle let fall a perpendicular upon the opposite side. Let these perpendiculars be **BD** and **FH**.

Now $$\frac{BD}{FH} = \frac{AC}{EG} \text{ (?)},$$

and $$\frac{\frac{1}{2}AC}{\frac{1}{2}EG} = \frac{AC}{EG} \text{ (?)}.$$

Multiplying the corresponding ratios together, we have

$$\frac{\frac{1}{2}AC \times BD}{\frac{1}{2}EG \times FH} = \frac{\overline{AC}^2}{\overline{EG}^2}.$$

But $$\tfrac{1}{2}AC \times BD = \text{area } ABC,$$

and $$\tfrac{1}{2}EG \times FH = \text{area } EFG \text{ (?)}.$$

Hence, $$\frac{\text{area } ABC}{\text{area } EFG} = \frac{\overline{AC}^2}{\overline{EG}^2}.$$

And, finally, as $$\frac{\overline{AC}^2}{\overline{EG}^2} = \frac{\overline{AB}^2}{\overline{EF}^2} = \frac{\overline{BC}^2}{\overline{FG}^2} \text{ (?)},$$

we have $$\frac{\text{area } ABC}{\text{area } EFG} = \frac{\overline{AC}^2}{\overline{EG}^2} = \frac{\overline{AB}^2}{\overline{EF}^2} = \frac{\overline{BC}^2}{\overline{FG}^2}. \quad \text{Q. E. D.}$$

PROPOSITION VIII.

409. Theorem.—*The areas of similar polygons are to each other as the squares of any two homologous sides of the polygons.*

---

* The only object in taking the *largest* angles is to make the perpendicular fall *within* the triangle. The demonstration is essentially the same when the perpendiculars fall upon the opposite sides *produced*.

Demonstration.

Let ABCDEF and *abcdef* be two similar polygons, the homologous sides being AB and *ab*, BC and *bc*, CD and *cd*, DE and *de*, EF and *ef*, FA and *fa*.

Let area **ABCDEF** $= P$,

and area *abcdef* $= p$.

Then is

$$\frac{P}{p} = \frac{\overline{BC}^2}{\overline{bc}^2},$$

or as the squares of any two homologous sides.

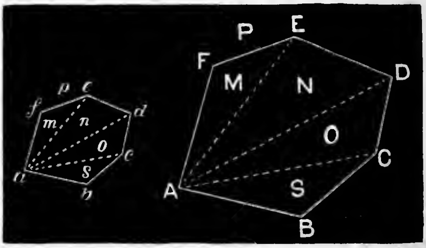

Fig. 210.

Draw the homologous diagonals **AC**, **AD**, **AE**, and *ac*, *ad*, and *ae*, dividing the polygons into the similar triangles **M** and *m*, **N** and *n*, **O** and *o*, and **S** and *s* **(388)**.

Now $$\frac{M}{m} = \frac{\overline{FE}^2}{\overline{fe}^2} \text{ (?)},$$

$$\frac{N}{n} = \frac{\overline{ED}^2}{\overline{ed}^2},$$

$$\frac{O}{o} = \frac{\overline{DC}^2}{\overline{dc}^2},$$

$$\frac{S}{s} = \frac{\overline{CB}^2}{\overline{cb}^2}.$$

But $$\frac{\overline{CB}^2}{\overline{cb}^2} = \frac{\overline{DC}^2}{\overline{dc}^2} = \frac{\overline{ED}^2}{\overline{ed}^2} = \frac{\overline{FE}^2}{\overline{fe}^2} \text{ (?)};$$

whence, $$\frac{M}{m} = \frac{N}{n} = \frac{O}{o} = \frac{S}{s}.$$

Taking this by composition, we have

$$\frac{M+N+O+S}{m+n+o+s} = \frac{P}{p} = \frac{M}{m} = \frac{\overline{CB}^2}{\overline{cb}^2}.$$

And as the ratio $\frac{\overline{CB}^2}{\overline{cb}^2}$ is the same as that of the squares of any two homologous sides, P and p are to each other as the squares of any two homologous sides.

Finally, as this argument can be extended to the case of any two similar polygons, the areas of any two similar polygons are to each other as the squares of any two homologous sides of the polygons. Q. E. D.

410. Corollary 1.—*Similar polygons* are to each other as the squares of their corresponding diagonals.*

In the demonstration we have $\frac{P}{p} = \frac{M}{m} = \frac{\overline{CB}^2}{\overline{cb}^2}$.

By (**388, 408**) we have $\frac{M}{m} = \frac{\overline{AE}^2}{\overline{ae}^2} = \frac{\overline{AD}^2}{\overline{ad}^2} = \frac{\overline{AC}^2}{\overline{ac}^2}$.

Hence $\frac{P}{p} = \frac{\overline{AE}^2}{\overline{ae}^2}$, etc.

411. Corollary 2.—*Regular polygons* of the same number of sides are to each other as the squares of their homologous sides.* [They are similar figures (?)].

412. Corollary 3.—*Regular polygons of the same number of sides are to each other as the squares of their apothems.*

For their apothems are to each other as their sides. Hence the squares of their apothems are to each other as the squares of their sides.

413. Corollary 4.—*Circles are to each other as the squares of their radii* (**390**), *and as the squares of their diameters.*

OF PERIMETERS AND THE RECTIFICATION OF THE CIRCUMFERENCE.

414. The **Rectification** of a curve is the process of finding its length.

The term *rectification* signifies making straight, and is applied as above, under the conception that the process consists in finding a straight line equal in length to the curve.

---

* This is a common elliptical form for "The *areas* of, etc."

PROPOSITION IX.

415. Theorem.—*The perimeters of similar polygons are to each other as their homologous sides, and as their corresponding diagonals.*

DEMONSTRATION.

Let a, b, c, d, etc., and A, B, C, D, etc., be the homologous sides of two similar polygons whose perimeters are p and P.

Then
$$\frac{p}{\mathsf{P}} = \frac{a}{\mathsf{A}} = \frac{b}{\mathsf{B}} = \frac{c}{\mathsf{C}}, \text{ etc.};$$

and r and R being corresponding diagonals,
$$\frac{p}{\mathsf{P}} = \frac{r}{\mathsf{R}}.$$

Since the polygons are similar,
$$\frac{a}{\mathsf{A}} = \frac{b}{\mathsf{B}} = \frac{c}{\mathsf{C}} = \frac{d}{\mathsf{D}}, \text{ etc.}$$

By composition,
$$\frac{a+b+c+d+\text{etc. (or } p)}{\mathsf{A}+\mathsf{B}+\mathsf{C}+\mathsf{D}+\text{etc. (or } \mathsf{P})} = \frac{a}{\mathsf{A}},$$

or as any other homologous sides. Also, as the homologous sides are to each other as the corresponding diagonals **(387)**,
$$\frac{p}{\mathsf{P}} = \frac{r}{\mathsf{R}}. \quad \text{Q. E. D.}$$

416. COROLLARY 1.—*The perimeters of regular polygons of the same number of sides are to each other as the apothems of the polygons* **(382)**.

417. COROLLARY 2.—*The circumferences of circles are to each other as their radii, and as their diameters* **(390)**.

PROPOSITION X.

418. Problem.—*To find the relation between the chord of an arc and the chord of half the arc in a circle whose radius is* r.

Solution.

Let O be the centre of the circle, AB any chord, and CB the chord of half the arc AB.

Let AB $= C$, **and CB** $= c$.

We are to find the relation between C and c.

Draw the radii **CO** and **BO**, and call each r.

CO is perpendicular to **AB** (?).

In the right-angled triangle **BDO**,

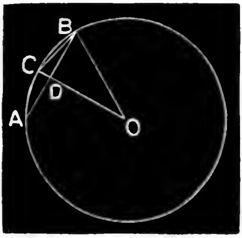

Fig. 211.

$$\mathrm{DO} = \sqrt{\overline{\mathrm{BO}}^2 - \tfrac{1}{4}C^2} \ (?),$$

or

$$\mathrm{DO} = \sqrt{r^2 - \tfrac{1}{4}C^2}.$$

Hence,

$$\mathrm{CD} = r - \sqrt{r^2 - \tfrac{1}{4}C^2}.$$

Again, in the right-angled triangle **CDB**,

$$\begin{aligned}\mathrm{CB} &= \sqrt{\overline{\mathrm{CD}}^2 + \overline{\mathrm{BD}}^2} \\ &= \sqrt{\left(r - \sqrt{r^2 - \tfrac{1}{4}C^2}\right)^2 + \tfrac{1}{4}C^2} \\ &= \sqrt{2r^2 - 2r\sqrt{r^2 - \tfrac{1}{4}C^2}} \\ &= \sqrt{2r^2 - r\sqrt{4r^2 - C^2}}.\end{aligned}$$

Therefore, $c = \sqrt{2r^2 - r\sqrt{4r^2 - C^2}}$ is the relation desired.

419. Scholium.—The formula

$$c = \sqrt{2r^2 - r\sqrt{4r^2 - C^2}}$$

is the value of the chord of half the arc in terms of the chord of the whole arc and the radius. From this we readily obtain

$$C = \frac{c}{r}\sqrt{4r^2 - c^2},$$

which is the value of the chord in terms of the chord of half the arc and the radius.

PROPOSITION XI.

420. Theorem.—*The circumference of a circle whose radius is* 1, *is* 2π, *the numerical value of* π *being approximately* 3.1416.

DEMONSTRATION.

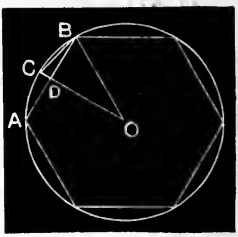

Fig. 212.

We will approximate the circumference of a circle whose radius is 1, by obtaining, 1st, the perimeter of the regular inscribed hexagon; 2d, the perimeter of the regular inscribed dodecagon; 3d, the perimeter of the regular inscribed polygon of 24 sides; then of 48, etc.

By varying the polygon in this manner, it is evident that the perimeter approaches the circumference as its limit (**282**, **354**), since at each bisection the sum of two sides of a triangle is substituted for the third side. Moreover, the perimeter can never pass the circumference, since a chord is always less than its arc.

Now let **AB** $= r$ (?) $= 1$ be the side of the inscribed hexagon. Then by the formula (**418**), we have

$$\mathbf{CB} = c = \sqrt{2 - \sqrt{4-1}} = .51763809,$$

which is therefore the side of a regular dodecagon. Hence the perimeter of the dodecagon is

$$.51763809 \times 12 = 6.21165708.$$

Again, let the side of the inscribed regular polygon of 24 sides be c', and we have

$$c' = \sqrt{2 - \sqrt{4 - c^2}} = \sqrt{2 - \sqrt{4 - (.51763809)^2}} = .26105238;$$

and the perimeter, $\quad .26105238 \times 24 = 6.26525722.$

Carrying the computation forward in this manner, we have the following:

No. Sides.	Form of Computation.	Length of Side.	Perimeter.
6	See (280)	1.00000000	6.00000000
12	$c = \sqrt{2 - \sqrt{4 - 1^2}} = \sqrt{2 - \sqrt{3}}$, or $(2 - 3^{\frac{1}{2}})^{\frac{1}{2}}$	.51763809	6.21165708
24	$c' = \{2 - [4 - (2 - 3^{\frac{1}{2}})]^{\frac{1}{2}}\}^{\frac{1}{2}} = [2 - (2 + 3^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}$	.26105238	6.26525722
48	$c'' = (2 - \{4 - [2 - (2 + 3^{\frac{1}{2}})^{\frac{1}{2}}]\}^{\frac{1}{2}})^{\frac{1}{2}} = \{2 - [2 + (2 + 3^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}}$ ..	.13080626	6.27870041
96	$c''' = (2 - \{2 + [2 + (2 + 3^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}})^{\frac{1}{2}}$	.06543817	6.28206396
192	$c^{\text{iv}} = [2 - (2 + \{2 + [2 + (2 + 3^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}$ *	.03272346	6.28290510
384	$c^{\text{v}} = \{2 - [2 + (2 + \{2 + [2 + (2 + 3^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}}$	.01636228	6.28311544
768	$c^{\text{vi}} = (2 - \{2 + [2 + (2 + \{2 + [2 + (2 + 3^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}})^{\frac{1}{2}}$	.00818121	6.28316941

* It will be found easy to write any one of these forms, if we observe that the number of 2's is 1 more than the index of c, and the number of sides of the corresponding polygon is 3×2 with an exponent 2 more than the index of c. Thus, to write the form for c^{v}, we observe that there will be 6 twos; hence we write, first,

$$2- \quad 2+ \quad 2+ \quad 2+ \quad 2+ \quad 2+3^{\frac{1}{2}},$$

leaving spaces for the brackets. Then commencing with the binomial $2+3^{\frac{1}{2}}$, we introduce ( ) first, [ ] next, and { } next, and then repeat. The number of sides of the corresponding polygon is $3 \times 2^7 = 3 \times 128 = 384$.

It now appears that the first four decimal figures do not change as the number of sides is increased, but will remain the same *how far soever we proceed.* When the foregoing process is continued till 5 decimals become constant, we have 6.28318+. We may therefore consider 6.28318 as *approximately* the circumference of a circle whose radius is 1,

Hence, letting 2π stand for the circumference, we have

$$2\pi = 6.28318+,$$

and $$\pi = 3.1416, \text{ nearly.} \quad \text{Q. E. D.}$$

421. Scholium.—The symbol π is much used in mathematics, and signifies, primarily, *the semi-circumference of a circle whose radius is* 1. $\frac{1}{2}\pi$ is therefore a symbol for a quadrant, 90°, or a right angle. $\frac{1}{4}\pi$ is equivalent to 45°, and 2π to a circumference, the radius being always supposed 1, unless statement is made to the contrary. The numerical value of π has been sought in a great variety of ways, all of which agree in the conclusion that it cannot be exactly expressed in decimal numbers, but is approximately as given in the proposition. From the time of Archimedes (287 B. C.) to the present, much ingenious labor has been bestowed upon this problem. The most expeditious and elegant methods of approximation are furnished by the Calculus. The following is the value of π extended to fifteen places of decimals: 3.141592653589793.

PROPOSITION XII.

422. Theorem.—*The circumference of any circle is* $2\pi r$, *r being the radius.*

Demonstration.

The circumferences of circles being to each other as their radii (**417**), and 2π being the circumference of a circle whose radius is 1, we have

$$\frac{2\pi}{\text{circf.}} = \frac{1}{r};$$

whence, $$\text{circf.} = 2\pi r. \quad \text{Q. E. D.}$$

423. Corollary.—*The circumference of any circle is* πD, *D being the diameter.*

AREA OF THE CIRCLE.

PROPOSITION XIII.

424. Theorem.—*The area of a circle whose radius is 1, is* π.

DEMONSTRATION.

The area of a circle is $\frac{1}{2}r \times$ circf. (**356**). When $r = 1$,

circf. $= 2\pi$ (**420**);

hence, area of circle whose radius is $1 = \frac{1}{2} \times 2\pi = \pi$. Q. E. D.

PROPOSITION XIV.

425. Theorem.—*The area of any circle is* πr^2, *r being the radius.*

DEMONSTRATION.

The areas of circles being to each other as the squares of their radii (**413**), and π being the area of a circle whose radius is 1, we have

$$\frac{\pi}{\text{area of any circle}} = \frac{1^2}{r^2};$$

whence, area of any circle $= \pi r^2$. Q. E. D.

426. SCHOLIUM 1.—Since the area of a sector is to the area of the circle of the same radius as its angle is to 4 right angles (**359**), if we represent the angle of the sector by $a°$, we have for its area $\frac{a\,\pi\,r^2}{360}$.

427. SCHOLIUM 2.—As the value of π cannot be exactly expressed in numbers, it follows that the area cannot. Finding the area of a circle has long been known as the problem of "Squaring the Circle;" *i. e.*, finding a square equal in area to a circle of given radius. Doubtless many hare-brained visionaries or ignoramuses will still continue the chase after the phantom, although it has long ago been demonstrated that the diameter of a circle and its circumference are incommensurable by any finite unit. It is, however, an easy matter to conceive a square of the same area as any given circle. Thus, let there be a rectangle whose base is equal to the circumference of the circle, and whose altitude is half the radius; its area is exactly equal to the area of the circle. Now, let there be a square whose side is a mean proportional between the altitude and base of this rectangle; the area of the square is exactly equal to the area of the circle.

PROPOSITION XV.

428. Theorem.—*If a perpendicular is let fall from any angle of a triangle upon the opposite side (or on the side produced), the difference of the squares of the segments is equivalent to the difference of the squares of the other two sides.*

DEMONSTRATION.

Let ABC be any triangle, and CD be the perpendicular let fall from C upon AB (or AB produced). Call the sides opposite the angles A, B, and C, a, b, and c, respectively; and let the segment BD = m, AD = n, and CD = p.

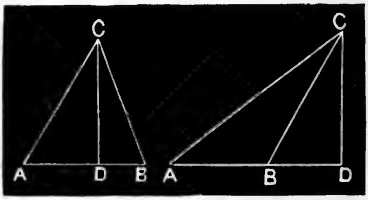

Fig. 213.

Then is $m^2-n^2 = a^2-b^2$.

For, from the right-angled triangle **BCD**,

$$a^2-m^2 = p^2.$$

Also, from **CDA**,

$$b^2-n^2 = p^2,$$

Whence, $$a^2-m^2 = b^2-n^2,$$

or $$m^2-n^2 = a^2-b^2. \quad \text{Q. E. D.}$$

429. COROLLARY.—Since

$$m^2-n^2 = (m+n)(m-n),$$

and $$a^2-b^2 = (a+b)(a-b),$$

we have $$\frac{m+n \text{ (or } c)}{a+b} = \frac{a-b}{m-n}$$

430. Scholium.—In case the perpendicular falls without, the distances **BD** and **AD** are still, for simplicity of expression, spoken of as segments.

---

431. A line is said to be divided in **Extreme and Mean Ratio** when it is so divided that the whole line is to the greater segment as the greater segment is to the less, *i. e.*, when the greater segment is a mean proportional between the whole line and the less segment.

PROPOSITION XVI.

432. Problem.—*To divide a line in extreme and mean ratio.*

Solution.

Let it be proposed to divide the line AB in extreme and mean ratio, *i. e.*, C being the point of division, so that

$$\frac{AB}{AC} = \frac{AC}{CB}.$$

At one extremity of **AB**, as **B**, erect a perpendicular **BO**, and make it equal to $\frac{1}{2}$**AB**.

From **O** as a centre, with **OB** as a radius, describe a circle.

Draw **AO**, cutting the circumference in **D**.

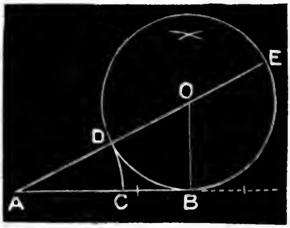

Fig. 214.

Then is **AD** the greater segment, and taking **AC** = **AD**, **AB** is divided in extreme and mean ratio at **C**.

Demonstration of Solution.

Produce **AO** to **E**.

Now $$\frac{AE}{AB} = \frac{AB}{AD} \text{ (?)},$$

or, by inversion, $$\frac{AB}{AE} = \frac{AD}{AB}.$$

By division, we have

$$\frac{\mathsf{AB}}{\mathsf{AE} - \mathsf{AB}} = \frac{\mathsf{AD}}{\mathsf{AB} - \mathsf{AD}}.$$

But, as $\mathsf{DE} = \mathsf{AB}$ (?),

$$\mathsf{AE} - \mathsf{AB} = \mathsf{AE} - \mathsf{DE} = \mathsf{AD} = \mathsf{AC};$$

and $\mathsf{AB} - \mathsf{AD} = \mathsf{AB} - \mathsf{AC} = \mathsf{CB}$.

Hence, substituting, $\frac{\mathsf{AB}}{\mathsf{AC}} = \frac{\mathsf{AC}}{\mathsf{CB}}$. Q. E. D.

PROPOSITION XVII.

433. **Problem.**—*To inscribe a regular decagon in a circle, and hence a regular pentagon, and regular polygons of 20, 40, 80, etc., sides.*

Solution.

Let it be required to inscribe a regular decagon in the circle whose centre is O and radius OA.

Divide the radius **OA** in extreme and mean ratio, as at (*a*).

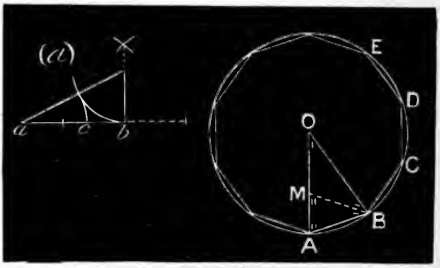

Fig. 215.

Then is *ac*, the greater segment, the side of the inscribed decagon, **ABCDE**, etc.

To prove this, draw **OA** and **OB**, and taking **OM** = *ac* = **AB**, draw **BM**.

Now $\frac{\mathsf{OA}}{\mathsf{OM}} = \frac{\mathsf{OM}}{\mathsf{MA}}$, by construction; and, as **OM** = **AB**, we have

$$\frac{\mathsf{OA}}{\mathsf{AB}} = \frac{\mathsf{AB}}{\mathsf{MA}}.$$

Hence, considering the antecedents as belonging to the triangle **OAB**, and the consequents to the triangle **BAM**, we observe that the two sides about the angle **A**, which is common to both triangles, are proportional; hence the triangles are similar (**373**).

Therefore, **ABM** is isosceles, since **OAB** is, and

angle **BMA** = **A** = **OBA**,

and **MB** = **BA** = **OM**.

This makes **OMB** also isosceles, and

the angle **O** = **OBM**.

Again, the exterior angle **BMA** = **O** + **OBM** = **2O**;

hence, **A** (which equals **BMA**) = **2O**.

Hence, also, **OBA** (which equals **A**) = **2O**.

Wherefore, **O** is $\frac{1}{5}$ the sum of the angles of the triangle **OAB**, or $\frac{1}{5}$ of 2 right angles, = $\frac{1}{10}$ of 4 right angles.

The arc **AB** is therefore the measure of $\frac{1}{10}$ of 4 right angles, and is consequently $\frac{1}{10}$ of the circumference. Hence **AB** is the chord of $\frac{1}{10}$ of the circumference, and if applied, as **AB**, **BC**, **CD**, **DE**, etc., will give an equilateral inscribed decagon.

Moreover this inscribed polygon is equiangular, and hence regular by (**272**).

To construct the pentagon, join the alternate angles of the decagon. To construct the regular polygon of 20 sides, bisect the arcs subtended by the sides of the decagon, etc.

MISCELLANEOUS EXERCISES.

434. 1. Show that if a chord of a circle is conceived to revolve, varying in length as it revolves, so as to keep its extremities in the circumference while it constantly passes through a fixed point, the rectangle of its segments remains constant.

2. The two segments of a chord intersected by another chord are 6 and 4, and one segment of the other chord is 3. What is the other segment of the latter chord?

3. Show how Propositions I, II, and III may be considered as different cases of one and the same proposition.

Suggestions.—By stating Propositions I and II thus, *The distances from the intersection of the lines to their intersections with the circumference,* what follows? In Fig. 204, if the secant **AO** becomes a tangent, what does **OD** become?

4. In a triangle whose sides are 48, 36, and 50, where do the bisectors of the angles intersect the sides?

5. In the last example, find the lengths of the bisectors.

6. A and B have farms of similar shape, with their homologous sides on the same road. A's is 150 rods on the road, and B's 200 rods. How does A's farm compare with B's in size?

7. Draw two similar triangles with their homologous sides in the ratio of 3 to 5, and divide them into equal partial triangles, showing that their areas are as 3^2 to 5^2, that is, as 9 to 25.

8. What are the relative capacities of a 5-inch and a 7-inch stove-pipe?

9. If a circle whose radius is 24 is divided into 5 equal parts by concentric circumferences, what are the diameters of the several circles?

Solve geometrically as well as numerically.

10. The projection of one line upon another in the same plane is the distance between the feet of two perpendiculars let fall from the extremities of the former upon the latter. Show that this projection is equal to the square root of the difference between the square of the line and the square of the difference of the perpendiculars.

11. The three sides of a triangle being 4, 5, and 6, find the segments of the last side made by a perpendicular from the opposite angle. *Ans.* 3.75 and 2.25.

12. Same as above, when the sides are 10, 4, and 7, and the perpendicular is let fall from the angle included by the sides 10 and 4. Draw the figure. Why is one of the segments negative?

13. What is the area of a regular octagon inscribed in a circle whose radius is 1? What is its perimeter? What if the radius is 10?

14. What is the side of an equilateral triangle inscribed in a circle whose radius is 1?

15. What is the side of a regular inscribed decagon in a circle whose radius is 4? What the side of the inscribed pentagon? What is the area of each?

16. Draw two squares, and construct two others, one equal to their sum, and the other to their difference.

17. Draw any two polygons, and construct two squares, one equivalent to their sum, and the other equivalent to their difference.

18. Show that the length of a degree in any circle is $\frac{\pi r}{180}$, and hence that the lengths of degrees in different circles are to each other as the radii of the circles.

19. What is the length of a minute on a circle whose radius is 10 miles?

20. Calling the equatorial radius of the earth 3962.8 miles, what is the length of a degree on the equator?

21. How many degrees in the arc of a circle which is equal in length to the radius?

22. Compute the area of the triangle whose sides are 20, 30, and 40.

Find the segments of the base (40) by (**428**). Hence the perpendicular.

23. Given the side of a regular inscribed pentagon, as 16, to find the side of the similar circumscribed polygon.

24. Prove that if a triangle is circumscribed about a given triangle by drawing lines through the vertices of the given triangle and parallel to the opposite sides, the area of the circumscribed triangle is four times that of the given triangle.

25. Prove that the bisectors of the angles of a triangle pass through a common point.

26. Prove that the perpendiculars to the three sides of a triangle at their middle points pass through a common point.

27. The three perpendiculars drawn from the angles of a triangle upon the opposite sides intersect in a common point.

Draw through the vertices of the triangle lines parallel to the opposite sides. The proposition may then be brought under the preceding.

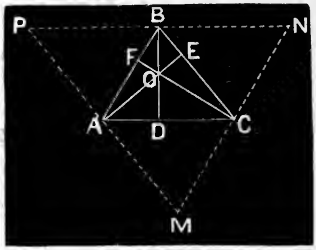

Fig. 216.

28. The following triangles are similar—viz., **BOE**, **BDC**, **AOD**, and **AEC**, each to each; also **BOF**, **BDA**, **DOC**, and **CFA**. Prove it.

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435. The **Medial Lines** in a triangle are the lines drawn from the vertices to the middle points of the opposite sides.

29. The three medial lines of a triangle mutually trisect each other, and hence intersect in a common point.

To prove that $OE = \frac{1}{3}BE$ (Fig. 217), draw **FC** parallel to **AD** until it meets **BE** produced. Then the triangles **AEO** and **FEC** are equal (?); whence

$$EF = OE.$$

Also, $$BO = OF \text{ (?)}.$$

Having shown that

$$OE = \tfrac{1}{3}BE,$$

by a similar construction we can show that

$$OD = \tfrac{1}{3}AD.$$

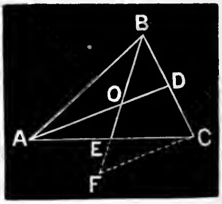

Fig. 217.

Finally, we may show that the medial line from **C** to **AB** cuts off $\frac{1}{3}$ of **BE**, and hence cuts **BE** at the same point as does **AD**.

Another Demonstration.—Lines through **O** parallel to the sides trisect the sides, etc.

Still Another.—Without **EF** and **FC**, draw **ED**, and prove by similar triangles.

CHAPTER II.

SOLID GEOMETRY.*

SECTION I.

OF STRAIGHT LINES AND PLANES.

436. Solid Geometry is that department of Geometry in which the magnitudes treated are not limited to a single plane.

437. A **Plane** (or a **Plane Surface**) is a surface such that a straight line joining any two points in it lies wholly in the surface.

PLANE, HOW DETERMINED.

438. A plane is said to be **Determined** by given conditions which fix its position.

All planes are considered as indefinite in extent, unless the contrary is stated.

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* In some respects, perhaps, "Geometry of Space" is preferable to this term; but, as neither is free from objections, and as this has the advantage of simplicity and long use, the author prefers to retain it.

PROPOSITION I.

439. Theorem.—*Three points not in the same straight line determine a plane.*

DEMONSTRATION.

Let A, B, and C be three points not in the same straight line.

Then one plane can be passed through them, and only one; *i. e.*, they determine the position of a plane.

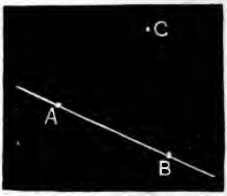

Fig. 218.

For, pass a straight line through any two of these points, as **A** and **B**. Now, conceive any plane containing these two points; then will the line passing through them lie wholly in the plane (**437**). Conceive this plane to revolve on the line as an axis until the point **C** falls in the plane. Thus we have one plane passed through the three points.

That there can be only one is evident, since when **C** falls in the plane, if the plane be revolved either way, **C** will not be in it. The same may be shown by first passing a plane through **B** and **C**, or **A** and **C**. There is, therefore, only one position of the plane in which it will contain the third point. Q. E. D.

440. COROLLARY 1.—*A line and a point without it determine a plane.*

441. COROLLARY 2.—*Through one line, or two points, an infinite number of planes can be passed.*

442. COROLLARY 3.—*The intersection of two planes is a straight line.*

For *two* planes cannot have even three points, *not in the same straight line*, common, much less an indefinite number, which would be required if we conceived the intersection (that is, the common points) to be in any other than a straight line.

---

443. The **Trace** of one plane in another is their intersection.

PROPOSITION II.

444. Theorem.—*Two intersecting lines determine the position of a plane.*

DEMONSTRATION.

For, the point of intersection may be taken as one of the three points requisite to determine the position of a plane, and any two other points, one in each of the lines, as the other two requisite points. Now, the plane passing through these points contains both the lines, for it contains two points in each. Q. E. D.

PROPOSITION III.

445. Theorem.—*Two parallel lines determine the position of a plane.*

DEMONSTRATION.

For, pass a plane through one of the parallels, and conceive it revolved until it contains some point of the second parallel. Now, if the plane be revolved either way from this position, the point will be left without it. Hence, it is the *only* plane containing the first parallel and this point in the second.

But parallels lie in the same plane (**120**, **121**), whence the plane of the parallels must contain the first line and the specified point in the second.

Therefore, the plane containing the first line and a point in the second is the plane of the parallels, and is fixed in position. Q. E. D.

446. SCHOLIUM.—When a plane is determined by two lines, according to either of the last two propositions, it is spoken of as the *Plane of the Lines.* In like manner, we may speak of the *Plane of Three Points.*

RELATIVE POSITION OF A LINE AND A PLANE.

447. A line may have one of three positions in relation to a plane: (*a*) It may be *perpendicular,* (*b*) *oblique,* or (*c*) *parallel.*

OF LINES PERPENDICULAR TO A PLANE.

448. A line is said to **Pierce** a plane at the point where it passes through it.

449. The point where a perpendicular meets, or pierces, a plane is called its **Foot.**

450. A **Perpendicular to a Plane** is a line which is perpendicular to all lines of the plane passing through its foot, and hence to every line of the plane. Conversely, the plane is perpendicular to the line.

451. The **Distance** of a point from a plane is the length of the perpendicular let fall from the point upon the plane.

PROPOSITION IV.

452. **Theorem.**—*A line which is perpendicular to two lines of a plane, at their intersection, is perpendicular to the plane.*

DEMONSTRATION.

Let PD be perpendicular to AB and CF at D.

Then is it perpendicular to **MN**, the plane of the lines **AB** and **CF**.

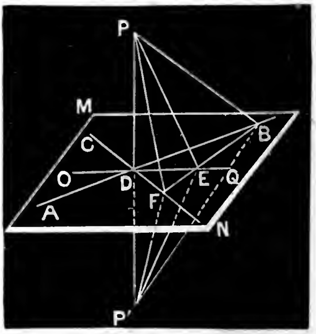

Fig. 219.

