ELEMENTARY PROJECTIVE GEOMETRY

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ELEMENTARY PROJECTIVE GEOMETRY

by

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PREFACE

THE development of the methods of Projective Geometry forms an important part of Modern Geometry, and the valuable results obtained justify the increasing attention which is being paid to this subject. I propose, therefore, to arrange in orderly sequence the elementary propositions of plane projective geometry, assuming a knowledge of the first six books of Euclid, or their equivalent, and I trust that this book will be of use to the Upper Forms of Schools, and to Junior Students at the Universities.

The projective unit is the cross-ratio of four collinear points or of four concurrent lines in a plane: from this I proceed to the study of projective rows and pencils, and the involutions of six points or lines, which play an important part in the solution of problems. I then deduce the properties of the curve of the second degree, defined as the locus of the intersections of corresponding rays of two projective pencils, first proving an important harmonic property of the tangent.

The chief properties of polars follow; and of inscribed and circumscribed polygons, with the construction of conics to satisfy five given conditions, and solutions of other problems connected with the conic.

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I conclude with the elements of polar reciprocation, and of plane homology, with brief notes on projection in space and the sections of a circular cone.

In some cases I have developed a point at greater length, as in the extension of Maclaurin's Theorem, and in the treatment of the harmonic conics of four-points and four-sides. The student should draw many figures in addition to those given in the book; the examples given at the ends of the several chapters include many of the questions recently set in this subject, and also propositions suggested by the text, and others chosen from various writers of the last century.

In an elementary treatment of the subject I have avoided dependence on the use of points at infinity and imaginary points and lines; these will find place in a more advanced treatise, and also the properties of curves of degree higher than the second, and of surfaces and curves in space of three dimensions.

In conclusion I must thank the Syndics of the University Press for undertaking the publication of this book, and the Readers of the Press for their carefulness in the revision of the proofs.

A. G. P.

28 *August*, 1909.

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Page 14, line 19.	*For*	AX, AY	*read*	AX : AY.
,, 41, ,, 2 from foot.	,,	BJ	,,	JB.
,, 42, ,, 7 ,, ,,	,,	*a′b*	,,	λ . *a′b*.
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,, 177, line 2 from foot.	,,	points	,,	tangents.

CHAPTER I

CROSS-RATIO

1. Projection in a plane. If we take a point S and a line KL, and join S to any point A in the plane, the line SA will cut the base KL at some point A′ which we call the projection of A on KL from the centre S.

Two points which are collinear with the centre will have the same projection.

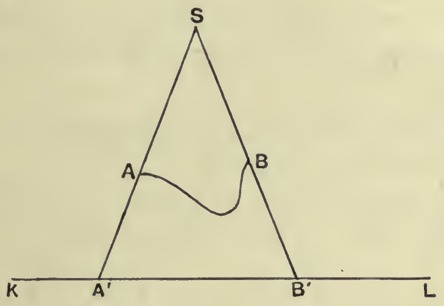


Fig. 1.

The only points in the plane which have no projections are points in a line through S parallel to the base; we may however give verbal completeness to our definition by supposing that two parallel straight lines meet at an infinitely distant point, and saying that if SA is parallel to the base KL, the projection of A is the point at infinity on KL.

The methods of parallel projection in which AA′ is drawn not through a fixed point S but parallel to a fixed direction, and orthogonal projection in which A′ is the foot of the perpendicular from A to KL, may be regarded as special cases of projection.

The projection of any line joining AB, whether straight or curved, is A′B′, where A′ and B′ are the projections of A and B. If AB is a straight line passing through S its projection has no magnitude, being the single point A′. The same is true if AB is an open polygon or a curved line with the two ends A, B collinear with S.

2. Projection in space. In the same way a point in space may be projected on a given plane from a given centre, by finding where the plane is cut by a straight line joining the centre to the given point. A perspective drawing is a projection of a landscape or collection of objects, the eye being the centre of projection.

Another example may be found in pinhole photography, the pinhole being the centre and the photographic plate the plane of projection.

3. Notation. Points are usually denoted by capital letters and straight lines by small letters.

Thus we speak of points A, B lying on a line k; that part of the infinite line k which begins at A and ends at B we call AB.

A **row** of points is a number of points A, B, C, ... lying on a straight line a.

A **pencil** of lines is a set of lines a, b, c, ... passing through one point A.

A **quadrangle** or four-point ABCD is the figure formed by joining the four points A, B, C, D: it has six sides.

A **quadrilateral** $abcd$ is the figure formed by the four lines a, b, c, d and their six intersections.

4. Vanishing points. If we project a row A, B, C, ... on a from centre S, into A′, B′, C′, ... on a'; then, with two exceptions, each point of a corresponds to one point of a', and *vice versa*.

The exceptions are I on a, such that SI is parallel to a'; and J′ on a', such that SJ′ is parallel to a.

I has no finite projection on a', and J′ is not the projection of any finite point of a. We may, however, give verbal completeness to our statement, without confusion, by saying that two parallel lines meet at infinity, and that the projection of I is a point at infinity on a', and that J′ is the projection of a point at infinity on a.

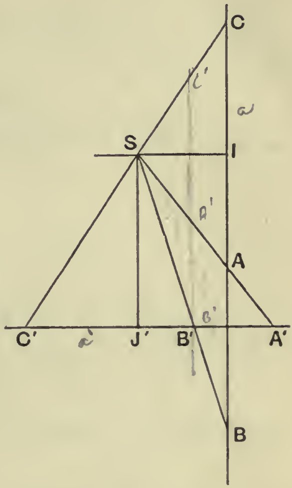


Fig. 2.

The points I, J′ at which lines through S parallel to a' and a, meet a, a' respectively are called the **vanishing points** of a, a' for the centre S.

5. The length of AB is not, in general, equal to its projection A′B′. If A, B, C on a are projected from centre S into A′, B′, C′ on a line a' parallel to AB, we have similar triangles SAB, SA′B′ and SBC, SB′C′, and hence we can prove that AB : BC as A′B′ : B′C′, *i.e.* the ratio of two segments of a line is not altered by projection on a parallel line. This is also true for parallel projection; but it is not true in Central Projection, when a, a' are not parallel: we shall find however relations between parts of lines which are not altered by projection, and the consideration of these relations will form the basis of much of our work in projective geometry.

6. Sign. If A′, B′ and C′ are the projections of three points A, B and C which are not in a straight line, the projection of the figure composed of AB and BC is A′C′, the projection of AB is A′B′, and of BC is B′C′.

In order that the projection of ABC may equal the sum of the projections of AB and BC in all cases, we must have

$$A'C' = A'B' + B'C',$$

whether B′ does or does not lie between A′ and C′.

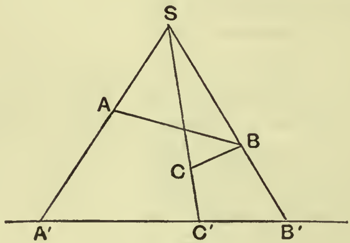


Fig. 3.

The similarity between this operation and the algebraic addition of positive and negative quantities suggests the use of the signs + and − to express opposite directions in the same straight line; and we say that $BA = -AB$; and that AB, CD are both positive or both negative if they are segments of the same line drawn in the same direction, but one is positive and the other negative if they are drawn in opposite directions in the same straight line; also $AC + CB = AB$, if C is collinear with A and B.

7. From the Rule of Signs it follows that the rectangle of two parts AC, CB of a line AB is positive when C lies between A and B, and negative when C is in AB produced either way.

Thus Euclid II. 4 and 7 are two cases of

$$AB^2 = AC^2 + BC^2 + 2AC \,.\, CB;$$

in the former C lies between A and B, in the latter C lies in AB produced. Similarly II. 5 and 6 are two cases of

$$AD \,.\, DB = MA^2 - MD^2,$$

where M is the middle point of AB.

Again the generalized form of Euclid II. 1 is

$$XY \,.\, AB = XY \,.\, AK + XY \,.\, KL + XY \,.\, LM + \ldots + XY \,.\, PB.$$

Hence

$$\begin{aligned} &AB \,.\, CD + BC \,.\, AD + CA \,.\, BD \\ &= (AD \,.\, CD - BD \,.\, CD) + (BD \,.\, AD - CD \,.\, AD) + (CD \,.\, BD - AD \,.\, BD) \\ &= 0, \end{aligned}$$

an important result which will be required presently.

Exercise 1. If O lies in the same straight line as A, B, C prove that

$$OA^2 \,.\, BC + OB^2 \,.\, CA + OC^2 \,.\, AB = -AB \,.\, BC \,.\, CA.$$

2. Prove that this is also true when A, B, C lie in a straight line, but O lies outside that line.

8. Ratio of segments of a straight line. The ratio of AC to CB will be positive when C lies between A and B, negative when C is in AB produced either way.

Thus the bisector of the vertical angle C of a triangle ACB meets the base AB at a point K, such that $AK : KB = AC : CB$; and the bisector of the exterior angle at C meets AB at L, such that $AL : LB = -AC : CB$. Hence $AK : KB = -AL : LB$.

Again, a line parallel to the base of a triangle divides the sides in equal positive ratios if the line lies between the base and the vertex, in equal negative ratios if the line cuts the sides produced through the vertex or beyond the base.

9. If C is a variable point in a straight line AB, there are no two positions for which the ratio $AC : CB$ has the same value. As C moves from A to B, the value of the ratio increases from 0 to ∞: as C moves from B along AB produced the value increases from $-\infty$ towards the value -1: as C moves from A along BA produced the value of AC : CB decreases from 0 towards -1.

Thus for each position of C there is a definite value of AC : CB, positive if C lies between A, B; otherwise negative; save that *at* B no meaning attaches to AC : CB for CB is zero, but however near C is to B there is a value of the ratio, positive on one side, negative on the other, numerically large in each case, thus giving the idea that $+\infty$ and $-\infty$ are the same.

[The sign ∞ stands for an infinite number, *i.e.* one too great for our comprehension. Point at infinity (∞) stands for a point too far away for our comprehension.]

Again to every numerical quantity, positive or negative, corresponds one, and only one, position of C such that AC : CB has that value, save the number -1, and as AC + CB is AB and not zero, therefore AC : CB is never -1: but when C is very distant towards either end of AB the value of AC : CB is very nearly -1, in fact we can find a position of C for any number which differs from -1 however slightly, in AB produced if the number is less than -1, in BA produced if slightly greater than -1.

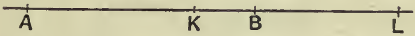
We obtain verbal completeness then by supposing a point 'at infinity' (towards either end indifferently), where AC : CB has the value -1.

Otherwise we may say that as a line SC turns round S cutting AB at C, for every position of the line there is one unique value of AC : CB, and for every value that can be given to AC : CB there is one unique position of SC; and that when SC is parallel to AB, the value of AC : CB is -1.

Harmonic Rows and Pencils.

10. If K is a point in the line AB, there exists another point L, such that AL : LB = − AK : KB; the two points K, L are said to be **harmonically conjugate** with respect to A, B; one of them must lie between A and B, the other must lie outside the segment AB. If however K is the middle point of AB, there is no finite position of K, for in this case AL : LB = – 1.

N.B. The **lengths** of AK, AB, AL are in harmonical progression.



$\frac{AL}{LB} = -\frac{AK}{KB}$

Fig. 4.

If K, L are harmonically conjugate with respect to A, B, then A, B are harmonically conjugate with respect to K, L.

For $\dfrac{AK}{KB} = -\dfrac{AL}{LB}$, hence $\dfrac{KA}{AL} = -\dfrac{KB}{BL}$. Q.E.D.

Two points A, B and their conjugates K, L form a harmonic range {ABKL}.

11. Theorem. If K, L are harmonic conjugates with respect to A, B and S a point outside AB, and if a line be drawn through K parallel to SL to cut SA, SB at D, E, then DK = KE.

For AK : AL = DK : SL,

and KB : BL = KE : SL.

But, by hypothesis,

AK : KB = – AL : LB,

∴ AK : AL = KB : BL,

and hence DK = KE.

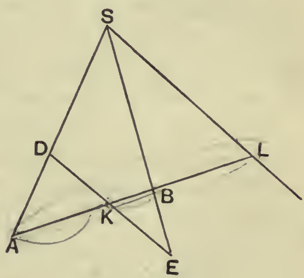


Fig. 5.

Conversely: If K is the middle point of a line DE, and S a point outside DE, then any line through K cuts a line through S parallel to DE at the harmonic conjugate of K with respect to the points where the line cuts SD and SE.

For $DK = KE$, hence $AK : AL = DK : SL = KE : SL = KB : BL$, and therefore $\{ABKL\}$ is harmonic.

12. Theorem. *The projection of a harmonic range is a harmonic range.* Let K, L be harmonically conjugate to A, B; project from centre S, giving K', L' and A', B'; through K, K' draw DE, $D'E'$ parallel to SL cutting SA, SB at D, E and D', E'.

$\{ABKL\}$ is harmonic,

$\therefore$ $DK = KE$. (§ 11)

But $D'E'$ is parallel to DE,

$\therefore$ $DK : KE = D'K' : K'E'$;

hence $D'K' = K'E'$,

and $\therefore$ $\{A'B'K'L'\}$ is harmonic.

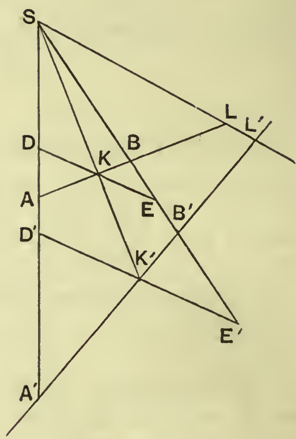


Fig. 6.

If, however, the base of projection is parallel to SL, we get as the projections of A, B, K points A', B', K' such that $A'K' = K'B'$ and L has no finite projection. But if the base is not quite parallel we shall have $A'K' : K'B'$ nearly equal to unity, and L' a very distant point so that $A'L' : L'B'$ is nearly equal to -1. Hence we are led to imagine that the base is parallel to SL, it is cut by SL at an infinitely distant point L', and then $A'K' : K'B' = 1$; $A'L' : L'B' = -1$ and $\{A'B'K'L'\}$ is harmonic. With this qualification we may *formally* state that the projection of a harmonic range on any straight line is always a harmonic range.

Corollary. The lines joining any point S to two points (A, B) and their harmonic conjugates (K, L) are such that any other line cuts them in a harmonic range: these four lines form a **harmonic pencil** at S.

Corollary. If a, b, k, l are four lines through the point S such that a, b cut off a segment on any transversal parallel to l, which is bisected by k, then k, l are harmonically conjugate to a, b.

Corollary. The internal and external bisectors of the angles between a, b are harmonically conjugate to a, b.

Corollary. If a, b are perpendicular lines, and k, l harmonically conjugate to them, then a, b are also harmonically conjugate to k, l: hence a line parallel to a cuts b, k, l at B, K, L such that $KB = BL$, and hence k, l make equal angles with b.

13. Theorems. If $AK : KB = -AL : LB$,

I. $AK . BL = KB . AL$,

$\therefore AL . BL - KL . BL$
$= KL . AL - BL . AL$,

$\therefore 2AL . BL = KL . BL + KL . AL$,

$\therefore 2AL . BK = -KL . AL + KL . BL$
$= LK . AB$;

hence $AB : BK =$ twice $AL : LK$.

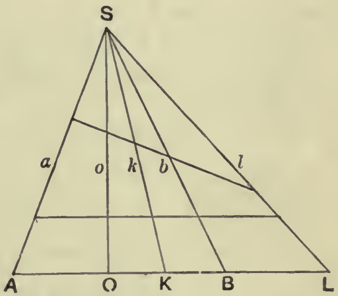


Fig. 7.

II. $$\frac{KB}{AK} + \frac{LB}{AL} = 0,$$

$$\therefore \frac{AB}{AK} + \frac{AB}{AL} = 2,$$

or $$\frac{1}{AK} + \frac{1}{AL} = \frac{2}{AB}.$$

III. If O is the middle point of AB,

$$\because AK . BL = KB . AL,$$

$$\therefore (OK - OA)(OL - OB) = (OB - OK)(OL - OA),$$

i.e. $$(OK + OB)(OL - OB) + (OK - OB)(OL + OB) = 0,$$

since $OA = -OB$.

$$\therefore 2 . OK . OL - 2 . OB^2 = 0, \quad i.e.\ OB^2 = OK . OL.$$

IV. If SK, SL are harmonically conjugate to SA, SB then, drawing a transversal perpendicular to SA, we get from II. that

$$\cot \widehat{ASK} + \cot \widehat{ASL} = 2 \cot \widehat{ASB}, \quad \text{or} \quad \cot \widehat{ak} + \cot \widehat{al} = 2 \cot \widehat{ab}.$$

V. Similarly from III. by drawing a transversal perpendicular to SO, the bisector of angle ASB, we get

$$\tan \widehat{OSK} \,.\, \tan \widehat{OSL} = \tan^2 \widehat{OSB}, \ \textit{i.e.}\ \tan \widehat{ok} \,.\, \tan \widehat{ol} = \tan^2 \widehat{ob}.$$

Cross-Ratio of Four Points.

14. Definition. If on the line AB we take two points K, L the ratio of $\frac{AK}{KB} : \frac{AL}{LB}$ is called the **cross-ratio** of ABKL and is denoted by $\{ABKL\}$.

If we fix A, B and K and vary the position of L, the cross-ratio $\{ABKL\}$ has various values.

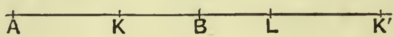


Fig. 8.

Thus, taking K between A and B, we find the value of $\{ABKL\}$ when L is on the left of A is negative and between $-\frac{AK}{KB}$ and $-\infty$; from A to K its value diminishes from ∞ to $+1$; from K to B the value decreases from $+1$ to 0; and as L moves from B in the direction AB its value decreases from 0 to $-\frac{AK}{KB}$.

[When L is at K′, the harmonic conjugate of K with respect to A, B, $\{ABKL\} = -1$.]

Thus $\{ABKL\}$ may have any value, except the value $-\frac{AK}{KB}$, for this would correspond to $\frac{AL}{LB} = -1$, which is not true for any finite position of L: if however we imagine an infinitely distant point (∞) such that $\frac{A\infty}{\infty B} = -1$, we may say $\{ABK\infty\} = -\frac{AK}{KB}$:

similarly $\{AB\infty L\} = -1 \div \frac{AL}{LB}$.

Exercise. Trace the changing value of $\{ABKL\}$ as L moves along the line, when K lies outside AB.

15. It is important to notice the order in which the letters A, B, K, L are written in the symbol $\{ABKL\}$: thus

$$\{BKAL\} = \frac{BA}{AK} : \frac{BL}{LK},$$

which is not the same as $\{ABKL\}$.

Four letters may be written in 24 different orders, *i.e.* there are 24 cross-ratios of four points in a line, dependent on the order in which the points are taken: these 24 values are however closely related. We call the first or the second pair **associated points.**

Theorems. I. If we exchange a pair of associated points, the value of the new cross-ratio is the reciprocal of the former value.

Let $\{ABKL\}$, *i.e.* $\frac{AK}{KB} : \frac{AL}{LB} = c$,

then $\{BAKL\} = \frac{BK}{KA} : \frac{BL}{LA} = \frac{KB}{AK} : \frac{LB}{AL} = \frac{1}{c}$,

also $\{ABLK\} = \frac{AL}{LB} : \frac{AK}{KB} = \frac{1}{c}$.

Corollary. $\{BALK\} = 1 \div \{BAKL\} = \{ABKL\}$.

II. The exchange of the middle letters changes the value from c to $1 - c$.

For $AB \,.\, KL + BK \,.\, AL + KA \,.\, BL$

$$= (AL \,.\, KL - BL \,.\, KL) + (BL \,.\, AL - KL \,.\, AL) + (KL \,.\, BL - AL \,.\, BL) = 0,$$

$$\therefore \quad \mathsf{AB}\,.\,\mathsf{KL} = \mathsf{KB}\,.\,\mathsf{AL} - \mathsf{AK}\,.\,\mathsf{LB},$$

$$\therefore \quad \frac{\mathsf{AB}\,.\,\mathsf{LK}}{\mathsf{BK}\,.\,\mathsf{AL}} = 1 - \frac{\mathsf{AK}\,.\,\mathsf{LB}}{\mathsf{AL}\,.\,\mathsf{KB}},$$

i.e.

$$\{\mathsf{AKBL}\} = 1 - \{\mathsf{ABKL}\} = 1 - c.$$

Corollary. Since $\{\mathsf{ABKL}\} = \{\mathsf{BALK}\}$,

$$\therefore \quad \{\mathsf{AKBL}\} = \{\mathsf{BLAK}\} = \{\mathsf{LBKA}\};$$

hence $\{\mathsf{LBKA}\}$ also has the value $1 - c$, $\therefore$ $\{\mathsf{LKBA}\} = c$, thus the exchange of the outer pair, and the inner pair, brings no change in value.

Applying I. and II. we get

$$c = \{\mathsf{ABKL}\} = \{\mathsf{BALK}\} = \{\mathsf{KLAB}\} = \{\mathsf{LKBA}\},$$

$$\frac{1}{c} = \{\mathsf{ABLK}\} = \{\mathsf{BAKL}\} = \{\mathsf{KLBA}\} = \{\mathsf{LKAB}\},$$

$$1 - c = \{\mathsf{AKBL}\} = \{\mathsf{BLAK}\} = \{\mathsf{KALB}\} = \{\mathsf{LBKA}\}.$$

Applying I. to the ratios equal to $1 - c$, we get

$$\frac{1}{1-c} = \{\mathsf{AKLB}\} = \{\mathsf{BLKA}\} = \{\mathsf{KABL}\} = \{\mathsf{LBAK}\}.$$

Applying II. to the second set, we get

$$1 - \frac{1}{c} = \{\mathsf{ALBK}\} = \{\mathsf{BKAL}\} = \{\mathsf{KBLA}\} = \{\mathsf{LAKB}\},$$

and from either of the two sets last obtained, we get

$$1 - \frac{1}{1-c} \equiv \frac{-c}{1-c} = \{\mathsf{ALKB}\} = \{\mathsf{BKLA}\} = \{\mathsf{KBAL}\} = \{\mathsf{LABK}\}.$$

Thus the 24 ratios are arranged in sets of 4 with the values

$$c,\ 1 - c,\ \frac{1}{c},\ \frac{1}{1-c},\ \frac{c-1}{c},\ \frac{c}{c-1}.$$

Corollary. Since $\{\mathsf{LKBA}\} = \{\mathsf{ABKL}\}$ the cross-ratio of four points is unaltered if we reverse the line. This was not true of the single line, for its sign was changed, nor of the ratio $\mathsf{AK} : \mathsf{KB}$, for it was changed into its reciprocal.

16. Note 1. If $x^2 - x = \mathsf{X}$, $c^2 - c = \mathsf{C}$, the sextic

$$(x-c)(x-\overline{1-c})\left(x-\frac{1}{c}\right)\left(x-\frac{1}{1-c}\right)\left(x-\frac{c-1}{c}\right)\left(x-\frac{c}{c-1}\right)=0$$

reduces to
$$(\mathsf{X}-\mathsf{C})\left(\mathsf{X}+\frac{\mathsf{C}}{c^3}\right)\left(\mathsf{X}-\frac{c^3}{\mathsf{C}^2}\right)=0.$$

$$\therefore\ (\mathsf{X}-\mathsf{C})\left(\mathsf{X}^2+\mathsf{X}\frac{\mathsf{C}^3-c^6}{c^3\mathsf{C}^2}-\frac{1}{\mathsf{C}}\right)=0,$$

$$\therefore\ (\mathsf{X}-\mathsf{C})\left(\mathsf{X}^2-\mathsf{X}\frac{3\mathsf{C}+1}{\mathsf{C}^2}-\frac{1}{\mathsf{C}}\right)=0,$$

$$\therefore\ \mathsf{C}^2(\mathsf{X}^3+3\mathsf{X}+1)=\mathsf{X}^2(\mathsf{C}^3+3\mathsf{C}+1),$$

and hence
$$\mathsf{C}^2(\mathsf{X}+1)^3=\mathsf{X}^2(\mathsf{C}+1)^3.$$

Note 2. If AB, KL are overlapping segments the semicircles on AB, LK will intersect at a point O. Let C, M be the centres, and let the angle COM be 2θ.

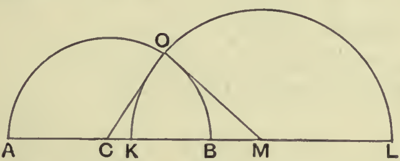


Fig. 9.

Let $\mathsf{AB} = 2x$, $\mathsf{KL} = 2y$, $\mathsf{CM} = 2z$.

Taking the triangle COM,

$$\therefore\ -\tan^2\theta = -\frac{(-x+y+z)(x-y+z)}{(x+y+z)(x+y-z)} = \frac{\mathsf{LB}\,.\,\mathsf{AK}}{\mathsf{AL}\,.\,\mathsf{KB}} = \{\mathsf{ABKL}\},$$

$$\sin^2\theta = \frac{(-x+y+z)(x-y+z)}{4xy} = \frac{\mathsf{BL}\,.\,\mathsf{AK}}{\mathsf{AB}\,.\,\mathsf{KL}} = \{\mathsf{ALKB}\}.$$

Similarly $\cos^2\theta = \{\mathsf{AKLB}\}$, and we can get similarly $\sec^2\theta$, $\cot^2\theta$, $\operatorname{cosec}^2\theta$ expressed as cross-ratios.

Corollary. If the circles are orthogonal $\{\mathsf{ABKL}\} = -1$.

Theorem. If $\{AB, KL\} = \{ABLK\}$, then K, L either coincide or are harmonically conjugate to AB.

For in this case $c = \frac{1}{c}$, $\therefore c^2 = 1$, $\therefore c = \pm 1$.

17. Theorem. If A, B, K, L are four points in a straight line, and S any point outside the line, and a line through A parallel to SB cuts SK, SL at X, Y respectively, then

$$\{AB, KL\} = AX : AY.$$

From the similar triangles AKX, BKS

$$AK : KB = AX : SB.$$

Similarly $AL : LB = AY : SB$.

Moreover when K is between A and B, $AK : KB$ is positive, and AX lies in the direction SB; but when K is outside AB, on either side, AX is opposite in direction to SB: and similarly for $AL : LB$ and AY.

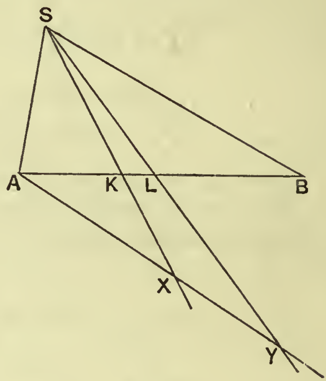


Fig. 10.

Hence $\frac{AK}{KB} : \frac{AL}{LB} = AX, AY$ in magnitude and sign.

Problem. Given 3 points A, B, K, find L so that $\{ABKL\} = c$.

Theorem. *The cross-ratio of four points is unaltered by projection.*

For, if four points A, B, K, L be projected from centre S into points A′, B′, K′, L′ on A′B′, we may draw AXY parallel to SB to cut SK, SL at X, Y, and similarly A′X′Y′ from A′ (fig. 11).

Then $$AX : A'X' = SA : SA' = AY : A'Y',$$

$$\therefore AX : AY = A'X' : A'Y',$$

$$\therefore \{AB, KL\} = \{A'B', K'L'\}.$$

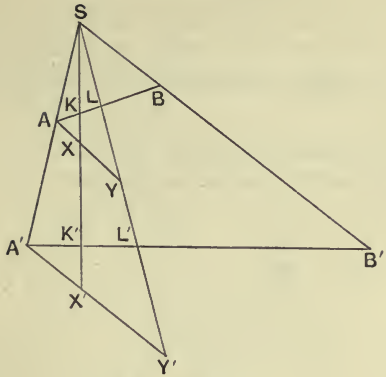


Fig. 11.

18. Thus if we have a pencil of four rays a, b, k, l at a point S, any transversal cuts them in a row of four points whose cross-ratio is always the same: this value may be called the cross-ratio of the pencil, and denoted by $\{abkl\}$. There are 24 cross-ratios of four concurrent lines, dependent upon the order in which they are taken, and these arrange themselves in sets of 4, having values c, $1-c$, $\frac{1}{c}$, $1-\frac{1}{c}$, $\frac{c}{c-1}$: this follows by taking any transversal.

Note. But if the transversal is parallel to l, cutting a, b, k at A, B and K, then $\{abkl\} = -\frac{\mathrm{AK}}{\mathrm{KB}}$.

Note. If $\widehat{ak}$ denotes the angle between a and k, due regard being paid to sign, so that $\widehat{ka} = -\widehat{ak}$: and if ABKL is a transversal of the pencil $abkl$ at S such that SA = SB (fig. 12),

$$\mathrm{AK} : \mathrm{SK} = \sin \widehat{ak} : \sin \mathrm{SAK},$$

$$\mathrm{SK} : \mathrm{KB} = \sin \mathrm{S\hat{B}K} : \sin \widehat{kb},$$

$$\therefore\ \mathrm{AK} : \mathrm{KB} = \sin \widehat{ak} : \sin \widehat{kb}.$$

Similarly $$\mathsf{AL} : \mathsf{LB} = \sin \widehat{al} : \sin \widehat{lb}.$$

$$\therefore\ \{abkl\} = \{\mathsf{ABKL}\} = \frac{\sin \widehat{ak}}{\sin \widehat{kb}} : \frac{\sin \widehat{al}}{\sin \widehat{lb}},$$

which shews the relation between the cross-ratio of four lines forming a pencil, and the angles between the lines.

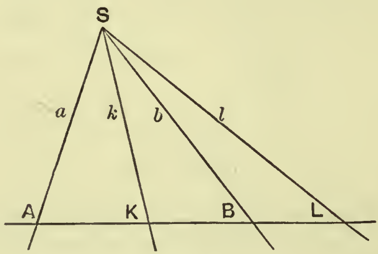


Fig. 12.

Exercise. Deduce theorems IV. and V. of page 10, for the case of a harmonic pencil, *i.e.* when $\{abkl\} = -1$.

Theorem. If $\{abkl\} = \{ablk\}$, then k, l either coincide, or are harmonically conjugate to a, b.

Corollary. If we have three lines a, b, k meeting at S we can find a line l through S, such that $\{abkl\} =$ a given quantity c.

Harmonic Properties of Quadrilaterals and Quadrangles. Constructions.

19. Four straight lines, of which no three are concurrent, form a quadrilateral or four-side: the lines intersect in 6 points so that a four-side has three pairs of vertices [A, B; C, D; E, F]; the joins of these introduce three new lines, called **diagonals** [AB, CD, EF], forming the **diagonal triangle** [KLM].

Theorem. If A, B are a pair of opposite intersections of a four-side, the points K, L where AB is cut by the other two diagonals are harmonically conjugate with respect to A and B.

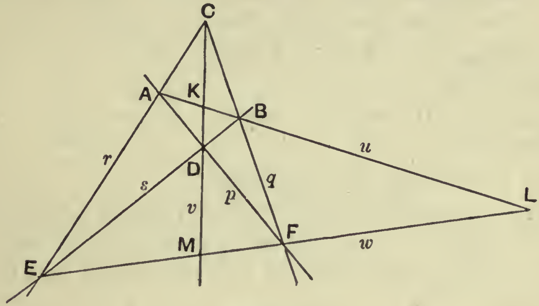


Fig. 13.

Let diagonal CD cut AB at K, and diagonal EF cut AB at L and CD at M.

Then, by projection from C,

$$\{ABKL\} = \{EFML\},$$

but, projecting from D back to the line AB we get

$$\{EFML\} = \{BAKL\} = \{ABLK\};$$

hence $\{ABKL\} = \{ABLK\}$; and therefore, as K, L do not coincide, they must be harmonically conjugate to A, B.

Corollary 1. If AB is parallel to EF so that L is at infinity, $AK = KB$.

Corollary 2. The diagonals of a parallelogram bisect each other.

Problem. To construct the harmonic conjugate of K, with respect to A, B. Through K draw any line, and on it take any two points C, D. Let AC, BD meet at E; AD, BC at F; and let EF cut AB at L.

Then L is the required point harmonically conjugate to K.

Problem. Given a line AB and its middle point M, to draw a line through any point O parallel to AB.

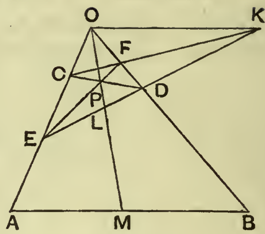


Fig. 14.

Join OA, OB, OM; take any point P on OM and draw through P lines CD, EF meeting OA, OB at C, E and D, F respectively.

Then if CF, ED meet at K, OK is the line required.

For, if OP cuts ED at L, since OP, CF, DE are the diagonals of a four-side formed by OA, OB, CD, EF, therefore $\{EDKL\} = -1$, hence the pencil OA, OB, OM, OK is harmonic; but the transversal AB is bisected at M, and therefore OK is parallel to AB.

Problem. Through a point O lying between AB, AC draw a line, such that O is the middle point of the segment cut off by AB, AC.

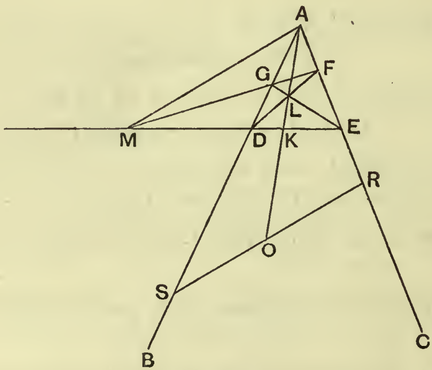


Fig. 15.

Join AO and draw any line to cut AB, AC, AO at D, E, K; take any point L on AK and let DL meet AC at F, EL meet AB at G.

Join FG, and let it cut DE at M.

Join AM and through O draw RS parallel to AM : this will be the line required.

Taking the four-side AG, AF, LG, LF the diagonal DE is cut by the diagonals AL, GF at K and M, hence $\{DEKM\} = -1$.

Therefore the pencil AB, AC, AO, AM is harmonic, and RS is drawn parallel to the ray AM. Hence RS is bisected at O.

20. Definition. Four points, no three of which are collinear, form a **quadrangle or four-point.** They can be joined by six lines, so that a four-point has three pairs of sides; the intersections of these six lines are the given four points, and three other points, called **diagonal points**, forming the **diagonal triad.**

Thus, if P, Q, R, S are four points we get PR, QS or a, b; PS, QR or c, d; RS, PQ or e, f the three pairs of sides; and U, V, W the intersections ab, cd, ef forming the diagonal triad. UV, UW, VW, are k, l, m.

Note 1. Just as the line joining two points A, B is represented by AB, so the intersection of two lines a, b is written ab.

Note 2. Compare this notation with that for a four-side, a corresponds to A, etc. etc.

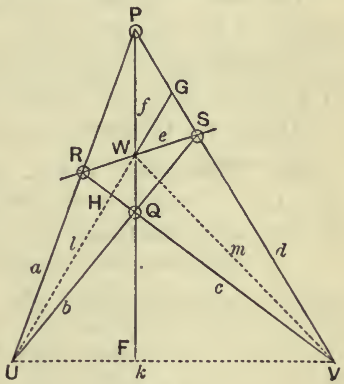


Fig. 16.

Theorem. *At a diagonal point of a four-point the lines which join it to the other two diagonal points are harmonically conjugate with respect to the two sides which intersect there.*

Proof 1. Adopting the notation of figure 16, let l or UW cut RQ at H and PS at G. Then $\{abkl\} = \{RQVH\}$ on RQ, and also equals $\{PSVG\}$ on PS.

But, projecting from W, $\{RQVH\} = \{efml\} = \{SPVG\}$;
and $\{SPVG\} = \{PSGV\} = \{ablk\}$;
hence $\{abkl\} = \{ablk\}$, and therefore each equals -1.

Proof 2. If PQ meets UV at F, by the properties of a four-side, W and F are harmonically conjugate to P and Q, hence the lines UW and UF or UV are harmonically conjugate to UP and UQ.

Corollary 1. VW cuts PR, QR at the harmonic conjugates of U with respect to P, R and Q, S respectively.

Corollary 2. The diagonals and diameters of a parallelogram form a harmonic pencil.

Hence, also, by making the parallelogram a rectangle we find, as before, that two lines and the bisectors of the angles between them form a harmonic pencil.

Problem. To find the harmonic conjugate of SK with respect to SA and SB.

Through any point K on SK draw any line to cut SA, SB at P, Q respectively; and another to cut them at R, S.

Then if PS, QR intersect at L, SL is the line required.

21. Theorem. If K′, L′, M′, N′ are the harmonic conjugates of K, L, M, N with respect to A, B, then $\{K'L'M'N'\} = \{KLMN\}$.

For, if O is the middle point of AB, we have

$$OA^2 = OK . OK' = OL . OL' = OM . OM' = ON . ON';$$

hence $$\{K'L'M'N'\} = \frac{K'M'}{M'L'} : \frac{K'N'}{N'L'} = \frac{OM' - OK'}{OL' - OM'} : \frac{ON' - OK'}{OL' - ON'}$$

$$= \frac{\dfrac{OA^2}{OM} - \dfrac{OA^2}{OK}}{\dfrac{OA^2}{OL} - \dfrac{OA^2}{OM}} : \frac{\dfrac{OA^2}{ON} - \dfrac{OA^2}{OK}}{\dfrac{OA^2}{OL} - \dfrac{OA^2}{ON}}$$

$$= \frac{\dfrac{MK}{OM . OK}}{\dfrac{LM}{OL . OM}} : \frac{\dfrac{NK}{ON . OK}}{\dfrac{LN}{OL . ON}}$$

$$= \frac{MK}{LM} : \frac{NK}{LN} = \frac{KM}{ML} : \frac{KN}{NL} = \{KLMN\}.$$

Corollary. If k', l', m', n' are the harmonic conjugates of k, l, m, n with respect to a, b, then $\{k'l'm'n'\} = \{klmn\}$.

Definitions. The **polar** of a point K with respect to two lines SA, SB is the harmonic conjugate SL of SK with respect to SA and SB.

It is the locus of the harmonic conjugate of K on any line through K with respect to the points in which that line cuts SA, SB.

The **pole** of a line with respect to two points is the harmonic conjugate with respect to those points of the intersection of their join by the line.

If we take any point on the line and join it to the two points, the harmonic conjugate of the line with respect to those joins always passes through the pole of the line.

From the theorem just proved it follows that the cross-ratio of any four lines forming a pencil is equal to the cross-ratio of their four poles, and conversely.

EXAMPLES. I.

1. If M is the mid-point of AB and C any other point on the line, prove that rect. AC . CB $=$ MA^2 $-$ MC^2, whether C is in AB or AB produced.

2. If AB, CD are any two segments of the same straight line and M, N their mid-points, prove that 4AC . BD $=$ 4MN^2 $-$ (AB $-$ CD)^2, and 4AD . BC $=$ 4MN^2 $-$ (AB $+$ CD)^2. Also deduce that AD . BC $+$ BD . CA $+$ CD . AB $= 0$.

3. Verify these results by numerical calculation when

(*a*) AB $=6$, AC $=9$, AD $=19$; (*b*) AB $=60$, AC $=30$, AD $=22$;
(*c*) AB $=20$, AC $=-14$, AD $=6$.

4. In each of the three cases given in the previous question find the values of the ratios AC : CB, AD : DC and CB : BD.

5. In each of the three cases find the position of the harmonic conjugate of D with respect to B and C.

6. The bisectors of the interior and exterior angles at the vertex C of a triangle ABC meet the base AB at K and L, and M is the mid-point of the base AB; prove that $MK \cdot ML = MA^2$.

7. If K, L are two points in AB harmonically conjugate with respect to AB, prove that A and B are harmonically conjugate with respect to K and L.

8. If A, B, K, L are four points in a straight line and $AB=4$, $AK=6$, find the values of $\{ABKL\}$ when AL has successively the values 1, 2, 3, 4, 5 and 6. Also find the values of $\{ABLK\}$ and $\{AKBL\}$ in each of the six cases.

9. Find the values of $\{ABKL\}$, $\{ABLK\}$, $\{AKBL\}$, $\{AKLB\}$, $\{ALBK\}$ and $\{ALKB\}$ by direct calculation when $AB=6$, $AK=5$, $AL=10$.

Shew that the equation whose roots are these six values can be written

$$36\,(x^2-x+1)^3=343\,(x^2-x)^2.$$

10. Given three points A, B, K on a straight line such that $AB=6$, $AK=4$, find by a geometrical construction a point L on the line such that $\{ABKL\}=3$.

11. From the harmonic properties of a complete four-side deduce that the diagonals of a parallelogram bisect each other.

12. K, L are the mid-points of the sides AC, AB of the triangle ABC, and BK, CL meet at G. Prove that AG bisects BC at M.

Also prove that KL bisects AM and deduce that KL trisects AG, and G trisects AM.

13. Given two parallel straight lines AB and CD, bisect AB and CD by the use of a ruler only.

14. A line AB is divided harmonically by P and P′, by Q and Q′, and by R and R′. Prove that $\{PP'QR\}=\{P'PQ'R'\}$.

15. If AP, BQ, CR are the tangents drawn to any circle from three collinear points A, B, C, prove that

$$AP^2 \cdot BC + BQ^2 \cdot CA + CR^2 \cdot AB = -BC \cdot CA \cdot AB.$$

16. If O, A, B, C are collinear, prove that

$$(1)\quad \frac{1}{AB \cdot AC}+\frac{1}{BA \cdot BC}+\frac{1}{CA \cdot CB}=0,$$

$$(2)\quad \frac{OA}{AB \cdot AC}+\frac{OB}{BA \cdot BC}+\frac{OC}{CA \cdot CB}=0.$$

17. If A, B, C, D, P, Q, R are collinear, prove that

(1) $$\frac{1}{\mathrm{AB.AC.AD}}+\frac{1}{\mathrm{BA.BC.BD}}+\frac{1}{\mathrm{CA.CB.CD}}+\frac{1}{\mathrm{DA.DB.DC}}=0,$$

(2) $$\frac{\mathrm{PA}}{\mathrm{AB.AC.AD}}+\frac{\mathrm{PB}}{\mathrm{BA.BC.BD}}+\frac{\mathrm{PC}}{\mathrm{CA.CB.CD}}+\frac{\mathrm{PD}}{\mathrm{DA.DB.DC}}=0,$$

(3) $$\frac{\mathrm{PA.QA}}{\mathrm{AB.AC.AD}}+\frac{\mathrm{PB.QB}}{\mathrm{BA.BC.BD}}+\frac{\mathrm{PC.QC}}{\mathrm{CA.CB.CD}}+\frac{\mathrm{PD.QD}}{\mathrm{DA.DB.DC}}=0,$$

(4) $$\frac{\mathrm{PA.QA.RA}}{\mathrm{AB.AC.AD}}+\frac{\mathrm{PB.QB.RB}}{\mathrm{BA.BC.BD}}+\frac{\mathrm{PC.QC.RC}}{\mathrm{CA.CB.CD}}+\frac{\mathrm{PD.QD.RD}}{\mathrm{DA.DB.DC}}=-1.$$

18. Prove that $\Sigma\,\dfrac{\mathrm{P_1A_1.P_2A_1.P_3A_1.\,\ldots\ldots P_{n-k}A_1}}{\mathrm{A_1A_2.A_1A_3.\,\ldots\ldots A_1A_n}}$ has the value 0, when k has any integral value from 2 to $n-1$, and the value $(-1)^n$ when k is 1.

CHAPTER II

INVOLUTION

22. *A pair of points* A, A′ *are in involution with two other pairs* B, B′ *and* C, C′ *if* $\{AA'BC\} = \{AA'C'B'\}$.

If this relation holds it follows that

$$\frac{AB}{BA'} : \frac{AC'}{C'A'} = \frac{AC}{CA'} : \frac{AB'}{B'A'}, \textit{ i.e. } \{AA'BC'\} = \{AA'CB'\} \;\ldots\ldots(1).$$

Also
$$\frac{AB \,.\, AB'}{A'B \,.\, A'B'} = \frac{AC \,.\, AC'}{A'C \,.\, A'C'} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Again
$$\{ABA'C'\} = \{A'B'AC\};$$

hence
$$\frac{AA'}{A'B} \cdot \frac{C'B}{AC'} \cdot \frac{AB'}{A'A} \cdot \frac{A'C}{CB'} = 1, \quad \therefore \quad \frac{AB'}{B'C} \cdot \frac{BC'}{C'A} \cdot \frac{CA'}{A'B} = -1 \ldots(3).$$

Hence

$$\frac{BB'}{B'A} : \frac{BC'}{C'A} = \frac{B'B}{BA'} : \frac{B'C}{CA'}, \textit{ i.e. } \{BAB'C'\} = \{B'A'BC\},$$

and $\therefore$ $\{BB'AC'\} = \{BB'CA'\}$, *i.e.* B, B′ are in involution with the two pairs A, A′; C, C′. Similarly C, C′ are in involution with A, A′; B, B′……………………………………………………(4).

If we divide AA′ at O, such that

$$OA : OA' = \frac{AB \,.\, AB'}{A'B \,.\, A'B'} = \frac{AC \,.\, AC'}{A'C \,.\, A'C'},$$

then
$$OA \,.\, OA' = OB \,.\, OB' = OC \,.\, OC' \;\ldots\ldots\ldots\ldots(5).$$

For $\{AA'BO\} = \dfrac{A'B'}{AB'}$, $\therefore$ $\{ABA'O\} = 1 - \dfrac{A'B'}{AB'} = \dfrac{AA'}{AB'}$,

$$\therefore\ AO : OB = AB' : A'B = AO - AB' : OB - A'B = B'O : OA',$$

$$\therefore\ OA \,.\, OA' = OB \,.\, OB', \textit{ et similiter.}$$

Conversely. If $OA \,.\, OA' = OB \,.\, OB' = OC \,.\, OC'$, then A, A'; B, B'; C, C' are in involution ....(6).

For $$OA : OB = OB' : OA' = AB' : BA',$$

$$\therefore\ \{ABA'O\} = \frac{AA'}{A'B} : \frac{AB'}{A'B} = \frac{AA'}{AB'},$$

$$\therefore\ \{AA'BO\} = 1 - \frac{AA'}{AB'} = \frac{A'B'}{AB'},$$

$$\therefore\ \frac{AO}{A'O} = -\frac{AB}{BA'} \cdot \frac{AB'}{A'B'} = \frac{AB \,.\, AB'}{A'B \,.\, A'B'}.$$

Similarly $$\frac{AO}{A'O} = \frac{AC \,.\, AC'}{A'C \,.\, A'C'}.$$

Hence $$\frac{AB \,.\, AB'}{A'B \,.\, A'B'} = \frac{AC \,.\, AC'}{A'C \,.\, A'C'},$$

and therefore A, A'; B, B'; C, C' are in involution.

23. Theorem. If A, A'; B, B'; C, C' form an involution on a line, and S is any point outside the line, then a line through

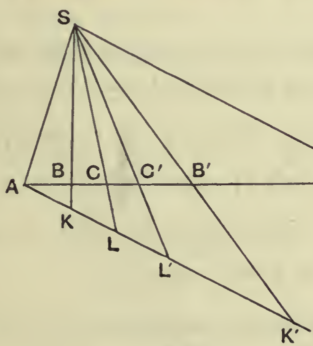


Fig. 17.

A parallel to SA′ will cut SB, SB′, SC, SC′ at points K, K′, L, L′ respectively, so that AK . AK′ equals AL . AL′.

And, conversely:

For $$AK : AL = \{AA'BC\},$$

and $$AK' : AL' = \{AA'B'C'\}.$$

But $$\{AA'BC\} = \{A'AB'C'\},$$

$$\therefore \; AK : AL = AL' : AK',$$

$$\therefore \; AK \,.\, AK' = AL \,.\, AL'.$$

And, conversely, if

$$AK \,.\, AK' = AL \,.\, AL',$$

then $$AK : AL = AL' : AK',$$

$$\therefore \; \{AA'BC\} = \{A'AB'C'\},$$

and hence A, A′ ; B, B′ ; C, C′ are in involution.

Corollary. *The projection of an involution is an involution.*

(Cf. the proof in § 17 that a cross-ratio is unaltered by projection.)

Definition. Three pairs of straight lines through a point form an involution if any transversal cuts them in an involution.

1. If a, a' ; b, b' ; c, c' form a pencil in involution,

$$\{aa'bc\} = \{aa'c'b'\} \quad \text{and} \quad \{aa'bc'\} = \{aa'cb'\}.$$

2. If $\{abcd\} = \{a'b'c'd'\}$ and $\{abcd'\} = \{a'b'c'd\}$,

then aa', bb', cc' form an involution.

3. The pairs of harmonic conjugates to two lines form an involution.

24. Theorem. *The three pairs of joins of four points cut any transversal in an involution.*

For if a transversal cuts PQ, RS at A, A′; PR, QS at B, B′; PS, QR at C, C′ respectively, and PQ, RS intersect at T:
projecting from P we get

$$\{AA'BC\} = \{TA'RS\};$$

and projecting from Q we get

$$\{TA'RS\} = \{AA'C'B'\};$$

hence $$\{AA'BC\} = \{AA'C'B'\},$$

i.e. AA′, BB′, CC′ are in involution.

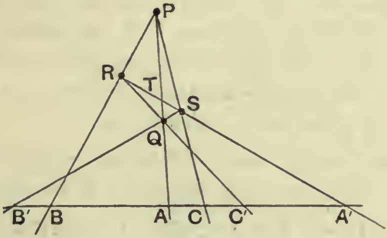


Fig. 18.

Problem. Given five points A, A′, B, B′, C in a line, to find C′ such that AA′, BB′, CC′ form an involution.

25. Theorem. If AA′, BB′, CC′ form an involution, and DD′ are another pair of points such that $\{ABCD\} = \{A'B'C'D'\}$, then DD′ form an involution with any two of the other three pairs: and conversely.

For $$\frac{AC}{CB} : \frac{AD}{DB} = \frac{A'C'}{C'B'} : \frac{A'D'}{D'B'},$$

but $$\{ABCA'\} = \{A'B'C'A\},$$

$$\therefore \frac{AC}{CB} : \frac{AA'}{A'B} = \frac{A'C'}{C'B'} : \frac{A'A}{AB'},$$

$$\therefore \frac{AD}{DB} : \frac{AA'}{A'B} = \frac{A'D'}{D'B'} : \frac{A'A}{AB'},$$

i.e. $\{ABDA'\} = \{A'B'D'A\}$,

i.e. DD′ are in involution with AA′ and BB′.

Corollary. If O is the centre of the involution

$$OD \,.\, OD' = OA \,.\, OA' = OB \,.\, OB' = OC \,.\, OC'.$$

Corollary. If DD′ are in involution with AA′ and BB′, they are also in involution with AA′ and CC′.

Rows in involution. In any straight line containing AA′ and BB′ we may take C, D, E,... and find conjugate points C′, D′, E′,... so that CC′, DD′, EE′ are severally in involution with AA′ and BB′. [For if O is the centre of AA′, BB′ we have only to make $OC' \,.\, OC = OA' \,.\, OA$, etc.] Thus we get a continuous double row of points, of which (by the above theorem and its corollary) any three pairs of points are in involution.

If $OA \,.\, OA'$ is positive, there will be two points X, Y such that $OX^2 = OY^2 = OA \,.\, OA'$; thus X is conjugate to itself, so also is Y.

These double points X, Y are called the **foci** of the involution.

Since $OX^2 = OA \,.\, OA'$, it follows that A, A′ are harmonic conjugates to X, Y.

Pencils in involution may similarly be obtained.

26. Problem. *Given two pairs of points* (AA′, BB′) *on a line, to construct the centre of their involution; the double points (if any); and the conjugate of any point* C′.

Take any point G and construct the circles AGA′, BGB′.

If these circles have a common tangent at G cutting the line at O, then $OG^2 = OA \,.\, OA' = OB \,.\, OB'$; hence O is the centre of the involution; the double points are X, Y where $OX = OY = OG$; and a circle touching OG at G and passing through C cuts the line at C′ such that $OC \,.\, OC' = OG^2$.

But if not, the circles intersect again at some point H; let GH cut the line at O.

