ELEMENTARY

STATICS

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BY THE

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PREFACE.

It is hoped that this little book will be found to be a suitable text-book for students preparing for the Cambridge Previous Examination, for Woolwich, for the Oxford and Cambridge Certificate, for the London Matriculation, for the Local, and for other Examinations of a similar nature. At the same time I have endeavoured not to lose sight of the importance of the subject as an *introduction* to the study of Physics and of Practical Mechanics.

A knowledge of the 'Trigonometry of one Angle'* is assumed in some parts of the book—it will be found, however, that considerable portions may be read without any acquaintance with Trigonometry.

The truth of the Parallelogram of Forces is assumed, and the student who has not already read some elementary Dynamics is recommended to postpone the consideration of the *proof* until he reaches that subject.

I have therefore based the whole subject on Newton's Laws of Motion, a method which in my opinion greatly simplifies the subject. The accustomed proofs of the fundamental propositions based upon the principle of the Transmissibility of Force are given in a separate Chapter.

The use of the word Resolute, as the proper abbreviation for 'Resolved Part', will I hope be found useful in emphasizing the importance of the idea.

* This is required in the Additional Subjects of the Cambridge Previous Examination. See Examination Papers at the end of this book.

I have added a chapter on Graphic Statics and have reserved for that Chapter the consideration of the 'Triangle of Forces'; as I venture to think that the method of solution based upon purely geometrical principles is best kept distinct from that based upon the Resolution of Forces.

The Examples have been made as simple as possible; the collection of 100 Miscellaneous Examples at the end of the book will be found somewhat more difficult.

For this SECOND Edition (which has been stereotyped), the whole work has been very carefully revised. Thanks to the assistance of many friends and teachers many defects have been removed. I have particularly to thank my friend Mr J. C. Trautwine, C.E., of Philadelphia, U.S.A., for most valuable criticisms of almost every page, and I am much indebted also to Mr T. H. Kirby, M.A., of Clifton, for many corrections and suggestions, particularly with regard to the examples and answers.

The general character of the work is unchanged; but I have slightly enlarged its scope by the insertion of some illustrative problems worked out in pages 231 to 238, together with a carefully graduated set of interesting examples for exercise. I have also increased the miscellaneous examples at the end by the addition of problems selected and adapted from those set in Cambridge in the last two or three years.

Any corrections or suggestions will be gratefully received by the Author or the Publishers.

J. B. LOCK.

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STATICS.

CHAPTER I.

Force.

1. We derive our first notion of 'Force' from our muscular sense.

We press something with our hand; the something resists the pressure. We apply with our hand *force* to the something; the something applies an opposite force called resistance to our hand.

2. When an *agent* [D. 183] produces similar effects to those which are produced by our own muscular exertions, we conclude that the agent itself exerts force.

Example. When a *Steam Engine* presses against a carriage, it produces an effect on the carriage similar to that which can be produced by the exertion of the muscles of men or of horses.

The *explosion of gunpowder* propels a cannon ball by force applied continuously for a short interval to the cannon ball, just as a bowler propels a cricket ball by the force applied continuously for a small interval by his hand to the ball.

3. When our muscular exertion is resisted or impeded by a thing we are said to exert **force** upon that thing. This *resistance* is a necessary accompaniment of the existence of force.

Example. A man cannot press with his hand unless he has *something* to which to apply pressure.

4. *DEFINITION.* Any thing to which force can be applied, or which can offer resistance to force, is called **matter.**

DEF. The amount of matter in a thing is called its **mass.**

We decide whether a certain thing is *matter* by an appeal, direct or indirect, to our muscular sense. The eye may easily be deceived with regard to the mass of an object; for the mass of a body is proportional to its capacity for resisting force, and not to its shape, or colour, etc.

5. The true effect of **force** upon matter was first clearly enunciated by Sir Isaac Newton in his great work the 'Principia,' written when in residence as a Fellow of Trinity College, Cambridge, and published in the year 1687.

Newton pointed out that **force** tends to produce in matter an increase of velocity; the increase being in the direction in which the force acts and proportional to it.

The tracing out of all the consequences of this law of nature belongs to the great science of **Dynamics** [*Δύναμις, force*]. The science of **Statics** [Sto, *I stand still*] is that part of Dynamics which treats of masses which are at rest and which remain at rest when under the action of forces.

Thus, in Statics, our treatment of force is limited; and the knowledge of the nature of force to be obtained by the consideration of statical problems only, is consequently limited.

For the purposes of Statics the definition of force takes the following form,

Def. **Force** when applied to a mass at rest produces in that mass a tendency to move; this tendency is in the direction of the force and is proportional to it.

6. Now in order to treat of Force as a measureable quantity we must *first* be able to test whether two given forces **are equal**, and secondly we must have a **unit** force with which to measure our forces.

PROP. Two forces are equal, which are such that they can be applied to the same mass in opposite directions without causing the mass to begin to move.

For each force tends to produce motion in the mass in its own direction, and the tendency is proportional to the force. But if the mass does not begin to move then tendencies which are exactly opposite to one another must be equal to each other.

And therefore the forces which are proportional to them must also be equal.

7. It is convenient to use the word **body** to denote any separate portion of matter.

When a body is so small that its size is unappreciable and its shape of no importance it is called a **particle**.

Bodies of definite size and shape are supposed to be made up of a very large number of particles.

8. It is a fact, made evident by our muscular sense, that every body with which we are acquainted is acted on by a force vertically downwards towards the earth; which force is called the **weight** of the body.

It is proved in Dynamics that the weight of a body is, for all practical purposes, the same at any place in the British Isles.

It is slightly different in different latitudes. The weights of the same body at one of the poles and at the equator are in the ratio $\frac{32255}{32091}$.

Hence if we choose a lump of matter of some durable substance which does not corrode or waste, we can use its *weight* as our *unit force*.

9. The **unit force** in Statics (and in practical engineering) all over the English speaking world is **the weight of a certain lump of platinum** which is kept in London.

The amount of matter in this lump is called one pound, or, 1 lb.

Thus the Static Unit force is the **weight of 1 lb.**

Since, in Statics, the number of lbs. in a mass is of importance only because it decides the *weight* of the mass, it will be sufficient in what follows to speak of a force as *a number of lbs.* (omitting the word *weight* as unnecessary).

10. Let us consider some way in which we can apply force to matter.

Take two coins, *A* and *B*, one in each hand, and press their edges one against the other.

The coin *A* is pressed by the hand holding it and by the coin *B*. The hand applies one force, and the coin *B* applies a second force, and since the coin *A* remains at rest these two forces are equal and opposite.

Similarly the coin *B* is acted on by two equal and opposite forces.

Now consider what takes place at the point at which the two coins touch each other. At this point there are **two forces** acting one on the coin *A* and the other on the coin *B*, and these two forces are equal and opposite.

11. It will be found that when one body *A* applies a force to a second body *B* at any point, then *B* applies at the same point an equal and opposite force to *A*.

It can be shewn that this must be so by Art. 6.

For suppose that the two coins *A* and *B* considered above are just kept apart by a particle of dust. This particle of dust is at rest and remains at rest, and therefore if it is acted on by two forces only (the pressures applied to it by *A* and *B* respectively) these two forces must be equal and opposite.

This law of force is usually referred to as **Newton's Third Law.** Action and Reaction are equal and opposite.

12. Since then a force cannot be applied to a mass without that mass replying with an equal and opposite force, it follows that, in nature, forces *always* occur in pairs; each pair consisting of two *equal* and *opposite* forces.

Such a pair of forces is called a **stress.**

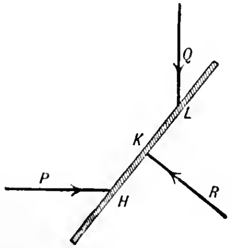
13. When a body is acted on by other bodies so that forces are applied to it, we consider in Statics that the arrangement of the forces must be such that the body remains at rest. So that we may say that the effect of the forces upon the body is practically nothing.

14. But the actual effect of forces acting on a body which remains at rest is to cause a slight change in the arrangements of the particles of which the body is composed and consequently to place the particles in a state of **strain.**

Consider a stick *HKL* placed on a smooth horizontal table. Let it be pressed by three horizontal forces applied at three different points of its length.

When the forces are first applied, the particles of the wood are strained, their relative positions being altered by the forces.

Let the forces be such that the stick remains at rest. As long as the forces continue in action the particles of the wood remain in a state of strain, and the stick is distorted. If the stick is naturally straight and is to be considered straight in the problem, the distortion must be slight.



15. In considering a problem such as the above we shall, in what follows, always assume that *the shape of the body is unaltered.*

The consideration of the change of shape in body consequent on the action of force is very difficult and complicated. Hence in commencing the study of the effect of force on mass we choose those bodies in which the change of shape is imperceptible.

The only part of the above experiment which provides us with a question in Elementary Statics is, the relation which must exist between the three forces, in order that the stick considered as a whole may remain at rest; and we should in discussing this problem leave out altogether any consideration of the slight change of shape which the stick may undergo.

But it must not be forgotten that there is *some* change of shape however slight; and that the particles of the body are in a state of strain.

16. *DEF.* When we wish to call attention to the fact that the change of shape in the body under consideration, caused by the forces acting on it, is so small that we neglect it, we say that the body is **rigid.**

17. Although in the present treatise we do not propose to consider the change of shape in a body due to the action of force, yet it is necessary to point out that the change of shape caused by force affords a very convenient means of measuring forces.

It is found that a piece of coiled steel wire can be compressed to a certain observed distance by the action of a certain force and that the compression is repeatedly the same for the same force.

Hence we conclude that two forces which compress the same steel spring equally, are equal forces.

Thus the action of a steel spring gives us a means of doubling, trebling, etc. a force, and hence of measuring any force whose magnitude we wish to know.

Geometrical Representation of Force.

18. Force has **magnitude.**

We say that a force is so many times the weight of 1 lb.

Force has **direction.**

A given force produces tendency to motion in a certain direction.

We apply a pressure to a body in a certain direction.

Force when acting upon a body is **applied** at a certain **point.**

Each of these, *magnitude*, *direction* and *point of application* must be known before the effect which the force produces in the body on which it acts can be known.

They can all be indicated geometrically by a finite straight line with some mark on it to indicate the point of application.

For a finite straight line has *magnitude*, namely its length.

Its magnitude is measured by the *number* of units of length which it contains. Hence when a line represents a force each unit of length represents a certain proportion of a unit force. *For example;* half an inch may represent 1 lb.

Thus the *lengths* of the lines, in a diagram representing a system of forces, represent the *magnitudes* of the corresponding forces on one uniform scale.

A finite straight line has *direction*, provided it is understood that it is drawn *from* some point; and this point can serve to indicate the *point of application.*

Thus when we speak of the force OA we shall mean that the force is represented in length by the length of OA, and in direction by the direction from O to A; and that the point of application is O.

It is convenient to indicate by an arrow that the force acts in the direction OA and not in the direction AO.

19. It is important to notice that the word *direction* means more than *line* of action. For any given *line* has **two directions,** the one exactly opposite to the other.

Thus, the line terminated by the point O and A has the two directions; namely, (i) from A to O, (ii) from O to A.

Example i. *Three lines of* 3 in., $2\frac{1}{2}$ in. *and* 4 in. *drawn from a point O in given directions represent* three forces; *the smallest force is* 5 lbs.; *what are the magnitudes of the others?*

Here $2\frac{1}{2}$ in. represents 5 lbs.

Therefore 1 in. represents 2 lbs.
Therefore 3 in. represents 6 lbs.
and 4 in. represents 8 lbs.

Therefore the other two forces are 6 lbs. and 8 lbs.

Example ii. *Two forces of* 4 lbs. *and* 5 lbs. *respectively act at a point. We draw a line at random parallel to the first force and take it to represent that force; on measuring this line we find it measures* 3·2 inches; what line *will represent* the second force?

The line must be drawn parallel to the second force; and since a force of 4 lbs. is represented by 3·2 inches, a force of 5 lbs. will be represented by $\frac{5}{4}$ of 3·2 inches, that is, by 4 inches.

The required line is a line parallel to the 5 lbs. 4 inches long.

EXAMPLES. I.

1. I wish to represent 3 forces of 2 lbs., 3 lbs., and 4 lbs. by straight lines drawn parallel to the forces; if I represent the force of 2 lbs. by a line $\frac{1}{2}$ in. long, with what lengths must I represent the other two forces?

2. Three forces are proportional to the sides of a triangle whose sides are 3 in., 4 in., and 5 inches; the smaller force is $7\frac{1}{2}$ lbs.; what are the others?

3. Assuming that a line of 3 inches represents a force of 4 lbs.: find the lengths of the lines to represent 10 lbs., 12 lbs. and 25 lbs.

4. Assuming that a force of 50 lbs. is represented by a line of $6\frac{1}{4}$ inches: find the force represented by a line of 2 ft. and by a line of 14 inches.

5. In a right-angled triangle the sides containing the right angle are 3 in. and 4 in.; taking the hypotenuse to represent a force of $2\frac{1}{2}$ lbs., what force would be represented by the sides?

6. Forces of 3 lbs. towards the North and 3 lbs. towards the West are represented by lines OA, OB respectively; what force is represented by the line AB?

7. ABC is an equilateral triangle, and AD is drawn perpendicular to BC; DA represents a force 3 lbs. northwards; what forces do AB and DB represent?

CHAPTER II.

FORCES ACTING AT ONE POINT.

20. *DEF.* A **particle** is a portion of matter which is so small that it may be treated practically as a geometrical point.

Hence, when forces act on a particle they are to be considered as each acting at the point at which the particle is placed.

Practically, we may consider those masses to be particles in which we do not take into account any tendency to *rotation*, or in which all the points of the mass tend to move together in the same direction.

NOTE. In practice a force is applied to a body distributed over a small *area;* a force cannot be actually applied at a geometrical *point*, nor can a force act in a geometrical *line*.

Even a force applied to a body by means of a fine thread is really applied to a small area. But such a force approximates to a force having a *line* of action and a *point* of application.

As the student proceeds with the subject he will see that a single force acting in a geometrical line may in many cases be considered as the *statical representation* of a great number of minute parallel forces and that the *point* of application may be considered to be the *centre* of the system of parallel forces which the single force represents.

21. *DEF.* When the forces acting upon a particle at rest are such that the particle continues at rest, the forces are said to be **in equilibrium**.

22. *DEF.* The **resultant** of a number of forces acting upon a *particle* is the single force which can produce in the particle the same effect as the forces acting together.

Any number of forces acting on a Particle at rest, produce in that particle a tendency to motion. This tendency can be only in one direction and must be of definite magnitude. Hence, any number of forces acting at a point must have *one and only one* resultant.

Since the forces considered in Statics are always in equilibrium, we may with advantage express the above definition as follows.

DEF. The **resultant** of any number of forces acting at a point is the force equal and opposite to that force which when acting at the same point will form with the given forces a system in equilibrium.

The force which forms with any given system of forces a system in equilibrium is called the **anti-resultant** of that system of forces.

A Particle acted on by One Force.

23. When a single force acts upon a mass, it produces motion in the mass.

But in Statics we only consider those arrangements of forces which do not produce motion.

Therefore we do not consider the effect of a single force acting alone on a mass.

The discussion of the effect of a single force will be found in Dynamics.

A Particle acted on by Two Forces.

24. When two forces act upon a mass, each force produces its own tendency to motion in the mass.

When two forces acting at one point upon the same mass together produce rest, the tendencies to motion which they produce must be equal and opposite.

Therefore the *forces* must also be equal in magnitude, in the same line, and opposite in direction. Hence,

PROP. When two forces acting at the same point are in equilibrium they must be of equal magnitude, they must act in the same line and they must act in opposite directions.

A Particle acted on by Several Forces having the same Line of Action.

25. Consider a mass at rest when under the action of several forces all of which have the same line of action.

Since the tendency to motion produced by all the forces together is zero, therefore the sum of the tendencies in the one direction must be equal to the sum of the tendencies in the other direction.

Therefore the sum of all the forces acting in one direction must be equal to the sum of the forces acting in the opposite direction.

Example. Forces of 3 lbs. and 4 lbs. towards the North act on a particle and forces of 2 lbs. and 5 lbs. towards the South act on the same particle. The particle is in equilibrium because the sum of the two forces 3 lbs. + 4 lbs. are equal to the sum of the opposite forces of 2 lbs. + 5 lbs.

26. Also, in order to find the resultant [Art. 22] of any number of forces acting on a particle having the same line of action, we must add together all those forces which act in one direction and subtract from them the sum of all the forces which act in the contrary direction; the difference will be the *resultant* of the forces under consideration.

Example. Find the resultant *of forces of* 5 lbs., 6 lbs. and 7 lbs. *acting on a particle towards the East and* 3 lbs. *and* 8 lbs. *acting on the same particle towards the West.*

We have $5+6+7=18$ lbs. in one direction opposed by $3+8=11$ lbs. in the opposite direction.

Hence their resultant is $18-11=7$ lbs. towards the East.

27. Further, we shall obtain the desired result even if we *subtract the forces whose sum is the greater from the forces whose sum is the less,* provided we record the result as a force *opposed* to the lesser force.

Example. In the above example we might have said that the resultant *towards the West* is $11-18=-7$ lbs.; that is, 7 lbs. in a direction *exactly opposite* to the direction considered.

28. This suggests that we should define the opposing forces as **negative forces**, and agree *that the sign* $-$ *applied to a force shall indicate that it is opposed to the force to which the sign* $+$ *is applied.*

29. *DEF.* It is convenient to use the word **sense** to indicate that forces in the same line are or are not opposed to each other.

Thus the word *sense* in Geometry corresponds to the word *sign* as commonly used in Algebra.

We shall always indicate the sense of a line in a figure by the order of the letters thus, $AA'=-A'A$.

30. In future when we speak of the **sum** of a number of forces or of lines representing them, it must be understood that in forming the *sum* due attention must be paid to *sign*.

In the above example the sum of forces acting towards the East on the particle is $(+5+6+7-3-8)$ lbs., that is $+7$ lbs.

With this understanding the condition for the equilibrium of a single particle when acted on by forces in one line may be stated thus.

31. *PROP. Any number of forces acting in the same line at the same point are in equilibrium when the* **sum** *of those forces is zero.*

Example i. *Arrange the forces* 2 lbs., 3 lbs., 4 lbs. *and* 6 lbs. *acting in the same line, that their resultant may be as small as possible.*

When we oppose the forces of 2 lbs. and 6 lbs. to those of 3 lbs. and 4 lbs. the resultant is 1 lb.

It will be found on trial that the resultant of any other arrangement is greater than 1 lb.

Hence, the arrangement required is $+2$ lbs., -3 lbs., -4 lbs., $+6$ lbs.

Example ii. *Can the forces* 1 lb. 2 lbs. 3 lbs. and 6 lbs. *acting on a particle in the same line be so arranged as to be in equilibrium?*

The *sum* of the forces must be zero and the sum of the forces $+1$ lb. $+2$ lbs. $+3$ lbs. -6 lbs. is zero.

Therefore when the forces 1 lb., 2 lbs., 3 lbs. act in the same sense and the 6 lbs. in the opposite sense the forces are in equilibrium.

EXAMPLES. II.

1. Find the resultant of forces 2 lbs. and 5 lbs. acting towards the north and 6 lbs. and 3 lbs. acting towards the south.

2. Find the resultant of the forces 4 lbs., 5 lbs. and 2 lbs. acting upwards, and 3 lbs., 7 lbs. and 8 lbs. acting downwards.

3. Find the resultant of the forces $+3$ lbs., $+5$ lbs., $+7$ lbs. and -4 lbs., -6 lbs., -8 lbs. acting all towards the north.

4. $ABCD$ is a straight line, $AB = 2$ ft., $BC = 3$ ft., $CD = 4$ ft.; find the resultant of the forces represented by AB, BC, CD and DB.

5. As in Question **4**, find the resultant of the forces represented by AC, BD, DA and CB.

6. As in Question **4**, find the resultant of the forces represented by AD, BD, CD, $-BC$, $-AB$.

7. A weight of 36 lbs. is on a horizontal plane, a man applies an upward pull of 30 lbs. to the weight. What is the pressure of the weight on the plane?

8. 3 men endeavour to lift a stone of 2 cwt.; one man pulls upwards with a force of 50 lbs., another with a force of 80 lbs., and there is a pressure upwards of the ground on the stone of 20 lbs.; what is the pull of the third man?

9. A cart is on the side of a hill, and it is known that a force of 3 cwt. up the hill is necessary to make it move up the hill; three men push up the hill each with a force of 28 lbs.; and there are three horses, one pulls upwards with a force of 150 lbs., another with a force of 200 lbs. upwards. What is the pull of the third horse if the cart remains at rest?

10. A, B, C, D are points on a straight line; shew that, for any arrangement whatever of the points, forces represented by AB, BD, DC, CA are in equilibrium.

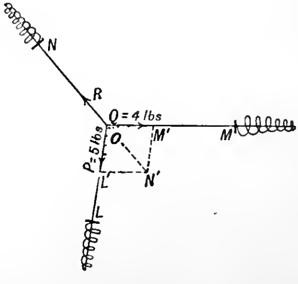
CHAPTER III.

THE PARALLELOGRAM OF FORCES.

32. WHEN two forces act in **different directions** upon a particle, each force produces in the particle a tendency to move. But the particle can only tend to move in one direction; so that the combined effect of the two forces must be a tendency to move in some one direction.

Suppose for example we have two coiled springs made of steel wire arranged each with an indicator, like a spring balance, and we connect each of them by means of a string to a small ring *O*. And suppose we have also a third spring, like the other two, also connected with the ring by a third string.

Then we can apply to the ring two known forces *P* lbs. and *Q* lbs. in directions indicated by the first two strings; and we can keep the ring from moving by a force *R* lbs. applied by the third spring. We shall thus have two forces, *P* and *Q*, and their anti-resultant *R*.



Now the magnitude and direction of R must depend upon those of P and Q, and therefore there must be some rule by which the direction and magnitude of R can be found by calculation from the known magnitudes of P and Q.

The following rule for finding the *direction* of R will be found to be true.

Along the direction of P measure OL from O containing as many inches as there are lbs. in P; along the direction of Q measure OM from O containing as many inches as there are lbs. in Q.

Draw MN' parallel to OL and LN' parallel to OM; join ON'.

Then $N'O$ produced *is the direction of* R.

The rule for finding the magnitude of R is as follows.

With the same construction as before, draw the parallelogram $OMN'P$, and find how many inches there are in ON'.

Then the *number of inches in* ON' = the number of lbs. in R.

This rule may be enunciated as follows.

33. *PROP.* **The Parallelogram of Forces.** *When two forces acting at a point are represented in magnitude and direction by two lines* OA, OB, *then their resultant is represented in magnitude and direction by the diagonal* OC *of the parallelogram* OACB.

34. The truth of the Parallelogram of Forces is deduced from the dynamical definition of force. [See *Dynamics*, Art. 110.]

35. A static proof (depending on certain assumptions with regard to the nature of matter) is given below in Chapter XIV. and this proof can if it be thought desirable be read at this point.

The student however who has not yet read any Dynamics is advised to postpone the consideration of the theoretical *proof* of the proposition for the present, and to follow out in example and experiment the *results* which can be deduced from the proposition.

36. The student should notice that the parallelogram of forces includes the case of two forces acting in the same straight line.

For suppose OA and OB be two forces including a very small angle [Fig. i.], then OC becomes very nearly equal to $OA + AC$, that is to $OA + OB$.

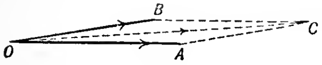


Fig. i.

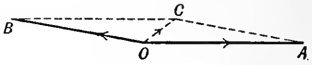
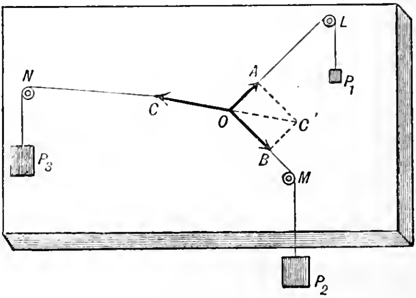


Fig. ii.

Again, [Fig. ii.] suppose OA and OB to include nearly two right angles, then OC becomes very nearly equal to $OA - CA$, that is to $OA - BO$.

37. The truth of the parallelogram of forces may be roughly tested by some such apparatus as the following:

A board has three small smooth pulleys L, M, N fixed to it.



Three strings knotted together at O have their other extremities fastened to weights P_1, P_2, P_3; these strings are then arranged as in the figure, the plane of the board being vertical.

Now considering the equilibrium of the knot at O, we have three forces applied to it by the strings in the directions OL, OM and ON respectively, the forces being equal to the weights of P_1, P_2, P_3 respectively.

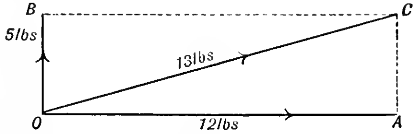
Suppose for example $P_1 = 3$ lbs. and $P_2 = 4$ lbs.; draw on the board lines OA, OB parallel to OM, ON making $OA = 3$ in., $OB = 4$ in., and complete the parallelogram $OAC'B$.

Then if we measure OC', it will be found that the number of inches in OC' will be equal to the number of pounds in P_3.

38. To find the resultant of **two forces at right angles to each other** we have to find the length and direction of the diagonal of a parallelogram, which in this particular case is a rectangle, whose sides are given.

Example i. *Find the* resultant *of two forces of* 5 lbs. *and* 12 lbs. *at right angles to each other.*

Let OA represent 12 lbs. and OB represent 5 lbs. the angle AOB being a right angle.



Complete the parallelogram $OACB$ and join OC.

Then OC represents the required resultant.

Now $OC^2 = OA^2 + AC^2 = OA^2 + OB^2 = 12^2 + 5^2 = 169 = 13^2$.

Therefore OC represents 13 lbs. .... (i),

and the angle AOC is such that $\tan AOC = \frac{AC}{OA} = \frac{5}{12} = \cdot416663\ldots$ (ii).

Whence [from the Tables or by actual measurement] $AOC = 22^\circ\ 28'$ nearly.

Example ii. *Find the direction and magnitude of the* resultant *of forces of* 8 lbs. *and* 9 lbs. *at right angles to each other.*

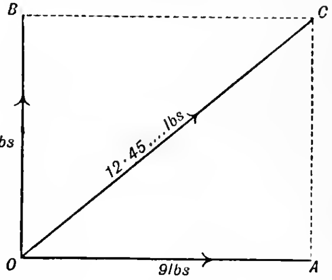
Let OA represent 9 lbs., and OB 8 lbs.

Complete the parallelogram $OACB$.

Join OC. Then OC represents the required resultant.

Now $OC^2 = OA^2 + AC^2 = OA^2 + OB^2 = 9^2 + 8^2$

$= 155 = (12 \cdot 04 \ldots)^2$.



Therefore OC represents 12·04...lbs. .... (i),

and $\tan AOC = \frac{AC}{OA} = \frac{8}{9} = \cdot 888888$........... (ii).

Whence [from the Tables] $AOC = 41° 22'$ nearly.

EXAMPLES. III.

Find the direction and magnitude of the resultant of the following pairs of forces; each pair being at right angles (working out the surds to 4 significant figures).

1. 3 lbs. and 4 lbs. **2.** 15 lbs. and 10 lbs.

3. 25 lbs. and 60 lbs. **4.** 6 lbs. and 7 lbs.

5. 10 lbs. and 11 lbs. **6.** 10 lbs. and 20 lbs.

7. 4 lbs. and 4 lbs. **8.** 1 lb. and $\sqrt{3}$ lbs.

9. $(1 + 2\sqrt{2})$ lbs. and $2\sqrt{2}$ lbs.

10. $(3 + \sqrt{3})$ lbs. and 1 lb.

11. $(1 + \sqrt{3} + \sqrt{2})$ lbs. and $(1 + \sqrt{2})$ lbs.

12. $(2 + \sqrt{2} + 3 + \sqrt{3})$ lbs. and $(2\sqrt{2} + 5)$ lbs.

13. $3a$ lbs. and $4a$ lbs. **14.** a lbs. and b lbs.

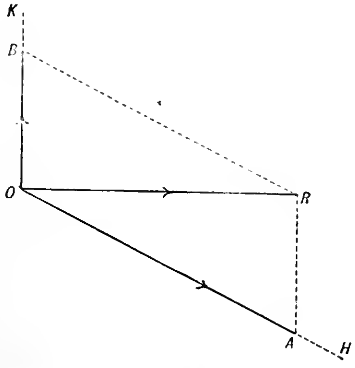
39. We may [Art. 33] at any time replace two forces which act upon a particle by a single force called their resultant; and the effect of this single force on the particle will be the same as the effect of the two forces together.

Conversely we may, if we please, at any time replace the single force (the resultant) by the two forces which together have the same effect.

40. When we replace a single force acting on a particle by two forces whose joint effect is the same, the two forces are called **components** of the single force.

41. *PROP. A force can be resolved into two components in any assigned directions.*

Let OR be the given force and OH, OK the given

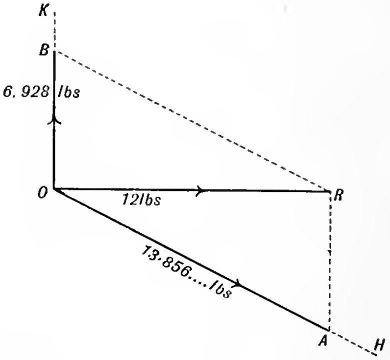


directions, from R draw RA parallel to KO and RB parallel to HO.

Then by the parallelogram of forces we may replace the force OR by the two forces represented by OA and OB.

42. It is however seldom required to resolve a force into components, except in the case when the two components are at right angles to each other. The following examples may be omitted on first reading the subject.

**Example. Resolve the* force of 12 lbs. *into two others, making the* angles 30° and 90° *with it, one on each side.*



Let OR represent 12 lbs.

Draw the angles $HOR = 30°$, $ROK = 90°$.

Through R draw RA parallel to KO and RB parallel to HO. Then OA and OB represent the two components required.

Now $$\frac{OA}{OR} = \frac{\sin ORA}{\sin OAR} = \frac{1}{\sin 60°} = \frac{2}{\sqrt{3}}.$$

Hence $$OA = \frac{2}{\sqrt{3}} \times OR = \frac{2\sqrt{3}}{3} \times 12 \text{ lbs.}$$
$$= 8 \times 1{\cdot}7320\ldots\text{lbs.} = \underline{13{\cdot}8560\ldots\text{lbs.}}$$

Again, $$\frac{OB}{OR} = \frac{\sin ORB}{\sin OBR} = \frac{\sin 30°}{\sin 60°} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Hence, $$OB = \frac{\sqrt{3}}{3} \times OR = \frac{1}{3}\sqrt{3} \times 12 \text{ lbs.} = 4 \times 1{\cdot}7320\ldots\text{lbs.}$$
$$= \underline{6{\cdot}928\ldots\text{lbs.}}$$

*EXAMPLES. IV.

1. Resolve the force of 36 lbs. into two others making the angles 90° and 30° with it, one on each side.

2. Resolve the force of 10 lbs. into two others making the angles 30° and 60° with it, one on each side.

3. Resolve the force of 1 lb. into two others each making the angle 60° with it.

4. Resolve the force of 100 lbs. into two others each making the angle 30° with it.

5. Resolve the force of 25 lbs. into two others each making the angle 45° with it.

6. Resolve the force of 100 lbs. into two equal forces, one of which makes an angle of 60° with it.

7. Resolve a force of 20 lbs. into two equal forces, one of which makes an angle of 45° with it.

8. Resolve a force of 45 lbs. into two equal forces, one of which makes an angle of 30° with it.

9. A force is resolved into two forces P and P', each making 45° with it, and also into two forces Q and Q' each making an angle 30° with it; shew that $Q = \frac{1}{2}\sqrt{6}$ of P.

10. When a force is resolved into two components, the greater component is that which makes the smaller angle with it.

11. Shew that a force can be resolved into two components one of which can have any assigned magnitude and any assigned direction.

Find the following components by means of a scale and protractor.

12. Resolve the force 12 lbs. into two equal forces, one making the angle 25° with it.

13. Resolve the force 16 lbs. into two forces at right angles, one of them making 40° with it.

14. Resolve the force 10 lbs. into two forces making angles 35° and 50° respectively with it.

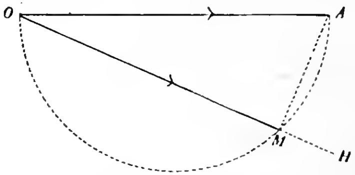
15. Resolve the force 15 lbs. into two forces, one of which is 20 lbs. and makes the angle 42° with it.

43. *DEF.* When a force is resolved into two components at right angles to each other each component is called the **resolute** of the force in its own direction.

The **resolute of a force in a given direction** is an idea with which the student must make himself perfectly familiar.

44. To find the resolute of a force in a given direction we proceed thus.

Let OA represent the given force, OH the given direction;



from A draw AM perpendicular to OH; then OM represents the required resolute in the direction OH.

45. When the given direction makes an angle α with the given force so that $AOH = \alpha$,

then, since $$\frac{OM}{OA} = \cos\alpha,$$

we have $$OM = OA\cos\alpha,$$

or, the resolute of a force P in a direction making the angle α with it is $$P\cos\alpha.$$

Also the resolute of OA perpendicular to OM is $$OA\sin\alpha.$$

NOTE. That which we have here called the *resolute* of a force is in most books on Statics called **the** *resolved part* of a force.

The idea is so important in the subject that a definite *name* will be found useful.

Example i. *Find the* resolute *of a force of* 10 lbs. *in the direction making the* angle 60° *with it.*

Here OA represents 10 lbs. and $AOM = 60°$,
therefore $OM = 10 \cos 60° = \underline{5 \text{ lbs.}}$

Example ii. *Find the* resolute *of a force of* 321 lbs. *in the direction making the* angle whose cosine is ·647 *with it.*

Here OA represents 321 lbs. and $\cos AOM = \cdot 647$,
therefore $OM = 321 \times \cdot 647$.

EXAMPLES. V.

Find the resolute of each of the following forces in the direction making the given angle with it.

1. 30 lbs., at the angle 60° with it.

2. 50 lbs., at the angle 30° with it.

3. $4\sqrt{2}$ lbs., at the angle 45° with it.

4. 125 lbs., at the angle whose cosine is $\frac{3}{5}$ with it.

5. 300 lbs., at the angle whose cosine is ·7 with it.

6. 437 lbs., at the angle whose cosine is ·125 with it.

7. 237 lbs., at the angle whose cosine is ·794 with it.

8. 34·7 lbs., at the angle 35° 48′ 30″ [cos 35° 48′ 30″ = ·811].

Find the following resolutes by means of a scale and protractor.

9. 25 lbs. at the angle 36°.

10. 3 lbs. at the angle 45° 30′.

11. 22 lbs. at the angle 40°.

12. 12 lbs. at the angle 52°.

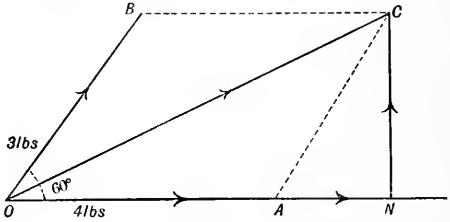
13. 18 lbs. at the angle 20°.

14. 10 lbs. at the angle 25°.

46. To find the resultant of **any two given forces** we may proceed as follows.

Example. Find the resultant *of two forces of* 4 lbs. *and* 3 lbs. *acting on a particle inclined to each other at an angle of* 60°.

Let OA represent 4 lbs. wt., let OB represent 3 lbs. wt., and let the angle $AOB = 60°$.



Complete the parallelogram $OACB$, then OC represents the required resultant.

We have to find the force represented by OC.

Let the unit length represent 1 lb.

Then the length of $OB = 3$, the length of $OA = 4$.

Also $AN = AC \cos 60° = 3 \times \frac{1}{2}$,

$$NC = AC \sin 60° = 3 \times \tfrac{1}{2}\sqrt{3},$$

$$OC^2 = (OA + AN)^2 + NC^2 = (4 + 1\tfrac{1}{2})^2 + (\tfrac{3}{2}\sqrt{3})^2$$

$$= \tfrac{121}{4} + \tfrac{27}{4} = 37;$$

$$\therefore\ OC = \sqrt{37} = 6 \cdot 08 \ldots \text{lbs.}$$

EXAMPLES. VI.

Find the resultants of the following forces.

1. Forces of 5 lbs. and 5 lbs., inclined to each other at the angle whose cosine is $\frac{3}{5}$.

2. Forces of 6 lbs. and 3 lbs., inclined to each other at the angle whose cosine is $\frac{4}{5}$.

3. Forces of 7 lbs. and 12 lbs., inclined to each other at the angle whose cosine is $\frac{5}{13}$.

4. Forces of 3 lbs. and 4 lbs., inclined at the angle 30°.

5. Forces of 3 lbs. and 8 lbs., inclined at 60°.

6. Forces of 5 lbs. and 10 lbs., inclined at 45°.

7. Forces of 1 lb. and $3\sqrt{2}$ lbs., inclined at 45°.

8. Forces of 10 lbs. and $10\sqrt{3}$ lbs., inclined at 30°.

9. A force of 20 lbs. towards the North, and a force of 30 lbs. towards the North-west.

10. A boat is pulled by a force of 30 lbs. towards the South and by a force of 20 lbs. towards the North-west. What other force is acting upon it if it remains at rest?

11. Two horses pull at a block of stone, one with a horizontal force of 100 lbs. the other with a horizontal force of 130 lbs., the forces being inclined to each other at the angle whose cosine is $\frac{12}{13}$; what is the force which keeps the stone at rest?

12. Two forces P lbs. and $\sqrt{2}P$ lbs. act at a point; P acts towards the East and $\sqrt{2}P$ towards the N.W.; find their resultant.

13. Two forces 2 lbs. and 4 lbs. inclined to each other at an angle of 120° act at a point. What is their resultant?

14. Two forces of 2 lbs. each acting at an angle of 60° have the same resultant as two equal forces acting at right angles; what is the magnitude of these two forces?

47. It will be seen that the method employed above is simply a method of finding the third side of a triangle of which two sides and the included angle are given.

$$
\begin{aligned}
\text{Thus } OC^2 = ON^2 + NC^2 &= (OA + AN)^2 + NC^2 \\
&= OA^2 + AN^2 + NC^2 + 2OA \,.\, AN \\
&= OA^2 + AC^2 + 2OA \,.\, AC \cos NAC \\
&= OA^2 + AC^2 - 2OA \,.\, AC \,.\, \cos OAC.
\end{aligned}
$$

[The method is thus identical with the use of the Trigonometrical Formula $a^2 = b^2 + c^2 - 2bc \cos A$.]

This result may be written $R^2 = P^2 + Q^2 + 2PQ \cos \widehat{PQ}$.

48. The student must distinguish clearly between *the resolute of a force in a given direction* and a *component* of the force in the same direction when the two components are not at right angles.

49. As the student proceeds in the subject he will find that *the resolute* of a given force *in a given direction* may be said to be that which represents **the effect** of the given force **in that direction.**

Example. A force may be said to have no effect in the direction perpendicular to itself. The resolute of a force in the direction perpendicular to the force is zero. A force may have a *component* of any given magnitude in the direction perpendicular to itself.

50. By Art. 45 the resolute of a force P lbs. in the direction making the angle α with it, is $P\cos\alpha$ lbs..

We notice that $\cos\alpha$ is a *number;* so that $P\cos\alpha$ lbs. is a *force.*

We notice that $\cos\alpha$ is never greater than 1, so that the resolute of a force is always less than the force itself,

Except when it is in the direction of the force, and then it is the force itself.

In the figure on Page 24 if a circle be described on OA as diameter, the chord of this circle drawn through O in any direction represents the resolute in that direction.

51. The student must notice that two separate resolutes of the same force are not a pair of *components* of the force unless these two resolutes are *at right angles to each other.*

52. There is one direction and one direction only in which the resolute of a force is zero; namely, the direction perpendicular to the force itself.

Hence no force can have its resolutes in two different directions each separately zero. A fact which may be stated thus;

When the resolutes of a force in two different directions are each separately zero that force must be itself zero.

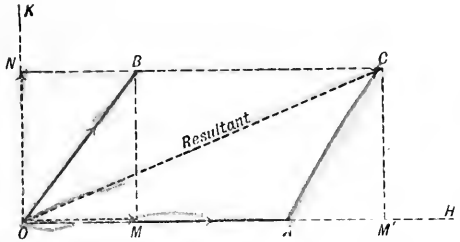
This fact is most important, and will often be used in subsequent parts of the book.

53. The method of Art. 46 should be considered as an example of the resolution of forces.

54. *PROP. To find the resultant of two given forces.*

Let OA, OB represent the forces. Resolve one of them, OB, into its two rectangular components OM, ON, along and perpendicular to the line of action of the other OA.

Then compound the resulting forces.



In the figure $OM' = OM + OA$ and the resultant

$$OC^2 = ON^2 + OM'^2$$
$$= ON^2 + (OM + OA)^2.$$

To find the square of the resultant of two forces, find the sum of the resolutes of the forces in each of two directions at right angles to each other and add their squares.

Example. Let OA and OB represent two forces.

Draw OK perpendicular to OA.

Draw BM, BN perpendiculars on OA and OK.

Then the force OB may be replaced by the two forces OM and ON, where $OM = OB \cos MOB$, $ON = MB = OB \sin MOB$.

Hence the two forces OA and OB may be replaced by the two forces $OA + OM$ along OA and along OK perpendicular to OA.

The resultant of these two is OC where

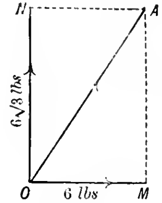
$$OC^2 = (OA + OB \cos MOB)^2 + OB^2 \sin^2 MOB \text{ .... (i)},$$

and

$$\tan NOC = \frac{NC}{ON} = \frac{OB \sin MOB}{OA + OB \cos MOB} \text{ .... (ii)}.$$

Example i. A force has its resolute towards the North equal to $6\sqrt{3}$ lbs., the other component is 6 lbs. towards the East; find the magnitude and direction of the force.

Let OM represent the force of 6 lbs. and ON that of $6\sqrt{3}$ lbs.



Complete the rectangle $NOMA$.

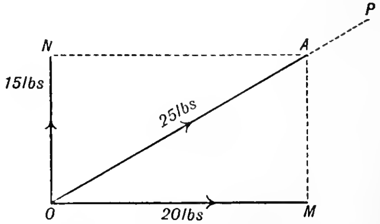
Then OA represents the required force and

$$OA = \sqrt{\{6^2 + 6^2 \times 3\}} \text{ lbs.} = \underline{12 \text{ lbs.}}$$

Also, since $\dfrac{OM}{OA} = \dfrac{6}{12} = \dfrac{1}{2}$, the angle $\underline{MOA = 60^\circ}$.

Example ii. *The resolute of a force in the direction making the* angle whose sine is $\frac{3}{5}$ *with the force is* 20 lbs.; *find the* force *and the* other component *of the force.*

Let OM represent the given resolute.



Draw the angle MOP whose sine is $\frac{3}{5}$, draw MA perpendicular to OM cutting OP in A.

Then OA represents the required force.

Complete the rectangle $OMAN$; then ON is the other component of the force.

Now, since $\sin MOA = \dfrac{3}{5}$, $\therefore \dfrac{AM}{OA} = \dfrac{3}{5}$ and $\dfrac{OM}{OA} = \dfrac{4}{5}$;

$$\therefore OA = \tfrac{5}{4} \text{ of } OM = \tfrac{5}{4} \text{ of } 20 \text{ lbs.} = \underline{25 \text{ lbs.}},$$

also $$MA = \tfrac{3}{4} \text{ of } OM = \tfrac{3}{4} \text{ of } 20 \text{ lbs.} = \underline{15 \text{ lbs.}}$$

EXAMPLES. VII.

1. Find the resultant of forces of 4 lbs. and 3 lbs. inclined to each other at an angle of $45°$.

2. The rectangular components of a force in magnitude are 3 lbs. and 4 lbs. respectively; what is the force?

3. The rectangular components of a force are each 5 lbs.; what is the force?

4. The resolute of a force which makes an angle of $60°$ with the force is 10 lbs.; what is the force?

5. The resolute of a force which makes an angle of $30°$ with it is $\sqrt{3}$ lbs.; what is the force? and find the other component.

6. The resolute of a force making an angle of $45°$ with it is 20 lbs.; what is the other component?

7. The resolute of a force making an angle whose tangent is $\frac{3}{4}$ with the force is 12 lbs.; find the other component.

8. If two forces are represented in magnitude and direction by the side (of length l ft.) and a diagonal of a square, prove that their resultant will be represented either by a line of length $\sqrt{5}l$ or by a line of length l.

9. ABC is an equilateral triangle; forces P lbs., P lbs., Q lbs. act at a point parallel to AC, CB, AB respectively; shew that their resultant is $(P+Q)$ lbs.

10. Forces of 25 lbs., 24 lbs. and 7 lbs. acting at a point are in equilibrium; shew that two of them are at right angles.

11. Forces of 40 lbs., 41 lbs. and 9 lbs. acting at a point are in equilibrium; shew that two of them are at right angles.

12. A force of 10 lbs. acts along the side AB of an equilateral triangle; what is the resolute of this force along the side AC? find also its resolute in the direction parallel to CB.

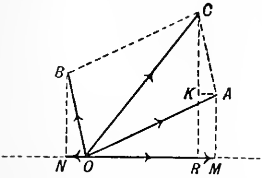
13. Shew that the resultant of the two resolutes in **12** is 5 lbs.

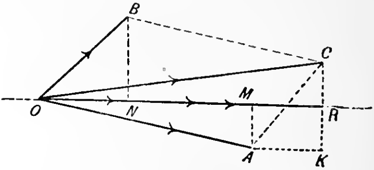
14. D is the middle point of the side AB of the equilateral triangle ABC; prove that the resultant of two forces represented respectively by AD, AC is represented in magnitude by $\sqrt{7}$ times AD.

15. Prove that if OA represent a force and a circle be described on AB as diameter, the resolute of the force in any direction is represented by the chord of the circle, drawn in that direction through O.

55. *PROP. The sum of the resolutes in any chosen direction of two forces is equal to the resolute of their resultant in that direction.*

Let OA, OB, OC represent the component forces and their resultant respectively.





Take any line OR; draw AM, BN, CR perpendicular from A, B, C on the line.

Then OM, ON, OR represent the resolutes of OA, OB, OC respectively in the direction OR.

We have to prove that $OR = OM + ON$.

Through A draw AK parallel to OR to meet CR produced, when necessary, in K; then the triangles OBN, ACK have their sides parallel and $OB = AC$;

$$\therefore\ ON = AK = MR.$$

Therefore $OR = OM + MR = OM + ON$. Q.E.D.

N.B. In the above proposition due attention must be paid to the signs of the lines MR and ON.

Thus in the left-hand figure we may say that

$$OR = OM - RM = OM - NO;$$

but this is the same thing as saying that

$$OR = OM + MR = OM + ON.$$

Example. Two forces not vertical support a weight of 10 lbs.

The resultant of these two forces is therefore a force of 10 lbs. acting vertically upwards.

The sum of the vertical resolutes of these two forces is therefore equal to 10 lbs.

The sum of the horizontal resolutes of these two forces is zero.

56. *PROP. The sum of the resolutes in any chosen direction of any number of forces acting at a point is equal to the resolute in the same direction of their resultant.*

For the sum of the resolutes of any two of the forces is equal to the resolute of their resultant.

The sum of the resolutes of this resultant and of a third force is equal to the resolute of *their* resultant;

and so on.

57. From this we deduce the following very important proposition.

58. *PROP. The sum of the resolutes in any chosen direction of any number of forces acting at a point which are in equilibrium, is zero.*

For the sum of their resolutes in any direction is equal to the resolute of their resultant.

But since the forces are in equilibrium their resultant is zero, and therefore its resolute in any direction is zero.

59. *PROP. When the sums of the resolutes in each of two different directions of any number of forces acting at a point are each separately zero, the forces are in equilibrium.*

For it follows by Art. 56 that the resolute of their resultant in two different directions must be each separately zero. This [Art. 52] can only be true when their resultant is itself zero.

60. To find the resultant of any number of forces acting at a point, we make use of the proposition of Art. 56. We find the sums of the resolutes of all the forces in each of two directions mutually at right angles and thus obtain the resolutes of the resultant in each of those directions:

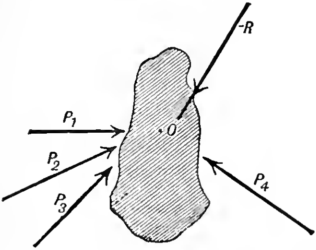
We observe that two resolutes of a force which are mutually at right angles, are rectangular components of that force.

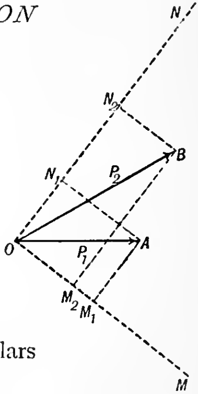
61. *PROP. To find the resultant of any number of forces acting at a point.*

Let *OA*, *OB*, *OC*... represent the forces

P_1 lbs., P_2 lbs., P_3 lbs.... respectively.

Take any line *OM*; draw *ON* perpendicular to *OM*.





Draw AM_1, AN_1 perpendiculars to *OM* and *ON* respectively.

Then we may replace the force *OA* by its two rectangular components OM_1, ON_1, that is by

$(P_1 \cos AOM)$ lbs. along *OM*, and $(P_1 \sin AOM)$ lbs. along *ON*.

Treating *OB*, *OC* in a similar manner, we replace the forces *OA*, *OB*, *OC*,... by

$$(P_1 \cos AOM)\text{lbs.} + (P_2 \cos BOM)\text{lbs.} + (P_3 \cos COM)\text{lbs.} + \ldots = \Sigma\,(P \cos \alpha)$$

acting along *OM*, and

$$(P_1 \sin AOM)\text{lbs.} + (P_2 \sin BOM)\text{lbs.} + (P_3 \sin COM)\text{lbs.} + \ldots = \Sigma\,(P \sin \alpha)$$

acting along *ON*.

The resultant of these is a force *OR*,

where $OR^2 = (P_1 \cos AOM + P_2 \cos BOM + P_3 \cos COM \ldots)^2 + (P_1 \sin AOM + P_2 \sin BOM + P_3 \sin COM \ldots)^2$,

or, $OR^2 = [\Sigma\,(P \cos \alpha)]^2 + [\Sigma\,(P \sin \alpha)]^2$

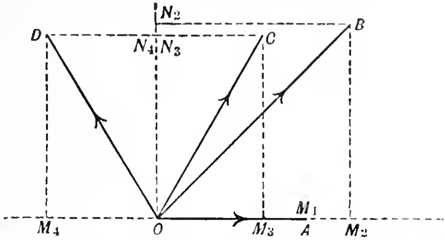
and

$$\tan ROM = \frac{P_1 \sin AOM + P_2 \sin BOM + P_3 \sin COM\ldots}{P_1 \cos AOM + P_2 \cos BOM + P_3 \cos COM\ldots}$$

$$= \frac{\Sigma(P \sin \alpha)}{\Sigma(P \cos \alpha)}.$$

Example. Four forces OA, OB, OC, OD *of* 3, $4\sqrt{2}$, $2\sqrt{3}$, $2\sqrt{3}$ lbs. *respectively act at* O, *so that the* angle BOA$=45°$, *the* angle COB$=15°$ *and the* angle DOC$=60°$; *find their* resultant.

Resolve each force along and perpendicular to OA.



The resolutes along OA are OA, OM_2, OM_3, OM_4 in the figure, which $= 3$ lbs., $4\sqrt{2}\cos 45°$ lbs., $2\sqrt{3}\cos 60°$ lbs., $2\sqrt{3}\cos 120°$ lbs. respectively,

therefore their *sum* is $\left(3 + 4\sqrt{2} \times \frac{1}{\sqrt{2}} + 2\sqrt{3} \times \frac{1}{2} - 2\sqrt{3} \times \frac{1}{2}\right)$ lbs.,

that is $(3 + 4 + \sqrt{3} - \sqrt{3})$ lbs.$= 7$ lbs.

The resolutes perpendicular to OA are 0, ON_2, ON_3, ON_4; that is 0, $4\sqrt{2}\sin 45°$, $2\sqrt{3}\sin 60°$, $2\sqrt{3}\sin 120°$ respectively, therefore their *sum* is $(4 + 3 + 3)$ lbs. $= 10$ lbs.

The required resultant OR is such that

$$OR = \sqrt{\{7^2 + 10^2\}} = \sqrt{\{149\}} = 12{\cdot}4 \text{ lbs.} \qquad \text{(i)},$$

and

$$\tan ROA = \tfrac{10}{7} \qquad \text{(ii)}.$$

EXAMPLES. VIII.

Find the resultant of each of the following sets of forces.

1. OA, OB, $OC = 3$, 4, 5 lbs. respectively; the angle $AOB = 60°$, the angle $BOC = 30°$.

2. $OA = 4\sqrt{2}$ lbs., $OB = 6$ lbs., $OC = 6$ lbs.; the angle $AOB = 45°$, the angle $BOC = 90°$.

3. $OA = 4$ lbs., $OB = 3\sqrt{3}$ lbs. $= OC$; the angle

$$AOB = 60° = BOC.$$

4. $OA = 4$ lbs., $OB = 5$ lbs., $OC = 8$ lbs.; the angle AOB such that its cosine is $\frac{4}{5}$; the angle $AOC = 90°$.

5. $OA = 100$ lbs., $OB = 200$ lbs., $OC = 300$ lbs.; OA acts towards the East, OC towards the N.-W. and OB at an angle 30° to the East of North.

6. $OA = 10$ lbs. $= OB = OC = OD$; the angle

$$AOB = 60° = BOC = COD.$$

7. Forces of 4 lbs., 5 lbs. and 6 lbs. acting on a particle each inclined to the other at the angle of 120°.

8. Forces of 4 lbs., 5 lbs. and 6 lbs. acting at a point parallel respectively to the sides of an equilateral triangle taken in order.

9. Three forces each of 10 lbs., acting at a point, parallel respectively to the sides taken in order of a right-angled triangle whose sides are 3 ft., 4 ft., 5 ft. respectively.

10. Forces of 6 lbs., 8 lbs. and 10 lbs. respectively acting at a point parallel respectively to the sides taken in order of a triangle whose sides are 3 ft., 4 ft. and 5 ft. respectively.

11. Forces of a lbs., b lbs., c lbs. respectively acting at a point parallel respectively to the sides taken in order of a triangle whose sides are a ft., b ft. and c ft. respectively.

12. Forces of 1 lb., 2 lbs., 3 lbs., 4 lbs. and 5 lbs. acting at a point parallel to the sides AB, BC, CD, AD and the diagonal AC of a square $ABCD$.

13. $ABCD$ is a square, and forces represented in direction by the lines AB, BD, DA and AC, and in magnitude by the numbers 1, $2\sqrt{2}$, 3, and $\sqrt{2}$ act at a point; find their resultant.

14. Forces of 1, 2, 4, 6, 8 lbs. respectively act from the centre of a pentagon to the corners. Find the resultant.

15. Forces of 1, 2, 3, 4, 5, 6 lbs. respectively act from the centre of a regular hexagon to its corners; find their resultant.

16. $ABCDEF$ is a regular hexagon; find the resultant of forces represented by AB, AC, AD, AE, AF.

17. P, Q and R act at a point parallel to the sides taken in order of an equilateral triangle; shew that their resultant is

$$\sqrt{(P^2 + Q^2 + R^2 - QR - RP - PQ)}.$$

18. The arc of a quadrant is divided into four equal parts; forces of 1, 2, 3, 4, 5 lbs. respectively act from the centre to the extremities of those parts; find their resultant.

MISCELLANEOUS EXAMPLES. IX.

1. Forces of 1, 3, 6, 15 lbs. act at a point and in directions parallel to the sides of a square taken the same way round; find the magnitude of their resultant.

2. Forces of $2Q$, $3Q$, $4Q$ act at a point in directions parallel to the sides of an equilateral triangle taken the same way round; find their resultant.

3. Three forces are represented in direction and magnitude by the three sides of a triangle taken the same way round; prove that the algebraical sum of their resolutes in any chosen direction is zero.

4. The resultant of two forces P and Q acting at right angles is R; if P and Q be each increased by 3 lbs. R is increased by 4 lbs., and is now equal to the sum of the original values of P and Q; find P, Q and R.

5. ABC is an equilateral triangle, and D is the middle point of AB; prove that the line of action of the resultant of forces represented by AD, AC cuts off one third part of BC.

6. Forces of 40, 41, and 9 lbs. acting at a point produce equilibrium; shew that two of them are at right angles.

7. If the angle between two given forces be increased, shew that their resultant is consequently diminished.

8. Forces acting at a point are represented in magnitude by $2AB$, $2BC$, CA where ABC is a triangle, and in direction by AB, BC, CA; find the magnitude and direction of their resultant.

9. Two forces P and Q of given magnitude act at a given point A and the direction of P is fixed; shew that if the direction of Q change the extremity of a straight line drawn from A representing the resultant of P and Q will lie on the circumference of a fixed circle.

10. If two forces be inclined to one another at an angle of three halves of a right angle, find the ratio of their magnitudes when their resultant equals the less.

11. $ABCD$ is a quadrilateral figure whose angles DAB, BCD are right angles; forces represented by AB, AD act at A, and forces represented by CB, CD act at C. Shew that the direction of their resultant bisects the angle AEC where E is the middle point of BD.

12. Two forces which are to each other as 2 to $\sqrt{3}$ when compounded produce a force equivalent to half the greater; find the angle at which they are inclined to each other.

13. Forces of 8 lbs. and 10 lbs. acting at a point have a resultant of 14 lbs.; find the cosine of the angle between the forces.

14. Forces of 5 lbs. and 4 lbs. act at a point, the cosine of the angle between their directions being $\frac{1}{5}$; find the magnitude of their resultant.

15. A box is carried by a strap attached to handles at its ends; shew that the longer the strap the less will be its tension.

16. *ABCD* is a quadrilateral and forces acting at a point are represented in direction and magnitude by *BA*, *BC*, *DA*, *CD*; find their resultant.

17. Three forces keep a particle at rest; one acts towards the East, another towards the N.-W. and the third towards the South; if the magnitude of the first be 3 lbs., find the magnitudes of the other two.

18. Three forces 6 lbs., 7 lbs., 10 lbs. act at a point in directions such that each if produced would bisect the angle between the other two; find the magnitude of their resultant.

19. Three forces P, Q, R act at a point; the angle between the directions of P and Q is 90°, and the direction of R bisects the angle between the other two; if $P=1$ lb., $Q=\sqrt{3}$ lbs. and $R=3\sqrt{2}$ lbs., find the magnitude of the resultant of the three forces.

20. The resultant of two forces inclined to each other at 45° makes an angle 30° with the smaller; if the smaller force be 6 lbs. what is the greater?

21. Forces are represented by the sides *AB*, *AC* of a triangle *ABC*; if the resultant passes through the centre of the circle described about *ABC*, prove that the triangle must be either right-angled, or isosceles.

