AN ELEMENTARY TREATISE

ON

MODERN PURE GEOMETRY.

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BY

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PREFACE.

THE object of this treatise is to supply the want which is felt by Students of a suitable text-book on geometry. Hitherto the study of Pure Geometry has been neglected; chiefly, no doubt, because questions bearing on the subject have very rarely been set in examination papers. In the new regulations for the Cambridge Tripos, however, provision is made for the introduction of a paper on "Pure Geometry;—namely, Euclid; simple properties of lines and circles; inversion; the elementary properties of conic sections treated geometrically, not excluding the method of projections; reciprocation; harmonic properties, curvature." In the present treatise I have brought together all the important propositions—bearing on the simple properties of lines and circles—that might fairly be considered within the limits of the above regulation. At the same time I have endeavoured to treat every branch of the subject as completely as possible in the hope that a larger number of students than at present may be induced to devote themselves to a science which deserves as much attention as any branch of Pure Mathematics.

Throughout the book a large number of interesting theorems and problems have been introduced as examples to illustrate the principles of the subject. The greater number have been taken from examination papers set at Cambridge and Dublin; or from the *Educational Times.* Some are original, while others are taken from Townsend's *Modern Geometry,* and Casey's *Sequel to Euclid.*

In their selection and arrangement great care has been taken. In fact, no example has been inserted which does not admit of a simple and direct proof depending on the propositions immediately preceding.

To some few examples solutions have been appended, especially to such as appeared to involve theorems of any distinctive importance. This has been done chiefly with a view to indicate the great advantage possessed by Pure Geometrical reasoning over the more lengthy methods of Analytical Work.

Although Analysis may be more powerful as an instrument of research, it cannot be urged too forcibly that a student who wishes to obtain an intimate acquaintance with the science of Geometry, will make no real advance if the use of Pure Geometrical reasoning be neglected. In fact, it might well be taken as an axiom, based upon experience, that every geometrical theorem admits of a simple and direct proof—by the principles of Pure Geometry.

In writing this treatise I have made use of the works of Casey, Chasles, and Townsend; various papers by Neuberg and Tarry,—published in *Mathesis*;—papers by Mr A. Larmor, Mr H. M. Taylor, and Mr R. Tucker—published in the *Quarterly Journal*, *Proceedings of the London Mathematical Society*, or *The Educational Times*.

I am greatly indebted to my friends Mr A. Larmor, fellow of Clare College, and Mr H. F. Baker, fellow of St John's College, for reading the proof sheets, and for many valuable suggestions which have been incorporated in my work. To Mr Larmor I am especially indebted for the use which he has allowed me to make of his published papers.

R. LACHLAN.

CAMBRIDGE,
11th *February*, 1893.

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CHAPTER I.

INTRODUCTION.

Definition of a Geometrical Figure.

1. A PLANE geometrical figure may be defined as an assemblage of points and straight lines in the same plane, the straight lines being supposed to extend to infinity. Usually either the point or the straight line is regarded as the element, and then figures are treated as assemblages of points or assemblages of straight lines respectively. To illustrate this remark let us consider the case of a circle. Imagine a point P to move so that its distance from a fixed point O is constant, and at the same time imagine a straight line PQ to be always turning about the point P so that the angle OPQ is a right angle. If we suppose the point P to move continuously we know that it will describe a circle; and if we suppose the motion to take place on a plane white surface and all that part of the plane which the line PQ passes over to become black, there will be left a white patch bounded by the circle which is described by the point P.

There are here three things to consider:—

i. The actual curve which separates the white patch from the rest of the plane surface.

ii. The assemblage of all the positions of the moving point P.

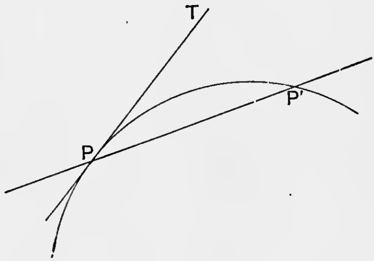
iii. The assemblage of all the positions of the moving line PQ.

It is usual to say that the curve is the *locus* of all the positions of the moving point, and the *envelope* of all the positions of the moving line. But it is important to observe that the three things are distinct.

2. Let us consider now the case of any simple plane figure consisting of a single curved line. Such a figure may be conceived as traced out by the motion of a point. Hence we may regard a simple figure as the locus of an assemblage of positions of a moving point.

The conception of a curve as an envelope is less obvious, but it may be derived from the conception of it as a locus. It will be necessary however to define a tangent to a curve.

Let a point P' be taken on a curve near to a given point P, and let PT be the limiting position which the line PP' assumes when P' is made to approach indefinitely near to P; then the straight line PT is said to *touch* the curve at the point P, and is called the *tangent* at the point.



If now we suppose a point P to describe continuously a given curve, and if for every position of P we suppose the tangent to the curve to be drawn, we may evidently regard these straight lines as the positions of a straight line which turns about the point P, as P moves along the curve. Thus we obtain the conception of a curve as the envelope of positions of a straight line.

3. It remains to consider two special cases. Firstly, let us suppose the point P to describe a straight line: in this case the assemblage of lines does not exist, and we may say that the straight line is the locus of the positions of the point. Secondly, let us suppose the point P to be fixed: in this case there is no assemblage of points, and we may say that the point P is the envelope of all the positions of a straight line which turns round it.

4. It follows that any plane figure consisting of points, lines, and curves, may be treated either as an assemblage of points or as an assemblage of straight lines. It is however not always necessary to treat a figure in this way; sometimes it is more convenient to consider one part of a figure as an assemblage of points, and another part as an assemblage of straight lines.

Classification of Curves.

5. Curves, regarded as loci, are classified according to the number of their points which lie on an arbitrary straight line. The greatest number of points in which a straight line can cut a curve is called the *order* of the curve. Thus a straight line is an assemblage of points of the *first* order, because no straight line can be drawn to cut a given straight line in more than one point. The assemblage of points lying on two straight lines is of the *second* order, for not more than two of the points will lie on any arbitrary straight line. A circle is also a locus of the second order for the same reason.

On the other hand it is easy to see that every assemblage of points of the first order must lie on a straight line.

6. Curves, regarded as envelopes, are classified according to the number of their tangents which pass through an arbitrary point. The greatest number of straight lines which can be drawn from an arbitrary point to touch a given curve is called the *class* of the curve. Thus a point is an envelope of the *first* class, because only one straight line can be drawn from any arbitrary point so as to pass through it. A circle is a curve of the *second* class, for two tangents at most can be drawn from a point to touch a given circle.

On the other hand, an assemblage of straight lines of the first class must pass through the same point; but an assemblage of straight lines of the second class do not necessarily envelope a circle.

The Principle of Duality.

7. Geometrical propositions are of two kinds,—either they refer to the relative positions of certain points or lines connected with a figure, or they involve more or less directly the idea of measurement. In the former case they are called *descriptive*, in the latter *metrical* propositions. The propositions contained in the first six books of Euclid are mostly metrical; in fact, there is not one that can be said to be purely descriptive.

There is a remarkable analogy between descriptive propositions concerning figures regarded as assemblages of points and those concerning corresponding figures regarded as assemblages of straight lines. Any two figures, in which the points of one correspond to the lines of the other, are said to be *reciprocal* figures. It will be found that when a proposition has been proved for any figure, a corresponding proposition for the reciprocal figure may be enunciated by merely interchanging the terms 'point' and 'line'; 'locus' and 'envelope'; 'point of intersection of two lines' and 'line of connection of two points'; &c. Such propositions are said to be *reciprocal* or *dual*; and the truth of the reciprocal proposition may be inferred from what is called the *principle of duality.*

The principle of duality plays an important part in geometrical investigations. It is obvious from general reasoning, but in the present treatise we shall prove independently reciprocal propositions as they occur, and shall reserve for a later chapter a formal proof of the truth of the principle.

The Principle of Continuity.

8. The principle of continuity, which is the vital principle of modern geometry, was first enunciated by Kepler, and afterwards extended by Boscovich; but it was not till after the publication of Poncelet's "Traité des Propriétés Projectives" in 1822 that it was universally accepted.

This principle asserts that if from the nature of a particular problem we should expect a certain number of solutions, and if in any particular case we find this number of solutions, then there will be the same number of solutions in all cases, although some of the solutions may be imaginary. For instance, a straight line can be drawn to cut a circle in two points; hence, we state that every straight line will cut a circle in two points, although these may be imaginary, or may coincide. Similarly, we state that two tangents can be drawn from any point to a circle, but they may be imaginary or coincident.

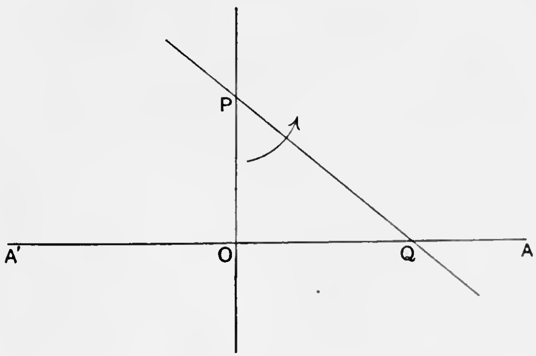
In fact, the principle of continuity asserts that theorems concerning real points or lines may be extended to imaginary points or lines.

We do not propose to discuss the truth of this principle in the present treatise. We merely call attention to it, trusting that the reader will notice that certain propositions, which will be proved, might be inferred from earlier propositions by the application of the principle.

It is important however to observe that the change from a real to an imaginary state can only take place when some element of a figure passes through either a zero-value, or an infinite value. For instance, if a pair of points become imaginary, they must first coincide; that is, the distance between them must assume a zero-value. Imagine a straight line drawn through a fixed point to cut a given circle in two real points, and let the line turn about the fixed point:—as the line turns, the two points in which it cuts the circle gradually approach nearer and nearer, until the line touches the circle, when the points coincide, and afterwards become imaginary.

Points at infinity.

9. Let AOA' be an indefinite straight line, and let a straight line be drawn through a fixed point P cutting the given line AA' in the point Q. If now we suppose the line PQ to revolve continuously about the point P, the point Q will assume every position of the assemblage of points on the line AA'. Let O be



the position of the point Q when the line PQ is perpendicular to the line AA', and let us suppose that PQ revolves in the direction indicated by the arrow-head in the figure. Then we see that the distance OQ increases from the value zero, and becomes indefinitely great as the angle OPQ becomes nearly a right angle. When the angle OPQ is a right angle, PQ assumes a position parallel to OA, and as the line PQ continues to revolve about P the point Q appears at the opposite extremity of the line $A'A$. We say then that when the line PQ is parallel to OA, the point Q may be considered as situated on the line OA at an infinite distance from the point O, and may be considered as situated on either side of O. That is to say, on the hypothesis that the line PQ always cuts the line OA in one real point, the line OA must be considered as having one point situated at infinity, that is at an infinite distance from every finite point on the line.

It follows also that any system of parallel lines, in the same plane, must be considered as intersecting in a common point at infinity. And conversely every system of straight lines drawn through a point at infinity is a system of parallel straight lines.

10. Since every straight line has one point situated at infinity, it follows that all the points at infinity in a given plane constitute an assemblage of points of the first order. Hence, all the points at infinity in a given plane satisfy the condition of lying on a straight line. This straight line is called the *line at infinity* in the plane.

CHAPTER II.

MEASUREMENT OF GEOMETRICAL MAGNITUDES.

Use of the signs + and − in Geometry.

11. IN plane geometry, metrical propositions are concerned with the magnitudes of lengths, angles, and areas. Each of these, as we shall see, is capable of being measured in two opposite directions. Consequently it is convenient to use the algebraic signs + and − to distinguish between the directions in which such magnitudes as have to be compared are measured. It is usual to consider magnitudes measured in some specified direction as positive, and those measured in the opposite direction as negative; but it is seldom necessary to specify the positive direction, since it is always possible to use such a notation for any kind of magnitude as shall indicate the direction in which it is measured.

Measurement of lengths.

12. If A and B be two points on a straight line, the length of the segment AB may be measured either in the direction from A towards B, or in the opposite direction from B towards A.

When the segment is measured from A towards B its length is represented by AB, and when it is measured from B towards A its length is represented by BA.

Consequently, the two expressions AB and BA represent the same magnitude measured in opposite directions. Therefore we have $BA = - AB$, that is $AB + BA = 0$.

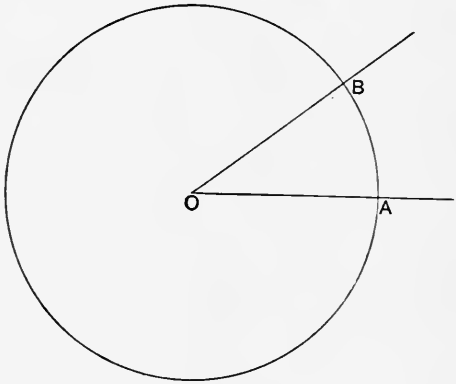
13. The length of the perpendicular drawn from a point A to a straight line x, is represented by Ax when it is measured from the point A towards the line, and by xA when it is measured from the line x towards the point A.

Consequently, the two expressions Ax, Bx will have the same sign when the points A and B are on the same side of the line, and different signs when the points are on opposite sides of the line.

14. Segments measured on the same or parallel lines may evidently be compared in respect of both direction and magnitude, but it must be noticed that segments of lines which are not parallel can only be compared in respect of magnitude.

Measurement of angles.

15. Let AOB be any angle, and let a circle, whose radius is equal to the unit of length, be described with centre O to cut OA, OB in the points A and B. Then the angle AOB is measured by



the length of the arc AB. But the length of this arc may be measured either from A towards B, or from B towards A. Consequently, an angle may be considered as capable of measurement in either of two opposite directions.

When the arc is measured from A towards B, the magnitude of the angle is represented by AOB; and when the arc is measured from B towards A, the magnitude of the angle is represented by BOA.

Thus the expressions AOB, BOA represent the same magnitude measured in opposite directions, and therefore have different signs. Therefore
$$AOB + BOA = 0.$$

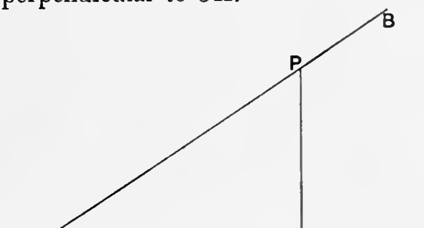
16. Angles having different vertices may be compared in respect of sign as well as magnitude. For if through any point O' we draw $O'A'$ parallel to and in the same direction as OA, and $O'B'$ in the same direction as OB; the angles AOB, $A'O'B'$ are evidently equal, and have the same sign.