Then OG . OH = OA . OA′ = OB . OB′: hence O is the centre of the involution; if O lies in GH produced, the power is positive

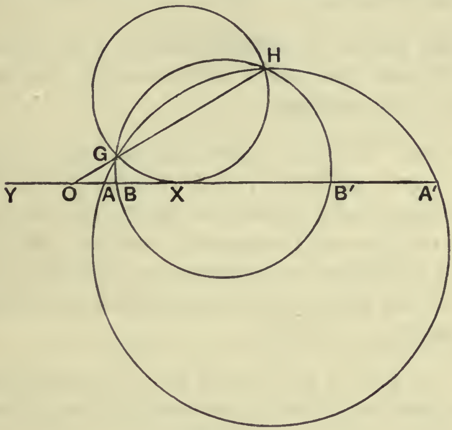


Fig. 19.

and the double points are X, Y where OX = OY = the tangent from O to either circle; also the circle GHC cuts the line at C′ the conjugate of C, for OC . OC′ = OG . OH.

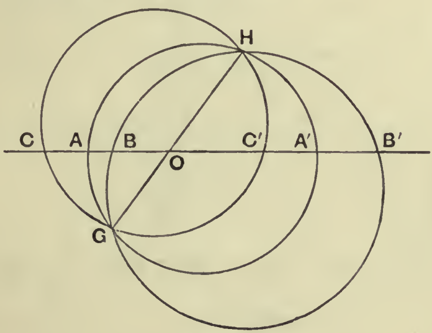


Fig. 20.

27. Corollary. If the power is positive either BB′ both lie on the other side of O from AA′, or if on the same side and OB > OA, OB′ < OA′.

Hence one of the segments AA′, BB′ lies entirely without or entirely within the other.

But **if the power is negative** let A, B be on the same side of O and OB < OA, then OB′ > OA′ in absolute length, hence **the segments** AA′, BB′ **overlap.**

Otherwise. If A and A′ are both outside the segment BB′, in passing along the arc AGA′ from A we enter the circle BGB′ at G, and we must cross it again before reaching A′, hence G, H lie on the same side of the line and the power is positive. But if AA′ overlaps BB′ we must cross the circle BGB′ on each side of the line to get from A to A′; G, H lie on opposite sides of the line, and the power is negative.

Exercise. The locus of the centre of the involution determined on any straight line cutting two circles by the pairs of points of section is the radical axis.

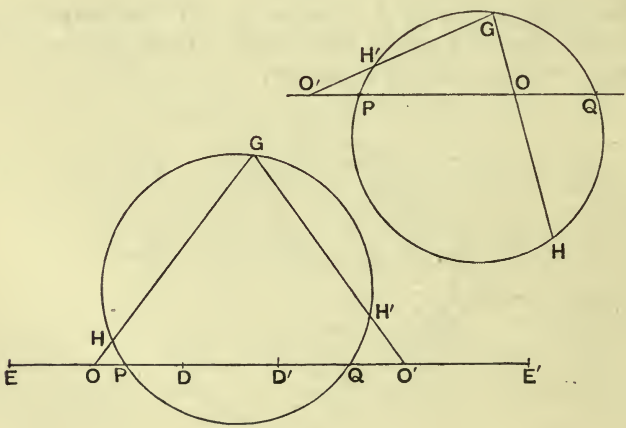


Fig. 21.

28. Problem. Given two independent rows in involution on the same line to find a common pair of conjugate points.

Let O, O′ be the centres, take any point G not on the line, and on OG, O′G find points H, H′ such that OG . OH and O′G . O′H′ are equal to the powers of the respective rows. The circle GHH′ will cut OO′ at the required points P, Q.

For $\quad OP \,.\, OQ = OG \,.\, OH$ and $O'P \,.\, O'Q = O'G \,.\, O'H'$.

If the power for O is negative, G, H lie on opposite sides of the line, and the circle GHH′ must cut the line, whether the power for O′ is positive or negative.

If both powers are positive let D, E and D′, E′ be the (real) double points of the two involutions: the required points are harmonically conjugate to DE and to D′E′, hence they are the double points of the involution given by DE, D′E′.

They are real or imaginary according as segments DE, D′E′ do not or do overlap.

29. Problem. Given five rays a, a', b, b', c through a centre S, to find a sixth ray c' such that $aa'bb'cc'$ may be in involution.

We wish to have $\quad \{aa'bc\} = \{a'ab'c'\}$

and $\quad \therefore\ \{aa'bc\} = \{aa'c'b'\}$.

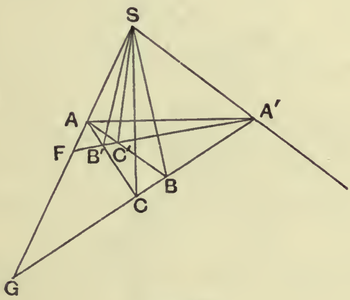


Fig. 22.

We shall then find a pencil at another centre which is in perspective with both of these.

Draw any two lines through a point A′ on a', one cutting a, b, c at G, B, C, the other cutting a, b' at F, B′.

Join CB′ to cut a at A; join AB cutting A′F at C′; then SC′ is the ray required.

For $\{aa'bc\}$ is in perspective with A {GA′BC}, *i.e.* A {FA′C′B′} which is in perspective with $\{aa'c'b'\}$.

Or: $$\{aa'bc\} = \{\mathrm{GA'BC}\} = \{\mathrm{FA'C'B'}\} \text{ (by projection from A)},$$ $$= \{aa'c'b'\} = \{a'ab'c'\}.$$

Theorem. *The lines joining any point to the three pairs of vertices of a (complete) four-side are in involution.*

For, if the lines AB, AB′ cut A′B, A′B′ at B, C, C′, B′ so that A, A′; B, B′; C, C′ are opposite vertices, we have the pencil {SA, SA′, SB, SC} which we may write S {AA′BC} equal on BC to the row {GA′BC} where SA cuts BC at G; and, by projection from A, this row = {FA′B′C′} = pencil S {AA′C′B′} = S {A′AB′C′}, hence SA, SA′; SB, SB′; SC, SC′ are in involution.

30. Problem. Two pairs of lines a, a'; b, b' pass through

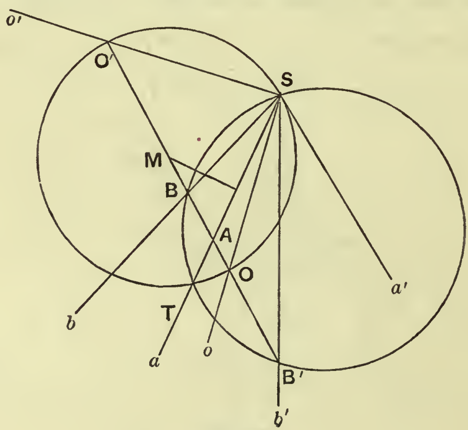


Fig. 23.

the point S, to draw two lines through S in involution with a, a' and b, b' and perpendicular to each other.

Draw a line parallel to a', to cut a, b, b' at A, B, B′ respectively, and construct the circle SBB′, cutting a again at T.

Let the perpendicular bisector of ST cut BB′ at M, and construct a circle with M as centre passing through S and T.

If this circle cuts BB′ at O and O′, SO and SO′ are the required lines.

For $$AO \cdot AO' = AS \cdot AT = AB \cdot AB'.$$

$\therefore$ SO, SO′ are in involution with a, a', b, b'.

Also OSO′, being the angle in a semicircle, is a right angle.

31. Theorem. If, in a pencil in involution, there are two pairs of conjugate rays each consisting of two lines at right angles to each other, then every ray is perpendicular to its conjugate.

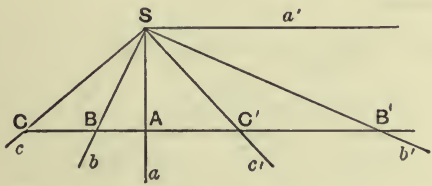


Fig. 24.

Let a be perpendicular to its conjugate a', and b to its conjugate b'.

Draw a line parallel to a' to cut a, b, b' at A, B, B′ respectively: and let this line cut any other conjugate rays c, c' at C, C′.

A lies between B, B′ and BSB′ is a right angle,

$$\therefore \; AB \cdot AB' = -AS^2.$$

But $AC \cdot AC' = AB \cdot AB'$ (§ 23), $\therefore$ $AC \cdot AC' = -AS^2$, hence CSC′ is a right angle.

Corollary. If a right angle be turned about its vertex its arms cut any transversal in an involution.

Theorem. The middle points of the diagonals of a four-side are collinear.

Let AA′, BB′, CC′ be the three pairs of opposite vertices. Construct circles on AA′, BB′ as diameters intersecting at S; then in the involution S{AA′BB′CC′} two pairs of rays SA, SA′; SB, SB′ are perpendicular, hence SC, SC′ are also perpendicular, and hence S lies on the circle whose diameter is CC′.

Similarly T the other intersection of the two former circles lies also on this circle. Hence the three circles are coaxal, and therefore their centres, *i.e.* the middle points of AA′, BB′, CC′, are collinear.

32. There is no single ray corresponding to the centre of a linear involution, but the pair of perpendicular rays possess similar properties.

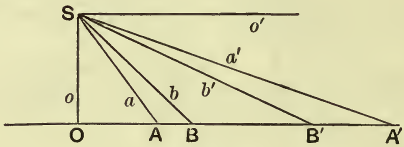


Fig. 25.

Thus, if o, o' are lines at right angles in involution with aa' and bb', and we draw a line parallel to o' to cut o, a, a', b, b' at O, A, A′, B, B′ we have OA . OA′ = OB . OB′, but SO is perpendicular to OA,

$$\therefore\ \tan \widehat{oa} \,.\, \tan \widehat{oa'} = \tan \widehat{ob} \,.\, \tan \widehat{ob'}.$$

Conversely. If $\tan \widehat{oa} \,.\, \tan \widehat{oa'} = \tan \widehat{ob} \,.\, \tan \widehat{ob'}$, then a, a' and b, b' are in involution with o and a line perpendicular to o.

Double rays. There will be two double or focal rays x, y such that $\tan^2 \widehat{ox} = \tan^2 \widehat{oy} = \tan \widehat{oa} . \tan \widehat{oa'}$; and these will be real or imaginary according as $\tan \widehat{oa} . \tan \widehat{oa'}$ is positive or negative.

By a proof similar to that of § 27, or by deduction from the result of § 27, we may shew that the double rays are real or imaginary according as the angle between a, a' does not or does overlap the angle between b and b'.

EXAMPLES. II.

1. If A, A′; B, B′; C, C′ form an involution, and D, D′ are in involution with A, A′ and B, B′, prove that they are also in involution with A, A′, C, C′, and with B, B′, C, C′.

2. If $\{ABCD\} = \{A'B'C'D'\}$ and $\{ABCD'\} = \{A'B'C'D\}$, then the three pairs of points A, A′; B, B′; C, C′ form an involution. Also D, D′ are a pair of points in involution with any two of the other three pairs.

3. Three pairs of harmonic conjugates to two given points on a straight line form an involution.

4. Any straight line cuts three circles passing through the same two points in an involution.

5. A transversal cuts a system of coaxal circles in pairs of points in involution. Find, also, the centre and the double points of the involution.

6. If the circles of a coaxal system do not cut their radical axis the double points of the involution traced on the line of centres are the point circles of the system.

7. If the coaxal circles have a real common chord find the position of the double points of the involution on the line through the centres of the circles, and the value of the power.

8. The opposite pairs of sides of a parallelogram, and the two diagonals cut any transversal in the pairs of points A, A′; B, B′; C, C′; prove, by similar triangles, that $\{ABCC'\} = \{B'A'CC'\}$, and deduce that AA′, BB′, CC′ form an involution.

9. A, B, C, D, E, ... and A′, B′, C′, D′, E′, ... are two sets of points in a straight line such that the cross-ratio of any four points of the first set equals the cross-ratio of the four corresponding points of the second set, *e.g.* $\{ABCD\} = \{A'B'C'D'\}$, and this remains true when a pair of the points are exchanged, viz. $\{ABCD'\} = \{A'B'C'D\}$, etc., prove that A, A′; B, B′; C, C′; ... are pairs of points in involution.

10. Any transversal is cut by the sides BC, CA, AB of a triangle at L, M, N respectively; and L′, M′, N′ are three other points on the transversal, such that L, L′; M, M′; N, N′ form an involution. Prove that AL′, BM′, CN′ are concurrent.

11. The cross-ratio of four points ABCD on one line is equal to that of A′B′C′D′ on another line; and A′B′C′D′ are projected on to the first line, using the intersection of AB′ and A′B as centre of projection. Prove that an involution is obtained.

CHAPTER III

PROJECTIVE ROWS AND PENCILS

33. Two rows of points A, B, C, D, E, F, ... on a, and A′, B′, C′, D′, E′, F′, ... on a' are **in perspective** if AA′, BB′, CC′, etc. all pass through the same point S; and S is the centre of perspective.

It has been proved that $\{ABCD\} = \{A'B'C'D'\}$, etc.

Problem. If ABC on a are not in perspective with A′B′C′ on a', to find a point D′ on a' corresponding to D on a, such that $\{A'B'C'D'\} = \{ABCD\}$.

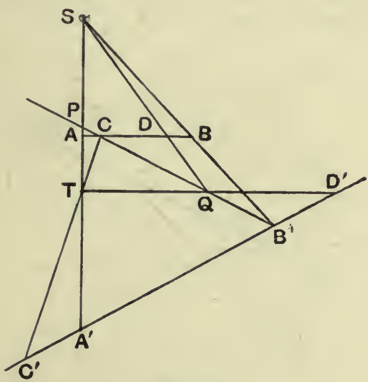


Fig. 26.

Join BB′ cutting AA′ at S, and CC′ cutting AA′ at T.

Let CB′ cut AA′ at P and SD at Q, then TQ will cut A′B′ at the required point D′.

For, by projection from S, $\{ABCD\} = \{PB'CQ\}$ and, by projection from T, $\{PB'CQ\} = \{A'B'C'D'\}$.

Exercise. Find another solution by converting the ratio $\{ABCD\}$ into the ratio AK : AL, as in the proof that a cross-ratio is projective (§ 17).

Theorem. If we take A, B, C on a and A′, B′, C′ on a', and find K′, L′, M′, N′ on a' corresponding to K, L, M, N on a, so that $\{ABCK\} = \{A'B'C'K'\}$, etc., etc., etc., then shall

$$\{KLMN\} = \{K'L'M'N'\}.$$

For K, L, M, N are in perspective with four points on CB′, and these points are in perspective again with K′, L′, M′, N′ respectively.

Definition. Two rows of points such that the cross-ratio of any four points of one is equal to the cross-ratio of the four corresponding points of the other are called **projective rows.**

Definition. If $AC : CB = A'C' : C'B'$, then $AK : KB = A'K' : K'B'$, and $KL : LM = K'L' : L'M'$; in this case the rows are said to be **similar.**

34. Problem. To find a third row in perspective with each of two given projective rows.

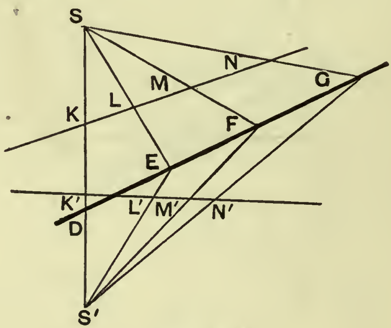


Fig. 27.

Let KLMN correspond to K′L′M′N′, take *any* point E, and let EL, EL′ meet KK′ at S, S′; let SM, S′M′ meet at F.

Then, if EF meets KK′ at D, and SN at G, we have

$$\{KLMN\} = \{DEFG\} = \{K'L'M'N'\},$$

hence S′G cuts K′L′ at N′: and therefore the row on EF is in perspective with the rows on KL and on K′L′.

There are an infinite number of positions of the line EF.

Note. This is a solution of the previous problem, the solution using CB′ being a special case of the general construction here given.

If the two lines a, a' intersect at O, we shall find a point P′ on a' such that $\{ABCO\} = \{A'B'C'P'\}$, so that O regarded as a point on the first row corresponds to some other point P′ on the second row, but when we take O to be a point of the second row we get some other point N on the first row.

35. Theorem. If two projective rows are such that the intersection corresponds to itself the rows are in perspective.

Let the intersection O correspond to itself, also A, B, K to A′, B′, K′. Join AA′, BB′ intersecting at S.

Then pencil $S\{OABK\} = \{OABK\} = \{OA'B'K'\}$, hence SK passes through K′; hence and similarly the join of any pair of corresponding points passes through S.

Therefore the rows are in perspective with S as centre.

Theorem. If the joins of three pairs of corresponding points of two projective rows are concurrent, the rows are in perspective.

For, if AA′, BB′, CC′ meet at S; and K is any other point on AB. Then SK meets A′B′ at K″, so that $\{A'B'C'K''\} = \{ABCK\}$. Hence K″ coincides with the point K′ on A′B′, corresponding to K on AB; *i.e.* KK′ passes through S.

36. Projective axis of two rows. If projective rows be taken on two straight lines which intersect at P, then P is a point of both rows. Let P on the first row correspond to B′ on the second, and let A on the first correspond to P on the second.

Take any two points K, L of the first row, and the corresponding points K′, L′ of the second, so that $\{APKL\} = \{PB'K'L'\}$.

But $\{PB'K'L'\} = \{B'PL'K'\}$, hence $\{APKL\} = \{B'PL'K'\}$, and these are equal cross-ratios on two lines, with the intersection P corresponding to itself, hence they are in perspective,

$$\therefore \; KL',\; K'L \text{ intersect on } AB'.$$

Hence, if K, L on one row correspond to K′, L′ on a projective row, the locus of the intersection of KL′ and K′L is a fixed straight line, which we may call the projective axis of the two rows.

37. Vanishing Points. If we have three points A, B, C on a, and corresponding points A′, B′, C′ on a', we can find a point I′ on a' such that $\{A'B'C'I'\} = -AC : CB$.

Take any point E (fig. 28) and let EB, EB′ meet AA′ at S, S′; and let SC, S′C′ meet at F. Draw SK parallel to AB to meet EF at K, let S′K cut A′B′ at I′. Let EF cut AA′ at D.

Then $\{A'B'C'I'\} = \{DEFK\}$, which is the cross-ratio of the pencil formed at S by SA, SB, SC, SK.

But SK is parallel to the transversal AB, hence the cross-ratio of $S\{ABCK\} = -AC/CB$. $\therefore$ a point I′ is found such that

$$\{A'B'C'I'\} = -AC/CB.$$

There is no finite point I on AB, such that $\{ABCI\} = -AC/CB$, and we may say either that I′ is the one point of a' to which there is no point of a to correspond, or that I′ corresponds to a point at infinity on a (such that $AI : IB = -1$). I′ is called the **vanishing point.**

Similarly we get a **vanishing point** J on a, corresponding to the point at infinity I′ on a'.

Corollary. In the case of similar rows this construction fails. For $A'C' : C'B' = AC : CB$.

$\therefore$ the cross-ratio of $S'\{DEFK\} = -A'C' : C'B'$.

$\therefore$ $A'B'$ is parallel to $S'K$, and I' is at infinity.

Hence the vanishing points of two similar rows are at infinity.

38. Theorem. The product of the distances of corresponding points from the vanishing points is constant.

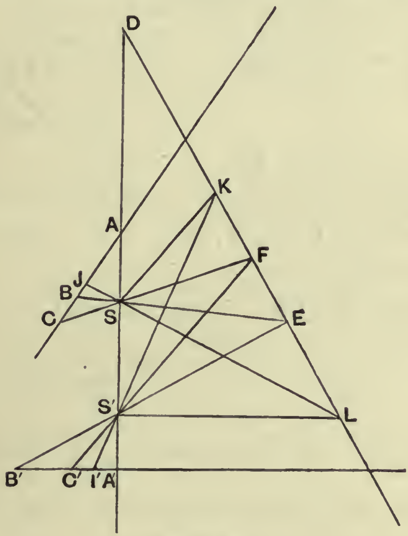


Fig. 28.

If SK parallel to AB cuts EF at K, $S'L$ parallel to $A'B'$ cuts EF at L; we have

$$\{DEKL\} = -1 : \frac{AJ}{BJ},$$

and

$$\{DEKL\} = -\frac{A'I'}{I'B'}.$$

Hence $$\frac{JA}{JB} \cdot \frac{I'A'}{I'B'} = 1,$$

or $$JB \,.\, I'B' = JA \,.\, I'A';$$

similarly $$JC \,.\, I'C' = JA \,.\, I'A';$$

and generally if P corresponds to P′, we have $JP \,.\, I'P' = JA \,.\, I'A'$.

Special case. When the rows are in perspective; SJ is parallel to A′B′ and SI′ to AB. Hence, O being the intersection of the rows,

$$JA : JS = I'S : I'A',$$

$$\therefore \quad JA \,.\, I'A' = I'O \,.\, JO,$$

similarly

$$JO \,.\, I'O = JB \,.\, I'B' = JP \,.\, I'P'.$$

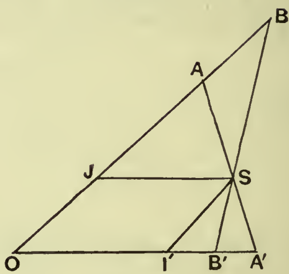


Fig. 29.

Algebraically. Let O, A, B, C, X correspond to O′, A′, B′, C′, X′, let

$OA = a$, $OB = b$, $OC = c$, $OX = x$;

$$O'A' = a', \quad O'B' = b', \quad O'C' = c', \quad O'X' = x',$$

$$\therefore \quad \frac{c-a}{b-c} : \frac{x-a}{b-x} = \frac{c'-a'}{b'-c'} : \frac{x'-a'}{b'-x'},$$

$$\therefore \quad \frac{x-a}{b-x} = \lambda \,.\, \frac{x'-a'}{b'-x'}, \text{ where } \lambda = \frac{c-a}{b-c} : \frac{c'-a'}{b'-c'},$$

$$\therefore \quad xx'(1-\lambda) - x(b' - \lambda a') - x'(a - \lambda b) + ab' - a'b = 0,$$

or $$xx' + l'x + lx' + ll' = k,$$

or $$(x+l)(x'+l') = k.$$

If $OJ = -l$ and $O'I' = -l'$, then $JX \,.\, I'X' = k$.

Also when X′ is at infinity JX is o, hence J is the point which corresponds to the point at infinity of A′B′: similarly I′ corresponds to the point at infinity on AB.

39. Pencils. When two pencils are formed by joining the various points of a line to two vertices, *i.e.* when intersections of corresponding rays are collinear, the pencils are said to be **in perspective**, and the line of intersection is the **axis** of perspective.

The cross-ratio of four rays of one pencil equals the cross-ratio of the four rays corresponding in the other, being the cross-ratio of the four points in which they cut the axis.

Problem. Given three lines a, b, c through S, and three lines a', b', c' through S′, such that aa', bb', cc' are not collinear points, to construct a line l' through S′ corresponding to l through S, such that $\{a'b'c'l'\} = \{abcl\}$.

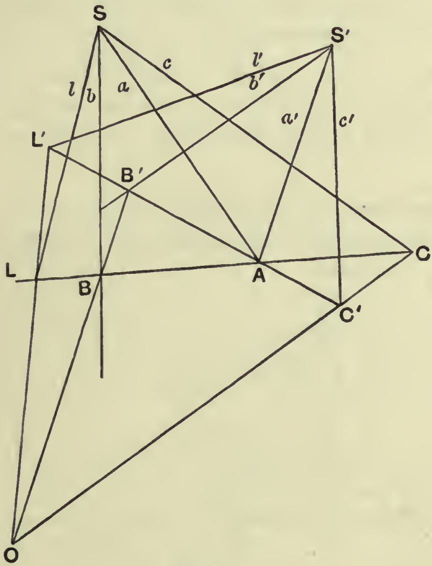


Fig. 30.

Let aa' intersect at A, through A draw any line cutting b, c, l at B, C, L; and any line cutting b', c' at B′, C′. Let BB′, CC′ meet at O, join OL to cut AB′ at L′. Then S′L′ is the required ray l'. For $\{abcl\} = \{ABCL\}$ and $\{a'b'c'l'\} = \{AB'C'L'\}$, but, by projection from O, we have $\{ABCL\} = \{AB'C'L'\}$.

Corollary. Take any four lines k, l, m, n through S, and the points K, L, M, N where they cut AB, from centre O project these on AB′ giving K′L′M′N′; and thus, by joining K′L′M′N′ to S′, get rays $k'l'm'n'$ through S′ corresponding to $klmn$. Then $\{klmn\} = \{KLMN\}$ and $\{k'l'm'n'\} = \{K'L'M'N'\}$; but, by projection from O, $\{KLMN\} = \{K'L'M'N'\}$.

Hence the cross-ratio of any four rays through S equals the cross-ratio of the corresponding rays through S′.

Definition. Two pencils, such that the cross-ratio of any four rays of the one is equal to that of the corresponding rays of the other, are called **projective** pencils.

Corollary. From the above construction it follows that, when two pencils are projective, we can, in an infinite number of ways, find a third pencil which is in perspective with each of them.

40. Theorem. If the line SS′ regarded as a ray of the pencil at S corresponds to the same line in the pencil at S′, the rows are in perspective.

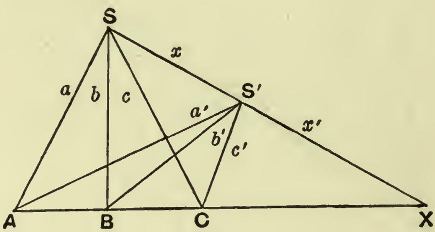


Fig. 31.

Let SS′ be x through S, and x' through S′, and let a, b through S meet a', b' through S′ at A, B.

Join AB cutting SS′ at X, and let c through S cut AB at C.

Then $\{abcx\} = \{\mathrm{ABCX}\}$.

But S′A is a'; S′B is b'; S′X is x'; hence S′C is c'; *i.e.* any two corresponding rays c, c' intersect on AB.

Theorem. If the intersections of three pairs of corresponding rays are collinear the pencils are in perspective.

For if a, b, c meet a', b', c' on the line ABC cutting SS′ at X, we have $\{abcx\} = \{\mathrm{ABCX}\} = \{a'b'c'x'\}$: so that SS′ corresponds to S′S.

Corresponding Pairs of Perpendicular Rays.

41. Problem. In two pencils in perspective to find a pair of perpendicular rays of the one, such that the corresponding rays of the other are also perpendicular.

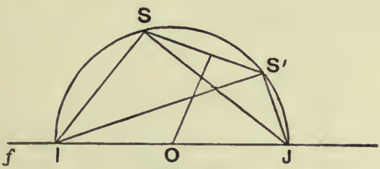


Fig. 32.

Let S, S′ be the vertices of the pencils and f the axis. Bisect SS′ at right angles by a line meeting f at O, and make a circle with O as centre and OS as radius (which will pass through S′), cutting f at I, J.

Then SI, SJ are perpendicular, and so also are the corresponding rays S′I, S′J. If SS′ is perpendicular to f, SS′ and SI, S′I′ perpendicular to SS′ give the solution. There is only one solution of the problem in any case.

Problem. In two projective pencils to find the pair of perpendicular rays of the one such that the corresponding rays of the other are also perpendicular.

Let S, S′ be the vertices, and let the ray at S′ corresponding to SS′ at S be S′T; turn the S′ pencil round S′ as a centre until S′T is in a straight line with SS′. The two pencils will then be in perspective, and the solution may be found by using the previous problem.

Theorem. If ij are perpendicular rays at S corresponding to perpendicular rays $i'j'$ at S′, and x, x' are any other pair of rays, and if we draw lines parallel respectively to i and j' cutting j, x at J, X and i', x' at I′, X′, the value of JX . I′X′ is constant.

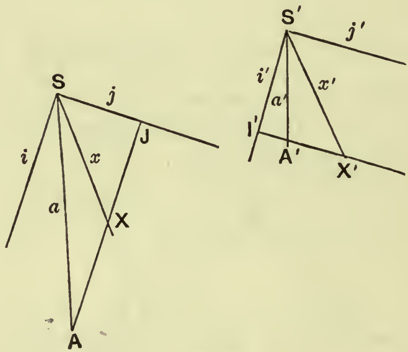


Fig. 33.

Let any fixed corresponding rays a, a' cut these lines at A, A′. Then $\{axji\} = \mathrm{JA} : \mathrm{JX}$ and $\{a'x'j'i'\} = \mathrm{I'X'} : \mathrm{I'A'}$; hence

$$\mathrm{JX} \,.\, \mathrm{I'X'} = \mathrm{JA} \,.\, \mathrm{I'A'} = \text{constant}.$$

42. **(Trigonometrical.)** Let rays $abcx$ at S make angles α, β, γ, θ with a line through S, and the corresponding rays $a'b'c'x'$ at S′ make angles α', β', γ', θ' with a line through S′.

Then
$$\frac{\sin \widehat{ax}}{\sin \widehat{xb}} : \frac{\sin \widehat{ac}}{\sin \widehat{cb}} = \frac{\sin \widehat{a'x'}}{\sin \widehat{x'b'}} : \frac{\sin \widehat{a'c'}}{\sin \widehat{c'b'}},$$

$$\therefore \quad \frac{\sin \overline{\theta - \alpha}}{\sin \overline{\beta - \theta}} = k \,.\, \frac{\sin \overline{\theta' - \alpha'}}{\sin \overline{\beta' - \theta'}},$$

i.e. $\sin \overline{\theta - \alpha} \sin \overline{\theta' - \beta'} = k . \sin \overline{\theta' - \alpha'} . \sin \overline{\theta - \beta}$,

$\therefore (\tan\theta - \tan\alpha)(\tan\theta' - \tan\beta') = k . (\tan\theta' - \tan\alpha')(\tan\theta - \tan\beta)$,

hence $\tan\theta . \tan\theta' + l \tan\theta' + l' \tan\theta + m = 0$.

Corollary. This may be written

$$(\tan\theta + \tan\lambda)(\tan\theta' + \tan\lambda') = c(1 - \tan\theta\tan\lambda)(1 - \tan\theta'\tan\lambda'),$$

where
$$\frac{1 - c\tan\lambda\tan\lambda'}{1} = \frac{\tan\lambda' + c\tan\lambda}{l'} = \frac{\tan\lambda + c\tan\lambda'}{l} = \frac{\tan\lambda . \tan\lambda' - c}{m},$$

three equations which determine λ, λ' and c.

Hence $\tan(\theta + \lambda) . \tan(\theta' + \lambda') = c$.

If we measure the angles θ, θ' from another pair of lines through S, S′ at angles λ, λ' to those originally used, this becomes $\tan\phi . \tan\phi' = c$.

Further, when $\phi = \frac{\pi}{2}$, $\phi' = 0$; when $\phi = 0$, $\phi' = \frac{\pi}{2}$. So that if i, j' are the lines through S, S′ from which the angles ϕ, ϕ' are measured; the line at S′ corresponding to i is a line i' perpendicular to j'; while j' corresponds to a line j at S perpendicular to i.

Hence we have two rays ij at S at right angles to each other, and their corresponding rays $i'j'$ at S′ are also perpendicular: and if x, x' make angles ϕ, ϕ' with i and j' we have $\tan\phi . \tan\phi' = c$.

43. Desargues' Theorem. If two triangles have the lines joining corresponding vertices concurrent, then the intersections of corresponding sides are collinear.

Let ABC, DEF be two triangles having AD, BE, CF meeting at O: and let BC, EF meet at K; CA, FD at L; AB, DE at M. Let KL cut OA, OB, OC at R, S, T respectively.

Then, by projection from K, we have {BESO} = {CFTO}; and, by projection from L, we have {ADRO} = {CFTO}; hence

{BESO} = {ADRO},

and these projective rows have a common homologous point O, hence BA, ED, RS are concurrent; but BA, ED meet at M, hence M lies on RS, *i.e.* is collinear with K and L.

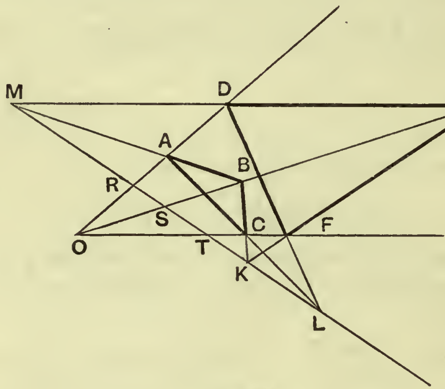


Fig. 34.

Conversely. If the points K, L, M of intersection of BC, EF; CA, FD; AB, DE are collinear the joins AD, BE, CF are concurrent.

Let BE, CF meet at O.

Join OK, OL, OM.

Then the pencil MB, ME, MK, MO is in perspective with KB, KE, KM, KO; which is in perspective with LC, LF, LM, LO.

Hence MA, MD, ML, MO is projective with LA, LD, LM, LO; two projective pencils in which the ray ML corresponds to ML, hence they are in perspective, *i.e.* A, D, O are collinear.

44. Two triangles whose corresponding sides meet in three points lying on a straight line *s*, and whose corresponding vertices are joined by three straight lines which meet at a point S, are said to be **in homology** or **in plane perspective**: S is the centre, and s the axis of homology.

Two sets of points in a plane are homologous if the joins of corresponding points are concurrent; and then the line joining two points of the one set meets the line joining the corresponding points of the other set on a fixed axis of homology.

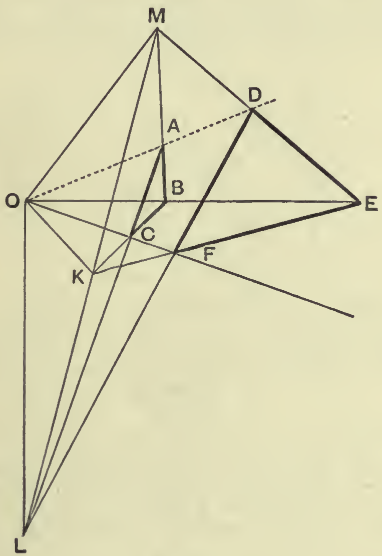


Fig. 35.

Conversely, two sets of points are in homology if the intersections of *all* possible joins of the one set, with corresponding joins of the other set, are collinear, by Desargues' Theorem.

Similarly, two sets of lines are homologous if the intersections of corresponding lines are collinear; and then the join of the intersections of two lines of the first set to the intersection of the corresponding lines of the second set passes through a fixed centre of homology. Conversely, if two sets of lines are such that every possible intersection of two lines of the first set, and the corresponding intersection of the second set, are collinear with a fixed point, then the two sets of lines are in homology, by Desargues' Theorem.

The extension of Desargues' Theorem to rectilinear figures requires care. Thus if we have two sets of points A, B, C, D and A′, B′, C′, D′ such that AB, A′B′; BC, B′C′; CD, C′D′; DA, D′A′ meet at four collinear points, the figures are not generally in homology.

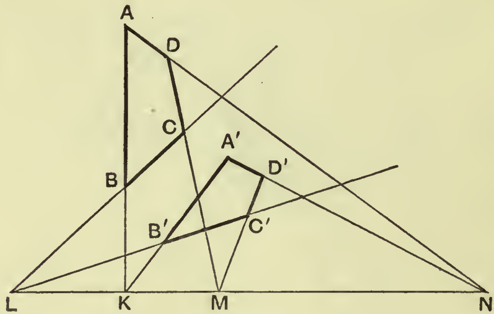


Fig. 36.

In fact, if AB, BC, CD, DA cut s at K, L, M, N we might turn C′D′ round M without altering the position of A′, B′ and only in one position would CC′ pass through the intersection of AA′ and BB′.

45. Theorem. If A, B, C, D and A′, B′, C′, D′ are two sets of four-points such that the five intersections of corresponding sides AB, A′B′; AC, A′C′; AD, A′D′; BC, B′C′; BD, B′D′ lie on a straight line, then the sixth intersection, viz. of CD, C′D′, will lie on the same straight line.

For, by Desargues' Theorem, the triangles ABC, A′B′C′ are in homology; and also ABD, A′B′D′.

Hence AA′, BB′, CC′, DD′ are concurrent.

Hence triangles ACD, A′C′D′ are in homology, and therefore the intersection of CD, C′D′ is collinear with the intersections of AC, A′C′ and AD, A′D′.

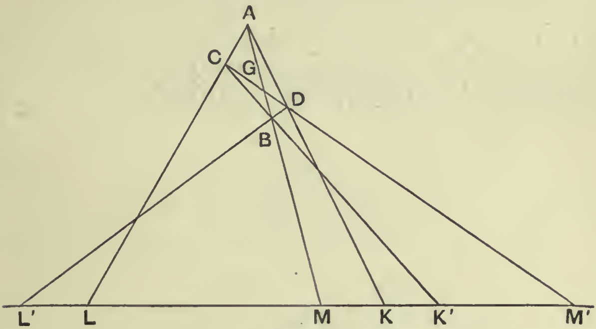


Fig. 37.

Problem. Given five points K, K′, L, L′, M on a straight line to find the sixth point M′ such that, if a pair of opposite sides of a four-point pass through K, K′, and a second pair through L, L′, the third pair may pass through M, M′.

Join any point A to K, L, M; on AM take any point B and join to K′, L′. Let BK′ cut AL at C, and BL′ cut AK at D: then CD will cut the line KK′ at the required point M′.

The preceding theorem shews that when K, K′, L, L′, M are given the position of M′ is fixed, *i.e.* we should get the same point M′, wherever we placed the point A.

Corollary. Using A, B successively as centres of projection, we have

$$\{KLMM'\} = \{DCGM'\} = \{L'K'MM'\};$$

hence $$\{KLMM'\} = \{K'L'M'M\}.$$

Exercise 1. Consider the special case in which K, K′ and also L, L′ coincide.

Exercise 2. State and prove the corresponding theorem and problem relating to two sets of four lines, and the six joins of the corresponding vertices.

Projective Rows on the same straight line.

46. We may take on the same straight line two sets of three points A, B, C; A′, B′, C′ and to any point K find a corresponding point K′ such that {ABCK} = {A′B′C′K′}.

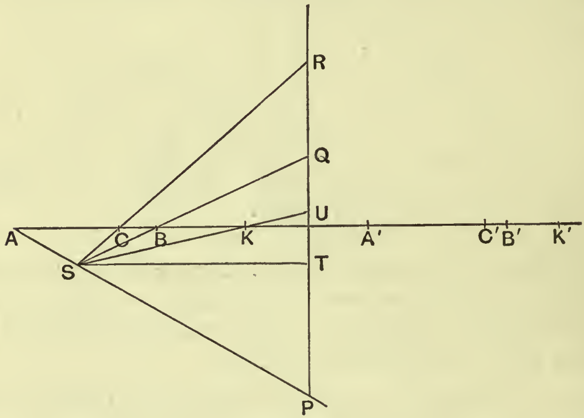


Fig. 38.

To construct the point K′ we take any other line and from centre S project A, B, C on this line, giving P, Q, R.

Also project K giving U.

Now taking PQRU find K′ on the original line such that {PQRU} = {A′B′C′K′} by the construction of § 30.

Then $\qquad\qquad$ {ABCK} = {PQRU} = {A′B′C′K′}.

From the construction referred to (§ 30) we find that if U, V, W, X are four successive positions of U (projections of K, L, M, N), giving positions K′, L′, M′, N′ for K′, then {K′L′M′N′} = {UVWX}.

But, by projection, $\{UVWX\} = \{KLMN\}$;

hence $\{K'L'M'N'\} = \{KLMN\}$.

47. There is one position Y of U which corresponds to no finite position on A′B′, and if the projection of this on AB is J, we have $\{ABCJ\} = -A'C' : C'B'$.

Similarly, if we draw ST parallel to the line to cut PQ at T, and $\{PQRT\} = \{A'B'C'I'\}$, then I′ corresponds to no finite point of the A, B, C row.

Also $\{A'B'C'I'\} = \{PQRT\}$ which is the cross-ratio of a pencil whose vertex is S formed by SP, SQ, SR, ST, and AB is parallel to one ray of this pencil, and hence $\{PQRT\} = -AC : CB$.

Hence $\{A'B'C'I'\} = -AC : CB$.

J and I′ are the vanishing points of the two rows.

Further $\{ABCJ\} = -\dfrac{A'C'}{C'B'}$; hence $\dfrac{JA}{JB} = \dfrac{AC}{CB} \div \dfrac{A'C'}{C'B'}$;

also $\{A'B'C'I'\} = -\dfrac{AC}{CB}$; hence $\dfrac{I'B'}{I'A'} = \dfrac{AC}{CB} \div \dfrac{A'C'}{C'B'}$;

and $\therefore\ \dfrac{JA}{JB} = \dfrac{I'B'}{I'A'}$, or $JB \,.\, I'B' = JA \,.\, I'A'$.

Similarly, if K, K′ are corresponding points of the two rows, $JK \,.\, I'K' = JA \,.\, I'A'$, *i.e.* is a constant k.

This constant k is called the **power** of the rows.

Double points. If possible let E be a double point, *i.e.* a point such that $\{ABCE\} = \{A'B'C'E\}$, so that the point of the second row corresponding to E on the first row is the point E itself. Let O be the middle point of I′J.

Then $JE \,.\, I'E = JA \,.\, I'A = k$. But $JE \,.\, I'E = OE^2 - OJ^2$ (Euc. II. 5, 6).

$$\therefore\ OE^2 = k + OJ^2.$$

Hence, if $(k + \mathrm{OJ}^2)$ is positive, there are two real double points E, F equally distant from O;

if $k = -\mathrm{OJ}^2$, the two double points coincide at O;

if $k < -\mathrm{OJ}^2$, the double points are imaginary.

48. Problem. From the vanishing points J, I′, and a pair of corresponding points A, A′ to construct the point corresponding to any point K.

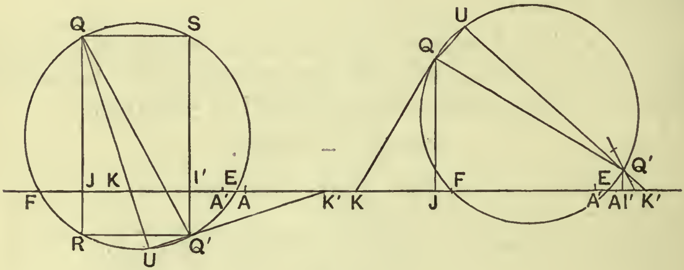


Fig. 39.

At J erect a perpendicular JQ equal to JA, and at I′ a perpendicular I′Q′ equal to I′A′, placing them on opposite sides of the line if JA, I′A′ is positive, but on the same side if negative. On QQ′ as diameter make a circle. Join QK cutting the circle in U, then UQ′ cuts the line at the required point K′.

For the angle QUQ′ being the angle in a semicircle is a right angle: hence UKI′ is the supplement of UQ′I′.

Hence $\mathrm{Q}\hat{\mathrm{K}}\mathrm{J} = \mathrm{I'Q'K'}$, and so the triangles QJK, K′I′Q′ are similar, viz. $\mathrm{JK} : \mathrm{JQ} = \mathrm{I'Q'} : \mathrm{I'K'}$,

and $$\therefore\ \mathrm{JK} \cdot \mathrm{I'K'} = \mathrm{JQ} \cdot \mathrm{I'Q'} = \mathrm{JA} \cdot \mathrm{I'A'}.$$

Further, if QJ, Q′I′ cut the circle again at R, S respectively, in the first case,

(1) when JK is negative, U lies in semicircle RQS, hence K′ lies to the left of I′, and I′K′ is negative;

(2) when JK is positive, U lies in semicircle $RQ'S$, hence $I'K'$ is positive.

So that in all cases $JK \cdot I'K'$ is positive.

A similar investigation will shew that, in case (2), $JK \cdot I'K'$ is always negative.

Corollary. The double points are the points at which the circle cuts the line.

Theorem. If two lines turn about a common point, so that they include a constant angle, *i.e.* so that the lines turn always through equal angles in the same direction, they trace out two equiangular pencils; hence they trace two projective rows on any straight line.

Also, if the two straight lines turn in opposite directions always through equal angles, they trace out two projective rows on any straight line.

49. Problem. To find, if possible, a row which is in perspective with both the A, B, C and A', B', C' rows.

If the new row is on a line cutting the line containing the rows at X, then X on the new row must correspond to itself regarded as a point of either the A, B, C or A', B', C' rows. [For the intersection of two rows in perspective corresponds to itself.]

Hence X must be at one of the double points E, F of the rows.

If, then, through E or F we draw any straight line, and project the A, B, C row on this line, so that A, B, C project into K, L, M,

then $\qquad \{ABCE\} = \{KLME\}$, but $\{ABCE\} = \{A'B'C'E\}$:

hence $\{A'B'C'E\} = \{KLME\}$, and therefore $A'K$, $B'L$, $C'M$ are collinear, and hence rows $K, L, M \ldots, A', B', C' \ldots$ are in perspective.

50. Two projective pencils with a common vertex may be drawn by taking two corresponding sets of three lines a, b, c and a', b', c'. There will be one pair of rays i, j at right angles in the first set, whose corresponding rays i', j' are also perpendicular; and if on i' and j we take unit distances SI′ and SJ and draw through I′ and J lines parallel to j' and i respectively to cut k' and k at K′ and K, then JK . I′K′ is constant.

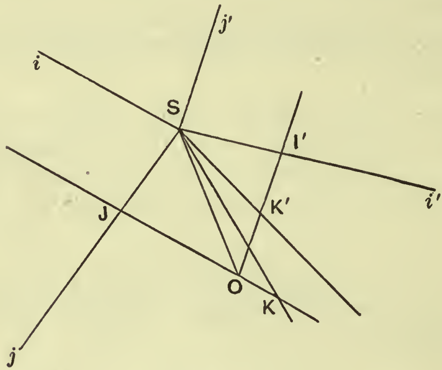


Fig. 40.

Trigonometrically. Tan JSK . tan I′SK′ = constant k.

Let $\quad JSO = OSI' = \alpha; \quad OSK = \theta; \quad OSK' = \phi;$

$$\therefore \tan(\alpha + \theta) \cdot \tan(\phi - \alpha) = k.$$

In order that SK, SK′ may coincide, *i.e.* $\theta = \phi$, we must have

$$\tan(\theta + \alpha) \tan(\theta - \alpha) = k,$$

$$\tan^2\theta - \tan^2\alpha = k(1 - \tan^2\theta \cdot \tan^2\alpha),$$

or

$$\tan^2\theta(1 + k\tan^2\alpha) = k + \tan^2\alpha;$$

$$\therefore \tan^2\theta = \frac{1}{\tan^2\alpha}\left(1 - \frac{1 - \tan^4\alpha}{1 + k\tan^2\alpha}\right);$$

this gives two real and different, coincident, or imaginary double rays, according as

$$1 - \tan^4\alpha \lesseqgtr 1 + k\tan^2\alpha,$$

i.e. as

$$k \gtreqless -\tan^2\alpha.$$

EXAMPLES. III.

1. A line KL parallel to AB cuts TA, TB at K, L, and P is taken on TA. By means of a common perspective row find Q on TB, such that

$$\{TBLQ\} = \{ATKP\}.$$

2. If K, L are the mid-points of TA, TB respectively, find a point Q on TB corresponding to any point P on AT, so that {TBLQ} may be equal to {ATKP}. Also prove that AP : PT as TQ : QB.

3. If K, K′ and L, L′ be corresponding pairs of points on two projective rows KL, K′L′, prove that the locus of the intersection of KL′ and K′L is a straight line; and find where this line cuts the two rows.

4. If k, l in a pencil correspond to k', l' respectively in another pencil projective with the former one, prove that the join of the intersections of k, l' and k', l passes through a fixed point.

Find the joins of this projective centre to the vertices of the pencils.

5. On two lines intersecting at T two projective rows are taken in which T, A of one correspond respectively to B, T of the other, and K on TA to K′ on BT. Construct the point L′ on TB corresponding to any point L on AT.

6. In the figure of the previous question prove that if

$$TK : KA = BK' : K'T,$$

then

$$TL : LA = BL' : L'T.$$

7. Through a given point draw a straight line which would, if produced, pass through the inaccessible intersection of two given lines.

8. Construct the vanishing points of two rows determined by three points A, B, C on AB and the corresponding points A′, B′, C′ on A′B′.

9. Two straight lines intersect at T, and T, A, K on one correspond to B, T, L on the other. Construct the vanishing points I, J of the two rows. Also prove that IJ is parallel to AB.

10. If the vertices of a triangle lie on three concurrent lines, and two sides pass through fixed points, the third side will always pass through another fixed point collinear with the two given points.

11. If A, B, C, D respectively lie on four given lines which are collinear, and AB, BC, CD respectively pass through three given points K, L, M, then the other joins AD, AC, BD will respectively pass through three other fixed points N, O, P. Also the six points K, L, M, N, O, P are the vertices of a complete quadrilateral.

12. Find the relations between two quadrilaterals in order that they may be in homology.

13. How many conditions must be satisfied in order that two complete pentagons may be in homology?

14. If one of two triangles in homology be turned about the axis of homology into another plane, prove that the joins of corresponding vertices will meet at a point coplanar with pairs of corresponding sides.

15. State and prove the correlative proposition.

16. Three triangles have their bases on one straight line and their vertices on another. The intersection of one side of one triangle with one side of another is joined to the intersection of the other side of the first triangle with the other side of the second triangle. Prove that the six lines thus obtained form a complete quadrangle.

17. A, B, C, D are four points on a straight line, and P any other point on the line; construct the point Q such that $\{ABCP\} = \{ABDQ\}$.

Find also the vanishing points and the double points of the two rows.

18. Two lines fixed at right angles to each other turn about the vertex of the right angle; prove that they trace projective rows on any given straight line. Also find the vanishing points and double points of the rows.

19. $\{ABCD\}$ and $\{ACDE\}$ are two harmonic ranges on a straight line; construct their vanishing points and the other double point.

20. The join of A to any point on BC is bisected at M, and points E, F are taken on it, such that ME equals MF. If BE, CM meet at X, and BM, CF at Y, prove that A, X, Y are collinear.

21. Two lines AC, BD meet at E, and a transversal cuts them at K, L respectively. AD, BC meet at X; AL, BK at Y; and CL, DK at Z. Prove that each of the pencils $X\{LDZC\}$ and $X\{LDYC\}$ equals $\{EAKC\}$; and hence that X, Y, Z are collinear.

22. Construct a quadrilateral with its vertices on four given straight lines, two sides parallel to given directions, and the other two sides passing through fixed points. Shew that there are two solutions.

CHAPTER IV

THE CIRCLE

51. Cross-ratios in the circle. If two pencils at the vertices S, S′ are such that corresponding rays intersect on a circle passing through S and S′, the angles between corresponding rays at S, S′ are equal, so the pencils are exactly equal, and hence projective.

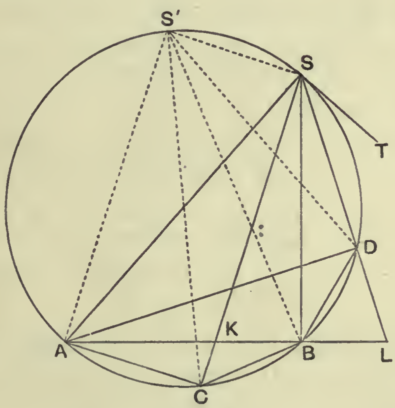


Fig. 41.

Conversely, if we take two points S, S′ and three other points A, B, C concyclic with S and S′, and construct at the vertices S, S′ pencils in which SA, SB, SC correspond to S′A, S′B, S′C, and take any other line at S cutting the circle at D, then S $\{ABCD\} = S'\{ABCD\}$; hence the ray at S' corresponding to SD is $S'D$, or all corresponding rays intersect on the circle.

Again, if ST is the tangent at S, angle $AST = AS'S$; hence ST is the ray at S corresponding to the ray $S'S$ at S'.

The cross-ratio of the pencil joining four points A, B, C, D on a circle to any vertex S lying on the circle may be called the cross-ratio of A, B, C, D and written $\{ABCD\}$.

52. Theorem. If A, B, C, D lie on a circle

$$\{ABCD\} = \frac{AC}{CB} : \frac{AD}{DB},$$

where AC, AD, etc. are chords.