22. Two forces P and Q acting at an angle of 60° have a resultant R; prove that $2Q+P=\sqrt{(4R^2-3P^2)}$.

23. If two forces acting at a point be each multiplied by the same number, shew that their resultant is multiplied by the same number, and is unchanged in direction.

24. Shew that the resolute of a force *OA* in any direction is represented by the chord drawn from *O* in that direction of the circle whose diameter is *OA*.

CHAPTER IV.

MOMENTS.

62. *DEF.* The **moment** *of a force about a point* is that which varies as the force and also as the perpendicular distance of the point from the line of action of the force.

That is, if the force is doubled or trebled...the moment of the force about a given point is doubled or trebled...; also if the perpendicular distance of the point from the line of action of a given force is doubled or trebled...the moment of the given force about the point is doubled or trebled.

$\therefore$ Its moment $= K \times$ the force $\times$ perpendicular.

63. We shall assume as our **unit** *moment* that of unit force about a point whose distance from the line of action is the unit distance.

The unit force being 1 lb. weight, the unit distance being a foot, the unit moment would be a **foot-pound weight***.

Hence the moment of a force about a point is *measured* by the product of the number of lbs. in the force by the number of feet in the perpendicular distance of the line of action of the force from the point; or shortly,

The moment of a force about a point is *numerically equal* to the product of the force into its perpendicular distance from the point.

* NOTE.—The unit work in Dynamics is called a *foot-pound-weight*; but the student should notice carefully that the foot-pound-weight of work is the product of a force into *a movement*, *viz. a foot passed over in the direction of the force by the point of application.* This product involves motion; no work is done unless movement has actually taken place.

Thus the *unit work* is essentially different from the *unit moment* which is the product of *a force into a distance* perpendicular to the force.

Example. Three forces of 3 lbs., 4 lbs. *and* 5 lbs. *act along the two sides* AB, AD *and the diagonal* AC *of the square* ABCD *each of whose sides is* 2 ft.; *find the* measures of the moments *of each of these forces about the point* D.

The moment of the force 3 lbs. along AB is numerically equal to

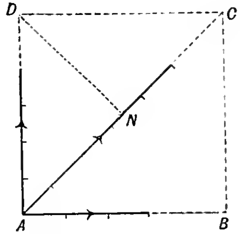
$$3 \times AD = 3 \times 2 = \underline{6}.$$

The moment of the force 5 lbs. along AC is numerically equal to $5 \times DN$, where DN is perpendicular to AC.

$$DN^2 + AN^2 = AD^2 = 4 \text{ and } DN = AN;$$

$$\therefore \; DN^2 = 2; \; \therefore \; DN = \sqrt{2},$$

the required measure $= 5 \times \sqrt{2} = \underline{7{\cdot}071\ldots}$



The moment of the force along AD about D is zero; for the perpendicular from D on AD is zero.

EXAMPLES. X.

1. $ABCD$ is a square each of whose sides is 5 feet; forces of 5 lbs., 6 lbs., and 7 lbs. act along AB, AC and AD, find the moment of each of these forces about D.

2. Find the moment of each of the forces of Question 1 about the middle point of AC.

3. Find the moment of each of the forces of Question 1 about the middle point of DC.

4. $ABCD$ is a square each of whose sides is 18 inches; forces of 3 lbs., 4 lbs., 5 lbs. and 6 lbs. act along AB, BC, CD and DA respectively; find the moment of each of these forces about a point Q in AD produced so that $DQ = 6$ in.

5. Find the moment of each of the forces in Question 4 about a point Q on AC produced so that $CQ = 6\sqrt{2}$ inches.

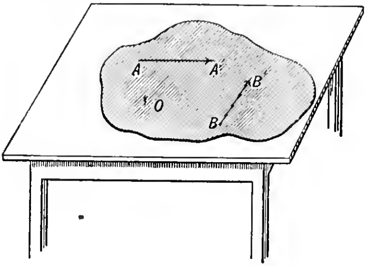
6. A and B are two points 1 foot apart; a force of 5 lbs. acts at A perpendicular to AB and a force of 7 lbs. acts at B parallel to the first force; find the point in AB about which the moments of these forces are equal in magnitude.

7. Find the point in AB produced about which the moments of the two forces in Question **6** are equal.

8. ABC is an equilateral triangle and forces of 4 lbs. and 5 lbs. act at A along AB and AC respectively, find the point in BC about which the moments of these forces are equal.

64. The following experiment is important.

Take a thin piece of board having a small hole in it at O. Let the board be placed on a smooth horizontal table, and let a small tack be



driven into the table at the point O so that the board can turn freely about O.

Let two small tacks be fastened to the board at any two points A and B, and let two horizontal forces represented by AA', BB' be applied (by strings fastened to the tacks) to the board at A and A'.

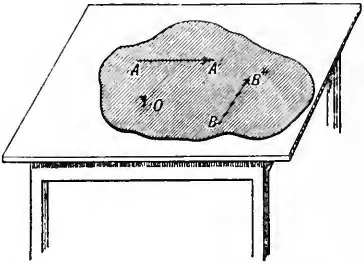
It will be found that when the board is in equilibrium under the action of these forces,

(i) the moment of the force AA' about O is equal to the moment of the force BB' about O.

(ii) that if each of the forces in them be allowed to act alone for a moment they will each exhibit a power of causing the board to turn round O, but *in opposite senses.*

These two conditions are expressed shortly thus. *When a board one point* O *of which is fixed, is in equilibrium, the moments about* O *of the forces which act upon the board must be equal in magnitude and of opposite senses.*

65. The manner in which the force AA', as drawn above, tends to produce rotation about O is said to be **clockwise**; the manner in which the force BB', as drawn above, tends to produce rotation is said to be **counter-clockwise**.



Being respectively like and unlike the rotation of the hands of a clock when laid face upwards on a table.

66. It will be found convenient to indicate that two moments are of the same or of contrary senses by the aid of the signs + and −.

And in what follows when use is made of the **sum** of the moments about a point of any number of forces, the sum will be understood to mean their algebraical sum, the moment of each force having its proper sign according to its sense.

The necessary condition that the board in the experiment of Art. 64 should be in equilibrium may now be stated thus. The sum of the moments of the two forces acting upon it about the point O must be zero.

It must be clearly understood that the experiment of Art. 64 does not *prove* this statement. Its truth will be deduced later on from the laws of motion.

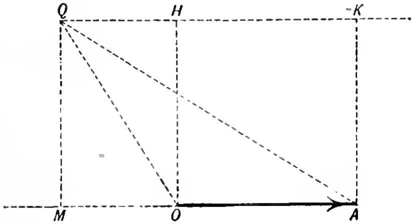
It is perhaps more usual to consider counter-clockwise rotation as positive.

67. Since the moment of a force about a given point is numerically equal to the *product* of the force into the perpendicular distance from the point to the line of action of the force, it follows that

The moment of a force about a point is *numerically equal* to twice the area of the triangle formed by joining the point to the extremities of the line which represents the force.

[Provided the unit force is represented by the same unit of scale as the unit distance.]

For let OA represent a given force in magnitude, direction and line of action; let Q be any chosen point.



Draw QM perpendicular to the line of OA.

Then the moment of the force OA about Q is measured by the **product** of the *number* of lbs. in the force OA, by the *number* of linear units in QM.

But the number of lbs. in the force is the number of linear units in OA.

Hence the moment of the force OA about the point Q is numerically equal to $QM \times OA$.

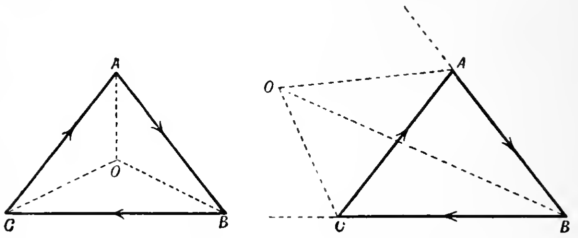
Through Q draw QHK parallel to OA, and draw OH, AK perpendicular to OA.

Then $QM \times OA = HO \times OA$.

And $HO \times OA$ numerically equals the area of the rectangle $HOAK$, which is double of the area of the triangle QOA,

$\therefore$ $QM \times OA =$ double of the area of the triangle QOA.

Example. Three forces are represented in direction, magnitude and line of action by the three sides of a triangle taken the same way round; shew that the sum of the moments of these forces about any point whatever in the plane of the triangle is numerically equal to twice the area of the triangle.



Let ABC be the triangle, and let the forces be taken in the directions indicated by the arrows. Then the sense of each of the moments of AB, BC, CA about any point O within the triangle is clockwise; therefore their sum is numerically equal to twice the sum of the areas of the triangles AOB, BOC, COA, that is to twice the area of the triangle ABC (clockwise).

When the point O is outside the triangle but within the angle ABC, as in the figure, the sense of the moment of CA about O is counterclockwise; while the sense of the moments of AB and BC about O is clockwise.

Hence the *sum* of the moments of AB, BC, CA about O is numerically equal to twice (the area of AOB + area of BOC − area of COA) clockwise; that is, equal to twice the area ABC (clockwise).

Similarly the proposition may be proved to be true for any other position of the point O in the plane of ABC.

EXAMPLES. XI.

1. Prove that the sum of the moments of two equal parallel and opposite forces, not in the same line of action about any point in their plane, is the same for all positions of that point.

2. Forces are represented in position, magnitude and sense by the sides of a closed polygon taken the same way round; prove that the sum of the moments of these forces about any point in their plane is numerically equal to twice the area of the polygon.

3. Three forces act along the sides of an equilateral triangle taken the same way round; shew that the sum of their moments about a point in their plane is different for different positions of the point, except when the forces are equal.

4. Find the locus of a point in a plane such that the moment of a given force about it may be constant.

5. Three forces are represented in position, magnitude and direction by the three sides of a given triangle not taken the same way round; find the locus of a point which moves so that the sum of the moments of the three forces about it may be constant.

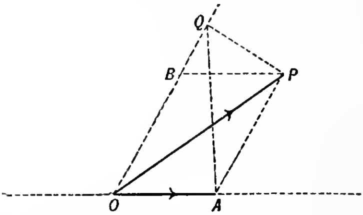
6. Prove that the moment of a given force about each of three points cannot be zero unless the three points lie in the same straight line.

7. ABC is an equilateral triangle; the moments of a force about the points A, B, C are respectively 6, -6, 0; find the line of action of the force; if a side of the triangle is 2 ft., what is the magnitude of the force?

8. ABC is an isosceles right-angled triangle whose equal sides AB, AC are 4 ft. each; the moments of a force about the points A, B, C are respectively 8, 8 and 16, find the magnitude and line of action of the force.

68. *PROP. The moment of a force about a given point is equal to the moment of one of any two components into which it can be resolved, provided the other component passes through the given point.*

Let OP represent the force. Let Q be the given point.



Join OQ, and draw a line OA, making any finite angle with OQ.

From P draw PA parallel to QO and PB parallel to AO.

Then OA and OB represent any two components into which OP can be resolved, of which one component passes through Q.

We have to prove that the moment of OP about Q is equal to the moment of OA about Q.

The moments have the same sense, and their magnitudes are represented by the triangles QOP, QOA which are upon the same base OQ and between the same parallels $OQ . AP$; and are therefore equal.

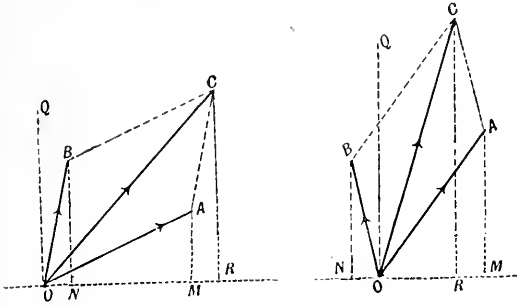
This proves the proposition.

69. It is worthy of notice that the necessary condition that the moment of OA about Q should be equal to the moment of OP about Q is, that A should lie on the line through P parallel to OQ.

And that this is the necessary condition that OA should be one of the components of OP, of which components one passes through Q.

70. *PROP. The sum of the moments of two forces acting at a point about any given point is equal to the moment of their resultant about the same point.*

Let OA, OB be the forces; OC their resultant.



Let Q be the given point.

Then we have to prove that the area of

twice ($\triangle\, QOA + \triangle\, QOB$) = twice $\triangle\, QOC$.

Draw a line ONM at right angles to OQ.

Draw AM, BN, CR each parallel to QO.

Then, twice $\triangle\, QOA$ = the rectangle $OM \times OQ$,
twice $\triangle\, QOB$ = the rectangle $ON \times OQ$,
twice $\triangle\, QOC$ = the rectangle $OR \times OQ$.

Now OM, ON, OR are the resolutes of the forces OA, OB, OC in the direction OM.

Therefore as in Art. 55,

$$OM + ON = OR,$$

$$\therefore\ OM \times OQ + ON \times OQ = OR \times OQ.$$

That is

twice $\triangle\, QOA$ + twice $\triangle\, QOB$ = twice $\triangle\, QOC$. Q. E. D.

71. *PROP. The sum of the moments of a number of forces acting at a point in a plane about a given point in their plane is equal to the moment of their resultant about the same point.*

For the sum of the moments of any two of the forces about the point is equal to the moment of their resultant about that point.

The sum of the moments of this resultant and a third force about the point is equal to the moments of *their* resultant about the same point;

and so on.

So that the sum of the moments of all the forces about the point is equal to the moment of the last resultant about the same point; the last resultant being the resultant of all the forces. Q. E. D.

72. *PROP. When a number of forces in one plane acting at a point are in equilibrium, then the sum of their moments about every single point in their plane is zero.*

For the sum of the moments of the forces about the point is equal to the moment of their resultant about the same point.

The resultant of the forces is *zero;* for the forces are in equilibrium.

Therefore the moment of the resultant about any point is zero.

73. There is always a series of points about any one of which the sum of the moments of any given system of forces acting at a point is zero; namely, the series of points which lie in the line of action of the resultant of the given system of forces.

For the sum of the moments of a given system of forces about any chosen point Q is equal to the moment about Q of their resultant.

When Q is on the line of action of this resultant, the perpendicular QM drawn from Q to this line of action is zero; and therefore the product $QM \times OA$ is, in this particular case, zero.

74. *PROP. The moments of a force about each of three points not on the same straight line cannot be each separately zero.*

For the moment of a force about a point is only zero when its line of action passes through the point; and a straight line cannot pass through each of three points not in the same straight line.

75. *PROP. When a system of forces acting at one point in a plane are such that the sum of their moments about each of three points not in the same straight line, is each separately zero, the system of forces must be in equilibrium.*

For the sum of the moments of the forces about any one point Q is equal to the moment about Q of their resultant.

Therefore the moments of the resultant about each of three points not in the same straight line are each separately zero.

This by Art. 74 is impossible unless the resultant is itself zero. That is, the system of forces is in equilibrium.

Q. E. D.

Example. It is observed of a system of forces acting at a point that the sums of their moments about each of two given points are equal. What does this prove of their resultant?

The resultant of the forces either (i) is parallel to the line joining those two points or (ii) passes through the point of bisection of the line joining the two given points. For its perpendicular distances from each of these two points must be equal.

EXAMPLES. XII.

1. It is observed that the sums of the moments of two forces acting at a point about each of two points in their plane are in the ratio of 2 to 1; what do you infer with regard to the resultant of the two forces?

2. The sum of the moments of certain forces acting at a point O is found to be zero about a certain point Q; what can be inferred from that fact with regard to their resultant?

3. Shew that the sum of the moments of the forces in Question **2** about any point in the line OQ is zero.

4. If the sum of the moments of the forces in Question **2** is also zero about some point not in the line OQ, shew that the forces must be in equilibrium.

5. Forces of 3 lbs. and 4 lbs. act along the sides AB, AD of a square $ABCD$ such that $AB = 5$ ft.; shew that the distance of their resultant from the point B is 4 ft., and from the point C is 1 ft. and from the point D is 3 ft.

6. ABC is a triangle and D, E, F are points on the sides BC, CA, AB such that $\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB}$; prove that the sums of the moments of forces represented by AD, BE, CF about A, B, C respectively are all equal.

7. A and B are fixed points on the circumference of a circle; AP and AQ are any two chords at right angles to each other on opposite sides of AB; shew that if AP, AQ represent forces, the difference of their moments about B is constant.

8. Forces act at P the intersection of perpendiculars ABC of a triangle represented by the perpendiculars AD, BE, CF from ABC on the opposite sides; shew that the sums of their moments about each of the angular points cannot be zero unless the triangle is equilateral. Can the forces be in equilibrium?

9. The sum of the moments of forces acting at a point in a plane cannot have the same value at each of three points unless the three points lie in one straight line parallel to a certain straight line.

10. The sum of the moments of forces represented by the bisectors of the sides about an angular point of the triangle is zero; hence shew that these three forces are in equilibrium.

11. The sum of the moments of forces represented by the bisectors of the angles of a triangle cannot be zero about an angular point of the triangle unless the triangle is isosceles; hence shew that these three forces cannot be in equilibrium unless the triangle is equilateral.

12. The sum of the moments of forces acting along the perpendiculars from the angles on the opposite sides of a triangle, each force being proportional to the side to which it is perpendicular about an angular point of the triangle, is zero; hence shew that these forces are in equilibrium.

13. A point O is taken within a triangle ABC; and forces are represented by OA, OB, OC; prove that the sum of the moments of these forces about A is not zero unless AO produced bisects BC; hence prove that the forces cannot be in equilibrium unless O is the point of intersection of the bisectors of the sides of the triangle.

14. If the sides of a triangle be taken two and two to represent forces acting in each case from the point of their intersection; prove that there is one point about which the sum of the moments of each pair of forces is zero. What is that point?

COUPLES.

76. *DEF.* A **couple** consists of two equal parallel forces of opposite senses, not in the same line of action.

When a workman applies to the head of a screw a twist (without pressing in the direction of the screw) he applies *a couple.*

When a man winds up his watch or a clock with a symmetrical key he does it by means of *a couple.*

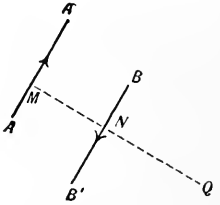
NOTE. Parallel forces are said to be **like** or **unlike** according as they act in the *same* or in *opposite* senses.

77. *PROP. The sum of the moments of the two forces forming a couple, about any point in their plane has a* constant value; *this value is numerically equal to* the product of one of the forces into the perpendicular distance between them.

Take any point Q in the plane containing the two equal parallel and opposite forces AA', BB'.

Draw QMN perpendicular to the lines of action of the forces cutting them in M and N.

Then, the sum of the moments of these two forces about Q is numerically equal to



$$QN \times BB' - QM \times AA'$$
$$= (QM - QN) \times AA' \qquad [\text{for } AA' = BB']$$
$$= NM \times AA',$$

which is independent of the position of Q.

78. *DEF.* The **moment of a couple** is numerically equal to the product of one of the forces into the perpendicular distance between the forces.

79. The moment of two equal parallel unlike forces is not zero except:

(i) when the forces are each zero;

(ii) when they are in the same line of action;

that is to say, when the two forces do not form a couple.

Hence, **the moment of a couple is never zero.**

Example. Three forces, not parallel to each other, are such that if they acted at one point they would be in equilibrium; prove that they are equivalent to a couple.

The resultant of two of the forces must be equal, parallel and of opposite sense to the third force; for if this resultant and the third force acted at a point they would be in equilibrium.

This resultant and the third force therefore form a couple.

EXAMPLES. XIII.

1. Four forces in the same plane not all parallel are such that if they acted at one point they would be in equilibrium; prove that they are equivalent to a couple.

2. Three forces are represented in direction, magnitude, and line of action by the sides of a triangle taken the same way round. Prove that they form a couple.

3. Prove that the moment of the couple of Question **2** is numerically equal to twice the area of the triangle.

4. Forces are represented in direction, magnitude, and line of action by the sides of a square taken in order; prove that the sum of their moments about each point in the plane of the square is constant.

5. Prove that the forces in Question **4** are equivalent to a couple.

6. Prove that the moment of the couple in Question **5** is numerically equal to twice the area of the square.

CHAPTER V.

The Equilibrium of Systems of Particles.

80. In the present treatise we propose to consider only systems of forces acting in one plane.

81. *DEF.* A **body** is a limited portion of matter, which we consider as composed of a very large number of particles.

A **rigid body** is such that the external forces acting upon it have so small an effect upon its shape that it may be considered to retain its shape unchanged notwithstanding the forces which act upon it. [See Arts. 15, 16.]

82. The bodies whose equilibrium we propose to discuss will generally be either **rods,** or **laminæ,** or **strings.** [See Art. 120.]

A **rod** is a *rigid* body whose length is so great compared with its breadth and its thickness that the last two (breadth and thickness) need not be considered.

A **light** *rod* is a rod whose weight is inconsiderable compared with the other forces acting upon it.

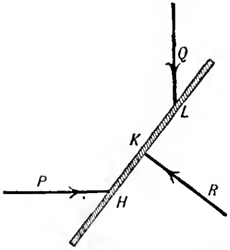
A **lamina** is a *rigid* body whose surface is so great compared with its thickness that the last (thickness) need not be considered.

A **string** is a body whose breadth and thickness are very small compared with its length; which is incapable of exerting any stress except in the direction of its length; this stress is of one kind only, namely, a '*pull.*'

An **inextensible** *string* is such that its length is considered to be unchanged by the action of the external forces acting upon it.

Strings will always be considered *inextensible* unless the contrary is expressly stated.

83. Consider a body acted on by external forces.



To fix our ideas let us consider a rod HK placed on a smooth horizontal plane acted on at points H, L, K by forces P, Q, R, as in the figure.

The force P is applied to a particle at H.

This particle acts on the neighbouring particles of the stick, and those on the next particles, and so on.

Thus, besides the external forces P, Q, R, there are a multitude of internal forces forming the mutual actions of the particles of the stick: these mutual actions consist in all cases of *pairs* of equal and opposite forces, actions and their reactions.

Each particle of the body acts upon the next particle in the body and is acted upon by it. The force which one particle composing the body under consideration applies to a neighbouring particle of the body is called an **internal** force.

Since action and reaction are equal and opposite, *all internal forces may be grouped into pairs of equal and opposite forces in the same line of action.*

The force which an external particle applies to a particle composing the body is called an **external** force.

[Corresponding to each external force there is a reaction which is applied to the external body, but when we are not considering the equilibrium of that external body we do not need to consider this reaction.]

Now when the body is in equilibrium each particle of the body must be in equilibrium.

Hence, considering the forces which act on any *one particle*, they must be such that

(I) The sum of their resolutes in *any* chosen direction whatever is zero. [By Art. 55.]

(II) The sum of their moments about *any* chosen point whatever is zero. [By Art. 72.]

These two statements are true of the forces acting upon each particle.

Therefore they are true of all the forces acting upon all the particles; that is,

(I) The sum of the resolutes in *any* chosen direction whatever of all the forces internal and external which act on the particles of the body is zero.

(II) The sum of the moments about *any* chosen point whatever in their plane of all the forces internal and external which act on the particles of the body is zero.

But the *internal* forces always occur in *pairs*, each pair consisting of *two equal and opposite forces* (namely, an action and its reaction).

The above two statements (I) and (II) are true for every such pair of forces taken separately.

Therefore they are true for all the *internal* forces taken by themselves.

And consequently they must be true for the *external* forces taken by themselves. Hence

When a body acted on by any external forces in one plane is in equilibrium,

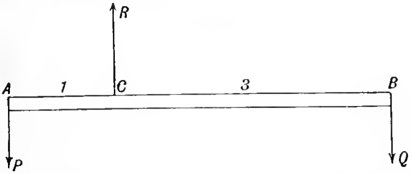
(I) **the sum of the resolutes in any chosen direction whatever of those external forces is zero,**

(II) **the sum of the moments about any chosen point whatever in their plane of the external forces is zero.**

84. The above two statements are true of *every* body or collection of bodies which is in equilibrium.

We shall presently prove that when they are true for a '*rigid*' body, that body must be in equilibrium.

Example. A rod AB 4 feet *long is placed on a smooth horizontal table and is acted on by three parallel horizontal forces P, Q* and *R*; of which *P and Q act one at each end, and R acts at the point C* 1 foot *from A. Find conditions of equilibrium.*



Since the sum of the resolutes in the direction parallel to the forces must be zero, we have

$$P+Q+R=0 \quad \text{.................(i)}.$$

Since the sum of their moments about C must be zero, we have

$$P\times AC=Q\times BC,$$

$$P\times 1=Q\times 3 \quad \text{.................(ii)},$$

$$P=3Q,$$

$$R=-(P+Q)=-(3Q+Q)=-4Q \text{ [from (i)]};$$

$$\therefore \quad \frac{P}{3}=\frac{Q}{1}=-\frac{R}{4}.$$

Thus if $P=3$ lbs., then $Q=1$ lb. and $R=4$ lbs. in the opposite direction.

85. Consider a rigid body, such as a lamina, which can only move on one plane; say the plane surface of a table.

For example, Let the rigid body be a thin piece of metal and suppose it to be placed on the surface of the table and prevented from moving except along the surface of the table.

Suppose now one point of the lamina to be fixed to the table; the only subsequent movement possible for the lamina is a turning about that fixed point as centre.

And if then some other point is fixed to the table the lamina cannot move at all.

Moreover if a body is rigidly fastened to the lamina this body is also incapable of motion ; hence,

When a *rigid* body is at liberty to move only in some assigned plane, then the freedom which it has to move will be entirely taken away from it, provided we fix any two chosen points of the body which are in that plane.

86. *PROP.* To prove that *whatever forces in one plane may be acting upon a single rigid body, that rigid body can be kept from moving, by two external forces; which forces may be applied at any two chosen points in the body, provided the two forces have the proper magnitude and direction ; also, that the direction of one of the forces may be taken at right angles to the line joining the two chosen points.*

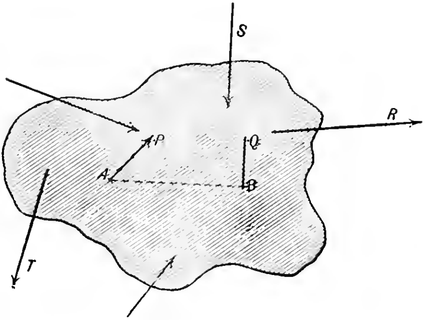
Since the forces act in one plane, the tendency to motion of the lamina will be only in that plane.

Consider a lamina in the plane of the forces.

A thin piece of metal or of card-board placed on a smooth horizontal table acted on by any system of horizontal forces in the plane of the table.

Any assigned particle A in the lamina being on the point of motion, its tendency to motion can only be in one direction; and therefore by choosing the right force in the right direction, and applying it to the particle *A*, we can cause the particle *A* of the lamina to have no tendency to motion. Suppose the force to be *P*.

Imagine a small hole in the card-board at *A* through which a pin's point passes and is fastened to the table. The card-board can press upon the pin in one direction only with a definite amount of force in that direction. This single force is sufficient to keep the point *A* at rest.



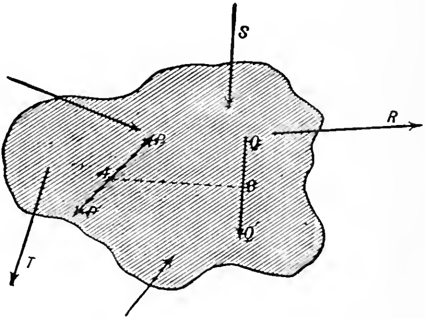
The particles of the body *are rigidly connected* with the particle *A* and therefore the only motion possible for any other particle *B* is along the circle whose centre is *A* and radius *AB*.

Hence, the only tendency to motion which any other particle *B* of the lamina can now have is in the direction perpendicular to *AB*.

Therefore by applying to the particle *B* the proper force (*Q* suppose) in the line perpendicular to *AB* we can cause the particle *B* to have no tendency to move.

And since every particle of the body is rigidly connected with *B*, the body cannot turn about *A*. The body is therefore incapable of any motion whatever. Q. E. D.

87. Consider a rigid body acted on by some assigned system of forces *R*, *S*, *T*... in the same plane. Let the force *P* applied to the body at some chosen point *A*, and the force *Q*, applied to the body at some other chosen point *B* and perpendicular to *AB* be the two forces necessary to keep the body from moving.



Then two other forces *P′* and *Q′*, equal and opposite respectively to *P* and *Q*, when applied to the body at *A* and *B* respectively are **statically equivalent** to the system *R*, *S*, *T*, ... ; for the system *R*, *S*, *T*, ... and the forces *P′*, *Q′*, can either of them be kept in equilibrium by the forces *P*, *Q*.

88. It is important that the student should clearly understand the meaning of the words *statically equivalent*.

If we were taking account of the *internal* forces in the body the effect of a system of external forces on a body, and the effect of forces which are statically equivalent to that system would be very different. They would set up different internal stresses between the particles of the body to which they are applied.

89. *DEF.* Two systems of forces are statically equivalent, which, when applied in turn to the same single rigid body, can be counteracted (or the rigid body can be kept in equilibrium) by the same system of forces.

Since we do not take account of the internal changes in a body due to the action of the forces, two systems of forces which are statically equivalent are also practically equivalent.

When we have found the simplest system of forces which is statically equivalent to an assigned system, the simplest system is often called the **resultant** of the assigned system.

It must be noticed that this resultant is not strictly a *resultant* in the sense of the definition of Art. 22. It does not produce in a body the *same* effect as the system; it sets up different internal forces.

90. Since the forces R, S, T, ... P, Q in Art. 87 form a system of forces in equilibrium, therefore the sum of their resolutes in any direction is zero.

Therefore the sum of the resolutes in any direction of R, S, T,... are equal and opposite to the sum of the resolutes of P and Q in the same direction.

But the sum of the resolutes of P', Q', P, Q in any direction is also zero.

Therefore the sum of the resolutes of the system of forces R, S, T, ... in any direction is equal to the sum of the resolutes in the same direction of P', Q'.

Similarly we may shew that the sum of the moments about any point of R, S, T, ... is equal to the sum of the moments about the same point of P', Q'.

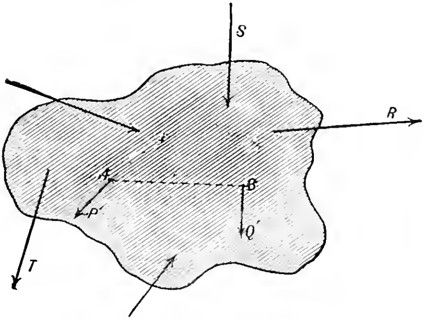
91. *PROP. A rigid body acted on by any assigned system of forces in one plane is in equilibrium provided*

I. the sum of the resolutes of the forces in any chosen direction is zero,

II. the sum of the resolutes of the forces in a second chosen direction is zero,

III. the sum of the moments of the forces about any chosen point is zero.

Let the forces R, S, T, ... be the assigned system of forces acting on a rigid body.



By Art. 85 the forces R, S, T, ... acting on a rigid body if not in equilibrium, are statically equivalent to two forces P', Q' acting at any two chosen points A, B in the body, Q' being perpendicular to AB.

And by Art. 90 the sum of the resolutes in any direction, and the sum of the moments about any point of R, S, T,... and of P', Q' are equal respectively.

Therefore the conditions I, II, III, must be true of the forces P', Q' acting on the body at A and B, Q' being perpendicular to AB.

Conditions I. and II. require that P' and Q' (if each of them is not zero) should be equal and parallel of opposite senses.

Condition III. shews that P' and Q' can have no magnitude if equal parallel and of opposite senses. [Art. 79.]

Therefore P' and Q' are each of them zero.

Therefore when the above conditions are satisfied, the rigid body remains at rest without the application of any additional external force.

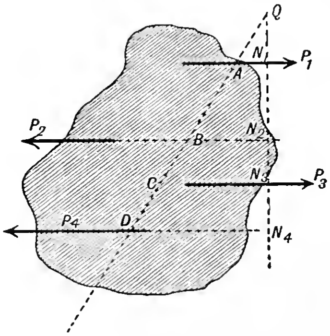
92. We have now shewn (Art. 91) that any system of forces which act on a single rigid body and which satisfy conditions I, II, III, are in equilibrium. We have also proved (Art. 83) that every system of forces which acting on a rigid body is in equilibrium must satisfy conditions I, II, III.

93. We proceed to apply these important results to particular cases.

CHAPTER VI.

PARALLEL FORCES.

94. *PROP. To find the conditions necessary for the equilibrium of a rigid body when acted on by parallel forces.*



Let P_1, P_2, P_3 ... be the parallel forces acting upon the rigid body.

Take any point Q in the plane of the forces, and draw $QN_1N_2N_3$... perpendicular to the forces P_1, P_2, P_3... cutting their lines of action in N_1, N_2, N_3 ... respectively.

By Art. 91, I, II, the sums of the resolutes of the forces P_1, P_2, P_3 ... in any two chosen directions must be zero.

Choose the two directions parallel and perpendicular to the forces, then Art. 91, II, is satisfied, since the resolute of each force in the direction perpendicular to itself is zero.

From Art. 91, I, we have

$$P_1 + P_2 + P_3 + \ldots = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

since the resolute of a force in its own direction is the force itself.

Due regard being paid to the *sense* of each force.

By Art. 91, III, the sum of the moments of the forces $P_1, P_2, P_3, \ldots$ about any point must be zero.

Take Q for the point, then

$$P_1 \times QN_1 + P_2 \times QN_2 + P_3 \times QN_3 + \ldots = 0 \quad \ldots\text{(ii)}.$$

Due regard being paid to the *sense* of each moment.

These two conditions (i) and (ii) are the necessary and sufficient conditions that any number of *parallel* forces acting on a rigid body in one plane should be in equilibrium.

95. From Q draw a line in any direction cutting the lines of action of the forces in $A, B, C \ldots$.

Then $$\frac{QN_1}{QA} = \frac{QN_2}{QB} = \frac{QN_3}{QC} = \ldots.$$

Let the common value of these fractions be k, then

$$QN_1 = k \cdot QA, \quad QN_2 = k \cdot QB, \quad QN_3 = k \cdot QC, \text{ etc.},$$

whence substituting in (ii) above, and dividing by k

$$P_1 \times QA + P_2 \times QB + P_3 \times QC + \ldots = 0 \ldots\ldots \text{(ii)}.$$

Hence in using Art. 91, III, in the case of parallel forces, we may write down 'the equation of moments' thus,

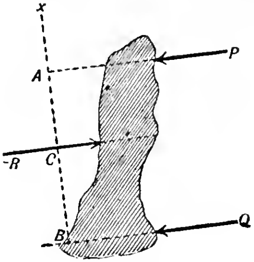
$$P_1 \times QA + P_2 \times QB + P_3 \times QC + \ldots = 0,$$

whether the line $QABC$ be perpendicular to the line of action of the forces or not.

That is, when we *take the moments of a system of* **parallel** *forces* about a point, the distances of the forces may be measured in any direction which may be convenient.

Example. When there are three parallel forces in equilibrium, viz. P, Q, and $-R$, then if a line cut these forces in A, B, C respectively,

$$P \times AC = Q \times BC.$$



Taking moments about any point X, we have

$$P \times AX + Q \times BX - R \times CX = 0,$$

also $$R = P + Q,$$

$$\therefore \quad P \times AX + Q \times (BC + CA + AX) - (P + Q)(CA + AX) = 0.$$

$$\therefore \quad Q \times BC = P \times AC.$$

96. *PROP. To find the resultant of two parallel forces.*

Let the forces be P lbs. and Q lbs. applied to a rigid body at points A and B respectively. [Fig. Art. 97.]

Join AB. Let the force R be the resultant of P and Q then the force $-R$ is the anti-resultant of the forces P and Q; so that the forces P, Q, $-R$ are in equilibrium.

Since the resolutes of the forces P and Q in the direction perpendicular to them are both zero, the resolute of $-R$ in that direction must also be zero.

That is, $-R$ is parallel to P and Q ....(I).

Again, since the sum of the resolutes of these parallel forces in their own direction is zero;

$$\therefore \quad -R + P + Q = 0,$$

or $$R = P + Q.$$

Therefore the resultant of two parallel forces is equal to their sum ....(II).

Let the line of action of $-R$ cut AB in C, then taking moments about C, we have, as in Art. 95,

$$-R \times 0 + P \times CA + Q \times CB = 0,$$

or
$$P \times CA + Q \times CB = 0.$$

Hence the point C in AB, through which the line of action of the resultant passes, is such that the moment of P about C is equal and opposite to the moment of Q about C..(III).

97. **First**, let P and Q be **like** parallel forces [Fig. I.].

Then, in order that the moment of P about C may be (equal and) *opposite* to the moment of Q about C, it follows that C must be between A and B.

Also in this case, R (which equals the *sum* of the forces) is the *numerical* sum of P and Q.

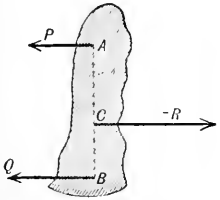


Fig. I

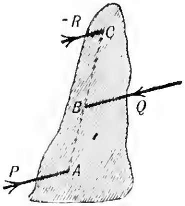


Fig. II

Next, let P and Q be **unlike** parallel forces [Fig. II.].

Then in order that the moment of P about C may be (equal and) *opposite* to the moment of Q about C, it follows that C must *not* be between A and B.

Also, in this case R (which equals the *sum* of the forces) is *numerically* equal to the *difference* between Q and P.

Moreover, if Q is greater than P, then

$$(\text{since } P \times CA = -Q \times CB),$$

CA must be greater than CB. That is, of the three parallel forces P, Q, $-R$, **the greatest is always between the other two.**

98. *PROP. If three forces acting on a rigid body are in equilibrium, they must either be all parallel or must all intersect in the same point.* [N.B. In the present work we confine our attention to forces in one plane.]

Consider two of the forces; they must either be parallel or they must meet in a point.

First, suppose that two of the forces are parallel.

The resolute of each of these forces in the direction perpendicular to itself is zero.

And by Art. 83 the sum of the resolutes of all three forces in this direction is zero.

Therefore the resolute of the third force in this direction is zero. That is, the third force is parallel to the other two.

Next, suppose that two of the forces intersect in the point O.

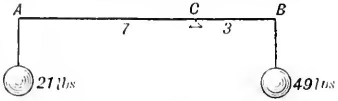
The moment of each of these forces about O is zero. And by Art. 83 the sum of the moments of all three forces about O is zero.

Therefore the moment of the third force about O is zero.

That is, the third force passes through O, the point of intersection of the other two. Q. E. D.

The above proposition is of frequent use in the geometrical method of solving statical problems [see examples at the end of the book].

Example i. *A light rod* 10 ft. *long has weights of* 21 lbs. *and* 49 lbs. *fastened one at each end. A man wishes to carry them both, where must he take hold of the rod and what weight will he lift?*



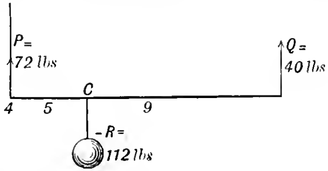
Here, considering the equilibrium of the rod, we have two like parallel forces acting upon it. Their resultant is therefore equal to their sum.

The man must therefore lift (21 + 49) lbs. = 70 lbs.

Also if AB be the rod and C be the point at which the resultant acts (which is the point at which the man must take hold of the rod),

we have $$AC + CB = 10 \text{ ft.}$$
and $$21 \times AC = 49 \times CB,$$
$$\therefore\ 49\,AC + 49\,BC = 490 \text{ ft.}$$
$$\therefore\ 49\,AC + 21\,AC = 490 \text{ ft.}$$
$$\therefore\ \left.\begin{array}{r} AC = 7 \text{ ft.} \\ \text{and } BC = 3 \text{ ft.}\end{array}\right\}.$$

Example ii. *Two men carry a weight of* 1 cwt. *slung on a light pole* 14 ft. *long, each holding one end of the pole; if the weight be placed at a point* 5 ft. *from one end, what weight does each man carry?*

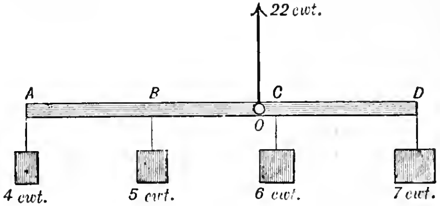


Consider the forces acting on the rod.

In this case $-R$ is 112 lbs. downwards; P and Q are the forces applied by the men's hands to the rod; the weight of the rod itself is neglected.

$$AC \text{ is } 5 \text{ ft., } BC \text{ is } 9 \text{ ft.;}$$
$$P + Q = R = 112 \text{ lbs. upwards} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i).}$$
Also $$P \times 5 = Q \times 9 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii),}$$
$$\therefore\ 5P + 5Q = 112 \text{ lbs.} \times 5 = 560 \text{ lbs.}$$
$$\therefore\ 9Q + 5Q = 560 \text{ lbs.}$$
$$\therefore\ Q = \underline{40 \text{ lbs.}}$$
$$\text{and } P = (112 \text{ lbs.} - Q) = \underline{72 \text{ lbs.}}$$

Example iii. *Weights of* 4 cwt., 5 cwt., 6 cwt., 7 cwt. *are suspended from points* ABCD *of a light horizontal rigid rod which are* 1 foot *apart; find* where *a single upward force must be applied to the rod that it may be in equilibrium.*



The rod is *light:* therefore its weight is neglected. The only forces acting on the rigid body, the rod, are the weights; these weights are vertical forces acting downwards.

Since the sum of the resolved parts in any direction of a system of forces in equilibrium is zero, we see, by considering the resolved parts in the vertical direction, that the upward force must

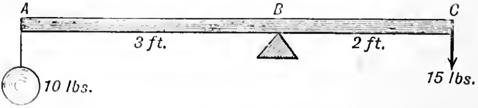
$$= (4+5+6+7) \text{ cwt.} = 22 \text{ cwt.}$$

By taking the sum of the moments about the point A of all the forces, we have, if OA contain x feet,

$$4 \times 0 + 5 \times 1 + 6 \times 2 + 7 \times 3 - 22 \times x = 0,$$

or
$$x = \tfrac{38}{22} = 1\tfrac{8}{11} \text{ ft.} = OA.$$

Example iv. *A light rigid rod* 5 ft. long *has a* weight of 10 lbs. *suspended at one end and is supported at a point* 3 ft. *from that end by the upward pressure of a man's shoulder; what* force *must be exerted by the man's hand on the other end of the rod to keep it in equilibrium?*



Let AC be the rod, B the position of the man's shoulder.

Let x lbs. be the vertical pressure upwards on the rod at B.

Consider the equilibrium of the rod.

Then since the forces of 10 lbs. and x lbs. are parallel, and the force of x lbs. is the resultant of two like parallel vertical forces acting downwards, x lbs. = the sum of these forces. Therefore the other force is $(10 - x)$ lbs.; hence

By taking the sum of the moments about C of all the forces we have

$$10 \times AC - x \times BC + (x - 10) \times 0 = 0,$$

or

$$10 \times 5 - x \times 2 = 0;$$

$$\therefore \quad x = 25.$$

Hence the required force, which is $(x - 10)$ lbs., is $(25 - 10)$ lbs., or 15 lbs. downwards.

EXAMPLES. XIV.

1. Find the resultant of two like parallel forces of 3 lbs. and 2 lbs. acting at points 5 ft. apart.

2. Find where a force must be applied to a rigid bar to keep it in equilibrium when acted on by two like parallel forces of 3 lbs. and 5 lbs. acting at points on the bar 4 ft. apart.

3. Two men carry a weight of 1 cwt. by means of a light rigid rod 14 ft. long, each having one end of the rod on his shoulder: find what each man carries when the weight is suspended from a point 6 ft. from one end.

4. Two men carry a barrel weighing 80 lbs. by means of a rigid pole 10 ft. long, each supporting one end of the rod and the barrel is slung at a point on the rod 4 ft. from one end: what weight does each man carry?

5. A man carries two weights by means of a rod 12 ft. long supported by his shoulder; if the rod presses on his shoulder with a force of 36 lbs., and the point of the rod on his shoulder is 4 ft. from one end; what are the two weights?

6. A man carries two buckets of water by means of a pole which he holds in his hand at a point three-fifths of its length from one end; if the total weight carried is 40 lbs., how much do the buckets weigh respectively?

7. A rigid bar 4 ft. long is acted on by three like parallel forces of 3 lbs., and 3 lbs. at each end and one of 4 lbs. at the middle point; what force is necessary to maintain equilibrium?

8. A rigid bar 4 ft. long is acted on by equal like parallel forces of 4 lbs. at each end, and by an unlike parallel force of 5 lbs. at a point 1 foot from one end; find the force necessary to maintain equilibrium.

9. A man can just lift 112 lbs.; a rod 3 ft. long which can turn about one end has a weight fastened at a distance of 1 ft. from that end; what is the greatest magnitude of the weight if the man is to lift it by applying a lifting force to the other end of the rod?

10. A horizontal rod 15 ft. long can turn about one extremity which is fixed; a force of 10 lbs. acts upwards at the other end and one of 20 lbs. is applied downwards at a point between; find where the 20 lbs. is applied.

11. Three like parallel forces of 5 lbs., 7 lbs. and 9 lbs. act in lines whose distances apart are 3 ft. and 4 ft.; find their resultant.

12. Two like parallel forces of 5 lbs. and 9 lbs. and an unlike parallel force of 7 lbs. are such that the line of action of the 7 lbs. is between the other and distant 3 ft. from the former and 4 ft. from the latter force; find their resultant.

13. Two men, one stronger than the other, have to remove a block of stone weighing 300 lbs. with a light pole whose length is 6 ft.; the weaker man cannot carry more than 100 lbs.; where must the stone be fastened to the pole so as just to allow him his full share of the weight?

14. Two men, one stronger than the other, have to remove a block of stone weighing 270 lbs. with a light pole whose length is 12 ft.; the stronger man is just able to carry 180 lbs.; how must the stone be suspended from the pole so as to allow him his maximum weight?

15. A light rigid bar 30 ft. long has suspended from its middle point a weight of 700 lbs., and rests on two walls 24 ft. apart, so that 1 foot of it projects over one of them; a weight of 192 lbs. is suspended from a point 2 ft. from the other end; what is the pressure borne by each of the walls?

16. A light plank 20 ft. long rests on the top of a wall; at one end is a man weighing $12\frac{1}{2}$ stone, at the other a boy weighing $6\frac{1}{2}$ stone, and 2 ft. from that end a basket of eggs weighing 7 lbs.; how much of the plank is on each side of the wall?

17. The resultant of two unlike parallel forces is 2 pounds and acts at distances 6 in. and 8 in. from them; find the forces.

18. A light rigid rod 10 ft. long can turn freely about a point 4 ft. from one end, at which end a weight of 210 lbs. is hung; if there is also a weight of 140 lbs. suspended at the middle point of the rod, what weight must be suspended at the other end to maintain equilibrium?

19. A light rigid rod 12 ft. long turns freely about a point 9 ft. from one end at which a weight of 100 lbs. is suspended; at the middle point is suspended a weight of 50 lbs.; what weight must be suspended from the other end to maintain equilibrium?

20. A light rigid rod 20 ft. long is supported in a horizontal position on two posts 9 ft. apart, one post is 4 ft. from the end of the rod; from the middle point of the rod a weight of 63 lbs. is suspended: find the pressures on the posts.

21. Unlike parallel forces of 3 lbs. and 7 lbs. act at points of a bar 10 ft. apart; find the least length of the bar that it may be capable of being kept in equilibrium by a single force acting upon it.

22. A rod 3 ft. long is suspended by two vertical strings one attached to each end of the rod; two equal weights are suspended from the rod at points distant 9 in. and 21 in. respectively from one end of the rod; find the greatest possible magnitude of the equal weights in order that neither of the forces exerted by the strings may exceed 1 cwt.

23. If in Question **22** the rod is 2 ft. long and the distances of the points of suspension of the weights 4 in. and 16 in. respectively from one end of the rod; find how great the equal weight may be if the strings will break under any force exceeding 1 cwt.

24. Six like parallel forces of 1 lb., 2 lbs., 3 lbs., 4 lbs., 5 lbs., 6 lbs. respectively are applied to a rigid rod at points one inch apart; find their resultant.

25. Six like parallel forces of 7 lbs., 6 lbs., 5 lbs., 4 lbs., 3 lbs., 2 lbs. are applied to a rigid rod at points 1 foot apart; find their resultant.

26. Weights of 3 lbs., 5 lbs. 7 lbs. and 9 lbs. are suspended from a light rigid rod 8 ft. long at points equally distant from each other; find where a force must be applied to the rod to support it.

27. Weights of 2, 4, 6, and 8 lbs. are suspended from a light rigid rod 12 ft. long at points equally distant from each other; find where the rod must be supported that it may be in equilibrium when horizontal.

28. A light horizontal rigid rod 3 ft. long has a weight of 15 cwt. suspended from a point on it, and it is supported by four strings, which apply forces to it which are in the ratio of 1 : 2 : 4 : 8, and which are fastened to the rod at points each 1 foot apart; where is the weight fastened, and what force does each string apply to the rod?

29. If two unlike parallel forces of 70 lbs. and 30 lbs. be altered to forces of 100 lbs. and 60 lbs. (the lines of action being unaltered), shew that the distance of the new resultant from the force 100 lbs. is double that of the old resultant from the same line (i.e. the line of action of the 70 lbs.).

30. If two like parallel forces of 20 lbs. and 30 lbs. be changed to 40 lbs. and 10 lbs. respectively (the lines of action being unaltered), shew that the distance from the line of action of the 30 lbs. of the new resultant is double that of the first resultant.

31. A, B, C, D are points in a rigid rod 1 foot apart; forces of 7 lbs. and 9 lbs. act at A and C upwards, and forces of 3 lbs. and 20 lbs. act at B and D downwards; find where a single force applied to the rod can keep it in equilibrium.

32. A, B, C, D are points in a light rigid rod $AB = 1$ ft., $BC = 2$ ft., $CD = 3$ ft.; forces of 8 lbs., 6 lbs. and 4 lbs. are applied at A, B and D downwards, and an upward force of 20 lbs. is applied at C; find what force will keep the rod in equilibrium, and where it must be applied.

33. A light rigid rod AE is divided in the points B, C, D so that $AB : BC : CD : DE = 1 : 3 : 5 : 7$, and weights of 1, 2, 3, 4 lbs. are placed at the points B, C, D, E. The rod is supported in a horizontal position fastened at G; prove that

$$AG : GE = 5 : 3.$$

34. A light rigid rod AF is divided in the points B, C, D, E, so that $\frac{AB}{1} = \frac{BC}{3} = \frac{CD}{5} = \frac{DE}{7} = \frac{EF}{9}$, and weights of 1, 2, 3, 4, 5 lbs. are suspended from the points B, C, D, E, F; shew that if the proper upward force be applied at G, where $\frac{AG}{3} = \frac{GF}{2}$, the rod will be in equilibrium.

35. A pair of nut-crackers is 5 inches long, and when a nut is placed $\frac{1}{2}$ of an inch from the hinge a pressure of $3\frac{1}{2}$ lbs. applied at the end will crack it; what weight if simply placed on the top of the nut would crack it?

CHAPTER VII.

The Centre of Parallel Forces.

99. In the preceding chapters we have learned how to determine not only the magnitude of the resultant of a given system of parallel forces and the sense in which it acts, but also its position relatively to the forces of the system. This position is determined when we know any single point in the line of action of the resultant. The resultant itself is parallel to the forces and its line of action passes through every such point.

Now suppose the forces of such a system (still remaining parallel to each other) to be turned each about its point of application; it is the purpose of this chapter to prove that their resultant (still remaining parallel to the system) will also turn about a certain fixed point in its line of action.

This fixed point is called **the Centre of the Parallel Forces.**

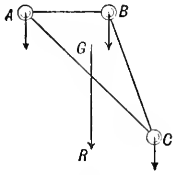


Fig. i.

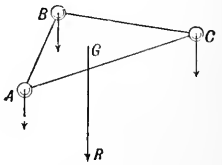


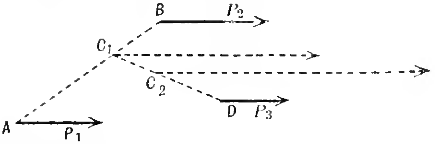
Fig. ii.

For example, let ABC (Fig. i.) be three heavy iron balls fixed at the corners of a light frame of rigid rods.

The weights of these balls are three parallel downward forces, and their resultant R is a vertical downward force equal to the sum of the three weights, whose position relatively to ABC may be found by Art. 95. If now the system of bodies be moved from the position shewn in Fig. i. to that shewn in Fig. ii., the directions of the three original forces will be changed as regards the relative positions of the balls, but it will be found that there is a point G, whose position relative to A, B, C is *fixed*, through which R *must pass*.

100. *PROP. To prove the existence of a Centre of Parallel Forces.*

Let $P_1 P_2$ be the given magnitudes of two parallel forces applied to a rigid body at given points A and B.



Their resultant is parallel to P_1 and P_2, and is equal in magnitude to $P_1 + P_2$, and its line of action cuts AB in a point C_1, such that $C_1A \times P_1 + C_1B \times P_2 = 0$.

This point C_1 will be the same for all *directions* of the given parallel forces provided their points of application are unchanged.

Hence, the parallel forces $P_1 P_2$ may be replaced by a single parallel force whose magnitude is $P_1 + P_2$ applied at the *fixed* point C_1.

Let P_3 be a third parallel force and let it be applied to the rigid body at the given fixed point D.

Then we have two forces, $(P_1 + P_2)$ acting at C_1, and P_3 acting at D.

As before, their resultant is parallel and equal to

$$(P_1 + P_2) + P_3,$$

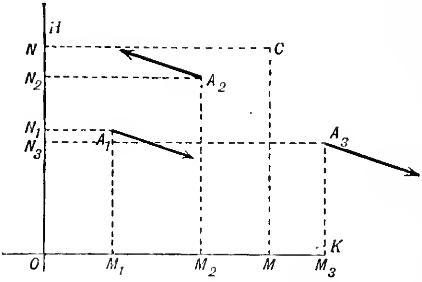
and its line of action cuts C_1D in a fixed point C_2, such that

$$C_2C_1 \times (P_1 + P_2) + C_2D \times P_3 = 0.$$

Hence the *three* parallel forces P_1, P_2, P_3 may be replaced by a single parallel force whose magnitude is $(P_1 + P_2 + P_3)$ applied at the *fixed* point C_2; and so on.

101. *PROP. To find the* Centre *of a given system of Parallel Forces.*

Let $A_1A_2A_3$ ... be the given fixed points at which the given parallel forces P_1, P_2, P_3 ... are applied to a rigid body in one plane.



In the plane of the forces take any two lines OH, OK mutually at right angles.

Draw A_1M_1, A_2M_2... perpendicular to OK and draw A_1N_1, A_2N_2... perpendicular to OH.

The resultant of the system is a force of magnitude $(P_1 + P_2 + P_3 + ...)$ parallel to the system and it always passes through the *Centre* of the system.

Let C be the centre of the system.

Draw CM, CN perpendicular to OK and OH respectively.

Now by Art. 100 the position of C depends on the positions of A_1, A_2, A_3,... and on the *magnitudes* of P_1, P_2, P_3,... and it does *not* depend on the direction of the forces P_1, P_2, P_3... ; hence

We shall first suppose the forces be all parallel to the line OH.

Then since the force $-(P_1+P_2+P_3+\ldots)$ acting through C is in equilibrium with the forces P_1, P_2, P_3 ... acting at A_1, A_2, A_3 ..., we have, by taking moments about O (remembering that all the forces are parallel to OH),

$$-(P_1+P_2+P_3+\ldots)\times OM+P_1\times OM_1+P_2 OM_2+P_3\times OM_3+\ldots = 0;$$

$$\therefore \quad OM=\frac{P_1\times OM_1+P_2\times OM_2+P_3\times OM_3+\ldots}{P_1+P_2+P_3+\ldots}$$

$$=\frac{\Sigma[P_1\times OM_1]}{\Sigma[P_1]}.$$

Next let the system of forces be parallel to OK, then similarly,

$$ON=\frac{P_1\times ON_1+P_2\times ON_2+P_3\times ON_3+\ldots}{P_1+P_2+P_3+\ldots}$$

$$=\frac{\Sigma[P_1\times ON_1]}{\Sigma[P_1]}.$$

Thus the position of C is found.

NOTE.—The student must notice carefully that the position of the centre of a system of parallel forces depends (i) on their points of application, (ii) on their relative magnitudes; it does not depend on the direction in which they act.

Moreover since it depends on their *relative* magnitudes, it follows that if each force were doubled or tripled... the position of the Centre of Parallel Forces would not be changed.

This also appears from the formula

$$OM=\frac{\Sigma[P_1\times OM_1]}{\Sigma[P_1]}.$$

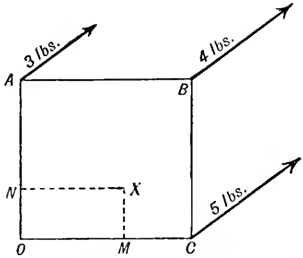
For if the forces P_1, P_2, P_3... are each multiplied by the same number the value of OM is unchanged.

Again, it appears from the *symmetry* of the formula

$$OM = \frac{\Sigma [P_1 \times OM_1]}{\Sigma [P_1]},$$

that in whatever *order* we take the forces, the value of OM is the same; so that the formula can give only *one* position for the Centre.

Example. Three parallel forces of 3 lbs., 4 lbs., 5 lbs. *act at the angular points* A, B, C *of a square* ABCO *whose side is* 4 in.; *find* the Centre of these parallel forces.



Let X be the required centre; draw XM, XN perpendicular to OC, OA respectively.

Then, by Art. 101,

$$ON = \frac{3 \times 4 \text{ in.} + 4 \times 4 \text{ in.} + 5 \times 0 \text{ in.}}{3+4+5}$$

$$= \frac{28}{12} \text{ in.} = \underline{2\tfrac{1}{3} \text{ in.}}$$

$$OM = \frac{3 \times 0 \text{ in.} + 4 \times 4 \text{ in.} + 5 \times 4 \text{ in.}}{3+4+5}$$

$$= \frac{36}{12} \text{ in.} = \underline{3 \text{ in.}}$$

EXAMPLES. XV.

Find the centre of the ten following systems of parallel forces :

1. Forces of 4 lbs., 5 lbs., 6 lbs. acting at A, B, C in the square of p. 80.

2. Forces of 6, 7, 8, 9 lbs. acting at A, B, C, O on p. 80.

3. Forces of 10, 10, 12 lbs. acting at B, C, O of a square $ABCO$, where $AB = 1{\cdot}6$ inches.

4. Forces of 10, 10, 20 lbs. acting at A, B, C in an equilateral triangle ABC, where $AB = 10$ inches.

5. Four equal like parallel forces acting at the angular points of a given parallelogram.

6. Three equal like parallel forces acting at the angular points of a given triangle.

7. Three like parallel forces acting at the three angular points of a given triangle, proportional respectively to the opposite sides.

8. Three equal parallel forces (not all like) acting at the three angular points of a given triangle.

9. Three parallel forces (not all like) acting at the three angular points of a given triangle, proportional respectively to the opposite sides.

