ELEMENTARY PLANE GEOMETRY

BAKER

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ELEMENTARY PLANE GEOMETRY

INDUCTIVE AND DEDUCTIVE

BY

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PREFACE.

THE geometry of Euclid is deductive. Yet the processes of all sciences, other than pure mathematics, involve both induction and deduction. All the knowledge which we have of life, with its varied phenomena, is reached by induction and deduction. Any science, then, which permits the student, from a number of observations, to reach a general result, and again from such generalization to draw conclusions, must have distinct educational value. The present little book is an attempt to make the processes of elementary geometry both inductive and deductive. I feel that in making this attempt I am adapting the study of Geometry to immature minds. The mind of youth receives its knowledge in the form of isolated facts; it is for the educator to point out that isolated facts fall into groups and may be crystallized into general conclusions. Special opportunities present themselves in elementary geometry for following this method. Thus, if a number of triangles be accurately constructed with bases of 45 millimetres and angles at the bases 75° and 62°, by actual measurement the learner finds that all the sides opposite to the angles of 75° are equal, and likewise those opposite to the angles of 62°, and that the remaining angles of the triangles have the same magnitude. Analogous constructions and measurements being repeated in a number of cases, the learner, as a matter of inductive observation, feels himself justified in making the generalization expressed in the enunciation of Euclid I., 26. In the process the intellectual interest and curiosity of the pupil are excited, and in reaching the conclusion he feels almost as if he had made the discovery himself. If, subsequently, geometrical forms are presented to him where he can utilize his previous conclusion, he feels with keenness the value of his previous work. He has, in fact, been going through the process of induction and deduction,—the process through which every scientific discoverer goes—with, in miniature, the emotions of the investigator.

It is justly claimed for Euclid that he inculcates accuracy of thought. Most admirably in this respect he does his work. It too often happens, however, that in the class-room triangles are alleged to be equal which are ridiculously unlike, and lines are *proved* to be equal which the eye tells us differ in length by several inches. In fact, in spite of accuracy of thought, the utmost contempt for physical accuracy is often inculcated. The whole spirit of the following pages is accuracy of construction. Only by exact drawing can results be attained, and the pupil will find that inaccuracy means failure. *My object is to make the class-room in geometry a sort of workshop, where exactness in drawing lines of required length, in measuring lines that are drawn, in constructing angles of given magnitude, in measuring angles that are constructed, and generally in constructing all figures, is insisted on. The attitude of the pupil towards his geometrical figures should be that of the skilled mechanic towards an instrument or machine of precision which he is making, where inaccuracy in measurement would mean loss of time and of material, and would be considered evidence of stupidity.*

I do not suggest this book as a substitute for Euclid, but as an introduction to the study of the work of the great geometer, or of some work covering the same ground. Hence I have included the leading geometrical facts reached in Euclid's elements, and have introduced them in nearly Euclid's order. Teachers will find here about one year's work for a class of beginners. If the pupils pursue the subject of geometry no further, I humbly trust that the practical work they have done in connection with this course will have impressed the leading facts of elementary geometry indelibly on their minds; if on the other hand they take up the study of deductive geometry, I hope they will the better, from following this concrete course, appreciate the absolutely general and irrefragable character of Euclid's methods.

UNIVERSITY OF TORONTO, A. B.

May, 1903.

CONTENTS.

(At the close of Chapter VIII. the suggestion is made that Chapters XIX., XX., and XXI., relating to similar triangles, may at once be proceeded with.)

INSTRUMENTS.

In the pages that succeed, the following instruments are essential:

1. A ruler or straight-edge,

on which are marked inches divided into sixteenths, and on which also is a scale giving millimetres.

This is used for drawing straight lines; for making them of any required length; and for measuring straight lines that are drawn.

2. A pair of compasses,

one leg of which is furnished with a pencil.

This is used for describing circles; also, with the help of the ruler, for laying off required distances; and for measuring distances that are laid off.

3. A protractor.

This is used for constructing angles of any given number of degrees; and for measuring the number of degrees in any given angle. It may also be used for determining whether one angle is greater than, equal to, or less than another.

For the more rapid and more accurate construction of figures, the following instruments are also desirable:

4. A pair of dividers,

both the legs of which terminate in fine points. These more accurately than the compasses will enable the pupil to measure and to transfer distances.

5. A set-square.

The right angle has very frequently to be constructed, and its construction can be more rapidly effected with the set-square than with the protractor.

6. A bevel.

This enables us very rapidly to determine the equality or inequality of angles, and to construct an angle equal to another.

7. Parallel rulers.

While for drawing lines parallel to each other nothing more is essential than a ruler along which the set-square is made to slide, or a ruler and an instrument for measuring angles, or a ruler and compasses, these methods become tedious from the frequency with which the construction has to be made. Parallel rulers make the construction rapidly and accurately.

Care should be taken to use a pencil with a hard fine point, so that lines drawn may be narrow and well defined.

Smooth paper will be found better than rough.

Points and the ends of lines should be marked by indentations made with a needle or with the sharp points of the dividers.

A piece of smooth, perfectly flat board, about a foot square, will be found useful as a drawing board.

In all cases the pupil should construct for himself the necessary figures, and not content himself with those in the book, which are merely intended as suggestions. It will be usually found desirable to make figures on a larger scale than those in the text.

The chapters on similar triangles may be taken up, if thought desirable, as soon as the pupil has obtained an acquaintance with parallel lines, and knows that the opposite sides and angles of parallelograms are equal. Prominence may then be given to Exercise 17, Chapter xxi., which suggests a demonstration of the 47th, Book I., Euclid.

CHAPTER I.

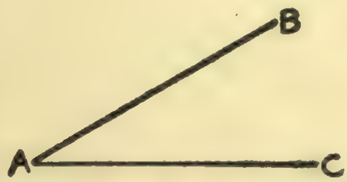
Geometrical Elements.

A straight line:

It is evidently the shortest distance between its ends.

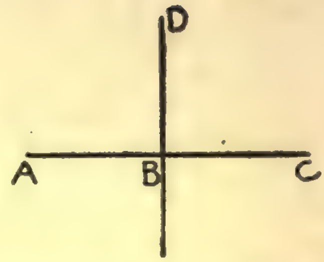
A broken line: 

A curved line: 

An angle: 

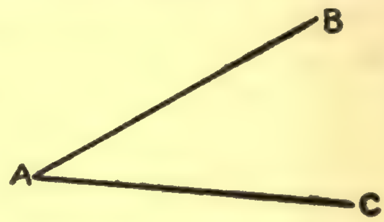
The size of the angle does not depend on the lengths of the bounding lines **AB** and **AC**, but on the *amount of divergence* of these lines from one another. Thus the angle **P** is greater than the angle **Q**, and the angle **R** is less than the angle **Q**.

It is usual to indicate an angle by using one letter, as the angle **P**, or by using three letters, as the angle **BAC**. In the latter case the letter at the angle itself is in the middle, and the other two letters lie on the *arms* of the angle.

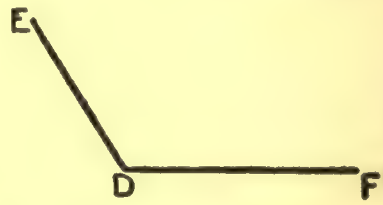


If **ABC** be a straight line, and the angles **DBA, DBC** be equal, then each of them is called a **right angle,** and the lines **DB** and **ABC** are said to be **perpendicular** to each other.

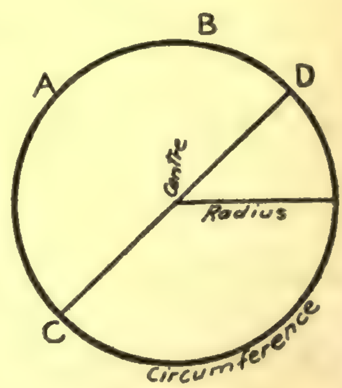
Evidently at the point **B** there are four right angles.



An angle which is less than a right angle, as **BAC,** is called an **acute** angle.



An angle which is greater than a right angle, as **EDF,** is called an **obtuse** angle.



A **circle** is the usual figure described on a flat surface by means of the compasses.

Note the parts called **centre, radius,** and **circumference.**

All radii of the same circle are equal, since the ends of the compass legs remain the same distance apart while the circle is being described.

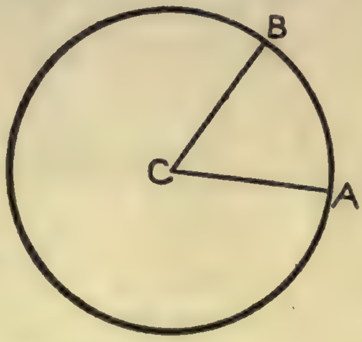
A line through the centre and terminated both ways by the circumference is called a **diameter,** as **CD.**

The part of the circle on each side of a diameter is called a **semicircle.**

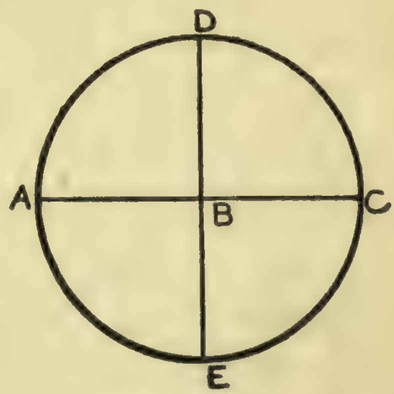
A part of the circumference, as **AB,** is called an **arc** of the circle. The straight line joining **A** and **B** is called a **chord.**

Any line drawn from a point without the circle and cutting it, is called a **secant.**

The circumference of any circle is supposed to be divided into 360 equal parts, each part being called a **degree.**

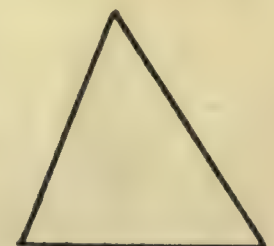


If the arc **AB** contains 60 degrees, then the angle **ACB** at the centre is an angle of 60 degrees, expressed by 60°.



The lines **AC, DE,** through the centre, being perpendicular, each of the arcs **AD, DC, CE, EA** must contain 90°, and the angles **ABD, DBC,** . . . . are angles of 90°.

A semicircle contains 180°, and the straight angle **ABC** contains 180°.



A **triangle:**

It has three sides and three angles.

A **quadrangle:**

It has four angles. Having four sides, it is also called a **quadrilateral.**

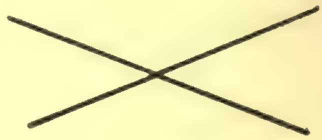
A straight line joining two opposite corners of a quadrilateral is called a **diagonal.**

Figures contained by more than four straight lines are called **polygons.**

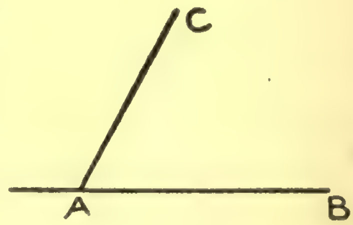
A straight line has evidently throughout its entire length the same *direction.*

Two straight lines which have the same direction are said to be **parallel** to one another.

Parallel straight lines cannot intersect. For if they did, at the point of intersection they would have different directions, and would therefore have different directions throughout their entire lengths, and hence would not be parallel.



To construct with the protractor at the point **A** in the line **AB** an angle of any required magnitude, say 63°: Place the centre of the protractor at **A**, and let the line joining the centre with the point on the circumference which indicates 0°, rest along **AB**. At the point where the 63° line meets the circumference make a fine mark, **C**, on the paper. Removing the protractor, join **AC**. The angle **BAC** is of magnitude 63°.



Exercises.

All figures in this and succeeding exercises must be accurately constructed with instruments.

1. With the dividers (or compasses) take off on the ruler distances 8, 11, 17, 34 . . . . millimetres. With the points of the dividers mark on your paper points at these distances from each other. With the ruler draw straight lines joining each pair of points, thus getting straight lines of lengths 8, 11, 17, 34 . . . millimetres.

2. With the compasses describe circles having radii of lengths 5, 7, 10, . . . sixteenths of an inch.

3. With the protractor construct angles of magnitude 10°, 15°, 25°, 30°, 37°, 43°, ...

4. With the bevel construct a second set of angles of the foregoing magnitudes, using these angles to set the bevel.

5. Draw five straight lines of different lengths, and with the dividers and rule measure their lengths in inches and sixteenths of an inch. Measure also their lengths in millimetres.

6. Construct five angles, and, using the bevel, determine which is greatest and which least. Arrange them in order of magnitude. Using the protractor, measure their magnitude to the nearest degree.

7. Draw five straight lines of different lengths, and with the eye endeavor to judge their lengths (1) in inches and fractions of an inch, (2) in millimetres. Afterwards test the correctness of your judgment by actually measuring the lines.

8. Construct five angles of different magnitudes, and with the eye endeavor to judge the number of degrees in each. Afterwards test the correctness of your judgment by actually measuring the angles with the protractor.

9. With the eye endeavor to judge the lengths or heights of various objects in the room, at a distance from you. Afterwards test the correctness of your judgment by actually measuring the lengths or heights.

10. A and B being two distant objects and your eye being at C, endeavor with the eye to judge the angle which these objects subtend at your eye, *i.e.*, the angle ACB. Afterwards sight the inside edges of the legs of the bevel towards A and B, and then placing the bevel on the protractor, roughly measure in this way the angle ACB, so correcting, if necessary, your judgment.

11. Draw any two lines of different lengths, and draw a line equal to their difference.

12. Draw any line, and draw another line three times as long as the former.

13. Construct two angles of different magnitudes, and with the bevel constructing two adjacent angles equal to them, form an angle equal to their difference. Measure with the protractor the number of degrees in the original angles and in the difference, and compare.

14. Construct two angles of different magnitudes, and with the bevel constructing two adjacent angles equal to them, form an angle equal to their sum. Measure with the protractor the number of degrees in the original angles and in the sum, and compare.

15. Construct an angle of 30°. With the bevel construct two other angles equal to it, one on each side of the first, the three bounding lines radiating from the same point. What positions do the outside lines of your figure occupy with respect to each other, and why? Test with an instrument.

16. Construct an angle of 60°. With the bevel construct five other angles equal to it, each adjacent to the preceding, the bounding lines all radiating from the same point. What positions do the first and last lines of these angles occupy with respect to each other, and why?

17. In the figure of the preceding question, if O be the point from which the lines radiate, measure off with the dividers on these lines equal lengths, OA, OB, OC, OD, OE, OF. What do you observe as to the lengths AB, BC, CD, DE, EF, FA?

18. Fold a piece of paper so as to get a straight crease. Fold the crease over on itself. How many degrees in each of the four angles so obtained, and why?

19. With a needle mark two points. Join them, using ruler and a fine pencil. Turn the ruler over to the other side of the two points and again join them. What quality in the ruler may you test in this way?

20. At points on your paper some distance from one another, construct two angles, as nearly as you can judge, equal. Test with an instrument the correctness of your judgment.

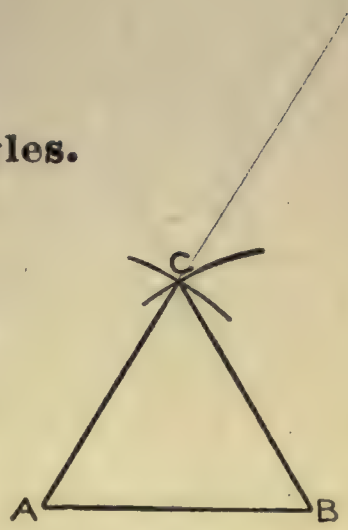
21. Through what angle does the minute-hand of a clock move in 20 minutes? Through what angle does the hour-hand move in the same time?

22. Describe a circle, and, supposing it intended for the face of a clock, mark the points where the usual Roman numerals should be placed.

23. One side of a piece of paper being a straight line, tear the remaining boundary into any irregular shape. With your protractor convert this paper into a protractor, so as to mark angles at intervals of 10°, the markings being on the irregular edge of the paper.

CHAPTER II.

Construction of Triangles.



1. Take a line **AB** of any length. First with **A** as centre, then with **B** as centre, and in both cases with the same radius **AB**, describe portions of circles so that they intersect, as indicated, at **C**. Then the three lines **AB, BC, CA** are all equal. The triangle **CAB**, which has thus all its sides equal, is called an **equilateral** triangle.

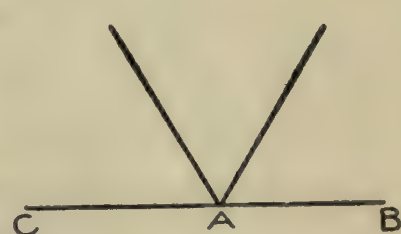
Adjust the bevel to each of the angles of this triangle, and compare their magnitudes.

Construct equilateral triangles whose sides are 14, 21, 30, 40 . . . sixteenths of an inch.

Apply the bevel to all the angles of these triangles, and compare their magnitudes.

Cut accurately any one of these equilateral triangles from the paper, and, clipping off the angles, fit them on one another, and on the angles of the other equilateral triangles, so as to compare their magnitudes.

The result of our observations is that **the angles in an equilateral triangle are equal to one another, and are equal to the angles in any other equilateral triangle.**



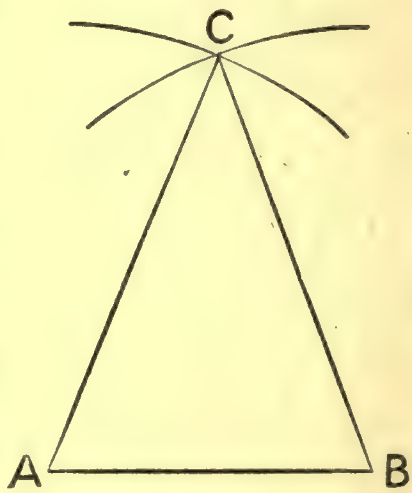
Using the bevel, construct three angles adjacent to one another, in the way indicated in the annexed figure, each angle being equal to the angle of an equilateral tri-

NOTE.—It is well to mark on lines and angles their magnitudes, when known.

angle. Applying the ruler, it will be found that **CA** and **AB** are in the same straight line. Hence it appears that the three angles of any equilateral triangle are together equal to 180°, and any one of the angles in such a triangle is 60°.

Measure the angles in several of the equilateral triangles with the protractor to verify this.

2. Take a line **AB** of, say, 25 millimetres in length, and with centres **A** and **B** describe portions of circles intersecting as indicated at **C**, each circle having the same radius, say 35 millimetres. Draw lines from **C** to **A** and **B**. Then the triangle **CAB** has two sides equal. A triangle with two of its sides equal is called an **isosceles** triangle.



Adjusting the bevel to the angles **CAB** and **CBA**, compare their magnitudes.

Compare also the sizes of these angles by accurately cutting the triangle out of the paper, and placing the triangle reversed in the vacant space left in the paper, so that the angle **B** rests in the space **A**.

Compare also the sizes of these angles by folding the triangle along the line from **C** to the middle of **AB**.

Construct the following isosceles triangles:

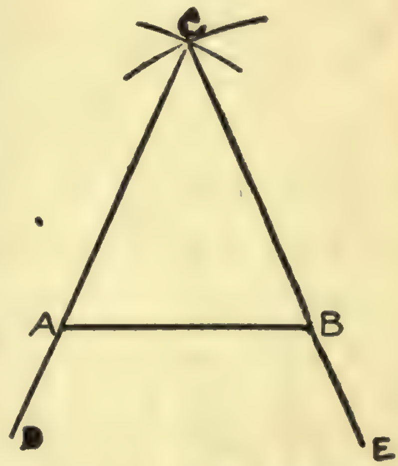
Base 1 in., each side 2 in.
Base 3 in., each side 2 in.
Base $2\frac{1}{2}$ in., each side $2\frac{15}{16}$ in.

. . . .

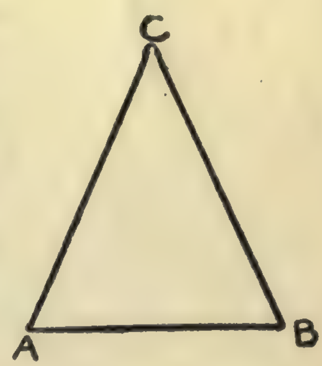
In each case compare the magnitudes of the angles at the base.

The result of our observations is that **the angles at the base of an isosceles triangle are equal.**

Of course it would follow from this that all the angles in an equilateral triangle are equal, as we have already seen.



Prolong the sides **CA, CB,** and adjusting the bevel to the angles **BAD, ABE,** on the other side of the base, they will be found to be equal. This may also be reasoned out as follows: The angles on one side of a straight line at any point in it make up 180°. But the angles **CAB** and **CBA** are equal. Therefore the remaining angles **BAD** and **ABE** are also equal.



3. Taking any line **AB,** with the bevel or protractor construct equal angles at **A** and **B,** and produce the bounding lines of these angles to meet in **C.** Then employing the dividers or compasses, compare the lengths of the sides **CA, CB,** of the triangle **CAB.**

Construct the following triangles:

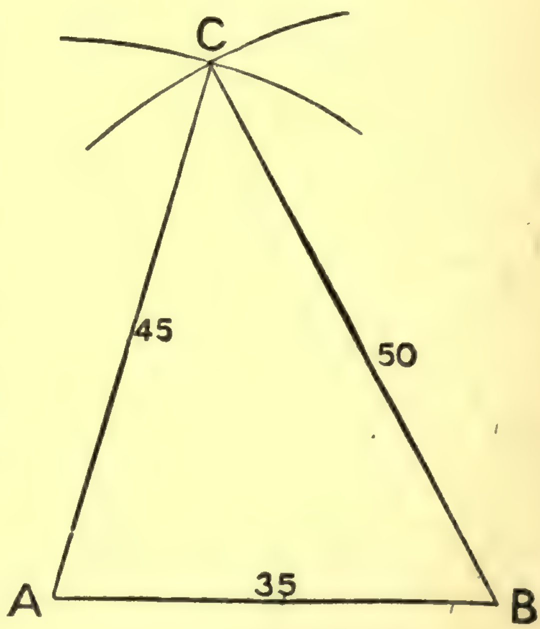
Base 25 millimetres, each of angles at base 75°.
Base 70 millimetres, each of angles at base 30°.
Base 3 in., each of angles at base 45°.

. . . .

In each case compare the magnitudes of the sides adjacent to the equal angles.

The result of our observations is that **if two angles of a triangle are equal, the sides opposite to these angles are also equal.**

In the case of each of the above triangles measure the size of the angle at the top, or **vertex,** of the triangle, and find the total number of degrees in the three angles of each triangle.

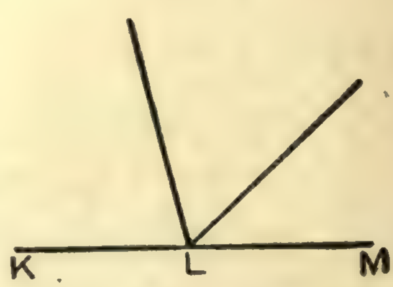


4. Take a line **AB** of length 35 millimetres, and with centres **A** and **B,** and radii 45 and 50 millimetres respectively, describe portions of circles, so that they intersect at **C.** Join **CA** and **CB.** We have thus a triangle **CAB** whose sides are unequal, called a **scalene** triangle.

Construct the following triangles:

With sides 3, 5 and 6 inches.
With sides 70, 80 and 100 millimetres.
With sides $3\frac{1}{2}$, $4\frac{1}{2}$ and $2\frac{1}{2}$ inches.

. . . .



With the bevel lay off three angles adjacent to one another, equal to the angles of each triangle, in the way indicated in the adjacent figure; and determine the positions of the initial and final lines, **LM, LK,** with respect to one another.

What conclusion do you draw as to the total number of degrees in the three angles of each of these triangles?

Can you construct a triangle with sides of 30, 50 and 90 millimetres, or with sides of 2, 3 and 6 inches? Attempt the construction.

What relation must exist between the given lengths, that a triangle may be constructed with sides of such lengths?

Exercises.

Teachers are advised to have their classes work but a few of the exercises at the close of each chapter. The time of pupils should be chiefly occupied in verifying the geometric truths reached in the text.

1. At a given point in a straight line construct an angle of 60°, using only compasses and ruler.

2. Construct an isosceles triangle, and produce the base both ways. What do you note as to the magnitudes of the exterior angles so formed?

3. Construct a triangle with sides 30, 50, 70 millimetres. With the bevel or protractor determine which is the greatest angle and which the least.

4. The angle at the vertex of an isosceles triangle is 75°, and each of the equal sides is 2 inches. Construct the triangle.

5. At A in the line AB construct the angle BAD of 40°, and at B the angle ABC of 120°. Produce AD, BC to meet. Measure the size of the third angle of this triangle. Which is the greatest side and which the least?

6. On one side of BC describe an equilateral triangle ABC, and on the other side of BC describe an isosceles triangle DBC. Join AD. Take a number of points E, F, G, . . . in AD. What do you note as to the lengths of EB and EC; of FB and FC; of GB and GC, . . . ?

7. Make the same construction as in the preceding question, but with the isosceles triangle on the same side of BC as the equilateral. Produce AD both ways. What again do you note as to the distances of any point in AD, or AD produced, from B and C?

8. On BC describe an equilateral triangle ABC, and on the other side of BC describe a scalene triangle DBC. Join AD. Take a number of points E, F, G, . . . in AD. What do you note as to the lengths of EB and EC; of FB and FC; of GB and GC, ... ?

9. Repeating the figure of 6, take in BC, and on the same side of AD, a number of points K, L, M, N. . . . What do you note as to the lengths of AK, AL, AM, AN, . . . ? Do they seem to follow any law as to magnitude?

10. Describe an equilateral triangle ABC. On BC describe an equilateral triangle DBC; on CA an equilateral triangle ECA; and on AB an equilateral triangle FAB. Join AD, BE, CF. What do you observe as to the positions of the lines DC, CE with respect to one another; of EA, AF; and of FB, BD?

11. In the preceding question mark all the angles that are equal to one another; also all the lines that are equal to one another.

What triangles are isosceles?

Do you observe any equilateral four-sided figures?

How many equilateral triangles are there?

12. With centre A, outside a straight line, describe a circle of such radius as to cut the line in two points, B and C. What sort of triangle is ABC?

13. In the figure of the preceding question find on the side of BC remote from A, a point D, such that a circle with D as centre can be described to pass through both B and C.

14. B and C being two points in a line, find on either side of the line points K, L, M, N, . . . such that a circle may be described, with any one of them as centre, to pass through B and C. What do you observe as to the positions of K, L, M, N, . . . with respect to one another?

15. Construct a scalene triangle ABC, and on the side of BC away from A, describe a triangle DBC, with DB = AB, and DC = AC. Join AD. What triangles in the figure are isosceles? What inference can you draw as to the angles BAC, BDC? Is any line in the figure bisected? What are the angles at the intersection of BC and AD? (Apply set-square.)

16. Construct a scalene triangle ABC, and on the side of BC remote from A, describe a triangle DBC with DB = AC, and DC = AB. Join AD. How do AD and BC appear to divide each other?

Repeat the construction several times with different pairs of triangles, and note whether the same peculiarity of division occurs in each case.

17. Construct a scalene triangle ABC. On the other side of BC construct DBC with DB = AB, and DC = AC; on the other side of AC construct EAC with CE = CB, and AE = AB; on the other side of AB construct FAB with BF = BC, and AF = AC. Join AD, BE, CF. What lines in the figure are bisected? What triangles are isosceles? What angles are right angles? How many right-angled triangles are there?

18. Construct a triangle ABC (BC = 47, CA = 40, AB = 27 millimetres). On the other side of BC construct DBC with DB = AC, and DC = AB; on the other side of AC construct EAC with EC = AB, and EA = BC; on the other side of AB construct FAB with FA = BC, and FB = AC. Join AD, BE, CF. What are the positions of DC, and CE with respect to each other; also EA, AF; and FB, BD?

What lines in the figure are bisected? Has any line the third part cut off? Has any line the sixth?

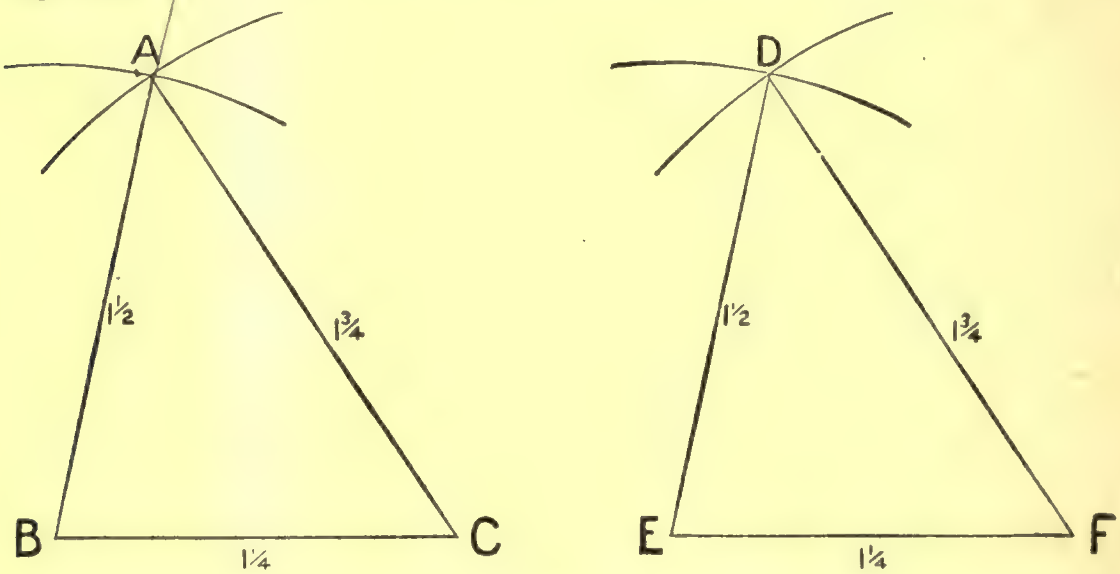
19. If two sides of a triangle are unequal, the angles opposite to them are unequal. (Suppose the angles equal and prove an absurdity.)

20. If two angles of a triangle are unequal, the sides opposite to them are unequal.

CHAPTER III.

Equality of Triangles.

1. Construct two triangles, each with sides of lengths $1\frac{1}{4}$, $1\frac{1}{2}$ and $1\frac{3}{4}$ inches, as indicated in the adjacent figures.



Adjust the bevel, or the protractor, to the angle **A**, and also to the angle **D**, and carefully compare the magnitudes of these angles. In like manner compare the magnitudes of the angles **B** and **E**, and also the magnitudes of the angles **C** and **F**.

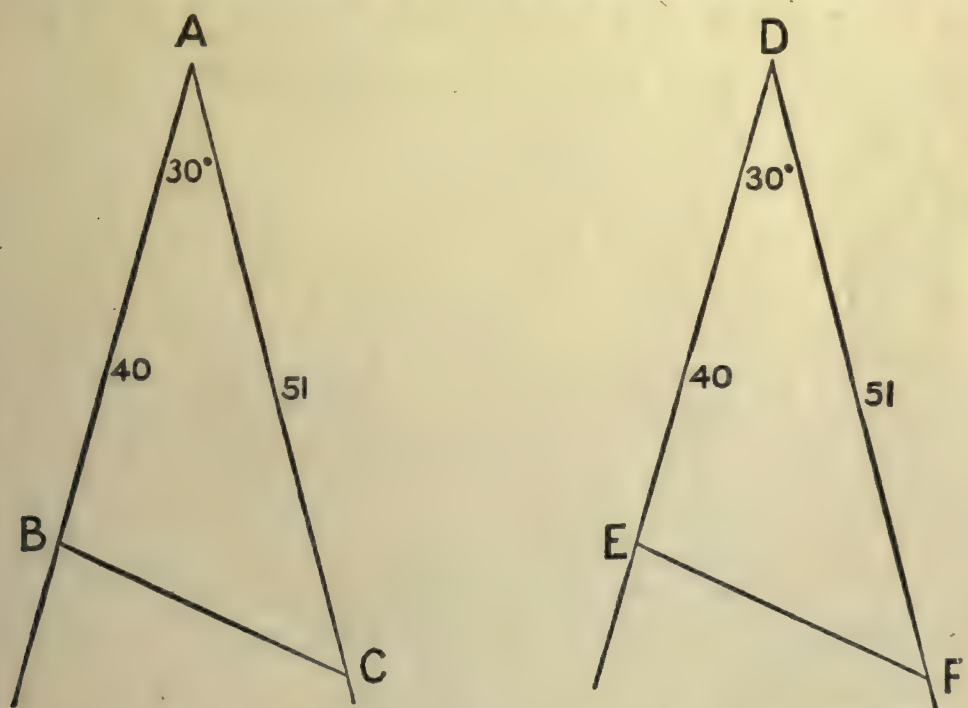
Next cut both triangles from the paper, and place one triangle upon the other so that the corresponding angular points coincide. From this superposition what conclusion do you draw as to the areas of the triangles?