For angles SAC, SBC are equal or supplementary, hence

$$\triangle ASC : \triangle BSC = \frac{AC \,.\, AS}{CB \,.\, SB};$$

but, if SC cuts AB at K, $\triangle ASC : \triangle BSC = AK : KB$;

$$\therefore \quad \frac{AK}{KB} = \frac{AC}{CB} \cdot \frac{AS}{SB},$$

similarly, if SD cuts AB at L,

$$\frac{AL}{LB} = \frac{AD}{DB} \cdot \frac{AS}{SB},$$

hence $$\frac{AK}{KB} : \frac{AL}{LB} = \frac{AC}{CB} : \frac{AD}{DB};$$

i.e. $$\{ABCD\} = \frac{AC}{CB} : \frac{AD}{DB}.$$

If $\{ABCD\}$ is positive, C, D both lie on the same arc AB, but if negative, C, D lie one in the smaller and one in the greater arc AB: thus the sign will be consistent in the above equation if we consider AC : CB to be positive or negative as C does or does not lie in the arc which runs from A to B in the positive (counter-clockwise) direction.

In figure 41, AC : CB is positive, AD : DB is negative. Also

AK : KB is positive, AL : LB is negative, but if S were taken in the positive arc AB, then K would lie in AB produced and L between A and B, but the double change would not alter the sign of {ABCD}.

Corollary. Since $\{ABCD\} = 1 - \{ACBD\}$ (§ 15, Theorem II, page 11),

$$\therefore \quad \frac{AC \, . \, DB}{CB \, . \, AD} = 1 - \frac{AB \, . \, DC}{BC \, . \, AD};$$

hence
$$AB \, . \, CD + BC \, . \, AD + CA \, . \, BD = 0,$$

$$\therefore \quad AB \, . \, CD = AC \, . \, BD + CB \, . \, AD.$$

Hence the rectangle of the diagonals of a cyclic quadrilateral is equal to the sum of the rectangles of pairs of opposite sides. [Ptolemy's Theorem.]

53. Theorem. Any chord which passes through a point P cuts the circle at points which are harmonically conjugate with respect to the points of contact of tangents from P.

Let the tangents from P be PA and PB, and let the chord of contact AB cut any secant CD, passing through P, at O.

Then the cross-ratio of the four points A, B, C, D on the circle equals the cross-ratio of the pencil

$$AP, AB, AC, AD = \{POCD\},$$

and also equals the cross-ratio of the pencil

$$BA, BP, BC, BD = \{OPCD\}.$$

Hence $\{OPCD\} = \{OPDC\}$;

$\therefore$ each $= -1$;

$\therefore$ $\{ABCD\} = -1$.

Corollary 1. Any chord through P is divided harmonically by P and the chord of contact of tangents from P.

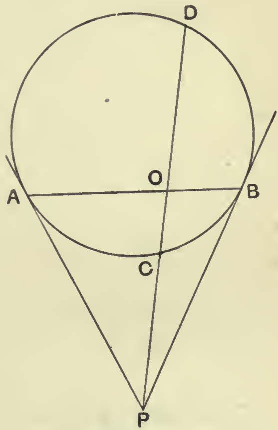


Fig. 42.

Corollary 2. The tangents at C, D intersect on AB since

$$\{CDAB\} = -1;$$

thus AB is the locus of intersection of tangents at the extremities of any chord through P.

Corollary 3. The tangent at any point C of the circle cuts AB at the harmonic conjugate of the point at which PC cuts AB.

54. Theorem. If through any point O on a chord AB passes another chord CD, the line joining the intersections E, F of AD, BC and AC, BD passes through the intersection T of tangents at A, B and the harmonic conjugate P of O with respect to A, B.

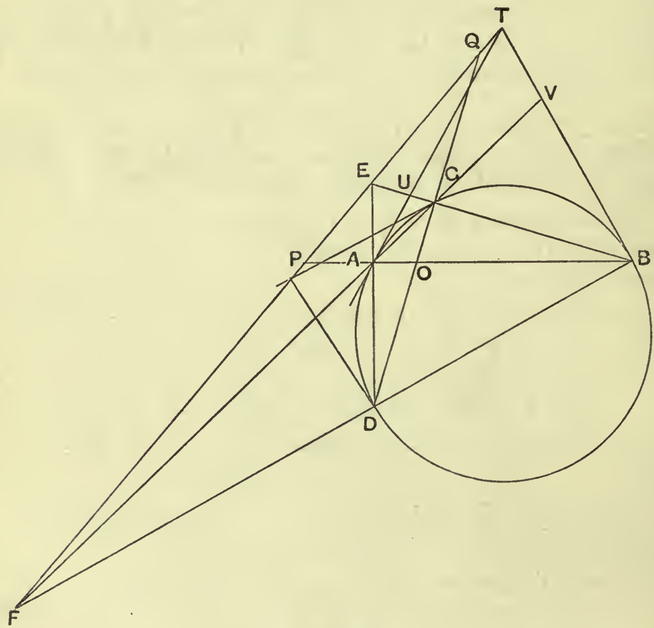


Fig. 43.

For the cross-ratio of the points A, B, C, D on the circle is equal to that of the pencil AT, AB, AC, AD and also of BA, BT, BC, BD.

Take transversals on BC, AC respectively, then

$$\{UBCE\} = \{AVCF\},$$

two projective rows with their intersection corresponding to itself, and hence in perspective, ∴ UA, VB, EF are collinear, *i.e.* EF passes through T.

Again, AB, CD, EF are the diagonals of the four-side AC, BD, BC, AD ; ∴ CD, EF divide AB harmonically at O, P.

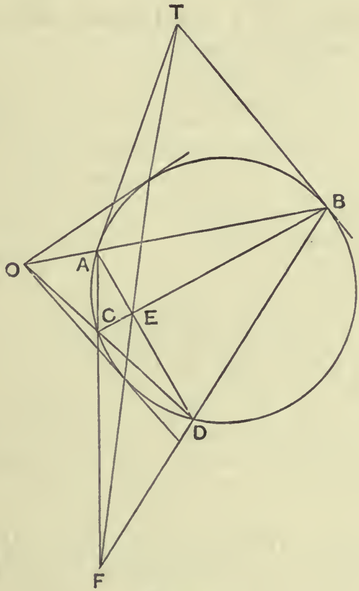


Fig. 44.

Corollary. On the same line lie the intersection of tangents at C, D and also the harmonic conjugate of O with respect to C, D.

Hence on one and the same straight line lie (1) the harmonic conjugate of O with respect to the extremities of any chord through O, (2) the intersection of tangents at the extremities of any chord through O and (3) the intersection of the joins of the extremities of any two chords through O.

This line is called the polar of O.

When O lies outside the circle the polar is the chord of contact of tangents from O. Cf. corollaries 1 and 2 of the previous proposition.

55. Theorem. If B is on the polar of A, then A is on the polar of B. Through B draw any chord KL, let AK, AL cut the curve again at M, N respectively.

Then KL, MN intersect on the polar of A and ∴ MN also passes through B; hence MK, NL intersect on the polar of B, *i.e.* A lies on the polar of B.

If the polar of A is BC, and of B is AC, then the polar of C is AB, and ABC forms a self-polar triangle. It may be shewn that in every case one vertex will lie within the circle and two outside.

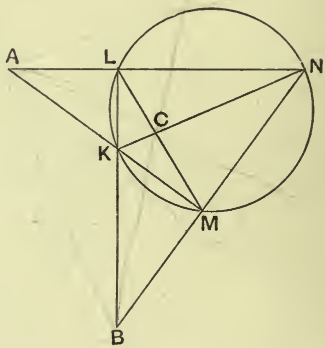


Fig. 45.

56. Problem. If the triangle ABC is self-polar to a circle of which K is a given point, construct the circle.

Join BK cutting AC at P, and find the harmonic conjugate L of K with respect to B and P; also join CK cutting AB in Q and find the harmonic conjugate N of K with respect to C and Q; then L, N also lie on the circle, and the three points K, L, N being known the circle can be constructed.

Exercise. Prove that BN and CL meet on the circle.

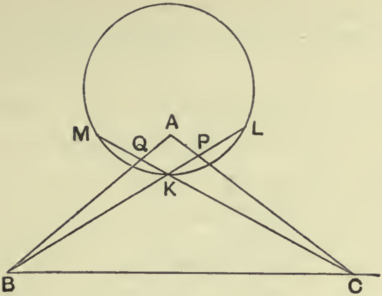


Fig. 46.

57. Theorem. The polar of any point P is a line perpendicular to the join OP of the point to the centre O, and cuts OP at a point Q such that rectangle OP . OQ equals the square on the radius.

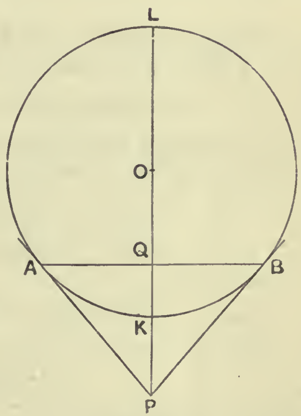


Fig. 47.

I. Take P outside the circle: its polar is then the chord of contact AB.

Now $PA = PB$, hence a line PQ perpendicular to AB bisects AB: but a line which bisects AB at right angles passes through the centre O.

Again, if PQ cuts the circle at K, L, we have P, Q and K, L forming a harmonic range, hence

$$OP \cdot OQ = OK^2.$$

II. If P is within the circle, its polar is without the circle.

Take any point R on the polar and draw its polar passing through P and cutting the polar of P at S, then PR is also the polar of S. Also, by I., RO is perpendicular to PS, and SO to

PR, hence PO is perpendicular to RS, and O is the orthocentre of triangle PRS. The second part follows as before.

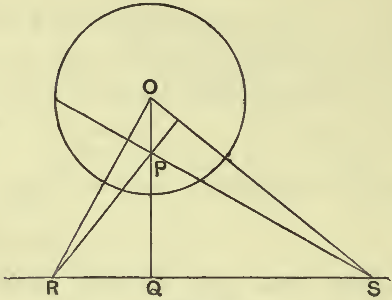


Fig. 48.

58. Theorem. Any three lines drawn through a point O cut a circle in six points in involution.

Let OAA′, OBB′, OCC′ be the lines.

CA′, C′A; CA, C′A′; CB′, C′B intersect at points K, L, M on the polar of O: let CC′ cut that polar at N.

Then $C\{A'AB'C'\} = \{KLMN\} = C'\{AA'BC\}$.

Hence $\{AA'BC\} = \{A'AB'C'\}$, *i.e.* AA′, BB′, CC′ form an involution on the circle.

Conversely, if $\{AA'BC\} = \{A'AB'C'\}$, then AA′, BB′, CC′ are collinear. For if AA′ meets BB′ at O, and OC cuts the circle at C″, then

$$\{AA'BC\} = \{A'AB'C''\},$$

hence C″ is identical with C′.

The point O is called the pole of the involution AA′, BB′, CC′ on the circle.

When O is outside the circle this proposition follows at once from the fact that A, A′ are harmonically conjugate on the circle to the points of contact D, E of tangents from O: and D, E are the double points of the involution.

Corollary. The common points of two involutions on a circle are the real or imaginary points at which the line joining their poles O, O′ cuts the circle.

59. Problem. Given two pairs of points on a line and one other point, complete the involution.

Let AA′, BB′ and C be the given points. Join them to any point P on a circle, let PA, PA′, PB, PB′, PC cut the circle at K, K′, L, L′, M respectively. Join KK′, LL′ meeting at R, and let RM cut the circle again at M′. Then PM′ cuts the line at the required point C′. To find the centre of the involution draw PX parallel to the line to cut the circle at X, and let RX cut the circle at X′, then PX′ cuts the line at the centre of the involution.

The double points D, E are such that PD, PE cut the circle at the points of contact of tangents from R.

If the circle passes through A, A′, then K, K′ coincide with A, A′ and R lies on the line.

If B′S touches the circle, then LL′ coincides with SB.

Hence we obtain the following **simpler solution.**

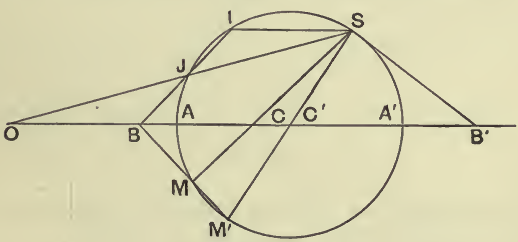


Fig. 49.

Choose the pair of points B, B′ so that one, at least, lies outside the segment AA′, viz. B′.

Describe a circle through AA′, and draw a tangent B′S to the circle.

Let SC cut the circle at M : join BM to cut the circle at M′.

Then SM′ cuts the line at C′ the point conjugate to C.

To find the centre draw SI parallel to AA′, let IB cut the circle again at J, then SJ cuts the line at the centre O of the involution.

To find the double points draw tangents BP, BQ to the circle, then SP, SQ cut the line at the double points D, E.

If the segments AA′, BB′ overlap, B is within the circle and there are no real double points; in any other case the double points are real.

60. Theorem. The cross-ratio of the four points in which four tangents cut any variable fifth tangent is equal to the cross-ratio of the points of contact on the circle.

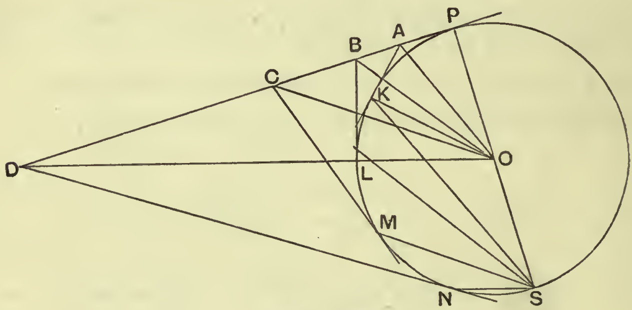


Fig. 50.

Let the tangents at K, L, M, N cut the tangent at P in the four points A, B, C, D, then shall $\{ABCD\} = \{KLMN\}$.

Join P to the centre O and produce to cut the circle at S.

Join KS, LS, MS, NS and KO.

$\because$ AK, AP are two tangents to the circle,

$\therefore$ $A\hat{O}P = \frac{1}{2}K\hat{O}P = K\hat{S}P$ (angle at the circumference).

$\therefore$ AO is parallel to KS.

Similarly BO, CO, DO are respectively parallel to LS, MS, NS.

Hence pencil O {ABCD} = pencil S {KLMN}.

∴ {ABCD} on AD = {KLMN} on the circle.

Corollary. Any four tangents to a circle cut two other tangents in two sets of four points which are projective: in other words, a variable tangent describes projective rows on two fixed tangents.

N.B. But it is not always true that the joins of corresponding points on two projective rows are all tangents to one circle; we shall investigate this envelope in the next chapter.

61. To find two projective rows such that the joins of corresponding points all touch one circle.

Take TI, TJ of equal length, and from C the middle point of IJ draw CA, CB perpendicular to TI, TJ respectively.

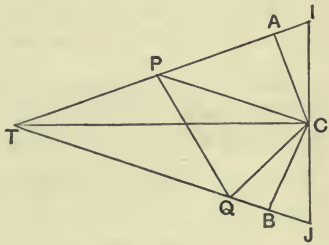


Fig. 51.

Construct two rows with vanishing points I, J, having A, T corresponding to T, B. Let P, Q be corresponding points, such that

$$IP \,.\, JQ = IA \,.\, JT \; (= IT \,.\, JB).$$

Then $IP \,.\, JQ = IC^2$ and

$$\therefore \quad IP : IC = JC : JQ;$$

also $\quad C\hat{I}P = Q\hat{J}C;$

hence triangles PIC, CJQ are similar,

$$\therefore \quad I\hat{P}C = J\hat{C}Q \quad \text{and} \quad \therefore \quad P\hat{C}Q = P\hat{I}C$$

(three angles equal to a straight angle); also from the similar triangles we have

$$PC : CQ = PI : CJ = PI : CI;$$

hence triangles PCQ, PIC are similar, and $I\hat{P}C = C\hat{P}Q$.

∴ PQ touches the circle whose centre is C, radius CA. Q.E.D.

N.B. There are two conditions to be satisfied,

(1) $TI = TJ$ or $TA = TB$.

(2) $IA \cdot IT = \frac{1}{4} \cdot IJ^2$ or in other words $\frac{IA}{AT} = \tan^2(\frac{1}{2}\hat{T})$.

62. Propositions deduced from the tangent cross-ratio property of a circle.

The propositions here enunciated are correlative with those deduced from the projective relation between two pencils in the circle.

If any line p cuts a circle at A, B the tangents from any point on p are harmonically conjugate with the tangents at A and B; *i.e.* the four tangents cut any other tangent in a harmonic range.

The chord of contact of tangents from any point on AB passes through the intersection O of the tangents at A, B.

The point of contact of any tangent c to the circle is joined to O by a line which is the harmonic conjugate with respect to OA, OB of the join of O to the intersection of c with AB.

If any line u be drawn through the intersection (O) of tangents a, b (at A, B), and c, d are the tangents from any point P on u (touching the circle at C, D), and e, f the joins of the intersections (K, L) of ad, bc and (M, N) of ac, bd respectively cut one another at U, then U lies on the chord of contact AB (since $\{OMAK\} = \{BLON\} = \{ONBL\}$ and $\therefore$ MN, KL, AB are collinear): also U is on the line v which is harmonically conjugate to u with respect to a and b (by the theory of four-sides).

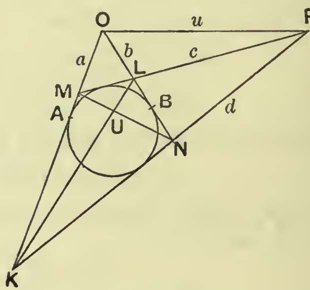


Fig. 52.

Corollary. Through the same point passes CD, and also the harmonic conjugate of u with respect to c, d.

Hence through U passes the harmonic conjugate of u with respect to tangents from any point on it; the chord of contact of tangents from any point on u; and the join of the intersections of any two pairs of tangents from points on u. U is called the pole of u.

If u cuts the circle, then U is the intersection of tangents at the points where u cuts the circle.

In all cases u is the polar (as previously defined) of U.

If a passes through the pole of b, then b passes through the pole of a.

If the pole of a lying on b and of b lying on a be joined by c, then the pole of c is the intersection of a, b and a, b, c form a self-polar triangle.

63. The tangents from points lying on any line form an involution on any other tangent. For if a, a'; b, b'; c, c' are three pairs of tangents, let K, L, M be the points where a, a', b' cut c, and K′, L′, M′ where a', a, b cut c'; then KK′, LL′, MM′ pass through O the pole of the line. Join O to N the intersection of c, c'.

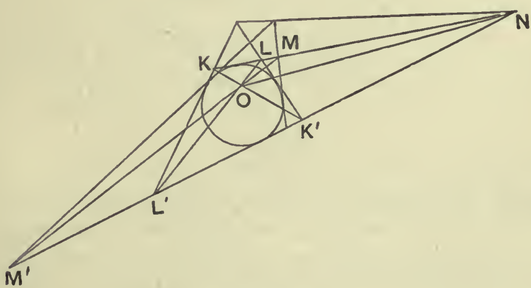


Fig. 53.

Then by projection from O we get $\{KLMN\} = \{K'L'M'N\}$.

$\therefore$ the row traced on c by $aa'b'c'$ = row traced on c' by $a'abc$.

$\therefore$ on any tangent, $\{aa'b'c'\} = \{a'abc\}$, *i.e.* aa', bb', cc' form an involution.

When o cuts the conic at D, E this proposition is otherwise deducible from the fact that aa', bb', cc' are pairs of harmonic conjugates to the tangents d, e at the points D, E : also d, e are the double tangents of the involution.

Conversely. If an involution of tangents be drawn to a circle, pairs of conjugate tangents intersect in collinear points.

64. Problem. Given two pairs of conjugate rays at a point and a fifth ray, to complete the involution.

Given five rays a, a', b, b', c at a vertex S, to find a sixth ray through S, such that aa', bb', cc' are in involution.

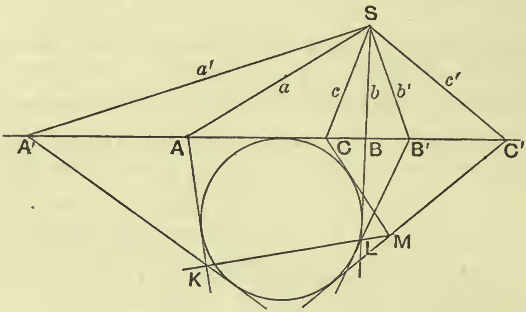


Fig. 54.

Let a, a', b, b', c cut any tangent to any circle at A, A′ ; B, B′; C ; draw the second tangents from A, A′ to the circle intersecting at K ; and from B, B′ intersecting at L ; let the second tangent from C cut KL at M, then the other tangent from M will cut AA′ at a point C′ whose join to the vertex of the pencil is the required sixth ray C′.

The conjugate point to that at which the circle touches the line is the point where KL cuts the line. Hence the construction may be simplified by taking a circle to touch the line at C.

Also if the circle touches a, a', then K becomes the vertex S of the pencil.

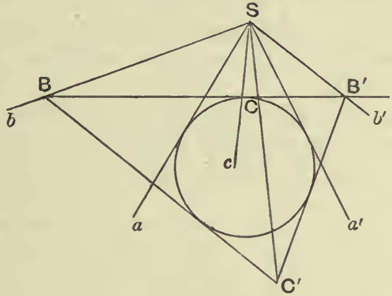


Fig. 55.

Hence draw a circle to touch a, a', let c cut the circle at C and draw tangent at C cutting b, b' at B, B′.

Draw tangents BC′, B′C′ from B, B′ intersecting at C′. Then SC′ is the required sixth ray.

EXAMPLES. IV.

1. Shew that the length of a chord of circle of diameter d, which subtends an angle α at the circumference, is $d \sin \alpha$; and hence prove that the cross-ratio of the pencil formed by joining four points A, B, C, D on the circle to any other point on the circle is

$$\frac{AC}{CB} \bigg/ \frac{AD}{DB}.$$

2. The sum of the rectangles of pairs of opposite sides of a cyclic quadrilateral is equal to the rectangle of the diagonals.

3. If the chord CD passes through the intersection of the tangents at A and B, the rectangles of pairs of opposite sides of the quadrilateral ACBD are equal.

4. C is the centre of a circle and N the mid-point of a chord PQ; prove that the tangents at P, Q meet on CN at a point T such that CN . CT equals the square of the radius CA. Prove also that the tangents at the ends of any other chord passing through N meet on a line drawn through T perpendicular to CT.

5. Find the condition that a circle can be drawn to pass through two given points and have a given pole and polar.

6. Prove that the orthocentre of a triangle self-polar to a circle is the centre of the circle: and that every self-polar triangle is obtuse angled.

7. The inscribed circle of a triangle touches the sides BC, CA, AB at D, E, F respectively; and the tangent at any point P of the circle cuts them at K, L, M respectively. Prove that $\{KLMP\}$ on PK equals $\{DEFP\}$ on the circle.

8. The incircle of a triangle ABC touches BC, CA, AB at D, E, F and EF cuts BC at K; prove that D, K are harmonic conjugates with respect to A and B.

9. Four lines form a harmonic pencil; prove that their poles with respect to a given circle are collinear and form a harmonic range.

10. PK is the perpendicular from P to the polar of Q, and QL is the perpendicular from Q to the polar of P for the same circle; prove that

$$PK : QL \text{ as } OP : OQ.$$

11. Any tangent is drawn to a circle whose centre is C and radius CA, P is its pole with respect to a circle whose centre is O, and PM the perpendicular from P to the polar of C with respect to the same circle. Prove that OP : PM has the constant value OC : CA.

12. The incentre of a triangle ABC is I, and any tangent meets lines through I perpendicular to IA, IB, IC at K, L, M respectively. Find the poles of AK, BL, CM, and prove that these lines are concurrent.

13. Find the centre and radius of the circle to which a triangle ABC is self-polar.

Prove that the radical axis of this circle and the nine-point circle of ABC cuts BC, CA, AB at the points where they are met respectively by the sides EF, FD, DE of the pedal triangle.

14. Pairs of tangents are drawn to a circle from three collinear points; prove that their points of contact form an involution on the circle.

15. If A, B, C, A′, B′, C′ lie on a circle, prove that the pencils A {A′B′C′C} and B {A′B′C′C} cut CA′, CB′ in rows in perspective. Hence shew that the three intersections of AB′ and A′B, AC′ and A′C, BC′ and B′C are collinear.

16. If A, B, C, D are four points on a circle and A′, B′, C′, D′ four others on the same circle such that {ABCD} = {A′B′C′D′}, prove that the intersections of AB′ and A′B, AC′ and A′C, AD′ and A′D are collinear.

17. Two pencils with a common vertex are projective, and a fixed circle passing through the vertex cuts two rays of one pencil at K, L and the corresponding rays of the other pencil at K′, L′. Prove that the locus of the intersection of KL′ and K′L is a straight line.

Use this projective axis to find the common rays of the two pencils.

18. Given three points of a row on a straight line and the three corresponding points of a projective row on the same line, construct the common points of the rows.

19. Circles are drawn each bisecting the circumferences of two given circles. Prove that the polars with respect to them of any given point pass through another fixed point.

20. A quadrilateral KLMN is inscribed in one circle, and its sides touch another circle at P, Q, R, S. Prove that PR and QS are perpendicular. Prove that KM, LN intersect at the same point as PR, QS.

21. Two circles are such that a quadrilateral can be inscribed in one and circumscribed to the other. If P, Q, R, S are the points of contact of the four sides KL, LM, MN, NK, prove that the four intersections of PQ, RS; PS, QR; KL, MN; KN, LM lie on a fixed straight line.

22. From any point K on the circumcircle of a triangle tangents are drawn to the incircle and they meet the circumcircle again at L and M. Shew that A {BCKL} and M {BCKL} describe projective rows on KL and BC respectively, and deduce that LM touches the incircle.

23. A line is drawn to cut two non-intersecting circles; find two points on this line such that each is the intersection of the two polars of the other with respect to the two circles.

CHAPTER V

THE CONIC

65. Definition. A conic is the locus of intersections of corresponding rays of two projective pencils not in perspective.

If we take two vertices S, T and three other points A, B, C and draw SA, SB, SC from S and TA, TB, TC from T, these will determine two projective pencils at S and T, and we may construct successive positions of the point P, such that the cross-ratio of SA, SB, SC, SP equals the cross-ratio of TA, TB, TC, TP.

If A, B, C lay on a straight line, the rows would be in perspective, and P would lie on that line or on ST; if one of the points, say A, were collinear with S and T, the pencils would have a common ray, and would be in perspective with BC as base, and P would lie on BC, or on ST. Hence, if any three of the five points are collinear, the pencils are in perspective, and we get, not a conic, but two straight lines.

When the angles ASB, ASC respectively equal the angles ATB, ATC, the locus of P becomes a circle through S, T, A, B, C.

66. To construct a conic having given two vertices S, T and three points A, B, C, no three of the five points being collinear.

Let SB cut AC at K, and TC cut AB at L. Through the intersection O of SB, TC draw any straight line to cut AC, AB at X, Y respectively, and join SX, TY intersecting at P, then P is a point of the locus.

For the pencil

$$S\{ABCP\}=\{AKCX\} \text{ and pencil } T\{ABCP\}=\{ABLY\};$$

but, by projection from O, we have

$$\{AKCX\}=\{ABLY\}.$$

Hence $S\{ABCP\}=T\{ABCP\}$. By varying the position of the line XY passing through O, we can get the successive positions of P, the intersection of XS, TY.

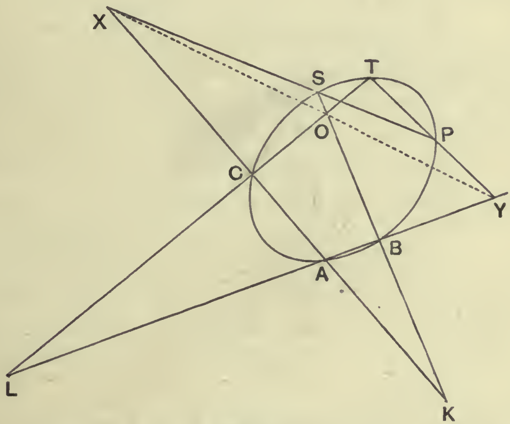


Fig. 56.

67. We shall prove presently (§ 86) that the lines which join corresponding points of two projective rows not in perspective all touch a conic, and that every tangent to the conic cuts the two rows in corresponding points; in other words the conic is the **envelope** of joins of corresponding points of two projective rows not in perspective. The converse proposition is also proved in this chapter so that the two classes of curves are identical, and either definition might have been taken.

In § 217 *et seq.* it is proved that the conic so defined is equivalent to the curve got by taking a plane section of a circular cone and conversely. In the following proposition we now prove that this property might be deduced from the focus definition of a conic, and the converse is proved in § 177.

68. Theorem. If a point P moves so that its distance from a fixed point S (called the focus) bears a constant ratio to its distance from a fixed line (called the directrix), the pencils formed by joining any two points A, B of the locus to the variable point P will be projective.

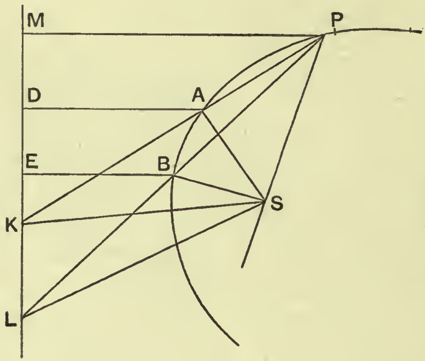


Fig. 57.

Let AD, BE be the perpendiculars from A, B to the directrix, so that SA : AD = SB : BE = e, and PM the perpendicular from P, so that SP : PM = e.

Join AP, BP cutting the directrix at K, L respectively: join SK, SL.

Then SP : SA = PM : AD = PK : AK;

hence SK bisects an angle between SA, SP.

Similarly SL bisects one of the angles at S between SB and SP. (In figure 57 the angles are exterior angles.)

Hence as SP turns round S through any angle, each of the lines SK, SL turns through half the angle at S.

∴ SK, SL describe equiangular pencils at S, and hence the rows described by K, L on the directrix are projective.

But AP cuts the directrix at K, BP cuts the directrix at L.

∴ pencil described by AP at A is projective with the pencil of BP at B.

69. Theorem. The conic described with vertices S_1, S_2 to pass through three points A, B, C also passes through the two vertices: and the tangent at a vertex S_1 is the ray at S_1 corresponding to the ray S_2S_1 at S_2, and *vice versa.*

For if S_1T at S_1 corresponds to S_2S_1 at S_2, and S_1P, S_2P are corresponding rays nearly coincident with S_1T, S_2S_1 respectively, the tangent at S_1 is the ultimate position of the chord S_1P when P is brought to coincide with S_1, but when P coincides with S_1, S_2P coincides with S_2S_1, and hence S_1P coincides with S_1T.

Corollary. Three points and the tangents at two of them completely determine a conic. For the three rays S_1T, S_1S_2, S_1A at S_1 are given corresponding to the three rays S_2S_1, S_2T, S_2A at S_2; hence to any other ray at S_1 we can find the corresponding ray at S_2.

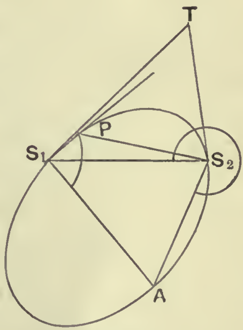


Fig. 58.

70. We shall now prove that we get the same conic from five given points if we take any pair of the points as vertices.

Theorem. The pencil formed at any point of a conic by its joins to the various points on the conic is projective with the pencils formed by joining those points to the original vertices.

First Proof. Let A, B be the vertices; AT the tangent at A; K, C two other points, and P a variable point on the conic.

To prove that pencil of CP at C is projective with the pencil of AP at A. Join KP cutting AT at E, AB at F, and AC at G.

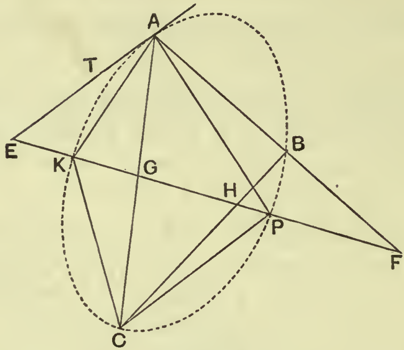


Fig. 59.

Then $$A\{KPTC\} = B\{KPAC\} \text{ (by definition)};$$

$$\therefore \{KPEG\} = \{KPFH\},$$

viz. $$\frac{KE}{EP} : \frac{KG}{GP} = \frac{KF}{FP} : \frac{KH}{HP};$$

$$\therefore \frac{KE}{EP} : \frac{KF}{FP} = \frac{KG}{GP} : \frac{KH}{HP},$$

viz. $$\{KPEF\} = \{KPGH\},$$

i.e. $$\text{pencil } A\{KPTB\} = \text{pencil } C\{KPAB\};$$

$$\therefore A\{TBKP\} = C\{ABKP\};$$

thus AP, CP are corresponding rays of projective pencils determined by AT, AB, AK at A and CA, CB, CK at C; and

$$A\{P_1P_2P_3P_4\} = C\{P_1P_2P_3P_4\} \text{ (§ 39, Cor.)}. \qquad \text{Q. E. D.}$$

Second Proof. Let the conic be defined by the points A, B, the tangents AT, BT and point C. Let P be a variable point of the conic.

Let AP cut BC at U; BP cut AC at V; CP cut AB at L.

$$\text{Pencil } A\{BTCP\} = B\{TACP\} \text{ by definition,}$$

$$\therefore\ A\{BTCP\} = B\{ATPC\}$$

two projective pencils with a common ray; hence T, U, V are collinear.

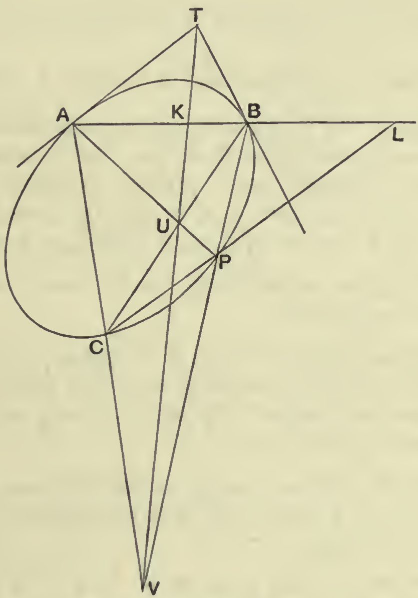


Fig. 60.

Let TUV cut AB at K, then by the theory of the complete four-side, K and L are harmonic conjugates with respect to AB.

Hence as P describes the conic, K and L describe projective rows on AB.

But the row of K on AB projected from the fixed point T on the fixed line BC gives the row of U on BC, which is projective with the pencil of AP at A: and the row of L on AP is projective with the pencil formed at C by CP.

Hence if P, Q, R, S are four points on the conic the cross-ratio of the pencil C {PQRS} = the cross-ratio of A {PQRS} and B {PQRS}; and CA, CB and the tangent at C correspond to AT, AB, AC at A, and to BA, BT, BC at B.

Similarly, if we join to any other point of the conic D, we shall get

$$D\{PQRS\} = C\{PQRS\}, \text{ and } D\{ABPQ\} = C\{ABPQ\}.$$

Corollary 1. If P is a point on a conic obtained from the vertices A, B and three points C, D, E, so that A {CDEP} = B {CDEP}, then C {DBEP} = A {DBEP}, and C {ABEP} = D {ABEP}; hence P also lies on the conic with vertices A, C passing through B, D, E and on the conic with vertices C, D passing through A, B, E.

Hence five given points determine the same conic, whichever pair are taken as vertices.

Corollary 2. The cross-ratio of the pencil formed by joining four points P, Q, R, S on a conic to any other point on the conic has a constant cross-ratio. We may call this value the cross-ratio of {PQRS} on the conic.

Corollary 3. If we describe a conic through five points P, Q, R, S, A and K is a point such that K {PQRS} = A {PQRS}, then K must lie on the conic.

If not let KP cut the conic at K′, join K′ to PQRS. Then K′ {PQRS} = A {PQRS}. Hence K′ {PQRS} = K {PQRS} and we have two projective pencils at K, K′ with a common (self-corresponding) ray KK′P, hence they are in perspective, and therefore Q, R, S are collinear, which is contrary to hypothesis: as no three of the five points which determine a conic may be collinear.

Hence if a point moves so that its joins to four given points, no three of which are collinear, form a pencil of constant cross-ratio, its locus is a conic.

71. Mechanical construction of a conic. Take four rods loosely jointed together at a point P, and another rod to which are attached rings at four points K, L, M, N through which the four rods can slide freely. To the surface on which the conic is to be drawn, fix four rings A, B, C, D and slide the rods PK, PL, PM, PN through A, B, C, D.

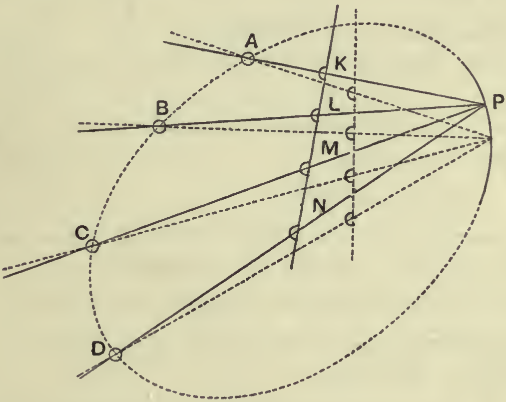


Fig. 61.

The point P will be constrained to describe the conic which passes through A, B, C, D and in which the cross-ratio of P {ABCD} is {KLMN}.

72. Theorem. A conic is a curve of the second order, *i.e.* any straight line cuts it in two points, real and different, or coincident or imaginary.

For any straight line cuts two projective pencils in two projective rows of points, and a double point of these rows is a point on the conic: but two projective rows on the same straight line have two real and different, or coincident, or imaginary double points.

73. Theorem. If the six points A, B, C, A′, B′, C′ lie on a conic the three intersections of AB′ and A′B, AC′ and A′C, BC′ and B′C will be collinear (Pascal).

Let these three intersections be K, L, M respectively: also let AB′ cut CA′ at X, and BA′ cut B′C at Y.

The cross-ratios of A {A′B′C′C} and B {A′B′C′C} are equal, by the definition of the conic.

Hence, taking transversals A′C and B′C respectively we have

$$\{A'XLC\} = \{YB'MC\}.$$

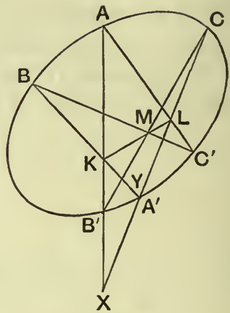


Fig. 62.

But these rows have a common point C, hence they are in perspective, *i.e.* A′Y, XB′, LM are concurrent.

But A′Y, XB′ meet at K, and therefore K lies on LM.

Construction. Given five points of a conic to construct other points on the conic.

Let A, B, C, A′, B′ be the five points, join AB′, A′B intersecting at K; and through K draw any line to cut CA′, CB′ at L and M respectively.

The intersection of AL and BM will be a point of the conic.

74. Taking any points A, B, C and A′, B′, C′ on the conic, we may find other pairs of points, such as D, D′ such that {ABCD} on the conic equals {A′B′C′D′} on the conic, thus forming two projective rows on the conic.

Then A {A′B′C′D′} = A′ {ABCD}, two pencils with a common ray;

∴ AB′, AC′, AD′ respectively meet A′B, A′C, A′D in three points on a straight line.

But by the Theorem of § 73, the intersections of BC′ and B′C, BD′ and B′D, CD′ and C′D lie on the straight line.

Hence, if K, L are any two points in the row A, B, C, ... and K′, L′ the corresponding points in the row A′, B′, C′, ... the intersection of KL′, K′L lies on a fixed straight line. It is called the **projective axis** of the two rows.

Corollary. If this straight line cuts the conic at two real points these are the double points of the two rows on the conic.

Chords.

75. Theorem. *If* AT, BT *be the tangents at* A, B *to a conic* (§ 69) *and* P, Q *two other points on the conic, the intersections* U *of* AP, BQ, *and* V *of* AQ, BP *will be collinear with* T. *Also* UV, PQ *divide* AB *harmonically.*

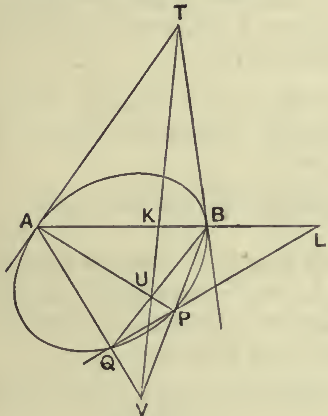


Fig. 63 (1).

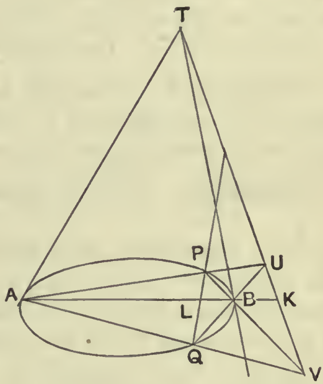


Fig. 63 (2).

For in the projective pencils at A, B, we have AB, AT at A corresponding to BT, BA at B,

$$\therefore\ A\{BTPQ\} = B\{TAPQ\},$$

$$\therefore\ A\{BTPQ\} = B\{ATQP\};$$

and these are two projective pencils with a common (self-corresponding) ray AB.

∴ they are in perspective, and hence T, U, V are collinear.

Again, by § 19, the diagonals UV, PQ of the four-side formed by AP, AQ, BP, BQ divide the third diagonal AB harmonically at K, L.

Corollary 1. TK also passes through the harmonic conjugate of L with respect to P, Q. Similarly, if any other chord be drawn through L, the harmonic conjugate of L with respect to the ends of the chord lies on TK.

Corollary 2. If a tangent passes through L, its point of contact lies on TK.

Corollary 3. If PQ, RS are two chords through L, and PR, QS meet at M and PS, QR at N, then MN passes through the harmonic conjugate of L, with respect to P, Q and with respect to R, S (§ 20, Cor. 1). Hence M, N lie on TK.

Corollary 4. When RS and PQ (Cor. 3) coincide, PR, QS become the tangents at P, Q. Hence the tangents at the ends of any chord through L intersect on TK.

Definition. TK (thus obtained) is called the **polar** of L.

76. Theorem. The middle points of parallel chords are collinear. In fig. 63, § 75, let PQ become parallel to AB. Then UV bisects AB and PQ. Hence the middle point of PQ lies on a line through T and the middle point M of AB.

Similarly TM contains the mid-point of any other chord parallel to AB.

Definition. A line which bisects all chords parallel to a given direction is called a **diameter.**

Corollary 1. The joins of the ends of two chords parallel to AB intersect on TM.

Corollary 2. The tangents at the ends of any chord parallel to AB intersect on TM.

Corollary 3. A tangent parallel to AB has its point of contact on TM.

Corollary 4. When the infinity directions are real and coincident all diameters are parallel.

TANGENTS.

77. Theorem. If a conic touches two lines AT, BT at A, B respectively, the tangent at any other point P is the harmonic conjugate of PT with respect to PA, PB. We may prove as in § 75, that UV passes through T, and UV, PQ divide AB harmonically.

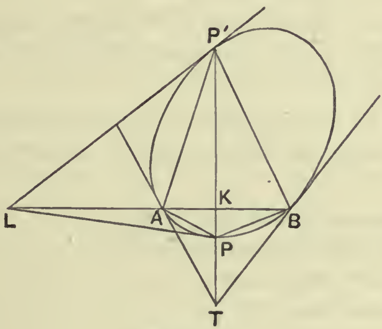


Fig. 64.

If in fig. 63 (1), the chord PQ be turned about P, until Q coincides with P, then PQ will become the tangent at P, and UV will become TP.

Hence the tangent PL (fig. 64), and the line PT cut AB harmonically at L, K.

(Also cf. Cor. 4 to the next Theorem.)

Theorem. Any chord PP′ which passes through the intersection T of tangents at A, B is divided harmonically by T and AB.

Let PP′ (fig. 64) cut AB at K.

Then $A\{PP'BT\} = B\{PP'TA\}$, $\therefore$ $\{PP'KT\} = \{PP'TK\}$,

$\therefore$ K, T are harmonic conjugates with respect to P, P′.

Corollary 1. If $TP = PK$, P′ is infinitely distant: and conversely.

Corollary 2. If $TP < PK$, P′ is on the other side of T from P.

Corollary 3. Since $A\{TBPP'\}$ is harmonic, the pencil joining any other point of the conic to A, B, P, P′ is harmonic, *i.e.* P, P′ are harmonic conjugates *on the conic* to A, B.

Corollary 4. If the tangent at P meets AB at L, then $P\{LP'AB\} = A\{PP'TB\}$ which is harmonic. Hence L is the harmonic conjugate of K with respect to A, B.

Corollary 5. Tangents at P, P′ intersect on AB.

Corollary 6. P and L are harmonic conjugates to the points at which PL cuts TA, TB.

Corollary 7. If the join of T to M, the mid-point of AB, cuts the conic at C, the tangent at C is parallel to AB (cf. § 76, Cor. 3).

78. Theorem. Any four tangents cut a fixed tangent in four points whose cross-ratio is equal to the cross-ratio on the conic of their four points of contact. Also any variable tangent cuts two fixed tangents in projective rows.

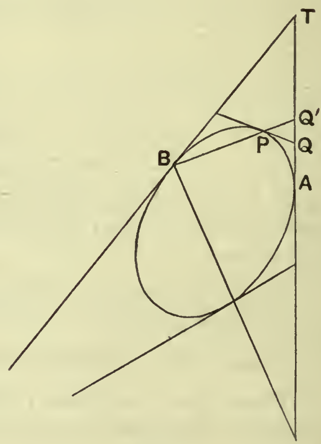


Fig. 65.

Consider the conic which touches AT, BT at A, B and passes through P.

Let the tangent at P cut TA at Q, and let BP cut TA at Q′.

Then PQ, PT are harmonic conjugates with respect to PA, PB (§ 77).

$$\therefore \ \{TQAQ'\} = -1,$$

$$\therefore \ \{ATQ'Q\} = -2,$$

hence
$$\frac{AQ'}{Q'T} = 2 \cdot \frac{AQ}{QT}.$$

If we take successive positions P_1, P_2, ... ; Q_1, Q_2, ... ; Q_1', Q_2', ... we have therefore

$$\frac{AQ_1}{Q_1T} : \frac{AQ_2}{Q_2T} = \frac{AQ_1'}{Q_1'T} : \frac{AQ_2'}{Q_2'T},$$

i.e.
$$\{ATQ_1Q_2\} = \{ATQ_1'Q_2'\},$$

hence
$$\{Q_1Q_2Q_3Q_4\} = \{Q_1'Q_2'Q_3'Q_4'\}.$$

$\therefore$ the cross-ratio of the row described by Q equals the cross-ratio of the pencil described by BP, which is the cross-ratio of the pencil joining P_1, P_2, P_3, P_4 to any point of the conic.

Similarly the cross-ratio of the row described by the tangent on BT has the same value. Hence the variable tangent describes projective rows on the two fixed tangents TA, TB.

Corollary 1. If any point be joined to the two projective rows traced by a variable tangent on two fixed tangents, two projective pencils with a common vertex will be obtained. The double rays of these two pencils will be those tangents to the conic which pass through the point.

Corollary 2. Two pencils at a point have two real and different, coincident, or imaginary double rays. Hence two tangents can, in general, be drawn from any point to touch a conic. Thus **a conic is a curve of the second class.**

79. If pencils be formed by joining two points A, B on the conic to other points of the conic, and if in these two pencils any ray AP is parallel to the corresponding ray BQ at B, then any transversal parallel to AP will be cut by the two pencils in similar rows. Hence one double point will be at infinity, and the line will cut the conic in one and only one finite point.

Hence, also, if we take the pencil at any other point of the conic the ray at that point corresponding to AP at A will be parallel to AP.

If through T, the intersection of tangents at A, B, we draw lines parallel to the rays of the two pencils at A, B, we get two projective pencils at the one vertex T, and the infinity directions become the double rays of the two pencils.

Hence there are two real and different, or coincident, or imaginary infinity directions in a conic.

80. The two pencils at T cut AB in two projective rows, of which A, B are the vanishing points.

1. If the power AU . BV is positive there are two real and different double points (§ 47). In this case (fig. 66 (1)) lines through A, B parallel to TU, TV respectively intersect on the other side of TA from B, or on the other side of TB from A, and the curve consists of two branches outside the angle ATB.

Conversely, if any point of the curve lies in the supplementary angle at T to ATB, then AU . BV is positive.

2. If AU . BV is negative, but $AM^2 + AU \cdot BV$ is positive there are real and different infinity directions (§ 47).

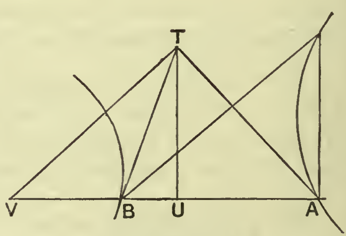


Fig. 66 (1).

Take AU, BV of equal length in directions BA, AB respectively (fig. 66 (2)).

Then AU is less than MA, and $MA : AU = MB : BV$.

∴ lines from A, B parallel to TU, TV meet at a point C on TM; and $TC < CM$.

In this case TM cuts the curve again at C′ the harmonic conjugate of C with respect to TM, but $TC < CM$, ∴ C′ is on the other side of T from M.

Hence the curve has two branches, one lying in the angle ATB, and the other in the vertically opposite angle at T.

In case 1, a tangent which cuts AT internally cuts BT externally, and *vice versa*, hence the vanishing points I, J of the rows on TA, TB lie between T, A and T, B respectively.

In case 2, let the tangent at C cut TA at P and TB at P′.

Then $\{ATPI\} = -TP'/P'B$; but $TP' < P'B$ and $AP > PT$;

$\therefore$ $IA : IT$ is positive and greater than 1, hence I is on the other side of T from A.

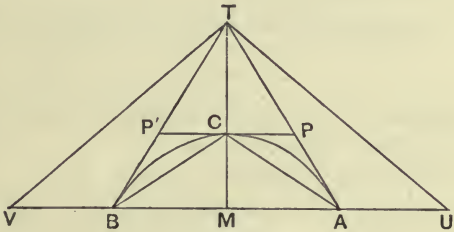


Fig. 66 (2).

3. If $AU \cdot BV = -AM^2$, the infinity directions are real and coincident, and are parallel to TM.

In this case $TC = CM$. Also the tangent at C bisects TA, TB, hence the rows traced on TA, TB by a variable tangent are similar.

4. If $AU \cdot BV + AM^2$ is negative, the infinity directions are imaginary.

In this case $TC > CM$.

Also $TP' > P'B$ and $AP < PT$, $\therefore$ $IA : IT$ is positive and less than 1, $\therefore$ A lies between I, T.

Definitions. When a variable tangent traces similar rows on two tangents the conic is a **parabola** (case 3).

When the vanishing points of the rows traced on two tangents

TA, TB by a variable tangent lie on the other side of AB from T, the conic is an **ellipse.**

When the vanishing points lie on the same side of AB as T, the conic is a **hyperbola.**

Corollary. In § 68 it is shewn that, in the locus there described, SK bisects the exterior angle between AS, SP.

Hence when SA, SP coincide with a line SL parallel to the directrix, the tangent at L cuts the directrix at X, the foot of the perpendicular from S to the directrix. By symmetry LS meets the curve again at L′, so that LS = SL′ and the tangent at L′ passes through X.

The locus is therefore an ellipse, parabola or hyperbola as the constant ratio $\lesseqqgtr 1$.

81. Theorem. Lines drawn through four points of a conic parallel to an infinity direction have the same cross-ratio on any transversal as the joins of any point on the conic to those four points.

Draw CK, DL parallel to the infinity direction to cut AB at K, L. Let CD cut AB at E, and the tangent at C cut AB at T.

Then CA, CB, CT, CK at C correspond to DA, DB, DC, DL at T; hence, taking transversals on AB, we have

$$\{ABTK\} = \{ABEL\},$$
$$\therefore\ \{ABTE\} = \{ABKL\};$$

but {ABTE} is the cross-ratio of C {ABTD}, and therefore equals the cross-ratio of the joins of A, B, C, D to any point of the conic; and {ABKL} equals the cross-ratio of the lines through A, B, C, D parallel to the infinity direction on AB or any other transversal, being unaltered by parallel projection.