10. Four equal like parallel forces acting at the angular points of a given quadrilateral.

11. Use the result of Question **10** to prove that the lines joining the middle points of opposite sides of a quadrilateral bisect each other.

12. $ABCD$ is a parallelogram ; like parallel forces proportional to 6, 10, 14, 10 act at A, B, C, D respectively ; shew that the centre and resultant of the parallel forces will remain unchanged if, instead of these forces, the parallel forces 8, 12, 16, 4 act at the points of bisection of the sides AB, BC, CD, DA respectively.

13. $ABCD$ is a square whose side is 17 inches and E the intersection of its diagonals ; like forces proportional to 3, 8, 7, 6 and 10 act at the points A, B, C, D, E respectively ; prove that the distances of their centre from AB and AD are 9 inches and 10 inches respectively.

CHAPTER VIII.

Centre of Gravity.

102. We have said that *force* is applied to mass. There are **three ways** in which force may be applied to **mass.**

I. As a **Pressure.**

II. As a **Tension.**

III. As an **Attraction** or *repulsion.*

103. A force is called a **pressure** when it is so applied that it causes the external particles of the mass to tend to compress the neighbouring particles of the mass closer together.

Suppose, for example, we take a piece of stick, such as a penholder, and press with two of our fingers one on each end of the stick in opposite directions. Then we are applying a force at each end of the stick; and these forces tend to compress the particles of the stick closer together. The particles *are* pressed closer together in consequence of the action of the two forces and the particles of the stick are in a state of *strain.* The amount of compression is however too small to be observed.

104. A force is called a **tension** when it is so applied to a mass that it causes the external particles of the mass to tend to *extend* the neighbouring particles of the mass so as to occupy a larger space.

Suppose, for example, we take a piece of stick, or of thread, and taking hold of each end with a finger and thumb, pull on the stick (or thread) in opposite directions with each hand. Then we are applying a force at each end of the stick; and these forces tend to pull the particles of the stick further apart. The particles *are* extended more widely in consequence of the action of the two forces; and the particles of the stick are in a state of *strain*. The amount of extension is however too small to be observed.

105. A force is called an **attraction** when there is a force acting upon each particle of the mass, which is quite independent of the internal actions and reactions of the neighbouring particles of the mass.

Such a force is said to act *from a distance* because it is not communicated to each particle by the action of neighbouring particles.

Such a force is the force called **weight**. Every particle of matter on the surface of the earth is attracted by the earth in such a way that each of its particles is pulled towards the centre of the earth by a force which is proportional to its mass and inversely proportional to the square of its distance from the centre of the earth.

Thus if the distance of a particle from the earth's centre is changed, its weight is changed. And since the shape of the earth is not exactly a sphere it happens that the weight of the same particle is different at different places on the earth's surface. See *Dynamics*, p. 38.

106. It can be shewn that the weight of a given mass at the same point of the earth's surface is always the same. Hence we are justified in taking the weight of a pound at Greenwich as our unit Force in Statics.

A certain pressure (or a certain tension) is said to be *equal* to the unit force [Art. 34] when it is such that if applied vertically to the mass 1 lb. under the action of *gravity* (that is, the earth's attraction) it will keep it at rest.

NOTE.—A *pressure*, a *tension*, a *weight* are not each a different *kind of force*; each is a force [DEF. Art. 6], but the manner of their application to mass is different.

WEIGHT.

107. The centre of the earth is at so great a distance from its surface when compared with the greatest linear dimension of any mass upon which we can make any experiment that we may consider the weights of the particles of any mass to be **parallel vertical** forces, each force being proportional to the mass of the particle.

The distance of the centre of the earth is about 4000 miles. A mass whose greatest dimension is a few yards is perhaps as large a mass as it is possible to consider or treat as a rigid body; hence in considering the weights of the particles of a body to be *parallel* forces we are at most neglecting an angle AOB when AB = say 10 yards and $OA = 4000 \times 1760$ yards, that is we are neglecting an angle of less than $\frac{3}{10}$ of a second. An angle of $\frac{3}{10}$ of a second is as small an angle as can possibly be observed with accuracy, even under the most favourable conditions.

108. Consider now two particles A and B rigidly connected by a light straight rod.

The weights of these particles are two parallel vertical forces acting the one at A the other at B.

These two forces are statically equivalent to a single force equal in magnitude to their sum, parallel and of like sense, acting at a point C between A and B such that $AC \times$ the force at $A = BC \times$ the force at B. [Art. 96.]

Hence the two particles A and B when under the action of gravity may be kept in equilibrium by an upward vertical force acting at C whose magnitude is equal to the sum of the weights of the particles A and B.

The point C is the centre of the parallel forces A and B, namely, the weight of the two particles.

109. Suppose then we fasten a light string to the rod at C and apply this upward vertical force to the rod, then the two particles will be in equilibrium under the action of this force and of their own weight, no matter what angle the line AB makes with the string.

Hence the weights of the two particles A and B are statically equivalent to the weight of a single particle at C whose weight is equal to the sum of the weights of A and B.

The point C is called the **Centre of Gravity** of the particles A and B.

DEF. The **Centre of Gravity** of a system of particles is the centre of the system of parallel forces which consists of the weights of the particles.

110. When the system of particles is in the form of a rigid body the Centre of Gravity is a fixed point in or near the body; and a downward force applied at that fixed point is statically equivalent to the weights of all the particles for all positions of the body, hence

111. The Centre of Gravity of a *rigid body* is a point such that when that point is supported, the body will be in equilibrium under the action of gravity in *whatever position* the body may be placed.

The Centre of Gravity in many cases is not within the substance of the rigid body. It must of course in any case be rigidly connected with the body for the purpose of the above statement.

The student is recommended to take some rigid body whose centre of gravity is not *within* its substance, say a Windsor chair, (or a box without a lid,) and find G its centre of gravity, rigidly connecting it with the chair by a system of fine wires, so that the chair can be supported by another wire fastened to the wires meeting at G. It will be found that in whatever position the chair is placed when suspended from G, it will rest without any tendency to move.

112. Consider a solid body; it occupies space. The amount of space which it occupies is called its **volume.**

It consists of matter.

The amount of matter of which it consists is called its mass; the mass of a body is proportional to its weight.

113. *DEF.* The average **density** of a solid body is that which varies directly as the mass of the body and inversely as the volume.

Thus, suppose we have two bodies of equal volume, one of which has double the mass (and therefore double the weight) of the other, then the density of the first body is double the density of the other.

Or again, suppose we have two bodies of equal mass (and therefore of equal weight) one of which has double the volume of the other, then the density of the first body is half the density of the other.

114. In order to measure the densities of different bodies we must select some substance whose density shall be our standard or unit density.

The substance usually selected is pure water at a temperature of 4 degrees centigrade, at which temperature pure water has its greatest density.

The average density of any given body is the ratio of its weight to the weight of an equal volume of the standard substance.

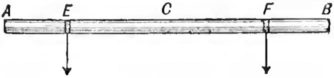
115. *DEF.* A body is said to be of **uniform** *density* when the average density of any portion of it however small is the same as the average density of the whole.

116. The weight of any portion of a body of uniform density is proportional to its volume.

For, the mass of a body is proportional to its volume and its density; therefore when the density is uniform the mass is proportional to the volume, and its mass is proportional to its weight.

117. A **uniform** rod is a straight rod made of material of uniform density and whose breadth and thickness are the same at every point of its length.

118. *PROP. The Centre of Gravity of a* uniform rod *is at its middle point.*



Let AB be a uniform rod, let C be its middle point.

Since the rod is uniform the rod is symmetrical about C; that is, corresponding to any particle E of the rod between C and A there is an exactly equal particle F between C and B such that $CE = CF$.

The centre of gravity of every pair of such particles is at C.

Hence the centre of gravity of all the particles of the rod is at C.

119. Hence the weight of an uniform rod is statically equivalent to a single force acting vertically downwards at its middle point C, whose magnitude is equal to the sum of the weights of all the particles of the rod.

Example. A uniform rod AB 3 ft. *long weighs* 3 lbs.; *at the point* A *a force of* 3 lbs. *is applied to the rod in a vertical direction downwards; shew how to apply a force to the rod which will keep it in equilibrium.*

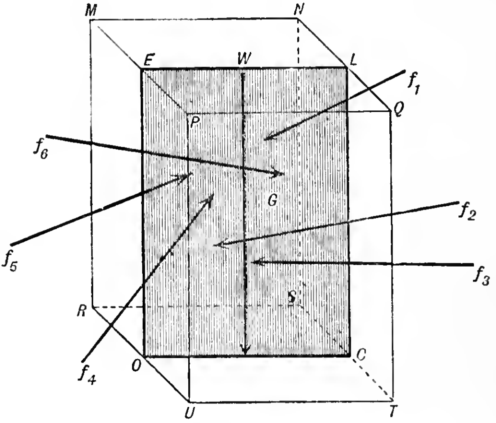
Let C be the middle point of the rod. The rod being uniform its weight is equivalent to a force of 3 lbs. acting vertically downwards at C.

Hence we have to find a force and its point of application such that it is in equilibrium with the two parallel forces of 3 lbs. each, acting at A and C respectively.

The required force is the anti-resultant of the two like parallel forces of 3 lbs. each. It is therefore a force of 6 lbs. acting vertically upwards at the middle point of AC.

120. The forces which occur in *nature* are applied to bodies either at a portion of their *surface* or at a part of their *volume* (as in the case of weight).

Consider a pressure applied to a portion of the surface of a body, such as the cube in the figure. We may look upon



it as a solid bundle of minute forces whose lines of action form a cylinder. Suppose these forces all symmetrical about a certain plane, say the plane $ELCO$. Then if we select any minute force on one side of the plane, there is an exactly symmetrical equal force on the other side of the plane, and their resultant will act *in the plane*.

Hence, the whole bundle of forces will be equivalent to a series of forces acting in the plane about which they are symmetrical. This system of forces will have a resultant acting in this plane; and it is this *resultant* of a bundle of forces which is the *force* of Statics and which has *line* of action and *point* of application.

Such a force as the theoretical linear force of statics does not exist in nature, but it is a convenient *representation* of the cylindrical or conical or solid force of nature.

121. In practical problems the various forces (viz. pressures, tensions, weights) which act on a body are in very many cases all *symmetrical about the same plane;* in such cases the corresponding theoretical linear forces are all in the same plane.

These symmetrical problems form the very numerous class of problems which can be solved by the consideration of *forces in one plane.*

Illustration.—Suppose $MPQN$, $RUTS$ to represent a cubical block of stone; suppose it to rest on a horizontal plane. The forces acting on it are:

First, its *weight;* this force is applied to each particle of the stone; we therefore consider the weight to be a multitude of very small forces all parallel and symmetrical about the plane $EOCL$; for to each particle on one side of this plane there is an equal particle similarly placed on the other side of this plane. Therefore the weight considered as the resultant of this multitude of parallel forces, acts in the plane $EOCL$.

Next, the *pressure* of the plane on $RUTS$, the base of the stone. This is applied to the surface of the stone, and is one part of the stress set up between the stone and the plane. If the plane and the stone be perfect planes the pressure will be *uniformly* distributed over the *surface.* Thus we have a multitude of vertical forces upwards, forming, if we represent their magnitudes by finite straight lines, a portion of a solid vertical cylinder. It is the theoretical resultant of this solid cylinder of force which is the linear force of Statics called the pressure on the plane.

Next, suppose we apply *a force* f_1 to the stone in any direction. This force would practically be applied to part of the *surface* of the cube, but if applied symmetrically about the plane, may be represented by a linear force *in the plane* $OCLE$.

And so on, for other forces applied to the cube, whether by means of rods or strings or whether caused by the pressure of other solid bodies.

EXAMPLES. XVI.

1. A uniform rod weighing 10 lbs. has a weight of 10 lbs. fastened to one extremity; find the centre of gravity of the two.

2. A uniform rod weighing 5 lbs. has weights of 3 lbs. and 5 lbs. fastened to its extremities; where must an upward force be applied to the rod which will support it?

3. A ladder 22 ft. long and weighing 44 lbs. has its centre of gravity 10 ft. from one end; if it is carried by two men lifting it at each end, what weight do they each lift?

4. A pole 40 ft. long weighing 20 lbs. has its centre of gravity 16 ft. from one end; it is carried by two men, one of whom lifts it at its heavier end; where must the other lift it that he may support an equal weight with the first man?

5. A ladder 50 ft. long and weighing 100 lbs. is carried by two men, one lifts it at one end and the other at a point 2 ft. from one end; the first carries two-thirds of the weight which the second carries: where is the centre of gravity of the ladder?

6. A pole 8 ft. long weighing 10 lbs. has weights of 8 lbs. and 4 lbs. fastened one to each of its ends; the centre of gravity of the whole is at the middle of the rod: where is the centre of gravity of the pole?

7. A pole 10 ft. long weighing 20 lbs. has a weight of 12 lbs. fastened to one end; the centre of gravity of the whole is 4 ft. from that end: where is the centre of gravity of the pole?

8. A pole weighing 20 lbs. is 12 ft. long, and its centre of gravity is 4 ft. from one end; if the pole is supported at its middle point, find where a weight of 10 lbs. must be fastened to it that it may be in equilibrium.

9. A pole 6 ft. long is found to balance about a point 2 ft. from one end; when a weight of 1 lb. is fastened to that end, and a weight of 3 lbs. is fastened to the other end, the pole is found to balance about its middle point: find the weight of the pole.

10. A ladder 20 ft. long has its centre of gravity at a point 8 ft. from one end; when a weight of 10 lbs. is fastened to that end and a weight of 20 lbs. to the other end, it is found to balance about a point 8 ft. from this other end: find the weight of the ladder.

11. Four weights of 7 lbs., 1 lb., 3 lbs. and 5 lbs. respectively are placed a foot apart on a uniform rod 3 ft. long weighing 8 lbs.; find the point on which the rod will balance.

12. Four weights 1 lb., 4 lbs., 5 lbs. and 3 lbs. respectively are placed 2 ft. apart on a rod 6 ft. long weighing 3 lbs. whose centre of gravity is 2 ft. from the end at which is the 1 lb. weight: find the centre of gravity of the whole.

13. Weights of 3 lbs., 5 lbs., 7 lbs., 9 lbs. are fastened to a rod 6 ft. long at intervals of 2 ft.; the rod weighs 24 lbs. and the centre of gravity of the whole system is at the middle point of the rod; where is the centre of gravity of the rod?

14. A uniform bar is 18 inches long, weighs 3 lbs. and can turn about a fixed point 3 in. from one extremity; what weight must be fastened to this extremity that the bar may be in equilibrium, when a weight of 2 lbs. is fastened to the other extremity?

15. A uniform rod OA 12 inches long is suspended by two vertical strings attached to the rod at O and A; weights of 2 lbs. and 7 lbs. are fastened to the rod at points distant 1 inch from O and 2 inches from A respectively; if the strings break when subjected to a strain of more than 7 lbs., find the greatest weight the rod can have without breaking either of the strings.

16. A uniform rod 2 ft. long is suspended by two vertical strings attached to the ends of the rod; weights of 7 lbs. and 1 lb. are attached to the rod at points distant 4 in. from the ends. If the strings break if subjected to a strain of more than 10 lbs. find the greatest weight the rod can have.

17. A uniform beam weighing 700 lbs. and 30 ft. long rests on two walls 24 ft. apart so that 1 foot of it projects over one of them; a man weighing 192 lbs. stands 2 ft. from the other end: find the pressure on each of the walls.

18. A uniform plank 20 ft. long weighing 14 lbs. rests on the top of a wall; at one end is a man weighing 168 lbs., at the other a boy weighing 98 lbs., and two feet from that end a weight of 7 lbs., the whole balances: how much of the plank is on each side of the wall?

19. Two weights of 12 lbs. and 2 lbs. hanging from the extremities of a uniform rod 3 ft. long which can turn about a fixed point, keep it at rest; if each weight is increased by 1 lb. the fixed point must be moved three-eighths of an inch nearer the less weight: find the weight of the rod.

20. A uniform lever 15 ft. long has its fulcrum at one end; a force of 30 lbs. acts upwards at the other end, and one of 20 lbs. is applied downwards at a point between; the lever is then in equilibrium; if the lever were without weight it would be in equilibrium if the points of application of the two forces were interchanged: what is its weight?

21. A uniform bar $7\frac{1}{2}$ ft. long weighing 17 lb. rests on a horizontal table with one end projecting $2\frac{1}{2}$ ft. over the edge; find the greatest weight that can be hung at this end without making the beam topple over.

22. A uniform bar 8 ft. long rests with one end on a horizontal table; a weight of 10 lbs. is placed on that end; it is found that when the bar projects $6\frac{1}{2}$ ft. over the edge it is on the point of toppling over: what is the weight of the bar?

23. A uniform bar weighing 20 lbs. rests with one end on a horizontal table; a weight of 6 lbs. is placed on that end and a weight of 4 lbs. is placed on the end which projects over the edge; when the bar projects 8 ft. over the edge it is on the point of toppling over: what is its length?

24. If 5 cubic inches of silver weigh as much as 21 cubic inches of glass, and silver is 10·5 times as dense as pure water, shew that the density of the glass is 2·5.

25. The radii of two spheres are 2 inches and 3 inches and their weights 8 lbs. and 10 lbs. respectively: shew that the ratio of their densities is 27 to 10.

26. A cubic foot of oak weighs 100 times as much as a cubic inch of metal, shew that their densities are as 1 : 17·28.

27. A lump of matter whose density is ·865 weighs 432·5 grains; and an equal volume of different stuff weighs 9 grains less; shew that the density of the latter is ·847.

CHAPTER IX.

Centre of Gravity of a Lamina.

122. A material lamina has thickness.

Consider, however, a geometrical plane; and suppose that on each side of this plane, material of some kind (iron, or wood, or cardboard ...) is symmetrically arranged.

(Forming a sheet of iron, or a board, or a card ....)

Then the centre of gravity of the body so formed must lie in the geometrical plane about which it is symmetrical.

This follows as in Art. 120.

When the thickness of the substance on each side of the geometrical plane is small, then we have a body which we call a *lamina*.

When the substance is spread uniformly over the plane then we have a *uniform* lamina.

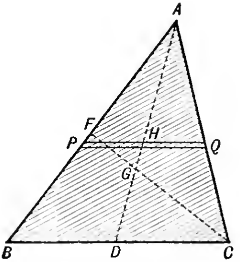
When we speak of the centre of gravity of a *triangle*, a *circle*, a *quadrilateral*, we mean the centre of gravity of a uniform lamina whose boundary is a triangle, a circle, or a quadrilateral, etc.

When a uniform lamina is symmetrical about a *point*, that point must be its Centre of Gravity.

For example; the C. G. of a Circle is its centre. The C. G. of a rectangle is its middle point.

If the substance is not uniformly spread over the plane, yet if it is spread *symmetrically*, the centre of gravity of the body so formed must lie in the plane under consideration; in such a case we have a lamina which is not uniform.

123. *PROP. To find the* Centre of Gravity *of a uniform* triangular *lamina.*



Consider the lamina ABC to be made up of a series of very thin parallel rods such as PQ, each parallel to one of the sides BC of the triangle, and formed by sections of the triangle made parallel to the side BC.

Since the lamina is uniform, each rod will be uniform.

Therefore the C.G. of each rod will be at its middle point. [Art. 118.]

Hence, the weight of each rod may be considered to be condensed at its middle point.

Now bisect BC in D and join AD cutting PQ in F; then AD bisects PQ in H.

For, by the similar triangles ADC, AHQ

$$DC : HQ = DA : HA,$$

and by the similar triangles ABD, APH

$$BD : PH = DA : HA;$$

therefore $DC : HQ = BD : PH$.

But $BD = DC$, $\therefore HQ = PH$.

Therefore the C.G. of each rod making up the lamina lies in the line AD.

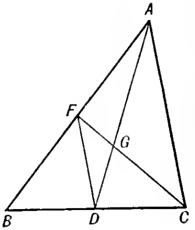
But the Centre of a series of parallel forces whose points of application are in a straight line, lies also in that straight line [Art. 110].

Therefore the C.G. of the whole lamina is at some point on the line AD.

In a similar manner it can be shewn that the C.G. of the lamina is at some point on the line joining the point C to F, the middle point of the line AB.

Therefore the Centre of Gravity of the lamina must be at G, the point of intersection of the lines AD, CF.

124. *PROP. To prove that the* C.G. *of a uniform triangular lamina cuts the line joining an angular point to the middle point of the base, in the ratio 2 to 1.*



Let ABC be the triangle.

Bisect BC, AB in D and F;

join FD;

join AD, CF intersecting in G.

Then, by Art. 123, G is the C.G. of the lamina. Now, since

$$AF : FB = CD : DB,$$

$$\therefore\ FD \text{ is parallel to } AC,$$

$$\text{and } \therefore\ ACB,\ FDB \text{ are similar triangles;}$$

$$\therefore\ AC : FD = AB : FB = 2 : 1.$$

Also, since AGC, DGF are similar triangles,

$$AG : GD = AC : FD$$

$$= 2 : 1;$$

$$\therefore\ AG = \text{twice } GD, \qquad \text{Q. E. D.}$$

$$\text{and } AD = \text{thrice } GD.$$

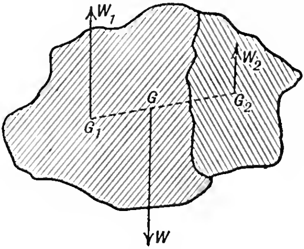
NOTE.—The C.G. of a triangular lamina coincides with that of three equal particles placed one at each of its angular points.

For the C.G. of two equal particles of weight W placed at B and C respectively, is at D; we may therefore replace these two particles by a particle of weight $2W$ at D.

The C.G. of $2W$ at D and W at A is at G where $AG = 2GD$.

NOTE.—The C.G. of a parallelogram is at the point of intersection of its diagonals. This may be deduced from Art. 118 or from Art. 123.

125. *PROP. Having given the weight and* C.G. *of a body, and also the weight and* C.G. *of a portion of the same body, to find the* C.G. *of the remaining portion.*



Let G be the C.G. of the given body and W its weight.

Let G_1 be the C.G. of the given portion and W_1 its weight.

If G_2 be the required C.G. of the remainder and W_2 its weight, then $W_2 = W - W_1$.

Also G is the centre of the two parallel forces W_1, W_2 applied at the points G_1, G_2;

$\therefore$ G_1, G, G_2 are points in a straight line,

and $GG_2 \times W_2 = G_1G \times W_1$;

$$\therefore\ GG_2 = \frac{W_1}{W - W_1} \times G_1G.$$

Hence G_2 is found by producing G_1G a distance given by the above equation.

126. *A heavy body is suspended from a fixed body by a string; prove that the direction of the string must be vertical and must pass through the Centre of Gravity of the body.*

The weight of the body is equivalent to a single vertical force acting vertically downwards and applied to the body at its centre of gravity.

The only other force acting on the body is the tension of the string.

Therefore these two forces, the weight and the tension, must be equal, opposite, and in the same line of action.

But the tension acts in the direction of the string.

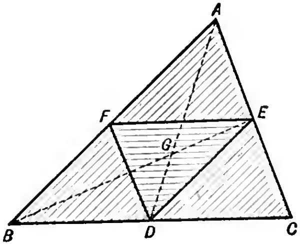
Therefore the direction of the string must be in the same line as that in which the weight acts. Q. E. D.

127. To find the Centre of Gravity of a number of heavy bodies whose weight is known, and the position of whose Centres of Gravity is given, we proceed as in Art. 100 or as in Art. 101.

For the weight of a heavy body is a force acting in a fixed direction, viz. vertically downwards, and it is applied at the Centre of Gravity of the body.

Thus the problem is the same as that of finding the position of the centre of a given system of like Parallel Forces having given points of application.

Example i. *The angular points D, E, F of one triangular piece of cardboard are so placed that each is at the middle point of one of the sides of another triangular piece of cardboard ABC. Shew that their* C.G.s *are superposed.*



The line EF joining the middle points E and F of two sides of a triangle is parallel to the third side BC of the triangle and is bisected by the line AD joining the middle point D of the third side to the opposite angular point A.

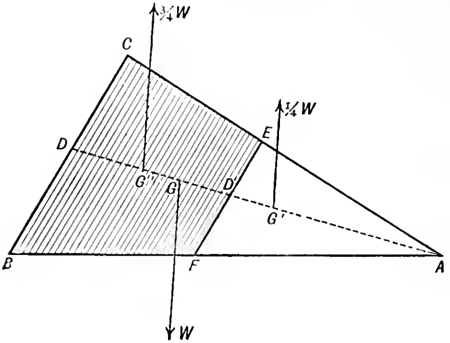
Hence if DEF be the middle points of the sides BC, CA, AB then the line AD contains the C. G. of the triangle ABC and also of the triangle DEF.

For AD bisects BC and EF.

Similarly BE contains both Centres of Gravity.

∴ the Centre of Gravity of each triangle is at the point of intersection of AD and BE.

Example ii. ABC *is a triangle and* E, F *are the middle points of the sides* AC, AB; *find the* C.G. *of the quadrilateral* CBFE.



The triangle AEF is one-fourth of the triangle ABC.

Let D' be the middle point of EF and G' the C.G. of the triangle AEF; then D, G, D', G', A are in the same straight line.

Let G'' be the C.G. required. It is in the line GG' [Art. 125].

Now
$$DG = \tfrac{1}{3}\, DA;$$
$$D'A = \tfrac{1}{2}\, DA;$$
$$D'G' = \tfrac{1}{3}\, D'A = \tfrac{1}{6}\, DA.$$

Let W be the weight of the triangle ABC.

Then $\frac{1}{4}W$ is the weight of the triangle AFE; $\frac{3}{4}W$ is the weight of $CBFE$;

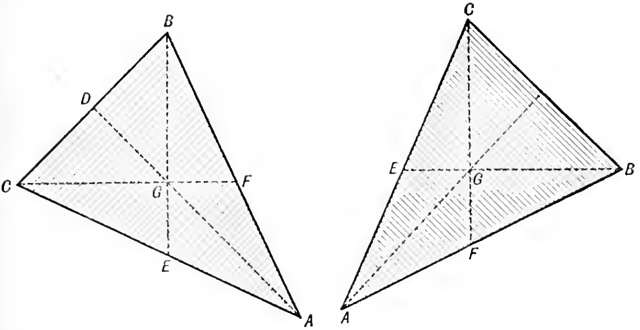
hence a force of $\frac{3}{4}W$ acting upwards at G'' and a force of $\frac{1}{4}W$ acting upwards at G' are in equilibrium with a force W acting downwards at G; hence taking moments about D, we have

$$\tfrac{3}{4}W \times DG'' + \tfrac{1}{4}W \times DG' - W \times DG = 0;$$

$$\therefore\ DG'' = \frac{W \times DG - \frac{1}{4}W \times DG'}{\frac{3}{4}W} = \tfrac{4}{3}\left\{DG - \tfrac{1}{4}DG'\right\}$$
$$= \tfrac{4}{3}\left\{\tfrac{1}{3}DA - \tfrac{1}{4}\left(\tfrac{1}{2}DA + \tfrac{1}{6}DA\right)\right\}$$
$$= \tfrac{4}{3}\left\{\tfrac{1}{3} - \tfrac{1}{8} - \tfrac{1}{24}\right\} DA$$
$$= \tfrac{2}{9}\, DA.$$

Thus DG'' is two-thirds of DG.

Example ii. *An isosceles triangular board* ABC *is suspended successively from* B *and* C *the angular points of the base; shew that the two positions which the base successively occupies will be perpendicular to each other, provided the base of the triangle is two-thirds of its altitude.*



Let the figures represent the two positions of the board.

EF are the middle points of AC and AB; G is the C.G.

Now BE must be vertical when the triangle is suspended at B; for the only forces acting on the triangle are (i) its weight which may be considered to be concentrated at G and acts vertically downwards, and (ii) the action of the constraint at B. This action must be equal, opposite, and in the same line as the weight; $\therefore$ BE in the left-hand figure is vertical.

When the triangle is suspended at C, as in the figure on the right hand, we may shew similarly that then CF is vertical.

Suppose the triangle turned about G from its first position into its second; then, considering one figure only, the line BE is turned until it comes into the position CF.

The line CB will be turned about the same angle. We have from the mechanical data of the problem that this angle is a right angle.

Therefore the angle CGB is a right angle.

Join AGD. Then AGD bisects CB at right angles.

And since CGB is a right angle, a circle described about CGB will have its centre at D.

$\therefore$ $DG=DB$, but $DG=\frac{1}{3}DA$, $\therefore$ $DB=\frac{1}{3}DA$. Q.E.D.

Therefore BC is two-thirds of DA.

EXAMPLES. XVII.

1. A cross is made up of six equal squares; find its centre of gravity.

2. Weights of 1 lb. are placed at each of three corners of a square and a weight of 2 lbs. at the fourth corner; find the C.G. of the four weights.

3. Weights of 2 lbs. are placed at each of three corners of a square and a weight of 1 lb. at the fourth corner; find their C.G.

4. If the triangle formed by joining the middle points of the sides of a triangular lamina be removed, prove that the C.G. of the remainder coincides with that of the original triangle.

5. Two rectangular pieces of card-board of lengths 6 and 8 inches and breadths 2 and $2\frac{1}{2}$ in. respectively are placed touching but not overlapping one another on a table to form a T-shaped figure, the former piece forming the cross bar. Find the position of the centre of gravity.

6. From the corner of a square piece of cardboard whose side is 6 inches another square whose side is 2 inches is cut away; find the C.G. of the remaining piece.

7. *ABCD* are the corners of a square piece of cardboard; *AC*, *BD* meet in *E*; if the triangle *AEB* be cut away, find the C.G. of the remainder.

8. A quarter of a triangle is cut off by a line drawn parallel to one of its sides bisecting each of the other sides, find the C.G. of the remainder.

9. A circular plate of wood has a circular hole cut in it whose diameter is half that of the plate and whose centre divides a diameter of the plate in the ratio 2 : 3; find the position of its centre of gravity.

10. A circular board has two equal circular holes cut in it, the centres of these holes being at the middle points of two radii at right angles to each other; the radius of each hole is one-third that of the board; find the centre of gravity of the remainder.

11. Two equal rods *AB*, *BC* are fixed upon a circular board so as to coincide with the chords of two adjacent quadrants; the weight of each rod is equal to that of the board; find the centre of gravity of the whole body.

12. The diagonals of a square plate $ABCD$ intersect in O, the triangle BOC is cut out and placed so as to fit upon the triangle AOD and the two portions are firmly connected in this position; find the centre of gravity of the body thus formed.

13. $ABCD$ is a square plate; E and F being the middle points of AB and BC; the plate is bent along EF so that the triangle EBF lies flat on the other side of the plate; find the centre of gravity in this position.

14. A square is described on the base of an isosceles triangle. What is the ratio of the altitude of the triangle to its base when the C.G. of the whole figure is at the middle point of the base?

15. If a triangle have a side upon which it will not stand upon a horizontal plane with its plane vertical, that side is the shortest and is not half the length of the longest side.

16. On a uniform triangular plate straight lines are drawn joining the middle points of the sides. Two of the triangular portions marked off between these lines and the angle of the triangle are covered with triangular plates of the same material and thickness as the given triangle; find the centre of gravity of the whole.

17. A square is described on the base of an equilateral triangle; find the C.G. of the whole figure.

18. A system of three equal particles connected by rigid wires without weight forms a triangle and when hung up by the middle point of one side rests with that side horizontal; prove that the triangle is isosceles.

19. An isosceles right-angled triangle is described on the side of a square as hypotenuse, and its vertex turned away from the square; find the C.G. of the whole figure.

20. $ABCD$ is a rectangle; A is joined to E the middle point of CD; find the C.G. of $ABCE$.

21. A triangular lamina EDC of the same weight as a square lamina $ABCD$ (E being the middle point of AB) is laid upon the square lamina in the position indicated by the letters; find the C.G. of the system.

22. If weights be placed at the angular points of a triangle, respectively proportional to the sum of the sides which meet at those points, prove that their C.G. will coincide with that of the perimeter of the triangle.

23. The C.G. of the two complements which are about the diagonal of any parallelogram is in that diagonal.

24. ABC is a triangle, D is a fixed point in BC; a triangle PBC is cut away whose vertex P is in AD; prove that whatever be the position of P the C.G. of the remainder lies on a fixed straight line.

25. If there are two triangles on the same base and between the same parallels, prove that the distance between their centres of gravity is one-third the distance between their vertices.

26. Find the centre of gravity of four equal heavy particles in one plane; thence shew that the lines bisecting pairs of opposite sides of any quadrilateral bisect each other.

27. A triangular lamina is hung up by one of its angular points and when in equilibrium the opposite side is horizontal; prove that the triangle is isosceles.

28. Find the angles of an isosceles triangle when the angle between the two positions of the base, the triangle having been suspended freely from each of the equal angles in turn, is a right angle.

29. Find the tangent of the angle between the positions of the base when a right-angled isosceles triangle is suspended (1) from the right angle, (2) from one of the equal angles.

30. Two uniform rods OA, OB of equal weight whose lengths are a and b are rigidly connected at O so that AOB is a right angle; they are hung up by a string attached to O, prove that if θ be the inclination of OA to the horizon, then

$$\sin\theta = \frac{a}{\sqrt{(a^2+b^2)}}.$$

31. Two uniform rods OA and OB of equal length, whose weights are in the ratio of m to n, are rigidly connected at O, so that AOB is a right angle; prove that, if they are suspended by a string attached at O, and if θ be the inclination of OA to the horizon, then $\tan\theta = \frac{m}{n}$.

32. A piece of uniform wire is bent into the form of three sides of the square $ABCD$ of which the side AD is wanting; prove that if it be hung up by the two points A and B successively, the angle between the two positions of BC is $\tan^{-1} 18$.

33. Shew that the centre of gravity of a plane quadrilateral does not coincide with that of four equal particles placed at its angular point except the quadrilateral be a parallelogram.

34. Shew by the method of Art. 122 that the centre of gravity of a lamina in the form of a parallelogram is at the point of intersection of the diagonals.

35. The C.G. of a lamina in the form of a trapezium $ABCD$ is in the line joining the middle points of the parallel sides AB, CD of the figure, and divides in the ratio $2AB+CD:2CD+AB$.

36. A piece of uniform wire in the shape of three sides of a rectangle is suspended by one of its angular points; shew that when the ratio of two adjacent sides is 1 to $\sqrt{3}-1$ (the missing side being the longer) the sides will be all equally inclined to the horizon.

37. A right-angled triangle suspended from either of the points of bisection of the hypotenuse will rest with one side horizontal.

38. A triangle suspended from a point of trisection of a side rests with one side vertical.

39. Prove the following construction for finding the C.G. of three equal weights at the points A, B, C; bisect BC in D and divide AD in G so that $AG=2GD$; G is the required C.G.

40. The C.G. of three particles placed one at each of the angular points A, B, C of a triangle such that the weight of each is proportional to the opposite side, is at the centre of the circle inscribed in the triangle.

41. Three particles placed at the angular points A, B, C of a triangle are proportional to the areas of the triangles OBC, OCA, OAB respectively, where O is the centre of the circumscribing triangle; shew that their C.G. is O.

42. The C.G. of three uniform rods forming a triangle A, B, C is at the centre of the inscribed circle of the triangle formed by joining the middle points of the sides of the triangle ABC.

43. Two uniform rods AB, AC are firmly joined at A and are suspended from a fixed point by a string fastened to A; prove that the tangent of the angle which AB makes with the horizon is $\dfrac{AC^2+AB^2\cos BAC}{AB^2\sin BAC}$.

CHAPTER X.

Bodies on a Horizontal Plane.

128. When a body stands on a plane, its **base** is the area enclosed on the plane by a string drawn tightly round all the points in which the body touches the plane.

PROP. A rigid body under the action of gravity only, standing on a horizontal plane is in equilibrium provided the vertical line through its centre of gravity cuts the plane at some point within its base.

The proof of the above general proposition involves the consideration of parallel forces *not* all in the same plane and therefore cannot be fully treated here. Its truth depends on the following.

I. When a *rigid* body rests on a horizontal plane the plane is supposed rigid also, and therefore capable of applying to the body at *any* point of its contact the *vertical* force **upwards** that may be necessary to maintain equilibrium.

II. The line of action of the resultant of any number of **like** vertical forces applied to the rigid body cuts the horizontal plane at a point *within* the base; and by properly arranging the magnitudes of the vertical forces, the line of action of the resultant may be made to cut the plane at *any* chosen point within the base.

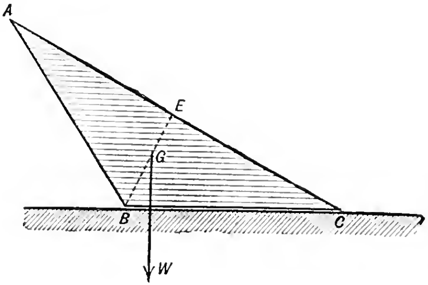
Assuming the above we can prove the proposition. For the weight of any body is equivalent to a single force downwards.

If the line of action of this vertical force downwards cuts the plane *within* the base, the upward vertical actions of the plane on the body will be so arranged that the line of action of their resultant coincides with that of the weight of the body and the magnitude of this resultant will just equal the weight of the body. Therefore the forces acting on the body will be in equilibrium.

If the line of action of the weight cuts the plane outside the base, then the upward actions of the plane on the body cannot be so arranged that the line of action of their resultant shall coincide with that of the weight of the body; in this case the forces acting on the body cannot be in equilibrium and the body will '*topple over.*'

When a rigid body and the forces acting upon it are symmetrical about a vertical plane, the problem of its stability may be treated as if its base were a *straight line;* viz. the straight line in which this vertical plane cuts the *base* of the rigid body. Compare Art. 120.

129. When the base of the body is a line BC then



I. The actions of the rigid plane on the rigid base are supposed to apply to the body, at points between B and C inclusive, any force that may be necessary to maintain equilibrium.

Hence we may, if necessary, consider any force whatever P to be applied to the body vertically upwards at B and any other force Q at C.

II. The resultant of two *like* forces P and Q at B and C is a like force $(= P + Q)$, acting upwards at some point **between** B and C [Art. 97]; moreover by properly arranging the magnitudes of P and Q this resultant may be made to act at *any point* between B and C.

Thus if the weight of the body cuts BC in any point between B and C, the actions of the plane on the body (by II.) can be, and therefore (by I.) will be, in equilibrium with the weight. But if the weight cuts the line BC *produced*, it cannot be in equilibrium with these forces whose resultant is a vertical force which cuts BC *between* B and C (by II.).

Stable and Unstable Equilibrium.

130. When a body, which is acted on by forces which are in equilibrium, is slightly displaced from its position, one of three things must happen to it.

I. Either, the forces acting on it in its new position are such that they are *not* in equilibrium, but have a resultant which tends to *restore* the body to its original position.

In this case the original position of the body is said to be one of **stable** equilibrium for that displacement.

II. Or, the forces acting on the body in its new position are such that they are *not* in equilibrium, but have a resultant which tends to make the body *move further* from its original position.

In this case the original position of the body is said to be one of **unstable** equilibrium for that displacement.

III. Or, the forces acting on the body in its new position are in equilibrium.

In this case the original position of the body is said to be one of **neutral** equilibrium for that displacement.

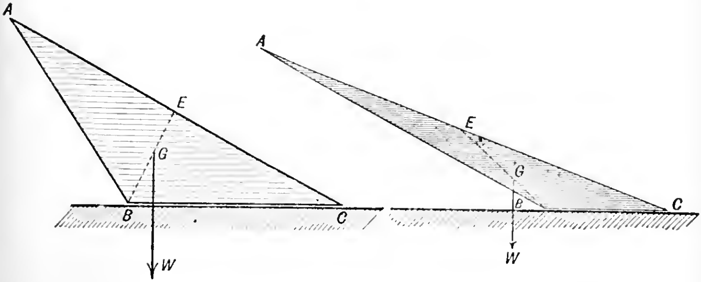
Illustration.—Take a sphere of a hard substance (a wooden ball will do), place it in a large bowl; it will come to rest at the lowest point; and if it be slightly displaced in any direction it will tend to return to its position: such a position is one of *stable* equilibrium.

Now put the same sphere on the highest point of another sphere (on the highest point of an inverted bowl); it is theoretically in a position of equilibrium; but if it be displaced ever so slightly it will tend to go further from the position of equilibrium: such a position is one of *unstable* equilibrium.

Now put the same sphere on a horizontal plane; then wherever it be placed it will be in a position of equilibrium; and if it be slightly displaced, and then left at rest, it will still be in a position of equilibrium, and will have no tendency to go further away from or to return to its original position: such a position is one of *neutral* equilibrium.

Example i. *An obtuse-angled triangle* ABC *is placed with its plane vertical and kept so, with its shortest side* BC *resting on a horizontal plane; find the condition that it shall not tumble over.*

Since the plane of the triangle is constrained to remain vertical the only way in which the triangle can tumble over is in its own plane.



Let B be the obtuse angle, E the middle point of AC, G the centre of gravity.

The vertical through the C.G. will cut BC *between* B and C unless the angle EBC is obtuse.

∴ the angle EBC must not exceed a right angle. This is the required condition.

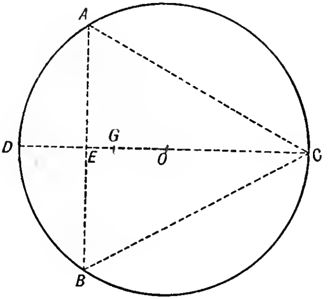
Example ii. *A round board* ABC *weighing ten pounds is made into a table by having three equal legs (of no weight) fastened to it at right angles, at points* A, B, C *on its circumference equidistant from each other; the table is placed on a smooth horizontal plane; find the least weight which placed on the top of the table can cause it to tumble over.*

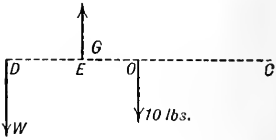
The legs of the table touch the plane in three points vertically under ABC; therefore the *base* on which the table stands is the triangle vertically under ABC.

Draw $COED$ through the centre O bisecting AB at right angles in E.

Then E is one of the nearest points in the boundary of the base to the centre of the table.

Let a weight W be placed on the table and let G be the Centre of Gravity of the table and W together; then the condition that the table should tumble over is that G shall be *outside* the triangle ABC.





The further from O we place W, the further is G from O.

$\therefore$ G is furthest from O when W is on the edge of the table.

Hence the *least* magnitude of W which will bring G on the boundary of the triangle, is when W is at D and G at E.

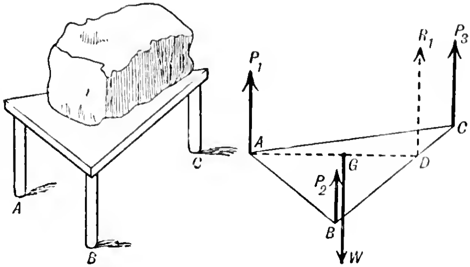
Now $OE = \frac{1}{2}OC = \frac{1}{2}OD = DE$.

$\therefore$ when G is at E, then $W = 10$ lbs.

Hence, when W exceeds 10 lbs. the table will tumble over if W is placed at D.

Example iii. *A body is placed on a three-legged stool, which stands on a horizontal plane; shew how to find the* pressure *of each leg on the plane.*

Let A, B, C be the points in which the legs of the stool touch the plane.



Let G be the point in which the vertical line through the Centre of Gravity of the stool and body together cuts the plane.

Then, three parallel vertical forces applied at the points A, B, C upwards, and the weight W (of the body and stool together) acting vertically downwards through G, are the only external forces acting on the stool.

These parallel forces are therefore in equilibrium.

Join AG and produce it to cut BC in D.

Let P_1, P_2, P_3 be the pressures acting at A, B, C.

The resultant of P_2 and P_3 must be equal and opposite to, and must therefore cut the plane ABC in the same point with, the resultant of P_1 and W.

The line of action of the first of these resultants cuts the plane somewhere in the line BC, the line of action of the second resultant cuts the plane somewhere in the line AG;

and, since they must each cut the plane in the *same* point, that point must be D.

Therefore we have $W = P_1 + P_2 + P_3$ .... (i),

$P_1 : W = DG : DA$ .... (ii),

$P_2 : P_3 = DC : DB$ .... (iii).

These equations give P_1, P_2, P_3 when G, A, B, C and W are known.

EXAMPLES. XVIII.

1. A flat board ABC in the shape of a triangle right-angled at A stands with its plane vertical and its side AC in contact with a horizontal plane; D is the middle point of AC; if the triangular portion ABD be cut away, shew that the remainder will be just on the point of falling, it being supposed that the board is constrained to remain in a vertical position.

2. A triangular lamina is placed upon a horizontal table; how far can its vertex be made to project over the side of the table when the base is kept parallel to the side of the table?

3. A parallelogram whose height is equal to its base will just stand on that base when placed vertically on a horizontal plane, find the angles of the parallelogram.

4. A uniform triangular lamina ABC is placed upon a horizontal table with the side BC on the table and parallel to the edge, and one-ninth of the area of the triangle overhangs the table. Shew that if a weight be placed at A greater than the weight of the triangle itself, the triangle will be upset.

5. A five-sided figure consisting of a square $ABCD$ with an isosceles triangle upon the side BC as base is cut out of one piece of board; find the greatest height of the triangle that the figure may stand with its side DC on a horizontal plane without tumbling over.

6. If a table stand on three legs, shew that in whatever way weights are placed upon it without upsetting it, the centre of gravity of the table and weights together will be vertically above a given triangle.

7. A circular board weighing 10 lbs. is made into a table by the addition of four legs without weight fixed perpendicular to its plane at equidistant points on its circumference; find the least weight with which it is possible to upset the table by placing the weight on it.

8. An equilateral triangular board is made into a table by the addition of three legs without weight fixed at right angles to the board at the middle points of the sides. Shew that it is possible to upset the table by putting on it a weight which is just heavier than one-third of the weight of the table.

9. A square board weighing 20 lbs. is made into a table by inserting 4 equal legs into it one at the middle point of each side. Three weights of 20 lbs. each are placed at three of the corners of this table as it stands on a horizontal floor. Find the greatest weight that can be placed upon the fourth corner without overturning the table, neglecting the weight of the legs.

10. A square board *ABCD* weighing 18 lbs. is made into a table by inserting three equal legs in it, one at the corner *C* and the others at the middle points of the sides *AB*, *AD*. Find how great a weight may be placed upon the corner *A* as the table stands on a horizontal floor without overturning it, the weight of the legs not being taken into account.

11. A number of bricks, each 9 inches long, 4 inches wide, and 3 inches thick, are placed one upon another in such a way that whilst their narrowest surfaces (or thicknesses) are in the same vertical plane, each brick overlaps the other by half an inch of its length, the lowest brick resting on a horizontal plane; how many bricks may be so piled without falling?

12. A triangular lamina *ABC* obtuse-angled at *C*, stands vertically with its side *AC* in contact with a table: shew that the least weight which suspended from *B* will overturn it is

$$\tfrac{1}{3}W\frac{a^2+3b^2-c^2}{c^2-a^2-b^2}$$

where W = the weight of the lamina. Interpret the above when $c^2 > a^2+3b^2$.

13. A table stands on three legs on a horizontal plane. Shew that if the C.G. of the table be vertically above the centre of gravity of the triangle formed by joining the three points in which the legs touch the plane, the pressure of the legs on the plane will be equal.

14. Find the pressure of each leg of the table in Question **10** on the plane on which it stands when no additional weight is placed on the table.

15. A weight *W* is placed at *O* on a triangular table *ABC* supported on a horizontal plane by three vertical legs at *A*, *B*, *C*; shew that the portions of *W* supported by the legs are proportional to the areas of *BOC*, *COA*, *AOB*.

16. A hemisphere, and a circular cylinder of the same material and having a common base of radius r, are cemented together; shew that when the height of the cylinder is equal to $\frac{1}{2}r\sqrt{2}$ the equilibrium is neutral if the hemisphere is placed with its curved surface on a horizontal plane. [The area of a circle $=\pi r^2$; the volume of a sphere $=\frac{4}{3}\pi r^3$; the C.G. of a hemisphere is distant $\frac{3}{8}r$ from centre.]

CHAPTER XI.

A Rigid Body with one point fixed.

131. When a rigid body is said to have *one point* **fixed**, it is understood that there is some constraint, (a hinge, or fastening of some kind) which can and does apply to the rigid body at that point whatever force is necessary to prevent the body from moving as a whole away from that point.

132. *PROP. To find the conditions for the equilibrium of a given system of forces acting upon a rigid body having one point fixed.*

If a body, which has one point C fixed, requires no additional force to keep it from moving, the given system of forces acting upon it must be in equilibrium with the force applied to the rigid body by the constraint at C.

In other words, in order that the given system of forces may be in equilibrium, their resultant must either be zero or must be a single force passing through the fixed point C.

The condition that this is so, is that **the sum of their moments about the fixed point must be zero.**

For in this case the resultant cannot be a *couple*, and it cannot be a force *not* passing through C.

Hence, any system of forces acting on a rigid body which has one point C fixed, must be in equilibrium with the force applied by the constraint at C, provided *the sum of their moments about* C *is zero.*

133. The above result will be found of very great importance in the theory of many machines.

For an illustration see Art. 64.

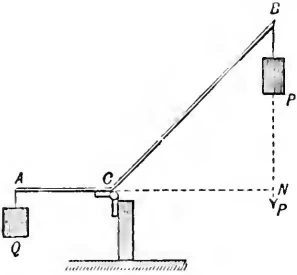
134. *PROP. A rigid body which has one point* C *fixed and is acted on by two forces, has no tendency to turn about* C *provided that the moment of one of the forces about* C *is equal and opposite to the moment of the other force about* C.

For the sum of the moments about C of the forces acting on the rigid body is zero.

The **tendency** of a given force **to turn** a rigid body about a chosen point may be measured by *the moment about that point of the given force.*

Example i. ACB *is a rigid rod without weight having a fixed point at* C; AC *is horizontal and* CB *is inclined at an angle* α *to the horizon, a* weight Q *is suspended at* A *and a* weight P *is suspended at* B. *What is the necessary condition for equilibrium?*

Draw CN perpendicular to the vertical through B.



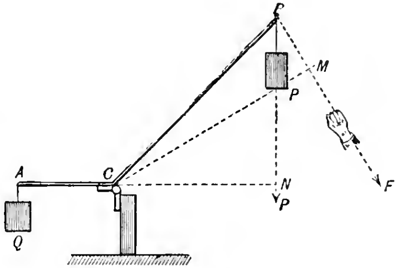
The forces acting on ACB will be in equilibrium with the force applied by the hinge at C, provided the sum of the moments of the forces about C is zero. [Art. 134.]

The required condition is

$$AC \times Q = CN \times P = CB \cos \alpha \times P.$$

Example ii. *In* Example i. *if the forces are not in equilibrium, what* force *must I apply in the direction making an angle* β *with* CB *to maintain equilibrium?*

Let F be the required force; draw CM perpendicular to F.



Then by Art. 132,

$Q \times AC = P \times CN + F \times CM$;

$$\therefore\ F = \frac{Q \times AC - P \times CN}{CM} = \frac{Q \times AC - P \times CB \cos \alpha}{CB \sin \beta}.$$

Example iii. *What* force *is applied to the rigid rod by the constraint at* C in Example i., *and* Example ii., *respectively?*

In Example i., it is supposed that the rigid rod is in equilibrium under the action of the constraint and of the weights Q and P respectively. The force applied by the hinge at C is therefore the anti-resultant of the two vertical forces downwards Q and P;

The required force is therefore $(Q+P)$ acting vertically upwards.

In Example ii., let the force of the constraint be R inclined to the horizontal line CA at an angle θ.

Then the sum of the horizontal resolutes of all the forces acting on the rigid body is zero;

$$\therefore\ R \cos \theta - F \sin NBF = 0.$$

Also the sum of the vertical resolutes of all the forces acting on the rigid body is zero;

$$\therefore\ R \sin \theta - Q - P - F \cos NBF = 0.$$

These two equations give

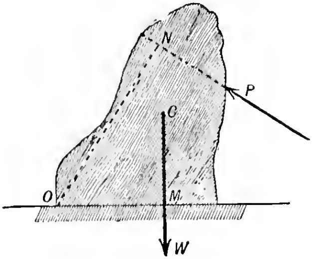
$$\tan \theta = \frac{Q + P + F \cos NBF}{F \sin NBF},$$

$$R^2 = F^2 \sin^2 NBF + (Q + P + F \cos NBF)^2.$$

[It will be seen that $NBF = CBF - CBN = \beta - 90^\circ + \alpha$.]

Example iv. *A heavy body of weight* W *rests on a horizontal plane, find the force in a given direction necessary to turn it over.*

Let the picture represent a plane section of the body about which it is symmetrical.



Let O be the point about which the body is to turn.

Let P be the required force, C the Centre of Gravity of the body.

Draw ON perpendicular to the force, let the vertical through C cut the horizontal plane in M.

In such a question it is understood that the point O of the body is practically fixed, either by the roughness of the ground or by some obstacle.

When the body is on the point of turning about O, the only force acting on the body besides P and W is the reaction of the constraint at O; hence the condition that the body should be on the point of turning is that the moment about O of P = the moment about O of W; hence

$$P \times ON = W \times OM:$$

$$\therefore \ P = W \times \frac{OM}{ON}, \text{ the required value.}$$

Suppose the body to have turned about O through some finite angle; the perpendicular distance of W from O will be diminished; consequently the moment about O of OW will also be diminished.

Hence, the force necessary to keep the body tilted up through a finite angle if it acts along the fixed line NP is less than the above value of P.

Hence, if a force slightly greater than the P found above be applied to the body in a fixed direction the body will be tilted quite over.

EXAMPLES. XIX.

In the **ten following** Examples a piece of thin cardboard is placed on a smooth table and on it is drawn a square $ABCD$ each of whose sides is 2 inches.

1. If the point A is fixed and a force of 1 lb. acts on the cardboard along the side BC, what force must act along CD to keep the cardboard at rest?

2. The point A being fixed, forces of 6 lbs. and 5 lbs. act along BC and CD, find the least force which acting at D will maintain equilibrium.

3. The point A being fixed, a force of 5 lbs. acts at B parallel to AC, and a force of 4 lbs. acts at C parallel to DB; what force acting along DB will maintain the cardboard at rest?

4. The point A is fixed; forces of 3 lbs., 4 lbs., 5 lbs. and 6 lbs. each act along AB, BC, CD, DA respectively; where must a force of 1 lb. parallel to DB be applied in AC (produced if necessary) to keep the cardboard at rest?

5. If the point E, where AC and BD intersect, is fixed, instead of A in **4**, where must the force of 1 lb. be applied?

6. The point A being fixed, and forces of 3 lbs., 4 lbs., 5 lbs., 6 lbs., 7 lbs. acting along BC, CD, DB, through C parallel to DB, through the middle points of AB and DC respectively, find the point in AD, produced if necessary, nearest to A at which a force of 2 lbs. will keep the cardboard at rest.

7. Forces of 1 lb. and 2 lbs. act along AB and DC respectively; what point of the cardboard in AD must be fixed that equilibrium may be maintained?

8. Forces of 3 lbs., 4 lbs. and 5 lbs. act along AB, BC, CD; find the points of the cardboard (i) in AD produced if necessary, (ii) in DC produced if necessary, which must be fixed in order to keep the cardboard from moving.

9. Forces of 4 lbs., 5 lbs. and 6 lbs. act along AB, BC, CA; find the points in which the line of action of their resultant cuts AD and DC.

10. Forces of 2 lbs., 2 lbs. and $2\sqrt{2}$ lbs. act along AB, BC and CA respectively; shew that it is impossible to keep the cardboard at rest by fixing one point only.

11. An equilateral triangle ABC is drawn on a piece of cardboard placed on a smooth table, each side being 3 inches; forces of 3 lbs., 4 lbs. and 5 lbs. act along AB, BC and CA respectively; if the centre of gravity of the triangle be fixed, what is the least force which acting at A will keep the triangle at rest?

12. In Question **11** find a point in the perpendicular through A to BC which if fixed will keep the cardboard on which the three forces act at rest.

13. Forces of 3 lbs. each act along AC, CB, BA of the triangle ABC in Question **11**; shew that the cardboard cannot be kept at rest by fixing one point only.

14. If in Question **11** the points A and B are fixed, what forces perpendicular to the line AB act on the constraints?

15. A rod AOB, such that AOB is a right angle, is fixed at O and is in a vertical plane; it is in equilibrium when weights P and Q are suspended from A and B, and OA is inclined at an angle of 60° to the horizon; find what change will require to be made in the force at B so that the rod may rest with OA inclined to the horizon at an angle of 30°.

16. A straight uniform rod ACB of weight W has the point C fixed, and weights P and Q are fastened at A and B respectively; shew that if the rod be at rest

$$AC : CB = 2Q + W : 2P + W.$$

17. A triangular board ABC weighing 3 lbs. with its plane vertical is hinged at A; what vertical force must act at the middle point of BC to keep it at rest?

18. The centre of gravity of an equilateral triangle is fixed and its plane is vertical; weights of 3 lbs., 4 lbs., 5 lbs. are fixed to the angular points; find the horizontal force which must act along the side joining the 4 lbs. and 5 lbs., that the 3 lbs. may rest vertically above the centre of gravity.

19. A ladder AB weighing 60 lbs. whose C. G. is 10 ft. from A has the point A fixed to the ground; what force must a man 6 ft. high be able to apply to the ladder to raise it to a vertical position supposing he applies the force at the point in the ladder which is 6 ft. from the ground?

20. If the sides of a triangle be taken two and two, to represent forces, acting in each case from the point of intersection of the sides, prove that there is one point about which each of the three pairs will balance and find the point.

CHAPTER XII.

TENSIONS OF STRINGS AND OF RODS.

135. A string or a rod will bear a tension which is very great when compared with its own weight; hence the weight of a string in a machine is usually neglected.

Such strings are said to be **light**. It will be always understood that a string is *light* unless it is expressly stated to be otherwise.

136. The tension of a string exemplifies in a remarkable way the truth of Newton's third Law.

Take a piece of light string and pull at one end with each hand. Then the string applies a force, say for example of 2 lbs., to each hand. Fix your attention upon any point in the string. The sting is said to be *tight* at that point; there is a *stress* at that point consisting of two equal and opposite forces each equal to 2 lbs. weight. This is the case at *every point* of the string. This *stress* is the *tension* of the string.

137. At the point at which a string is attached to a body the stress acts between the body and the string; one of the two equal and opposite forces of this stress urges the body towards the string; the other urges the string towards the body. We usually confine our attention to the first of these two forces.

138. Similar remarks hold good with regard both to the pull and the thrust of a light rod.

139. When a string is **heavy** the stress at each point is not the same for all points in its length.

Therefore when a string is heavy its tension is **not constant** along its whole length.

Example. One end of a uniform chain 25 ft. long *which weighs* 4 lbs. per linear foot, *is attached to a hook; the chain hanging vertically down supports a weight of* 500 lbs. *attached to its lower end; what are the tensions of the string at each end and at its middle point?*

Let ABC be the chain, A its highest, B its middle and C its lowest point.