Repeat the same construction, measurement, and superposition with two triangles whose sides are 4, 2 and $4\frac{1}{2}$ inches; with two whose sides are 50, 80 and 100 millimetres; etc.

The result of our observations in these cases is that **if two triangles have their sides equal, the angles which are opposite to equal sides are equal, and the areas are equal.** In other words two such triangles are the same triangle in different positions.

Another way of stating the fact is to say that if the sides of a triangle are fixed, the angles are fixed, and the area is fixed.

2. Construct two angles, **BAC** and **EDF**, each of 30°. On sides of these angles measure off distances **AB** and **DE**, each of length 40 millimetres; and also distances **AC** and **DF**, each of length 51 millimetres. Join **BC** and **EF**, thus forming two triangles, **ABC** and **DEF**.



Adjust the bevel, or protractor, to the angle **B**, and also to the angle **E**, and carefully compare the magnitudes of these two angles. In like manner compare the magnitudes of the angles **C** and **F**. With the dividers compare the magnitudes of the sides **BC** and **EF**.

Further, cut one triangle from the paper, and place it upon the other. From this superposition what conclusion do you draw as to the areas of the two triangles?

Repeat the same construction, measurement, and superposition with the following triangles:

Two whose sides are $1\frac{3}{4}$ and $2\frac{1}{4}$ inches, and included angle 30°.

Two whose sides are 30 and 110 millimetres, and included angle 78°.

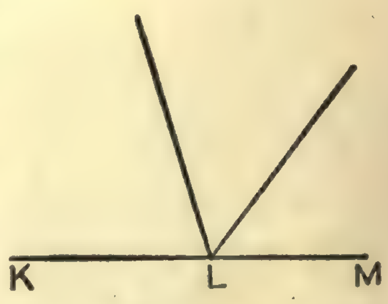
Two whose sides are $1\frac{1}{2}$ and 2 inches, and included angle 135°

. . . .

The result of our observations in all these cases is that **if two triangles have two sides in each equal, and the angles included by these two sides equal, then the remaining sides are equal, and the angles opposite to equal sides are equal, and the triangles are equal in area.** In other words two such triangles are the same triangle in different positions.

Another way of stating the fact is to say that if two sides and the included angle of a triangle are fixed, the remaining side and angles are fixed, and the area is fixed.

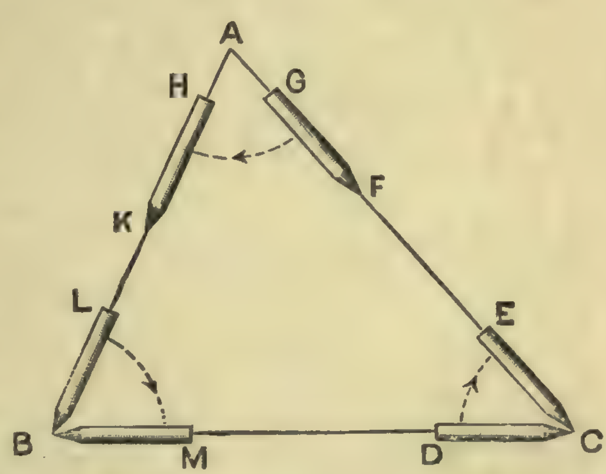
3. In the case of all the triangles in 1 and 2, lay off, with the bevel, three angles adjacent to one another, equal to the three angles of each triangle, in the way indicated in the figure. Determine the positions of the initial and final lines, LM and LK, with respect to one another.



The result of such an examination will be found to be that the lines **KL** and **LM** are in the same straight line, *i.e.*, **the sum of the three angles in each of these triangles is two right angles, or** 180°.

4. It is proposed to show that the sum of the three angles of any triangle must be two right angles, or 180°:

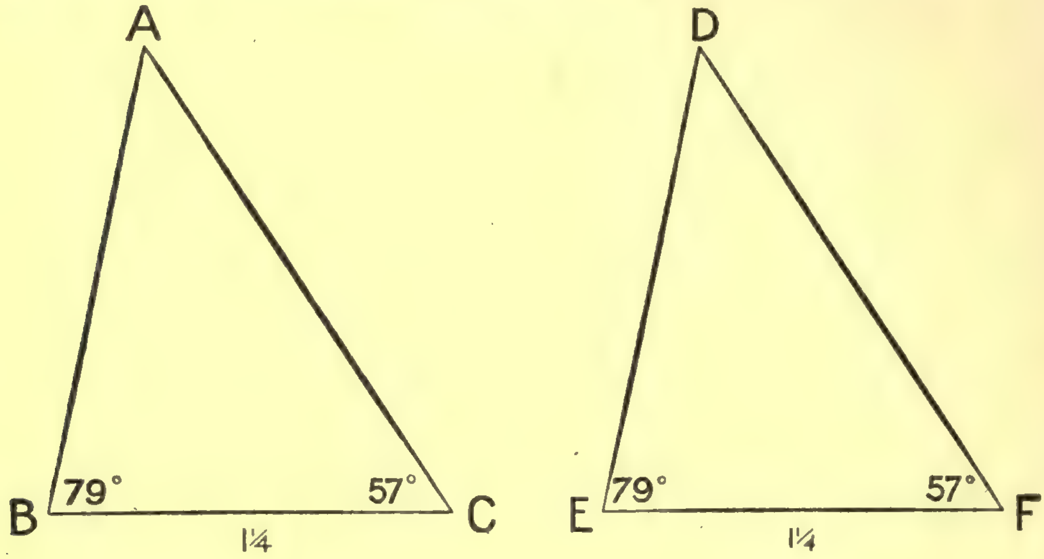
Construct a triangle **ABC**, and place a pencil in the position **DC**. Turn the pencil through the angle **BCA**, in the direction indicated by the arrow head, to the position **EC**. Slide it along **CA**, towards **A**, to the position **FG**, and turn it through the angle **CAB**, to the position **HK**. Slide it along **AB** to the position **BL**, and turn it through the angle **B**, to the position **BM**.



The pencil has rotated through all the angles of the triangle. But in its final position **BM** it points in a direction just opposite to its first position **DC**, and therefore must have rotated through 180°. Hence **all the angles of this (which is any) triangle must together equal** 180°, **or two right angles.**

It follows that **if two triangles have two angles in the one equal to two angles in the other, the third angle in one triangle is equal to the third angle in the other.**

5. Construct two triangles, **ABC, DEF,** each with base $1\frac{1}{4}$ inch, and angles at the base 79° and 57°.



It follows, from 4, that the remaining angles at **A** and **D** are equal, each being 44°. Putting the points of the dividers on **A** and **B,** and carrying the dividers, so adjusted, to **DE,** compare the magnitudes of **AB** and **DE.** In like manner compare the magnitudes of **AC** and **DF.**

Next, cutting one of the triangles from the paper, place it upon the other. From this superposition what conclusion do you draw as to the areas of the triangles?

Repeat the same construction, measurement, and superposition with the following triangles:

Two whose bases are $1\frac{3}{4}$ in., and angles adjacent to base 38° and 110°.

Two whose bases are 90 millimetres, and angles adjacent to base 89° and 57°.

Two whose bases are $3\frac{1}{2}$ in., and angles adjacent to the base 49° and 95°.

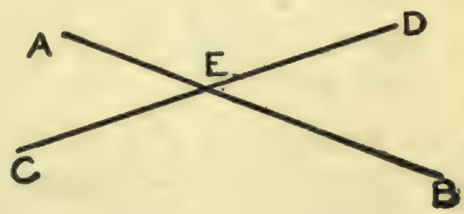
. . . .

The result of our observations in all these cases is that **if two triangles have their bases equal, and angles adjacent to the bases equal, the remaining angles are equal, and the sides opposite to equal angles are equal, and the areas are equal.** In other words they are the same triangle in different positions.

Another way of stating the fact is to say that if a side of a triangle and the angles adjacent to this side are fixed, then the remaining angle and sides are fixed, and area is fixed.

6. The following fact, demonstrated in Chapter VI., may be of service in connection with the succeeding exercises:

The vertically opposite angles AEC and BED are equal; and also the vertically opposite angles AED and BEC.



Exercises.

In numerical exercises, such as the first twelve, the teacher should solve the triangles by the usual trigonometrical formulæ, that he may inform the class as to the closeness of their approximations reached by instrumental methods.

1. The sides of a triangle are 35, 52 and 63 millimetres. Construct the triangle; and with the protractor measure the angles to the nearest degree.

2. The sides of a triangle are 36, 48 and 60 millimetres. Construct the triangle; and with the protractor measure the angles to the nearest degree.

3. The sides of a triangle are 66, 90 and 31 millimetres. Construct the triangle; and measure the angles to the nearest degree.

4. Two sides of a triangle are $2\frac{1}{2}$ and $1\frac{1}{2}$ inches, and the included angle is 47°. Construct the triangle; and measure the remaining side to the nearest sixteenth of an inch, and the remaining angles to the nearest degree.

5. Two sides of a triangle are 50 and 68 millimetres, and the included angle is 94°. Construct the triangle; and measure the remaining side to the nearest millimetre, and the remaining angles to the nearest degree.

6. Two sides of a triangle are $5\frac{1}{2}$ and $6\frac{1}{2}$ inches, and the included angle is 54°. Construct the triangle; and measure the remaining side to the nearest sixteenth of an inch, and the remaining angles to the nearest degree.

7. Two angles of a triangle are 55° and 65°, and the side adjacent to them is 27 millimetres. Construct the triangle; and measure the remaining angle to the nearest degree, and the remaining sides to the nearest millimetre.

8. Two angles of a triangle are 107° and 27°, and the side adjacent to them is 50 millimetres. Construct the triangle; and measure the remaining angle to the nearest degree, and the remaining sides to the nearest millimetre.

9. Two angles of a triangle are 53° and 66°, and the side adjacent to them is 4 inches. Construct the triangle; and measure the remaining angle to the nearest degree, and the remaining sides to the nearest sixteenth of an inch.

10. The sides of a triangle are 4, 6 and 7 inches. Construct the triangle; and measure the angles to the nearest degree.

11. Two sides of a triangle are 90 and 70 millimetres, and the included angle is 58°. Construct the triangle; and measure the remaining side to the nearest millimetre, and the remaining angles to the nearest degree.

12. Two angles of a triangle are 30° and 128°, and the side adjacent to them is $2\frac{1}{2}$ inches. Construct the triangle; and measure the remaining angle to the nearest degree, and the remaining sides tc the nearest sixteenth of an inch.

13. Two lines AB and CD intersect in E, and, with the dividers, AE and EB are taken equal to one another, and also CE and ED equal to one another. Join AC, CB, BD, DA. What lines, angles and triangles are equal to one another? Give proof.

14. A triangle ABC is described, and on the other side of BC the triangle DBC is constructed with DB=AB and DC=AC. AD is joined. What lines, angles and triangles are equal to one another? Give proof. What angles are right angles?

15. A triangle ABC is described, and on the other side of BC the triangle DBC is constructed with DB=AC and DC=AB. AD is joined. What lines, angles and triangles are equal to one another? Give proof.

16. A triangle ABC is described, and on the same side of BC another triangle DBC is described with DB=AC and DC=AB. AD is joined. What lines, angles and triangles in the figure are equal? Give proof.

If BA and CD be produced to meet in E, what are the triangles EAD, EBC? Give reasons.

17. From two lines diverging from A, equal lengths AB, AC are cut off, and also equal lengths AD, AE. CD, BE, BC and DE are joined. What lines, angles and triangles in the figure are equal? Give proof.

18. Two circles have the same centre O. AOB is a diameter of one, and COD a diameter of the other. AC and BD are joined. What lines, angles and triangles in the figure are equal?

19. Equal lines AB, AC are drawn, making equal angles with AE on opposite sides of it. At B and C equal angles ABF, ACG are constructed towards the same side. If AE, BF and CG be produced, will they hit the same point? Give proof.

20. Describe ABC, DBC, two isosceles triangles on the same base BC, but on opposite sides of it. How does AD divide the angles BAC, BDC? Give proof.

21. With centres A and B two circles are described, intersecting at C and D. How are the angles CAD, CBD divided by AB? How is CD divided by AB? What are the angles at the intersection of AB and CD? Give proof.

22. Construct an equilateral triangle ABC. At B and C construct equal angles GBC, GCB. Join AG. How does AG divide the angle BAC? Give proof.

23. With O as centre describe a circle, and, with the dividers, take three points on the circumference, A, B, C, such that the chords AB, BC are equal. How does OB divide the angles ABC, AOC? How does OB divide AC, and what are the angles at the point of intersection? Give proof.

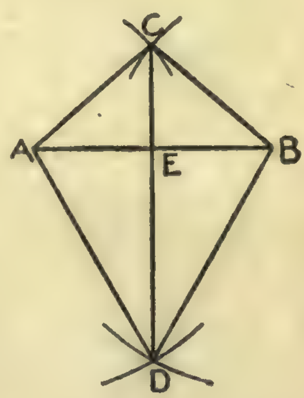
24. ABC is any triangle. The side BC is produced to D, CA to E, and AB to F. How many degrees are there in the sum of the angles ACD, BAE, CBF? Verify by measurement and addition.

CHAPTER IV.

Bisection of Lines and Angles. Perpendiculars.

1. To bisect a straight line.

Suppose **AB** the line to be bisected. With **A** and **B** as centres describe portions of circles with equal radii intersecting at **C**, and with the same centres describe portions of circles with equal radii, intersecting at **D**. Then if **CD** be drawn, it bisects **AB** at right angles.



For, using the dividers, it will be found that **AE** and **EB** are equal; and, using the protractor or set-square, all the angles at **E** will be found to be 90°.

Or again, we would conclude that **AE** and **EB** are equal, and that the angles at **E** are right angles, from the **symmetry** of the figure with respect to the line **CD**—the figure on one side of this line being just the same as the figure on the other side, but turned in the opposite direction.

Or again, we may "reason out" the equality of **AE** and **EB**, and that the angles at **E** are right angles, as follows: Since the triangles **ACD, BCD** have their sides equal, they are equal in all respects (Ch. III., 1). Hence the angles at **C** are equal; also the sides about these angles, **AC, CE,** and **BC, CE,** are equal; therefore (Ch. III., 2) the triangles **ACE** and **BCE** are equal in all respects. Hence **AE** is equal to **BE**; also the angle **AEC** is equal to the angle **BEC**; therefore each is 90°.

In practice it is not necessary to draw the lines **AC, BC, AD, BD, CD.** Having found the points **C** and **D,** placing the ruler on these points, we may mark the point **E** in **AB.**

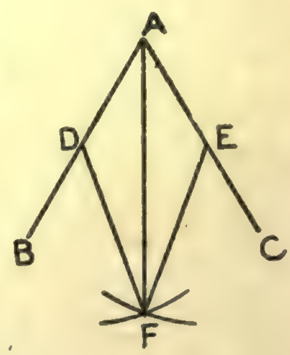
Subsequently, when the subject of parallel lines comes to be dealt with, another and possibly readier way of finding the middle point of a line will be given.

A number of exercises should now be given in bisecting lines of different lengths, the dividers being used in each case to determine whether the point reached is the middle point.

It is suggested that the pupil be given exercises in estimating with the eye the middle points of lines of various lengths, these points being afterwards accurately determined by geometrical construction.

2. To bisect an angle.

Let **BAC** be the angle. Place one of the points of the dividers or compasses at **A,** and mark off equal lengths **AD, AE** in **AB** and **AC.** With centres **D** and E describe portions of circles with equal radii, intersecting at **F.** Then drawing **AF,** the angle is bisected by it.



For, adjusting the bevel to either of the angles at **A,** it will be found equal to the other.

Or again, we would conclude that the angles at **A** are equal from the symmetry of the figure with respect to the line **AF**—the figure on one side of this line being just the same as the figure on the other side, but turned in the opposite direction.

Or again, we may prove the equality of the angles as follows: The triangles **DAF, EAF** have their sides equal. Hence (Ch. III., 1) the angles **DAF, EAF** are equal.

In practice it is not necessary to draw the lines **DF, EF.**

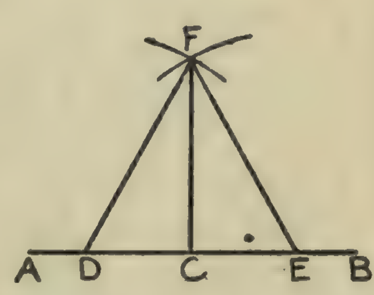
A number of exercises should be given in bisecting angles of various magnitudes, the bevel being used in each case to determine whether the bisection is accurate.

The protractor may also be used for bisecting angles.

It is suggested that the pupil be given exercises in estimating with the eye the bisecting lines of a number of angles, the bisection being afterwards accurately reached by geometrical construction.

Greater accuracy is likely to be secured in bisecting an angle, by making **AD, AE** and **DF, EF** of considerable length. The point **F** is then remote from **A,** and any trifling error in locating the exact point where the circles intersect, has less effect on the angle at **A** through being on the circumference of a large circle (radius **AF**).

3. From a point in a line to draw a line at right angles to it.



If **C** be the point in **AB** from which the perpendicular is to be drawn, place one point of the dividers or compasses at **C,** and mark off equal lengths **CD** and **CE.** Then with centres **D** and **E** describe portions of circles with equal radii, intersecting at **F.** Draw **FC**: it is perpendicular to **AB.**

For, applying the set-square or protractor, the angles at **C** will be found to be right angles.

Or again, from the symmetry of the figure with respect to **CF**, we may conclude that the angles at **C** are right angles.

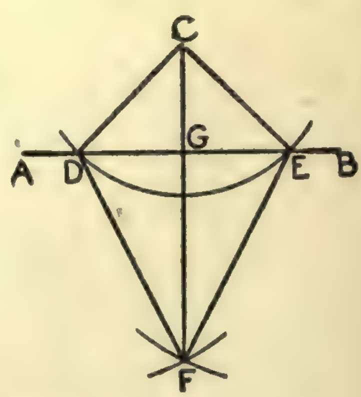
Or again, since the sides of the triangles **DCF**, **ECF** are equal, therefore (Ch. III., 1) these triangles are equal in all respects, and the angles at **C** are equal. Hence the angles at **C** are right angles.

In practice the lines **FD** and **FE** need not be drawn.

A number of exercises should be given in drawing lines at right angles to others from points in the latter, the correctness of the constructions being tested by using the set-square or protractor.

In future, in the various constructions that are to be made, where a line is to be drawn at right angles to another from a point in the latter, the set-square or protractor should in general be used instead of this construction.

4. To draw a line perpendicular to another from a point without the latter.



Let **C** be the point without **AB** from which the perpendicular is to be drawn to **AB**. With **C** as centre describe a circle cutting **AB** in **D** and **E**. With **D** and **E** as centres describe portions of circles with equal radii, intersecting at **F**. Join **CF**, cutting **AB** in **G**. **CG** is the perpendicular from **C** on **AB**.

For, applying the set-square or protractor, the angles at **G** will be found to be right angles.

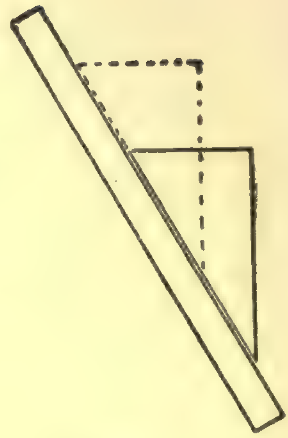
Or again, from the symmetry of the figure with respect to **CF**, we may conclude that the angles at **G** are right angles.

Or again, since the sides of the triangles **DCF**, **ECF** are equal, therefore, (Ch. III., 1) the angles at **C** are equal. Also since in the triangles **DCG**, **ECG** the angles at **C** are equal, and the sides about these angles equal, therefore (Ch. III., 2) these triangles are equal in all respects, and the angles at **G** are equal. Hence the angles at **G** are right angles.

In practice the lines **CD**, **CE**, **FD**, **FE**, **GF** need not be drawn.

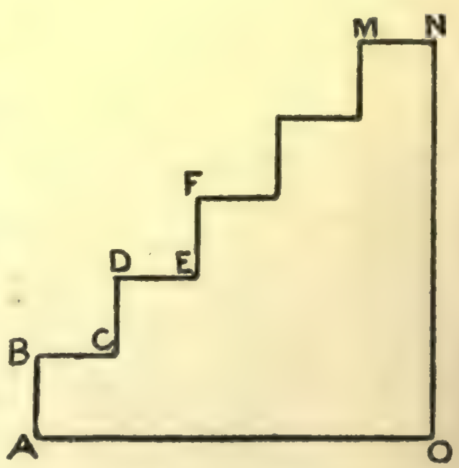
A number of exercises should be given in drawing lines perpendicular to others from points without the latter, the correctness of the constructions being tested by using the set-square or protractor.

5. In future, where a line is to be drawn perpendicular to another from a point without the latter, the set-square or protractor should in general be used instead of the preceding construction. When for this purpose the protractor is used, the edge of the ruler is to be placed over the centre-point of the protractor and over the 90° mark; the base of the protractor is then to be slid along the line until the edge of the ruler is over the given point without the line. The centre-point of the protractor then marks the foot of the perpendicular on the line.



The annexed diagram illustrates how, by sliding the set-square along the ruler, lines may be drawn parallel to each other; and also how a line may be drawn perpendicular to another from any point, whether the point be without or on the latter line. In drawing a perpendicular to a line by placing an edge of the right angle of set-square against the latter, we are often unable to bring the perpendicular up to the line by reason of the right angle of the set-square having become rounded through use.

Sometimes a convenient way of drawing a perpendicular through a point is to draw a perpendicular in any position, and then a parallel to this through the given point, the former perpendicular being afterwards erased.



Draw a number of equal and perpendicular lines **AB, BC, CD,** . . . . . **MN,** and finally draw **AO** perpendicular to **AB,** and **NO** perpendicular to **MN,** as indicated in the figure. The accuracy of the series of constructions may be tested by the conditions that **AO** is both equal and perpendicular to **NO.**

Exercises.

1. What is meant by the distance of a point from a line?

2. AOB is any angle and OC bisects it. What do you observe as to the distances of any point in OC from OA and OB? Give proof.

What do you observe as to the angles which OC makes with these distances to OA and OB? Give proof.

3. AB and CD intersect in O. Bisect the angles AOC and BOD by OE and OF. What position do OE and OF occupy with respect to each other? Give proof.

Bisect the angle AOD by OG. What position do OG and OE or OF occupy with respect to each other? Give proof.

4. Construct a triangle ABC, and find a point in the base BC such that the perpendiculars from it on AB and AC are equal.

5. Taking any two points, A and B, in the plane of the paper, draw a line such that the distances from any point on it to A and B are equal.

6. Find a point equidistant from two given points A and B, and one inch from a third given point C. Is it always possible to do this?

7. In a given straight line find a point which is equidistant from two given points not lying in the line.

8. Take two points, A and B, in the plane of the paper, and describe a circle of radius two inches which shall pass through A and B.

9. Construct a triangle ABC with sides 4, $3\frac{1}{2}$ and 3 inches. Bisect the angles ABC, ACB by BD, CD meeting in D. What do you observe as to the distances of D from the three sides? Give proof.

10. Construct a triangle ABC. Bisect AB in D, and AC in E. Join DE, producing it both ways. In the base BC, or BC produced, take points F, G, H, . . . , and join AF, AG, AH . . . . . What do you observe as to the division of the lines AF, AG, AH . . . by DE or DE produced?

11. Draw two straight lines AB, AC, and with the set-square draw any two lines at right angles to them. What relation do you observe between the acute angle between the lines and the acute angle between the perpendiculars? Give proof.

12. Construct a triangle one of whose angles is a right angle. How many degrees do you find in the sum of the other two angles? Give reason.

13. Construct a triangle ABC with C a right angle. At C make the angle BCD equal to the angle CBA, CD meeting AB in D. What do you note as to the magnitudes of the lines DA, DB, DC? Give reason.

14. ABC is an isosceles triangle with AB = AC. Produce BA to D, making AD equal to AB or AC. Join CD. What is the magnitude of the angle BCD? Give reason.

15. AB and CD are any two straight lines. Find a point E such that EAB and ECD are both isosceles triangles.

16. With ruler and compasses (*i.e.*, not using set-square) draw from a point at the extremity of a given line another line at right angles to it, without producing the given line.

17. Construct an equilateral triangle ABC, with side two inches. Draw AD to the bisection of the base BC. How many degrees are there in each of the angles of the triangle ABD?

18. Construct an equilateral triangle ABC, and draw AD perpendicular to the base BC. On AD describe another equilateral triangle EAD. How many degrees does each of the sides of EAD make with each of the sides of ABC?

19. At the points A and B in the line AB draw equal lines AC, BD at right angles to AB and on the same side of it. Join CD, and produce it and AB both ways. From other points E, F, G . . . in AB draw perpendiculars EK, FL, GM . . . . to CD. Compare the lengths of EK, FL, GM . . . with AC or BD. What are the angles at E, F, G . . . ?

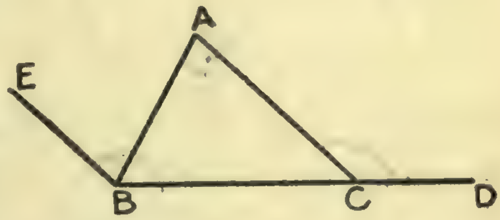
20. Construct a triangle ABC, and bisect the sides BC, CA, AB at D, E and F respectively. Join DE, EF, FD. What relations exist between the lengths of the sides of the triangles ABC and DEF? What relations exist between the angles of the two triangles? Make three different figures, the triangles being of different shapes, and examine whether the same relations exist in the three cases.

CHAPTER V.

Respecting Angles of a Triangle.

1. We have seen (Ch. III., 4) that the sum of the three angles of any triangle is two right angles, or 180°.

Definition: In any rectilineal figure, an *exterior angle* is an angle contained by any side and an adjacent side produced.

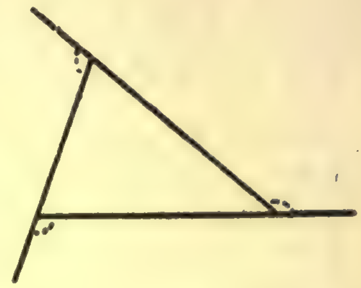


Produce the side **BC** to **D.** With the bevel or protractor lay off the angle **ABE** equal to the angle at **A.** Using the bevel, examine now the relation existing between the angle **EBC,** which is equal to the sum of the angles **A** and **B** of the triangle, and the *exterior* angle **ACD.**

Repeat the construction and examination in the case of a triangle of different shape, say one in which the angle at **B** is an obtuse angle.

If the constructions and measurements have been accurately made, it will be found that **the exterior angle (ACD) is equal to the sum of the two interior and opposite angles at A and B.**

We may show that this is always the case as follows: The three angles of the triangle make up 180°; but the angles **ACB, ACD** also make up 180°; hence the angle **ACD** must be equal to the sum of the angles **A** and **B.**

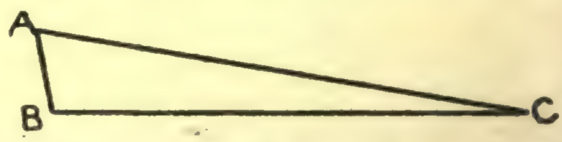


2. Lay off about a point, and adjacent to one another, angles equal to the three exterior angles of the triangle; or, with the protractor, measure the number of degrees in each of these angles. What is their sum? Give reasons for this sum being what it is.

3. Of course, since the exterior angle of any triangle is equal to the sum of the two interior and opposite angles, it follows that **the exterior angle of any triangle is greater than either of the interior and opposite angles.**

Also, since the sum of the three angles of any triangle is equal to two right angles, **the sum of any two angles of a triangle is less than two right angles.**

We may, however, by elongating the triangle, make the sum of two of its angles but little short of two right angles. Thus the sum of **A** and **B** will be found, on using the protractor, to be but little less than 180°; and, by still further removing **C,** we may still further increase their sum.

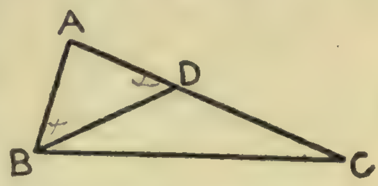


4. Construct a triangle with sides 50, 70 and 90 millimeters; and, by adjusting the bevel to the angles, find out which is the greatest angle, which is next in magnitude, and which is least.

Repeat the same examination in the case of the triangle whose sides are 2, 4 and 5 inches.

What position do you observe the greatest, intermediate and least angles occupy with respect to the greatest, intermediate and least sides respectively?

We shall find for all triangles a definite answer to the preceding question in the following proof: Let **AC** be greater than **AB**, and let **AD** be equal to **AB**. Then (Ch. II., 2) the angles **ABD** and **ADB** are equal. But the angle **ABC** is greater than **ABD**, and the angle **ACB** is less than **ADB** (Ch. V., 3). Therefore the angle **ABC** is greater than the angle **ACB**. That is, **in any triangle, the greater side has the greater angle opposite it.**

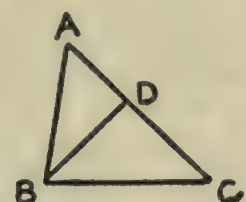


5. Construct a triangle with angles 40°, 60°, 80°, and, using the dividers or compasses, arrange the sides in order of magnitude.

Make the same examination in the case of a triangle whose angles are 100°, 50°, 30°.

What position do you observe the greatest, intermediate and least sides occupy with respect to the greatest, intermediate and least angles respectively?

We shall find for all triangles a definite answer to the preceding question in the following proof: Let the angle **ABC** be greater than the angle **ACB**; then the side **AC** is greater than the side **AB**. For, with bevel or protractor, construct the angle **CBD** equal to the angle **ACB**, so that **DBC** is an isosceles triangle. Then **AC** = **AD** + **DC** = **AD** + **DB** > **AB**, the straight line **AB** being the shortest distance between **A** and **B**. Hence **in any triangle the greater angle has the greater side opposite to it.**



Exercises.

1. Construct a quadrilateral figure. With the protractor measure the number of degrees in each of the angles, and add them. What is the sum? Deduce this sum also from geometrical truths already reached.

2. Produce the sides of the quadrilateral, and measure the exterior angles. What is their sum? Deduce this also from knowing the sum of the interior angles.

3. Construct a polygon with any number of sides, ABCDE . . . . Taking the sides in order, produce each from the preceding angle, as in the figure of 2, Ch. V. Placing your pencil along AB, turn it through the exterior angle at B into coincidence with BC; then through the exterior angle at C; and so on, until it has been turned through all the exterior angles.

How much has the pencil been turned? What, therefore, do you conclude the sum of all the exterior angles of any polygon is?

Verify this by measurement with protractor.

4. From the result reached in the previous question, show that all the interior angles of any polygon are equal to twice as many right angles as the figure has angles (or sides), less four right angles.

5. How many right angles is the sum of all the angles in a pentagon (5 sides) equal to? If the angles be equal, how many degrees are there in each?