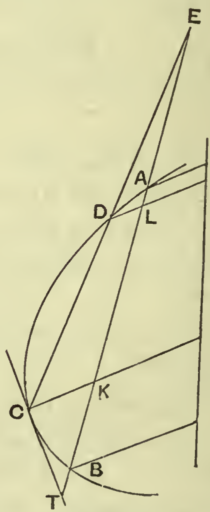


Fig. 67.

Exercise. If we have a pencil at a point which is projective with a row on a line, the locus of intersection of rays of the pencil with lines drawn parallel to any given direction through the corresponding points on the row is a conic with real infinity directions.

Envelope of the Joins of Projective Rows.

82. Theorem. If projective rows, not in perspective, be taken on two given lines the envelope of the joins of corresponding points touches each of the given lines, and the point of contact with one line corresponds to the intersection of the two lines regarded as a point of the other line.

Let the intersection of the lines be T, and let T on TB correspond to A on TA. Let A′ be a point near A on TA, and T′ the corresponding point on TB.

Then as A′ approaches A, T′ approaches T, and ultimately the tangent T′A′ assumes the position TA; also A′ is the intersection of TA with a near tangent, and when those two tangents are made to coincide A′ assumes the position A.

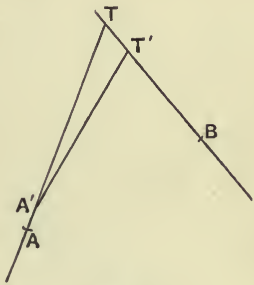


Fig. 68.

Hence A is the point of contact of the tangent TA.

Theorem. From any point two real or coincident or imaginary tangents can be drawn.

83. Theorem. If we join corresponding points on two projective rows, the row traced on any one of these joins by the others will be projective with the rows they trace on the original lines.

Let TA, TB be the rows of which A corresponds to T, C to D, K to L, P to Q.

Let KL, PQ cut CD at M and R respectively; and cut each other at O.

Then $$\{KPAC\} = \{LQTD\},$$

$$\therefore \text{ pencil } O\{KPAC\} = O\{LQTD\},$$

hence $$O\{KPAT\} = O\{LQCD\},$$

hence $$\{KPAT\} = \{MRCD\},$$

two rows on CA and CD; if now we keep other lines fixed but vary the position of PR, it follows that if we get successively the points $P_1P_2P_3P_4$ and $R_1R_2R_3R_4$, then $\{P_1P_2P_3P_4\} = \{R_1R_2R_3R_4\}$. Q. E. D.

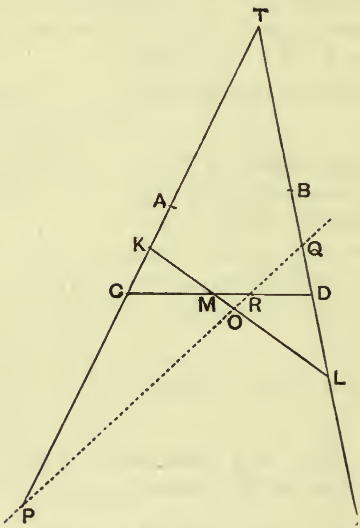


Fig. 69.

Hence the joins of four pairs of corresponding points on two projective rows cut any other join in four points of constant cross-ratio.

Hence, also, five given lines give the same envelope whichever pair we take as the basis of projective rows.

84. Problem. To find where any join of corresponding points of two projective rows touches its envelope.

On two lines TP, TQ take projective rows, in which T, P correspond respectively to Q, T: and let A, B correspond to A′, B′, so that

$$\{PTAB\} = \{TQA'B'\},$$

$$\therefore \ \{TPAB\} = \{TQB'A'\},$$

two equal cross-ratios with a common point T, hence PQ, AB′, A′B are collinear, *i.e.* AB′, A′B intersect at a point L on PQ.

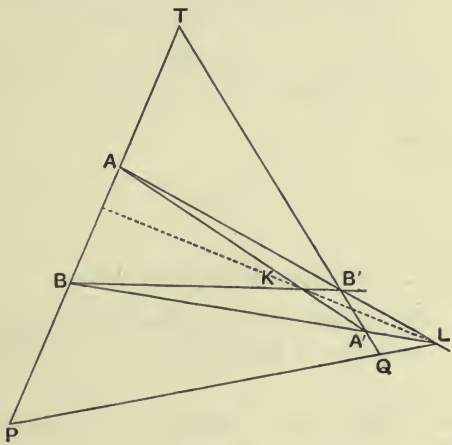


Fig. 70.

Let AA′, BB′ intersect at K, then from the harmonic property of a four-side KL is divided harmonically by TP, TQ.

If we now bring B into coincidence with A (and B′ with A′), L will become the point at which AA′ meets PQ, and K being the intersection of AA′ with a consecutive tangent will become the point of contact of AA′.

Hence the point of contact K of AA′ with its envelope is the harmonic conjugate of the point L at which it is cut by the line PQ.

Corollary. If AA′ is parallel to PQ it is bisected at its point of contact.

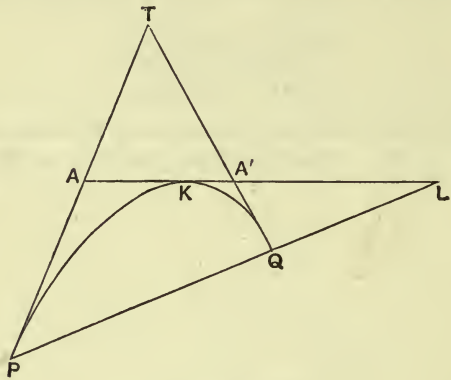


Fig. 71.

ANOTHER METHOD OF PROOF.

85. A variable join of two projective rows describes on any fixed join a row which is projective with either of the original rows.

Let CD be a given join of rows on TA, TB and PQ a variable join such that

$$\{TACP\} = \{BTDQ\},$$

and let CD cut PQ at R.

Let CD cut AB at F, and take on CD a point E such that

$$\{CDEF\} = -1.$$

A transversal cuts the sides of triangle TCD at P, R, Q,

$$\therefore \ \frac{TP}{PC} \cdot \frac{CR}{RD} \cdot \frac{DQ}{QT} = -1, \qquad \therefore \ \frac{CR}{RD} = \frac{-PC}{TP} \cdot \frac{QT}{DQ}.$$

Similarly $\frac{\mathsf{TA}}{\mathsf{AC}}\cdot\frac{\mathsf{CF}}{\mathsf{FD}}\cdot\frac{\mathsf{DB}}{\mathsf{BT}}=-1,\qquad \therefore\ \frac{\mathsf{CE}}{\mathsf{ED}}=\frac{\mathsf{AC}}{\mathsf{TA}}\cdot\frac{\mathsf{BT}}{\mathsf{DB}},$

$$\therefore\ \{\mathsf{CDER}\}=\frac{\mathsf{AC}}{\mathsf{TA}}\cdot\frac{\mathsf{BT}}{\mathsf{DB}}\Big/\frac{\mathsf{CP}}{\mathsf{TP}}\cdot\frac{\mathsf{QT}}{\mathsf{PC}}.$$

But
$$\frac{\mathsf{TB}}{\mathsf{BD}}\Big/\frac{\mathsf{TQ}}{\mathsf{QD}}=\frac{\mathsf{AT}}{\mathsf{TC}}\Big/\frac{\mathsf{AP}}{\mathsf{PC}},$$

$$\therefore\ \{\mathsf{CDER}\}=\frac{\mathsf{AC}}{\mathsf{CT}}\Big/\frac{\mathsf{AP}}{\mathsf{PT}}=\{\mathsf{ATCP}\}.$$

Hence P, R describe projective rows on CA, EC, in which C, A correspond to E, C and T to D. Q. E. D.

Corollary. The point of contact of CD is E, the harmonic conjugate with respect to C, D of the point F at which CD cuts AB.

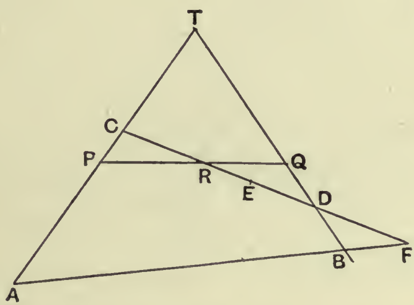


Fig. 72.

86. Theorem. *The envelope of the joins of two projective rows, not in perspective, is a conic.*

Let TA, TB be the bases of the rows, AB being their point of contact. Let PQ cut AB at K and touch the envelope at C; RS cut AB at L and touch the envelope at D: also let AC, AD cut BT at U, V respectively.

Then $\{PQCK\} = -1$,

$\therefore$ by projection from A, $\{TQUB\} = -1$.

$$\therefore \frac{TQ}{QB} = 2 \cdot \frac{TU}{UB}, \text{ similarly } \frac{TS}{SB} = 2 \cdot \frac{TV}{VB},$$

$$\therefore \{TBQS\} = \{TBUV\} = \text{pencil } A\{TBCD\}.$$

Similarly $\{ATPR\} = \text{pencil } B\{ATCD\}$,

but by hypothesis $\{TBQS\} = \{ATPR\}$,

hence pencil $A\{TBCD\} = B\{ATCD\}$;

hence C and D lie on the same conic with vertices A, B, tangents at vertices AT, BT; hence and similarly all points of contact of the joins with their envelope lie on the same conic; which is therefore identical with the envelope.

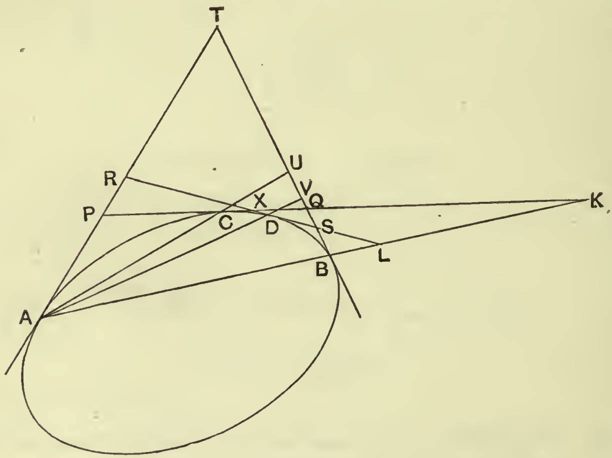


Fig. 73.

Corollary 1. Five tangents uniquely determine a conic.

Corollary 2. Two conics cannot have more than four common tangents.

Corollary 3. From any point two real or coincident or imaginary tangents can be drawn to a conic: *i.e.* a conic is a curve of the second class or degree. For the joins of any point X to the two rows form two projective pencils at X: and these have two double rays. Also when the double rays coincide X lies on the conic.

Definition. A point may be said to be outside, on, or inside a conic, as the tangents from it are real and different, or coincident, or imaginary.

Corollary 4. Three tangents and the points of contact of two of them determine a conic uniquely.

Thus we have proved that a conic might equally be defined as the envelope of the joins of corresponding points of two projective rows, not in perspective. (If the rows were in perspective the joins would all pass through a point.)

87. Theorem. If we describe a conic to touch five given lines p, q, r, s, a, and k is another line such that the row described on k by the lines p, q, r, s is projective with the row described on a by p, q, r, s, then k touches the conic.

For, if not, another line k' can be drawn through the intersection T of k and p, to touch the conic; and hence p, q, r, s describe on k' a row projective with the row they describe on a. Hence we have on k and k' two projective rows having a common point T; and therefore in perspective, so that q, r, s are collinear, which is contrary to hypothesis, for the projective rows on a, p must not be in perspective.

Thus the envelope of a line on which four given lines, no three of which are concurrent, form a row of constant cross-ratio is a conic.

We have already proved that the cross-ratio of the row described on any tangent to a conic by four given tangents p, q, r, s is constant: this may be called the cross-ratio of the four tangents with respect to that conic, and denoted by $\{p, q, r, s\}$. It will have a different value for each conic which touches the four lines.

We see also that from five given lines the same conic will be obtained, whichever pair we take as the bases of the projective rows.

Exercise **1.** Construct a conic touching four given lines, and for which the four lines have a given cross-ratio.

2. Prove that the cross-ratio of four tangents is harmonic, if two of them intersect on the chord of contact of the other two.

3. Given three tangents p, q, r to a conic, construct a fourth tangent s, such that $\{pqrs\} = -1$, with respect to that conic.

88. Maclaurin's Theorem. If a variable triangle be such that its three sides pass through three given fixed points, and two of its vertices lie on two given lines, then the locus of the third vertex is a conic.

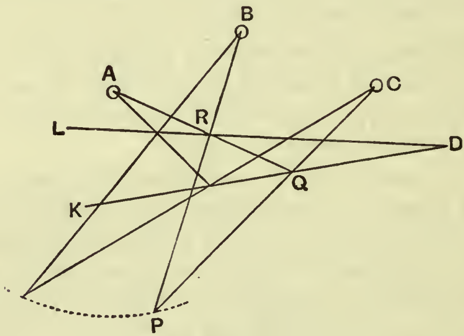


Fig. 74.

Let PQR be the triangle of which Q, R respectively lie on the two lines DK, DL, and QR, RP, PQ pass respectively through the fixed points A, B, C. Then shall the locus of P be a conic.

For, if $P_1Q_1R_1$, $P_2Q_2R_2$, $P_3Q_3R_3$, $P_4Q_4R_4$ be four successive positions of the triangle, the pencil B $\{P_1P_2P_3P_4\} = \{R_1R_2R_3R_4\}$ on the transversal DL: and C $\{P_1P_2P_3P_4\} = \{Q_1Q_2Q_3Q_4\}$ on DK.

But, by projection from A, we have $\{R_1R_2R_3R_4\} = \{Q_1Q_2Q_3Q_4\}$.

Hence B $\{P_1P_2P_3P_4\} = C\{P_1P_2P_3P_4\}$, and therefore the locus of P is a conic.

Corollary. This conic passes through B and C.

89. Theorem. If a triangle moves so that its three vertices lie on three fixed lines, and two sides pass through two given points, its third side will envelope a conic.

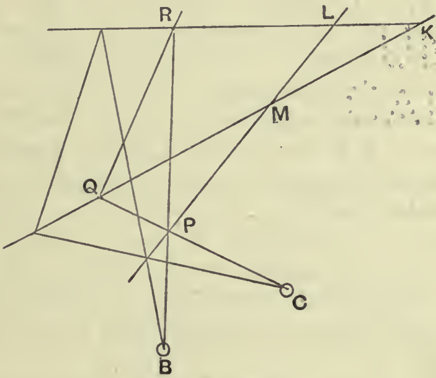


Fig. 75.

Let the triangle PQR move so that P always lies on LM, Q on MK, R on KL, and PR, PQ pass through B, C respectively, then the envelope of QR is a conic.

For the pencils described by BP at B, and CP at C, are projective, being in perspective; hence the rows described by Q on KM, and R on KL are projective. Hence QR envelopes a conic.

Corollary. The conic touches KL and KM.

These theorems may be extended as follows.

90. Theorem. If a triangle moves so that its three sides QR, RP, PQ pass always through three given points A, B, C respectively, and Q lies on a conic through A, C and R lies on a conic through A, B; then the locus of P is a conic passing through B, C.

For since BP passes through R, ∴ pencil described by BP at B is equal to pencil described by AR at A: and pencil described by CP at C, *i.e.* pencil described at C by CQ = pencil described at A by AQ. But AQ, AR are the same line,

∴ pencil B{P ...} = C{P ...},

and ∴ the locus of P is a conic passing through B and C.

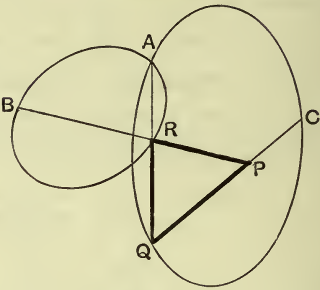


Fig. 76.

Thus, if from an intersection of two conics we draw a line to cut the conics at R, Q respectively; and join R, Q to two given points B, C on their respective conics, the locus of the intersection of BR, CQ is a conic.

Corollary. The locus of P passes through any other intersection of the conics AB ..., AC ..., and hence if these conics intersect in four points it is completely determined for it passes through the three intersections other than A, and through the two points B, C.

Thus we get a system of three conics passing through three points X, Y, Z and intersecting in pairs at A, B, C, concerning which we have proved that if P on BCXYZ, Q on CAXYZ, R on ABXYZ, are such that PR passes through B, and PQ through C, then QR passes through A.

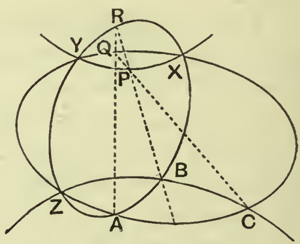


Fig. 77.

Thus we get a system of co-related triads P, Q, R on the three conics.

91. Theorem. If a triangle PQR moves so that its vertices P, Q, R lie respectively on LM, MK, KL, and PR touches a conic touching LM and LK, and PQ touches a conic touching ML and MK, then the envelope of QR is a conic touching KL and KM.

For the row described by Q on MK = row described by P on ML = row described by R on KL.

Hence QR is the join of projective rows on KM and KL, and ∴ its envelope is a conic which touches KL and KM.

Corollary. The envelope of QR touches the other common tangents of the two given conics. Hence, if they have four common tangents, it is determined by KL, KM and three of these common tangents other than LM.

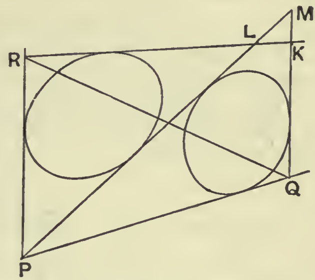


Fig. 78.

Thus we get a set of three conics each touching the three sides of the triangle ABC; and touching in pairs, the sides KL, LM, MK

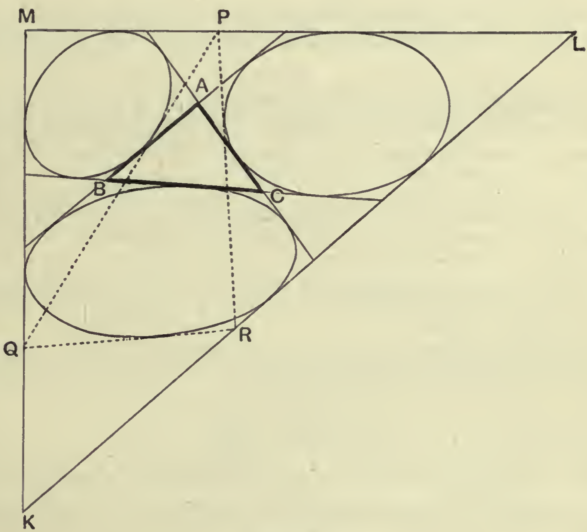


Fig. 79.

of triangle KLM: and we have proved that if from any point P on LM we draw tangents PQ to the conic touching MK, ML meeting MK at Q; and PR to the conic touching LK, LM meeting LK at R, then QR touches the third conic. Thus we get a set of co-related triangles PQR.

Corollary. *Three* special forms of the triangle PQR on the lines BC, CA, AB.

EXAMPLES. V.

1. Through A a line is drawn parallel to the side BC of a quadrilateral ABCD, and the joins of two fixed points on this line to a variable point on BC cut AB at K, L; prove that the locus of the intersection of CK, DL is a conic passing through A, B, C, D.

Also find the cross-ratio of {ABCD} on this conic.

2. A conic touches two lines TA, TB at A and B, and passes through a point C. Find the lines through T which meet the curve at infinity.

3. Through corresponding points on two projective rows lines are drawn parallel to two given directions, prove that the locus of their intersection is a conic.

4. A parabola touches TA, TB at A, B, and M is the middle point of AB. Prove that the locus of the intersection of the perpendicular from T to a tangent touching the parabola at K and a line through K parallel to TM is a conic.

5. If the chord CD of a conic passes through the intersection of the tangents at A, B prove that C, D are harmonic conjugates to A, B on the conic.

6. Three straight lines through a point cut a conic in six points forming an involution on a conic.

7. If a chord PQ of a conic cuts a chord AB at K and the tangents AT, BT at D, E, prove, from the definition, that

$$\frac{KD}{DQ} : \frac{KE}{EP} = -\frac{KP}{KQ}.$$

Deduce that when P coincides with Q then {KPDE} is harmonic.

8. Construct a conic circumscribing ABCD, with respect to which {ABCD} is harmonic.

9. Prove that the chord of contact of tangents to a conic from any point on a given straight line passes through a fixed point.

10. Construct a conic to touch four given straight lines, so that the cross-ratio of the four tangents shall have a given value.

11. A conic is constructed by joining corresponding points on two rows TA, TB of which T and A correspond to B and T. Two of the joins PQ, P′Q′ cut AB at R, R′ respectively and intersect at X, and TX cuts AB at Y. Prove that

$$\frac{AY}{YB}:\frac{AR'}{RB}=\frac{RY}{YR'}.$$

Deduce that, when P′Q′ coincides with PQ, X becomes the point of contact of PQ, and {ABYR} is harmonic.

12. Two conics are drawn through four points A, B, C, D and any line through A cuts the conics at Q, R; find the locus of the intersection of BQ and CR.

13. A conic touches TA, TB at A and B, and the joins of A, B to any point P of the conic cut TB, TA respectively at K, L. Prove that KL envelopes a conic, which also touches TA, TB at A, B.

14. In the figure of the previous question prove that the point at which KL touches its envelope lies on TP. Also prove that KL, AB and the tangent at P are concurrent.

15. In the same figure prove that TP is harmonically divided by AB and KL. Also find the points at which the line joining T to the middle point of AB cuts the envelope of KL.

16. A variable tangent to a conic cuts two fixed tangents TA, TB at K, L respectively, prove that the locus of the intersection of AL, BK is a conic having double contact with the given conic.

17. Any line through a given point K in the side BC of a triangle ABC cuts CA at L, and AB at M, and BL, CM cut AK at X and Y respectively; prove that the locus of the intersection of CX and BY is a conic.

18. Any point P is taken in the side of a triangle ABC, and BQ, CR parallel to AP meet CA, BA respectively at Q, R; if PQ cuts AB at M and PR cuts AC at L, prove that LM envelopes a conic.

19. Any point Q of a fixed straight line is joined to two given points S and S′, and SP, S′P are drawn making given angles with QS, QS′ respectively and meeting at P. Prove that the locus of P is a conic.

20. If ABC, A′B′C′ lie on a conic prove that the intersections of AB′ and A′B, AC′ and A′C, BC′ and B′C are collinear.

21. If two projective pencils have a common vertex lying on a conic which cuts any two rays of the first pencil at K, L and the corresponding rays of the second pencil at K', L', prove that the locus of the intersection of KL' and $K'L$ is a straight line.

22. Construct the points at which a given straight line is cut by the conic passing through five given points.

23. A conic touches TA at A and TB at B, and a variable tangent to the conic cuts TA at P and TB at Q; P' is the mid-point of TP, and Q' of TQ. Prove that the locus of the intersection of AQ', BP' is a conic.

24. Tangents from three collinear points touch a conic at six points forming an involution on the conic.

25. A chord of a conic subtends an angle at a given point of the conic whose bisector is fixed; prove that the chord always passes through a given fixed point.

26. Two conics are such that a triangle is inscribed in one and circumscribed to the other, prove that an infinite number of such triangles exist.

27. The tangents at A, B to a conic meet on the normal at C; prove that AC, BC are equally inclined to the normal at C.

28. If the tangents from P to a conic cut a given line AB at K, L so that $AK \,.\, AL$ bears a fixed ratio to $BK \,.\, BL$ prove that the locus of P is a conic, which passes through the intersections of the tangents from A, B to the original conic, and divides AB harmonically.

29. On a given tangent to a conic two other tangents, which meet at T, cut off a segment which subtends a constant angle at a given point on the conic. Prove that the locus of T is a conic, having double contact with the given conic.

Prove, also, that the chord of contact of the two conics is independent of the size of the constant angle.

Discuss the special case where the constant angle is a right angle.

CHAPTER VI

POLARS

92. Theorem. If any point K be taken within or without a conic, there exists a straight line which contains the intersection of tangents at the extremities of any chord through K, the harmonic conjugate of K with respect to the extremities of any chord through K, and the intersections of the joins of the extremities of any two chords through K. This line is called the polar of K.

Through K draw a chord AB, and let the tangents at A, B meet at T: find the harmonic conjugate L of K with respect to A and B: then TL is the polar of K.

Draw any other chord XY through K, let AX, BY meet at V and AY, BX at U. It has been proved that UV passes through T (§ 74).

But XY, UV, AB are the diagonals of a quadrilateral, hence UV, XY divide AB harmonically, *i.e.* UV passes through L. Hence U and V lie on TL.

Similarly UV contains the harmonic conjugate P of K with respect to X, Y; and the intersection Q of tangents at X, Y. Hence P, Q lie on TL.

Hence if X′Y′ be any other chord, the intersections of XX′, YY′ and of XY′, YX′ lie on PQ, and therefore on TL.

Corollary. If K is outside the conic its polar is the chord of contact of tangents from K. For it has been proved (§ 75, Cor.) that the points of contact of tangents from K lie on TL.

Construction. *To draw tangents from* K *to a conic.*

Draw through K two lines to cut the conic at A, B and X, Y respectively; let AY, BX meet at U and AX, BY at V; if UV cuts the conic at C, C′ then KC, KC′ are the tangents from K.

If UV does not cut the conic, K lies within the conic.

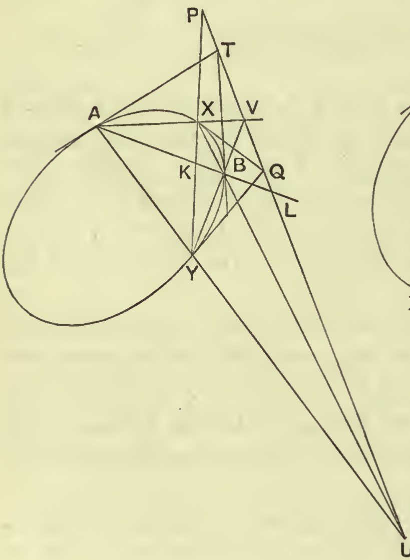


Fig. 80.

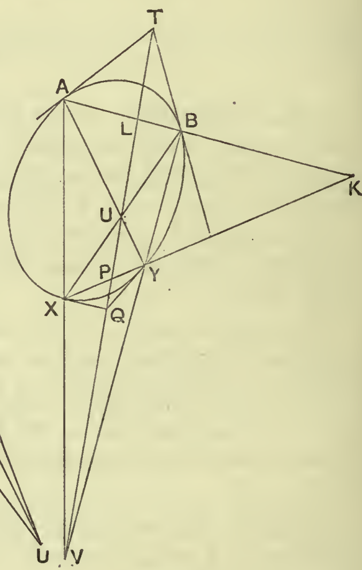


Fig. 81.

93. Theorem. If B is on the polar of A, then A is on the polar of B.

1. If AB cuts the conic at E, F, B is the harmonic conjugate of A with respect to E, F; and therefore A lies on the polar of B.

2. Through B draw any chord KL: join AK, AL to cut the conic in M, N.

Then KL, MN intersect on the polar of A, but KL cuts that polar at B, hence MN passes through B.

Thus KL, MN are two chords through B, and therefore the intersection A of KM, LN lies on the polar of B.

Corollary. The pole of AB is the intersection C of KN and LM.

Two points each lying on the polar of the other are called conjugate points.

A has a conjugate point on every line through it, their locus being the polar of A.

If A, B are conjugate points on a line whose pole is C, then A is the pole of BC, and B of AC.

A triangle of which each side is the polar of the opposite vertex is called a **self-polar** or **self-conjugate** triangle.

Corollary. The diagonal points of any inscribed four-point are the vertices of a self-polar triangle.

Corollary. One vertex of any self-polar triangle lies within the conic, and two vertices lie outside the conic.

94. Problem. Given a self-polar triangle to a conic and one point on the conic, to construct three other points of the conic.

Let ABC be the self-polar triangle, and K the given point. Join AK to cut BC at P, and find the harmonic conjugate L of K with respect to A and P.

Since BC is the polar of A, it follows that A and P are conjugate points on the line AP, therefore AP cuts the conic in points which are harmonically conjugate with respect to A and P. But one point of section is K, therefore the other is L.

Similarly two other points M, N may be obtained by joining BK to cut AC at Q, and taking M the harmonic conjugate of K with respect to B, Q; and joining CK to cut AB at R, and taking N the harmonic conjugate of K with respect to C, R.

Corollary. If two conics have a common self-polar triangle they intersect in four points or not at all.

For if they intersect at one point, the above construction gives three other points, each of which lie on both conics.

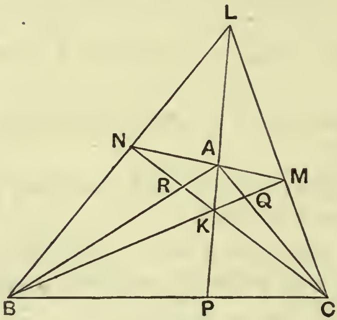


Fig. 82.

95. *A system of conics passing through four points* A, B, C, D *have a common self-polar triangle, whose vertices are the diagonal triad* U, V, W *of the four-point* ABCD.

Also V, W are conjugate points (on their join) with respect to any conic of the system; hence the conics trace an involution on VW. To this involution belong also the points at which VW is cut by those common chords which pass through U, since these are harmonically conjugate to UV and UW.

Involutions are also traced on UV and UW. Any one conic of the system, however, cuts two and only two of the lines in real points (§ 93, Cor.).

96. Theorem. If two four-points have a common diagonal triad, their eight vertices either lie on a conic or on two straight lines through a diagonal point.

Let K, L, M, N and K′, L′, M′, N′ have a common diagonal triad A, B, C (see fig. 82).

1. If K′ lies on KL, then L′ also lies on KL; since K′L′, KL both pass through A.

Again AM is the harmonic conjugate of AK with respect to AB and AC, and AM′ of AK′. But AK′ lies along AK, therefore M′ lies on AM.

But MN, M′N′ both pass through A, therefore M′, N′ both lie on MN.

Hence and similarly, if any one of the second set of four points lies on the join of two points of the first set, then another lies on that join, and the other two on the other join which passes through the same diagonal point.

2. If K′ is not collinear with any two of the four points K, L, M, N a conic can be described through these five points. Also ABC will be a self-polar triangle to this conic, and hence L′, M′, N′ lie on the conic.

97. Theorem. The chords of contact of four tangents to a conic pass through the vertices of the diagonal triangle.

Let KL, MN, OP, the diagonals of the four-side formed by tangents at A, B, C, D, form the triangle UVW (fig. 83).

The tangents KA, KC touch the conic at A, C; intersect at K; and are cut by two other tangents at M, O and P, N respectively.

$$\therefore \{KAMO\} = \{CKPN\},$$

$$\therefore \{KAMO\} = \{KCNP\};$$

two rows with a common corresponding point K, therefore AC, MN, OP are concurrent; *i.e.* AC passes through U.

Similarly BD passes through U : AB and CD through V : and AD, BC through W.

Corollaries. 1. The diagonal triad of four points on a conic are the vertices of the diagonal triangle of the four lines which touch the conic at those four points.

2. The diagonals of a circumscribing quadrilateral form a self-polar triangle.

3. Given a self-polar triangle and one tangent three others can be constructed (cf. § 94).

4. Given a self-polar triangle and two tangents, the conic can be constructed, except in certain special cases.

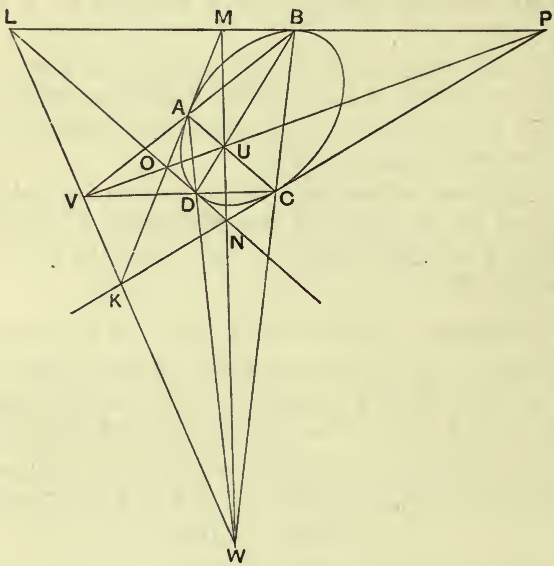


Fig. 83.

5. If two four-sides have a common diagonal triangle, the eight lines either touch one conic, or pass through two points lying on one of the diagonals (cf. § 96).

6. If two conics have a common self-polar triangle they have either four common tangents or none. For if they have one common tangent we can construct three others by Cor. 3.

7. Two or more conics touching four given lines have a common self-polar triangle. Also pairs of tangents from a vertex of this triangle form a pencil in involution (cf. § 95).

Pole Theorem.

98. If VW is any straight line there exists a point U through which pass the chord of contact of tangents from any point of VW; the harmonic conjugate of VW with respect to tangents to the conic from any point of VW; and the joins of intersections of the tangents from any two points of VW. This point U is called the **pole** of VW.

If VW cuts the conic, its pole is the intersection of tangents drawn at the points of section.

If U is the pole of VW, then VW is the polar of U as defined in § 92.

If we join any point V on a line VW to U the pole of VW, then VW contains the pole of VU. Two lines through a point each of which contains the pole of the other are called **conjugate lines.**

If U is the pole of VW, and W of VU, then V is the pole of UW: and UVW is a self-polar triangle.

99. Theorem. *As a line turns round a point, its pole moves along the polar of the point describing a row projective with the pencil of the line.*

Let AP be the line, turning round A, and cutting a, the polar of A, at P.

Take any fixed line through A cutting the conic at L and M.

Let PL, PM cut the conic again at X, Y respectively.

P is on the polar of A, therefore XY passes through A.

Therefore LY, MX meet at P′, the pole of AP.

Hence as P moves along a, P′ also moves along a, and (L, M being fixed) the pencil described by AP at A = the pencil of LP or LX at L = the pencil of MX at M (by the definition of a conic) = the row described by P′ on the transversal *a*.

Corollary. Conjugate points (P, P′) describe projective rows on a straight line: again when P is at P′, P′ is at P, since P is the pole of AP′, hence $\{PP'QR\} = \{P'PQ'R'\}$, *i.e.* the double row of conjugate points on a line are in involution.

When the line cuts the conic these results follow directly from the fact that P, P′ are harmonically conjugate to the points of intersection, which are the double points of the involution.

Conjugate lines at any point describe a pencil in involution. If the point is outside the conic the tangents from it are the double rays of the involution.

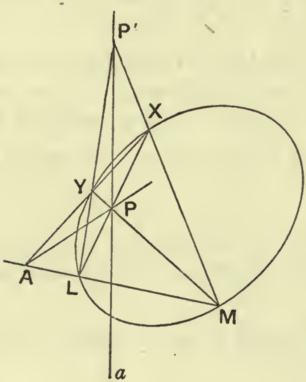


Fig. 84 (i).

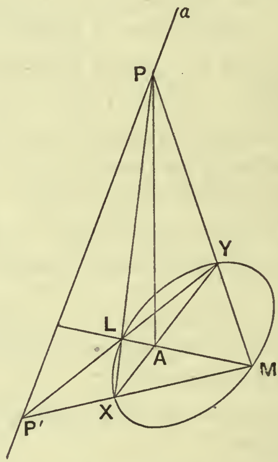


Fig. 84 (ii).

100. Theorem. If P, Q and A, B be two pairs of conjugate points, then the intersections K, L of PA, QB and PB, QA respectively are also conjugate points.

Let R, C be the poles of PQ and AB. Draw PR to cut BC at G, and AC at H; QR to cut BC at E; and PB to cut AC at Y.

Then E lies on the polars of A and P, hence it is the pole of AP; similarly G is the pole of AQ, so B, C, E, G are the poles of AC, AB, AP, AQ,

and $\therefore$ {BCEG} = pencil A {CBPQ} = transversal {YBPL},

therefore {BCEG} = {BYLP} by a double exchange of terms: and these are two projective rows with a common point B, therefore CY, EL, GP are concurrent: *viz.* EL passes through H.

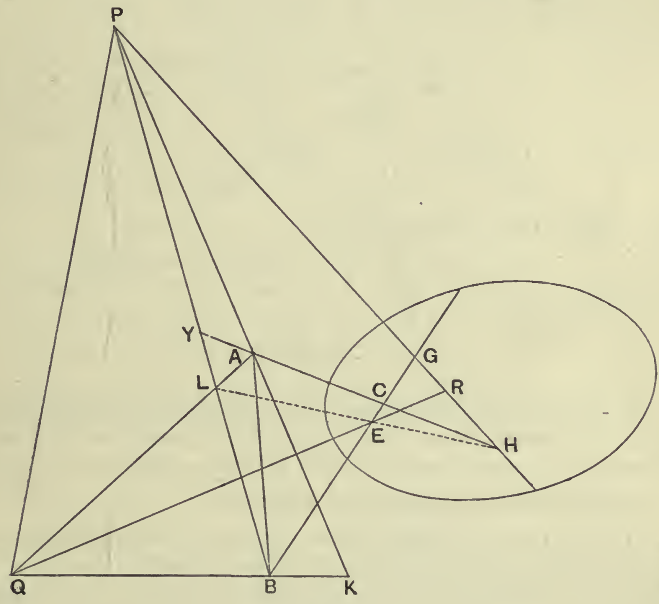


Fig. 85.

Now E is the pole of AP, and H is the pole of BQ (being the intersection of the polars of B and Q): therefore EH is the polar of K.

Hence the polar of K passes through L. Q. E. D.

101. Given a self-conjugate triangle ABC and a point P and its polar p, to find the conic.

Join PC to cut p at Z, and AB at F, then C, F and P, Z are two pairs of conjugate points on PC, and hence we may find the double points K, L of the involution on PC. These will be the points at which PC cuts the conic, similarly we may find two points on PA, and two on PB.

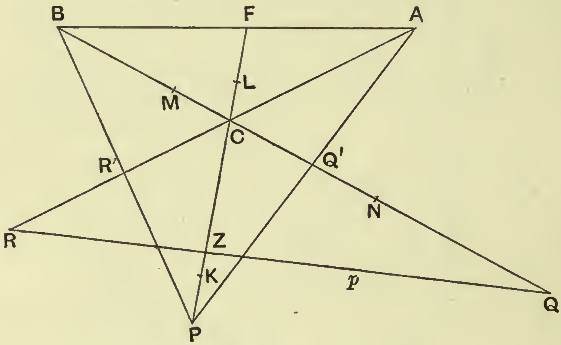


Fig. 86.

Now one vertex of a self-conjugate triangle lies inside the conic, and two outside. Hence one at least of the lines PA, PB, PC must cut the conic, so that we have found at least one pair of real points.

Again let BC cut p at Q, and PA at Q′.

Then Q lies on p and BC, hence its polar passes through P and A, therefore Q, Q′ are conjugate points on BC, and so also are B, C, hence the involution on BC is determined, and its double points, M, N, are a pair of points on the conic. But two sides of a self-conjugate triangle cut the conic and the other does not. Hence we get two pairs of real points, and one imaginary pair.

Hence we have got at least six real points of the conic (*viz.* four on the sides of triangle ABC, and two on one of the lines PA, PB, PC), and the conic is completely determined.

Corollary. If P lies on a side AB of the triangle ABC, its polar is a line CQ through C, and the construction fails.

In this case let CQ cut AB at Q. Then PCQ is also a self-polar triangle. P, Q and A, B determine the involution on AB; let G, H be the double points of this involution, then any conic which touches CG, CH at G, H satisfies the conditions.

102. *Two conics cannot have more than one common self-conjugate triangle* [unless they touch two given lines OG, OH at the same points G, H, in which case any triangle whose vertices are O and two harmonic conjugates P, Q with respect to G, H is a common self-conjugate triangle].

If U, V are two points which have common polars for the conics, and these polars intersect at W, then W has a common polar UV and UVW is the common self-polar triangle.

If there exists a point U which has the same polar QR for both conics, the conjugate points on QR for each conic trace an involution, these involutions have one pair of common points, which may or may not be real: if they are real, *viz.* V and W, then UVW is a self-polar triangle common to the conics.

Hence two conics have not more than one common pole and polar, except when they have a common self-polar triangle.

Common Conjugate Points and Lines for two Conics.

103. If the polars of P for two conics intersect at P′, the polars of P′ intersect at P, and P′ is the conjugate of P for both conics: P has only one common conjugate point P′, except when it has a common polar for the two conics, in which case it is the common conjugate of any point on that polar.

If P, P′ and Q, Q′ are two pairs of common conjugate points, the intersections of PQ, P′Q′ and PQ′, P′Q are also common conjugate points (§ 100).

If P is an intersection of the conics it coincides with its common conjugate P′.

If P lies on a common chord AB its common conjugate also lies on AB and is the harmonic conjugate of P for A and B; if P is the intersection of two common chords it has a common polar for the two conics.

On any line the conjugate points for either conic trace an involution, and two non-coincident involutions on a line have one and only one pair of common points, real or imaginary, hence on any line which is not a common chord there is one and only one pair of common conjugates (real or imaginary) : if on a line there are two pairs of common conjugates (the involutions coincide, and) the line is a common chord cutting both conics at the (real or imaginary) double points.

On a common tangent to two conics the common conjugate points are the points of contact.

104. Conjugate lines. If C, D are the poles of AB for two conics, the poles of CD lie on AB, and AB, CD are common conjugate lines for the two conics : AB has one conjugate line CD, except when it is a line having the same pole for each, in which case it is conjugate for both conics to any line through the common pole.

If AB is a common tangent it coincides with its common conjugate.

If AB passes through the intersection O of two common tangents, its common conjugate also passes through O and is the harmonic conjugate of AB with respect to the common tangents.

Through any point which is not an intersection of common tangents pass one, and only one, pair of common conjugate lines (real or imaginary). If more than one pair of common conjugates can be drawn through a point, that point is an intersection of common tangents, *viz.* the (real or imaginary) double rays of the involution.

105. Theorem. If U has a common polar for two conics, and P, P′ are a pair of common conjugate points, the common conjugate of any point Q on UP lies on UP′.

Proof. Let the poles of UP for the two conics be S, T (lying on the polar of U), the polars of P are SP′, TP′: also if UP cuts ST at F, the polars of F are SU, TU. Let the polars of Q be SQ′, TQ′.

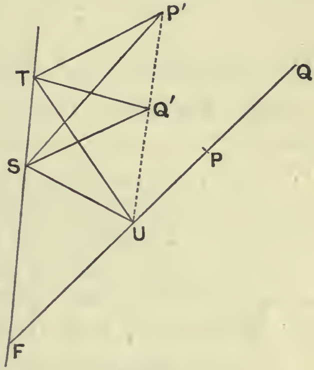


Fig. 87.

Then each of the two pencils formed at S and T by joining to F, U, P′, Q′ is projective with {UFPQ}, since the cross-ratio of four points on a line is equal to that of their four polars at the pole of the line (§ 99).

Thus at S, T we have two projective pencils with a common ray SF (the polar of U), hence they are in perspective, and therefore U, P′, Q′ are collinear.

Hence as P moves along a line UP its common conjugate moves along another line through U.

If UP′ cuts ST at F′, then {F′UP′Q′} = {UFPQ} = {FUQP}, hence P′Q and PQ′ intersect on FF′, *i.e.* on ST.

106. Conversely: if P, Q, R are three collinear points whose common conjugates for two conics P′, Q′, R′ are also collinear, the intersection of PQ and P′Q′ has a common polar for the two conics. Let S, T be the poles of PQ for the two conics, then SP′, SQ′, SR′ and TP′, TQ′, TR′ are the polars of P, Q, R: also if P′Q′ cuts ST at F′, the pole of SF′ for the one conic is a point U on PQ such that {PQRU} = {P′Q′R′F′}, and the polar of TF′ for the other conic is the same point U.

Hence ST has the same pole U (lying on PQ) for both conics, and hence by the theorem above the common conjugates P′, Q′, R′ of P, Q, R lie on one line through U.

Corollary. The common conjugate of any other point of PQ lies on P′Q′.

Conjugate Conic of a Line with respect to two Conics.

107. Problem. To find the locus of the common conjugate points for two conics, of all points lying on a given line.

1. If the line has the same pole U for both conics, the common conjugate to any point on the line is at U (but if there is a self-polar triangle UVW the points V, W of the line are conjugate to any points on UW and VW respectively): also, if the conjugates of two points K, L of the line are the same point that point is the pole of KL for both conics.

2. If the line passes through a common pole the locus of the conjugate points is another line through that pole: and if the conjugate points K′, L′, M′ of three points K, L, M are collinear, the intersection of KL, K′L′ is a common pole, and the common conjugate of any point of KL lies on K′L′; the locus is a straight line.

3. If the above special conditions are not satisfied, let K′, L′, M′ be the conjugates of three points K, L, M of the line, and S, T the poles of the line for the two conics, so that SK′, TK′ are the respective polars of K, etc., etc.

Take any other point P of the line, and its polars SP′, TP′.

Because the cross-ratio of four points on a line equals the cross-ratio of their polars at the pole of the line (§ 99),

$$\therefore\ \mathrm{S}\,\{\mathrm{K'L'M'P'}\} = \{\mathrm{KLMP}\} = \mathrm{T}\,\{\mathrm{K'L'M'P'}\},$$

and K′, L′, M′ are not collinear, therefore the locus of P′ is a conic given by the five points S, T, K′, L′, M′. It is called the **conjugate conic** of the line.

Corollary 1. The conjugate conic cuts the line at the pair of common conjugate points which lie on the line.

Corollary 2. If U is a point which has a common polar QR for the two conics it lies on the conjugate conic of any line, for it is conjugate to the point where QR cuts the line: if the conics have a common self-polar triangle its vertices lie on the conjugate conic of any line.

Corollary 3. If A, B on the line have common conjugates A′, B′ and A′B′ cuts AB at C, the common conjugate of C is the intersection C′ of AB′ and BA′ (§ 100).

Corollary 4. If P is any other point on AB, and P′ the common conjugate of P, and K is any other point of the conjugate conic of AB, we have $K\{A'B'C'P'\} = S\{A'B'C'P'\} = \{ABCP\}$; but A, A′, B, B′, C, C′ are pairs of vertices of a four-side, therefore the joins to K are in involution, hence KP and KP′ describe pencils in involution at K.

If the common conjugate points are real points X, Y (§ 103) this may be proved as follows.

The pencil $K\{P' \ldots\} = S\{P' \ldots\} = \{P \ldots\}$; hence KP′, KP describe two projective pencils at K. Also, in these pencils, KX corresponds to KY and KY to KX, hence the pencils are in involution.

Corollary 5. If U is a point which has the same polar for both conics, and a common chord through U cuts AB at P, the common conjugate P′ also lies on UP. Hence the common chords through U are the real or imaginary double rays of the involution described by UP, UP′ at U; and UP, UP′ are harmonically conjugate with respect to the two common chords.

108. Theorem. If two conics have a common self-polar triangle they have either one or three pairs of common chords.

Let UVW be the self-polar triangle, through each vertex pass two, real or imaginary, common chords. If two pairs are real, there are four real intersections of the conics, and the third pair of common chords are real.

Also, by § 94, Cor., if the conics intersect at all, they intersect in four points, and therefore there are three pairs of common chords.

Suppose, however, that the common chords through V and W are imaginary, and take a point P within the triangle UVW. Let P′ be the common conjugate of P.

Since the double rays of the involution at V are imaginary, VP and VP′ lie in supplementary angles at V (§ 32); and, in the same way, WP and WP′ lie in supplementary angles at W.

Hence P′ lies either in the angle between VU produced and WU produced, or in the space bounded by UV produced, UW produced and VW. In either case UP′ and UP lie in the same or opposite angles at U, and therefore the double rays of the involution at U are real (§ 32), forming a pair of real common chords.

Hence the conics either intersect in four points, and have three pairs of real, common chords, or they do not intersect at all, but still have one pair of real, common chords, *i.e.* a pair of lines which, although they do not cut either conic, still satisfy a certain test which applies to common chords.

109. Theorem. *Any two conics have at least one common pole and polar.*

Take two lines KL, KM and find their conjugate conics for the two conics: these intersect at K′, the common conjugate of K, hence they intersect in at least one other point U.

Let Q be the point of KL which corresponds to U, and R the point of KM; then QR is the polar of U for both conics.

Corollary. If two conics intersect at A, we can find a point U which has the same polar QR for both conics, and another common point of the conics will be the harmonic conjugate on UA of A with respect to U and the point where UA cuts QR.

110. On the polar QR of U conjugate points with respect to either conic trace an involution whose double points are the points where QR cuts the conic. Now two involutions on a line have one and only one pair of common points, but these are imaginary if, and only if, the double points are the ends of real and overlapping segments (§ 28).

Let V, W be the common pair of conjugate points on the polar of U, if they are real UVW is a self-polar triangle: and therefore the conics intersect in four points or not at all.

Conversely: *if two conics do not intersect at all* (and U, QR are the common pole and polar), one conic must lie entirely within the other or else each outside the other, in either case QR cuts the conics in segments which are one or both imaginary, or if both real not overlapping, and therefore V, W are real. [In this case there are two real common chords (§ 108).]

Hence it follows that if V, W are imaginary the conics must intersect in one and only one pair of points, and (conversely) if they intersect in one pair of points only, V, W are imaginary. In this case the line QR cuts both conics, and therefore its pole U is outside both conics.

If U is a common pole to a common polar QR, and AB is a common tangent meeting QR at A, then another common tangent is the harmonic conjugate of AB with respect to QR and AU. Hence two conics have 4, 2, or 0 common tangents.

If U, V, W are real the conics have four common tangents or none, and conversely, if they have either four or no common tangents then V, W are real.

Hence when U is real, but V and W are imaginary, there are two and only two common tangents: conversely also if there are two and only two common tangents, V, W are imaginary (and the conics intersect in two points).

Problem. To construct the common chords of two conics which do not intersect.

Take two points A, B and their common conjugates A′, B′; and construct the conjugate conics of the lines AB, A′B′. These will intersect at C, the intersection of AB′ and A′B, and also at three other real points U, V, W forming the self-polar triangle UVW.

We can now construct the involutions joining U, V, W to A, A′, B, B′, C, C′ and in one of these the double rays will be real, and these are the common chords.

111. Theorem. If any straight line be drawn to cut the six joins of a four-point, and the harmonic conjugate be taken on each join of the point where the line cuts it, these six conjugates lie on a conic, which also passes through the diagonal triad of the four-point.

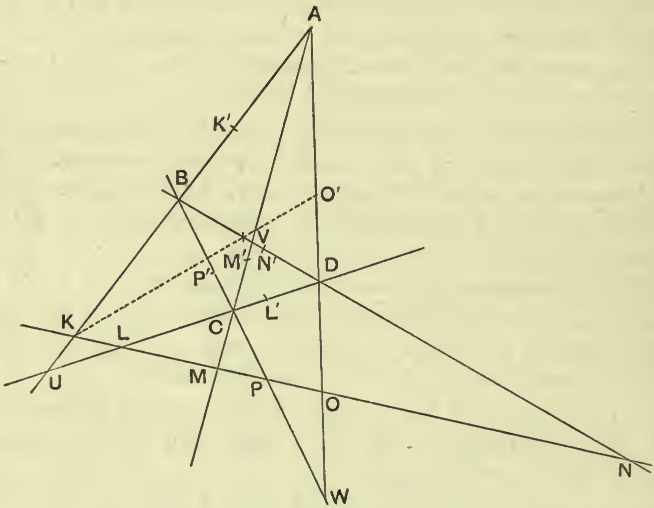


Fig. 88.

Let AB cut the line at K, and find K′ the harmonic conjugate of K on AB.

Similarly let CD, AC, BD, AD, BC cut the line at L, M, N, O, P and L′, M′, N′, O′, P′ be the respective harmonic conjugates.

Let AB, CD meet at U ; AC, BD at V ; AD, BC at W,

$$\{ABKK'\} = \{ACMM'\},$$

hence BC, KM, K′M′ are concurrent, *i.e.* K′M′ passes through P.

Similarly $K'N'$, $K'O'$, $K'P'$ pass through O, N, M respectively.

Hence $\quad K'\{M'N'O'P'\} = \{PONM\} = \{MNOP\}$.