The tension at C must be the force necessary to keep the weight W 500 lbs. in equilibrium.

The weight is a rigid body acted on only by its own weight vertically downwards and by the pull of the string; which must therefore be a force of 500 lbs. vertically upwards.

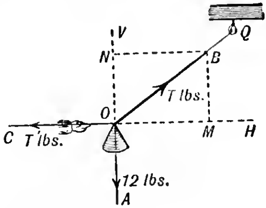
To find the tension at the middle point of the string, consider the lower half of the string and the weight W as forming one rigid body, (the equilibrium would not be disturbed if they were actually to become rigid). Then as before the pull of the string upon its lower half must be 500 lbs. + the weight of $12\frac{1}{2}$ ft. of chain; that is 550 lbs. This is the required tension.

To find the tension at the highest point, consider the whole string and the weight as a single rigid body and we find the tension to be 500 lbs. + the weight of the string; that is, 600 lbs.

140. It should be noticed, that when a system of particles or bodies is in equilibrium, we may choose any portion of the system for consideration; then the external forces acting on that portion must satisfy the conditions of Art. 91.

That this is so is perhaps made more evident by pointing out, that the equilibrium when it exists, would not be disturbed if the selected portion of the system were *actually to become rigid.*

Example i. *A weight of* 12 lbs. *is suspended from a fixed hook by a string; I tie a second string to the weight and by pulling horizontally cause the first string to be inclined to the vertical at an angle whose* cosine is $\frac{3}{5}$. *Find the* forces *applied by the strings.*



Let O be the point of the weight to which the strings are attached; let Q be the hook, P the hand holding the second string. Then we have three forces acting at O: (i) the weight of 12 lbs. acting vertically downwards; represent this by OA:

(ii) the tension of the string along OQ; represent this by OB, and let it be T lbs.:

(iii) the tension of the string along OC which is horizontal; represent this by OC, and let it be T' lbs.

Then these three forces are in equilibrium.

Therefore the sum of their resolutes in any direction is zero.

Draw OH and OV horizontally and vertically.

Then $\cos BOV = \frac{3}{5}$, and $\therefore \cos BOH = \frac{4}{5}$.

Take the sum of the horizontal resolutes of the forces.

The resolute of OC along OH is itself; viz. $-T'$ lbs.

The resolute of OA .... is zero.

The resolute of OB .... is $(T \cos BOH)$ lbs. $= \frac{4}{5} T$ lbs.;

$\therefore$ by Art. 58, $\quad -T' + \frac{4}{5}T = 0$.

Take the sum of the vertical resolutes of the forces.

The resolute of OC along OV is zero.

The resolute of OA .... is -12 lbs.

The resolute of OB .... is $(T \cos BOV)$ lbs. $= \frac{3}{5} T$ lbs.;

$$\therefore \quad -12 + \tfrac{3}{5}T = 0;$$

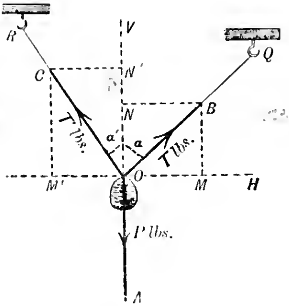
$$\therefore \quad T = 20,$$

and
$$T' = \tfrac{4}{5}T = 16;$$

$\therefore$ the tension of OQ is 20 lbs.: the tension of OP is 16 lbs.

[NOTE.—We here look upon a string as simply a mechanical means of applying force to the 10 lbs. weight. The force applied by a string to a mass is always **in the direction of the string**; also the magnitude of the force has nothing to do with the length of the string.]

Example ii. *A weight* O *of* P lbs. *is supported by two strings* OQ *and* OR *making angles* α *and* α' *with the vertical; find the tensions of the strings.*



Let OA vertically downwards represent the weight of P lbs.
let OB represent the tension (T lbs.) of OQ,
let OC .... (T' lbs.) of OR.
The three forces OA, OB, OC are in equilibrium;
∴ the sum of their resolutes in any direction is zero.
Taking their vertical resolutes, we have

$$-P+T\cos\alpha+T'\cos\alpha'=0.$$

Taking their horizontal resolutes, we have

$$T\sin\alpha-T'\sin\alpha'=0,$$

whence

$$T'=\frac{T\sin\alpha}{\sin\alpha'};$$

$$\therefore\quad -P+T\cos\alpha+\frac{T\sin\alpha\cos\alpha'}{\sin\alpha'}=0;$$

$$\therefore\quad T\left(\cos\alpha+\frac{\sin\alpha\cos\alpha'}{\sin\alpha'}\right)=P;$$

$$\therefore\quad T=\frac{P\sin\alpha'}{\cos\alpha\sin\alpha'+\cos\alpha'\sin\alpha},$$

and

$$T'=\frac{P\sin\alpha}{\cos\alpha\sin\alpha'+\cos\alpha'\sin\alpha}.$$

and we have found T and T'. Q. E. F.

EXAMPLES. XX.

N.B.—The *length* of a string has of itself nothing whatever to do with its tension.

1. A weight of 140 lbs. hangs at the end of a rope 12 ft. long, whose other end is fixed; if the whole rope weighs 3 lbs., find the tension of the rope at a point 4 ft. from the weight.

2. A weight 112 lbs. hangs at the end of a rope 10 ft. long, whose other end is fixed; if the rope weighs 1 lb. per linear foot, find the tension of the rope at the point 4 ft. from the weight.

3. A chain weighing 3 lbs. per linear foot passes over a smooth small fixed pulley which is 20 feet from the ground; one end of the chain is coiled up on the ground vertically under the pulley, the other hangs vertically and is 10 ft. from the ground; what weight must be fastened to the chain to keep it at rest?

4. Shew that in Question **3** if the weight fastened to the end of the chain be too small it will run up to the pulley, if it be too large it will descend to the ground.

5. A chain weighing 1 lb. per foot passes over a smooth small fixed pulley* 50 feet from the ground, and one end is coiled up on the ground; the other end is held by a man at a point 30 ft. below the pulley; if the man pulls the chain over the pulley, what force must he exert? what is the pressure on the pulley?

6. Shew that in Question **5** when the end of the chain has left the ground the man's pull will gradually diminish; when will it cease altogether?

7. If a heavy body hang supported by three strings, two of which are vertical, prove that the third string must be vertical.

8. A heavy chain of length 12 ft. 8 in., and weighing 19 lbs., has a weight of 3 lbs. attached to one end, and hangs in equilibrium over a smooth peg; what length of it hangs on each side?

9. An endless chain 20 feet long, weighing 20 lbs., passes round a smooth fixed peg; two weights of 4 lbs. and 2 lbs. are fastened to points in the chain 5 ft. apart; what positions will the weights take up when hanging vertically under the peg in equilibrium, and what will be the tensions of the chain at the points 6 ft. from the peg?

10. An endless chain 16 ft. long, weighing 32 lbs., passes round a fixed smooth peg; three weights of 10 lbs., and 5 lbs. and 2 lbs. are fastened to the chain at points 2 ft. apart; where will the weights be when the chain hangs vertically from the peg in equilibrium? Find the tension of the chain at its lowest point, and at points 4 ft. from the peg.

* NOTE.—A smooth pulley or peg is here to be considered a mechanical means of altering the *direction* of the tension of a string without altering its magnitude.

11. A weight of 12 lbs. is fastened by a string to a hook; if I pull horizontally with another string fastened to the weight until the first string makes an angle whose cosine is $\frac{4}{5}$ with the vertical, with what force do I pull?

12. With a similar arrangement to that in Question **11**, if the weight is 16 lbs. and my horizontal pull is 12 lbs., find the tension of the other string.

13. If in a similar arrangement to that in Question **11** the inclination of the first string to the vertical is an angle whose cosine is $\frac{5}{13}$; when I pull horizontally with a force of 12 lbs., what is the weight?

14. A weight of 12 lbs. is supported by two strings each making an angle whose cosine is $\frac{3}{5}$ with the vertical; find the tensions of the strings.

15. A weight of 84 lbs. is supported by two strings, one making an angle whose cosine is $\frac{3}{5}$, and the other an angle whose cosine is $\frac{4}{5}$ with the vertical; find their tensions.

16. A weight of 204 lbs. is supported by two strings, making a right angle with each other, one of which makes an angle whose cosine is $\frac{5}{13}$ with the vertical; find their tensions.

17. A weight O of 84 lbs. is supported by two strings OA, OB fastened to two fixed hooks A and B such that $OA = 4$ ft., $OB = 3$ ft. and AB is horizontal and $= 5$ ft.: find the tensions of the strings.

18. A weight O of 204 lbs. is supported by two strings OA, OB fastened to two fixed points A and B, so that $OA = 5$ ft., $OB = 12$ ft. and $AB = 13$ ft. and is horizontal; find the tensions of the strings.

19. A weight O is suspended from a fixed hook Q by a string OQ; shew that if I apply a horizontal force to the weight O, I increase the tension of the string OQ.

20. Two forces P lbs. and Q lbs. have a resultant which is vertical and downwards; the force P lbs. makes the angle α with the line drawn vertically upwards, and the force Q lbs. makes the angle β with the same line on the other side of it; prove that $P \sin \alpha = Q \sin \beta$, and that the resultant is

$$(P \cos \alpha + Q \cos \beta) \text{ lbs.}$$

21. A heavy rod has one end of a light string twice as long as itself fastened to it; the loop of the string passes over a smooth fixed peg; prove that if the rod is uniform it is in equilibrium when it hangs from the peg when the rod is horizontal and when it is vertical, and in no other position.

22. If the rod is not uniform, what is the other position besides the vertical in which the rod is in equilibrium?

23. A heavy uniform rod of weight W is fastened to a fixed point by two strings of length a and b; find the tension of the strings when the rod hangs in equilibrium.

24. A heavy uniform rod is hinged to a fixed point, and is supported at an angle θ to the vertical by a horizontal string; find the tension of the string.

25. Two equal uniform rods each of weight W are hinged together at one end, and have their middle points connected by a string whose length is equal to half that of each rod; the rods and string stand on a smooth horizontal plane in the form of an A; find the tension of the string.

26. Two equal light rods replace the heavy ones of Question 25 and a weight W is fastened to the hinge at the vertex; find in this case the tension of the string.

27. A uniform heavy rod is placed in a smooth hemispherical bowl whose radius is equal to the length of the rod; what will be its position of equilibrium?

28. If the rod of Question 27 be not uniform, where will it rest and what will be the forces acting upon it?

29. A string each of whose ends is fastened to fixed points has a heavy smooth bead sliding upon it; what will be the position of the bead when in equilibrium?

30. A heavy uniform rod is supported from two fixed points by two strings fastened one to each of its extremities; shew that if when in equilibrium the rod is horizontal the tensions of the strings must be equal.

Shew how to find the tensions when the rod is not horizontal.

31. Prove that if a uniform rod be suspended from a smooth peg by the loop of a string whose ends are fastened to the ends of the rod, the rod can only rest in a vertical or in a horizontal position.

32. A uniform rod of length a is supported in a horizontal position by two equal strings of length l; the ends of each string are fastened to an end and to the middle point of the rod, and the loops each pass over a smooth small peg; shew that the tension of the strings is $\frac{1}{4}W\frac{l^2-a^2}{l^2}$.

33. A heavy uniform rod has two strings fastened one to each of its ends, the other ends of the strings being fastened one to each of the ends of another uniform rod whose middle point is fixed; prove that when the rods are at rest either the strings or the rods are parallel.

CHAPTER XIII.

Machines.

141. We proceed to describe and as far as may be explain the principles of the following machines:

The Lever, the Pulley, the Wheel and Axle, the Inclined Plane, the Screw.

Levers.

142. *DEF.* A **lever** is a rigid rod moveable in one plane, about a fixed point in the rod.

143. The fixed point is called the **fulcrum.**

144. It is understood that two forces act on the lever besides the reaction of the fulcrum; these two forces are often called the **power** and the **weight.**

145. The **arms** of a lever are the *perpendicular distances* of the lines of action of power and the weight from the fulcrum. [See Art. 132.]

146. The weight of the lever itself is generally inconsiderable compared with the forces acting upon it and accordingly the weight of the lever is often neglected.

It is then said to be *light.*

147. *PROP. To prove the principle of the Lever;* namely, that when a lever is in equilibrium

the Power × its arm is equal to the Weight × its arm.

When the Power and the Weight are two parallel forces and the lever is straight the arms may be measured along the lever. Otherwise, the arm is the *perpendicular* distance of the line of action of the force from the fulcrum.

The point of the lever in contact with the fulcrum is *fixed;* therefore the fulcrum will apply to the lever whatever force is necessary to keep that point of the lever at rest.

The remaining forces will be in equilibrium with this constraining force provided the sum of the moments about that point is zero. [See Art. 134.]

That is, provided

the Power × its arm = the Weight × its arm.

148. It has been customary to divide levers into three classes.

In the **first class**, the fulcrum is between the Power and the Weight.

In the **second class**, the Weight is between the Power and the fulcrum.

In the **third class**, the Power is between the Weight and the fulcrum.

But these *classes* have no importance beyond the historical interest.

Examples of the first class: a crowbar; a poker raising the coals using the bar of a grate as a fulcrum; the handle of a pump. Double levers: scissors; pliers, nippers and pincers.

Examples of the second class: a wheelbarrow; an oar (in which the blade is the fulcrum and is supposed to be practically at rest). Double levers: nutcrackers.

Examples of the third class: the human arm, when put into a horizontal position with the palm of the hand upwards and supporting a weight, is a good example; the power is the biceps muscle which is attached to the arm near the elbow, and the elbow joint is the fulcrum. Double levers: a pair of fire tongs, as usually used; sugar tongs.

On Mechanical Advantage.

149. One of the most important problems which an Engineer has to solve, is how to raise heavy weights.

With a lever, the ratio of the two forces, the Power and the Weight, may be made (by properly arranging the length of the arms) as small or as great as we please.

Hence we may with a lever *support* as great a weight as we please with as small a force as we please; provided the fulcrum is *fixed* and the lever is strong enough.

150. *DEF.* The capacity for counteracting a large force with a smaller one is called **mechanical advantage.**

It is customary to say that a machine possesses mechanical advantage when the *weight* supported is greater than the *power* by which it is supported.

The mechanical advantage of such a machine is said to be *measured* by the ratio of the weight to the power.

For example, a lever of the first class has mechanical advantage when the power-arm is longer than the weight-arm. A lever of the second class has always mechanical advantage. A lever of the third class is always at a **disadvantage** mechanically.

151. It must however be noticed that the advantage possessed by a machine is only *statical.* When a lever raises a weight it will be found that the *Work* done on the Weight is exactly equal to the work done by the Power.

For example, if a weight of 100 lbs. is supported by a force of 5 lbs. with a lever of the first class, the arm of the power is 20 times as long as the arm of the weight, and it therefore must move through 20 times the distance through which the weight moves.

What is gained in force is lost in distance moved.

EXAMPLES. XXI.

1. A lever with the fulcrum between the Power and the Weight has its arms 5 ft. and 10 ft. respectively; if a power of 20 lbs. acts at the extremity of the longer arm, what weight can the lever support?

2. A lever with the fulcrum at one end has arms such that one is 3 ft. longer than the other; if the power is 10 times the weight, what is the length of the lever?

3. A lever with the power in the middle (i.e. *between* the fulcrum and the Weight); the power is $3\frac{1}{2}$ times the weight and the pressure on the fulcrum is 18 lbs.; what is the weight?

4. A heavy uniform rod 10 ft. long is used as a lever, and the fulcrum is 3 ft. from the end; the power is 1 lb. and the weight is 4 lbs.; what is the weight of the rod?

5. A wheel-barrow is 5 ft. long and exclusive of the wheel it weighs 20 lbs., the centre of gravity being 2 ft. from the axle of the wheel; how near the axle must a weight of 30 lbs. be placed that a man may gain mechanical advantage by using the wheel-barrow?

6. A cube weighing 1 ton stands on a horizontal plane. A man of 12 stone wishes with a crow-bar 5 ft. long to raise the middle of one edge of the cube from the ground; how near the end of his crow-bar must he put his fulcrum?

7. A pair of nutcrackers is $4\frac{1}{2}$ inches long, and a nut is placed $\frac{5}{8}$ of an inch from the hinge; what pressure applied at the ends will crack the nut if a weight of $20\frac{1}{4}$ lbs. when simply placed on the top of the nut will crack it?

8. A heavy uniform beam 7 ft. long rests on two supports, one at one end and the other $5\frac{1}{2}$ feet from that end; the greatest weight that can be hung on at the other end without disturbing the equilibrium is 16 lbs.; find the weight of the beam.

9. Explain two different ways of arranging a rod 15 ft. long as a lever which will lift a weight of 24 lbs. with a force of 12 lbs.

10. A straight rod AF without weight is divided in points B, C, D, E so that $AB : BC : CD : DE : EF$ as $1 : 3 : 5 : 7 : 9$, and weights of 1, 2, 3, 4, 5 lbs. are placed at the points B, C, D, E, F respectively; shew that if G be the point in the rod at which a fulcrum would support it, $AG : GF = 3 : 2$.

11. Find the length of a lever of the second kind that a power of 5 lbs. may support a weight of 12 lbs., and that their points of application may be 1 ft. apart.

12. The shorter arm of a lever is 7 inches long; the lever is in equilibrium when weights of 5 lbs. and 8 lbs. are suspended from its arms; find the length of the other arm.

13. The arms of a light lever of the first kind are unequal, and a body when suspended from one end is balanced by a weight of 18 lbs. at the other; when suspended from the longer end it is balanced by a weight of 20 lbs.; what is the ratio of the arms of the lever, and what is the weight of the body?

14. A straight lever whose length is 5 ft. and weight 10 lbs. has its fulcrum at one end; weights of 3 lbs. and 6 lbs. are fastened to it at distances 1 ft. and 3 ft. from the fulcrum, and it is kept horizontal by a vertical force at the other end; find the pressure on the fulcrum.

15. If in Question **14** the force keeping the lever horizontal were inclined at an angle of 30° to the horizon, what would the force be, and what would be the pressure on the fulcrum?

16. AOB is a bent lever whose fulcrum is O, and whose arms OA, OB are equal and straight; it is in equilibrium with OA horizontal when weights P and Q are suspended from A and B; find the change which must be made in the weight suspended at B that the lever may be in equilibrium when P is suspended at A and OB is horizontal.

17. A lever has its fulcrum in the middle (i.e. between the Power and the Weight), and a weight W fastened to one end is supported by a force of P lbs. at the other; if the ends are interchanged the necessary force to balance W is a force of Q lbs.; prove that $W=\sqrt{(PQ)}$.

18. The arms of a bent lever are 2 ft. and 3 ft. respectively; what force acting at an angle of 30° to the longer arm will balance a force of 30 lbs. acting at right angles to the shorter arm?

19. A bent lever whose arms are inclined at right angles to each other, and are 3 ft. and 4 ft. long respectively, is at rest under the action of forces of 16 lbs. and 12 lbs. acting at the extremities of the shorter and longer arms respectively; if the force of 16 lbs. acts at right angles to its arm, at what angle must the other force act and what is the pressure on the fulcrum?

20. A bent lever consists of two heavy uniform straight arms, whose lengths are 3 ft. and 5 ft. respectively; if the beam weighs 10 lbs. per foot, what weight must be suspended from the extremity of the shorter arm that the lever may balance with its arms equally inclined to the horizon?

21. *AB* and *DEC* are two light horizontal levers arranged so that *B* is vertically above *C*, and connected with it by a light inextensible string. The fulcrum of *DC* is at *D*; the weight *W* is placed at *E* so that *DE* is $\frac{1}{9}$th of *EC*; the 'weight' of *AB* is the tension of the string; the fulcrum of *AB* is at *H*, so that *BH* is $\frac{1}{10}$th of *AH*; what power acting vertically downwards at *A* will support 100 lbs. placed at *E*?

NOTE.—The above **compound lever** illustrates the principle of the weighing-machine.

The Common Balance.

152. The common balance is a machine for testing whether two bodies have equal weights.

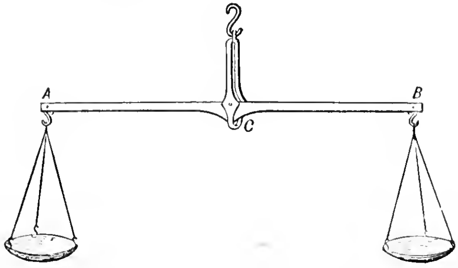
It is practically a lever with the fulcrum exactly in the middle of the beam.

153. The '**weights**' of a balance are bodies whose weights are some multiple or submultiple of the standard weight.

The weights are bodies weighing respectively 112 lbs., 56 lbs., 28 lbs., 14 lbs., 7 lbs., 4 lbs., 2 lbs., 1 lb., $\frac{1}{2}$ lb., $\frac{1}{4}$ lb., 1 oz., $\frac{1}{2}$ oz., $\frac{1}{4}$ oz., etc.

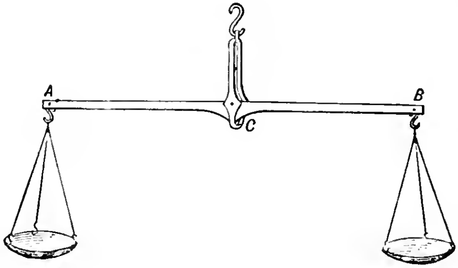
The weight of a body is ascertained by testing with the balance which of these weights will '*balance*' it.

154. The following is a description of the balance and the method of using it.



The figure gives a simple form of the balance. The *beam AB* is supported by a fulcrum at *C*. The fulcrum is a short bar of steel or other hard substance whose section is wedge-shaped; this is fixed to the beam with its sharp edge downwards and passes through round holes in the support from which the beam is suspended; thus the beam is supported by the upward pressure of the support on its sharp edge called the *knife-edge*. Knife-edges are fixed to the beam at *A* and *B* and from them are suspended the scale-pans; so that the upward pressure of each of these knife-edges supports the weight of a scale-pan and of anything placed on it. The short bars forming the knife-edges are at right angles to the plane of the picture.

The body to be weighed is placed in one scale-pan, say *A*, which at first rests on a table, and weights are placed in the other scale-pan *B* also on the table; the beam is then raised by the hook until it is just horizontal, and both scale-pans just touch the table; it is then quickly raised a little further, so as to lift both scale-pans off the table. Then,



if the scale-pan *A* falls, the weights are too small; if the scale-pan *B* falls, the weights are too great; the trials are continued until the beam *hesitates* to turn one way or the other.

155. The **requisites** of a good balance are

(I) it must be **true**; the arms must be *exactly* equal, and the scale-pans of equal weight;

(II) it must be **sensitive**†; that is, it must *quickly* shew when the sum of the weights, which are placed in the scale-pans opposite to that in which is placed the body to be weighed, is *not* equal to the weight of the body;

The discussion of the means by which a balance may be made sensitive is really a *dynamical* question.

It will be found that in order that a balance may be sensitive it is necessary

(i) The knife edges of the beam and of the pans must be very hard and must rest on a hard surface.

(ii) The arms must be as long as possible.

(iii) The beam must be as light as possible.

(iv) The C. G. of the beam must be near the knife edge on which it rests.

† The word *sensible!* is often used in this senso.

(III) it must be **stable**; that is, it must not move very far from its mean position, when the weight of the body and of the weights differ.

This is also a dynamical question; but we may point out here that, when the knife-edge on which the beam rests is above the straight line joining the knife-edges of the scale-pans (as it usually is), then the nearer the knife-edge of the fulcrum is to the straight line joining the knife-edges of the scale-pans, the more sensitive and the less stable will the balance be.

The further *above* this line the fulcrum is, the more stable it will be. Hence for purposes requiring great accuracy the three knife-edges are made nearly or exactly in a straight line; while for the rougher purposes of weighing heavy goods the knife-edge of the fulcrum is placed a sensible distance above the line joining the knife-edges supporting the scale-pans.

156. It is not sufficient for the truth of a balance that the beam should remain horizontal when raised with the scale-pans empty, for the scale-pans might in that case be of unequal weight and the arms also unequal.

157. To test the truth of a balance, we first see that the beam balances when the pans are empty; then taking two equal weights we put one in each of the scale-pans; if the beam again balances, then the balance is true.

For, let a, b be the lengths of the arms; W, W' the weights of the scale-pans; ω the weight of the equal weights; then by the first experiment

$$W \times a = W' \times b,$$

by the second

$$(W + \omega) \times a = (W' + \omega) \times b;$$

$$\therefore\ \omega \times a = \omega \times b;$$

$$\therefore\ a = b, \text{ and } \therefore \text{ also } W = W'.$$

Or, we may (having first tested when the pans are empty) find the weight of a body Q when placed in one scale-pan to be W_1, and when placed in the other to be also W_1; then the balance is true.

EXAMPLES. XXII.

1. In a balance which rests in a horizontal position when unloaded, but whose arms are of different lengths, a body when weighed in one scale appears to weigh W lb. and when weighed in the other W' lb.; prove that its true weight is $\sqrt{(WW')}$.

2. A grocer has to weigh out to a customer a certain weight of tea; he knows that his balance is in the condition described in Question **1**, so he weighs one half of the quantity of tea required, and then puts his weights into the other scale and weighs the other half in the other pan; has he got the proper weight of tea, or is he a gainer or loser?

3. In a balance such as that described in Question **1**, the apparent weights of a body are $42\frac{1}{4}$ lbs. and 49 lbs., and the whole length of the beam is $2\frac{1}{4}$ ft.; find the length of each arm.

4. In a false balance, the arms being of unequal length, a weight is measured in one scale by P lbs. and in the other by Q lbs.; shew that the arms are to one another as $\sqrt{P} : \sqrt{Q}$.

5. If the arms of a false balance are 2 ft. and 2 ft. 1 in. respectively, what is the true weight of a body which appears to weigh 10 lbs. when placed in the scale at the end of the shorter arm?

6. When one of the scales of a common balance (which is otherwise true) is loaded, a body appears to weigh W lbs. when placed in one scale and W' when placed in the other scale; prove that its true weight is the Arithmetic mean between W and W'.

7. If the apparent weights in Question **6** are 18 oz. and 20 oz., find the weight with which one of the scales is loaded.

8. Shew that a balance is (other things being equal) more sensitive as the arms are longer.

9. If the beam of a false balance is uniform and heavy, shew that the arms are proportional to the differences between the true weight and the apparent weights of a body.

10. The arms of a false balance are without weight, and one arm is longer than the other by one-ninth part of the shorter arm; and in using it the substance weighed is put as often into one scale as in the other, shew that the seller loses five-ninths per cent. on his transactions.

11. A tradesman's balance has arms whose lengths are 11 in. and 12 in. respectively, and it rests horizontally when the scales are empty; if he sells two separate pounds of tea each at 2*s*. 9*d*. per lb., putting his weights into different scales for each transaction, find whether he gains or loses, and how much.

12. One pound is weighed at each end of a false balance and the sum of the apparent weights is 2 lb. 2 oz.; what is the ratio of the lengths of the arms?

13. A balance has its arms unequal in length and in weight. A certain article appears to weigh Q_1 lbs. when placed in one scale pan, and Q_2 lbs. when placed in the other; a certain other article appears to weigh R_1 lbs., R_2 lbs. respectively; shew that the weight of the article which appears to weigh the same in whichever scale it is put is

$$\frac{Q_1 R_2 - Q_2 R_1}{Q_1 - Q_2 - R_1 + R_2} \text{ lbs.}$$

158. **The common steelyard.** The common steelyard is an instrument for weighing goods.

It consists of a straight steel lever AB having a fulcrum at a fixed point C near one end A.

A weight of P lbs. is arranged as a ring D which can slide along the arm CB. At A is a hook or scale-pan on which can be placed the article whose weight is to be ascertained. The arm CB has certain marks and numbers engraved on it.

The weight of an article is obtained as follows:

The fulcrum is firmly supported and the steelyard is held in a horizontal position while the article is placed in the scale-pan. Then the ring D is shifted until by repeated trials the position is found for it at which, when the whole is left to the support of the fulcrum alone, the steelyard *hesitates which way to turn*. The mark at D engraved on the rod OB indicates the weight of the article.

The **graduation** of the steelyard is the process by which the maker ascertains where to put the proper marks on the rod CB.

159. *PROP. To graduate the common steelyard.*

Let the weight of the rod and scale-pan together be M lbs.; and let their joint centre of gravity be at G.

[The weight of the scale-pan acts upon the rod exactly as the weight of an equal body *fixed to the rod* at A.]

We first obtain the zero point O by trial.

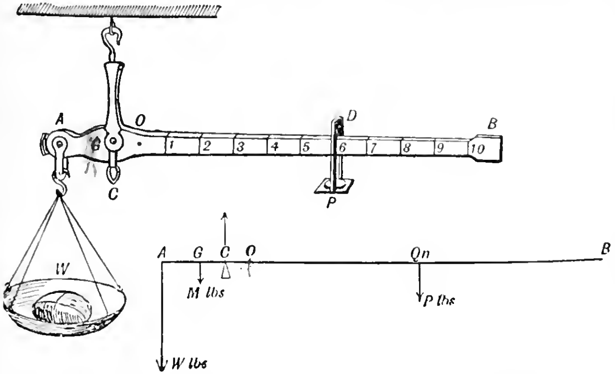
That is, the ring D, which weighs P lbs., is shifted until a point O is found, at which the steelyard balances when no article at all is in the scale-pan.

Then $$M \times GC = P \times OC' \quad \text{(i)}.$$

Now suppose a weight of W lbs. to be placed in the scale-pan.

Let the ring D be shifted to a point Q_n at which it is found that the steelyard balances.

Then we have a force of W lbs. acting vertically downwards on the steelyard at A, a force of M lbs. acting verti-



cally downwards at G and a force of P lbs. acting vertically downwards at Q_n.

These forces P, M, W and a vertical force upwards at C are in equilibrium.

Taking the moments about C of all these forces we have

$$W \times AC + M \times GC - P \times CQ_n = 0 \ldots\ldots\ldots \text{(ii)}.$$

But by our first experiment we have

$$M \times GC - P \times OC = 0 \ldots\ldots\ldots\ldots\ldots \text{(iii)};$$

$\therefore$ putting $P \times OC$ for $M \times GC$ in (i), we have

$$W \times AC + P \times CO - P \times CQ_n = 0,$$

that is $$P(CQ_n - CO) = W \times AC,$$

or $$P \times OQ_n = W \times AC,$$

$$\therefore \; OQ_n = \frac{W \times AC}{P}.$$

From this, since AC is known, and O has been found, the different positions of Q_n for different values of W can be found.

160. Suppose now that $W = 1$ lb.; then $OQ_1 = \frac{AC}{P}$;

$\therefore$ to find Q_n for n lbs., we have $OQ_n = n \times OQ_1$.

That is, in order to graduate a steelyard to weigh lbs., we must find by trial the point O of zero graduation, and also by a second trial the point Q_1 for 1 lb., and mark their positions on the rod. Next we find by measurement points Q_2, Q_3, Q_4, etc. such that $OQ_1 = Q_1Q_2 = Q_2Q_3$, and so on, and engrave on the rod the numbers 2, 3, 4, etc. at those points.

Then, when an article weighing a number of lbs. say 4, is placed in the scale-pan, the steelyard balances when the ring is at the point on the rod which is marked 4.

NOTE.—The student should observe that if the lengths OQ_1, Q_1Q_2,... be each subdivided into any number of equal parts, say 16, then the steelyard is graduated for sixteenths of a lb.

Example. In a certain Common Steelyard the weight of the rod and scale-pan is 4 lbs.; *the length of AC is* 2 inches; *G lies between A and C, and GC is* ½ in.; $P = 2$ lbs.; *graduate the steelyard.*

We first find O.

We have $\quad M = 4$ lbs.; $GC = \frac{1}{2}$ in.; $P = 2$ lbs.;

$\therefore$ by (i) $\quad 4 \times \frac{1}{2} = 2 \times OC$,

$\therefore \ OC = 1$ inch.

The point O must be marked on the bar at a distance 1 inch from C.

Next we find OQ_n when $n = 1$ lb.

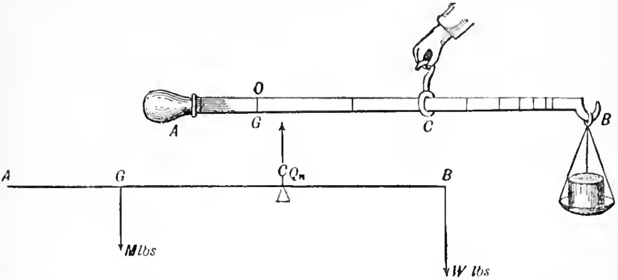
We have $\quad OQ_n = \frac{n \times AC}{P} = \frac{1 \times 2}{2} = 1.$

Thus the distance between each graduation for lbs. is 1 inch.

Hence marks are engraved on the bar at equal distances of an inch from the point O, and the numbers 1, 2, 3, etc. placed near them. These spaces are then subdivided into sixteenths and the machine is then graduated to weigh lbs. and oz.

161. **The Danish steelyard.** The Danish steelyard consists of a lever AB whose fulcrum C is moveable.

At one end A is a lump of metal, and at the other B, a hook or scale-pan.



The weight of an article placed in the scale-pan is ascertained by observing the point at which the fulcrum must be placed in order that the whole should balance when supported by the fulcrum.

162. *PROP.* *To graduate the Danish steelyard.*

The position of O, the zero point of graduation, is the centre of gravity G of the rod and scale-pan. [For when the ring is at G the steelyard balances when there is no weight in the scale-pan.] This must be ascertained by trial and carefully marked on the rod.

Let M lbs. be the weight of the rod and scale-pan.

Let W lbs. be placed in the scale-pan, and let Q_n be the corresponding position of the fulcrum.

Then the three forces M, W and a vertical force upwards at C are in equilibrium.

Hence, taking the moments of all these forces about C, we have $$W \times Q_nB - M \times GQ_n = 0.$$

From this result, since M is known, the position of Q_n for different values of W can be determined.

It will be seen that Q_n divides GB into parts which are in the ratio of W to M.

Example. A Danish steelyard and its pan weighs 4 lbs., shew how to graduate it so as to weigh lbs.

We have to find the points on the rod at which we must put the marks 1, 2, 3, etc. which shall indicate that when the steelyard balances with the fulcrum say at 2 the weight in the scale-pan is 2 lbs.

We have $$W \times Q_nB - M \times GQ_n = 0,$$

or $$GQ_n = \frac{W}{M} \times Q_nB.$$

Also $$M = 4 \text{ lbs.}$$

First, let $W = 1$ lb.; then $GQ_n = \frac{1}{4} Q_nB = \frac{1}{5}$ of GB;

hence, if we divide GB into *five* equal parts, GQ_n contains one of those parts.

Next, let $W = 2$ lbs.; then $GQ_n = \frac{2}{4} Q_nB = \frac{2}{6}$ of GB;

hence, if we divide GB into *six* equal parts, GQ_n contains two of those parts.

Next, let $W = 3$ lbs.; then $GQ_n = \frac{3}{4} Q_nB = \frac{3}{7}$ of GB;

hence, if we divide GB into *seven* equal parts, GQ_n contains three of those parts. And so on;

When $W = 4$ lbs., $GQ_n = \frac{4}{8}$ of GB;

when $W = 5$ lbs., $GQ_n = \frac{5}{9}$ of GB; etc.

NOTE.—In the Balance and the Steelyards the scale-pans are treated as forming part of the beam; the student should notice that it would make no difference to the argument if the scale-pan were replaced by an equal weight *fixed to the beam at the point of suspension.*

The student will notice that in the Balance and Steelyards it is important that the *points of suspension* of the scale-pans should be *fixed in the beam.* Hence in the steelyards the scale-pan should be suspended from a knife-edge *fixed to the beam* as in the balance.

EXAMPLES. XXIII.

1. A common steelyard is formed of a uniform rod 1 foot long, the fulcrum being 1 inch from one end; the sliding weight and the weight of the rod are each 1 lb.; find the least weight that can be weighed with it, and the distance between the graduations for pounds.

2. If in the steelyard of Question 1 the sliding weight be changed into one of 2 lbs., find what error will be made by using the old graduations as 2 lb. graduations.

3. If the distance of the C. G. of the beam of a common steelyard from the fulcrum is 2 inches, the moveable weight 4 oz., and the weight of the beam 2 lbs., find the distance of the zero of graduations from the centre of gravity.

4. A uniform rod 2 feet long, weighing 3 lbs., is to be used as a steelyard; the fulcrum is 2 inches from one end of the rod and the sliding weight is 1 lb.; find the greatest and least weight that can be determined by the machine, and the distance between the 1 lb. graduations.

5. The errors of a certain false steelyard are these: the distance of the zero point from the fulcrum is too great by a distance a, and every one of the distances between consecutive graduations is too long by a distance b; shew that the only weight which is correctly indicated by the instrument is $\frac{a}{b}P$ where P is the moveable weight.

6. A common steelyard is 3 ft. long, the C.G. of the steelyard is 5 inches from one end, and the fulcrum is 6 inches from the same end; the weight of the steelyard is 8 lbs., and the moveable weight is $1\frac{1}{2}$ lbs.; find the position of the zero of graduation and the distance between the lb. graduations.

7. If the moveable weight for which a common steelyard is constructed is 1 lb., and a tradesman substitutes a weight of 2 lbs., using the same points of graduation as before, but doubling the value indicated by the marks, shew that he defrauds his customers if the C.G. of the steelyard is in the longer arm, and himself if it is in the shorter arm.

8. Where must be the C. G. of a common steelyard that any moveable weight may be used with it, the marks of graduation indicating multiples of the moveable weight?

9. In a Danish steelyard the zero of the graduations is 12 inches from the end at which the body to be weighed is attached, and the weight of the beam is 8 oz.; find the position of the graduation corresponding to a weight of 16 oz.

10. If the bar of a Danish steelyard balance when the fulcrum is halfway between the first and second graduations, shew that the weight then in the scale is $\frac{7}{5}$ of the weight of the bar.

11. A weight of 4 oz. is in equilibrium on a Danish steelyard when the fulcrum is 6 inches from the end to which the weight is attached; a weight of 8 oz. is in equilibrium when the fulcrum is 4 inches from the end; find the C.G. and the weight of the instrument.

12. The weight of a Danish steelyard is 1 lb. and the nearest distance of the fulcrum from the end from which the weight is to be suspended is 1 inch, and the distance between the above position of the fulcrum and the C.G. is 3 ft.; what is the greatest weight that can be weighed with the machine?

13. There are no graduations on a certain Danish steelyard and its weight is not known, but by suspending from the end B in succession weights of P lb. and Q lb. respectively it is found that the corresponding distances of the fulcrum from B are a inches and b inches respectively; shew that the C.G. of the instrument is $\frac{(P-Q)\,ab}{Pa-Qb}$ inches from B, and that its weight is $\frac{aP-Qb}{b-a}$ lbs.

14. If the common steelyard be correctly constructed for a moveable weight P, shew that it may be made a correctly constructed instrument for a moveable weight nP by fixing at the C.G. of the steelyard a weight equal to $(n-1)$ times the weight of the steelyard.

15. If the beam of a common steelyard be uniform, and its weight be m times the moveable weight, and the fulcrum be one n^{th} part of the length of the beam from the end, shew that the greatest weight that can be weighed is $\frac{1}{2}\{2n-2+m(n-2)\}$ times the moveable weight.

16. If a common steelyard be 17 in. long and with the scale-pan weigh 3 lbs., their common centre of gravity being 1 in. from the end at which the scale-pan is suspended, the fulcrum being 2 in. from the same end and the moveable weight being 1 lb., find the distances between the graduations of half pounds.

17. When weights of P lbs. and Q lbs. are successively placed in the scale-pan of a common steelyard, the moveable weight is at distances a and b from the fulcrum; prove that, if the moveable weight be equal to that of the machine, the distance of the C.G. of the machine from the fulcrum is

$$\frac{Pb - Qa}{P - Q}.$$

18. Prove that in the common steelyard the distances of the marks of graduation from a certain point are in A.P., and that in the Danish steelyard they are in H.P.

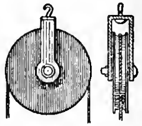
19. A common steelyard made of a uniform bar is 40 inches long; the weight of the beam is equal to the moveable weight, and the greatest weight that can be weighed with it is four times the moveable weight; find the place of the fulcrum.

20. In a common steelyard the distance between two successive marks of graduation for lbs. is 1 inch; the distance of the fulcrum from the scale-pan end is 2 inches; shew that the moveable weight is half-a-pound.

21. In a weighing machine constructed on the principle of the common steelyard the pounds are read off by graduation from 0 to 14, and the stones (of 14 lbs. each) by weights hung at the *end* of the arm. The weight corresponding to one stone is 7 oz.; the moveable weight is $\frac{1}{2}$ oz.; the length of the arm is one foot. Prove that the distances between the graduation are $\frac{3}{4}$ in. each.

Pulleys.

163. A Pulley is a machine for changing the direction of the pull of a string without sensibly changing its magnitude.



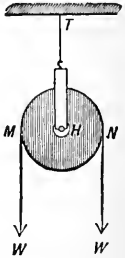
It consists of a small wheel of metal or wood whose axle is held by a frame called a **block**.

The word *pulley* is often used to indicate *the pulley and the block;* so that when we speak of a *fixed pulley*, we shall mean a pulley in a fixed block.

A moveable pulley generally indicates that the block of the pulley is not fixed, but has a string fastened to it; the pull of this string is in equilibrium with the pulls of the string which passes through the pulley.

It is usual to consider that the weight of the string is so small, compared with the other weights considered, that it may be omitted in our calculations.

Example i. *To find the conditions of equilibrium of a single fixed pulley, the strings being parallel.*



First, let the weight of the pulley be neglected.

Let the string passing round the pulley support a weight W at each extremity. These weights must be equal, because the pulley is smooth.

The block is fastened to a support by a string; let the tension of this string be T.

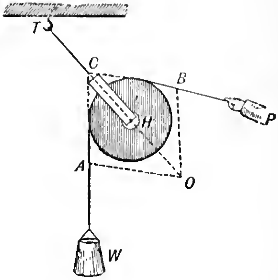
The pulley is now acted on by three parallel forces W, W, T which are in equilibrium.

Therefore $T = W + W = 2W$.

Next, consider the weight of the pulley to be w.

Then there are four parallel forces, three of which are like, viz. W, W and w; $\therefore\ T = 2W + w$.

Example ii. *The block* H *of a pulley is fastened to a fixed beam* T *by a string; the end of another string, which goes over the pulley, is fastened to a weight* W; *the other end I hold in my hand, as in the figure; with what force do I pull? and what is the tension of the strings?*



The tension of the string which is fastened to the weight, is equal to W, the weight itself.

This tension is unaltered after passing round the pulley; therefore the force with which I pull is equal to W.

First, suppose the weight of the pulley to be neglected.

Then the only forces acting on the pulley are the tensions of the strings.

These forces are in the direction of the strings.

Therefore if a parallelogram $ACBO$ be drawn as in the figure, AC represents W; BC represents W, and CO represents the tension of the string TC.

Then if the angle $BCA = 2\theta$, we have $CO = 2AC\cos\theta$.

Therefore the tension of CT is $2W\cos\theta$.

Next, suppose the weight of the pulley taken into consideration.

In this case there are *four* forces acting on the pulley: (i) its weight, (ii) and (iii) the two equal tensions of the strings PB, WA; (iv) the tension of the string TC. If the first three forces are given we can by Art. 91 find the magnitude and direction of the tension of the string TC.

164. In the simpler problems on pulleys, the blocks are usually so arranged that the strings are all vertical; in which case the problem is one on parallel forces only.

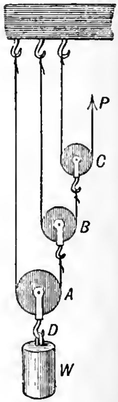
There are three arrangements of pulleys usually explained in elementary text-books on Statics. They consist of separate blocks so arranged that the tension of one string caused by a force called the **Power**, supports a larger vertical force called the **Weight.**

We shall consider the strings to be all vertical.

165. *PROP. To find the relation between the* Power *and the* Weight *in a system of pulleys, in which each pulley hangs in the loop of a separate string, one end of which is fastened to a fixed beam; all the strings being parallel.*

[This system is called THE FIRST SYSTEM OF PULLEYS.]

The system is shewn in the figure.



First, neglect the weight of each pulley.

Let P be the Power, and W the Weight.

Let A, B, C... be the pulleys.

Let T_1, T_2, T_3... be the tensions of the strings which go round the pulleys A, B, C... respectively.

Then, the forces on the pulley A are three parallel forces, W downwards and T_1 and T_1 upwards;

$$\therefore \; 2T_1 = W.$$

The forces acting on the pulley B are three parallel forces, T_1 downwards and T_2 and T_2 upwards;

$$\therefore \; 2T_2 = T_1.$$

Similarly, $$2T_3 = T_2,$$
and so on.

$$\therefore \; W = 2T_1 = 4T_2 = 8T_3 = \ldots = 2^n T_n.$$

And when there are n pulleys $P = T_n$;

therefore $$W = 2^n T_n = 2^n P;$$

or $$P = \frac{W}{2^n}.$$

Next, let the weights of the pulleys A, B, C... be w_1, w_2, w_3 ....

Then, if there are n pulleys, in order to support the pulley A, the power must be increased by $\frac{w_1}{2^n}$;

to support the pulley B, the power must be increased by

$$\frac{w_2}{2^{n-1}};$$

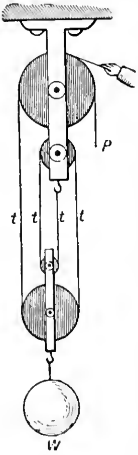
and so on.

Hence, the power P^1 necessary to support both W and the n pulleys is given by

$$P^1 = \frac{W}{2^n} + \frac{w_1}{2^n} + \frac{w_2}{2^{n-1}} + \ldots + \frac{w_{n-1}}{2^2} + \frac{w_n}{2}.$$

166. *PROP. To find the relation between the* Power *and the* Weight *in the system of pulleys in which all the pulleys are arranged in two blocks, one block fixed, the other moveable; the same string going round all the pulleys, and all the strings being parallel.*

[This system is called THE SECOND SYSTEM OF PULLEYS.]



The figure gives two pulleys to each block.

Since the *same* string goes round each pulley, the tension of every portion of the string is the same. Let t be the tension of the string. Then $t = P$ (the Power). Let W be the Weight, and let w be the weight of the lower block. The equilibrium would not be changed if the lower block, the weight, and the strings touching it were all to be considered as one rigid body. This rigid body is acted on by four equal parallel forces, each equal to t acting vertically upwards, and by its weight $W + w$ acting vertically downwards;

$$\therefore \; W + w = 4t = 4P.$$

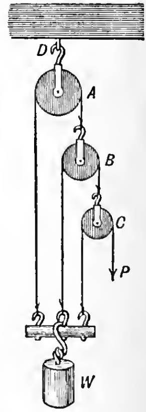
Similarly, when there are a number of pulleys so arranged that n strings pass upwards from the lower block, we have

$$W + w = nP.$$

NOTE.—In practice the blocks are usually made so that the pulleys are all side by side. In this case the strings cannot be all exactly parallel. When the distance between the blocks is not very small, the angle between any two strings will be very small, and the results obtained from considering the strings parallel will be sensibly correct.

167. *PROP. To find the relation between the* Power *and the* Weight *in a system of pulleys in which each pulley is supported by a separate string, which, after passing over the pulley above it, has its other end fastened to the weight; all the strings being parallel.*

[This system is called THE THIRD SYSTEM OF PULLEYS.]



The figure gives a system of three pulleys.

Let P be the Power and W the Weight.

First, neglect the weights of the pulleys.

Let T_1, T_2, T_3,... be the tensions of the strings passing round A, B, C... respectively.

Let T be the strain on the hook at D from which the pulley A is suspended.

Then, considering the whole system as a single rigid body, the forces acting upon it are the pull T vertically upwards and W and P vertically downwards;

$$\therefore\ T = W + P.$$

But $T = 2T_1$, $T_1 = 2T_2$, $T_2 = 2T_3$, and so on, as in the First System.

∴ when there are n pulleys,

$$T = 2^n T_n = 2^n P;$$

$$\therefore\ 2^n P = W + P;$$

$$2^n P - P = W,$$

or

$$(2^n - 1)P = W;$$

$$\therefore\ P = \frac{W}{2^n - 1}.$$

Next, let the weights of the pulleys B, C ... be w_2, w_3, ...

Then the *Weight* which can be supported by the weight of w_2 is $(2-1)w_2$;

the *Weight* which can be supported by the weight of w_3 is $(2^2 - 1)w_3$; and so on.

For W_3 takes the position of P in a system of two pulleys.

Hence we have the relation between P and W' (the Weight which can be supported by weights of the pulleys and P together)

$$W' = (2-1)w_2 + (2^2 - 1)w_3 + \ldots + (2^{n-1} - 1)w_n + (2^n - 1)P.$$

168. The student should notice that the First and Third Systems of Pulleys are practically the same; the one is simply the other inverted.

If the page be inverted in either case this statement will be evident.

169. This System (I. and III.) is not much used for the raising of weights; it is too complicated; it is however often used on board ships where it is necessary to maintain a considerable tension on a certain rope.

The Second System of Pulleys is much used for the raising of weights in conjunction with the wheel and axle. The spare rope as it is pulled in from the pulleys is conveniently coiled on the wheel.

170. The **mechanical advantage** of a system of pulleys is measured by the ratio of the Weight to the Power.

It must not be supposed that any more *work* is done on the Weight than is done by the Power.

In the first system, in which (neglecting the weights of the pulleys) $W = 2^n P$, it will be found that for W to ascend 1 foot, P must descend 2^n feet.

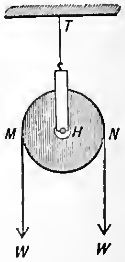
Therefore $W \times$ (distance ascended by W) $= P \times$ (distance descended by P).

This is an instance of the principle that no machine can produce more work than is put into it from outside.

171. We have assumed that when a string passes round a pulley, the tensions of the parts of the string on opposite sides of the pulley are equal. This will be strictly true when the pulley is at rest, provided there is *no friction* at the axle of the pulley and the pulley is perfectly *circular*.

For, consider the forces acting on the pulley (not the block) in the figure; the forces all pass through the centre of the wheel except the two tensions.

The tensions may be supposed to be applied to the wheel at M and N, for equilibrium would not be disturbed if the string were to become fastened to the wheel at M and N.



Taking moments about the centre of the wheel we have by III. Art. 91, one tension $\times$ radius of wheel $=$ the other tension $\times$ radius of the wheel; therefore the tensions are equal provided the radii of the wheel are all equal.

EXAMPLES. XXIV.

1. In a system of 4 pulleys of Art. 165, a weight of 16 lbs. is suspended from the lowest pulley, what is the power, the weight of the pulleys being neglected?

2. In a system of pulleys in which each pulley hangs by a separate string, the power is 10 lbs.; what is the weight if there are 3 pulleys whose weight may be neglected?

3. In the First System of Pulleys, $P = 28$ lbs. and $W = 16$ cwt.; find n.

4. In a system of three pulleys in which the string which passes round any pulley has one extremity fixed and the other fastened to the pulley next above it, a weight of 12 lbs. is suspended from the lowest pulley, which weighs 4 lbs.; the next pulley weighs 2 lbs., and highest 1 lb.; what weight will maintain equilibrium?

5. In a system of pulleys in which each pulley hangs from a fixed point by a separate string there are two pulleys, the upper weighing 1 lb., the lower 2 lbs.; what power would support a weight of 3 cwt.?

6. In the system of pulleys in which each pulley hangs from a fixed support by a separate string, the weights of three moveable pulleys, beginning with the lowest, are 4 lbs., 5 lbs. and 6 lbs. respectively; what weight will a power of 1 cwt. support?

7. How many pulleys each weighing 1 lb. must be used in a system in which each pulley has a separate string, one end of which is fastened to a fixed support, that a power of 2 lbs. may support a weight of 65 lbs. If the weight be allowed to descend 1 inch, how much will the power ascend?

8. In the system of pulleys of Art. 165, if there are three moveable pulleys, the lowest of which weighs 5 lbs., the middle one 4 lbs. and the highest 3 lbs., and the weight is 7 lbs., find the power.

9. If there be 3 pulleys in a system of Art. 165, and the pulleys each weigh W, find the power necessary to support the system when no weight is attached to the lowest pulley; what weight can be suspended from the lowest pulley when the power just found is doubled?

10. Find the weight of each of 4 equal pulleys in the system of Art. 165 that P may be equal to W when $W = 10$ lbs.: in this system what additional power must be applied in order to support a weight of 1 cwt. instead of 10 lbs.?

11. If the strings in a system of 3 weightless pulleys of Art. 165 are attached to a beam at points 1 foot apart, find the centre of the parallel forces applied by the strings to the beam.

12. A man standing on the floor pulls at the power end of a system of 3 pulleys of the First System; if the weight be four times the weight of the man, what is the pressure of his feet on the floor (neglecting the weights of the pulleys)?

13. If there are 3 pulleys (in the First System) of equal weight, find the weight of each pulley in order that a weight of 56 lbs. attached to the lowest pulley may be supported by a power of 7 lbs. 14 oz.?

14. If in the First System there be three pulleys each of weight w and $W=2P$; find w.

15. In the system of pulleys in which there is only one string and there are 9 pulleys, the lower block containing five pulleys and weighing 10 lbs., what force will support a weight of 1 cwt.?

16. The cable by which Great Paul, the bell weighing 18 tons, was lifted to its place in the Cathedral tower, passed four times through each of two blocks of pulleys; find the lowest possible breaking strain of the cable.

17. In the system of pulleys in which one string only is used, the string is fastened to the lower block, which weighs 6 lbs. and contains 3 pulleys, what weight will a power of 10 lbs. support?

18. If in the system of the last Question the weight is 1 ton and the end of the string to which the power is usually applied is fastened to the lower block, what will be the tension?

19. What weight can be supported by a force of 2 lbs. by means of a system of 8 pulleys, 4 in one fixed block and 4 in a moveable block, the string passing over all of them when the lower block weighs 1 lb.?

20. In the system of pulleys in which the same string passes round all the pulleys the weight of the moveable block is 5 lbs., and the fixed and moveable block each contain two pulleys; find what weight a power of 1 cwt. will support.

21. Shew that in the system of pulleys in which there is only one string the tension is least when all the strings are parallel; shew also that all the strings cannot be parallel except one only.

22. How many times his own weight can a man raise with two blocks, one containing 4 pulleys and the other 5, each block weighing $\frac{1}{10}$th of the man's weight?

23. A man raises a weight of 1 ton by means of two blocks each containing three pulleys each weighing 10 lbs.; find the pull on the beam from which the upper block is suspended, and the least weight of the man.

24. Shew that in the Second System of Pulleys, unless the ratio of the weight of the lower block to the suspended weight be less than the number of strings in the lower block diminished by unity, there is no mechanical advantage.

Third System.

25. In the system of pulleys in which each pulley has a separate string, one end of which is attached to the weight, if the power be 8 lbs. and the weight of the pulleys neglected, what is the weight when there are three pulleys?

26. If in the Third System there be three pulleys such that the weight of each pulley is equal to the power, shew that the power will support a weight 11 times as great as itself.

27. Shew that in a Third System of two pulleys, if each pulley weighs 1 lb., a power of 3 lbs. supports a weight of 10 lbs.

28. In a Third System of pulleys, if the weights of the pulleys are 1 lb., 2 lbs., 3 lbs. respectively, find the greatest weight and the least weight which can be kept in equilibrium by a power of 7 lbs., it being understood that the pulleys may be arranged in any order.

29. In the Third System if the weight be 2 cwt. 1 lb., and the power be 15 lbs. (neglecting the weight of the pulleys), what is the number of the pulleys?

30. In the system of pulleys in which each string is attached to the weight, there are three moveable pulleys of weights w_1, w_2, w_3, beginning with the lowest, and the force P then balances a weight W; when the first and second pulleys are interchanged then a force P' balances W; shew that $P' - P = \frac{4}{15}(w_1 - w_2)$.

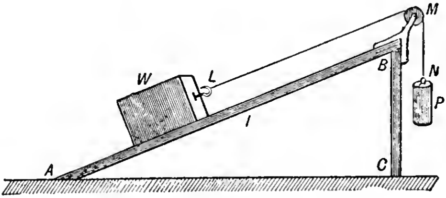
31. There is one system of pulleys in which the weight of the pulleys increases the mechanical advantage; which is the system?

The Smooth Inclined Plane.

172. An **Inclined Plane** is a rigid plane making a finite angle with the horizon.

In what follows we consider the inclined plane to be a machine in which all the forces acting are in one plane; this plane is a vertical plane perpendicular to the intersection of the inclined plane with the horizon, and therefore cutting the inclined plane in a *line of greatest slope.*

173. The representation on paper of an inclined plane is the section made with it by the vertical plane in which all the forces are supposed to act.



Thus in the figure AB is a line of greatest slope of the plane.

An inclined plane is often supposed to be of a finite **length** AB, in which case the lines BC and AC being drawn vertically and horizontally through B and A, BC is called the **height** and AC is called the **base.**

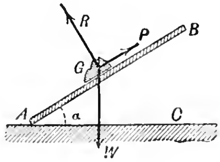
174. In the present Chapter we shall suppose the inclined plane perfectly *smooth;* so that it can only apply to a body a pressure in the direction perpendicular to its surface. [See Art. 203.]

175. *To find the relation between the* Power *and the* Weight, *when the Power is parallel to the plane.*

Let α be the inclination of the plane to the horizon.

The Weight W is the weight of a body placed on the plane acting at G its Centre of Gravity.

The Power P is a force applied to the body by a string or otherwise up the plane parallel to it.



Let R be the pressure of the plane on the body.

R is therefore a force perpendicular to the plane acting on the body through its Centre of Gravity.

These three forces W, P, R which act on the body are in equilibrium.

Taking the sum of the resolutes along the plane, of all the forces,

we have $$W \sin \alpha - P = 0 \qquad \text{(i)}.$$

Taking the sum of the resolutes perpendicular to the plane, of all the forces,

we have $$W \cos \alpha - R = 0 \qquad \text{(ii)}.$$

Example. The height of an inclined plane is to its base as 5 : 12 : *find the* Power P *parallel to the plane which will support a Weight of* 13 cwt.

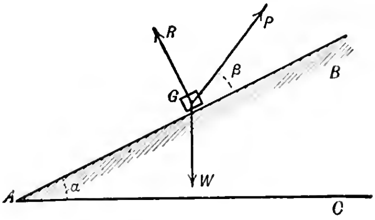
Let the height CB of the plane = 5 m.; then the base AC = 12 m. and the length AB is $\sqrt{(12^2+5^2)}$ m., that is 13 m. [See fig. on p. 154.]