Let **OQ** be any other line of the plane **MN**, passing through **D**. Draw **FB** intersecting the three lines **AB**, **CF**, and **OQ** in **B**, **E**, and **F**. Produce **PD** to **P′**, making **P′D** = **PD**, and draw **PF**, **PE**, **PB**, **P′F**, **P′E**, **P′B**.

Then is **PF** = **P′F**,

and **PB** = **P′B**,

since **FD** and **BD** are perpendicular to **PP′**, and

PD = **P′D** (96)

Hence, the triangles **PFB** and **P′FB** are equal (**305**); and if **PFB** be revolved upon **FB** till **P** falls at **P′**, **PE** will fall in **P′E**.

Therefore, **OQ** has **E** equally distant from **P** and **P′**, and as **D** is also equidistant from the same points, **OQ** is perpendicular to **PD** at **D** (**98**).

Now, as **OQ** is *any* line, **PD** is perpendicular to any line of the plane passing through its foot, and consequently perpendicular to the plane (**450**). Q. E. D.

453. Corollary.—*If one of two perpendiculars revolves about the other as an axis, its path is a plane perpendicular to the axis, and this plane contains all the perpendiculars to the axis at the common point.*

Thus, if **AB** revolves about **PP′** as an axis, it describes the plane **MN**, and **MN** contains all the perpendiculars to **PP′** at **D**. For, if there could be a perpendicular to **PP** at **D** which did not lie in the plane **M N**, there would be two perpendiculars to **PP′** at **D**, both lying in the same plane, which is impossible (**88**).

PROPOSITION V.

454. Theorem.—*At any point in a plane one perpendicular can be erected to the plane, and only one.*

Demonstration.

Let it be required to show that one perpendicular, and only one, can be erected to the plane MN at D.

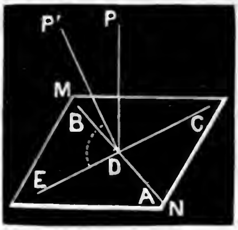

Fig. 220.

Through **D** draw two lines of the plane, as **AB** and **CE**, at right angles to each other. **CE** being perpendicular to **AB**, let a line be conceived as starting from the position **ED** to revolve about **AB** as an axis. It will remain perpendicular to **AB** (**453**). Conceive it to have passed to **P′D**. Now, as it continues to revolve, **P′DC** diminishes continuously, and at the same rate as **P′DE** increases; hence, in one position of the revolving line, and in only one, as **PD**, **PDE** = **PDC**, and **PD** is perpendicular to **CE** (**86**).

Again, any line which is perpendicular to **MN** at **D** is perpendicular to **AB** and **CE** (**450**). But the plane of the lines **PD** and **DE** contains all lines perpendicular to **AB** at **D**. Hence, **PD** is perpendicular to the plane (**452**), and is the only perpendicular. Q. E. D.

PROPOSITION VI.

455. Theorem—*From a point without a plane one perpendicular can be drawn to the plane, and only one.*

DEMONSTRATION.

Let it be required to show that one perpendicular can be drawn from P to the plane MN, and only one.

Take **RS** as an auxiliary plane, and at any point as **C** erect **DC** perpendicular to **RS**.

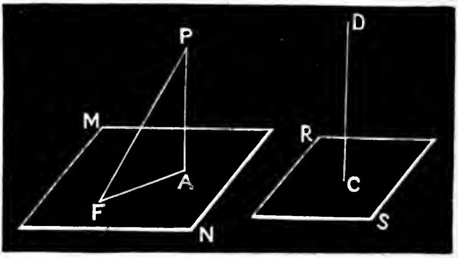

Fig. 221

Now place the plane **RS** in coincidence with **MN**, and move it in **MN** till the perpendicular **DC** passes through **P**.

Then **DC**, which passes through **P** and is perpendicular to **RS**, is perpendicular to **MN**, with which **RS** is coincident. Q. E. D.

To prove that there can be but one perpendicular from **P** to **MN**, suppose that there could be two, as **PA** and **PF**.

Draw **FA**.

Then since **FA** is a line of the plane, and **PF** and **PA** are perpendiculars to the plane, **PFA** and **PAF** are both right angles (?), and the triangle **PFA** has two right angles, which is absurd. Hence there can be but one perpendicular from **P** to **MN**. Q. E. D.

456. COROLLARY.—*The perpendicular is the shortest line that can be drawn to a plane from a point without.*

Thus, let **PA** be a perpendicular and **PF** any oblique line.

$$PA < PF \text{ (?)}.$$

PROPOSITION VII.

457. Theorem.—Conversely to the last, *Through a given point in a line, one plane can be passed perpendicular to the line, and only one.*

DEMONSTRATION.

Let D be the point in the line PG.

Pass two lines through **D**, as **EF** and **AB**, each perpendicular to **PD**; the plane of these lines is perpendicular to **PD**. Q. E. D.

To show that but one plane can be passed through **D** perpendicular to **PG**, assume that **M′N′** is another plane passing through **D**, and perpendicular to **PG**, but not containing **BD**. Through **PD** and **BD** pass a plane, and let **B′D** be its intersection with **M′N′**. Then, on the hypothesis that **M′N′** is perpendicular to **PG**, **B′DP** is a right angle, and we have two lines in the same plane with **PG**, and perpendicular to it at the same point, which is absurd. Hence there can be but one plane perpendicular to **PG** and passing through **D**. Q. E. D.

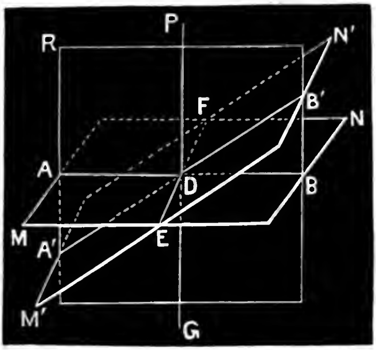

Fig. 222.

PROPOSITION VIII.

458. Theorem.—*If from the foot of a perpendicular to a plane a line is drawn at right angles to any line of the plane, and their intersection is joined with any point in the perpendicular, the last line is perpendicular to the line of the plane.*

DEMONSTRATION.

From the foot of the perpendicular PD (Fig. 223) let DE be drawn perpendicular to AB, any line of the plane MN, and E joined with O, any point of the perpendicular.

Then is **OE** perpendicular to **AB**.

Take **EF** = **EC**, and draw **CD**, **FD**, **CO**, and **FO**. Now,

CD = **DF** (?),

whence **CO** = **FO** (?),

and **OE** has **O** equally distant from **F** and **C**, and also **E**. Therefore, **OE** is perpendicular to **AB** (?). Q. E. D.

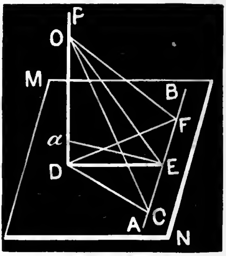

Fig. 223.

459. Corollary.—*The line* **DE** *measures the shortest distance between* **PD** *and* **AB**.

For a line drawn from **E** to any other point in **PD** than **D**, as **E**a, is longer than **DE** (?).

Again, if from any other point in **AB**, as **C**, a line be drawn to **D**, it is longer than **DE** (?); and if drawn from **C** to a, any other point in **PD** than **D**, **C**a is longer than **CD** (?), and consequently longer than **DE** (?).

PROPOSITION IX.

460. Theorem.—*If one of two parallels is perpendicular to a plane, the other is perpendicular also.*

Demonstration.

Let AB be parallel to CD and perpendicular to the plane MN.

Then is **CD** perpendicular to **MN**.

For, drawing **BD** in the plane **MN**, it is perpendicular to **AB** (?), and consequently to **CD** (?). Through **D** draw **EF** in the plane and perpendicular to **BD**, and join **D** with any point in **AB**, as **A**; then is **EF** perpendicular to **AD** (?).

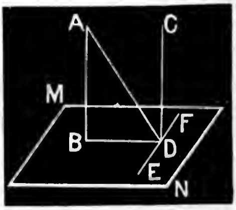

Fig. 224.

Now, **EF** being perpendicular to two lines, **AD** and **BD**, of the plane **ABDC**, is perpendicular to the plane, and hence to any line of the plane passing through **D**, as **CD**.

Therefore, **CD** is perpendicular to **BD** and **EF**, and consequently to the plane **MN** (?). Q. E. D.

461. Corollary.—*Two lines which are perpendicular to the same plane are parallel.*

Thus, **AB** and **CD** being perpendicular to the plane **MN** are parallel. For, if **AB** is not parallel to **CD**, draw a line through **B** which shall be. By the Proposition, this line is perpendicular to **MN**, and hence must coincide with **AB** (454).

PROPOSITION X.

462. Theorem.—*Two lines parallel to a third not in their own plane are parallel to each other.*

Demonstration.

Let AB and CD be parallel to EF.

Then are they parallel to each other.

For, through **F′**, any point in **EF**, pass a plane **MN** perpendicular to **EF**.

Now **AB** and **CD** are respectively perpendicular to **MN** (?), and hence are parallel to each other (?). Q. E. D.

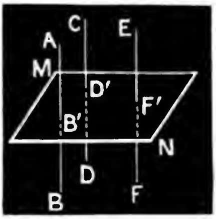

Fig. 225.

OF LINES OBLIQUE TO A PLANE.

463. An **Oblique Line** is a line which pierces the plane (if sufficiently produced), but is not perpendicular to the plane.

464. The **Projection of a Point** on a plane is the foot of the perpendicular from the point to the plane.

465. The **Projection of a Line** upon a plane is the locus of the projection of the point which generates the line.

PROPOSITION XI.

466. Theorem.—*The projection of a straight line upon a plane is a straight line.*

DEMONSTRATION.

Let AB be any line and MN the plane upon which it is projected.

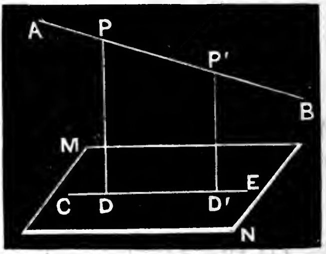

Fig. 226.

Then is the projection of **AB** in **MN** a straight line.

Let **P** be a point in **AB**, and **D** its projection in **MN**.

Pass a plane, **S**, through **AB** and **PD** (**444**), and let **CE** be its trace in **MN**.

Now let **P′** be any point in **AB** other than **P**, and let **D′** be its projection in **MN**.

As **PD** and **P′D′** are perpendicular to **MN**, they are parallel to each other (**461**), and a plane may be passed through them (**445**).

But the plane of **PD** and **P′D′** is S, since it contains **PD** and **P′** (**440**).

Therefore **D′** lies in **S**, and as it lies in **MN**, it is in the trace of **S** in **MN**, which trace is a straight line (**442**).

Hence, as **P′** is any point in **AB**, the projection of every point of **AB** is in a straight line. Q. E. D.

467. COROLLARY.—*The projection of a line upon a plane is the trace of a plane containing the line and the projection of any point of the line.*

---

468. The **Projecting Plane** is the plane of a line and its projection upon another plane.

469. The **Plane of Projection** is the plane upon which a point or a line is projected.

470. The **Inclination of a Line** to a plane is the angle included between the line and its projection.

PROPOSITION XII.

471. Theorem.—*If from any point in a perpendicular to a plane, oblique lines are drawn to the plane, those which pierce the plane at equal distances from the foot of the perpendicular are equal; and of those which pierce the plane at unequal distances from the foot of the perpendicular, those which pierce at the greater distances are the greater.*

DEMONSTRATION.

Let PD be a perpendicular to the plane MN, and PE, PE′, PE″, and PE‴ be oblique lines piercing the plane at equal distances ED, E′D, E″D, and E‴D from the foot of the perpendicular.

Then **PE = PE′ = PE″ = PE‴**.

For each of the triangles **PDE, PDE′**, etc., has two sides and the included angle equal to the corresponding parts in the other.

Again, let **FD** be longer than **E′D**.

Then is **PF > PE′**.

For, take **ED = E′D**; then **PE = PE′**, by the preceding part of the demonstration. But **PF > PE**, by (113). Hence, **PF > PE′**. Q. E. D.

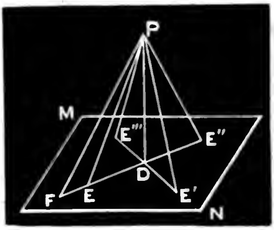

Fig. 227.

472. COROLLARY 1.—*The angles which oblique lines drawn from a common point in a perpendicular to a plane, and piercing the plane at equal distances from the foot of the perpendicular, make with the perpendicular, are equal; and the inclinations of such lines to the plane are equal.*

Thus, the equality of the triangles, as shown in the demonstration, shows that

EPD = E′PD = E″PD = E‴PD,

and **PED = PE′D = PE″D = PE‴D.**

473. COROLLARY 2.—Conversely, *If the angles which oblique lines drawn from a point in a perpendicular to a* *plane, make with the perpendicular, are equal, the lines are equal, and pierce the plane at equal distances from the foot of the perpendicular.*

Thus, let $E'PD = E''PD$;

then the right-angled triangles PDE' and PDE'' are equal (?). Hence,

$$PE' = PE'', \text{ and } DE' = DE''.$$

474. Corollary 3.—*Lines drawn from the same point in a perpendicular, and equally inclined to the plane, are equal, and pierce the plane at equal distances from the foot of the perpendicular.*

475. Corollary 4.—*Equal oblique lines from the same point in the perpendicular, pierce the plane at equal distances from the foot of the perpendicular, are equally inclined to the plane, and also to the perpendicular.*

Since the right-angled triangles PDE' and PDE'' have their altitudes and hypotenuses equal, the triangles are equal (**309**), and

$$DE' = DE'', PE'D = PE''D, \text{ and } E'PD = E''PD.$$

OF LINES PARALLEL TO A PLANE.

476. A Line is Parallel to a Plane when it is parallel to its projection in that plane.

PROPOSITION XIII.

477. Theorem.—*A line parallel to a plane is everywhere equidistant from the plane, and hence can never meet the plane;* and, conversely, *a straight line which cannot meet a plane is parallel to it.*

Demonstration.

The distance between a point in the line and the plane being the perpendicular (**451**), is also the distance between the point and the projection of the line (**464**). But this is everywhere the same (**476**, **136**). Hence a line parallel to a plane is everywhere equidistant from it, and therefore can never meet it. Q. E. D.

Conversely; A line which meets a plane meets it in the projection of the line in the plane, since the projecting plane contains all the perpendiculars, or shortest lines, from the line to the plane. Hence a line which never meets a plane is parallel to its projection in that plane, that is, to the plane itself (**476**). Q. E. D.

PROPOSITION XIV.

478. Theorem.—*Either of two parallel lines is parallel to every plane containing the other.*

DEMONSTRATION.

Let AB and CD be two parallel lines, and MN a plane containing CD.

Then is **AB** parallel to the plane **MN**.

Since **AB** and **CD** are in the same plane (?), and as the intersection of their plane with **MN** is **CD** (?), if **AB** meets the plane **MN**, it must meet it in **CD**, or **CD** produced. But this is impossible (?).

Whence **AB** is parallel to **MN** (**477**). Q. E. D.

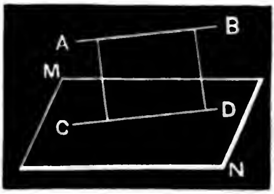

Fig. 228.

479. COROLLARY 1. *A line which is parallel to a line of a plane is parallel to the plane.*

480. COROLLARY 2.—*Through any given line a plane may be passed parallel to any other given line not in the plane of the first.*

For, through any point of the line through which the plane is to pass, conceive a line parallel to the second given line. The plane of the two intersecting lines is parallel to the second given line (?).

481. COROLLARY 3.—*Through any point in space a plane may be passed parallel to any two lines in space.*

For, through the given point conceive two lines respectively parallel to the given lines; then is the plane of these intersecting lines parallel to the two given lines (?).

PROPOSITION XV.

482. Theorem.—*Of two lines perpendicular to each other, if one is perpendicular to a plane the other is parallel to the plane.*

DEMONSTRATION.

Let AB and PD be perpendicular to each other, and PD perpendicular to the plane MN.

Then is **AB** parallel to **MN**.

If **AB** does not intersect **PD**, through any point in **PD**, as **G**, draw **A′B′** parallel to **AB**; then is it perpendicular to **PD** (**32**, foot-note).

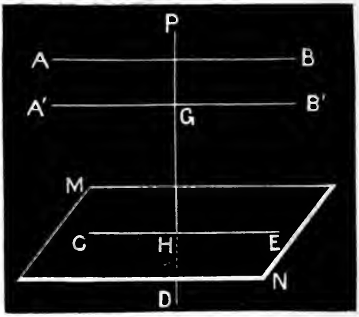

Fig. 229.

Let **CE** be the projection of **A′B′** in the plane **MN**. Then is **H** the point where **PD** pierces the plane in **CE** (?).

Hence **A′B′** is parallel to its projection **CE** (?), and consequently parallel to the plane **MN**.

Therefore **AB** is parallel to **CE** (?), and consequently to the plane **MN** (**479**). Q. E. D.

483. COROLLARY.—*A line and a plane which are both perpendicular to the same line are parallel.*

RELATIVE POSITION OF TWO PLANES.

OF PARALLEL PLANES.

484. Parallel Planes are such that either is parallel to any line of the other.

485. The **Distance between Two Parallel Planes** at any point is measured by the perpendicular.

PROPOSITION XVI.

486. Theorem.—*Parallel planes are everywhere equidistant and hence can never meet.*

DEMONSTRATION.

Let P and Q be two parallel planes.

Then are they everywhere equidistant, and hence can never meet.

Let **A** and **B** be any two points in **P**, and pass a line through them.

Since **Q** is parallel to **P**, it is parallel to the line **AB** (484). And since it is parallel to **AB** it is everywhere equidistant from **AB**.

Hence **A** and **B**, any two points in **P**, are equidistant from **Q**, and consequently **P** and **Q** can never meet. Q. E. D.

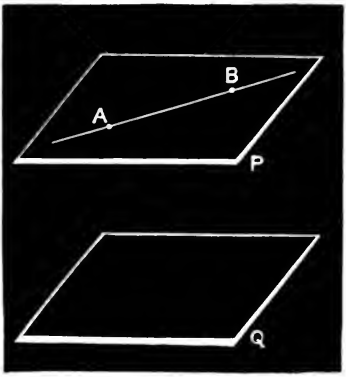

Fig. 230.

PROPOSITION XVII.

487. Theorem.—*Two planes perpendicular to the same line are parallel to each other.*

DEMONSTRATION.

Let P and Q be two planes perpendicular to the line AB.

Then are **P** and **Q** parallel.

For any line in one plane is parallel to its projection in the other, since any line in either plane is perpendicular to **AB** (?).

Hence either plane is parallel to any line of the other (476), and therefore the planes are parallel to each other. Q. E. D.

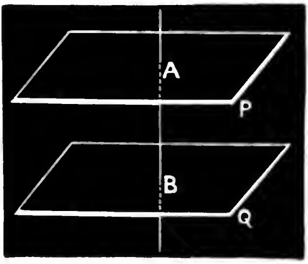

Fig. 231.

PROPOSITION XVIII.

488. Theorem.—*If a plane intersects two parallel planes, the lines of intersection are parallel.*

Demonstration.

Let RS intersect the parallel planes MN and PQ in AB and CD.

Then is **AB** parallel to **CD**.

For, if **AB** and **CD** could meet, the planes **MN** and **PQ** would meet, as every point in **AB** is in **MN**, and every point in **CD** in **PQ**. Hence, **AB** and **CD** lie in the same plane, and do not meet how far soever they be produced (**132**); they are therefore parallel. Q. E. D.

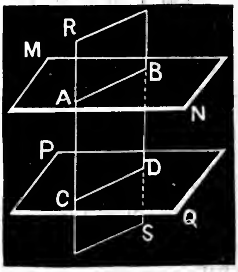

Fig. 232.

489. Corollary.—*Parallel lines intercepted between parallel planes are equal.*

Thus, **AC** = **BD**, if they are parallel. For, the intersections **AB** and **CD**, of the plane of these parallels, are parallel (?), and the figure **ABDC** is a parallelogram; whence, **AC** = **BD** (?).

PROPOSITION XIX.

490. Theorem.—*A line which is perpendicular to one of two parallel planes, is perpendicular to the other also.*

Demonstration.

Let MN and PQ be two parallel planes; and let AB be perpendicular to PQ.

Then is **AB** perpendicular to **MN**.

For, pass any plane through **AB**, and let **AC** and **BD** be its intersections with **MN** and **PQ** respectively Then are **AC** and **BD** parallel (?). Now, **AB** is perpendicular to **BD** (?), and hence to **AC** (?). Thus, **AB** is shown to be perpendicular to any line of **MN** passing through its foot, and hence perpendicular to **MN** (?). Q. E. D.

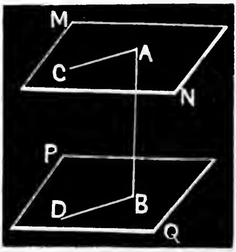

Fig. 233.

PROPOSITION XX.

491. Theorem.—*Through any point without a plane, one plane can be passed parallel to the given plane, and only one.*

DEMONSTRATION.

Let MN be a plane, and B any point without MN.

Let **BA** be a perpendicular from **B** upon **MN**.

Through **B** draw **DE** and **FG** perpendicular to **AB**. Then is the plane of **DE** and and **FG** parallel to **MN** (452, 487). Q. E. D.

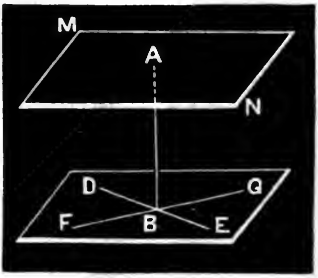

Fig. 234.

Again, as any plane parallel to **MN** is perpendicular to **AB**, and as only one plane can be passed through **B** perpendicular to **AB** (457), only one plane can be passed through **B** parallel to **MN**. Q. E. D.

PROPOSITION XXI.

492. Theorem.—*Two angles lying in different planes, but having their sides parallel and extending in the same direction, or in opposite directions, are equal, and their planes are parallel.*

DEMONSTRATION.

Let A and A′ lie in the different planes MN and PQ, and have AB parallel to A′B′, and AC to A′C′.

Then **A** = **A′**, and **MN** and **PQ** are parallel.

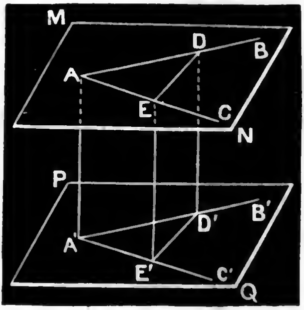

Fig. 235.

For, take **AD** = **A′D′**, and **AE** = **A′E′**, and draw **AA′**, **DD′**, **EE′**, **ED**, and **E′D′**. Now, **AD** being equal and parallel to **A′D′**,

AA′ = **DD′** (?)

For like reason,

$$AA' = EE';$$

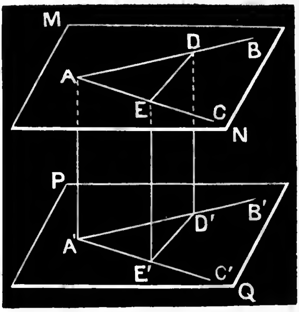

Fig. 235.

therefore $EE' = DD'$. Again, since EE' and DD' are respectively parallel to AA', they are parallel to each other (?); whence $EDD'E'$ is a parallelogram (?), and $ED = E'D'$. Hence the triangles ADE and $A'D'E'$ are mutually equilateral, and A, opposite ED, is equal to A', opposite $E'D'$, equal to ED. Q. E. D.

Again, the plane of the angle BAC, MN, is parallel to PQ, the plane of $B'A'C'$. For, let a plane be passed through A and revolved until it is parallel to PQ. It must cut DD' which is parallel to AA', and EE' which also is parallel to AA', so that DD' and EE' shall equal AA' (?); hence it must pass through D. Hence the planes of the angles are parallel. Q. E. D.

493. Corollary 1.—*If two intersecting planes are cut by parallel planes, the angles formed by the intersections are equal.*

Thus, AB' and AC' being cut by the parallel planes MN and PQ, AD is parallel to $A'D'$ (?), and extends in the same direction from vertex A that $A'D'$ does from A'; and the same may be said of AC and $A'C'$. Hence, $BAC = B'A'C'$ (?).

494. Corollary 2.—*If the corresponding extremities of three equal parallel lines not in the same plane are joined, the triangles formed are equal, and their planes parallel.*

Thus, if $AA' = DD' = EE'$, the sides of the triangle AED are equal to the sides of $A'E'D'$, since the figures AD', DE', and EA' are parallelograms (?), and the corollary comes under the proposition (?).

PROPOSITION XXII.

495. Theorem.—*If two lines are cut by three parallel planes, the corresponding intercepted segments are proportional.*

Demonstration.

Let AB and CD be cut by the three parallel planes M, N, and P, AB piercing the planes in A, E, and B, and CD in C, F, and D.

Then is $\frac{AE}{EB} = \frac{CF}{FD}$.

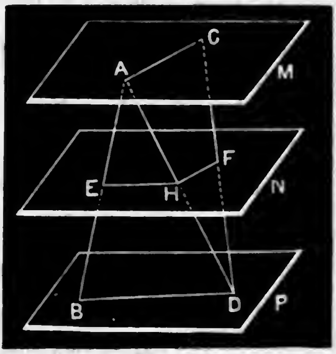

Fig. 236.

Join the points A and D by the straight line AD, and conceive planes passing through AD and DC, and through AB and AD.

Let EH and BD be the intersections of the planes N and P with the plane BAD, and AC and HF the intersections of M and N with ADC.

Now, since EH is parallel to BD (?),

$$\frac{AE}{EB} = \frac{AH}{HD} \text{ (?)}.$$

In like manner, by reason of the parallelism of HF and AC,

$$\frac{CF}{FD} = \frac{AH}{HD}.$$

Hence, by equality of ratios,

$$\frac{AE}{EB} = \frac{CF}{FD}. \quad \text{Q. E. D.}$$

[Note.—Planes perpendicular or oblique to each other give rise to one species of solid angles; hence their consideration is reserved for the next Section.]

EXERCISES.

496. 1. Designate any three points in the room, as one corner of the desk, a point on the stove, and some point in the ceiling, and show how you can conceive the plane of these points.

2. Show the position of two lines which will not meet, and yet are not parallel.

3. Conceive two lines, one line in the ceiling and one in the floor, which shall not be parallel to each other.

4. The ceiling of my room is 10 feet above the floor. I have a 12-foot pole, by the aid of which I wish to determine a point in the floor directly under a certain point in the ceiling. How can I do it?

SUGGESTION.—Consult Proposition XII.

5. Upon what principle in this section is it that a stool with three legs always stands firm on a level floor, when one with four may not?

6. By the use of two carpenter's squares you can determine a perpendicular to a plane. How is it done?

7. If you wish to test the perpendicularity of a stud to a level floor, on how many sides of it is it necessary to measure the angle which it makes with the floor? By applying the right angle of the carpenter's square on *any* two sides of the stud, to test the angle which it makes with the floor, can you determine whether it is perpendicular or not?

8. If a line is drawn at an inclination of 23° to a plane, what is the greatest angle which any line of the plane, drawn through the point where the inclined line pierces the plane, makes with the line? Can you conceive a line of the plane which makes an angle of 50° with the inclined line? Of 80°? Of 15°? Of 170°?

SECTION II.

OF SOLID ANGLES.

497. A **Solid Angle** is the opening between two or more planes, each of which intersects all the others. The lines of intersection are called **Edges,** and the planes, or the portion of the planes between the edges where there are more than two, are called **Faces.**

498. Solid Angles are of **Three Species,** viz., **Diedral, Triedral,** and **Polyedral,** according as they have two, three, or more than three faces.

OF DIEDRALS.

499. A **Diedral Angle,** or simply a **Diedral,** is the opening between two intersecting planes.

500. A **Diedral (Angle) is Measured** by the plane angle included by lines drawn in its faces from any point in the edge, and perpendicular thereto.

A diedral angle is called **Right, Acute,** or **Obtuse,** according as its measure is right, acute, or obtuse.

Two diedrals are said to be **Supplementary,** when their measures are supplementary.

Of course the magnitude of a solid angle is independent of the distances to which the edges may chance to be produced.

Illustrations.—The opening between the two planes **CABF** and **DABE** (Fig. 237) is a *Diedral* (angle), **AB** is the *Edge*, and **CABF** and **DABE** are the *Faces*. Let **MO** lie in the plane **AF**, perpendicular to the edge; and **NO** in **AE**, and also perpendicular to the edge; then the plane angle **MON** is the measure of the diedral.

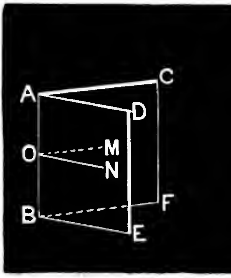

Fig. 237.

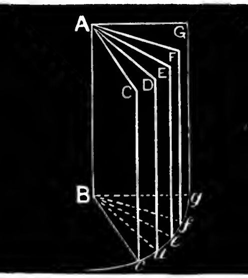

Fig. 238.

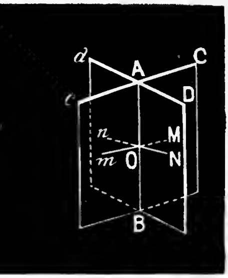

Fig 239.

501. A diedral may be read by the letters on the edge, when there would be no ambiguity, or otherwise by these letters and one in each face.

Thus, the diedral in Fig. 237 may be designated as **AB**, or as **C-AB-D**.

502. A diedral may be considered as generated by the revolution of a plane about a line of the plane, and hence we may see the propriety of measuring it by the angle included by two lines in its faces perpendicular to its edge, as stated in the preceding article.

Illustration.—Let **AB** (Fig. 238) be a line of the plane **GB**. Conceive *g***B** perpendicular to **AB**. Now, let the plane revolve upon **AB** as an axis, whence *g***B** describes a circle (?); and at any position of the revolving plane, as *f***BAF**, since *f***B***g* measures the amount of revolution, it may be taken as the measure of the diedral *f*-**BA**-*g*. When *g***B** has made $\frac{1}{4}$ of a revolution, the plane will have made $\frac{1}{4}$ of a revolution, and the diedral will be right.

503. When two planes intersect, four diedrals are formed, any two of which are either **Adjacent** to each other, or **Opposite.**

504. Adjacent Diedrals are on the same side of one plane, but on the opposite sides of the other.

As **D-AB-C** and **D-AB-***c*, or *c*-**AB-D** and *c*-**AB**-*d* (Fig. 239).

Opposite Diedrals are on opposite sides of both planes.

As **D-AB-C** and *d*-**AB**-*c*, or **D-AB**-*c* and *d*-**AB-C** (Fig. 239).

PROPOSITION I.

505. Theorem.—*When two planes intersect, the opposite diedrals are equal, and the adjacent ones are supplementary.*

DEMONSTRATION.

Let the planes DE and CF intersect in AB.

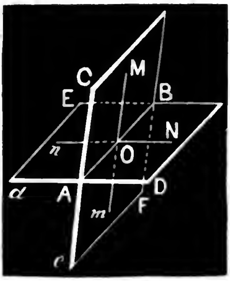

Fig. 240.

Then $\quad$ D-AB-C $= d$-AB-c,

and $\quad$ D-AB-$c = d$-AB-C;

and $\quad$ D-AB-C + D-AB-c* $= 180°$,

c-AB-D + c-AB-$d = 180°$, etc.

Through O, any point in AB, let Mm be drawn in the plane CF, and Nn in the plane DE, each perpendicular to AB. Then is MON, the measure of D-AB C (?), $= m$On, the measure of d-AB-c (?), etc. Q. E. D.

Also, $\quad$ MON + NOm $= 180°$ (?),

NOm + mOn $= 180°$, etc. Q. E. D.

PROPOSITION II.

506. Theorem.—*Any line in one face of a right diedral, perpendicular to its edge, is perpendicular to the other face.*

DEMONSTRATION.