6. How many right angles is the sum of all the angles in a hexagon (6 sides) equal to? If the angles be equal, how many degrees are there in each?

7. Construct an isosceles triangle ABC (AB = AC). In AB take any point D. With the dividers or compasses determine whether D is nearer to B or to C. Give reason.

8. ABCD is a right-angled equilateral four-sided figure. AC is joined. Any point E is taken within the triangle ABC. Is E nearer to B or to D? Give reasons.

9. A triangle can have only one angle either equal to or greater than a right angle, *i.e.*, at least two of the angles of a triangle must always be acute angles.

10. The perpendicular is the least line that can be drawn from a

given point to a given line; and any line nearer to the perpendicular is less than one more remote.

11. ABCD is a four-sided figure. How does the sum of the exterior angles at A and C compare in magnitude with either of the interior angles B or D? Give reasons.

12. ABC is a triangle, and O is a point within it. Is the angle BOC greater than, equal to, or less than the angle BAC? Give reasons.

13. Can more than two equal straight lines be drawn to a straight line from a point without it? Give reasons.

14. Use the result obtained in the previous question to show that a circle cannot cut a straight line in more than two points.

15. In a right-angled triangle the hypotenuse is the greatest side.

16. In the triangle ABC can you find a point D, such that AD is equal to or greater than the greater of the sides AB, AC?

17. In any triangle can you find a point such that the distance from it to any one of the angles is equal to or greater than the greatest of the sides?

18. Describe two circles with the same centre, *i.e.*, concentric. Take a point A on the circumference of one, and a point B on the circumference of the other. When will the line AB be least? Give reasons.

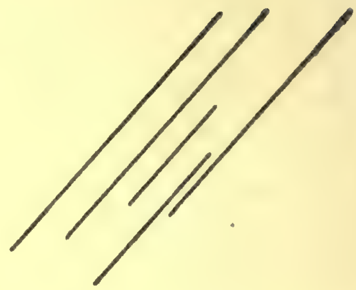
19. A, B, C are three points on a line, at any intervals apart. Rotate the line about A in a direction contrary to the motion of the hands of a clock through 30°; *i.e.*, draw a new line through A, making an angle of 30° with the original line, and locate B and C on it at same intervals as before. Rotate the line about B from its new position, in the same direction, through 20°. Rotate the line about C from its new position, in the same direction, through 15°. What angle does the line in its final position make with its original position?

20. The same problem as the preceding, there being, however, four angles, 45°, 60°, 30° and 90°.

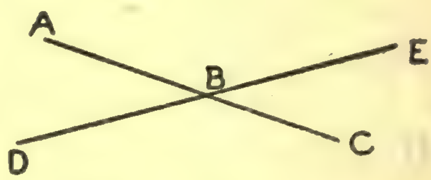
The point in the last two questions is that if a line rotates through various angles and about different points in it, the aggregate rotation is the same as if it all took place about a single fixed point in the line.

CHAPTER VI.

Parallel Lines.



1. Parallel lines were defined to be such as have the same direction. Thus the lines in the figure, though differing in position, have all the same direction, and are parallel.



2. **AC** and **DE** are straight lines. Using the bevel, what do you observe with reference to the magnitudes of the **vertically opposite** angles **ABD** and **CBE**? What with reference to the magnitudes of the angles **ABE, DBC**?

Draw other intersecting straight lines and note the magnitudes of vertically opposite angles.

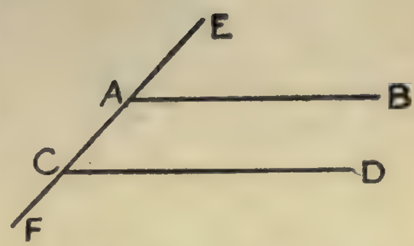
We may demonstrate the relation between such angles as follows:

$\angle ABD + \angle ABE = 2$ rt. angles $= \angle EBC + \angle EBA$, and dropping from both sides the angle **ABE**, we have $\angle ABD = \angle EBC$.

Hence **if two straight lines cut one another, the vertically opposite angles are equal.**

Yet such a proposition scarcely needs demonstration; for, as was said in Chapter I., a straight line has the same direction throughout its entire length. Hence the two lines **ABC, DBE** must deviate from one another as much to the left of **B** as to the right of **B**, and thus the angles **ABD, EBC** are equal.

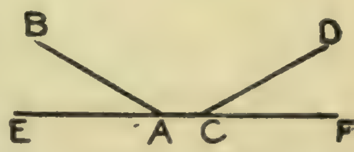
3. Straight lines which deviate from the same straight line by the same amount, *i.e.*, which make equal angles with this straight line in the same direction, must have the same direction, and therefore must be parallel.



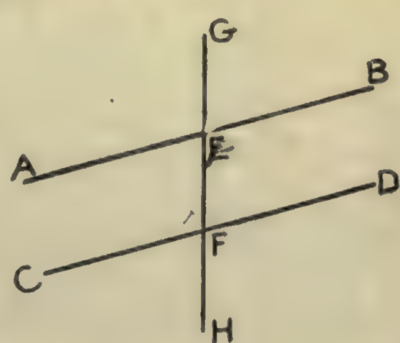
Thus if the directions, or lines, **AB** and **CD** deviate equally from the same direction, or line, **EF**, *i.e.*, if the angles **EAB** and **ACD** are equal, then **AB** and **CD** have the same direction, and are said to be parallel.

ACD is said to be the **interior and opposite** angle with respect to the **exterior** angle **EAB.**

It is to be noted that the parallel lines are inclined to the cutting line equally and *in the same direction.* Thus though **AB** and **CD** deviate equally from **EF**, they deviate in *opposite* directions, and therefore are not parallel.



4. It is understood, then, that if **AB** and **CD** are any two parallel lines, and any line **GH** cuts them, **the exterior angle GEB is equal to the interior and opposite angle EFD.**



5. The angles **AEF**, **EFD** are called **alternate** angles.

By actual measurement, with the bevel or protractor, show that they are equal.

We may also prove their equality thus:

$\angle$**GEB** = $\angle$**EFD**, because the lines are parallel; also $\angle$**GEB** = $\angle$**AEF**, because these are vertically opposite angles; $\therefore$ $\angle$**AEF** = $\angle$**EFD.**

6. The angles **BEF, EFD** are called **interior** angles.

By measurement with the protractor, or by laying off, with the bevel, two adjacent angles equal to them, show that the sum of **BEF** and **EFD** is 180°.

We may also prove this thus:

$$\angle \mathbf{GEB} = \angle \mathbf{EFD};$$

$$\therefore \quad \angle \mathbf{GEB} + \angle \mathbf{BEF} = \angle \mathbf{EFD} + \angle \mathbf{BEF}.$$

$$\text{But } \angle \mathbf{GEB} + \mathbf{BEF} = 2 \text{ rt. angles};$$

$$\therefore \quad \angle \mathbf{BEF} + \angle \mathbf{EFD} = 2 \text{ rt. angles}.$$

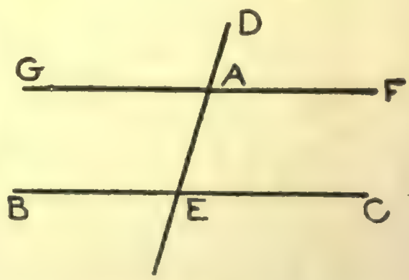
7. There is no difficulty in verifying by actual measurement, or in proving the following equalities:

$$\angle \mathbf{BEF} = \angle \mathbf{EFC}$$

$$\angle \mathbf{HFD} = \angle \mathbf{FEB}$$

$$\angle \mathbf{AEF} + \angle \mathbf{EFC} = 2 \text{ rt. angles}.$$

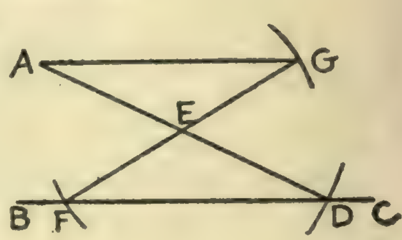
8. To draw a straight line through a given point **A** parallel to a given straight line **BC**.



Through **A** draw **DAE**, cutting **BC**, and make the angle **DAF** equal to the angle **AEC**. Then **AF** is parallel to **BC**. **FA** may then be produced, if necessary, to **G**.

Of course we could have drawn **GA** parallel to **BC** by making the angle **GAE** equal to the alternate angle **AEC**.

The line through **A** parallel to **BC** can also be drawn without measuring any angle, as follows:

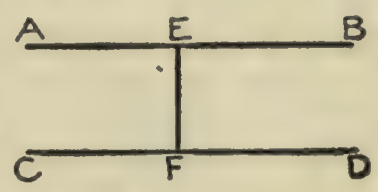


With **A** as centre and radius, say, of 2 inches, describe a portion of a circle cutting **BC** in **D**. Measure off on this a distance **AE** of 1 inch, so that **E** is the middle point of **AD**. With centre **E** and any radius of sufficient length to reach **BC**, describe a portion of a circle cutting **BC** in **F**; and let the diameter of this circle, through **F**, meet the circle again in **G**. Then **AG** is parallel to **BC**. For the sides **AE, EG** of the triangle **AEG** are equal to the sides **DE, EF** of the triangle **DEF**. Also the angles **AEG, DEF** are equal. Hence these triangles are equal in all respects, and the angle **GAE** is equal to the angle **EDF**. **AG** is therefore parallel to **BC**.

A number of exercises should be given pupils in drawing lines through given points parallel to lines in given positions, using both the preceding methods. At the end of each construction the accuracy of the work may be tested with the parallel rulers, or with ruler and set-square (Ch. IV., 5), or by examining whether lines drawn perpendicular to each pair of parallels are equal in length. (See 9 and 10, following.)

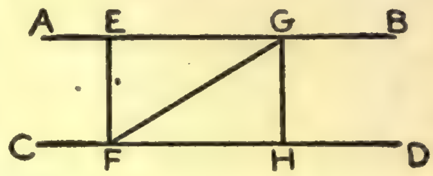
For the most part, in future, in drawing parallel lines parallel rulers are to be used, or ruler and set-square (Ch. IV., 5).

9. A straight line which is perpendicular to one of two parallel lines, is also perpendicular to the other. The truth of this should be tested by drawing with the set-square a line perpendicular to one of the parallels, and examining, with the set-square, whether it is also perpendicular to the other.



Of course this is only a particular case of the truth, that parallel lines have each the same direction with respect to any third line that cuts them.

Or we may *prove* it as follows: If **DFE** is a right angle, then since **DFE** + **FEB** = 2 rt. angles, **FEB** must also be a right angle.



10. Two parallel lines are, of course, throughout their lengths at the same distance from one another. For, with the set-square or protractor, draw lines **EF**, **GH**, . . . perpendicular to **AB** and **CD**. Then, if the dividers be adjusted to the length **EF**, they will be found to be adjusted to the other lengths **GH**, . . .

We may *prove* that this is always the case, as follows: **EF** and **GH** are parallel to one another because they have the same direction with respect to the third line **AB** (or **CD**).

Again, $\angle$**EGF** = $\angle$**GFH**, being alternate angles;
$\angle$**EFG** = $\angle$**HGF**, " " "
Side **FG** is common to the two triangles;
∴ (Ch. III., 5) **EF** = **GH**.

We have everywhere illustrations of this. Thus we say that an ordinary board or ruler, whose sides are parallel, is of the same *width* throughout its length.

11. The method of drawing a line parallel to another by sliding the set-square along the ruler (Ch. IV., 5) receives its justification in the first paragraph of § 3 of this chapter. The line **EACF** corresponds to the edge of the ruler; the lines **AB**, **CD** to the edge of the set-square in its two positions; and the angles **EAB**, **ECD** to the angle of the set-square in its two positions, the angle of the set-square being of course always the same.

It may be added that, in drawing parallel lines, some prefer the ruler and set-square to parallel rulers. The cost of an instrument is saved. If the edges of ruler and set-square are perfectly straight, the method gives absolutely correct results. Parallel rulers possibly work more rapidly and conveniently.

Exercises.

1. Draw a line through A parallel to a line BC, as follows: Join AB, AC. With B as centre, and radius equal to AC, describe a circle. With C as centre, and radius equal to AB, describe a circle. Let D be the point where the circles intersect on the same side of BC as A. Then AD is parallel to BC.

Test with parallel rulers.

Examine the equality of alternate angles.

Examine whether the sum of interior angles on the same side is 180°.

Prove that the alternate angles ADB, DBC are equal, and that, therefore, the lines are parallel.

2. If AB, CD intersect in O, and AO = OB, and CO = OD, what position do AD, CB occupy with respect to each other; and what do AC and DB? Apply tests with instruments. Give reasons, i.e., *proof.*

3. If AB, CD intersect in O, and AO = OD, CO = OB, what position do AD, CB occupy with respect to each other? Apply tests. Give reasons.

4. Construct a quadrilateral with two sides equal, and the other two parallel and unequal.

5. In the receding question produce the equal sides to meet, and by applying tests determine the character of the two triangles so formed.

6. The two interior angles on the same side which one line makes with two others are 105° and 70°. Infer from 6 of Ch. VI. that the lines meet. On which side of the cutting line, and why?

7. AD and BF are parallel lines. From A draw equal lines AB, AC to BF; and also equal lines DE, DF, less than the former. Show that AC meets DE and DF on one side of the parallel lines, and AB meets them on the other side.

8. A, B are the extremities of the diameter of a circle, and parallel lines AC, BD are drawn, terminated by the circle. What is the relation of AC, BD as to magnitude? Give reasons.

9. A is a point not lying in the straight line BC. From A draw lines AD, AE, AF . . . to BC, and produce them to K, L, M, . . .

making DK = AD, EL = AE, FM = AF, ... What do you observe as to the positions of the points K, L, M, . . ? Give reasons.

10. Two parallel lines are 3 inches apart, and a point A is taken 2 inches from one line and 1 inch from the other. Lines are drawn through A terminated by the parallels. By measurement determine how these lines are divided at A.

11. Construct a triangle with sides 2, 3 and 4 inches. Bisect the sides and join the points of bisection. What do you observe as to the direction of the sides of the new triangle? What as to magnitudes of its angles and sides?

12. Construct a triangle, and through its angular points, with the parallel rulers draw lines parallel to the opposite sides. Four new triangles are thus constructed. Compare their sides and angles with those of the original triangle, and give results of comparison.

13. Construct a triangle ABC, and through any points D, E, F in the plane of the paper draw lines parallel to BC, CA, AB. Compare the angles of the new triangle with those of the original.

14. If through a point A any two lines be drawn, and through any point B lines be drawn parallel to the former two, *prove* that the angles at A and B are equal.

15. ABC, CDE are two triangles with AB, CD equal and parallel, and also BC, DE equal and parallel. What position do AC, CE occupy with respect to each other?

16. Make an irregular drawing on the paper to represent a pond, or other obstruction, and on opposite sides of it take points A and B. By a line construction about the pond, with measurements, obtain a line at A which if produced would pass through B, without placing the ruler on AB.

17. Draw two lines, both parallel to the same straight line. What is their position with respect to each other?

18. The side BC of a triangle ABC is produced to D. Bisect the angles BAC, ACD. Can the bisecting lines be parallel to one another?

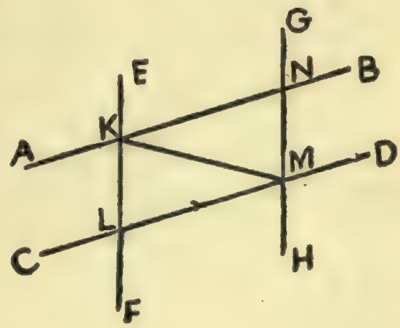
19. On any line AC as diagonal, construct a quadrilateral ABCD with its opposite sides equal. How are the opposite sides placed with respect to each other? Test and give proof.

20. Two lines make an angle of 63° with each other. Place a straight line 2 inches long with its ends resting on them, and making an angle of 80° with one of them.

CHAPTER VII.

Parallelograms, Rectangles and Squares.

1. With the parallel rulers, or by other means, draw a pair of parallel lines **AB, CD,** and also another pair of parallel lines **EF, GH,** inclined to the former pair at any angle.



The figure **KLMN** is called a **parallelogram,** *i.e.*, **a parallelogram is a four-sided figure whose opposite sides are parallel.**

With the dividers compare the lengths of **KL** and **NM**; also the lengths of **KN** and **LM.** With the bevel compare the magnitudes of the angles **NKL** and **NML**; also the magnitudes of the angles **KLM** and **KNM.**

Construct two or more parallelograms with sides of different lengths, and angles of different magnitudes; and in the case of each compare the magnitudes of the opposite sides and angles.

The result of such observations will be that **the opposite sides and angles of parallelograms are equal.**

We may also *prove* this as follows: Draw **KM,** the **diameter** or **diagonal,** as it is called, of the parallelogram. We have in the two triangles **NKM, LMK,**—

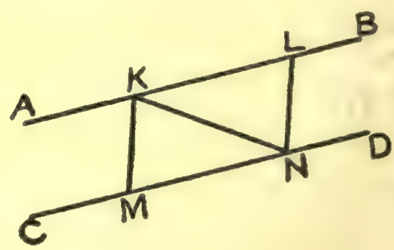
The side **KM** common to both,
the alternate angles **NKM, LMK** equal,
" " " **NMK, LKM** "

Hence (Ch. III. 5) these triangles are equal in all respects, *i.e.*,

$$KN = ML,$$
$$KL = MN,$$
$$\angle KNM = \angle KLM.$$
$$\text{Also } \angle NKM = \angle LMK,$$
$$\text{and } \angle LKM = \angle NMK;$$
$$\text{therefore adding } \angle NKL = \angle NML.$$

Of course the triangle **KNM,** if cut out, can be fitted on the triangle **MLK,** and is equal to it, *i.e.*, **the diagonal of a parallelogram bisects it.**

2. Draw a pair of parallel lines, **AB** and **CD**. In **AB** take any length **KL,** and, adjusting the dividers to it, in **CD** mark off an equal length **MN.** Join **K, M** and **L, N.**



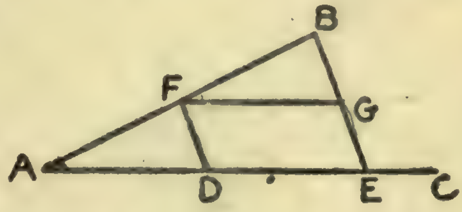
Using the dividers, what do you note with reference to the lengths of **KM** and **LN**? Using the bevel or parallel rulers, what do you note with reference to the position of **KM** and **LN** with respect to one another?

Draw other parallel lines, mark off on them equal lengths, join the extremities of these equal lengths, and repeat the examination as to the lengths and relative position of the joining lines.

The result of such observations will be that **the straight lines joining the extremities of equal and parallel straight lines are themselves equal and parallel.**

We may *prove* this as follows: In the triangles **LKN, MNK,** the sides **LK, KN** are equal to the sides **MN, NK;** and the angles **LKN, MNK** are equal. Hence these triangles are equal in all respects. Therefore **LN** and **MK** are equal. Also the alternate angles **LNK** and **MKN** are equal, and therefore **LN** and **MK** are parallel.

3. With the power of drawing parallel lines we have another means of bisecting a line, indeed of dividing a line into any number of equal parts:

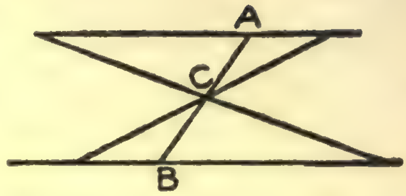


Let **AB** be the line to be bisected. Draw through **A** any other line **AC**, and with the dividers mark off on it equal lengths **AD, DE.** Join **BE,** and with the parallel rulers draw **DF** parallel to **BE. F** is the bisection of **AB.** For, drawing **FG** parallel to **AC,** the triangles **ADF, FGB** are evidently equal, and **AF** is equal to **FB.**

In employing this method of bisecting a line, we may avoid altogether drawing the lines **AC,** &c. For, place the edge of the ruler against **A,** and, close to the edge of the ruler, with the sharp points of the dividers, mark the points **D** and **E** (the distances **AD, DE** being equal). Then place the edge of the parallel rulers against the points **B** and **E,** and move the edge, parallel to itself, back to **D.** The point in which the edge cuts **AB** is its middle point, and can be marked with a point of the dividers.

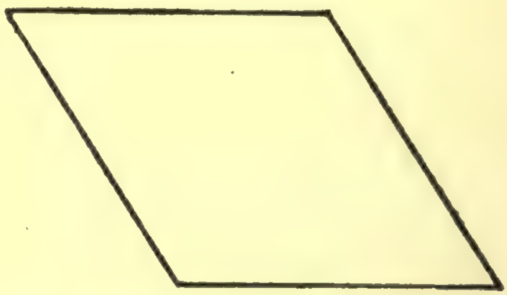
It is well to so place **AC** and the points **D** and **E,** that the lines **DF, EB** cut **AB** at nearly 90°. The point **F** is thus located with most definiteness.

We leave to the pupil to discover for himself, following the suggestion here given, a means of dividing a straight line into any number of equal parts.



4. Having drawn two parallels, adjust the points of the dividers to a distance of, say, 1 inch from one another. Place one point at **A**, and let the other point of the dividers meet the other parallel at **B.** Then ½ inch from **A** gives the middle point of **AB.** Draw a number of lines through **C,** and terminated by the parallels. Using the dividers, **C** will be found to be the middle point of all these lines.

5. If the angles of a parallelogram are right angles, it is called a **rectangle.**



If the adjacent sides, and therefore all the sides, of a parallelogram are equal, it is called a **rhombus.**

If the angles of the rhombus are right angles, the figure is called a **square,** *i.e.*, a square is a four-sided figure with all its sides equal, and all its angles right angles.

Construct the following parallelograms:

Sides 50 and 80 millimetres, and included angle 45°.
Sides 40 and 110 millimetres, and included angle 110°.
Sides 2 and 3 inches, and included angle 58°.

. . . .

In all cases test the equality of the opposite sides and angles.

Construct the following rhombuses:

Sides 70 millimetres, and one angle 60°.
Sides 5 inches, and one angle 75°.
Sides $3\frac{1}{2}$ inches, and one angle 15°.

. . . .

Construct the square whose side is 50 millimetres; whose side is $4\frac{1}{2}$ inches; whose side is 70 millimetres; . . . .

Exercises.

1. With two equal triangles, cut out of paper, form a parallelogram.

2. Draw a number of straight lines of various lengths, and, by the method of § 3, bisect them, using points only in your construction. With the dividers test the accuracy of your construction.

3. Draw a number of straight lines of various lengths, and, by the method of § 3, trisect them, using points only in your construction. With the dividers test the accuracy of your construction.

4. Draw both diagonals in a number of parallelograms, and examine how the point in which the diagonals intersect divides them. Give proof.

5. Draw two lines whose intersection bisects both, and show by using parallel rulers that the lines joining the extremities of the bisected lines are parallel in pairs. Give proof.

6. Two equal and parallel lines are joined towards opposite parts. How do the joining lines divide each other? Apply tests. Give proof.

7. With compasses and ruler only, construct a four-sided figure with opposite sides equal. How are opposite sides placed with respect to each other? Apply tests. Give proof.

This exercise explains the principle of the construction of parallel rulers.

8. With protractor and ruler, construct a four-sided figure with one pair of opposite angles equal and each 75°, and the other pair of opposite angles equal and each 105°. What is the figure? Apply tests. Give proof.

9. Can you construct a four-sided figure with opposite angles equal, such pairs of angles having any magnitude? If there be any restriction, what is it?

10. Show how to bisect a straight line by means of a set-square (or other triangular shape) and ruler.

11. Using protractor and ruler, on a given diagonal AB construct a four-sided figure, such that AB bisects the angles at A and B, these angles being equal. What is the figure? Apply tests. Give proof.

12. Give proof that the diagonals of a parallelogram or rhombus are in general unequal. When are they equal?

13. At what angle do the diagonals of a rhombus intersect? Apply test. Give proof.

14. Draw AB, CD intersecting in O, and make OA, OB, OC, OD all equal to one another. What is the figure OCBD? Apply tests and give proof.

15. If two railway tracks of the same gauge cross one another at any angle, what special kind of parallelogram is formed by the rails? Apply tests. Give proof.

16. In the preceding question, if the tracks be of different gauges, can this special kind of parallelogram be formed?

17. Describe a circle, and drawing any two diameters, join their extremities. What is the figure so formed? Apply tests and give proof.

18. Construct a parallelogram with angles 120° and 60°, and sides 110 and 50 millimetres. Bisect the angles of the parallelogram. What is the figure formed by the bisecting lines? Apply test. Give proof.

19. Show that every straight line through the intersection of the diagonals of a parallelogram divides the parallelogram into two equal areas.

20. D is any point lying in the angle BAC. Construct a parallelogram ABEC, such that D may be the intersection of the diagonals.

21. D is any point lying in the angle BAC. Through D draw a line bisected at D and terminated by AB, AC.

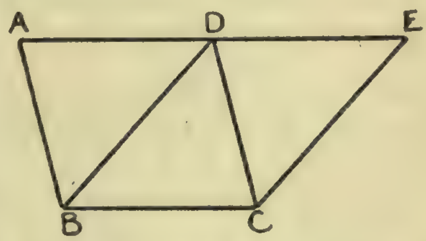
22. On any line, with the dividers mark off equal lengths AB, BC, CD, DE . . . . ; and through A, B, C, D, ... draw, with the parallel rulers, in any direction, parallel lines cutting any line in K, L, M, N, ... What do you note as to the lengths of KL, LM, MN. . . . . ?

CHAPTER VIII.

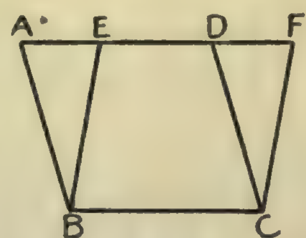
Certain Relations in Area between Parallelograms and Triangles.

1. **If two parallelograms have equal bases and equal heights, or altitudes, they are equal in area.**

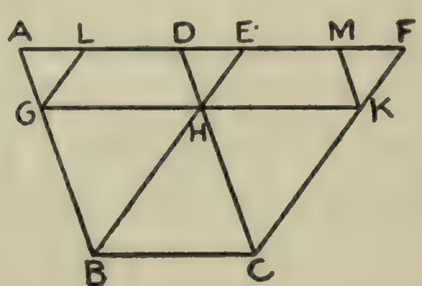
For if such be placed, or constructed, on the same base, we shall get one of the three following cases:



(1) The parallelograms may lie as **ABCD** and **DBCE,** and the triangle **EDC** can be cut out, pushed to the left, and made to cover exactly the triangle **DAB.** Thus the area **DBCE** is made to coincide with the area **ABCD,** and they are equal.



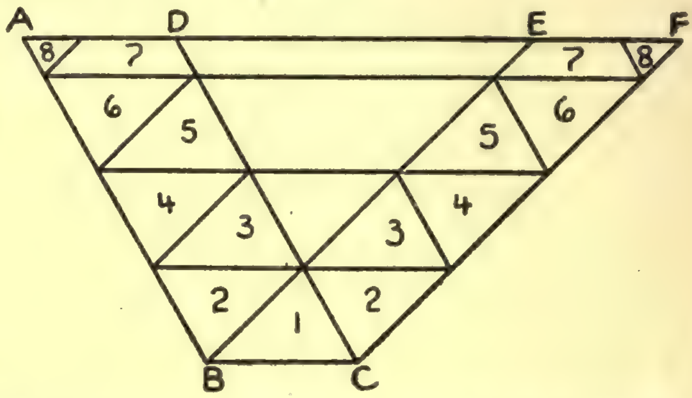
(2) The parallelograms may lie as **ABCD** and **EBCF,** and the triangle **FDC** can be cut out, pushed to the left, and made to cover exactly the triangle **EAB.** Thus the area **EBCF** is made to coincide with the area **ABCD,** and they are equal.



(3) The parallelograms may lie as **ABCD** and **EBCF.** Draw **GHK** parallel to **BC** or **AF.** The triangle **KHC** can be cut out, pushed to the left, and made to cover exactly the triangle **HGB.**

Next draw **GL** parallel to **BE** or **CF,** and **KM** parallel to **AB** or **CD.** The figure **EHKM** can now be cut out, pushed to the left, and made to cover exactly the figure **LGHD**; and the triangle **FMK** can be cut out, moved to the left, and made to cover exactly the triangle **LAG.** Thus the area **EBCF** is made to coincide with the area **ABCD,** and they are equal.

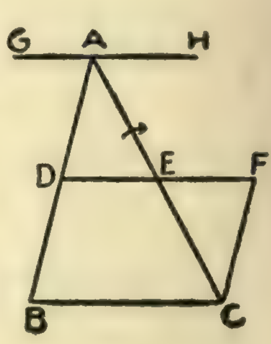
If the parallelograms be much inclined from one another, more than one line corresponding to **GHK** must be drawn. The accompanying figure



illustrates how **EBCF** must be cut up so that its sections may exactly make up **ABCD.** Corresponding numbers are placed on the figures, which are to be placed on one another. It will be noticed, however, that all the triangles, on both sides, numbered from 1 to 6, can be made to coincide with one another.

Several pairs of parallelograms should be constructed, each pair with the same base and between the same parallels, and the cutting just described should be done so as to show that each pair may be made to coincide. Care should be taken to illustrate the different cases that may occur. The figures, of course, must be accurately and completely drawn before the cutting is proceeded with.

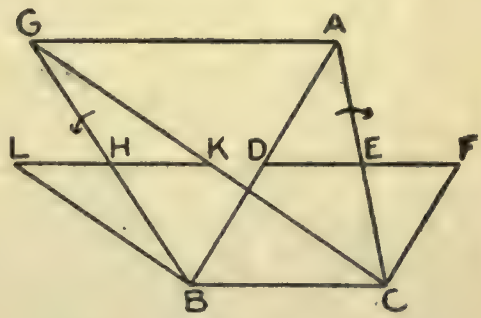
2. If any triangle be cut along a straight line through the centres of two of its sides, the two parts of the triangle can be formed into a parallelogram, of course equal in area to the triangle. For, let **D** and **E** be the middle points of two sides, so that



the cutting is made along **DE.** Then let the triangle **ADE** be turned about **E,** through 180°, in the direction indicated by the arrow head. The point **A** arrives at **C,** and **D** at **F.** Since the alternate angles **ADE, EFC** are the same, **DB** is parallel to **CF,** and they are equal. Hence (Ch. VII., 2) **DBCF** is a parallelogram.

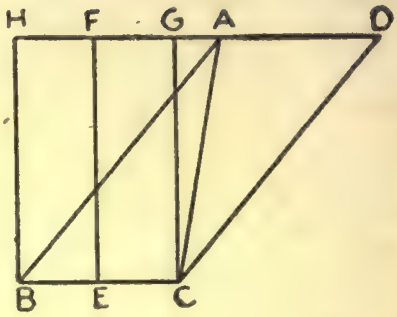
Since **D** is the middle point of **AB,** it is (Ch. VII., 4) mid-way on the perpendicular through **D** to each of the parallels **GAH** and **BC.** Hence the triangle **ABC** has twice the altitude of the parallelogram **DBCF** into which it has been converted.

3. **If two triangles have the same or equal bases, and equal altitudes, they are equal in area.**



For, let the triangles **ABC, GBC,** upon the same base **BC,** have the same altitude. The triangle **ABC** can, by a section along **DE,** be converted into the parallelogram **DBCF,** whose altitude is half that of the triangle. Also, the triangle **GBC** can, by a section along **HK,** be converted into the parallelogram **LBCK,** whose altitude is half that of the triangle. Hence the parallelograms **DBCF, LBCK,** being on the same base and with equal altitudes, are (Ch. VIII., 1) equal in area. Hence the triangles are equal in area.

The parallelograms **DBCF, LBCK** may be cut up (Ch. VIII., 1) so that the one exactly coincides with the other. Hence, in so cutting and placing the parallelograms, the original triangles are made to exactly coincide with each other.