Now P is equal to the resolute of W along the plane; that is,

$$P = W \sin \alpha = \tfrac{5}{13} W;$$

$$\therefore\ P = \tfrac{5}{13} \text{ of } 13 \text{ cwt.} = \underline{5 \text{ cwt.}}$$

176. *PROP. To find the relation between the* Power *and the* Weight, *when the Power makes any given angle* β *with the inclined plane.*



Let W be the weight of the body on the plane; let P be the Power; and let R be the pressure of the plane on the body.

Let α be the inclination of the plane to the horizon.

Then the forces W, P, R are in equilibrium, therefore the sum of their resolutes parallel to the plane is zero.

$$\therefore \quad W \sin\alpha - P\cos\beta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

The sum of their resolutes perpendicular to the plane is zero.

$$\therefore \quad W\cos\alpha - R - P\sin\beta = 0 \quad \ldots\ldots\ldots\ldots \text{(ii)}.$$

Hence $$P = W\frac{\sin\alpha}{\cos\beta},$$

and $$R = W\cos\alpha - P\sin\beta$$

$$= W\cos\alpha - W\frac{\sin\alpha\sin\beta}{\cos\beta}$$

$$= W\frac{\cos(\alpha+\beta)}{\cos\beta}.$$

NOTE. **For instance**; Let P act horizontally, then $\beta = -\alpha$

and $$P = W\tan\alpha,$$

$$R = W\sec\alpha.$$

EXAMPLES. XXV.

1. Prove that when a body is kept in equilibrium on a smooth inclined plane and the power acts along the plane, the Power is to the Weight as the height of the plane is to the length.

2. When a body is kept in equilibrium on an inclined plane and the power is horizontal, then the Power is to the Weight as the height of the plane is to the base.

3. If the inclination to the horizon of a plane be 60°, find the force which acting horizontally would support a weight of 12 lbs.

4. A weight of 12 lbs. resting on a smooth plane at an angle of 30° to the horizon is fastened to a cord which passes up the plane and over the top; find what weight must be attached to the cord to preserve equilibrium.

5. Find the inclination of a plane on which a horizontal force will support a weight equal to itself.

6. Find the inclination of a plane on which a power parallel to the plane will support double its own weight.

7. The angle of an inclined plane is 30°, and a force P acting horizontally keeps a weight in equilibrium; if P acts in a direction making the angle 30° with the plane and above it, shew that it will still maintain equilibrium and that the pressure on the plane will be reduced one half.

8. AB, AC are two smooth planes inclined to the horizon at 60° and 30° respectively; a weight P on AB and a weight Q on AC are connected by a string which passes over a pulley at A. If P and Q are in equilibrium, what is the ratio of their weights?

9. An inclined plane makes the angle 30° with the horizon, and a weight W is supported on it by a force P such that $2P^2 = W^2$; in what direction does P act?

10. An inclined plane makes the angle 45° with the horizon and $2W^2 = 3P^2$; in what direction does P act?

11. The Powers, which when acting horizontally and parallel to the plane respectively will support a given weight, are in the ratio of 2 to 1; what is the angle of the plane?

12. A body weighing 9 lbs. is in equilibrium upon an inclined plane under the action of a horizontal force of $3\sqrt{3}$ lbs.; what is the inclination of the plane and the pressure on it?

13. A heavy weight is fastened by means of a string to a fixed point, and rests on a smooth plane inclined to the horizon at the angle 30°, the direction of the string making the angle 60° with the horizon; shew that the tension of the string is equal to the pressure on the plane.

14. $W : P : R = \sqrt{3} : 1 : 1$; find the inclination of the plane and the direction of P.

15. A weight $2P$ is kept in equilibrium on an inclined plane by a horizontal force P and a force P acting parallel to the plane; find the ratio of the base of the plane to the height and the pressure on the plane.

16. Two planes AB, AC having a common height are inclined to the horizon at angles α and β respectively. Two weights, one on each plane, are kept in equilibrium by a string attached to the weights and passing over A. Find the ratio of the weights.

17. A power P acting parallel to the plane can support W, and acting horizontally can support W'; prove that

$$P^2 = W^2 - W'^2.$$

18. A railway train of 160 tons is supported on an incline of 1 in 80 by means of a rope parallel to the plane; find the tension of the rope.

19. If R be the pressure when P acts horizontally, and R' when it acts parallel to the plane, then $RR' = W^2$.

20. If a horizontal power P supports W, and P' parallel to the plane supports W, then $\frac{1}{P'^2} - \frac{1}{P^2} = \frac{1}{W^2}$.

21. A force P acting along a given plane can support a weight W_1, and acting horizontally can support a weight W_2; prove that $P^2 = W_1^2 - W_2^2$.

22. W can be supported on a given smooth inclined plane by the force P along the plane, or by the horizontal force $2P$; if R_1, R_2 be the pressure on the plane in these two cases, prove that $R_1 R_2 = \frac{4}{3} P^2$.

23. For a given value of P and a given value of W shew that there are generally two inclinations in which P may act so as to support W on a given smooth inclined plane.

24. What weight can be supported on a smooth inclined plane of angle 45° by a horizontal force of 3 lbs. and a force of 4 lbs. parallel to the plane acting together?

25. Three forces of 9 lbs., 8 lbs. and 10 lbs. respectively support W on a smooth plane which rises 1 foot in 20 feet measured along the plane; the forces make with the plane upwards angles whose cosines are $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$; shew that

$$W = 140 \text{ lbs.}$$

26. From the foot of a smooth plane inclined at the angle 45° to the vertical rises a smooth vertical wall; find in what position a plank must be placed so that it may rest in equilibrium against the two surfaces.

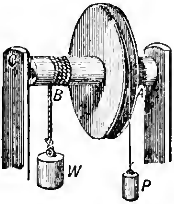
27. If the weight, power and pressure on a smooth inclined plane are in the ratio $\sqrt{3} : 1 : 1$, prove that the inclination of the plane is 30°.

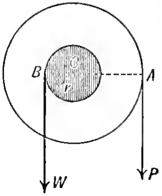
28. Prove that when two weights, placed one on each of two inclined planes having a common vertex and connected by a string, are in equilibrium, then if they are moved in their planes, the string being taut, their common C. G. moves in a horizontal line.

The Wheel and Axle.

177. In this machine the **axle** is a *cylinder* with a fixed *axis*, and the **wheel** is fixed to the cylinder and turns about the same axis.

Thus in the figure, a rope coiled round the cylinder B supports a weight W; and a cord coiled round the wheel in the opposite direction supports a weight or force P.





178. The forces acting on the rigid body composing the wheel and axle are not all in one plane, nor are they symmetrical about any one plane; therefore their theory is not properly within the scope of this work.

We shall assume however that the force applied by the rope to the axle at B acts just as if it were in the plane of the wheel; as in the second figure.

With this assumption the theory is simply that of a rigid body acted on by forces in one plane with one point (the centre of the wheel), fixed.

179. *PROP. To find the relation between the* Power *and the* Weight *in the Wheel and Axle.*

Let the plane of the paper be the central plane of the wheel and let C be the centre of the circle in which the wheel cuts that plane.

Let the cord passing round the wheel touch this circle at A, and let the rope passing round the axle touch the axle in B.

Let R be the radius of the wheel and r the radius of the axle.

Then we have by Art. 91, III (the only necessary condition of equilibrium, since the point C is fixed)

$$W \times r = P \times R,$$

or $W \times$ radius of axle $= P \times$ radius of wheel.

180. It will be seen that the practical effect of the arrangement of the wheel and axle is, that it is a kind of continuous lever of the first class—the axis is the fulcrum; the radius of the wheel is the longer arm; the radius of the axle is the shorter arm of the lever.

The mechanical advantage [Art. 170] of this machine is

$$\frac{\text{radius of wheel}}{\text{radius of axle}}$$

and this may theoretically be made as large as we please; but practically its magnitude is limited (i) by the fact that a very large wheel is costly and unwieldy, (ii) a very small axle cannot be made of sufficient strength to sustain any very great strain. See Examples **XXVI.** 10, 11.

EXAMPLES. XXVI.

1. Shew that the power may be applied to the wheel in any direction provided its magnitude is unchanged and provided it is kept always perpendicular to the axis.

2. Shew that if the weight is n times the power in the wheel and axle, then the cords round the wheel must be unwound n feet in order to wind the rope 1 foot round the axle.

3. Four sailors each exerting a force capable of supporting 116 lbs., lift an anchor by means of a capstan, whose radius is 1 ft. 2 in. and whose spokes are 8 feet long (measured from the axis); what is the weight of the anchor?

4. To raise a block of stone of 1 ton a single pulley is used at the top and a windlass worked by two men at the bottom; if the crank of the windlass is 2 feet and the radius of the axle 4 inches what force perpendicular to the cranks must each man exert at the least? Does it make any difference if the direction of the chain passing from the pulley to the windlass is not vertical?

5. If the string from an axle of radius 3 inches passes round a moveable pulley and has its end fixed to a beam above (that part of the string which does not touch the axle or pulley being vertical) and if a weight of 2 tons (fastened to the moveable pulley) is supported by a power of 1 cwt. (applied to the wheel), find the radius of the wheel.

6. The radius of the wheel is 6 feet, the radius of the axle 6 inches; a weight of 8 lbs. is fastened to a rope coiled round the axle and a weight of 4 lbs. is fastened to a rope nailed to the rim of the wheel; find the position of equilibrium.

7. In what direction must the power act in order that the pressure on the axle may be (i) the least possible, (ii) the greatest possible?

8. If the axis of the axle do not coincide with the centre of the wheel, shew that the sum of the weights which are the greatest and least with which a given power can be in equilibrium is double the weight which this power would support if the axis of the same wheel did coincide with the axis of the axle.

9. A wheel is made square, the side of the square being 2 ft.; the axle is round and its radius is 3 inches; find the position which the machine will assume when the weight is 5 lbs. and the power 1 lb.

10. To the same axis are fixed a wheel of radius R, a smaller wheel of radius r, and an axle of radius r_2. A rope is coiled round the axle and after passing over a single moveable pulley is coiled the other way round the smaller wheel. The power P acts in the proper direction on a string coiled round the larger wheel and the weight W is attached to the moveable pulley; prove that $$W = P \times \frac{2R}{r_1 - r_2}.$$

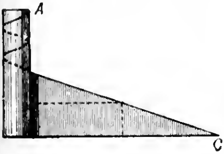
The above arrangement is called the **Differential Wheel and Axle.**

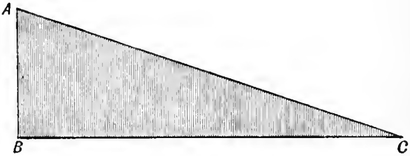
11. Shew that the mechanical advantage of the Differential Wheel and Axle can be made as large as we please without unduly either enlarging the wheel or diminishing the axle.

THE SCREW.

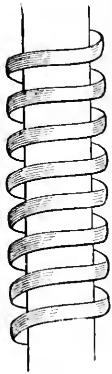
181. The **screw** may be described as an *inclined plane* wound round a cylinder.

Take a piece of paper in the form of a right-angled triangle ABC as in the figure and wind it round a cylindrical pencil; the slant side of the triangle will give the trace of a screw on the surface of the pencil.





182. A screw is acted on by a couple which tends to make it turn about its axis. The thread of the screw is fitted into a counter part so that as the screw turns, the cylinder on which the screw is traced moves in the direction of its axis.



If we suppose the screw perfectly smooth, and assume the principle of Work (viz. that what is lost in force is gained in distance moved), we can find the relation between P and W.

183. Let a force P be applied at the end of a lever, whose arm is l inches long, so as to cause the screw to tend to turn; and suppose that this tendency to turn is counteracted by a pressure W against the head of the screw in the line of its axis, opposing its consequent tendency to move parallel to the axis.

Now suppose the distance between the threads of the

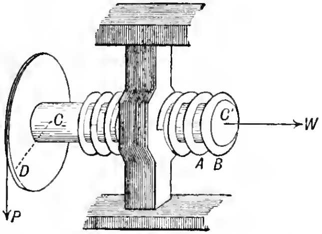
screw to be n inches; then for each complete turn of the screw the axis would move forward n inches;
also, if the axis were vertical and the pressure against the screw were caused by a weight W resting on it, each complete turn of the screw would raise W a height of n inches.

But in doing this, the force P, applied at right angles to the lever, would have to work through the whole circumference of the circle of radius l inches; that is, it would have to work through the distance $2\pi l$ inches.

Therefore, by the Principle of Work, viz. that in a machine in which the constraints are all *smooth*, the work done on the machine is equal to the work done by the machine, we have

$$P \times 2\pi l = W \times n.$$

In practice a great deal of this work $P2\pi l$, even under the most favourable circumstances, is consumed in overcoming the friction of the screw; so that in practice Wn is considerably less than $2P\pi l$.



The figure here given shews a combination of wheel and screw which will help the student to understand, that while W works through the distance between two threads of the screw, the distance which P works through is the circumference of the wheel.

Suppose a cord coiled round the wheel, and a weight of P lbs. attached to it; for each complete turn of the screw, this weight P would descend a distance equal to the circumference of the wheel; that is, it would descend about twice $\frac{22}{7}$ths of the radius ($2\pi l$).

In the figure $CD=l$, $AB=n$.

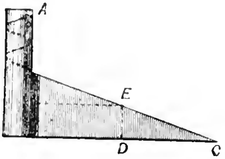
The work done by P is thus $2\pi l \times P$.

The work done on $W = W \times$ distance between two threads $= n \times W$.

By the principle of the indestructibility of work [see *Dynamics*, page 165] (since the pressures of the constraints do no work) we have

$$2\pi l \times P = n \times W.$$

184. The **pitch** of a screw is the angle of inclination of the inclined plane which when wound round a cylinder forms the screw. Thus in the figure the angle ECD is the



pitch. If DC be the circumference of the cylinder then DE is the distance between two consecutive threads, and

$$DE = DC \tan i,$$

where i is the *pitch* of the screw.

Hence, the distance between the threads of a screw = the circumference of the screw × tan i.

The following is a static proof of the proposition:

In the screw the ratio of the power to the weight is equal to the ratio of the distance between two consecutive threads of the screw to the circumference of the circle described by P.

A vertical screw may be regarded as a combination of

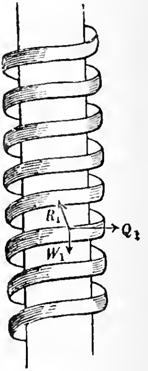
(i) **an inclined plane** whose inclination to the horizon is equal to i the pitch of the screw, and

(ii) **a wheel and axle** in which the radius of the wheel is that of the circle described by P and the radius of the axle is the radius of the screw. We shall denote by Q

the force which acting at the extremity of the radius of the screw balances the force P.

So that $Q \times$ circumference of the screw
$= P \times$ circumference of circle described by P...(ii).

The pressure of the companion screw on the screw itself may be divided up into small portions R_1, R_2, etc. corresponding to small parts of the upper surface of the screw.



Each of these reactions R_1, R_2,... is in equilibrium with a portion W_1, W_2, etc. of W and a portion Q_1, Q_2, etc. of the force Q.

Hence, Q_1 is a horizontal force supporting W_1 on an inclined plane of angle i.

Therefore by Art. 176 note,

$$Q_1 = W_1 \tan i;$$

similarly $$Q_2 = W_2 \tan i,$$

and so on.

Hence

$$Q = (Q_1 + Q_2 + \ldots) = (W_1 + W_2 + \ldots) \tan i$$
$$= W \tan i \ldots\ldots\ldots\ldots \text{(i)}.$$

But, $P \times$ circumference of its circle

$= Q \times$ circumference of the screw ....(ii)

$= W \tan i \times$ (circumference of screw),

and the circumference of the screw $\times \tan i$

= the distance between the threads [from above];

$\therefore$ $P \times$ circumference of its circle

$= W \times$ distance between the threads.

Example. What must be the length of the power arm of a screw having 6 threads to an inch that its mechanical efficiency may be 216?

The mechanical efficiency of a machine is the ratio of the Weight to the Power.

In the screw $P \times 2\pi$ times the power arm $= W \times$ distance between the threads;

$$\therefore \; 2\pi \text{ times the power arm} = 216 \times \tfrac{1}{6} \text{ of an inch.}$$

Take π as $= \frac{22}{7}$; then

$$\text{the power arm} = \frac{216 \times 7}{6 \times 44} \text{ inches} = 5{\cdot}73\ldots \text{inches} = \underline{5\tfrac{3}{4} \text{ inches.}}$$

EXAMPLES. XXVII.

1. In a Screw the circumference of the circle described by the extremity of the power arm is 2 feet and the distance between the threads $\frac{1}{2}$ in.; what weight will a power of 28 lbs. support?

2. If in a Screw the power arm is 2 ft. long and the distance between the threads 1 inch, what power will support 1 ton?

3. If the mechanical advantage of a Screw is 1000 and the length of the power arm is 25 inches, what is the distance between the threads?

4. If the extremity of the power arm describes a circle of 10 ft. and a force of 1 lb. supports 1 ton, what is the distance between the threads of the Screw?

CHAPTER XIV.

The Transmissibility of Force.

185. The conditions of Art. 91 for the equilibrium of forces acting on a single rigid body shew, that when *two forces* act on a rigid body, these two forces are in equilibrium provided they are *equal*, *opposite*, and *in the same line of action.* They may each of them be *applied* to the rigid body *at any point whatever*, provided the points of application are in their line of action.

Therefore, *the statical effect of a force on a* **rigid** *body is not altered when its point of application is transferred from one point to any other point in its line of action.*

The force must be applied to the rigid body; but we may, if we please, suppose the extent of the rigid body to be increased by a system of light rigid rods, so as to include points outside it.

The above result is usually referred to as
the Principle of the **Transmissibility of Force.**

The student must notice that Force is transmitted through a rigid body by the internal stresses which are set up by the force; and, although it makes no difference to the external effect of a force on a '*rigid*' body, at what point in its line of action it is applied, it *does* make a difference in the internal stresses which transmit the force.

186. In order to avoid any reference to the Dynamical Proof of the Parallelogram of Forces, many treatises on Statics assume the truth of the above principle and deduce the parallelogram of forces from it.

187. The usual proof is given here, as it is interesting historically, and is a very ingenious piece of inductive reasoning. The student of Statics is recommended, when he has satisfied himself of its logical correctness, not to bestow further thought upon it; it illustrates no mechanical principle—its interest belongs to the domain of history and of pure mathematics.

188. The proof depends (i) on the Principle of the Transmissibility of force and (ii) on the assumption that the resultant of two *equal* forces acting at a point *bisects* the angle between them.

This assumption is justified by the remark that two equal forces are *symmetrical* about the line bisecting the angle between them—so that if they had a resultant on one side of this bisecting line, they would also have another resultant on the other side of this line; in which case two forces would have two separate resultants; which is impossible.

Now, when a parallelogram has two equal adjacent sides, its diagonal bisects the angle between them.

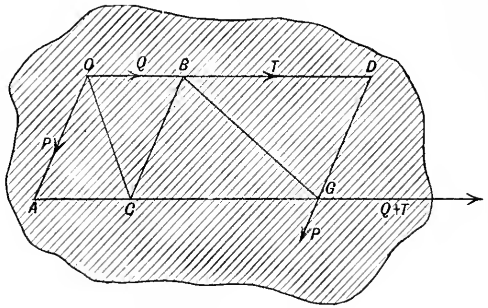
It follows therefore that the parallelogram of forces is true as far as regards the *direction* of the resultant, when the two forces are *equal.*

189. We proceed to shew that the parallelogram of forces is true as far as regards the **direction** of the resultant for *any* two forces.

Proof. Assume that the proposition is true (as regards *direction* only) for two chosen forces P and Q;

Assume also that the proposition is true (as regards *direction* only) for the force P and another chosen force T;

We proceed to shew from the principle of the transmissibility of force that on these assumptions it is true for the forces P and $(Q+T)$.



Let the forces be applied to a rigid body.

Let O be the point of application of the forces.

Let the force P act along OA and let Q and T act along OB.

Let OA, OB represent P and Q in magnitude; produce OB to D, so that BD represents T in magnitude. The force T, which is applied at O, may be supposed (without altering its statical effect) to be applied to the rigid body at any point in the line OBD; [Art. 185.] we shall suppose that it is applied at B.

Complete the parallelogram $AOBC$.

The resultant of P and Q, by our assumption, *acts in the line* OC.

Also, by the principle of transmissibility, this resultant may be supposed to be applied to the rigid body at any point in OC.

Let it be applied at C.

This force, which is the resultant of P and Q, is now applied at C; but we should get the same effect if we replaced this resultant by two forces equal and parallel to P and Q, acting on the rigid body at C. Let this be done.

The forces now acting on the rigid body are P and Q at C, (their lines of action being BC and AC), and T at B, (its line of action being BD).

But P at C may be applied to the rigid body at any point in its line of action; let it be applied at B. It is then represented in direction and magnitude by BC. Complete the parallelogram $CBDG$.

Then, by our assumption concerning P and T, BG is the line of action of the resultant of P and T; and, as before, this resultant may be supposed to be applied to the rigid body at any point in BG.

Suppose it applied at G; and then replaced by its two components P and T, which are thus now applied at G.

Q is applied at C and its line of action is CG; it may therefore be considered to be applied to the rigid body at G.

Thus, finally, our three forces P, Q and T, originally applied to the rigid body at O, are shewn to have the same statical effect on the rigid body when they are supposed to act at G instead of at O.

But suppose we replaced P, Q, T when acting at O by their resultant; and also when acting at G.

This resultant can be supposed to be applied at any point in its own line—*and at no point not on that line.*

But we have shewn that the resultant may be applied at G; $\therefore$ the line OG must be the line of the resultant of P, Q, T acting at O.

But OG is the diagonal of the parallelogram whose sides represent P and $Q+T$.

Hence we have this **result**; if the proposition be true as regards *direction* for two chosen forces P and Q, and also for P and another chosen force T, then it must be true as regards *direction*, for the forces P and $(Q+T)$.

From this result we proceed to deduce the truth of the parallelogram of forces as far as the *direction* of the resultant is concerned.

First, choose $Q=P$ and $T=P$.

We know the proposition to be true for two equal forces P and P, and for P and P; therefore by the above result it must be true for P and $(P+P)$; that is, it is true for P and $2P$.

Next (making use of the above result), choose $Q=2P$, and $T=P$.

Therefore it is true for P and $(2P+P)$; **and so on.**

Therefore it is true for P and nP.

As before, we can now shew it to be true for $2P$ and nP, and hence for mP and nP.

Therefore it is true for any two commensurable forces.

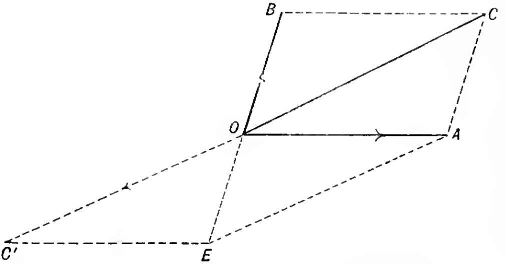
It must also be true for two incommensurable forces; for we can find two commensurable forces which approximate as closely as we please to the two incommensurable forces and the proposition is true for these last two forces.

Therefore the proposition is true generally.

190. The above proof of the Parallelogram of Forces as far as regards direction is known as **Duchayla's** Proof. We proceed to shew that if the proposition be true for direction it must be true for magnitude, and conversely. These proofs depend only on the principle that when three forces acting at a point are in equilibrium, any one of them is the anti-resultant of the other two.

191. *PROP. Assuming that the diagonal* OC *of the parallelogram* OACB *indicates the* direction *of the resultant of any two forces* OA, OB, *prove that it must also represent the* magnitude *of their resultant.*

Draw OC' to represent in magnitude and direction the anti-resultant of OA, OB.



Then, by hypothesis, *COC' is a straight line.*

Also, OA, OB, OC' represent three forces in equilibrium; so that any one of them, OB, is the anti-resultant of the other two, OA, OC'.

Complete the parallelogram $AOC'E$;

Then, by hypothesis, OE represents the *direction* of the resultant of OA, OC'.

Therefore EOB is a straight line;

But OB is parallel to AC, by construction.

Therefore EO is parallel to AC.

Also, EA was drawn parallel to $C'O$, and therefore to OC.

Thus, $AEOC'$ is a four-sided figure, whose opposite sides are parallel; therefore its opposite sides are equal.

Therefore $OC = EA$; and EA is equal to $C'O$, the opposite side in the parallelogram $AOC''E$.

Therefore OC is *equal* to CO, the anti-resultant of OA, OB. That is, OC represents the *magnitude* of the resultant of OA, OB. Q. E. D.

192. *PROP. Assuming that the diagonal* OC *of the parallelogram* AOCB *represents the* magnitude *of the resultant of any two forces* OA, OB, *prove that it must also indicate the* direction *of their resultant.*

Draw OC' to represent in direction and magnitude the anti-resultant of OA, OB. [See Figure on p. 172.]

Then, by hypothesis, $OC' = OC$.

Also OA, OB, OC' represent three forces in equilibrium, so that any one of them, OB, is the anti-resultant of the other two, OA, OC'.

Complete the parallelogram $AOC'E$.

Then, by hypothesis, OE represents the *magnitude* of the resultant of OA, OC'.

Therefore $EO = OB$;

But $OB = AC$, by construction.

Therefore $EO = AC'$.

Also, $AE = OC'$ (the opposite side of the parallelogram $AOC'E$) and $OC' = CO$.

Therefore $AE = CO$.

Thus, $AEOC$ is a four-sided figure whose opposite sides are equal; therefore it is a parallelogram.

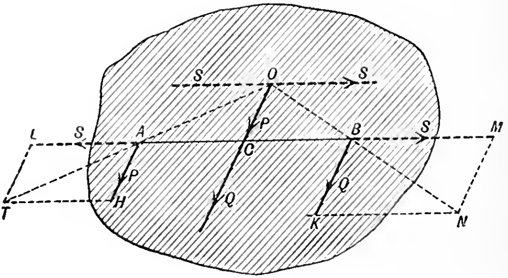
Therefore OC is parallel to EA, which was drawn parallel to $C'O$.

Therefore OC is in *the same straight line* with OC', the anti-resultant of OA and OB. That is, OC represents the *direction* of the resultant of OA, OB. Q. E. D.

193. We shall now prove by the principle of the Transmissibility of Force the result of Art. 96.

194. *To find the resultant of two like parallel forces acting on a rigid body.*

Let A, B be the points of application of the two parallel forces P and Q; let AH, AK represent them in direction and magnitude.



Let two equal and opposite forces each of magnitude S be applied to the rigid body along the line AB.

Let AL represent one force S and BM represent the other force S.

Since the body is rigid these two forces will not alter the external effect of the forces P and Q. [Art. 185.]

Complete the parallelograms $HALT$, $KBMN$.

Then, since the angles HAB, KBA are together equal to two right angles, the angles TAB, NBA are together greater than two right angles. Therefore TA and NB will if produced meet on the other side of AB.

Let them meet at O.

Draw OC parallel to AH or BK, to cut AB in C.

Now the two forces AH, AL may be replaced by the force AT; which may be supposed to act at O; and may there be replaced by two forces equal and parallel to AH and AL; that is, to P and S.

Similarly, the two forces BK, BM may be replaced by the force BN; which may be supposed to act at O; and may there be replaced by two forces equal and parallel to BK, BM; that is, to Q and S.

Thus finally, instead of the two forces P, Q applied to the rigid body at A and B respectively, we now have the four forces applied at O; namely two equal and opposite forces each equal to S, and two like forces P and Q acting in the line OC; the two forces each equal to S are in equilibrium and may be omitted.

Hence, the resultant of the two like parallel forces, is a single force acting in the line OC parallel to the two forces, and applied at any point in OC; also its magnitude is $P+Q$.

Now the triangles HAT and COA are similar,

$$\therefore \frac{AH}{HT}=\frac{OC}{CA}; \text{ that is, } \frac{P}{S}=\frac{OC}{CA}.$$

Also the triangles KBN, COB are similar,

$$\therefore \frac{KN}{BK}=\frac{CB}{OC}; \text{ that is } \frac{S}{Q}=\frac{CB}{OC},$$

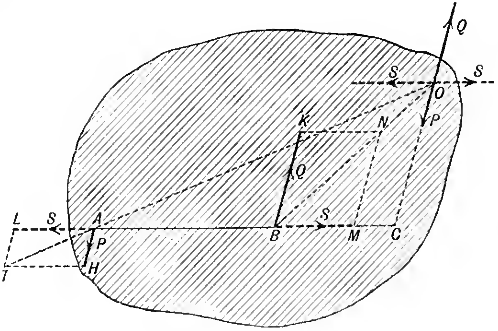
$$\therefore \frac{P}{S}\times\frac{S}{Q}=\frac{OC}{CA}\times\frac{CB}{OC}.$$

Hence, $$\frac{P}{Q}=\frac{CB}{CA}; \text{ that is, } P\times CA=Q\times CB.$$

Q. E. D.

195. *PROP. To find the resultant of two unlike parallel forces acting on a rigid body.*

With a similar construction, the proof is word for word the same as that of the last proposition; except that AT and BN do not in this case *always* intersect; the exception being when $P = Q$.



The resultant will be the '*sum*' of the two forces; which, as the forces are unlike, will *numerically* be equal to their difference, and will act in the direction of the greater force.

The figure given is for the case in which Q is greater than P.

For when Q is greater than P, since $BM = AL$, the angle NBM is greater than LAT; that is, than OAB; so that OBA is correctly drawn as a *triangle*.

When the forces P and Q are equal and unlike the lines AT, NB in the above figure are always parallel, and the construction fails. In fact, in this case P and Q together form a couple and are not capable of a single resultant. In Art. 197 below we shew that a couple can be counteracted by an equal and opposite couple.

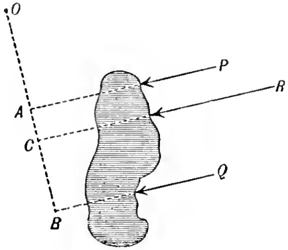
196. *PROP.* *To prove that the sum of the moments of any two parallel forces about any chosen point in their plane is equal to the moment of their resultant about the same point.*

In the figure let P and Q be the forces and R their resultant;

let O be the chosen point.

Draw $OACB$ cutting the lines of action of the forces PQR in A, B and C.

Then $$P \times AC = Q \times CB$$
and $$R = P + Q.$$ [Art. 194.]



Now
$$P \times OA + Q \times OB = P \times (OC - AC) + Q \times (OC + CB)$$
$$= (P + Q) \times OC - P \times AC + Q \times CB$$
$$= R \times OC.$$ Q. E. D.

It follows that the sum of the moments about any chosen point of any number of parallel forces is equal to the moment of their resultant.

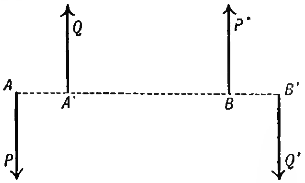
For as far as moments are concerned we may replace any two forces by their resultant and so on.

It also follows that the sum of the moments about any chosen point of any system of parallel forces which are in equilibrium, is zero; for their resultant is itself zero.

197. *PROP. To prove that two couples in the same plane whose moments are equal and opposite are in equilibrium.*

First, let the forces P, P' and Q, Q' of the couples be parallel and equal.

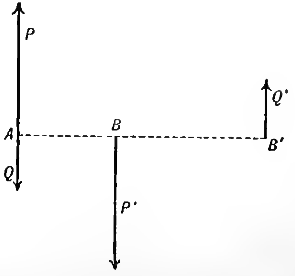
Then, if a line $ABA'B'$ cuts the lines of action of the forces respectively in A, B and A', B' at right angles, since the forces of the couples are equal and their moments equal, it follows that their arms must be equal; $\therefore AB = A'B'$; the resultant of P, Q' ($P + Q'$ acting at C the middle point of AB') is equal and opposite to the resultant $(P' + Q)$ of Q, P' which acts at the middle point of $A'B$, which middle point coincides with that of AB'.



Thus, the couple P, P' acting at A, B is equivalent to an exactly similar couple Q, Q' acting at A', B'.

Hence a couple may be moved parallel to itself from one position to another in its own plane, without altering its effect upon a rigid body.

Next, let the forces P and Q be parallel but unequal.



By what we have just proved, one of the couples P, P' may be transferred parallel to itself to a position in its own plane so that P is in the line of action of Q. Let this be done.

Then the forces at A are equivalent to $P-Q$;

and the forces will be in equilibrium provided

$$P' \times AB = Q' \times AB';$$

that is, provided the moments of the couples are equal.

Hence, a couple may be replaced by any couple in its own plane of equal moment, whose forces are parallel to its own forces.

Lastly, let the couples not have their forces parallel.

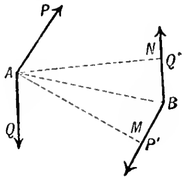
By what we have just proved, one of the couples may be replaced by another of equal moment having its forces parallel, so that the forces P, P' of this new couple are equal to the forces Q, Q' of the second couple.

Let the forces P, Q intersect in A and P', Q' intersect in B.

Draw AN, AM perpendicular to P', Q' respectively.

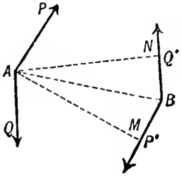
Then, since the moments of the couples are equal,

$\therefore P \times AN = Q \times AM$; but $P = Q$; $\therefore AN = AM$.



Therefore the right-angled triangles ABN, ABM are equal in all respects;

Hence, AB bisects $P'BQ'$, and also bisects the exterior angle PAQ; and $\therefore$ also BA bisects the angle PAQ itself.



Therefore, since $P' = Q'$ and $P = Q$, the resultant of P' and Q' is equal and opposite to the resultant of P and Q.

For the resultants are equal and each of them acts in the line AB bisecting each of the angles PAQ, $P'AQ'$.

That is, the two couples are in equilibrium.

Hence any two couples in the same plane whose moments are equal, have the same effect on a rigid body.

198. The results of this Chapter all follow at once from Art. 91. But we have here given the proofs which are usually given in books on Elementary Statics, since they are often asked for in Examination Papers.

CHAPTER XV.

FRICTION.

199. We propose in this Chapter to consider the nature of the stress set up between the surfaces of two separate *rigid* bodies which press the one on the other.

200. We shall confine our attention to the case of a fixed plane and a body placed upon it.

201. When a *rigid* body has one of its surfaces in contact with a fixed rigid plane, then, so long as the surface remains in contact with the plane, the only motion of which the body is capable is in some direction parallel to the plane.

For, whatever may be the forces acting upon the body, the plane, being rigid, can and does exert upon it whatever force is necessary to prevent motion in the direction perpendicular to the plane.

This force, applied by the plane to the body in the direction perpendicular to its surface, is equal and opposite to the perpendicular **pressure** of the body on the plane.

202. If there were a rigid substance whose surface could be made *perfectly* smooth, then the pressure applied by such a surface to any other surface pressing against it would be *exactly in the direction at right angles to the smooth surface.*

Although no such substance is known, yet it is often convenient to imagine such a substance for the purposes of theoretical statics. For the surfaces of some substances can be rendered very much more smooth than those of others, and hence we obtain results which are approximately true for surfaces which are comparatively smooth.

Moreover, for the purposes of explanation, we at first make our problems as simple as possible; advancing from the simple problems of theory to the more complex problems of practical mechanics.

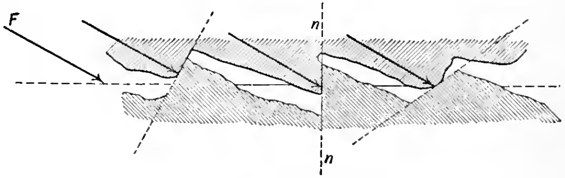
Accordingly we use the words *smooth* and *rough* in the following technical sense.

203. *DEF.* A surface is said to be **smooth** which is understood to be incapable of exerting pressure on any other body in contact with it *except in the direction perpendicular to itself.*

204. *DEF.* A surface is said to be **rough** which is capable of exerting pressure on a body in contact with it *in other directions* besides that perpendicular to itself.

It will be seen that a smooth surface may be said to be one which offers no resistance to the motion along it of a body which is pressing against it.

A rough surface does offer some resistance to the motion along it of a body pressing against it.



For all actual surfaces, even those that appear to be perfectly plane, would if sufficiently magnified be seen to consist of minute projections and depressions as in the Figure. These always interlock more or less with another surface in contact with it, and oppose resistance to a force such as the horizontal component of F which tends to slide one surface over the other.

Sometimes the projections interlock so that the sliding cannot take place without scratching as at n, n in the figure.

205. *DEF.* A **perfectly rough** surface is one which is capable of offering whatever resistance may be necessary to prevent the motion along it of a body pressing against it.

Thus a *perfectly* rough surface is a theoretical surface along which bodies pressing on it cannot move.

206. It is as we have said impossible to get a surface *smooth* as defined above. It must however be understood that when we speak of a 'rough' plane surface we mean a surface which has been made as nearly smooth as the nature of the substance will permit.

For instance, if the material is of *metal* it is understood that the surface has been carefully polished; if of *wood*, that it has been planed; if of *stone*, that it has been rubbed down, and brought to as smooth a surface as the particular kind of stone will allow. Its *surface* will then depend only on the material; it will be smooth to the eye and touch, but not *statically* smooth.

207. Consider now the action exerted by a fixed 'rough' plane upon a body which presses against it.

Since the plane is rough, the force applied to the body is not necessarily perpendicular to the plane.

We resolve this force into two rectangular components, one at right angles to the plane, the other along the plane.

The component force at right angles to the plane is called the **pressure** between the plane and the body.

The component force along the plane is called the **friction** between the plane and the body.

The friction is the part of the action of the plane on the body which prevents the movement of the body along the plane.

When a body is at rest on a fixed plane the action of the plane upon it is equal and opposite to the resultant of all the external forces acting on the body. The friction is therefore equal and opposite to the part of the external forces which tends to cause motion of the body on the plane. Hence we have the following definition.

208. *DEF.* When a body rests on a rough plane and forces act on the body tending to cause it to slide along the plane, a force is called into play which acts on the body in the direction contrary to that in which it tends to move, and tends to prevent motion.

This force is called **Friction.**

209. The **General Laws of Friction** are

I. When the surfaces of two bodies are in contact and at rest relatively to each other friction always acts upon each body in the direction exactly opposite to that in which the body tends to move along the surface on which it presses.

II. When there is no relative motion, the amount of friction which is called into action is just as much as is necessary to prevent relative motion—and no more.

Example i. *A mass of weight* W *rests on a rough horizontal plane; it is pulled in a certain direction by a horizontal force of* P lbs.; *the mass does not move.* Why is this?

The mass is under the action of the following forces:

(i) its own weight downwards,

(ii) the pressure of the plane on it upwards,

(iii) the force P horizontally,

(iv) the force of friction, acting between the surface of the mass and of the plane; this force is horizontal and must be equal to the only other force acting, namely P lbs., and must act in exactly the opposite direction.

If the plane in the present case were perfectly smooth, in other words, if we suppose friction not to exist, it is clear that the force P would cause the mass to move in its own direction.

Example ii. *A heavy mass resting on a horizontal plane is acted on by a force of* 4 lbs. *towards the North and a force of* 3 lbs. *towards the West. The mass is kept at rest by the friction between itself and the plane. What is the force of friction called into play?*

The resultant of the two horizontal forces is a force of 5 lbs. acting in a direction somewhere about N.W.N.

The force of friction must therefore be a force of 5 lbs. acting exactly opposite to this resultant.

210. A little thought will enable the student to understand how great a part friction plays in practical mechanics.

It is friction which keeps the furniture of a room in its place; which enables us to walk and move at will. It is interesting to try and realize for a moment how different the world would be if this force of friction ceased to act. The student will find that very few problems would then be *statical* problems. Almost every problem which could be proposed in mechanics would necessarily involve motion.

EXAMPLES. XXVIII. a.

1. A body rests on a horizontal plane and is acted on by the force of 10 lbs. in the direction making the angle 60° with the plane. What amount of friction is called into play?

2. A body rests on a horizontal plane and is acted on by the force of 40 lbs. in the direction making the angle 45° with the plane. What friction is acting on the body?

3. A body weighing 40 lbs. rests on an inclined plane whose inclination to the horizon is 30°. What amount of friction is acting between the body and the plane?

4. What will the friction be if the plane in question **3** be inclined at the angle 45° to the horizon?

5. A body weighing 50 lbs. rests on an inclined plane whose angle is 30° and a force of 10 lbs. is acting upon it up the plane; what friction is there?

6. If the body in question **5** weighs 112 lbs. and the force up the plane is 112 lbs.; what friction is called into play?

7. A body weighing 1 ton is placed on an inclined plane of inclination 60° and is acted on by a force of 1 ton up the plane; what friction is acting on it?

8. A body rests on a horizontal plane and is pulled horizontally by two men who exert forces of 30 lbs. towards the North and 40 lbs. towards the East respectively; what friction is acting on the body?

9. A mass of one cwt. lies on an inclined plane of angle 30° and two men push against it, one up the plane with the force 56 lbs. and the other down the plane with the force 84 lbs. What friction is called into play?

Limiting Friction.

211. It is most important that the student should realize that in any particular case in which there is no motion, the amount of friction actually called into play is only just so much as is necessary to prevent sliding motion.

In most cases however if the force tending to cause sliding motion be increased, it will be found that there is a **limit** to the amount of friction which can be called into play. So that when the force necessary to prevent sliding motion is greater than a certain amount, the surfaces slide the one on the other.

212. *Illustration.* Suppose we take a plane such as the surface of a piece of board. Let the board have a hinge so that it can be tilted up at different inclinations to the horizon and let its surface be made as smooth as the nature of the substance of which it is made will permit.

Now take a rigid body such as a lump of iron; let its face be made as smooth as possible and let it be placed without restraint on the plane.

First let the plane be horizontal.

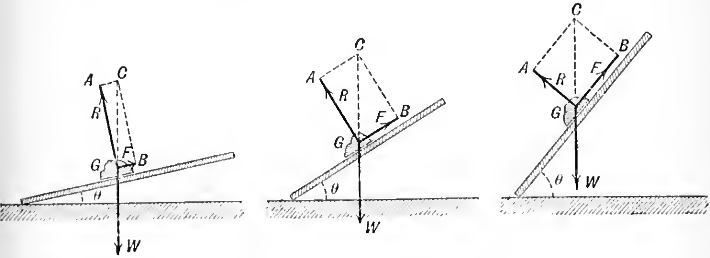
The body will be at rest; as the only forces acting upon it are its weight downwards and an equal pressure of the plane on the mass upwards.

Next let the plane be tilted up so as to make a small angle (θ) with the horizon; and suppose that the body still remains at rest.

The forces now acting upon the body are its weight vertically downwards and the action of the plane on it; these two forces must be equal and opposite. Hence the action of the plane on the body must still be vertical; that is, the action is no longer perpendicular to the plane.

We resolve this action into two rectangular components, (i) perpendicular to the plane, which we call the **pressure** (R), (ii) along the plane which we call the **friction** (F).

These two (i) pressure (R), and (ii) friction (F), are the



rectangular components of a constant vertical force upwards, equal to (W) the weight of the body.

It will be seen from the figure that as the angle of inclination θ of the plane is increased R diminishes and F increases.

For $R = W\cos\theta$, $F = W\sin\theta$, and $\therefore \frac{F}{R} = \frac{W\sin\theta}{W\cos\theta}$ $= \tan\theta$.

Hence with a small inclination of the plane the ratio $\frac{F}{R}$ is small, but this ratio increases with θ and it can be made as great as we please (for $\tan 90^0 = \infty$).

Now it will be found (if the experiment be performed carefully) that with a plane made of any chosen kind of material, say box wood, and a body of any other chosen material, say soft iron, motion will always begin when (θ) the inclination of the plane reaches a certain magnitude (α). That is, when the ratio of $\frac{F}{R}$ attains a certain maximum value ($\tan\alpha$).

Experiments indicate that provided the surfaces in contact have been made as smooth as possible, this angle α at which motion begins, is for most practical purposes, independent (i) of the weight of the mass, and therefore of the magnitude of the pressure R, (ii) of the size of the surface in contact.

Thus if masses of the *same materials* but of different sizes and shapes be used for repetitions of the experiment, the angle α will be found to be approximately the same in every case.

213. It is from the results of a careful series of experiments such as the one described above that we obtain the following

LAWS OF LIMITING FRICTION.

I. When two bodies whose surfaces are in contact have a tendency to relative motion it is found that—when the surfaces have been made as smooth as possible—the amount of friction which can be called into play *cannot exceed a certain limiting value.*

II. This limiting value is a certain fraction of the normal pressure.

This fraction *depends on* the nature of the bodies in contact, and is called the **coefficient of friction** of those bodies.

Thus the coefficient of friction for soft iron and brass is about ·17.

This fraction *is independent of* the size and shape of the surfaces in contact.

It is to be noticed however that the surface must be large enough to prevent actual scratching of the surfaces.

III. When motion actually is taking place the above laws are still approximately true as to the friction between sliding bodies; but the fraction $\frac{F}{R}$ is rather less in each particular case when the bodies are sliding than when just *on the point* of motion.

It seems as if the bodies when at rest have time to interlock their projections more completely than when in motion the one over the other; so that it takes rather more force to start them than to keep them moving uniformly.

SOME COEFFICIENTS OF FRICTION.

From MORIN.

Substances in Contact.	Coef.	Angle.
Oak on oak, all the fibres parallel to the motion	·48	25°·38′
,, ,, moving fibres on end, resting fibres parallel to the motion	·19	10°·46′
Wrought iron on brass	·17	9°·39′
Steel on cast iron	·20	11°·19′
Steel on polished glass	·11	6°·17′
Polished marble on polished marble	·16	9°· 6′
Polished marble on common birch	·44	23°·45′
Common birch on common birch	·64	32°·38′
Wrought iron on oak (fibres parallel to motion)	·62	31°·47′
,, ,, elm ,, ,, ,,	·25	14°· 3′

The above are quoted from a very full list given on page 373 of Trautwine's *Engineer's Pocket-book*. Ed. 1888.

NOTE.—In the above table the coefficient of Limiting Friction is the tangent of the angle at which the body will begin to slide when on an inclined plane as in Art. 212. This will be clear if the student will notice that in the experiment the action of the plane on the body is always *vertical*, and therefore this action makes the same angle with the perpendicular to the plane which the plane makes with the horizon. It is usual to use the letter μ to denote a coefficient of friction. Thus,

$$F = \mu R.$$

But, if α is the limiting angle,

$$F = R \tan \alpha;$$

$$\therefore \quad \mu = \tan \alpha.$$

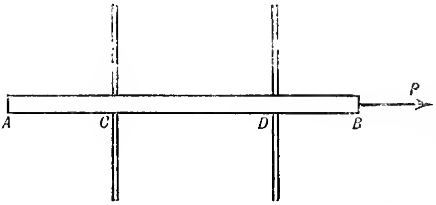
Example i. *An iron girder weighing* 1 ton *lies on* horizontal *steel rails; what horizontal force must a horse apply to the girder to make it* *move forward along the rails and in the direction of its length, if the coefficient of friction for iron and steel is ·175?*

The whole pressure of the girder on the rails is 1 ton = 2240 lbs.; this pressure is distributed over the surface of that part of the girder in contact with the steel rails.

When a horizontal force has been applied to the girder so that it is on the point of motion, then at every part of the surfaces at which there is pressure there is a horizontal force of friction opposite to the direction in which there is a tendency to motion. This friction is in all cases ·175 of the pressure.

For the direction in which the girder is to be pulled is in the direction of its length, so that if any of the points of contact be on the point of motion, *every* point must be on the point of motion.

[Thus if AB be the girder, resting on two horizontal steel rails at C and D, the pressures on the rails at C and D may be unequal; suppose them to be R_1 and R_2.



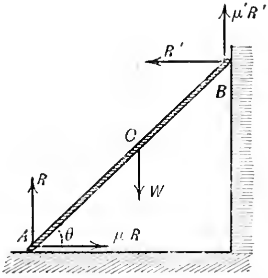
Then $$R_1 + R_2 = 1 \text{ ton},$$
and the limiting friction due to $R_1 = \cdot175 \times R_1$ and the limiting friction due to $R_2 = \cdot175 \times R_2$. So that the girder will be on the point of motion when a force $\cdot175 R_1 + \cdot175 R_2$ is applied to it in the direction AB, that is ·175 times 1 ton.]

Therefore the whole amount of the friction is ·175 of the whole amount of the pressure.

∴ the limiting friction = ·175 × 2240 lbs.
= 392 lbs.

Therefore if a horizontal force of any magnitude exceeding 392 lbs. be applied to the girder in the direction of its length it will begin to move in the direction of the force.

Example ii. *A uniform ladder rests with one end on a horizontal stone pavement, the other leaning against a vertical brick wall; find the limiting position of equilibrium; the coefficients of friction being respectively* μ *and* μ'.



With the notation of the figure, in which the ladder AB is supposed to be on the point of slipping down, we get by [Art. 91] considering the conditions for the equilibrium of the ladder, and

I. resolving horizontally,

II. resolving vertically, and

III. taking moments about C we have

I. $$R' - \mu R = 0.$$

II. $$R + \mu' R' - W = 0.$$

III. $$R\cos\theta - \mu R\sin\theta - R'\sin\theta - \mu' R'\cos\theta = 0;$$

$$\therefore \quad \frac{R}{R'} = \frac{\sin\theta + \mu'\cos\theta}{\cos\theta - \mu\sin\theta}$$

$$= \frac{1}{\mu} \text{ [from I.]},$$

$$\therefore \quad \mu\sin\theta + \mu\mu'\cos\theta = \cos\theta - \mu\sin\theta;$$

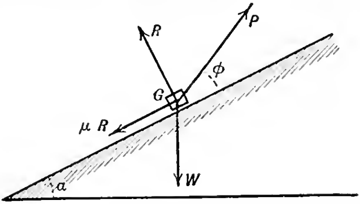
$$\therefore \quad \tan\theta = \frac{1 - \mu\mu'}{2\mu}.$$

That is, the angle of inclination of the ladder to the horizon must not be less than the angle whose tangent is $\frac{1 - \mu\mu'}{2\mu}$.

For instance if $$\mu = \mu' = \tfrac{1}{2};$$

$$\tan\theta = \tfrac{3}{4}.$$

Example iii. *Let a mass* W *resting on a rough inclined plane, be acted on by a force* P *making an angle* ϕ *with the plane, as in the figure. Find the relation between* P, W, ϕ.



With the notation of the figure, **first** we shall find the relation between P, W, ϕ when the mass is on the point of moving *up* the plane.

In this case, the pressure on the plane being R the friction called into play is μR and acts *down* the plane as drawn.

We get by [Art. 91] considering that the mass is in equilibrium, and resolving I. along, and II. perpendicular to the plane

I. $W \sin a + \mu R - P \cos \phi = 0$.

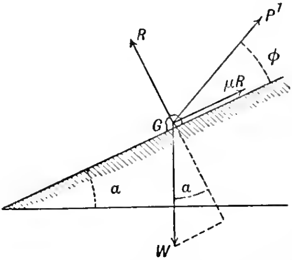
II. $R + P \sin \phi - W \cos a = 0$.

Multiplying II. by μ and subtracting this from I. we have

$$W \sin a - P \cos \phi - \mu P \sin \phi + \mu W \cos a = 0,$$

or $$W(\sin a + \mu \cos a) = P(\cos \phi + \mu \sin \phi).$$

This is the equation which will give the magnitude of the force necessary to cause the mass to be on the point of motion up the plane when W, a, μ and ϕ are given.

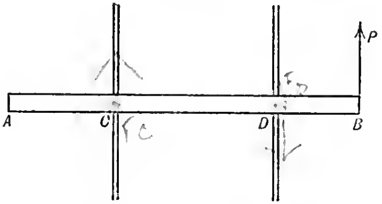


Next, supposing the mass on the point of moving down the plane, the force μR must be reversed as in the figure; the effect of this will be to change the *sign* of μ in the above equations.

Example iv. *An iron girder* 8 ft. long *weighing* 1 ton *is placed symmetrically at right angles to two fixed steel rails which are* 4 feet *apart. What horizontal force applied at right angles to the girder at one extremity will cause it to be on the point of motion?*

Let AB be the girder resting on the rails at C and D. The vertical pressures at C and D are each equal to 1120 lbs.

The coefficient of friction between iron and steel is ·175.



Consider the horizontal forces acting on the girder. They are the force P at B and the frictions F_C and F_D at D and C. F_C and F_D may either of them have any value *between* 0 and 196 lbs. ($\cdot175 \times \frac{1}{2}$ ton) each; and they act opposite to the tendency to relative motion at C and D respectively.

Supposing motion to ensue, there are four possible hypotheses. Either

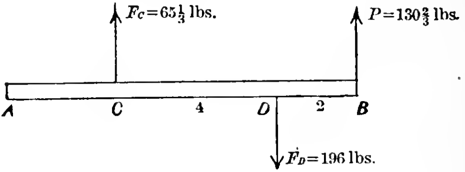
(i) the girder slides at D and turns about C;
or, (ii) the girder slides at C and turns about D;
or, (iii) the girder slides at both C and D in directions parallel to P;
or, (iv) the girder slides at D and at C in some other directions.

We proceed to shew that (i) is the correct hypothesis.

If the girder begins to turn about C the direction of relative motion at D is perpendicular to the girder so that F_D is perpendicular to CD and is 196 lbs. Therefore F_C is the anti-resultant of the two parallel forces at D and B, and therefore is (i) parallel to them, (ii) less than the force at D, i.e. less than 196 lbs.; thus hypothesis I. is possible. We may note that P is also less than the force at D, i.e. is less than 196 lbs.

If (ii) the girder begins to turn about D, then the direction of relative motion at C is perpendicular to CD; so that F_C is parallel to P; and F_D which is the anti-resultant of F_C and P must also be parallel to P; when three parallel forces are in equilibrium the force which lies *between* the other two is the sum of those two. Hence in this case F_D must be greater than F_C, i.e. *greater than* 196 lbs., which it cannot be.

If (iii) and (iv), the girder were to begin to slide at both points at once, F_C and F_D would each be 196 lbs.; hence in the case of (iii) P is the anti-resultant of two equal parallel forces; in which case CB ought to be equal to CD, which it is not.



In the case of (iv) the directions of F_C and F_D must meet on the line of action of P; and since F_C and F_D are *equal*, P must bisect the angle between F_C and F_D; but this is impossible when P is perpendicular to CD except when P bisects CD.

Hence, the only possible hypothesis is (i), that the girder is on the point of turning about C.

The friction at D is 196 lbs. and is parallel to P.

Taking the sum of the moment about C of all the forces acting on the girder, we have

$$P \times 6 - 196 \text{ lbs.} \times 4 = 0,$$

or,

$$P = \tfrac{2}{3} \text{ of } 196 \text{ lbs.} = \underline{130\tfrac{2}{3} \text{ lbs.}}$$

EXAMPLES. XXVIII.

1. A weight placed on a rough inclined plane whose inclination is 30° is just on the point of sliding down; what is the coefficient of friction?

2. A mass of iron weighing 1 ton placed on a horizontal plane made of brass, what horizontal force is necessary to move it?

3. A body placed on a rough inclined plane whose angle is α is just on the point of motion; what horizontal force is necessary to move a body of 1 cwt. of the same material placed on the plane when horizontal?

4. A body placed on an inclined plane of angle 30° is just on the point of moving upwards when acted on by a horizontal force equal to its own weight; find the coefficient of friction.

5. A body placed on a horizontal plane requires a horizontal force equal to its weight multiplied by $\sqrt{3}$ to cause it to be on the point of motion; at what angle must the plane be tilted that the body may then be on the point of motion?

6. A body placed on a rough horizontal plane is on the point of motion when acted on by a force equal to its own weight inclined to the horizon at an angle of 60°; find the coefficient of friction.

7. If the inclination to the horizon of the force in Question **6** were 45°, what would be the coefficient of friction?

8. A mass of 1 lb. placed on an inclined plane is fastened to the plane by a thread which is parallel to the plane; the coefficient of friction being ·2, what must be the least tension of the string when the inclination of the plane is 45°?

9. A heavy beam rests with one end on a horizontal pavement and the other in contact with a vertical wall; if the coefficient of friction of the ground and wall be equal to ·3 and the beam is uniform, find the inclination of the beam to the horizon when it is on the point of slipping down.

10. A ladder 20 ft. long without weight rests with one end on the ground which is rough ($\mu = \cdot 4$) and the other against a vertical wall which is smooth and is inclined to the horizon at an angle of 45°; a man commences to walk up the ladder; how far will he go before the ladder slips?

11. Shew that if in Question **10** the ground is smooth and the wall rough, the ladder will slip wherever it is placed, directly the man attempts to ascend it.

12. A uniform ladder 20 ft. long weighing 56 lbs. rests with one end on the ground which is rough (coefficient of friction $= \mu$) and the other against a smooth vertical wall; find the least angle to the horizon at which it can rest; and find how far a boy of 1 cwt. could walk up the ladder when at that angle, without causing it to slip.

13. It is said that the force required to keep a train moving uniformly on an incline of 1 in 80 is four times the force necessary on horizontal rails; assuming that the friction varies as the normal pressure, find the coefficient of friction.

14. Taking the same coefficient of friction as in Question **13**, find how many times the force required on a horizontal rail is required on an incline of 1 in 60.

15. A particle of weight W placed on a rough horizontal plane whose coefficient of friction is μ, is fastened to a fixed point by a string which is extended to its full length; a horizontal force is applied to the particle in the direction which makes the angle $135°$ with the string; what is the magnitude of the force when the particle is on the point of motion?

16. A cubical block of stone is placed on rollers. The force necessary to make the stone move with the rollers is ·01 of the weight; the force necessary to make the stone slide over the rollers themselves is ·4 of the weight. A force is applied to the stone making an angle θ with the rollers; find the least magnitude θ can have that the stone may move with the rollers (without sliding).

17. A four-wheeled waggon with its wheels all parallel is such that if it be pulled on a horizontal plane parallel to the wheels it requires a force of ·0125 of the weight to cause it to be on the point of motion; it requires a force of ·5 of its weight in the direction perpendicular to the wheels to cause it to slip; what is the greatest angle which a horizontal force may make with the wheels so as to cause the waggon to move in the direction of the wheels? Find the force in the limiting case, when the waggon weighs 1 ton.

18. A force of 10 lbs. can just cause a weight on an inclined plane of angle $30°$ to be on the point of moving up the plane and a force of 1 lb. can just cause it to be on the point of moving down the plane, both forces acting parallel to the plane; what force would just cause it to be on the point of motion on a horizontal plane?

19. A cube of wood stands on a rough horizontal plane; a horizontal force acts at the middle point of one of its upper edges perpendicular to the edge. Find the coefficient of friction that it may be just on the point of toppling and of sliding at the same time.

20. A body is placed on a rough horizontal plane and a line is drawn upward from the plane, making with it an angle equal to the angle of friction. Shew that this is the direction of the smallest force which will make the body slide.

21. Two given weights W, W' of different material are laid on a given inclined plane, and connected by a string in a state of tension in the line of greatest slope of the plane; the coefficients of friction being μ, μ'. Shew that the angle of the plane θ when both weights are on the point of motion is given by

$$\tan\theta = \frac{\mu W + \mu' W'}{W + W'},$$

and that the tension of the string is $(W\sin\theta - \mu W\cos\theta)$.