Similarly $\quad L'\{M'N'O'P'\} = \{MNOP\}$,

therefore K', L' lie on the same conic through M', N', O', P'.

Again UM', UN', UO', UP' are the harmonic conjugates of UM, UN, UO, UP with respect to UA and UC.

$$\therefore\ U\{M'N'O'P'\} = \{MNOP\} = K'\{M'N'O'P'\},$$

therefore U also lies on this conic, and similarly V and W.

Corollary. Since $K'M'$, $L'N'$ meet at P and $K'N'$, $L'M'$ at O, therefore O, P are two diagonal points of $K'L'M'N'$; hence $K'L'$, $M'N'$ intersect at X, the harmonic conjugate with respect to K', L' of the point where $K'L'$ cuts the given line; similarly $O'P'$ cuts $K'L'$ at the same point X.

Also X is the pole of the given line for the conic.

Again K', L', M', N', O', P' form an involution on the conic: which is otherwise proved since

$$U\{K'L'M'O'\} = \{KLMO\} = \{LKNP\} = U\{L'K'N'P'\}$$

because KL, MN, OP form an involution on the line.

This conic may be called **the harmonic or nine-point conic** of the line with respect to the four-point.

112. Theorem. *The nine-point conic of a line with respect to four points is the locus of the poles of the line with respect to a system of conics through the four points.*

Let S be the pole of the line with respect to any conic through A, B, C, D. Then (using the last figure) the polars of M, N, O, P are SM', SN', SO', SP', therefore $S\{M'N'O'P'\} = \{MNOP\}$, therefore S lies on the conic $UVWK'L'M'N'O'P'$.

Corollary. The nine-point conic of a line with respect to four points is the conjugate conic of the line with respect to any two conics through the four points.

For K, K′ are conjugate points with respect to any two conics through AB, etc., etc., hence K′, L′, M′, N′, O′, P′ are six points on the conjugate conic of the line, which therefore coincides with the nine-point conic.

Corollary. The polars of a point Y with respect to a system of conics passing through four points A, B, C, D are concurrent.

Through Y draw any straight line, and construct its nine-point conic. Let S be the pole of the line for any conic of the system, and SY′ the polar of Y, cutting the nine-point conic at S and Y′, then Y′ is conjugate to Y for all conics of the system.

113. Correlative theorems on the conjugate conic of lines through a point.

If the poles of AB for two conics are C and D, the poles of CD lie on AB, and AB, CD are conjugate lines for both conics.

If a line turns round a point its common conjugate for two given conics envelopes a conic conjugate to the point.

If the conics have a common self-polar triangle its sides touch the conjugate conic of any point.

If any point be joined to the six intersections of a four-side, and the harmonic conjugate be taken at each vertex with respect to the two sides of the four-side which meet at that vertex, these six conjugates touch a conic which also touches the sides of the diagonal triangle of the four-side.

The intersections of the three pairs of these conjugate lines drawn through pairs of opposite vertices lie on a straight line; which is the polar of the given point with respect to the conic.

This nine-tangent conic is the envelope of the polar of the point with respect to a system of conics touching the four lines which form the four-side.

It also coincides with the conjugate conic of the point for any pair of conics touching the four lines.

The poles of any line with respect to a system of conics touching four given lines are collinear.

EXAMPLES. VI.

1. Prove that any line through the pole of a chord AB cuts the lines joining A and B to any other point C of the conic in conjugate points.

2. Through a given point two lines are drawn to cut a conic at four points which lie on a circle. Prove that the given point has the same polar for each circle thus obtained. Also prove that the circles are coaxal.

3. Prove that the diagonal points of a four-point inscribed in a conic are the vertices of a self-polar triangle.

4. A system of conics through four points trace an involution on each side of the diagonal triangle of the four points.

5. The three diagonals of a quadrilateral circumscribing a conic form a self-polar triangle.

6. If a line be drawn through a fixed point, and its pole with respect to a given conic be joined to another fixed point, prove that the locus of the intersection of the two lines is a conic, which passes through the two fixed points.

7. Through a given point conjugate lines are drawn with respect to a given conic; prove that they form a pencil in involution.

8. If two pairs of sides of a four-point are conjugate lines with respect to a conic, the third pair of joins will also be conjugate.

9. If ABC, A′B′C′ are two self-polar triangles for a conic, and AB, AC cut B′C′ at K, L respectively, and A′B′, A′C′ cut BC at M, N respectively, prove that $\{B'C'KL\} = \{MNBC\}$.

Also prove (1) that A, B, C, A′, B′, C′ lie on a conic; and

(2) that the six sides of the two triangles touch a conic.

10. A conic is inscribed in a triangle which is self-polar to another conic; KL is any tangent to the inscribed conic, and M is the pole of KL with respect to the other conic. Prove that the two tangents from M to the inscribed conic form with KL a triangle which is self-polar to the other conic.

11. If two conics be such that a triangle can be inscribed in one which is self-polar for the other, then an infinite number of such triangles can be described.

12. If a system of conics touch four given lines, pairs of tangents drawn to them from an intersection of two diagonals of the four-side will form an involution.

13. Find the conjugate conic of a straight line with respect to the two conics which respectively touch the line at two given points, and pass through three other given points.

14. Prove that two conics intersect in an even number of points.

15. Find the common chords of two ellipses, one of which is entirely within the other.

16. The poles with respect to two given conics of a straight line which passes through a fixed point are P, Q. Prove that the join PQ envelopes a fixed conic inscribed in the common self-polar triangle.

17. The locus of the centres of conics inscribed in a four-side is a straight line.

18. The locus of the centres of conics which pass through four given points is a conic.

19. Tangents are drawn to a conic from any two points; prove that the four points of contact and the two given points lie on one conic.

20. Find the nine-point conic of a tangent to a circle with respect to the four ends of two diameters of the circle.

21. Find the nine-tangent conic of the focus of a parabola with respect to four lines which touch the parabola.

22. Prove that the polars of any point with respect to a system of conics passing through four given points are concurrent.

CHAPTER VII

POLYGONS. CONSTRUCTIONS

114. In this chapter we consider:

1. The properties of inscribed and circumscribed triangles, hexagons, pentagons and quadrilaterals.

2. Some properties of systems of conics satisfying four conditions (*e.g.* the system of conics which touch two lines at two given points),—in many of the cases there is an involution of points on a transversal or of tangents from an external point.

3. The construction of conics to satisfy given conditions, and other problems relating to conics.

Conics are constructed:

(*a*) to touch 5 lines (§ 119);

(*b*) to pass through 4 points and touch 1 line (§ 121);

(*c*) to touch 4 lines and pass through 1 point (§ 123);

(*d*) to touch 1 line at a given point, and pass through 3 other points (§ 124);

(*e*) to touch 1 line at a given point, 1 other line and pass through 2 points (§ 127);

(*f*) to touch 1 line at a given point, and 3 other lines (§ 128);

(*g*) to touch 1 line at a given point, 2 other lines and pass through 1 point (§§ 130, 138);

(*h*) to touch 2 lines at given points, and pass through 1 point (§§ 132, 143);

(*i*) to touch 2 lines at given points, and 1 other line (§§ 135, 139);

(*j*) to touch 2 lines and pass through 3 points (§ 137);

(*k*) to touch 3 lines and pass through 2 points (§ 144).

For the sake of convenience we may add to this list the construction in § 94 of a conic with a given self-polar triangle and passing through 2 points; and, in § 101, of a conic when a self-polar triangle is given, and a point (not on the conic) and its polar.

Circumscribed and Inscribed Triangles.

115. Theorem. If a conic touches the three sides of a triangle then the lines joining the vertices to the points of contact of opposite sides are concurrent.

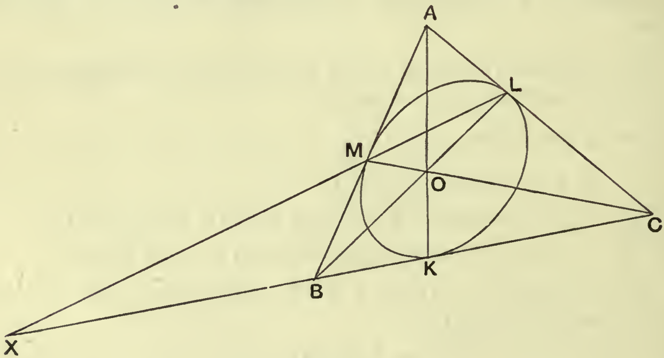


Fig. 89.

Let the conic touch BC at K, CA at L, AB at M.

Let BL, CM intersect at O; and LM cut BC at X.

Then, by § 77, Cor. 6, K, X are harmonic conjugates with respect to B, C.

But the diagonals AO, LM of the four-side AMOL, divide the third diagonal BC harmonically, hence AO passes through K. Q. E. D.

Otherwise. It has been proved (§ 77) that pencils

$$M\{KLCA\},\ L\{KMBC\}$$

are harmonic, and they have a common ray LM, hence the intersections of MK, MC, MA with LK, LB, LC respectively, *viz.* K, O, A, are collinear.

Corollary 1. $\dfrac{AM}{MB} \cdot \dfrac{BK}{KC} \cdot \dfrac{CL}{LA} = -1.$

Corollary 2. If LM is parallel to BC, the conic touches BC at its middle point, and conversely.

Problem. Given three tangents to a conic, and the points of contact of two of them, find the point of contact of the third.

116. Theorem. If a triangle be inscribed in a conic the tangents at the vertices meet the opposite sides in three collinear points.

Let the tangents at A, B, C respectively be LM, MK, KL and let them meet the sides BC, CA, AB of the triangle ABC in X, Y, Z respectively.

It has been proved that {LMAX} is harmonic; also that {LKCZ} is harmonic.

Hence these rows, having a common point L, are in perspective,

i.e. MK, CA, XZ are concurrent, and Y lies on XZ.

[Otherwise from the previous theorem, since ABC, KLM are in perspective.]

Problem. Given three points on a conic and the tangents at two of them, find the tangent at the third point.

Theorem. If the sides of a triangle KLM touch a conic at

A, B, C, and if AK, BL, CM intersect at O and BC, LM; CA, MK; AB, KL at X, Y, Z respectively, then the line XYZ is the polar of O.

For X lies on BC, the polar of K, hence the polar of X passes through K; it also passes through A, and so it is KA, which passes through O.

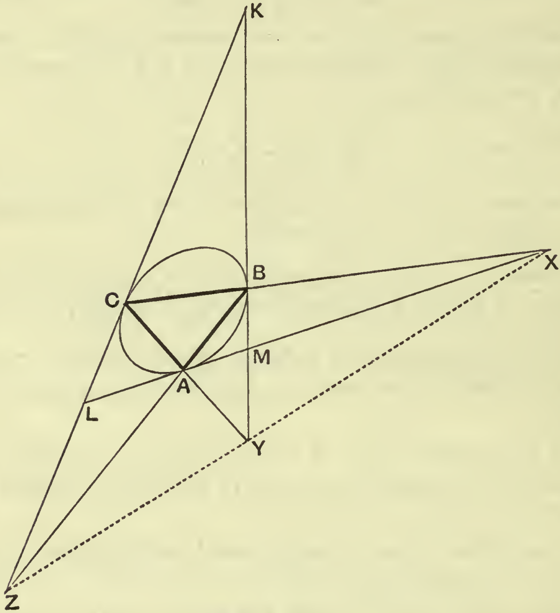


Fig. 90.

Similarly the polars of Y and Z are LB, MC respectively.

Hence the polars of X, Y, Z all pass through O. Q.E.D.

This is another proof that, if AK, BL, CM are concurrent, then X, Y, Z are collinear.

We may call O and XY the pole and polar of the inscribed triangle ABC or of the circumscribed triangle KLM with respect to the conic.

117. Complementary triangles. Let the sides LM, MK, KL of a triangle KLM touch a conic at A, B, C, and let O and XYZ be the pole and polar of the triangle.

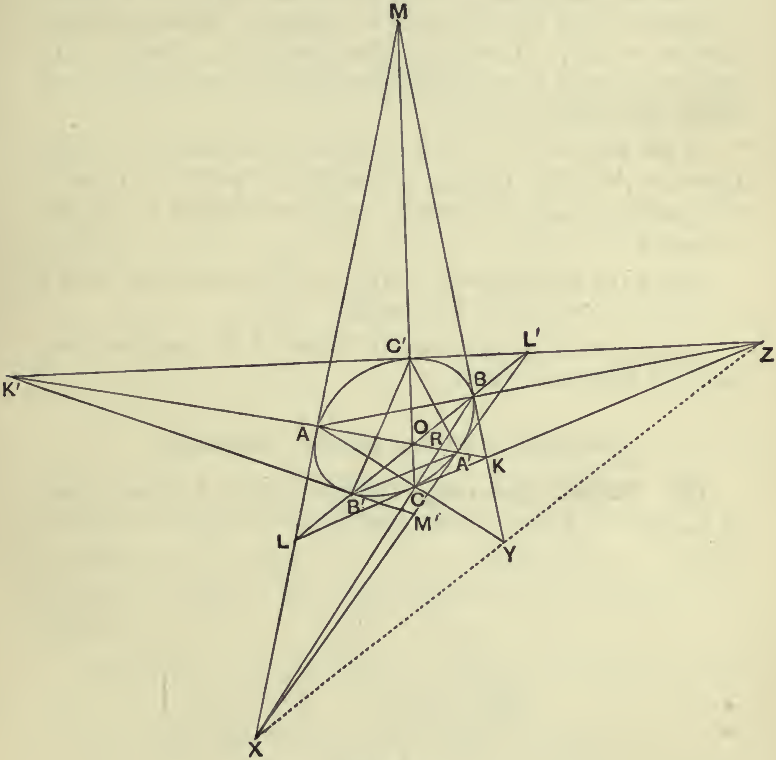


Fig. 91.

Let KA cut the conic again at A′, then the tangent L′M′ at A′ passes through X; and if LB, MC cut the conic at B′, C′ respectively, the tangents M′K′, K′L′ at B′, C′ pass respectively through Y and Z.

Hence we get a triangle K′L′M′ formed by the tangents at A′, B′, C′ which is in homology with KLM, the axis of homology being the polar XYZ.

Further, considering the tangents from K and K′, we find that the lines BB′, CC′ joining points of contact of opposite tangents intersect on KK′. Hence KK′ passes through O. Similarly LL′, MM′ pass through O, and therefore O is the centre of homology of the triangles KLM, K′L′M′.

If AA′ cuts BC at R, we have proved that $\{BCRX\} = -1$ and therefore, projecting from O, we find that $\{L'M'A'X\} = -1$, hence B′C′ passes through X, similarly C′A′ passes through Y, and A′B′ through Z.

Hence the triangles ABC, A′B′C′ are also in homology with O as centre and XY as axis of homology.

Also O and XY are the pole and polar of the complementary triangles A′B′C′ and K′L′M′.

Inscribed and Circumscribed Hexagons.

118. Pascal's Theorem. The intersections of the three pairs of opposite sides of a hexagon inscribed in a conic are collinear.

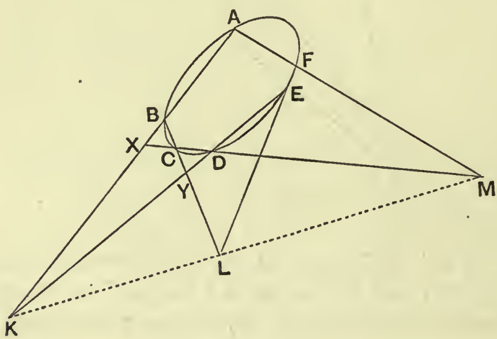


Fig. 92 (1).

Let ABCDEF be the hexagon, and let AB, DE meet at K; BC, EF at L; CD, FA at M.

Also let AB meet CD at X, and BC meet DE at Y.

Then A{BCDF} = E{BCDF} by definition; hence, taking transversals CD and BC,

$$\{XCDM\} = \{BCYL\},$$

two rows with a common point C, hence BX, DY, ML are concurrent; but BX, DY intersect at K, hence K, L, M are collinear. Q.E.D.

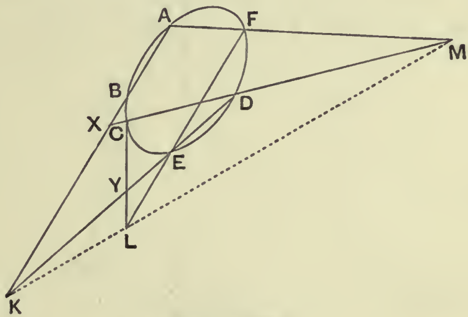


Fig. 92 (2).

Theorem. If the three pairs of opposite sides of a hexagon intersect in collinear points, the vertices of the hexagon lie on one conic.

Problem. Given five points on a conic, to find where any line through one of them cuts the conic again.

Pascal lines of a hexagon inscribed in a conic:—

Given six points on a conic we can form 60 different hexagons ($\frac{1}{2}\underline{|5}$) by joining them in various ways, and each of these gives a different line of intersection of pairs of opposite sides, hence 6 given points furnish 60 Pascal lines.

If A, B, C, D, E, F are six points forming a convex hexagon, the 60 hexagons arrange themselves in 12 distinct types, as follows, (the number after each is the number of hexagons of that type): ABCDEF (1), ABDCEF (6), ABDCFE (6), ABEDCF (3), ABDFCE (6), ABDECF (12), ABDFEC (6), ABECFD (6), ABEDFC (6), ABEFCD (2), ABFDEC (3), ACFDBE (3).

119. Problem. Construct the conic which touches five given lines.

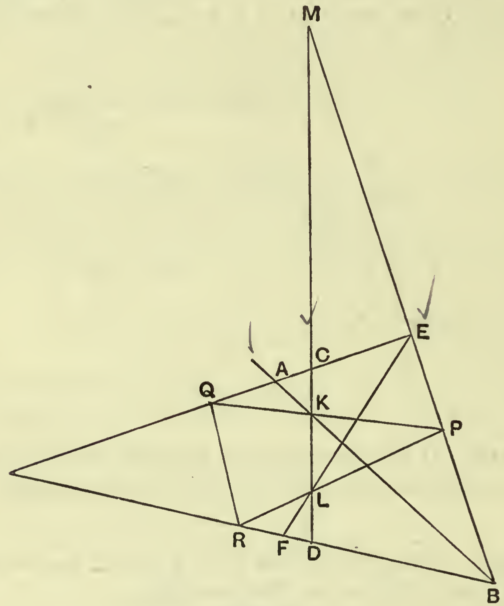


Fig. 93.

Let three of the tangents cut the other two at A, B; C, D; E, F respectively.

Let AB, EF cut CD at K, L respectively.

Take any point P on BE, and let PK, PL cut AC, BD at Q, R, then QR is a tangent to the conic. Let CD cut EB at M.

Then, projecting from K, we have $\{ACEQ\} = \{BMEP\}$ and, projecting from L, we have $\{BMEP\} = \{BDFR\}$,

hence $$\{ACEQ\} = \{BDFR\},$$

and therefore QR is a tangent to the conic.

Thus, as P moves along EB, we get a series of positions of QR enveloping the conic.

Corollary. We might have taken P on AF, instead of on EB.

120. Brianchon's Theorem. If the six sides of a hexagon touch a conic, the lines joining the three pairs of opposite vertices are concurrent.

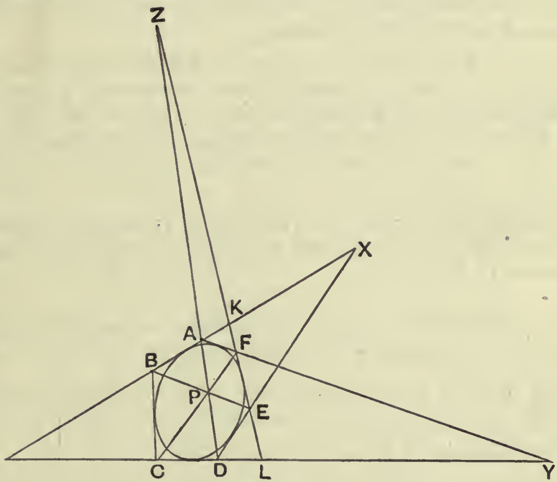


Fig. 94.

Let AF cut CD at Y, AB cut DE at X.

Let EF cut AB, CD at K, L.

Then $\{BAKX\} = \{CYLD\}$, hence pencil $E\{BAKX\} = F\{CYLD\}$, and these have a common ray EK or FL, hence the intersections of

EB, FC; EA, FY; EX, FD are collinear; hence EB, FC, DA are concurrent.

Corollary. Six tangents to a conic form ($\frac{1}{2}\,\underline{|5}=$) 60 Brianchon hexagons, and therefore furnish 60 "Brianchon points" of intersection of joins of opposite vertices.

Theorem. If the three joins of opposite vertices of a hexagon are concurrent, all the sides touch the same conic.

Problem. Given five tangents to a conic, and a point on one of them, draw the other tangent from that point.

Exercise 1. If two triangles are in homology, the six points of intersection of the sides of the one with the non-corresponding sides of the other lie on a conic; also the six joins of the vertices of one with the non-corresponding vertices of the other touch a conic.

2. If P, Q, R, S are four points on a conic and PQ, RS intersect at K, and if A, B are two other points on the conic, then through K pass the joins of the intersections of AP, BR and AS, BQ; AP, BS and AR, BQ; AQ, BR and AS, BP; AQ, BS and AR, BP; and similarly four joins pass through L, and four through M, the other two diagonal points of the four-point P, Q, R, S.

3. If the points K on AB, L on BC and M on CD are collinear, then any line through L cuts KD and MA in two points lying on the same conic through A, B, C, D.

121. Theorem. Any straight line cuts a system of conics through four given points in an involution.

Let A, B, C, D be the points, and LPP′ the line cutting BC at L and one of the conics at P, P′: if PD cuts AB at Q and P′A cuts CD at R, then Q, R are collinear with L.

Therefore Q, R describe projective rows on AB and CD: hence projecting from D, A respectively we find that P, P′ describe projective rows on the given line.

If the point P moves to the point where CD cuts the line, then P′ becomes the point where AB cuts the line: hence and similarly if AB, BC, CA cut the line at K, L, M and CD, AD, BD cut it at K′, L′, M′ then to K, L, M of the row described by P correspond K′, L′, M′ of the row described by P′, so that

$$\{KLMP\} = \{K'L'M'P'\}.$$

But K, K′, L, L′, M, M′ form an involution [being the points at which the line is cut by the six joins of a four-point], hence P, P′ describe a double row in involution.

Corollary 1. An involution has two, real or imaginary, double points: hence two, real or imaginary, conics of the system touch the line.

Also, if a common tangent to two conics of the system cut any other conic at P, Q then P, Q are harmonic conjugates with respect to the two points of contact.

Corollary 2. On any line there is one and only one pair of points which are common conjugate points for any two conics of the system, *viz.* the double points of the involution.

Corollary 3. Two diagonal points of a four-point are harmonically conjugate to the points at which their join is cut by any conic through the four points.

Problem. Describe a conic to pass through four points and touch a given line.

Let A, B, C, D be the four points. Their joins cut the line in three pairs of points in involution, find E, F the double points of this involution. Then A, B, C, D, E and A, B, C, D, F give the solutions: which are two real and different or coincident or imaginary conics.

122. Theorem. If a conic touches 4 lines KA, AB, BC, CK and a line from A to a point D in CK, cuts a line from C to any point E on AK at O, the tangents from D, E intersect on BO; *i.e.* the joins of E, D to any point on BO touch the same conic touching the 4 lines.

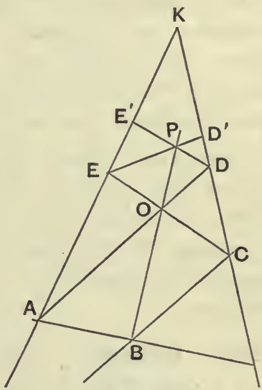


Fig. 95.

Hence if we take any conic touching 4 lines KA, AB, BC, CK and draw any line through B, and from any point P on the line we draw tangents PD to cut CK at D, PE to cut AK at E, then shall AD and CE intersect on the line BP.

Theorem. The tangents from a fixed point to a system of conics touching four given lines form a pencil in involution; and the joins of the given point to the three pairs of intersection of opposite sides are pairs of corresponding rays of the involution.

Take the 4 lines KA, AB, BC, CK, and the fixed point P.

Let tangents from P cut CK, AK at D, E respectively, and let AD, CE intersect at O, which lies on BP.

Pencil described by PD = row of D on CK = row of O on BP (by projection from A) = row of E on AK (by projection from C) = pencil described by PE, hence tangents PD, PE describe projective pencils.

Also if PE cuts CK at D′ and PD cuts AK at E′, we have PD′ of the first pencil corresponding to PE′ of the second pencil, *i.e.* PE of the first pencil corresponds to PD of the second, so that PE, PD are interchangeable.

Hence the double pencil at P forms an involution.

Further as O approaches very near to P, the line PD approaches the position PA, and PE the position PC, so that ultimately PA, PC are a pair of rays of the involution, which proves the latter part of the theorem.

Corollary. When PD, PE coincide then DE is a tangent touching the conic at P. Now an involution has two real different, or coincident, or imaginary double rays, hence there are two conics of the system which pass through P, and their respective tangents at P are the double rays of the pencil in involution.

123. Problem. Given 4 straight lines and a point construct a conic to touch the 4 lines and pass through the point.

Let P be the point, and let the sides intersect in pairs at A, B; C, D; E, F; find the double rays PK, PL of the involution determined by PA, PB; PC, PD; PE, PF. Then the 4 given lines and either PK or PL determine a conic which satisfies the conditions.

There are two real and different, or coincident, or imaginary solutions.

Theorem. If through a given point P pass two conics which touch 4 given lines, and their tangents at P are PK and PL, then the tangents from P to any other conic touching the 4 lines are harmonic conjugates with respect to PK and PL.

Theorem. From a diagonal point of the complete four-side formed by four lines the tangents drawn to any inscribed conic are harmonic conjugates with respect to the lines which join that diagonal point to the other two.

For UV, UW are the double rays of the involution formed by joining U to A, B, C, D, E, F.

Pentagon.

124. Problem. Construct a conic to touch a given line at a given point, and to pass through three other points.

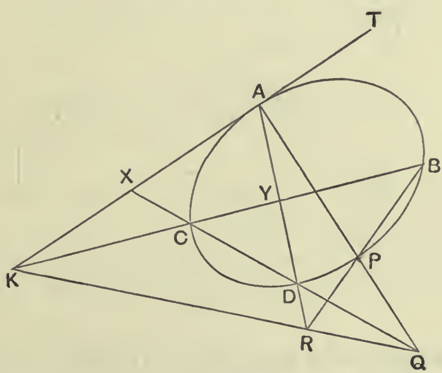


Fig. 96.

Let AT be the line, which is to touch the conic at A, and B, C, D the other points.

Then we have rays AT, AC, AD at A to correspond to BA, BC, BD at B and the problem can be uniquely solved.

Construction. Let BC cut AT at K, draw any line through K to cut AD at R, CD at Q, and let AQ, BR intersect at P.

Let CD cut AK at X, and AD cut BC at Y.

Then $$A\,\{TCDP\} = \{XCDQ\}$$
$$= \{AYDR\} \text{ by projection from } K$$
$$= B\,\{ACDP\},$$

hence P lies on the conic, and as the line KR turns round K, the point P describes the conic.

Now APBCD is a pentagon inscribed in the conic, and so we get the following theorem.

125. Theorem. If a pentagon ABCDE be inscribed in a conic, the tangent at A meets the opposite side CD in a point collinear with the intersections of AB with DE, and AE with CB.

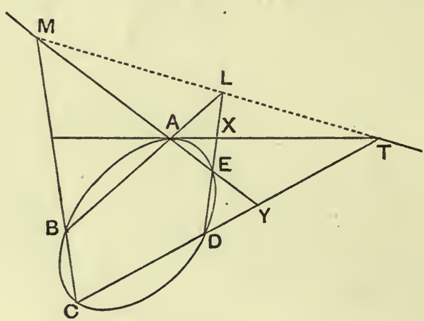


Fig. 97.

Let the tangent at A meet CD at T, and let AB, DE and AE, BC intersect at L, M respectively.

Let DE cut AT at X, and AE cut CD at Y.

Then A {TDEB} = C {ADEB} because the points lie on the conic, hence {XDEL} = {AYEM}, which are two rows with a common point E, hence XA, YD and LM are collinear, *viz.* T is collinear with L, M.

Corollary. Given five points on a conic, we can draw the tangents at those points.

This theorem may also be deduced from Pascal's Theorem by making two of the vertices of the hexagon coincide, and supposing the vanishing side to become a tangent.

Problem. Given 4 points A, B, C, D and the tangent AT, to find where a line AP cuts the conic.

Pentagon lines. Five points give ($\frac{1}{2}\,\underline{|4} =$) 12 different pentagons. These, if the pentagon is convex, give 4 distinct types ABCDE (1), ABDCE (5), ACBED (5), ACEBD (1); yielding respectively 1, 3, 3, 1 kinds of lines.

The tangent at A meets CD in a point collinear with (1) the intersections of AB, DE and AE, CB; and (2) the intersections of AB, CE and AE, DB; the tangent at A meets similarly each of the 6 joins of the 4 points B, C, D, E, and through each intersection pass 2 pentagon lines, giving in all 12 pentagon lines corresponding to the tangent at A. Hence the pentagon furnishes 60 pentagon lines.

There are 15 intersections of pairs of opposite sides, and 8 lines pass through each.

126. Theorem. Let A, B, E be given and a line AT, on AE take any point M and on AB any point L, let LM cut AT at K, then shall any line through K cut MB and LE at points lying on the same conic of the system of conics which touch AT at A, and pass through B and E.

Theorem. Let A, B, C be given and a line AT, let any line through C cut AT at T, and a line through A cut BC at M, and let AB cut TM at L, then any line through L cuts AM and CT at points lying on the same conic of the system of conics which touch AT at A, and pass through B and C.

Theorem. Any straight line cuts a system of conics each of which touches a given line at a given point, and passes through two other given points in an involution.

For if AT is the given tangent at A, and B, C the other two points, and if the given line cuts AB at K, and one of the conics at X, Y and XC cuts AT at Q, and YA cuts BC at R, QR are collinear with K. Hence rows described by Q on AT, R on BC are projective, and hence rows described by X, Y on the given line are projective: also X, Y are interchangeable, hence the two rows form an involution.

Corollary 1. The points where AT, BC cut the line are a pair of points of the involution. So also are the points where AB, AC cut the line.

Corollary 2. Two conics of the system (real and different, or coincident, or imaginary) touch any given straight line, their points of contact being the double points of the involution.

Corollary 3. If a common tangent to two conics of the system touches these two conics at P, Q it cuts any other conic of the system in points harmonically conjugate with respect to P, Q.

127. Problem. Given two points and two lines to construct a conic to pass through the two points, touch one line at a given point, and also touch the other line.

To construct a conic to touch TA at A, pass through B, C and touch TD.

Let BC cut TD at U, and AB, AC cut TD at K, L.

Take any point M on TD and draw a conic through A, B, M, C touching AT at A (§ 124); let it cut the line again at N.

Find X, Y the double points of the involution determined by T, U; K, L; M, N.

Then either of the conics touching AT at A and passing through B, C, X or B, C, Y satisfies the required conditions.

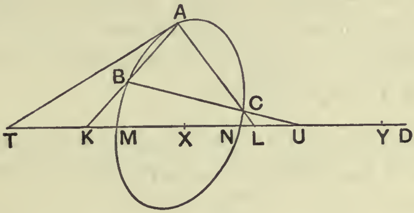


Fig. 98.

128. Problem. Given four tangents to a conic, and the point of contact of one of them, to construct the conic.

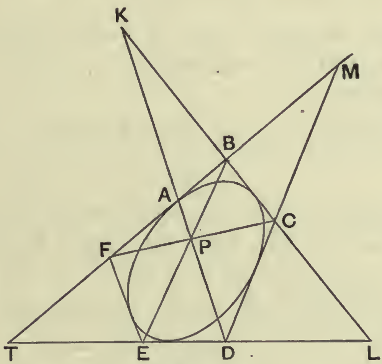


Fig. 99.

To construct a conic to touch TB at A, and to touch BC, CD, DT.

Join AD, on it take any point P, let BP, CP cut TD, TA at E, F respectively. Then as P moves along AD, the line EF will envelop the required conic.

Let BC cut AD at K and TD at L; and DC cut TA at M.

Then, by projection from C, {ABMF} = {AKDP} and, by projection from B, {AKDP} = {TLDE}, therefore {ABMF} = {TLDE}, hence BL, MD, FE describe projective rows on TA, TD with A on TA corresponding to D on TD. Hence they all touch the same conic touching TA at A, and TD.

129. Theorem. If a conic touches five lines BC, CD, DE, EF, FB and A is the point of contact with FB, then AD, BE, CF are concurrent.

For {TEDL} = {AFMB}, where FB cuts DE at T, DC at M, and BC cuts ED at L, therefore pencil B {TEDL} = C {AFMB}, but BL is the same line as CB, hence these pencils are in perspective, *viz.* the intersections of BT, CA; BE, CF; BD, CM are collinear, hence A, D are collinear with the intersection P of BE, CF.

Note. This theorem might be deduced from Brianchon's Theorem, by supposing two of the sides to coincide, but the direct proof is preferable.

Exercise. Find the number of pentagon points of five tangents to a conic.

130. Theorem. If a system of conics touch a given line at a given point and also touch two other lines, the tangents from a given point form an involution.

Let the conics touch AB, AC, BC, touching BC at K, and P be the given point.

If tangents from P to one of the conics cut AB, AC at M, L respectively; then, by the previous theorem, BL, CM intersect at a point O on PK.

Then the pencil described by PL equals the pencil described by BL, and the pencil described by PM equals the pencil described by CM, but the pencils of BL at B and CM at C each equal the row of O on PK, hence PL, PM describe projective pencils at P.

Also PM, PL are interchangeable; hence PL, PM form a pair of corresponding rays of an involution at P.

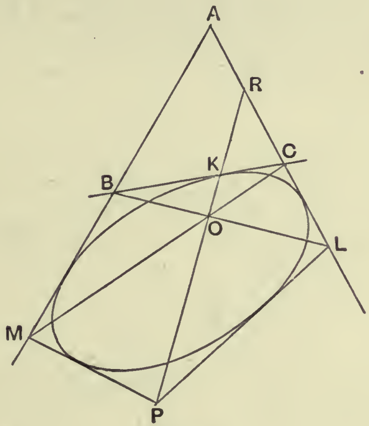


Fig. 100.

Corollary 1. If PK cuts AC at R, then when O is very near R, L is very near R and M is very near A, so that PK, PA are two corresponding rays of the involution.

Corollary 2. By taking O at K we see that PB, PC are corresponding rays of the involution.

Corollary 3. Two conics of the system pass through any point, their tangents at that point being respectively the double rays of the involution formed there by tangents to the various conics of the system.

Corollary 4. If two conics of the system pass through a given point P, and touch PX, PY respectively; then tangents from P to any other conic of the system are harmonically conjugate with respect to PX and PY.

Problem. Construct a conic to touch a given line at a given point, to touch two other given lines and to pass through a given point.

131. Theorem. If a system of conics touch four lines, KF, FB, BC, CK, the pencil described by the tangent from a fixed point E on KF is projective with the row described on FB by the point of contact of the conic.

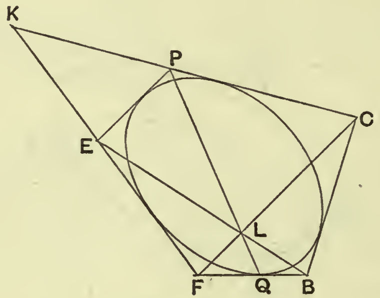


Fig. 101.

Let EB, FC intersect at L.

Then if we take one of the conics which touches FB at Q, and whose tangent from E cuts CK at P, PQ always passes through L.

Hence the pencil described by EP

= the row described by P on CK

= the row described by Q on BF. Q.E.D.

Problem. Given five tangents to a conic, find their points of contact. Hence construct the conic as a point locus.

QUADRILATERALS.

132. Problem. Given two tangents and their points of contact, also a third point, construct the conic.

Let TA, TB be tangents at A, B and let C be another point on the conic. Join BC, cutting AT at K; AC, cutting BT at L; and let any line through T cut BC and AC at Q, R respectively; then AQ, BR will intersect at a point D on the conic.

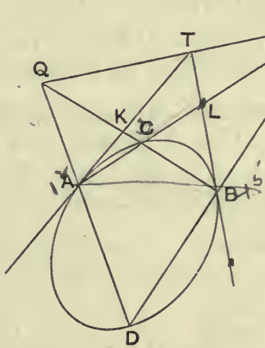


Fig. 102.

For

$$A\{TBCD\} = \{KBCQ\} = \{ALCR\}$$

(by projection from T)

$$= B\{ATCD\},$$

hence D lies on the conic determined by AT, AB, AC at A, and corresponding rays BA, BT, BC at B. Also, as QR turns round T, D describes the conic.

Theorem. If A, B, C, D are points on a conic, the intersection of tangents at A, B is collinear with the intersections of AC, BD and AD, BC.

Let AC, BD intersect at R; AD, BC at Q and tangents from A, B at T; also let BC cut AT at K and AC cut BT at L.

Then $A\{TBCD\} = B\{ATCD\}$, hence $\{KBCQ\} = \{ALCR\}$, two rows with their intersection C corresponding to itself, and hence KA, LM, QR are collinear, *viz.* T lies on QR.

133. Theorem. If A, B, C, D lie on a conic, the intersections of the tangent at A with BC, and the tangent at B with AD are collinear with the intersection of AB, CD.

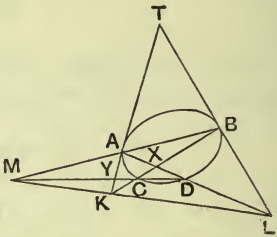


Fig. 103.

Let K, L, M be the respective intersections; also let AD meet BC at X, AT meet DC at Y.

Then A {TBCD} = B {ATCD},

hence {YMCD} = {ALXD}, ∴ YA, ML, XC are concurrent, *i.e.* K lies on LM.

Corollary. Through M also passes the join of the intersections of the C tangent with DA, and the D tangent with CB: also of the C tangent with DB and the D tangent with CA: also of the A tangent with BD and the B tangent with AC.

Corollary. This affords another solution of the problem above, to construct a conic to touch TA at A, TB at B and pass through C.

Let BC cut AT at K, through K draw *any* straight line to cut BT at L and AB at M; the intersection of CM, AL determines a point D of the conic.

Note. These theorems might have been deduced from Pascal's theorem by making two pairs of vertices coincide.

134. Problem. Given four points A, B, C, D on a conic and the tangent AT at one of them, to construct the tangent at another of the points.

First Construction. Let AC, BD meet at K, AD, BC at L. Join KL to cut AT at M. Then BM is the required tangent at B.

Second Construction. Let AB, CD intersect at K, AT, BC at L. Join KL to cut AD at M. Then BM is the required tangent at B.

135. Theorem. A system of conics touching two lines at given points trace an involution on any straight line which cuts them.

Let the system of conics touch TA at A, TB at B; and let XY be the fixed line cutting AB at M, and any one of the conics at C, D.

Let BC meet TA at K, AD meet TB at L.

Then K, L, M are collinear (§ 133), ∴ K, L describe projective rows on TA, TB.

But if we project the K row from centre B on XY we get the row described by C; similarly L projects into D, from centre A.

Hence C, D describe projective rows on X, Y. Also they are interchangeable; for when C is transferred to D, BD cuts TA at a point K′, and K′M cuts BT at L′, and AL′ cuts XY at C. Hence C, D are pairs of an involution on X, Y.

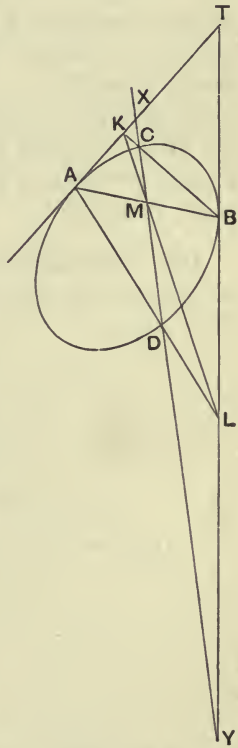


Fig. 104.

Corollary 1. When C is near X, K is near X, hence L is near Y, and therefore D is near Y, so that ultimately X, Y are a pair of the involution.

Corollary 2. When C is near M, K is near A, hence L is near B, and therefore D is near M: so that ultimately M is a double point of the involution.

Corollary 3. The other double point is the harmonic conjugate M′ of M with respect to X, Y.

Hence if a fixed line cuts AB at M, and two other lines TA, TB at X, Y, and M′ is the harmonic conjugate of M with respect to X, Y; then any conic touching TA, TB at A and B cuts the line in points harmonically conjugate to M and M′.

Problem. To describe a conic to touch two given lines (TA, TB) at given points (A, B) and to touch another given line. Find the double points M, M′ as in the previous corollaries. No conic can pass through A, B, M because they are collinear; hence there is only one solution, *viz.* the conic touching TA, TB at A, B and passing through M′.

Cf. § 77 where it was proved that, if a conic touches two sides TX, TY of a triangle TXY at A, B, it touches the third side XY at a point M′ which is the harmonic conjugate of the point M, at which AB cuts XY.

136. Problem. To describe conics to touch two given lines and pass through two given points.

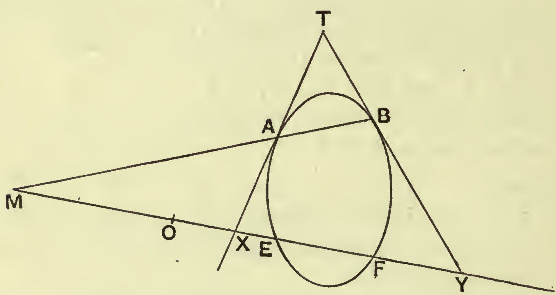


Fig. 105.

Let E, F be the given points and TX, TY the given lines cutting EF at X, Y.

Find the centre O of the involution determined on EF by the pairs of points E, F and X, Y; and hence the double point M (such that $OM^2 = OE . OF = OX . OY$).

Through M draw any line to cut TX, TY at A, B. Then the conic which touches TX, TY at A, B and passes through E will also pass through F and will be a conic of the system.

An infinite system of conics can be obtained by turning the line MA round the fixed point M.

A second system may be got by taking instead of M the other double point of the involution.

137. Problem. To construct a conic to touch two given straight lines and pass through three given points. Shew that there are four such conics.

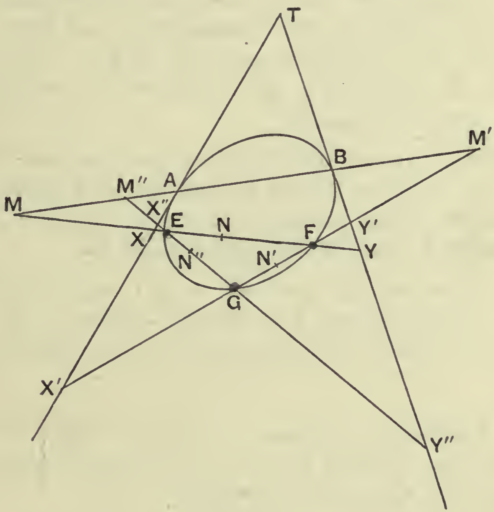


Fig. 106.

Let E, F, G be the three points and TX, TY the given lines cutting EF at X, Y; FG at X′, Y′; and EG at X″, Y″.

Find M a double point of the involution EF, XY and M′ a double point of FG, X′Y′.

Join MM′ cutting TX, TY in A, B respectively.

Then because AB passes through M, the conic which touches TA, TB at A, B and passes through F also passes through E. And because AB passes through M′ this conic passes through G.

If M, N are the two double points on EF and M′, N′ on FG, there will be four conics given by the lines MM′, MN′, NM′, NN′.

Hence there are four solutions to the problem.

N.B. Two belong to each of the two systems of conics which touch TA, TB and pass through E, F.

Since the MM′ conic passes through E and G, the chord AB must pass through M″ one of the two double points of the involution, NN′ will also pass through M″; and MN′, M′N will pass through the other double point N″ on EG. Thus the three pairs of double points will be the vertices of a four-side (of which EFG is the diagonal triangle).

Corollary. If MN, M′N′, M″N″ are the three pairs of vertices of a four-side and any straight line cut MN, M′N′, M″N″ at X, X′, X″, then the harmonic conjugates of X with respect to MN, of X′ to M′N′, and of X″ to M″N″ are on a straight line.

Also if these two straight lines cut any one of the four sides at A, B the conic through A, B and the vertices of the diagonal triangle EFG of the four-side touches those two straight lines at A and B.

138. Problem. Construct a conic to touch a line at a given point, pass through two other points and touch one other line. Shew that there are two solutions.

[Let TX, TY be the given lines, A the given point on TX, and E, F be the other two given points. Let E, F cut TX, TY at X, Y. Find a double point M on EF, XY and join MA cutting TY at B.]

139. Problem. Given three tangents to a conic and the points of contact of two of them, to construct the conic.

To construct a conic to touch TK at A, TL at B, and also touch KL.

Take any point R on AB, and join KR, LR cutting TL, TK at M, N respectively. Then MN is a tangent, and as R moves along AB, MN envelopes the required conic.

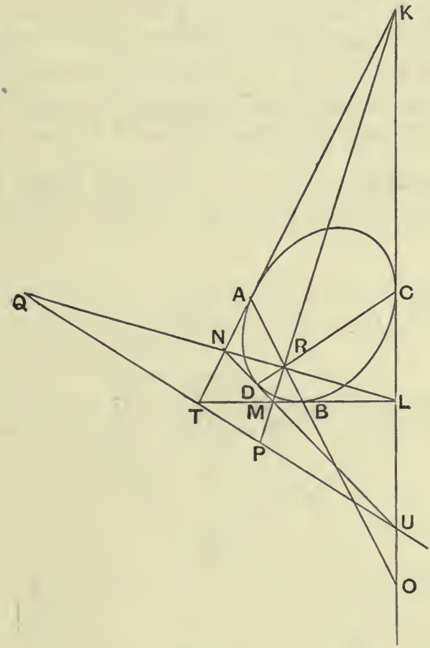


Fig. 107.

For, if AB cuts KL at O, $\{TAKN\} = \{BAOR\}$, by projection from L, $\{BTLM\} = \{BAOR\}$, by projection from K.

$\therefore$ $\{TAKN\} = \{BTLM\}$,

$\therefore$ MN is a tangent to the conic which touches TA at A, etc.

Theorem. If a conic touches four lines KL, LM, MN, NK, the join of the points of contact of LM and KN is concurrent with KM, LN.

Let the points of contact be A, B.

Then $\{TAKN\} = \{BTLM\} = \{TBML\}$ (by a double interchange), $\therefore$ AB, KM and LN are concurrent.

Corollary 1. The join of the points of contact C, D of KL, MN also passes through the intersection of KM and LN.

Corollary 2. If KN, LM meet at T, and KL, MN at U, then AC, BD both pass through the intersection Q of LN, TU; also BC, AD both pass through the intersection P of TU and KM.

140. Theorem. If a, b, c, d are four tangents to a conic of which a touches the conic at A, b at B, the joins of A to

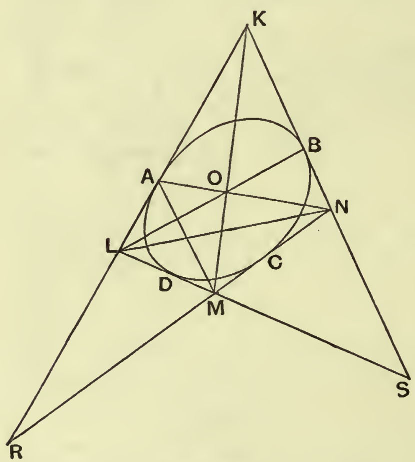


Fig. 108.

intersection bc, and B to intersection ad, are concurrent with the join of intersections ab and cd.

Let KL, KN touch the conic at A, B respectively; and let LM, MN be the other two tangents, which cut KB, KA respectively at S, R.

Then $\{LKAR\} = \{SBKN\}$, hence the pencil M $\{LKAR\} = L\,\{SBKN\}$; but ML coincides with LS, hence the intersections of MK, LB; MA, LK; and MR, LN are collinear; *viz.* MK, LB intersect on AN.

Q. E. D.

Corollary 1. This leads to another solution of the problem, given two tangents (KL, KS) and their points of contact (A, B), also a third tangent (LS), to construct the conic as an envelope.

Take any point M on LS, join KM to cut LB at O, and let AO cut KS at N, then MN is a tangent to the conic; and as M moves along LS, MN envelopes the conic.

Corollary 2. KM also passes through the intersection of AS, BR: also of CL, DN, and CS, DR, where C, D are the points of contact of NM and ML.

Note. These theorems on circumscribed quadrilaterals might have been deduced from Brianchon's Theorem, by making two pairs of tangents coincide.

141. Problem. Given four tangents to a conic and the point of contact of one of them, to find the points of contact of the other three.

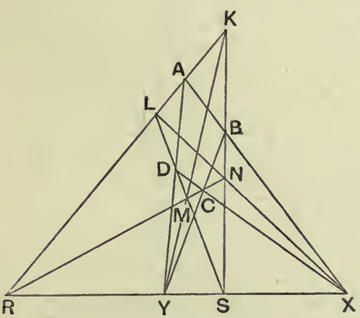


Fig. 109.

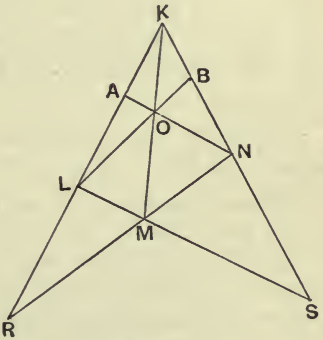


Fig. 110.

Let KL, LM, MN, NK be the four tangents (and let KL, MN meet at R; KN, LM at S).

Given that KL touches the conic at A, to find where KN touches the conic.

First Construction. Let LN, RS meet at X, AX will cut KN at B, the required point of contact, fig. 109.

If KM cuts RS at Y, then AY will cut LM at D, the point of contact of LM; and DX or BY will cut MN at C, the point of contact of MN.

Second Construction. Let AN cut KM at O, then LO will cut KN at its point of contact B, fig. 110.

If AM cuts LN at P, then KP will cut LM at its point of contact D; also if BM cuts LN at Q, then KQ will cut MN at its point of contact C.

142. Theorem. A system of conics touches two given lines at given points, prove that the tangents drawn from any other given point form an involution.

Let TA, TB be tangents at A, B: and P a given point. Join PT. Let the tangents from P to any conic of the system cut TA, TB at K, L.

Then BK, AL intersect at a point O on PT.

Hence the row of K on TA = the row of O on TP = row of L on TB, and hence PK, PL describe projective pencils; also PK, PL are interchangeable (as shewn in the dotted lines of the figure), and hence the double pencil at P forms an involution.

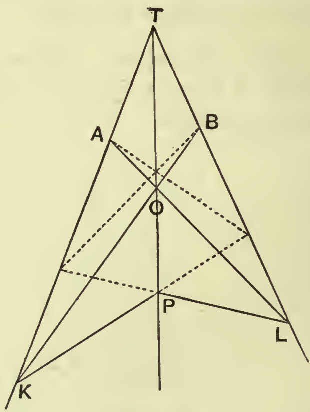


Fig. 111.

Corollary 1. PA, PB are a pair of rays of the involution.

Corollary 2. PT is a double ray of the involution.

Corollary 3. Hence the other double ray is the harmonic conjugate of PT, with respect to PA, PB.

Corollary 4. If PT cuts AB at X, and $\{ABXY\} = -1$, then the tangents from P to any conic which touches TA, TB at A, B are harmonically conjugate with respect to PX and PY.

This result has been established previously.

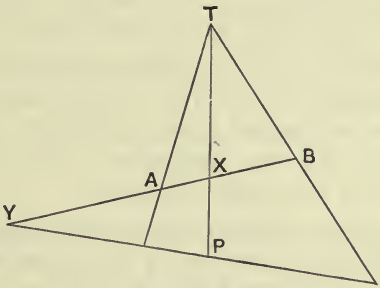


Fig. 112.