17. Straight lines are often represented by single letters; and the expressions ab, ba are sometimes used to represent the angles between the lines a and b. But since two straight lines form two angles of different magnitudes at their point of intersection, this notation is objectionable. If however we have a series of lines meeting at a point O, and if we represent the lines OA, OB by the letters a, b, the use of the expression ab as meaning the same thing as the expression AOB is free from ambiguity. In this case we shall evidently have $ab = -ba$.

The Trigonometrical Ratios of an angle.

18. In propositions concerning angles it is very often convenient to use the names which designate in trigonometry certain ratios, called the trigonometrical ratios of an angle.

Let AOB be any angle, let any point P be taken in OB and let PM be drawn perpendicular to OA.

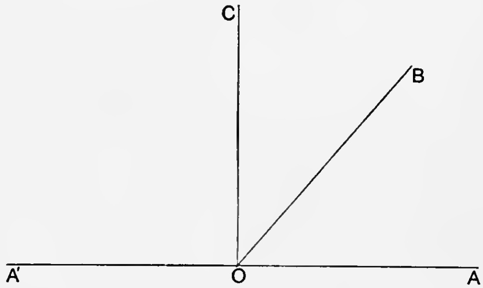


The ratio of $MP : OP$ is called the *sine* of the angle AOB;
the ratio of $OM : OP$ is called the *cosine* of the angle AOB;
the ratio of $MP : OM$ is called the *tangent* of the angle AOB;
the ratio of $OM : MP$ is called the *cotangent* of the angle AOB;
the ratio of $OP : OM$ is called the *secant* of the angle AOB;
the ratio of $OP : MP$ is called the *cosecant* of the angle AOB.

These six ratios are called the *trigonometrical ratios* of the angle AOB.

Let us now consider the line OA to be fixed, and let OB revolve round the point O. For different positions of OB the length OP is taken to be of invariable sign, but the lengths OM and MP will vary in magnitude as well as in sign. Since OB may be drawn so as to make the angle AOB equal to any given angle, the trigonometrical ratios of angles are easily compared in respect of magnitude and sign.

19. The following useful theorems are easily proved, and may be found in any treatise on Trigonometry.



Let AOB be any angle, and let AO be produced to A'; then

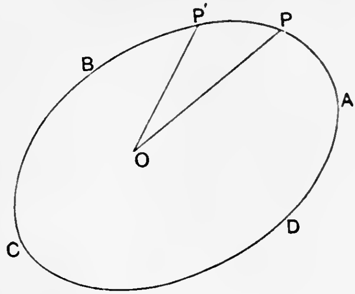
$$\begin{aligned}
\sin AOB &= -\sin BOA = \sin BOA' = -\sin A'OB;\\
\cos AOB &= \cos BOA = -\cos BOA' = -\cos A'OB;\\
\tan AOB &= -\tan BOA = -\tan BOA' = \tan A'OB;\\
\cot AOB &= -\cot BOA = -\cot BOA' = \cot A'OB;\\
\sec AOB &= \sec BOA = -\sec BOA' = -\sec A'OB;\\
\operatorname{cosec} AOB &= -\operatorname{cosec} BOA = \operatorname{cosec} BOA' = -\operatorname{cosec} A'OB.
\end{aligned}$$

Also if OC be drawn perpendicular to OA,

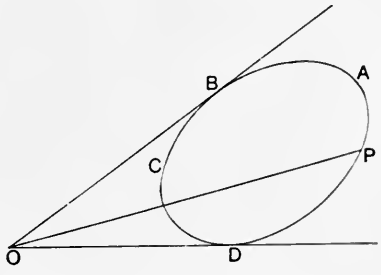
$$\begin{aligned}
\sin AOB &= \cos COB = \cos BOC;\\
\cos AOB &= -\sin COB = \sin BOC;\\
\tan AOB &= -\cot COB = \cot BOC;\\
\cot AOB &= -\tan COB = \tan BOC;\\
\sec AOB &= -\operatorname{cosec} COB = \operatorname{cosec} BOC;\\
\operatorname{cosec} AOB &= \sec COB = \sec BOC.
\end{aligned}$$

The measurement of areas.

20. Let $ABCD$ be any contour, and let O be a point within it. Let P be any point on the contour, and let P be supposed to move round the contour in the direction $ABCD$. The area enclosed by the contour is said to be traced out by the radius OP. For if we take consecutive radii such as OP, OP' the magnitude of the area $ABCD$ is evidently the sum of the elementary areas OPP'.



Now suppose the point O to lie without the contour $ABCD$; and let OB, OD be the extreme positions of the revolving line OP. The area enclosed by the contour is now evidently the difference of the area $ODAB$ traced out by OP as it revolves in one direction from the position OD to the position OB, and the area $OBCD$ traced out by OP as it revolves in the opposite direction from the position OB to the position OD.



We may thus regard the magnitude of the area enclosed by any contour such as $ABCD$, as capable of measurement in either of two opposite directions. And if we represent the magnitude of the area by the expression $(ABCD)$ when the point P is supposed to move round the contour in the direction $ABCD$, and by the expression $(ADCB)$ when the point P is supposed to move in the direction $ADCB$; we shall have

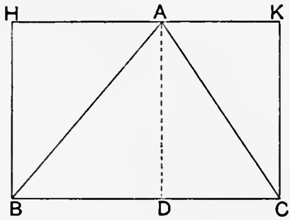
$$(ABCD) + (ADCB) = 0.$$

Areas may evidently be compared in respect to sign as well as magnitude wherever they may be situated in the same plane.

It should be noticed that the expression for the magnitude of an area will have the same meaning if the letters be interchanged in cyclical order.

21. If a closed contour be formed by a series of straight lines a, b, c, d, ..., the magnitude of the area enclosed by them may be represented by the expression $(abcd...)$, without giving rise to ambiguity, provided that $(abcd)$ be understood to mean the area traced out by a point which starting from the point da moves along the line a towards the point ab, and then along the line b towards the point bc, and so on.

22. Let ABC be any triangle, and let AD be drawn perpendicular to BC. Through A let HAK be drawn parallel to BC, and let the rectangle $BHKC$ be completed.



It is proved in Euclid (Book I., prop. 41) that the area of the triangle ABC is half the area of the rectangle $HBCK$.

That is $$(ABC) = \tfrac{1}{2}(HBCK).$$

Therefore the area (ABC) is equal in magnitude to

$$\tfrac{1}{2} HB \,.\, BC, \text{ i.e., } \tfrac{1}{2} AD \,.\, BC.$$

And since $$DA = BA \sin CAB,$$

the area (ABC) is equal in magnitude to

$$\tfrac{1}{2} BA \,.\, BC \,.\, \sin ABC;$$

or by symmetry to $$\tfrac{1}{2} AB \,.\, AC \,.\, \sin BAC.$$

It is often necessary to use these expressions for the area of a triangle, but when the areas of several triangles have to be compared it is generally necessary to be careful that the signs of the areas are preserved. Two cases occur frequently:

(i) When several triangles are described on the same straight line, we shall have $$(ABC) = \tfrac{1}{2} AD \,.\, BC,$$

where each of the lengths AD, BC is to be considered as affected by sign.

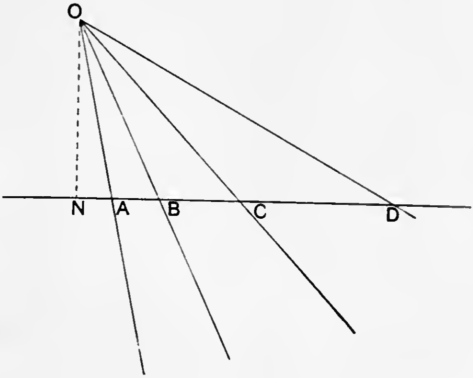
(ii) When several triangles have a common vertex A, we shall have

$$(ABC) = \tfrac{1}{2} AB \,.\, AC \,.\, \sin BAC,$$

where the lengths AB, AC are to be considered as of invariable sign, but the angle BAC as affected by sign.

From these two values for the area of the triangle ABC, we have the theorem $$AD \,.\, BC = AB \,.\, AC \,.\, \sin BAC,$$

which is very useful for deriving theorems concerning the angles formed by several lines meeting in a point, from theorems concerning the segments of a line.



Thus let any line cut the lines OA, OB, OC,..., in the points A, B, C,..., and let ON be drawn perpendicular to the line AB. Then we have,

$$AB \, . \, ON = OA \, . \, OB \, . \sin AOB,$$

$$AC \, . \, ON = OA \, . \, OC \, . \sin AOC,$$

where the segments AB, AC, &c...., of the line AB, and the angles AOB, AOC,..., are affected by sign, but the lengths ON, OA, OB,... are of invariable sign.

Ex. 1. If A, B, C, D be any four points in a plane, and if AM, BN be drawn parallel to any given straight line meeting CM, DN drawn perpendicular to the given straight line, in M and N, show that

$$(ABCD) = \tfrac{1}{2}(AM \, . \, ND + NB \, . \, MC).$$

Ex. 2. On the sides AB, AC of the triangle ABC are described any parallelograms $AFMB$, $AENC$. If MF, NE meet in H, and if BD, CK be drawn parallel and equal to HA, show that the sum of the areas $(AFMB)$, $(ACNE)$ will be equal to the area $(BDKC)$.

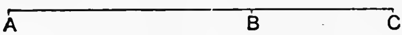
CHAPTER III.

FUNDAMENTAL METRICAL PROPOSITIONS.

Relations between the segments of a line.

23. *If A, B, C be any three points on the same straight line, the lengths of the segments BC, CA, AB are connected by the relation*

$$BC + CA + AB = 0.$$



Let the point B lie between the points A and C. Then AB, BC, AC represent lengths measured in the same direction, and

$$AC = AB + BC.$$

But $$AC + CA = 0,$$

therefore $$BC + CA + AB = 0 \qquad \text{.....................(1).}$$

Since this is a symmetrical relation, it is obvious that it must be true when the points have any other relative positions. Therefore the relation must hold in all cases.

This relation may also be stated in the forms:

$$BC = AC - AB \qquad \text{.......................(2).}$$

$$BC = BA + AC \qquad \text{.......................(3).}$$

24. Ex. 1. If A, B, C, ... H, K be any number of points on the same straight line, show that

$$AB + BC + \ldots + HK + KA = 0.$$

Ex. 2. If A, B, C be any three points on the same straight line, and if O be the middle point of BC, show that

$$AB + AC = 2AO.$$

Ex. 3. If A, B, C, D be points on the same line, and if X, Y be the middle points of AB, CD respectively, show that

$$2XY = AC + BD = AD + BC.$$

Ex. 4. If A, B, C be points on the same line, and if A', B', C' be respectively the middle points of the segments BC, CA, AB, show that

$$BC' = C'A = A'B'.$$

Show also that the middle point of $A'B'$ coincides with the middle point of CC'.

25. *If A, B, C, D be any four points on the same straight line, the lengths of the six segments of the line are connected by the relation* $\quad BC \,.\, AD + CA \,.\, BD + AB \,.\, CD = 0.$

By the formulae (2) and (3) of § 23, we have

$$BD = AD - AB,$$

$$CD = CA + AD.$$

Hence

$$CA \,.\, BD + AB \,.\, CD = CA \,.\, AD + AB \,.\, AD = AD \,.\, (CA + AB).$$

Therefore $\quad CA \,.\, BD + AB \,.\, CD = AD \,.\, CB;$

that is $\quad BC \,.\, AD + CA \,.\, BD + AB \,.\, CD = 0.$

This result may also be very easily proved by means of Euclid, Book II., prop. 1.

26. A number of points on the same straight line are said to form a *range*. Instead of saying that the points A, B, C ... are on the same straight line, it is usual to speak of the range $\{ABC \ldots\}$. Thus the proposition in the last article is usually stated:

The lengths of the six segments of any range $\{ABCD\}$ are connected by the relation

$$BC \,.\, AD + CA \,.\, BD + AB \,.\, CD = 0.$$

27. Ex. 1. If $\{ABCD\}$ be a range such that C is the middle point of AB, show that $\quad DA \,.\, DB = DC^2 - AC^2.$

Ex. 2. Show also that

$$DA^2 - DB^2 = 4DC \,.\, CA.$$

Ex. 3. If $\{ABCD\}$ be any range, show that

$$BC \,.\, AD^2 + CA \,.\, BD^2 + AB \,.\, CD^2 = -BC \,.\, CA \,.\, AB.$$

Ex. 4. Show that the last result is also true when D is not on the same straight line as A, B, and C.

Ex. 5. If $\{AA'BB'CC'P\}$ be any range, and if L, M, N be the middle points of the segments AA', BB', CC', show that

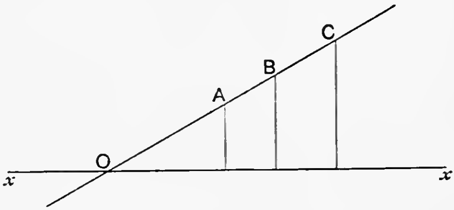
$$PA \,.\, PA' \,.\, MN + PB \,.\, PB' \,.\, NL + PC \,.\, PC' \,.\, LM$$

has the same value whatever the position of the point P on the line.

By Ex. 1, we have $PA \,.\, PA' = PL^2 - AL^2$. Hence this expression, by Ex. 3, is independent of the position of P.

28. *If $\{ABC\}$ be any range, and if x be any straight line, then*

$$Ax \,.\, BC + Bx \,.\, CA + Cx \,.\, AB = 0.$$



Let the straight line AB cut the given straight line x in the point O.

Then, by § 25, we have

$$OA \,.\, BC + OB \,.\, CA + OC \,.\, AB = 0.$$

But since Ax, Bx, Cx are parallel to each other, we have

$$Ax : Bx : Cx = OA : OB : OC.$$

Therefore $\quad Ax \,.\, BC + Bx \,.\, CA + Cx \,.\, AB = 0.$

Ex. 1. If C be the middle point of AB, show that

$$2\,Cx = Ax + Bx.$$

Ex. 2. If G be the centre of gravity of equal masses placed at the n points A, B, ... K, show that

$$n\,Gx = Ax + Bx + \ldots + Kx,$$

where x denotes any straight line.