22. A sphere is placed upon a rough inclined plane, coefficient of friction μ; find the position of the centre of gravity of the sphere that it may be on the point of sliding and of rolling down the plane simultaneously.

23. A uniform circular cylinder, 5 inches in diameter and 16 inches high, is on the point of sliding and of toppling over simultaneously when the inclination of the plane on which it stands is gradually increased; find the coefficient of friction.

24. If the force which acting parallel to an inclined plane of angle a is just sufficient to draw a weight up, be n times the force which will just let it be on the point of sliding down, shew that $\tan a = \mu \frac{n+1}{n-1}$.

25. A uniform cube of stone stands on a rough horizontal plane and is acted on by a force at right angles to one of its topmost edges inclined 45° to the horizon; the stone is on the point of slipping and also of toppling over; what is the coefficient of friction?

26. A cubical block of stone rests with one of its edges on a horizontal plane and another against a vertical wall; find the limiting position of equilibrium, the coefficients of friction being μ and μ'.

27. Two equal uniform bars are hinged together at the extremity and placed on a rough horizontal plane so as to make an inverted V; find the greatest angle between them when the coefficient of friction is μ.

28. The centre of gravity G of a bicycle and its rider (exclusive of the front wheel) is distant 2 ft. from O the centre of the front wheel and OG makes an angle whose sine is $\frac{1}{6}$ with the vertical; the radius of the front wheel 28 inches; the weight of the rider and machine exclusive of the front wheel is 2 cwt.; what pressure can the rider apply to the front wheel with his brake without being thrown forward, the coefficient of friction between the wheel and the brake being $\frac{4}{3}$?

29. A uniform girder 12 ft. long weighing 2 tons is placed at right angles across two parallel horizontal iron bars; if it be placed so that the pressures on the bars are equal, shew that if it be acted on by a horizontal force parallel to the bars it will always begin to turn about the bar which is farthest from the force.

30. If the girder in Question **29** be so placed that the pressure on one bar is double that on the other, find where to apply a single force parallel to the bars that the girder may begin to slide along both bars simultaneously.

31. A boy of weight W stands on a sheet of ice, balancing himself by means of a chair of weight W', but not leaning any of his weight on it. Shew that if the chair be heavier than the boy he may incline his body to the vertical at any angle less than $\tan^{-1} 2\mu$; but that if the boy is heavier than the chair he can not incline it at a greater angle than $\tan^{-1} 2\mu \frac{W'}{W}$; μ being the coefficient of friction between the boy and the ice and also between the chair and the ice.

32. A smooth mass of 1 cwt. lies on an inclined plane in the form of a wedge whose angle is 30° and weight 1 ton and which rests on a horizontal plane. The mass is kept from sliding down the plane by a horizontal force. Prove that the coefficient of friction between the wedge and the plane must be greater than $\frac{1}{63}\sqrt{3}$.

CHAPTER XVI.

THE GRAPHIC METHOD.

214. WE propose to give in the present Chapter a short account of the Graphic Method of treating Statical Problems.

215. The Graphic Method is a method by which many statical problems may be solved by the aid of diagrams in which lines and angles drawn to scale replace the calculations of Arithmetic and Trigonometry.

216. A force may be said to be *known*, when its magnitude and direction are known. When it is applied to a rigid body we must (in order to know what its effect will be) also be told its line of action.

Also if it is one of a system of parallel forces having a centre, we must also be told its point of application.

We shall in what follows often use **diagrams** formed of lines representing forces: representing them, that is, in direction and magnitude; but having *no reference to their line of action.*

217. The student must distinguish between (i) *a diagram* representing forces, whose positions in the diagram have *no reference to their point of application*, and (ii) a figure representing forces *as they actually* act on a body. Such diagrams and figures form the basis of the application of geometrical problems to Statics.

Example. We are about to prove that the force represented by one of the sides of a triangle is the resultant of the two forces represented by the other two.

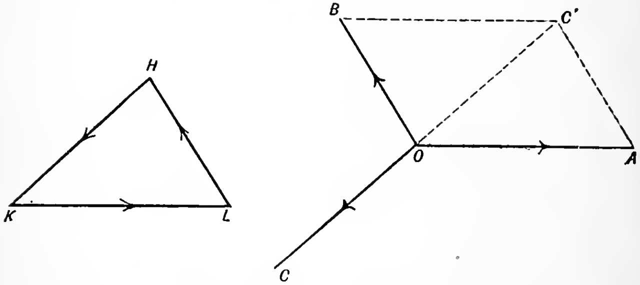
That is to say, if these three forces act *at a point* in directions parallel to the sides of the triangle in the proper senses they will be in equilibrium.

The triangle is here used as a *diagram*.

218. The following is the Graphic Form of the Parallelogram of Forces.

THE TRIANGLE OF FORCES.

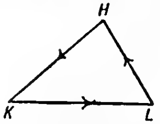
When three forces OA, OB, OC *in one plane act at a point, and a triangle* KLH *has two of its sides*, KL, LH, *taken the same way round*†, *parallel and proportional to two of the forces* OA, OB, *then the necessary and sufficient condition that the forces* OA, OB, OC *may be in equilibrium is, that the third force* OC *is parallel and proportional to the third side, taken the same way round*, HK *of the triangle* KLH.



Construction. Complete the parallelogram $OAC'B$; join OC'.

Since in the triangles OAC', KLH, by this construction,

† NOTE.—'**Taken the same way round**' means that the senses in which the forces represented by the sides are to be taken, are such as would be indicated by a *continuous tour* of the sides.



Thus the forces may be those represented by KL, LH and HK in the figure; or by LK, KH, HL; but not by KL, LH and KH. The same thing is sometimes expressed by the phrase *Taken in order.*

two sides OA, AC' are parallel and proportional to the two sides KL, LH, it follows that the third side OC' is parallel and proportional to HK the third side of the triangle KLH.

Proof. The Parallelogram of Forces states that the necessary and sufficient condition that the three forces OA, OB, OC may be in equilibrium is that OC is equal and opposite to OC'.

In other words, that OC is parallel and proportional to HK.

219. The Triangle of Forces may also be stated thus:

Let OA, OB, OC be three forces acting at a point; from any point K draw KL parallel to OA and equal to $\lambda \times OA$ (where λ is any number); from L draw LH parallel to OB and equal to $\lambda \times OB$; from H draw HM parallel to OC and equal to $\lambda \times OC$; then OA, OB, OC are, or are not, in equilibrium according as M does, or does not, coincide with K.

NOTE. In Graphic Statics we often speak of a line as representing a force when it does not represent it in position, but only represent it in direction and magnitude.

For example, we shall often speak of three forces OA, OB, OC as represented by the three sides of the triangle KLH.

220. *PROP. When* OA, OB, OC *represent three forces in equilibrium, then* $\dfrac{\text{OA}}{\sin \text{BOC}} = \dfrac{\text{OB}}{\sin \text{COA}} = \dfrac{\text{OC}}{\sin \text{AOB}}$.

Complete the parallelogram $OAC'B$; join OC'.

Then* $$\frac{OA}{\sin OC'A} = \frac{AC'}{\sin C'OA} = \frac{C'O}{\sin OAC'},$$

but, $$\sin OC'A = \sin BOC' = \sin BOC$$

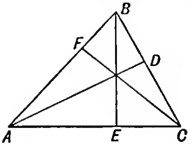
$$\sin C'OA = \sin COA$$

and $\sin OAC' = \sin AOB$, and the result follows.

* See *Elem. Trigonometry*, p. 197.

Example. ABC *is a triangle; three forces parallel respectively to* BC, CA *and* AB *and inversely proportional to the perpendiculars* AD, BE, CF *drawn from* A, B, C *on the opposite sides* BC, CA, AB *act at a point. Prove that these three forces are in equilibrium.*

In order to be in equilibrium the forces must be proportional to the sides of the triangle to which they are parallel.



That is, we must prove that the perpendiculars are inversely proportional to the sides to which they are perpendicular. But $AD \times BC$ $\doteq$ twice area of triangle $ABC = BE \times CA = CF \times AB$,

or $$\frac{BC}{\frac{1}{AD}} = \frac{CA}{\frac{1}{BE}} = \frac{AB}{\frac{1}{CF}} = 2 \text{ area of } ABC. \quad \text{Q.E.D.}$$

EXAMPLES. XXIX.

1. If one of three forces in equilibrium is at right angles to the second, prove that it is less than the third.

2. One of three forces in equilibrium is at right angles to the second, and is half the third: find their inclinations to each other.

3. Three forces in equilibrium are proportional to 1, 1, $\sqrt{2}$: find their inclinations to each other.

4. Prove that of three forces in equilibrium the sum of any two must be greater than the third.

5. If two forces of given magnitude have a resultant of given magnitude, then they must always act at the same inclination to each other.

6. Three forces represented in magnitude by the numbers 2, 3, 5 act at a point and are in equilibrium: what are their mutual inclinations?

7. Three forces of 3 lbs., 3 lbs. and 5 lbs. act at a point parallel to the side of an equilateral triangle taken in order; find their resultant.

8. Two forces of given magnitude act at a point P, and pass one through each of two given points A, B: prove that if their resultant is of constant magnitude P lies on a fixed circular arc.

9. The sides of a triangle taken in order are 3, 4, 5 inches respectively; forces of 12, 16, and 19 pounds act along them taken the same way round; what is the magnitude of their resultant?

10. Three forces 5 lbs., 13 lbs. and 20 lbs. act at a point in directions such that if produced each would bisect the angle between the other two; find the resultant of the three forces.

11. Prove that if two forces acting at a point O be represented by lines OA, OB, of which OB is twice OA, and if A be joined to the middle point C of OB, and AC be divided at D in the ratio of 2 to 1, the resultant of OA and OB will be $3OD$.

12. Prove that if two forces OA, OB be such that $OB=2OA$, and if OA be produced to C so that $OC=OB$, and BC be divided in D in the ratio 1 to 2, the resultant will be $\frac{3}{2}OD$.

13. If three forces in equilibrium be in the ratio 5 : 12 : 13, find the angles between their directions.

14. Two forces which act on a particle which are to one another as $2:\sqrt{3}$ have a resultant equal to half the greater force; find the angle between the forces.

15. $ABCDEF$ is a regular hexagon and three forces act at the point A in the directions AC, AF, DA; the force in the direction AF is 2 pounds; find the other forces if the system be in equilibrium.

16. AD, BE, CF are the perpendiculars drawn from the angular points of a triangle ABC to the opposite sides. Forces act at D in the directions DE, DF; find their relative magnitudes if they be kept in equilibrium by a force in the direction AD.

17. Two forces of 5 and 6 pounds respectively act at a point; find the cosine of the angle between them supposing the resultant to be 8 pounds.

18. Give a geometrical construction for resolving the force represented by the diagonal of a square into three forces, each equal to the side of the square, one of the forces being coincident with a side of the square.

19. Shew that a given force may in general be resolved into three forces each equal to a given force, the direction of one of the forces to which they are equal being given.

20. If three forces keep a particle at rest, prove that the angle between the two greatest is larger than that between any other two.

21. *A* and *B* are fixed points on the circumference of a circle, *P* is any other point on the circumference; shew that if two forces of constant magnitude act along *PA* and *PB*, their resultant will pass through a certain point for all positions of *P*.

22. *C* and *B* are fixed points; *CA* and *CB* represent two forces; prove that when *A* moves along a given straight line, the extremity of the straight line which represents the resultant moves along a parallel straight line.

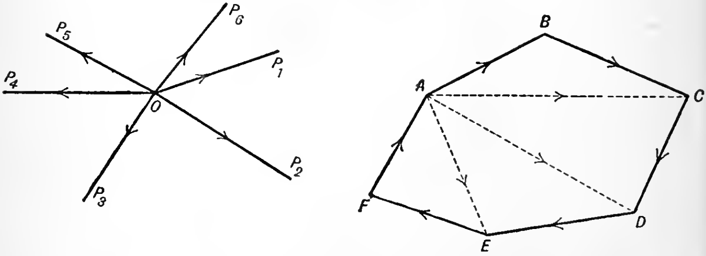
23. Forces represented by the sides of a polygon taken the same way round, one side being omitted, act at a point, shew that their resultant is represented by the omitted side taken the other way round.

24. A smooth circular ring is fixed in a horizontal position, and a small ring sliding upon it is in equilibrium when acted upon by two strings in the direction of the chords *PA*, *PB*; shew that if *PC* is a diameter of the circle the tensions of the strings are in the ratio *BC* to *AC*.

221. From the Triangle of Forces we deduce

THE POLYGON OF FORCES.

When any number of forces act at a point, they are, or are not, in equilibrium according as the polygon formed by drawing lines, the same way round, parallel and proportional to the forces, is, or is not, a **closed** *polygon.*



Construction. Let OP_1, OP_2, OP_3... be the series of forces acting at *O*. Take any point *A* and from *A* draw

AB parallel and proportional to OP_1; from B draw BC parallel and proportional to OP_2; from C draw CD parallel and proportional to OP_3; and so on.

Proof. By the Triangle of Forces AC represents the anti-resultant of forces represented by AB, BC. Therefore OP_1, OP_2 may be replaced by a force acting at O represented by AC.

Similarly the force acting at O represented by AC and OP_3, represented by CD, may be replaced by a force acting at O represented by AD; and so on.

Suppose that the last three forces left are OP_5, OP_6 and a force acting at O represented by AE; then by the triangle of forces, it follows that these three forces (which replace the original series) are, or are not, in equilibrium according as the extremity M of the final line FM drawn parallel and proportional to OP_6 does, or does not coincide with A;

That is, the series of forces is, or is not, in equilibrium according as the polygon $ABCD$... whose sides taken the same way round, represent the forces respectively, is, or is not, a *closed* polygon.

NOTE.—If the polygon is not closed the line which closes the polygon represents the resultant of the system of forces.

223. Any system of forces in one plane acting upon a rigid body at rest must fulfil the conditions of equilibrium which apply to forces acting at a point [see Art. 91, Conditions I., II.]. Hence the *magnitude* and *direction* of the resultant of any number of forces acting on a rigid body may be found by the polygon of forces. Its *line of action* cannot be so found. The method of finding the line of action of the resultant is given in Art. 227.

It may be noticed that when the system of forces is equivalent to a couple the *resultant* is zero.

EXAMPLES. XXX.

1. The sides of a quadrilateral taken the same way round are 1, 2, 9, 7 inches respectively; forces of 2, 4, 8, 14 lbs. respectively act parallel to them in the same way: what is their resultant (i) if they act at a point, (ii) if they act along the sides of the quadrilateral?

2. $ABCD$ is a square, and forces acting at a point are represented in direction and magnitude by AB, $2BC$, $3CD$, $4DA$; shew that their resultant is represented by $2CA$.

3. A straight line OA is at right angles to another straight line COB, and forces each of 7 lbs. act one in OB another in OA; a third in the bisector of the angle COA; find the magnitude of their resultant.

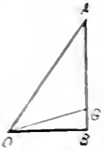
4. Prove that if four forces be fully represented by the sides of a quadrilateral figure taken in order, they cannot be in equilibrium.

5. The side BC of a square $OABC$ is bisected at E; find the resultant of forces represented in direction and magnitude by OA, OB, $2OE$.

6. $ABCD$ is a square, and forces represented in magnitude by the numbers 4, $2\sqrt{2}$, 3 and $\sqrt{2}$, and in direction by the lines AB, BD, DA and AC act at a point: what is their resultant?

7. The circumference of a circle is divided into any number of equal parts, and forces are represented by lines drawn from the centre to the point of division: shew (by the polygon of forces) that these forces are in equilibrium.

224. *PROP. The resultant of two forces, one represented by a line* OA, *the other represented by* n *times the line* OB *is represented by* (n + 1) *times the line joining* O *to the point which cuts* AB *in the ratio of* n *to* 1.



Let G divide AB so that $AG = n$ times GB.

Join OG.

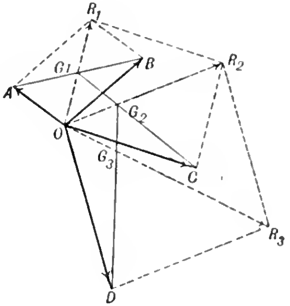
Then two forces represented by OA, AG if acting at O would have OG for their resultant [Art. 218] and two forces represented by n times OB and n times BG, if acting at O would have for their resultant n times OG.

Therefore the four forces, OA, AG, n times OB, n times BG, if acting at O, would have for their resultant $(n+1)$ times OG.

But $AG = n$ times GB; therefore the two forces AG and n times BG are in equilibrium.

Therefore the resultant of the two forces, OA and n times OB is $(n+1)$ times OG. Q. E. D.

225. Consider a system of forces OA, OB, OC, ... acting at a point O.



Join AB, bisect AB in the point G_1.

The resultant of OA, OB is twice OG_1, by Art. 224.

Join G_1C, and cut G_1C in G_2 so that $CG_2 : G_2G_1$ in the ratio of 2 : 1.

Then the resultant of twice OG_1 and OC is 3 times OG_2; and so on.

If there are n forces and G be the last point thus determined G is called the **centroid** of the n points A, B, C ...

Example. Prove that the centroid of three points A, B, C *is the centre of gravity of the triangular area* ABC.

To find the centroid of OA, OB, OC we bisect AB at F join CF and divide CF in G so that $CG : GF = 2 : 1$.

But a similar construction gives the centre of gravity of the triangular area ABC.

226. When the point O coincides with the centroid of the points A, B, C ... the forces OA, OB, OC ... are in equilibrium.

For the resultant is n times the distance between O and the centroid, which distance is in this case zero.

Example. Prove that the forces OA, OB, OC *are in equilibrium when* O *is the centre of gravity of the triangular area* ABC.

This follows at once; for the C. G. of ABC is the centroid of the forces OA, OB, OC.

EXAMPLES. XXXI.

1. Prove that the centroid of forces OA, OB, OC, OD where $ABCD$ is a parallelogram, is the intersection of the diagonals of the parallelogram.

2. Prove that if A_1, A_2, A_3, ... be points equidistant from each other on the circumference of a circle their centroid is the centre of the circle.

3. A, B, C, D, ... are the angular points of a regular polygon of n sides inscribed in a circle; prove that the resultant of the forces OA, OB, OC, OD, ... is n times OQ where Q is the centre of the circle.

4. $ABCD$ ... and $A'B'C'D'$ are the angular points of two regular polygons, each of n sides, one inscribed in, the other described about the same circle; prove that the system of forces OA, OB, OC, OD ... is equivalent to the system OA', OB', OC', OD'.

5. ABC is a triangle, D, E, F are the middle points of its sides; prove that the system of forces OA, OB, OC is equivalent to the system OD, OE, OF.

6. What is the locus of a point O which is such that the resultant of the forces represented by OA, OB, OC, where A, B, C are fixed points, is of given magnitude?

7. ABC, DEF are given triangles; find the locus of a point O such that the resultant of the forces OA, OB, OC is equal to the resultant of the forces OD, OE, OF.

8. If two forces are represented by m times OA and n times OB, prove that their resultant is represented by $(n+m)$ times OG where G is the point which cuts AB so that

$$AG \times m = BG \times n.$$

9. From the theorem of Question 8 find the line of action of the resultant of two like parallel forces.

10. Prove that if two forces acting at O are represented by OA and by n times OB *reversed*, the resultant will be $(n-1)$ times OG where G cuts AB *externally* so that $AG=n$ times BG.

11. Prove that if two forces acting at O are represented by m times OA and n times OB *reversed*, the resultant will be $(m-n)$ times OG where G cuts AB externally so that

$$m \text{ times } AG = n \text{ times } BG.$$

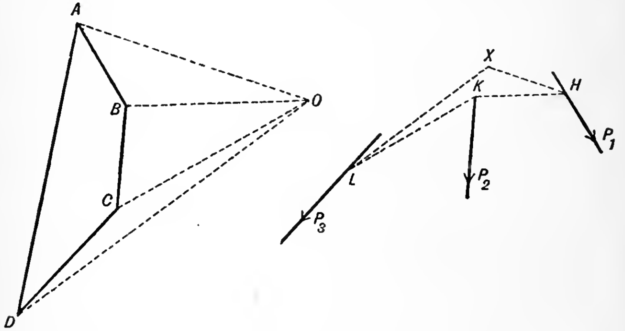
12. From the theorem of Question 11 deduce the rule for finding the resultant of two unlike parallel forces.

13. Prove that if lines be drawn from any point P in a plane to four fixed points in the same plane and these lines represent forces, the resultant of these forces will pass through a certain fixed point G and will be proportional to PG.

14. DEF are the middle points of the sides of the triangle ABC; the system of forces represented by OA, OB, OC is equivalent to the system represented by OD, OE, OF where O is any point in the plane ABC.

227. *PROP. To give a construction for finding the resultant of any number of forces acting on a rigid body in one plane.*

Let P_1, P_2, P_3 be three forces acting on a rigid body in one plane.



Draw the polygon $ABCD$ so that AB, BC, CD are equal and parallel to the lines representing the forces P_1, P_2, P_3.

Then the line AD is equal and parallel to the required resultant. [Art. 223.]

To find the line of action of the resultant we proceed thus.

Take *any* convenient point O and join OA, OB, OC, OD.

The point O is called the Pole.

Take any point H in the actual line of action of P_1 and draw HK parallel to OB to cut the line of action of P_2 at K.

B is the point in the other diagram in which lines representing P_1, P_2 intersect.

Through K draw KL parallel to OC to cut P_3 at L.

C is the point of intersection of the lines representing P_2, P_3.

Through L and H draw lines parallel to OD and OA respectively; we proceed to prove that their point of in-section, X, is a point in the required line of action of the resultant.

For, a force represented by AB may (by the triangle of forces) be replaced by forces represented by AO and OB, provided the forces indicated by AO and OB act at a point in the line of action of the force indicated by AB.

Replace P_1 by two forces equal to AO and OB acting along XH and HK.

Replace P_3 by two forces equal to CO, OD acting along LX and KL.

Passing through the point K we have forces represented by OB, BC and CO; these forces are in equilibrium, and may be omitted.

Hence the three forces P_1, P_2, P_3 are now replaced by two forces passing through X, equal and parallel to AO, OD respectively.

These two forces may be replaced by a force equal and parallel to AD.

The force through X equal and parallel to AD is therefore the resultant required.

228. The student should exercise himself by drawing the figure and extending the proof to the case of four forces, and so satisfy himself that the method is true generally.

We proceed to apply the method to the particular case of parallel forces.

We shall use the same lettering throughout.

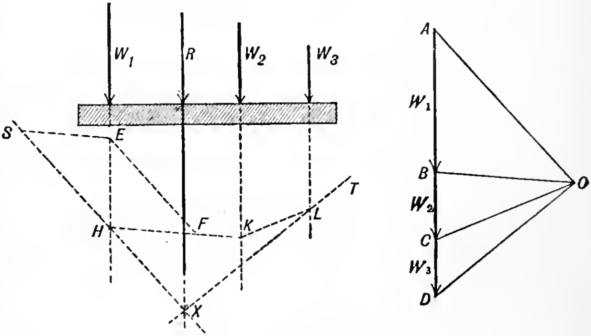
229. *PROP. To find the line of action of the resultant of a series of parallel forces acting on a rigid body.*

The polygon of forces is in this case a straight line.

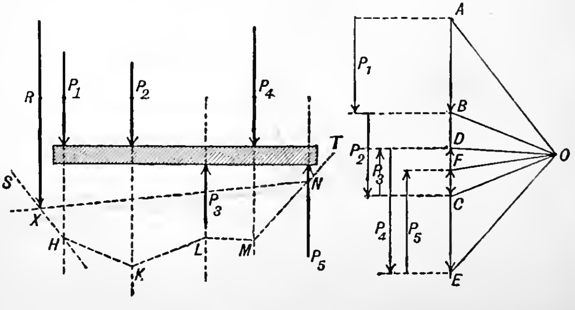
The construction is precisely the same as that of Art. 228.

The lines of action of the forces are of course all parallel.

The following is the figure for the case of three like parallel forces, W_1, W_2, $W_3 = AB$, BC, CD; their resultant is $R = AD$ acting through the point X.



230. The following is the figure for a case of five parallel forces, P_1, P_2, P_4 in one sense and P_3 and P_5 in the opposite sense; the 'Polygon of Forces' is $AB = P_1$, $BC = P_2$, $CD = P_3$, $DE = P_4$, $EF = P_5$. Hence the resultant is AF acting through the point X.



X is found by taking any convenient point O and joining OA, OB, OC, OD, OE, OF.

Then, in the line of action of P_1, take any convenient point H, and

from H, draw HK parallel to BO* to cut P_2 in K;
from K, draw KL parallel to CO to cut P_3 in L;
from L, draw LM parallel to DO to cut P_4 in M;
from M, draw MN parallel to EO to cut P_0 in N.

Lastly, from H and N draw lines parallel to OA and OF respectively, to intersect in X.

231. **A Funicular Polygon.** In the figure of Art. 229 let XH and XL be produced to S and T. Let SH, HK, KL, LT be a string, whose ends are fixed at S and T, to which weights W_1, W_2, W_3 are fastened at H, K, L respectively.

Then these weights would be supported in the positions indicated by the strings. The tensions of the strings will be represented by AO, BO, CO and DO respectively.

Take $HF = BO$, $HS = OA$; complete the parallelogram $SEFH$. Then the triangle HFE is equal in all respects to BOA; $\therefore EH = AB$ and HE represents W_1, which is the anti-resultant of HS and HF.

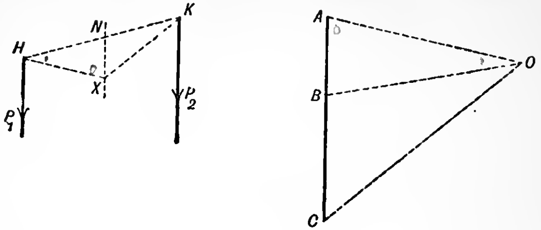
Such an arrangement of lines as $SHKLT$ is called a *funicular polygon.*

If the point O were taken on the left of the line AD, it would be found that if the lines SH, HK, KL, LT are replaced by rigid weightless *rods*, and as before, weights W_1, W_2, W_3 were fastened at HKL, the rods would form an *arch* by which the weights could be supported in the positions of the lines.

Such an arrangement of lines is called a **linear arch.**

* The lines drawn from the point in the line of action of each force must be parallel to the lines drawn from the pole O to the ends of the line which in the polygon of forces represents that particular force.

Example. Find by the graphic method the resultant of two parallel forces.



With the construction and lettering of Art. 227, we have HK parallel to BO, XK parallel to CO, and HX parallel to AO.

Draw XN parallel to P, to cut HK in N, then by similar triangles,

$$XN : NK = CB : BO,$$

and $$HN : XN = BO : BA;$$

$$\therefore\ HN : NK = CB : AB,$$

or $$HN \times P_1 = NK \times P_2.$$

This gives the position of the line of action of the resultant which is equal to AC or $P_1 + P_2$. Q. E. D.

232. *To find by the Graphic Method the* **Centre** *of a system of Parallel Forces applied at given points.*

For instance, to find the centre of gravity of bodies whose weights and centres of gravity are given.

Let G_1, G_2, G_3 be the points of application of three parallel forces W_1, W_2, W_3.

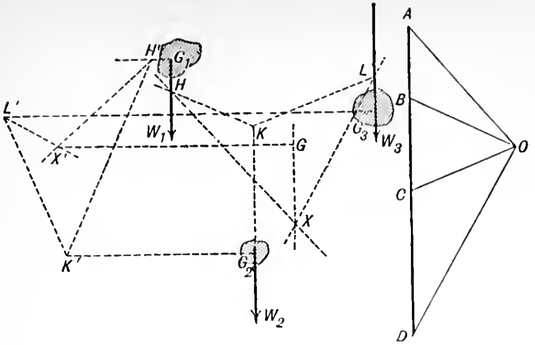
Draw the 'polygon of forces' A, B, C, D, and join A, B, C, D to any convenient chosen point O.

Choose any point H in the line of action of W_1.

First, draw HK parallel to BO to cut the line of action of W_2 in K.

Draw HX parallel to AO.

Through K draw KL parallel to CO to cut the line of action of W_3 in L.



Through L draw LX parallel to OD, cutting HX in X.

Then X is a point in the line of action of the resultant of the forces W_1, W_2, W_3.

Draw XG parallel to W_1.

Next, suppose the parallel forces to act in some other direction; we can then by a similar process find another point X' through which the line of action of the resultant of the parallel forces in their new direction passes.

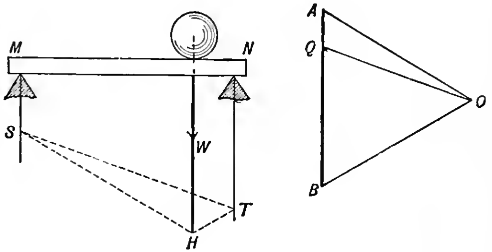
Draw through X' a line parallel to the forces; this line must cut XG in the required centre G.

NOTE.—It is convenient to take the new direction at right angles to the old. Then the same diagram of the polygon of forces will serve; we have only to draw $H'K'$ *perpendicular* to $B'O$; and so on; as in the figure.

In the second part of the construction, lines are drawn from G_1, G_2, G_3 *perpendicular* to $ABCD$; in G_1H' a point H' is chosen, and then lines $H'K'$, $K'L'$, $L'X'$, $H'X'$ are drawn *perpendicular* to BO, CO, DO, AO respectively; then through X' a line is drawn perpendicular to AD to cut XG in G.

Example i. *A beam supported at each end has placed upon it a weight* W; *neglecting the weight of the beam, find by the graphic method the pressure on each end.*

With the usual construction draw AB to represent W; join AO, BO; take any point H in the line vertically below W and draw HS, HT lines parallel to OA and BO, cutting vertical lines through M and N (the points of support) in S, T respectively; join ST; draw OQ parallel to ST; then AQ represents the pressure on M, one point of support, and QB represents the pressure on N, the other point of support.



The proof consists in shewing that W is the resultant of the forces AQ, QB acting in the vertical lines through S and T.

The proof we leave to the student. It may be abbreviated from that of the next Example.

Example ii. *A beam supported at each end is loaded at different points with three weights; find the pressure on the supports at the ends.*

Let W_1, W_2, W_3, be the weights.

Draw AB, BC, CD to represent W_1, W_2, W_3, respectively, and join OA, OB, OC, OD.

Take a convenient point H in the vertical line through W_1.

Draw HK, KL, LT, HS parallel to BO, CO, DO, AO respectively.

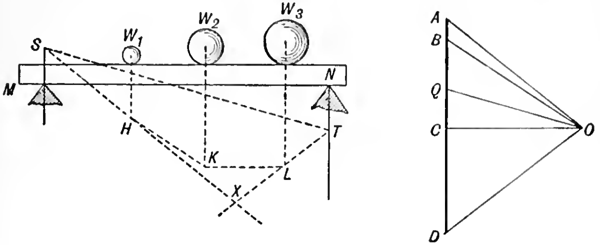
Let HS, LT intersect the vertical lines through the supports M and N in S and T respectively.

Join ST; through O draw OQ parallel to ST.

Then QA represents the pressure at M,

DQ represents the pressure at N.

The proof consists in shewing that forces represented by AB, BC, CD, DQ, QA acting vertically at points W_1, W_2, W_3, N, M are in equilibrium.



The student is recommended to work out the proof for himself before reading the following.

By the triangle of forces, the force QA at M may be replaced by forces represented by OA, QO, acting along HS and ST.

DQ at N may be replaced by forces represented by DO, OQ acting along LT and TS.

W_1 or AB may be replaced by forces represented by AO, OB acting along SH, KH.

W_2 or BC may be replaced by forces represented by BO, OC acting along HK, LK.

W_3 or CD may be replaced by forces represented by CO, OD acting along KL, TL.

It will be found that along each of the lines HK, KL, LT, TS, SH, a pair of equal and opposite forces are acting which are therefore in equilibrium.

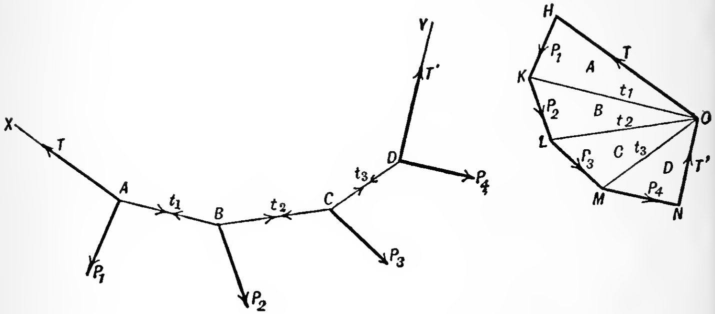
NOTE.—$SHKLT$ is a *funicular polygon.*

Let vertical forces act at S and T, equal respectively to QA and DQ respectively.

Let ST be a rod along which acts a stress represented by QO.

Then, if $SHKLT$ be a string having weights W_1, W_2, W_3 fastened at H, K, L respectively, the rod and the string will be in equilibrium and the stresses along the parts of the string SH, HK, KL, LT are represented by OA, OB, OC, OD respectively.

Example iii. *A light flexible string* XABCDY *is acted on by given forces* P_1, P_2, P_3, P_4 *at the points* A, B, C, D *of the string; find the conditions of equilibrium.*



Draw the polygon of forces $HKLMN$, where $HK=P_1$, $KL=P_2$, $LM=P_3$, $MN=P_4$; from H and N draw HO, NO parallel to the direction of the string XA, DY.

Join KO, LO, MO.

Then HO, KO, LO, MO, NO represent the tensions T, t_1, t_2, t_3, T' of the strings XA, AB, BC, CD, DY.

For, **first** consider $ABCD$ to be a rigid body acted on by the external forces T, P_1, P_2, P_3, P_4, T'.

These forces form a system in equilibrium, and HK, KL, LM, MN represent P_1, P_2, P_3, P_4 and HO is parallel to XA; NO is parallel to DY.

Therefore, by the polygon of forces, OH represents the force along AX and NO the force along DY.

Next. The point A of the string is in equilibrium under the action of the forces T, P_1, t_1, and the triangle HOK has its sides parallel to these forces. Also HK represents P_1;

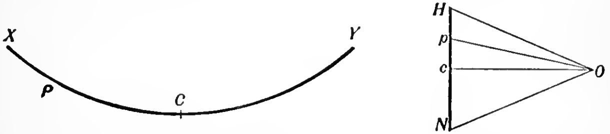
Therefore KO represents t_1.

Similarly LO represents t_2, and MO represents t_3.

NOTE.—It may be noticed that there is a triangle of forces for each point A, B, C, D; that there is a polygon $HKLMNO$ for the external forces; and that the internal forces t_1, t_2, t_3 are represented by internal lines. In fact the internal line OK represents a stress and belongs to two triangles, the A triangle and the B triangle.

Corollary. In the above figure suppose P_1, P_2, P_3, P_4 to act vertically downwards and to represent the weights of equal particles fastened to the string; then $HKLMN$ would be a vertical line in which $HK=KL=LM=MN$.

Suppose the weights to become more numerous, being still equal to each other and applied at equal intervals along the string. We have then a means of finding the tension in a **Catenary** (the curve described by a heavy uniform chain suspended from two fixed points).



Let HN represent the polygon of forces for the weights of the particles of the chain; then HN represents the weight of the chain. Draw HO parallel to the direction of the curve at X, and draw NO parallel to the direction of the curve at Y.

Then HO represents the tension of the string at X, and NO that at Y.

Let X and Y be in the same horizontal line, then the curve will be symmetrical and $HO=ON$.

Draw Oc perpendicular to HN; take P any point in the chain and let C be the middle point of the chain; then Oc represents the tension at C; let Hp represent the weight of XP; then Op represents the tension at P; and

$$Op^2=Oc^2+cp^2;$$

$\therefore$ sq. of tension at P=sq. of tension at c+sq. of weight of CP.

Also the direction of the tangent to the catenary at P is the same as that of Op, the tension at P; therefore it makes an angle with the horizon whose tangent is

$$\frac{pc}{co}=\frac{\text{weight of } PC}{\text{tension at } C}.$$

Again, the horizontal resolute of the tension Op at any point P is Oc, which is the same for all points, and is the tension at C.

CHAPTER XVII.

COUPLES.

233. IN this chapter we call attention to a few points concerning Couples, using the principles of Chapter V.

A **couple** consists of two forces which are equal parallel and of opposite senses but not in the same line.

If the forces forming a couple were in the same line of action they would be in equilibrium.

234. *PROP. Let a system of forces acting in one plane on a rigid body not in equilibrium be such that if an exactly equal and parallel system of forces all passed through the same point, they would be in equilibrium; then this system of forces is equivalent to* a couple.

For, by Art. 86, the system of forces is equivalent to two forces applied at two chosen points A and B, one of the forces (that at B) being perpendicular to AB.

Since the system of forces is such, that if the forces acted at a point they would be in equilibrium, therefore the sum of their resolutes in any direction is zero.

Therefore the sum of the resolutes in any direction, of the equivalent forces at A and B, is zero.

Therefore the force at A must be equal parallel and of opposite sense to the force at B.

In other words, the system of forces is statically equivalent to a couple.

235. *PROP. When a system of forces in one plane acting on a rigid body is not in equilibrium, but is such that an exactly equal and parallel system of forces, passing all through one point, is in equilibrium, then the sum of the moments of these forces about any point in their plane is* constant.

This system of forces is statically equivalent to a couple. [Art. 234.]

The moment of a couple is constant for any point in its plane [Art. 77]; which proves the proposition.

236. *PROP. Two couples are statically equivalent which are in the same plane and have equal like moments.*

For two couples which are in the same plane and have equal and opposite moments are in equilibrium.

For they satisfy the conditions of Art. 91.

237. *A number of couples in the same plane are statically equivalent to a single couple whose moment is the sum of the moments of the couples.*

For each couple can be replaced by a couple of equal moment having an arm common to all.

Example. Forces represented in direction, magnitude and line of action by the sides of a triangle taken the same way round form a couple.

Let ABC be the triangle, from A draw AD perpendicular to BC; the sum of the resolutes along BC of the forces represented by AB, BC, CA are BC, CD, DB whose sum is zero.

Similarly the sum of the resolutes along AB is zero.

Therefore the forces represented by AB, BC, CA if they acted at one point would be in equilibrium.

Therefore they are equivalent to a couple. Q. E. D.

Therefore the sum of their moments about any point is constant.

The sum of their moments about A is represented by twice the area of the triangle ABC.

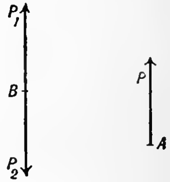
238. The student should endeavour to get a practical idea of a couple by experiment.

The twist applied to a screw driver (without pressure on the head of the screw) is a couple.

The two forces applied by the fingers to the handle of a tap form a couple.

The full effect of a couple can hardly be understood without some knowledge of rigid dynamics; just as the full effect of a single force cannot be realized without the study of linear dynamics.

239. *PROP. A force* P *acting at any point* A *of a rigid body is equivalent to an equal parallel force acting at any other point* B *of the rigid body and a couple whose moment is equal to* P × *the perpendicular distance between the forces.*



At the point B let two opposite forces P_1, P_2 each equal and parallel to P be applied to the rigid body.

These two forces will not disturb the equilibrium, and the three forces are statically equivalent to P.

But P_2 and P are a couple whose moment is equal to $P \times$ the perpendicular distance between P and P_2.

Thus P is statically equivalent to P_1 and this couple.

N.B. This couple is statically equivalent to a couple of equal moment in the same plane acting anywhere on the rigid body.

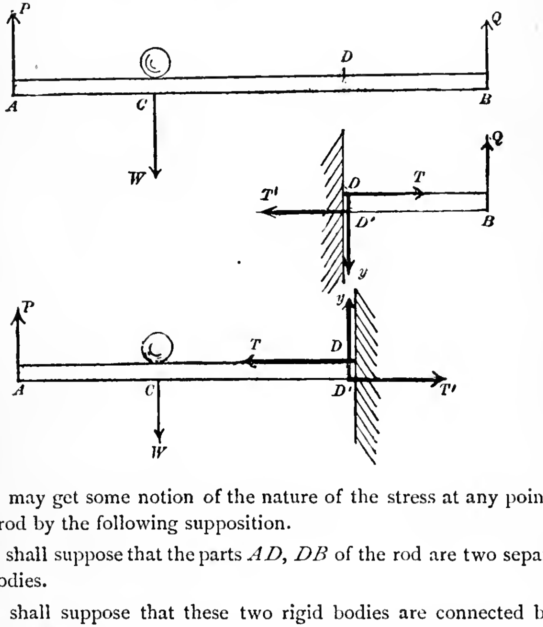
Thus the point of application of a force may be moved from any point of a rigid body to any other point whatever, without changing its effect on the rigid body as a whole, provided that at the same time the proper couple be applied to the rigid body on which it acts.

240. In the following examples we consider the nature of the strain at a point of a rigid rod.

Example i. *A light horizontal rigid rod* AB *is supported at its ends* A *and* B *by rigid supports, and a weight* W *is fixed at a point* C *of the rod; find* **the tendency to break** *at a chosen point* D *of the rod.*

Since the rod is in equilibrium we have, if P and Q are the upward pressures on the rod at A and B respectively,

$$P+Q=W \text{ and } P\times AC=Q\times BC.$$



We may get some notion of the nature of the stress at any point D of the rod by the following supposition.

We shall suppose that the parts AD, DB of the rod are two separate rigid bodies.

We shall suppose that these two rigid bodies are connected by a small hinge at D, the highest point, and by a very short wire at D', the lowest point, of the section of the rod at DD'.

Now since each part is in equilibrium, it would make no difference, so far as the rod DB is concerned, if the rod AD were replaced by a fixed wall to which the rod DB is attached by the hinge D and the wire D' respectively.

We shall suppose then that the rod DB is fastened to a vertical wall in this way.

We proceed to consider what stresses in the hinge at D and in the wire at D' are necessary to keep the part of the rod DB at rest.

Let the action of the hinge D on the rod DB be a vertical force downwards Y and a thrust T.

Let the action of the wire D' on the rod DB be a pull T'.

Then (i) resolving horizontally, (ii) resolving vertically, and (iii) taking moments about D, we have

$$T = T' \quad \text{.................................(i)},$$

$$Y = Q \quad \text{.................................(ii)},$$

$$T \times DD' = Q \times BD' \quad \text{.........................(iii)}.$$

Hence the hinge and wire must apply to the beam BD a vertical force downwards $= Q$, and a couple whose moment

$$= Q \times BD'.$$

Similarly by considering the beam BD to be replaced by a fixed wall we shall find that the rods must apply to the beam AD a vertical force upwards

$$= W - P = Q,$$

and a couple T, T', such that

$$T \times DD' = P \times AD - W \times CD = Q \times BD.$$

Hence, the stresses in the short connecting hinge and wire at D consist

(i) of a tendency on the part AD to slide down DD' measured by the force Q; this part of the stress is called a **shearing stress**,

(ii) of a tension T in the hinge D', and a compression T in the wire D, such that

$$T \times DD' = Q \times BD.$$

It should be noticed that when DD', the thickness of the rod at D is diminished, the tension and compression T and T' increase.

And that when DD' is small compared with BD, T is large when compared with Q.

If the stresses are investigated for other points than D of the rod the shearing stress will be found to be uniform throughout the portion CB of the rod, while throughout AC the shearing stress

$$= P = W - Q.$$

We have seen that T the tension at D' is to Q as $BD : DD'$ and in a *thin rod* when D is near the middle the ratio would be supposed to be large. Hence in the case of a rod it is customary to consider that its tendency to break is measured by the magnitude of T.

Now T is proportional to $Q \times BD$, that is to the moment of the force acting on *one point of the rod* about the point D.

In actual iron beams with heavy top and bottom flanges and light connecting webs, as in the case of bridge girders, rolled iron beams etc., the slight resistance of the web is often neglected for convenience in calculation and to leave a margin for safety. The flanges are then regarded as acting like the hinge and wire at D, D' in the figure, and their stresses are found in the same way. [See Ex. **14**, p. 229.]

But in solid rectangular beams (as of timber, etc.) this method is not sufficiently approximate.

In such beams the resistance of the cross section is much more complex, the fibres throughout offering resistances which vary with their distance from the central axis of the beam.

But it is still approximately true that the tendency to break at any point increases proportionally to the sum of the moments about that point of all the forces acting at either of the parts of the beam into which it is divided at that point. Hence

The **tendency to break** at a given point of a rigid rod is measured by the sum of the moments of all the forces acting on one of the parts into which the rod is divided at that point.

Most materials used for beams, such as well-seasoned wood, iron, stone, exert greater resistance to crushing than to tension. Hence when a beam breaks the fracture most frequently occurs first in the part under tension; which, in a beam loaded between its supports as in the above example, is the lower part of its section.

When a beam is loaded at its *ends* and supported between them, or when it is fixed at one end and loaded at the other, the upper rod at D would be in *tension*, and the lower one in *compression*.

Example ii. *A* heavy *uniform rod* AB, *of weight* w, *is supported by fixed supports at each end and is loaded by a weight* W *at* C; *find at what point in its length the tendency to break is greatest.*

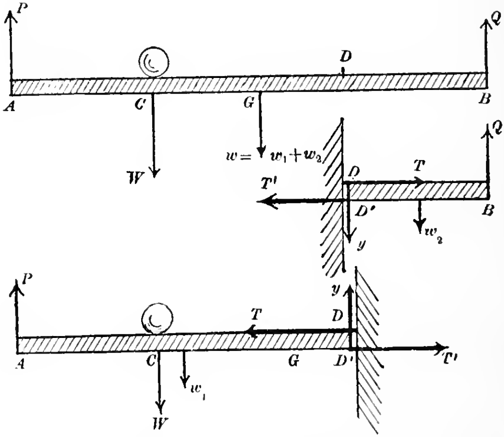
Let G be the middle point of the rod and let C be between A and G. We shall use the method and notation of Example i.

Since the rod AB is in equilibrium we have

$$P+Q=W+w,$$

and $$Q\times BA=w\times\tfrac{1}{2}AB+W\times AC.$$

Also since the rod is uniform the weight, w_1, of any portion BD of the rod $=\dfrac{BD}{AB}w$.



I. Consider the tendency to break at a point in the part CB of the rod.

Take a point D in CB; then with the notation of *Example* i. we have

$$T=T',$$

$$Y=Q-w_1 \quad \text{[where } w_1 \text{ is the weight of the rod } BD\text{]};$$

$$T\times DD'=Q\times BD-w_1\times\tfrac{1}{2}BD,*$$

$$=Q\times BD-w\times\tfrac{1}{2}\frac{BD^2}{AB}.$$

Now we have explained that the tendency of the rod to break is supposed to be measured by the magnitude of T.

Therefore this tendency is greatest for different positions of D, when

$$Q\times BD-w\times\tfrac{1}{2}\frac{BD^2}{AB} \text{ is greatest};$$

* The student should notice that this is **the moment-sum about D of all the external forces which act on the part BD** of the rod.

that is, when $\frac{w}{2AB}\left\{\frac{2Q}{w}AB \cdot BD - BD^2\right\}$ is greatest;

that is, when $\left\{\frac{Q^2}{w^2}AB^2 - \left(\frac{Q}{w}AB - BD\right)^2\right\}$ is greatest;

that is, when $\left(\frac{Q}{w}AB - BD\right)$ is least;

that is, when $\left(\frac{1}{2}AB + \frac{W}{w} \cdot AC - BD\right)$ is least;

that is, when $\left(\frac{W}{w}AC - GD\right)$ is least,

supposing that D now lies between C and G.

Therefore the tendency to break in the part CB is greatest at D when $GD = \frac{W}{w}AC$; unless $\frac{W}{w}AC$ is greater than GC, in which case *the tendency to break* is greatest at C.

II. It will be found that the tendency to break in the part AC is, in any case, greatest at C.

Thus the tendency to break in the rod is greatest at C unless $\frac{W}{w}AC$ is less than GC, in which case the tendency to break is greatest at the point D between C and G such that $GD = \frac{W}{w}AC$.

241. We may here **sum up** the results which have been proved with regard to a system of forces acting on a rigid body in one plane.

I. Any such system of forces **not** in equilibrium is statically equivalent to a single force, *or* to a couple.

II. If it is equivalent to *a couple*, then

The sum of their moments about any point in the plane is constant.

The sum of their resolutes in any direction is zero.

If forces equal and parallel to these acted at a point, they would be in equilibrium.

III. If it is equivalent to *a single force*, then

The sum of the resolutes of the given forces in any direction is equal to the resolute of this single force in that direction;

Also the sum of their moments about any point is equal to the moment of this resultant about that point.

IV. A system of forces acting on a rigid body **is** in equilibrium when it is equivalent neither to a force, *nor* to a couple.

For *Example*. If the sums of their moments about three given points not in the same straight line are each zero, the system is in equilibrium.

For this statement cannot be true of a single force nor of a couple.

Again. If the sums of the moments about any two points A and B are each zero and the sum of the resolutes parallel to AB is zero the system is in equilibrium.

EXAMPLES. XXXII.

1. Forces represented in magnitude and line of action by the sides of two triangles taken opposite ways round are in equilibrium provided the triangles are of equal area.

2. Forces represented in direction and line of action by the sides of a polygon taken the same way round are equivalent to a couple whose moment is represented by twice the area of the polygon.

3. ABC is a triangle; D, E, F are points in the sides such that $\frac{BD}{DC}=\frac{CE}{EA}=\frac{AF}{FB}=k$; shew that the sum of the moments about A of forces represented by AD, BE, CF is represented by $\frac{1 \sim k}{1+k}$ of twice the area of the triangle ABC; hence shew that the forces cannot be in equilibrium unless $k=1$.

4. Prove that a system of forces in one plane acting on a rigid body is in equilibrium if the sums of their moments about two points A and B are each zero, and the sum of the resolutes in any direction, not perpendicular to AB, is also zero.

5. What is the resultant of two couples, and how is it found?

6. In the example of page 223 find at what point of the rod there is the greatest tendency to break.

7. A heavy rod is supported in a horizontal position by two rigid supports one at each end; shew that the point at which there is the greatest tendency to break is the centre of gravity.

8. A man weighing 160 lbs. walks across a uniform heavy plank weighing 100 lbs. Compare the tendency of the plank to break (i) when he is a quarter of the way across and (ii) when he is half way across with its tendency to break when the man is not on the plank.

9. Find the tendency to break of the rod in Example iii, p. 70, at the points B, O, and C.

10. Find the tendency to break of the rod in Example iv, p. 70, at the point B and at the middle point of the rod.

11. A uniform heavy rod of weight W and length $2a$ is supported in a horizontal position by a rigid support at each extremity; shew that the shearing force and bending moment at a point distant x from one end are

$$\tfrac{1}{2}W\left(1-\frac{x}{a}\right) \text{ and } \tfrac{1}{2}Wx\left(1-\frac{x}{2a}\right) \text{ respectively.}$$

12. Shew that other things being equal, a lever of the second class is less likely to break than a lever of the first class in supporting the same weight.

13. Three straight tobacco pipes with long tubes rest upon a table with their bowls mouth downwards in the angles of an equilateral triangle, the tubes being supported in the air in an approximately horizontal position by crossing symmetrically each under the second and over the third so as to form another equilateral triangle; shew that the mutual pressure of the tubes, considered vertical, varies inversely as the side of the last triangle.

14. A rolled iron joist 20 feet long, whose section is $ABCDEF$, such that AB and CD are each 6 inches by $\frac{1}{2}$ inch and EF is 1 foot long, is supported by its ends in a horizontal position; find what load can be safely placed at any point in its length by the method of p. 225; assuming that the strength of the web EF is sufficient to support the weight of the joist itself and that rolled iron can just sustain a tensile strain of 20 tons per square inch.

15. If in the last example AB and CD are each 8 inches by 1 inch, $EF=10$ inches, the length of the joist 16 feet, and the material steel, capable of bearing a tensile stress of 45 tons per square inch, what is then the load which the joist can safely bear?

242. **In Solving a Statical Problem, remember,**

That conditions I and II of Art. 83 are true of the *external* forces acting on any body (or bodies), whether rigid or not, when the body is in equilibrium;

That when conditions I, II, III of Art. 91 are satisfied by the *external* forces which act upon a *rigid* body, that body must be in equilibrium.

Hence, in solving a Statical Problem we fix our attention on a rigid body, then, drawing a figure,

I. We draw lines to represent the forces.

When we know the direction and magnitude of a force we can represent it by a straight line of definite length.

When we know the direction and not the magnitude we draw an indefinite line in the known direction and represent the unknown magnitude by some letter P, or Q....

When we know neither the direction nor the magnitude, we choose two convenient directions at right angles to each other and represent the rectangular components of the force each by some letter x, or y....

II. We consider carefully whether from the conditions of the question the direction of any reaction on the rigid body is known.

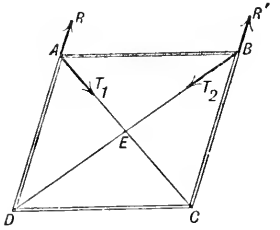
III. We then write down three equations derived from the conditions I, II, III of Art. 91.

243. If in any question there is more than one rigid body we consider each rigid body in turn.

It will be found by experience that a great deal depends on the choice of the direction in which we resolve forces, and also on the choice of the point about which we take moments. A convenient choice will often simplify the subsequent equations, and thus enable us to avoid awkward algebraic transformations.

244. We proceed to give a few examples of the application of the above methods and principles to statical problems.

Example i. *Four equal rods hinged together in the form of a rhombus* ABCD, *are placed on a horizontal table, their extremities* A, C *and* B, D *being connected by two strings each under tension.* Compare these tensions.



We may consider the extremities of the strings to be attached to the rods AB and DC; so that the rod BC is acted on only by the actions of the rods at its extremities; hence the actions on the rod BC must be equal and opposite, i.e. along the rod BC; similarly the actions on the rod AD must be along the rod AD; hence,

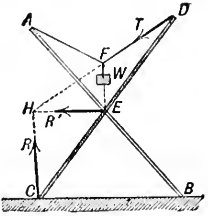
Considering the forces acting on the rod AB, they are as drawn.

The resultant of the parallel forces, R, R' is a parallel force and must pass through the intersections of the tensions at E. The rod AB would therefore be at rest when under the action of the two tensions T_1, T_2 and of a force $=(R+R')$ applied at the right point in the direction CB; these forces, therefore, if they acted at one point would be in equilibrium. Hence, the sides CB, BE, EC of the triangle BCE, being parallel to this resultant and the two tensions, are proportional to them respectively.

Therefore $$T_1 : T_2 = EC : EB.$$

NOTE. When a rod is acted on by forces which are applied *only* at each of its ends, these forces must be equal, opposite and along the rod.

Example ii. *Two equal light rods* AEB, CED *are hinged together at their middle points* E *and are placed in a vertical plane with their ends* CB *on a smooth horizontal plane; a string carrying a weight* W *at its middle point* F, *has its ends fastened to the ends* A *and* D *of the rod.* Prove that in the position of equilibrium F bisects the perpendicular from E on AD.



Since the whole figure is symmetrical about the vertical through E the forces must be symmetrical about that line. *Therefore the stress at* E, *the hinge, must be perpendicular to* FE.

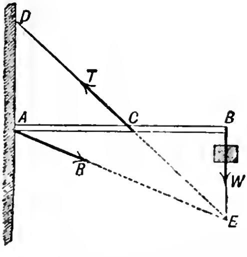
Consider the forces acting on the rod CD.

They are as drawn in the figure; and the three forces meet in a point H. HC is parallel to FE and $CE=ED$;

$$\therefore \ FE=\tfrac{1}{2}HC. \qquad \text{Q. E. D.}$$

Example iii. *A light rod* ACB, *which is hinged at* A *to a vertical wall* AD, *and which is supported in a horizontal position by a string fastened to the rod at* C *and to the wall at* D, *supports a weight* W *fastened to its extremity* B; find the stress between the wall and the rod at A.

Method I. *Geometrical.*



Consider the forces acting on the rod ACB.

They are W vertically downwards at B,
a tension T along the string CD,
the action R at the hinge.

The directions of these three forces meet in a point.

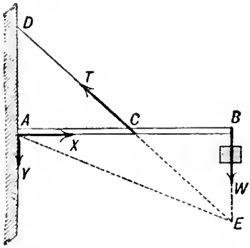
Let DC produced meet the vertical through B in E.

Then the direction of R is along AE.

AED is a triangle whose sides are parallel to R, T, W;

$$\therefore\ R : T : W = AE : ED : DA.$$

Method II. *By resolution.*



With the same notation as above let R be resolved into two rectangular components, X horizontal, Y vertical;

let $ACD = \alpha$.

Then taking the sums of (i) the horizontal resolutes, (ii) the vertical resolutes, (iii) the moments about A, we have

$$X - T\cos\alpha = 0 \qquad \text{(i)}$$

$$T\sin\alpha - Y - W = 0 \qquad \text{(ii)}$$

$$T \times AC\sin\alpha - W AB = 0 \qquad \text{(iii).}$$

Whence

$$X = T\cos\alpha = W\frac{AB}{AC}\cot\alpha = W\frac{AB}{AC}\times\frac{AC}{AD} = W\frac{AB}{AD},$$

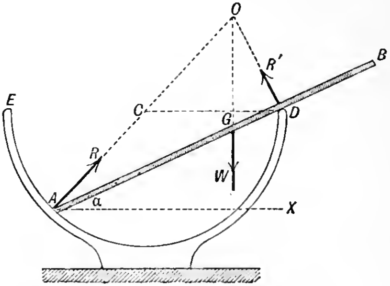
$$Y = T\sin\alpha - W = W\left(\frac{AB}{AC} - 1\right) = W\frac{CB}{AC} = W\frac{BE}{AD}$$

[by similar triangles CBE, CAD],

$$\therefore\ R = \sqrt{(X^2 + Y^2)} = W\sqrt{\frac{AB^2 + BE^2}{AD^2}} = W\frac{AE}{AD}.$$

This agrees with the first solution.

Example iv. *A smooth uniform rod* ABC *of length* 2a *and weight* W *is placed in a smooth hemispherical bowl* DAE *of radius* r *which is fixed so that its rim is horizontal.* Find the position of equilibrium.



Consider the forces acting on the rod; they are, (i) its weight acting at G its middle point; (ii) the action R' of the rim on the rod at B perpendicular to the rod; (iii) the action R of the bowl on the rod at A which is perpendicular to the bowl and passes through the centre C of the hemisphere.

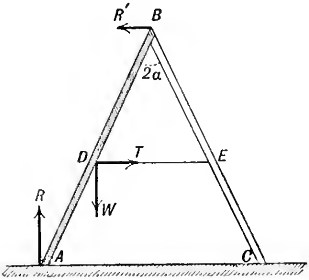
[The directions of these three forces must meet in a point and the problem could be solved geometrically.]