4. The triangle **ABC** is half of the parallelogram **ABCD**, and, therefore, half of the rectangle **HBCG**. Hence if we find **E**, the bisection of **BC**, and draw **EF** perpendicular to **BC**, the triangle **ABC** is equal to either of the rectangles **HBEF** or **FECG**.

Hence **to construct a rectangle equal to a triangle, bisect the base of the triangle, and on the half-base construct a rectangle of the same altitude as the triangle.**

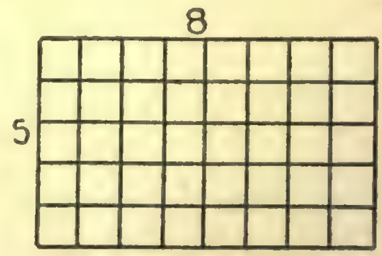
Construct rectangles equal in area to the following triangles:

Sides 80, 90, 140 millimetres.

Sides 70, 100 millimetres, and included angle 50°.

Base 80 millimetres, and angles at base 45° and 75°.

. . . .



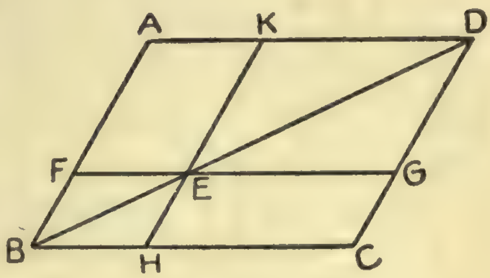
5. To find the area of a rectangle we multiply the length by the breadth. Thus the adjoining rectangle being 8 units in length, and 5 units in breadth, the area is evidently $8 \times 5 = 40$ square units.

Since the area of a triangle is half that of the rectangle on the same base and with same altitude, **we may find the triangle's area by multiplying the base by the perpendicular height and dividing by 2.**

Calculate approximately, in square millimetres, the areas of the triangles in 4, by finding the lengths of the sides of the rectangles which have been constructed equal to the triangles.

It will be interesting for the teacher to calculate the areas of such triangles by the usual trigonometrical formulæ, that he may inform the class as to the closeness of their approximations reached by instrumental methods.

6. In the adjoining figure **BED** is the diagonal of each of the parallelograms **ABCD, FBHE, KEGD,** and therefore bisects each of them. Hence we have

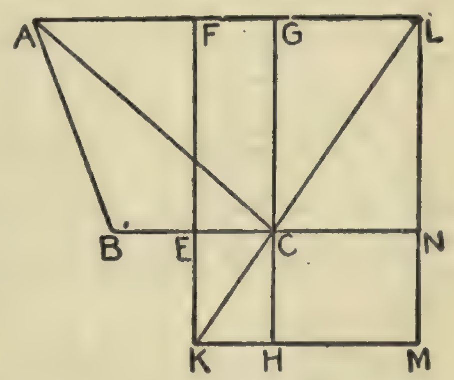


triangle **ABD** = triangle **CBD,**
" **FBE** = " **HBE,**
" **KED** = " **GED.**

Therefore **the parallelogram AFEK is equal to the parallelogram CHEG.** Note the position of these parallelograms with respect to the diagonal **BD.**

7. In constructing a rectangle equal to a given triangle (Ch. VIII., 4), one of the sides of the rectangle is half the base of the triangle. We may, however, construct a rectangle equal to any triangle, and give to one of the sides of the rectangle any length we choose.

Thus having constructed **FECG** equal to the triangle **ABC,** suppose we wish to make a rectangle equal to the triangle, with one of its sides of length **CH.** Complete the rectangle **EKHC,** and let **KC, FG** meet in **L.** Draw the remaining lines as indicated in the figure. Then the rectangle **CM,** whose side **CH** is of the required length, is



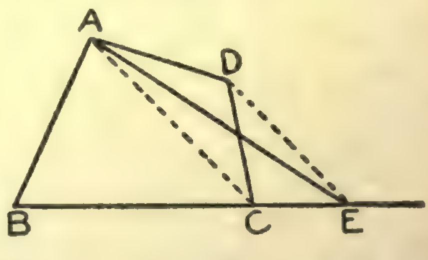
equal to the rectangle **FC** (Ch. VIII., 6), which is equal to the original triangle **ABC**.

Construct rectangles, each with a side of 50 millimetres, equal to the triangles in 4.

The sides of a triangle are 2, 3 and 4 inches. Construct a rectangle equal to it, having one side of $2\frac{1}{2}$ inches. Measuring the other side of the rectangle, calculate approximately the area of the rectangle, *i.e.*, of the triangle.

The sides of a triangle are 3 and 4 inches, and the included angle is 50°; construct a rectangle equal to it, one of whose sides is 2 inches. Measuring the other side of the rectangle to the nearest sixteenth of an inch, calculate approximately the area of the rectangle, *i.e.*, of the triangle.

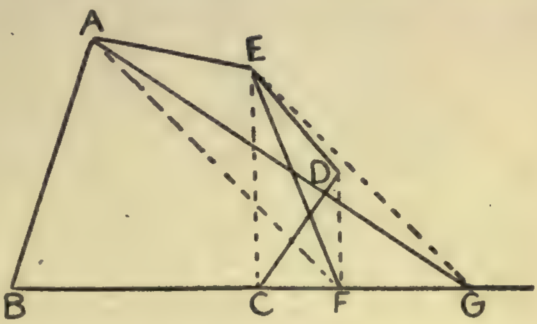
8. If we wish to construct a rectangle equal in area to a polygon, and thence, if necessary, calculate the area of the polygon, it is well first to construct a triangle equal to the polygon by the following method:



Let **ABCD** be a quadrilateral whose area we wish to calculate. Place the edge of the parallel rulers along **AC**, and slide one bar out until the edge reaches **D**, and mark the point **E** in **BC** produced. **AC** is parallel to **DE**, and therefore the triangle **ACE** is equal to the triangle **ACD** (Ch. VIII., 3); and therefore the triangle **ABE** is equal to the quadrilateral **ABCD**.

We may then measure the perpendicular height of **ABE** and its base **BE**: their product divided by 2 gives the area of **ABE** (Ch. VIII., 5), and therefore of **ABCD**.

Suppose we wish to find the area of the pentagon **ABCDE**. Place the edge of the parallel rulers along **CE**, and slide one bar out until the edge reaches **D**, and mark the point **F** in **BC** produced. **DF** is parallel to **CE**, and therefore the triangle **ECF** is equal to the triangle **ECD**. Thus the quadrilateral **ABFE** is equal to the pentagon **ABCDE**.



Again, place the edge of the parallel rulers along **AF**, and slide one bar out until the edge reaches **E**, and mark the point **G** in **BC** produced. **EG** is parallel to **AF**, and therefore the triangle **GAF** is equal to the triangle **EAF**; and therefore the triangle **ABG** is equal to the quadrilateral **ABFE**, and to the pentagon **ABCDE**.

We may then measure the perpendicular height of **ABG** and its base **BG**: their product divided by 2 gives the area of **ABG** and therefore of **ABCDE**.

In the preceding figures the dotted lines need not be drawn. The point **E** in the former figure, and the points **F** and **G** in the latter, are where the edge of the parallel rulers cuts **BC**.

Had we selected **AB** as our base, instead of **BC**, our resulting triangle would have had a different height and base, but would necessarily have been of the same area as **ABE** or **ABG**.

The sides **AB**, **BC**, **CD** of a quadrilateral are 70, 60 and 50 millimetres; the angles **ABC**, **BCD** are 70° and 60°. Construct a triangle equal to it in area, and thence calculate its area. Here use BC and next AB as bases; and, by comparing the areas of the resulting triangles, obtain a test of the accuracy of your construction.

Construct several quadrilaterals and pentagons, and find triangles equal to them in area. In each case construct the triangle in two different ways (as in the preceding example) and, by comparing the areas of such triangles, obtain a test of the accuracy of your construction.

In all cases where numerical measurements are made, such measurements are necessarily approximate, and therefore in examples such as the preceding the areas will be found only approximately. Hence where a numerical area has been reached by two different ways, we are to expect only approximate agreement.

Chapters XIX., XX., and XXI., relating to similar triangles, may now be taken up if thought desirable.

Exercises.

1. If two triangles have the same base and equal areas, what relation exists between their altitudes?

If their vertices be joined, what position does it occupy with respect to the common base?

2. If D and E be the middle points of the sides AB, AC of the triangle ABC, what relation exists between the areas of the triangles DBC, EBC? What do you infer as to the position of DE with respect to BC?

3. Construct a quadrilateral, and bisect the sides. What positions do the lines joining the bisections of adjacent sides occupy with respect to the diagonals?

What is the figure formed by joining in succession the points of bisection?

4. Construct a quadrilateral, and bisect the sides. How do the lines joining the bisections of opposite sides divide each other? Give reason.

5. Two sides of a quadrilateral are parallel and of lengths $2\frac{1}{2}$ and 3 inches. The distance of these sides apart is $\frac{3}{4}$ of an inch. What is the area of the quadrilateral? (Join two opposite corners, and find area of each triangle.)

6. The sides of a rectangle are 2 and 3 inches. Find by geometrical construction a rectangle equal to it in area, one of whose sides is $2\frac{1}{2}$ inches. Test by measurement and numerical calculation the accuracy of your construction.

7. The sides of a triangle are 3, $4\frac{1}{2}$ and 5 inches. Construct a rectangle equal to it in area with one side $2\frac{1}{2}$ inches. Construct also a rectangle equal to it in area, one of whose sides is 3 inches.

8. The base of a triangle is 70 millimetres, and the angles at the base 30° and 50°. Construct a rectangle equal to it in area, one of whose sides is 45 millimetres.

9. The sides of a rectangle are 30 and 40 millimetres. Construct a parallelogram equal to it in area, one of whose sides is 30 millimetres, and one of whose angles is 60°.

10. On a base of 35 millimetres construct two parallelograms of equal area, one having a side of 55 millimetres and an angle of 75°, and the other an angle of 120°.

11. The sides of a triangle are 2 and 3 inches, and the included angle 45°. Construct a rectangle equal to it in area, one of whose sides is $2\frac{1}{2}$ inches.

12. In the previous question, construct a parallelogram equal to the triangle, with one of its angles 45°.

13. In a quadrilateral ABCD, AB = 35, BC = 45, CD = 55 millimetres; ABC = 60°, BCD = 75°. Construct a triangle and also a rectangle equal to it in area. Hence calculate its area, approximately, in square millimetres.

14. A quadrilateral ABCD has AB (2 in.) and CD ($3\frac{1}{2}$ in.) parallel, and $1\frac{1}{2}$ in. apart. Construct a rectangle equal to it in area, one of whose sides is $1\frac{1}{2}$ in.

15. ABC is a triangle, and D, E the middle points of AB, AC. BE, CD intersect in O. Join AO, and show that the triangles OAB, OBC, OCA are equal in area.

16. In the previous question, if F be the middle point of BC, and OF be joined, what relation holds between the areas of the six triangles OAD, ODB, . . . with vertex at O?

17. In the same question, what is the position of AO, OF with respect to each other? Test and give reasons.

18. Construct two equal triangles on the same base and on opposite sides of it. What is the only restriction as to the positions of their vertices? If the vertices be joined, how is the joining line divided by the base, or base produced?

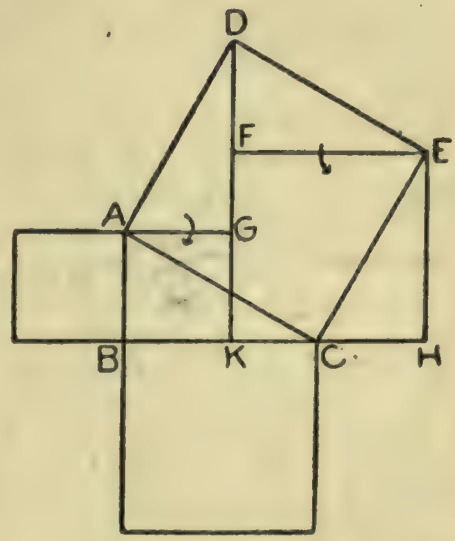
19. From any point in an equilateral triangle draw perpendiculars to the sides. What relation exists between their sum and the altitude of the triangle? Give reasons.

20. The sides of a right-angled triangle are 3, 4 and 5 inches. If a point within the triangle be 1 inch from each of the sides containing the right angle, how far is it from the hypotenuse?

21. In a quadrilateral ABCD, $AB=2$, $BC=3$, and $CD=1\frac{1}{2}$ inches. $ABC=35°$, $BCD=100°$. Construct a triangle and also a rectangle equal to it in area. Hence calculate the area of ABCD, approximately, in square inches.

CHAPTER IX.

Squares on Sides of a Right-angled Triangle.



1. Let the angle **B** of the triangle **ABC** be 90°. Describe squares on the sides of **ABC**, as in the figure. Draw the lines **AG**, **EF**, **CH** parallel to **BC**; and the lines **DK**, **EH** parallel to **AB**.

Then measurement (with dividers for lines, and bevel for angles) will show that the triangle **AGD** is in all respects equal to the triangle **ABC**; and cutting out the triangle **AGD**, it may be turned about **A**, in the direction indicated by the arrow head, into the position **ABC**. Measurement will also show that the triangle **EFD** is in all respects equal to the triangle **EHC**; and cutting out the triangle **EFD**, it may be turned about **E**, in the direction indicated by the arrow head, into the position **EHC**. We thus have the square on **AC** converted into **ABKG** and **FKHE**, which will be found to be the squares on **AB** and **BC**.

Repeat the same construction, measurements, and superposition in the case of the following triangles:

AB = 35, **BC** = 50 millimetres; **ABC** = 90°.

AB = $1\frac{1}{2}$ in., **BC** = $2\frac{1}{2}$ in.; **ABC** = 90°.

AB = 2 in.; **ABC** = 90°, **BAC** = 60°.

The result of these observations may be stated thus: **In any right-angled triangle the square which is described on the side subtending the right angle, is equal to the sum of the squares described on the sides containing the right angle.**

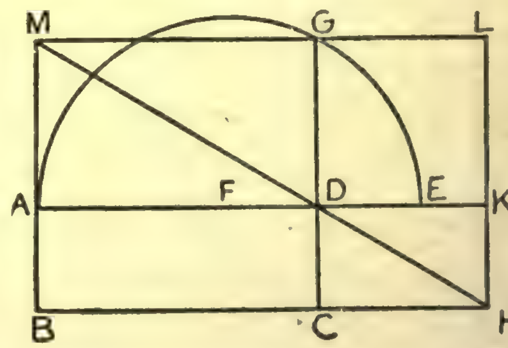
Two sides of a right-angled triangle about the right angle, are 3 and 4. What is the length of the third side?

If a string or rope of length 12 be broken into lengths 3, 4 and 5, and these be formed into a triangle, such triangle is right-angled.

If the lengths of the pieces of rope be 30, 40 and 50, the triangle formed with them is also right-angled.

2. A square may be constructed equal in area to any rectangle, as follows:

Let **ABCD** be the rectangle. Make **DE** equal to **DC,** and find **F** the middle point of **AE.** Describe the semicircle, and produce **CD** to **G.** Then the square on **DG** is equal to the rectangle **ABCD.**



For, describe the square **DGLK** on **DG,** and let **LK** and **BC** meet in **H.** Then, if the figure has been accurately constructed, on producing the lines **LG, HD** and **BA,** they will be found to all pass through one point, **M.** Hence (Ch. VIII., 6) the square **GDKL** is equal to the rectangle **ABCD.**

In the succeeding constructions it is of course absolutely necessary that the three lines corresponding to **LG, HD** and **BA** pass through the same point **(M).**

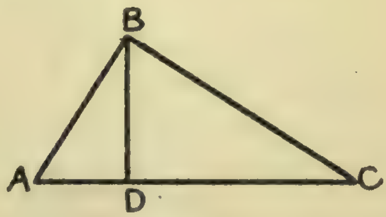
Describe the rectangle whose sides are 40 and 90 millimetres. Construct, as above, the square equal to it. Measure in millimetres the side of the square, and thence verify the accuracy of your construction.

Proceed similarly with the rectangle whose sides are 1 and 4 inches.

Also with the rectangle whose sides are 9 and 16 sixteenths of an inch.

Also with the rectangle whose sides are 18 and 32 sixteenths of an inch. The sides of this rectangle are twice those of the former: note the number of times its area is greater than that of the former; note the same with respect to the resulting squares.

ABC is a right-angled triangle, **ABC** being the right angle; and **BD** is perpendicular to **AC**.



Construct the rectangle whose sides are **CA, AD**; by the preceding method construct the square equal to it, and show that it is the square on **AB**.

Similarly by construction show that the rectangle contained by **AC, CD** is equal to the square on **BC**.

Also that the rectangle contained by **AD, DC** is equal to the square on **BD**.

Exercises.

1. Three straight lines, of lengths 3, 4, 5, forming a right-angled triangle, what sort of triangle is formed by lines of lengths 6, 8, 10, or 9, 12, 15, or 12, 16, 20, etc.?

2. Construct triangles with sides as follows: 3, 4, 5 inches; 30, 40, 50 millimetres; 36, 48, 60 millimetres; $3\frac{3}{4}$, 5, $6\frac{1}{4}$ inches. Compare the angles of these triangles, and state the result of such comparison. What relation do the sides of one triangle bear to the sides of another?

3. Given

$$(2n+1)^2+(2n^2+2n)^2=(2n^2+2n+1)^2$$

by assigning to n in succession the values 1, 2, 3, . . . , form a series of whole numbers, in groups of three, such that each group gives the lengths of the sides of a right-angled triangle.

4. The side of an equilateral triangle is 2. What is the length of the perpendicular from any angle on the opposite side?

5. Draw two lines CA, CB at right angles to each other and each of length one inch. What is the area of the square on AB?

6. In the figure of the preceding question, draw AD (=1 in.) perpendicular to AB. What is the area of the square on DB?

7. In the same figure draw DE (= 1 in.) perpendicular to DB. What is the area of the square on EB? Test by measuring the length of EB.

8. Construct a square which shall contain 13 square inches.

9. Test the accuracy of the construction in the preceding question by drawing, at right angles to the side of the square, a line equal to the side of a square containing 3 square inches, joining the ends of the lines, and measuring the hypotenuse of the right-angled triangle so obtained.

10. Describe squares on the sides of a right-angled triangle. Construct another triangle with sides equal to the diagonals of these squares. What is this latter triangle?

11. In the preceding question by what multiplier can you obtain the sides of one triangle from those of the other?

Compare the angles of the two triangles and state the result of such comparison.

12. Describe a triangle such that the square on one side is greater than the sum of the squares on the two other sides, say with sides of 2, 3 and 4 inches. What relation does the angle opposite the greatest side bear to a right angle? Measure it with protractor.

13. Construct a triangle with sides of 30, 40 and 55 millimetres ($55^2 > 30^2 + 40^2$). What sort of angle is that opposite the greatest side?

14. Describe a triangle such that the square on one side is less than the sum of the squares on the two other sides, say with sides of

40, 60 and 65 millimetres. What relation does the angle opposite the greatest side bear to a right angle?

15. Construct a triangle with sides of 2, 3 and $3\frac{1}{2}$ inches $(3.5^2 < 2^2 + 3^2)$. What is the angle opposite the side of $3\frac{1}{2}$ inches?

16. Construct any quadrilateral with its diagonals at right angles to each other. Show that the sum of the squares on two opposite sides is equal to the sum of the squares on the other two sides.

17. Describe a square ABCD, and in the sides take points E, F, G, H, such that AE = BF = CG = DH. What is the figure EFGH. Apply tests. Give reasons.

18. Two squares being given, say of 9 and 16 square inches, show how to draw a line the square on which shall be equal to the difference of these given squares.

19. ABC, A′B′C′ are right-angled triangles with the hypotenuses AB, A′B′ equal, and also the sides BC, B′C′ equal. Show that the remaining sides AC, A′C′ are equal.

20. The sides of a triangle are $1\frac{1}{2}$, 2, $2\frac{1}{2}$ inches. Construct a square equal to it.

21. The side of an equilateral triangle is 2 inches. Construct a square equal to it.

22. The sides of a rectangle are 24 and 54 sixteenths of an inch. Construct a square equal to it. Measure the side of the square, and thence verify the accuracy of your construction.

23. If a right-angled triangle have one of the acute angles double the other, divide it into two triangles, one equilateral and the other isosceles.

24. Bisect the hypotenuse of a right-angled triangle. What relation between the distances of the point so obtained from the three angles?

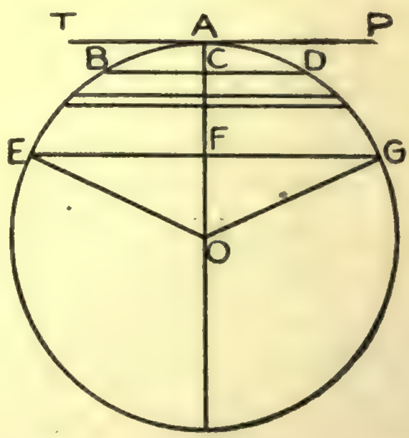
25. ABC is a right-angled triangle, and CD is drawn perpendicular to the hypotenuse. Examine the relations between the angles of the three triangles ABC, ACD, BCD. Give reasons.

CHAPTER X.

The Circle. Its Symmetry. Tangents. Finding of Centre.

1. The fundamental quality of the circle, next to the equality of its radii, is its **symmetry.**

In the first place, every line drawn through the centre from circumference to circumference (*i.e.*, every diameter) is bisected at the centre. This is called **central symmetry.**



In the second place, every chord drawn at right angles to a diameter is bisected by that diameter. This is called **axial symmetry.** Thus the chord **EFG** being perpendicular to **OA,** the parts **EF, FG** are equal. Measurement will establish the equality of these parts. Or we may *prove* it thus:

The rt. $\angle^{les}$ at **F** are equal.

Because **OE** = **OG**, $\therefore$ $\angle$ **OEF** = $\angle$ **OGF.**

Hence $\angle^{les}$ at **O** are equal.

Also sides **EO, OF** = sides **GO, OF.**

$\therefore$ (Ch. III., 2) **EF** = **FG.**

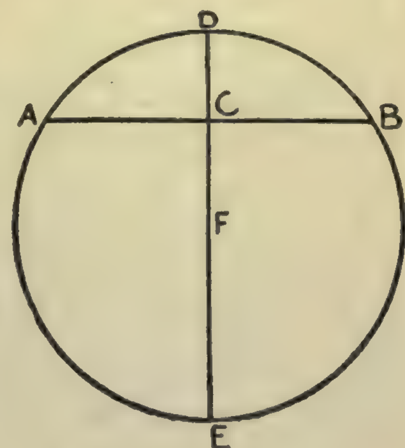
And hence all chords perpendicular to a diameter are bisected by it.

2. As the chord **BCD** moves parallel to itself down to **A,** since the parts on each side of the diameter are always equal, when one part vanishes, the other vanishes

also. Thus the line **TAP**, through **A** parallel to **BCD**, while it **touches** the circle, does not cut it. Such a line (**TAP**) is called the **tangent** to the circle at **A**. That is to say, **a tangent is a line drawn through the extremity of a diameter, and at right angles to it.**

The tangent is evidently a straight line which meets the circle, but does not cut it: this is sometimes given as the definition of a tangent.

3. Since a diameter bisects every chord to which it is at right angles, therefore a line drawn through the bisection of a chord and at right angles to it, must be a diameter. Hence if the centre of any circle be not indicated, we may reach it by the following construction:

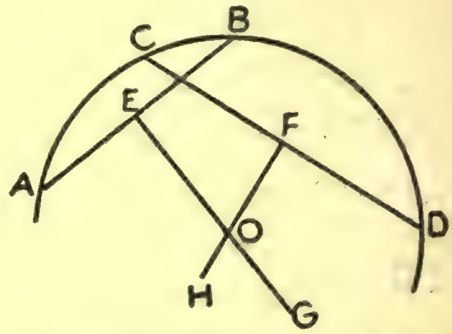


Draw any chord **AB**. Bisect it at **C**. Draw **DCE** perpendicular to **AB**. **DE** must pass through the centre. Hence, bisecting **DE** at **F**, **F** must be the centre of the circle.

We may describe circles without marking their centres by placing a piece of thin wood or cardboard under the stationary point of the compasses, removing this piece of wood or cardboard when the circle is described.

Circles being thus described, or being obtained by marking with the pencil about a round object placed on the paper (coin, bottom of ink bottle, plate, &c.), attempts should be made to locate the centre by the eye's judgment. We may afterwards test the correctness of this by making the preceding construction, and finally test the accuracy of the construction by trying with such centre to reproduce the circle by using the compasses.

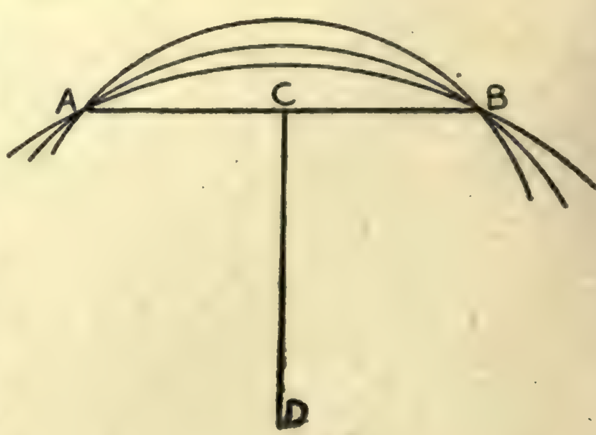
It will be found, of course, that the greater the circle, the greater will be the difficulty of locating, with the eye's judgment, the position of the centre. The same difficulty occurs in locating the bisection of a straight line with the eye.



4. If only an arc of the circle be given, we may find the centre, and complete the circle, as follows:

Draw two chords **AB** and **CD**; find their middle points **E** and **F**; through these middle points draw perpendiculars **EG** and **FH**. The centre of the circle must lie on each of the lines **EG** and **FH** (Ch. X., 3), and therefore must be at **O**.

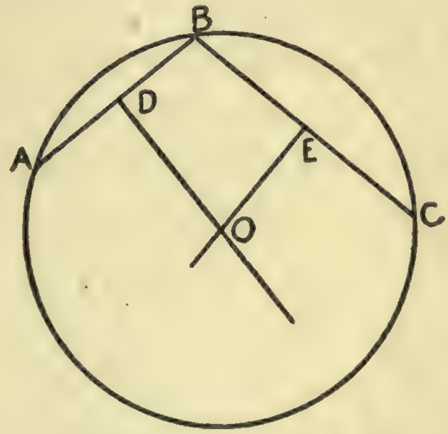
Arcs of circles should be described without marking the centres, by the method suggested in § 3. The positions of the centres should then be judged with the eye; afterwards constructed for, and the accuracy of the construction tested by attempting, with the compasses, to describe the arc with the centre so obtained.



5. If any line **AB** be bisected at **C**, and **CD** be drawn perpendicular to it, then all points in **CD** are equally distant from **A** and **B**. Hence if we place the sharp point of the compasses at *any* point on **CD**, and the

pencil end at **A**, and describe a circle, it will also pass through **B**. We thus get an unlimited number of circles through **A** and **B**, all of which have their centres at different points on **CD**.

Draw a line **AB** of 50 millimetres, and describe circles passing through **A** and **B**, with radii 30, 40, 50 and 60 millimetres.



6. **We can readily obtain a method for describing a circle to pass through any three points:**

Let **A, B, C** be the three points. Draw **DO** from the middle point of **AB** at right angles to it; and draw **EO** from the middle point of **BC** at right angles to it. Then all points in **DO** are equally distant from **A** and **B**; and all points in **EO** are equally distant from **B** and **C**. Hence **O** is equally distant from **A, B** and **C**; and placing the sharp point of the compasses at **O** and the pencil end at **A**, and describing a circle, it will pass through **B** and **C**, if the construction has been accurate.

AB is 1 inch, **BC** is 2 inches, and angle **ABC** is 120°. Describe a circle to pass through **A, B** and **C**.

AB is 40 and **BC** 60 millimetres, and the angle **ABC** is 75°. Describe a circle to pass through **A, B** and **C**.

AB is $1\frac{1}{2}$ and **BC** $2\frac{1}{2}$ inches, and the angle **ABC** is 90°. Describe a circle to pass through **A, B** and **C**. Show that its centre bisects **AC**.

Mark sets of three points in various positions with respect to one another, and describe a circle to pass through each set.

Exercises.

1. Describe a circle; draw in it any chord; join the centre to the extremities of the chord; and drop a perpendicular from centre on chord.

What is the relation between the angles the radii make with the chord? What between the angles the radii make with the perpendicular? What between the segments of the chord made by the foot of the perpendicular?

2. With the bevel or protractor construct two equal angles at the centre of a circle, and draw the chords which subtend these angles. What is the relation between these chords? Apply test. Give reasons.

3. Describe a circle, and with dividers and ruler place two equal chords in it. Join the ends of the chords to the centre. What is the relation between the angles these equal chords subtend at the centre? Apply test. Give reasons.

4. As in the previous question, in a circle place two equal chords, and from the centre drop perpendiculars on them. What is the relation between these perpendiculars? Apply test. Give reasons.

5. Describe a circle of radius 3 inches, from the centre draw two equal lines of length 2 inches, and through the extremity of each draw a line at right angles to it, so obtaining two chords at equal distances from the centre. What is the relation between the lengths of these chords? Apply test. Give reasons.

6. The sides of a triangle are $2\frac{1}{2}$, 3 and $3\frac{1}{2}$ inches. Describe a circle passing through the angular points.

7. The sides of a triangle are 2, 3 and 4 inches. Describe a circle passing through the angular points.

8. The sides of a triangle are 3, 4 and 5 inches. Describe a circle passing through the angular points.

9. Two chords of a circle with one end of each common, are of lengths 2 and 3 inches, and make an angle of 60° with each other. Describe the circle.

10. Two chords of a circle make angles of 50° and 60° with a third chord whose length is $2\frac{1}{2}$ inches, and are inclined towards one another. Describe the circle.

11. The sides of a rectangle are 40 and 60 millimetres. Describe a circle passing through all the angular points.

12. Describe a parallelogram ABCD, not being a rectangle. Can a circle be described passing through its angular points? (Every circle through A and B has its centre on the line which bisects AB at right angles.)

13. The diameter of a circle is 30 inches, and a chord is 24 inches. How far is the chord from the centre?

14. The radius of a circle is $3\frac{1}{4}$ inches. What is the length of a chord whose distance from the centre is $1\frac{1}{4}$ inches?

15. The equal sides AB, AC, of an isosceles triangle ABC, are 50 millimetres, and they contain an angle of 45°. A circle with centre A, and radius 70 millimetres, cuts BC produced in D and E. What is the relation between the lengths of DB and CE? Apply test. Give reasons.

16. Describe a circle; draw a diameter, producing it; and from a point A in the produced diameter draw two lines on opposite sides of it, making equal angles with it. What do you observe as to the lengths of the segments of these lines between A and the points of section by the circle? What as to the parts within the circle? Apply tests. Give reasons.

17. The same question as the preceding, but with A within the circle.

18. Construct two intersecting circles, join their centres, and through either of the points of intersection, draw a line parallel to the line joining centres, and terminated by the circumferences. What relation in length between the second line drawn and the line joining the centres? Apply test. Give reasons.

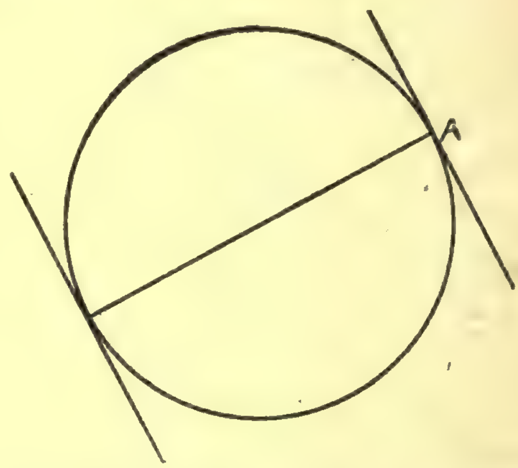
19. AB, CD are two parallel chords in a circle. What relation exists between the lengths of the chords AC, BD? Apply test. Give reasons.