Ex. 3. If any straight line x cut the sides of the triangle ABC in the points L, M, N, show that

$$Bx \,.\, Cx \,.\, MN + Cx \,.\, Ax \,.\, NL + Ax \,.\, Bx \,.\, LM = 0.$$

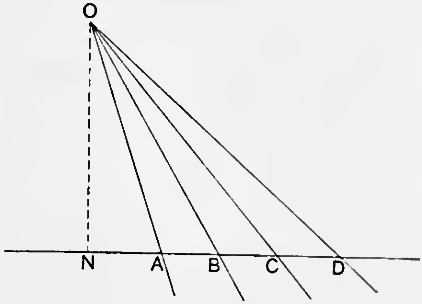
[Trin. Coll., 1892.]

Relations connecting the angles of a pencil.

29. If several straight lines be drawn in the same plane through a point O, they are said to form a *pencil.* The point O is called the *vertex* of the pencil, and the straight lines are called the *rays* of the pencil. The pencil formed by the rays OA, OB, OC,... is usually spoken of as the pencil $O\{ABC\ldots\}$.

30. *The six angles of any pencil of four rays $O\{ABCD\}$ are connected by the relation*

$$\sin BOC \cdot \sin AOD + \sin COA \cdot \sin BOD + \sin AOB \cdot \sin COD = 0$$



Let any straight line be drawn cutting the rays of the pencil in the points A, B, C, D. Then, by § 25, we have

$$BC \cdot AD + CA \cdot BD + AB \cdot CD = 0.$$

But if ON be the perpendicular from the vertex of the pencil on the line AB, we have, from § 22,

$$NO \cdot AB = OA \cdot OB \cdot \sin AOB,$$

and similar values for $NO \cdot AD$, $NO \cdot CD$, &c.

Substituting these expressions for the segments AB, AC, &c., in the above relation, we obtain the relation

$$\sin BOC \cdot \sin AOD + \sin COA \cdot \sin BOD + \sin AOB \cdot \sin COD = 0.$$

This relation is of great use. It includes moreover as particular cases several important trigonometrical formulae.

31. Ex. 1. If $O\{ABC\}$ be any pencil, prove that

$$\sin AOC = \sin AOB \cdot \cos BOC + \sin BOC \cdot \cos AOB.$$

Let OD be drawn at right angles to OB. Then we have

$$\sin AOD = \sin\left(\frac{\pi}{2} + AOB\right) = \cos AOB, \ \sin BOD = 1,$$

and $$\sin COD = \sin\left(\frac{\pi}{2} - BOC\right) = \cos BOC.$$

Making these substitutions in the general formula for the pencil $O\{ABCD\}$ the required result is obtained.

Ex. 2. In the same way deduce that

$$\cos AOC = \cos AOB \cdot \cos BOC - \sin AOB \cdot \sin BOC.$$

Ex. 3. If in the pencil $O\{ABCD\}$ the ray OC bisect the angle AOB, prove that

$$\sin AOD \,.\, \sin BOD = \sin^2 COD - \sin^2 AOC.$$

Ex. 4. If $O\{ABCD\}$ be any pencil, prove that

$$\sin BOC \,.\, \cos AOD + \sin COA \,.\, \cos BOD + \sin AOB \,.\, \cos COD = 0,$$

and $$\cos BOC \,.\, \cos AOD - \cos COA \,.\, \cos BOD + \sin AOB \,.\, \sin COD = 0.$$

Ex. 5. If a, b, c denote any three rays of a pencil, and if P be any point, show that

$$Pa \,.\, \sin (bc) + Pb \,.\, \sin (ca) + Pc \,.\, \sin (ab) = 0.$$

Elementary theorems concerning areas.

32. *If ABC be any triangle, and if O be any point in the plane, the area (ABC) is equal to the sum of the areas (OBC), (OCA), (OAB).*

That is $$(ABC) = (OBC) + (OCA) + (OAB) \quad \text{...........(1).}$$

This result evidently follows at once from the definition of an area considered as a magnitude which may be measured in a specified direction.

If A, B, C, D be any four points in the same plane, then

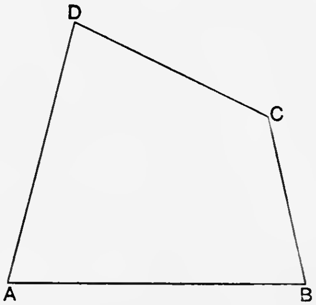
$$(ABC) - (BCD) + (CDA) - (DAB) = 0 \quad \text{.........(2).}$$

This result is merely another form of the previous result, since

$$(CDA) = -(CAD) = -(DCA).$$

33. The second relation given in the last article may be obtained otherwise.

(i) Let us suppose that the points C and D lie on the same side of the line AB. Then the expression $(ABCD)$ clearly represents the area of the quadrilateral $ABCD$.

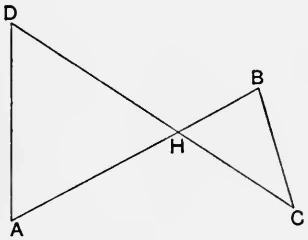


But the quadrilateral $ABCD$, may be regarded either as made up of the two triangles ABC, CDA; or as made up of the two triangles BCD, DAB.

Hence, we have
$$(ABCD) = (ABC) + (CDA)$$
$$= (BCD) + (DAB).$$

Therefore
$$(ABC) - (BCD) + (CDA) - (DAB) = 0.$$

(ii) If the points C and D lie on opposite sides of the line AB, let AB cut CD in the point H. Then the expression $(ABCD)$ is clearly equal to the difference of the areas of the triangles (AHD) and (HCB).



That is
$$(ABCD) = (AHD) - (HCB)$$
$$= (ABD) - (DCB)$$
$$= (ABD) + (DBC).$$

Similarly we may show that
$$(ABCD) = (ABC) + (CDA).$$

Hence, as before,
$$(ABC) + (CDA) = (ABD) + (DBC);$$
that is,
$$(ABC) - (BCD) + (CDA) - (DAB) = 0.$$

34. Ex. 1. If a, b, c, d be any four straight lines in the same plane, show that
$$(abcd) = (abc) + (cda).$$

Ex. 2. Show also that
$$(abc) = (dbc) + (dca) + (dab).$$

35. *If A, B, C be any three points on a straight line, and P, Q any other points in the same plane with them,*
$$(APQ) \,.\, BC + (BPQ) \,.\, CA + (CPQ) \,.\, AB = 0.$$

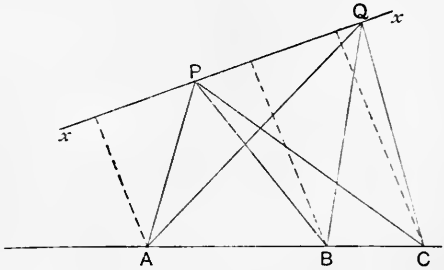
Let x denote the straight line PQ. Then, by § 28, we have
$$Ax \,.\, BC + Bx \,.\, CA + Cx \,.\, AB = 0.$$

But, by § 21,
$$(APQ) = \tfrac{1}{2} Ax \,.\, PQ.$$

Therefore

$$Ax : Bx : Cx = (APQ) : (BPQ) : (CPQ).$$

Hence $\quad (APQ) . BC + (BPQ) . CA + (CPQ) . AB = 0$ ......(1).



This relation may also be written in the forms:

$$(APQ) . BC = (BPQ) . AC + (CPQ) . BA \quad ....(2),$$

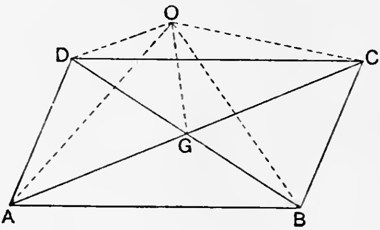
$$(APQ) . BC = (BPQ) . AC - (CPQ) . AB \quad ....(3).$$

36. Ex. 1. If A, B, C be any three points on a straight line, x any other straight line, and O any given point; show that

$$(OBC) . Ax + (OCA) . Bx + (OAB) . Cx = 0.$$

Ex. 2. If $ABCD$ be a parallelogram, and if O be any point in the same plane, show that

$$(OAC) = (OAB) + (OAD).$$



Let the diagonals meet in G. Then G is the middle point of BD. Hence, by § 35 (2), we have

$$2 (OAG) = (OAB) + (OAD).$$

But since $\quad AC = 2AG, \ (OAC) = 2 (OAG).$

Ex. 3. Prove the following construction for finding the sum of any number of triangular areas (POA), (POB), (POC), &c. From A draw AB' equal and parallel to OB, from B' draw $B'C'$ equal and parallel to OC, and so on. Then (POB') is equal to $(POA) + (POB)$; (POC') is equal to $(POA) + (POB) + (POC)$; and so on.

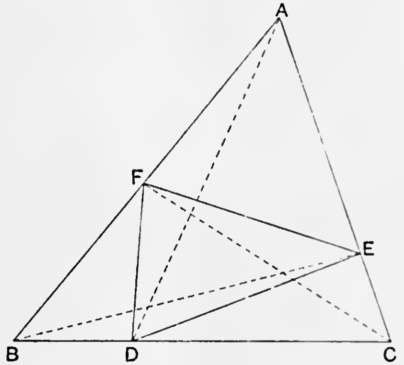
Ex. 4. If $A, B, C, \ldots K$ be n points in a plane, and if G be the centroid of equal masses placed at them, show that

$$\Sigma (POA) = n (POG).$$

Ex. 5. If A, B, C, D be any four points in a plane, find a point P on the line CD such that the area (PAB) shall be equal to the sum of the areas (CAB), (DAB).

Ex. 6. If three points D, E, F be taken on the sides BC, CA, AB of a triangle, prove that the ratio of the areas (DEF), (ABC) is equal to

$$\frac{BD \cdot CE \cdot AF - CD \cdot AE \cdot BF}{BC \cdot CA \cdot AB}.$$



By § 35 (3) we have

$$(DEF) \cdot BC = (CEF) \cdot BD - (BEF) \cdot CD.$$

But $\quad (CEF) : (CAF) = CE : CA,$

and $\quad (CAF) : (ABC) = AF : AB.$

Therefore $\quad (CEF) : (ABC) = CE \cdot AF : CA \cdot AB.$

Similarly $\quad (BEF) : (ABC) = BF \cdot AE : BA \cdot AC.$

Hence $\quad \dfrac{(DEF)}{(ABC)} = \dfrac{BD \cdot CE \cdot AF - CD \cdot BF \cdot AE}{BC \cdot CA \cdot AB}.$

It follows from this result, that when the points D, E, F are collinear,

$$BD \cdot CE \cdot AF = CD \cdot BF \cdot AE;$$

and conversely, that if this relation hold, the points D, E, F must be collinear.

Ex. 7. Points P and Q are taken on two straight lines AB, CD, such that

$$AP : PB = CQ : QD.$$

Show that the sum of the areas (PCD), (QAB) is constant.

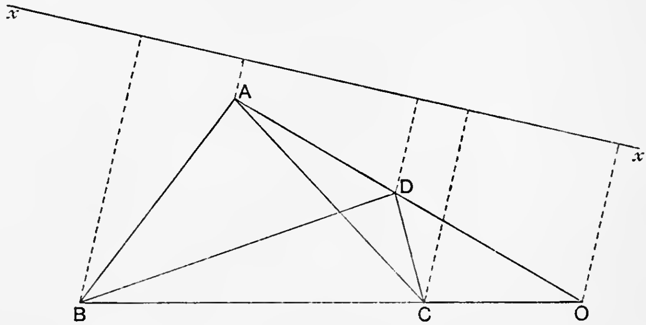
Ex. 8. The sides BC, CA, AB of a triangle meet any straight line in the points D, E, F. Show that a point P can be found in the line DEF such that the areas (PAD), (PBE), (PCF) are equal. [St John's Coll. 1889.]

Ex. 9. If A, B, C, D be any four points on a circle and P be any given point, show that

$$PA^2 \cdot (BCD) - PB^2 \cdot (CDA) + PC^2 \cdot (DAB) - PD^2 \cdot (ABC) = 0.$$

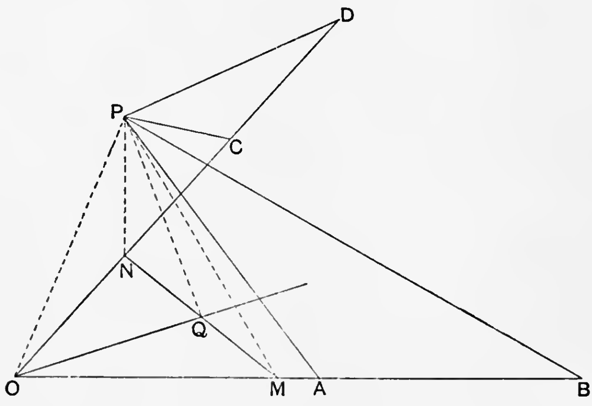
Let AC, BD meet in O, and apply the theorem given in § 27, Ex. 4, to each of the ranges $\{AOC\}$, $\{BOD\}$.

Ex. 10. If A, B, C, D be any four points, and x any straight line, prove that $$(BCD).Ax-(CDA).Bx+(DAB).Cx-(ABC).Dx=0.$$



Let AD cut BC in the point O, then, by § 28, we have
$$BC.Ox+CO.Bx+OB.Cx=0,$$
and
$$AD.Ox+DO.Ax+OA.Dx=0.$$
Hence $$DO.BC.Ax-CO.AD.Bx-OB.AD.Cx+OA.BC.Dx=0;$$
or $$DO.BC.Ax+DA.CO.Bx-BO.DA.Cx-BC.AO.Dx=0.$$
But
$$(BCD)=\tfrac{1}{2}DO.BC.\sin BOD,$$
$$(CDA)=\tfrac{1}{2}DA.CO.\sin AOC,$$
$$(DAB)=\tfrac{1}{2}DA.BO.\sin AOC,$$
$$(ABC)=\tfrac{1}{2}BC.AO.\sin BOD.$$
Also
$$\sin BOD=-\sin AOC.$$
Hence we have the required result.

37. *If A, B, C, D be any four points in a plane, the locus of a point P, which moves so that the sum of the areas (PAB), (PCD) is constant, is a straight line.*



Let the straight lines AB, CD meet in the point O, and let M and N be two points on these lines respectively, such that $OM = AB$, and $ON = CD$.