Let the rod make the angle $\alpha = BAX$ with the horizontal line AX; then
$$CAD = CDA = BAX = \alpha.$$

Taking the sum of the resolutes along the rod, we have
$$R \cos\alpha - W \sin\alpha = 0.$$

Taking the sum of the moments about B, we have
$$R \times BA \sin\alpha - WGB \cos\alpha = 0;$$
$$\therefore\ R \times 2r \sin\alpha = W(2r\cos\alpha - a)\cos\alpha;$$
$$\therefore\ W \times 2r \sin^2\alpha = W(2r\cos\alpha - a)\cos\alpha;$$
$$\therefore\ 2r - 4r\cos^2\alpha + a\cos\alpha = 0;$$
$$\therefore\ \cos\alpha = \frac{a \pm \sqrt{(a^2 + 32r^2)}}{8r}.$$

Example v. *Two equal uniform beams* ADB, BCE *each of weight* W *hinged at* B *stand inclined to each other at an angle* 2α *in a vertical plane, on a smooth horizontal plane. Their middle points* D, E *are joined by a light string.* Find the conditions of equilibrium.



I. Consider the forces acting on the beam ADB.

The beams act upon each other at B.

Let R' be the action of the beam CEB upon ADB.

The reaction of ADB upon CEB is equal and opposite to R'. The beams are symmetrical about the vertical line through B; therefore the action between the beams at B must be symmetrical with the vertical line, hence the action at B must be *horizontal.*

The action R between the beam and the vertical plane at A must be perpendicular to the plane because it is smooth. The tension T of the string DE is along the string which is horizontal.

Thus, the forces acting on the beam AB are (i) R' horizontally at B; (ii) T horizontally at D; (iii) W vertically downwards at D; (iv) R vertically upwards at A.

Hence taking the sum of the horizontal resolutes, we have

$$R' - T = 0.$$

Taking the sum of the vertical resolutes, we have

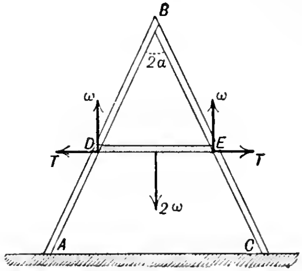
$$R - W = 0.$$

Taking the sum of the moments about D, we have

$$R \sin \alpha = R' \cos \alpha.$$

Whence $$T = R' = R \tan \alpha = W \tan \alpha.$$

Example vi. *In the above example let the string be replaced by a beam whose weight is* 2w. Find the conditions of equilibrium.



I. Consider the forces acting on the beam *DE*.

Let the action of the beam *AB* upon it be resolved into two forces, one *T* along the beam, and the other *X* vertically.

Let the action of the beam *BEC* upon *DE* be similarly represented by *T′* and *X′*. The only other force acting on *DE* is its own weight.

Hence, from the horizontal resolute sum we have

$$T - T' = 0;$$

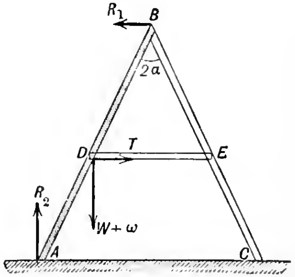
from the vertical resolute sum we have

$$X + X' - 2w = 0;$$

and the sum of the moments about the middle point gives us that

$$X - X' = 0.$$

Hence $$X = w.$$



II. Consider the forces acting on the rod *ADB*.

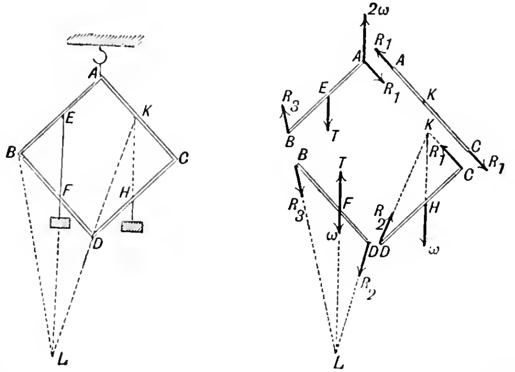
With the notation of the last example we have the action R_1 at *B* of *BEC* on *ADB* is horizontal; R_2 is vertical; the action of *DE* on *ADB* consists of *w* vertically downwards, and *T* horizontal towards the right.

Then, as before, we have

$$T - R_1 = 0 \quad \ldots\ldots \text{(i)}$$
$$R_2 - W - w = 0 \quad \ldots\ldots \text{(ii)}$$
$$R_2 \sin \alpha - R_1 \cos \alpha = 0 \quad \ldots\ldots \text{(iii)}.$$

Whence $\quad T = R_1 = R_2 \tan \alpha = (W + w) \tan \alpha.$

Example vii. *A framework in the form of a square* ABDC *made of four weightless rods hinged at their extremities, is suspended from* A; *the middle points of* AB, BD *are joined by a string, and equal weights* ω, ω *are suspended from* F, H *the middle points of* BD, DC; find the direction of the stresses at the angular points of the square.



We may suppose the string at A to be fastened to the rod AB; then considering the forces acting on each rod separately, we have

(i) the rod AC is acted on only by the two actions of the other rods at its extremities. These actions R_1, R_1 must be equal and opposite, and therefore must act along the rod as drawn;

(ii) the rod DC is acted on by the weight ω at H; by R_1 at C as drawn and by R_2 the action at D. The two forces ω and R_1 meet at K the middle point of AC; therefore R_2 acts along DK;

(iii) the rod BD, is acted on by R_2 as drawn passing through K; by $T - \omega$ vertically upwards at F; if these forces intersect in L, the direction of the stresses at B must be along BL.

The magnitudes of the stresses can now be found from the geometry. The values are

$$R_1 = \tfrac{1}{4}\sqrt{2}\,\omega; \quad R_2 = \tfrac{1}{4}\sqrt{10}\,\omega; \quad R_3 = \tfrac{1}{4}\sqrt{26}\,\omega; \quad T = (1 + \sqrt{2})\,\omega.$$

Example viii. *An isosoceles triangular lamina* ABC *is placed in a vertical position with its base* BC *against a smooth vertical wall and is supported in that position by a smooth horizontal rod* O *parallel to the wall.* Find the limits of the distance of the rod from the wall.

Suppose the triangle on the point of toppling over.

Then the pressure on the wall must be all concentrated at an angular point of the base.

The two cases are indicated in the figures and the forces acting on the triangles are as drawn.

Let the angle $BAC = 2a$.

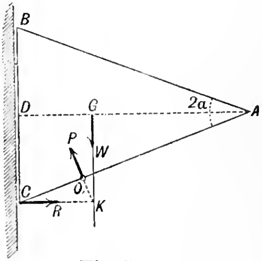


Fig. I.

Then in Fig. I. the perpendicular from O on the wall

$$= CO \cos a = CK \cos^2 a = \tfrac{1}{3} DA \cos^2 a.$$

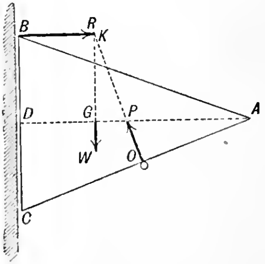


Fig. II.

In Fig. II. the perpendicular from O on the wall

$$= DG + OK \sin a \text{ [and } OK = BC \cos a - BK \sin a]$$

$$= \tfrac{1}{3} DA + DC \sin a \cos a - \tfrac{1}{3} DA \sin a = DA\{\tfrac{1}{3} - \tfrac{1}{3}\sin a + 2\sin^2 a\}.$$

EXAMPLES. XXXIII. PROBLEMS.

1. Two equal uniform rods AEB, BFC are hinged at B and their middle points E, F are joined by a string whose length is less than that of AB; they are suspended from a fixed point by a second string attached to the point A; shew that the tension of the first string is double that of the second.

2. Two equal uniform rods ABC, BDF each of weight W are hinged at their middle points and placed in a vertical plane with their extremities A, D on a smooth horizontal plane so that the angle $ABD=2\alpha$, and are kept in that position by a string joining A and D; prove that the stress at the hinge $= W \tan \alpha$.

3. Two equal uniform rods AB, BC each of weight W are hinged at B and are placed in a vertical plane with their extremities A, C on a rough horizontal plane so that they are inclined to each other at the angle α; prove that the stress at the hinge = the amount of friction called into play $=\frac{1}{2} W \tan \alpha$.

4. A uniform rod ABC has a string attached to the point B, where $BC=3\,AB$, of length equal to AB, the other end of the string is fastened to a point D in a smooth vertical wall; the rod is placed with the end A against the wall and is supported by the string, the string and the rod being in a vertical plane. Shew that the rod will be in *neutral* equilibrium.

5. A uniform rod of length $2l$ rests with one end against a smooth vertical wall and is placed across a smooth horizontal bar fixed at the distance a from the wall; prove that if α be the inclination of the rod to the horizon $\cos^3 \alpha = \frac{a}{l}$.

6. A heavy sphere of weight W rests between two planes inclined at angles α_1, α_2 to the horizon; if the pressures on the planes are R_1, R_2, prove that

$$\frac{W}{\sin(\alpha_1+\alpha_2)} = \frac{R_1}{\sin \alpha_2} = \frac{R_2}{\sin \alpha_1}.$$

7. A ladder $(a+b)$ feet long whose C. G. is a feet from the lower end, is placed with one end *on the top* of a smooth wall, the other end on the ground which is rough; shew that the amount of friction called into play is $\frac{W a \sin^2 \alpha + b \cos^2 \alpha}{a+b}$, where α is the inclination of the ladder to the horizon.

8. Two equal heavy uniform beams AB, BC each of weight W hinged together at B rest in a vertical plane with their ends A, C supported by the tops of two walls in the same horizontal line; prove that the horizontal thrust on each wall is

$$\tfrac{1}{4} W \frac{AC}{\sqrt{(AB^2 - \frac{1}{4} AC^2)}}.$$

9. If the beams of Question 8 are placed on a smooth horizontal plane, prove that the bending moment of the strain on the hinge is $\frac{1}{4} W \times AC$.

10. Two equal heavy uniform beams AB, BC are hinged together at B and their other ends rest on a smooth horizontal plane, the plane ABC being vertical. The rods are kept from slipping by a string joining A to the middle point of BC. Prove that when ABC is an equilateral triangle, the tension of the string is half the weight of one of the rods.

11. If the beams in Question 10 are kept from slipping by a string joining any point to AB to any point in BC, prove that the stress at the hinge is parallel and equal to the tension of the string.

12. Two equal uniform beams AEB, BFC hinged at B, stand in a vertical plane with their ends A, C on a smooth horizontal plane; their middle points E, F are joined by a beam of like material so that BEF is an equilateral triangle, and a weight equal to a quarter that of the beam AB is suspended from B; prove that the horizontal stress along the horizontal rod is $\sqrt{3}$ times its weight.

13. A pair of steps AB, BC consisting of two uniform heavy ladders each of length a and weight W_1, freely hinged together at their upper ends and inclined to each other at the angle 2α, are placed in a vertical plane with their ends AC on a rough horizontal floor. A man of weight W ascends the ladder AB to a distance c from A. Prove that the end C will slip before B; and that if θ be the inclination to the vertical of the stress at the hinge

$$\tan\theta = \tan\alpha\left(\frac{aW_1}{cW} + 1\right).$$

14. A pair of steps in the form of two equal uniform ladders ABC, CDE, each of length a and weight W, freely jointed at C, stand in a vertical position with their feet A, E on a smooth horizontal plane, and the points B, D are connected by a string when $CB = CD = b$; a man of weight W' walks a distance c up one of the ladders; prove that the tension of the string is

$$\frac{Wc + W'a}{2b}\tan\tfrac{1}{2}ACE.$$

15. Two equal uniform heavy beams each of length $2a$ are freely hinged at one extremity and are placed astride on a smooth circular cylinder of radius b whose axis is horizontal. Shew that the inclination α of each beam to the vertical is given by the equation

$$a \sin^2 \alpha - b \cos \alpha = 0.$$

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16. A square lamina rests with its plane perpendicular to a smooth wall, one corner resting against the wall and another corner being attached to a point in the wall by a string whose length is equal to a side of the square. Shew that the distances of its angular points from the wall are as

$$0 : 1 : 3 : 4.$$

17. A picture frame, rectangular in shape, is suspended against a smooth vertical wall by two strings, each equal to the height of the frame, attached to two points A, B in the upper edge of the back of the frame and to two points C, D in the wall in the same horizontal line such that $AB = CD$; shew that if the C. G. of the frame coincides with its centre of figure, it will rest against the wall at the inclination $\tan^{-1} \dfrac{b}{3a}$ where a = the height and b = the thickness of the frame.

18. An isosceles triangular lamina of height h and vertical angle 2α is placed in a vertical plane with its base against a smooth vertical wall and is supported by a string attached to its vertex and to a point in the wall; prove that the greatest possible length of the string is

$$h\sqrt{(1 + \tfrac{9}{4}\tan^2 \alpha)}.$$

19. An equilateral triangular lamina is suspended against a smooth vertical wall by means of a string fastened to the middle point of one side of the triangle and to a point in the wall, shew that when there is equilibrium the tension of the string is independent of its length; and that equilibrium is impossible if the string be longer than the height of the triangle.

20. A uniform right circular cylinder of height $2b$, and radius a, with its base resting against a smooth vertical plane, is supported by a string fastened to its curved surface whose distance from the vertical plane is h. Shew that h must be greater than $b - 2a \tan\theta$ and less than b; where θ is the angle which the string makes with the vertical.

21. A right circular cone of height h and semivertical angle a is placed with its circular base resting against a smooth vertical wall, and is supported by a string attached to its vertex and to a point in the wall; prove that the greatest possible length of the string is

$$h\sqrt{(1+\tfrac{16}{9}\tan^2 a)}.$$

[N.B. The C.G. of a cone divides the height in the ratio 1 to 3.]

22. A uniform hemispherical bowl, whose C.G. is at the middle point of its axis, is placed with its circular base against a smooth vertical wall, and is supported by a string attached externally to the extremity of its horizontal radius, or axis, and to a point in the vertical plane; shew that the angle which the string makes with the vertical lies between $\tan^{-1}\frac{3}{4}$ and $\frac{1}{2}\pi$.

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23. A uniform rod of length $32a$ rests partly within and partly without a smooth circular cylindrical cup of radius a; prove that in the position of equilibrium the rod makes the angle 60° with the horizon.

24. Shew that the cup in the last question will topple over unless its weight is at least six times the rod.

25. Shew that if in the last question the length of the rod is l, the angle which it makes with the horizon is the angle whose cosine is $\sqrt[3]{\frac{2a}{l}}$.

26. A uniform upright circular cylindrical cup of diameter d, height h and weight W, open at the top and bottom, is set on a smooth horizontal plane; a smooth uniform rod of length l and weight W', rests partly within and partly without the cylinder; one end of the rod resting both against the plane and against the cylinder. Prove that for equilibrium $w(h^2+d^2)^{\frac{3}{2}}$ must be greater than $W'lh^2$.

27. Two equal cubes each of weight W are placed near each other on a rough horizontal plane whose coefficient of friction is $\tan\lambda$ so that the line joining their centres is at right angles to the adjacent vertical faces. An isosceles prism of weight W' and semivertical angle a is supported symmetrically between them, resting on an edge of each cube. Prove that if the cubes are about to slip

$$\tan\lambda\tan(a+\lambda)=\frac{W'}{2W+W'}.$$

28. A uniform rod is placed inside a rough hollow fixed circular cylinder whose axis is horizontal so that the rod is perpendicular to the axis; prove that the greatest inclination to the horizon at which the rod can rest is

$$\tan^{-1} \frac{\mu}{\cos^2 \alpha - \mu \sin^2 \alpha},$$

where μ is the coefficient of friction and 2α the angle subtended by the rod at the centre of the cylinder.

29. Two equal spheres of the same material each of weight W are suspended from a fixed point by two equal strings attached to points on their surfaces each string inclined at the angle β to the vertical. A sphere of weight ω is placed symmetrically between them in the vertical plane containing the strings, and the line joining the centre of the sphere to the centre of each of the other spheres is inclined at the angle α to the vertical. Prove that the two equal spheres will not separate provided $\frac{\tan \beta}{\tan \alpha}$ is greater than $\frac{\omega}{2W + \omega}$.

30. An equilateral triangular lamina, suspended from a smooth peg by the loop of a string whose ends are fastened to two of its angular points, rests with one of its sides vertical; shew that the length of the string is double of the altitude of the triangle.

31. A uniform heavy rod AB has one end A fastened to a fixed point C by a string and has the other resting on a rough horizontal plane; prove that if the angle CAB is a right angle and the inclination of the rod and the plane is α, then the coefficient of friction must exceed $\frac{\tan \alpha}{1 + 2 \tan^2 \alpha}$.

---

32. AB, BC are two uniform rods of equal weight but unequal length, hinged at B. A string, one end of which is attached to A and the other to a smooth ring moveable along BC, is hung over a small pulley. When the system is in equilibrium shew (i) that one part of the string will be perpendicular to the rod BC, (ii) that the portions of the string on either side of the pulley will be equally inclined to the horizon, (iii) that the actions at B will be horizontal.

33. Two uniform heavy beams AB, BC each of length a are freely jointed at B and rest in a vertical plane symmetrically over two smooth small pegs CD in the same horizontal line; prove that each rod is inclined to the horizon at the angle

$$\cos^{-1} \left(\frac{CD}{AB}\right)^{\frac{1}{3}}.$$

34. Four equal rods AB, BC, CD, DA freely jointed at A, B, C, D are placed on a smooth horizontal table to which BC is fixed. The middle points of AD, DC are joined by a string which is tight when the rods form a square; shew that the moment of the couple, which acting upon AB produces on the string a tension T, is $\frac{1}{2}T \cdot AB\sqrt{2}$.

35. Three equal uniform heavy beams AEB, BC, CFD are freely jointed at B and C, and rest in a vertical plane on two smooth small pegs E and F in the same horizontal line; shew that EB must be equal to $\frac{1}{3}AB\cos^2\beta$, and that $3\tan\alpha\tan\beta = 1$, where α, β are the inclinations to the horizon of the stress at either hinge, and of the rod AB.

36. Two equal heavy uniform beams AB, BC are freely jointed at B and their extremities A, C can slide by means of smooth small rings each on one of two fixed rods DA, DC each equally inclined to the vertical; prove that the inclination of each rod to the vertical is

$$\tan^{-1}(2\cot\tfrac{1}{2}ADC).$$

37. Four equal uniform rods AB, BC, CD, DA freely jointed together to form a rhombus are hung over a smooth small peg at A and the hinges BD are kept apart by a light rod; prove that the stress on the rod is to the weight of the frame as $BD : 2AC$.

38. Four equal uniform rods each of length $2a$ are freely jointed at their extremities so as to form the rhombus $ABCD$. They hang symmetrically on two smooth pegs in the same horizontal line, A being the highest point. Shew that when the pegs are $a\sqrt{2}$ apart the rods form a square and that then the actions at the joints are as $3 : \sqrt{5} : 1$.

39. Seven equal uniform rods each of length $2a$ are freely jointed to each other and the two free ends are fixed to two pegs in the same horizontal line, about which they are free to turn, and the whole system hangs in a vertical plane; shew that if the rods beginning with the highest make the angles α_1, α_2, α_3 with the vertical $3\tan\alpha_1 = 2\tan\alpha_2 = \tan\alpha_3$.

40. Four uniform rods are jointed throughout so as to form a quadrilateral, and placed horizontally on a smooth horizontal table; four forces act one at each angular point each bisecting the angle at which it acts and proportional to the cosine of half the angle at which it acts; shew that the system is in equilibrium for all shapes of the quadrilateral.

41. Four uniform equal rods AB, BC, CD, DE are jointed together; the ends A and E are fixed at two points in the same horizontal line, and the joints B and D are connected by a string; the system hangs in a vertical plane so that ABC and CDE are each a straight line; prove that the tension of the string is to the weight of a rod as half AE is to the vertical distance of C below AE.

42. Three equal uniform rods AB, BC, CD are freely jointed at B and C; A and D are hinged to two points in the same horizontal plane whose distance apart is twice the length of the rods; prove that in the symmetrical position of equilibrium the stresses at the hinges are as $\sqrt{31}$ to $\sqrt{7}$.

43. A rhombus formed of four equal uniform rods of length a jointed together is hung over a smooth sphere of radius r so as to rest symmetrically in a vertical plane; shew that the angle θ each rod makes with the vertical is given by

$$\cot^2\theta + \cot\theta = \frac{a}{r};$$

shew also that the stresses at the joints which are not horizontal make the angle $\tan^{-1}(\frac{1}{2}\tan\theta)$ with the vertical.

44. A weightless rod AB of length l is hinged at A to a vertical wall and to the other end B is attached a string which passes over a pulley fixed at a height l vertically above A and supports a weight w. Between the rod and the wall a smooth circular cylinder of weight w and radius a is placed; shew that if θ be the inclination of the rod to the vertical

$$l\sin\theta\sin\tfrac{1}{2}\theta = a.$$

45. The centre of a heavy circular disc of weight W and radius a, and a heavy particle, of weight w, are connected by a string of length l. The particle rests against the smooth edge of the disc; the loop of the string passes over a smooth small peg and the system hangs in a vertical plane. Prove (i) that the two parts of the string are equally inclined to the vertical, (ii) that the pressure between the particle and the disc is to the tension of the string as $a : l$, (iii) that $w : W < l + a : l - a$.

46. Six equal heavy uniform rods are jointed so as to form a regular hexagon and one side is held in a horizontal position, the hexagon being kept in shape by a weightless horizontal rod forming the horizontal diagonal of the hexagon; prove that the strain along the diagonal is $\frac{5}{12}\sqrt{3}$ times the weight of one of the rods.

47. A regular hexagon is formed of six equal uniform heavy rods connected by hinges at their ends; shew that when hung up by a corner it will keep its shape unchanged provided a rod of proper length be placed horizontally between the two vertical rods so that it divides the vertical sides in the ratio of 5 to 1.

48. Four rods are jointed at their extremities so as to form a quadrilateral which can be inscribed in a circle; each pair of opposite angles is joined by a string; prove that the tension of each string is inversely proportional to its length.

49. A rectangular framework $ABCD$ formed of four uniform rods jointed together, has its opposite corners joined by strings; prove that if the framework be hung up from one corner the difference between the tensions of these strings is half the weight of the rods.

50. Four equal uniform rods are jointed together so as to form a square frame, and the middle point of one side is joined by strings to the middle points of two adjacent sides; shew that if the weights of the rods be neglected the tensions of these strings must be equal; also if the rods are heavy and the frame be suspended from one corner the difference between the tensions will be equal to the weight of the frame.

51. Four equal uniform rods are hinged at their extremities so as to form the rhombus $ABCD$. The rhombus is suspended from A and the shape is retained by the tension of a string joining the middle point of AB, BC; prove that if θ and ϕ are the inclinations to the vertical of the stresses at B and C

$$4 \tan \theta = 2 \tan \phi = \cot \tfrac{1}{2} ABC.$$

52. Three rods jointed at their extremities are laid on a smooth horizontal table, and forces are applied at the middle points of their sides of the triangle formed by the rods, and perpendicular to them respectively. Shew that if these forces produce equilibrium, the strains at the joints will be equal to each other, and their directions will touch the circle circumscribing the triangle.

MISCELLANEOUS EXAMPLES. XXXIV.

1. *ABCD* is a square; forces of 1 lb., 6 lbs. and 9 lbs. act in directions *AB*, *AC*, *AD* respectively; find the magnitude of their resultant correct to two places of decimals.

2. Find a point within a triangle from which if straight lines be drawn to the angular points of the triangle, the forces represented by these lines shall be in equilibrium.

3. A number of uniform tiles each 10 in. by 6 in. by $\frac{1}{2}$ in. are placed one upon another in such a way that whilst their narrowest surfaces (thicknesses) are in the same vertical plane, each tile overlaps the next by an inch of its length; the lowest tile resting on a horizontal plane; how many tiles may be so piled without falling?

4. Two weights of 8 oz. and 4 oz. are in equilibrium at opposite ends of a straight lever without weight; if 2 oz. be added to the greater weight, the fulcrum must be moved through $\frac{4}{7}$ of an inch for equilibrium; find the length of the lever.

5. If the resultant R of the two forces P and Q inclined to each other at any given angle make the angle θ with P, prove that the resultant of the forces $(P+R)$ and Q at the same angle makes the angle $\frac{1}{2}\theta$ with $(P+R)$.

6. A wedge of angle 60° is placed upon a smooth table, and a weight of 20 lbs. on the slant side is supported by the tension of a string parallel to that face of the wedge; what horizontal force is necessary to keep the wedge from moving?

7. A smooth circular hoop is supported in a horizontal plane in a fixed position. Three weights of 3 lbs., 4 lbs. and 5 lbs. are suspended over its rim by three strings meeting in the centre. What must be the position of the strings that the weights may be in equilibrium?

8. Two equal weights W_1, W_2 are connected by a string which passes over two small pegs A and B, and supports a weight W_3 which hangs from a smooth small ring through which the string passes; prove that when AB is horizontal the depth of the ring below AB is

$$\frac{W \cdot AB}{2\sqrt{(4P^2 - W^2)}}.$$

9. If a quadrilateral be such that one of its diagonals divides it into two equal triangles, the centre of gravity of the quadrilateral is in that diagonal.

10. The resultant of two forces P and Q is the same when their directions are inclined at the angle θ as when they are inclined at the angle $(\frac{1}{4}\pi - \theta)$; shew that $\tan\theta = \sqrt{2} - 1$.

11. A weight is supported by two strings equally inclined to the vertical; shew that when instead of one of these strings we substitute a string pulling horizontally so as not to disturb the position of the weight, the tension of the other string will be doubled.

12. Two uniform cylinders of equal diameter whose lengths are as 2 : 5 and whose weights are in the ratio of 3 to 7 are joined together so as to form one cylinder; find their centre of gravity.

13. The resultant of P and Q is equal to P; shew that if P be doubled, the new resultant is at right angles to Q.

14. If R be the resultant of two forces P and Q, and S be the resultant of P and R, prove that the resultant of S and Q is $2R$.

15. A heavy right-angled triangle is suspended by its right angle and the inclination of its hypotenuse to the horizon is 40°; find the angles of the triangle.

16. Two strings have each one end fastened to a fixed peg and the other to the ends of a uniform rod; shew that when the rod hangs in equilibrium the tension of the strings are in the ratio of their lengths.

17. Two uniform rods are hinged together and have each a smooth ring at their other extremities which rings are passed through a rigid horizontal wire; shew that in the position of equilibrium the shorter rod must be vertical.

18. A uniform ladder rests with one end on a rough horizontal plane and the other end leans at an angle of 45° against a smooth vertical wall, prove that the friction required to prevent slipping is one-half the weight of the pole.

19. A material particle P is attracted to three points A, B, C by forces proportional to the distances PA, PB, PC; prove that the resultant of these forces is in the direction of the centre of gravity G of the triangle ABC, and is proportional to PG.

20. A heavy uniform rod AB is supported in a position inclined 60° to the vertical by a string OA fastened to a fixed point O and by a horizontal force acting at the lower end B. Find the inclination of the string to the vertical.

21. A smooth sphere is supported in contact with a smooth vertical wall by a string fastened to a point on its surface, the other end being fastened to a point on the wall; the length of the string is equal to the radius of the sphere; find the position of equilibrium, the tension of the string, and the pressure on the wall.

22. Three forces act along the sides of a triangle; prove that if the sum of two of the forces is equal to the third force, the resultant of the three forces will pass through the centre of the inscribed circle of the triangle.

23. A smooth solid sphere rests upon two parallel bars in the same horizontal plane, the distance between the bars being equal to the radius of the sphere; find the pressure on each bar.

24. The horizontal roadway of a bridge is 30 ft. long and weighs 6 tons, and its ends rest on similar supports; what pressure is borne by each support when a carriage weighing 2 tons is one-third of the way across the bridge?

25. ABC is a triangle; AE, BF, CD are lines drawn from the angles to the points of bisection of the opposite side; prove that forces represented by AE, BF, CD are in equilibrium.

26. Three heavy particles are placed at the angles A, B, C of a triangle, their weights being proportional to the opposite sides a, b, c; prove that the distance of the centre of gravity of the particles from A is

$$\frac{2bc}{a+b+c}\cos\frac{A}{2}.$$

27. A common steelyard supposed uniform is 40 inches long, the weight of the beam is equal to the moveable weight, and the greatest weight that can be weighed by it is four times the moveable weight; find the place of the fulcrum.

28. O is the centre of the circumscribing circle; OD, OE, OF are perpendiculars from O upon the sides of the triangle ABC; prove that the six forces represented by AO, BO, CO, OD, OE, OF are in equilibrium.

29. A body consists of two parts, A and B, whose C. G.'s are at P and Q respectively; the part B is moved so that its C. G. comes to Q', the part A remaining fixed; prove that the C. G. of the whole body moves through a distance parallel and proportional to QQ'.

30. A steelyard is constructed so that for each complete stone (14 lbs.) placed in the weighing pan an additional weight of m oz. has to be suspended at the end of the arm one foot in length; the odd pounds are measured by a weight of n oz. sliding along the arm; prove that the distance between the graduations of the arm for successive pounds must be

$$\frac{6m}{7n} \text{ inches.}$$

31. The weights of two spherical balls are as 4 : 3 and their densities as 9 : 2; compare their diameters.

32. Forces of 2, $\sqrt{3}$, 5, $\sqrt{3}$, 2 lbs. respectively act at one of the angular points of a hexagon towards the five others; find their resultant.

33. A hollow circular cylinder of weight W of thin metal open at both ends of radius a and height $4a$ stands with its axis vertical on a smooth horizontal plane. Inside it are placed two smooth spheres each of radius r and weight w one above the other, $2r$ being $>a$ and $<2a$; shew that the cylinder will just stand without toppling over when

$$aW = 2w(a-r).$$

34. O is the point of intersection of two straight lines bisecting the opposite sides of a quadrilateral $ABCD$; shew that forces represented by OA, OB, OC, OD are in equilibrium.

35. A uniform lever is 18 inches long; each inch weighs one ounce; it balances about its fulcrum when there is a weight of 27 ozs. at one end and 9 ozs. at the other; if the smaller weight be doubled how much must the position of the fulcrum be shifted to preserve equilibrium?

36. A uniform beam rests with one end against the junction of the horizontal ground and a vertical wall; it is supported by a string fastened to the other end of the beam and to a staple in the vertical wall; find the tension of the string; and shew that it is equal to half the weight of the beam when the length of the string is equal to the height of the staple above the ground.

37. The wind makes an acute angle with the forward direction of a ship's course, and is received on a sail spread in a direction between that of the wind and that of the ship's course; explain generally why the ship advances.

38. An inclined plane of angle 30° is 3 ft. long; weights of 7, 5, 4, 8 lbs. are placed on the plane in order one foot apart, the weight 8 being at the top of the plane; find the distance of the C. G. of the weights from the base of the inclined plane.

39. In the Danish steelyard if a_n represent the distance of the fulcrum from the end of the steelyard at which the weight is suspended, when the weight is n lbs., prove that

$$\frac{1}{a_{n+1}} + \frac{1}{a_{n-1}} = \frac{2}{a_n}.$$

40. In each of the three systems of pulleys if P and W are displaced together their centre of gravity remains in the same position. (The weights of the pulleys are neglected.)

41. Find the centre of gravity of a uniform wire 16 inches long bent into a right angle of which one arm is 6 inches.

42. Explain how a man by walking slowly on the surface of a large rough sphere can make it roll up an inclined plane.

43. A uniform wire is bent so as to form three sides AB, BC, CD of a regular polygon; the C. G. of the wire is at the point of intersection of AC and BD; prove that the polygon has six sides.

44. A right-angled triangle is suspended successively from its acute angles, and when at rest the side opposite the point of suspension in each case makes angles θ, ϕ with the vertical; shew that $\tan\theta \tan\phi = 4$.

45. Three smooth pegs A, B, C stuck in a wall form an equilateral triangle, A being the highest and BC horizontal; a string of length 4 times BC passes round them and each end is fastened to a weight W which hangs in equilibrium below BC; find the pressure on each peg.

46. A number of telegraph wires which are all practically horizontal are supported by a pole A; they all come from one pole B to the pole A and some of them pass on to a pole C and some to a pole D; shew that the pole B must have a stay to keep it vertical, if the tensions of the wires are all equal.

47. A boat with a deep keel is fastened by a long rope to a point in the middle of the bed of a rapid river; explain how by keeping the boat's keel in the proper direction the boat may be made to cross from one side of the river to the other.

48. A horizontal telegraph wire carried by a vertical post makes at that post a turn through 60°. The post is kept vertical by a wire stay fastened to the ground and to the middle point of the post, making the angle 30° with the post. Shew that the tension of the stay is 4 times that of the telegraph wire.

49. A square uniform lamina $ABCD$ of weight W has heavy particles P and Q fixed to it at B and D; it is suspended by a string fastened at A; prove that AC is inclined to the horizon at the angle whose tangent is

$$\frac{P-Q}{P+Q+W}.$$

50. A pack of cards is laid on a table and each projects in the direction of the length of the pack as far as possible beyond the one below it beginning at the top; prove that the distances between the extremities of successive cards will form a Harmonic Progression.

51. Shew that the least force which will move a weight W along a rough horizontal plane is $W \sin \phi$ where ϕ is the limiting angle of resistance.

52. Two equal heavy particles on two equally rough inclined planes, of the same height, are connected by a string passing over the top of the planes; shew that when the particles are on the point of moving the limiting angle of resistance will be half the difference of the inclination of the planes.

53. A uniform right circular cylinder is placed with its base on an inclined plane, the coefficient of friction being $\frac{1}{3}\sqrt{3}$; find the ratio of the height of the cylinder to the radius of the base that it may be just on the point of sliding and of toppling over at the same time.

54. A uniform ladder rests between a vertical wall and the horizontal ground, both being rough; the coefficient of friction for the ladder and wall is $\frac{1}{3}$, and for the ladder and ground $\frac{3}{7}$; find the angle which the ladder makes with the ground when it is just on the point of sliding.

55. A square uniform lamina $ABCD$, of weight W, rests in a vertical plane with its side BC on a line of greatest slope of a perfectly rough plane inclined at the angle a to the horizon; a string AP is attached to A the highest point of the lamina, and passing over a smooth pulley at P the top of the plane supports a weight w; AP is horizontal; shew that the lamina will be on the point of turning about B when $w = \frac{1}{2} W(1 + \tan a)$.

56. AB is a smooth inclined plane of angle a; at A the lower end is a smooth hinge, to which is fastened one end of a smooth uniform heavy plank AC of length $2a$ and weight W. Between the plank and the plane is placed a cylinder of radius r and weight w, which is prevented from sliding down the plane by the pressure of the plank on it from above. Shew that the angle θ between the plane and the plank in equilibrium is given by the equation

$$wr \sin a = aW \cos (a + \theta)(1 - \cos \theta).$$

57. A heavy rod rests with its extremities on a rough circular hoop fixed in a vertical plane; the rod subtends 90° at the centre of the hoop; find its inclination to the horizon when in the limiting position of equilibrium.

58. Find the C. G. of three weights placed at the centres of the escribed circles of a triangle and inversely proportional to their radii.

59. Shew that if a man sitting in one scale of a common balance presses with a stick against any point of the beam between the point from which the scale is suspended and the fulcrum, he will appear to weigh more than before.

60. A heavy insect of weight w crawls on the lower circumference of the wheel in a wheel and axle, and so just raises a weight $5w$, the ratio of the radii of the wheel and axle being 10 to 1; shew that the radius of the wheel passing through the insect makes 30° with the vertical; and shew that the insect is in a position of stable equilibrium, but that if it were on the point of the circumference of the wheel vertically above its present position its equilibrium would be unstable.

61. A uniform ladder 70 ft. long is equally inclined to a vertical wall and the horizontal ground; the weight of a man with his burden ascending the ladder is 2 cwt., and the ladder weighs 4 cwt.; how far up the ladder can the man ascend before it slips? the tangent of the angle of resistance for the wall is $\frac{1}{3}$ and for the ground $\frac{1}{2}$.

62. A heavy particle P is placed on a rough inclined plane of angle α and is fastened by a string at its full length to a fixed point A on the plane; AB is the line of greatest slope; prove that when the particle is on the point of slipping

$$\sin PAB = \mu \cot \alpha,$$

and interpret the result when $\mu \cot \alpha > 1$.

63. Two equal smooth circular cylinders are placed with their axes parallel and in the same horizontal plane in the loop of a thread whose ends are then held so that the free parts of the string are parallel; prove that the pressure between the cylinders is equal to the weight of one of them.

64. A uniform rod rests with one extremity against a rough vertical wall (coefficient of friction $\frac{7}{3}$), the other extremity being supported by a string three times the length of the rod, one end of which is fastened to it and the other attached to a point in the wall; shew that the tangent of the angle which the string makes with the wall in the limiting position of equilibrium is $\frac{5}{27}$, or $\frac{1}{3}$.

65. A solid circular cylinder rests on a rough horizontal plane with one of its flat ends on the plane, and is acted on by a horizontal force through the centre of its upper end; if this force is just sufficient to move the solid, prove that it will begin to slide or turn over, according as the coefficient of friction is less or greater than the ratio of the radius to the height.

66. A square figure $ABCD$ is formed by four equal uniform rods joined together, and the system is suspended from the joint A, and kept in the form of a square by a string joining A and C; prove that the tension of the string is half the weight of the four rods, and shew that the action at the joint B is horizontal and equal to the weight of a rod.

67. Two uniform beams of equal weight but of unequal length are placed with their lower ends in contact on a smooth horizontal plane and their upper ends against two parallel vertical planes; shew that in the position of equilibrium the beams are equally inclined to the horizon.

68. Three equal rods AB, BC, CD without weight are connected by hinges at B and C and are moveable about hinges at A and D, the distance AD being twice the length of each rod; a force P acts at the middle point of each rod and at right angles to it; shew that the pressure on each of the hinges A and D will be $\frac{1}{3}P\sqrt{3}$, and that its direction will make 60° with AB.

69. A uniform heavy cube balances on the highest point of a fixed sphere of radius r; the surfaces are sufficiently rough to prevent sliding; the side of the cube is $\frac{1}{2}\pi r$; shew that the cube may be made to rock through a right angle without falling.

70. Four heavy rods, equal in all respects, are freely jointed at their extremities so as to form the rhombus $ABCD$. If this rhombus is suspended by two strings attached to the middle points of AB and AD, each string being inclined at the angle θ to the vertical, prove that in the position of equilibrium the angles of the rhombus will be 2θ and $\pi - 2\theta$.

71. A plane of small slope rises one foot vertical for n feet horizontal, and the coefficient of friction is μ; shew that the force which will just move a weight W up the plane is nearly $W\left(\frac{1}{n} + \mu\right)$.

72. A light rod rests wholly within a smooth hemispherical bowl of radius r, and a weight W is clamped on to a rod at a point whose distances from the ends are a, b. Shew that θ the inclination of the rod to the horizon in the position of equilibrium is given by the equation $2\sqrt{(r^2 - ab)} \sin\theta = a - b$.

73. The poles supporting a lawn-tennis net are kept in a vertical position by guy-ropes, one to each pole, which pass from the top of the poles, round pegs distant 2 ft. from the poles; the coefficient of limiting friction between the ropes and pegs is $\frac{4}{3}$; the poles are 4 ft. high; shew that the inclination of the pegs to the vertical must not be less than $\tan^{-1}\frac{2}{11}$.

74. A uniform heavy rod is placed across a smooth horizontal rail, and rests with one end of the rod against a smooth vertical wall; the distance of the rail from the wall being one-sixteenth of the length of the rod, prove that the rod will rest at the angle 60° with the horizon.

75. A heavy sphere hangs from a horizontal bar by a string whose length is equal to the radius, and it rests against a parallel smooth bar vertically below the former, the distance between the bars being equal to the diameter of the sphere; prove that the tension of the string is double the pressure on the bar.

76. A cylindrical shell without a bottom stands on a horizontal plane, and two smooth spheres are placed within it whose diameters are each less while their sum is greater than that of the surface of the shell: shew that the cylinder will not upset if the ratio of its weight to that of the upper sphere is greater than $2c-a-b : c$, where a, b, c are the radii of the spheres and cylinder.

77. A smooth circular ring of radius a and weight W rests on two parallel horizontal bars which pass through the ring. The shortest distance between the bars is $2b$, and it makes the angle α with the vertical. Prove that the stresses between the ring and the bars are

$$W \sin(\alpha-\theta) \operatorname{cosec} \theta \text{ and } W \sin(\alpha+\theta) \operatorname{cosec} \theta \text{ where } \cos\theta = \frac{b}{a}.$$

Explain what is indicated by this result when $b > a$.

78. A square lamina is placed with its plane vertical resting on two smooth parallel bars in the same horizontal plane, shew that when it is in equilibrium the inclination of one of its edges to the horizon is half the angle whose sine is $\frac{a^2-c^2}{a^2}$, $2a$ being the length of a side of the square and c the distance between the bar.

79. Two rods AB, BC of equal weight but unequal lengths are hinged together at B, and their other extremities are attached to two fixed hinges A and C in the same vertical line; prove that the line of action at the hinge B bisects the straight line AC.

80. A three-legged stool stands on a horizontal plane, the coefficient of friction being the same for each of the three feet. A small horizontal force is applied to one of the feet in a given direction, and is gradually increased until the stool begins to move. Shew that this force will be greatest when its direction intersects the vertical through the Centre of Gravity of the stool.

81. An inextensible string binds tightly together two smooth cylinders, radii r_1, r_2; shew that the ratio of the pressure between the cylinders and the pressure by which it is produced is $\frac{4\sqrt{(r_1 r_2)}}{r_1+r_2}$.

82. A roof of given span is to be constructed with two equal beams which are connected at the vertex by a single smooth pin; the weight of the roof is proportional to the length of the beams; shew that when the pressures on the walls are the least possible their direction makes the same angle with the vertical which the beams make with the horizon.

83. One end of a string of length l is fastened to a point A in a smooth vertical wall, and the other to the middle point of one of the edges (of length h) of a cube; shew that the distance of the edge of the cube which rests against the wall from A is $\sqrt{\frac{l^2-2h^2}{3}}$.

84. A sphere of radius a and weight W is suspended by a string of length l from a given point A; another body of weight W' is also suspended from A by a string so long that W' hangs below the sphere; shew that the angle which the first string makes with the vertical is the angle whose sine is

$$\frac{Wa}{(W+W')(a+l)}.$$

85. A uniform equilateral triangular lamina has the ends of a string fastened to two corners; the loop of the string passes over a smooth horizontal bar; shew that when the length of the string is double the height of the triangle the stable position of equilibrium of the triangle is with one side vertical.

86. Four equal rods are hinged together so as to form a rhombus $ABCD$; the points A and C are joined by a string whose tension is P; and the points B and D are joined by a string whose tension is Q; prove that $\cos BAD = \frac{P^2-Q^2}{P^2+Q^2}$.

87. The ends of a string are fastened to two fixed points and weights are suspended from different points in the string: shew that the horizontal resolute of the tension of the string is constant throughout its length, and that when the weights are all equal the tangents of the angles which the successive portions of the string make with the horizon are in Arithmetic Progression.

88. Two light small rings are capable of sliding along a rough horizontal rod; a smooth string of length $4l$ passes through each of the rings and has both ends attached to a weight; shew that the greatest possible distance between the rings is $l\frac{\mu^2-1}{\mu^2}$.

89. A uniform rod rests wholly within a rough circular tube, is perpendicular to the axis of the tube, and subtends the angle β at the centre; shew that θ the inclination of the rod to the horizon in its limiting position is given by

$$2\tan\theta = \sin 2a \sec(\beta + a) \operatorname{cosec}(\beta - a),$$

where $\tan a$ is the coefficient of friction.

90. Three equal spheres placed in contact with each other on a horizontal plane support a fourth equal sphere; prove that when the spheres and the plane are all of the same material their coefficient of friction cannot be less than $\frac{1}{4}(\sqrt{3}-\sqrt{2})$.

91. A rectangular table stands on a rough inclined plane with two sides horizontal; the distance between the legs is $2a$, and the height of the C. G. is h; the coefficient of friction of the lower pair of legs with the plane is μ and of the higher μ'; prove that the angle θ of the plane when the table is on the point of sliding is given by $\tan\theta = \dfrac{a(\mu_1+\mu_2)}{2a+(\mu_1+\mu_2)h}$.

92. Shew that no smooth uniform rod can rest partly within and partly without a smooth hemispherical bowl at an inclination to the horizon greater than the angle whose sine is $\frac{1}{3}\sqrt{3}$.

93. Two equal uniform beams AB, AC of weight W, connected by a smooth hinge at A, are placed in a vertical plane with their extremities B and C resting on a smooth horizontal plane. Two strings connect B and C respectively with the middle point of the opposite beam. Shew that the tension of each string is $\frac{1}{8}W\sqrt{(9\tan^2\alpha+1)}$, where 2α is the angle between the beams.

94. A uniform rod rests symmetrically in a horizontal position inside a rough right circular cone, whose axis is vertical and vertex downwards. The semi-vertical angle of the cone is 60°, and the coefficient of friction between the rod and the cone unity. Prove that the cone may be tilted through an angle $\tan^{-1}2$ in a vertical plane passing through the rod without causing the cone to slide.

95. If a uniform wire is bent into the form of a triangle and at the middle points of the sides there are placed three beads whose weights are proportional to the sides on which they are, prove that when the beads are moved with equal velocities the same way round along the sides there is no change in the position of the Centre of Gravity of the whole system.

96. ABC is a triangle of jointed rods; BC is held fixed and AB, AC are acted on by forces at their middle points perpendicular and proportional to them. Prove that the reaction at A is along the tangent at A to the circumscribing circle.

97. Two uniform rods AB, AC each of length $2a$, are freely jointed at A and have their other extremities connected by a string. The rod AB is placed so as to be in contact with two horizontal bars D and E in the same horizontal plane; the rod is perpendicular to the bars and passes over D (which is nearer A) and under E. Prove that equilibrium will not be possible unless $AE > a \cos^2 \frac{1}{2} BAC$.

98. A uniform rod of weight W rests in a horizontal position with its ends on the circumference of a rough fixed vertical circle, and subtends an angle $2a$ at the centre. An insect of weight w starts from the middle point of the rod and crawls gently towards one end. Prove that if ϵ the angle of friction be less than $45°$, it will be able to reach the end of the rod without disturbing the equilibrium, provided $(W+w) \sin 2\epsilon > w \sin 2a$.

99. A regular pentagon $ABCDE$, formed of five equal heavy rods each of weight W, jointed together, is suspended from the joint A, and the regular pentagonal form is maintained by a rod without weight joining the middle points KL of BC and DE. Prove that the stress at K is $2W \cot 18°$.

100. O is any point on the circle circumscribing the triangle ABC; OL, OM, ON are the perpendiculars on its sides. The line LMN meets the perpendiculars from ABC on the opposite sides in P, Q, R respectively. Forces OL, OM, ON, OP, OQ, OR act at O; prove that their resultant is $3OK$, where K is the ortho-centre.

101. In a triangular lamina ABC, AD, BE, CF are the perpendiculars on the sides, and forces represented by the lines BD, CD, CE, AE, AF, BF are applied to the lamina; prove that their resultant will pass through the centre of the circle described about the lamina.

102. A solid circular cylinder rests with the centre of its base in contact with the highest point of a fixed sphere, and the height of the cylinder is one quarter of the circumference of a great circle of the sphere; supposing the surfaces in contact to be sufficiently rough to prevent sliding in all cases, shew that the cylinder may be made to rock through $90°$, but not more, without falling off the sphere.

103. A uniform rod AB of weight W, rests with one end A against a rough vertical plane, and the other end B supported by a string which passes over a small smooth pulley vertically above A and supports a weight P. Prove that equilibrium is impossible unless P is greater than $W \cos a$, where $\tan a$ is the coefficient of friction.

104. A small weight is attached by a string to a point or rough inclined plane, angle α; shew that the greatest angle the direction of the string can make with the line of greatest slope is given by

$$\sin\theta = \mu \cot\alpha.$$

105. A body lies on a rough inclined plane of angle α (whose inclination is such that the body cannot be sustained by friction alone) sustained by a force P; prove that the greatest angle which the resolute of P in the plane can make with the line of greatest slope is $\sin^{-1}(\mu \cot\alpha)$, μ being the coefficient of friction.

106. A heavy uniform rod of weight W rests in contact with a rough fixed circular cylinder, whose axis is horizontal and whose diameter is equal to the length of the rod. The rod is maintained in its position by a smooth string which passes round the cylinder and is attached to the ends of the rod. Shew that the tension of the string must not be less than $W\dfrac{\sin(\theta-\alpha)}{2\sin\alpha}$, $\tan\alpha$ being the coefficient of friction, and θ the inclination of the rod to the vertical.

107. The C. G. of four equal particles in any position is the same as that of four other equal particles each of which is the C. G. of three of the former particles.

108. ABC is a triangle and DEF the feet of the perpendiculars from A, B, C on the opposite sides; prove that if we place masses at A, B, C proportional to EF, FD, DE respectively their centre of mass is at the centre of the circumscribing circle.

109. Forces act at the angular points of a plane quadrilateral each being proportional and perpendicular to the diagonal which does not pass through its point of application; prove that they are in equilibrium.

110. A particle of weight W is attached by a string to a fixed point on a rough inclined plane (coefficient μ, inclination to horizon α); it is pulled away from the line of greatest slope of the plane, keeping the string tight until it is on the point of slipping back. Shew that if the particle is now left free the tension of the string is $W\sin\alpha\sqrt{(1-\mu^2\cot\alpha)}$.

The resultant of forces represented by $\lambda . PA$ and $\mu . PB$ acting on a particle at P is a force represented by $(\lambda+\mu) PG$, where G is the Centre of Gravity of masses proportional to λ, μ placed at A and B.

111. Six uniform rods each of weight W are jointed together so as to form a hexagon; they are hung up from a corner. The two middle rods are connected by a light horizontal rod; shew that if these rods rest vertically the horizontal rod divides them in a ratio which is independent of its length.

If the horizontal rod be heavy and uniform in length and material with the others, shew that the ratio is 6 : 1 and that the stress in the horizontal rod is $\frac{7}{6} W \sqrt{3}$.

112. Two equal rods AB, AC each of length l and weight w, smoothly jointed at A, are placed over a smooth horizontal circular cylinder of radius a and the ends B and C are connected by a tight string of length b; prove the tension if the string is

$$w\left\{\frac{4cl}{b^2} - \tfrac{1}{4}b\,(l^2 - \tfrac{1}{4}b^2)^{-\frac{1}{2}}\right\}.$$

113. A tipping basin whose interior surface is spherical is free to turn about an axis at the distance c below the centre of the sphere, and at the distance a above the C. G. of the basin; a heavy ball is laid in the bottom of the basin. Prove that the basin will tip over if the weight of the ball exceeds the fraction a/c of the weight of the basin.

114. The triangle ABC is formed of three rods hinged together and lies on a horizontal plane; strings AD, BE connect A and B with points on the opposite sides; these strings cross at F and tensions act along them proportional to AF and BF respectively; prove that the stress at C is parallel to AB.

115. The handles of a drawer are equidistant from the sides of the drawer, and distant c from each other; prove that it is impossible to pull the drawer out, by a straight pull on one handle only, unless the depth of the drawer exceeds μc, where μ is the coefficient of friction.

116. A uniform frame $ABCD$ in the shape of a parallelogram, sides a and b and hinged at its angular points, is suspended at A; A and C are connected by a string so that the angles of the frame are right angles: prove that the stress at each of the hinges B and D is to the weight of the frame as

$$ab : 2\sqrt{2}\,(a+b)\sqrt{(a^2+b^2)}.$$

117. The arms of a balance are of unequal lengths a and b. Prove that when a body of weight W is weighed, first in one scale and then in the other, the difference of its apparent weight is $\dfrac{a^2 - b^2}{ab} W$.

118. A regular pentagon $ABCDE$, formed by five equal strings with four particles each of weight W at the angular points B, C, D, E, is suspended from A; the regular pentagonal form being maintained by a horizontal rod BE without weight having its ends fastened to the particles B, E; prove that the stress of the rod : $W = \sqrt{(10+2\sqrt{5})} : \sqrt{5} - 1$.

119. Four rods are hinged at their extremities so as to form a parallelogram $ABCD$ whose sides are in the ratio 7 : 1. A and C are joined by a string of such length that the frame forms a rectangle, which is hung up by the angle A. Prove that the tension of the string is a half and that the action at each of the hinges B and D is $\frac{7}{100}$ of the weight of the frame.

120. A uniform rod of weight W rests in a limiting position of equilibrium with one end on a rough horizontal plane and the other end on an equally rough inclined plane of angle α; prove that if θ be the angle of inclination of the rod to the horizon

$$\tan\theta = \frac{\sin(\alpha - 2\lambda)}{2\sin\lambda\sin(\alpha-\lambda)},$$

where $\tan\lambda$ is the coefficient of friction.

121. $ABCD$ is a rectangular framework of weightless rods hinged together, AB is fixed in a vertical position, B uppermost, and B, D are connected by a string; a weight W is fixed at some point on BC; shew that the tension of the string is unaltered if W be placed on the lower rod vertically under its former position.

122. A sphere of radius a whose C.G. is at the distance b from its geometrical centre rests in limiting equilibrium on a perfectly rough inclined plane of angle α. It can be turned through the angle 2θ and will then be again in limiting equilibrium; prove that $b\cos\theta = a\sin\alpha$.

123. Two equal rods AB, BC, each of weight W, jointed at B, have their middle points connected by a string of such a length that when it is straightened ABC is a right angle. If the rods are suspended from A, prove that the inclination of AB to the vertical is $\cot^{-1} 3$, and that the tension of the string is $\frac{3}{5}W\sqrt{5}$, and that the action at the joint is $W\sqrt{\frac{2}{5}}$.

124. Four rods are hinged at their ends so as to form a parallelogram $ABCD$ whose sides are as 7 : 1. A and C are joined by a string of such a length that the frame is rectangular; shew that when the frame, whose weight is W, is suspended from A the tension of the string is $\frac{1}{2}W$, and the stress at the hinges B, D is $\frac{7}{100}W$.

125. Of four forces in a plane in equilibrium, one is given absolutely, a second and third have their lines of action given, while the fourth has its magnitude given; prove that the line of action of this fourth force must touch a fixed circle.

126. A uniform rod of length $2a$ presses with its lower end on a rough vertical wall and rests on a smooth horizontal peg at a distance b from the wall. If the coefficient of friction is $\tan\lambda$, prove that the equation giving the limiting values of θ the inclination of the rod to the wall is

$$\frac{b}{a}=\frac{\sin^2\theta\sin(\theta\pm\lambda)}{\cos\lambda}.$$

127. $ABCDEF$ is a regular hexagon; prove that equal forces acting along AB, CD, EF, AF, CB, ED are in equilibrium.

128. A weightless string is suspended from two fixed points and at given points on the string equal weights are attached; prove that the tangents of the inclinations to the horizon of the different portions of the string are in A. P.

129. A uniform heavy rod of length $2a$ turns freely on a pivot at a point in it, and suspended by a string of length l fastened to the ends of the rod, hangs a bead of equal weight, which slides on the string. Prove that the rod cannot rest in an inclined position unless the distance of the pivot from the middle point of the rod is less than $\frac{a^2}{l}$.

130. A symmetrical three-legged stool stands on a rough horizontal plane; a string is attached to one foot and pulled in the plane parallel to the vertical plane through the other two feet; shew that the smallest tension that will move the stool is $2W\sin\gamma$; $3W$ being weight of the stool and $\tan\gamma$ the coefficient of friction.

131. A uniform rod rests horizontally on two pegs, one at one end of the rod; find where the other peg must be placed that the bending moment at a given point of the rod may be zero.

132. Seven equal thin straight rods without weight are freely jointed together, so as to form three equilateral triangles. The frame so constructed is placed in a vertical plane with the straight lines ABC and DE horizontal, the former being uppermost. The points A and C are fixed and a given weight W is attached to B. Find the tension of the rod DE (1) by the polygon of forces, (2) by resolution of forces, (3) by the principle of virtual work.

133. A rough circular cylinder of weight W lies with its axis horizontal on a plane inclined to the horizon at the angle α, and a man of weight W' stands upon the cylinder and keeps it at rest. The man's feet are at A and a vertical line through A meets the plane at B; prove that the angle θ subtended by AB at the centre of the cylinder is given by

$$\frac{\sin(\theta+\alpha)}{\sin\alpha}=1+\frac{W}{W'},$$

the friction being sufficient to prevent sliding.

134. The sum of the weights of a certain steelyard $ABCD$ and of its moveable weight is ω, the fulcrum is at C and the body to be weighed at A. The steelyard is graduated, and after graduation the fulcrum is shifted towards A to C'. A body is then weighed the old graduation being used and the apparent weight is W. Prove that the true weight is

$$W+(W+\omega)\frac{CC'}{AC'}.$$

135. Four uniform rods AB, BC, CD, DE each of weight ω and length a are jointed together; to the joint C a weight W is fastened; B and D are connected by a string of length a, and A and E are fixed in the same horizontal line so that $AE=(1+\sqrt{3})a$; prove that the tension of the string is

$$\tfrac{1}{3}(W+4\omega)\sqrt{3}.$$

136. Four uniform equal rods AB, BC, CD, DE each of weight W are jointed together; their ends A and E are fixed at points in the same horizontal line and the joints B and D are connected by a string and the system hangs in a vertical plane so that ABC and CDE are straight lines, and ACE is an equilateral triangle. Prove that the tension of the string is $\frac{1}{3}W\sqrt{3}$.

137. Prove that if in the last example a weight is fastened to the hinge at C it will not alter the tension of the string.

133. A uniform lamina in the shape of a right-angled triangle ABC rests in a vertical plane upon two smooth pegs in the same horizontal line, with the right angle AB below the pegs. Shew that if θ be the angle which the bisector of the right angle makes with the vertical in the position of equilibrium then

$$3h\sin 2\theta=a\sin\{\theta+\tfrac{1}{2}(B\sim C)\},$$

where a is the hypotenuse of the triangle, and h the distance between the pegs.

139. A rhombus constructed of four uniform rods freely jointed together is hung symmetrically over two smooth pegs in the same horizontal line. Shew that each peg must divide the rod with which it is in contact in the ratio $\cos^2\alpha : \sin^2\alpha$, where 2α is the angle between the two rods in contact with the pegs.

140. Two equal straight rods AC, BC are connected by a smooth joint at C; the rod BC is free to turn about a smooth fixed point at A and the end B free to move along a smooth straight line AD; forces $2P$ and P in the plane of ABC act at the middle point of and perpendicular to the rods AC, BC respectively both being directed towards the inside of ACB; prove that in the position of equilibrium ACB is equilateral.

141. Two equal cylinders of radius a rest with their axes parallel on a rough table; a third equal rough cylinder is placed so as to be in contact with them along generating lines; prove that if $\mu > 2 - \sqrt{3}$, equilibrium is possible so long as the distance between the axes of the lower cylinder is less than $8\mu a/(1+\mu^2)$ where μ is the coefficient of friction of all the surfaces.