20. In the previous question prove angle ABD = angle BAC; also angle ACD = BDC; also chord AD = chord BC.

CHAPTER XI.

Tangents to Circles, and Circles Touching One Another.

1. To draw the tangent at any point **A** on the circumference of a circle, draw the diameter through **A**, and draw at **A** the perpendicular to this diameter. The perpendicular is a tangent to the circle (Ch. X., 2).



Evidently the tangents at opposite ends of a diameter are parallel to one another.

Construct a circle of radius 55 millimetres. Draw radii at intervals of 30°, and draw the tangents at the ends of these radii, producing each both ways until it meets the adjacent tangents.

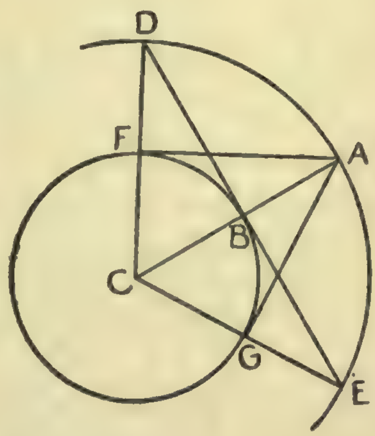
Construct a circle of radius 49 millimetres. Draw radii at intervals of 45°, and draw tangents at the ends of these radii, producing each both ways until it meets the adjacent tangents.

Construct a circle of radius $1\frac{5}{16}$ in. Draw radii at intervals of 72°, and draw tangents at the ends of these radii, producing each both ways until it meets the adjacent tangents.

In each of the three preceding constructions, the resulting figure about the circle should have equal sides and equal angles. The equality of the sides (measured with the dividers) and the equality of the angles (measured with the bevel) may be regarded as a test of the accuracy of the construction.

Any two diameters in a circle are drawn, inclined at an angle of, say, 30° to each other, and tangents at the ends of these diameters are constructed. What quadrilateral figure about the circle do the tangents form? Measure its sides.

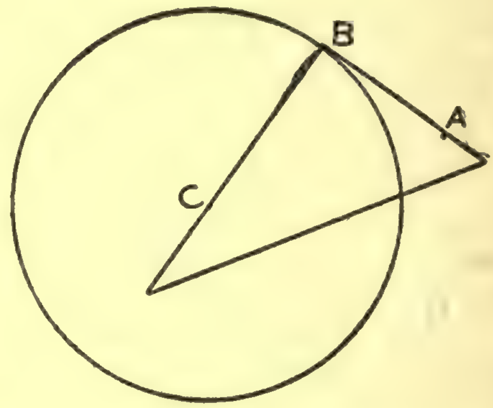
2. From a point without a circle, evidently two tangents can be drawn to the circle. To draw those from **A** to the circle **FBG**:



Join **AC**, cutting the circle in **B**. Describe a second circle **DAE**, with centre **C** and radius **CA**. Draw **DBE** perpendicular to **CB**. Join **CD** and **CE**, cutting the small circle in **F** and **G**. Then **AF** and **AG** are the tangents from **A**.

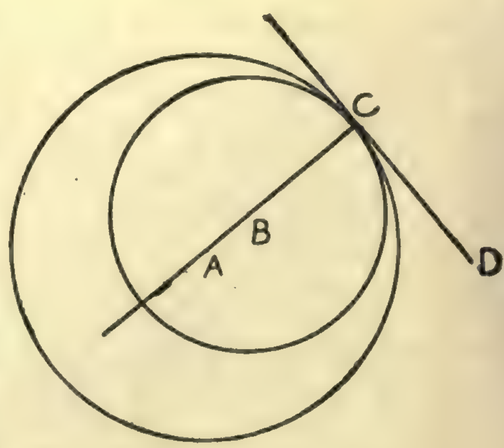
For, the triangles **ACF** and **DCB** are equal. But the angle **CBD** is a right angle; therefore the angle **CFA** is a right angle, and **AF** is a tangent to the circle (Ch. X., 2). In the same way we may prove that **AG** is a tangent.

Symmetry suggests that the tangents **AF, AG** are equal in length, and that they make equal angles with **AC**. The truth of this may be tested by measurement. It may also be *proved* as follows: Because **CDE** is an isosceles triangle, and the angles at **B** right angles, therefore the triangles **CDB, CEB** are equal in all respects. But the triangle **CAF** is equal in all respects to **CDB**; and the triangle **CAG** is equal in all respects to the triangle **CEB**. Therefore the triangles **CAF** and **CAG** are equal in all respects. Hence **AF, AG** are equal, and the angles at **A** are equal.

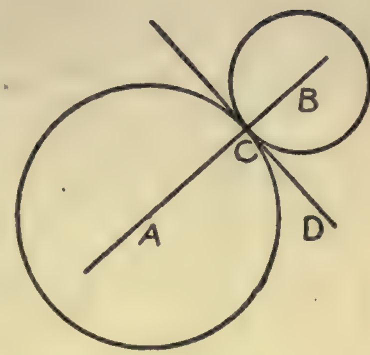


In practice, an easy way to draw a tangent from any point **A**, outside the circle, is as follows: Place the set-square so that one of its sides passes through **A** and the other through **C**, the centre of the circle. Then so adjust the instrument that the right angle rests on the circumference at, say, **B**. **AB**, a tangent through **A**, may then be drawn.

Construct a circle of radius $1\frac{1}{2}$ in., and draw any line through its centre. From points on this line at distances from the centre 2, $2\frac{1}{2}$, 3 in., draw tangents to the circle.



3. Let a circle be described with centre **A**, and the tangent at any point **C** be drawn; and let, with centre **B**, on **AC**, and radius **BC**, another circle be drawn. Then both circles have **CD** for tangent. Both *touching* the same line at the same point, they are said to *touch* one another,—in this case *internally*.



Let a circle be described with centre **A**, and the tangent at any point **C** be drawn; and let, with centre **B**, in **AC** produced, and radius **BC**, another circle be described. Then both circles have **CD** for tangent. Both *touching* the same line at the same point, they are said to *touch* one another,—in this case *externally.*

Evidently, whether circles touch internally or externally, the straight line joining their centres passes through the point of contact.

Describe circles of radii 34 and 56 millimetres to touch (1) externally, (2) internally.

Construct a series of circles of radii 20, 17, 14, 11, . . . millimetres, their centres being in the same straight line, and each circle touching the preceding (and succeeding) externally.

Describe circles of radii as in preceding, but each circle touching the others at the same point, internally.

Two circles of radii 30 and 40 millimetres touch one another externally. Describe a circle of radius 20, to touch both of them externally. (This involves the construction of a triangle with sides 70, 60 and 50 millimetres.)

Make the same construction as in the preceding question, when the first two circles have radii 25 and 35, and the third a radius of 15 millimetres.

The sides of a triangle are 75, 60 and 45 millimetres. With the angular points of this triangle as centres, describe three circles with radii 15, 30 and 45 millimetres, so that each may touch the other two.

When the sides of the triangle are 100, 75 and 65 millimetres, discover the circles whose radii are such that in like manner each will touch the other two, the angular points of the triangle being centres of the circles.

Exercises.

1. Describe a circle of radius 40 millimetres; draw two diameters at right angles to one another; and draw tangents at ends of the diameters, and produce them so that they intersect. What do you observe as to lengths of tangents? What angles do they make with one another? Apply tests with dividers and set-square.

2. Describe a circle of radius $1\frac{1}{2}$ in.; draw diameters at intervals of 60°; and draw tangents at ends of diameters. What do you observe as to lengths of tangents? What angles do they make with one another? Apply tests.

3. Describe a circle of radius $1\frac{1}{4}$ in.; draw any line in plane of paper; draw a tangent parallel to this line. (From centre drop perpendicular on line, and at point of intersection with circle draw tangent.)

4. Describe a circle of radius 35 millimetres; draw any line in plane of paper; draw a tangent to circle which shall be perpendicular to this line.

5. Draw any line and draw circles of radii 1, $1\frac{1}{2}$ and 2 inches, touching the line at any points.

6. Describe two circles of radii 1 inch and $2\frac{1}{4}$ inches, so as to touch any line at points 3 inches apart. Do the circles touch one another?

7. A tangent of length 4 inches is drawn from a point to a circle of radius 3 inches. How far is the point from the centre of the circle?

8. A tangent is drawn to a circle of radius 1 inch, and another circle, concentric with the former, is described of radius 2 inches. What is the length of the tangent between the point where it is intercepted by the second circle and the point of contact? What angle does the intercepted portion of the tangent subtend at the common centre?

9. A circle has a radius of 30 millimetres, and a tangent of length 40 millimetres is drawn to it. What line (curved) represents all the points, outside the circle, from which this tangent may be drawn?

10. From four points, equidistant from one another, on a circle of radius 2 inches, draw tangents to a concentric circle of radius 1 inch.

11. Describe two circles of radii 1 and $1\frac{1}{2}$ inches, to touch one another; and describe a circle of radius $2\frac{1}{2}$ inches to touch both, and contain both.

12. The preceding problem with each circle external to the other two.

13. Describe three circles of radii $2\frac{1}{4}$, 3 and $3\frac{1}{4}$ inches, so that each may touch the other two.

14. Describe two concentric circles of radii 1 and 3 inches, and describe a number of circles touching both of them.

15. Two circles touch internally at A, and ABC is drawn to meet the circles at B and C. What is the position of radii to B and C with respect to each other? Apply test. Give reasons.

16. Two circles touch externally at A, and ABC is drawn to meet the circles at B and C. What is the position of radii to B and C with respect to each other? Apply test. Give reasons.

17. OA, OB are drawn through the centre of a circle at right angles to each other, and a tangent to the circle meets these lines at A and B. Two other tangents are drawn to the circle from A and B. What is the position of these latter tangents with respect to each other? Apply test. Give reasons.

18. Draw two tangents to a circle from an external point, and join the points of contact What is the relation between the angles this "chord of contact" makes with the tangents? Apply test. Give reasons.

19. Two circles touch externally and parallel diameters are drawn. Lines are drawn from opposite ends of these diameters to the point of contact: what position do they occupy with respect to each other?

20. Two circles touch internally and parallel diameters are drawn. Lines are drawn from corresponding ends of these diameters to the point of contact: what position do they occupy with respect to each other?

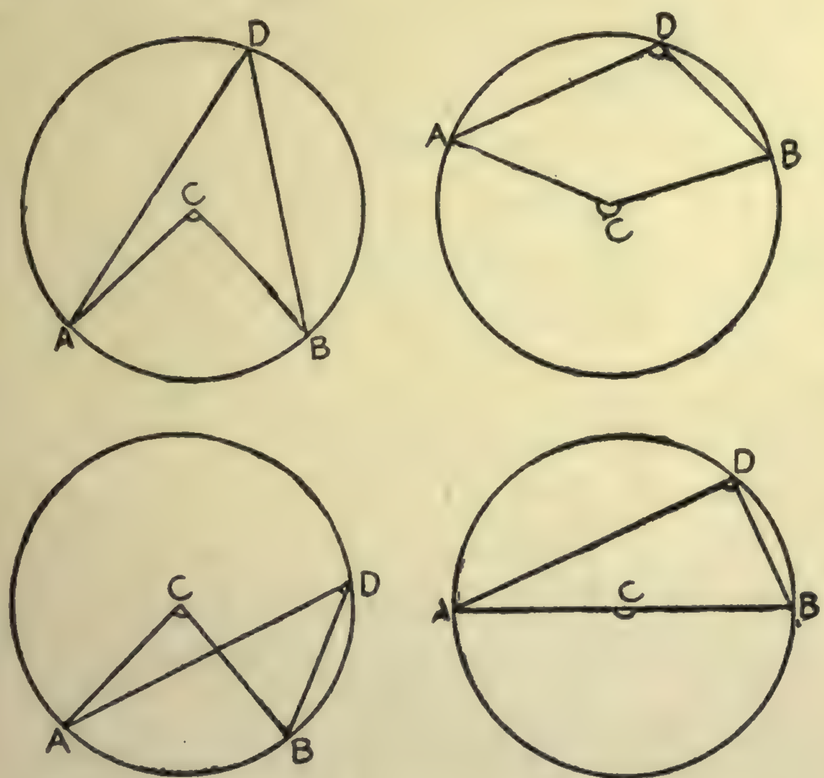
21. Describe two circles with radii $1\frac{1}{4}$ in. and $\frac{1}{2}$ in., respectively, their centres being 3 in. apart. Concentric with the larger, describe a third circle of radius $\frac{3}{4}$ in. $(1\frac{1}{4}-\frac{1}{2})$; and from the centre of the smallest circle draw a tangent to this third circle. Draw a line parallel to this tangent, and at distance $\frac{1}{2}$ in. from it. What is this last line with respect to the first two circles? Apply tests.

22. Describe two circles with radii $1\frac{1}{4}$ in. and $\frac{1}{2}$ in., respectively, their centres being 3 in. apart. Concentric with the larger circle, describe a third circle of radius $1\frac{3}{4}$ in. $(1\frac{1}{4}+\frac{1}{2})$; and from the centre of the smallest circle draw a tangent to this third circle. Draw a line parallel to this tangent, and at distance $\frac{1}{2}$ in. from it. What is this last line with respect to the first two circles? Apply tests.

CHAPTER XII.

Angles in a Circle.

1. The angles **ACB, ADB** stand on the same arc **AB**, the one being at the centre and the other at the circumference.

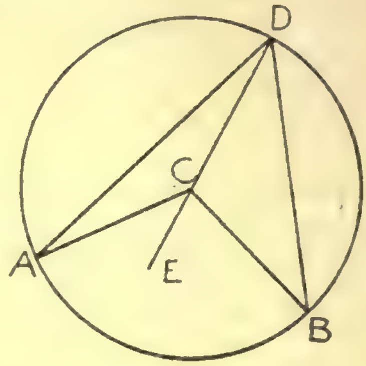


Measure the number of degrees in each, and compare these numbers.

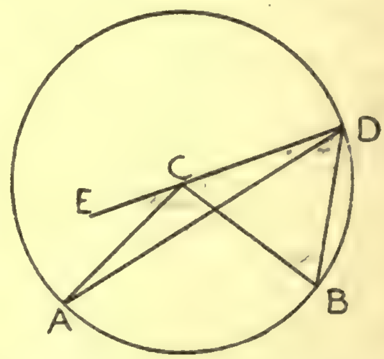
Make the same constructions in the case of two or three other circles, and repeat the measurements and comparison.

What is your conclusion as to the size of the angle at the centre, compared with the size of the angle at the circumference?

The relation between these angles may be *reasoned out* as follows:

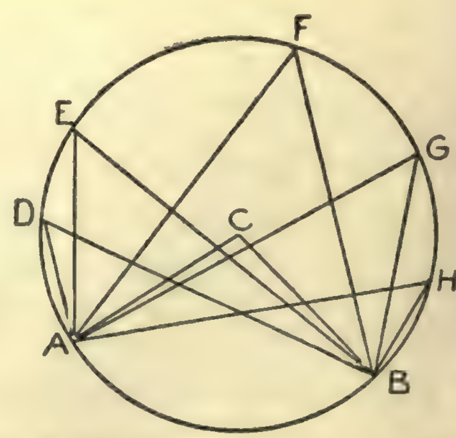


CAD is an isosceles triangle; and therefore the angles **CAD, CDA** are equal. Hence the exterior angle **ACE,** which is equal to their sum (Ch. V., 1), must be twice **ADC.** Similarly **BCE** is twice **BDC.** Therefore the sum (or difference, see second figure) **ACB** is twice **ADB.**



That is, **the angle at the centre of a circle is double the angle at the circumference which stands upon the same arc** (here **AB**).

The truth of this should be tested by describing a number of circles, constructing, in each case, an angle at the centre and another at the circumference on the same arc, and using the protractor to determine the magnitudes of these angles.

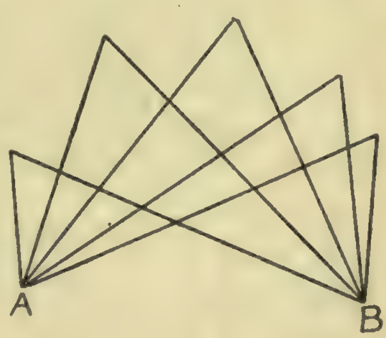


2. Construct such a figure as the annexed, where an angle **ACB** at the centre, and a number of angles **ADB, AEB,** . . . . at the circumference, stand on the same arc **AB.** Then, adjusting the bevel to the angles at the circumference, compare their magnitudes. The result of such a comparison might have been anticipated, since each of

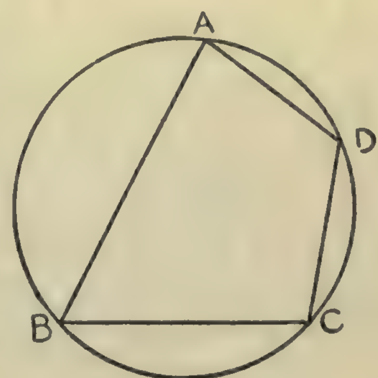
the angles at the circumference is half the same angle, **ACB**, at the centre.

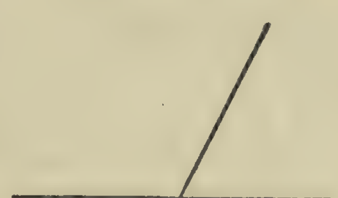
Hence **angles described in the same segment of a circle,** *i.e.*, **angles standing on the same arc of a circle, being on the circumference, are equal to one another.**

Using the bevel, construct a number of angles as in the annexed figure, all of the same magnitude, and with the sides of each passing through the points **A** and **B.** Then taking any three of the angular points, and, by the method of Ch. X., 6, constructing for the circle through these three points, show, by describing the circle, that it passes through the other angular points, and also through the points **A** and **B.**

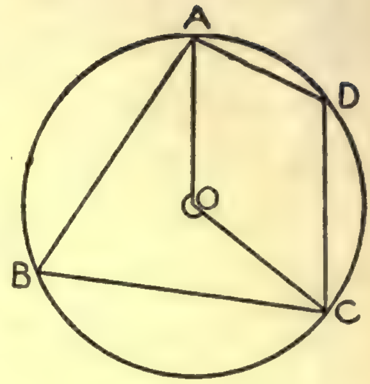


3. Take any four points, **A, B, C, D,** on the circumference of a circle, and join them as in the figure, so constructing a quadrilateral in the circle. Adjust the bevel to the opposite angles **B** and **D,** and construct angles equal to them adjacent to one another. What do you observe with reference to the sum of the angles **B** and **D**? What with reference to the sum of the angles **A** and **C**?



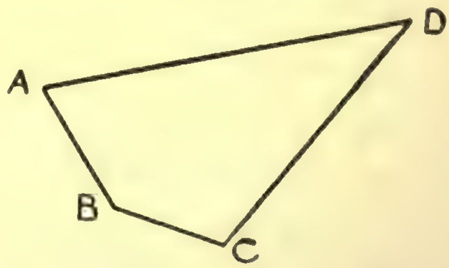


Repeat this measurement with respect to the opposite angles of other quadrilaterals in circles.



The annexed figure suggests what conclusion should be reached with respect to the sum of the angles at **B** and **D** and at **A** and **C.** For the angle marked at **O** is double of the angle **ADC** (Ch. XII., 1); and the other angle at **O** is double the angle **ABC.** Therefore the angles at **O** are together double the sum of the angles **ADC** and **ABC.** But the angles at **O** make up four right angles. Hence the angles **ABC, ADC** are together equal to two right angles.

Hence **the opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.**



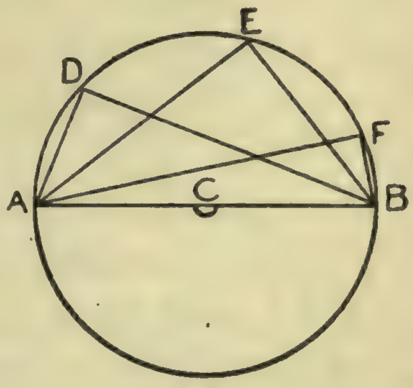
Using the protractor, construct a quadrilateral with two of its opposite angles together equal to two right angles. Taking any three of the angular points, and, by the method of Ch. X., 6, constructing for the circle through these three points, and describing the circle, note the position of the quadrilateral with respect to the circle.

Repeat the construction for several such quadrilaterals.

The result of such observations is that **if the opposite angles of a quadrilateral are together equal to two right angles, a circle can be described about it.**

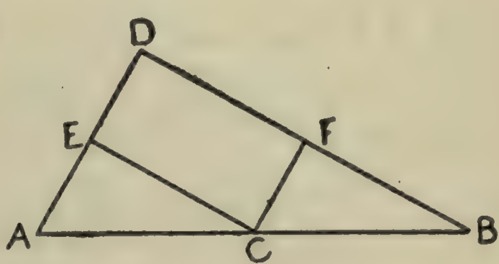
Since a quadrilateral can be divided into two triangles by joining its opposite angles, the sum of all the angles of any quadrilateral is four right angles. Hence if the sum of a pair of opposite angles be two right angles, the sum of the other pair is two right angles also.

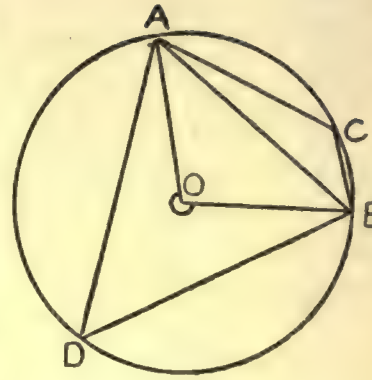
4. Describe a circle, and in the semicircle construct a number of angles as indicated in the figure. Adjust the protractor to the angles **ADB**, **AEB**, ... What is the magnitude of these angles?



The magnitude of the angle in a semicircle may be *proved* thus: The *straight* angle **ACB** at the centre is (Ch. XII., 1) double any of the angles at the circumference. But the straight angle **ACB** is 180°. Hence **the angle in a semicircle is** 90°.

ADB being a right-angled triangle, find the centre of the circle through **A**, **D** and **B**, by bisecting **AD**, **BD** and drawing the perpendiculars **EC**, **FC**. Note that these perpendiculars intersect in **AB**; and note also that **C**, being the centre of the circle through **A**, **D** and **B**, the centre of the hypotenuse of a right-angled triangle is equidistant from the three angles of the triangle.

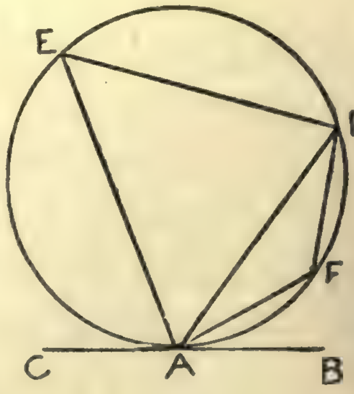




5. A chord, such as **AB**, which does not pass through the centre, divides the circle into two segments, one of which, **ADB**, is greater, and the other, **ACB**, less than a semicircle. Evidently the marked angle **AOB** is greater than two right angles, and therefore the angle **ACB**, which is half of the marked angle **AOB**, is greater than one right angle. Similarly the angle **ADB**, being half the other angle at **O**, is less than a right angle.

Hence **the angle in a segment of a circle less than a semicircle is greater than a right angle; and the angle in a segment of a circle greater than a semicircle is less than a right angle.**

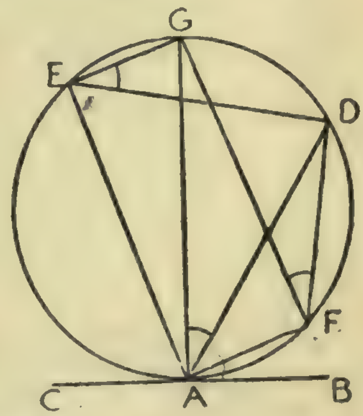
AC, CB contain an angle which has, in succession, the magnitudes 80°, 85°, 89°, 91°, 95°. Construct in the different cases the circles through **A, C** and **B**, and note the positions of the centre with respect to the side **AB**.



6. Draw with accuracy the tangent **CAB** at any point **A** on the circumference of a circle. From **A** draw any chord **AD**, and construct the angles **AED, AFD** in the segments into which **AD** divides the circle. Then, using the bevel, discover the relation in size between the angle **CAD** and the angle **AFD** in the alternate segment; and the relation between the angle **BAD** and the angle **AED** in the alternate segment.

Repeat the same examination in the case of different circles, drawing the chord at various inclinations to the tangent.

As a result of these observations we are led to the conclusion that **if from the point of contact of any tangent to a circle, a chord be drawn cutting the circle, the angles the chord makes with the tangent are equal to the angles in the alternate segments of the circle.**



We may establish the same result in the following way: Let **AG** be the diameter through **A.** The angles **GED, GAD, GFD,** are equal to one another because they stand on the same arc **GD.** Also the angles **CAG, BAG, AEG, AFG** are right angles. Hence

$$\angle \mathbf{BAG} - \angle \mathbf{DAG} = \angle \mathbf{AEG} - \angle \mathbf{DEG},$$

$$\text{or } \angle \mathbf{BAD} = \angle \mathbf{AED}, \text{ in alternate segment.}$$

Again, $\angle \mathbf{CAG} + \angle \mathbf{DAG} = \angle \mathbf{AFG} + \angle \mathbf{GFD},$

$$\text{or } \angle \mathbf{CAD} = \angle \mathbf{AFD}, \text{ in alternate segment.}$$

It will be noticed that, as **AD** revolves to the right about **A,** the angles **BAD, AED,** have just as much taken from them as **CAD, AFD** have added to them, the points **E** and **F** being supposed to remain stationary.

Placing the centre of the protractor on the circumference of a circle, and marking the initial line of protractor as a chord, we may place in the circle an angle of any required magnitude, *i.e.*, we may cut off from the circle a segment containing an angle of any size.

Exercises.

1. Describe a circle of radius $1\frac{1}{2}$ in., and in it place an angle of 60°. In it also describe a triangle of vertical angle 60° and altitude 2 in.

2. Describe a circle of radius 35 millimetres. From it cut off a segment containing an angle of 50°, and describe in it a triangle with angles 50°, 30° and 100°.

3. Describe a circle of radius 40 millimetres, and in it describe a triangle with angles 50°, 55° and 75°.

4. Describe a circle of radius 2 in. Draw a chord AB, cutting off a segment containing an angle of 120°, and a chord BC, cutting off a segment containing an angle of 100°. What is the angle contained in the segment cut off by CA? Apply test. Give reason.

5. Describe a circle of radius 50 millimetres, and in it draw a chord cutting off a segment containing an angle of 55°. What angle is contained in the segment which forms the rest of the circle? Apply test. Give reason.

6. Describe a circle of radius $1\frac{3}{4}$ in. Draw in it a chord AB, dividing the circle into two segments, ACB, ADB, containing angles of 70° and 110° respectively. Construct in the circle an angle CAD of 50°. What is the angle CBD? Mark on the quadrilateral ACBD the size of each angle.

7. Describe a circle of radius 40 millimetres, and in it construct a quadrilateral with angles 55°, 75°, 125°, 105°.

8. Describe a circle of radius 45 millimetres, and in it draw a number of chords, AB, CD, EF, . . . all cutting off angles of 60°. Are the chords all of the same length? Apply test. Give reasons.

9. Describe a circle of radius $1\frac{1}{2}$ in., and in it construct a triangle with angles 30°, 70°, 80°. Does the size of the triangle vary according as it happens to be placed in the circle? Give reasons.

10. Describe a circle of radius 2 in., and in it construct a quadrilateral with angles 45°, 120°, 135°, 60°. Show that the size and shape of the quadrilateral can be made to vary. What lines belonging to the quadrilateral remain constant?

11. ABCD is a quadrilateral in a circle, and the side AB is produced to E. To what angle of the quadrilateral is the exterior angle CBE equal? Apply test. Give reasons.

12. AB is a line of length $2\frac{1}{4}$ in. If on it a segment of a circle is to be constructed containing an angle of 60°, what angle will AB subtend at the centre C? What are the angles of the triangle CAB? Find C by construction, and then describe the circle.

13. AB is a line of length 60 millimetres. Following the method suggested in the previous question, construct on it a segment of a circle containing an angle of 70°. Test the accuracy of your construction by measuring an angle in the segment.

14. AB is a line of length $2\frac{1}{2}$ in.; to construct on it a segment of a circle containing an angle of 70°: Make BAC = 90°, ABC = 90° − 70° = 30°. Then ACB = 70°. Bisect BC at O, and with O as centre and OA, OB, or OC as radius, describe a circle. The segment ACB contains an angle of 70°, and it stands on AB.

15. Construct a triangle with sides 60, 75 and 85 millimetres. On these sides, and within the triangle, construct segments containing angles of 120°. Should these segments all pass through the same point within the triangle?

16. AB, CD are two chords, perpendicular to each other, in a circle whose centre is O. Of what angles are the angles AOC, BOD double? What, therefore, is their sum?

17. AB, CD are two chords of a circle, intersecting in E. Show that the triangles AEC, DEB are equiangular.

18. ABCD is a quadrilateral in a circle, and the sides AB, CD, produced, meet in E. Show that the triangles EBC, EDA are equiangular.

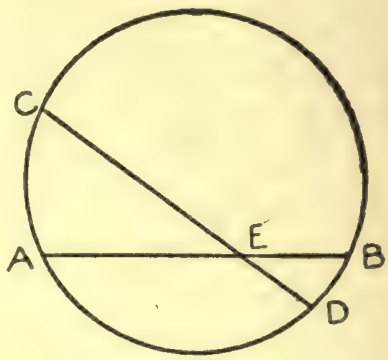
19. AB, AC are tangents to a circle whose centre is O. Show that BOC = 180 − A; also that the angle in the segment BC, between the tangents, contains an angle $90° + \frac{1}{2}$ A.

20. AD, BE are drawn perpendicular to the opposite sides of the triangle ABC. Show that a circle can be described about AEDB, and describe it. How are the angles ABC, DEC related? Apply test. Give reasons.

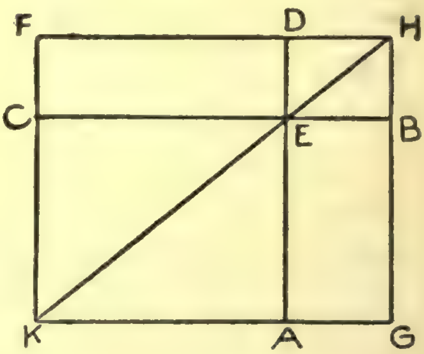
CHAPTER XIII.

Relation Between Segments of Intersecting Chords.

1. **AEB** and **CED** are any two chords in a circle, intersecting at **E**.



In the second figure **CEB** and **AED** are any two lines drawn perpendicular to each other, and and on these we lay off the following distances with the dividers:



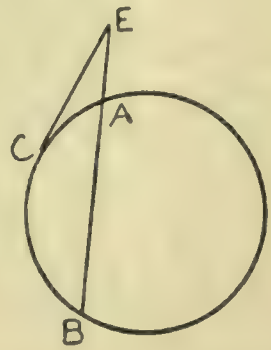
AE = **AE** of circle
EB = **EB** " "
CE = **CE** " "
ED = **ED** " "

Complete the rectangles **CEDF** and **AEBG**, and let **FD** and **GB** meet in **H**. Then produce the lines **FC, HE** and **GA,** and note how nearly they come to passing through the same point (at **K**). Go over the measurements and construction with extreme care, getting rid of all inaccuracies. Do these lines (**FC, HE, GA**) all pass through the same point? If they do, how do the areas **CEDF, AEBG** compare in size (Ch. VIII., 6), and therefore the rectangles **AE.EB, CE.ED,** contained by the segments of the chords?