Then we have

$$(PAB) = (POM), \text{ and } (PCD) = (PON).$$

Let Q be the middle point of MN. Then, by § 35 (2), we have

$$2(POQ) = (POM) + (PON).$$

Therefore $\qquad 2(POQ) = (PAB) + (PCD);$

that is, the area represented by (POQ) is constant.

Hence the locus of P is a straight line parallel to OQ.

38. Ex. 1. Let A, B, C, D be any four points, and let AB, CD meet in E, and AC, BD in F. Then if P be the middle point of EF, show that

$$(PAB) - (PCD) = \tfrac{1}{2}(ABDC).$$

Ex. 2. Show that the line joining the middle points of AD and BC passes through P, the middle point of EF.

Ex. 3. If A, B, C, D be any four points, show that the locus of a point P, which moves so that the ratio of the areas (PAB), (PCD) is constant, is a straight line passing through the point of intersection of AB and CD.

Ex. 4. If $ABCD$ be a quadrilateral circumscribing a circle, show that the line joining the middle points of the diagonals AC, BD passes through the centre of the circle.

CHAPTER IV.

HARMONIC RANGES AND PENCILS.

Harmonic Section of a line.

39. WHEN the straight line joining the points A, B is divided internally in the point P, and externally in the point Q, in the same ratio, the segment AB is said to be divided *harmonically* in the points P and Q.

Thus, the segment AB is divided harmonically in the points P and Q, when $$AP : PB = AQ : BQ.$$

The points P and Q are said to be *harmonic conjugate* points with respect to the points A and B; or, the points A and B are said to be *harmonically separated* by P and Q.

40. *If the segment AB is divided harmonically in the points P and Q, the segment PQ is divided harmonically in the points A and B.*

For by definition, we have

$$AP : PB = AQ : BQ;$$

and therefore $$PB : BQ = AP : AQ.$$

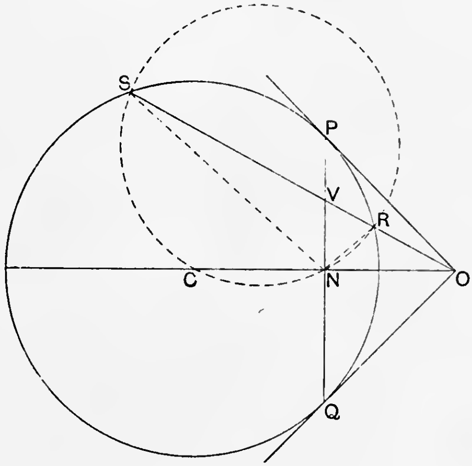
Thus, A and B are harmonic conjugate points with respect to P and Q.

41. When the segment AB is divided harmonically in the points P and Q, the range $\{AB, PQ\}$ is called a *harmonic* range; and the pairs of points A, B; P, Q; are called *conjugate* points of the range.

It will be found convenient to use the notation $\{AB, PQ\}$ for a harmonic range, the comma being inserted to distinguish the pairs of conjugate points.

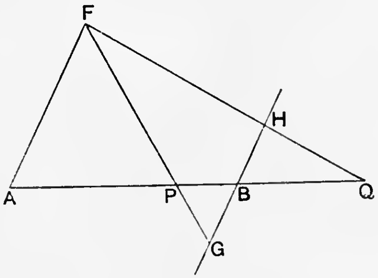
42. Ex. 1. If ABC be any triangle, show that the bisectors of the angle BAC divide the base BC harmonically.

Ex. 2. If tangents OP, OQ be drawn to a circle from any point O, and if any straight line drawn through the point O cut the circle in the points R and S and the chord PQ in the point V, show that $\{OV, RS\}$ is a harmonic range.



Let C be the centre of the circle, and let OC cut PQ in the point N. Then we have $OR \,.\, OS = OP^2 = ON \,.\, OC$. Therefore the points S, R, N, C are concyclic. But C is evidently the middle point of the arc SNR; therefore NC, NP bisect the angle SNR. Hence $\{OV, RS\}$ is a harmonic range.

43. *If $\{ABP\}$ be any range, to find the harmonic conjugate of P with respect to the points A, B.*



Through A and B draw a pair of parallel lines AF, BH. And through P draw a straight line FPG in any direction meeting AF in F and BH in G. In BH take the point H, so that B is the middle point of GH, and join FH.

Then FH will meet AB in the point Q, which will be the point required.

For
$$AQ : BQ = AF : BH$$
$$= AF : GB$$
$$= AP : PB.$$

That is $\{AB, PQ\}$ is a harmonic range.

It should be noticed that the solution is unique, that is, there is only one point Q which corresponds to a given point P.

44. Ex. 1. If P be the middle point of AB, show that the conjugate point Q is at infinity.

In this case it is easy to see that FH is parallel to AB.

Ex. 2. If $\{ABC\}$ be any range, and if P be the harmonic conjugate of A with respect to B and C, Q the harmonic conjugate of B with respect to C and A, and R the harmonic conjugate of C with respect to A and B; show that A will be the harmonic conjugate of P with respect to Q and R.

Harmonic Section of an angle.

45. When the angle AOB is divided by the rays OP, OQ so that
$$\sin AOP : \sin POB = \sin AOQ : \sin BOQ,$$
the angle AOB is said to be divided *harmonically* by the rays OP, OQ.

The rays OP, OQ are said to be *harmonic conjugate* rays with respect to the rays OA, OB.

46. *If the angle AOB be divided harmonically by the rays OP, OQ, the angle POQ is divided harmonically by the rays OA, OB.*

For since OP, OQ divide the angle AOB harmonically,
$$\sin AOP : \sin POB = \sin AOQ : \sin BOQ.$$
Therefore
$$\sin POB : \sin BOQ = \sin AOP : \sin AOQ.$$
Thus, the rays OA, OB are harmonic conjugate rays with respect to OP and OQ.

47. When the rays OP, OQ of the pencil $O\{ABPQ\}$ are harmonic conjugates with respect to the rays OA, OB, the pencil is called a *harmonic pencil;* and each pair of rays, namely OA, OB; and OP, OQ; are called *conjugate* rays of the pencil.

It will be found convenient to use the notation $O\{AB, PQ\}$ for a harmonic pencil, the comma being inserted to distinguish the pairs of conjugate rays.

48. Ex. 1. If the rays OP, OQ bisect the angle AOB, show that the pencil $O\{AB, PQ\}$ is harmonic.

Ex. 2. If the pencil $O\{AB, PQ\}$ be harmonic, and if the angle AOB be a right angle, show that OA, OB are the bisectors of the angle POQ.

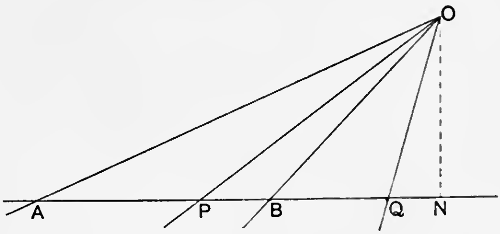
Ex. 3. If the pencil $O\{AB, PQ\}$ be harmonic, and the angle AOB a right angle, and if a line be drawn perpendicular to OP meeting OA, OB in A' and B', show that the line drawn through O perpendicular to OQ will bisect $A'B'$.

Ex. 4. If A, B, C, D, O be five points on a circle, such that the pencil $O\{AB, CD\}$ is harmonic, and if P be any point on the same circle, show that the pencil $P\{AB, CD\}$ will be harmonic.

Ex. 5. In the same case, show that, if the tangents at A, B, C and D intersect the tangent at the point P in the points A', B', C', D' respectively, the pencil $H\{A'B', C'D'\}$ will be harmonic—where H is the centre of the circle.

It is easy to show that the angles $A'HC'$, APC are equal or supplementary. Hence this theorem follows from the last.

49. *Any straight line is cut harmonically by the rays of a harmonic pencil.*



Let any straight line cut the rays of the harmonic pencil $O\{AB, PQ\}$ in the points A, B, P, Q; then the range $\{AB, PQ\}$ is harmonic.

Let ON be drawn perpendicular to the line AB, then we have

$$NO \,.\, AP = OA \,.\, OP \sin AOP,$$

$$NO \,.\, AQ = OA \,.\, OQ \sin AOQ,$$

$$NO \,.\, PB = OP \,.\, OB \sin POB,$$

$$NO \,.\, BQ = OB \,.\, OQ \sin BOQ.$$

But since $O\{AB, PQ\}$ is a harmonic pencil,

$$\sin AOP : \sin POB = \sin AOQ : \sin BOQ.$$

Therefore $$AP : PB = AQ : BQ.$$

Hence, $\{AB, PQ\}$ is a harmonic range.

Conversely, we may prove that if $\{AB, PQ\}$ be a harmonic range, and if O be any point not on the same line, then the pencil $O\{AB, PQ\}$ will be harmonic.

50. Ex. 1. If a straight line be drawn parallel to any ray of a harmonic pencil, show that the conjugate ray will bisect the segment intercepted by the other two rays.

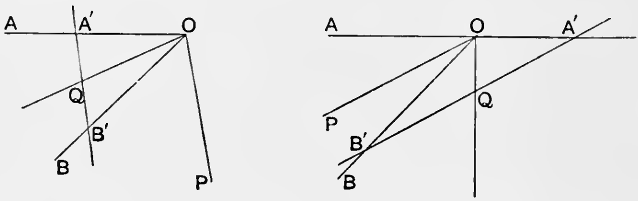
Ex. 2. Hence show that when a pair of conjugate rays of a harmonic pencil are at right angles, they bisect the angles between the other pair of conjugate rays.

Ex. 3. If P', Q' be respectively the harmonic conjugate points of P and Q with respect to A and B, show that the segments PQ, $Q'P'$ subtend equal or supplementary angles at any point on the circle described on AB as diameter.

Ex. 4. If P, Q, R, S be any four points on the line AB, and if P', Q', R', S' be their harmonic conjugates with respect to A and B; show that when the range $\{PQ, RS\}$ is harmonic, so also is the range $\{P'Q', R'S'\}$.

Take any point X on the circle described on AB as diameter. Then AX, BX are the bisectors of each of the angles PXP', QXQ', &c. Hence it is easily shown that when the pencil $X\{PQ, RS\}$ is harmonic, so also is the pencil $X\{P'Q', R'S'\}$.

51. *If $O\{ABP\}$ be any pencil, to find the ray which is conjugate to the ray OP with respect to the rays OA, OB.*



Draw any straight line parallel to the ray OP, meeting the rays OA, OB in A' and B'. Let Q be the middle point of $A'B'$, then OQ will be the ray conjugate to OP.

For $A'B'$ meets OP at infinity, and the point conjugate to the point at infinity with respect to the points $A'B'$ is the middle point of the segment $A'B'$, that is the point Q.

Therefore, by § 49, $O\{AB, PQ\}$ is a harmonic pencil.

Relations between the segments of a harmonic range.

52. If $\{AB, PQ\}$ be a harmonic range, we have by definition

$$AP : PB = AQ : BQ,$$

that is
$$AP \,.\, BQ = PB \,.\, AQ,$$

or
$$AP \,.\, BQ + AQ \,.\, BP = 0.$$

But since A, B, P, Q are four points on the same straight line, we have, by § 25,

$$AB \,.\, PQ + AP \,.\, QB + AQ \,.\, BP = 0.$$

Hence we have

$$AB \,.\, PQ = 2AP \,.\, BQ = 2AQ \,.\, PB.$$

Conversely, when segments of the range $\{ABPQ\}$ are connected by this relation, it is obvious that the range $\{AB, PQ\}$ is harmonic.

53. Again, since

$$AP \,.\, BQ + AQ \,.\, BP = 0,$$

we have
$$AP\,(AQ - AB) + AQ\,(AP - AB) = 0.$$

Therefore
$$2\,AP \,.\, AQ = AB \,.\, (AQ + AP),$$

that is
$$\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}.$$

Similarly we may obtain the relations

$$\frac{2}{BA} = \frac{1}{BP} + \frac{1}{BQ},$$

$$\frac{2}{PQ} = \frac{1}{PA} + \frac{1}{PB},$$

$$\frac{2}{QP} = \frac{1}{QA} + \frac{1}{QB}.$$

Conversely, when the segments of the range $\{ABPQ\}$ are connected by any one of these four relations, it follows that the range $\{AB, PQ\}$ is harmonic.

54. Ex. 1. If $\{AB, PQ\}$ be a harmonic range, and if C be the middle point of AB, show that

$$PA \,.\, PB = PQ \,.\, PC.$$

Ex. 2. Show that $PA \,.\, PB + QA \,.\, QB = PQ^2.$

Ex. 3. Show that $CP : CQ = AP^2 : AQ^2.$

Ex. 4. If R be the middle point of PQ, show that

$$PQ^2 + AB^2 = 4CR^2.$$

Ex. 5. Show that

$$AP : AQ = CP : AC = AC : CQ.$$

Ex. 6. If $\{AB, PQ\}$ be a harmonic range, and O any point on the same straight line, show that

$$2\frac{OB}{AB} = \frac{OP}{AP} + \frac{OQ}{AQ}.$$

Ex. 7. Show also that

$$OA \,.\, BP + OB \,.\, AQ + OP \,.\, QB + OQ \,.\, PA = 0.$$

55. *If $\{AB, PQ\}$ be a harmonic range, and if C be the middle point of AB, then*

$$CP \,.\, CQ = CA^2 = CB^2.$$

For since $$AP : PB = AQ : BQ,$$

therefore $$AP + PB : AP - PB = AQ + BQ : AQ - BQ;$$

that is $$AB : AP + BP = AQ + BQ : AB.$$

But since C is the middle point of AB,

$$AP + BP = 2CP,\ AQ + BQ = 2CQ,$$

and $$AB = 2AC.$$

Therefore $$AC : CP = CQ : AC;$$

that is $$CP \,.\, CQ = AC^2.$$

Conversely, if this relation holds, it may be easily proved that the range $\{AB, PQ\}$ is harmonic.

56. Ex. 1. If $\{AB, PQ\}$ be a harmonic range, and if C and R be the middle points of AB and PQ, show that

$$CA^2 + PR^2 = CR^2.$$

Ex. 2. If O be any point on the same line as the range, show that

$$OA \,.\, OB + OP \,.\, OQ = 2OR \,.\, OC.$$

Ex. 3. If $\{AB, PQ\}$ be a harmonic range, and if E be the harmonic conjugate of any point O with respect to A, B, and T the harmonic conjugate of O with respect to P, Q; show that

$$\frac{1}{OA \,.\, OB} + \frac{1}{OP \,.\, OQ} = \frac{2}{OE \,.\, OT}.$$

Let C, R be the middle points of AB, PQ. Then, by § 54, Ex. 1, we have $OA \,.\, OB = OE \,.\, OC$, and $OP \,.\, OQ = OT \,.\, OR$, consequently this result may be deduced from that in Ex. 2.