In the face CB of the right diedral C-AB-D, let MO be perpendicular to the edge AB.

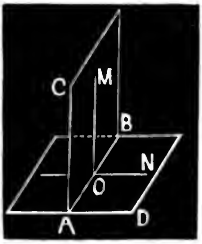

Fig. 241.

Then is MO perpendicular to the face DB.

For, draw ON in the face DB, and perpendicular to AB. Now, since the diedral is right, and MON measures its angle, MON is a right angle; whence MO is perpendicular to two lines of the plane DB, and consequently perpendicular to the plane. Q. E. D.

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* By this is meant the measure of the diedral.

507. Corollary 1.—Conversely, *If one plane contains a line which is perpendicular to another plane, the diedral is right.*

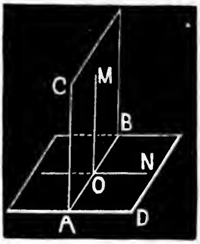

Fig. 241.

Thus, if **MO** is perpendicular to the plane **DB**, **C-AB-D** is a right diedral. For **MO** is perpendicular to every line of **DB** passing through its foot (?); and hence is perpendicular to **ON**, drawn at right angles to **AB**. When **C-AB-D** is a right diedral, for it is measured by a right plane angle.

508. Two planes are **Perpendicular** to each other when they intersect so as to make the adjacent diedrals equal. In this case, all four of the diedrals are right.

509. Corollary 2.—*The plane which projects a line upon a plane* (**468**) *is perpendicular to the plane of projection.*

PROPOSITION III.

510. Theorem.—*If each of two intersecting planes is perpendicular to a third, their intersection is perpendicular to the third plane.*

Demonstration.

Let EF and CD be two planes perpendicular to the third plane MN, and let AB be the intersection of EF and CD.

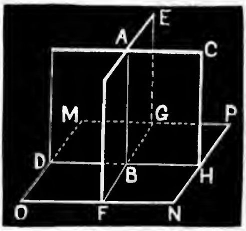

Fig. 242.

Then is **AB** perpendicular to **MN**.

For, **EF** being perpendicular to **MN**, **D-FG-E** is a right diedral, and a line in **EF** perpendicular to **FG** at **B** is perpendicular to **MN**; also a line in the plane **CD**, and perpendicular to **DH** at **B**, is perpendicular to **MN** (?).

Hence, as there can be one and only one perpendicular to **MN** at **B**, and as this perpendicular is in both planes, **CD** and **EF**, it is their intersection. Q. E. D.

PROPOSITION IV.

511. Theorem.—*The angle included by perpendiculars drawn from any point within a diedral to its faces, is the supplement of the diedral.*

DEMONSTRATION.

Let P′ be any point within the diedral F-AB-C, and let the perpendiculars P′D′ and P′E′ be drawn to the faces.

Then is **D′P′E′** the supplement of **F-AB-C.**

From **P**, any point in the plane which bisects the diedral **F-AB-C**, draw **PD** and **PE** perpendicular to the same faces respectively as **P′D′** and **P′E′**. Then is **DPE = D′P′E′**.

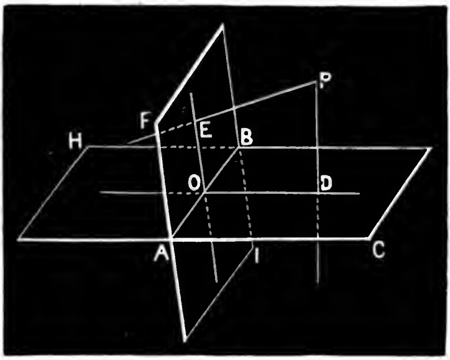

Fig 243.

Now pass a plane through **PE** and **PD**, and let **EO** and **DO** be its intersections with **FB** and **CB** respectively. Then, by (507), **FB** and **CB** are perpendicular to the plane **PEOD**. Hence, **AB** is perpendicular to **PEOD** (?), and **EOD** is the measure of **F-AB-C** (?). But in the quadrilateral **PEOD**, **P** is the supplement of **EOD** (?), and hence of **F-AB-C.**

Hence, **D′P′E′** is the supplement of **F-AB-C**. Q. E. D.

512. COROLLARY 1.—*If from a point in the edge of a diedral perpendiculars are erected to the faces on the same sides of the planes respectively as the perpendiculars let fall from a point within, the included angle is the supplement of the angle of the diedral.*

513. COROLLARY 2.—*The angle* **DPE** *is the supplement of the opposite diedral* **H-AB-I**, *and equal to each of the adjacent diedrals* **C-AB-I** *and* **F-AB-H**.

PROPOSITION V.

514. Theorem.—*Between any two lines not in the same plane one line, and only one, can be drawn which shall be perpendicular to both, and this line is the shortest distance between them.*

Demonstration.

Let AB and CD be two lines not in the same plane.

Then one line, as **HG**, and only one, can be drawn which is perpendicular to both **AB** and **CD**, and **HG** measures the shortest distance between **AB** and **CD**.

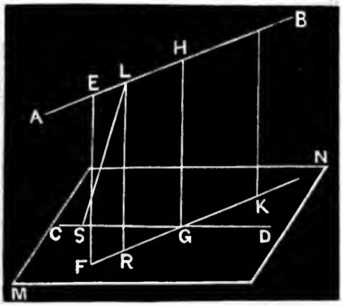

Fig. 244.

Through either line, as **CD**, pass a plane **MN** parallel to **AB** (480). From any point in **AB**, as **E**, let fall **EF** perpendicular to **MN**.

Let **EK** be the plane of the lines **EF** and **EB**, and let **FK** be its trace in **MN**.

Now, as **AB** and **CD** are not in the same plane, **EK**, and hence its trace **FK**, cuts **CD** in some point, as **G**.

From **G** draw **GH** perpendicular to **AB**.

1st. **GH** lies in the plane **EK** (?), which is perpendicular to **MN** (?), and being perpendicular to **AB** is perpendicular to **FK** (?), and hence to the plane **MN** (508).

Therefore, **GH**, which is perpendicular to **AB**, is perpendicular to **CD** (?).

2d. **GH** is the only line which is perpendicular to both **AB** and **CD**.

For any line which is perpendicular to **AB** and **CD** is perpendicular to **FK** (?), and hence to **MN** (?).

Now every perpendicular from **AB** to the plane **MN** meets this plane in **FK** (?).

But **FK** and **CD** have only one point common, viz., **G**. Hence, **GH** is the only perpendicular from **AB** to **CD**.

3d. **GH** is the shortest distance between **AB** and **CD**. For a line from any point in **AB** to any other point in **CD**, as **LS**, would be oblique to **MN** (?), and hence longer than the perpendicular **LR**, = **HG**.

PROPOSITION VI.

515. Theorem.—*If one of two parallel planes is perpendicular to a third plane, the other is also.*

DEMONSTRATION.

Let PD and QE be two parallel planes; and let PD be perpendicular to the third plane MN.

Then is **QE** perpendicular to **MN.**

Through **PD** and **QE** pass the plane **RS** perpendicular to **MN**, and let **FK** be its trace in **QE**, and **HI** in **PD**.

Then is **FK** perpendicular to **MN** (?).

And, as **HI** is parallel to **FK** (?), it is perpendicular to **MN** (460).

Hence, **QE** is perpendicular to **MN** (507). Q. E. D.

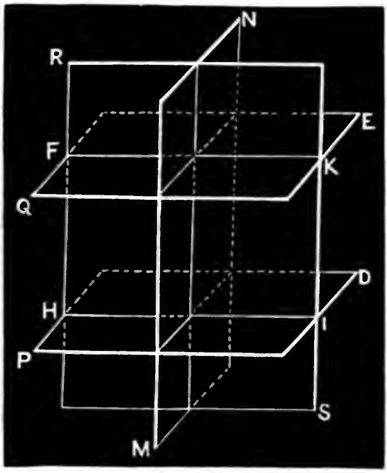

Fig. 245.

OF TRIEDRALS.

516. As diedrals result from the intersection of *two* planes, so triedrals result from the intersection of *three* planes.

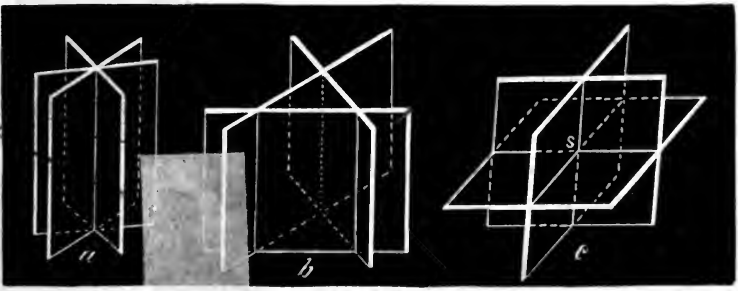

Fig. 246.

517. Three planes may intersect in three principal ways:

1st. Their intersections may all coincide, as in (*a*).

2d. They may have three parallel intersections, as in (*b*).

3d. They may have three non-parallel intersections, as in (*c*). In this case the three intersections meet in a common point, as at S.

In the first case the three planes have an *infinite* number of common points. In the second case they have *no* common point. In the third case they have but *one* common point.

The third case gives rise to **Triedrals.**

518. A **Triedral** is the opening between three planes which meet in a common point.

519. When three planes meet so as to form one triedral, they form also eight, as planes are to be considered indefinitely extended, unless otherwise stated.

520. The planes enclosing a particular triedral are called its **Faces,** and their intersections its **Edges.** The common point is called the **Vertex.**

521. A triedral may be designated by naming the letter at the vertex and then three other letters, one in each edge.

Thus, in the figure, the opening between the three planes ASC, CSB, and BSA is the triedral S-ABC. The *faces* are ASC, CSB, and BSA.

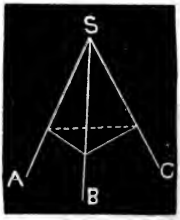

Fig. 247.

522. The plane angles enclosing a solid angle are called **Facial Angles.**

523. In every particular triedral there are six parts, **Three Facial Angles** and **Three Diedrals.**

524. Our study of triedrals will be confined to the relations of the facial angles and the diedrals, and the comparison of different triedrals.

525. Triedrals are **Rectangular, Bi-rectangular,** or **Tri-rectangular,** according as they have one, two, or three right diedral angles.

ILLUSTRATION.—The corner of a cube is a *Trirectangular* triedral, as **S-ADC**. Conceive the upper portion of the cube removed by the plane **ASEF**; then the angle at **S**, *i. e.*, **S-AEC**, is a *Bi-rectangular* triedral, **A-SC-E** and **A-SE-C** being right diedrals.

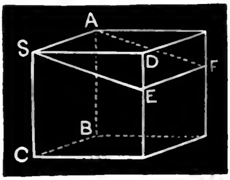

Fig. 248.

526. An **Isosceles Triedral** is one that has two of its facial angles equal. An **Equilateral Triedral** is one that has all three of its facial angles equal.

527. Opposite Triedrals are such as lie on opposite sides of each of the intersecting planes, as **S-ABC** and **S-*abc***.

Opposite triedrals have mutually equal facial and equal diedral angles, but these being differently disposed, such triedrals are not in general capable of superposition.

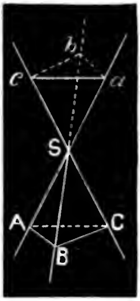

Fig. 249.

ILLUSTRATION.—Let the edges of the triedral **S-ABC** be produced beyond the vertex, forming the opposite triedral **S-*abc***. Now, the faces are equal plane angles, but disposed in a different order. Thus, **ASB** = ***a*S*b***, **ASC** = ***a*S*c***, and **BSC** = ***b*S*c***, and the diedrals are also equal; but the triedrals cannot be superimposed, or made to coincide. To show this fact, conceive the upper triedral detached, and the face ***a*S*c*** placed in its equal face **ASC**, **S*a*** in **SA**, and **S*c*** in **SC**. Now the edge **S*b***, instead of falling in **SB**, in front of **ASC**, will fall behind the plane **ASC**.

Or, otherwise, if **S-*abc*** be revolved on **S** by bringing it forward and turning it down on **S-ABC**, since the diedrals **A-SB-C** and ***c*-S*b*-*a*** are equal, they will coincide; but, as facial angle ***a*S*b*** is not necessarily equal to **CSB**, **S*a*** will not necessarily fall in **SC**. For a like reason, **S*c*** will not necessarily fall in **SA**.

528. Symmetrical Triedrals are triedrals in which each part in one has an equal part in the other; but the equal parts not being similarly disposed, the triedrals may not be capable of superposition.

Symmetrical solids are of frequent occurrence: the two hands form an illustration; for, though the parts may be exactly alike, the hands cannot be placed so that their like parts will be similarly situated; in short, the left glove will not fit the right hand.

529. Adjacent Triedrals are such as lie on *different* sides of *one* of the intersecting planes, and on the *same* side of *two* of them.

Thus, **S-ADE** is adjacent to **S-DRE.**

In adjacent triedrals, two of the facial angles of one are the supplements of two of the other, each to each, and one is equal in each.

Thus, in the adjacent triedrals **S-DRE** and **S-ADE**, **ASE** and **ASD** are supplements respectively of **ESR** and **DSR**, while **DSE** is common to both.

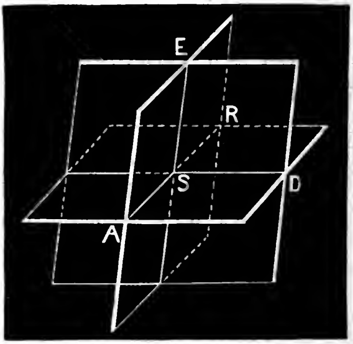

Fig. 250.

530. Of the eight triedrals formed by the intersection of three planes, each has its **Opposite** or **Symmetrical** triedral, and each has three **Adjacent** triedrals.

531. Two triedrals are **Supplementary** when the facial angles of the one are the supplements of the measures of the corresponding diedrals of the other.

532. Equality, as has been before defined, means, in Geometry, equality in all respects; and two figures that are said to be equal are capable of being so applied the one to the other that they will coincide throughout. This absolute equality is hence often called **Equality by Superposition,** in distinction from **Equality by Symmetry.**

533. Two figures are said to be **Equal by Symmetry,** or **Symmetrically Equal,** or simply **Symmetrical,** when each part in one has an equal part in the other; but these equal parts being differently arranged in the two figures, the one may not be capable of being superimposed upon the other. (See **527.**)

PROPOSITION VII.

534. Theorem.—*Opposite triedrals are equal and may be symmetrical.*

DEMONSTRATION.

Let S-ABC and S-*abc* be two opposite triedrals.

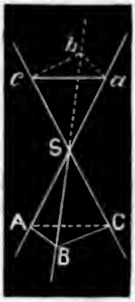

Fig. 251.

Then are the triedrals equal or symmetrical.

For the facial angle **ASC** = the facial angle *a***S***c* (?); also, **BSC** = *b***S***c*, and **ASB** = *a***S***b*.

Again, the diedral **A-SB-C** = *a*-**S***b*-*c*, since they are opposite diedrals.

For like reason, **B-SA-C** = *b*-**S***a*-*c*, and **A-SC-B** = *a*-**S***c*-*b*.

Hence all the parts in one triedral have equal parts in the other.

But, in general, these triedrals cannot be superimposed. (See illustration, **527.**)

If, however, **ASB** = **CSB**, then *a***S***b* = *c***S***b*, and the triedrals can be superimposed.

Thus, conceive the triedral **S**-*abc* revolved on **S**, being brought over towards the observer until **S***b* falls in **SB**.

Then, since **CSB** = **ASB** = *a***S***b*, *a***S***b* may be made to coincide with **BSC**, and as the diedrals **A-SB-C** and *a*-**S***b*-*c* are equal, *c***S***b* will fall in **ASB**, and the triedrals will coincide, and will be equal.

Hence, opposite diedrals are equal and may be symmetrical. Q. E. D.

535. COROLLARY 1. — *Opposite isosceles triedrals are equal.*

PROPOSITION VIII.

536. Theorem.—*Two symmetrical triedrals may always be conceived to be placed as opposite triedrals.*

DEMONSTRATION.

Let S-ABC and S′-A′B′C′ be two symmetrical triedrals, B and B′ being in front of the planes ASC and A′S′C′, ASB = A′S′B′, ASC = A′S′C′, BSC = B′S′C′, A-SB-C = A′-S′B′-C′, A-SC-B = A′-S′C′-B′, and B-SA-C = B′-S′A′-C′.

Then may S-ABC and S′-A′B′C′ be placed as opposite triedrals.

Produce the edges of either triedral, as S′-A′B′C′, beyond the vertex, forming the opposite triedral S′-*abc*.

Then can S-ABC be superimposed upon S′-*abc*, and the latter fulfills the requirements of the proposition.

The application is made as follows:

Since B′ is in front of the plane A′S′C′, *b* is behind the plane *a*S′*c*.

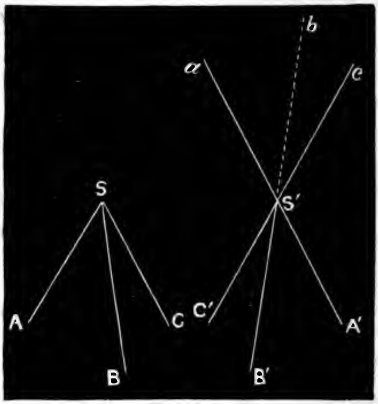

Fig. 252.

Now conceive S-ABC inverted and reversed so that B shall fall behind the plane ASC.

Then apply ASC to its equal *a*S′*c*, SA falling in S′*a*, and SC in S′*c*.

By reason of the equality of A-SC-B and *a*-S′*c*-*b* (= A′-S′C′-B′), the plane BSC will fall in *b*S′*c*, and for a like reason ASB will fall in *a*S′*b*; and since the planes coincide, their intersections SB and S′*b* must coincide.

Hence, S-ABC = S′-*abc*, the opposite to S′-A′B′C′. Q. E. D.

PROPOSITION IX.

537. Theorem.—*Two triedrals which have two facial angles and the included diedral equal, each to each, are either equal or symmetrical.*

DEMONSTRATION.

Let 1, 2, 3, be triedrals having the facial angle ASC = A′S′C′ = *a*S″*c*, CSB = C′S′B′ = *c*S″*b*, and A-SC-B = A′-S′C′-B′ = *a*-S″*c*-*b*.

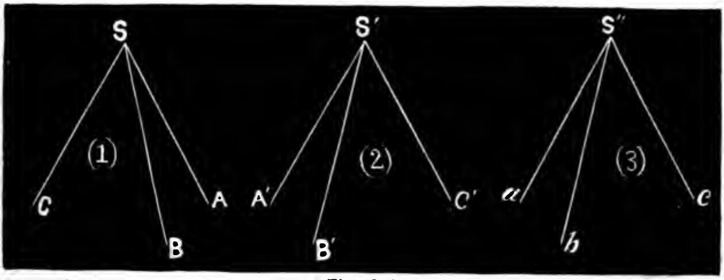

Fig. 253.

Then are the triedrals either equal or symmetrical.

1st. When the equal facial angles are on the same sides of the respective equal diedrals, as in Figs. 2 and 3, the triedrals may be applied the one to the other.

Thus, let the facial angle A′S′C′ be placed in its equal *a*S″*c*, A′S′ in *a*S, and S′C′ in S″*c*; whence, by reason of the equality of the diedrals A′-S′C′-B′ and *a*-S″*c*-*b*, and since the facial angles B′S′C′ and *b*S″*c* lie on the same sides respectively of their diedrals A′-S′C′-B′ and *a*-S″*c*-*b*, the plane of B′S′C′ falls in the plane of *b*S″*c*, and since angle B′S′C′ = angle *b*S″*c*, B′S′ falls in *b*S″, and A′S′B′ coincides with *a*S″*b*.

Hence the triedrals coincide and are equal. Q. E. D.

2d. But if the equal facial angles lie on different sides of the equal diedrals, as in Figs. 1 and 3, let the opposite of S-ABC be drawn (527), and call it S-*a′b′c′*. Then may 1 be applied to S-*a′b′c′*.

[Let the student draw the figure and make the application.]

PROPOSITION X.

538. Theorem.—*Two triedrals which have two diedrals and the included facial angles equal each to each, are either equal or symmetrical.*

DEMONSTRATION.

[Same as preceding. Let the student draw figures like those for the preceding, and go through with the details of the application.]

539. COROLLARY.—*In equal or in symmetrical triedrals, the equal facial angles are opposite the equal diedrals.*

PROPOSITION XI.

540. Theorem.—*The sum of any two facial angles of a triedral is greater than the third.*

DEMONSTRATION.

This proposition needs demonstration only in case of the sum of the two smaller facial angles as compared with the greatest (?).

Let **ASB** and **BSC** each be less than **ASC**; then is

$$ASB + BSC > ASC.$$

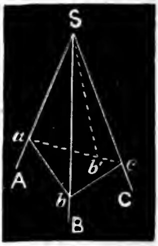

Fig. 254.

For, in the face **ASC**, make the angle $ASb' = ASB$, and $Sb' = Sb$, and pass a plane through b and b', cutting **SA** and **SC** in a and c.

The two triangles aSb and aSb' are equal (?), whence

$$ab' = ab.$$

Now, $\qquad ab + bc > ac$ (?),

and subtracting ab from the first member, and its equal ab' from the second, we have $bc > b'c$.

Whence the two triangles bSc and $b'Sc$ have two sides in the one equal to two sides in the other, each to each, but the third side $bc >$ than the third side $b'c$, and consequently angle $BSC > bSC$. Adding **ASB** to the former, and its equal ASb' to the latter, we have

$$ASB + BSC > ASC. \quad \text{Q. E. D.}$$

541. COROLLARY.—*The difference between any two facial angles of a triedral is less than the third facial angle* (?).

PROPOSITION XII.

542. Theorem.—*Two triedrals which have two facial angles of the one equal to two facial angles of the other, each to each, and the included diedrals unequal, have the third facial angles unequal, and the greater facial angle belongs to the triedral having the greater included diedral.*

DEMONSTRATION.

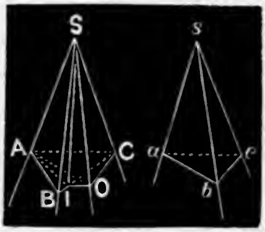

Fig. 255.

Let **ASC** = *asc*, and **ASB** = *asb*, while the diedral **C-SA-B** > *c-sa-b*.

Then **CSB** > *csb*.

For, divide the diedral **C-SA-B** by a plane **ASO**, making the diedral **C-SA-O** = *c-sa-b*; and taking **ASO** = *asb*, bisect the diedral **O-SA-B** with the plane **ISA**. Conceive the planes **OSI** and **OSC**.

Now, the triedrals **S-AOC** and *s-abc* are equal or symmetrical, having two facial angles and the included diedral equal each to each (537).

For a like reason, **S-AIO** and **S-AIB** are symmetrical, and the facial angle **OSI** = **ISB**.

Again, in the triedral **S-IOC**,

OSI + **ISC** > **OSC** (540),

and substituting **ISB** for **OSI**, we have

ISB + **ISC** (or **CSB**) > **OSC**, or its equal *csb*. Q. E. D.

543. COROLLARY.—Conversely, *If the two facial angles are equal, each to each, in two triedrals, and the third facial angles unequal, the diedral opposite the greater facial angle is the greater.*

That is, if **ASB** = *asb*, and **ASC** = *asc*,

while **BSC** > *bsc*,

the diedral **B-AS-C** > *b-as-c*.

For, if **B-AS-C** = *b-as-c*, **BSC** = *bsc* (537, 539);

and if **B-AS C** < *b-as-c*, **BSC** < *bsc*, by the proposition.

Therefore, as **B-AS-C** cannot be equal to nor less than *b-as-c*, it must be greater. Q. E. D.

PROPOSITION XIII.

544. Theorem.—*Two triedrals which have the three facial angles of the one equal to the three facial angles of the other, each to each, are either equal or symmetrical.*

Demonstration.

Let **A**, **B**, and **C** represent the facial angles of one, and a, b, and c the corresponding facial angles of the other. If $\mathbf{A} = a$, $\mathbf{B} = b$, and $\mathbf{C} = c$, the triedrals are equal or symmetrical.

For **A** being equal to a, and **B** to b, if, of their included diedrals, **SM** were greater than sm, **C** would be greater than c (?); and if diedral **SM** were less than diedral sm, **C** would be less than c (?). Hence, as diedral **SM** can neither be greater nor less than diedral sm, it must be equal to it.

Therefore the triedrals have two facial angles and the included diedral equal, each to each, and are consequently equal or symmetrical. Q. E. D.

PROPOSITION XIV.

545. Theorem.—*If from any point within a triedral perpendiculars are drawn to the faces, they will be the edges of a supplementary triedral.*

Demonstration.

From S′ within the triedral S-ABC, let S′A′ be drawn perpendicular to ASB, S′B′ to ASC, and S′C′ to BSC.

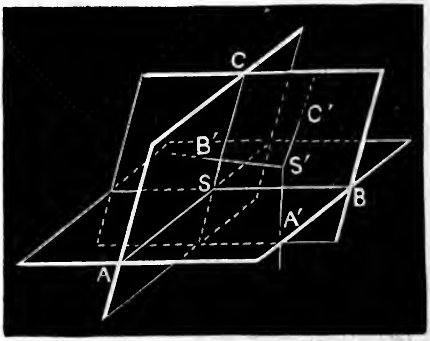

Fig. 256.

Then is **S′-A′B′C′** supplementary to **S-ABC**.

For the facial angle **A′S′B′** is the supplement of the diedral **B-AS-C** (511); and for like reason **B′S′C′** is the supplement of **A-SC-B**, and **A′S′C′** of **A-SB-C**.

Again, since **S′A′** is perpendicular to the face **ASB**, and **S′B′** is perpendicular to **ASC**, the plane of **S′A′** and

S′B′ is perpendicular to ASB and ASC, and therefore to SA. Hence SA is perpendicular to the face A′S′B′.

For a similar reason, SC is perpendicular to B′S′C′. Hence ASC is the supplement of A′-S′B′-C′.

In like manner, it may be shown that BSC is the supplement of A′-S′C′-B′, and ASB of B′-S′A′-C′. Q. E. D.

546. Scholium 1.—If perpendiculars were drawn from the point S, or any other point, parallel to those from S′, and in the same directions respectively from S that S′A′, etc., are from S′, they would also be perpendicular to the faces of the diedral, and would form a supplementary triedral.

547. Scholium 2.—The triedral S′-A′B′C′ is also supplementary to the triedral opposite to S-ABC.

548. Scholium 3.—The triedral S′-A′B′C′ will not be supplementary to the triedral *adjacent* to S-ABC, but one facial angle will be supplementary to the corresponding diedral in the other, and the other facial angles will be equal to their corresponding diedrals.

549. Scholium 4.—One triedral adjacent to S′-A′B′C′ will be supplementary to one of those adjacent to S-ABC.

PROPOSITION XV.

550. Theorem.—*In an isosceles triedral the diedrals opposite the equal facial angles are equal;* and,

Conversely, *If two diedrals of a triedral are equal, the triedral is isosceles.*

Demonstration.

In the triedral S-ABC, let ASC = CSB.

Then is C-SA-B = C-SB-A.

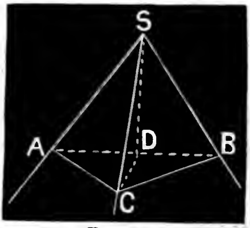

Fig. 257.

For, pass the plane CSD through the edge SC, bisecting the diedral A-SC-B. Then the two triedrals S-ACD and S-CBD have two facial angles of one equal to two facial angles of the other, each to each; that is, ASC = CSB, by hypothesis, and CSD common; and the included diedrals equal by construction. Hence the triedrals are symmetrical, and

$$\text{C-SA-B} = \text{C-SB-A} \ (537, 539). \quad \text{Q. E. D.}$$

Conversely, if $\text{C-SA-B} = \text{C-SB-A}$,

$$\text{ASC} = \text{CSB}.$$

For the supplementary triedral is isosceles; whence the diedrals opposite those equal facial angles are equal. But **ASC** and **CSB** are the supplements of these equal diedrals, and hence equal. Q. E. D.

551. COROLLARY 1.—*The plane which bisects the angle included by the equal facial angles of an isosceles triedral is perpendicular to the opposite face, and bisects the opposite facial angle.*

552. COROLLARY 2.—*If the three facial angles of a triedral are equal, each to each, the diedrals are also equal, each to each, and conversely.*

PROPOSITION XVI.

553. Theorem.—*Two triedrals which have the three diedrals of the one equal to the three diedrals of the other, each to each, are equal or symmetrical.*

DEMONSTRATION.

In the two supplementary triedrals, the facial angles of the one are equal to the facial angles of the other, each to each, since they are supplements of equal diedrals (**545**). Hence, the supplementary triedrals are equal or symmetrical (**544**).

Now, the facial angles of the first triedrals are supplements of the diedrals of the supplementary; whence the corresponding facial angles, being the supplements of equal diedrals, are equal. Therefore, the proposed triedrals have their facial angles equal, each to each, and are consequently equal, or symmetrical. Q. E. D.

554. COROLLARY. — *All tri-rectangular triedrals are equal.*

PROPOSITION XVII.

555. Theorem.—*The sum of the facial angles of a triedral may be anything between zero and four right angles.*

DEMONSTRATION.

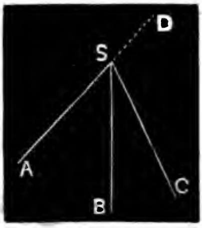

Fig. 258.

Let ASB, BSC, and ASC be the facial angles enclosing a triedral.

Then, as each must have some value, the sum is greater than zero, and we have only to show that **ASB + ASC + BSC** is less than 4 right angles.

Produce either edge, as **AS**, to **D**. Now, in the triedral **S-BCD**, **BSC** is less than **BSD + CSD** (?). To each member of this inequality add **ASB + ASC**, and we have

ASB + ASC + BSC less than **ASB + ASC + BSD + CSD** (?).

But **ASB + BSD** = 2 right angles (?),

and **ASC + CSD** = 2 right angles;

whence **ASB + ASC + BSD + CSD** = 4 right angles,

and consequently **ASB + ASC + BSC** is less than 4 right angles. Q. E. D.

PROPOSITION XVIII.

556. Theorem.—*The sum of the diedrals of a triedral may be anything between two and six right angles.*

DEMONSTRATION.

Each diedral being the supplement of a facial angle of the supplementary triedral (**531**), the sum of the three diedrals is 3 times 2 right angles, or 6 right angles, minus the sum of the facial angles of the supplementary triedral.

But this latter sum may be anything between 0 and 4 right angles (?). Hence the sum of the diedrals may be anything between 2 and 6 right angles. Q. E. D.

OF POLYEDRALS.

557. A **Convex Polyedral** is a polyedral none of the faces of which, when produced, enter the solid angle. A section of such a polyedral made by a plane cutting all its edges is a convex polygon. (See Fig. 259.)

PROPOSITION XIX.

558. Theorem.—*The sum of the facial angles of any convex polyedral is less than four right angles.*

DEMONSTRATION.

Let S be the vertex of any convex polyedral.

Then is the sum of the angles **ASB**, **BSC**, **CSD**, **DSE**, and **ESA** less than 4 right angles.

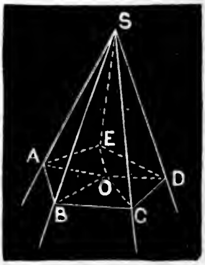

Fig. 259.

Let the edges of this polyedral be cut by any plane, as **ABCDE**, which section will be a convex polygon, since the polyedral is convex.

From any point within this polygon, as **O**, draw lines to its vertices, as **OA**, **OB**, **OC**, etc. There will thus be formed two sets of triangles, one with their vertices at **S**, and the other with their vertices at **O**; and there will be an equal number in each set, for the sides of the polygon form the bases of both sets.