142. A triangle formed of three rigid bars jointed together at their ends is in equilibrium under the action of three forces passing through the centre of the circumscribing circle, respectively perpendicular to the side; shew that the stresses at the corners are equal.

143. A yacht has to sail due North in a wind which blows steadily North-East; shew that, neglecting heeling, its speed should be greatest when the sail bisects the angle between the wind and the direction of motion.

144. A uniform hemisphere of radius a and weight W rests with its spherical surface on a horizontal plane and a rough particle of weight W' lies on the plane surface of the hemisphere; prove that the distance of the particle from the centre is not greater than $3W\mu a/8W'$, where μ is the coefficient of friction [the C.G. of a hemisphere is $\frac{3}{8}a$ from the centre].

145. Four equal and similar heavy rods are freely jointed so as to form a rhombus. The rhombus is suspended from one of its angular points and a uniform smooth circular cylinder whose radius is $\frac{9}{16}\sqrt{3}$ times the length of a rod and whose weight is four times that of the rhombus is passed through the rhombus and balanced. Shew that in the position of equilibrium each rod will make $30°$ with the vertical.

146. A cylinder whose height is h and radius a, is placed in a vertical position on a rough horizontal table, the nearest point of the base to the edge of the table being distant b from it. To the top of the cylinder is tied a light string which after passing over the edge of the table is attached to a weight which hangs freely. If this weight be gradually increased shew that the cylinder will slide before it turns provided the coefficient of friction is less than $\frac{ab}{(a+b)h}$.

147. Two equal uniform rods AB, BC each of weight ω are smoothly jointed at B and freely moveable about the point A which is fixed; C is attached to a weight W by means of a string, which passes over a small smooth fixed pulley D. If there is equilibrium when AB is horizontal prove that $2W$ cannot be less than 3ω, and that the string CD and the rod BC are inclined to the horizon at the angles

$$\sin^{-1}\frac{3\omega}{2W} \text{ and } \tan^{-1}\frac{2\omega}{(4W^2-9\omega^2)^{\frac{1}{2}}} \text{ respectively.}$$

148. A lamina whose Centre of Gravity is G is hung over a smooth peg by a string of length l whose ends are fastened to the lamina at points A and B; the angles GAB, GBA each $=\beta$ and $BC=c$; prove that in equilibrium the line AB must be either horizontal or inclined to the vertical at the angle

$$\sin^{-1}\left(\frac{l}{c}\sin\beta\right).$$

149. Three equal heavy rods being connected by two hinges a string is attached to the free ends and hung over a peg so that the system rests with the middle rod horizontal; prove that $2\tan\theta=3\tan\phi$, where θ is the inclination of each of two of the rods to the vertical and ϕ the inclination of each portion of the string.

150. A uniform rod of length $4a$ hangs from a pivot at one end, a sphere of equal weight is fastened to the same pivot by means of a string of length a, the string being attached to a point on the surface of the sphere; shew that the string and the rod are equally inclined to the vertical.

151. A uniform rod of length $4a$ is bent into two equal arms at right angles and placed over a rough circular cylinder of radius a (whose axis is horizontal) in a plane perpendicular to the axis. Shew that in the limiting position of equilibrium the inclination to the horizon of the arm joining the point of contact of the rod and the sphere is $2\tan^{-1}\mu$, where μ is the coefficient of friction.

152. Four uniform rods are jointed together to form the parallelogram $ABCD$. AB is fixed with A vertically above B and the middle points of CD and AD are connected by a string. Shew that the ratio of the tension of the string to the weight of the parallelogram is $AC : AB$.

153. A rod of length $2a$ with its C.G. at the distance b from its middle point is placed with its ends on a rough vertical circular hoop, and it subtends the angle β at the centre; shew that if the rod can just rest in a horizontal position

$$\lambda = \tfrac{1}{2} \sin^{-1} \frac{b}{a} \sin \beta.$$

154. Shew that a window frame may be made to rest in a raised position without weights, if the C.G. is distant c horizontally from the centre of the window provided the coefficient of friction is greater than $\frac{a}{2c}$, where a is the length of the window.

155. Three equal weights are connected by equal strings of length l, and the two extreme weights are joined by strings of length c to two fixed points in the same horizontal line, distance $2a$ apart; shew that the distance of the middle weight below the line joining the two fixed points is given by the equation

$$\frac{2lax}{(x^2+a^2)^{\frac{3}{2}}} = \sqrt{\left(4 - \frac{x^2+a^2}{c^2}\right)}.$$

156. Draw nine equal straight lines, AB, BC, CD, DE in the same horizontal line, CH vertically downwards, FG, GH, HJ, JK parallel to AE; join AG, GC, CJ, JE, FD, BH, HD, DK. Replace all these seventeen straight lines by rigid rods without weight hinged together at their extremities. A structure so formed rests on supports at F and K, and weights W, $2W$, $3W$ are placed at A, C, E respectively; prove that the tension of HJ is equal to $2W$ and that the tension of GH is zero.

EXAMINATION PAPERS.

I. Cambridge Previous Examination. Part III. Additional Subjects. *October*, 1886.

1. Explain the measurement of angles in degrees, minutes and seconds.

How many minutes of angle does the hour hand of a watch pass over in a minute of time?

2. Shew from the definitions of the trigonometrical functions that

$$\sin^2 A + \cot^2 A + \cos^2 A = \operatorname{cosec}^2 A.$$

Prove also that

$$\frac{\tan A + \sec A + 1}{\tan A + \sec A - 1} = \frac{\sec A + 1}{\tan A}.$$

3. If three forces acting at a point be in equilibrium, and any triangle be drawn having its sides parallel to the lines of action of the forces, the forces will be proportional in magnitude to the sides of the triangle.

A heavy particle is held at rest by means of two strings attached to it, one of which is horizontal. If the tension of one string is double that of the other find the inclination to the vertical of the string which is not horizontal.

4. Find the resultant of two unlike parallel forces.

Prove that any given force can be resolved into two parallel forces, one of which is double the given force; and exhibit in a diagram the relative positions of the force and its two components.

5. Find the line of action of the resultant of three forces which are completely represented by the sides AD, CB, CD, of a parallelogram $ABCD$.

6. Shew that the algebraical sum of the moments of two forces acting at a point about any point in the line of action of their resultant is zero.

If the moments of two given forces about a point in their plane be equal and in the same direction prove that the point must lie on a certain straight line.

7. Having given the weights and the centres of gravity of the whole of a body and of one part, shew how to find the centre of gravity of the remaining part.

ABCD is a parallelogram and *O* the intersection of its diagonals. If the triangle *AOB* be removed, find the centre of gravity of the remainder of the parallelogram.

8. A straight horizontal lever has for fulcrum a hinge at one end *A*, and at a point *B* is hung the weight *W*. If the strain on the hinge must not exceed $\frac{1}{3}W$ either upwards or downwards, prove that the power must act somewhere within a space equal to $\frac{3}{4}AB$.

9. Find the ratio of the power to the weight in any one system of moveable pulleys, the weights of the pulleys being neglected.

In that system in which each pulley hangs by a separate string there are four moveable pulleys whose weights in order, beginning with the lowest, are 4 lbs., 3 lbs., 2 lbs., and 1 lb. Find the power required to support a weight of 38 lbs.

II. Cambridge Previous Examination. Part III.

Additional Subjects. *December*, 1886.

1. From the definitions of the trigonometrical functions prove that

$$\cos^2 A + \sin^2 A = 1, \text{ and } \operatorname{cosec}^2 A \tan^2 A = 1 + \tan^2 A.$$

Prove that $4(\cos^6 A + \sin^6 A) - 3(\cos^4 A - \sin^4 A)^2 = 1$.

2. Investigate the values of cot 45° and cos 30°.

Two adjacent sides of a parallelogram are of lengths 18 and 25 and the angle between them is 120°; find the lengths of both diagonals.

3. Enunciate the Parallelogram of Forces, and assuming its truth prove that the resultant of two forces *P*, *Q* acting at an inclination α to one another is

$$\sqrt{P^2 + Q^2 + 2PQ\cos\alpha}.$$

A string, which passes over a smooth peg, has its ends attached to the ends of a uniform bar, the bar resting in a horizontal position. Shew that the tension of the string is diminished if its length be increased.

4. Explain what is meant by resolving a force, and shew that a force may be resolved into two components in an infinite number of ways.

A force P acts along the bisector of a right angle BAC; resolve it into components along AB, AC.

5. State the rule for finding the magnitude and line of action of the resultant of two parallel forces.

Parallel forces of 2, 3, 4, 8 lbs. act at points in a straight line distant 1 foot from each other, the first three forces acting in the same direction, and the last in the opposite direction; find the centre of the system.

6. If a body consist of two parts, whose weights and centres of gravity are given, find the centre of gravity of the whole body.

G is the centre of gravity of a plane lamina in the form of an isosceles triangle right-angled at A, and having the side BC of length a. The portion GBC being cut away, find the distance of the centre of gravity of the remaining piece from A.

7. If a heavy body be held up by three strings attached to one peg, prove that the peg and the centre of gravity of the body must be in the same vertical line.

8. Weights of $7\frac{1}{2}$ and $2\frac{1}{2}$ lbs. are suspended from the ends of a straight uniform bar of length 5 ft. and weight 10 lbs. If the bar be laid across a fulcrum distant 15 inches from the greater weight, where must a weight of 15 lbs. be suspended so as to produce equilibrium?

9. Describe the system of pulleys in which each string is attached to a bar to which the weight is attached; and when the weights of the pulleys are neglected, shew that $W = P(2^n - 1)$, where n is the number of pulleys.

Shew that by neglecting the weights of the pulleys in our calculation we are making the mechanical advantage appear too small.

III. Cambridge Previous Examination. Part III. Additional Subjects. *June*, 1887.

1. Define the sine and tangent of an angle; and shew how to find the sine and tangent of an angle whose cosine (m) is given.

If $\sin A = \tan B$, prove that

$$\cos^2 A \cos^2 B = (\cos B + \sin B)(\cos B - \sin B).$$

2. Trace the changes in the tangent of an angle as the angle changes from 180° to 270°.

If $\sin\theta = -\frac{3}{5}$, find $\tan\theta$; and explain, by means of a figure, the reason why there are two answers to this question.

3. If two forces P and Q act at the same point, and if their directions make with each other an angle θ, find the magnitude of their resultant.

A bullet weighing 4 oz., suspended from a fixed point by a string 25 in. long, is kept by a horizontal force in equilibrium at a distance of 15 in. from the vertical line through the point of suspension. Find the tension of the string.

4. Shew how to resolve a force into two forces, acting in the directions of any two given lines passing through its point of application.

The side BC of a square $ABCD$ is bisected at E, and a force P acts along AE. Resolve this force into two forces, acting respectively along AB and AC.

5. Find the resultant of two like parallel forces P and Q, acting on a body at two points A and B.

A uniform iron rod 2 feet long, whose weight is 7 lbs., is placed upon two nails, which are fixed at two points A and B in a vertical wall. AB is horizontal and 5 inches long. Find the distances to which the ends of the rod extend beyond the nails, if the difference of the pressures on the nails be 5 lbs.

6. ABC, ABD are equilateral triangles on opposite sides of the base AB. Forces of 2 lbs., 3 lbs., 5 lbs. act respectively along AC, CB, BA. Prove that their resultant passes through D, and find its magnitude.

7. Find the centre of gravity of a uniform triangular plate.

Find the position of the centre of gravity of a uniform heavy wire, bent into the form of a quadrilateral figure, having two of its opposite sides equal to one another and 4 inches long, and its other two sides parallel to each other and respectively 3 inches and 7 inches long.

8. A uniform straight lever, 2 feet long, weighs 3 lbs. A weight of 9 lbs. at one end of it balances a weight of 16 lbs. at the other end. Find the position of the fulcrum.

9. Find the ratio of the power to the weight in the case of a system of pulleys, round each of which a separate string passes and has one of its ends fixed to a horizontal bar to which the weight is attached; the strings being all parallel, and there being four pulleys, the weight of each of which is one-half that of the power.

IV. Cambridge Previous Examination. Part III.

Additional Subjects. *October*, 1887.

1. Define the cosine and cotangent of an angle and express each of them in terms of the cosecant.

Prove that

$$(\sec^2 A + \tan^2 A)(\text{cosec}^2 A + \cot^2 A) = 1 + 2 \sec^2 A \text{ cosec}^2 A.$$

2. Trace the changes in sign and magnitude of the sine of an angle as the angle increases from 0° to 180°.

If the sine of an angle be $> \frac{1}{\sqrt{2}}$ and the cosine of that angle be $> \frac{1}{2}$, between what limits does the angle lie?

3. Assuming the truth of the "parallelogram of forces" for the direction of the resultant of two forces, prove its truth for the magnitude of the resultant.

Prove that the resultant of two forces P and $P + Q$ acting at an angle of 120° is equal to the resultant of two forces Q and $P + Q$ acting at the same angle.

4. If three forces acting at a point be represented in magnitude and direction by the three sides of a triangle taken in order, prove that they are in equilibrium.

ABC is a triangle, D, E are points in AB and AC respectively, BE, CD cut in O; indicate the direction of the resultant of forces represented by CD, BE.

5. Prove that the algebraical sum of the moments of two forces about any point, outside the angle formed by their directions, is equal to the moment of their resultant about the same point.

6. Prove that the centre of gravity of a triangle coincides with that of three equal heavy particles at its angular points.

If the triangle ABC weigh 6 oz., what weight must be placed at A so that the centre of gravity of the whole may bisect the line joining A to the middle point of BC?

7. If a body be suspended by a string from a fixed point the centre of gravity will be vertically below the point of suspension.

8. Give examples of the different kinds of levers, pointing out those in which there is mechanical advantage or disadvantage.

A lever, 30 inches in length, has weights 3 lbs. and 15 lbs. fastened to its ends and balances about a point 9 inches from one end: what is the weight of the lever?

V. Cambridge Previous Examination. Part III. Additional Subjects. *December*, 1887.

1. Trace the changes in sign and magnitude of the tangent of an angle as the angle increases from 0° to 180°.

If $3 \sin A + 2 \cos A = 3$, find the value of $\tan A$.

2. Find the value of $\sin 45^\circ$, $\sin 60^\circ$, and $\sin 90^\circ$.

If $\sin(4A + 3B) = 2 \sin(3A + B) = 1$, find A and B.

3. Define the resolved part of a force in a given direction. Shew that the component of a given force in a given direction may be of any magnitude, and determine the direction and magnitude of the other required component, enunciating the proposition you assume.

4. If three forces acting at a point are in equilibrium they can be represented in magnitude and direction by the three sides of a triangle taken in order.

Find the resultant of two forces of 10 lbs. and 9 lbs. acting at an angle whose tangent is $\frac{4}{3}$.

5. Find the magnitude and line of action of the resultant of two unlike parallel forces.

The resultant of two unlike parallel forces of 10 lbs. and 18 lbs. acts in a line at a distance of 12 ft. from the line of action of the less force; what is the distance between the lines of action of the two forces?

6. Shew that every system of heavy particles has one and only one centre of gravity.

ABC is a triangle right-angled at *A*; *AB* and *AC* are 12 inches and 15 inches respectively. Weights of 2 oz., 3 oz., 4 oz. are placed at *A*, *C*, *B* respectively: find the distances of their centre of gravity from *B* and *C*.

7. One corner of a square sheet of paper, whose side is 1 foot, is folded down so as to coincide with the centre of the square. Find the distance of the centre of gravity of the paper from the centre.

8. Define a lever, and find the condition of equilibrium on a straight lever.

A uniform lever, 1 yard long, weighs 15 oz. The fulcrum is 3 in. from one end. What force will be required at either end to cause it to balance?

9. Find the condition of equilibrium on the system of pulleys in which all the strings, being parallel, are attached to the weight.

If there be 3 pulleys, each weighing 8 oz., what weight will a power of 3 lbs. support?

VI. Cambridge Previous Examination. Part III.

Additional Subjects. *June*, 1888.

1. Define the tangent and cosecant of an angle, and express each of them in terms of the cosine.

Prove that $\sec^2 A \operatorname{cosec}^2 A = \sec^2 A + \operatorname{cosec}^2 A$.

2. Arrange in order of magnitude $\sin 17°$, $\cos 17°$, $\sec 17°$ and $\operatorname{cosec} 17°$, proving that the order you give is correct.

Shew that there is an angle less than a right angle whose cosine is equal to its tangent. Find the value of the sine of this angle correct to three places of decimals.

3. Define the resultant of any number of forces. If the greatest resultant that two forces can have be P, and the least resultant they can have be Q, find what their resultant is when they act on the same particle in directions at right angles to each other.

4. If two forces, acting at a point O, be represented in direction and magnitude by the sides AB, BC of the triangle ABC; prove that the side AC will represent their resultant.

The side BC of an equilateral triangle ABC is bisected at D, and forces are represented in direction and magnitude by BA, BD. Find the magnitude of their resultant if the force along BD be equal to the weight of one pound.

5. Find the resultant of two unequal parallel forces acting towards the same parts.

Prove that the moment of the resultant of these two forces about any point, situated between their directions, is equal to the algebraic sum of the moments of the component forces.

6. Define the centre of gravity of a body, and shew how to find that of three unequal particles placed at the three angular points of a triangle.

Find the centre of gravity of six heavy particles, situated along a straight rod, the successive particles weighing 1, 4, 9, 16, 25, 36 grains respectively; the distance between the first and second particles being one inch, and the distances between the others being respectively 3, 5, 7 and 9 inches. The weight of the rod need not be taken into account.

7. If a body be placed upon a horizontal plane, what is the condition that it may stand upon it without falling?

In the side CD of a uniform square plate $ABCD$ a point E is taken and the triangle ADE is cut off. Find the length of DE so that the plate $ABCE$ may just be able to stand with its side CE on a horizontal plane, the side of the square being a inches long.

8. Find the condition of equilibrium in the system of pulleys, in which each pulley hangs by a separate string, one end of each string being attached to a beam above the pulleys. The strings may be considered all to be parallel, and the weights of all the pulleys to be the same.

VII. CAMBRIDGE GENERAL EXAMINATION FOR THE ORDINARY B.A. DEGREE. STATICS. *December*, 1886.

1. Define Force; shew that forces may be represented by straight lines; mention the principal classes of forces with which we are concerned in Elementary Mechanics.

2. Enunciate the proposition known as "the parallelogram of forces," and prove it so far as the direction of the resultant is concerned for commensurable forces.

Two forces P and Q of given magnitude act at a point A and the direction of P is fixed. Shew that if the direction of Q change, the extremity of a straight line drawn from A representing the resultant of P and Q will lie on the circumference of a fixed circle.

3. Prove that the moment of the resultant of two forces which act at a point about any point O in the plane of the forces is equal to the algebraical sum of the moments of the forces about the same point O in the case when the point lies within the angle between the directions of the forces.

4. Find the magnitude and line of action of the resultant of two parallel forces which act at given points in the same direction.

5. Define the term Centre of Gravity, and shew how to determine the centre of gravity of a system of heavy particles.

A straight rod AE without weight is divided in the points B, C, D, so that $AB : BC : CD : DE :: 1 : 3 : 5 : 7$, and weights of 1, 2, 3, 4 lbs. are placed at the points B, C, D, E, respectively: shew that if G be the centre of gravity of the system

$$AG : GE :: 5 : 3.$$

6. Shew how to find the centre of gravity of a plane triangle of uniform thickness and density.

Shew that the centre of gravity of a plane quadrilateral cannot coincide with that of four equal particles placed at its angular points unless the quadrilateral be a parallelogram.

7. Describe and explain the mode of graduating the Common Steelyard.

8. Find the relation between the power and the weight when there is equilibrium in that system of pulleys in which the strings are parallel and the extremity of each string is attached to a fixed beam. If there be four moveable pulleys the weight of each of which is 1 lb., find the power when the weight supported is 33 lbs. Find the whole strain on the fixed beam.

9. Find the condition of equilibrium when a weight is supported on a smooth inclined plane by a force acting parallel to the plane.

Two weights are connected by a fine string and supported upon a smooth double inclined plane whose base is horizontal, the string passing over the intersection of the planes; prove that the weights are proportional to the lengths of the planes on which they rest.

Find the parts of the whole weight supported at the points where the weights rest, and at the intersection of the planes, in the case when the inclinations of the planes to the horizon are respectively 60° and 30°.

VIII. Cambridge General Examination for the Ordinary B.A. Degree. Statics. *June*, 1887.

1. Define *force*, *resultant*, *component*.

Find the resultant of seven different forces acting in the same straight line on a point, three in one direction and four in the other; and shew that if one of them be reversed in direction, the change in the resultant will be twice the magnitude of this force.

2. Enunciate the proposition known as "the parallelogram of forces," and assuming it to be true as regards the direction of the resultant, complete the proof.

If two forces be inclined to one another at an angle of three halves of a right angle, find the ratio of their magnitudes when the resultant equals the less.

3. State and prove the polygon of forces.

ABC is a triangle; D, E are the middle points of AB, AC: shew that forces acting at a point represented in magnitude and direction by DB, BC, CE are equivalent to forces represented by DA, AE.

4. Find the resultant of two parallel forces that act in opposite directions.

Two men, one stronger than the other, have to remove a block of stone weighing 300 lbs. with a light plank whose length is 6 feet: the weaker man cannot carry more than 100 lbs., how must the stone be placed on the plank so as just to allow him that share of the weight?

5. Define the moment of a force about a point, and shew that, if the algebraical sum of the moments of two forces about a point is zero, that point is on their resultant.

6. Find the centre of gravity of a uniform triangular plate.

What is the form of a triangle, if its centre of gravity coincides with the centre of a circle circumscribing it?

7. How can the centre of gravity of a body be practically determined by suspending it by a string?

8. Find the ratio of the power to the weight in a system of n weightless pulleys in which a separate string passes round each and is attached to a moveable beam from which the weight is suspended, the strings being parallel.

If there be four pulleys arranged as described, and if the three moveable pulleys beginning with the lowest weigh P, $2P$, $4P$ respectively, shew that if the system be acted upon by a power P it will support a weight $32P$.

9. If a weight W rests on an inclined plane when acted upon by a horizontal force P, shew that $P : W ::$ height of plane : base of plane.

If the pressure on the plane be double the weight, what is the angle of the plane?

IX. Cambridge General Examination for the Ordinary B.A. Degree. Statics. *December*, 1887.

1. Define the terms "force" and "weight," and enumerate the chief forces with which we have to deal in Statics.

How may the weights of two bodies be compared?

2. Enunciate the "Parallelogram of Forces." Prove its truth so far as the direction of commensurable forces is concerned.

The resultant of two forces acting at an angle of $\frac{4}{3}$ of a right angle is perpendicular to the smaller component. The greater component is equal to a weight of 50 lbs. Find the other component and the resultant.

3. Shew that if two forces acting at a point be represented in direction and magnitude by two sides of a triangle taken in order, the third side of the triangle, not taken in the same order as the other two, represents their resultant.

Three forces are completely represented by the lines joining the angular points of a triangle with the middle points of the opposite sides. Shew that they are in equilibrium.

4. If three forces in the same plane be in equilibrium, prove that they are either all parallel to each other or all meet in a point.

A uniform rod has its lower end fixed to a hinge, and its other end attached to a string which is tied to a point vertically above the hinge. Shew that the direction of the action at the hinge bisects the string.

5. Define the centre of gravity of a body, and find the position of the centre of gravity of three equal weights placed at the corners of a triangle.

Weights of 1, 3, 5 and 7 lbs. are placed at the corners of a square taken in order. Shew that their centre of gravity is midway between one of the sides of the square and the intersection of the diagonals.

6. Describe a method by which the centre of gravity of a flat piece of metal of any shape may be practically found. Prove the principle on which the method depends.

7. What are the different classes of levers? Give an example of each.

A straight uniform lever, whose weight is 15 lbs. and length 3 yards, rests in equilibrium on a fulcrum when a weight of 3 lbs. is suspended from one extremity; find the position of the fulcrum and the pressure on it.

8. Find the condition of equilibrium in the system of pulleys in which the same string goes round all the pulleys and the parts of the string between the pulleys are parallel.

If a power of 3 lbs. will just support a weight of 11 lbs. suspended from the lower block, the number of strings being four, find the weight of the lower block.

9. A weight is supported on a smooth inclined plane by a power acting parallel to the plane. Find the relation between the power and the weight.

If the weight be 10 lbs., and the power 5 lbs., what is the pressure on the plane?

X. Cambridge General Examination for the Ordinary B.A. Degree. Statics. *June*, 1888.

1. Define a force. What three elements must be known in order to determine a force completely, and how can forces be represented by straight lines?

2. Enunciate the "Parallelogram of Forces," and prove it as to the direction of two commensurable forces.

The sum of two forces is 36 lbs., and the resultant, which is at right angles to the smaller of the two, is 24 lbs. Find the magnitude of the forces.

3. State and prove the polygon of forces.

Forces represented in magnitude and direction by the diagonals of a parallelogram act at one of the corners, what single force will counteract them?

4. When two parallel forces act in opposite directions shew that the moment of their resultant about any point in their plane is equal to the algebraical sum of the moments of the two forces about the same point.

A thin board in the form of an equilateral triangle, and weighing 1 lb., has one quarter of its base resting on the end of a horizontal table, and is kept from falling over by a string attached to its vertex and to a point on the table in the same vertical plane as the triangle. If the length of the string be double the height of the vertex of the triangle above the base, find its tension.

5. Shew that every system of heavy particles must have a centre of gravity, and find that of four equal weights placed at the corners of a parallelogram.

A uniform bar 4 yards long weighing 12 lbs. has three rings each weighing 6 lbs. upon it at distances 1 foot, 5 feet and 7 feet from one end. At what point will it balance?

6. If the centres of gravity of a whole body, and of a part of this body, be known, shew how to find the centre of gravity of the remainder.

One corner of a square is cut off by a straight line passing through the middle points of two adjacent sides. Find the position of the centre of gravity of the remainder.

7. Describe and graduate the common steelyard.

8. Find the condition of equilibrium in the system of pulleys in which each string is attached to the weight.

If there are five moveable pulleys each weighing half a pound, and the weight is 35 lbs., what is the power?

9. A force P acting up an inclined plane supports a weight W on it. If R be the reaction of the plane, prove that

$$P : W : R :: \text{height of plane} : \text{length} : \text{base}.$$

XI. Oxford and Cambridge Schools Examination. *June*, 1888.

Note.—(1) *In order to pass in Elementary and Additional Mathematics, a Candidate must satisfy the Examiners in Part I.*

(2) *Distinction in Mathematics will depend upon the Candidate's work in the whole of the Paper.*

Part I.

1. Assuming the parallelogram of forces as regards the *direction*, prove its truth for the *magnitude* of the resultant.

Two forces are represented in direction and magnitude by the diameters AC, BD of the parallelogram $ABCD$; find the resultant by means of a geometrical construction.

2. Find the resultant of two given like parallel forces.

A beam AB, of length 15 feet and weight 200 lbs., whose centre of gravity is at a distance 7 feet from A, is supported in a horizontal position by props at A and B. Find the pressure on the prop at A.

3. From a body, the centre of gravity of which is known, a given portion whose centre of gravity is also known is cut out: shew how the centre of gravity of the remainder may be determined.

On a radius of a given sphere as diameter another sphere is described and the latter sphere is cut out of the former. Find the centre of gravity of the remainder, assuming that the volume of a sphere varies as the cube of its radius.

4. A smooth uniform beam, of weight W lbs. and length 6 feet, rests with one point of it on the top of a fixed vertical post 3 feet high. The lower end of the beam is on the horizontal plane through the foot of the post and connected with the latter by a string 4 feet long. Find the tension of the string.

5. Find the conditions of equilibrium of a body of given weight supported on a smooth inclined plane by the action of a horizontal force.

Two particles A and B are connected by an inextensible string and their weights are such that when they are placed one on each of two inclined planes having a common altitude, they are in equilibrium: shew that the centre of gravity of the two masses is always in the same horizontal line for all positions of equilibrium.

PART II.

9. State the laws of limiting friction.

A body is placed upon a rough inclined plane of angle α, and is kept in equilibrium by a force, whose line of action makes the angle $\frac{\alpha}{2}$ with the line of greatest slope of the plane. Find the limiting values of the ratio of the weight of the body to the force, when the coefficient of friction is $\tan\frac{\alpha}{2}$.

The rest of the paper is on Dynamics.

XII. OPEN COMPETITION FOR ROYAL MILITARY ACADEMY, WOOLWICH. *December*, 1886.

1. How are forces measured in Statics? Can three forces, proportional to 9, 5, 3, respectively, acting in one plane on a particle, be so arranged as to be in equilibrium? State and prove any proposition that involves the answer to the question.

2. Two equal forces inclined to each other at a given angle act on a fixed point; find the pressure on the point.

Two equal weights (W) are attached to the extremities of a thin string which passes over three tacks in a wall arranged in the form of an isosceles triangle with the base horizontal, the vertical angle at the upper tack being 120°; find the pressure on each tack.

3. If a straight uniform rod is suspended by a thin string fastened to its middle point, and be kept in equilibrium by two weights on the opposite sides of the middle point, find the tension of the string by which the rod is suspended, and shew how the weights are related to each other.

A straight lever 2 feet long is moveable about a hinge at one end, and is kept in a horizontal position by a thin vertical string attached to the lever at a distance of 8 inches from the hinge, and fastened to a fixed point above the lever; if the string can just support a weight of 9 ounces without breaking, find the greatest weight that may be suspended from the other end of the lever.

4. Shew that any system of forces acting on a rigid body in one plane may be reduced to a single force and a single couple. A rod is placed in *any* given position with one end on a smooth floor and the other end against a smooth wall. Find a single force and a single couple which together will keep it at rest in that position.

5. If a right cone be cut by a plane bisecting its axis, find the distance of the vertex of the cone from the centre of gravity of the frustum thus cut off.

6. In that system of three pulleys (usually called the third system) where each string is attached to the weight, and the weights of the pulleys are all equal, find the relation of the power to the weight, when equilibrium is established. If each pulley weighs 2 ounces, what weight would be supported by the pulleys alone?

If the weight supported be 25 lbs. and the power 3 lbs., find what must be the weight of each pulley.

7. Find the force acting up and parallel to a given rough inclined plane that is just able to move a weight up the plane.

Two equal weights are attached to a string that is laid over the top of two inclined planes having the same altitude and placed back to back, the angles of inclination of the planes being 30° and 60° respectively, shew that the weights will be on the point of moving if the coefficient of friction between each plane and weight be $\frac{1}{2+\sqrt{3}}$.

8. State the principle of virtual velocities. Assuming the principle true, deduce from it the condition of equilibrium on a bent lever of unequal arms when acted on by weights suspended at the extremities of the arms, and shew that for an infinitely small displacement the centre of gravity of the weights will neither ascend nor descend.

9. A lamina in the form of an isosceles triangle, whose vertical angle is α, is poised upon a sphere, radius r, so that its plane is vertical and one of its equal sides (a) is upon the surface of the sphere; shew that the equilibrium will be stable in the plane of the triangle if $\sin\alpha$ be less than $\frac{3r}{a}$.

10. Define the unit of work, and explain how it varies with the units of length, mass, and time.

A chain weighing 8 lbs. per foot is wound up from a shaft by the expenditure of four million units of work; find the length of the chain.

11. A cylindrical shaft has to be sunk to a depth of 100 fathoms through chalk whose specific gravity is 2·3; the diameter of the shaft being 10 ft. What horse-power is required to lift out the material in 12 working days of 8 hours each? [The weight of a cubic foot of water is 62·5 pounds, and one horse-power is 33,000 foot-pounds a minute.]

ANSWERS.

I.

1. $\frac{3}{4}$ inch; 1 inch. **2.** 10 lbs.; $12\frac{1}{2}$ lbs. **3.** $7\frac{1}{2}$ in.; 9 in.; $18\frac{3}{4}$ in.

4. 192 lbs.; 112 lbs. **5.** $1\frac{1}{2}$ lbs.; 2 lbs. **6.** $3\sqrt{2}$ lbs. S. W.

7. *DB* represents a force of $\sqrt{3}$ lbs. westwards; *AB* represents a force of $2\sqrt{3}$ lbs. inclined at an angle of 30° to *AD*.

II.

1. Resultant 2 lbs. towards the south. **2.** Resultant 7 lbs. downwards.

3. Resultant 3 lbs. towards the south. **4.** 2. **5.** Zero.

6. 15. **7.** 6 lbs. **8.** 74 lbs.

9. The third horse must back with a force of 98 lbs.

III.

1. Resultant 5 lbs., angle 53° . 7′ nearly. [These angles are found from a table of tangents.] **2.** 18·02...lbs., 33° . 40′.

3. 65 lbs., 67°. 23′. **4.** 9·219 lbs., 49°. 24′. **5.** 14·86 lbs., 47°. 42′.

6. 22·36 lbs., 63° . 26′. **7.** 5·657 lbs., 45°. **8.** 2 lbs., 60°.

9. 4·76...lbs., 36° . 29′. **10.** 4·836...lbs., 11° . 55′.

11. 4·797 lbs., 30° . 12′. **12.** 11·31 lbs., 43°. 56′.

13. $5a$ lbs., 53° 7′. **14.** $\sqrt{a^2+b^2}$, $\tan^{-1}\frac{b}{a}$.

[The angle in each of these cases is that which the resultant makes with the direction of the first of the forces given.]

IV.

1. 20·784...lbs.; 41·568...lbs. **2.** 8·66...lbs.; 5 lbs.

3. 1 lb.; 1 lb. **4.** 57·7...lbs.; 57·7...lbs.

5. 17·675...lbs.; 17·675...lbs. **6.** 100 lbs.; 100 lbs.

7. 14·14...lbs.; 14·14 lbs. **8.** 25·98...lbs.; 25·98...lbs.

12. 6·6 lbs. **13.** $10\frac{1}{4}$ lbs.; $12\frac{1}{4}$ lbs.

14. $7\frac{3}{4}$ lbs.; $5\frac{3}{4}$ lbs. **15.** 13·4 lbs.

V.

1. 15 lbs. **2.** 43·3 lbs. **3.** 4 lbs. **4.** 75 lbs.

5. 210 lbs. **6.** 54·625 lbs. **7.** 188·178 lbs. **8.** 20·2995 lbs.

9. 20 lbs. **10.** 2·1 lbs. **11.** $16\frac{3}{4}$ lbs. **12.** $7\frac{2}{5}$ lbs.

13. 16·9 lbs. **14.** 9 lbs.

VI.

1. 8·94...lbs. 2. 8·47...lbs. 3. 16·05...lbs. 4. 6·76...lbs.
5. 9·848...lbs. 6. 14 lbs. very nearly. 7. 5 lbs. 8. 26·45...lbs.
9. 46·35...lbs. 10. 21·25...lbs. 11. 225·6...lbs.
12. P lbs. 13. $2\sqrt{3}$ lbs. 14. $\sqrt{6}$ lbs.

VII.

1. 6·48...lbs. 2. 5 lbs. 3. 7·07...lbs. 4. 20 lbs.
5. 2 lbs. : 1 lb. 6. 20 lbs. 7. 9 lbs. 12. 5 lbs.; -5 lbs.

VIII.

1. 9·83...lbs., [tan $ROA = 1·692...$].
2. 10·19...lbs., [tan $ROA = 1·5$].
3. 9·85...lbs. nearly, [tan $ROA = 2·25$].
4. 13·6...lbs., [tan $ROA = 1·375$].
5. 385·5...lbs., [tan $ROA = -31·86$].
6. 17·32...lbs. acting in direction perpendicular to OA.
7. $-1·732$...lbs., $\left[\tan ROA = \frac{-·866}{-1·5}\right]$; ∴ Resultant lies in the third quadrant. 8. $-1·732$...lbs., see Answer to 7.
9. 4·47...lbs., [tan $ROA = \frac{1}{2}$]. 10. Zero. 11. Zero.
12. 9·605...lbs., [tan $ROA = 6·25$]. 13. Zero.
14. $-7·96$...lbs.: if the direction of force of 1 lb. be the initial line the resultant force lies in the third quadrant, [tan $ROA = 1·75$].
15. 6 lbs. and the resultant makes an angle of 120° with the direction of force of 3 lbs.
16. 6 times the force represented by AB acting along AD, [tan $RAB = 2·15$].
18. 13·28 lbs., [tangent of angle which the resultant makes with the direction of the force of 1 lb. $= 3·09...$].

IX.

1. 13 lbs. 2. $Q\sqrt{3}$ making an angle of 150° with the direction of $2Q$. 4. $P = 5$ lbs., $Q = 12$ lbs., $R = 13$ lbs.
8. CA represents the resultant. 10. As $1 : \sqrt{2}$. 12. 150°.
13. $\cos^{-1}\frac{1}{8}$. 14. 7 lbs. 16. $2AB$. 17. $3\sqrt{2}$ lbs.; 3 lbs.
18. $\sqrt{13}$ lbs. 19. $\sqrt{\{4^2 + (3+\sqrt{3})^2\}}$ lbs., 6·19...lbs.
20. 11·6 lbs. nearly.

X.

1. 25; $15\sqrt{2} = 21 \cdot 2\ldots$; zero.
2. 12·5; zero; 17·5.
3. 25; $\frac{15}{2}\sqrt{2} = 10 \cdot 6\ldots$; 17·5.
4. 72; 72; 30; zero.
5. 72; 24; 30; 144.
6. 7 inches from *A*.
7. 42 inches from *A*.
8. The point is $\frac{5}{9}$ of *BC* from *B*.

XI.

4. A straight line parallel to the direction of the force.
5. A straight line drawn parallel to the force which does not act the same round as the other two.
7. The force passes through *C*, bisects *AB* and $= 6$ lbs.
8. 2 lbs. parallel to *AB* and distant 4 feet from *A*.

XII.

1. The resultant of the forces cuts the line joining these two points in the ratio of 2 to 1.
2. The resultant, if not zero, must pass through *Q*.
8. No, unless the triangle is equilateral.

XIV.

1. 5 lbs. acting 3 ft. from smaller force.
2. $2\frac{1}{2}$ ft. from smaller force.
3. 64 lbs.; 48 lbs.
4. 48 lbs.; 32 lbs.
5. 12 lbs.; 24 lbs.
6. 24 lbs.; 16 lbs.
7. An unlike parallel force of 10 lbs. acting at the middle point.
8. An unlike parallel force of 3 lbs. acting at a point $2\frac{2}{3}$ ft. from the other unlike force.
9. 3 cwt.
10. $7\frac{1}{2}$ ft. from fixed point.
11. An unlike parallel force of 21 lbs. acting at a point 4 ft. from smallest weight.
12. An unlike parallel force of 7 lbs. acting at a point 6 ft. from the smaller weight.
13. 2 ft. from the stronger man.
14. 4 ft. from him.
15. $267\frac{2}{3}$ lbs.; $624\frac{1}{3}$ lbs.
16. $7\frac{5}{39}$ ft.; $12\frac{34}{39}$ ft.
17. 8 lbs. and 6 lbs.
18. $116\frac{2}{3}$ lbs.
19. 350 lbs.
20. 42 lbs.; 21 lbs.
21. $17\frac{1}{2}$ ft.
22. 96 lbs.
23. 96 lbs.
24. 21 lbs. acting at a point $3\frac{1}{3}$ in. from end.
25. 27 lbs. .... $1\frac{23}{27}$ ft. ....
26. 24 lbs. .... $5\frac{1}{9}$ ft. ....
27. 8 ft. ....
28. $2\frac{4}{15}$ ft. from end; 1 cwt., 2 cwt., 4 cwt., 8 cwt. respectively.
31. A force of 7 lbs. acting at a point $6\frac{3}{7}$ ft. from *A*.
32. A force of 2 lbs. downwards at a distance 15 ft. from *A*.
35. 20 lbs.

XV.

1. OM in line $OC = 2\frac{14}{18}$ in., $ON = 2\frac{2}{5}$ in.
2. $OM = 2$ in., $ON = 1\frac{11}{18}$ in. 3. $OM = 1$ in., $ON = \frac{1}{2}$ in.
4. $CM = 3\frac{3}{4}$ in., $CN = 2{\cdot}165\ldots$ in.
5. The point of intersection of the diagonals.
6. The C. G. of the triangle. See p. 95.
7. At the centre of the Inscribed Circle. $CM = s - c$, $CN = \frac{S}{s} = r$, see *Trig.* p. 232.
8. If the force at C be unlike that of the other two then

$$CM = a + b \,.\, \cos c, \ CN = b \sin c.$$

If the parallelogram $ACBO$ be completed, O is the centre of the forces.

9. At the centre of an Escribed Circle. If the force at C be the unlike force, then $CM = s$, $CN = r_3$, see *Trig.* p. 233.
10. CM in direction $CD = \frac{1}{4}(2c - d \,.\, \cos D + b \cos C)$,

CN in direction perpendicular to $CD = \frac{1}{4}(b \sin C + d \sin D)$,

$$AB = a, \ BC = b, \ CD = c, \ DA = d.$$

XVI.

1. $\frac{1}{4}$ of length of rod from point of suspension of 10 lbs.
2. $\frac{15}{26}$ of length of rod from point of suspension of 3 lbs.
3. 24 lbs., 20 lbs. 4. 8 ft. from the other end.
5. $28\frac{4}{5}$ ft. from the first man.
6. $5\frac{3}{5}$ ft. from point of suspension of 8 lbs.
7. $6\frac{2}{5}$ ft. from point of suspension of 12 lbs.
8. 10 ft. from that end. 9. 6 lbs. 10. 10 lbs.
11. $1\frac{5}{12}$ ft. from point of suspension of 7 lbs.
12. $3\frac{1}{4}$ ft. from point of suspension of 1 lb.
13. $2\frac{1}{6}$ ft. from point of suspension of 3 lbs.
14. 16 lbs. 15. 2 lbs. 16. 8 lbs.
17. $267\frac{2}{3}$ lbs.; $624\frac{1}{3}$ lbs. 18. $7\frac{31}{41}$ ft.; $12\frac{10}{41}$ ft. 19. 16 lbs.
20. $33\frac{1}{3}$ lbs. 21. $8\frac{1}{3}$ lbs. 22. 6 lbs. 23. 15 ft.

XVII.

1. C. G. is $2\frac{1}{6}a$ from the foot of the cross [a = side of square].
2. C. G. is distant $\frac{2}{5}$ of the diameter measured from the point of suspension of 2 lbs.
3. C. G. is distant $\frac{4}{7}$ of the diameter measured from the point of suspension of 1 lb.
5. C. G. is $5\frac{7}{8}$ in. from the middle point of the lowest side of the figure.

6. C. G. is distant $\frac{13}{24}$ of the diameter from that corner.
7. C. G. is distant $\frac{2}{9}$ of the perpendicular from E on DC.
8. C. G. is distant $\frac{2}{9}$ of the line drawn from the middle of base to vertex.
9. C. G. is distant $\frac{1}{15}$ of the radius of the greater circle from its centre.
10. C. G. is distant $\frac{1}{14}\sqrt{2}$ of the radius bisecting the angle between the two radii from the centre.
11. C. G. is distant $\frac{1}{3}$ of the radius drawn from centre to B.
12. C. G. is distant $\frac{1}{3}$ of the line drawn from centre to the middle of AD.
13. C. G. is distant $\frac{1}{24}$ of OD.
14. As $\sqrt{3} : 1$.
16. C. G. is distant $\frac{5}{18}$ of the line drawn from the middle of the opposite side to the other angle.
17. C. G. is distant $\dfrac{3}{8+2\sqrt{3}}$ of the base from its middle point.
19. C. G. is distant $\frac{11}{30}$ of the side of the square from the middle point of the base.
20. Let the C. G. be at G; draw GM, GN perpendiculars to DA, DC; then $GM = \frac{11}{18}$ of DC; $GN = \frac{5}{9}$ of AD.
21. C. G. is distant from E $\frac{7}{12}$ of the length of the line joining the middle of DC with E.
28. The tangent of each base angle is 3. 29. 3.

XVIII.

2. Two thirds of the line joining the vertex with the middle point of the opposite side. 3. 45°, 135°.
5. $a\sqrt{3}$. [a = side of square.] 7. 24·142... lbs.
9. 120 lbs. 10. 18 lbs.
11. 17 bricks; for the C.G. of 17 bricks is vertically over the edge of the 18th brick.
12. If $c^2 > a^2 + 3b^2$ then a force $= \frac{1}{3} W \dfrac{c^2 - a^2 - 3b^2}{c^2 - a^2 - b^2}$ must be applied vertically upwards at B to maintain equilibrium. 14. 6 lbs.

XIX.

1. A force of − 1 lb. 2. A force of − 11 lbs. 3. A force of − 3 lbs.
4. AC must be produced to point Q, so that $AQ = 18$ inches.
5. 18 in. from E in the same direction.
6. 1·519... inches. 7. $1\frac{1}{3}$ in. from A.
8. (i) AD must be produced 7 in.
(ii) DC must be produced $1\frac{1}{2}$ in.

9. It cuts AD produced to M so that $AM = 41 \cdot 2$ in., and DC produced to N so that $DN = 12 \cdot 6$ in.

11. A force of 6 lbs. acting in a direction parallel to BC.

12. The resultant of the three forces is parallel to this perpendicular; so that no such point can be found.

14. $2\sqrt{3}$ lbs. at B, $\frac{5}{2}\sqrt{3}$ lbs. at A.

15. A weight equal to $2Q$ must be suspended from B.

17. 2 lbs. 18. $\sqrt{3}$ lbs. 19. 100 lbs.

XX.

1. 141 lbs. 2. 116 lbs. 3. A weight of 30 lbs.

5. (i) A force of 20 lbs. (ii) 100 lbs.

6. When the end of the chain has reached a point 20 feet from the ground. 8. 5 ft. 4 in.; 7 ft. 4 in.

9. (i) the 4 lb. weight is at the lowest point; (ii) 7 lbs., 5 lbs.

10. (i) the 10 lbs. weight is at the lowest point. (ii) The weight of 10 lbs. is at its lowest point and one part of the chain supports 7 lbs. more than the other; therefore the tensions of the two parts of the chain at the lowest point are $8\frac{1}{2}$ lbs. and $1\frac{1}{2}$ lbs. (iii) $16\frac{1}{2}$ lbs. 11. 9 lbs. 12. 20 lbs. 13. 5 lbs.

14. 10 lbs. 15. $67\frac{1}{5}$ lbs.; $50\frac{2}{5}$ lbs. 16. $78\frac{6}{13}$ lbs.; $188\frac{4}{13}$ lbs.

17. $50\frac{2}{5}$ lbs.; $67\frac{1}{5}$ lbs. 18. $188\frac{4}{13}$ lbs.; $78\frac{6}{13}$ lbs.

22. The vertical line from the peg downwards must pass through its C. G. Also since the peg is smooth the tension of the string is constant throughout; therefore the vertical through the C.G. bisects the angle between the strings. Hence the string is divided at the peg into two parts in the same ratio as the rod is divided at its C.G. [Euclid VI. 3].

23. $\frac{a}{c} \times W$; $\frac{b}{c} \times W$, where c = twice length of line joining the peg to middle point of the rod.

24. $\frac{1}{2}W \tan\theta$. 25. $\sqrt{3}W$. 26. $\frac{1}{3}\sqrt{3}W$.

27. The vertical downwards from centre of sphere will bisect the rod—perpendicular dist. $= \frac{1}{2}\sqrt{3}\, r$ [r = radius].

28. The vertical downwards will pass through the C. G. of rod. The 3 forces acting on the rod are the pressures at the two ends and the force of gravity at the C. G. These 3 forces meet at the centre of sphere.

29. The vertical through the bead bisects the angle between the strings.

30. If the strings supporting the extremities be produced they will meet the vertical drawn from the C. G. of the rod in one and the same point. If the parallelogram whose sides are along the strings and diagonal vertical be completed the tensions may be found.

XXI.

1. 40 lbs. **2.** $3\frac{1}{3}$ ft. **3.** $7\frac{1}{5}$ lbs. **4.** $2\frac{1}{2}$ lbs.
5. Any distance less than $3\frac{3}{8}$ ft.
6. $4\frac{8}{13}$ inches. **7.** $2\frac{13}{16}$ lbs. **8.** 12 lbs.
9. (i) Place the weight at one end with an arm of 5 ft. and the power at the other with an arm of 10 ft.
(ii) Place the weight $7\frac{1}{2}$ ft. from the fulcrum which is at one extremity and the power acting vertically upwards at the other end.
11. $1\frac{5}{7}$ ft. **12.** $11\frac{1}{5}$ ft.
13. (i) $\sqrt{9} : \sqrt{10}$; (ii) $6\sqrt{10}$ lbs. $= 18{\cdot}97...$ lbs.
14. $9\frac{4}{5}$ lbs. **15.** (i) $18\frac{2}{5}$ lbs. (ii) $\frac{1}{5}\sqrt{\{49^2+46^2\times 3\}}$ lbs.
16. Q must be multiplied by $\frac{P^2}{Q^2}$.
18. 40 lbs. **19.** 90°; 20 lbs. **20.** 26·7 lbs. **21.** 1 lb.

XXII.

2. The grocer loses $\frac{1}{2}\frac{(a-b)^2}{ab}$ of the nominal weight, where a, b are the lengths of the arms. **3.** 13 in.; 14 in. **5.** $10\frac{5}{12}$ lbs.
7. 1 oz. **11.** He loses $\frac{1}{4}d.$ on every 2 lbs. **12.** As 11·3 : 16.

XXIII.

1. 5 lbs.; 1 inch. **2.** The nominal weight is always 5 lbs. too great.
3. 18 in. **4.** 26 lbs.; 15 lbs.: 2 in. **6.** $5\frac{1}{3}$ in. from fulcrum; 4 in.
8. At the fulcrum. **9.** The fulcrum is 8 in. from the C.G. of the beam.
11. C.G. is distant 12 inches from the end to which the weight is attached: 4 oz. **12.** 36 lbs. **16.** 1 inch.

XXIV.

1. 1 lb. **2.** 80 lbs. **3.** 6 (pulleys). **4.** 3 lbs. **5.** 85 lbs.
6. 7 c. 2 qr. 18 lbs. (putting the 6 lbs. pulley highest).
7. 6 pulleys; 64 in. **8.** 4 lbs. **9.** $\frac{7}{8}W$; $7W$. **10.** 10 lbs.: $6\frac{3}{8}$ lbs.
11. The centre of the parallel forces is $\frac{4}{7}$ of a foot from point of suspension of third pulley. **12.** Half his weight.
13. 1 lb. **14.** $\frac{6}{7}P$. **15.** $12\frac{1}{3}$ lbs.
16. 2 tons [neglecting the weight of cable]. **17.** 64 lbs.
18. $280\frac{3}{4}$ lbs. **19.** 15 lbs. **20.** 443 lbs.
22. $8\frac{9}{10}$ times his own weight. **23.** 2635 lbs.; 375 lbs.
25. 56 lbs. **28.** 60 lbs. greatest; 54 lbs. least. **29.** Four.
31. The system in which each pulley is supported by a separate string.

XXV.

3. $12\sqrt{3}$ lbs. = 20·78 lbs. 4. 6 lbs. 5. 45°. 6. 30°.
8. As $1 : \sqrt{3}$. 9. 45°. 10. 30°. 11. 60°.
12. 30°; $6\sqrt{3}$ lbs. 14. 30°; 30°. 15. As 3 : 4; $2P$.
16. As $AC : AB$. 18. 1·998... tons = 2 tons very nearly.

XXVI.

3. 28 cwt. 1 qr. $17\frac{5}{7}$ lbs. 4. $186\frac{2}{3}$ lbs. No. 5. 5 ft.
6. The radius of the wheel which is drawn to the nail makes an angle with the vertical whose sine is $\frac{1}{6}$.
7. (i) when it acts vertically upwards; (ii) when it is vertically downwards.
9. The diagonal to the point of application of the power makes an angle with the vertical whose sine is $\frac{5}{8}\sqrt{2}$.

XXVII.

1. 12 cwt. 2. 14·85... lbs. 3. $\frac{1}{10}$ of an inch.
4. $\frac{3}{56}$ = ·054... of an inch.

XXVIII. a.

1. 5 lbs. 2. About $28\frac{1}{4}$ lbs. 3. 20 lbs.
4. About $28\frac{1}{4}$ lbs. 5. 15 lbs. 6. 56 lbs.
7. About 2 cwt. 76 lbs. 8. 50 lbs. 9. 84 lbs.

XXVIII.

1. $\frac{1}{3}\sqrt{3}$. 2. 380·8 lbs. 3. $112 \times \tan a$. 4. ·577....
5. 60°. 6. $2 \pm \sqrt{3}$. 7. $\sqrt{2} \pm 1$. 8. $\frac{2}{3}\sqrt{2}$ lbs.
9. $\tan^{-1}\frac{91}{60}$. 10. 8 ft. 12. $\tan^{-1}\frac{1}{2\mu}$; 10 ft.
13. $\frac{1}{240}$ very nearly. 14. Nearly 5 times. 15. $\mu\sqrt{2}W$.
16. The resolute of the force along the roller must be less than ·4 times W. The resolute perpendicular to the roller is ·01 times W, therefore the limiting angle must be the angle whose tangent is $\frac{1}{40}$.
17. $\tan^{-1}\frac{1}{40}$; very little more than 1120 lbs.
18. $W \times \mu = 9$ lbs. $\times \frac{11}{27}\sqrt{3} = 6·35...$ lbs. 19. $\mu = \frac{1}{2}$.
22. The distance of the C. G. from the centre $= \frac{r\mu}{\sqrt{1+\mu^2}}$, and the inclination of the plane must be $\tan^{-1}\mu$.
23. $\frac{5}{16}$. 25. 1.
26. The angle which the face between the wall and the plane makes with the horizon must not be greater than 45° nor less than the angle whose tangent is $\frac{1-\mu\mu'}{1+2\mu+\mu\mu'}$.

27. $\tan\alpha = 2\mu$, when 2α is the angle between the bars. **28.** 24 lbs.

29. If the girder begins to slip at both points at once the forces of friction are equal. Therefore it will only begin to slip at both points at once if the force be applied at the middle point. If it be nearer one bar than the other the greater of these parallel forces is in the middle, and the friction at the point at which it slips is greater than at the point where it does not.

30. The point of application must divide the part of the girder within the bars in the ratio of 2 : 1.

XXIX.

2. $150°$, $120°$, $90°$. **3.** $135°$, $135°$, $90°$.

6. They are in the same straight line.

7. 2 lbs. acting in the direction of the 5 lbs.

9. 1 lb. acting in a contrary direction to the 19 lbs.

10. The forces are parallel to the sides of an equilateral triangle taken the same way round; hence their resultant is that of 8 lbs. and 15 lbs. inclined at $120° = 13$ lbs.

13. $90°$, the angle whose sine is $\frac{12}{13}$ and the angle whose sine is $\frac{5}{13}$. [In each of these two cases the angle lies between $90°$ and $180°$.]

14. $150°$. **15.** The force along $AC = 2\sqrt{3}$ lbs. The force along $DA = 4$ lbs.

16. AD bisects the angle FDE, $\therefore$ the resultant is

$$2P\cos ADE = 2P\sin FDB = 2P\sin A.$$

$\therefore$ the forces are as $1 : 1 : 2\sin A$. **17.** $\frac{1}{20}$.

18. The two forces make an angle of $30°$ and $150°$ respectively with the side of the square.

XXX.

1. (i) A force of 10 lbs. acting in the direction opposite that of the 8 lbs.; (ii) A force of 10 lbs. acting along the 9 inch side opposite to the 8 lbs. and a couple whose moment is twice the area of the quadrilateral. **3.** $7\sqrt{3}$ lbs. **5.** $3\sqrt{2} \times OA$.

6. 4·6 and it lies in the fourth quadrant.

XXXI.

6. A circle. **7.** A line drawn perpendicular to the line joining the two "C.G.s" from the middle point.

XXXII.

5. A couple whose moment = the sum of the moments of the couples.

6. At C. **8.** The tendency to break in each of the three cases is $90 \times l$, $130 \times l$ and $50 \times l$ where l is half the length of the plank.

9. $4 \times 112 \times 1$; $448 \times 1\frac{8}{11} + 560 \times \frac{8}{11}$; 784×1.

10. 10×3; $10 \times 2\frac{1}{2}$. **14.** 12 tons at the middle point.

15. 54 tons at the middle point.

MISCELLANEOUS EXAMPLES. XXXIV.

1. $14 \cdot 24 \ldots$ lbs. **2.** The C. G. of the triangle.

3. Nine; for the C. G. of nine tiles is vertically above the outside edge of the tenth. **4.** One foot. **6.** $5\sqrt{3}$ lbs.

7. The two strings with weights 3 lbs. and 4 lbs. attached to them must be at right angles to one another, and the string attached to the weight of 5 lbs. must be in the direction of the diameter of the parallelogram formed by the other two strings.

12. The C. G. is in the axis of the longer cylinder, at a distance $= \frac{29}{100}$ of its length from the point of junction of its axis.

15. $90°$, $65°$, $25°$. **20.** $\tan^{-1} \frac{1}{2}\sqrt{3}$.

21. The direction of the string must pass through the centre of the sphere, it makes $30°$ with the vertical. Tension $= \frac{2}{3} W\sqrt{3}$. Pressure $= \frac{2}{3} W\sqrt{3}$. **23.** $\frac{1}{3} W\sqrt{3}$. **24.** $4\frac{1}{3}$ tons; $3\frac{2}{3}$ tons.

27. 10 in. from the scale-pan. **31.** As 2 : 3. **32.** 10 lbs.

35. $1\frac{5}{7}$ in. further from the 27 oz.

36. Tension : Weight = length of the string : twice the height of the staple.

37. The keel of the ship prevents sensible motion in the direction perpendicular to its length. In whatever direction the wind blows, the pressure which it applies to the sail is nearly perpendicular to the surface of the *sail*. The sail is so arranged that this pressure has a resolute in the direction of the ship's course.

38. $\frac{37}{48}$ of a foot.

41. Join the middle points of the two wires the C. G. will be distant $2 \cdot 186$ in. from the middle point of longer wire. Its distances from the wires are $3\frac{1}{3}$ in. and $1\frac{1}{3}$ in.

42. The friction permits him to arrange his body so that the C. G. of (his body and the ball) is outside the base, so that the joint body topples over up the plane.

45. $W : \frac{1}{3} W\sqrt{3} : \frac{1}{3} W\sqrt{3}$. **53.** $\sqrt{3} : 2$. **54.** $45°$.

57. The radius at one extremity of the rod makes the angle $\tan^{-1} \dfrac{1 - 2\mu - \mu^2}{1 + 2\mu - \mu^2}$ with the vertical.

58. Let I_1, I_2, I_3 be the centres of the escribed circles; then $r_2 : r_3 = I_2A : I_3A$; $\therefore$ the C. G. of the weights at I_2, I_3 is at A; $\therefore$ the C. G. of all three lines on AI_1; $\therefore$ their C. G. is at the centre of the inscribed circle.

61. 45 feet. **131.** AB is the rod; A, C the pegs; D the point; then $AC = \dfrac{AB^2}{AB^2 + DB^2}$. **132.** The tension is $\frac{1}{3} W\sqrt{3}$.

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