Ex. 4. If P', Q' be the harmonic conjugates of P, Q respectively with respect to A and B, prove that

$$PQ \,.\, PQ' : P'Q \,.\, P'Q' = AP^2 : AP'^2.$$

Let C be the middle point of AB, then

$$CP \,.\, CP' = CQ \,.\, CQ' = CA^2.$$

Hence $$CP : CQ = CQ' : CP' = PQ' : QP'.$$

Also $$CP : CQ' = PQ : Q'P'.$$

Therefore

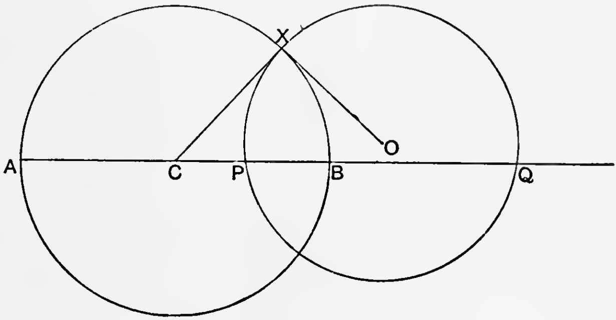
$$PQ \,.\, PQ' : QP' \,.\, Q'P' = CP^2 : CQ \,.\, CQ' = CP : CP'.$$

Whence, by § 54, Ex. 3, the result follows.

Ex. 5. Show also that

$$AP \,.\, AQ : AP' \,.\, AQ' = PQ : Q'P'.$$

Ex. 6. If $\{AB, PQ\}$ be a harmonic range, then every circle which passes through the points P, Q is cut orthogonally by the circle described on AB as diameter.



Let the circle described with centre O which passes through P and Q, cut the circle described on AB as diameter in the point X; and let C be the middle point of AB.

Then we have, by § 55,

$$CX^2 = CA^2 = CP \,.\, CQ.$$

Therefore CX touches the circle PXQ; and therefore CXO is a right angle.

Ex. 7. If two circles cut orthogonally, show that any diameter of either is divided harmonically by the other.

Relations between the angles of a harmonic pencil.

57. If $O\{AB, PQ\}$ be a harmonic pencil, we have by definition $$\sin AOP : \sin POB = \sin AOQ : \sin BOQ;$$

that is $$\sin AOP \,.\, \sin BOQ = \sin POB \,.\, \sin AOQ.$$

But, by § 30, we have

$$\sin AOB \,.\, \sin POQ + \sin AOP \,.\, \sin QOB + \sin AOQ \,.\, \sin BOP = 0.$$

Hence $$\sin AOB . \sin POQ = 2 \sin AOP . \sin BOQ$$
$$= 2 \sin AOQ . \sin POB.$$

Conversely, when this relation holds between the angles of the pencil, it follows that the pencil is harmonic.

58. *If OC bisect the angle AOB internally, then*
$$\tan COP . \tan COQ = \tan^2 COA = \tan^2 COB.$$

For by definition
$$\sin AOP : \sin POB = \sin AOQ : \sin BOQ.$$

Therefore $$\frac{\sin AOP + \sin POB}{\sin AOP - \sin POB} = \frac{\sin AOQ + \sin BOQ}{\sin AOQ - \sin BOQ},$$

that is $$\tan AOC . \cot COP = \tan COQ . \cot AOC,$$

or $$\tan COP . \tan COQ = \tan^2 AOC.$$

The same relation is true if OC bisect the angle AOB externally.

Conversely, when this relation is true, it follows that the pencil $O\{AB, PQ\}$ is harmonic.

59. Ex. 1. If $O\{AB, PQ\}$ be a harmonic pencil, prove that
$$2 \cot AOB = \cot AOP + \cot AOQ.$$

Ex. 2. If OX be any other ray, show that
$$2\frac{\sin BOX}{\sin AOB} = \frac{\sin POX}{\sin AOP} + \frac{\sin QOX}{\sin AOQ}.$$

Ex. 3. If $O\{AB, PQ\}$ be a harmonic pencil, and if OE be the conjugate ray to OX with respect to OA, OB, and OT the conjugate ray to OX with respect to OP, OQ, show that
$$\cot XOA . \cot XOB + \cot XOP . \cot XOQ = 2 \cot XOE . \cot XOT.$$

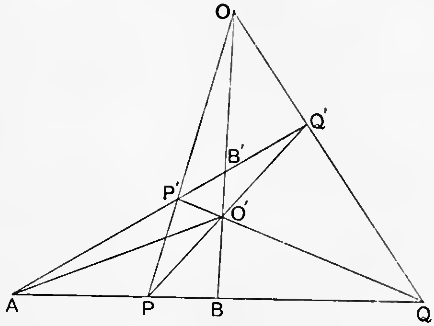
Ex. 4. If $O\{AB, PQ\}$ be a harmonic pencil, and if OC bisect the angle AOB, show that
$$\sin 2COP : \sin 2COQ = \sin^2 AOP : \sin^2 AOQ.$$

Ex. 5. If the rays OP', OQ' be the conjugate rays respectively of OP, OQ with respect to OA, OB, show that
$$\sin POQ . \sin POQ' : \sin P'OQ . \sin P'OQ' = \sin^2 AOP : \sin^2 AOP'.$$

Ex. 6. Show also that
$$\sin AOP . \sin AOQ : \sin AOP' . \sin AOQ' = \sin POQ : \sin Q'OP'.$$

Theorems relating to Harmonic Ranges and Pencils.

60. *If $\{AB, PQ\}$, $\{AB', P'Q'\}$ be two harmonic ranges on different straight lines, then the lines BB', PP', QQ' will be concurrent and the lines BB', PQ', $P'Q$ will be concurrent.*

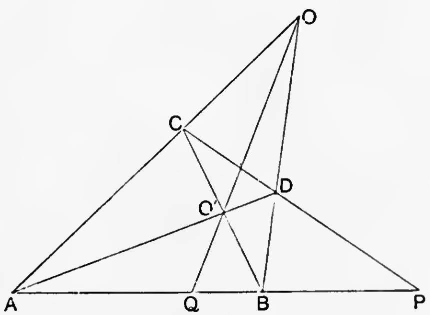


Let PP', BB' intersect in O, and join OA, OQ. Then since $O\{AB, PQ\}$ is a harmonic pencil, the line AB' will be divided harmonically by OP, OQ. But OP cuts AB' in P'. Hence OQ must cut AB' in Q' the point which is conjugate to P' with respect to A, B'.

Again, let $P'Q$ cut BB' in O', and join $O'A$, $O'P$. Then the pencil $O'\{AB, PQ\}$ is harmonic. Hence it follows as above that $O'P$ must pass through Q'.

61. This theorem furnishes an easy construction for obtaining the harmonic conjugate of a point with respect to a given pair of points.

Let A, B be any given points on a straight line, and suppose that we require the harmonic conjugate of the point P with respect to A and B.

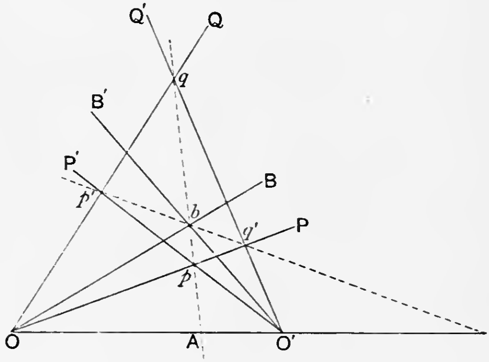


Let A and B be joined to any point O, and let a straight line be drawn through P cutting OA, OB in C and D respectively. Join AD, BC, and let them intersect in O'.

By the last article, the line joining the harmonic conjugates of P with respect to A, B; and C, D; must pass through O and O'.

Hence, if OO' meet AB in Q, Q must be the harmonic conjugate of P with respect to A and B.

62. If the pencils $O\{AB, PQ\}$, $O'\{A'B', P'Q'\}$ have one ray common, i.e. if OA and $O'A'$ are coincident, the three points in which the rays OB, OP, OQ intersect the rays $O'B'$, $O'P'$, $O'Q'$ respectively, are collinear; and likewise the three points in which the rays OB, OP, OQ intersect the rays $O'B'$, $O'Q'$, $O'P'$ respectively, are collinear.



Let OB, OP, OQ cut $O'B'$, $O'P'$, $O'Q'$ in the points b, p, q respectively; and let bp cut OO' in A. Then because the pencil $O\{AB, PQ\}$ is harmonic, OQ must cut the line Ab in the point which is the conjugate of p with respect to A, b. Similarly $O'Q'$ must cut Ab in the same point. Hence q the point of intersection of OQ, and $O'Q'$, must lie on Ab. That is, the points p, b, q are collinear.

In the same way, we can show that if OP cut $O'Q'$ in q', and if OQ cut $O'P'$ in p', then p', b, q' will be collinear.

63. Ex. 1. Show that if A, B, C, D be any four points in a plane, and if the six lines joining these points meet in the points E, F, G; then the two lines which meet in any one of these points are harmonically conjugate with the two sides of the triangle EFG which meet in the same point.

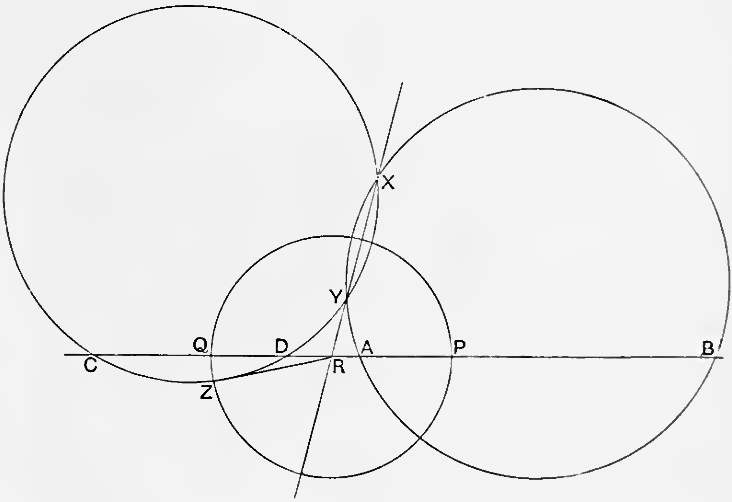
This follows from § 60.

Ex. 2. Deduce from § 62, the corresponding theorem when four straight lines are given.

Ex. 3. If through a fixed point O, two straight lines be drawn intersecting two fixed lines in the points A, B and C, D respectively, show that the locus of the point of intersection of AD and BC is a straight line.

Ex. 4. Show how to draw (with the aid of a ruler only) a straight line from a given point which shall pass through the point of intersection of two given straight lines which do not meet on the paper.

64. *Given any two pairs of points A, B and C, D on a straight line, to find a pair of points P, Q which shall be harmonically conjugate with respect to each of the given pairs of points.*



Take any point X not on the straight line, and describe the circles XAB, XCD intersecting again in the point Y.

Let the line joining X, Y, cut AB in R. Then if R does not lie within the circles, draw a tangent RZ to either, and with centre R and radius RZ describe a circle cutting AB in P and Q.

Then P and Q will divide each of the segments AB, CD harmonically.

For $\quad RP^2 = RZ^2 = RX \,.\, RY = RA \,.\, RB = RC \,.\, RD.$

The problem only admits of a real solution when R lies without each of the circles, that is when the segments AB and CD do not overlap.

65. Ex. 1. If A, B, C, D be four points taken in order on a straight line, show that the locus of a point at which the segments AB, CD subtend equal angles is a circle.

Let P, Q be harmonic conjugates with respect to A, D and B, C, then the locus is the circle described on PQ as diameter.

Ex. 2. Show that if A, B, C, D be four points taken in order on a straight line, two points can be found at each of which the segments AB, CD subtend equal angles, and the segments AD, BC supplementary angles.

Ex. 3. If the points P, Q be harmonically conjugate with respect to the points A, B, and also with respect to the points C, D; and if O, H, K be the middle points of the segments PQ, AB, CD; show that

$$XA \,.\, XB - XC \,.\, XD + 2HK \,.\, XO = 0,$$

where X is any point on the same line.

Ex. 4. Show also that if M, N be the conjugate points of O with respect to A, B and C, D respectively,

$$\frac{NP}{OH} + \frac{PM}{OK} + \frac{MN}{OP} = 0.$$

CHAPTER V.

THEORY OF INVOLUTION.

Range in Involution.

66. WHEN several pairs of points A, A'; B, B'; C, C'; &c.: lying on a straight line are such that their distances from a fixed point O are connected by the relations

$$OA \,.\, OA' = OB \,.\, OB' = OC \,.\, OC' = \&c.;$$

the points are said to form a range in *involution*.

The point O is called the *centre*, and any pair of corresponding points, such as A, A', are called *conjugate points* or *couples* of the involution.

The most convenient notation for a range in involution is

$$\{AA', BB', CC', \ldots\ldots\}.$$

67. Ex. 1. If $A', B', C', \ldots$ be respectively the harmonic conjugates of the points $A, B, C, \ldots$, with respect to the points S, S'; show that the range $\{AA', BB', CC', \ldots\}$ is in involution.

Ex. 2. If a system of circles be drawn through two fixed points A and B, show that any straight line drawn through a point O on the line AB will be cut by the circles in points which form a range in involution, the point O being the centre of the involution.

Ex. 3. If the range $\{AA', BB', CC'\}$ be in involution, and if L, M, N be the middle points of the segments AA', BB', CC'; show that

$$PA \,.\, PA' \,.\, MN + PB \,.\, PB' \,.\, NL + PC \,.\, PC' \,.\, LM = 0,$$

where P is any point on the same line.

By § 27, Ex. 5, the expression on the left-hand side must be equal to

$$OA \,.\, OA' \,.\, MN + OB \,.\, OB' \,.\, NL + OC \,.\, OC' \,.\, LM,$$

where O is the centre of the involution ; and this expression

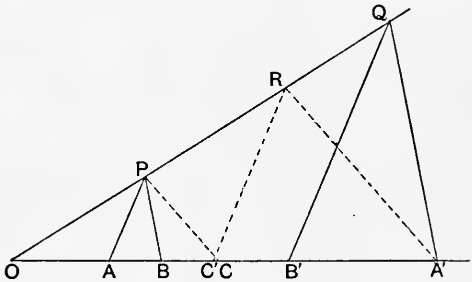
$$= OA \,.\, OA' \,.\, \{MN + NL + LM\} = 0.$$

Ex. 4. Show also that

$$LA^2 \,.\, MN + MB^2 \,.\, NL + NC^2 \,.\, LM = -MN \,.\, NL \,.\, LM.$$

This result follows from the previous result, by applying the theorem of § 27, Ex. 3.