Now, the sum of the angles of each of these two sets of triangles is the same. But the sum of the angles at the bases of the triangles having their vertices at **S** is greater than the sum of the angles at the bases of the triangles having their vertices at **O**, since **SBA** + **SBC** is greater than **ABC**, **SCB** + **SCD** is greater than **BCD**, etc. (**540**).

Therefore the sum of the angles at **S** is less than the sum of the angles at **O**, *i. e.*, less than 4 right angles. Q. E. D.

EXERCISES.

559. 1. I have an iron block whose corners are all square (edges right diedrals, and the vertices tri-rectangular, or right, triedrals). If I bend a wire square around one of its edges, as cS$'d$, at what angle do I bend the wire? If I bend a wire obliquely around the edge, as aSb, at what angle can I bend it? If I bend it obliquely, as eS$''f$, at what angle can I bend it?

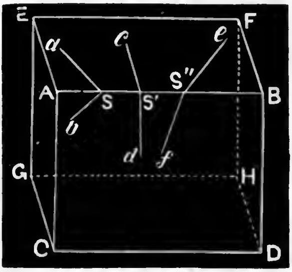

Fig. 260.

2. Fig. 260 represents the appearance of a rectangular parallelopiped, as seen from a certain position. Now, all the angles of such a solid are right angles: why is it that they nearly all *appear* oblique? Can you see a right parallelopiped from such a position that all the angles seen shall *appear* as right angles?

3. The diedral angles of crystals are measured with great care, in order to determine the substances of which the crystals consist. How must the measure be taken? If we measure obliquely around the edge, shall we get the true value of the angle?

4. Prove that if three planes intersect so as to make two traces parallel, the third is parallel to each of these.

5. From a piece of pasteboard cut two figures of the same size, like **ABCDS** and *abcds* (Fig. 261). Then drawing **SB** and **SC** so as to make 1 the largest angle and 3 the smallest, cut the pasteboard almost through in these lines, so that it will readily bend in them. Now fold the edges **AS** and **DS** together, and a triedral will be formed. From the piece *abcds* form a triedral in like manner, only let the lines *sc* and *sb* be drawn so as to make

the angles 1, 2, and 3 of the same size as before, while they occur in the order given in *abcds*. Now, see if you can slip one triedral into the other, so that they will fit. What is the difficulty?

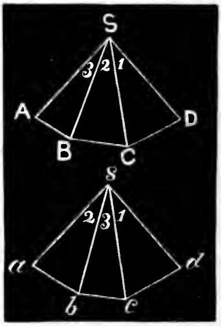

Fig. 261.

6. In the last case, if 1 equals $\frac{5}{6}$ of a right angle, $2 = \frac{1}{3}$ of a right angle, and $3 = \frac{2}{6}$ of a right angle, can you form the triedral? Why? If you keep increasing the size of 1, 2, and 3, until the sum becomes equal to 4 right angles, will it always be possible to form a triedral? How is it when the sum equals 4 right angles?

7. What is the locus of a point in space equidistant from three given points?

To demonstrate that such a locus is a straight line, pass a plane through the three points, and also a circumference. Now, 1st, a perpendicular to this circle at its centre has every point equidistant from the three points; and, 2d, any point out of the perpendicular is unequally distant from the points. Hence this perpendicular is the locus sought.

Notice that in demonstrating such a proposition the *two* points should both be proved.

8. The locus of a point equidistant from two planes is the plane which bisects the diedral included between them. [Give proof.]

9. What is the locus of a point in space equidistant from the faces of a triedral? [Give proof.]

10. If each of the projections of a line upon three intersecting planes is a straight line, the line is a straight line.

11. To find the point in a plane such that the sum of its distances from two given points without the plane, and on the same side of it, shall be a minimum.

Solution.—Let the two points be **P** and **P′**. Let fall a perpendicular from either point, as **P**, upon the plane, and call it **PD**. Produce **PD** on the opposite side of the plane to **P″**, making **P″D** = **PD**. Join **P″** and **P′**. The point where **P″P′** pierces the plane is the point sought. [Give proof.]

SECTION III.

OF PRISMS AND CYLINDERS.

560. A **Prism** is a solid, two of whose faces are equal, parallel polygons, while the other faces are parallelograms. The equal parallel polygons are the *Bases*, and the parallelograms make up the *Lateral* or *Convex Surface.* Prisms are triangular, quadrangular, pentagonal, etc., according to the number of sides of the polygon forming a base.

561. A **Right Prism** is a prism whose lateral edges are perpendicular to its bases. An **Oblique Prism** is a prism whose lateral edges are oblique to its bases.

562. A **Regular Prism** is a right prism whose bases are regular polygons; whence its faces are equal rectangles.

563. The **Altitude** of a prism is the perpendicular distance between its bases: the altitude of a right prism is equal to any one of its lateral edges.

564. A **Truncated Prism** is a portion of a prism cut off by a plane cutting the lateral edges, but not parallel to its base. A section of a prism made by a plane perpendicular to its lateral edges is called a *Right Section.*

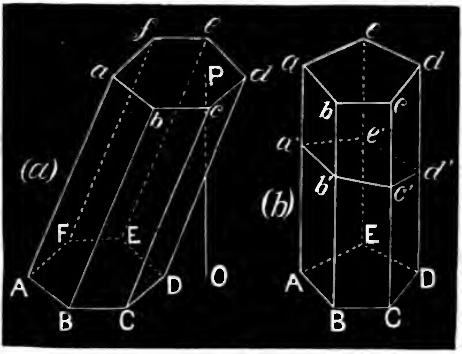

Fig. 262.

Illustrations.—In the figure, (*a*) and (*b*) are both prisms: (*a*) is oblique and (*b*) right. **PO** represents the altitude of (*a*); and any edge of (*b*), as *b*B, is its altitude. **ABCDEF** and *abcdef* are lower and upper bases, respectively. Either portion of (*b*) cut off by an oblique plane, as $a'b'c'd'e'$, is a truncated prism.

565. A **Parallelopiped** is a prism whose bases are parallelograms; its faces, inclusive of the bases, are consequently all parallelograms. If its faces are all rectangular, it is a *rectangular* parallelopiped.

566. A **Cube** is a rectangular parallelopiped whose faces are all equal squares.

567. The **Volume** or **Contents** of a solid is the number of times it contains some other solid taken as the unit of measure; or it is the ratio of one solid to another taken as the standard of measure.

In applied geometry the unit of volume is usually a cube described on some linear unit, as an inch, a foot, a yard, etc. To this the perch and the cord are exceptions.

PROPOSITION I.

568. Theorem.—*Parallel plane sections of any prism are equal polygons.*

Demonstration.

Let ABCDE and *abcde* be parallel sections of the prism MN.

Then are they equal polygons.

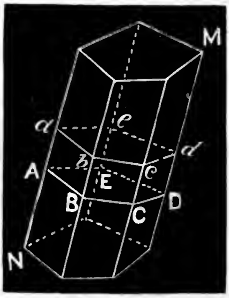

Fig. 263.

For, the intersections with the lateral faces, as ab and AB, etc., are parallel, since they are intersections of parallel planes by a third plane (**488**).

Moreover, these intersections are equal, that is, $ab = AB$, $bc = BC$, $cd = CD$, etc., since they are parallels included between parallels (**138**).

Again, the corresponding angles of these polygons are equal, that is, $a = A$, $b = B$, $c = C$, etc., since their sides are parallel and lie in the same direction (**492**).

Therefore the polygons **ABCDE** and *abcde* are mutually equilateral and equiangular; that is, they are equal. Q. E. D.

569. COROLLARY.—*Any plane section of a prism, parallel to its base, is equal to the base; and all right sections are equal.*

PROPOSITION II.

570. Theorem.—*If two prisms have equivalent bases, any plane sections parallel to the bases are equivalent.*

DEMONSTRATION.

Let M and N be any two prisms having equivalent bases B and B′; and let P and Q be sections parallel thereto.

Then, by the preceding proposition,

$$P = B,$$

and $$Q = B' = B,$$

whence, $$P = Q. \quad \text{Q. E. D.}$$

PROPOSITION III.

571. Theorem.—*If three faces including a triedral of one prism—complete or truncated—are equal respectively to three faces including a triedral of the other, and similarly placed, the prisms are equal.*

DEMONSTRATION.

In the prisms Ad and A′d' (Fig. 264), let ABCDE equal A′B′C′D′E′, ABba = A′B′$b'a'$, and BCcb = B′C′$c'b'$.

Then are the prisms equal.

For, since the facial angles of the triedrals B and B′ are equal, the triedrals are equal (**544**), and being applied they will coincide.

Now, conceiving A′d' as applied to Ad, with B′ in B, since the bases are equal polygons, they will coincide throughout; and for like reason aB will coincide with a'B′, and cB with c'B′.

Furthermore, since the bases coincide, **C′D′** falls in **CD**, and as **C′**c' falls in **C**c, and **D′**d' is parallel to **C′**c', and **D**d to **C**c (?), **D′**d' falls in **D**d.

In like manner, **E′**e' can be shown to fall in **E**e.

Finally, since the upper bases have the angles $a'b'c'$ and abc coincident, they coincide (**444**).

Hence the prisms can be superimposed, and are therefore equal. Q. E. D.

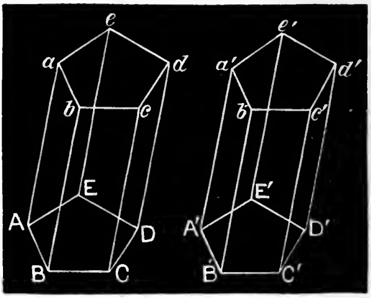

Fig. 264.

572. Corollary.—*Two right prisms having equal bases and equal altitudes are equal.*

If the faces are not similarly arranged, as the edges are perpendicular to the bases, one prism can be inverted and then superimposed on the other.

PROPOSITION IV.

573. Theorem.—*Any oblique prism is equivalent to a right prism, whose bases are right sections of the oblique prism, and whose edge is equal to the edge of the oblique prism.*

Demonstration.

Let LB be an oblique prism, of which *abcde* and *fghil* are right sections, and gb = GB.

Then is lb equivalent to **LB**.

For the truncated prisms l**G** and e**B** have the faces including any two corresponding triedrals, as **G** and **B**, respectively, equal and similarly placed (?), whence these prisms are equal (**571**).

Now, from the whole figure take away prism l**G**, and there remains the oblique prism **LB**; also, from the whole take away the prism e**B**, and there remains the right prism lb.

Therefore, the right prism lb is equivalent to the oblique prism **LB**. Q. E. D.

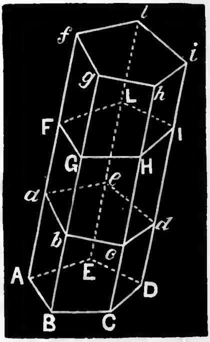

Fig. 265.

PROPOSITION V.

574. Theorem.—*The opposite faces of a parallelopiped are equal and parallel.*

DEMONSTRATION.

Let A*c* be a parallelopiped, AC and *ac* being its equal bases (560).

Then are its opposite faces equal and parallel.

Since the bases are parallelograms, **AB** is equal and parallel to **DC**; and, since the faces are parallelograms, *a***A** is equal and parallel to *d***D**. Hence,

angle *a***AB** = *d***DC**,

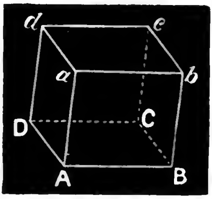

Fig. 266.

and their planes are parallel, since their sides are parallel and extend in the same directions.

Therefore, *a***B** and *d***C** are equal (322) and parallel parallelograms. In like manner it may be shown that *a***D** is equal and parallel to *b***C**. Q. E. D.

PROPOSITION VI.

575. Theorem.—*The diagonals of a parallelopiped bisect each other.*

DEMONSTRATION.

Let ABCD-*b* be a parallelopiped whose diagonals are *b*D, *d*B, *c*A, and *a*C.

Then do *b***D**, *d***B**, *c***A**, and *a***C** bisect each other.

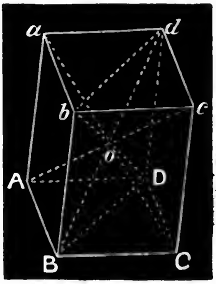

Fig. 267.

Pass a plane through two opposite edges, as *b***B** and *d***D**.

Since the bases are parallel (?), *bd* and **BD** will be parallel (488), and *b***BD***d* will be a parallelogram. Hence, *b***D** and *d***B** are bisected at *o* (?).

For a like reason, passing a plane through *dc* and **AB**, we may show that *d***B** and *c***A** bisect each other, and hence that *c***A** passes through the common centre of *d***B** and *b***D**.

So also *a***C** is bisected by *b***D**, as appears from passing a plane through *ab* and **DC**.

Hence, all the diagonals are bisected at *o*. Q. E. D.

576. COROLLARY.—*The diagonals of a rectangular parallelopiped are equal.*

PROPOSITION VII.

577. **Theorem.**—*The diagonal of a right parallelopiped is equal to the square root of the sum of the squares of the three adjacent edges of the parallelopiped.*

DEMONSTRATION.

Let a, b, c be the three adjacent edges of a right parallelopiped, d the diagonal of the face whose edges are b and c, and D the diagonal of the parallelopiped.

Then $$d^2 = b^2 + c^2 \text{ (?)},$$

and $$D^2 = a^2 + d^2 = a^2 + b^2 + c^2 \text{ (?)},$$

or $$D = \sqrt{a^2 + b^2 + c^2}. \quad \text{Q. E. D.}$$

578. COROLLARY.—*The diagonal of a cube is $\sqrt{3}$ times its edge.*

PROPOSITION VIII.

579. **Theorem.**—*The area of the lateral surface of a right prism is equal to the product of its altitude into the perimeter of its base.*

DEMONSTRATION.

The lateral faces are all rectangles, having for their common altitude the altitude of the prism (**563**). Whence the area of any face is the product of the altitude into the side of the base which forms its base; and the sum of the areas of the faces is the common altitude into the sum of the bases of the faces, that is, into the perimeter of the base of the prism. Q. E. D.

580. A **Cylindrical Surface** is a surface traced by a straight line moving so as to remain constantly parallel to its first position, while any point in it traces some curve. The moving line is called the *Generatrix,* and the curve traced by a point of the line the *Directrix.*

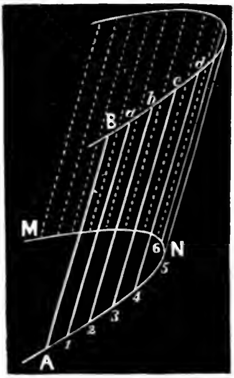

Fig. 268.

Illustration.—Suppose a line to start from the position **AB**, and move towards **N** in such a manner as to remain all the time parallel to its first position **AB**, while **A** traces the curve

A 1 2 3 4 5 6 . . . . **M.**

The surface thus traced is a *Cylindrical Surface;* **AB** is the *Generatrix*, and the curve **ANM** the *Directrix*.

581. A **Circular Cylinder,** called also a **Cylinder of Revolution,** is a solid generated by the revolution of a rectangle around one of its sides as an axis.

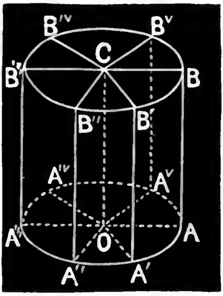

Fig. 269.

Illustration.—Let **COAB** be a rectangle, and conceive it revolved about **CO** as an axis, taking successively the positions $COA'B'$, $COA''B''$, etc.; the solid generated is a *Circular Cylinder*, or a cylinder of revolution. The revolving side **AB** is the generatrix of the surface, and the circumference $AA'A''$ (or $BB'B''$) is the directrix. This is the only cylinder treated in Elementary Geometry, and is usually meant when the word *Cylinder* is used without specifying the kind of cylinder.

582. The **Axis** of the cylinder is the fixed side of the rectangle. The side of the rectangle opposite the axis generates the **Convex Surface;** while the other sides of the rectangle, as **OA** and **CB**, generate the **Bases,** which in the cylinder of revolution are circles. Any line of the surface corresponding to some position of the generatrix is called an **Element** of the surface.

583. Any section of a cylinder of revolution made by a plane parallel to its base is equal to its base, since such a section would be a circle with a radius equal to **OA**.

584. A **Right Cylinder** is one whose elements are perpendicular to its base. In such a cylinder any element is equal to the axis. A *Cylinder of Revolution* (**581**) is *right.*

585. A prism is said to be inscribed in a cylinder, when the bases of the prism are inscribed in the bases of the cylinder, and the edges of the prism coincide with elements of the cylinder.

PROPOSITION IX.

586. Theorem.—*The area of the convex surface of a cylinder of revolution is equal to the product of its axis into the circumference of its base, i. e.,* $2\pi RH$, *H being the axis and R the radius of the base.*

DEMONSTRATION.

Let AD be a cylinder of revolution, whose axis HO $= H$, **and the radius of whose base is OB** $= R$.

Then is the area of its convex surface $2\pi RH$.

Let a right prism, with any regular polygon for its base, be inscribed in the cylinder, as *k-abcdef*.

The area of the lateral surface of the prism is **HO** ($= hb$) into the perimeter of its base, *i. e.*,

$$HO \times (ab + bc + cd + de + ef + fa).$$

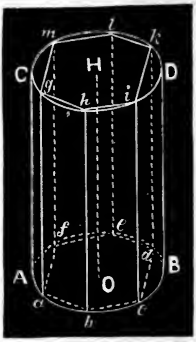

Fig. 270.

Now, bisect the arcs ab, bc, etc., and inscribe a regular polygon of twice the number of sides of the preceding, and on this polygon as a base construct the right inscribed prism with double the number of faces that the first had. The area of the lateral surface of this prism is

$$HO \times \text{the perimeter of its base.}$$

In like manner, conceive the operation of inscribing right prisms with regular polygonal bases continually repeated; it will *always* be true that the area of the lateral surface is equal to

$$HO \times \text{the perimeter of the base.}$$

By continually increasing the number of the sides of the inscribed polygon in this manner, the perimeter of the polygon may be made to differ from the circumference by less than any assignable quantity, *i. e.*, by an infinitesimal, which is therefore 0 in comparison with the perimeter (**341**), and the prism of an infinite number of faces is to be considered as the cylinder.

Therefore, the area of the convex surface of the cylinder is **HO** into the circumference of the base.

Finally, if R is the radius of the base, $2\pi R$ is its circumference. This multiplied by H, the altitude, *i. e.*, $H \times 2\pi R$, or $2\pi RH$, is the area of the convex surface of the cylinder. Q. E. D.

PROPOSITION X.

587. Theorem.—*Rectangular parallelopipeds are to each other* as the products of any three adjacent edges.*

DEMONSTRATION.

Let the adjacent edges of one rectangular parallelopiped, P, be three lines, which we will call A, B, and C, and of another, Q, the three lines a, b, c.

Then
$$\frac{\mathsf{P}}{\mathsf{Q}} = \frac{A \times B \times C}{a \times b \times c}.$$

For A, B, C, a, b, and c are at least commensurable by an infinitesimal unit. Let the common measure of the edges be i; and let it be contained in A m times, in B n times, in C p times, in a q times, in b r times, and in c s times, so that

$$m = \frac{A}{i}, \qquad n = \frac{B}{i}, \qquad p = \frac{C}{i},$$

$$q = \frac{a}{i}, \qquad r = \frac{b}{i}, \text{ and } s = \frac{c}{i}.$$

Now let A and B be the sides of the rectangular base of **P**, and C its altitude, and a, b, and c corresponding edges of **Q**. The base of **P** may be conceived as divided into mn units of surface. If upon each of these we conceive a cube described, there will be mn such cubes. Now, of these layers of cubes there will be p in the entire parallelopiped **P**. Hence **P** will be composed of mnp equal cubes. In like manner, **Q** may be shown

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* This means that their volumes are to each other.

to be composed of qrs equal cubes, each equal to one of the mnp cubes which compose **P**.

Hence, $$\frac{\mathbf{P}}{\mathbf{Q}} = \frac{mnp}{qrs},$$

and substituting their values for m, n, p, q, r, and s, we have

$$\frac{\mathbf{P}}{\mathbf{Q}} = \frac{\frac{A}{i} \times \frac{B}{i} \times \frac{C}{i}}{\frac{a}{i} \times \frac{b}{i} \times \frac{c}{i}} = \frac{A \times B \times C}{a \times b \times c}.$$ Q. E. D.*

PROPOSITION XI.

588. Theorem.—*The volume of a rectangular parallelopiped is equal to the product of its three adjacent edges.*

DEMONSTRATION.

Let P be any rectangular parallelopiped whose adjacent edges are A, B, and C, and let Q be the proposed unit of measure, whose edges are each 1.

Then, by the last proposition,

$$\frac{\mathbf{P}}{\mathbf{Q}} = \frac{A \times B \times C}{1 \times 1 \times 1},$$

or, $$\mathbf{P} = (A \times B \times C) \times \mathbf{Q}$$

Thus, **P** contains the unit **Q** $A \times B \times C$ times. Hence, $A \times B \times C$ is the volume of **P**. Q. E. D.

589. COROLLARY 1.—*The volume of a cube is the third power of its edge.*

590. SCHOLIUM.—This fact gives rise to the term *cube*, as used in arithmetic and algebra, for "third power."

591. COROLLARY 2.—*The volume of a rectangular parallelopiped is equal to the product of its altitude into the area of its base, the linear unit being the same for the measure of all its edges.*

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* For other demonstrations see Appendix.

PROPOSITION XII.

592. Theorem.—*The volume of any prism, or of any solid whose plane sections parallel to the base are all equal to the base, is equal to that of a rectangular parallelopiped having an equivalent base and the same altitude, and hence is equal to the product of its base into its altitude.*

DEMONSTRATION.

Let Q be any prism or solid whose plane sections parallel to its base are equal to its base, and P a rectangular parallelopiped of the same altitude, and whose base $B = B'$, the base of the first solid.

Then is volume **Q** = volume **P**.

If **Q** be a prism, any plane section parallel to its base is equal to its base (?); hence the case is the same whether **Q** be a prism or any other solid having its plane sections parallel to its base equal to its base.

Now conceive two planes to start from coincidence with B and B' at the same time, and move upward at the same rate, generating the solids **P** and **Q**. As these sections are always equivalent to each other, since each is constantly equal to B or B', they generate equal volumes in equal times, and by reason of the equal altitudes of the two solids, both volumes are generated in the same time. Hence the two volumes are equivalent. Q. E. D.

593. COROLLARY 1.—*The volume of a right prism is equal to the product of its edge into its base.*

594. COROLLARY 2.—*Prisms of the same altitude are to each other as their bases; and prisms of the same or equivalent bases are to each other as their altitudes; and, in general, prisms are to each other as the products of their bases and altitudes.*

PROPOSITION XIII.

595. Theorem.—*The volume of a cylinder of revolution is equal to the product of its base and altitude, i. e., $\pi R^2 H$, H being the altitude and R the radius of the base.*

DEMONSTRATION.

By (**592**) the volume of such a cylinder is equal to the product of its base into its altitude, since all plane sections parallel to its base are equal thereto (**583**).

But the base is a circle whose radius is R, the area of which is πR^2 (?). Hence the volume of the cylinder is $H \times \pi R^2$, or $\pi R^2 H$. Q. E. D.

596. COROLLARY.—*The volume of any cylinder is equal to the product of its base into its altitude.*

This can be demonstrated in a manner altogether analogous to the case given in the proposition.

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597. Similar Solids are such as have their corresponding solid angles equal and their homologous edges proportional.

598. Similar Cylinders of revolution are such as have their altitudes in the same ratio as the radii of their bases.

599. Homologous Edges of similar solids are such as are included between equal plane angles in corresponding faces.

ILLUSTRATION.—The idea of similarity in the case of solids is the same as in the case of plane figures, viz., that of *likeness of form*. Thus, one would not think such a cylinder as *one* joint of stovepipe similar to another composed of a hundred joints of the same pipe. One would be *long* and *very slim* in proportion to its length, while the other would not be thought of as *slim*. But, if we have two cylinders the radii of whose bases are 2 and 4, and whose lengths are respectively 6 and 12, we readily recognize them as of the same shape: they are similar.

PROPOSITION XIV.

600. Theorem.—*The altitudes of two similar prisms are to each other as any two homologous edges, and the areas of corresponding faces are to each other as the squares of any two homologous edges, or as the squares of the altitudes.*

Demonstration.

Let P and p be any two similar prisms, H and h their altitudes, Aa and A$'a'$ two homologous edges, and Ab and A$'b'$ two corresponding faces.

Then is $\frac{H}{h} = \frac{\mathrm{A}a}{\mathrm{A}'a'}$, or as any other two homologous edges;

and $\frac{\mathrm{A}b}{\mathrm{A}'b'} = \frac{\overline{\mathrm{A}a}^2}{\overline{\mathrm{A}'a'}^2} = \frac{H^2}{h^2}$, *i. e.*, as the squares of any other two homologous edges, or as the squares of the altitudes.

From the homologous vertices a and a' let fall the perpendiculars aI and a'I$'$, and draw AI and A$'$I$'$.

$a\mathrm{I} = H$, and $a'\mathrm{I}' = h$ (?).

Now, since the prisms are similar, they may be so placed that their homologous edges will be parallel; hence, let AB be parallel to A$'$B$'$, AE to A$'$E$'$, and aA to a'A$'$. Then is aI parallel to a'I$'$, and AI to A$'$I$'$, and the triangles aAI and a'A$'$I$'$ are similar.

Whence we have

$$\frac{H}{h} = \frac{\mathrm{A}a}{\mathrm{A}'a'},$$

or as any other two homologous edges, since by definition any two homologous edges bear the same ratio. Q. E. D.

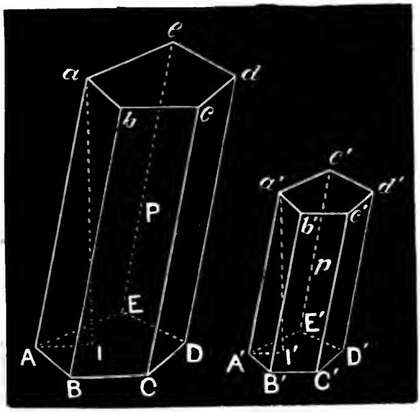

Fig. 271.

Again, since the corresponding faces **A**b and **A**$'b'$ have their homologous sides proportional (**597**), and their homologous angles, as a**AB** and a'**A**$'$**B**$'$, equal, being the homologous facial angles of equal triedrals, the faces are similar plane figures, and

$$\frac{\mathrm{A}b}{\mathrm{A}'b'} = \frac{\overline{\mathrm{A}a}^2}{\overline{\mathrm{A}'a'}^2} = \frac{H^2}{h^2},$$

or as the squares of any two homologous edges. Q. E. D.

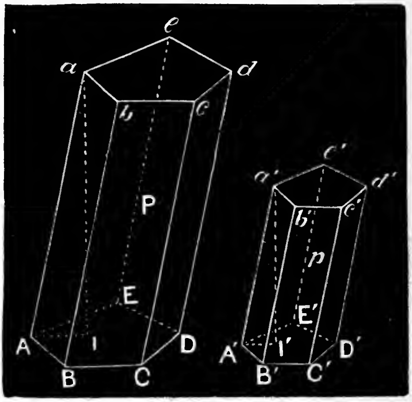

Fig. 271.

601. Corollary.—*The corresponding faces of any two similar solids are to each other as the squares of any two homologous edges of the solids.*

PROPOSITION XV.

602. Theorem.—*The lateral surfaces of similar prisms are to each other as the squares of any two homologous edges, or as the squares of the altitudes of the prisms.*

Demonstration.

Let A, B, C, D, etc., and a, b, c, d, etc., be the corresponding faces of two similar prisms, and M and m any two homologous edges, and H and h the altitudes.

By the last proposition,

$$\frac{A}{a} = \frac{M^2}{m^2}, \quad \frac{B}{b} = \frac{M^2}{m^2}, \quad \frac{C}{c} = \frac{M^2}{m^2}, \quad \frac{D}{d} = \frac{M^2}{m^2}, \text{ etc.}$$

Hence,

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = \frac{D}{d}, \text{ etc.} = \frac{M^2}{m^2}.$$

and, by composition,

$$\frac{A+B+C+D, \text{ etc.}}{a+b+c+d, \text{ etc.}} = \frac{M^2}{m^2} = \frac{H^2}{h^2} \text{ (?)}. \quad \text{Q. E. D.}$$

603. Corollary.—*The* ENTIRE *surfaces of* ANY *two similar solids are to each other as the squares of any two homologous edges.*

PROPOSITION XVI.

604. Theorem.—*The volumes of similar prisms are to each other as the cubes of their homologous edges, and as the cubes of their altitudes.*

Demonstration.

Let V and v be the volumes of any two similar prisms, M and m any two homologous edges, and H and h their altitudes.

Then is $$\frac{V}{v} = \frac{M^3}{m^3} = \frac{H^3}{h^3}.$$

Let B and b be the bases of the prisms; whence their volumes are $B\times H$ and $b\times h$ respectively (**592**).

By (**600**), $$\frac{B}{b} = \frac{M^2}{m^2} = \frac{H^2}{h^2}.$$

But $$\frac{H}{h} = \frac{M}{m} = \frac{H}{h} \text{ (?)}.$$

Multiplying, $$\frac{B\times H}{b\times h} = \frac{V}{v} = \frac{M^3}{m^3} = \frac{H^3}{h^3}. \quad \text{Q. E. D.}$$

PROPOSITION XVII.

605. Theorem.—*The convex surfaces of similar cylinders of revolution are to each other as the squares of their altitudes, and as the squares of the radii of their bases.*

DEMONSTRATION.

Let H and h be the altitudes, and R and r the radii of the bases of two similar cylinders.

The convex surfaces are $2\pi RH$ and $2\pi rh$ (586).

Now, $$\frac{2\pi RH}{2\pi rh} = \frac{RH}{rh} = \frac{R}{r} \times \frac{H}{h}.$$

By hypothesis, $$\frac{H}{h} = \frac{R}{r}.$$

Whence, by substitution, we have

$$\frac{2\pi RH}{2\pi rh} = \frac{H^2}{h^2},$$

and $$\frac{2\pi RH}{2\pi rh} = \frac{R^2}{r^2}.$$ Q. E. D.

PROPOSITION XVIII.

606. Theorem.—*The volumes of similar cylinders of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*

DEMONSTRATION.

Let H and h be the altitudes of two similar cylinders of revolution, R and r the radii of their bases, and V and v their volumes.

Then $$\frac{V}{v} = \frac{H^3}{h^3} = \frac{R^3}{r^3}.$$

For, by (595), $V = \pi HR^2$,

and $v = \pi hr^2$.

Hence, $\frac{V}{v} = \frac{\pi HR^2}{\pi hr^2} = \frac{HR^2}{hr^2}$;

and, since $\frac{H}{h} = \frac{R}{r}$ (?),

we have, by substitution, $\frac{V}{v} = \frac{R^3}{r^3} = \frac{H^3}{h^3}$. Q. E. D.

607. SCHOLIUM.—It is a general truth, that the surfaces of similar solids, *of any form*, are to each other as the squares of homologous lines; and their volumes are as the cubes of such lines. These truths will be further illustrated in the following section, but the methods of demonstration will be seen to be the same as used in this.

EXERCISES.

608. 1. A farmer has two grain bins which are parallelopipeds. The front of one bin is a rectangle 6 feet long by 4 high, and the front of the other a rectangle 8 feet long by 4 high. They are built between parallel walls 5 feet apart. The bottom and ends of the first, he says, are "square" (he means, it is a rectangular parallelopiped), while the bottom and ends of the other slope, *i. e.*, are oblique to the front. What are the relative capacities of the bins?

2. How many square feet of boards in the walls and bottom of the first bin mentioned in Ex. 1?

3. An average sized honey bee's cell is a right hexagonal prism, .8 of an inch long, with faces $\frac{3}{20}$ of an inch wide. The width of the face is always the same, but the length of the cell varies according to the space the bee has to fill. Are honey bees' cells similar? Is a honey bee's cell, of the dimensions given above, similar to a wasp's cell, which is 1.6 inches long, and whose face is .3 of an inch wide? What are the relative capacities of the wasp's cell and the honey bee's?