Measure the number of millimetres in each of the lines **AE, EB, CE, ED** in the circle, and examine whether the product of **AE** and **EB** is approximately equal to the product of **CE** and **ED.**

Describe other circles, draw two chords in each, and repeat in the case of each circle the construction of the second figure. Repeat also the measurements and multiplications.

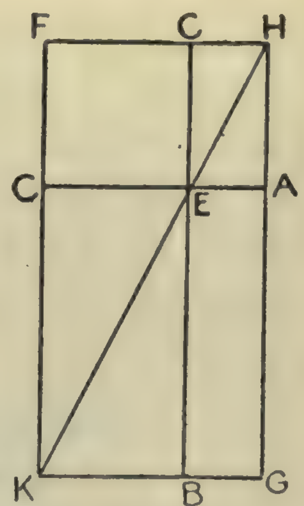
The result of our observations may be stated as follows: **If two chords of a circle cut one another within the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.**



2. Draw accurately the tangent **EC**; draw also the secant **EAB**.

In the second figure **CEA, BEC** are any two lines drawn at right angles to each other, and on these we lay off the following distances with the dividers:

EA = **EA** of circle
EB = **EB** " "
EC, EC = **EC** " "



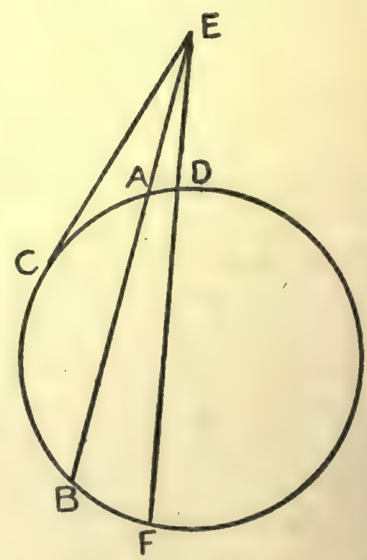
Complete the rectangle **EBGA** and the square **ECFC**, and let **FC, GA** meet in **H.** Then produce the lines **FC, HE** and **GB**, and note how nearly they come to passing through the same point (at **K**). Go over the measurements and construction with extreme care, getting rid of all inaccuracies. Do these lines (**FC, HE, GB**) all pass through the same point? If they do, how do the areas **EBGA, ECFC** compare in size (Ch. VIII., 6), and therefore the rectangle **EA.EB** and square on **EC** (see figure of circle)?

Measure the numbers of millimetres in each of the lines **EA, EB, EC** in the first figure, and examine whether the product of **EA** and **EB** is approximately equal to the square of **EC.**

Describe other circles, draw to each a secant and a tangent from the same point, and repeat in the case of each the construction of the second figure. Repeat also the measurements and multiplications.

The result of our observations may be stated as follows: **If from any point without a circle two straight lines be drawn, one a secant and the other a tangent, then the rectangle contained by the secant and the part of it without the circle is equal to the square on the tangent.**

If another secant **EDF** be drawn, since the rectangle contained by **EA** and **EB** is equal to the square on **EC,** and the rectangle contained by **ED** and **EF** is equal to the square on **EC,** therefore the rectangle contained by **EA** and **EB** is equal to the rectangle contained by **ED** and **EF.**



The segments of one chord are 3, 4, and of another 2, 6 quarters of an inch, the chords making an angle of 30° with one another, describe the circle through the extremities of the chords. If the segments of another line through the intersection of the chords be $1\frac{1}{2}$ and 8 quarters of an inch, do the ends of this necessarily rest on the circle? Place the line that its ends may so rest.

The tangent to a circle is 60 millimetres; a secant is 90, and the part of it without the circle 40 millimetres. These lines make an angle of 60° with one another. Describe the circle.

Exercises.

1. Two lines AB, CD intersect in E. AE = 30, EB = 40, CE = 20, ED = 60 millimetres, so that AE.EB = CE.ED. Show that a circle can be described to pass through the four points A, C, B, D, *i.e.*, that a circle through A, D, B, say, also passes through C.

2. Two lines, AB, CD cut one another in E. AE = $1\frac{1}{2}$, EB = 2, CE = 3, ED = 1 in., so that AE.EB = CE.ED. Describe a circle to pass through A, C, B, D.

3. Describe a circle of radius 2 in. Draw a diameter AB. Take in it a point C at distance 1 in. from centre, and draw chord DCE perpendicular to AB. By construction, as in text, show that rectangle AC.CB is equal to square on CD.

It may also be shown that CD = $\sqrt{3}$ in. by proving it equal to the altitude of an equilateral triangle whose side is 2 in.

4. Describe a circle of radius in $2\frac{1}{2}$ in. Draw a diameter AB. In it take a point C at distance $1\frac{1}{2}$ in. from centre, and draw chord DCE perpendicular to AB. What should be the length of CD? Measure it.

5. Two lines intersect at an angle of 30°. The segments of one 2 in. and $\frac{1}{2}$ in., of the other, both 1 in. Describe a circle to pass through the ends of the lines. With what inclination of the lines to one another would the longer line become a diameter?

6. Describe a circle of radius 3 in. In it place a chord of length 4 in., and take in the chord a point at distance 1 in. from an end. Through this point draw another chord whose segments shall be $1\frac{1}{2}$ in. and 2 in.

7. Describe a circle of radius 70 millimetres. In it place a chord of length 90 millimetres, and take a point in the chord at distance

40 millimetres from an end. Through this point draw two chords whose segments shall be 20 and 100 millimetres.

8. On a line take lengths, AB, AC, of 27 and 48 millimetres, in the same direction. Draw a line AD of 36 millimetres, making an angle of 45° with AC. Describe a circle through B, C, D. What is AD with respect to this circle?

9. Same problem as previous, but with $AB=36$, $AC=64$, $AD=48$ millimetres, and angle between AC, AD, 60°. Describe a circle through B, C and D. What position does AD occupy with respect to it?

10. AB, AC, measured along the same line, in the same direction, are 36 and 64 millimetres; and AD another line through A is 48 millimetres. Place AD so that the circle through B, C and D may have its centre in AC.

11. AB, AC measured along the same line, in the same direction, are 18 and 72 millimetres. Describe a number of circles through B and C, and from A draw a tangent to each. Measure the lengths of these tangents. What relation between the lengths and why?

12. Two lines AB, AC of length $\sqrt{3}$ in., both touch the same circle at B and C, and make an angle of 60° with one another. Construct the circle. What is its radius?

13. AB, AC measured along the same line in the same direction are 48 and 108 millimetres. Describe a circle on BC as diameter, and draw a line ADE cutting the circle in D and E, such that $AD=54$ millimetres. What is the length of AE? Draw a tangent to the circle from A. What is its length?

14. Describe two circles of radii 1 and 2 inches respectively, intersecting in A and B. Draw a straight line through A and B, and from any point in it, draw a tangent to each circle. Measure the tangents. What relation between their lengths? Give reason.

15. Describe two circles of radii 25 and 70 millimetres, intersecting in A and B. Draw a straight line through A and B, and from any point in it draw a tangent to each circle. Measure the tangents. What relation between their lengths? Give reason.

16. Describe three circles of radii 2, 3 and $3\frac{1}{2}$ inches, so that each intersects the other two. Through each pair of points of intersection draw straight lines. These three lines should pass through the same point.

17. If the tangents to two intersecting circles from any point be equal, that point must lie on the line joining the points of intersection of the circles.

18. The common chord of two intersecting circles on being produced, cuts a line that touches both circles. Show that the tangent line must be bisected.

19. ABC is a triangle right-angled at C, and from C a perpendicular CD is drawn to AB. By describing a circle about ABC, show that the rectangle AD.DB is equal to the square on CD.

20. ABC is a triangle right-angled at C, and from C a perpendicular CD is drawn to AB. By describing a circle about the triangle CDB, show that the rectangle AD.AB is equal to the square on AC.

21. In the previous question, describe a circle about the triangle ACD, and show that the rectangle BA.BD is equal to the square on BC.

22. The sides of a triangle are 3, 4, 5, and a perpendicular is dropped from the right angle in the hypotenuse. Find the lengths of the segments of the hypotenuse on each side of the perpendicular, and also the length of the perpendicular.

CHAPTER XIV.

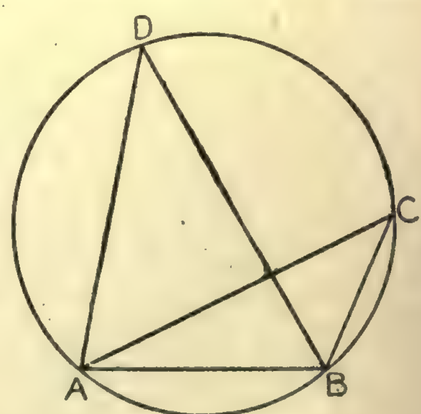
Triangles In and About Circles.

1. A triangle is said to be inscribed in a circle when the three angular points of the triangle rest on the circumference of the circle.

We evidently cannot in general construct in a circle of given size a triangle equal to a given triangle. In a small circle we could not place a large triangle. Indeed we have seen (Ch. X., 6) that there is but one circle which can be made to fit round a triangle of given size.

We can, however, always inscribe in any circle a triangle equiangular to another triangle, *i.e.*, a triangle with its angles of given size, their sum of course being 180°. Thus let it be required to construct in a given circle a triangle whose angles shall be 30°, 70°, 80°.

Using the protractor, adjust the bevel to an angle equal to any one of these, say, 30°. Place the angle of the bevel at any point **C** on the circumference, and with a needle mark the points, **A** and **B**, where the legs of the bevel cross the circumference. We have thus a segment **ACB** containing an angle of 30°, and all angles in the segment **ACB** are angles of 30°. With the protractor at **A** make the angle **BAD** of 80°. Join **BD.** Then



ADB is an angle of 30°. Hence the remaining angle **ABD** is 70°.

Of course the angle of 30° at **C** may be constructed with the protractor. The segment containing an angle of 30° may also be obtained by constructing at the centre an angle of 60°.

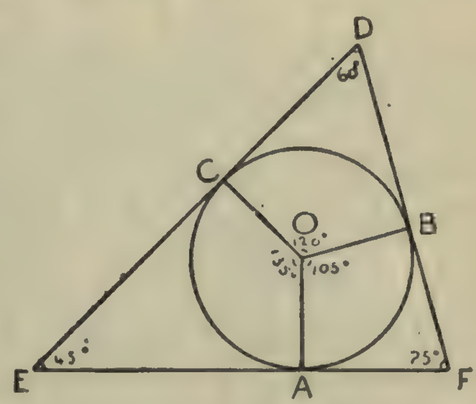
In a circle whose radius is 45 millimetres, construct a triangle whose angles are 75°, 45° and 60°.

In a circle whose radius is $1\frac{1}{2}$ in., construct a triangle whose angles are 65°, 75° and 40°.

2. **To construct a triangle whose sides shall be tangents to a given circle, and whose angles shall be of given magnitude,** say, 75°, 45° and 60°.

We can scarcely here proceed as in the previous case, adjusting the legs of the bevel to, say, the angle 75°, and placing them across the circle so as to be tangents to it. To assume that we can construct the tangent to a circle by laying the ruler against it and so drawing a line, is equivalent to assuming that we can lay off a right angle, using only the judgment of the eye.

It will be well to proceed thus: Find the angles which are the supplements of 75°, 45° and 60°, *i.e.*, 105°, 135° and 120°. Draw any radius **OA**, and make the angle **AOB** of 105°, and the angle **AOC** of 135°. The remaining angle **BOC** must be of 120°, since 105° + 135° + 120° = 360°. Draw lines (tangents) at **A, B** and **C** at right angles to the radii.



Since the angles of a quadrilateral make up four right angles, and the angles at **A** and **C** are right angles, therefore **AOC** + **AEC** = 180°. But **AOC** is 135°.

Therefore **AEC** is 45°, if **AOC** has been accurately constructed, and the tangents at **A** and **C** correctly drawn. Similarly the angles at **D** and **F** are 60° and 75° respectively.

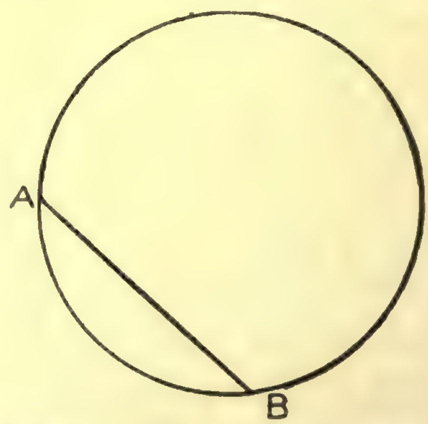
The triangle DEF is said to have been described about the circle.

About a circle whose radius is 20 millimetres, construct a triangle whose angles are 70°, 80° and 30°.

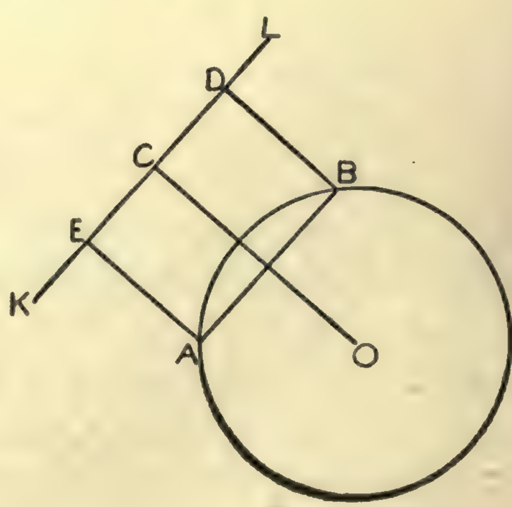
About a circle whose radius is 35 millimetres, construct a triangle whose angles are 90°, 30°, and 60°.

About a circle whose radius is $1\frac{1}{4}$ in., construct an equilateral triangle.

About a circle whose radius is $1\frac{1}{2}$ in., construct an isosceles triangle whose vertical angle is 30°.



3. In a circle we readily place a chord of any required length. For, take the length on the ruler with the points of the dividers, and place the points of the dividers on the circumference of the circle. The ends of the chord, **A, B** are thus marked, and the chord can be drawn.



We can without difficulty draw the chord in a required position, for example, parallel to a given line, **KL**: Draw **OC** perpendicular to **KL**, and mark off **CD, CE** each equal to half the length of the chord. Then draw **DB, EA,** parallel to **CO.**

The chord **AB** is equal to **ED**, and therefore is of the required length, and it is parallel to **KL**.

We may draw **EA** alone perpendicular to **KL**, and then draw **AB** parallel to **KL**, thus not using the point **D** or line **DB**.

Of course the chord can never be greater than the diameter of the circle in which it is to be placed.

In a circle whose radius is 55 millimetres, draw chords, with one end at the same point, of lengths 20, 25, 30, 35, 40, 45, 50, 55 and 110 millimetres.

In a circle of radius 1 inch, place ten chords of length ½ inch, such that each ends at the point where the next begins.

In a circle of radius 30 millimetres, place six chords each of length 30 millimetres, such that each ends where the next begins.

In a circle place a chord of given length so that it may be perpendicular to a given line.

Exercises.

1. In a circle of radius 45 millimetres, place an angle of 35°; also an angle of 145°.

2. In the circle of the previous question place these same angles so that the chord or chords on which they stand may be parallel to a line that makes 45° with the edge of your paper.

3. In a circle of radius 2 in., place an angle of 50°, so that the chord on which it stands may be perpendicular to a line that makes an angle of 60° with the edge of your paper.

4. In a circle of radius 1 in., place in succession four chords, AB, BC, . . . , each of length $\sqrt{2}$ in.

5. In a circle of radius 1½ in., construct an equilateral triangle.

6. In a circle of radius 2 in., construct an isosceles triangle, the angle at the vertex being 55°. (Construct at centre an angle of 110°. The symmetry of the circle suggests the rest of the construction.)

7. In a circle of diameter $3\frac{1}{2}$ in., construct an equilateral triangle, such that its base shall be parallel to the top or bottom of your paper. (Draw a line through centre perpendicular to top or bottom of paper, and at centre construct, on each side of this line, angles of 60°. Etc.)

8. Construct a triangle with angles of 55°, 65°, and 60°, and in a circle whose radius is $1\frac{3}{4}$ in. construct a triangle equiangular to this, its sides being also parallel to the sides of this triangle.

9. Describe a circle of radius 48 millimetres, and draw a line making an angle of 45° with the edge of your paper. Construct a triangle with angles 48°, 75°, and 57°, so that the side opposite 48° may be parallel to the line.

10. Describe a circle of radius 40 millimetres, and draw a line making an angle of 60° with the side of your paper. Draw a tangent to the circle parallel to this line. (From centre drop a perpendicular on the line. This gives point through which tangent is to be drawn.)

11. Describe a circle of radius 35 millimetres. Draw a line making an angle of 75° with the top or bottom of your paper, and draw a tangent to the circle perpendicular to this line. (Draw perpendicular to line, and then tangent parallel to this perpendicular.)

12. About a circle of radius 1 in. describe an equilateral triangle.

13. Describe a circle of radius 35 millimetres, and about it describe an equilateral triangle so that two of the sides may make angles of 60° with the side of your paper, the third side being parallel.

14. Describe a circle of radius 25 millimetres, and about it describe an isosceles triangle whose vertical angle is 40°, the base of the triangle being parallel to the top or bottom of your paper.

15. About a circle of radius $1\frac{1}{4}$ in. describe a triangle whose angles are 30°, 70° and 80°.

16. Draw any three intersecting lines. Describe a circle of radius $1\frac{3}{16}$ in., and about it describe a triangle whose sides are parallel to the lines. Test the accuracy of your construction by comparing the angles of the two triangles.

17. When a triangle ABC is inscribed in a circle, what are the magnitudes of the angles which the sides subtend at the centre compared with the magnitudes of the angles of the triangle?

18. Describe a circle of radius $1\frac{1}{2}$ in. In and about it describe two triangles with angles 50°, 60° and 70°, so that corresponding sides are parallel to each other.

19. An equilateral triangle is inscribed in a circle, and another is described about the circle. What relation exists between the lengths of the sides?

20. Describe a circle of radius 32 millimetres, and draw two tangents to it, such that the angle between them is 25°.

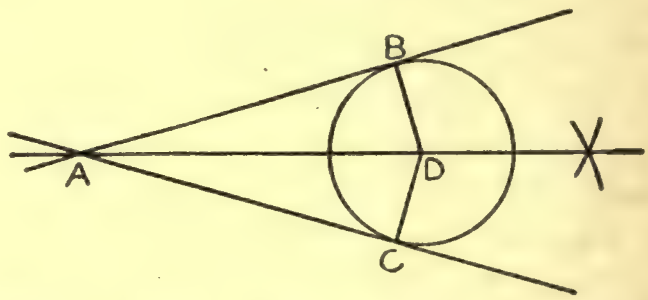
21. Describe a circle of radius 1 in., and from the same point draw two tangents to the circle, each of length 3 in.

22. Describe two circles of radii 1 in. and 2 in. In them describe triangles with angles of 45°, 65° and 70°. Compare the lengths of corresponding sides of the two triangles.

CHAPTER XV.

Circles In and About Triangles.

1. If the angle **BAC,** between two lines, be bisected, and, from any point **D** in it, perpendiculars **DB, DC** be drawn, these perpendiculars are evidently equal. If, then, a circle be described with centre **D,** and radius **DB** or **DC,** it will touch both the lines.



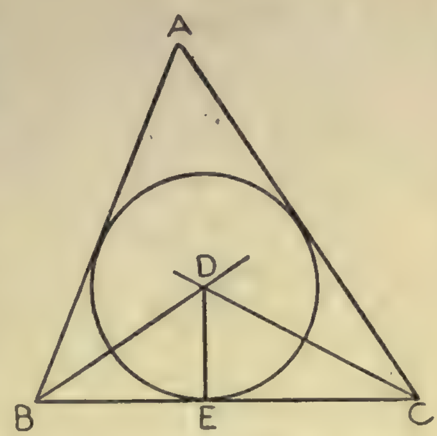
Thus all circles touching both lines have their centres in the straight line which bisects the angle between the lines.

Two lines make an angle of 120° with one another. Describe four circles, of different radii, touching both of them.

Two lines make an angle of 80° with one another. Describe a circle of radius $\frac{1}{2}$ in. to touch both of them; also of radius 1 in.

Two lines make an angle of 60° with one another. Describe a circle touching both of them; also a second circle touching the previous circle and the two lines.

2. **We may describe a circle touching the three sides of a triangle as follows:**



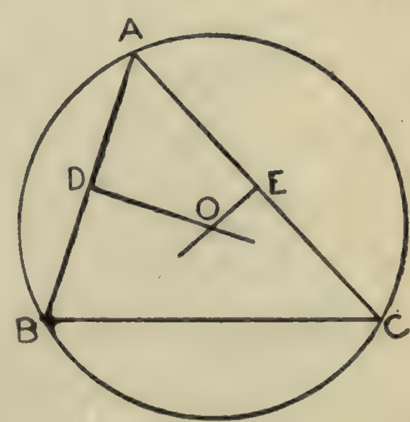
Bisect the angles at **B** and **C** by the lines **BD, CD.** Then **BD** contains the centres of circles touching **BA** and **BC**; and **CD** contains the centres of circles touching **CA** and **CB.** Hence **D** is the centre of a circle which touches all three sides. **DE,** perpendicular to **BC,** is the radius of this circle. Hence with centre **D** and radius **DE,** describe a circle. This is the circle *inscribed* in the triangle **ABC.**

The utmost care is to be exercised in accurately bisecting the angles; otherwise it may be found that, when the circle is described, it cuts a side, or falls short of one.

Inscribe a circle in the triangle whose sides are 75, 80 and 95 millimetres.

Describe a circle to touch the other side of **BC** (any triangle **ABC**), and the sides **AB** and **AC** produced.

The base of a triangle is 2 in., and the angles at the base are 40° and 110°. Inscribe a circle in it. Measure its radius.



3. We have already (Ch. X., 6), in effect, shown **how to describe a circle about any triangle,** *i.e.*, to pass through the angular points of the triangle. Two sides, say **AB** and **AC,** are bisected, and **DO, EO** are drawn through the points of bisection

perpendicular to **AB** and **AC**, respectively. Then all points in **DO** are equally distant from **A** and **B**; and all points in **EO** are equally distant from **A** and **C**. Hence **O** is equally distant from **A**, **B** and **C**; and if the sharp point of the compasses be placed at **O**, and the pencil end at **A**, or **B**, or **C**, and a circle be described, it will pass through **A**, **B** and **C**.

Here again the greatest care must be exercised in bisecting the sides, and in drawing the perpendiculars at the points of bisection; otherwise the circle will pass through the angle on which the pencil end of the compasses was placed, but may not pass through the two other angles.

Describe a circle about a triangle whose sides are 55, 70 and 90 millimetres. Measure its radius.

The side of an equilateral triangle is 3 in.; describe a circle about it.

Each of the equal sides of an isosceles triangle is 3 in., and the equal angles are each 75°. Describe a circle about it.

Should the course contained in this book prove too long for a year's work, it is suggested that Chapters XVI., XVII. and XVIII. be omitted, valuable though they may be as affording exercises in accurate geometrical construction.

Exercises.

1. Draw two lines making an angle of 50° with one another, and describe three circles touching both lines.

2. Two lines make an angle of 70° with one another. Describe a circle of radius $1\frac{1}{2}$ in. touching both of them. (Draw a perpendicular to either of the lines, of length $1\frac{1}{2}$ in., and through its end draw a line parallel to the line on which the perpendicular stands, producing this parallel until it meets the bisecting line.)

3. Two lines make an angle of 40° with one another. Describe a circle touching both of them; also a second circle touching the previous circle and the two lines. (At point where first circle cuts bisecting line, draw a line making an angle of 55° or 35° with it, according to cutting point selected.)

4. Describe a triangle with angles 30°, 60° and 90°, and hypotenuse 3 in., and in it inscribe a circle.

5. Describe a triangle with angles 30°, 60° and 90°, and hypotenuse 6 in., and in it inscribe a circle. Compare the length of the radius of this circle with length of the radius of circle in previous question.

6. Describe a triangle with sides 76, 68 and 44 millimetres, and in it inscribe a circle.

7. In the case of the triangle of the previous question, describe circles touching each side and the other two sides produced.

8. Having obtained the four circles of the two previous questions, through what points do the lines joining any two centres pass? What position does the line joining any two centres occupy with respect to the line joining the other two centres? Apply tests in both cases.

9. Two parallel lines are $1\frac{1}{2}$ in. apart, and a third line cuts them at an angle of 60°. Describe all the circles you can, each touching the three lines. What is the length of the radius?

10. In the previous question, what is the figure formed by joining the centres to the points where the parallels are cut by the third line? Apply test.

11. Is there any position which three lines can occupy, such that no circle can be described touching all?

12. Describe an equilateral triangle with side 2 in., and in it inscribe a circle. Express with exactness the radius of this circle.

13. Describe also a circle about the triangle of the previous question, and express with exactness its radius.

14. Construct a triangle with sides 40, 45 and 50 millimetres, and about it describe a circle.

15. Construct a triangle with sides 80, 90 and 100 millimetres, and about it describe a circle. Compare the length of radius of this circle with that of circle in previous question.

16. Is there any position which three points can occupy with respect to one another, such that a circle cannot be described to pass through all?

17. ABCD is a quadrilateral; $A=85°$, $B=80°$, $C=95°$; $AB=60$ and $BC=80$ millimetres. Construct the quadrilateral and describe a circle about it.

18. AB ($=3$ in.) and CD ($=2$ in.) are parallel and 1 in. apart. A line at right angles to one and through its bisection passes also through the bisection of the other. Describe a circle to pass through A, B, C, D.

19. A line AB is 3 in. long. Describe a circle of radius 3 in. to touch AB at A. Describe a second circle to touch the previous one and also AB at B.

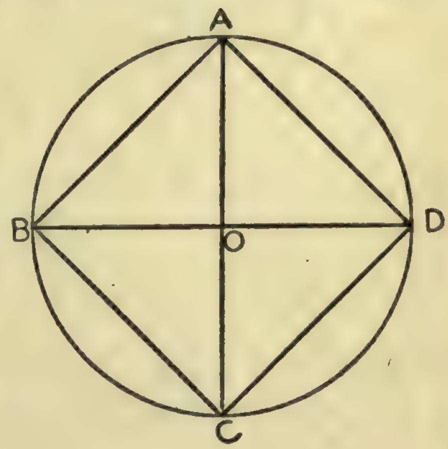
20. From the fact that two tangents from the same point to a circle are equal, what relation can you establish between the sums of the opposite sides of a quadrilateral whose sides touch a circle?

21. Construct a quadrilateral whose sides are 40, 30, 50 and 60 millimetres, and inscribe a circle in it.

CHAPTER XVI.

Squares and Circles In and About Circles and Squares.

1. To inscribe a square in a circle, draw two diameters at right angles to one another and join their extremities. The construction being accurately made, the set-square will show that the angles **A, B, C, D** are all right angles; and the equality of the sides **AB, BC, . . .** may be proved by using the dividers.



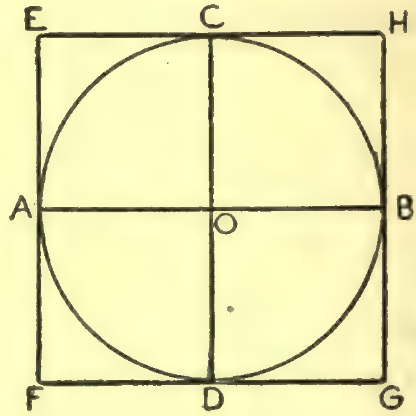
Of course the evident equality of the triangles **AOB, BOC, . . .** proves the equality of the sides, and the angles **ABC, BCD . . .** are all right angles, because they are angles in semicircles.

Inscribe a square in a circle of radius 40 millimetres. Test the accuracy of your construction by examining, with the dividers, the equality of the sides.

Inscribe a rectangle (which is not also a square) in a circle. Test the accuracy of your construction by examining, with the set-square, whether the angles are all right angles.

In a circle whose radius is 3 inches, inscribe a rectangle, one of whose sides is 1 inch. With instruments test the success of your construction,—the equality of opposite sides, the parallelism of opposite sides, the right-angledness of the figure.

2. **To describe a square about a given circle,** draw two diameters at right angles to each other, and through the ends of each diameter draw lines parallel to the other. The construction being accurately made, the set-square will show that the angles of **E, F, G, H** are all right angles, and the equality of the sides **EF, FG, . .** may be proved by using the dividers.



Evidently the figures **AOCE, AODF, ...** are equal squares, whence we readily prove that the sides of **EFGH** are all equal; and its angles are right angles.

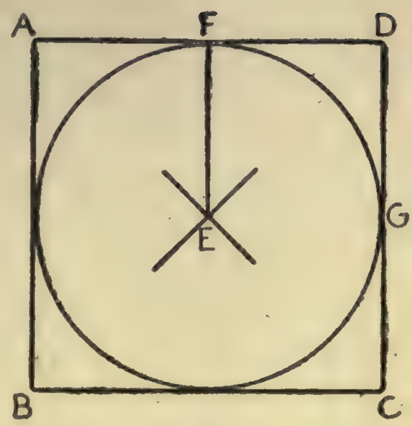
Describe a square about a circle whose radius is 30 millimeters. Test the accuracy of your construction by finding whether the sides are equal, using the dividers; and use the set-square to determine whether the angles are right angles.

Describe a square about a circle whose radius is $1\frac{1}{2}$ inches. As in the previous question, test the accuracy of your construction.

Draw two diameters in a circle not at right angles to each other, and draw tangents at their extremities. Determine the nature of the figure formed by the tangents by measuring the lengths of its sides.

3. **To inscribe a circle in a given square,** draw

portions of the diagonals of the square, so that they intersect, as at **E.** Draw **EF** perpendicular to one of the sides. With **EF** as radius, describe a circle. If the construction has been accurate the circle will touch the sides of the square.



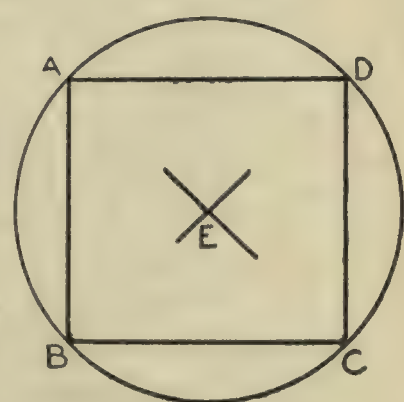
By drawing the complete diagonals it may readily be shown, from the equality of such triangles as **EFD, EGD,** that the perpendiculars from **E** on the sides are equal.

Describe a square with side of 4 inches, and in it inscribe a circle. Show, by measurement with dividers and set-square, that the lines joining the points of contact form a square. Show that the sides of this are perpendicular to the diagonals of the original square.

Inscribe a circle in the second square of the preceding question.

Inscribe a circle in a rhombus, each of whose sides is 4 inches, and one of whose angles is 60°.

4. **To describe a circle about a given square,** draw portions of the diagonals so that they intersect. Then, placing the sharp point of the compasses at **E,** where the diagonals intersect, and the pencil point on any one of the angles, and describing a circle, it will pass through the other angular points of the square.



The lines from **E** to the angles are equal if the square has been accurately constructed and the diagonals accurately drawn; for the diagonals of all parallelograms bisect each other, and the diagonals of a square are equal.

Construct a square whose side is 80 millimetres, and about it describe a circle.

Construct a square whose side is 40 millimetres, and about it describe a circle.

At the angular points of the square in the preceding question draw tangents to the circle, and, by measurement with the dividers and set-square, show that the tangents form a square.

About the square formed by the tangents in the preceding question describe a circle.

The sides of a rectangle are 80 and 35 millimetres. Describe a circle about it.