68. *Any two pairs of points on a straight line determine a range in involution.*



Let A, A'; B, B'; be two pairs of points on a straight line. Through A and B draw any two lines AP, BP intersecting in P; and through A', B' draw $A'Q, B'Q$ parallel to BP, AP respectively, meeting in Q. Let PQ meet AB in O.

Then since AP is parallel to $B'Q$, and BP parallel to $A'Q$, we have

$$OA : OB' = OP : OQ = OB : OA';$$

and therefore

$$OA \,.\, OA' = OB \,.\, OB'.$$

Hence, O is the centre of a range in involution of which A, A' and B, B' are conjugate couples.

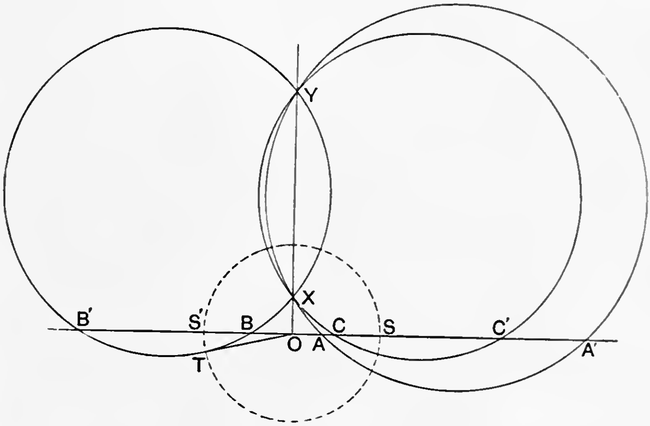
When the centre O has been found, we can find a point C' corresponding to any given point C on the line by a similar construction.

Thus, join CP, and draw $A'R$ parallel to CP meeting OP in R, and RC' parallel to PA meeting AB in C'.

Then we shall have

$$OC \,.\, OC' = OA \,.\, OA'.$$

69. We may also proceed otherwise. Let any two circles be drawn passing through the points A, A', and the points B, B', respectively; and let these circles intersect in the points X and Y. Then if the line XY meet the given straight line in the point O, this point will be the centre of the range.



For evidently

$$OA \,.\, OA' = OX \,.\, OY = OB \,.\, OB'.$$

To obtain the conjugate point to any point C', we have merely to draw the circle passing through the points X, Y, C. This circle will cut AB in C', the required point.

For $$OC \,.\, OC' = OX \,.\, OY = OA \,.\, OA'.$$

70. Ex. 1. If $\{AA', BB', CC' \ldots\}$ be a range in involution, whose centre lies between A and A', show that there are two points at which each of the segments AA', BB', CC', ... subtends a right angle.

Ex. 2. If $\{AA' BB'\}$ be any range such that the circles described on the segments AA', BB' as diameters meet in the point P, and if two points C, C' be taken on the line AB such that CPC' is a right angle, show that $\{AA', BB', CC'\}$ will be a range in involution.

Ex. 3. If $\{AA', BB'\}$ be a harmonic range, and if L, M be the middle points of the segments AA', BB', show that $\{AA', BB', LM\}$ will be a range in involution.

Ex. 4. If $\{AA', BB'\}$ be a harmonic range, and if Q, Q' be the harmonic conjugates of any point P with respect to the point-pairs A, A'; B, B'; show that $\{AA', BB', QQ'\}$ will be a range in involution.

Ex. 5. If A, A' be any pair of conjugate points of a range in involution, and if the perpendiculars drawn to OA, OA' at A and A' meet in P, where O is any point not on the same straight line, show that P lies on a fixed straight line.

If $\{AA', BB', \ldots\}$ be the range, the locus of P is a straight line parallel to the line joining the centres of the circles OAA', OBB',....

The Double Points.

71. When the points constituting any conjugate couple of a range in involution, lie on the same side of the centre, there exist two points, one on either side of the centre, each of which coincides with its own conjugate. These points are called the *double points* of the involution.

To find the double points, let OT be a tangent from O to any circle passing through a pair of conjugate points, such as A, A'. Then if with centre O, and radius OT, a circle be drawn cutting AA' in the points S and S' (see fig. § 69), we shall have

$$OS^2 = OS'^2 = OT^2 = OA \;.\; OA'.$$

Therefore S and S' are the double points.

When the points constituting a conjugate couple lie on opposite sides of the centre, the double points are imaginary.

72. It is evident that any pair of conjugate points of a range in involution are harmonic conjugates with respect to the double points of the involution.

We may also notice that there exists but one pair of points which are at once harmonically conjugate with respect to each pair of conjugate points of a range in involution.

73. Ex. 1. If S, S' be the double points of a range in involution; A, A', and B, B', conjugate couples; and if E, F be the middle points of AA', BB'; show that

$$PA \,.\, PA' \,.\, FS + PB \,.\, PB' \,.\, SE = PS^2 \,.\, FE,$$

where P is any point on the line.

Ex. 2. Show also that

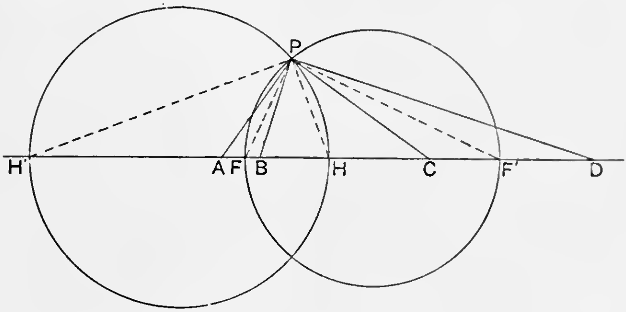
$$PA \,.\, PA' \,.\, SS' = PS'^2 \,.\, SE - PS^2 \,.\, S'E.$$

Ex. 3. Show also that

$$SA \,.\, SA' \,.\, SF = SB \,.\, SB' \,.\, SE.$$

Ex. 4. Show that four given points on a straight line determine three ranges in involution, and that the double points of any one range, are harmonically conjugate with the double points of the other two ranges.

Let A, B, C, D be the four given points. Then we shall have a range in which A, B and C, D are conjugate couples; a range in which A, C and B, D are conjugate couples; and a range in which A, D and B, C are conjugate couples. Let F, F'; G, G'; and H, H'; be the double points of these three ranges; and suppose A, B, C, D occur in order. Then by § 64, we see that F, F' and H, H' are real points, but G, G' imaginary.



Let the circles described on HH' and FF' as diameters meet in P. Then by § 48, Ex. 2, PH bisects the angles BPC, APD. Hence the angles APB, CPD are equal. But PF bisects the angle APB, and PF' the angle CPD; therefore the angle FPB is equal to the angle CPF'. Hence the angle FPH is equal to the angle HPF'. Therefore PH, PH' are the bisectors of the angle FPF'; and hence $\{HH', FF'\}$ is a harmonic range.

Again, it is easy to see that each of the angles APC, BPD is a right angle. Hence by § 70, Ex. 2, $\{AC, BD, FF', HH'\}$ is a range in involution; and therefore $\{FF', GG'\}$ and $\{HH', GG'\}$ are harmonic ranges.

Ex. 5. If M, N are the centres of the involutions $\{AC, BD\}$ and $\{AD, BC\}$, show that $\{MN, AB, CD\}$ is a range in involution.

Ex. 6. If $\{RP, CA, BD\}$ and $\{PQ, AB, CD\}$ be ranges in involution, show that $\{QR, BC, AD\}$ will be a range in involution.

Ex. 7. Show that any two ranges in involution on the same straight line have one pair of conjugate points common; and show how to find them.

Relations between the segments of a range in involution.

74. *If $\{AA', BB', CC'\}$ be any range in involution, the segments of the range are connected by the relation*

$$AB' . BC' . CA' + A'B . B'C . C'A = 0.$$

Let O be the centre of the range. Then

$$OA . OA' = OB . OB';$$

that is, $$OA : OB = OB' : OA'.$$

Therefore $$OA : OB = OB' - OA : OA' - OB$$
$$= AB' : BA'.$$

Similarly we shall have
$$OB : OC = BC' : CB',$$
$$OC : OA = CA' : AC'.$$

Hence, compounding these ratios, we have
$$AB' . BC' . CA' = BA' . CB' . AC';$$

which is equivalent to
$$AB' . BC' . CA' + A'B . B'C . C'A = 0.$$

In the same way, we may deduce the relations:
$$A'B' . BC' . CA + AB . B'C . C'A' = 0,$$
$$AB . B'C' . CA' + A'B' . BC . C'A = 0,$$
$$AB' . BC . C'A' + A'B . B'C' . CA = 0.$$

75. Conversely, if any one of these relations hold, then the range $\{AA', BB', CC'\}$ will be in involution.

For if not, let a range in involution be formed so that A, A', and B, B', are conjugate couples; and let C'' be the point conjugate to the point C.

Then if we have given the relation
$$AB' . BC' . CA' = - A'B . B'C . C'A,$$
we shall also have the relation
$$AB' . BC'' . CA' = - A'B . B'C . C''A.$$

Therefore $$BC' : C'A = BC'' : C''A.$$

Hence C' must coincide with C''; that is, the range
$$\{AA', BB', CC'\}$$
is in involution.

76. *If* $\{AA', BB', CC'\}$ *be any range in involution, then*
$$AB . AB' : A'B . A'B' = AC . AC' : A'C . A'C'.$$

Let O be the centre of involution. Then as in § 73, we have
$$OA : OB = AB' : BA'.$$

Similarly we shall have
$$OA : OB' = AB : B'A'.$$

Hence $$OA^2 : OB . OB' = AB . AB' : BA' . B'A'.$$

Therefore, since $OB \cdot OB' = OA \cdot OA'$,

$$OA : OA' = AB \cdot AB' : A'B \cdot A'B'.$$

Similarly we shall have

$$OA : OA' = AC \cdot AC' : A'C \cdot A'C'.$$

Hence $AB \cdot AB' : A'B \cdot A'B' = AC \cdot AC' : A'C \cdot A'C'.$

Conversely, if this relation is true, it may be proved that the range $\{AA', BB', CC'\}$ is in involution, by a similar method to that used in § 75.

77. Ex. 1. If $\{AA', BB', CC'\}$ be any range in involution, and if $\{AA', BC\}$ be a harmonic range, show that $\{AA', B'C'\}$ will be a harmonic range.

Ex. 2. If $\{AA', BC\}$, $\{AA', B'C'\}$ be harmonic ranges, show that $\{AA', BB', CC'\}$ and $\{AA', BC', B'C\}$ will be ranges in involution.

Show also that if F, F' and G, G' be the double points of these ranges, then each of the ranges $\{AA', FF'\}$, $\{AA', GG'\}$, $\{FF', GG'\}$ will be harmonic.

Ex. 3. If $\{AA', BC\}$, $\{AA', B'C'\}$ be harmonic ranges, and if M, N be the centres of the ranges in involution $\{AA', BB', CC'\}$ and $\{AA', BC', B'C\}$, show that $\{AA', MN\}$ will be a harmonic range.

Ex. 4. If $\{AA', BC\}$, $\{BB', CA\}$, $\{CC', AB\}$ be harmonic ranges, show that $\{AA', BB', CC'\}$ will be in involution.

Pencil in involution.

78. When several pairs of rays OA, OA'; OB, OB'; OC, OC'; &c.; drawn through a point O, are such that the angles which they make with a fixed ray OX are connected by the relation

$$\begin{aligned}\tan XOA \cdot \tan XOA' &= \tan XOB \cdot \tan XOB' \\ &= \tan XOC \cdot \tan XOC' \\ &= \&c.;\end{aligned}$$

they are said to form a pencil in *involution*.

If OX' be the ray at right angles to OX, it is easy to see that

$$\tan X'OA \cdot \tan X'OA' = \tan X'OB \cdot \tan X'OB' = \&c.$$

The rays OX, OX' are called the *principal* rays of the involution, and any pair of corresponding rays, such as OA, OA' are called *conjugate rays* of the pencil.

The notation used for a pencil which is in involution is

$$O\{AA', BB', \ldots\}.$$

79. If OX does not lie within the angle AOA' formed by any pair of conjugate rays it is evident that there will be two rays lying on opposite sides of OX, such that each of them coincides with its own conjugate. These rays are called the *double* rays of the pencil.

Let OS, OS' be the double rays, then we have

$$\tan^2 XOS = \tan^2 XOS' = \tan XOA \,.\, \tan XOA' = \&c.$$

Hence by § 58, we see that the double rays form with any pair of conjugate rays a harmonic pencil; and also that the principal rays are the bisectors of the angle between the double rays.

It should be noticed that the principal rays themselves constitute a pair of conjugate rays of a pencil in involution.

80. Ex. 1. Show that the rays drawn at right angles to the rays of a pencil in involution constitute another pencil in involution having the same principal rays.

Ex. 2. If $O\{AA', BB', \ldots\}$ be any pencil in involution, and if through any point O' rays $O'A$, $O'A'$, $O'B, \ldots$ be drawn perpendicular to the rays OA, OA', $OB\ldots$; show that the pencil $O'\{AA', BB', \ldots\}$ will be in involution.

Ex. 3. If OA', $OB', \ldots$ be the harmonic conjugate rays of OA, $OB, \ldots$ with respect to the pair of rays OS, OS', show that $O\{AA', BB', \ldots\}$ will be a pencil in involution, the double rays of which are OS and OS'.

Ex. 4. If the pencil $O\{AA', BB', CC'\}$ be in involution, and if the angles AOA', BOB' have the same bisectors, show that these lines will also bisect the angle COC'.

Ex. 5. When the double rays of a pencil in involution are at right angles, show that they bisect the angle between each pair of conjugate rays of the pencil.

Ex. 6. Show that any two pencils in involution which have a common vertex, have one pair of conjugate rays in common.

Ex. 7. Show that any pencil in involution has in general one and only one pair of conjugate rays which are parallel to a pair of conjugate rays of any other pencil in involution.