4. How many square inches of sheet iron does it take to make a joint of 7-inch stovepipe 2 feet 4 inches long, allowing an inch and a half for making the seam?

5. A certain water-pipe is 3 inches in diameter. How much water is discharged through it in 24 hours, if the current flows 3 feet per minute? How much through a pipe of twice as great diameter, at the same rate of flow?

6. What is the ratio of the length of a hogshead holding 125 gallons, to the length of a keg of the same shape, holding 8 gallons?

7. What are the relative amounts of cloth required to clothe three men of the same form (similar solids), one being 5 feet high, another 5 feet 9 inches, and the other 6 feet, provided they dress in the same style? If the second of these men weighs 156 lb., what do the others weigh?

8. If a man $5\frac{1}{2}$ feet high weighs 160 lb., and a man 3 inches taller weighs 180 lb., which is the stouter in proportion to his height?

9. I have a prismatic piece of timber, from which I cut two blocks, both 5 feet long measured along one edge of the stick; but one block is made by cutting the stick square across (a right section), and the other by cutting both ends of it obliquely, making an angle of 45° with the same face of the timber. Which block is the greater? Which has the greater lateral surface?

10. How many cubic feet in a log 12 feet long and 2 feet and 5 inches in diameter? How many square feet of inch boards can be cut from such a log, allowing one-quarter for waste in slabs and sawing?

11. How many square feet of sheet copper will it take to line the sides and bottom of a cylindrical vat (cylinder of revolution) 6 feet deep, if the diameter of the bottom is 4 feet? How many barrels does such a vat contain?

12. What are the relative capacities of cylinders of revolution of the same diameter, but of different lengths? What of those of the same length, but of different diameters?

SECTION IV.

OF PYRAMIDS AND CONES.

609. A **Pyramid** is a solid having a polygon for its base, and triangles for its lateral faces. If the base is also a triangle, it is called a triangular pyramid, or a tetraedron (*i. e.*, a solid with four faces). The vertex of the polyedral angle formed by the lateral faces is the *vertex* of the pyramid.

610. The **Altitude** of a pyramid is the perpendicular distance from its vertex to the plane of its base.

611. A **Right Pyramid** is a pyramid whose base is a regular polygon, and the perpendicular from whose vertex falls at the centre of the base. This perpendicular is called the *axis*.

612. A **Frustum** of a pyramid is a portion of the pyramid intercepted between the base and a plane parallel to the base. If the cutting plane is not parallel to the base, the portion intercepted is called a *Truncated* pyramid.

613. The **Slant Height** of a right pyramid is the altitude of one of the triangles which form its faces. The *Slant Height of a Frustum* of a right pyramid is the portion of the slant height of the pyramid intercepted between the bases of the frustum.

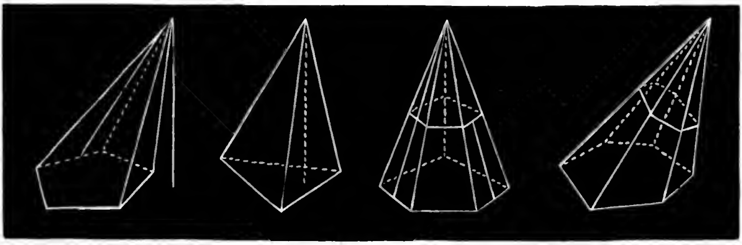

Fig. 272.

ILLUSTRATIONS.—The student will be able to find illustrations of the definitions in the above figures.

614. A **Conical Surface** is a surface traced by a line which passes through a fixed point, while any other point traces a curve. The line is the *Generatrix,* and the curve the *Directrix.* The fixed point is the *Vertex.* Any line of the surface corresponding to some position of the generatrix is called an *Element* of the surface.

615. A **Cone of Revolution** is a solid generated by the revolution of a right-angled triangle around one of its sides, called the *Axis.* The hypotenuse describes the *Convex Surface* of the cone, and corresponds to the generatrix in the preceding definition. The other side of the triangle describes the *Base.* This cone is *right,* since the perpendicular (the axis) falls at the centre of the base. The *Slant Height* is the distance from the vertex to the circumference of the base, and is the same as the hypotenuse of the generating triangle.

616. The terms **Frustum** and **Truncated** are applied to the cone in the same manner as to the pyramid.

617. A pyramid is said to be **Inscribed** in a cone when the base of the pyramid is inscribed in the base of the cone, and the edges of the pyramid are elements of the surface of the cone. The two solids have a common vertex and a common altitude.

618. If the generatrix be considered as an indefinite straight line passing through a fixed point, the portions of the line on opposite sides of the point will each describe a conical surface. These two surfaces, which in *general* discussions are considered but one, are called **Nappes.** The two nappes of the same cone are evidently alike.

ILLUSTRATION.—In Fig. 273, (*a*) represents a conical surface which has the curve **ABC** for its *directrix*, and **SA** for its *generatrix.* The numerals indicate the successive positions of the point **A**, as it passes around the curve, while the point **S** remains fixed. (*b*) represents a *Cone of Revolution*, or a right cone with a circular base. It may be considered as generated in the general way, or by the right-angled triangle **SOA** revolving about **SO** as an axis. **SA** describes the convex surface, and **OA** the

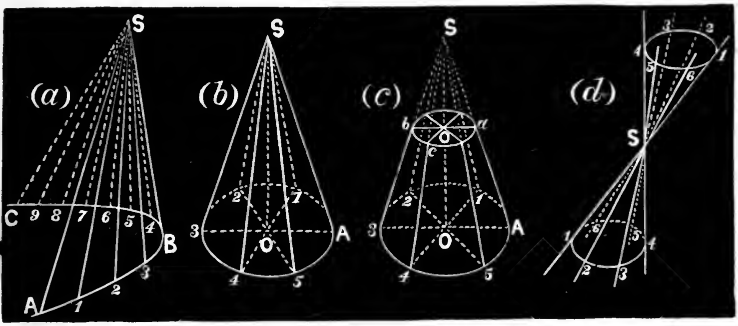

Fig. 273.

base. The figure (c) represents the *Frustum* of a cone, the portion above the plane abc being supposed removed. Figure (d) represents the two *Nappes* of an oblique cone.

PROPOSITION I.

619. Theorem.—*Any section of a pyramid made by a plane parallel to its base is a polygon similar to the base.*

DEMONSTRATION.

Let *abcde* be a section of the pyramid S-ABCDE made by a plane parallel to ABCDE.

Then is *abcde* similar to **ABCDE**.

Since **AB** and *ab* are intersections of two parallel planes by a third plane, they are parallel (?). So also *bc* is parallel to **BC**, *cd* to **CD**, etc. Hence, angle $b = \mathsf{B}$, $c = \mathsf{C}$, etc. (?), and the polygons are mutually equiangular. Again,

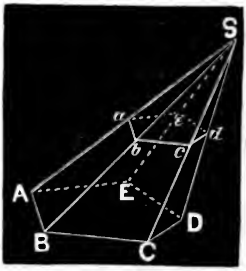

Fig. 274.

$$\frac{ab}{\mathsf{AB}} = \frac{\mathsf{S}b}{\mathsf{SB}}, \quad \text{and} \quad \frac{bc}{\mathsf{BC}} = \frac{\mathsf{S}b}{\mathsf{SB}} \ (?).$$

Hence
$$\frac{ab}{\mathsf{AB}} = \frac{bc}{\mathsf{BC}}, \quad \text{or} \quad \frac{ab}{bc} = \frac{\mathsf{AB}}{\mathsf{BC}} \ (?).$$

In like manner, we can show that

$$\frac{bc}{cd} = \frac{\mathsf{BC}}{\mathsf{CD}}, \text{ etc.}$$

Therefore, *abcde* and **ABCDE** are mutually equiangular, and have their corresponding sides proportional, and are consequently similar. Q. E. D.

PROPOSITION II.

620. Theorem.—*If two pyramids of equal altitudes are cut by planes equally distant from and parallel to their bases, the sections are to each other as the bases.*

DEMONSTRATION.

Let S-ABC and S′-A′B′C′D′E′ be two pyramids of the same altitude, cut by the planes *abc* and *a′b′c′d′e′*, parallel to and at equal distances from their bases.

Then is $$\frac{abc}{a'b'c'd'e'} = \frac{\mathsf{ABC}}{\mathsf{A'B'C'D'E'}}$$

For, conceive the bases in the same plane. Let SP and S′P′ be the equal altitudes, and $\mathsf{S}p = \mathsf{S'}p'$ the distances of the cutting planes from the vertices.

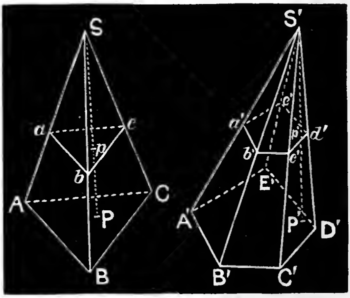

Fig. 275.

Conceive a plane passing through the vertices parallel to the plane of the bases. This plane, together with the plane in which the sections lie, and that in which the bases lie, make three parallel planes which cut the lines SA, SB, S′A′, S′B′, SP, and S′P′, whence

$$\frac{\mathsf{SB}}{\mathsf{S}b} = \frac{\mathsf{SP}}{\mathsf{S}p} = \frac{\mathsf{S'B'}}{\mathsf{S'}b'} = \frac{\mathsf{S'P'}}{\mathsf{S'}p'}.$$

Also, since the planes ASB and A′S′B′ are cut by parallel planes in AB, *ab*, A′B′, and *a′b′*, *ab* is parallel to AB, and *a′b′* to A′B′; whence,

$$\frac{\mathsf{AB}}{ab} = \frac{\mathsf{SB}}{\mathsf{S}b}, \quad \text{and} \quad \frac{\mathsf{A'B'}}{a'b'} = \frac{\mathsf{S'B'}}{\mathsf{S'}b'}.$$

Now $$\frac{\mathsf{ABC}}{abc} = \frac{\overline{\mathsf{AB}}^2}{\overline{ab}^2}\ (?) = \frac{\overline{\mathsf{S'B'}}^2}{\overline{\mathsf{S'}b'}^2}\ (?),$$

and $$\frac{\mathsf{A'B'C'D'E'}}{a'b'c'd'e'} = \frac{\overline{\mathsf{A'B'}}^2}{\overline{a'b'}^2} = \frac{\overline{\mathsf{S'B'}}^2}{\overline{\mathsf{S'}b'}^2}.$$

Hence, by equality of ratios,

$$\frac{\mathsf{ABC}}{abc} = \frac{\mathsf{A'B'C'D'E'}}{a'b'c'd'e'}, \quad \text{or} \quad \frac{abc}{a'b'c'd'e'} = \frac{\mathsf{ABC}}{\mathsf{A'B'C'D'E'}} \text{ (?)}. \qquad \text{Q. E. D.}$$

621. Corollary.—*If two pyramids having equivalent bases and equal altitudes are cut by planes parallel to and equidistant from their bases, the sections are equivalent.*

PROPOSITION III.

622. Theorem.—*The area of the lateral surface of a right pyramid is equal to the perimeter of the base multiplied by one-half the slant height.*

Demonstration.

The faces of such a pyramid are equal isosceles triangles (?), whose common altitude is the slant height of the pyramid (?).

Hence, the area of these triangles is the product of one-half the slant height into the sum of their bases. But this sum is the perimeter of the base.

Hence the area is equal to the perimeter of the base multiplied by one-half the slant height. Q. E. D.

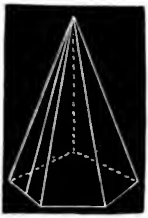

Fig. 276.

623. Corollary.—*The area of the lateral surface of the frustum of a right pyramid is equal to the product of its slant height into half the sum of the perimeters of its bases.*

The proof is based upon (**350**) and definitions.

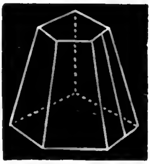

Fig. 277.

PROPOSITION IV.

624. Theorem.—*The area of the convex surface of a cone of revolution (a right cone with a circular base) is equal to the product of the circumference of its base and one-half its slant height, i. e.,* $\pi RH'$, *R being the radius of the base, and H' the slant height.*

Demonstration.

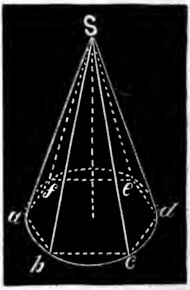

Fig. 278.

In the circle which forms the base of the cone, conceive a regular polygon inscribed, as *abcdef*. Joining the vertices of the angles of this polygon with the vertex of the cone, there will be constructed a right pyramid inscribed in the cone. Now, if the arcs subtended by the sides of this polygon be bisected, and these are again bisected, etc., and at every step a right pyramid is conceived as inscribed, it will *always* remain true that the lateral surface of the pyramid is the perimeter of its base into half its slant height.

But, as the number of faces of the pyramid is increased, the perimeter of the base approaches the circumference of the base of the cone as its limit, and hence the slant height of the pyramid approaches the slant height of the cone, and the lateral surface of the pyramid approaches the convex surface of the cone as their limits, and all reach their limits simultaneously.

Therefore, *at* the limit we still have the same expression for the area of the convex surface, that is, the circumference of the base multiplied by half the slant height.

Finally, if R is the radius of the base, its circumference is $2\pi R$, and H' being the slant height, we have for the area of the convex surface $2\pi R \times \frac{1}{2}H'$, or $\pi RH'$. Q. E. D.

625. Corollary 1.—*The area of the convex surface of a cone is also equal to the product of the slant height into the circumference of the circle parallel to the base, and midway between the base and vertex.*

This follows directly from the fact that the radius of the circle midway between the base and vertex is one-half the radius of the base, *i. e.*, $\frac{1}{2}R$ (?), whence its circumference is πR. Now, $\pi R \times H'$ is the area of the convex surface, by the proposition.

626. Corollary 2.—*The area of the convex surface of the frustum of a cone is equal to the product of its slant height into half the sum of the circumferences of its bases; i. e.,* $\pi(R+r)H'$, *R and r being the radii of its bases, and H' its slant height.*

From the corresponding property of the frustum of a pyramid, the student will be able to deduce the fact that $\frac{1}{2}(2\pi R+2\pi r)H'$ or $\pi(R+r)H'$ is the area of this surface by the same line of argument used in the demonstration of the main theorem.

627. Corollary 3.—*The area of the convex surface of the frustum of a cone is equal to the product of its slant height into the circumference of the circle midway between the bases.*

The radius of the circle midway between the bases is $\frac{1}{2}(r+R)$, whence its circumference is $\pi(r+R)$. Now, $\pi(r+R)\times H'$ is the area of the convex surface of the frustum, by the preceding corollary.

PROPOSITION V.

628. Theorem.—*Two pyramids having equivalent bases and the same altitudes are equivalent, i. e., equal in volume.*

First Demonstration.

Let S-ABCD and S′-A′B′C′D′E′ be two pyramids having the same altitudes, and base ABCD equivalent to base A′B′C′D′E′, *i. e.*, equal in area.

Then is pyramid **S-ABCD** equivalent to **S′-A′B′C′D′E′**, *i. e.*, equal in volume.

For, conceive the bases to be in the same plane, and a plane to start from coincidence with the plane of the bases, and move toward the vertices, remaining all the time parallel to the bases.

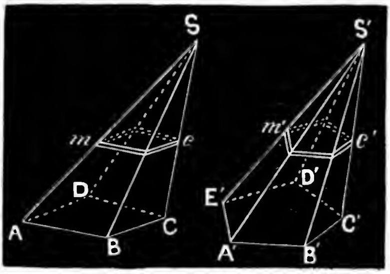

Fig. 279.

Now each of the sections of the pyramids made by this plane may be conceived as a varying polygon which generates its respective pyramid. And as these polygons are always equivalent, and move at the same rate, they generate equal volumes in equal times. Moreover, as the bases of the pyramids are in the same plane, and their altitudes are equal, the polygons generate their respective pyramids in the same time. Hence these volumes are equal. Q. E. D.

SECOND DEMONSTRATION.

Consider the pyramids divided into an infinite number of laminæ of equal but infinitesimal thickness, as mc, $m'c'$, parallel to the bases. Now each lamina in one will have a corresponding lamina in the other at the same distance from the base since the laminæ are of equal thickness, and hence equivalent to it.

Hence the pyramids are composed of an equal number of equivalent laminæ, and are consequently equivalent. Q. E. D.

PROPOSITION VI.

629. Theorem.—*The volume of a triangular pyramid is equal to one-third the product of its base and altitude.*

DEMONSTRATION.

Let S-ABC be a triangular pyramid, whose altitude is *H*.

Then is the volume equal to

$$\tfrac{1}{3}H \times \text{area } ABC.$$

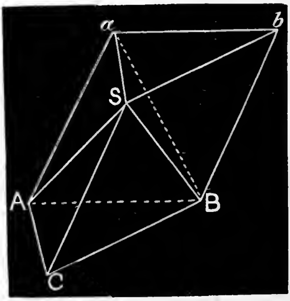

Fig. 280.

For, through **A** and **B** draw **A**a and **B**b parallel to **SC**; and through **S** draw **S**a and **S**b parallel to **CA** and **CB**, and join a and b; then **S**ab-**ABC** is a prism with its base equal to the base of the pyramid.

Now, the solid added to the given pyramid is a quadrangular pyramid with ab**BA** as its base, and its vertex at **S**.

Divide this into two triangular pyramids by drawing a**B** and passing a plane through **SB** and a**B**. These triangular pyramids are equivalent, since they have equal bases, a**AB** and ab**B**, and a common altitude, the vertices of both being at **S**.

Again, **S**-*ab***B** may be considered as having *ab***S** (equal to **ABC**) as its base, and the altitude of the given pyramid (equal to the altitude of the prism) for its altitude, and hence as equivalent to the given pyramid.

Thus the prism **ABC***ab***S** is divided into the three equivalent pyramids, **S-ABC**, **B**-*ab***S**, and **S**-*a***BA**.

Hence, the pyramid **S-ABC** is one-third the prism **S***ab*-**ABC**, which has the same base and altitude.

But the volume of the prism is

$$H \times \text{area } \mathbf{ABC}.$$

Therefore the volume of the pyramid **S-ABC** is

$$\tfrac{1}{3}H \times \text{area } \mathbf{ABC}. \quad \text{Q. E. D.}$$

630. Corollary 1.—*The volume of any pyramid is equal to one-third the product of its base and altitude.*

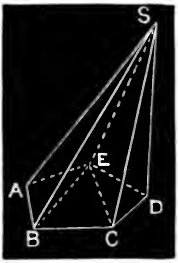

Fig. 281.

Since any pyramid can be divided into triangular pyramids by passing planes through any one edge, as **SE**, and each of the other edges not adjacent, as **SB** and **SC**, the volume of the pyramid is equal to the sum of the volumes of several triangular pyramids having the same altitude as the given pyramid, and the sum of whose bases is the base of the given pyramid.

631. Corollary 2.—*Pyramids having equivalent bases are to each other as their altitudes; such as have equal altitudes are to each other as their bases; and, in general, pyramids are to each other as the products of their bases and altitudes.*

Exercise.—A **Regular Tetraedron** is a triangular pyramid whose base is an equilateral triangle and each of whose lateral faces are equal to the base. What is the volume of such a tetraedron whose edge is 1 inch? *Ans.* $\frac{1}{12}\sqrt{2}$ *cu. in.*

What is the entire area of the surface of this tetraedron?

PROPOSITION VII.

632. Theorem.—*The volume of the frustum of a triangular pyramid is equal to the volume of three pyramids of the same altitude as the frustum, and whose bases are the upper base, the lower base, and a mean proportional between the two bases of the frustum.*

DEMONSTRATION.

Let *abc*-ABC (Fig. 282) be the frustum of a triangular pyramid.

Through *ab* and **C** pass a plane cutting off the pyramid **C**-*abc*. This has for its base the upper base of the frustum, and for its altitude the altitude of the frustum.

Again, draw **A***b*, and pass a plane through **A***b* and *b***C**, cutting off the pyramid *b*-**ABC**, which has the same altitude as the frustum, and for its base the lower base of the frustum.

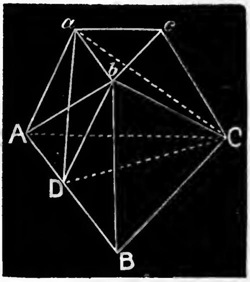

Fig. 282.

There now remains a third pyramid, *b*-**AC***a*, to be examined.

Through *b* draw *b***D** parallel to *a***A**, and draw **DC** and *a***D**.

The pyramid **D-AC***a* is equivalent to *b*-**AC***a*, since it has the same base and the same altitude (?). But the former may be considered as having **ADC** for its base, and the altitude of the frustum for its altitude, *i. e.*, as pyramid *a*-**ADC**.

We are now to show that **ADC** is a mean proportional between *abc* and **ABC**.

$$\frac{\mathbf{ABC}}{abc} = \frac{\overline{\mathbf{AB}}^2}{\overline{ab}^2} = \frac{\overline{\mathbf{AB}}^2}{\overline{\mathbf{AD}}^2} \ (?).$$

Also, $$\frac{\mathbf{ABC}}{\mathbf{ADC}} = \frac{\mathbf{AB}}{\mathbf{AD}} \ (?); \quad \text{or} \quad \frac{\overline{\mathbf{ABC}}^2}{\overline{\mathbf{ADC}}^2} = \frac{\overline{\mathbf{AB}}^2}{\overline{\mathbf{AD}}^2} \ (?).$$

By equality of ratios, $\dfrac{\mathbf{ABC}}{abc} = \dfrac{\overline{\mathbf{ABC}}^2}{\overline{\mathbf{ADC}}^2}$; whence, $\overline{\mathbf{ADC}}^2 = abc \times \mathbf{ABC}$;

i. e., **ADC** is a mean proportional between the upper and lower bases of the frustum.

Hence the volume of the frustum is equal to the volume of three pyramids, etc. Q. E. D.

X

633. Corollary.—*The volume of the frustum of any pyramid is equal to the volume of three pyramids having the same altitude as the frustum, and for bases, the upper base, the lower base, and a mean proportional between the two bases of the frustum.*

For, the frustum of any pyramid is equivalent to the corresponding frustum of a triangular pyramid of the same altitude and an equivalent base (?); and the bases of the frustum of the triangular pyramid being both equivalent to the corresponding bases of the given frustum, a mean proportional between the triangular bases is a mean proportional between their equivalents.

PROPOSITION VIII.

634. Theorem.—*The volume of a cone of revolution is equal to one-third the product of its base and altitude; i. e.,* $\frac{1}{3}\pi R^2H$, *R being the radius of the base and H the altitude.*

Demonstration.

The volume of a pyramid is equal to one-third the product of the base and altitude, and the cone being the limit of the pyramid, the volume of the cone is one-third the product of its base and altitude.

Now, R being the radius of the base of a cone of revolution, the base (area of) is πR^2, whence $\frac{1}{3}\pi R^2H$ is the volume, H being the altitude. Q. E. D.

635. Corollary 1.—*The volume of any cone is equal to one-third the product of its base and altitude.*

636. Corollary 2.—*The volume of the frustum of a cone is equal to the volume of three cones having the same altitude as the frustum, and for bases, the upper base, the lower base, and a mean proportional between the two bases of the frustum.*

The truth of this appears from the fact that the frustum of a cone is the limit of the frustum of a regular inscribed pyramid.

PROPOSITION IX.

637. Theorem.—*The lateral surfaces of similar pyramids are to each other as the squares of their homologous edges, or of their altitudes.*

Demonstration.

Let A, A′, A″, etc., and a, a', a'', etc., be homologous sides of the bases of two similar pyramids, E, E′, E″, etc., and e, e', e'', etc., homologous lateral edges, H and h the altitudes of the pyramids, and let S and s be the lateral surfaces.

Then is
$$\frac{S}{s} = \frac{A^2}{a^2} = \frac{A'^2}{a'^2} = \frac{A''^2}{a''^2}, \text{ etc.,}$$
$$= \frac{E^2}{e^2} = \frac{E'^2}{e'^2} = \frac{E''^2}{e''^2}, \text{ etc.,}$$
$$= \frac{H^2}{h^2}.$$

Since the pyramids are similar, the corresponding facial angles are equal, and the homologous edges proportional (**597**, **532**), hence the bases are similar polygons, and the corresponding lateral faces are similar triangles.

Now let F, F′, F″, etc., and f, f', f'', etc., be the corresponding lateral faces, of which triangles, A, A′, A″, etc., and a, a', a'', etc., are the bases respectively, and E, E′, E″, etc., and e, e', e'', etc., other homologous sides.

Then $\frac{A}{a} = \frac{A'}{a'} = \frac{A''}{a''}$, etc., $= \frac{E}{e} = \frac{E'}{e'} = \frac{E''}{e''}$, etc., $= \frac{H}{h}$ (?).

Whence $\frac{A^2}{a^2} = \frac{A'^2}{a'^2} = \frac{A''^2}{a''^2}$, etc., $= \frac{E^2}{e^2} = \frac{E'^2}{e'^2} = \frac{E''^2}{e''^2}$, etc., $= \frac{H^2}{h^2}$ (?)

Moreover, $\frac{F}{f} = \frac{A^2}{a^2}$, $\frac{F'}{f'} = \frac{A'^2}{a'^2}$, $\frac{F''}{f''} = \frac{A''^2}{a''^2}$, etc.

Whence
$$\frac{F + F' + F'', \text{ etc.}}{f + f' + f'', \text{ etc.}} = \frac{S}{s} = \frac{A^2}{a^2} = \frac{A'^2}{a'^2} = \frac{A''^2}{a''^2}$$
$$= \frac{E^2}{e^2} = \frac{E'^2}{e'^2} = \frac{E''^2}{e''^2}, \text{ etc.,} = \frac{H^2}{h^2}. \quad \text{Q. E. D.}$$

638. Corollary.—*The lateral surfaces of similar right pyramids are to each other as the squares of any homologous lines, as slant heights, altitudes, or of corresponding diagonals of the bases..*

PROPOSITION X.

639. Theorem. — *The convex surfaces of similar cones of revolution are to each other as the squares of their slant heights, the radii of their bases, or their altitudes; i. e., as the squares of any two homologous dimensions.*

Demonstration.

Let H' and h' be the slant heights of two similar cones of revolution, R and r the radii of their bases, and H and h their altitudes.

Their convex surfaces are $\pi RH'$ and $\pi rh'$.

Now, since the cones are similar,

$$\frac{R}{r} = \frac{H'}{h'} \text{ (?)}.$$

Multiplying the terms of this proportion by the corresponding terms of

$$\frac{\pi H'}{\pi h'} = \frac{H'}{h'},$$

we have
$$\frac{\pi RH'}{\pi rh'} = \frac{H'^2}{h'^2}.$$

Hence the convex surfaces are as the squares of their slant heights. Q. E. D.

But, as
$$\frac{H'^2}{h'^2} = \frac{R^2}{r^2} \text{ (?)} = \frac{H^2}{h^2},$$

$$\frac{\pi RH'}{\pi rh'} = \frac{R^2}{r^2} = \frac{H^2}{h^2}.$$

That is, the convex surfaces are to each other as the squares of the radii of the bases, or as the squares of the altitudes. Q. E. D.

PROPOSITION XI.

640. Theorem.—*The volumes of similar pyramids are to each other as the cubes of their homologous dimensions.*

SYNOPSIS OF DEMONSTRATION.

Let A and a be homologous sides of the bases of two similar pyramids, B and b their bases, and H and h their altitudes.

We have $$\frac{B}{b} = \frac{A^2}{a^2} = \frac{H^2}{h^2} \text{ (?)};$$

$$\frac{\frac{1}{3}H}{\frac{1}{3}h} = \frac{A}{a} = \frac{H}{h} \text{ (?)}.$$

$$\therefore \frac{\frac{1}{3}B \times H}{\frac{1}{3}b \times h} = \frac{A^3}{a^3} = \frac{H^3}{h^3} \text{ (?)}. \quad \text{Q. E. D.}$$

PROPOSITION XII.

641. Theorem.—*The volumes of similar cones are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*

SYNOPSIS OF DEMONSTRATION.

Let R and r be the radii of their bases, and H and h their altitudes.

We have $$\frac{R^2}{r^2} = \frac{H^2}{h^2} \text{ (?)},$$

and $$\frac{R^3}{r^3} = \frac{H^3}{h^3} \text{ (?)};$$

$$\frac{\frac{1}{3}\pi H}{\frac{1}{3}\pi h} = \frac{H}{h} \text{ (?)}.$$

$$\therefore \frac{\frac{1}{3}\pi R^2 \times H}{\frac{1}{3}\pi r^2 \times h} = \frac{H^3}{h^3} \text{ (?)},$$

or $$= \frac{R^3}{r^3}. \quad \text{Q. E. D.}$$

OF THE REGULAR POLYEDRONS.

642. A **Polyedron** is a solid bounded by plane surfaces. A *Regular Convex Polyedron* is a polyedron whose faces are all equal regular polygons, and each of whose solid angles is convex outward, and is enclosed by the same number of faces.

PROPOSITION XIII.

643. Theorem.—*There are five and only five regular convex polyedrons, viz.:*

The TETRAEDRON, *whose faces are four equal equilateral triangles;*

The HEXAEDRON, *or* CUBE, *whose faces are six equal squares;*

The OCTAEDRON, *whose faces are eight equal equilateral triangles;*

The DODECAEDRON, *whose faces are twelve equal regular pentagons; and*

The ICOSAEDRON, *whose faces are twenty equal equilateral triangles.*

DEMONSTRATION.

We demonstrate this proposition by showing—1st, that such solids can be constructed; and 2d, that no others are possible.

The Regular Tetraedron.—Taking three equilateral triangles, as **ASB**, **ASC**, and **BSC**, it is possible to enclose a solid angle, as **S**, with them, since the sum of the three facial angles is (what?) (**555**).

Then, since **AC** = **AB** = **CB** (?), considering **ACB** the fourth face, we have a regular polyedron whose four faces are equilateral triangles.

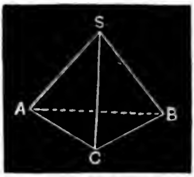

Fig. 283.

The Regular Hexaedron or Cube.—This is a familiar solid, but for purposes of uniformity and completeness we may conceive it constructed as follows: Taking three equal squares, as **ASCB**, **CSED**, and **ASEF**, we can enclose a solid angle, as **S**, with them (?).

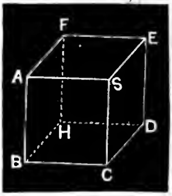

Fig. 284

Now, conceive the planes of **CB** and **CD**, **AB** and **AF**, **EF** and **ED** produced. The plane of **CB** and **CD** being parallel to **ASEF** (?), will intersect the plane of **EF** and **ED** in **HD** parallel to **FE** (?). In like manner, **FH** can be shown parallel to **ED**, **BH** to **CD**, and **HD** to **BC**. Hence the solid has for its faces six equal squares.

The Regular Octaedron.—At the intersection, **P**, of the diagonals of a square, **ABCD**, erect a perpendicular **SP** to the plane of the square, and making **SP** = **AP** (half of one of the diagonals) draw **SA**, **SD**, **SC**, and **SB**.

Fig. 285.

Making a similar construction on the other side of the plane **ABCD**, we have a solid having for faces eight equal equilateral triangles (?).

The Regular Dodecaedron.—Taking twelve equal regular pentagons, we may group them in two sets of six each, as in the figure. Thus, around **O** we may place five, forming five triedrals at the vertices of **O**. These triedrals are possible, since the sum of the facial angles enclosing each is $3\frac{3}{5}$ right angles (?)—*i.e.*, between 0 and 4 right angles (555).

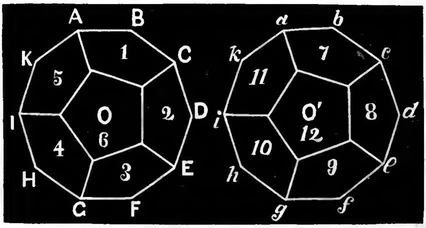

Fig. 286.