143. Problem. To describe a conic to touch TA at A and TB at B, and to pass through a point P.

Join PT to cut AB at X, find Y such that $\{ABXY\} = -1$; then PX, PY being the double rays of the involution the conic must touch PX or PY : but it cannot touch three lines TA, TB, TP ; hence the only solution is the conic which touches PY.

We have now two tangents and their points of contact, and one other tangent, and hence the conic can be uniquely described.

144. Problem. To describe conics to touch two given lines PQ, PR and pass through two given points A, B.

Find m a double ray of the pencil in involution determined by PA, PB ; PQ, PR, and on it take any point T. Join TA, TB, then the conic which touches TA at A, TB at B and touches PQ also touches PR. By varying the position of T on m we get a system of such conics; and we get a second system if, instead of m, we take n the other double ray of the involution.

Problem. To construct a conic to touch three given lines PQ, QR, RP and pass through two given points A, B. Shew that there are four solutions.

Find m a double ray of the involution determined by PA, PB and PQ, PR; and m' a double ray of the involution QA, QB and QP, QR, let them intersect at T. Then the conic which touches TA, TB at A and B, and also touches PQ, is the conic required.

If the other double ray n' of the second involution cuts m at U, then by taking U instead of T we get another solution, and two others may be obtained by taking V, W the points at which m' and n' cut n the other double ray of the first involution. These are the only four possible solutions.

Since the T conic touches RP and RQ and passes through A, B the line RT must be a double ray of the involution given by RA, RB and RP, RQ. Similarly W lies on the same double ray at R: and U, V on the other double ray at R.

Thus the three pairs of double rays at P, Q, R will be the sides of a four-point T, U, V, W (of which PQR is the diagonal triangle).

Corollary. If P, Q, R the diagonal points of a four-point T, U, V, W (P on TU, Q on TV and R on TW) be joined to any point A, then the harmonic conjugates of PA with respect to PT, PV; QA to QT, QU and RA to RT, RV also meet at a point B.

Also the conic which touches TA, TB and the three sides of the diagonal triangle PQR passes through A and B; and similarly if for T we substitute U, V, or W.

Problem. Construct a conic to touch a given line at a given point, touch two other given lines and pass through one other given point. Shew that there are two solutions.

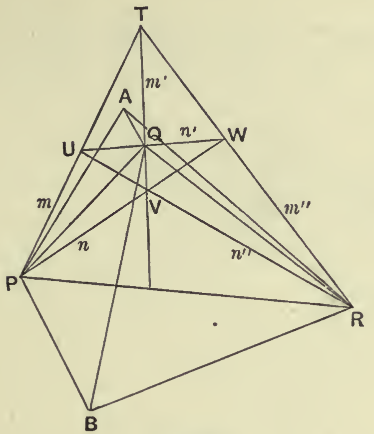


Fig. 113.

Let a be the given line and A its point of contact; let e, f be the other two given tangents meeting at P; and let B be the other given point. Join PA, PB and take a double ray m of the involution of rays through P, of which PA, PB and e, f are two pairs. If m meets a at T, the conic which touches TA, TB at A, B and touches e, also touches f, and is one solution.

145. Theorem. *The diagonal points of four points on a conic are the vertices of the diagonal triangle of the four tangents at those points.*

Let tangents at A, B, C, D be KM, ML, KN, NL and let KN, LM meet at P and KM, NL at O.

Let OP, MN meet at U; KL, OP at V; MN, KL at W, so that UVW is the diagonal triangle of the four tangents.

Then it has been proved (§ 139) that AC, BD both pass through U; AB, CD through V; AD, BC through W, hence U, V, W are the diagonal triad of A, B, C, D.

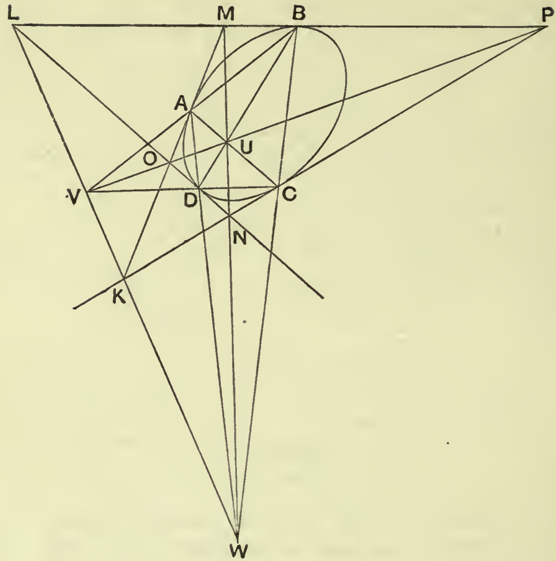


Fig. 114.

146. Corollary. If the points of contact of one pair of tangents are collinear with the intersection of the other pair; then the points of contact of the second pair are collinear with the intersection of the first pair.

If BD passes through K, then the tangents at B, D intersect on AC (§ 77, Cor. 5), *i.e.* A, C are collinear with L. In this case A, C are harmonic to B, D.

147. Theorem. If KM, KN touch a conic at A, D, and LM, LN touch it at C, B respectively, then the points R, R′ at which MB, NC and NA, MD intersect on KL are harmonic conjugates with respect to the points X, Y at which KL is cut by MN, and the other diagonal OP of KLMN.

Let MN cut BD at E.

Then, by projection from M, $\{XYRR'\} = \{EYBD\}$ and, by projection from N, $\{EYBD\} = \{XYLK\}$, but $\{XYLK\} = -1$ by the theory of four-sides.

Hence $\{XYRR'\} = -1$. Q. E. D.

Does CD pass through the intersection of LA and MB? Since the positions of B, L, C, M, A are independent of the position of D and its tangent KN, CD does not generally pass through the intersection of LA and MB.

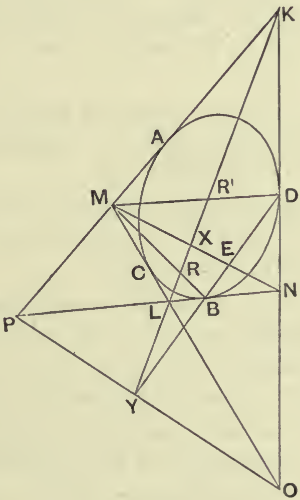


Fig. 115.

148. Theorem. K, L, M, N, O, P, A, B, C, D being defined as in the previous theorem, if CD passes through the intersection of LA and MB, it also passes through P; also the same line contains the intersections of NA and KB; also AB passes through O and contains the intersections of KC, MD and NC, LD.

For it has been proved (§ 115) that MB, LA intersect on PC, hence PC, CD are collinear; and NA, KB by the same proposition intersect on PD.

Again it has been proved (§ 146) that, if CD passes through P, AB passes through O, and the intersection of MD, KC lies on OA, and of LD, NC on OB.

149. Theorem. If a system of conics touch four lines KMP, LNP, KNO, LMO, that conic of the system which touches KM at the harmonic conjugate A of P with respect to K, M, also touches LN at the harmonic conjugate B of P with respect to L, N, also touches LM, NK at the harmonic conjugates C, D of O with respect to L, M and N, K; also AB passes through O and CD through P.

For if a conic touches the sides of a triangle OKM at A, C, D the line CD cuts KM at the harmonic conjugate of A with respect to K, M: hence CD passes through P.

But if CD passes through P, then AB passes through O (§ 146), also A, B, C, D are harmonic conjugates of P, O with respect to K, M; L, N; L, M; K, N respectively.

Definition. The conic which touches KMP, LNP, KNO, LMO at points harmonically conjugate to P, O with respect to K, M; L, N; K, N; L, M respectively may be called the **harmonic conic** of the four-side, with respect to the diagonal OP.

There are three harmonic conics of a four-side, *viz.* those which are harmonic with respect to the three diagonals OP, MN, KL.

Corollary 1. The points A, B, C, D are harmonically conjugate on the conic.

Corollary 2. If KMP, LNP, KNO, LMO are four given lines, XYZ the diagonal triangle, then XP, XO cut the sides in four points (A, A, A, A in the figure) where the first harmonic conic touches the four lines; similarly YM, YN determine the points B, B, B, B where the second harmonic conic touches the figure; and ZK, ZL determine the points of contact C, C, C, C of the third harmonic conic. X, Y, Z form the diagonal triad of each of the three four-points AAAA, BBBB, CCCC. Through X pass two AA lines, two BB lines, two CC lines, and these six lines form an involution whose double rays are XY and XZ. Similarly at Y and Z.

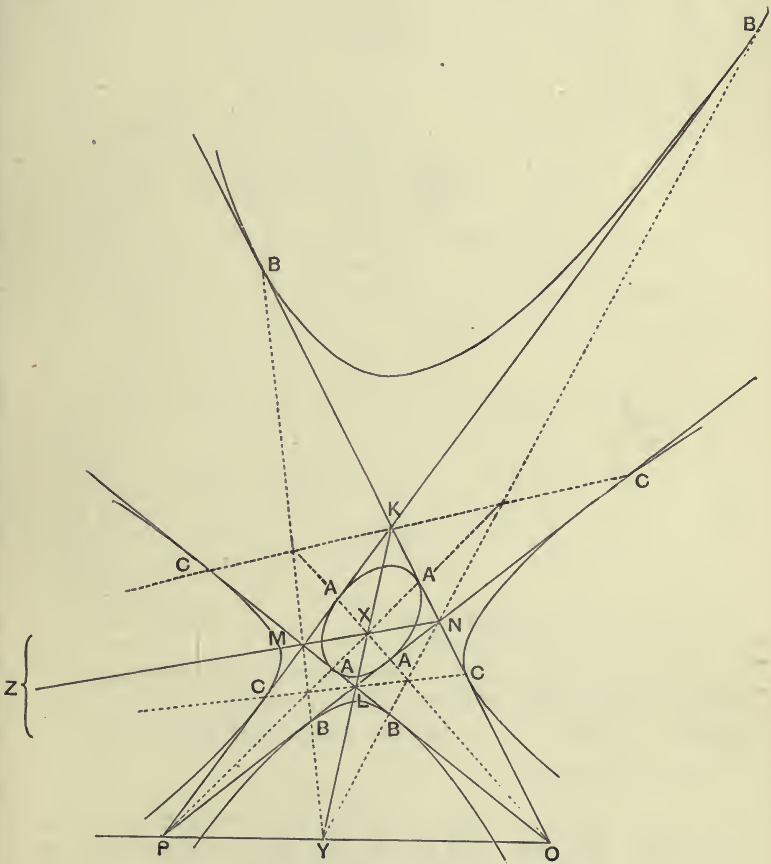


Fig. 116.

150. Theorem. If AB, CD intersect at X; AD, BC at Y; AC, BD at Z and K, L are the points where the C tangent and the B tangent cut AD: then XK, XL are harmonically conjugate with respect to XY, XZ.

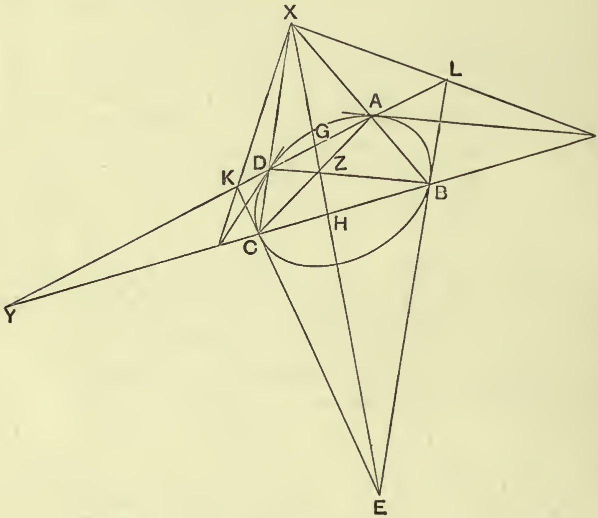


Fig. 117.

Let XZ cut AD, BC at G, H.

Then X {YZKL} = {YGKL}, but we have proved that tangents at B, C intersect on XZ, at E say, hence by projection from E we get {YGKL} = {YHCB}; and, by the theory of four-points, {YHCB} = − 1.

Hence X {YZKL} = − 1.

Note. XK also passes through the intersection of the D tangent with CB; and XL through the intersection of the A tangent with BC.

151. Theorem. If a system of conics pass through four points A, B, C, D that conic of the system which touches at A the harmonic conjugate of AB with respect to AC, AD, also touches at B the harmonic conjugate of BA with respect to BC and BD; and also touches at C, D the harmonic conjugates of CD for CA, CB and DA, DB.

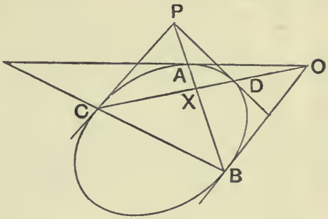


Fig. 118.

Let the tangent at A cut CD at O, also let AB, CD intersect at X.

Then, by hypothesis, AO, AX are harmonic conjugates with respect to AC, AD: hence $\{CDXO\} = -1$. Hence, by the theory of inscribed triangles, XA passes through P the intersection of tangents at C, D, *i.e.* tangents at C, D intersect on AB. Therefore tangents at A, B intersect on CD.

Again, $\{CDXO\} = -1$, hence BA, BO are harmonically conjugate to BC, BD.

Again, because tangents from O on CD touch the conic at A, B it follows that $\{ABXP\} = -1$, so that DP is the harmonic conjugate of DC with respect to DA, DB, and CP of CD with respect to CA and CB.

Definition. If X is a diagonal point of A, B, C, D, that conic through A, B, C, D which touches, at each, the harmonic conjugate of the line joining it to X with respect to the other two sides through it, may be called the **harmonic conic** of the four points with respect to X.

There are three harmonic conics of a four-point.

Corollary 1. The points A, B, C, D (and the tangents at them) are harmonically conjugate with respect to the conic.

Corollary 2. Let AB, CD intersect at X; AC, BD at Y; AD, BC at Z.

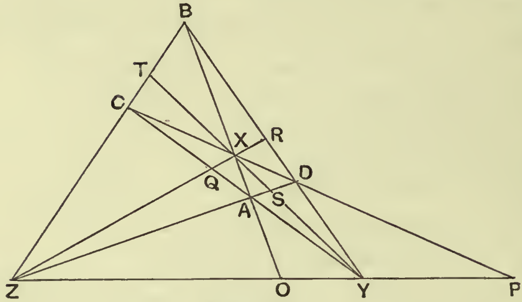


Fig. 119.

If YZ be cut by AB, CD at O, P then OC, OD, PA, PB are tangents to the first harmonic conic.

If XZ is cut by AC, BD at Q, R then QB, QD, RA, RC are tangents to the second harmonic conic.

If XY is cut by AD, BC at S, T then SB, SC, TA, TD are tangents to the third harmonic conic.

Corollary 3. Since {CAYQ} is harmonic, BQ cuts YZ at the harmonic conjugate of Y with respect to O, Z; similarly AR cuts YZ at the same point, so that BQ, RA intersect on YZ. So also do QD, RC. Similarly RA, QD and RC, CB on XY.

Hence XYZ is the diagonal triangle of the four tangents QB, QD, RA, RC.

Similarly it is the diagonal triangle of each of the other two sets of four tangents.

Corollary 4. Each set of four tangents intersect in pairs on YZ at points which are harmonically conjugate with respect to Y, Z.

EXAMPLES. VII.

1. If a conic touches three lines and two of them at given points, find where it touches the third.

2. Construct a conic, having given (*a*) five points, (*b*) four points and a tangent, (*c*) three points and two tangents, (*d*) two points and three tangents, (*e*) one point and four tangents, (*f*) five tangents.

How many solutions are there in each case?

3. Construct a conic having given a tangent and its point of contact and three other points or tangents.

4. Construct a conic having given two tangents and their points of contact and either (*a*) one other point or (*b*) one other tangent.

How many solutions are there in each case?

5. Shew that if A, B, C, D, E lie on one branch of a hyperbola and F lies on the other branch, 32 types of hexagons can be formed by joining the six points.

6. The six intersections of non-corresponding sides of two triangles in homology lie on a conic.

7. A, A′ and B, B′ are corresponding points of two projective rows on a conic, and the projective axis of the rows cuts the conic at X, Y, so that AB′, A′B intersect at P on XY; also A′Y, B′X meet at O. Prove that the intersection K of AA′ with the tangent at X is collinear with O, P.

Also prove that the intersection L′ of A′B′ with the tangent at Y lies on OP.

8. A, A′ and B, B′ are corresponding points of two projective rows on a conic and the projective axis cuts the conic at X, Y. If the tangents at X, Y are met by AA′ at K, K′ and by BB′ at L, L′, prove that KL′, K′L intersect on XY.

9. The joins of corresponding points of two projective rows on a conic envelope a conic, which has double contact with the given conic.

10. A conic touches BC, CA, AB at P, Q, R and QR, RP, PQ meet BC, CA, AB at K, L, M. Prove that K, L, M are collinear.

11. A conic touches BC, CA, AB at points P, Q, R such that AP, BQ, CR are parallel, and QR, RP, PQ meet BC, CA, AB respectively at K, L, M. Prove that KLM passes through the centre of the conic. Also prove that for conics satisfying this condition KLM envelopes a conic inscribed in the triangle ABC, and find where this conic touches the sides of ABC.

12. The lines which touch a conic at A, B, C meet the chords BC, CA, AB at K, L, M respectively; and the joins of A, B, C to any other point P of the conic cut BC, CA, AB at X, Y, Z respectively. Prove that KLM is a tangent to a conic which touches BC at X, CA at Y and AB at Z.

13. An ellipse whose centre is O is inscribed in a triangle ABC and the diameters conjugate to OA, OB, OC meet any tangent in D, E, F respectively; prove that AD, BE, CF are concurrent.

14. A circle touches the sides of an isosceles triangle, prove that the mid-points of the three pairs of tangents lie on a conic.

15. A parabola passes through A, B, C, D; AC meets the diameter through B at K, and BD meets the diameter through A at L. Prove that KL is parallel to CD.

16. Conics circumscribe a triangle ABC and touch a given line, prove that the polars of a given point P with respect to these conics envelope a conic. Also prove that, if AP cuts BC at K, this conic passes through the harmonic conjugate of P with respect to A, K.

17. Two conics through four given points touch a given line at K, L respectively; prove that the conic through the four points and the mid-point of KL has an asymptote parallel to KL.

18. Two conics through A, B, C and D have a common tangent touching them at K, L respectively; and KL cuts AC, BD at E, F. Prove that K, L are harmonic conjugates with respect to E, F.

Deduce a construction to find K, L for a given line.

Given four tangents to a parabola draw through any point on one of them the other tangent to the parabola from that point.

19. Four lines are given, find the points of contact of a parabola which touches them.

20. If AB is an asymptote, and BC, CD, DA tangents to a hyperbola, construct the other tangent through any point P on AD.

21. Prove that the pair of points at which a line cuts a conic through four points belong to the involution determined on the line by the six joins of the four points.

22. If the chords AB, BC, CD of a given conic respectively pass through three given collinear points, then will DA always pass through a fixed point collinear with the other three.

23. Given five points on a conic, construct the tangent at any one of them.

24. Any straight line cuts a conic at P, Q, two tangents at K, L and their chord of contact at X, prove that X has the same harmonic conjugate for P, Q that it has for K, L.

Deduce that the middle point of a chord of a hyperbola is also the middle point of the line intercepted on it by the asymptotes.

CHAPTER VIII

THE PARABOLA. CENTRAL CONICS

152. Definition. A **parabola** is the envelope of the join of corresponding points of two similar rows not in perspective. [Cf. § 80, Cor.]

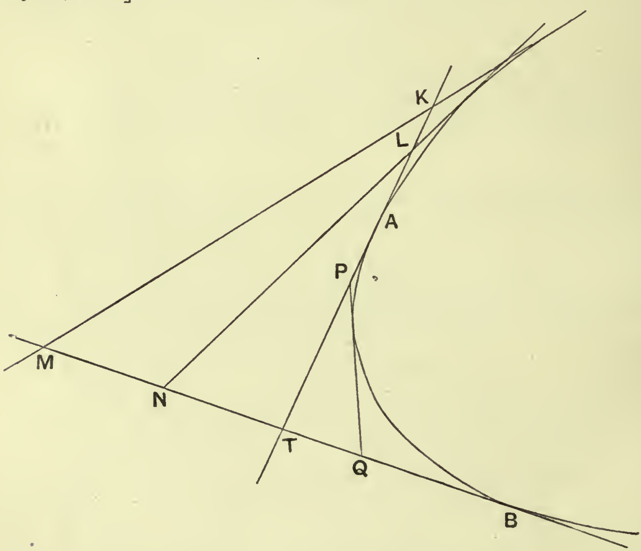


Fig. 120.

If on two lines we take segments KL and MN, and divide them similarly at P, Q, so that KP : PL = MQ : QN, then PQ will move in such a way that it is always tangent to a curve, and the point at which it touches the curve will be the ultimate position on PQ of its intersection by a near tangent P′Q′ when P′ is brought very near to P, and therefore Q′ to Q.

Let KL, MN intersect at T. Then, if KT : TL = MT : TN, the rows are in perspective, and KM, LN, PQ are all parallel: but if not there will be a point A in KL, such that KA : AL = MT : TN, and a point B in MN such that MB : BN = KT : TL. Also, when Q is very near to T, P is very near to A, and if QP is made to coincide with TA, their intersection is ultimately at A. Hence the line TA touches the curve at A; similarly TB is the tangent at B.

The points T, A, B completely determine the curve, for

$$AP : PT = TQ : QB.$$

153. Note. The property of a parabola thus taken as a definition can be deduced from the focal definition in the following manner, by those who prefer to commence with that definition, and as the converse is proved in § 158, the two classes of curves are proved to be identical.

Theorem. If a point P moves so that its distance from a fixed point is equal to its distance from a fixed line then any tangent to the locus of P will describe similar rows on any two given tangents to the locus.

I. Let S be the fixed point and AB the fixed line.

Take two points P, Q such that SP = PM, a perpendicular drawn to AB; and SQ = QN, perpendicular to AB.

The circles with P as centre, PS as radius, and centre Q, radius QS, will touch the line AB at M and N respectively.

Let the common chord SH cut AB at K and draw a line from K perpendicular to AB to cut PQ at V.

Then SH is a common chord, and is therefore perpendicular to PQ, also

$$KM^2 = \text{rect. } KH \,.\, KS = KN^2,$$

$$\therefore MK = KN.$$

But KV is parallel to MP and QN, hence PV = VQ.

Hence a line from S perpendicular to chord PQ meets the directrix AB at a point K, such that a line from K at right angles to the directrix bisects PQ.

If now we take any other chord parallel to PQ; the line perpendicular to it through S will still be SK.

Hence the middle points of any set of parallel chords lie on one straight line perpendicular to the directrix.

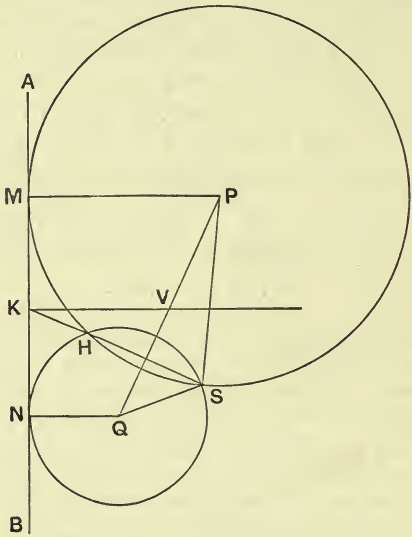


Fig. 121.

II. Let P′Q′ be another chord parallel to PQ, then PP′, QQ′ will meet on the line which passes through their middle points [*viz.* at a point Z such that ZV′ : ZV = P′V′ : PV].

But if P′Q′ is moved into coincidence with PQ, then PP′, QQ′ will simultaneously become the tangents at P and Q. Hence the tangents at P, Q meet on the diameter which bisects PQ.

III. Now take the two tangents TP, TQ and join T to V, the middle point of PQ. Let any other tangent touch the curve at R and cut TP, TQ at A and B. Draw AD, BE, RF parallel to TV to meet PQ at D, E, F.

Then AD, being parallel to TV, bisects PR, the chord of contact of tangents AP, AR, and is parallel to RF, therefore PD = DF. Similarly QE = EF.

Hence DE = $\frac{1}{2}$PQ, and therefore PD = VE and DV = EQ.

$$\therefore \text{PA} : \text{AT} = \text{PD} : \text{DV} = \text{VE} : \text{EQ} = \text{TB} : \text{BQ},$$

i.e. A and B divide PT, TQ similarly. Q. E. D.

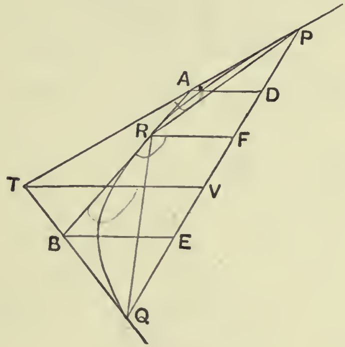


Fig. 122.

154. Theorem. *A tangent to a parabola describes on any given tangent a row similar to the rows which it describes on the original pair of lines.*

Let the parabola be determined by TA and TB, and take any line PQ, such that AP : PT = TQ : QB.

Draw any other tangent P′Q′ to cut TA, TB, PQ at P′, Q′, X, and to cut a line through P parallel to TB at K (fig. 123).

Then PX : XQ = PK : Q′Q.

Now PK : TQ′ = P′P : P′T = Q′Q : Q′B, since the rows on AT, TB are similar.

$$\therefore \text{PX} : \text{XQ} = \text{TQ}' : \text{Q}'\text{B} = \text{AP}' : \text{P}'\text{T}.$$

Hence X divides PQ in the same ratio in which P′ divides AT and Q′ divides TB. Q. E. D.

Corollary. The point R at which PQ touches the curve is given by PR : RQ = AP : PT.

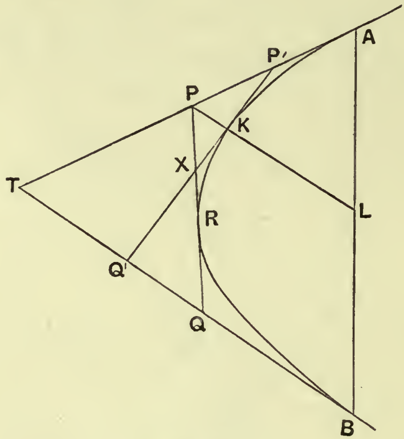


Fig. 123.

155. Theorem. The intercept PQ of any tangent by two given tangents TA, TB is divided harmonically by its point of contact R, and the point R′ at which it cuts AB.

Draw PL parallel to TB to meet AB.

$$PL : TB = AP : AT = TQ : TB. \quad \therefore\ PL = TQ.$$

Now $PR' : R'Q = -PL : QB = -TQ : QB = -PR : RQ$ (§ 154, Cor.).

$\therefore\ \{PQRR'\} = -1$, and the range is harmonic.

156. Theorem. The joins of two points A, B on a parabola to a variable point P on the parabola describe projective pencils at A, B.

Let the tangent at P cut the tangents TA, TB at K, L.

Join AP to cut TB at M, and BP cutting AT at N.

Then, if LK cuts AB at P′, the range {KLPP′} is harmonic; hence, projecting from A and B, we have two harmonic ranges {TLMB} and {KTNA}.

Hence TM : MB $= \frac{1}{2}$.TL : LB and AN : NT $= 2$.AK : KT.

$\therefore$ TM : MB $= \frac{1}{4}$.AN : NT. $\therefore$ {TBMM′} $=$ {ATNN′}, hence M and N describe projective rows on TB and AT, and therefore AP and BP describe projective pencils at A, B.

Also in these pencils AT, AB at A correspond to BA, BT at B.

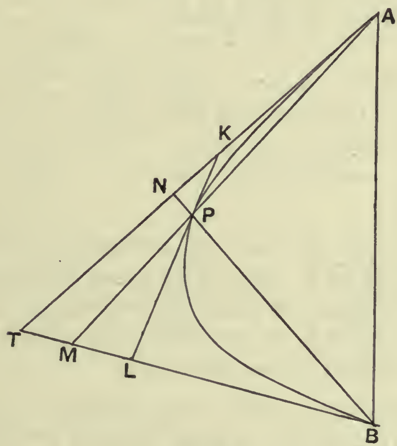


Fig. 124.

Corollary. When TM : MB $= -\frac{1}{2}$, AN : NT $= -2$, and AM, BN are both parallel to the line joining T to the mid-point of AB: hence P is at infinity. Also K, L are now at infinity, and the tangent KL is wholly at infinity.

157. Theorem. The line through the intersection of two points bisecting their chord of contact is constant in direction.

Let the tangent at P cut TA, TB at K, L, and draw KD, PC, LE parallel to the line TV which bisects AB at V.

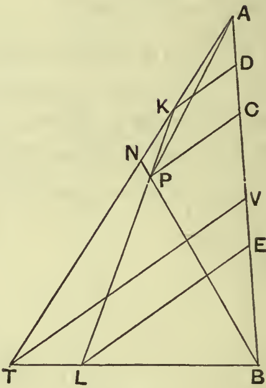


Fig. 125.

$$AK : KT = TL : LB.$$

$$\therefore\ AD : DV = VE : EB,$$

but $AV = VB$. $\therefore\ DE = \tfrac{1}{2} . AB = AV$.

But $KP : PL = AK : KT$.

$$\therefore\ DC : CE = AD : DV,$$

hence $AD = DC$ and $CE = EB$.

$\therefore$ KD parallel to TV bisects chord AP.

Similarly if Q is any other point on the curve the line through the intersection of tangents at P, Q and bisecting PQ is parallel to KD, and therefore to TV.

Corollary. $AC : CB = AD : CE = AD : DV = AK : KT = \tfrac{1}{2} . AN : NT$, where BP cuts AT at N. Hence the pencil described by BP at B is projective with the row described by C on AB, it is therefore projective with the row described by a line through P parallel to the constant direction on any transversal.

These parallel lines may be regarded as the pencil of lines joining successive positions of P to the point of the curve which is at infinity in the direction TV.

158. Theorem. The join of the intersection of two tangents to the mid-point of their chord of contact cuts the parabola at the point of contact of a tangent parallel to the chord, and is bisected there.

For the line KK′ joining the mid-points of TP and TP′ bisects TV at A; it is a tangent, since $PK : KT = TK' : K'P'$; and its point of contact is A, since $KA : AK' = PK : KT$.

Theorem. *The middle points of any set of parallel chords lie on a line parallel to a constant direction.*

For each lies on a line parallel to a constant direction through the point of contact A of the tangent which is parallel to the chords.

Definition. This line is called a **diameter.** The semichord (PV) is the **ordinate,** and AV is the **abscissa** of P with respect to the diameter through A.

The diameter which bisects that set of chords which are perpendicular to the direction of diameters is the **axis.**

Theorem. If PV, QW and AV, AW are the ordinates and abscissae of P and Q with respect to the diameter through A, then

$$PV^2 : QW^2 = AV : AW.$$

Draw QE parallel to AV, and join AQ to cut PV at D.

The row described on PP′ by transversals through P, P′, Q, A is $\{PP'EV\}$, and the pencil formed by joining A to the same points is AP, AP′, AD, AK.

$$\therefore \{PP'EV\} = -PD : DP'.$$

$$\therefore PE : EP' = -PD : DP',$$

i.e. $\{PP'DE\}$ is harmonic.

$$\therefore VD \,.\, VE = VP^2.$$

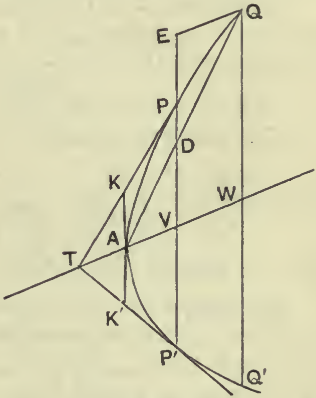


Fig. 126.

Hence $QW^2 : PV^2 = QW^2 : EV \,.\, DV = QW : DV \;(\because QW = EV)$
$= AW : AV$ (by similar triangles).

Corollary. If L is a point whose ordinate LR is double the abscissa AR, then $PV^2 = 4AR \,.\, AV$, and $QW^2 = 4AR \,.\, AW$.

Also the tangents at L and L′ meet at Z on RA such that $RZ = 2 \,.\, RA = RK = RK'$, hence LZL′ is a right angle.

159. If S is a point on the axis such that AS is half the ordinate SL perpendicular to the axis, and PN, AN are the ordinate and abscissa of P for the axis, we have $PN^2 = 4AS \cdot AN$. This point S is called the **focus**, and the chord LSL′ is the **latus rectum.**

The tangents at L, L′ meet at a point X on the axis produced such that $AX = AS$. The polar of S is a line through X parallel to the tangent at A, and hence perpendicular to the axis, this is the **directrix.**

Theorem. The distance of any point of the parabola from the focus equals its distance from the directrix.

For $SP^2 = PN^2 + SN^2 = 4AS \cdot AN + SN^2 = XN^2 = PM^2$.

Corollary 1. If the tangent at P meets the axis at T, we have $AT = AN$.

$\therefore$ $ST = XN$, hence $SP = ST$.

Corollary 2. Since

$$SP = ST,$$

the angle $SPT = S\hat{T}P = T\hat{P}M$,

i.e. the tangent bisects $S\hat{P}M$.

Corollary 3. If PT cuts the directrix at Z, the triangles SPZ, MPZ are equal by *s. a. s.* Hence PSZ is a right angle, and ZP bisects angle SZM.

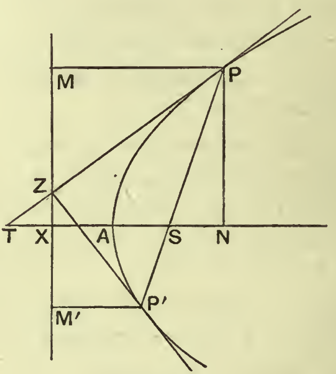


Fig. 127.

Again if P′ is the point where PS meets the curve again, P′SZ is a right angle, and hence P′Z is the tangent at P′, and it bisects $S\hat{Z}M'$.

Hence the tangents at the ends of any chord passing through the focus meet on the directrix and are perpendicular to each other.

Corollary 4. Since Z is the pole of SP, the lines SZ, SP are conjugate, and S *is a point at which all pairs of conjugate lines are perpendicular.*

Corollary 5. If KK′ is a focal chord whose middle point is M, and tangents at K, K′ meet at T (on the directrix and at right angles), TM cuts the parabola at C, such that $TC = CM = \frac{1}{2} . KM$. Also $T\hat{S}M$ is a right angle.

Hence $SC = CT = CM$; SC is called the **parameter** of the diameter CM. ∴ the focal chord (KK′) is four times the parameter of the diameter which bisects it.

Also, if PV, AV are the ordinate and abscissa to this diameter of any point P on the parabola, we have $PV^2 = 4CM . CV = 4SC . CV$.

Another Proof dependent on the Properties of Poles and Polars.

160. Let KK′ be a chord such that tangents KT, K′T are perpendicular.

Draw TS at right angles to KK′. Also draw TM to the middle point M of KK′ and TR at right angles to TM.

Then TS, TR make equal angles with TK; but TK, TK′ are the perpendicular rays of the involution of conjugate lines through T, hence TR, TS are conjugate lines.

Also $\{KK'RS\} = -1$. Hence TR is the polar of S.

Hence S is a point on the axis (since TR is parallel to the tangent at the end of the axis).

Also ST, SR are a pair of perpendicular conjugate lines at S; and so, also, are the axis and a line perpendicular to it through S; hence all pairs of conjugate lines at S are perpendicular.

Therefore S is a fixed point on the axis, and it is therefore that point on the axis whose ordinate is twice the abscissa, *viz.* the focus.

Hence all pairs of perpendicular tangents meet on the directrix, and their chords of contact pass through the focus, and each subtends a right angle at the focus.

161. Central Conics. In the projective rows described by a variable tangent PQ to a conic on two fixed tangents TA, TB when the rows are not similar, there are two vanishing points I, J such that IP . JQ has a constant value k (the power) [§§ 37, 38].

Also $\quad \text{IA . JT} = \text{IT . JB} = k; \quad \{\text{TBQJ}\} = -\text{AP : PT},$

and $\quad \{\text{TAPI}\} = -\text{BQ : QT}.$

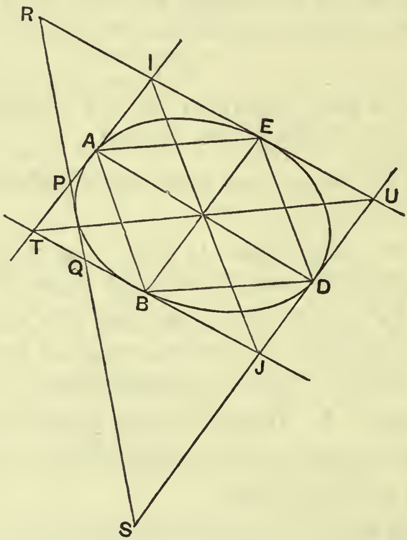


Fig. 128 (i).

Hence IA : IT = JB : JT, and therefore AB is parallel to IJ.

Also IU parallel to TJ and JU parallel to TI are two of the joins tangent to the curve.

If the variable tangent PQ, cutting IT at P and JT at Q, also cuts UI at R and UJ at S, by similar triangles IR : IP = JQ : JS, ∴ IR . JS = IP . JQ, hence I, J are the vanishing points of projective rows described by R on UI and S on UJ, and the power of the two rows, *viz.* IR . JS, has the same value k.

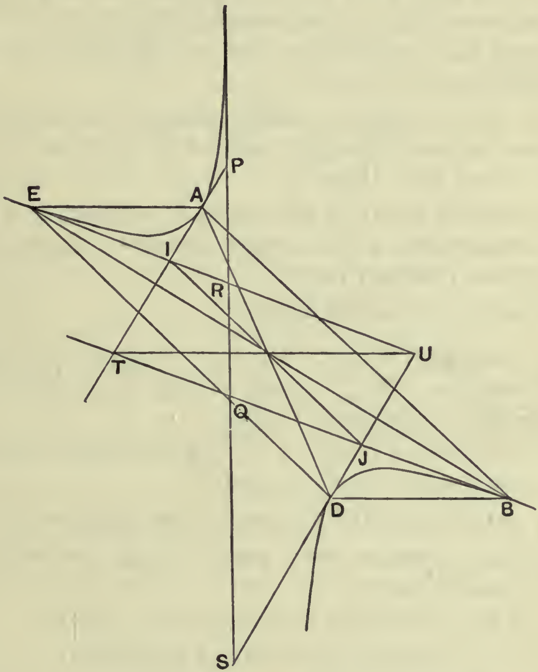


Fig. 128 (ii).

Again IP : IA = JT : JQ, ∴ IP : AP = JT : QT, ∴ IP . TQ = AP . TJ, but, by similar triangles, TP : IP = TQ : IR, ∴ TP . IR = IP . TQ.

∴ TP . IR = TJ . AP = UI . PA, to each add TP . UI.

∴ TP . UR = UI . TA.

Hence T, U are the vanishing points of projective rows described by P and R on IT and IU. Similarly T, U are vanishing points of projective rows described by Q, S on JT, JU; also the power TQ . US = TP . UR.

162. Since the power of R, S on UI, UJ has the same value as the power of P, Q on TI, TJ, it follows that if D, E are the points of contact of UI, UJ we have UE = BT and UD = AT. Hence DE is equal and parallel to BA; and AD, BE bisect each other at the intersection of UT and IJ.

If we keep the pair of parallel tangents TA, UD fixed, and vary the other pair, the middle point of BE will always be at the middle point of AD. Hence

The middle point of the chord of contact of a pair of parallel tangents is a fixed point, and every chord through that point is bisected there.

This point is called **the centre.**

163. Another Proof. Let P, P′ be points of contact of parallel tangents PT, P′T′; and A any point on the conic. Complete the parallelogram PAP′A′.

Then the pencils P {TAP′A′}, P′ {T′A′PA} have their corresponding rays parallel and are therefore equal.

∴ P {TAP′A′} = P′ {PAT′A′}, hence A′ lies on the conic.

Hence any chord AA′ which passes through the middle point of PP′ is bisected there.

Also if AU, A′U′ are the tangents at A, A′ we have

$$A' \{P'APU'\} = A \{P'UPA'\} = A \{PA'P'U\};$$

but A′P′ is parallel to AP, and A′P to AP′, hence A′U′ is parallel to AU.

Definition. The envelope of the join of projective rows on TA, BT, whose vanishing points are I, J, is **an ellipse** if A lies between T and I, and **a hyperbola** if A does not lie between T and I (cf. §§ 79, 80).

164. I. The Ellipse. Complete the parallelogram TIUJ, and let A, B, D, E be the points of contact of the sides.

[Then ABDE is a parallelogram whose sides are parallel to IJ and TU and diagonals intersect at C, the intersection of IJ, TU.]

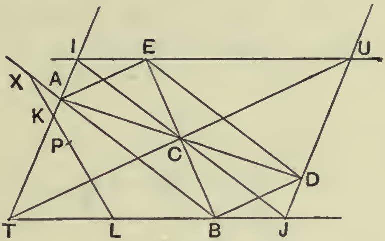


Fig. 129.

If a tangent KL cuts TA internally, then since $IT > IK > IA$, we have $JB < JL < JT$, hence L is also between T and B.

Hence LK cuts AB at a point X on AB produced.

Hence the polar of X, which passes through T, cuts AB internally, and hence it cuts KL internally.

Therefore the point of contact P of KL lies within the triangle TAB.

Similarly it may be proved that when K is between I and A, the contact is within the triangle IAE: and that when K is in the produced parts of IT towards I and T respectively, the point of contact is within triangles UDE and BJD respectively.

Also when KL is parallel to AB, P lies on TM. Now $IA : IK = JL : JT$ (by definition) $= IK : IT$ (by parallel projection).

$$\therefore\ AK : IK = KT : IT,$$

and hence $TK > KA$, $\therefore\ TP > PM$, which shews that there are no infinity directions (cf. § 79).

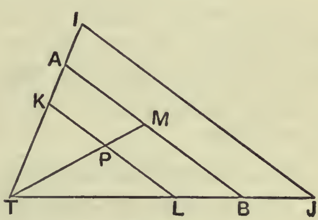


Fig. 130.

165. II. The Hyperbola. Let IA > IT (then JB > JT). As K moves from A to T, L moves from T to B, then KL and AB cut each other externally, hence the point of contact P is within the triangle ATB.

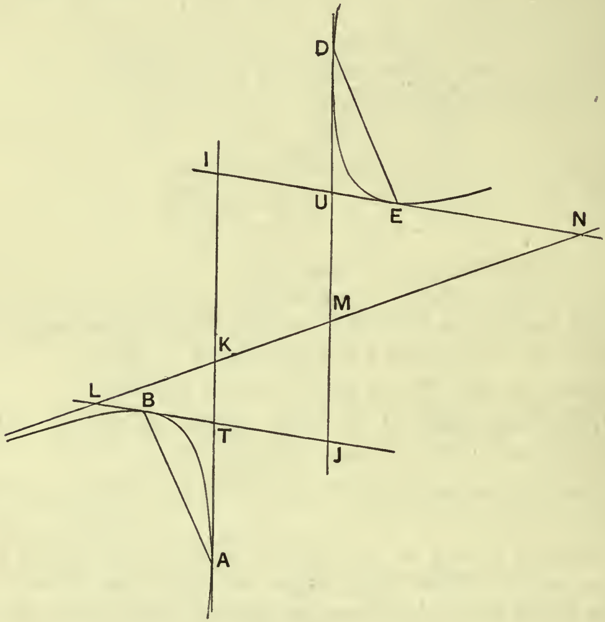


Fig. 131.

The parallel tangent to KL, by symmetry about the centre C, will envelope an arc in the triangle DUE, and will cut TI successively at all points from infinity to I.

As K moves from T to I, L moves from B to infinity: let KL cut IU at N.

Then AB cuts KL internally, hence P cuts KL externally.

Also TK . UN = TI . UE, and hence UN > UE.

∴ DE cuts LN internally, and hence P cuts LN externally.

Hence P is in LN produced, either in the angle ATB or DUE.

Similarly when K is between A and infinity, L is between T and J, N between I and U, and M is in UD produced; and P is again within angles BTA or EUD.

166. Asymptotes. On AI take points K, K′ such that $AK^2 = AK'^2 = AT \, . \, AI$, and draw K′L parallel to AB to cut TB at L.

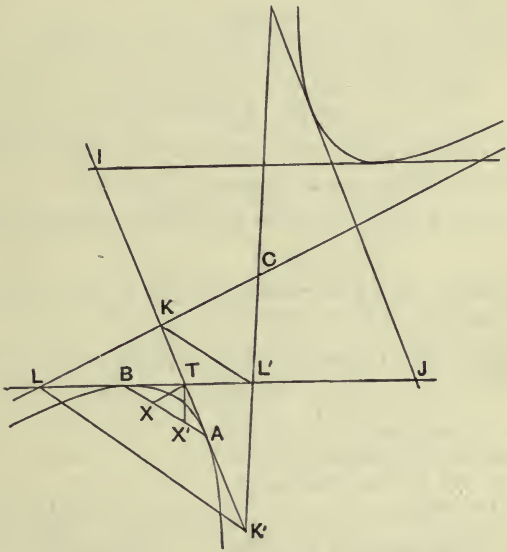


Fig. 132.

Then $$IK \, . \, IK' = IA^2 - AK^2$$
$$= IA^2 + IA \, . \, AT$$
$$= IA \, . \, IT.$$

But $$IK' : JL = IT : JT \text{ (by parallels).}$$
$$\therefore \; IK \, . \, JL = IA \, . \, JT,$$

and hence KL is a tangent.

But KA = AK′, hence AB bisects KL, and therefore the point of contact of KL is at infinity.

Similarly, if KL′ is drawn parallel to AB, we get another tangent K′L′ whose point of contact is at infinity.

167. Theorem. The asymptotes pass through the centre.

Since $\frac{TK}{KI} = \frac{K'T}{K'I}$ (because KK′TI is harmonic),

$$\therefore \quad \frac{TK}{KI} = \frac{LT}{LJ},$$

$$\therefore \quad \frac{TK}{\frac{1}{2}TI} = \frac{LT}{\frac{1}{2}(LT + LJ)},$$

hence (by similar triangles) LK passes through the centre C.

Similarly L′K′ also passes through the centre.

Thus there are two tangents at infinity, passing through the centre.

Corollary. If TX be drawn parallel to KL to cut AB at X, then BX : XA = CK : CL = IK : IK′ = JL′ : JL.

Similarly a line TX′ parallel to K′L′ cuts AB at X′, where

$$AX' = BX.$$

Exercise. Prove that X, X′ are the double points of the projective rows described on AB by lines through T parallel to PA, PB.

168. Theorems. I. The intercept of the asymptotes on any tangent is bisected at its point of contact.

For KK′ is bisected at A (fig. 132).

II. On any chord the intercepts between the asymptotes and the conic are equal.

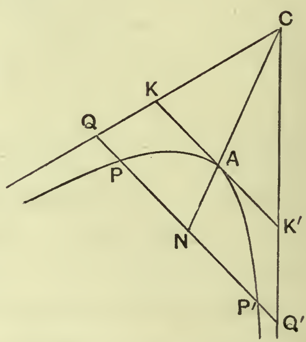


Fig. 133.

Let PP′ be a chord which cuts the asymptotes at Q, Q′, and A the point whose tangent is parallel to PP′, cutting the asymptotes at K, K′.

Then CA is the diameter which bisects PP′ at N. But KA = AK′, ∴ QN = NQ′.

Hence QP = P′Q′. Q. E. D.

169. Conjugate Diameters. If KLMN is a parallelogram whose sides touch a conic at A, B, D, E it has been proved that AB and DE are parallel to LN, and AE, BD to KM. Also that KM, LN, AD, BE intersect at the centre C.

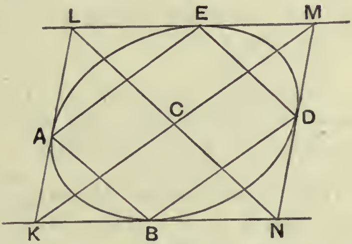


Fig. 134.

Further KM bisects AB, hence it contains that diameter which bisects all chords parallel to LN (§ 78); also LN bisects all chords parallel to KM.

Hence if we draw any diameter PCP′ bisecting all chords parallel to QQ′, the diameter QQ′ bisects all chords parallel to PP′.

Thus we get pairs of conjugate diameters.

The diagonals of a circumscribing parallelogram lie along conjugate diameters. The sides of an inscribed parallelogram are parallel to conjugate diameters.

170. Conjugate Diameters form an Involution. Let AD be any diameter through the centre C; take any point B on the conic and join to A and D.

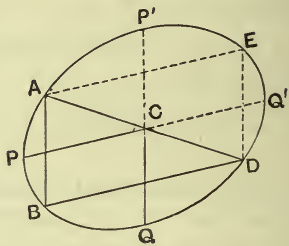


Fig. 135.

Then CP, CQ parallel to DB, AB are a pair of conjugate diameters.

Also pencil described at C by CP $\equiv$ pencil of DB at D = pencil of AB at A $\equiv$ pencil of the parallel line CQ at C, thus CP and CQ describe projective pencils at C.

Complete the parallelogram ABDE, then when B is at E, CP assumes the position CP′ in a straight line with CQ, and CQ becomes at the same time CQ′, which is in a line with CP.

Hence in the double pencil at C, the same ray CQ corresponds to CP whether we regard CP as belonging to the first or second pencil, hence the two pencils are in involution.

Corollary. If CP is a diameter the conjugate diameter CQ is parallel to the tangent at P. Hence when the tangent at P passes through the centre, the diameter coincides with its conjugate; and conversely.

Hence in an ellipse there are no real double rays of the involution formed by the pairs of conjugate diameters. [Hence the term "elliptic" involution.] Whereas in a hyperbola there are two double rays, *viz.* the asymptotes. ["Hyperbolic" involution.]

Otherwise: Let the tangent at any point P of a hyperbola cut the asymptotes in T, T′, then we have TP = PT′. But the diameter CQ conjugate to CP is parallel to TT′.

Hence CP and CQ are harmonically conjugate with respect to the asymptotes, and therefore form an involution, whose double rays are the asymptotes.

Hence also of any two conjugate diameters of a hyperbola, one and one only cuts the curve.

171. Conjugate Diameters at Right Angles. In an involution there is always one pair of perpendicular conjugate rays; and if there is more than one, then every pair of conjugate rays are perpendicular.

To construct the perpendicular conjugate diameters of a conic. If the tangent at P is perpendicular to CP, then CP and its conjugate are the required diameters.

If not describe a circle with radius CP to cut the conic again at Q. Then PC and QC meet both circle and conic again at P′ and Q′ respectively, and PQP′Q′ is a rectangle, which is inscribed in the conic.

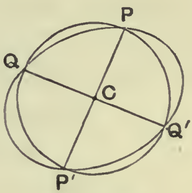


Fig. 136.

Hence diameters parallel to PQ and PQ′ are a pair of perpendicular conjugate diameters.

The perpendicular conjugate diameters are called **principal diameters** or **axes**.

The conic will be symmetrical with respect to each of its principal diameters. In the case of a hyperbola the principal diameters will bisect the angles between the asymptotes.

172. Segments of Diameters. Theorem. If tangents TQ, TQ′ be drawn to a conic whose centre is C, and CT cuts QQ′ at V, and the conic at P, then $CV \cdot CT = CP^2$.

If CV cuts the conic at P, P′, then T, V are conjugate points on PP′ and hence harmonically conjugate to P, P′, and C is the middle point of PP′, hence $CV \cdot CT = CP^2$.

But in the case of a hyperbola CT may not cut the conic in real points, however T, V being conjugate points still describe an involution as T moves along a fixed line through C, and C is the centre of the involution. Hence for all positions of T on a given line through C, the value of CV . CT is constant, but in this case the value is negative. We may find it convenient to speak of this quantity as the square of the semi-diameter along CT.

173. Theorem A. In an ellipse a chord QQ′ moves parallel to itself, so that its middle point V describes the diameter PP′, then $QV^2 : PV . VP'$ has the constant value $CD^2 : CP^2$, where CD is a semi-diameter parallel to QV.