Ex. 8. Show that if rays OA', $OB', \ldots$ be drawn perpendicular to the rays OA, $OB, \ldots$, the pencil $O\{AA', BB', \ldots\}$ will be in involution.

If AOA' is a right angle, then whatever the position of the line OX we have, $\tan XOA \tan XOA' = -1$.

81. When the double rays of a pencil in involution are real, it is easy to see that the rays of the pencil will cut any straight line in points which form a range in involution.

For, if $O\{AA', BB', \ldots\}$ be the pencil, and if OS, OS' be the double rays, let any straight line be drawn cutting the rays of the pencil in the points A, A', B, B',... and the double rays in the points S, and S'. Then since $O\{AA', SS'\}$ is a harmonic pencil, it follows from § 49, that $\{AA', SS'\}$ is a harmonic range. Similarly $\{BB', SS'\}$ is a harmonic range. Hence $\{AA', BB', \ldots\}$ is a range in involution whose double points are S and S''.

The converse of this theorem is also true, and follows immediately from § 49.

82. By the principle of continuity we could infer that this theorem is always true, whether the double rays are real or imaginary. The converse theorem, in fact, is often taken as the basis of the definition of a pencil in involution, and the properties of a pencil in involution are then derived from the properties of a range.

83. Ex. 1. If $O\{AA', BB', CC'\}$ be any pencil in involution, show that

$$\sin AOB' \,.\, \sin BOC' \,.\, \sin COA' + \sin A'OB \,.\, \sin B'OC \,.\, \sin C'OA = 0.$$

This is easily obtained from the theorem in § 74, by applying the method used in § 49.

Ex. 2. If $O\{AA', BB', CC'\}$ be any pencil in involution show that

$$\frac{\sin AOB \,.\, \sin AOB'}{\sin A'OB \,.\, \sin A'OB'} = \frac{\sin AOC \,.\, \sin AOC'}{\sin A'OC \,.\, \sin A'OC'}.$$

Ex. 3. If $O\{AA', BC\}$, $O\{AA', B'C'\}$ be harmonic pencils, show that the pencils $O\{AA', BB', CC'\}$ and $O\{AA', BC', B'C\}$ will be in involution, and that if OF, OF' and OG, OG' be the double rays of these pencils, then each of the pencils $O\{AA', FF'\}$, $O\{AA', GG'\}$, $O\{FF', GG'\}$ will be harmonic.

84. Instead of obtaining the connection between a pencil in involution and a range in involution, and deducing the properties of the pencil from the range, we may proceed otherwise, and obtain the properties of a pencil in involution directly from the definition given in § 78.

It will be convenient first to prove the following lemma: *If two chords AA', BB', of a circle meet in K, then*

$$KA : KA' = AB \,.\, AB' : A'B \,.\, A'B'.$$

Since the triangles KAB, $KB'A'$ are similar; therefore

$$KA : KB' = AB : B'A'.$$

Again, since the triangles KAB', KBA' are similar; therefore

$$KA : KB = AB' : BA'.$$

Hence $$KA^2 : KB . KB' = AB . AB' : B'A' . BA'.$$

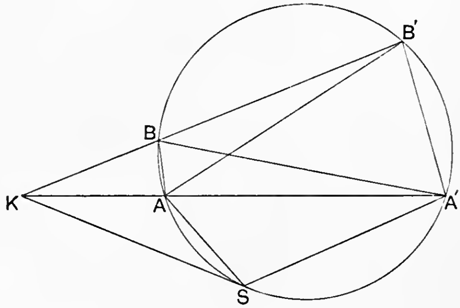
But $$KB . KB' = KA . KA'.$$

Therefore $$KA : KA' = AB . AB' : B'A' . BA',$$

that is $$KA : KA' = AB . AB' : A'B . A'B'.$$

Again, if KS be drawn to touch the circle, we shall have

$$KA : KA' = AS^2 : A'S^2.$$



For the triangles KAS, KSA' are similar, and therefore

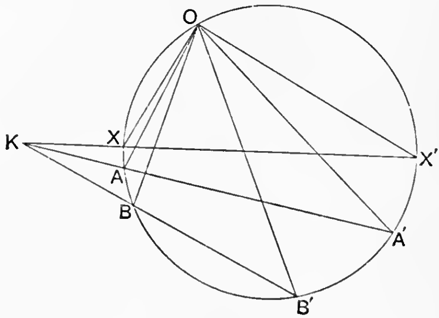
$$KA : KS = AS : SA';$$

that is $$KA^2 : KS^2 = AS^2 : SA'^2.$$

But $$KS^2 = KA . KA',$$

therefore $$KA : KA' = AS^2 : A'S^2.$$

85. *If $O \{AA', BB', CC', \ldots\}$ be any pencil in involution, and if a circle be drawn through the point O cutting the rays of the pencil in the points A, A', B, B', ..., the chords AA', BB', ... of this circle will pass through a fixed point.*



Let the circle cut the principal rays of the pencil in the points X, X', and let XX' meet AA' in the point K.

By § 84, we have

$$KX : KX' = XA \,.\, XA' : X'A \,.\, X'A'.$$

But if R be the radius of the circle,

$$2R = \frac{XA}{\sin XOA} = \frac{XA'}{\sin XOA'} = \&c.$$

Hence
$$\frac{KX}{KX'} = \frac{\sin XOA \,.\, \sin XOA'}{\sin X'OA \,.\, \sin X'OA'}$$
$$= \tan XOA \,.\, \tan XOA'.$$

Again, if BB' meet XX' in the point K', we shall have

$$\frac{K'X}{K'X'} = \tan XOB \,.\, \tan XOB'.$$

But by definition,

$$\tan XOA \,.\, \tan XOA' = \tan XOB \,.\, \tan XOB'.$$

Therefore
$$KX : KX' = K'X : K'X',$$

that is K and K' must coincide.

86. *Given any two pairs of conjugate rays of a pencil in involution, to find the principal rays, and the double rays.*

Let OA, OA' and OB, OB' be the given pairs of conjugate rays. Draw a circle passing through O and cutting these rays in the points A, A', and B, B' respectively.

Let AA' meet BB' in the point K, and let the diameter of the circle which passes through K meet the circle in X, X'. Then OX, OX' will be the principal rays of the pencil.

By § 84, we have

$$KX : KX' = XA \,.\, XA' : X'A \,.\, X'A'$$
$$= XB \,.\, XB' : X'B \,.\, X'B'.$$

Hence
$$\frac{\sin XOA \,.\, \sin XOA'}{\sin X'OA \,.\, \sin X'OA'} = \frac{\sin XOB \,.\, \sin XOB'}{\sin X'OB \,.\, \sin X'OB'}.$$

Therefore, since XOX' is a right angle,

$$\tan XOA \,.\, \tan XOA' = \tan XOB \,.\, \tan XOB'.$$

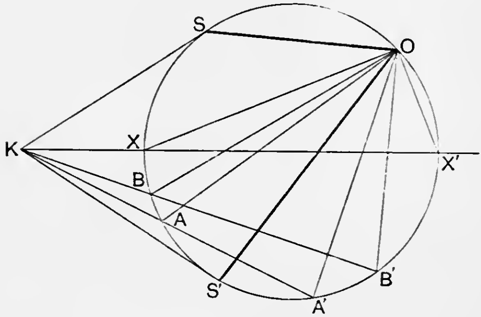
therefore OX, OX' are the principal rays.

To find the double rays, draw the tangents KS, KS' to the circle. By § 84, we have

$$KX : KX' = XS^2 : X'S^2,$$
$$= XA \,.\, XA' : X'A \,.\, X'A'.$$

Therefore $$\frac{\sin^2 XOS}{\sin^2 X'OS} = \frac{\sin XOA \, . \sin XOA'}{\sin X'OA \, . \sin X'OA'};$$

that is $$\tan^2 XOS = \tan XOA \, . \tan XOA'.$$



Similarly we may prove that

$$\tan^2 XOS' = \tan XOA \, . \tan XOA'.$$

Hence, OS, OS' are the double rays of the pencil.

The double rays will be real or imaginary according as K lies without or within the circle; that is according as AA' intersects BB' without or within the circle.

87. We infer from the above construction that a pencil in involution has in general one and only one pair of conjugate rays at right angles. The exceptional case occurs when the point K is the centre of the circle, that is when the two given pairs of conjugate rays are at right angles. In this case every line through K will be a diameter, and hence every pair of conjugate rays will be at right angles.

It follows that any pencil of rays $O\{AA', BB', \ldots\}$, in which each of the angles AOA', BOB', ... is a right angle, is a pencil in involution, of which any pair of conjugate rays may be considered as the principal rays.

88. Ex. 1. Show that if two pencils in involution have the same vertex, there exists one pair and only one pair of conjugate rays common to each pencil.

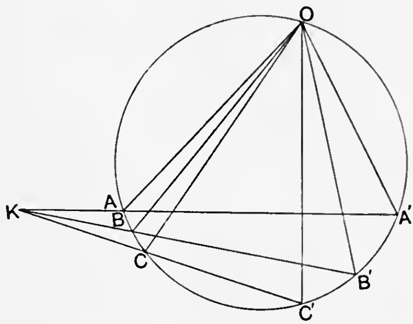
When is this pair of rays real?

Ex. 2. Show that any two pencils in involution have in general one and only one pair of conjugate rays which are parallel; and show how to construct these rays.

Ex. 3. Show that the straight line joining the feet of the perpendiculars drawn to a pair of conjugate rays of a pencil in involution from a fixed point, passes through another fixed point.

89. *If $O\{AA', BB', CC'\}$ be any pencil in involution,*

$$\frac{\sin AOB \cdot \sin AOB'}{\sin A'OB \cdot \sin A'OB'} = \frac{\sin AOC \cdot \sin AOC'}{\sin A'OC \cdot \sin A'OC'}.$$



Let any circle be drawn passing through O, cutting the rays of the pencil in the points A, A', B, B', C, C'. Then by § 85, AA', BB', CC' will meet in the same point K.

By § 84, we have

$$KA : KA' = AB \cdot AB' : A'B \cdot A'B',$$
$$= AC \cdot AC' : A'C \cdot A'C'.$$

Therefore $\quad AB \cdot AB' : A'B \cdot A'B' = AC \cdot AC' : A'C \cdot A'C'.$

But if R be the radius of the circle,

$$2R = \frac{AB}{\sin AOB} = \frac{AB'}{\sin AOB'} = \&c.$$

Hence $\quad \dfrac{\sin AOB \cdot \sin AOB'}{\sin A'OB \cdot \sin A'OB'} = \dfrac{\sin AOC \cdot \sin AOC'}{\sin A'OC \cdot \sin A'OC'}.$

90. *The rays of any pencil in involution cut any straight line in a series of points which form a range in involution.*

Let any straight line be drawn cutting the rays of the pencil $O\{AA', BB', CC'\}$ in the points A, A', B, &c.

Then if the pencil be in involution, we have by § 89,

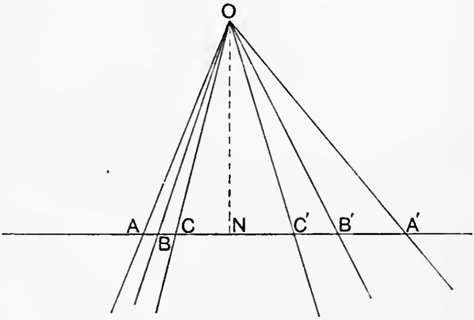
$$\frac{\sin AOB \cdot \sin AOB'}{\sin A'OB \cdot \sin A'OB'} = \frac{\sin AOC \cdot \sin AOC'}{\sin A'OC \cdot \sin A'OC'}.$$

Let ON be the perpendicular from O on the line AA'. Then we have

$$\sin AOB = \frac{NO \,.\, AB}{OA \,.\, OB},$$

$$\sin AOB' = \frac{NO \,.\, AB'}{OA \,.\, OB'},$$

and similar values for $\sin AOC$, &c.



Hence, we obtain the relation

$$\frac{AB \,.\, AB'}{A'B \,.\, A'B'} = \frac{AC \,.\, AC'}{A'C \,.\, A'C'}.$$

Therefore, by § 76, the range $\{AA', BB', CC'\}$ is in involution.

Conversely, it may be proved in a similar manner that if the points of a range in involution be joined to any point not on the same straight line, these lines will form a system of rays in involution.

91. Let $\{AA', BB', CC'\}$ be any range in involution, then by § 74, we have $AB' \,.\, BC' \,.\, CA' + A'B \,.\, B'C \,.\, C'A = 0$.

If now the points of the range be joined to any point O, it follows by the method used in the last article that the angles of any pencil in involution $O\{AA', BB', CC'\}$ are connected by the relation

$$\sin AOB' \,.\, \sin BOC' \,.\, \sin COA' + \sin A'OB \,.\, \sin B'OC \,.\, \sin C'OA = 0.$$

Conversely, by § 75, if this relation holds, we infer that the pencil $O\{AA', BB', CC'\}$ will be in involution.

92. Ex. 1. If $ABCD$ be a square, and if OX, OY be drawn through any point O parallel to the sides of the square, show that $O\{XY, AC, BD\}$ is a pencil in involution.

Ex. 2. If ABC be a triangle, and if through any point O, rays OX, OY OZ be drawn parallel to the sides BC, CA, AB, show that $O\{XA, YB, ZC\}$ will be a pencil in involution.

CHAPTER VI.

PROPERTIES OF TRIANGLES.

93. IN Euclid a *triangle* is defined to be a plane figure bounded by three straight lines, that is to say, a triangle is regarded as an area. In modern geometry, any group of three points, which are not collinear, is called a *triangle*. Since three straight lines which are not concurrent intersect in three points, a group of three straight lines may also be called a triangle without causing any ambiguity.

The present chapter may be divided into two parts. We shall first discuss some theorems relating to lines drawn through the vertices of a triangle which are concurrent, and also some theorems relating to points taken on the sides of a triangle which are collinear. Secondly we propose to deal with certain special points which have important properties in connection with a triangle, and the more important circles connected with a triangle.

In recent years the geometry of the triangle has received considerable attention, and various circles have been discovered which have so many interesting properties, that special names have been given them. We shall however at present merely consider their more elementary properties, reserving for a later chapter the complete discussion of them.