In like manner, the other six may be grouped by placing five of them about **O′**.

Now, conceiving the *convexity* of the group **O** in front and the *concavity* of group **O′**, we may place the two together so as to inclose a solid. Thus, placing **A** at *b*, the three faces 5, 7, 1, will inclose a triedral, since the diedral included by 5 and 1 is the diedral of such a triedral. Then will vertex **B** fall at *c*, and a like triedral will be formed at that point, and so of all the other vertices. Hence we have a polyedron having for faces twelve equal regular pentagons.

The Regular Icosaedron.—Taking twenty equal equilateral triangles, they can be grouped in two sets, as in the figure, in a manner altogether similar to the preceding case. The solid angles in this case are included by five facial angles whose sum is $3\frac{1}{3}$ right angles (?), which is a possible case (**555**). As before, conceiving the *convexity* of group **O** in front, and the *concavity* of **O′**, we can place them together by placing **A** at *a*, thus enclosing a solid angle with five faces, whence **B** will fall at *b*, etc. Thus we obtain a solid with twenty equal equilateral triangles for its faces.

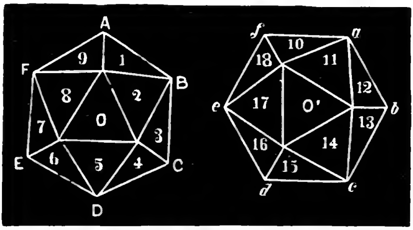

Fig. 287.

That there can be no other regular polyedrons than these five is evident, since we can form no other convex solid angles by means of regular polygons. Thus, with equilateral triangles (the simplest polygon) we have formed solid angles with three faces (the least number possible), as in the tetraedron; with four, as in the octaedron; and with five, as in the icosaedron. Six such facial angles cannot enclose a solid angle, since their sum is four right angles (?), and much less can any greater number. Again, with squares (the next most simple polygon) we have formed solid angles with three faces, as in the hexaedron, and can form no other, for the same reason as above. With regular pentagons we can enclose only a triedral, as in the dodecaedron, for a like reason. With regular hexagons we cannot enclose a solid angle (?), and much less with any regular polygon of more than six sides.

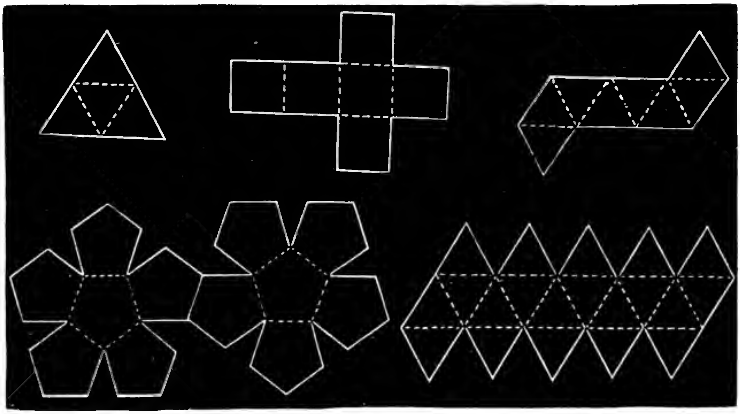

Fig. 288.

644. Scholium.—Models of the regular polyedrons are easily formed by cutting the preceding figures from cardboard, cutting half-way through the board in the dotted lines, and bringing the edges together as the forms will readily suggest.

PROPOSITION XIV.

645. Theorem.—*Any regular polyedron is inscriptible and circumscriptible by a sphere.*

Outline of Demonstration.

From the centres of any two adjacent faces, as c and c', let fall perpendiculars upon the common edge, and they will meet it in the same point o (?). The plane of these lines will be perpendicular to this edge (?), and perpendiculars to these faces from their centres, as c**S**, c'**S**, will lie in this plane (?), and hence will intersect at a point equally distant from these faces (?).

Fig. 289.

In like manner $c''\mathbf{S} = c'\mathbf{S}$, and the point **S** can be shown to be equally distant from all of the faces, and is therefore the centre of the inscribed sphere.

Joining **S** with the vertices, we can readily show that **S** is also the centre of the circumscribed sphere.

EXERCISES.

646. 1. What is the area of the lateral surface of a right hexagonal pyramid whose base is inscribed in a circle whose diameter is 20 feet, the altitude of the pyramid being 8 feet? What is the volume of this pyramid?

2. What is the area of the lateral surface of a right pentagonal pyramid whose base is inscribed in a circle whose radius is 6 yards, the slant height of the pyramid being 10 yards? What is the volume of this pyramid?

3. How many quarts will a can contain, whose entire height is 10 inches, the body being a cylinder 6 inches in diameter and $6\frac{1}{2}$ inches high, and the top a cone? How much tin does it take to make such a can, allowing nothing for waste and the seams?

4. If very fine dry sand is piled upon a smooth horizontal surface, without any lateral support, the angle of slope (*i. e.*, the angle of inclination of the sloping side of the pile with the plane) is about 31°. Suppose two circles be drawn on the floor, one 4 feet in diameter and the other 3, and sand piles be made as large as possible on these circles as bases, no other support being given. What is the relative magnitude of the piles?

5. In the case of sand piles, as given in the last example, the ratio of the radius of the base to the altitude of the pile is $\frac{5}{3}$. How many cubic feet in each of the above piles?

6. The frustum of a right pyramid was 72 feet square at the lower base and 48 at the upper; and its altitude was 60 feet. What was the lateral surface? What the volume? [Such a solid is called a **Prismoid.**]

7. Find the area of the surface, and the contents of a regular tetraedron, one of whose edges is 10 inches. What is the diameter of the inscribed sphere? Of the circumscribed?

647. A **Wedge** is a solid bounded by three quadrilaterals and two triangles.

Thus, **ABCD** is a *rectangle*, and is called the *Head* of the wedge, the two triangles **AED** and **FBC** are the *Ends*, and the two trapezoids **ABFE** and **DCFE** are the *Sides*. The *Altitude* is the perpendicular to the head from the edge opposite.

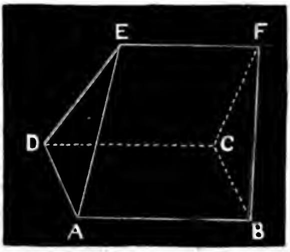

Fig. 290.

8. The base of a wedge being 18 feet by 9 feet, the edge 20 feet, and the altitude 6 feet, what are the contents?

Ans. 504 cu. ft.

SECTION V.

OF THE SPHERE.*

648. A Sphere is a solid bounded by a surface every point in which is equally distant from a point within called the *Centre.*

The distance from the centre to the surface is the *Radius,* and a line passing through the centre and limited by the surface is a *Diameter.* The diameter is equal to twice the radius.

CIRCLES OF THE SPHERE.

PROPOSITION I.

649. Theorem.—*Every section of a sphere made by a plane is a circle.*

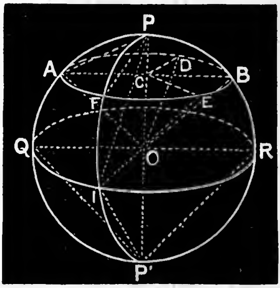

Fig. 291.

DEMONSTRATION.

Let AFEBD be a section of a sphere, whose centre is O, made by a plane; then is the section AFEBD a circle.

For, let fall from the centre **O** a perpendicular upon the plane **AFEBD**, as **OC**, and draw **CA, CD, CE, CB**, etc., lines of the plane, from the foot of the perpendicular to any points in which the plane cuts the

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* A spherical blackboard is almost indispensable in teaching this section as well as in teaching Spherical Trigonometry. A sphere about two feet in diameter, mounted on a pedestal, and having its surface slated or painted as a blackboard, is what is needed. It can be obtained of the manufacturers of school apparatus, or made in any good turning-shop.

surface of the sphere. Join these points with the centre, **O**, of the sphere.

Now **OA, OD, OB, OE**, etc., being radii, are equal; whence, **CA, CD, CB, CE**, etc., are equal; *i. e.*, every point in the line of intersection of a plane and surface of a sphere is equally distant from a point in this plane. Hence, the intersection is a circle. Q. E. D.

650. A circle made by a plane not passing through the centre is a **Small Circle;** one made by a plane passing through the centre is a **Great Circle.**

651. COROLLARY 1.—*A perpendicular from the centre of a sphere upon any small circle pierces the circle at its centre;* and, conversely, *a perpendicular to a small circle at its centre passes through the centre of the sphere.*

652. A diameter perpendicular to any circle of a sphere is called the **Axis** of that circle. The extremities of the axis are the **Poles** of the circle.

653. COROLLARY 2.—*The pole of a circle is equally distant from every point in its circumference.*

The student should give the reason.

654. COROLLARY 3.—*Every circle of a sphere has two poles, which, in case of a great circle, are equally distant from every point in the circumference of the circle; but, in case of a small circle, one pole is nearer any point in the circumference than the other pole is.*

655. COROLLARY 4.—*A small circle is less as its distance from the centre of the sphere is greater; hence the circle whose plane passes through the centre is the greatest circle of the sphere.*

For, its diameter, being a chord of a great circle, is less as it is farther from the centre of the great circle, which is also the centre of the sphere.

656. COROLLARY 5.—*All great circles of the same sphere are equal* (?).

PROPOSITION II.

657. Theorem.—*Any great circle divides the sphere into two equal parts.*

DEMONSTRATION.

Conceive a sphere as divided by a great circle, *i. e.*, by a plane passing through its centre, and let the great circle be considered as the base of each portion. These bases being equal, reverse one of the portions and conceive its base placed in the base of the other, the convex surfaces being on the same side of the common base. Since the bases are equal circles, they will coincide, and since all points in the convex surface of each portion are equally distant from the centre of the common base, the convex surfaces will coincide. Therefore, the portions coincide throughout, and are consequently equal. Q. E. D.

657, *a*.—A **Hemisphere** is one of the two equal parts into which a great circle divides a sphere.

PROPOSITION III.

658. Theorem.—*The intersection of any two great circles of a sphere is a diameter of a sphere.*

DEMONSTRATION.

The intersection of two planes is a straight line; and in the case of the two great circles, as they both pass through the centre of the sphere, this is one point of their intersection. Hence, the intersection of two great circles of a sphere is a straight line which passes through the centre. Q. E. D.

659. COROLLARY.—*The intersections on the surface of a sphere of two circumferences of great circles are a semi-circumference, or* 180°, *apart, since they are at opposite extremities of a diameter.*

DISTANCES ON THE SURFACE OF A SPHERE.

660. Distances on the surface of a sphere are always to be understood as measured on the arc of a great circle, unless it is otherwise stated.

PROPOSITION IV.

661. Theorem.—*The distance, measured on the surface of a sphere, from the pole of a circle to any point in the circumference of that circle, is the same.*

DEMONSTRATION.

Let P be a pole of the small circle AEB.

Then are the arcs **PA**, **PE**, **PB**, etc., which measure the distances on the surface of the sphere, from **P** to any points in the circumference of circle **AEB**, equal.

For, by (**653**), the straight lines **AP**, **PE**, **PP**, etc., are equal, and these equal chords subtend equal arcs, as arc **PA**, arc **PE**, arc **BB**, etc., the great circles of which these lines are chords and arcs being equal (**656**).

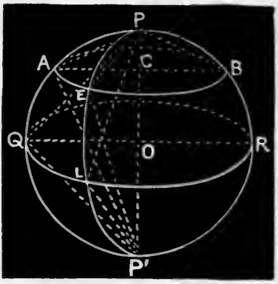

Fig. 292.

Thus, for like reasons,

arc **P′QA** = arc **P′LE** = arc **P′RB**, etc. Q. E. D.

622. COROLLARY.—*The distance from the pole of a great circle to any point in the circumference of the circle is a quadrant* (a quarter of a circumference).

Since the poles are 180° apart (being the extremities of a diameter), **PAQP′** = **PELP′** = a semi-circumference. But, in case of a great circle, chord **PL** = chord **P′L** (= chord **PQ** = chord **P′Q**), whence arc **PEL** = arc **P′L** = arc **PAQ** = arc **P′Q**. Hence, each of these arcs is a quadrant.

663. SCHOLIUM.—By means of the facts demonstrated in this proposition and corollary, we are enabled to draw arcs of small and great circles, in the surface of a sphere, with nearly the same facility that we draw arcs and lines in a plane. Thus, to draw the small circle **AEB** (Fig. 292), we take an arc equal to **PE**, and placing one end of it at **P**, cause a pencil held at the other end to trace the arc **AEB**, etc. To describe the circumference of a great circle, a quadrant must be used for the arc. By bending a wire into an arc of the circle, and making a loop in each end, a wooden pin can be put through one loop and a crayon through the other, and an arc drawn as represented in Fig. 293.

Fig. 293.

PROPOSITION V.

664. Problem.—*To pass a circumference of a great circle through any two points on the surface of a sphere.*

SOLUTION.

Let A and B be two points on the surface of a sphere, through which it is proposed to pass a circumference of a great circle.

From **B** as a pole, with an arc equal to a quadrant, strike an arc *on*, as nearly where the pole of the circle passing through **A** and **B** lies, as may be determined by inspection. Then, from **A**, with the same arc, strike an arc *st* intersecting *on* at **P**. Now, **P** is the pole of the great circle passing through **A** and **B** (?). Hence, from **P** as a pole, with a quadrant arc drawing a circle, it will pass through **A** and **B**; and it will be a great circle, since its pole is a quadrant's distance from its circumference.

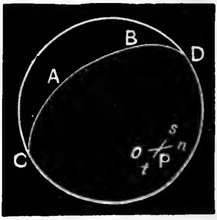

Fig. 294.

[The student should make this construction on the spherical blackboard.]

PROPOSITION VI.

665. Theorem.—*Through any two points on the surface of a sphere, one great circle* can always be made to pass, and only one, except when the two points are at the extremities of the same diameter, in which case an infinite number of great circles can be passed through the two points.*

DEMONSTRATION.

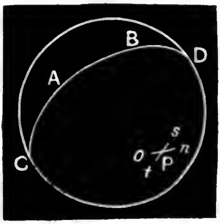

Fig. 295.

This proposition may be considered a corollary to the preceding. Thus, in general, the two great circles struck from **A** and **B** as poles, with a quadrant arc, can intersect in only two points (?), which are the poles of the same great circle (?).

But, if the two given points were at the extremities of the same diameter, as at **D** and **C**, the arcs *st* and *on* would coincide, and *any* point in this circumference being taken as a pole, great circles can be drawn through **D** and **C**.

[The student should trace the work on the spherical blackboard.]

666. SCHOLIUM.—The truth of the proposition is also evident from the fact that three points not in the same straight line determine the position of a plane. Thus, **A**, **B**, and the centre of the sphere, fix the position of one, and only one, great circle passing through **A** and **B**. Moreover, if the two given points are at the extremities of the same diameter, they are in the same straight line with the centre of the sphere, whence an infinite number of planes can be passed through them and the centre. The meridians on the earth's surface afford an example, the poles (of the equator) being the given points.

667. COROLLARY.—*If two points in the circumference of a great circle of a sphere, not at the extremities of the same diameter, are at a quadrant's distance from a point on the surface, this point is the pole of the circle.*

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* The word circle may be understood to refer either to the circle proper, or to its circumference. The word is in constant use in the higher mathematics in the latter sense.

PROPOSITION VII.

668. Theorem.—*The shortest distance on the surface of a sphere, between any two points in that surface, is measured on the arc less than a semi-circumference of the great circle which joins them.*

Demonstration.

Let A and B be two points in the surface of a sphere, AB the arc of a great circle joining them, and A*m*C*n*B any other path in the surface between A and B.

Then is arc **AB** less than **A*m*C*n*B**.

Let **C** be any point in **A*m*C*n*B**, and pass the arcs of great circles through **A** and **C**, and **B** and **C**. Join **A**, **B**, and **C** with the centre of the sphere. The angles **AOB**, **AOC**, and **COB** form the facial angles of a triedral, of which angles the arcs **AB**, **AC**, and **CB** are the measures.

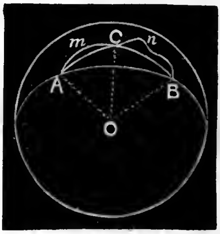

Fig. 296

Now, angle **AOB** < **AOC** + **COB** (540);

whence arc **AB** < arc **AC** + arc **CB** (?),

and the path from **A** to **B** is less on arc **AB** than on arcs **AC**, **CB**.

In like manner, joining any point in **A*m*C** with **A** and **C** by arcs of great circles, their sum will be greater than **AC**. So, also, joining any point in **C*n*B** with **C** and **B**, the sum of the arcs will be greater than **CB**.

As this process is indefinitely repeated, the path from **A** to **B** on the arcs of the great circles will continually increase, and also continually approximate the path **A*m*C*n*B**. Hence, arc **AB** is less than the path **A*m*C*n*B**. Q. E. D.

669. Corollary.—*The least arc of a circle of a sphere joining any two points in the surface, is the arc less than a semi-circumference of the great circle passing through the points; and the greatest arc is the circumference minus this least arc.*

Thus, let $AmBn$ be any small circle passing through A and B, and $ABDoC$ the great circle; then, as just shown, $ApB < AmB$.

Now, circf. $ABDoC >$ circf. $AmBn$ (655).

Subtracting the former inequality from the latter, we have $BDoCA > BnA$. Q. E. D.

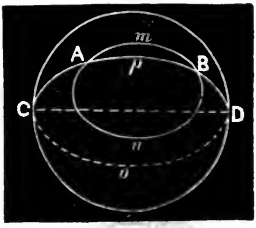

Fig. 297.

670. Two arcs of great circles are said to be perpendicular to each other when their circles are.

PROPOSITION VIII.

671. Theorem.—*If at the middle point of an arc of a great circle a perpendicular is drawn on the surface of a sphere, the distances being measured on great circles,*

1st. *Any point in this perpendicular is equally distant from the extremities of the arc.*

2d. *Any point out of the perpendicular is unequally distant from the extremities of the arc.*

Demonstration.

Let AB be any arc of a great circle, D its middle point, and PD a perpendicular.

Then is $PB = PA$, the arcs being all arcs of great circles.

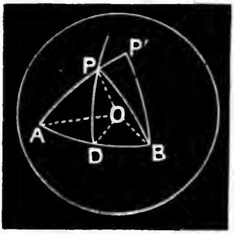

Fig. 298.

From O, the centre of the sphere, draw OP, OD, OB, and OA. The rectangular triedrals $O\text{-}PDB$ and $O\text{-}PDA$ are symmetrically equal (?); whence $PB = PA$. Q. E. D.

Again, let P' be a point out of PD. Pass arcs of great circles through P' and A, and P' and B, as $P'A$, $P'B$. From P, where one of these cuts PD, draw the arc of a great circle PB. Then is

$$P'B < P'P + PB \text{ (668)},$$

whence, $P'B < P'P + PA$ (?), and $P'B < P'A$ (?). Q. E. D.

672. COROLLARY 1.—*The perpendicular at the middle point of an arc contains all the points in the surface of the sphere which are equally distant from the extremities of the arc.*

673. COROLLARY 2.—*An arc which has each of two points, not at the extremity of the same diameter, equally distant from the extremities of another arc of a great circle, is perpendicular to the latter at its middle point.*

This is apparent, since by Corollary 1 such points are in the perpendicular, and two such points with the centre determine a great circle.

PROPOSITION IX.

674. Theorem.—*The shortest path on the surface of a hemisphere, from any point therein to the circumference of the great circle forming its base, is the arc not greater than a quadrant of a great circle perpendicular to the base, and the longest path, on any arc of a great circle, is the supplement of this shortest path.*

DEMONSTRATION.

Let P be a point in the surface of the hemisphere whose base is ADCBC′, and DPmD′ an arc of a great circle passing through P and perpendicular to ADCBC′.

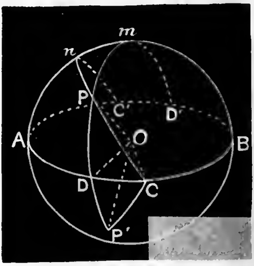

Fig. 299.

Then is **PD** the shortest path on the surface from **P** to circumference **ADCBC′**, and **P**m**D′** is the longest path from **P** to the circumference, measured on the arc of a great circle.

For, the shortest path from **P** to any point in circumference **ADBC′** is measured on the arc of a great circle (?). Now, let **PC** be any oblique arc of a great circle. We will show that

arc **PD** < arc **PC**.

Produce **PD** until $\mathbf{DP'} = \mathbf{PD}$; and pass a great circle through **P′** and **C**.

Then is the arc $\mathbf{PC} = $ arc $\mathbf{P'C}$.

And, since $$\mathbf{PC} + \mathbf{P'C} > \mathbf{PP'},$$

PC, the half of $\mathbf{PC} + \mathbf{P'C}$, is greater than **PD**, the half of **PP′**. Q. E. D.

Secondly, $\mathbf{P}m\mathbf{D'}$ is the supplement of **PD**, and we are to show that it is greater than any other arc of a great circle from **P** to the circumference **ADBC′**. Let $\mathbf{P}n\mathbf{C'}$ be any arc of a great circle oblique to **ADCBC′**. Produce $\mathbf{C'}n\mathbf{P}$ to **C**. Now $\mathbf{CP}n\mathbf{C'}$ is a semi-circumference and consequently equal to $\mathbf{DP}m\mathbf{D'}$. But we have before shown that

$$\mathbf{PD} < \mathbf{PC},$$

and subtracting these from the equals $\mathbf{CP}n\mathbf{C'}$ and $\mathbf{DP}m\mathbf{D'}$, we have

$$\mathbf{P}m\mathbf{D'} > \mathbf{P}n\mathbf{C'}.$$ Q. E. D.

675. Corollary.—*From any point in the surface of a hemisphere there are two perpendiculars to the circumference of the great circle which forms the base of the hemisphere; one of which perpendiculars measures the least distance to that circumference, and the other the greatest, on the arc of any great circle of the sphere.*

SPHERICAL ANGLES.

676. The angle formed by two arcs of circles of a sphere is conceived as the same as the angle included by the tangents to the arcs at the common points.

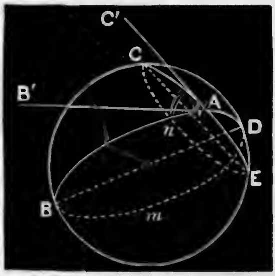

Fig. 300.

Illustration.—Let **AB** and **AC** be two arcs of circles of the sphere, meeting at **A**; then the angle **BAC** is conceived as the same as the angle **B′AC′**, **B′A** being tangent to the circle $\mathbf{BAD}m$, and **C′A** to the circle $\mathbf{CAE}n$.

677. A Spherical Angle is the angle included by two arcs of *great circles.*

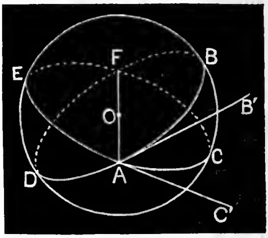

Fig. 301.

Illustration.—**BAC** is a spherical angle, and is conceived as the same as the angle **B'AC'**, **B'A** and **C'A** being tangents to the *great circles* **BADF** and **CAEF**. [The student should not confound such an angle as **BAC** Fig. 300) with a *spherical angle.*]

PROPOSITION X.

678. Theorem.—*A spherical angle is equal to the measure of the diedral included by the great circles whose arcs form the sides of the angle.*

Demonstration.

Let BAC be any spherical angle, and BADF and CAEF the great circles whose arcs BA and CA include the angle.

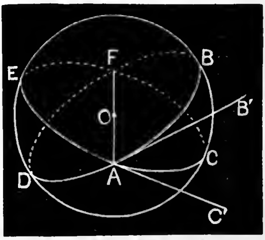

Fig. 302.

Then is **BAC** equal to the measure of the diedral **C-AF-B.**

For, since two great circles intersect in a diameter (?), **AF** is a diameter.

Now **B'A** is a tangent to the circle **BADF**, that is, it lies in the same plane and is perpendicular to **AO** at **A**.

In the like manner, **C'A** lies in the plane **CAEF** and is perpendicular to **AO**. Hence **B'AC'** is the measure of the diedral **C-AF-B** (?).

Therefore the spherical angle **BAC**, which is the same as the plane angle **B'AC'**, is equal to the measure of the diedral **C-AF-B**. Q. E. D.

PROPOSITION XI.

679. Theorem.—*If one of two great circles passes through the pole of the other, their circumferences intersect at right angles.*

DEMONSTRATION.

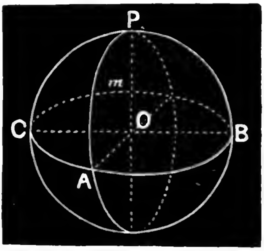

Fig. 303.

Thus, **P** being the pole of the great circle **CAB***m*, **PO** is its axis, and any plane passing through **PO** is perpendicular to the plane **CAB***m*. (?).

Hence, the diedral **B-AO-P** is right, and the spherical angle **PAB**, which is equal to the measure of the diedral, is also right. Q. E. D.

680. COROLLARY 1.—*A spherical angle is measured by the arc of a great circle intercepted between its sides, and at a quadrant's distance from its vertex.*

Thus, the spherical angle **CPA** is measured by **CA**, **PC** and **PA** being quadrants. For, since **PC** is a quadrant, **CO** is a perpendicular to **PO**, the edge of the diedral **C-PO-A**, and for the like reason **AO** is perpendicular to **PO**. Hence, **COA** is the measure of the diedral, and consequently **CA**, its measure, is the measure of the spherical angle **CPA**. Q. E. D.

681. COROLLARY 2.—*The angle included by two arcs of small circles is the same as the angle included by two arcs of great circles passing through the vertex and having the same tangents.*

Thus, **BAC** = **B″AC″**.

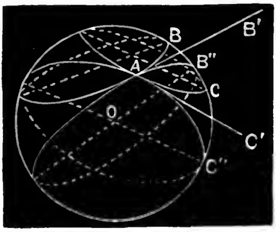

Fig. 304.

For the angle **BAC** is, by definition, the same as **B′AC′**, **B′A** and **C′A** being tangents to **BA** and **CA**. Now, passing planes through **C′A**, **B′A**, and the centre of the sphere, we have the arcs **B″A**, **C″A**, and **B′A**, **C′A** tangents to them. Hence, **B″AC″** is the same as **B′AC′**, and consequently the same as **BAC**. Q. E. D.

682. Scholium.—*To draw an arc of a great circle which shall be perpendicular to another; or, what is the same thing, to construct a right spherical angle.*

Let it be required to erect an arc of a great circle perpendicular to **CAB** at **A**. Lay off from **A**, on the arc **CAB**, a quadrant's distance, as **AP′**, and from **P′** as a pole, with a quadrant describe an arc passing through **A**. This will be the perpendicular required.

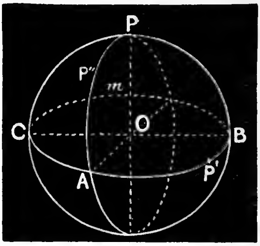

Fig. 305

In a similar manner we may let fall a perpendicular from any point in the surface, upon any arc of a great circle. To let fall a perpendicular from **P″** upon the arc **CAB**, from **P″** as a pole, with a quadrant describe an arc cutting **CAB**, as at **P′**. Then, from **P′** as a pole, with a quadrant describe an arc passing through **P″** and cutting **CAB**, and it will be perpendicular to **CAB**.

PROPOSITION XII.

683. Problem.—*To pass the circumference of a small circle through any three points on the surface of a sphere.*

Solution.

Let A, B, and C be the three points in the surface of the sphere through which we propose to pass the circumference of a circle.

Pass arcs of great circles through the points, thus forming the spherical triangle **ABC** (**664**).

Bisect two of these arcs, as **BC** and **AC**, by arcs of great circles perpendicular to each (**673**, **682**). The intersection of these perpendiculars, *o*, will be the pole of the small circle required (?).

Then from *o*, as a pole, with an arc *o***B** draw the circumference of a small circle: it will pass through **A**, **B**, and **C** (?), and hence is the circumference required.

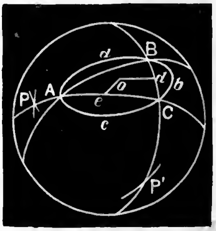

Fig. 306.

Query.—If the three given points chance to be in the circumference of a great circle, how will it appear in the construction?

OF TANGENT PLANES.

684. A Tangent Plane to a curved surface at a given point is the plane of two lines respectively tangent to two plane sections through the point.

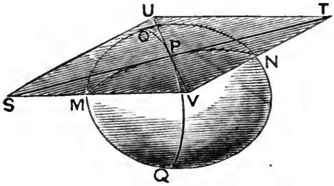

Fig. 307.

ILLUSTRATION.—Let **P** be any point in the curved surface. Pass any two planes through the surface and the point **P**, and let **OPQ** and **MPN** represent the intersections of these planes with the curved surface. Draw **UV** and **ST** in the planes of the sections, and tangent respectively to **OPQ** and **MPN** at **P**. Then is the plane of **UV** and **ST** the tangent plane at **P**.

PROPOSITION XIII.

685. Theorem.—*A tangent plane to a sphere is perpendicular to the radius at the point of tangency.*

DEMONSTRATION.

Let P be any point in the surface of a sphere; pass two great circles, as P*a*A, etc., and P*m*AR, through P, and draw ST tangent to the arc *m*P, and UV tangent to the arc *a*P.

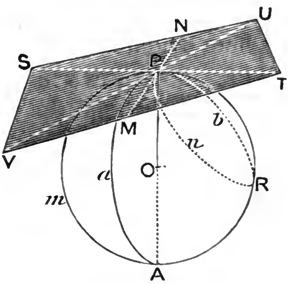

Fig. 308.

Then is the plane **SVTU** a tangent plane at **P**, and perpendicular to the radius **OP**.

For, a tangent (as **ST**) to the arc *m***P** is perpendicular to the radius of the circle, *i. e.*, to **OP**, and also a tangent (as **VU**) to the arc **AP** is perpendicular to the radius of *this* circle, *i. e.*, to **OP**.

Hence, **OP** is perpendicular to two lines of the plane **SVTU**, and consequently to the plane of these lines (?). Q. E. D.

686. Corollary 1.—*Every point in a tangent plane to a sphere, except the point of tangency, is without the sphere.*

For, **OP**, the perpendicular, is shorter than any line which can be drawn from **O** to any other point in the plane (?); hence any other point in the plane than **P** lies farther from the centre of the sphere than the length of the radius, and is, therefore, without the sphere.

687. Corollary 2.—*A tangent through* **P** *to* any *circle of the sphere passing through this point lies in the tangent plane.*

Thus, **MN**, tangent to the small circle **P***n***R***b* through **P**, lies in the tangent plane.

For, conceive the plane of the small circle extended till it intersects the tangent plane. This intersection is tangent to the small circle, since it touches at one point, but cannot cut it; otherwise the tangent plane would have another point than **P** common with the surface of the sphere.

But there can be only one tangent to a circle at a given point. Hence this intersection is **MN**, which is consequently in the tangent plane.

OF SPHERICAL TRIANGLES.

688. A **Spherical Triangle** is a portion of the surface of a sphere bounded by three arcs of great circles. In the present treatise these arcs will be considered as each less than a semicircumference; and the triangle considered will be the one which is less than a hemisphere.

The terms scalene, isosceles, equilateral, right-angled, and oblique-angled, are applied to spherical triangles in the same manner as to plane triangles.

PROPOSITION XIV.

689. Theorem.—*The sum of any two sides of a spherical triangle is greater than the third side, and their difference is less than the third side.*

DEMONSTRATION.

Let ABC be any spherical triangle.

Then is $BC < BA + AC$,

and $BC - AC < BA$;

and the same is true of the sides in any order.

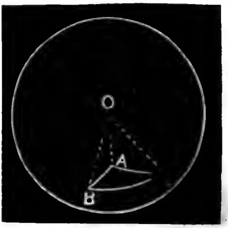

Fig. 309.