Let the tangent at Q cut CP at T and CD at U: draw QR parallel to CP to meet CD at W.

$$QV^2 : CD^2 = CW^2 : CD^2$$

$$= CW^2 : CW . CU$$

(by previous theorem)

$$= VQ : CU$$

$$= VT : CT = CV . VT : CP^2.$$

But $CV . VT = CV . CT - CV^2$

$$= CP^2 - CV^2 = PV . VP'.$$

$$\therefore\ QV^2 : PV . VP' = CD^2 : CP^2.$$

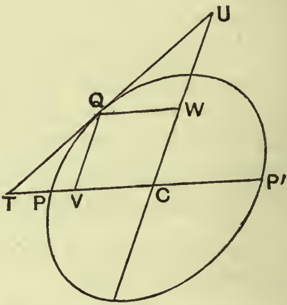


Fig. 137.

Theorem B. In a hyperbola, if the diameter which bisects the chord QQ′ at V meets the curve in real points P, P′, then $QV^2 : PV . VP'$ is constant, as V moves along PP′.

Let the tangent at Q cut CP at T and the conjugate diameter at U, and draw QW parallel to CP to cut CU at W.

Then CU . CW has a constant value d.

$$\therefore\ QV^2 : d = CW^2 : CW . CU$$

$$= CW : CU$$

$$= VQ : CU$$

$$= VT : CT$$

$$= CV . VT : CP^2.$$

But CV . VT = CV . CT − CV² = CP² − CV² = PV . VP′.

$$\therefore\ QV^2 : PV \,.\, VP' = d : CP^2.$$

[Here PV . VP′ and d are both negative.]

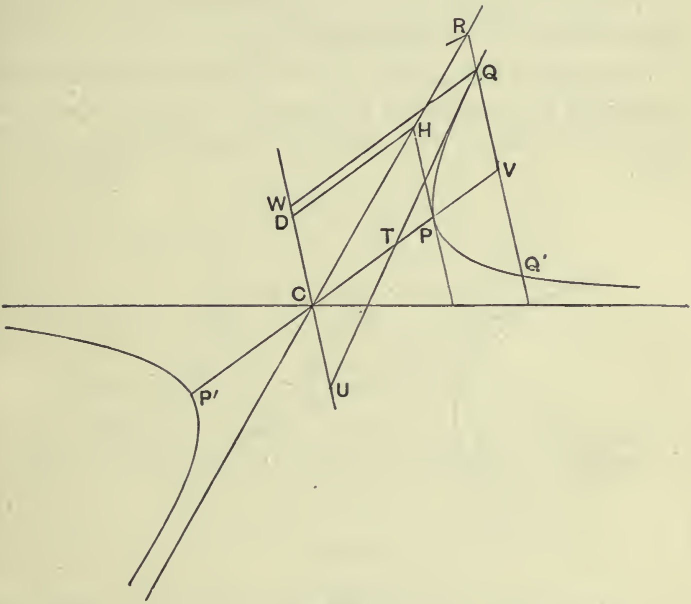


Fig. 138.

Corollary 1. If VQ cuts the asymptote at R, then RV : CV = ultimate value of QV : CV when CV is very large.

Now when CV is very large $QV^2 : CV^2$ differs by a very small amount from $QV^2 : CV^2 - CP^2$, which equals $-d : CP^2$,

$$\therefore\ RV^2 : CV^2 = -d : CP^2.$$

Corollary 2. If PH the tangent at P meets the asymptote at H, then $PH^2 = -d$: and if HD be drawn parallel to CP to meet the conjugate diameter at D, $CD^2 = -d$.

Corollary 3. Since $RV^2 : CV^2 = -d : CP^2 = QV^2 : CV^2 - CP^2$,

$$\therefore\ RV^2 - QV^2 : CP^2 = -d : CP^2,$$

hence $RV^2 - QV^2$, which equals $RQ \,.\, QR'$, is constant ($= PH^2$).

Hence as CV increases RQ continually decreases, or the curve continually approaches nearer to its asymptote.

Theorem C. If QQ′ is a chord such that the diameter which bisects it at V does not meet the hyperbola, but $CV \,.\, CT = p$, then $QV^2 : p - CV^2 = CD^2 : p$, where CD is the semi-diameter parallel to QV.

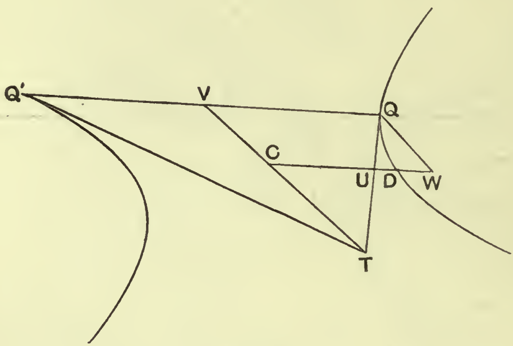


Fig. 139.

Let the tangent at Q cut CD at U and CV at T; draw QW parallel to CV to meet CD at W.

Then $QV^2 : CD^2 = CW^2 : CD^2$

$$= CW : CU = VQ : CU = VT : CT$$

$$= CV \,.\, VT : CV \,.\, CT$$

$$= CV \,.\, VT : p.$$

But $CV \,.\, VT = CV \,.\, CT - CV^2 = p - CV^2$,

$$\therefore\ QV^2 : p - CV^2 = CD^2 : p\,;$$

here $p - CV^2$ and p are both negative.

174. Theorem. If from a point T on the diameter PP′ of a conic, tangent TQ be drawn, and r, p are the squares of semi-diameters parallel to TQ and CT, then

$$TQ^2 : TP \,.\, TP' = r : p,$$

if PP′ cuts the conic in real points, but if not $TQ^2 : CT^2 - p = r : p$.

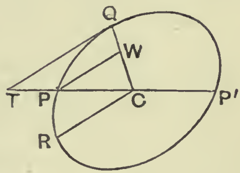


Fig. 140.

Case I. If CT cuts the conic.

Draw an ordinate PW to the semi-diameter CQ.

$$TQ^2 : PW^2 = CT^2 : CP^2$$

and $$PW^2 : r = CQ^2 - CW^2 : CQ^2 = CT^2 - CP^2 : CT^2,$$

$$\therefore\ TQ^2 : r = CT^2 - CP^2 : CP^2$$

or $$TQ^2 : TP \,.\, TP' = r : CP^2 = r : p.$$

Case II. Where CT does not cut the conic.

Let TQ cut the asymptotes at H, H′; and draw a line RR′ through Q and parallel to CT to meet the asymptotes at R, R′.

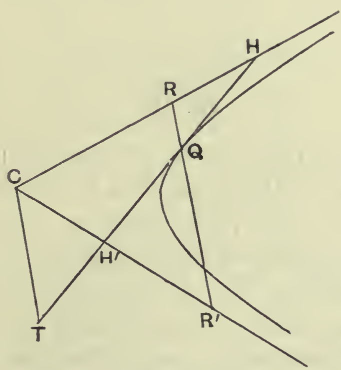


Fig. 141.

Then $$QH^2 = QH'^2 = -r$$
and $$RQ \,.\, QR' = -p.$$
But $$CT : HT = RQ : HQ,$$
$$CT : H'T = QR' : QH',$$
$$\therefore\ CT^2 : TQ^2 - QH^2 = RQ \,.\, QR' : HQ^2,$$
$$\therefore\ CT^2 : TQ^2 + r = p : r$$
and $$\therefore\ CT^2 - p : TQ^2 = p : r,$$
hence $$TQ^2 : CT^2 - p = r : p.$$

Corollary. If CQ, CR are conjugate diameters of an ellipse and tangents at Q, R meet at T, then $CT^2 = 2CP^2$, where CT cuts the conic at P. For TQ = CR.

175. Theorem. If O is any point on a chord QQ′ and r the square of the parallel semi-diameter, then $OQ \,.\, OQ' : r = CO^2 - p : p$, where p is the square of the semi-diameter along OC.

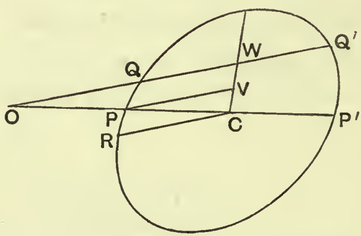


Fig. 142.

Case I. If p is positive, and OC cuts the conic at P, P′.

Draw a diameter to bisect QQ′, and let its power be k.

Then $$OW^2 : PV^2 = CW^2 : CV^2,$$
and $$QW^2 : PV^2 : r = k - CW^2 : k - CV^2 : k,$$
$$\therefore\ QW^2 - r : PV^2 - r = CW^2 : CV^2,$$
$$\therefore\ OW^2 - QW^2 + r : r = CW^2 : CV^2 = CO^2 : CP^2,$$
$$\therefore\ OW^2 - QW^2 : r = CO^2 - CP^2 : CP^2,$$
i.e. $$OQ \,.\, OQ' : r = CO^2 - p : p.$$

Case II. If OC does not cut the curve.

Let OQ cut the asymptotes at R, R′; draw a line through Q parallel to OC to cut the asymptotes at S, S′.

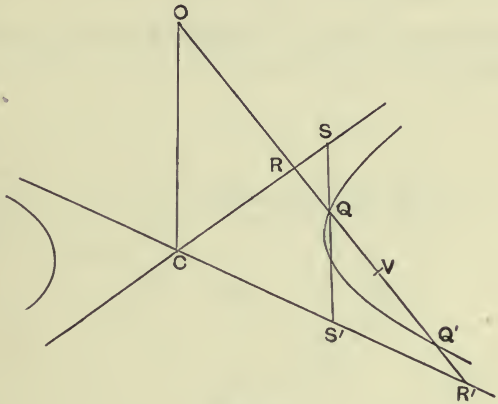


Fig. 143.

Then $$OC^2 : OR \,.\, OR' = SQ \,.\, QS' : RQ \,.\, QR'$$
$$= p : r,$$
$$\therefore\ OC^2 : p = OV^2 - RV^2 : r,$$
$$\therefore\ OC^2 - p : p = OV^2 - RV^2 - r : r,$$
but $$r = RQ \,.\, QR' = RV^2 - QV^2,$$
$$\therefore\ OC^2 - p : p = OV^2 - QV^2 : r$$
$$= OQ \,.\, OQ' : r.$$

Corollary. If two chords QQ′ and XX′ pass through a point O, and r, y are the squares of the parallel semi-diameters, then
$$OQ \,.\, OQ' : r = OX \,.\, OX' : y.$$

Hence if the directions of the chords are fixed, the ratio OQ . OQ′ : OX . OX′ is the same for all positions of O.

176. Focus. At any point N on the principal axes one pair of conjugate perpendicular lines are the axis and a chord PNP′ perpendicular to the axis: the pole of PP′ being a point T on the axis. Also $CN \cdot CT = CA^2$; and the polar of N is a line through T perpendicular to the axis.

If more than one pair of conjugate pairs are perpendicular, then all pairs are.

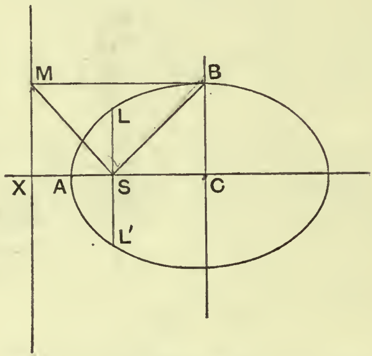


Fig. 144.

I. In the Ellipse. Let CB be the other principal axis, and take a point S on CA such that $CS^2 = CA^2 - CB^2$.

The tangent at B is parallel to CA, let it meet the polar XM of S at M. Join SM, SB.

Then $$CS \cdot SX = CS \cdot CX - CS^2 = CB^2.$$

$$\therefore\ CS : CB = XM : SX,$$

hence the angles XSM, CSB are complementary, and SB, SM are perpendicular.

But the point M is on the polar of S, and on the tangent at B, hence its polar is SB: *i.e.* SB, SM are a pair of (perpendicular) conjugate rays. Hence every pair of rays at S are perpendicular.

The point S is called a **focus.** There will be two foci on AA′, equally distant from C, so that $CS^2 = CS'^2 = CA^2 - CB^2$.

The line XM, the polar of S, is called a **directrix.**

The line LL′ through S perpendicular to S is the **latus rectum.**

Corollary. The intercept on any tangent between the point of contact and the directrix subtends a right angle at the focus.

Also the tangents at L, L′ pass through X.

177. If P is any point on an ellipse, focus S, and PM a line perpendicular to the directrix which is polar to S, then SP : PM is a constant ratio.

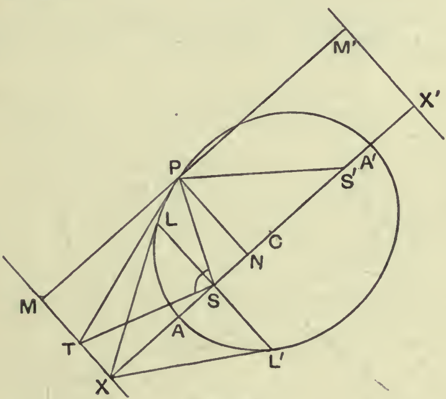


Fig. 145.

Draw PN perpendicular to the axis CS.

Then $$SP^2 = SN^2 + PN^2.$$

Now $$PN^2 : CA^2 - CN^2 = CB^2 : CA^2.$$

But if $CS = e \, . \, CA$, $CB^2 = CA^2 - CS^2 = (1 - e^2)\, CA^2$,

$$\therefore\; PN^2 = (1 - e^2)\,(CA^2 - CN^2).$$

Hence
$$SP^2 = (CS - CN)^2 + (1 - e^2)(CA^2 - CN^2)$$
$$= (e\,.\,CA - CN)^2 + (1 - e^2)(CA^2 - CN^2)$$
$$= CA^2 - 2e\,.\,CA\,.\,CN + e^2\,.\,CN^2,$$
$$\therefore\ SP = CA - e\,.\,CN$$
$$= e\,.\,CX - e\,.\,CN = e\,.\,NX = e\,.\,PM,$$
i.e. SP : PM has the constant value e.

Corollary. If PM′ is the perpendicular on the directrix of the other focus S′, $S'P = e\,.\,PM'$. Hence
$$SP + S'P = e\,.\,MM' = e\,.\,XX' = AA'.$$

Corollary. The tangents at L, L′ pass through X, hence the conic is an ellipse or hyperbola as $XA \gtrless AS$, *i.e.* as $e \lessgtr 1$.

178. II. In the Hyperbola. On the asymptote take CK equal to CA, draw KS perpendicular to CK to meet the axis at S, and draw KX perpendicular to the axis. Draw SI parallel to CK.

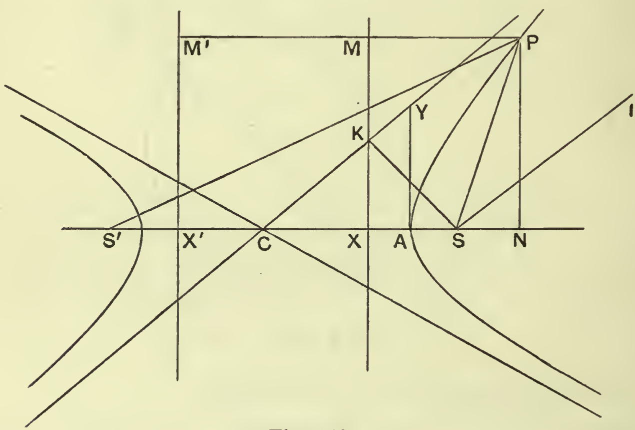


Fig. 146.

Then
$$CS\,.\,CX = CK^2 = CA^2.$$
$\therefore$ KX is the polar of S.

Since SI passes through S its pole is on KX, and since SI passes through the point of contact of CK its pole lies on CK, $\therefore$ K is the pole of SI, and SK, SI are a pair of conjugate lines at S, which are at right angles.

Hence all pairs of conjugate lines at S are perpendicular to each other.

S is the **focus** of the hyperbola. There will be two foci on the axis equally distant from C. XK is the **directrix** corresponding to S.

If we draw AY perpendicular to the axis to meet CK at Y, then by equality of the triangles CKS, CAY we have $\mathrm{CS} = \mathrm{CY}$.

$\mathrm{CS}^2 = \mathrm{CA}^2 + \mathrm{AY}^2 = \mathrm{CA}^2 - b$, where b is the (negative) square of the semi-diameter perpendicular to CA.

179. Theorem. $\mathrm{SP} = e\,.\,\mathrm{PM}$, where PM is a perpendicular from a point P of the conic to the directrix XM.

For $\quad \mathrm{SP}^2 = \mathrm{SN}^2 + \mathrm{PN}^2$, but $\mathrm{PN}^2 : \mathrm{CA}^2 - \mathrm{CN}^2 = b : \mathrm{CA}^2$;

now if $\mathrm{CS} = e\,.\,\mathrm{CA}$, $\quad b = \mathrm{CA}^2 - \mathrm{CS}^2 = (1 - e^2)\,\mathrm{CA}^2$,

$$\therefore\ \mathrm{PN}^2 = (1 - e^2)(\mathrm{CA}^2 - \mathrm{CN}^2),$$

$$\therefore\ \mathrm{SP}^2 = (e\,.\,\mathrm{CA} - \mathrm{CN})^2 + (1 - e^2)(\mathrm{CA}^2 - \mathrm{CN}^2)$$
$$= \mathrm{CA}^2 - 2e\,.\,\mathrm{CA}\,.\,\mathrm{CN} + e^2\,.\,\mathrm{CN}^2,$$

$$\therefore\ \mathrm{SP} = \mathrm{CA} \sim e\,.\,\mathrm{CN} = e\,(\mathrm{CX} \sim \mathrm{CN}) = e\,.\,\mathrm{XN} = e\,.\,\mathrm{PM}.$$

Corollary. If S′ is the other focus then $\mathrm{S'P} = e\,.\,\mathrm{PM'}$, hence

$$\mathrm{S'P} - \mathrm{SP} = e\,.\,\mathrm{MM'} = e\,.\,\mathrm{XX'} = \mathrm{AA'}.$$

180. Theorem. The tangent makes equal angles with the focal distances.

I. In the Ellipse.

$$\begin{aligned}\mathrm{SP} : \mathrm{S'P} &= \mathrm{NX} : \mathrm{X'N}\\ &= \mathrm{CX} - \mathrm{CN} : \mathrm{CX} + \mathrm{CN}\\ &= \mathrm{CX}\,.\,\mathrm{CT} - \mathrm{CA}^2 : \mathrm{CX}\,.\,\mathrm{CT} + \mathrm{CA}^2\\ &= \mathrm{CT} - e\,.\,\mathrm{CA} : \mathrm{CT} + e\,.\,\mathrm{CA}\\ &= \mathrm{CT} - \mathrm{CS} : \mathrm{CT} + \mathrm{CS} = \mathrm{ST} : \mathrm{S'T}.\end{aligned}$$

$\therefore$ PT bisects the exterior angle at P between SP and S′P.

II. In the Hyperbola.

$$\begin{aligned} SP : S'P &= XN : X'N \\ &= CN - CX : CN + CX \\ &= CA^2 - CX \,.\, CT : CA^2 + CX \,.\, CT \\ &= e \,.\, CA - CT : e \,.\, CA + CT \\ &= CS - CT : CS + CT = TS : S'T. \end{aligned}$$

$\therefore$ PT bisects the interior angle between SP and S′P.

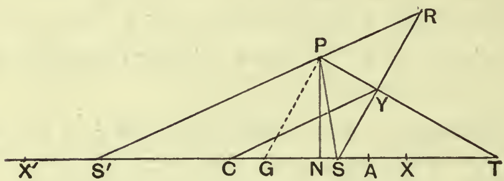


Fig. 147.

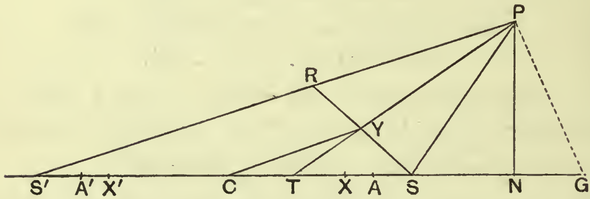


Fig. 148.

Corollary 1. The normal, *i.e.* the line drawn through P perpendicular to the tangent at P, bisects the other angle between SP, SP′, hence it meets the axis at G the harmonic conjugate of T with respect to S, S′, and $\therefore$ CG . CT $= CS^2 = e^2 . CA^2$.

Corollary 2. If the perpendicular SY from S to the tangent meets S′P at R, PR = PS and RY = YS. Hence S′R = the sum or difference of SP, S′P = AA′. Also CY $= \frac{1}{2}$S′R = CA.

Hence Y lies on a circle with AA′ as diameter; this is called the **auxiliary circle** of the conic.

Corollary 3. If the tangents at P, P′ meet at T, and R, R′ are the images of S, S′ in those tangents, then the triangles TRS′, TR′S are equal in every respect. But angle TSP = TRS′, and TS′P′ = TR′S.

Hence two tangents to a conic subtend equal (or supplementary) angles TSP, TS′P′ at the foci.

EXAMPLES. VIII.

1. SA meets a given line in K and TA meets a parallel line in L. Prove that if A moves along a straight line KL will envelope a parabola.

2. If ABCD is a square find the parabola which touches AB at B and AC at C.

Find where it intersects a parabola touching BA at A and BD at D.

3. Three parabolas are drawn each touching two sides of a triangle at the ends of the third side, find their points of intersection.

4. Two parabolas have a common focus and a common axis, prove that they intersect at right angles.

5. A line touches a parabola at K, cuts two other tangents at L, M and the diameter bisecting their chord of contact at N. Prove that LK equals MN.

6. A tangent to a parabola at K cuts two tangents TA, TB at L, M respectively, and the chord of contact AB at N. Prove that AN : BN as $LK^2 : KM^2$.

7. The foot of the perpendicular from a point to its polar with respect to a parabola is at the same distance from the focus as the point itself is from the directrix.

8. The vertices A, A′ and foci S, S′ of two parabolas are collinear, and AS, SA′, A′S′ are equal. If any focal chord PSQ of the first parabola meets it at P and Q and the normals PG, QH meet the axis at G, H, prove that G and H are the feet of ordinates of a focal chord of the second parabola.

9. Triangles are described self-polar to a given parabola, and having one vertex at a given point; prove that their nine-point circles form a coaxal system.

10. If a chord of a parabola passes through a fixed point O, the rectangle of the segments of the chord is equal to the rectangle of the parallel focal chord and the intercept, on the diameter through O, between O and the parabola.

11. The rectangles of the segments of two intersecting chords of a parabola are in the same ratio as the parallel focal chords.

12. In a parabola the rectangle of the abscissae with respect to a given diameter of the ends of a chord passing through a given point on the diameter is constant.

13. The mid-point of the chord of contact of tangents from P to a parabola lies on a fixed line, prove that the locus of P is a parabola. Also find its axis and focus.

14. Parabolas are described each passing through a given point and touching two given lines. Prove that the envelope of the diameter through one end of the chord of contact with the two lines is a hyperbola, and find its asymptotes.

Prove that the two hyperbolas thus obtained are of the same dimensions.

15. The two tangents drawn to a parabola from any point subtend equal angles at the focus.

16. Prove that the focus of a parabola lies on the circumcircle of the triangle formed by any three tangents; and find its pedal line with respect to the triangle.

17. A conic touches two lines TA, TB at A, B and passes through the centroid of the triangle TAB; and the joins of A, B to any point of the conic cut TB, TA at K, L. Prove that the envelope of KL is a parabola.

18. A conic touches TA, TB at A, B and the joins of A, B to any point P of the conic cut TB, TA at K, L. Prove that the envelope of KL is a conic, and find whether it is an ellipse, parabola or hyperbola. Also find its centre.

19. A variable tangent to a conic cuts at K, L two fixed tangents whose points of contact are A, B. Prove that the locus of the intersection of AL, BK is a conic. Find when this conic is a parabola, when an ellipse, and when a hyperbola. Find its centre.

20. A chord CD of a circle is bisected at K by a diameter AB, and the tangents at C, D intersect at L. Prove that any conic having its centre on AB and touching AC, AD, BC, BD divides KL harmonically.

21. Prove that two concentric conics have only one pair of common conjugate diameters, and that these are harmonically conjugate to the common chords of the two conics.

22. Prove that the common chords of a central conic and any circle are equally inclined to the principal diameters of the conic.

23. If AB is a diameter of a central conic, and the join BT of B to any point T on the tangent at A cuts the conic at P, shew that the tangent at P bisects AT.

24. Two parallel tangents to a central conic are cut by any other tangent at T, T′; prove that CT, CT′ lie along conjugate diameters.

Also prove that $TP \cdot PT' = CD^2$, where P is the point of contact of TT′, and CD is the semi-diameter conjugate to CP.

25. If CP, CD are conjugate semi-diameters of an ellipse, whose principal semi-diameters are CA, CB and foci S, S′; prove that $SP \cdot S'P = CD^2$; and that $CP^2 + CD^2 = CA^2 + CB^2$.

26. The portion of any tangent to an ellipse intercepted by a pair of conjugate diameters subtends supplementary angles at the foci.

27. A conic cuts the sides BC, CA, AB of a triangle at A_1 and A_2, B_1 and B_2, C_1 and C_2 respectively, prove that the product of

$$\frac{BA_1 \cdot BA_2}{CA_1 \cdot CA_2} \cdot \frac{CB_1 \cdot CB_2}{AB_1 \cdot AB_2} \cdot \frac{AC_1 \cdot AC_2}{BC_1 \cdot BC_2}$$

is unity. [Carnot's Theorem.]

28. If a conic cuts the sides BC, CA, AB of a triangle at K_1, K_2; L_1, L_2; M_1, M_2 respectively, and AK_1, BL_1, CM_1 are concurrent, so also are AK_2, BL_2, CM_2.

29. Prove that the lines which join the vertices of a triangle to any two given points cut the opposite sides in six points which lie on a conic.

30. A conic touches the sides of a triangle ABC at the feet of the perpendiculars from the opposite vertices, and the join of A to the centre of the conic cuts BC at K; prove that $BK : KC$ as $BA^2 : CA^2$.

31. The centroid of the triangle formed by the two tangents from P to a given conic and their chord of contact lies on the conic, find the locus of P.

32. A tangent to an ellipse whose foci are S, S′ is cut by a pair of parallel tangents at T, T′, prove that $ST \cdot ST' : S'T \cdot S'T' = SP : S'P$.

33. Tangents from any point T on an equiconjugate diameter of an ellipse touch the ellipse at A and B. Prove that the circle through T, A and B passes through the centre of the ellipse.

34. In an ellipse, centre C, CP is conjugate to the normal at Q; prove that CQ is conjugate to the normal at P.

35. Prove that the tangent to an ellipse at any point makes equal angles with the focal distances of the point.

If SY, S′Y′ are the perpendiculars from the foci S, S′ to any tangent to an ellipse whose major axis is AA′, prove that the pencil described by AY at A is projective with the row described by the tangent on the fixed tangent at A′. Also prove that the locus of the intersection of AY, A′Y′ is a conic having AA′ as one of its principal axes.

36. The directrix corresponding to a focus S of an ellipse cuts the chord of contact AB of two tangents AT, BT at K, and ST cuts AB at L, prove that K, L are harmonic conjugates to A, B; also prove that ST bisects the angle ASB.

37. An ellipse touches the sides of a triangle and has one focus at the orthocentre, find the position of the other focus.

38. Prove that the distance of a point P on a hyperbola from its focus is equal to a line drawn from P to the directrix parallel to an asymptote.

Also find the locus of the focus of a hyperbola which passes through two given points and has its asymptotes parallel to two given lines.

39. A line through a point P of a hyperbola parallel to the transverse axis cuts an asymptote at K, and the focal chord SP cuts the asymptote at L, prove that the sum of LP, LK is constant.

40. A tangent to a central conic meets the principal axes at T, T′ and the normal meets them at G, G′, prove that $CG \cdot CT = CS^2 = CG' \cdot CT'$.

41. Conics are drawn touching a given line KL, and having given parallel lines KX, LY as directrices. Prove that the locus of the focus corresponding to KX is a circle passing through K and bisecting KL.

42. Tangents are drawn to a set of confocal conics from a point on the common axis; prove that their points of contact lie on a circle.

43. Any point P on an ellipse is joined to the foci S, S′; prove that the loci of the centres of the escribed circles of the triangle SPS′ are two straight lines and an ellipse.

44. The polar of T with respect to a hyperbola cuts the asymptotes at K and L, and tangents from K, L touch the hyperbola at P, Q respectively. Prove that TP, TQ are parallel to the asymptotes.

45. A line touches a hyperbola at P and cuts an asymptote at K, a line parallel to the other asymptote through any point L on this tangent cuts the first asymptote at Q and the curve at R. Prove that $KP^2 : PL^2$ as $QL : LR$.

46. A line through a point A on a hyperbola parallel to one asymptote meets a chord BC at K, and a line through B parallel to the other asymptote meets the chord AC at L. Prove that KL is parallel to the tangent at C.

47. Two tangents from T to a hyperbola cut one asymptote at A, B, and the parallel tangents cut the other asymptote at A′, B′ respectively. Prove that AB′, A′B and CT are parallel.

48. If the asymptotes of a hyperbola are at right angles conjugate diameters are equal. Prove also that the orthocentre of any triangle inscribed in a rectangular hyperbola lies on the conic.

49. A circle cuts two fixed circles orthogonally and a diameter is drawn parallel to a given direction. Prove that the locus of its extremities is a rectangular hyperbola.

50. A point moves so that the perpendicular from it to a given line equals the distance of the foot of the perpendicular from a given point. Prove that the locus of the point is a rectangular hyperbola.

51. A tangent to a circle cuts two parallel sides of a circumscribing square at P, Q; and the parallel tangent cuts the other two sides of the square at R, S. Prove that P, Q, R, S and the centre of the circle lie on a rectangular hyperbola whose centre lies on the circle.

52. A circle cuts a rectangular hyperbola, centre C, at K, L, M, N, prove that $CK^2 + CL^2 + CM^2 + CN^2$ equals the square of the diameter of the circle.

A rectangular hyperbola passes through four concyclic points A, B, C, D and P is the orthocentre of the triangle ABC, prove that DP is a diameter.

53. The base of a triangle is given and the difference of the angles at the base, prove that the locus of the vertex is a rectangular hyperbola.

54. Given the asymptotes and one tangent to a hyperbola, construct the foci.

55. If parallel straight lines touch a series of confocal conics, their points of contact lie on a rectangular hyperbola.

56. Prove that the tangents from any point to a central conic make equal angles with the lines joining the point to the foci; and deduce that if a focus of a conic lies on the circumcircle of a triangle formed by three tangents the conic must be a parabola.

57. A triangle PQR circumscribes a conic whose centre is C, and ordinates are drawn from Q, R to the diameters CR, CQ respectively, prove that the join of the feet of these ordinates passes through the points of contact of PQ and PR.

58. A point moves so that the perpendicular drawn from it to the chord of contact of tangents to a given parabola passes through a given point, prove that its locus is a rectangular hyperbola.

59. Given one diameter of an ellipse in position and magnitude, and the sum of the squares of conjugate semi-diameters, prove that the ellipse touches a fixed ellipse whose foci are the ends of the given diameter, and that the common tangent is perpendicular to the conjugate diameter.

60. Two rectangular hyperbolas are concentric with, and each touches, a given hyperbola, prove that they intersect on the bisectors of the angles between the lines joining the centre to the points of contact.

CHAPTER IX

RECIPROCATION

181. Theorem. If a system of lines envelope a conic, the locus of their poles with respect to a given fixed conic is also a conic.

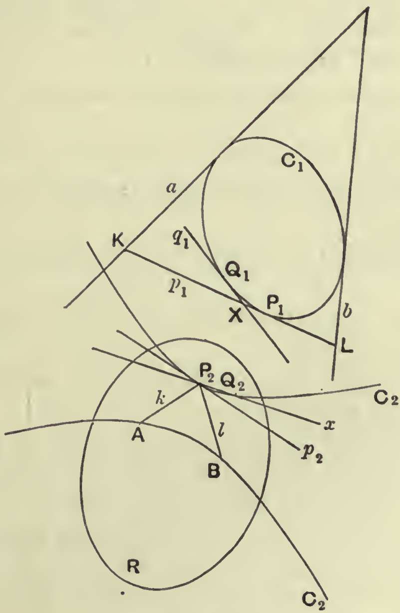


Fig. 149.

Let a, b be two tangents to a conic C_1, and A, B their poles with respect to a fixed conic R: let p_1 be any other tangent to C_1, and P_2 its pole with respect to R.

If p_1 cuts a, b respectively at K, L, then as p_1 envelopes the conic C_1, K, L describe two projective rows on a, b (§ 78).

Also AP_2 is the polar of K, hence AP_2 turns about A in a pencil projective with the row described by K on a (§ 99); similarly BP_2 describes a pencil with vertex B projective with the row of L on b.

Hence AP_2, BP_2 describe projective pencils; $\therefore$ the locus of P_2 is a conic passing through A and B.

Conversely. If a point P describes a conic, its polar p, with respect to a given fixed conic R, envelopes a conic.

182. The locus of the poles with respect to a conic R, of tangents to a conic C_1, is the same conic as the envelope of the polars with respect to R of points on C_1.

Let C_2 be the conic described by the pole with respect to R of tangents to C_1.

Let p_1, q_1 touch the conic C_1 at P_1, Q_1; and let P_2, Q_2 be their poles with respect to R. Then P_2Q_2 is the polar of the intersection X of p_1 and q_1.

But, if q_1 is made to coincide with p_1, X becomes the point of contact P_1.

At the same time Q_2 coincides with P_2, $\therefore$ P_2Q_2 becomes the tangent p_2 through C_2 at P_2.

Hence the polar of P_1 touches C_2: and as P_1 describes the conic C_1 its polar envelopes C_2.

Thus each conic is either the pole locus or polar envelope derived from the other, the two conics C_1, C_2 are called **polar reciprocal conics** with respect to the conic of reference R.

To each point P_1 of one conic C_1 corresponds one point P_2 of the other conic C_2, such that P_1 is the pole with respect to R of the line p_2 which touches C_2 at P_2, and P_2 is also the pole of the line p_1 which touches C_1 at P_1.

183. Theorem. If a conic C_1 lies entirely within the conic of reference R, its polar reciprocal C_2 lies entirely outside R.

For every tangent to C_1 cuts R in real points, hence its pole lies outside R.

Conversely. If R is entirely within C_1, then C_2 is entirely within R. [In this case, if R is a circle or ellipse, so also is C_2.]

But if C_1 is entirely outside R, C_2 is not necessarily entirely within R.

Common Points and Tangents. If C_1 and R have a common tangent touching R at L, then C_2 cuts R at L. Hence if C_1, R have four common tangents, then C_2, R have four real intersections; and if C_1, R have four common points, then C_2, R have four common tangents.

If C_1 touches R, then C_2 touches R at the same point.

If C_1 cuts C_2 at P, the polar of P is a common tangent to C_1, C_2. Hence the number of real intersections of C_1, C_2 is the same as the number of real common tangents.

Exercise. Draw figures to illustrate these statements in various cases.

Parallel tangents to C_1 reciprocate into points on C_2 which are collinear with the centre O of the conic of reference R.

For, if the parallel tangent to R touches it at K, the polars of the two lines lie on OK.

184. To find the centre of the reciprocal conic.

Let O be the centre of the conic of reference R, and x_1 its polar with respect to C_1, then the reciprocal of x_1 (*i.e.* its pole with respect to R) is the centre of the reciprocal conic C_2.

Draw any diameter KL of R, cutting the conic C_1 at A_1, B_1 and the line x_1 at X_1. The reciprocals of the points A_1, B_1, X_1 on OK are three lines a_2, b_2, x_2 parallel to the tangent to R at K; let them cut OK at D, E, Y respectively.

14—2

Then $\qquad \mathsf{OA_1 \cdot OD = OB_1 \cdot OE = OX_1 \cdot OY = OK^2}.$

But x_1 is the polar of O for the conic C_1; hence

$$\{OX_1A_1B_1\} = -1.$$

$$\therefore \quad \frac{1}{OA_1} + \frac{1}{OB_1} = \frac{2}{OX_1}, \qquad \therefore \quad OD + OE = 2 \cdot OY.$$

Hence x_2 is equidistant from the lines a_2, b_2 which are parallel tangents to the conic C_2, and therefore x_2 passes through the centre of C_2.

Similarly the reciprocal of each point of x_1 passes through the centre of C_2. Hence the centre of C_2 is the reciprocal X_2 of x_1.

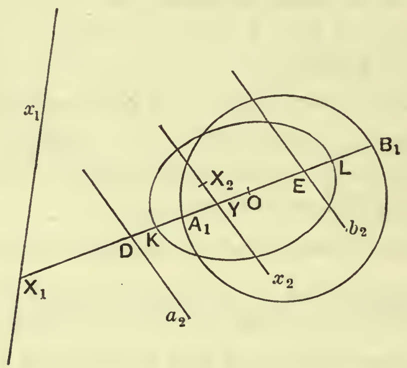


Fig. 150.

Corollary 1. If O lies on C_1, x_1 is the tangent to C_1 at O and is a diameter of R. Hence the centre of C_2 is an infinitely distant point on the diameter OK′ of R conjugate to x_1. Hence C_2 is a **parabola**, and its diameter is parallel to OK′.

Corollary 2. If two real tangents can be drawn from O to the conic C_1, these lie along two diameters OK, OL of R and their reciprocals are infinitely distant points on the diameters OK′, OL′ respectively conjugate to OK, OL.

Hence C_2 is a **hyperbola** whose asymptotes are parallel to OK', OL'.

If the tangents OK, OL from O to C_1 touch C_1 at X_1, Y_1, the **asymptotes** are the reciprocals of X_1, Y_1. They pass through the centre (since X_1Y_1 is the line x whose reciprocal is the centre of C_2), and touch the hyperbola at infinity.

185. Theorem. Conjugate diameters reciprocate into conjugate points on a straight line.

For the centre of C_1 reciprocates into the polar with respect to C_2 of O, the centre of the conic of reciprocation R. Hence diameters of C_1 reciprocate into points on this line.

Let p_1 be any diameter of C_1, A_1B_1 the conjugate diameter, and A_1K, B_1L the tangents (parallel to p_1) at A_1, B_1: these three lines, being parallel, reciprocate into three points P_2, A_2, B_2 collinear with O, the centre of R.

Let A_2K_2, B_2K_2 be the tangents to C_2 at A_2, B_2; they are the reciprocals of A_1, B_1; hence K_2 is the reciprocal of the diameter A_1B_1 of C_1.

But P_2 lies on the chord of contact A_2B_2 of tangents from K_2 to the conic C_2. Hence P_2, K_2 are conjugate points with respect to C_2.

Corollary. Pairs of conjugate points on a straight line describe an involution (§ 99, Cor.), hence pairs of conjugate diameters of any conic form an involution (cf. § 170).

186. Reciprocation with respect to a point. We may take a circle as the conic of reference R. The polar reciprocal of a figure with respect to a point O is the polar reciprocal with respect to a circle whose centre is at O.

If the reciprocals of points P_1, Q_1 with respect to O are the lines p_2, q_2, they are respectively perpendicular to OP_1, OQ_1. Hence the angle between p_2, q_2 equals the angle subtended at O by the join P_1Q_1.

Exercises. 1. Reciprocate a triangle with respect to its orthocentre.

2. If P, Q move along two fixed lines so that their join PQ subtends a constant angle at a given point O, prove that the reciprocal of PQ with respect to O describes a circle.

3. Prove that the reciprocal of a parabola with respect to a point on the directrix is a rectangular hyperbola. State the converse theorem.

4. Prove that the reciprocal of a central conic with respect to a point on the director circle is a rectangular hyperbola.

187. Theorem. The reciprocal of a conic with respect to a focus is a circle.

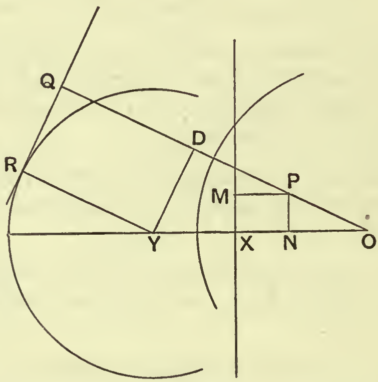


Fig. 151.

Let O be the focus, XM the directrix, OX the perpendicular to it from O, and PM from a point P of the conic, so that OP : PM = a constant e.

The reciprocal of P is a line QR perpendicular to OP through a point Q such that OP . OQ $= r^2$; the reciprocal of the directrix is a point Y on OX such that OX . OY $= r^2$.

Draw YR, YD perpendicular to QR, OQ respectively, and PN perpendicular to OX.

Then $\quad OP : OY = OX : OQ$,

but $\quad OP : OY = ON : OD$ (by similar triangles),

$\therefore \quad OP : OY = NX : DQ = PM : YR$,

$\therefore \quad OP : PM = OY : YR$;

hence YR is a constant, and the reciprocal of P envelopes a circle, whose centre Y is the reciprocal of the directrix.

Exercise. Find the reciprocals of the centre, the other focus and the other directrix.

188. Polar Reciprocation in general. If we take a "conic of reciprocation" R, and if we have any figure made up of straight lines and points we may, by taking the poles of the lines and polars of the points, obtain another figure with lines and points corresponding respectively to the points and lines of the first.

The intersection of two lines in the first figure has for its polar the join of the corresponding points of the second figure, and conversely.

If we consider the first figure as a system of points and take their polars to form the second figure, and thus obtain a system of points in which these polars intersect, and if we then take the polars of these intersections we shall get a system of lines whose intersections are the original points. Hence either system is got from the other by this process, the two figures are therefore called **polar reciprocal figures** with respect to R. **But the figures must be complete,** *i.e.* include all possible joins and intersections.

Concurrent straight lines reciprocate into collinear points; parallel straight lines into points collinear with the centre of the conic R of reciprocation. Conversely collinear points reciprocate either into concurrent lines or parallel lines according as they are not or are collinear with the centre of R.

A triangle reciprocates into a triangle.

A complete four-side reciprocates into a complete four-point, and the **descriptive** properties of a four-point follow from those of a four-side. *E.g.* from the proposition that "any point is joined to the six vertices of a four-side by six rays forming an involution" follows that "any straight line cuts the six sides of a four-point in an involution": and conversely.

189. Again a curve in the original figure may be regarded as the locus of a point, and if we take the polars of this **continuous** system of points we obtain a **continuous** system of lines enveloping a new curve. Or we may regard the original curve as enveloped by a system of lines and the locus of the poles of these lines will form a new curve.

The two curves so obtained are identical. For if we take two points A, B on the original curve whose polars are a, b, then the pole of the chord AB is the intersection of a, b: but if B is made to coincide with A, AB becomes the tangent at A, and at the same time b will coincide with a, and their intersection will become the point of contact of a, so that the pole of the tangent at A becomes the point of contact of a: hence the poles of the successive tangents to the first curve are the successive points of the second.

Thus each of the curves may be obtained from the other either by taking the pole locus or the polar envelope; the curves are hence called **polar reciprocal curves.**

Notice that in this case we do not take the complete set of joins and intersections.

The points in which a straight line cuts a curve become the tangents from its pole to the reciprocal curve. Hence the **degree or order** of one equals the **class** of the other, and conversely.

190. Duality. Any descriptive theorem relating to lines and their intersections furnishes by reciprocation a theorem relating to points and their joins, and conversely: any theorem about a curve, its tangents and their intersections becomes a theorem about the reciprocal curve, its points and the chords joining them.

Thus reciprocation doubles the number of descriptive theorems, and any theorem should be at once reciprocated,—the new theorem may or may not be already known.

Exercise. Reciprocate a four-side and its three harmonic inscribed conics (Chapter VI), with respect to one of those conics, and shew that the resulting conics are the conic of reciprocation and the other two harmonic conics circumscribed to the four-point formed by the points of contact of the original four lines.

EXAMPLES. IX.

1. Prove that the polar reciprocal of a conic with respect to (a circle whose centre is) the focus is a circle. Find the reciprocals of the other focus, the minor axis and the directrices.

2. Two conics have a common focus; prove that they cannot have more than two common tangents.

3. Two given conics have a common focus; if any other conic having the same focus touches them at P_1Q_1, prove that PQ passes through a fixed point. Also prove that the corresponding directrix of the variable conic has an envelope consisting of two conics, and find when one of these two conics is imaginary.

4. If PK touches a circle at K and subtends a right angle at a point S within the circle, prove that the locus of P is a polar reciprocal of an envelope of normals drawn to a conic section.

5. Prove that the locus of the pole of the tangent to a circle with respect to a concentric conic is a concentric conic; and that if these two conics cut orthogonally at their four points of intersection, then the tangents from any point on the circle to the given conic are perpendicular.

6. Reciprocate, with respect to the circumcentre, the theorem that if a conic touches the sides of a triangle and passes through the circumcentre its director circle touches the nine-point circle.

7. Prove that the reciprocal of a circle with respect to a circle whose centre lies on the circumference of the given circle is a parabola.

A system of parabolas has a common focus S and all touch a given line passing through a given point T; prove that the points of contact of the other tangents from T lie on a circle, which passes through S.

Reciprocate this theorem with respect to S.

8. Find the polar reciprocal of a system of confocal conics, with respect to a circle with centre at one of the common foci.

9. Prove by reciprocation that, if two confocal conics intersect, the tangents at their intersection are perpendicular.

10. Prove, and then reciprocate with respect to a focus:

If tangents be drawn to a system of confocal conics from a point on the common axis, their points of contact lie on a circle.

11. Reciprocate with respect to a focus, that if PT touches an ellipse at P, and TQ perpendicular to TP touches a confocal ellipse at Q, then CT bisects PQ, C being the centre.

12. Reciprocate: "Angles in the same segment of a circle are equal."

13. Prove that the poles of a given line with respect to a system of confocal conics are collinear.

14. S, S′ are two conics having a real and finite common self-conjugate triangle; S_1, S_1' are the polar reciprocals of S, S′ each with respect to the other; S_2, S_2' are similarly formed from S_1 and S_1'; and so on.

Shew that either one or both of the conics S_n, S_n' when n is infinite will be a pair of straight lines.

15. The polar reciprocal of an ellipse with respect to the circle on the major axis as diameter is a similar ellipse.

16. A circle passes through the centre of a hyperbola; find its reciprocal with respect to the hyperbola.

17. A system of circles pass through two points A, B. Find their reciprocals with respect to a rectangular hyperbola of which A, B are respectively the centre and one focus.

18. Reciprocate a triangle and its circumcircle, incircle and nine-point circle with respect to a rectangular hyperbola passing through the three vertices and the orthocentre.

19. Two conics have four given intersections and four given common tangents; find the conic with respect to which they are reciprocal.

20. Prove that two triangles which are reciprocal with respect to a given conic are in homology. Also shew that the six points in which the sides of one triangle intersect the non-corresponding sides of the other lie on a conic C_1, and the six lines joining the vertices of one to the non-corresponding vertices of the other touch a conic C_2, and prove that C_1, C_2 are reciprocals with respect to the given conic.

21. Prove that, if a rectangular hyperbola passes through the vertices of a triangle, it also passes through the orthocentre.

Reciprocate this proposition with respect to the orthocentre.

22. A chord of a conic moves so as to subtend a constant angle at the focus; find its envelope.

23. A chord of a conic subtends a constant angle at a given point on the conic; find its envelope.

CHAPTER X

HOMOLOGY

191. In connection with Desargues' Theorem (§§ 43—45) we have defined homology, axis of homology, and centre of homology.

We have seen that two quadrilaterals ABCD, A′B′C′D′ are not necessarily in homology when the four pairs of lines AB, A′B′ ; BC, B′C′ ; CD, C′D′ ; DA, D′A′ meet in points lying on one straight line (axis) : but if a fifth pair of joins of the two four-points, say AC, A′C′, meet on this axis, then the joins of corresponding vertices AA′, BB′, CC′, DD′ meet at one point (centre of homology), the sixth pair of joins of the four-points BD, B′D′ intersect on the axis and the two four-points are in homology.

If we take two four-sides a, b, c, d and a', b', c', d', the two triangles abc, $a'b'c'$ are in homology if the joins of ab, $a'b'$; ac, $a'c'$; bc, $b'c'$ pass through a centre S, and then the intersections aa', bb', cc' lie on an axis s. If now the joins of ad, $a'd'$ and bd, $b'd'$ pass through S, then the triangles abd, $a'b'd'$ are in homology, and d, d' intersect on s. Now we have the intersections bb', cc', dd' lying on s, hence the triangles are in homology, and the join of the vertices cd, $c'd'$ is concurrent with the joins of bc, $b'c'$ and bd, $b'd'$ and therefore passes through the centre S. Therefore two four-sides are in homology if five intersections of one are joined to the corresponding five intersections of the other by lines meeting at one point S; the join of the sixth vertices passes through S, and the four sides of one meet the corresponding sides of the other at four points lying on one axis s.

192. Let $A_1, A_2 \ldots A_n$; $A_1', A_2' \ldots A_n'$ be two sets of n-points in homology, so that the n-joins $A_1A_1', A_2A_2' \ldots A_nA_n'$ all pass through a centre of homology S, and each join A_kA_e of the first set meets the corresponding join $A_k'A_e'$ of the other at a point lying on a fixed axis of homology s.

Take another pair of corresponding points B, B' to satisfy the **two** conditions that BA_1, $B'A_1'$ meet on s, and BA_2, $B'A_2'$ meet on s.

Then the triangles BA_1A_2, $B'A_1'A_2'$ are in homology (by the converse of Desargues' Theorem); hence BB' also passes through S.

Again, since BB', A_1A_1', A_kA_k' pass through S, the triangles BA_1A_k, $B'A_1'A_k'$ are in homology; but BA_1, $B'A_1'$ meet on s, and also A_1A_k, $A_1'A_k'$; hence BA_k, $B'A_k'$ meet on s.

The two systems of $(n+1)$ points are therefore in complete homology.

Hence two more conditions are to be satisfied when one point is added to each of the two sets in homology.

Two systems of n-points are in homology if $(2n-3)$ joins of the one system, of which at least two pass through each of the n points, meet the $(2n-3)$ corresponding joins of the other system in points lying on one straight line s. If these conditions are satisfied, the remaining $\frac{1}{2}(n-2)(n-3)$ joins of the first system meet the corresponding joins of the second system at points also lying on s, and the n lines joining the points of one system to the corresponding points of the other meet at one centre S.

A set of n lines is in homology with another set of n lines, if $(2n-3)$ intersections of the first set of lines, of which each of the n lines contains at least two, are joined to the respectively corresponding intersections of the second set by lines passing through one centre S: in that case the n points in which corresponding lines of the two sets meet lie on one axis s, and all the joins of the $\frac{1}{2}n(n-1)$ intersections of the first n lines to the corresponding intersections of the second set pass through S.

Note that systems of lines are in homology if (a certain number of) their intersections are collinear with a centre S; systems of points are in homology if (a certain number of) their joins meet corresponding joins at points lying on an axis s.

193. Homology of Plane Curves. I. If to each point of one plane curve can be assigned a corresponding point of another plane curve, such that the line joining any two points of the one meets the corresponding join of the other always on a fixed line s, then the corresponding points of the two curves are collinear with a fixed point S, and the two curves are in homology with s as axis and S as centre of homology.

II. If to each tangent to one curve can be assigned a tangent to another curve such that the intersection of any pair of tangents of the one curve is joined to the corresponding intersection of tangents to the other curve by a line passing through a fixed point S, then the corresponding tangents meet on a fixed line s.

Since a tangent is the limit of a chord joining two points of a curve, and also the point of contact of a tangent is the limiting position on the tangent of the point at which it is met by an adjacent tangent, it follows that if the points of two curves are in homology (as in I) then the systems of tangents to the curves are in homology (as in II) with the same centre and axis; and conversely.

III. If P, Q, R, ... are consecutive points of one figure and P′, Q′, R′, ... of the other, and A, A′ two points connected with the curves, so that AP, AQ, AR, ... meet corresponding lines A′P′, A′Q′, A′R′, ... and also PQ, QR, ... meet P′Q′, Q′R′, ... at points lying on one straight line s, the two curves are in homology.

Let AA′, PP′ meet at S. The triangles APQ, A′P′Q′ are in homology, therefore QQ′ passes through S. Hence and similarly RR′, ... pass through S.