Starting with a square whose side is 100 millimetres, inscribe a circle in it, then a square within this circle, a circle within the last square, etc.

With the angular points of a square as centres, describe four circles, such that each touches two of the others. Describe a circle to touch these four circles.

If **ABCD** be a square, and from **AB, BC, CD** and **DA** equal lengths **AE, BF, CG, DH** be cut, what is the figure **EFGH**?

Exercises.

1. Inscribe a square in a circle of radius $\frac{3}{4}$ in. Test accuracy of construction.

2. Inscribe a square in a circle of radius $1\frac{1}{2}$ in. Test accuracy of construction. Compare length of side of square with that of side of square in previous question.

Compare area of square with that of square in previous question.

3. Describe a circle of radius $1\frac{3}{4}$ in. In it draw two diameters making an angle of 30° with one another, and join their extremities. What is the resulting quadrilateral? Apply tests.

4. Describe a circle of radius 30 millimetres, and in it construct a rectangle one of whose sides is 25 millimetres. Test accuracy of construction.

5. Describe a circle of radius 60 millimetres, and in it construct a rectangle one of whose sides is 50 millimetres. Test accuracy of construction.

Compare the length of the longer side of this rectangle with the length of the longer side of the rectangle in the preceding question. How are the areas of the rectangles related?

6. Describe a circle of radius $\frac{3}{4}$ in., and about it describe a square. Test accuracy of construction.

7. Describe a circle of radius 35 millimetres, and both in and about it construct squares.

8. What ratio always exists between the sides of squares about and in the same circle? What ratio between their areas?

9. Draw two diameters of a circle (radius 1 in.) at an angle of 30° to one another, and at their ends draw tangents. What is the resulting quadrilateral about the circle? Apply test.

10. About a circle of radius 35 millimetres construct a rhombus with angles 60° and 120°. Test accuracy of construction. Show that the length of each side must be $\frac{70}{\sqrt{3}}$ millimetres.

11. Why is it that a rectangle or parallelogram about a circle must always be a square or rhombus?

12. About a circle of radius $1\frac{1}{4}$ in. construct a rhombus with one angle three times the other. What is the length of the sides?

13. Construct a square with side 2 in., and in it inscribe a circle. Join points of contact, and show by tests that the resulting figure is a square. What is its side?

14. Construct a rhombus with sides 50 millimetres in length and angles 75° and 105°, and describe a circle touching the sides.

15. Construct a rhombus with diagonals of 60 and 80 millimetres, and in it inscribe a circle. Measure length of radius, and test accuracy of measurement by calculation.

16. Construct a square with side of 2 in., and about it describe a circle. At the angular points of the square draw tangents to the circle, and by tests show that the resulting figure is a square.

17. Construct a rectangle with sides 30 and 40 millimetres, and about it describe a circle. Measure radius of circle, and test accuracy of measurement by calculation.

18. Construct a rectangle such that when a circle is described about it, and tangents drawn at the angular points, the resulting rhombus shall have angles of 60° and 120°.

19. Beginning with a circle of radius 50 millimetres, inscribe a square in it, then a circle within the square, and finally a square within this latter circle. Test the accuracy of the final square.

What are the lengths of the sides of the squares, and the length of the radius of the second circle?

20. About a circle of radius $1\frac{3}{4}$ in. describe a quadrilateral with angles 60°, 150°, 110°, 40°.

Can you describe about a circle a quadrilateral equiangular to any given quadrilateral?

CHAPTER XVII.

Regular Polygons.

1. A **polygon** is a rectilineal figure contained by more than four straight sides.

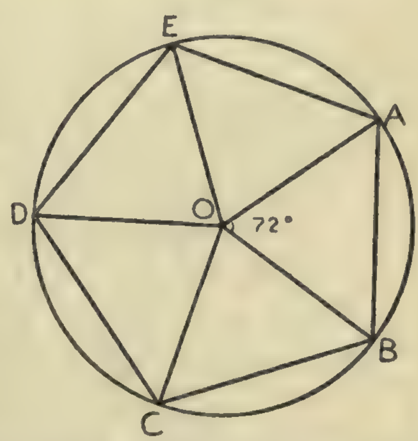
A **pentagon**	is a	figure	of 5	sides.
hexagon	"	"	6	"
heptagon	"	"	7	"
octagon	"	"	8	"
decagon	"	"	10	"
dodecagon	"	"	12	"
quindecagon	"	"	15	"

A polygon is said to be **regular** when all its sides are equal, and also its angles equal.

2. The angles at any point, for example, at the centre of a circle, make up 360°. We can divide this interval, by means of the protractor, into a number, 5, 6, 8, ... , of equal angles. If we prolong the sides of these angles until they intersect the circumference of the circle, and join the successive points of intersection, we have a **regular polygon** of 5, 6, 8, ... sides, as the case may be.

3. **To describe a regular pentagon in a circle:**

A pentagon having five sides, the angle subtended at the centre of the circle by the side of a regular pentagon inscribed in the circle, will be $\frac{1}{5}$ of $360° = 72°$. Using then the protractor, or adjusting the bevel to an angle of 72°, lay off at the centre 5 angles, each of this magnitude.

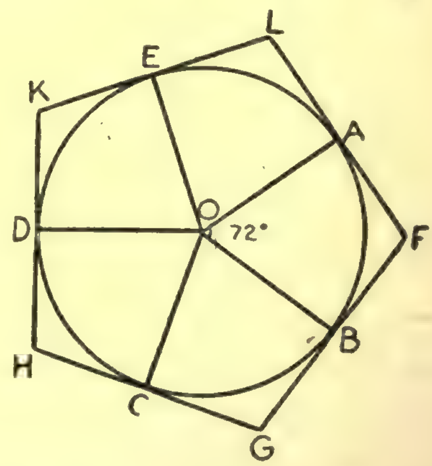


Produce the sides of the angles to meet the circumference, and join the succeeding points of intersection. The construction being accurately made, the bevel will show the equality of the angles **ABC, BCD, . . . ,** and the dividers will show the equality of the sides **AB, BC. . . .**

Of course, the evident equality of the isosceles triangles **OAB, OBC, . . . ,** proves the equality of the sides and angles of the pentagon.

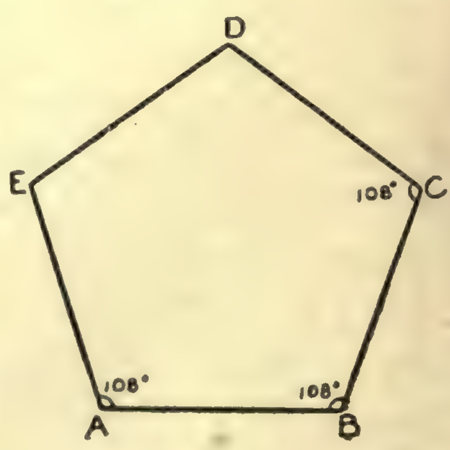
The angle at the vertex of each isosceles triangle in the figure being 72°, each angle at the base must be 54°; and therefore each of the angles (**ABC, BCD, . . .**) of a regular pentagon is 108°.

4. **If tangents to the circle be drawn at the angular points of the pentagon ABCDE, the tangents form another regular pentagon, which is said to be about the circle.**



The equality of the sides **FG, GH, . . .** may be tested with the dividers, and the equality of the angles **FGH, GHK, . . .** with the bevel.

5. If we wish **to construct on a given straight line** (AB), **as side, a regular pentagon,** at the points **A** and **B,** with the protractor we mark off angles **BAE, ABC** of 108°, and with the dividers make **BC** and **AE,** each equal to **AB.** At **C** we again make an angle **BCD** of 108°, and mark off **CD** equal to **AB.**



Joining **E** and **D**, we have a regular pentagon **ABCDE**. Using the bevel, we shall find that the angles at **E** and **D** are equal to the three other angles, and the dividers will prove the side **DE** to be equal to the other sides.

The radius of a circle being 36 millimetres, inscribe in it a regular pentagon. With the dividers and bevel prove the accuracy of your construction,—that the sides and angles are equal.

Describe also about the same circle a regular pentagon. With the dividers and bevel prove the accuracy of your construction.

On a line of length 2 inches, as side, construct a regular pentagon. With instruments prove the accuracy of your construction.

Exercises.

1. In a circle of radius 32 millimetres, inscribe a regular pentagon. Test equality of sides with dividers, and equality of angles with bevel or protractor.

2. In a circle of radius $1\frac{3}{4}$ in., inscribe a regular pentagon. Test accuracy of construction.

3. About a circle of radius $1\frac{1}{2}$ in., describe a regular pentagon. Test accuracy of construction.

4. About a circle of radius $\frac{3}{4}$ in., describe a regular pentagon. Test accuracy of construction.

5. In the two preceding questions, where the radius of one circle is twice that of the other, examine the relation between the lengths of all corresponding lines that can be drawn in the two figures,—sides, lines joining non-adjacent angles, segments of these lines by their intersection.

6. Inscribe two regular pentagons in any two circles of different radii. With the bevel examine the relation between all corresponding angles that can be formed in the two figures.

7. Describe an irregular equilateral pentagon, each side being 1 in.

8. About a circle of radius $1\frac{1}{2}$ in., describe a pentagon with angles 80°, 110°, 145°, 70°, and 135°.

9. Describe a regular pentagon with side of 1 in. Test accuracy of construction.

10. Describe a regular pentagon with side of 2 in. Test accuracy of construction.

11. In the two preceding questions, what is the relation between the radii of the two circles about the pentagons?

12. Hence if you have in a circle (radius OA) a regular pentagon with side 30 millimetres, how many times OA should you make the radius of a second circle, that the side of a regular pentagon in it may be 45 millimetres?

13. ABCDE being a regular pentagon, what sort of triangles are ACD, and ABC? What are the magnitudes of the angles CAD, ACD, CBD?

14. In the figure of the preceding question, join each angle to the other angles. Is the pentagon thus obtained, in the centre of the figure, regular? Apply tests. Measure each angle of the figure, formed by intersecting lines, and assign to it its magnitude in degrees.

15. Since the side of a regular pentagon subtends an angle of 72° at the centre of the circle about it, what angle should a side subtend at the circumference? Hence assign to each angle at circumference in question 14, its proper magnitude, and deduce values of all other angles in the figure.

16. In the figure of question 14, indicate all lines that are equal to one another; also all triangles that are isosceles.

17. In the same figure erase the circumference, and sides of the pentagon, so obtaining a star-shaped figure. Show how such a figure (called a pentagram) could be described without taking the pencil from the paper.

18. Without describing a circle, construct a pentagram, the line corresponding to AC being 3 in. Test accuracy of construction by determining lengths AB, BC, . . . , and angles ABC, BCD, ...

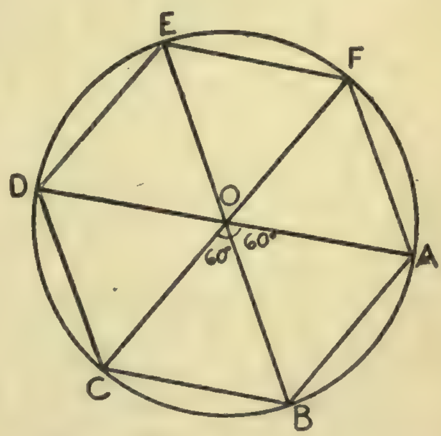
19. In the figure of question 14, how many rhombuses are there?

20. With respect to how many lines is a regular pentagon symmetrical? Has it central symmetry?

CHAPTER XVIII.

Regular Polygons (Continued).

1. To inscribe a regular hexagon in a circle:

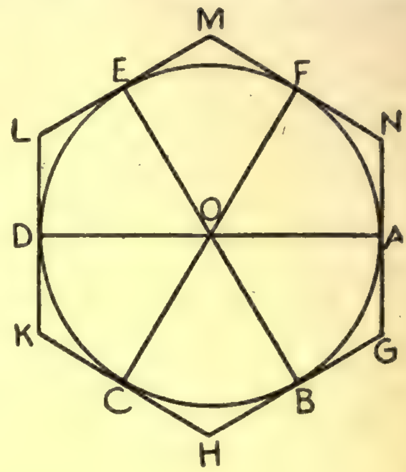


A hexagon having six sides, the angle subtended at the centre of the circle by the side of a regular hexagon inscribed in the circle, will be $\frac{1}{6}$ of $360° = 60°$. Using then the protractor, or adjusting the bevel to an angle of 60°, lay off at the centre two angles of 60°. Produce the three sides of these angles both ways to the circumference, and join the succeeding points of intersection. The construction being accurately made, the bevel will show the equality of the angles **ABC, BCD,** . . . , and the dividers will show the equality of the sides **AB, BC,** . . . . .

Since, however, each of the triangles in the figure is equilateral, having its sides equal to the radius, the sides of the hexagon are equal to the radius of the circle. Hence the easiest way to describe a hexagon in a circle is to measure off, with the dividers, six chords in succession, each equal to the radius.

Evidently the angle of a regular hexagon is 120°.

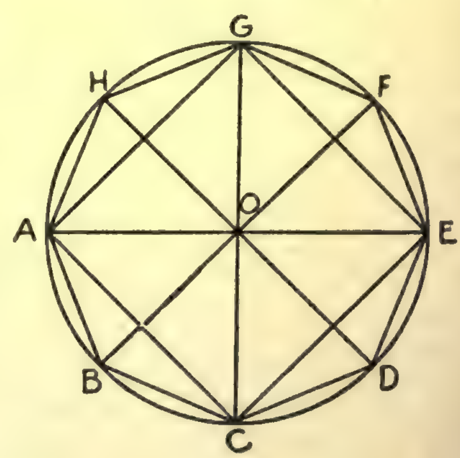
2. **If tangents to the circle be drawn at the angular points of the hexagon ABCDEF, the tangents form another hexagon, which is said to be about the circle.** The equality of the sides **GH, HK, ...** may be tested with the dividers, and the equality of the angles **GHK, HKL, . . .** with the bevel.



3. If we wish to construct a regular hexagon with sides of given length, we describe a circle with radius of this length, and in it inscribe a regular hexagon as in § 1.

4. **To inscribe a regular octagon in a circle:**

We may construct at the centre eight angles, each of 45°, and join the ends of consecutive radii bounding these angles; or, perhaps more conveniently, we may proceed as follows: Draw two diameters at right angles to one another and join their extremities. We thus have a square in the circle. Through the centre, using parallel rulers, draw diameters parallel to the sides of the square. The quadrants are thus bisected, and we get eight equal angles at the centre. Joining ends of the successive radii which bound these angles, we have an octagon inscribed in the circle. The accuracy



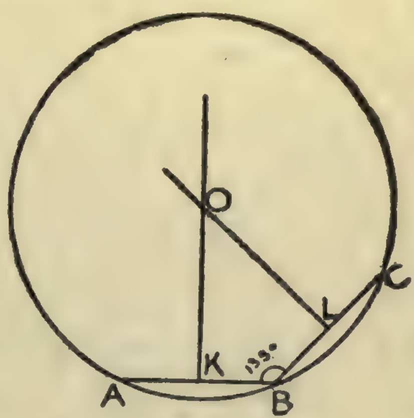
of the construction may be tested by using the dividers to determine whether the sides are equal, and the bevel to determine whether the angles are equal.

Each of the angles at the centre is 45°. Hence each of the angles at the base of any of the isosceles triangles, **OAB**, **OBC**, . . . is $67\frac{1}{2}$°, and the angle of a regular octagon is 135°.

5. **If tangents be drawn at the angular points of the octagon ABCDEFGH, the tangents form another regular octagon which is said to be about the circle.**

6. **To describe a regular octagon with side, AB, of given length** we may proceed as follows:

Construct the angle **ABC** of 135°, and make **BC** = **AB**. Bisect **AB** and **BC** in **K** and **L**, and draw **KO**, **LO** perpendicular to **AB** and **BC**. With **O** as centre, and radius **OA**, **OB** or **OC** describe a circle. On this lay off with the dividers six chords equal to **AB** or **BC**, beginning at the point **C** or **A**. That the rest of the circle is exactly taken up with six such chords affords a test of the accuracy with which the angle **ABC** (135°) is constructed, **AB** and **BC** are bisected, and the perpendiculars **KO** and **LO** are drawn.



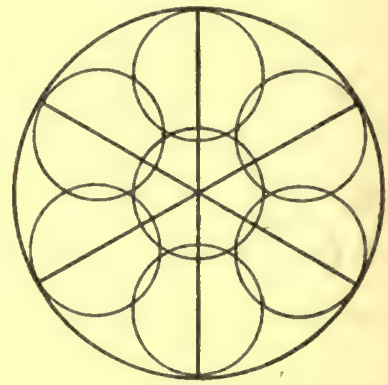
7. The pupil may continue these exercises, constructing regular decagons, dodecagons, etc., in a way quite analogous to the preceding constructions.

The radius of a circle being $1\frac{3}{4}$ in., inscribe in it a regular hexagon. Test the accuracy of your construction by testing the equality of all the angles.

Describe a regular hexagon about the circle in the preceding question, testing the equality of sides and angles of the figure.

Construct a regular hexagon with sides $1\frac{1}{2}$ in.

Construct a figure similar to that annexed, in which the outer circle touches six smaller ones.



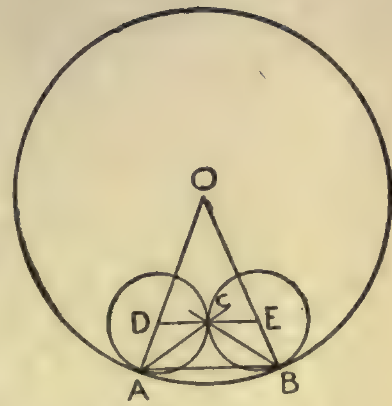
Construct the figure also so that the six small circles touch one another, and are all touched by the outer (large) and inner (small) circles. (Radius of small circles should be one-third radius of large circle.)

Describe a regular octagon in a circle whose radius is 43 millimetres. Test the accuracy of your construction by testing the equality of the sides (using dividers), and by examining whether each of the angles of the octagon is 135°.

Construct a regular octagon whose side is 2 inches. Examine the accuracy of your construction by testing, with the dividers, the equality of the sides, and, with the bevel, the equality of the angles.

Describe eight circles of the same radius, each touching two others of the set, and the entire eight lying within and being touched by a ninth circle of given radius.

The general way of solving such a problem as the



preceding is as follows: Suppose the number of small circles is to be 8, 9, . . . Let **AOB** be the 8th, 9th, . . . , as the case may be, part of 360°. Bisect the angles **OAB, OBA** by **AC, BC**. Through **C** draw **DCE** parallel to **AB**. Then evidently **DA, DC, EB, EC** are all equal, and the circle described with **D** as centre, and **DA** or **DC** as radius, will touch the circle described with **E** as centre, and **EB** or **EC** as radius; and both circles will touch the large one.

Exercises.

1. In a circle of radius $1\frac{1}{2}$ in., inscribe a regular hexagon.

2. Describe a regular hexagon, the sides being 35 millimetres.

3. Describe a regular hexagon with side of 2 in. Join alternate angles, so obtaining a star-shaped figure with six points. What is the six-sided figure at centre of this? Apply tests. What are the various triangles in the figure? Apply tests.

4. In the figure of the preceding question, at what various angles are the sides of the hexagon at centre inclined to any side of the original hexagon?

5. About a circle of radius 40 millimetres describe a hexagon with angles 90°, 100°, 110°, 130°, 140°, 150°.

6. A regular hexagon is described about a circle of radius 2 in. Show that the side of the hexagon is $\frac{4}{\sqrt{3}}$ in.

7. The side of a regular hexagon is 2 in. What is the length of the radius of the circle inscribed in it?

8. Inscribe a regular octagon in a circle of radius 32 millimetres. Test accuracy of construction.

9. In a circle of radius 50 millimetres, inscribe a regular octagon, ABCDEFGH. Join AD, DG, GB, ... , each time passing over two angles, and so obtaining a star-shaped figure with eight points. What is the figure formed at centre? Apply tests.

10. In the preceding figure, what are the various triangles formed? At what various angles are the sides of the octagon at centre inclined to any side of the original octagon?

11. In the same figure, what angles alone occur? How many rhombuses are there in the figure?

12. Construct a regular octagon whose side is 35 millimetres. Test the accuracy of your construction.

13. With the angular points of a regular octagon as centres, describe eight circles of equal radii, so that each touches two others of the set.

14. With respect to how many lines is a regular hexagon symmetrical? Has it central symmetry?

15. With respect to how many lines is a regular octagon symmetrical? Has it central symmetry? Has a regular heptagon central symmetry?

16. In a circle of radius 37 millimetres inscribe a regular dodecagon.

17. What is the ratio of the sides of two regular hexagons, one inscribed in, and the other described about, the same circle?

18. ABCDEF is a regular hexagon. Show that its area is twice that of the equilateral triangle ACE.

19. In a circle the angle ABC is equal to the angle BCD. How are the chords AB, CD related?

20. An equiangular polygon inscribed in a circle has its alternate sides equal.

21. At B, a point on a circle, construct an angle ABC of 108° (the angle of a regular pentagon), the sides AB, BC not being equal. At C make BCD of 108°; at D make CDE of 108°; and so on. Shall we at length reach accurately the point A? If so, after how many times about the circle? Has a regular pentagon been described? Can other regular pentagons be obtained from the figure by producing lines or otherwise?

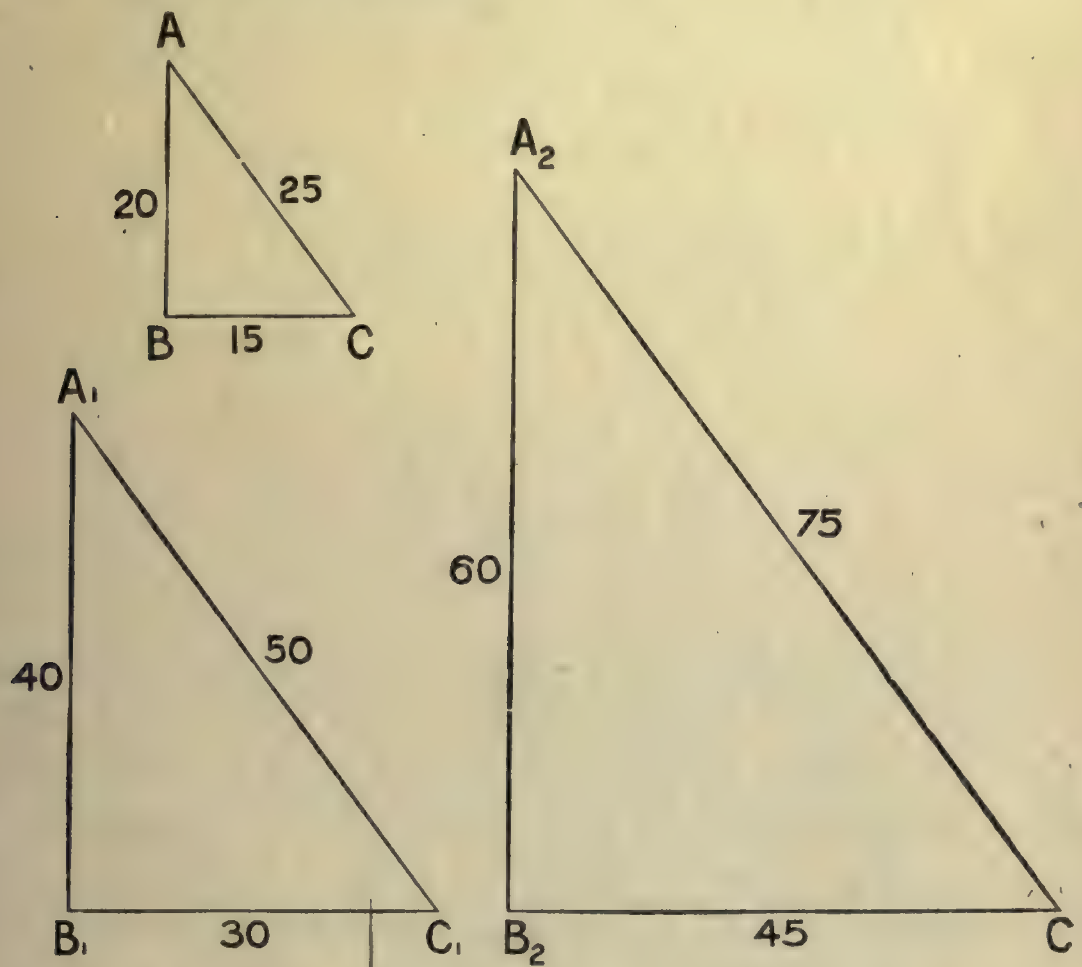
CHAPTER XIX.

Similar Triangles.

1. Two triangles are *similar* when the angles of one triangle are equal to the angles of the other, the sides not necessarily being equal.

Thus if two triangles of different sizes have their angles 45°, 65° and 70°, they are similar.

In the following article a remarkable property of such triangles is reached.



2. On a base **BC** of 15 millimetres construct a triangle with sides **AB, AC** of 20 and 25 millimetres.

Draw two other bases $\mathbf{B_1C_1}$, $\mathbf{B_2C_2}$ of lengths 30 and 45 millimetres. At $\mathbf{B_1}$ and $\mathbf{B_2}$ make angles $\mathbf{C_1B_1A_1}$, $\mathbf{C_2B_2A_2}$, each equal to **CBA**; and at $\mathbf{C_1}$ and $\mathbf{C_2}$ make angles $\mathbf{B_1C_1A_1}$, $\mathbf{B_2C_2A_2}$, each equal to **BCA**. It follows (Ch. III., 4) that the angles at **A**, $\mathbf{A_1}$, $\mathbf{A_2}$ are equal to one another. Hence the three triangles are equiangular and *similar*.

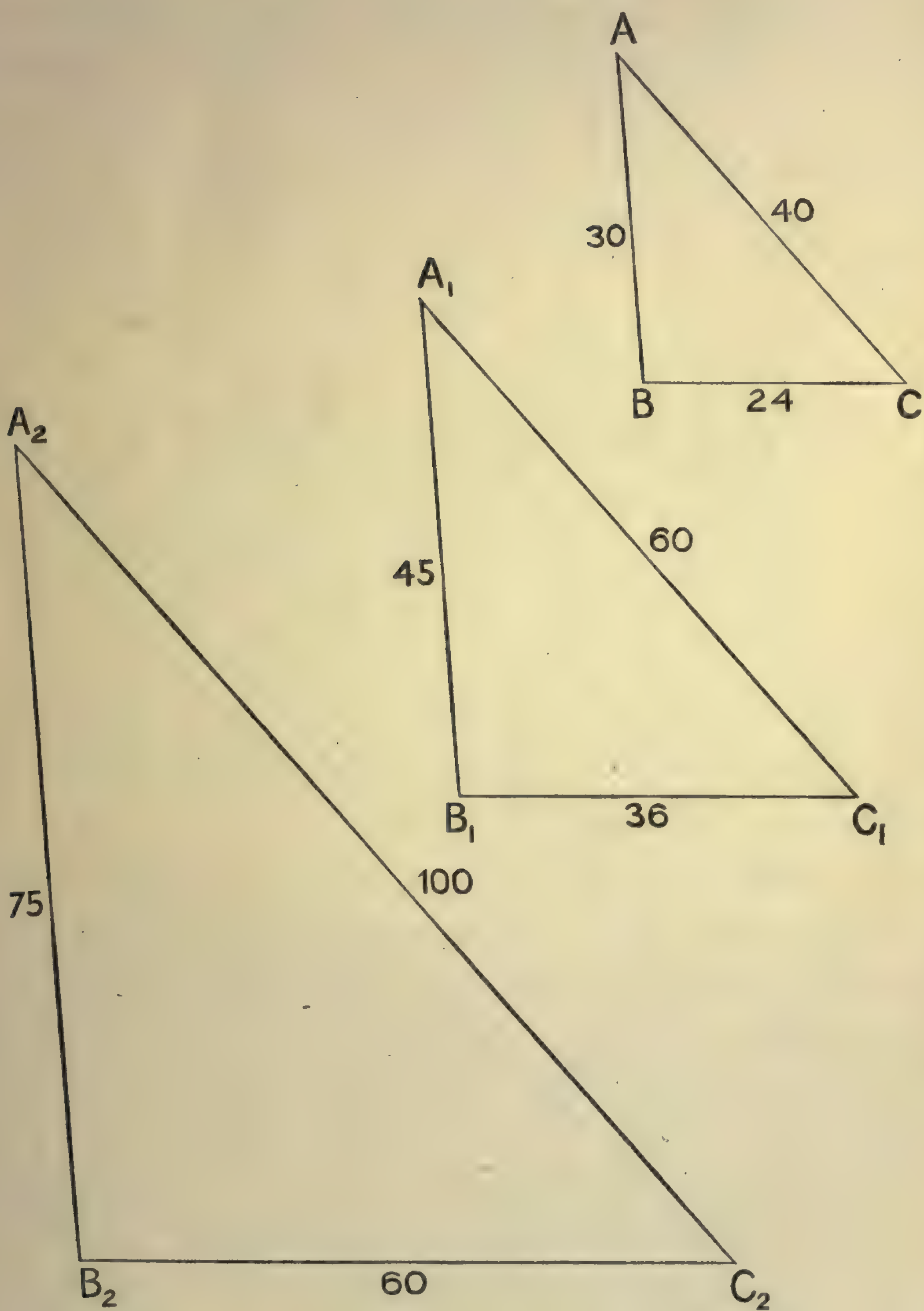
Now measure the lengths of the sides of the triangles $\mathbf{A_1B_1C_1}$ and $\mathbf{A_2B_2C_2}$. If the constructions have been accurately made, we shall have the following numerical values:

$$\mathbf{BC}=15 \qquad \mathbf{B_1C_1}=30 \qquad \mathbf{B_2C_2}=45$$
$$\mathbf{AB}=20 \qquad \mathbf{A_1B_1}=40 \qquad \mathbf{A_2B_2}=60$$
$$\mathbf{AC}=25 \qquad \mathbf{A_1C_1}=50 \qquad \mathbf{A_2C_2}=75$$

Then calling those sides *corresponding sides* which are opposite to equal angles, we observe that corresponding sides about equal angles are proportional, *i.e.*,

$$\tfrac{15}{20} = \tfrac{30}{40} = \tfrac{45}{60}$$
$$\tfrac{20}{25} = \tfrac{40}{50} = \tfrac{60}{75}$$
$$\tfrac{15}{25} = \tfrac{30}{50} = \tfrac{45}{75}$$

3. Again, construct a triangle **ABC**, whose base **BC** is 24, and sides **AB** and **AC**, 30 and 40 millimetres. Draw two other bases $\mathbf{B_1C_1}$ and $\mathbf{B_2C_2}$ of lengths 36 and 60 millimetres. At $\mathbf{B_1}$ and $\mathbf{B_2}$ make angles $\mathbf{C_1B_1A_1}$, $\mathbf{C_2B_2A_2}$, each equal to **CBA**; and at $\mathbf{C_1}$ and $\mathbf{C_2}$ make angles $\mathbf{B_1C_1A_1}$, $\mathbf{B_2C_2A_2}$, each equal to **BCA**. It follows (Ch. III., 4) that the angles at **A**, $\mathbf{A_1}$, $\mathbf{A_2}$ are equal to one another. Hence the three triangles are equiangular and *similar*.



Now measure the lengths of the sides of the triangles $A_1B_1C_1$, $A_2B_2C_2$. If the constructions have been accurately made, we shall have the following numerical values:

$$\begin{array}{lll} BC = 24 & B_1C_1 = 36 & B_2C_2 = 60 \\ AB = 30 & A_1B_1 = 45 & A_2B_2 = 75 \\ AC = 40 & A_1C_1 = 60 & A_2C_2 = 100 \end{array}$$

And we again find that corresponding sides about equal angles are proportional, *i.e.*,

$$\frac{24}{30} = \frac{36}{45} = \frac{60}{75}$$
$$\frac{30}{40} = \frac{45}{60} = \frac{75}{100}$$
$$\frac{24}{40} = \frac{36}{60} = \frac{60}{100}$$

4. The pupil may repeat this experiment with equiangular triangles, and, the constructions being accurately made, he will always reach the same conclusion as to the proportionality of the corresponding sides about equal angles.