For, join the vertices A, B, and C with the centre of the sphere, by drawing AO, BO, and CO. There is thus formed a triedral O-ABC, whose facial angles are measured by the sides of the triangle (188). Now, angle BOC is less than BOA + AOC (?), whence BC is less than BA + AC; and substracting AC from each member, we have BC − AC less than BA. Q. E. D.

PROPOSITION XV.

690. Theorem.—*The sum of the sides of a spherical triangle may be anything between* 0 *and a circumference.*

DEMONSTRATION.

The sides of a spherical triangle are measures of the facial angles of a triedral whose vertex is at the centre of the sphere. Hence their sum may be anything between 0 and the measure of 4 right angles, as these are the limits of the sum of the facial angles of a triedral (?). Q. E. D.

691. SCHOLIUM.—As the sides of a spherical triangle are arcs, they can be measured in degrees. Hence, we speak of the side of a spherical triangle as 30°, 57°, 115°, 10′, etc. In accordance with this, we say that the limit of the sum of the sides of a spherical triangle is 360°.

PROPOSITION XVI.

692. Theorem.—*The sum of the angles of a spherical triangle may be anything between two and six right angles.*

DEMONSTRATION.

The sum of the angles of a spherical triangle is the same as the sum of the measures of the diedrals of a triedral having its vertex at the centre of the sphere, as in (?). Now the limits of the sum of the measures of these diedrals are 2 and 6 right angles (?). Hence the sum of the angles of any spherical triangle may be anything between 2 and 6 right angles. Q. E. D.

693. COROLLARY.—*A spherical triangle may have one, two, or even three right angles; and, in fact, it may have one, two, or three obtuse angles; since, in the latter case, the sum of the angles will not necessarily be greater than* 540°.

694. A **Trirectangular Spherical Triangle** is a spherical triangle which has three right angles.

695. SCHOLIUM.—It will be observed that the sum of the angles of a spherical triangle is not constant, as is the sum of the angles of a plane triangle. Thus, the sum of the angles of a spherical triangle may be 200°, 290°, 350°, 500°, anything between 180° and 540°.

696. **Spherical Excess** is the amount by which the sum of the angles of a spherical triangle exceeds the sum of the angles of a plane triangle; *i. e.*, it is the sum of the spherical angles — 180°, or π.

EXERCISE.—Prove that if from any point within a spherical triangle arcs of great circles be drawn to the extremities of any side, the sum of these two arcs is less than the sum of the other two sides of the triangle.

PROPOSITION XVII.

697. Theorem.—*The trirectangular triangle is one-eighth of the surface of the sphere.*

Demonstration.

Pass three planes through the centre of a sphere, respectively perpendicular to each other. They will divide the surface into eight trirectangular triangles, any one of which may be applied to any other.

Thus, let ABA′B′, ACA′C′, and CBC′B′ be the great circles formed by the three planes, mutually perpendicular to each other. The planes being perpendicular to each other, the diedrals, as A-CO-B, C-BO-A, C-AO-B, etc., are right, and hence the angles of the eight triangles formed are all right.

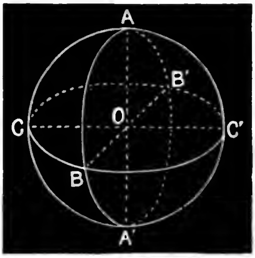

Fig. 310.

Also, as AOB is a right angle, AB is a quadrant; as BOC is a right angle, CB is a quadrant, etc. Hence, each side of every triangle is a quadrant.

Whence any one triangle may be applied to any other. [Let the student make the application.]

Hence the trirectangular triangle is one-eighth of the surface of the sphere. Q. E. D.

698. Corollary. — *The trirectangular triangle is equilateral and its sides are quadrants.*

Exercise 1. What is the spherical excess in a spherical triangle whose angles are 117°, 84°, and 96°, expressed in degrees? Expressed in right angles? Expressed in π?

Ans. 117°, $1\frac{3}{10}$, and $\frac{13}{20}\pi$.

2. Can there be a spherical triangle whose sides are 78°, 113°, and 31°? Can there be one whose sides are 152°, 136°, 148°?

3. Can there be a spherical triangle whose sides are 52°, 126°, and 140°?

PROPOSITION XVIII.

699. Theorem.—*In an isosceles spherical triangle, the angles opposite the equal sides are equal;* and, conversely, *If two angles of a spherical triangle are equal, the triangle is isosceles.*

DEMONSTRATION.

Let ABC be an isosceles spherical triangle, in which AB = AC.

Then angle **ABC** = **ACB.**

For, draw the radii **AO, CO,** and **BO,** forming the edges of the triedral **O-ABC.**

Now, since **AB** = **AC,** the facial angles **AOB** and **AOC** are equal, and the triedral is isosceles. Hence the diedrals **A-OB-C** and **A-OC-B** are equal (**550**), and consequently the spherical angles **ABC** and **ACB** are equal (**678**). Q. E. D.

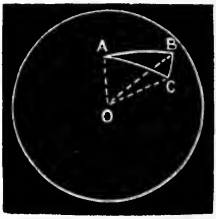

Fig. 311.

Again, if angle **ABC** = angle **ACB,** side **AC** = side **AB.** For in the triedral **O-ABC,** the diedrals **A-OB-C** and **A-OC-B** are equal, whence the facial angles **AOB** and **AOC** are equal (**550**), and consequently the sides **AB** and **AC,** which measure these angles. Q. E. D.

700. COROLLARY.—*An equilateral spherical triangle is also equiangular;* and, conversely, *An equiangular spherical triangle is equilateral.*

QUERIES.—1. What is the greatest angle which an equilateral spherical triangle can have?

2. What is the greatest side which an equilateral spherical triangle can have?

PROPOSITION XIX.

701. Theorem.—*On the same sphere, or on equal spheres, two isosceles triangles having two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, can be superimposed, and are consequently equal.*

DEMONSTRATION.

In the triangles ABC and AB′C′, let AB = AC, AB′ = AC′; and let AB = AB′, BC = B′C′, and angle ABC = AB′C′.

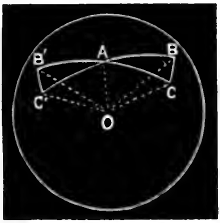

Fig. 312.

Then can the triangle **AB′C′** be superimposed upon **ABC**.

For, since the triangles are isosceles, we have

angle **ABC = ACB**,

AB′C′ = AC′B′ (699),

and, as by hypothesis

ABC = AB′C′,

these four angles are equal, each to each.

For a like reason, **AB = AC = AB′ = AC′**.

Now, applying **AC′** to its equal **AB**, the extremity **A** at **A**, and **C′** at **B**, with the angle **B′** on the same side of **AB** as **C**, the convexities of the arcs **AC′** and **AB** being the same, and in the same direction, the arcs will coincide. Then, as

angle **AC′B′ = ABC**,

C′B′ will take the direction **BC**, and since these arcs are equal by hypothesis, **B′** will fall at **C**. Hence **B′A** will fall in **CA**, as only one arc of a great circle can pass between **C** and **A**, and the triangle **AB′C′** is superimposed upon **ABC**; wherefore they are equal. Q. E. D.

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702. Symmetrical Spherical Triangles are such as have the parts of one respectively equal to the parts of the other, but arranged in a different order; hence such triangles are not capable of superposition.

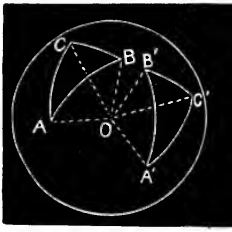

Fig. 313.

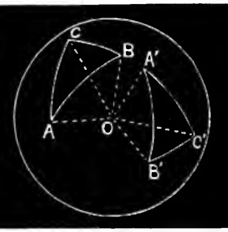

Fig. 314.

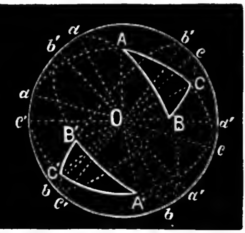

Fig. 315.

Illustration.—In Fig. 313, **ABC** and **A′B′C′** represent symmetrical spherical triangles. In these triangles,

$$\mathsf{A} = \mathsf{A'},\qquad \mathsf{B} = \mathsf{B'},\qquad \mathsf{C} = \mathsf{C'},$$

$$\mathsf{AC} = \mathsf{A'C'},\qquad \mathsf{AB} = \mathsf{A'B'},\quad \text{and}\quad \mathsf{BC} = \mathsf{B'C'};$$

nevertheless we cannot conceive one triangle superimposed upon the other. Thus, were we to make the attempt by placing **A′B′** in its equal **AB**, **A′** at **A**, and **B′** at **B**, the angle **C′** would fall on the opposite side of **AB** from **C**. Now, we cannot revolve **A′C′B′** on **AB** (or its chord), and thus make the two coincide, for this would bring their convexities together. Nor can we make them coincide by reversing **A′B′C′**, and placing **B′** at **A**, and **A′** at **B**. For, although these two arcs will thus coincide, as the angle **B′** is not equal to **A**, **B′C′** will not fall in **AC**; and, again, if it did, **C′** would not fall at **C**, since **B′C′** and **AC** are not equal.

But, considering the triangles **ABC** and **A′B′C′** in Fig. 314, in which

$$\mathsf{A} = \mathsf{A'},\qquad \mathsf{B} = \mathsf{B'},\qquad \mathsf{C} = \mathsf{C'},$$

$$\mathsf{AC} = \mathsf{A'C'},\qquad \mathsf{AB} = \mathsf{A'B'},\quad \text{and}\quad \mathsf{BC} = \mathsf{B'C'},$$

we can readily conceive the latter as superimposed upon the former. [The student should make the application.] Now, the two triangles are equivalent in each case, as will subsequently appear; and the former are equal. Such triangles as those in Fig. 313 are called *symmetrically equal*, while the latter are said to be equal by *superposition*.

Fig. 315 represents the same triangles as Fig. 314, and exhibits a complete projection* of the semi-circumferences of which the sides of the

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* To understand what is meant by the projection of these lines, conceive a hemisphere with its base on the paper, and represented by the circle *abc*, and all the arcs raised up from the paper as they would be on the surface of such a hemisphere. Thus, considering the arc *a***AB***b* (Fig; 315), the ends *a* and *b* would be in the paper just where they are, but the rest of the arc would be off the paper, as though you could take hold of **B** and raise it from

triangles are arcs. The student should become perfectly familiar with it, and be able to draw it readily. Thus, a**AB**b is the projection of the semi-circumference of which **AB** is an arc, a**AC**c of the semi-circumference of which **AC** is an arc, etc., etc.

PROPOSITION XX.

703. Theorem. — *Two symmetrical spherical triangles are equivalent, i. e., equal in area.*

Demonstration.

Let ABC and A′B′C′ be two symmetrical spherical triangles, with AB = A′B′, AC = A′C′, BC = B′C′, A = A′, B = B′, and C = C′.

Then are they equivalent.

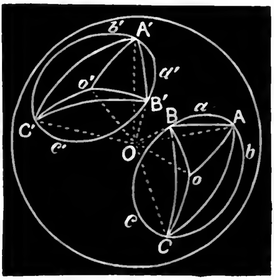

Fig. 316.

Pass circumferences of small circles through the vertices **A**, **B**, **C**, and **A′**, **B′**, **C′**, as abc and $a'b'c'$, of which o and o' are the respective poles.

Now, by reason of the mutual equality of the sides,

the chord **AC** = chord **A′C′**,

chord **AB** = chord **A′B′**,

and chord **BC** = chord **B′C′**,

and as the small circles are circumscribed about the equal plane triangles **ABC** and **A′B′C′**, these circles are equal. Hence,

$$oA = o'A' = oB = o'B' = oC = o'C',$$

and the triangles **A**o**B** and **A′**o'**B′**, **B**o**C** and **B′**o'**C′**, **A**o**C** and **A′**o'**C′** are isosceles.

Now call **O** the centre of the sphere, and draw the radii **OA**, **OB**, **OC**, **O**o, **OA′**, **OB′**, **OC′**, and **O**o'.

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the paper while a and b remain fixed. The lines in the figure are representations of lines on the surface of such a hemisphere, as they would appear to an eye situated in the axis of the circle abc, and at an infinite distance from it; that is, just as if each point in the lines dropped *perpendicularly* down upon the paper. Arcs of great circles perpendicular to the base are projected in straight lines passing through the centre, and oblique arcs are projected in ellipses. See *Spherical Trigonometry* (**97–109**).

Considering the triedrals **O-AoB** and **O-A$'o'$B$'$**, their facial angles are equal, being measured by equal arcs; hence the diedral A-oO-B = A$'$-o'O$'$-B$'$ (?), and the spherical angle AoB = A$'o'$B$'$ (?). Therefore, the isosceles triangle AoB = A$'o'$B$'$ (701).

In like manner, we may prove the triangles oBC and o'B$'$C$'$ equal, as also AoC and A$'o'$C$'$.

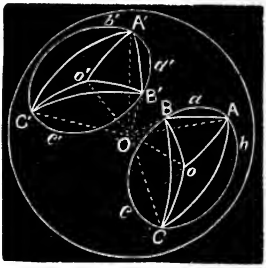

Fig. 316

Hence, ABC is equivalent to A$'$B$'$C$'$, as the two are composed of parts respectively equal. Q. E. D.

If the poles of the small circles fell without the given triangles, ABC would be equivalent to the sum of two of the partial triangles minus the third. What if the pole fell in a side?

PROPOSITION XXI.

704. Theorem.—*On the same sphere, or on equal spheres, two spherical triangles having two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, are equal, or symmetrical and equivalent.*

Demonstration.

Let ABC and A$'$B$'$C$'$ be two spherical triangles, having B = B$'$, BA = B$'$A$'$, and BC = B$'$C$'$.

Then are they either equal, or symmetrical and equivalent.

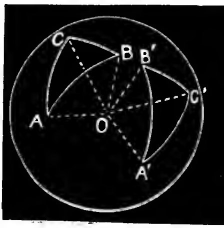

Fig. 317.

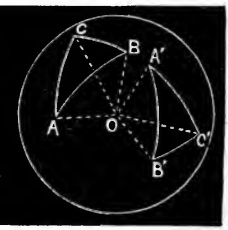

Fig. 318.

For, passing planes through the sides of each triangle and the centre of the sphere, two absolutely or symmetrically equal triedrals will be formed (?). Whence the facial angles AOC and A$'$OC$'$ are equal, and consequently AC = A$'$C$'$ (?). Also, the diedrals C-OA-B and C$'$OA$'$-B are equal, and B-OC-A = B$'$-OC$'$-A$'$ (?). Whence A = A$'$ and C = C$'$ (?).

Hence the parts of ABC are respectively equal to the parts of A$'$B$'$C$'$, and the triangles are equal, or symmetrical and equivalent, according as the equal parts are arranged in the same or in a different order. Q. E. D.

PROPOSITION XXII.

705. Theorem.—*On the same sphere, or on equal spheres, two spherical triangles having two angles and the included side of the one equal to two angles and the included side of the other, each to each, are equal, or symmetrical and equivalent.*

This is a direct consequence of a proposition concerning triedrals. Let the student give the deduction.

PROPOSITION XXIII.

706. Theorem.—*On the same sphere, or on equal spheres, if two spherical triangles have two sides of the one equal to two sides of the other, each to each, and the included angles unequal, the third sides are unequal, and the greater third side belongs to the triangle having the greater included angle.*

Conversely, *If the two sides are equal, each to each, and the third sides unequal, the angles included by the equal sides are unequal, and the greater belongs to the triangle having the greater third side.*

Demonstration.

In the triangles ABC and A′B′C′, let AB = A′B′, AC = A′C′, and A > A′.

Then is BC > B′C′.

For, join the vertices with the centre, forming the two triedrals O-ABC and O-A′B′C′.

In these triedrals, AOB = A′OB′, AOC = A′OC′, being measured by equal arcs; and C-AO-B > C′-A′O-B′, having the same measure as A and A′ (678). Hence COB > C′OB′ (?).

Therefore CB, the measure of COB, > C′B′, the measure of C′OB′.

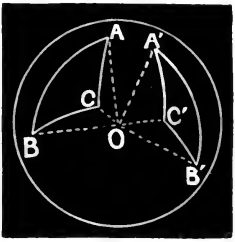

Fig. 319.

In like manner, the same sides of the triangles, and consequently the same facial angles of the triedrals, being granted equal, and BC > B′C′, A > A′. For, BC being greater than B′C′, COB > C′OB′; whence B-AO-C > B′-A′O-C′ (?), or A is greater than A′.

PROPOSITION XXIV.

707. Theorem.—*On the same sphere, or on equal spheres, two spherical triangles having the sides of the one respectively equal to the sides of the other, or the angles of the one respectively equal to the angles of the other, are equal, or symmetrical and equivalent.*

DEMONSTRATION.

The sides of the triangles being equal, the facial angles of the triedrals at the centre are equal, whence the triedrals are equal or symmetrical (?). Consequently, the angles of the triangles are equal, and the triangles are equal, or symmetrical and equivalent.

Again, the triangles being mutually equiangular, the triedrals have their diedrals mutually equal; whence the triedrals are equal or symmetrical (?). Therefore, the sides of the triangles are mutually equal, and the triangles are equal, or symmetrical and equivalent. (See Figs. 313, 314.)

PROPOSITION XXV.

708. Theorem.—*On spheres of different radii, mutually equiangular triangles are similar* (not equal).

DEMONSTRATION.

Let ABC and *abc* be two mutually equiangular spherical triangles on spheres whose radii are respectively R and r, and let angle A $= a$, B $= b$, C $= c$.

Then is $$\frac{\mathbf{AB}}{ab} = \frac{\mathbf{BC}}{bc} = \frac{\mathbf{CA}}{ca}.$$

For, joining the vertices of the triangles with the centres of the spheres, **O** and **O**$'$, the triedrals **O-ABC** and **O**$'$-*abc* have their diedrals mutually equal (?), whence their facial angles are mutually equal (?). Therefore sector **AOB** is similar to sector a**O**$'b$, sector **BOC** to b**O**$'c$, and sector **COA** to c**O**$'a$.

From the similarity of these sectors, we have

$$\frac{\mathsf{AB}}{ab} = \frac{R}{r}\,(?),\qquad \frac{\mathsf{BC}}{bc} = \frac{R}{r},\qquad \frac{\mathsf{CA}}{ca} = \frac{R}{r};$$

and hence,
$$\frac{\mathsf{AB}}{ab} = \frac{\mathsf{BC}}{bc} = \frac{\mathsf{CA}}{ca}.\quad \text{Q. E. D.}$$

709. Scholium.—In Spherical Trigonometry we are taught to find the sides of a spherical triangle having the angles given. But in such a case the sides are found in degrees, etc., which does not determine their absolute lengths. The length of an arc of any number of degrees is not known unless the radius of the sphere is known.

POLAR OR SUPPLEMENTAL TRIANGLES.

710. One spherical triangle is **Polar** to another when the vertices of one are the poles of the sides of the other, and the corresponding vertices lie on the same side of the side opposite. (For illustration, see **713.**)

Such triangles are also called *supplemental*, since the angles of one are the supplements of the sides opposite in the other, as will appear hereafter.

PROPOSITION XXVI.

711. Problem.—*Having a spherical triangle given, to draw its polar.*

Solution.

Let **ABC** (Fig. 320) be the given triangle.* From **A** as a pole, with

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* This should be executed on a sphere. Few students get clear ideas of polar triangles without it. Care should be taken to construct a variety of triangles as the given triangle, since the polar triangle does not always lie in the position indicated in the figure here given. Let the given triangle have one side considerably greater than 90°, another somewhat less, and the third quite small. Also, let each of the sides of the given triangle be greater than 90°.

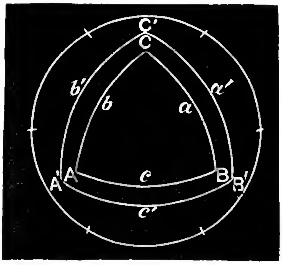

Fig. 320.

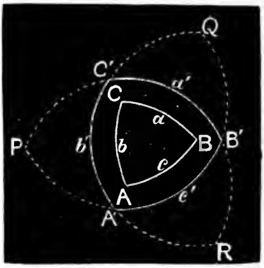

Fig. 321.

a quadrant strike an arc, as **C′B′**. From **B** as a pole, with a quadrant strike the arc **C′A′**; and from **C**, the arc **A′B′**. Then is **A′B′C′** polar to **ABC**.

712. Corollary.—*If one triangle is polar to another,* conversely, *the latter is polar to the former; i. e., the relation is reciprocal.*

Thus, **A′B′C′** (Fig. 320) being polar to **ABC**, reciprocally, **ABC** is polar to **A′B′C′**; that is, **A′** is the pole of **CB**, **B′** of **AC**, and **C′** of **AB**. For every point in **A′B′** is at a quadrant's distance from **C**, and every point in **A′C′** is at a quadrant's distance from **B**. Hence, **A′** is at a quadrant's distance from the two points **C** and **B** of **CB**, and is therefore its pole.

[In like manner, the student should show that **B′** is the pole of **AC**, and **C′** of **AB**.]

713. Scholium.—By producing (Fig. 321) each of the arcs struck from the vertices of the given triangle sufficiently, *four* new triangles will be formed, viz., **A′B′C′**, **QC′B′**, **PC′A′**, and **RA′B′**. Only the first of these is called polar to the given triangle. Thus, in **A′B′C′**, **A′**, corresponding to **A**, lies on the same side of **CB** or **C′B′** that **A** does, and so of any other corresponding vertices.

It is easy to observe the relation of any of the parts of the other three triangles to the parts of the polar. Thus,

$$\mathrm{QC'} = 180^\circ - b',$$

$$\mathrm{QB'} = 180^\circ - c',$$

$$\mathrm{QC'B'} = 180^\circ - \mathrm{B'C'A'},$$

$$\mathrm{QB'C'} = 180^\circ - \mathrm{C'B'A'},$$

and

$$\mathrm{Q} = \mathrm{A'} = 180^\circ - a,$$

as will appear hereafter.

PROPOSITION XXVII.

714. Theorem.—*Any* ANGLE *of a spherical triangle is the supplement of the* SIDE *opposite in its polar triangle; and any* SIDE *is the supplement of the* ANGLE *opposite in the polar triangle.*

FIRST DEMONSTRATION.

Let ABC and A′B′C′ be two spherical triangles polar to each other; and let the sides of each be designated as a, b, c, a', b', c', a being opposite A, a' opposite A′, b opposite B, etc.

Then
$$A = 180° - a',$$
$$B = 180° - b',$$
$$C = 180° - c',$$
$$a = 180° - A',$$
$$b = 180° - B',$$
and
$$c = 180° - C'.$$

Let **O** be the centre of the sphere, and draw **OA**, **OB**, **OC**, **OA′**, **OB′**, and **OC′**.

The angles **B′OA** and **B′OC** being right (?), **B′O** is perpendicular to the face **AOC** (?).

For like reasons, **C′O** is perpendicular to the face **AOB**.

Hence **B′OC′** is the supplement of the diedral **B-AO-C** (512).

But a' is the measure of **B′OC′**, and **B-AO-C** has the same measure as **A**.

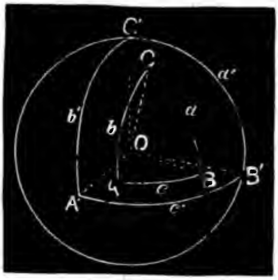

Fig. 322.

Hence,
$$A = 180° - a'.$$

In like manner, we may show that
$$B = 180° - b', \quad \text{and} \quad C = 180° - c'. \quad \text{Q. E. D.}$$

Again, since the edges **AO**, **BO**, and **CO** are perpendicular to the faces **B′OC′**, **A′OC′**, and **A′OB′**, we can show in like manner that
$$a = 180° - A',$$
$$b = 180° - B',$$
and
$$c = 180° - C'. \quad \text{Q. E. D.}$$

SECOND DEMONSTRATION.

Let ABC and A′B′C′ be two polar triangles. Let BC, CA, and AB be represented by a, b, and c respectively, and B′C′, C′A′, and A′B′ by a', b', and c'.

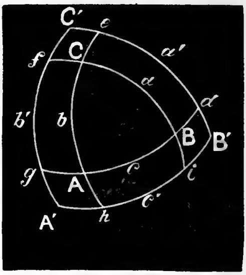

Fig. 323.

To show $\mathbf{A} = 180° - a'$, produce b and c, if necessary, till they meet the side a' of the triangle polar to **ABC** in e and d.

Now **A** is measured by ed (?). But, since $\mathbf{B'}e = 90°$, and $\mathbf{C'}d = 90°$,

$$\mathbf{B'}e + \mathbf{C'}d, \quad \text{or} \quad \mathbf{B'C'} + ed = 180°;$$

whence, transposing, and putting a' for **B′C′**, we have

$$ed = \mathbf{A} = 180° - a'.$$

In like manner, $\mathbf{C'}g + \mathbf{A'}f = \mathbf{C'A'} + fg = 180°;$

whence $fg = \mathbf{B} = 180° - \mathbf{C'A'}, \text{ or } 180° - b'.$

So, also, $\mathbf{C} = 180° - c'.$

To show that $\mathbf{A'} = 180° - a$, consider that **A′** being the pole of **CB**, fi is measure of **A′**.

Now $\mathbf{B}f = 90°$ (?), and $\mathbf{C}i = 90°$;

whence, $\mathbf{B}f + \mathbf{C}i = 180°.$

But $\mathbf{B}f + \mathbf{C}i = fi + a$, wherefore $fi + a = 180°$;

and transposing, and putting **A′** for fi, we have $\mathbf{A'} = 180° - a$.

In like manner, we may show that

$$\mathbf{B'} = 180° - b, \quad \text{and} \quad \mathbf{C'} = 180° - c. \qquad \text{Q. E. D.}$$

[The student should give the details.]

714, a. COROLLARY.—*The sum of the supplements of any two angles of a spherical triangle is greater than the supplement of the third angle.* (Consider **714, 689.**)

QUADRATURE OF THE SURFACE OF THE SPHERE.

715. The **Quadrature*** of a surface is the process of finding its area. The term is applied under the conception that the process consists in finding a *square* which is equivalent to the given surface.

PROPOSITION XXVIII.

716. Lemma.—*The surface generated by the revolution of a regular semi-polygon of an even number of sides, about the diameter of the circumscribed circle as an axis, is equivalent to the circumference of the inscribed circle multiplied by the axis.*

DEMONSTRATION.

Let ABCDE be one-half of a regular octagon, AE being the diameter of the circumscribing circle.

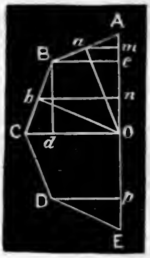

Fig. 324.

If the semi-perimeter **ABCDE** be revolved about **AE** as an axis, the surface generated is $2\pi r \times$ **AE**, r being the radius of the inscribed circle, as a**O**, or b**O**.

This surface is composed of the convex surfaces of cones and frustums of cones. Thus, **AB** generates the surface of a cone, **BC** the frustum of a cone, etc.

Let a and b be the middle points of **AB** and **BC** respectively, and draw am, **B**c, bn, and **CO** perpendicular to the axis, and **B**d parallel to it. Also draw the radii of the inscribed circle, a**O** and b**O**. Indicate the surfaces generated by the sides as Surf. **BC**, etc. The areas of these surfaces are:

$$\text{Surf. } \mathbf{AB} = 2\pi \times am \times \mathbf{AB}\ (?), \qquad (1)$$

$$\text{Surf. } \mathbf{BC} = 2\pi \times bn \times \mathbf{BC}, \text{ etc. } (?). \qquad (2)$$

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* Latin *quadratus*, squared.

Now, from the similar triangles $\mathbf{O}am$ and $\mathbf{BA}c$, we have

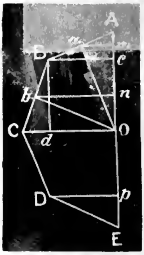

Fig. 324.

$$\frac{a\mathbf{O}}{\mathbf{AB}} = \frac{am}{\mathbf{A}c},$$

or
$$\frac{2\pi \times a\mathbf{O}}{\mathbf{AB}} = \frac{2\pi \times am}{\mathbf{A}c};$$

whence,
$$2\pi \times am \times \mathbf{AB} = 2\pi r \times \mathbf{A}c,$$

putting r for $a\mathbf{O}$.

Also, from the similar triangles $\mathbf{O}bn$ and $\mathbf{CB}d$,

we have
$$\frac{b\mathbf{O}}{\mathbf{BC}} = \frac{bn}{\mathbf{B}d\ (=c\mathbf{O})},$$

or
$$\frac{2\pi \times b\mathbf{O}}{\mathbf{BC}} = \frac{2\pi \times bn}{c\mathbf{O}};$$

whence,
$$2\pi \times bn \times \mathbf{BC} = 2\pi r \times c\mathbf{O},$$

putting r for $b\mathbf{O}$.

Substituting these values in (1) and (2), we obtain

$$\text{Surf. } \mathbf{AB} = 2\pi r \times \mathbf{A}c,$$

$$\text{Surf. } \mathbf{BC} = 2\pi r \times c\mathbf{O}.$$

And, in like manner,
$$\text{Surf. } \mathbf{CD} = 2\pi r \times \mathbf{O}p,$$

and
$$\text{Surf. } \mathbf{DE} = 2\pi r \times p\mathbf{E}.$$

Adding,
$$\text{Surf. } \mathbf{ABCDE} = 2\pi r\,(\mathbf{A}c + c\mathbf{O} + \mathbf{O}p + p\mathbf{E})$$
$$= 2\pi r \times \mathbf{AE}.$$

Finally, since the same course of reasoning is applicable to the semi-polygons of 16, 32, 64, etc., sides, the truth of the proposition is established.

717. Scholium.—This proposition is only a particular case of surfaces generated by *any* broken line revolving about an axis; and the general proposition can be established in a manner altogether similar to the method given above. But this case is all that we need for our present purpose.

PROPOSITION XXIX.

718. Theorem.—*The surface of a sphere is equivalent to four great circles; that is, to $4\pi R^2$, R being the radius of the sphere.*

DEMONSTRATION.

Let the semi-circumference ABCDE revolve upon the diameter AE, and thus generate the surface of a sphere.

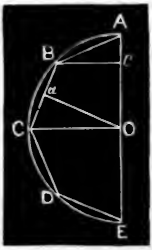

Fig. 325.

Conceive the half of a regular octagon inscribed in the semicircle **ABCDE**; and let both the semi-polygon and the semi-circumference be revolved about **AE** as an axis.

Call the radius of the inscribed circle, as a**O**, r, and let **AO** $= R$.

The surface generated by the broken line **ABCDE** is, by the last proposition, $2\pi r \times 2R = 4\pi rR$.

Now, conceive the arcs **AB**, **BC**, etc., bisected, and the chords drawn, and let r' be the radius of the circle inscribed in the regular polygon thus formed. The surface generated by the revolution of this semi-polygon is $4\pi r'R$.

By repeating the bisections, the broken line approximates to the semi-circumference, the radius of the inscribed circle to R, and the surface generated to the surface of the sphere, the three quantities reaching their limits at the same time. Hence, *at* the limit we have

$$\text{Surf. of sphere} = 2\pi R \times 2R = 4\pi R^2. \quad \text{Q. E. D.}$$

719. COROLLARY 1.—*The area of the surface of a sphere is equivalent to the circumference of a great circle multiplied by the diameter, that is, to $2\pi R \times 2R$, as above.*

720. COROLLARY 2.—*The surfaces of spheres are to each other as the squares of their radii.*

Thus, if R and R' are the radii of two spheres, the surfaces are $4\pi R^2$ and $4\pi R'^2$. Now,

$$\frac{4\pi R^2}{4\pi R'^2} = \frac{R^2}{R'^2}.$$

721. A **Zone** is the portion of the surface of a sphere included between the circumferences of two parallel circles of a sphere. The *altitude* of a zone is the distance between the parallel circles whose circumferences form its bases.

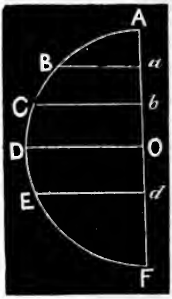

Fig. 326.