Also AA′, PP′, ZZ′ pass through S, therefore the triangles

APZ, A′P′Z′ are in homology. But AP, A′P′ and AZ, A′Z′ meet at points lying on *s*, hence corresponding chords PZ, P′Z′ meet on *s*.

Hence two curves are in homology, if the tangent at each point of one meets the corresponding tangent at a point lying on a fixed axis *s*, and also the join of each point of one curve to a certain point meets the join of the corresponding point of the second curve to a second fixed point on the same axis.

IV. If A, B are related to one curve PQ..., and A′B′ to another curve P′Q′..., and if to each point P of one curve we can assign a point P′ of the other, so that AP, A′P′ and also BP, B′P′ intersect always on a fixed line *s* which passes through the intersection of AB, A′B′; then the curves are in homology, with that line *s* as axis, and with the intersection S of AA′, BB′ as centre of homology.

[For AP, AQ, BP, BQ, AP meet the corresponding five joins of the four-point A′, B′, P′, Q′ at points lying on *s*; hence PQ, P′Q′ intersect on *s*.]

194. Problem. To find a curve passing through two points A′, B′, and in homology with a given conic.

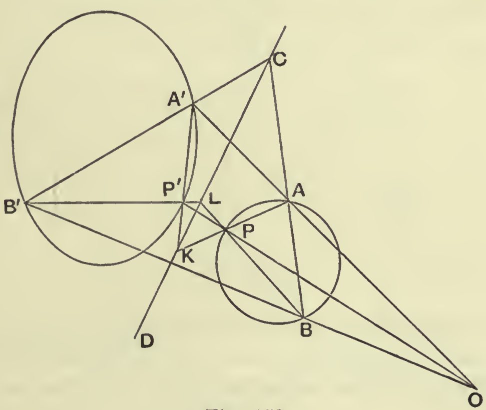


Fig. 152.

Take any chord AB of the conic meeting A′B′ at C, draw any line CD through C.

Take any point P of the conic, and let AP, BP cut CD at K, L respectively. Join A′K, B′L intersecting at P′.

The locus of P′ is the curve required.

Also the pencil $A\{P \ldots\} = B\{P \ldots\}$,

$\therefore$ the row $\{K \ldots\} = \{L \ldots\}$,

$\therefore$ the pencil $A'\{P' \ldots\} = B'\{P' \ldots\}$;

hence the locus of P′ is a conic passing through A′, B′.

195. The vanishing line. If O is the centre of homology, and XY the axis, and PA, PA′ corresponding lines (intersecting at a point P on XY), the point A′ on the second line corresponding to A on the first is the point at which OA meets the second line. Again, if Q is any other point of XY, QA corresponds to QA′.

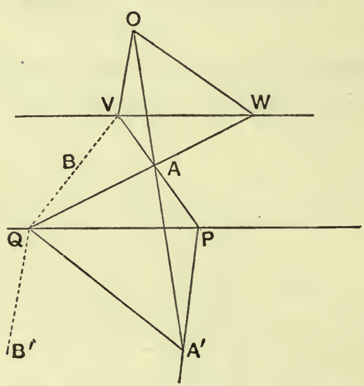


Fig. 153.

If now OV is drawn parallel to PA′ to meet PA at V, then V is the one point of PA which has no finite corresponding point on PA′. Further the line corresponding to QV must be a line QB′ parallel to PA′.

Thus to a system of lines meeting at V in the first figure corresponds a system of parallel lines in the second figure. V is called the **vanishing point** of lines parallel to PA′.

The vanishing point W of lines parallel to any other direction QA′ is found by drawing OW parallel to QA′ meeting QA at W.

Then, from the similar triangles OVA, A′AP, we have

$$VA : AP = OA : AA'$$

and, similarly $$WA : AQ = OA : AA'.$$

Hence $VA : WA = AP : AQ$, also $V\hat{A}W = PAQ$, hence the triangles VAW, PAQ are similar.

∴ VW is parallel to PQ.

Hence, and similarly, the vanishing points of all directions in the homologous figure lie on a line through V parallel to the axis. This line is called the **vanishing line** of the second figure.

Since VW passes through V, the homologous line to VW does not meet OV; since VW is parallel to the axis, the homologous line does not meet the axis; there is no finite line of the homologous figure to satisfy these conditions.

We may call the line homologous to VW, the **line at infinity**, each point of it is an intersection of parallel lines in the homologous figure.

Problems. **1.** Given the centre O, axis XY, and a vanishing point V, to find the homologue of a point A.

[If VA cuts the axis at P, a line through P, parallel to OV, will cut OA at the required point A′.]

2. Given the centre O, axis XY, and vanishing line VW, to find the homologue of a line *a*.

[If *a* cuts VW at V, and XY at P, the homologue is the line through P parallel to OV.]

3. Given the centre O, axis XY, and two homologous points A, A′, to construct the vanishing line.

[Take any point P on XY; a line OV, parallel to PA′, will meet PA at a point V on the vanishing line.

If the line VW (parallel to XY) cuts OA at W, and OA cuts XY at K, then $WA : AK = VA : AP = OA : AA'$.]

If we draw OV′ parallel to PA, meeting PA′ at V′, then all lines parallel to PA in the original figure become, in the homologous figure, lines passing through V′; and V′ of the second figure is the vanishing point of all lines parallel to PA.

We thus get, in the homologous figure, a **secondary vanishing line** V′W′.

Since OVPV′ is a parallelogram, the perpendicular from O to V′W′ equals the distance of VW from the axis.

196. Homology of Conics. Homologous pencils are projective, for they cut the axis in the same row.

Homologous rows are projective, for they are transversals of one pencil, whose vertex is the centre of homology.

If A, B, K, L, M, N on a conic are homologous to A′, B′, K′, L′, M′, N′; the pencils A {KLMN}, A′ {K′L′M′N′}, B {KLMN}, B′ {K′L′M′N′} are all projective. Hence K′, L′, M′, N′ lie on a conic through A′, B′.

Hence the homologous curve is a conic.

Also the cross-ratio of four points on a conic equals the cross-ratio of the homologous points on the homologous conic.

197. Problem. Given a conic, and a centre and axis of homology, to construct the homologous conic passing through a given point A′.

Let OA′ cut the given conic at A.

Join A to any point P of the conic, cutting the axis at K; join A′K, cutting OP at P′.

As P moves round the conic, P′ will describe the homologous conic.

Since OA′ cuts the conic in two real and different, or coincident, or imaginary points, there are two solutions of the problem.

If O lies inside the conic the two solutions are always real.

If O lies outside the conic the solutions are real and different if A′ and the conic lie in the same or opposite angles formed at O by tangents to the conic; coincident if A′ lies on one of those tangents; imaginary if A′ and the conic lie in supplementary angles formed by the tangents from O.

Problem. Given a conic and a centre and axis of homology, to construct the homologous conic which touches a given line a'. Shew that there are two solutions, and find when they are real.

198. Problem. Given the centre and axis of homology and the vanishing line, to construct the conic homologous with a given conic.

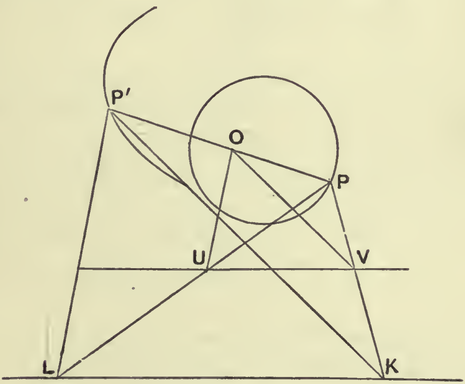


Fig. 154.

Take any point V on the vanishing line; and join a point P of the conic to V, and produce PV to cut the axis at K.

Draw KP′ parallel to OV, cutting PO at P′.

As P moves round the conic, P′ will describe the homologous conic.

If we take any other point U on the vanishing line, and PU cuts the axis at L,

UV is parallel to KL, $\therefore$ PU : UL = PV : VK,

but VO is parallel to KP′, $\therefore$ PV : VK = PO : OP′,

hence PU : UL = PO : OP′, $\therefore$ LP′ is parallel to UO.

Hence we get the same point P′, if we take U in place of V; *i.e.* there is one solution only.

Exercise. Construct the homologous conic by drawing tangents to the given conic, and finding the homologous tangents (and points of contact).

199. Theorem. If through the centre of homology O we can draw a tangent OG to the given conic, then OG also touches the homologous conic.

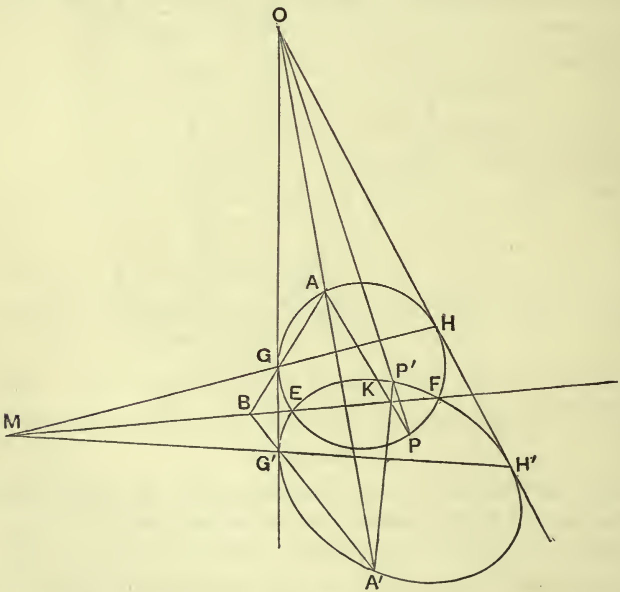


Fig. 155.

Also if GH be the chord of contact of tangents from O to the given conic, then OG, OH touch the homologous conic at G′H′ such that GH, G′H′ meet on the axis.

If A′ is a given point on the homologous conic, and we join AG to meet the axis at B, then A′B cuts OG at G′: and if GH meets the axis at M, then MG′ cuts OH at H′. Hence the homologous conic is the conic which touches OG, OH at G′, H′ and passes through A′.

The polar of M with respect to the given conic passes through O and is the harmonic conjugate of OM with respect to OG, OH, hence it is also the polar of M with respect to the homologous conic.

200. Theorem. If the given conic cuts the axis at E, F then the homologous conic also passes through E, F; and the tangents at E, F to the two conics meet at points T, T′ collinear with O.

We thus obtain another construction for the homologous conic which touches a given line a'.

Let a' cut the axis and from this point draw a tangent a to the given conic; let a cut ET at B, join OB to cut a' at B′, then B′E is the tangent at E to the homologous conic. If B′E cuts OT at T′, then T′F is the tangent at F. Hence the homologous conic is that conic which touches T′E, T′F at E, F and also touches a'.

The pole of TT′O with respect to either conic lies on the axis and is the harmonic conjugate with respect to EF of the point where TT′ cuts EF.

201. To find the centre of the homologous conic. If we take a tangent to the given conic parallel to the axis, the homologous tangent is also parallel to the axis. Hence if we draw tangents AU, BV to the given conic parallel to the axis, and A′, B′ are the points homologous to A, B, then A′B′ is a diameter of the homologous conic, and its middle point is the centre (cf. § 203, Cor. 1).

202. Theorem. If the vanishing line cuts the given conic in two real points the homologous conic is a hyperbola: if it touches, a parabola; if it does not cut it, an ellipse: and conversely.

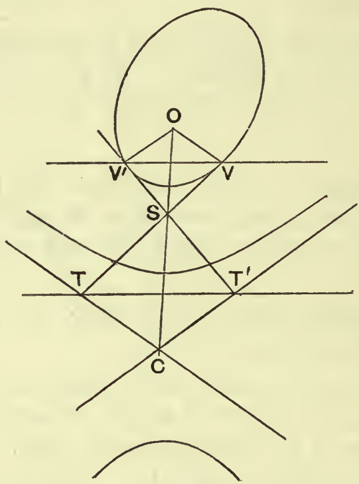


Fig. 156.

I. Let the given conic cut the vanishing line at V, V′. Draw a tangent VT to cut the axis at T and draw TC parallel to OV.

Then TC is the tangent homologous to TV, and its point of contact (the homologue of V) is at infinity.

Hence TC is an asymptote: the other asymptote CT′ may be similarly obtained from the tangent at V′.

Their intersection is the centre C.

II. If the vanishing line VU touches the conic at V. There is only one infinity direction of the homologous conic: and the homologous tangent to VU is entirely at infinity. Hence the homologous conic is a parabola whose axis is parallel to OV.

III. If the vanishing line does not cut the conic there are no infinity directions.

Corollary. If the given conic is a hyperbola the secondary vanishing line cuts the homologous conic.

203. Theorem. A point and its polar with respect to a conic project into a pole and polar with respect to the homologous conic. Also conjugate lines through a point project into conjugate lines in the homologous figure.

Corollary 1. The centre of the homologous conic is the point homologous to the pole of the vanishing line with respect to the given conic.

Corollary 2. If the centre of homology is a focus of the given conic, conjugate lines at that point are perpendicular, and these lines project into the same lines in the homologous figure, hence the centre of homology is also a focus of the homologous conic.

Corollary 3. If the vanishing line is a directrix of the given conic, the corresponding focus projects into the centre of the homologous conic.

Corollary 4. If a focus S be taken as centre of homology and its directrix as vanishing line, S is the centre of the homologous conic, and all conjugate diameters are perpendicular, hence the homologous conic is a circle.

Corollary 5. The homologue of a circle whose centre is the centre of homology is a conic whose focus is at that centre, and the corresponding directrix is the secondary vanishing line. If the circle touches the primary vanishing line the homologous conic is a parabola.

Corollary 6. A system of concentric circles project into a system of conics with a common focus and directrix.

204. To find the eccentricity of the conic homologous to a circle whose centre O is the centre of homology.

Draw OA perpendicular to the axis, cutting the primary vanishing line at W, and the secondary vanishing line at X.

Take any point P on the circle, join PW, cutting the axis at K; and draw KP′ parallel to OA, cutting PO at P′, and the secondary vanishing line at M. P′ is homologous to P.

Also $OW = XA = MK$,

and $PO : OW = PP' : P'K$,

$\therefore\ PO : OW = OP' : P'M$.

Hence $OP' : P'M$ has a constant value: and P′ describes a conic with O as focus and XM as directrix.

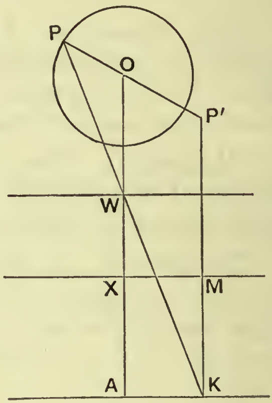


Fig. 157.

Corollary. This is another proof that the distance of a point from its focus bears a constant ratio to its distance from the corresponding directrix; and that the ratio is greater than, equal to, or less than unity according as the conic is a hyperbola, parabola or ellipse.

205. To find when the homologous conic is a circle. Let P be the pole of the vanishing line with respect to the given conic. Draw ON perpendicular to the vanishing line; join PN, cutting the axis at K; draw KC parallel to ON to meet PO at C.

C is the centre of the homologous conic.

Draw any pair of conjugate lines through P, cutting the vanishing line at A, B; and the axis at G, H.

Then CG, CH are conjugate diameters; hence, if the homologous conic is a circle, these lines are perpendicular; therefore the parallel pair of lines OA, OB are perpendicular, and $AN \cdot NB = ON^2$.

Hence the homologous conic is a circle, if the involution traced on the vanishing line by pairs of points conjugate with respect to the given conic has its centre at N the foot of the perpendicular from O, and its power is $-ON^2$.

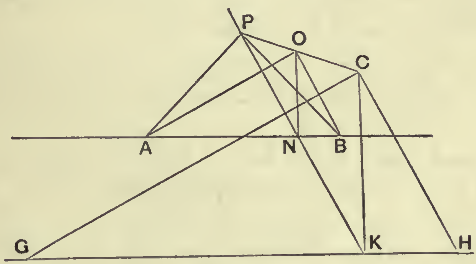


Fig. 158.

HOMOLOGY OF TWO GIVEN CONICS.

206. Theorem. If two conics touch the same line at the same point they are in homology with that point as centre.

Let OT be the common tangent at O.

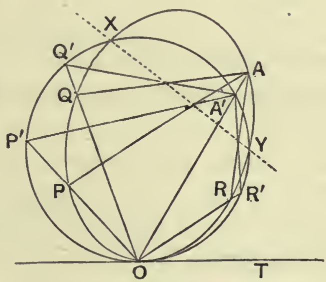


Fig. 159.

Draw OAA′ to cut the conics at A, A′ respectively.

Let any other lines through O cut the conics at P, P′; Q, Q′; R, R′, respectively.

Hence, from the definition of the conic, the pencil

$$A\ \{OPQR\} = O\ \{TPQR\}, \text{ and } A'\ \{OP'Q'R'\} = O\ \{TP'Q'R'\}.$$

∴ A {OPQR} = A′ {OP′Q′R′}, and these are two projective pencils with a common self-corresponding ray AA′O: ∴ the two pencils are in perspective, *i.e.* the intersections of AP with A′P′, AQ with A′Q′, etc. lie on a straight line *s*.

Again the triangles APQ, A′P′Q′ are in homology with O as centre, and AP, A′P′, AQ, A′Q′ intersect on *s*, hence PQ, P′Q′ meet on *s*.

Hence and similarly the join of any two points on one conic meets the join of the corresponding points of the other conic on *s*.

Hence the conics are in homology with O as centre and *s* as axis.

Corollary. If the conics intersect in two other points X, Y, then XY is the axis *s* of homology.

207. Theorem. If two conics touch a line at the same point they are in homology with that line as axis.

From any point T on the common tangent TA draw tangents TB, TB′ to the two conics.

From any points P, Q on TA draw tangents to the two conics to cut *a* at K, L and *a′* at K′, L′ respectively.

Then, since a variable tangent traces projective rows on two fixed tangents (§ 78)

$$\{ATPQ\} = \{TBKL\}, \text{ and } \{ATPQ\} = \{TB'K'L'\}.$$

∴ {TBKL} = {TB′K′L′}, and in these projective rows, the intersection T corresponds to itself, hence the rows are in perspective, ∴ B′B, L′L, K′K are concurrent.

Hence all the joins KK′, LL′, etc. cut BB′ at the same point O.

Again, if PK, QL intersect at U, and PK′, QL′ at U′, the triangles UKL, U′K′L′ have corresponding sides intersecting on TA, ∴ UU′ passes through the intersection O of KK′, LL′.

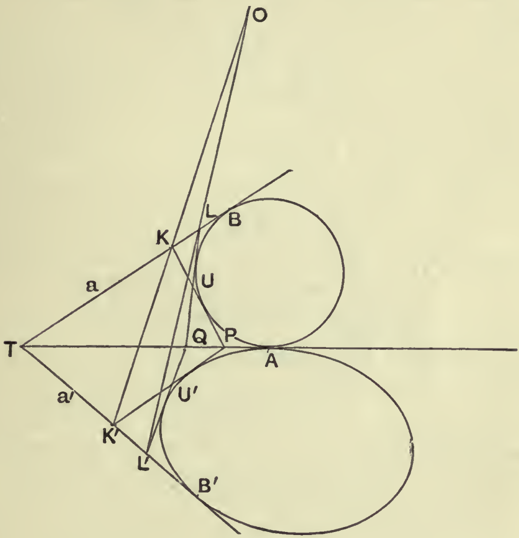


Fig. 160.

Similarly the intersections of any pairs of corresponding tangents are collinear with O.

But when the points P, Q coincide U, U′ become the points on the conics, where the tangents from P touch them.

Hence the conics are in homology with O as centre and TA as axis.

208. Theorem. If two conics pass through two given points E, F they are in homology with EF as axis: unless the segment EF is within one conic, and without the other (the latter in that case being a hyperbola with E, F on different branches).

From any point T on EF draw tangents TA, TA′ to the two conics.

Take any points K, L on EF, and let AK, AL cut the A conic again at P, Q; and A′K, A′L cut the A′ conic again at P′, Q′ respectively.

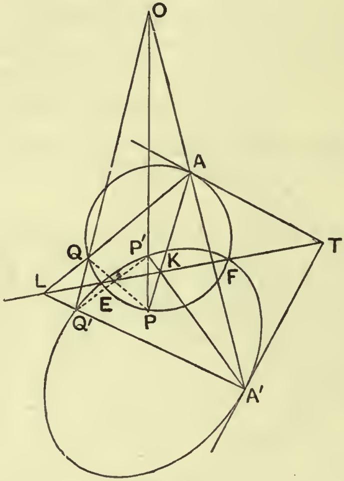


Fig. 161.

Then $$P\,\{AEFQ\} = A\,\{TEFQ\} = \{TEFL\},$$

and $$P'\,\{A'EFQ'\} = A'\,\{TEFQ'\} = \{TEFL\},$$

$$\therefore\ P\,\{AEFQ\} = P'\,\{A'EFQ'\}.$$

But PA, P′A′ cut EF at the same point K, ∴ PQ, P′Q′ intersect on EF.

Hence and similarly the joins of any pair of corresponding points intersect on EF.

Again the triangles APQ, A′P′Q′ being in homology (with EF as axis), the lines AA′, PP′, QQ′ are concurrent.

∴ joins PP′, QQ′, ..., cut AA′ at a fixed point O.

Hence the conics are in homology with EF as axis and O as centre.

Corollary 1. If the tangents at P, P′ cut EF at U, U′ we have

$$\{EFKU\} = A\ \{EFTP\} = \{EFTK\},$$

and similarly $$\{EFKU'\} = \{EFTK\},$$

hence U, U′ coincide, *i.e.* the tangents at P, P′ meet on EF.

Corollary 2. If TA, TB are the tangents from T to one conic, and TA′, TB′ to the other, then TA, TA′ being chosen the solution proceeds without ambiguity, giving a centre O, as above [and TB will correspond to TB′ (Cor. 1)].

But TB, TA′ will give a different solution, with another centre of homology O′ [and TA will correspond to TB′].

Thus the two conics are in homology with respect to the axis EF, and each of two different centres of homology O, O′: and these are two of the diagonal points of A, A′, B, B′, the third being the harmonic conjugate to T on EF.

Corollary 3. Any common tangent must pass through one, but not both, of the centres of homology.

209. If two conics intersect in four points they are in homology in either four or twelve different ways. Let A, B, C, D be the four points.

The segment AB can only be external to the conic if A, B lie on different branches of a hyperbola. Let A, B lie on one branch and C, D on the other branch of a hyperbola; if the other conic is an ellipse or parabola then AB, CD are possible axes, the other pairs AC, BD; AD, BC are not.

If the other conic is a hyperbola with ABCD all on one branch the same holds.

If a hyperbola with AC on one branch and BD on another, then neither AB, CD nor AC, BD are possible, but AD and BC are possible being external to both conics.

Hence either one or three pairs of the joins of A, B, C, D may be used as axes of homology, and each join may be taken with either of two centres, giving either four or twelve different homologies.

The two centres corresponding to AB are the same as those of the opposite join CD: so that there are two or six centres respectively. If there are four common tangents then the six centres are the vertices of the four-side which they form.

Since the self-polar triangle is real (being the diagonal triangle of the four points), if there is one common tangent there must be three others, and six centres of homology.

If one conic lies entirely within the other, we can construct two real common chords (§ 110), and there are a double pair of centres giving homology in four different ways.

210. Theorem. If two conics have a pair of common tangents, their intersection is a centre of homology, provided the conics lie in the same or opposite angles formed by the two tangents.

Through O draw a line to cut the conics at A, A′ and draw tangents at A, A′ cutting the common tangents at BC and B′C′.

Draw any line through O to cut BC, B′C′ at T, T′ and draw the second tangent TP from T cutting OB, OC at Q, R and from T′, tangent T′P′ cutting OB, OC at Q′, R′.

Because tangents describe projective rows on any two tangents

$$\therefore \{PQRT\} = \{TBCA\}, \text{ and } \{P'Q'R'T'\} = \{T'B'C'A'\};$$

but, by projection from O, $\{TBCA\} = \{T'B'C'A'\}$;

hence $\{PQRT\} = \{P'Q'R'T'\}$, but QQ′, RR′, TT′ meet at O, hence P, P′ are collinear with O.

Hence the conics are in homology with O as centre, and the locus of intersection of tangents PQ, P′Q′ is a straight line, on which also intersect AP and A′P′, and all other corresponding pairs of chords.

Corollary 1. If OA cuts the first conic at A and D, and the second at A′, D′, then A, A′ being taken to correspond (as above) the solution proceeds without further ambiguity, D corresponding to D′. We shall get one, and only one other homology with centre O, by taking A to correspond to D (in which case D corresponds to A′).

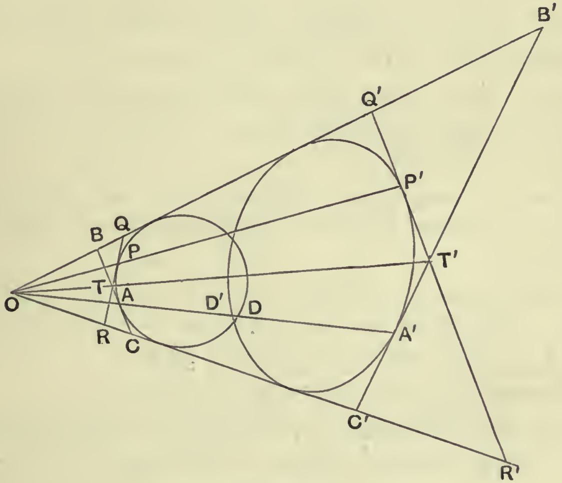


Fig. 162.

Thus the conics are in homology with respect to centre O and each of two axes: which are two of the diagonals of the four-side formed by the tangents at A, A′, D, D′. [The third diagonal passes through O, and is the harmonic conjugate of OA with respect to OB, OC.]

Corollary 2. Any point common to the two conics must lie on one or other of the axes of homology.

Corollary 3. If two conics have four common tangents they are in homology in either four or twelve different ways.

Either one or three of the pairs of vertices of the four-side formed by the common tangents will be possible.

The two axes corresponding to any vertex will be the same as those corresponding to the opposite vertex, thus there are two or six axes of homology respectively. If the conics intersect in four points, the sides of the four-point are the six axes.

211. If the chords of contact of the tangents from O meet at M, then each axis passes through M. (See fig. 155.)

If the axis cuts the conics at E, F then M is a double point of the involution determined by E, F and the points where the tangents cut the axis.

M has the same polar with respect to each conic, *viz.* the harmonic conjugate of OM with respect to the two tangents OG, OH from O.

Let this polar cut the first conic at a point P, join GP to cut MG′ at L; let OP cut the second conic at P′Q′ and G′P′, G′Q′ cut GP at X, Y. Then MX, MY are the two axes.

But at G′ the pencil G′ {OMP′Q′} is harmonic.

∴ {GLXY} is harmonic; ∴ M {GG′XY} is a harmonic pencil, *i.e.* the axes are harmonic conjugates with respect to the chords of contact.

If from any point on the axis we draw the two tangents TP, TR to one conic and TP′, TR′ to the other, then PP′ and RR′ will pass through O, but PR′ and P′R will pass through a second centre of homology O′, so that the conics are also in homology with respect to O′ and this axis. If there are two other common tangents then O′ is their intersection.

Exercises. 1. The second centre for MY will be the same point O′.

2. The polars of O′ pass through M and are harmonic conjugates to MX, MY.

3. The intersection M of the axes is the pole for either conic of the join OO′ of the vertices.

212. If the diameters DE, D′E′ of two conics meet at M, and the conjugate diameters AB, A′B′ are parallel to a line MX, and if also $MD . ME : MD' . ME'$ as $\frac{DE^2}{AB^2} : \frac{D'E'^2}{A'B'^2}$, the two conics are in homology with MX as axis.

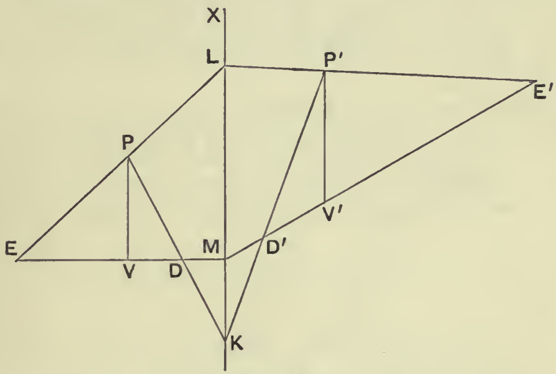


Fig. 163.

Take any point P on the first conic, and let DP, EP cut MX at K, L.

Join KD′, LE′ intersecting one another at P′.

Draw PV parallel to MX to meet DE at V, and P′V′ parallel to MX to meet D′E′ at V′.

Then $$PV^2 : EV . VD = AB^2 : DE^2.$$

But $$PV : VD = KM : MD;\ \ PV : EV = ML : ME,$$

$$\therefore\ PV^2 : EV . VD = KM . ML : MD . ME,$$

$$\therefore\ MD . ME : KM . ML = \frac{DE^2}{AB^2},$$

$$\therefore\ MD' . ME' : KM . ML = \frac{D'E'^2}{A'B'^2}.$$

But $$P'V'^2 : EV' . V'D' = K'M . ML : M'D' . M'E',$$

$$\therefore\ P'V'^2 : EV' . V'D' = A'B'^2 : D'E'^2,$$

therefore P' lies on the second conic.

Hence, and similarly, to each point P of the first conic corresponds a point P' of the second so that DP, $D'P'$ meet on MX, and also EP, $E'P'$. Hence, by § 193 (IV), the conics are in homology with MX as axis, and the intersection of DD', EE' as centre of homology.

Corollary. The line MX is a common chord of the two conics, cutting them in the same two (imaginary) points.

213. Problem. Given a conic and an axis of homology, to construct the homologous conic passing through three given points.

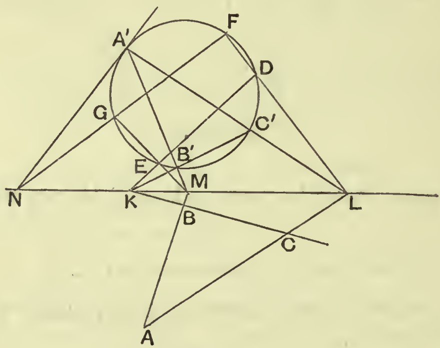


Fig. 164.

Let A, B, C be the three given points; and let BC, CA, AB meet the axis at K, L, M respectively.

Draw a line through K to cut the given conic at D, E and draw LD, ME, cutting the conic again at F, G. Join FG cutting the axis at N.

If a quadrilateral inscribed in a conic has three of its sides passing through three fixed points lying on one straight line, the fourth side will pass through a fourth point on that line. Hence if we turn FG round N until it becomes a line NA′ touching the conic at A′, and A′M, A′L cut the conic at B′, C′ respectively, B′C′ will pass through K.

We have now constructed a triangle A′B′C′ in homology with ABC with the given line as axis, hence AA′, BB′, CC′ meet at a point S, which is the centre of the required homology; and the homologous conic can be completely determined (§ 197).

Corollary. The two tangents from N to the given conic give two centres of homology; but two conics are in homology with a given line as axis with two different centres.

Hence the two centres determine only one homologous conic.

Problem. Given a conic and a centre of homology S, to construct the homologous conic passing through three given points A, B, C.

Join SA, SB, SC cutting the given conic at A′, B′, C′; the triangles ABC, A′B′C′ determine the axis of homology. Hence the homology is completely determined.

Since there are two positions, in general, of each of the points A′, B′, C′ there are eight different positions of the axis, giving four different homologous conics.

Corollary. If S is outside the given conic, the problem is identical with the construction of a conic to pass through three points and touch two lines (the tangents from S).

Problem. Given five points on a conic to construct the centre.

Let A, B, C, D, E be the five points. Describe a circle passing through D, E and construct the centre of homology S for the circle, the axis DE, and the three points A, B, C. The conic ABCDE is the conic homologous to the circle, with S as centre and DE as axis of homology, hence its centre is given by the construction of § 201.

Problem. Find the pole of a given line with respect to the conic which passes through five given points.

[Pole and polar are homologous to pole and polar.]

Exercises. 1. Given a conic and an axis of homology, construct the homologous conic,

(*a*) passing through two given points and touching a given line;

(*b*) passing through one given point and touching two given lines;

(*c*) touching three given lines.

2. Given a conic and a centre of homology, construct the homologous conic to satisfy the condition (*a*) (*b*) or (*c*) of exercise 1.

3. Construct the centre of the conic which passes through n points and touches $(5-n)$ lines, where n is 0, 1, 2, 3 or 4.

4. Construct the asymptotes and axes of a conic which satisfies five given conditions.

214. Projection in Space. A point A may be projected from a centre S on to a plane α, by joining SA by a line cutting α at A′.

A line a is projected by drawing a plane through S and a to cut the plane α in a line a', which is the projection of a.

Figures in a plane α may be projected into figures in another plane α'. The straight line in which the two planes intersect is called the **axis** of projection; any line and its projection meet on the axis, and lie in one plane passing through the centre of perspective.

Collinear points project into collinear points, and concurrent lines into concurrent lines.

We may regard the figure as made of a system of points, and project by means of a **sheaf** of lines all passing through the centre S; or we may regard it as a set of lines, and project by means of a **sheaf** of planes all passing through the centre S.

A curve is projected either (a) by means of a sheaf of lines through S and the system of points which form the curve, or (*b*) by means of a sheaf of planes through S and the system of tangents which envelope the curve, the sheaf cutting the plane α' in a system of lines whose envelope is the projected curve. The curves obtained by the two methods are identical.

If we project from a plane on to a **parallel** plane, any line AB will project into a parallel line A′B′, and the length AB : A′B′ as the perpendiculars from S to the two planes. Hence the projection will be similar and similarly situated to the original figure. *E.g.* the projection of a circle on a parallel plane is also a circle.

Straight lines intersecting at a point V in α project into straight lines meeting at the point V′ where SV meets α'. If V lies on the line v in which the plane α is cut by a plane through S parallel to α', the lines which meet at V project into parallel lines in the plane α'.

Also parallel lines in α project into lines meeting at a point lying on the line v' in which α' is cut by a plane through S parallel to α.

These two lines (v, v') are called **the vanishing lines of the projection**; they are parallel to the axis s. Each point of the vanishing line v is the **vanishing point** of lines parallel to one direction in α'. The vanishing point of lines in α' perpendicular to the axis s is N the foot of the perpendicular from S to v; the vanishing point of lines making an angle A with that perpendicular direction is a point V such that angle NSV equals A; if A is 45° then NV equals NS.

Exercises. 1. Given the axis s, and NV the vanishing line, of which N is the foot of the perpendicular from S, and NV = NS, construct the figure whose projection is a square with one side AB lying in the axis s.

[Take BK along s equal to AB. Join KV cutting BN at C; draw CD parallel to BA cutting AN at D.

ABCD is the figure required. For the projection of KC makes an angle of 45° with KB, hence CBK projects into an isosceles right-angled triangle, and the projection of BC equals BK.]

2. Find the figure whose projection is a square with one side parallel to s.

3. Find the figure whose projection is
   (*a*) a square with one side inclined to s at 30°;
   (*b*) a hexagon with one side lying along s;
   (*c*) a hexagon with one side perpendicular to s.

The construction of figures by the use of vanishing points and lines is called Perspective Drawing, and the student may refer to books on that subject for further illustrations of the applications of the theory to the solution of practical problems. We may note that if, in place of figures lying in one plane α' we have figures in several planes, *e.g.* the faces of a solid body, the corresponding figures in α will have various vanishing lines, but figures in parallel planes will have one and the same vanishing line.

215. If A, B, C, D are four collinear points in the plane α, their projectors lie in one plane through S and AB, hence their projections A′, B′, C′, D′ lie in a straight line lying in the same plane and in the plane α'. Also A′B′ and AB are two transversals of a plane pencil, hence $\{ABCD\} = \{A'B'C'D'\}$.

Hence **a row of points on a straight line projects into a projective row.**

If a, b, c, d are four concurrent lines in the plane α, meeting at T, their projecting planes form an axial pencil with ST as axis, and the projections are four lines a', b', c', d' meeting at the point T′ which is the projection of T. Also the planes Sa, Sb, Sc, Sd will cut the axis s at four points A, B, C, D which also lie on the lines a', b', c', d'; and ABCD is a transversal of both pencils, hence $\{abcd\} = \{a'b'c'd'\}$.

Hence **a plane pencil projects into a plane pencil projective with the original pencil.**

The Projection of a Conic is a Conic. First Proof. The conic is the locus of the intersections of corresponding rays of two projective pencils; but these pencils project into two pencils projective with the original pencils and therefore with each other. Hence the points forming the conic project into the intersections of corresponding rays of two projective pencils, and these form a conic. [See also § 217.]

Second Proof. The conic is the envelope of the joins of two linear projective rows in the plane α, and these project into two linear projective rows in the plane α', hence the projection is a conic.

Corollary 1. The cross-ratio $\{ABCD\}$ of four points on the conic equals the cross-ratio $\{A'B'C'D'\}$ of the corresponding points on the projected conic.

Corollary 2. Parallel tangents do not, in general, project into parallel tangents; hence the centre does not, in general, project into the centre of the projected conic.

216. Connection between Projection in Space and Homology in a Plane. If a figure in a plane α is projected into a figure in a plane α', corresponding lines meet on the intersection s of the two planes. If now the plane α' be turned about the axis s, and brought into coincidence with α; corresponding lines still meet on s, and the figures become figures in homology with s as axis; also S will become the centre of homology.

Conversely two plane figures in homology become two figures in projection if a plane containing one of the figures be turned about the axis of homology, corresponding lines continuing to meet on the axis, and the joins of corresponding points meeting now at one point S (lying outside both planes).

Locus of S. The vanishing lines of the homology become the vanishing lines of the projection, and a plane through S will cut the axis s and the two vanishing lines v, v' at points A, W, X (cf. fig. 157), such that SWAX is a parallelogram. Hence S describes a circle with centre W, and radius equal to AX, lying in a plane perpendicular to the axis.

217. Plane Sections of a Cone whose Base is a Circle. Theorem. Any plane section of a cone on a circular base is a curve such that the joins of any two points on it to a variable point on it describe two projective pencils with those two points as vertices.

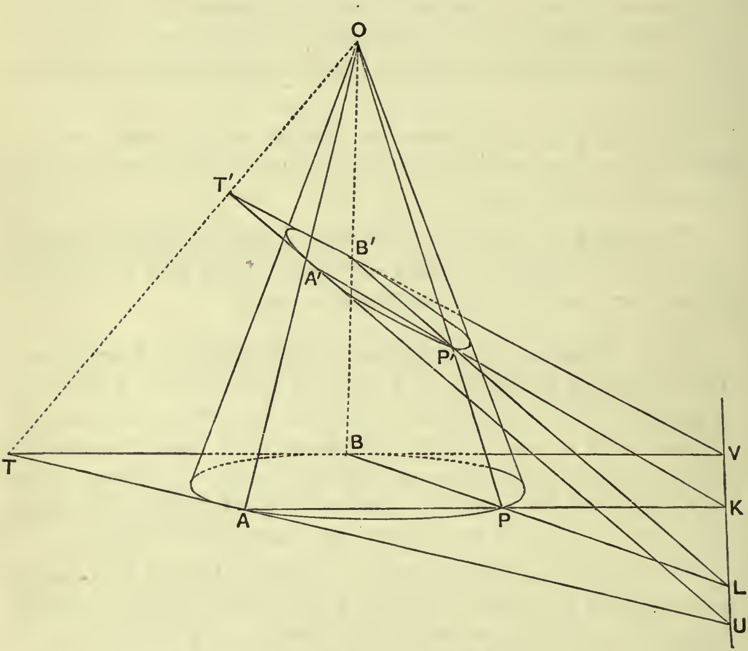


Fig. 165.

Let O be the vertex of a cone on a circular base ABP, and A′B′P′ any plane section.

Join OA′, OB′ and produce to meet the circle at A, B.

Join any other point P′ on the section to O by a line cutting the circle at P.

Let the planes OAP, OBP cut at K, L respectively the line in which the plane of the section cuts the plane of the circular base.

Then AP, A′P′ both pass through K, and BP, B′P′ both pass through L.

Now as P moves round the circle AP, BP turn through equal angles at A, B and therefore describe two projective pencils; hence K, L describe projective rows on KL.

Therefore, in the plane A′B′P′, A′P′ and B′P′ describe projective pencils about the vertices A′, B′.

Corollary 1. If the tangents TA, TB meet KL at U, V respectively, the planes OTA, OTB touch the sides of the cone, and therefore contain the tangents T′A′, T′B′ to the section, also passing through U, V. But in the circle AB at A corresponds to BT at B.

Hence, in the section, A′B′ at A′ corresponds to B′T′ at B′, and similarly B′A′ to A′T′.

Therefore the tangents at A′, B′ to the section are those rays of the pencils which correspond respectively to B′A′ and A′B′.

Corollary 2. The section is a hyperbola, parabola or ellipse according as a parallel plane through O cuts the sides of the cone in two real and different lines, or touches the cone or cuts it only at O.

Corollary 3. This proposition is equally true whether the cone is right or oblique.

218. Theorem. A tangent to any plane section of a circular cone describes projective rows on any two given tangents to that section.

Let T′A′, T′B′ be two given tangents to the plane section, and A′B′ any other tangent cutting these two at A′, B′ respectively.

The planes OT′A′, OT′B′, OA′B′ touch the sides of the cone; hence they cut the plane of the circular base in lines TA, TB, AB which touch the circle.

Now a variable tangent AB to a circle describes projective rows on two given tangents, therefore A, B describe projective rows on TA, TB.

But, in the plane OTA, we have two transversals TA, TA′ of a pencil whose vertex is O, hence the row described by A′ is projective with the row described by A.

Similarly the row described by B′ is projective with the row described by B.

Hence A′, B′ describe projective rows on TA′, TB′.

Corollary. On the tangents to the circle T on TA corresponds to B on TB. Hence T on TA′ corresponds to B′ on TB′. Similarly T on TB′ corresponds to A′ on TA′.

Exercise. Find the point of TB′ corresponding to a point at infinity on TA′, for different positions of the plane TA′B′.

219. Theorem. Every curve which is an envelope of the joins of two projective rows on two straight lines in a plane is a section of some cone which has a circular section.

Let TA, TB be the two fixed lines, touching the envelope at A, B. In any plane through TA describe a circle to touch TA at A, and draw the second tangent TB′ to this circle. Let PQ be any other tangent to the envelope cutting TA, TB at P, Q (corresponding points of the projective rows); and draw a second tangent PQ′ to the circle, meeting TB′ at Q′; similarly take another tangent RS to the envelope, and RS′ to the circle.

Then $\{TBQS\} = \{ATPR\}$; also in the plane of the circle we have $\{TB'Q'S'\} = \{ATPR\}$.

Hence $\{TBQS\} = \{TB'Q'S'\}$, two projective rows with a common point T, therefore SS′ always passes through the point O where QQ′ meets BB′.

If then we keep P, Q, Q′ fixed and vary R, S, S′, the point O will be fixed and RS will lie in a plane ORS′ which touches the cone whose vertex is O and whose base is the circle.

Hence the envelope of RS is a section of that cone.

Corollary. By suitable choice of A, the plane through TA and the circle in that plane, the cone may be made a right circular cone.

220. Theorem. Any locus of intersections of corresponding rays of two projective pencils is a section of some cone of which one section is a circle.

Take two points E, F on the locus (but not lying on different branches if the locus is a hyperbola) and in a plane through EF draw a circle passing through E and F.

From any point T on EF produced draw tangents TA, TA′ to the circle and the locus. Join A, A′ to any point K on EF, let AK meet the circle again at B, A′K meet the locus again at B′.

Then BB′ lies in the plane AKA′, and therefore meets AA′ at some point O.

Let P be any other point of EF, and let AP, A′P meet the circle and conic respectively again at Q, Q′.

Then $$\mathrm{B}\,\{\mathrm{AEFQ}\} = \mathrm{A}\,\{\mathrm{TEFQ}\} = \{\mathrm{TEFP}\},$$

and $$\mathrm{B'}\,\{\mathrm{A'EFQ'}\} = \mathrm{A'}\,\{\mathrm{TEFQ'}\} = \{\mathrm{TEFP}\},$$

$$\therefore\ \mathrm{B}\,\{\mathrm{AEFQ}\} = \mathrm{B'}\,\{\mathrm{A'EFQ'}\},$$

but BA, B′A′ cut EF at the same point K, hence BQ, B′Q′ also intersect on EF, and therefore lie in one plane.

Hence QQ′ meets BB′, but it also meets AA′, hence it passes through O.

Hence, and similarly, each point of the locus lies on a line joining O to some point of the circle; and the locus is a section of the circular cone.

Corollary 1. The proof holds equally well if for the circle we substitute any conic passing through E, F.

Corollary 2. If two conics cut the intersections of their planes at the same two points, two and only two cones can be drawn to pass through them; one vertex will lie in each pair of opposite dihedral angles between the planes.

221. Problem. Given the vertex O and a circular section of a cone to find a second set of circular sections (not parallel to the given circle).

Make a plane through O perpendicular to the given circle and passing through its centre C, let AB be the diameter of the circle lying in this plane.

Divide CO at N so that $CN^2 - ON^2 = CA^2$, and draw NP perpendicular to CO to meet CA at P. In the plane of the circle draw a line PX perpendicular to CP.

Then $PO^2 = PC^2 - CA^2 = PA \, . \, PB = PC \, . \, PP'$ (where P' is the pole of PX).

Take any pair of conjugate points Q, R on PX; P' is the orthocentre of the triangle CQR, therefore $QP \, . \, PR = PC \, . \, PP' = PO^2$.

Hence QR always subtends a right angle at O.

If PE, PF are the tangents from P to the circle, and K is any point on the circle, KE and KF describe projective pencils on PX, in which P is the vanishing point of both rows, hence KE, KF cut PX at conjugate points Q, R.

Now take a section of the cone through EF parallel to the plane OPX, let OK cut the section at L.

Then EL, FL are parallel to QO, OR; hence ELF is a right angle, and therefore the section is a circle of which EF is a diameter.

Hence the sections parallel to the plane OPX are circles.

Corollary. There are only two sets of circular sections of the cone, *viz.* those parallel to the given circular section and those parallel to the plane OPX.

EXAMPLES. X.

1. State and prove the condition that two sets of n points may be in homology.

2. Six lines a, b, c, d, e, f touch a conic, prove that another set of six lines a', b', c', d', e', f' in homology with them also touch one conic.

3. If A, B be two fixed points and P any point lying on a conic; and if PA cuts a fixed line at Q, the line through Q parallel to AB cuts PB at a point whose locus is a conic.

4. Given the centre, axis and vanishing line of homology, find the condition that the conic homologous to a given conic shall be a rectangular hyperbola.

5. If the vanishing line touches a given conic construct the focus of the homologous parabola.

6. A given parabola cuts the vanishing line of homology, find the centre of the homologous conic.

7. Draw the figure of § 208 for the case of two hyperbolas in each of which E, F lie on different branches.

8. Construct the two centres of homology described in § 208, Cor. 2.

9. (*a*) Two conics have common tangents meeting at O, and a line through O cuts the conics at A, A′ respectively. If points P, P′ move from A, A′ round the conics so that O, P, P′ are always collinear, prove that the locus of the intersection of AP, A′P′ is a straight line.

(*b*) If, in the same figure, we replace A, A′ by P, P′ we get the same line.

(*c*) Tangents at P, P′ meet on the same line.

(*d*) GP, GP′ intersect on this line, G, G′ being the points where the conics touch one of the common tangents through O.

(*e*) The chords of contact of the two fixed tangents from O intersect on this same line.

(*f*) If OA cuts the second conic again at D′, and we take A, D′ in place of A, A′ we get another line with similar properties.

10. Two conics have common tangents GG′, HH′ meeting at O, and any line through O cuts the conics at P, P′ respectively. If GP, G′P′ intersect at K, and HP, H′P′ at L, prove that K, L lie on an axis of homology of the conics, and that they describe projective rows on that axis.

11. Find the polar of a given straight line with respect to the conic which passes through five given points.

12. Find the nature of the conic which (*a*) touches five given lines, (*b*) touches two given lines and passes through three given points.

If the curve is a hyperbola construct its asymptotes.

13. Find the foci of the conic which passes through five given points.

14. Find the centre of homology of the two conics which pass through four given points and touch a given straight line.

15. Prove that a given conic can be projected into a circle and at the same time a given line to infinity.

16. Prove that a given conic can be projected into a circle and at the same time a given point into its centre.

17. Prove that a conic can be projected so that two given points become foci.

18. Prove that two conics can be projected into two confocal conics.

19. Generalize by projection that the angles between pairs of tangents from a given point to a system of confocal conics have a common bisector.

20. Generalize by projection that if a conic touches two given lines and has a given focus, the locus of the other focus is a straight line.

21. Prove that a system of conics touching four given lines can be projected into a system of confocal conics.

22. The join of PQ is divided harmonically by two opposite edges of a tetrahedron, and the join PR is divided harmonically by another pair of opposite edges; prove that QR meets the two remaining edges and is divided harmonically by them.

23. A plane turns round a line OA, and another plane turns round OB so that the two planes are always perpendicular to each other, prove that their intersection describes a cone of the second degree, which is cut by any plane perpendicular to OA or OB in a circular section. Also that any plane perpendicular to the plane AOB cuts the cone in a section of which one principal axis lies in the plane AOB.

24. Two projective ranges A, B, C, ..., A′, B′, C′, ... lie on two non-intersecting lines in space; shew that AA′, BB′, ... all intersect an infinite number of other fixed lines.

25. A set of planes have a common line of intersection, and a straight line cuts them at A, B, C, ..., prove that the pencils formed by joining any two points on the common axis to A, B, C, ... are projective.

26. Two parallel straight lines cut an axial pencil in projective rows.

27. Any two non-intersecting lines are cut by a system of axial planes in two projective pencils.

28. If two triangles in perspective be taken in two planes, and one of the planes with its triangle be turned about the line of intersection of the planes, the centre of perspective will describe a circle.

29. The axis of an axial pencil of planes $(\alpha, \beta, \gamma, \delta)$ intersects the axis of a projective axial pencil $(\alpha', \beta', \gamma', \delta')$; prove that a plane through the two lines of intersection of α, β' and α', β cuts the planes in a pencil in involution.

30. The latus rectum of a section of a right circular cone is proportional to the perpendicular from the vertex of the cone to the plane of the section.

31. Find the circular sections of a right elliptic cone.

32. The latus rectum of a parabolic section of a right circular cone is a third proportional to the distance of its vertex from the vertex of the cone and the diameter of the circular section through its vertex.

33. Determine whether a given line can be the directrix of any section of a given right circular cone. Also shew that, when the necessary condition is satisfied, there are two such sections, and that their latera recta are proportional to their eccentricities.

34. Given a parabola, construct a circle touching it at the vertex so that the circle can be turned about the axis of homology to make the cone of projection right circular.

35. Given a central conic, construct a circle touching it at the end of a major axis, and such that it can be turned about the common tangent to a position in which the cone of projection is a right circular cone.

36. If a point S be taken within a cone at a constant distance from the vertex, two sections containing S will have S as focus and the diameters of the corresponding focal spheres inscribed in the cone and touching those sections at S, will contain a constant rectangle.

37. (*a*) Prove that those chords common to a conic and a system of circles touching the conic at a given point S, which are conjugate to the common tangent, are parallel to each other.

(*b*) Hence, also, construct the circle of curvature, *i.e.* that circle which has three of its intersections with the conic coinciding at the point S of the conic, giving a contact of the second order.

(*c*) If S is at the end of a principal axis, prove that the contact is of the third order, and construct the circle of curvature in that case.

38. If corresponding vertices of two tetrahedra lie on four concurrent lines, prove that the six edges of one meet the corresponding edges of the other in six points lying three by three on four straight lines. Deduce that these six points lie on a plane, and that the four pairs of corresponding faces of the tetrahedra meet in four coplanar lines.

39. Three conics in different planes are such that each two have a common chord (along the intersection of their two planes), prove that these three common chords are concurrent; also prove that two cones pass through each pair of conics, and that the six vertices of the three pairs of conics lie three by three on four straight lines and are therefore coplanar.

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