(The easiest way to secure the equality of the angles is to place with the parallel rulers B_1C_1 parallel to **BC**, and then with the same rulers draw B_1A_1 parallel to **BA**, and C_1A_1 parallel to **CA**.)

The result of these observations may be stated thus:

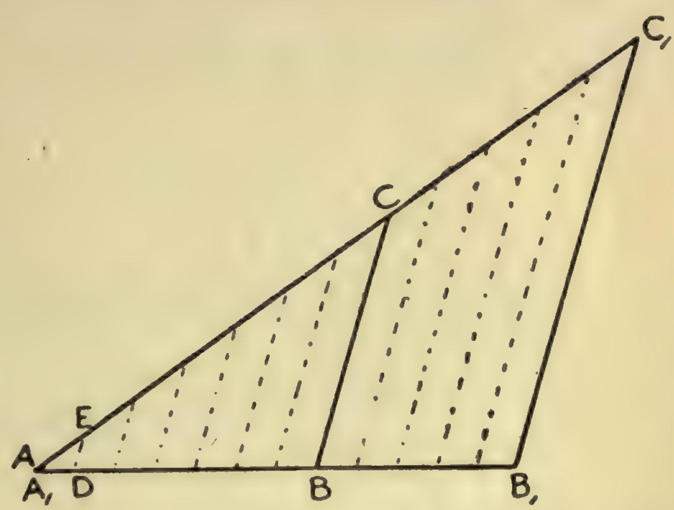
The sides about the equal angles of equiangular triangles are proportionals; and corresponding sides, *i.e.*, those which are opposite to equal angles, are the antecedents or consequents of the ratios.

(Note: In the ratio a : b, a is called the *antecedent*, and b the *consequent*.)

This is the most important proposition in Geometry: indeed, one of the most important results of all science. Through it, in effect, all measurements are made when we cannot actually go over the distance to be measured with a rule, a surveyor's chain, or other measuring instrument.

5. The result reached in the preceding article may be demonstrated more generally as follows:

Let ABC, $A_1B_1C_1$ be similar triangles, and let them be placed so that AB rests on A_1B_1, and AC on A_1C_1, as in the figure. Then BC is parallel to B_1C_1. Suppose AB and A_1B_1 commensurable, and let AB contain n units, and A_1B_1 contain n_1 units. Suppose



A_1B_1 divided into its units, and through the points of division draw lines parallel to BC or B_1C_1. Evidently the divisions of A_1C_1 are all equal to one another, though not necessarily equal to those of A_1B_1. Then also AC contains n parts equal to AE, as AB contains n parts equal to AD; and A_1C_1 contains n_1 parts equal to AE, as A_1B_1 contains n_1 parts equal to AD. Hence

$$\frac{AB}{A_1B_1} = \frac{n}{n_1} = \frac{AC}{A_1C_1}$$

In like manner the proportionality of the sides about the other equal angles may be shown.

6. On the other hand, if the lengths of the sides of one triangle may be obtained from the lengths of the sides of another by multiplying or dividing each by the same number; that is, if the sides of two triangles, taken in order, are proportional, what relation exists between the angles of the two triangles?

Construct and examine the following triangles, and see if you can supply an answer to the question:

(1) Sides 20, 30, 40, and 40, 60, 80 millimetres.

(2) Sides 1, $1\frac{1}{2}$, $1\frac{3}{4}$, and $1\frac{1}{2}$, $2\frac{1}{4}$, $2\frac{5}{8}$ inches.

(3) Sides 24, 36, 40, and 42, 63, 70 millimetres.

Exercises.

1. The sides of two triangles are 20, 30, 40, and 40, 60, 80 millimetres, respectively. Construct them, and, using the bevel, show that they are equiangular.

2. The sides of two triangles are 20, 30, 40 and 30, 45, 60 millimetres, respectively. Construct them, and show that they are equiangular.

3. The bases of two triangles are 35 and 60 millimetres, and the angles adjacent to each base are 75° and 70°. Construct the triangles, and show that corresponding sides are as 35 : 60.

4. Construct two triangles of different sizes with angles 35°, 45° and 100°. On a line AB lay off lines equal to the sides of one triangle; and on another line AC lay off lines equal to the sides of the other triangle. Let the ends of corresponding lengths on AB, AC be joined. What position do these joining lines occupy with respect to each other? Apply test. What is the inference?

5. The angles of two triangles are 60°, 75° and 45°. Construct the triangles, and, after the manner suggested in question 4, test the proportionality of the sides.

6. The angles of two triangles are 110°, 30° and 40°, and the sides opposite angle of 30° in each are 40 and 55 millimetres. Construct the triangles, and, after the manner suggested in question 4, test the proportionality of the sides.

7. The angles at the vertices of two triangles are both 36°. The sides adjacent to the vertex of one triangle are $1\frac{3}{4}$ in. and 2 in., and adjacent to the vertex of the other $2\frac{5}{8}$ in. and 3 in. Construct the triangles. Show by measurement that angles opposite corresponding sides are equal, and that the remaining sides are in ratio 1 : $1\frac{1}{2}$.

8. The angles at the vertices of two triangles are both 67°, and the sides about these angles are 40, 60 and 44, 66 millimetres. Construct the triangles. Show by measurement that triangles are equiangular, and that the remaining sides are as 10 : 11.

9. Construct an angle BAC of 39°, and from P in AC draw PN perpendicular to AB. Measure the lengths of AP, AN, PN in millimetres, and find the numerical values to two places of decimals of the ratios

$$\frac{PN}{AP}, \quad \frac{AN}{AP} \quad \text{and} \quad \frac{PN}{AN}.$$

10. In the preceding question, keeping to the angle of 39°, take the point P in different positions on AC, drop the perpendicular PN, for each position of P repeat the measurements and calculate to two decimal places the values of the preceding ratios. Compare values with those already obtained.

11. Keeping to same angle 39°, take the point P in AB and drop PN perpendicular on AC. Again calculate these ratios.

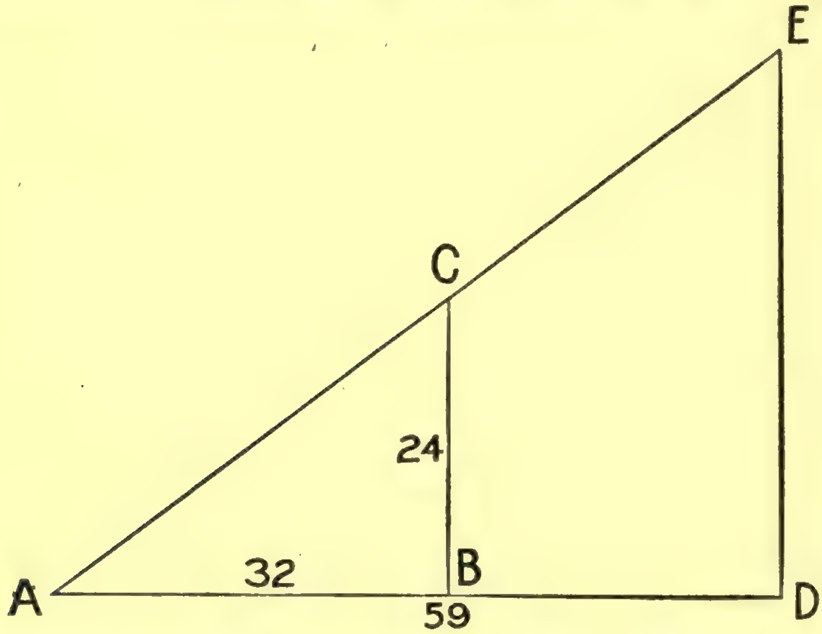
State your conclusion as to the values of these ratios,—perp. to hyp.; base to hyp.; perp. to base—so far as the angle 39° is concerned.

12. BC of a right-angled triangle ABC (C = 90°) is found to be 748 ft., and the angle ABC is 39°. Use the results of the three preceding questions to find approximately the lengths of AC and AB in feet.

CHAPTER XX.

Similar Triangles. (Continued).

1. In the annexed figure the triangles **ABC, ADE** are similar. Suppose the values of the lines are



AD = 59, **AB** = 32, **BC** = 24, and that **DE** is unknown. The property of similar triangles gives

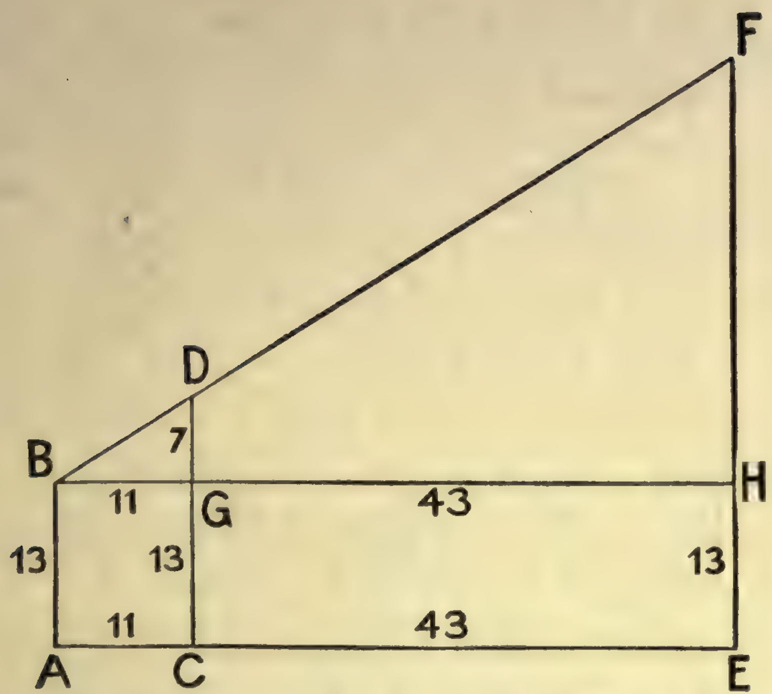
$$\frac{\mathbf{DE}}{59} = \frac{24}{32}$$

$$\mathbf{DE} = \frac{24}{32} \times 59 = 44\tfrac{1}{4}.$$

2. If level ground can be found extending out from the base of a tree, or other vertical object, its height may be found as follows:

Let two rods, **AB** and **CD**, be placed upright in the ground, at such distance apart that the eye sees the tops (**B** and **D**) of the rods and the top (**F**) of the tree in the same straight line.

The heights of the rods being measured, their difference **DG** is known. Let also the lengths **AC** (*i.e.*, **BG**) and **CE** (*i.e.*, **GH**) be measured.



Suppose **AC** = **BG** = 11, **CE** = **GH** = 43,
AB = 13, **CD** = 20.

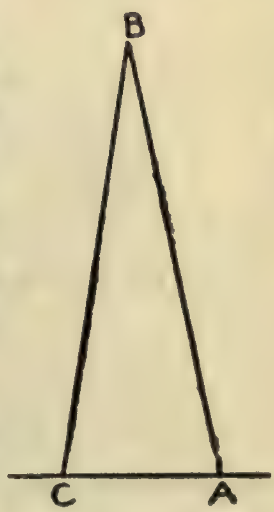
Then by similar triangles **BGD, BHF**

$$\frac{\mathbf{HF}}{43+11} = \frac{7}{11}; \quad \mathbf{HF} = \frac{7}{11} \times 54 = 34\frac{4}{11}$$

Then height of object, **EF** $= 34\frac{4}{11} + 13 = 47\frac{4}{11}$.

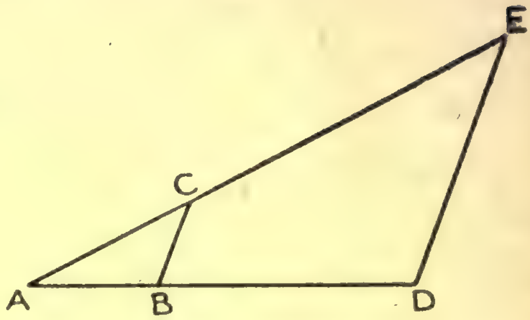
3. Suppose we wish to find the distance of an object **B** from **A**, without going over the distance **AB** with a surveyor's chain or other instrument for measuring.

Measure a *base line*, **AC**, of, say, 250 feet, and note the angles **CAB, ACB.** Then, on paper, construct a triangle $\mathbf{A_1B_1C_1}$, equiangular to **ABC**, but with a base line $\mathbf{A_1C_1}$ of, say, 1 foot. Measure the length, in feet, of $\mathbf{A_1B_1}$. The line **AB** will be 250 times the length of $\mathbf{A_1B_1}$.



This example embodies the principle of the range-finder, so much used in military and naval operations.

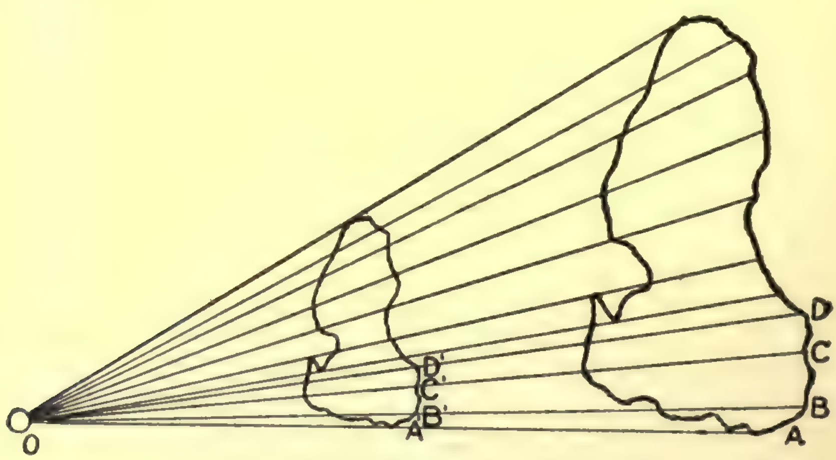
4. Diagrams such as the following should be constructed with accuracy, where **DE** is parallel to **BC**, and therefore the triangles **ABC** and **ADE** similar. **AB**, **BC** and **AD** should then be measured, and **DE** calculated from the proportion



$$\frac{\mathbf{DE}}{\mathbf{AD}} = \frac{\mathbf{BC}}{\mathbf{AB}}, \text{ or } \mathbf{DE} = \frac{\mathbf{BC}}{\mathbf{AB}} \times \mathbf{AD},$$

and the accuracy of the construction, measurements and calculation tested by measuring **DE** with the dividers and scale.

5. The proportionality of the sides of similar triangles may be employed to reduce or enlarge a figure to any scale.



Suppose we wish to obtain a figure the same shape as **ABC** . . . , but with linear dimensions half those of **ABC** . . . Take a line **OA′A**, with **OA′** = **A′A**. From **O** draw a number of lines **OA**, **OB**, . . . With the parallel rulers obtain **B′**, through **A′B′** being parallel to **AB**; also **C′**, through **A′C′** being parallel to **AC**;

also **D′**, through **A′D′** being parallel to **AD**; and so on. Then, with the judgment of the eye, fill in the contour between **A′** and **B′** similarly to that between **A** and **B**; between **B′** and **C′** similarly to that between **B** and **C**; and so on. Any two points in the larger figure should be just twice as far apart as the two corresponding points in the smaller, and this may be used to test the accuracy of the drawing.

Maps may, in this way, be reduced or enlarged, the first drawing being obtained by using translucent paper, or by tracing against a window pane. Of course the drawing of all maps is, in part, a question of the construction of similar figures.

Exercises.

1. Draw any line AB and divide it in the ratio of 7 to 8 by drawing another line ACD, inclined to AB at any angle, such that AC = 28 and CD = 32 millimetres, completing construction with parallel rulers. Verify result by measuring segments of AB.

2. Divide a line 4 in. long in the ratio 3.4 to 4.1.

3. There are three lines of lengths 27, 39 and 64 millimetres. Construct geometrically for a fourth proportional to them, and verify result by calculation and measurement.

4. A line is $4\frac{1}{2}$ in. in length. Divide it into three parts, such that they shall be to one another as 7 : 8 : 9.

5. Draw a line AB an inch long. Draw another line AC of length 50 millimetres, inclined to former at any angle. Divide the inch line into tenths.

6. Divide an inch into twelfths.

7. Draw AB, AC, making an angle of 47° with one another. In either of them take a point P and drop a perpendicular PN on the other. Measure the lengths of the sides of APN, and obtain the numerical values of the following ratios to two decimal places,—

$$\frac{PN}{AP}, \quad \frac{AN}{AP} \quad \text{and} \quad \frac{PN}{AN}.$$

(Most accurate results will be obtained by taking P at some distance from A, and measuring in millimetres.)

8. Take P in other line, at different distance from A, make similar construction, measure sides of APN, and again find, to two decimal places, the values of the above ratios for 47°.

9. Calling the side opposite 47° the perpendicular, the side opposite the right angle the hypotenuse, and the remaining side the base, whether it be on the upper or lower line, are the above ratios, *i.e.*,

$$\frac{\text{perp.}}{\text{hyp.}}, \quad \frac{\text{base}}{\text{hyp.}} \quad \text{and} \quad \frac{\text{perp.}}{\text{base}},$$

always the same for 47°, or do they depend on where the point P is taken?

10. With the explanation in the preceding question, find the values of these same ratios

$$\frac{\text{perp.}}{\text{hyp.}}, \quad \frac{\text{base}}{\text{hyp.}} \quad \text{and} \quad \frac{\text{perp.}}{\text{base}},$$

for an angle of 63°, to two decimal places.

11. It is required to find the distance of a point C from an object B on the other side of a chasm. For this purpose a line CA is run at right angles to BC. AC is found to be 278 feet, and the angle to A to be 47°. What is the distance of B from C?

12. In the preceding question, if AC be 344 feet, and the angle at A be 63°, what is the distance of B from C? Find also the length of AB.

13. To find how far a distant object C is from A, a base line AB is measured of 400 ft. and the angles at A and B are found to be 75° and 80°. Then on paper a line DE of length 3 in. is drawn, and angles EDF, DEF are constructed of 75° and 80°, respectively,—and FD is measured in inches and fractions of an inch. What, approximately, is the length of CA?

14. If, in the preceding question, AB be 250 feet, and the angles at A and B be 65° and 77°, respectively, by constructing a similar triangle on paper and measuring the sides, determine approximately the distances AC and BC.

15. In triangle ABC, AC = 372 feet, A = 48°, C = 90°. Find approximately the length of BC, having previously found for 48° the ratio $\frac{\text{perp.}}{\text{base}}$.

16. Draw an irregular quadrilateral, and construct another of same shape and with linear dimensions half those of former. Verify equality of corresponding angles, and ratio of sides and of diagonals.

17. Draw an irregular pentagon, and construct another of same shape and with linear dimensions one-third those of former. Verify equality of corresponding angles, and ratio of sides and of diagonals.

18. Make a map of Lake Superior with linear dimensions half those of map given in your atlas, properly placing islands. Verify correctness by finding ratio of distances of pairs of corresponding points.

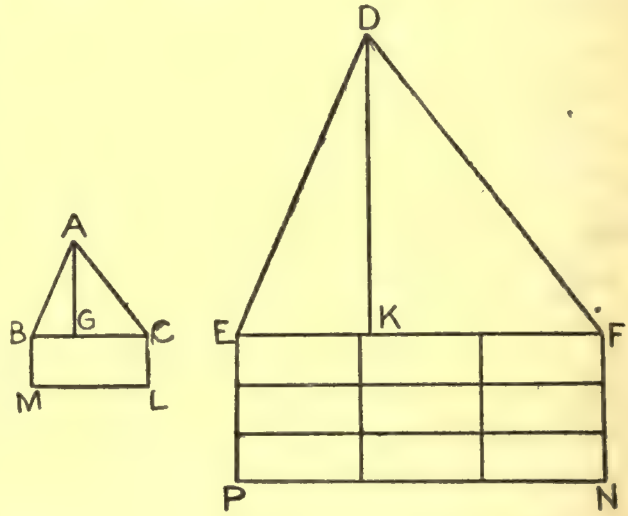
19. Make a map of Mackenzie River from Great Slave Lake to Arctic Ocean, half the size of that given in your atlas. Test correctness by finding ratio of distances between pairs of corresponding points.

20. Construct a triangle with sides 50, 30 and 48 millimetres. Bisect the angle opposite the last side. In what ratio are the segments into which this bisecting line divides this side? Does the same ratio exist elsewhere in the figure?

CHAPTER XXI.

Similar Triangles. (Continued).

1. Let **ABC** and **DEF** be similar triangles, having the base **EF** three times the base **BC.** The other sides of **DEF** are therefore three times the corresponding sides of **ABC.** If **DK** and **AG** be the perpendiculars to the bases, the triangles **ABG** and **DEK** are equiangular, and therefore, since **DE** is three times **AB, DK** is also three times **AG.**



If rectangles be constructed on the bases equal to the triangles, the heights of these rectangles are half the heights of the triangles (Ch. VIII., 5). Hence **FN,** which is half of **DK,** is three times **CL,** which is half of **AG.**

So that the rectangle **EFNP** (which is equal to the triangle **DEF**) is three times as long and three times as high as the rectangle **BCLM** (which is equal to the triangle **ABC**). Hence the rectangle **EFNP** is nine times the rectangle **BCLM,** and, therefore, the triangle **DEF** is nine times the triangle **ABC.**

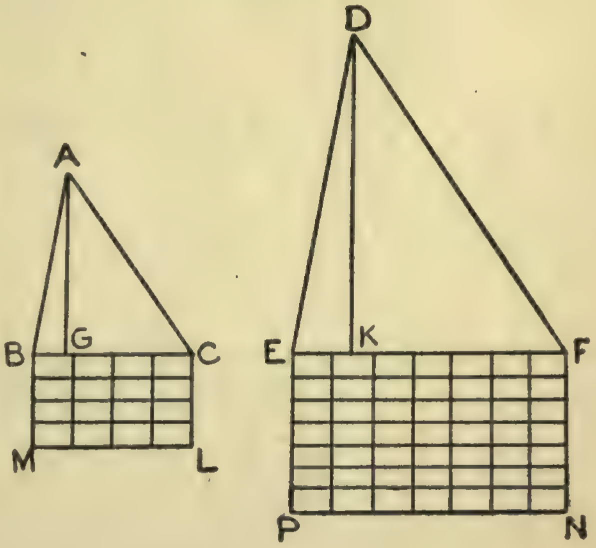
That is, when

side **BC** : side **EF** $= 1:3$,

then, triangle **ABC** : triangle **DEF** $= 1:3^2$,

the triangles being, of course, similar.

2. Again, let **ABC** and **DEF** be similar triangles, having the base **EF** one and three-quarter times the base **BC.** That is, the base **BC** is to the base **EF** as 4 is to 7, since $1:1\frac{3}{4}=4:7$. Since the angles are similar, the other sides of **DEF** are $1\frac{3}{4}$ times the corresponding sides of **ABC.** If **AG** and **DK** be the perpendiculars to the bases, the triangles **ABG** and **DEK** are equiangular, and, therefore, since **DE** is $1\frac{3}{4}$ times **AB, DK** is also $1\frac{3}{4}$ times **AG.**



If rectangles be constructed on the bases equal to the triangles, the heights of these rectangles are half the heights of the triangles (Ch. VIII., 5). Hence **FN,** which is half of **DK,** is $1\frac{3}{4}$ times **CL,** which is half of **AG.**

So that the rectangle **EFNP** (which is equal to the triangle **DEF**) is $1\frac{3}{4}$ times as long and $1\frac{3}{4}$ times as high as the rectangle **BCLM** (which is equal to the triangle **ABC**). That is, of such parts as **EF** contains 7, **BC** contains 4; and of such parts as **FN** contains 7, **CL** contains 4. Hence of such small areas as the rectangle **EFNP** contains $7^2=49$, the rectangle **BCLM** contains $4^2=16$. And therefore the triangle **ABC** is to the triangle **DEF** as 16 is to 49.

That is, when

$$BC:EF=1:1\tfrac{3}{4}=4:7,$$

$$\text{then, triangle } ABC: \text{triangle } DEF=16:49=4^2:7^2 \text{ or } 1:(1\tfrac{3}{4})^2.$$

3. Make figures as in § 1 and § 2 for the following problems:

Two similar triangles, **ABC** and **DEF**, have their corresponding sides **BC** and **EF**, 1 and 2 inches in length respectively; show that their areas are as 1 to 4, *i.e.*, as 1 to 2^2.

Two similar triangles, **ABC** and **DEF**, have their corresponding sides **BC** and **EF**, 1 and $1\frac{1}{2}$ inches in length respectively; show that their areas are as 4 to 9, *i.e.*, as 1 to $(1\frac{1}{2})^2$.

Two similar triangles, **ABC** and **DEF**, have their corresponding sides **BC** and **EF**, 30 and 50 millimetres in length respectively; show that their areas are as 9 to 25, *i.e.*, as $(30)^2$ to $(50)^2$.

(For the three preceding constructions, the method of article 4, which follows, should also be employed.)

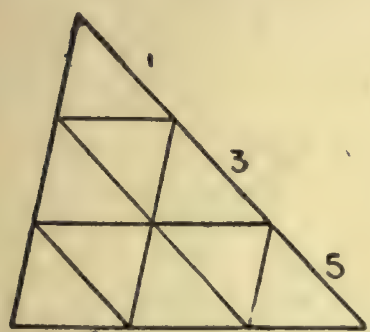
The result of our observations in such cases as the preceding may be stated thus:

Similar triangles are to one another as the squares of corresponding sides.

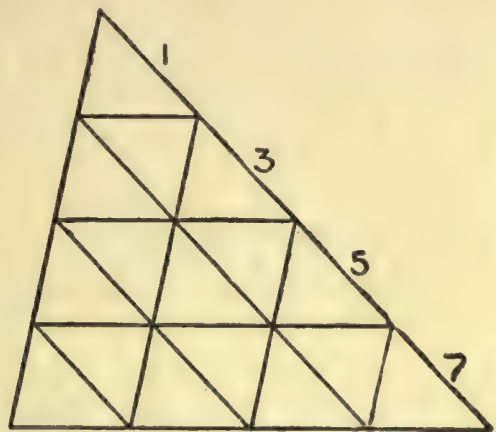
Note: In the preceding examples it will be observed that the lengths of the corresponding sides are supposed *commensurable*, *i.e.*, a unit of length can be found that is contained in both an exact number of times. All lines are not commensurable, though the preceding statement in black-face is true of all similar triangles, whether the corresponding sides be commensurable or not.

4. The following is possibly a more striking way of presenting the preceding proposition:

Let any side, say the base, of a triangle be divided into as many parts as it contains units of length. Through the points of division draw lines parallel to



$1+3+5=3^2$ Small Triangles



$1+3+5+7=4^2$ Small Triangles

the sides, and, through the points of intersection of these lines, draw lines parallel to the base. The triangle is thus divided into a number of triangles equal to one another in all respects, and all similar to the original triangle. It will be observed that, considering these triangles in rows, the rows contain 1, 3, 5, 7, . . triangles, respectively. Hence if the base be 2 units in length, the large triangle contains $1+3=2^2$ small triangles; if 3 units in length, $1+3+5=3^2$ small triangles; if 4 units in length, $1+3+5+7=4^2$ small triangles; and so on. Thus if there be two similar triangles, the base of one containing 3 units of length, and the base of the other 4 units of length, the number of small triangles in one will be 3^2, and in the other 4^2, all such triangles being equal to one another. Hence the areas of the triangles are as 3^2 to 4^2, *i.e.*, as the squares of the bases.

Exercises.

1. Construct two angles, the sides of one being 36, 48 and 50, and the sides of the other 54, 72, 75 millimetres. On the base of each construct a rectangle equal to it; and divide up the rectangles so as to show that the triangles are as $(36)^2$ to $(54)^2$.

2. Divide the triangles of the preceding question into smaller triangles, all equal to one another. Hence show that the original triangles are as $(48)^2$ to $(72)^2$.

3. Draw two straight lines which are to one another as these triangles.

4. Divide a line $3\frac{1}{2}$ in. in length into two segments, such that, when equilateral triangles are described on the segments, one triangle shall be four times the other.

Construct the equilateral triangles, and divide the greater into four triangles, each equal to the smaller.

5. Construct two triangles on bases of 45 and 75 millimetres, with angles adjacent to each base 70° and 50°. Divide the triangles into smaller ones, all equal to one another, showing that the areas of the triangles are as $(45)^2$ to $(75)^2$.

6. Draw a line AB of length 1 in., and produce it to C so that AB may be to BC as the areas of the two triangles in the preceding question.

7. Describe an irregular pentagon, and, after the manner of § 5, Ch. XX., construct another pentagon with linear dimensions half those of former. Divide each pentagon into three triangles by lines drawn from corresponding angles.

How are the sides and angles of corresponding triangles related? Test with bevel and dividers.

How many times is a triangle in the first pentagon greater than the corresponding triangle in the second?

How many times is one pentagon greater than the other?

8. ABC is any triangle, and in AB a point D is taken such that AD is one-quarter of AB. DE is drawn parallel to BC. What fractional part is ADE of the whole triangle? What ratio does ADE bear to the rest of ABC?

9. Construct an equilateral triangle with sides of $1\frac{1}{2}$ in., and construct another with area twice the former.

10. Construct a right-angled triangle with sides 30, 40 and 50 millimetres. On the sides describe equilateral triangles. Divide the triangles into smaller ones, so that the smaller ones may all be equal to one another. What relation do you discover between the area of the triangle on the hypotenuse and the areas of the two other triangles?

11. In the preceding question, instead of equilateral triangles, construct triangles with angles adjacent to the sides of 50° and 80°, so that the three triangles are similar. Again compare areas of smaller triangles with area of greatest.

12. Any line being taken as unity, construct for other lines which shall represent $\sqrt{2}$ and $\frac{1}{\sqrt{2}}$.

Hence draw lines parallel to the base of any triangle so as to form with sides, or sides produced, triangles half and twice the original.

13. The areas of the provinces of the Dominion being,—P. E. I., 2000; N. S., 20600; N. B., 28200; M., 73956; O., 222000; Q., 347350; B. C., 383300 square miles; and the square roots of these numbers being 45, 144, 168, 272, 471, 589, 619, or approximately as 5, 14, 17, 27, 47, 59, 62; construct seven equilateral triangles, all with same vertex, whose areas shall represent proportionately the areas of the provinces.

14. Draw also seven parallel lines, near one another, and all terminated at one end by the same straight line to which they are perpendicular, so that these lines may approximately represent the areas of the provinces.

15. Given the following areas,—England, 50867; Ireland, 32583; Scotland, 29785; Wales, 7442, construct four squares, with one angle in common, which shall represent proportionately and approximately the areas of these countries.

16. Draw also four parallel lines, as in 14, which shall represent approximately the areas of the countries of the United Kingdom.

17. ABC is a right-angled triangle ($C=90°$), and CD is drawn perpendicular to AB.

(1) Prove that triangles CAD and BAC are equiangular; also CBD and ABC equiangular.

(2) Hence show that

$$\frac{AC^2 + BC^2}{AB^2} = 1.$$

18. The cheese exports of Canada being, in 1871, 8271000 lbs.; in 1881, 49255000 lbs.; in 1891, 106202000 lbs.; in 1901, 195926000 lbs.; construct equilateral triangles whose areas shall approximately represent these exports, the side of the first triangle being 29 millimetres.

19. What will be the sides of the triangles if their perimeters are to represent the exports, the side of the first being again 29 millimetres?

20. Construct two triangles, the sides of one being twice the sides of the other, and ascertain the following:

(1) The ratio of perpendiculars from corresponding angles on opposite sides.

(2) The ratio of corresponding segments (of sides) made by feet of perpendiculars.

(3) The ratio of lines from corresponding angles to bisections of opposite sides.