Illustration.—The surface generated by **CB**, or any arc of the circle **ABCDE**, etc., as the semicircle revolves about **AE** as an axis, conforms to the definition, and is a *zone*. Such a portion of the surface as is generated by **AB** is called a *zone with one base*, the circle whose circumference would form the upper base having become tangent to the sphere. The altitude of the zone generated by **CB** is ab, and of that generated by **AB** the altitude is **A**a.

PROPOSITION XXX.

722. Theorem.—*The area of a zone is equal to $2\pi aR$, a being the altitude of the zone and R the radius of the sphere.*

Demonstration.

It is evident that in passing to the limit, the surface generated by such a portion of the broken line as lies between **C** and **B**, Fig. 326, is measured by the circumference of the inscribed circle multiplied by ab. Hence, at the limit, the zone generated by arc **BC** is measured by

$$2\pi R \times ab, \quad \text{that is,} \quad 2\pi aR,$$

representing ab by a. Q. E. D.

723. Corollary.—*On the same sphere, or on equal spheres, zones are to each other as their altitudes, and any zone is to the surface of the sphere as the altitude of the zone is to the diameter of the sphere.*

OF LUNES.

724. A **Lune** is a portion of the surface of a sphere included by two semi-circumferences of great circles.

The surface $AmBn$ is a lune.

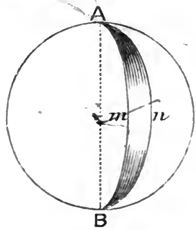

Fig. 327.

725. The **Angle of the Lune** is the angle included by the arcs which form its sides; or, what is the same thing, the measure of the diedral included between the great circles.

Thus, the spherical angle mAn, or the measure of the diedral m-AB-n is the angle of the lune $AmBn$.

726. An **Ungula, or Spherical Wedge,** is that portion of a sphere included between two semi-great-circles, as AmB and AnB. It has a lune for its convex surface and a diameter for its edge.

PROPOSITION XXXI.

727. Theorem.—*On the same sphere, or on equal spheres, lunes which have equal angles are equal.*

Demonstration.

[This is readily effected by applying one to the other. Let the student make the application.]

Exercise.—Can there be a spherical triangle whose angles are 152°, 136°, and 148°? One whose angles are 152°, 136°, and 168°? (See **714**, *a*.)

PROPOSITION XXXII.

728. Theorem.—*The area of a lune is to the area of the surface of the sphere on which it is situated as the angle of the lune is to four right angles.*

First Demonstration.

Let S represent the area of the surface of the sphere generated by the revolution of the semicircle MAN about MN as an axis, and L the area of the lune whose angle is AMD, or AOD.

Then is $\frac{L}{S} = \frac{\text{AOD}}{4 \text{ right angles}}$.

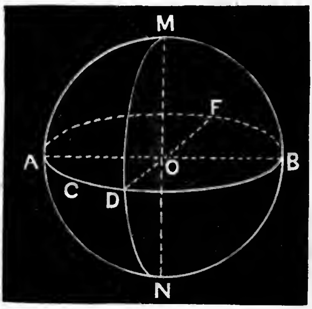

Fig. 328.

In the generation of S and L by the semi-circumference **MAN**, the middle point, **A**, of the semi-circumference generates the great circle **ACDBF**, on which the angles of the lunes are measured (?).

Now **A** generates equal and coincident parts of arc **AD** and circumference **ACDBFA**, in the same time that **MAN** generates corresponding equal and coincident parts of L and S.

Hence, if $$\frac{\text{arc AD}}{\text{circf. ACDBF}} = \frac{1}{n},$$

$$\frac{L}{S} = \frac{1}{n},$$

and $$\frac{L}{S} = \frac{\text{arc AD}}{\text{circf. ACDBF}} = \frac{\text{AOD}}{4 \text{ right angles}} \text{ (?)}.$$ Q. E. D.

Second Demonstration.

Let S represent the area of the surface of the sphere generated by the revolution of the semicircle MAN about MN as an axis, and L the area of the lune whose angle is AMD, or AOD.

Now the angles **AOD** and the sum of the four right angles **AOD, DOB, BOF, FOA** are at least commensurable by an infinitesimal unit. Let i be their common measure, and let it be contained in **AOD** n times, and in the four right angles m times, so that

$$\frac{\mathbf{AOD}}{4 \text{ right angles}} = \frac{n}{m}.$$

Now conceive the circumference divided into m equal parts, and radii drawn to the points of division; and through their extremities let semi-circumferences be drawn. Then is L divided into n lunes, each equal to one of the m equal lunes into which S is divided (**727**), so that

$$\frac{L}{S} = \frac{n}{m}.$$

Hence, $$\frac{L}{S} = \frac{\mathbf{AOD}}{4 \text{ right angles}}. \quad \text{Q. E. D.}$$

Third Demonstration.

Let S be the surface of the sphere, and ACEB $= L$ be a lune whose angle is the spherical angle CAB, or what is the same thing, the plane angle BOC measured by the arc CB, of which A is the pole.

Then is $$\frac{L}{S} = \frac{\mathbf{CAB}}{4 \text{ right angles}}.$$

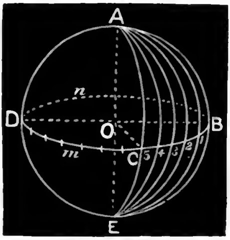

Fig. 329.

For, first, suppose the arc **CB** commensurable with the circumference **BC**m**D**n, and suppose that they are to each other as 5 : 24.

Divide **CB** into five equal arcs, and the entire circumference **BC**m**D**n into twenty-four arcs of the same length, and pass arcs of great circles through **A** and these points of division. Thus the lune is divided into five equal lunes, and the entire surface into twenty-four equal lunes of the same size. These lunes are equal to each other (**727**).

Hence, $$\frac{L}{S} = \frac{5}{24}.$$

Now, $$\frac{\mathbf{COB}}{4 \text{ right angles}} = \frac{\mathbf{CB}}{\mathbf{BC}m\mathbf{D}n} = \frac{5}{24}.$$

Therefore $$\frac{L}{S} = \frac{\mathbf{COB} \text{ (or } \mathbf{CAB})}{4 \text{ right angles}}. \quad \text{Q. E. D.}$$

If the angle of the lune is incommensurable with four right angles, or, what is the same thing, if the arc **BC** is not commensurable with the circumference, let us assume

$$\frac{L}{S} = \frac{\mathsf{BL}}{\mathsf{BC}m\mathsf{D}n}, \qquad (1)$$

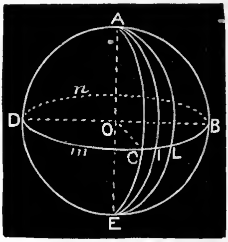

Fig. 330.

in which $\mathsf{BL} < \mathsf{BC}$.

Conceive the circumference **BC***m***D***n* divided into equal parts, each of which is less than **CL**, the assumed difference between **BC** and **BL**. Then conceive one of these equal parts applied to **BC** as a measure, beginning at **B**. Since the measure is less than **LC**, one point of division, at least, will fall between **L** and **C**. Let **I** be such a point, and pass the arc of a great circle through **A** and **I**.

Now
$$\frac{\text{lune } \mathsf{AIEB}}{S} = \frac{\mathsf{BI}}{\mathsf{BC}m\mathsf{D}n}, \qquad (2)$$

since the arc **BI** is commensurable with the circumference. In (1) and (2), the consequents being equal, the antecedents should be proportional;

hence we should have
$$\frac{L}{\text{lune } \mathsf{AIEB}} = \frac{\mathsf{BL}}{\mathsf{BI}}.$$

But this is absurd, since lune **ACEB** > lune **AIEB**, whereas $\mathsf{BL} < \mathsf{BI}$, that is, an improper fraction equals a proper fraction.

In a similar manner, we may reduce the assumption to an absurdity, if we assume $\mathsf{BL} > \mathsf{BC}$.

Hence, as the ratio of $\frac{L}{S}$ can neither be greater nor less than $\frac{\mathsf{BC}}{\mathsf{BC}m\mathsf{D}n}$, it is equal thereto, and

$$\frac{L}{S} = \frac{\mathsf{BC}}{\mathsf{BC}m\mathsf{D}n} = \frac{\mathsf{BOC}}{4 \text{ right angles}}. \qquad \text{Q. E. D.}$$

729. Scholium.—*To obtain the area of a lune whose angle is known,* find the area of the sphere, and multiply it by the ratio of the angle of the lune (in degrees) to 360°. Thus, R being the radius of the sphere, $4\pi R^2$ is the surface of the sphere; and the lune whose angle is 30° is $\frac{30}{360}$ or $\frac{1}{12}$ the surface of the sphere, *i. e.*, $\frac{1}{12}$ of $4\pi R^2 = \frac{1}{3}\pi R^2$.

730. Corollary.—*The sum of several lunes on the same sphere is equal to a lune whose angle is the sum of the angles of the lunes; and the difference of two lunes is a lune whose angle is the difference of their angles.*

731. Corollary.—*Ungulas bear the same ratio to the volume of the sphere that the corresponding lunes do to the area of the surface.*

PROPOSITION XXXIII.

732. Theorem.—*If two semi-circumferences of great circles intersect on the surface of a hemisphere, the sum of the two opposite triangles thus formed is equivalent to a lune whose angle is that included by the semi-circumferences.*

Demonstration.

Let the semi-circumferences CEB and DEA intersect at E on the surface of the hemisphere whose base is CABD.

Then the sum of the triangles **CED** and **AEB** is equivalent to a lune whose angle is **AEB.**

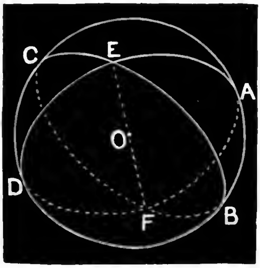

Fig. 331.

For, let the semi-circumferences **CEB** and **DEA** be produced around the sphere, intersecting on the opposite hemisphere, at the extremity **F** of the diameter through **E**.

Now, **FBEA** is a lune whose angle is **AEB.**

Moreover, the triangle **AFB** is equivalent to the triangle **DEC**; since

angle **AFB** = **AEB** = **DEC**,

side **AF** = side **ED**,

each being the supplement of **AE**; and

BF = **CE**,

each being the supplement of **EB**.

Hence, the sum of the triangles **CED** and **AEB** is equivalent to the lune **FBEA**. Q. E. D.

PROPOSITION XXXIV.

733. Theorem.—*The area of a spherical triangle is to the area of the surface of the hemisphere on which it is situated, as its spherical excess is to four right angles, or 360°.*

Demonstration.

Let ABC be a spherical triangle whose angles are represented by A, B, and C; let T represent the area of the triangle, and H the area of the surface of the hemisphere.

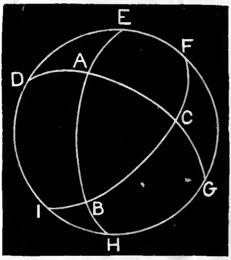

Fig. 332.

Then is $$\frac{T}{H} = \frac{A+B+C-180^\circ}{360^\circ}.$$

Let lune **A** represent the lune whose angle is the angle **A** of the triangle, *i. e.*, angle **CAB**, and in like manner understand lune **B** and lune **C**.

$$\begin{array}{lrcl}
\text{Now,} & \text{triangle } AHG + AED & = & \text{lune } A \ (732), \\
 & BHI + BEF & = & \text{lune } B, \\
 & CGF + CDI & = & \text{lune } C. \\
\hline
\text{Adding,} & 2ABC + \text{hemisphere} & = & \text{lune } (A+B+C)^*, \quad (1)
\end{array}$$

by (730), and since the six triangles **AHG**, **AED**, **BHI**, **BEF**, **CGF**, and **CDI** make the whole hemisphere and 2**ABC** besides, **ABC** being reckoned *three* times

From (1) we have, by transposing, and remembering that a hemisphere is a lune whose angle is 180° (730), and dividing by 2,

$$ABC = \tfrac{1}{2} \text{ lune } (A + B + C - 180^\circ).\dagger$$

But, by (728),

$$\frac{\frac{1}{2} \text{ lune } (A + B + C - 180^\circ)}{H} = \frac{A + B + C - 180^\circ}{360^\circ}.$$

Therefore, $$\frac{T}{H} = \frac{A + B + C - 180^\circ}{360^\circ}. \quad \text{Q. E. D.}$$

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* This signifies the lune whose angle is **A + B + C**, which is of course the sum of the three lunes whose angles are **A**, **B**, and **C**.

† This signifies one-half the lune whose angle is **A + B + C — 180°**.

734. Scholium 1.—*To find the area of a spherical triangle* on a given sphere, the angles of the triangle being given, we have simply to multiply the area of the hemisphere, *i. e.*, $2\pi R^2$, by the ratio of the spherical excess to 360°. Thus, if the angles are

$$\mathsf{A} = 110^\circ, \qquad \mathsf{B} = 80^\circ, \quad \text{and} \quad \mathsf{C} = 50^\circ,$$

we have

$$\text{area } \mathsf{ABC} = 2\pi R^2 \times \frac{\mathsf{A} + \mathsf{B} + \mathsf{C} - 180^\circ}{360^\circ} = 2\pi R^2 \times \frac{60}{360} = \tfrac{1}{3}\pi R^2.$$

735. Scholium 2.—This proposition is often stated thus: *The area of a spherical triangle is equal to its spherical excess multiplied by the trirectangular triangle.* When so stated, the spherical excess is to be estimated in terms of the right angle; *i. e.*, having subtracted 180° from the sum of its angles, we are to divide the remainder by 90°, thus getting the spherical excess in right angles. In the example in the preceding scholium, the spherical excess estimated in this way would be

$$\frac{110^\circ + 80^\circ + 50^\circ - 180^\circ}{90^\circ} = \frac{2}{3}$$

and the area of the triangle would be $\frac{2}{3}$ of the trirectangular triangle. Now, the trirectangular triangle being $\frac{1}{8}$ of the surface of the sphere (?), is $\frac{1}{8}$ of $4\pi R^2$, or $\frac{1}{2}\pi R^2$. This multiplied by $\frac{2}{3}$ gives $\frac{1}{3}\pi R^2$, the same as above.

The proportion

$$\frac{\mathsf{ABC}}{\text{surf. of hemisph.}} = \frac{\mathsf{A} + \mathsf{B} + \mathsf{C} - 180^\circ}{360^\circ},$$

is readily put into a form which agrees with the enunciation as given in this scholium. Thus,

$$\text{surf. of hemisph.} = 2\pi R^2;$$

whence,

$$\mathsf{ABC} = 2\pi R^2 \times \frac{\mathsf{A} + \mathsf{B} + \mathsf{C} - 180^\circ}{360^\circ} = \tfrac{1}{2}\pi R^2 \times \frac{\mathsf{A} + \mathsf{B} + \mathsf{C} - 180^\circ}{90^\circ}.$$

VOLUME OF SPHERE.

PROPOSITION XXXV.

736. Theorem.—*The volume of a sphere is equal to the area of its surface multiplied by one-third of the radius, that is, $\frac{4}{3}\pi R^3$, R being the radius.*

DEMONSTRATION.

Let OL = R be the radius of a sphere.

Conceive a circumscribed cube, that is, a cube whose faces are tangent planes to the sphere. Draw lines from the vertices of each of the polyedral angles of the cube to the centre of the sphere, as **AO**, **BO**, **DO**, **CO**, etc. These lines are the edges of six pyramids, having for their bases the faces of the cube, and for a common altitude the radius of the sphere (?). Hence the volume of the circumscribed cube is equal to its surface multiplied by $\frac{1}{3}R$.

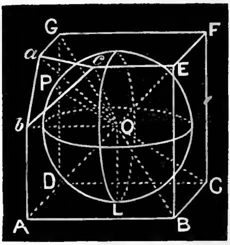

Fig. 333.

Again, conceive each of the triedral angles of the cube truncated by planes *tangent to the sphere.* A new circumscribed solid will thus be formed, whose volume will be nearer that of the sphere than is that of the circumscribed cube. Let *abc* represent one of the tangent planes. Draw from the polyedral angles of this new solid, lines to the centre of the sphere, as a**O**, b**O**, and c**O**, etc.; these lines will form the edges of a set of pyramids whose bases constitute the surface of the solid, and whose common altitude is the radius of the sphere (?). Hence the volume of this solid is equal to the product of its surface (the sum of the bases of the pyramids) into $\frac{1}{3}R$.

Now, this process of truncating the angles by tangent planes may be conceived as continued indefinitely; and, to whatever extent it is carried, it will *always* be true that the volume of the solid is equal to its surface multiplied by $\frac{1}{3}R$. Therefore, as the sphere is the limit of this circumscribed solid, we have the volume of the sphere equal to the surface of the sphere, which is $4\pi R^2$ multiplied by $\frac{1}{3}R$, *i. e.*, to $\frac{4}{3}\pi R^3$. Q. E. D.

737. Corollary.—*The surface of the sphere may be conceived as consisting of an infinite number of infinitely small plane faces, and the volume as composed of an infinite number of pyramids having these faces for their bases, and their vertices at the centre of the sphere, the common altitude of the pyramids being the radius of the sphere.*

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738. A **Spherical Sector** is a portion of a sphere generated by the revolution of a circular sector about the diameter around which the semicircle which generates the sphere is conceived to revolve. It has a zone for its base; and it may have as its other surfaces one or two conical surfaces, or one conical and one plane surface.

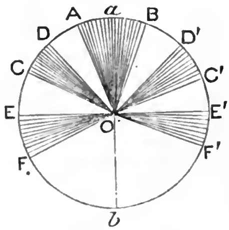

Fig. 334.

Illustration.—Thus, let *ab* be the diameter around which the semicircle *a*E*b* revolves to generate the sphere. The solid generated by the circular sector **AOB** will be a spherical sector having the zone generated by **AB** for its base; and for its other surface, the conical surface generated by **AO**. The spherical sector generated by **COD** has the zone generated by **CD** for its base; and for its other surfaces, the concave conical surface generated by **DO**, and the convex conical surface generated by **CO**. The spherical sector generated by **EOF** has the zone generated by **EF** for its base, the plane generated by **EO** for one surface, and the concave conical surface generated by **FO** for the other.

739. A **Spherical Segment** is a portion of the sphere included by two parallel planes, it being understood that one of the planes may become a tangent plane. In the latter case, the segment has but one base; in other cases, it has two. A spherical segment is bounded by a zone and one, or two, plane surfaces.

PROPOSITION XXXVI.

740. Theorem.—*The volume of a spherical sector is equal to the product of the zone which forms its base into one-third the radius of the sphere.*

DEMONSTRATION.

A spherical sector, like the sphere itself, may be conceived as consisting of an infinite number of pyramids whose bases make up the base of the sector, and whose common altitude is the radius of the sphere. Hence, the volume of the sector is equal to the sum of the bases of these pyramids, that is, the surface of the sector, multiplied by one-third their common altitude, which is one-third the radius of the sphere. Q. E. D.

741. COROLLARY.—*The volumes of spherical sectors of the same sphere, or of equal spheres, are to each other as the zones which form their bases; and, since these zones are to each other as their altitudes* (**723**), *the sectors are to each other as the altitudes of the zones which form their bases.*

PROPOSITION XXXVII.

742. Theorem.—*The volume of a spherical segment of one base is $\pi A^2(R - \frac{1}{3}A)$, A being the altitude of the segment, and R the radius of the sphere.*

DEMONSTRATION.

Let AO $= R$, and CD $= A$.

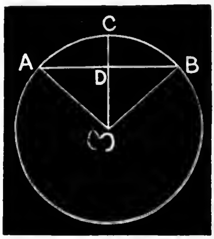

Fig. 335.

Then is the volume of the spherical segment generated by the revolution of **ACD** about **CO** equal to $\pi A^2(R - \frac{1}{3}A)$.

For, the volume of the spherical sector generated by **AOC** is the zone generated by **AC**, multiplied by $\frac{1}{3}R$, or $2\pi AR \times \frac{1}{3}R = \frac{2}{3}\pi AR^2$. From this we must subtract the cone, the radius of whose base is **AD**, and whose altitude is **DO**.

To obtain this, we have

$$\mathbf{DO} = R - A;$$

whence, from the right-angled triangle **ADO**,

$$\mathbf{AD} = \sqrt{R^2 - (R - A)^2} = \sqrt{2AR - A^2}.$$

Now, the volume of this cone is $\frac{1}{3}\mathbf{OD} \times \pi\overline{\mathbf{AD}}^2$, or

$$\tfrac{1}{3}\pi (R - A)(2AR - A^2) = \tfrac{1}{3}\pi (2AR^2 - 3A^2R + A^3).$$

Subtracting this from the volume of the spherical sector, we have

$$\begin{aligned} \tfrac{2}{3}\pi AR^2 - \tfrac{1}{3}\pi (2AR^2 - 3A^2R + A^3) &= \pi (A^2R - \tfrac{1}{3}A^3) \\ &= \pi A^2 (R - \tfrac{1}{3}A). \end{aligned}$$ Q. E. D.

743. Scholium.—The volume of a spherical segment with two bases is readily obtained by taking the difference between two segments of one base each. Thus, to obtain the volumes of the segment generated by the revolution of b**CA**c about a**O**, take the difference of the segments whose altitudes are ac and ab.

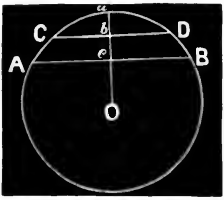

Fig. 336.

SPHERICAL POLYGONS AND SPHERICAL PYRAMIDS.

744. A **Spherical Polygon** is a portion of the surface of a sphere bounded by several arcs of great circles.

745. The **Diagonal** of a spherical polygon is an arc of a great circle joining any two non-adjacent vertices.

746. A **Spherical Pyramid** is a portion of a sphere having for its base a spherical polygon, and for its lateral faces the circular sectors formed by joining the vertices of the polygon with the centre of the sphere.

747. The elementary properties of spherical polygons and spherical pyramids are so readily deduced from the corresponding properties of polyedral angles, spherical triangles, etc., that we leave them for the pupil to demonstrate, merely stating a few fundamental theorems.

748. Theorem.—*The angles of a spherical polygon and its sides sustain the same general relations to each other as the diedral and facial angles of a polyedral angle having for its edges the radii of the sphere drawn to the vertices of the polygon.*

749. Theorem.—*The sum of the sides of a convex spherical polygon may be anything between* $0°$ *and* $360°$.

750. Theorem.—*The sum of the angles of a spherical polygon may be anything between* $2n - 4$ *and* $6n - 12$ *right angles, n being the number of sides.*

751. The **Spherical Excess** of a spherical polygon is the excess of the sum of its angles over the sum of the angles of a plane polygon of the same number of sides.

752. Theorem.—*The spherical excess of a spherical polygon of n sides, the sum of whose angles is S, is*

$$S + 360° - n \cdot 180°.$$

753. Theorem.—*The area of a spherical polygon is to the area of the surface of the hemisphere on which it is situated as its spherical excess is to four right angles.*

754. Theorem.—*The volume of a spherical pyramid is the area of its base multiplied by one-third the radius of the sphere on which it is situated.*

EXERCISES.

755. 1. What is the circumference of a small circle of a sphere whose diameter is 10, the circle being at 3 from the centre?

Ans. 25.1328.

2. Construct on the spherical blackboard a spherical angle of 60°. Of 45°. Of 90°. Of 120°. Of 250°.

SUGGESTIONS.—Let **P** be the point where the vertex of the required angle is to be situated. With a quadrant strike an arc passing through **P**, which shall represent one side of the required angle. From **P** as a pole, with a quadrant strike an arc from the side before drawn, which shall measure the required angle. On this last arc lay off from the first side the measure of the required angle,* as 60°, 45°, etc. Through the extremity of this arc and **P** pass a great circle (?).

3. On the spherical blackboard construct a spherical triangle **ABC**, having **AB** = 100°, **AC** = 80°, and **A** = 58°.

4. Construct as above a spherical triangle **ABC**, having **AB** = 75°, **A** = 110°, and **B** = 87°.

5. Construct as above, having **AB** = 150°, **BC** = 80°, and **AC** = 100°. Also having **AB** = 160°, **AC** = 50°, and **CB** = 85°.

6. Construct as above, having **A** = 52°, **AC** = 47°, and **CB** = 40°.

SUGGESTIONS.—Construct the angle **A** as before taught, and lay off **AC** from **A** equal to 47°, with the tape. This determines the vertex **C**. From **C**, as a pole, with an arc of 40°, describe an arc of a small circle; in this case this arc will cut the opposite side of the angle **A** in two places. Call these points **B** and **B′**. Pass circumferences of great circles through **C**, and **B**, and **B′**. There are two triangles, **ACB** and **ACB′**.

7. Construct on the spherical blackboard a spherical triangle **ABC**, having **A** = 59°, **AC** = 120°, and **AB** = 88°.

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* For this purpose, a tape equal in length to a semi-circumference of a great circle of the sphere used, and marked off into 180 equal parts, will be convenient. A strip of paper may be used.

8. Construct a triangle whose angles are 160°, 150°, and 140°.

9. Can there be a spherical triangle whose angles are 85°, 120°, and 150°? Try to construct such a triangle by first constructing its polar.

10. What is the area of a spherical triangle on the surface of a sphere whose radius is 10, the angles of the triangle being 85°, 120°, and 110°? *Ans.* 235.6+.

11. What is the area of a spherical triangle on a sphere whose diameter is 12, the angles of the triangle being 82°, 98°, and 100°?

12. A sphere is cut by five parallel planes at 7 from each other. What are the relative areas of the zones? What of the segments?

13. Considering the earth as a sphere, its radius would be 3958 miles, and the altitudes of the zones, North torrid = 1578, North temperate = 2052, and North frigid = 328 miles. What are the relative areas of the several zones?

SUGGESTION.—The student should be careful to discriminate between the *width* of a zone and its altitude. The altitudes are found from their widths, as usually given in degrees, by means of Trigonometry.

14. The earth being regarded as a sphere whose radius is 3958 miles, what is the area of a spherical triangle on its surface, the angles being 120°, 130°, and 150°? What is the area of a trirectangular triangle on the earth's surface?

15. In the spherical triangle **ABC**, given **A** = 58°, **B** = 67°, and **AC** = 81°; what can you affirm of the polar triangle?

16. What is the volume of a globe which is 2 feet in diameter? What of a segment of the same globe included by two parallel planes, one at 3 and the other at 9 inches from the centre, the centre of the sphere being without the segment? What if the centre is within the segment?

17. Compare the convex surfaces of a sphere and its circumscribed cylinder.

18. Compare the volumes of a sphere and its circumscribed cube, cylinder, and cone, the vertical angle of the cone being 60°.

19. If a and b represent the distances from the centre of a sphere whose radius is r, to the bases of a spherical segment, show that the volume of the segment is $\pi\ [r^2\,(b-a) - \frac{1}{3}(b^3 - a^3)]$.

APPENDIX

THE INFINITESIMAL METHOD.

THE author is a firm believer in both the logical soundness and the practical advantages of the strict infinitesimal method. Hence he has introduced it—though generally as an alternative method—in those cases in which the incommensurability of geometrical magnitudes by a finite unit makes the old demonstrations cumbrous.

As to the logical soundness of the method, he has not the shadow of a doubt. The well-known logical principle, that, if we create a certain category of concepts, under certain definite laws, use them in our argument in accordance with these laws, and finally eliminate them, the argument being conducted according to correct logical principles, the final results are correct, covers the entire case. Now the two essential laws of infinitesimals are, (1) Infinitesimals of the same order have the same relations among themselves as finite quantities; and (2) Infinitesimals in comparison with finites, are zero.

But the simple exposition given in the text (**340–342**) is quite adequate to show that the method can introduce no conceivable error. Thus, if $\frac{m}{n}=a$, all the quantities being finite, and if i is an infinitesimal, $\frac{m+i}{n}=a$ must be true, and i must be 0 in the relation. Otherwise solving the equation we have $i=an-m$, a finite quantity, unless $a=\frac{m}{n}$.

Of the immense practical utility of the method there can be no question. All, from Lagrange down, have acknowledged it. I know of no extended treatise which does not in some way imply it. Why, then, should not the pupil become familiar with it early in his course?

As to the method of limits it is not at all difficult to show that it is identical with the infinitesimal method, in its fundamental principles. Moreover, there is a sort of jugglery in the very first step in the method of limits which quite transcends any difficulty that the method of infinitesimals presents. Thus, we give the variable an increment, assume that the function takes a related increment, manipulate the function, and then make the increment of the variable zero (whence the increment of the function becomes zero), and, *presto,* we have a finite relation between two zeros! And this is the "*simple*" fundamental conception which the tyro is supposed to see at a glance!

NOTE ON (182), (343), (587), (628), AND (728).

These propositions are of a class in which the incommensurability by a finite unit of certain lines introduces particular difficulty, which difficulty disappears at once if we admit, as in the infinitesimal theory, that these lines are commensurable by an infinitesimal unit. Also, by the introduction of the principle of the generation of one magnitude by the motion of another, very simple demonstrations are afforded.

In the text the author has given illustrations of the three sorts of demonstrations. In (**182**) we have the old method of avoiding the difficulty which grows out of the incommensurability, by the *reductio absurdum.* The objection to this is not any objection to the *reductio absurdum* as a method of reasoning. But why use so cumbrous a method, when other exceedingly simple methods are at hand, and methods involving principles so necessary to subsequent use?

In (**728**) the three methods are given. In (**343**) and (**628**) the methods involving generation by motion, and the infinitesimal method, are given.

NOTE ON (182).

1. To prove this proposition by means of the conception of the generation of magnitudes by the motion of other magnitudes, we do not need the Lemma. Thus, referring to Fig. 85, p. 89, we are to prove that $\frac{\mathsf{AOB}}{\mathsf{DOE}} = \frac{\text{arc } \mathsf{AB}}{\text{arc } \mathsf{DE}}$.

Let the sector **AOB** be applied to **DOE**, **OA** being placed in **OD**. By reason of the equality of the circles the arc **AB** will fall in **DE**.

Conceive the angles **AOB** and **DOE** as generated by a radius moving from the position **OA** (which is now also **OD**) to **OB** and **OE**, with uniform motion. Let the time of generating **AOB** be r, and that of generating **DOE** be s. Whence $\frac{\mathsf{AOB}}{\mathsf{DOE}} = \frac{r}{s}$ (**48, 49**).

Again, the extremity of the radius, as **A** (or **D**), describes equal and (as far as the less extends) coincident parts of **AB** and **DE** in equal times, whence $\frac{\text{arc } \mathsf{AB}}{\text{arc } \mathsf{DE}} = \frac{r}{s}$. Hence, by equality of ratios, we have $\frac{\mathsf{AOB}}{\mathsf{DOE}} = \frac{\text{arc } \mathsf{AB}}{\text{arc } \mathsf{DE}}$.

2. To prove the same proposition by the infinitesimal method, we proceed exactly as in Case II., pp. 89, 90, simply conceiving m as infinitesimal when the angles are incommensurable by a finite unit, and for 5 putting the indefinite number r, and for 8 the indefinite number s.

NOTE ON (343).

By the old method the Lemma on which this demonstration is based is proved in two cases. 1st. When the bases are commensurable ; 2nd. When the bases are incommensurable. Dividing the bases into equal parts and erecting perpendiculars at the points of division the argument in the first case proceeds exactly like the argument in Case II. of (**182**). When the bases are incommensurable, we apply *abcd* to **ABCD** placing *ad* in its equal **AD**, whence *ab* falls in **AB**, as far as it extends, and *dc* in **DC**. Then assume that, if $\frac{\mathsf{ABCD}}{abcd}$ is not equal to $\frac{\mathsf{AB}}{ab}$, it is equal to $\frac{\mathsf{AB}}{ag}$, *ag* being either greater or less than *ab*. Now divide the base **AB** into equal parts, each of which is less than *bg*, and erect perpendiculars at each of the points of division. We may then show, as in Case III. of (**182**), the absurdity of supposing *ag* greater or less than *ab*.

FINIS.

and a right angle is called
an ... angle.

"a

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