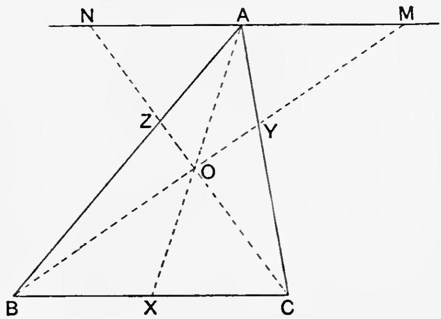
Concurrent lines drawn through the vertices of a triangle.

94. *If the straight lines which connect the vertices A, B, C of a triangle with any point O meet the opposite sides of the triangle in the points X, Y, Z, the product of the ratios*

$$BX : XC, \quad CY : YA, \quad AZ : ZB$$

is equal to unity.

Through A draw the straight line NAM parallel to BC, and let it cut BO, CO in M and N.



By similar triangles,

$$BX : XC = AM : NA,$$
$$CY : YA = BC : AM,$$
$$AZ : ZB = NA : BC.$$

Hence we have,

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

95. *If X, Y, Z are points on the sides of a triangle such that*

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1,$$

the lines AX, BY, CZ will be concurrent.

For let BY, CZ meet in the point O, and let AO meet BC in X'. Then we have

$$\frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

Therefore $\quad BX' : X'C = BX : XC.$

Therefore X' must coincide with X, or what is the same thing, AX must pass through O.

96. Ex. 1. Show that the lines joining the vertices A, B, C of a triangle to the middle points of the sides BC, CA, AB are concurrent.

Ex. 2. Show that the perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.

Ex. 3. If a straight line be drawn parallel to BC, cutting the sides AC, AB in Y and Z, and if BY, CZ intersect in O, show that AO will bisect BC.

Ex. 4. If points Y and Z be taken on the sides AC, AB of a triangle ABC, so that $CY : YA = BZ : AZ$, show that BY, CZ will intersect in a point O such that AO is parallel to BC.

Ex. 5. Show that the straight lines drawn through the vertices B and C of the triangle ABC, parallel respectively to the sides CA, AB, intersect in a point on the line which connects the point A to the middle point of BC.

Ex. 6. If the inscribed circle of a triangle touch the sides in the points X, Y, Z, show that the lines AX, BY, CZ are concurrent.

Ex. 7. If the escribed circle of the triangle ABC, opposite to the angle A, touch the sides in the points X, Y, Z, show that AX, BY, CZ are concurrent.

Ex. 8. If the pencils $O\{AA', BC\}$, $O\{BB', CA\}$, $O\{CC', AB\}$ be harmonic, show that the pencil $O\{AA', BB', CC'\}$ will be in involution.

This follows from Ex. 6, 7, by the aid of § 85.

Ex. 9. If any circle be drawn touching the sides of the triangle ABC in the points X, Y, Z, show that the lines joining the middle points of BC, CA, AB to the middle points of AX, BY, CZ respectively, are concurrent.

Ex. 10. A circle is drawn cutting the sides of a triangle ABC in the points X, X'; Y, Y'; Z, Z'; show that if AX, BY, CZ are concurrent, so also are AX', BY', CZ'.

Ex. 11. If the lines connecting the vertices of any triangle ABC to any point O, meet the opposite sides in the points D, E, F, show that the pencil $D\{AC, EF\}$ is harmonic.

Conversely, if D, E, F be three points on the sides of the triangle ABC, such that the pencil $D\{AC, EF\}$ is harmonic, show that AD, BE, CF are concurrent.

97. The theorem of § 94 may be proved otherwise. We have

$$\begin{aligned} BX : XC &= (ABO) : (AOC) \\ &= (AOB) : (COA), \\ CY : YA &= (BOC) : (AOB), \\ AZ : ZB &= (COA) : (BOC). \end{aligned}$$

Hence, as before, $$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

Ex. If the lines AO, BO, CO meet the sides of the triangle ABC in the points X, Y, Z, show that

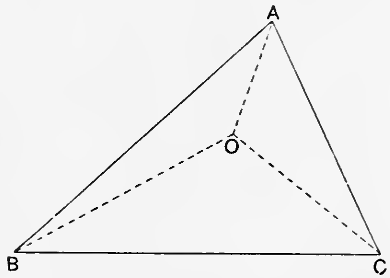
$$\frac{AO}{AX} + \frac{BO}{BY} + \frac{CO}{CZ} = 2.$$

98. *If ABC be any triangle, and O any point, then*

$$\frac{\sin BAO}{\sin OAC} \cdot \frac{\sin CBO}{\sin OBA} \cdot \frac{\sin ACO}{\sin OCB} = 1.$$

From the triangle BOC, we have

$$\sin CBO : \sin OCB = OC : OB.$$



Similarly from the triangles COA, AOB,

$$\sin ACO : \sin OAC = OA : OC,$$

$$\sin BAO : \sin OBA = OB : OA.$$

Hence, $$\frac{\sin BAO}{\sin OAC} \cdot \frac{\sin CBO}{\sin OBA} \cdot \frac{\sin ACO}{\sin OCB} = 1.$$

99. *If points X, Y, Z be taken on the sides of a triangle ABC, such that*

$$\frac{\sin BAX}{\sin XAC} \cdot \frac{\sin CBY}{\sin YBA} \cdot \frac{\sin ACZ}{\sin ZCB} = 1,$$

the lines AX, BY, CZ will be concurrent.

Let BY, CZ meet in the point O. Then by the last article we have

$$\frac{\sin BAO}{\sin OAC} \cdot \frac{\sin CBY}{\sin YBA} \cdot \frac{\sin ACZ}{\sin ZCB} = 1.$$

Therefore

$$\sin BAO : \sin OAC = \sin BAX : \sin XAC.$$

Hence it follows that the line AX must coincide with the line AO; that is, the lines AX, BY, CZ are concurrent.

100. Ex. 1. Show that the internal bisectors of the angles of a triangle are concurrent.

Ex. 2. Show that the internal bisector of one angle of a triangle, and the external bisectors of the other angles are concurrent.

Ex. 3. The tangents to the circle circumscribing the triangle ABC, at the points B and C, meet in the point L. Show that

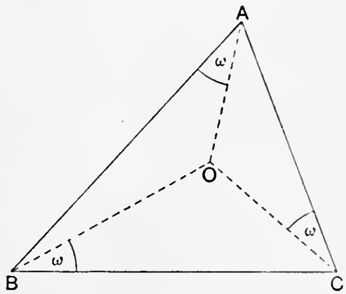
$$\sin BAL : \sin LAC = \sin ACB : \sin CBA.$$

Ex. 4. If the tangents at the points A, B, C to the circle circumscribing the triangle, meet in the points L, M, N, show that the lines AL, BM, CN will be concurrent.

Ex. 5. If on the sides of a triangle ABC similar isosceles triangles LBC, MCA, NAB be described, show that the lines AL, BM, CN will be concurrent.

Ex. 6. If the perpendiculars drawn from the points A, B, C to sides $B'C'$, $C'A'$, $A'B'$ of the triangle $A'B'C'$ are concurrent, show that the lines drawn from A', B', C' perpendicular to the sides BC, CA, AB of the triangle ABC will also be concurrent.

Ex. 7. Show that, connected with a triangle ABC, a point O can be found such that the angles BAO, ACO, CBO are equal.



Denoting the angle BAO by ω, and the angles BAC, ACB, CBA by A, B, C, we have from § 98,

$$\sin^3\omega = \sin(A-\omega)\sin(B-\omega)\sin(C-\omega);$$

whence by trigonometry,

$$\cot\omega = \cot A + \cot B + \cot C.$$

Thus there is but one value for the angle ω, and consequently only one point O which satisfies the given condition.

There is obviously another point O' such that the angles CAO', ABO', BCO' are each equal to the same angle ω.

Ex. 8. The vertices of a triangle ABC are joined to any point O; and a triangle $A'B'C'$ is constructed having its sides parallel to AO, BO, CO. If lines be drawn through A', B', C' parallel to the corresponding sides of the triangle ABC; show that these lines will be concurrent.

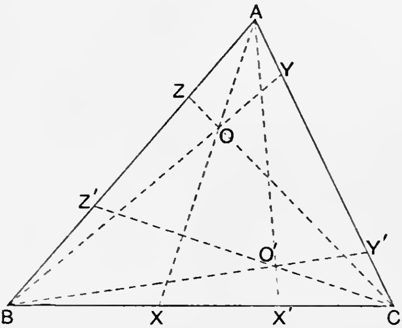
101. Any two lines AX, AX', drawn so that the angle XAX' has the same bisectors as the angle BAC are said to be *isogonal conjugates* with respect to the angle BAC.

Let AX, BY, CZ be the straight lines connecting the vertices of the triangle ABC to any point O, and let AX', BY', CZ' be their isogonal conjugates with respect to the angles of the triangle.

We have then

$$\frac{\sin BAX}{\sin XAC} \cdot \frac{\sin CBY}{\sin YBA} \cdot \frac{\sin ACZ}{\sin ZCB}$$

$$= \frac{\sin X'AC}{\sin BAX'} \cdot \frac{\sin Y'BA}{\sin CBY'} \cdot \frac{\sin Z'CB}{\sin ACZ'}.$$



But since AX, BY, CZ are concurrent, the latter product is equal to unity. Hence by § 99, it follows that AX', BY', CZ' are also concurrent.

Thus: *when three lines drawn through the vertices of a triangle are concurrent, their isogonal conjugates with respect to the angles at these vertices are also concurrent.*

If the lines AX, BY, CZ meet in the point O, and their isogonal conjugates in the point O', the points O, O' are called *isogonal conjugate points* with respect to the triangle ABC.

102. Ex. 1. Show that the orthocentre of a triangle and the circumcentre are isogonal conjugate points.

Ex. 2. If O, O' be any isogonal conjugate points, with respect to the triangle ABC, and if OL, $O'L'$ be drawn perpendicular to BC; OM, $O'M'$ perpendicular to CA; and ON, $O'N'$ perpendicular to AB; show that

$$OL \,.\, O'L' = OM \,.\, O'M' = ON \,.\, O'N'.$$

Show also that the six points L, M, N, L', M', N' lie on a circle whose centre is the middle point of OO', and that MN is perpendicular to AO'.

Ex. 3. If D, E, F be the middle points of the sides of the triangle ABC, show that the isogonal conjugate of AD with respect to the angle BAC, is the line joining A to the point of intersection of the tangents at B and C to the circle circumscribing ABC.

103. If two points X, X' be taken on the line BC so that the segments XX', BC have the same middle point, the points X, X' are called *isotomic conjugates* with respect to the segment BC.

Ex. 1. If X, Y, Z be any three points on the sides of a triangle ABC, and X', Y', Z' the isotomic conjugate points with respect to BC, CA, AB respectively, show that if AX, BY, CZ are concurrent so also are AX', BY', CZ'.

If AX, BY, CZ meet in the point O, and AX', BY', CZ' in the point O', the points O and O' are called *isotomic conjugate points* with respect to the triangle ABC.

Ex. 2. If the inscribed circle of the triangle ABC touch the sides in the points X, Y, Z, show that the isotomic conjugate points with respect to the sides of the triangle, are points of contact of the escribed circles of the triangle.

Ex. 3. In Ex. 1, show that the areas (XYZ), $(X'Y'Z')$ are equal, and that

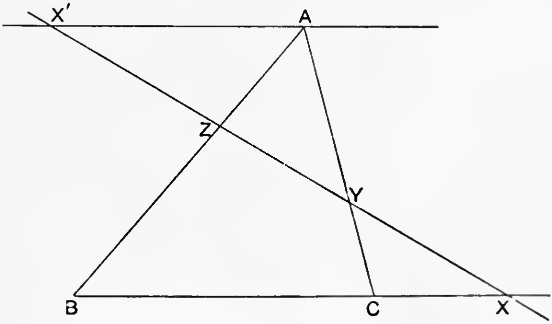
$$(BOC') \,.\, (BO'C) = (COA) \,.\, (CO'A) = (AOB) \,.\, (AO'B).$$

Collinear points on the sides of a triangle.

104. *If a straight line intersect the sides of a triangle ABC in the points X, Y, Z, the product of the ratios*

$$BX : CX ; \quad CY : AY ; \quad AZ : BZ ;$$

is equal to unity.



Through A draw AX' parallel to BC to cut the straight line XYZ in the point X'.

Then by similar triangles,

$$CY : CX = AY : AX' ;$$

$$BX : BZ = AX' : AZ.$$

Therefore $$\frac{BX}{CX} \cdot \frac{CY}{BZ} = \frac{AY}{AZ};$$

or $$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1.$$

This formula may also be written

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1,$$

and should be compared with the formula given in § 94.

When a straight line cuts the sides of a triangle it is often called a *transversal*. Thus, if X, Y, Z be collinear points on the sides BC, CA, AB, respectively, of the triangle ABC, the line on which they lie is referred to as the transversal XYZ.

105. *If X, Y, Z are points on the sides of a triangle ABC such that*

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1,$$

the points X, Y, Z are collinear.

Let the line joining the points Y and Z cut BC in the point X'. By the last article, we have

$$\frac{BX'}{CX'} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1.$$

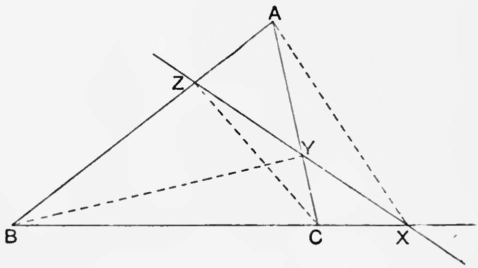
Hence, we must have

$$BX' : CX' = BX : CX.$$

Therefore X must coincide with X'; that is, the point X lies on the line YZ.

106. *If any straight line cut the sides of the triangle ABC in the points X, Y, Z, then*

$$\frac{\sin BAX}{\sin CAX} \cdot \frac{\sin CBY}{\sin ABY} \cdot \frac{\sin ACZ}{\sin BCZ} = 1.$$



This relation is easily deduced from that given in § 104, for we have

$$BX : CX = AB \,.\, \sin BAX : AC \,.\, \sin CAX,$$
$$CY : AY = BC \,.\, \sin CBY : BA \,.\, \sin ABY,$$
$$AZ : BZ = CA \,.\, \sin ACZ : CB \,.\, \sin BCZ.$$

Hence,

$$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}=\frac{\sin BAX}{\sin CAX}\cdot\frac{\sin CBY}{\sin ABY}\cdot\frac{\sin ACZ}{\sin BCZ}.$$

But by § 104, the former product is equal to unity. Therefore the theorem is true.

107. Conversely, if points X, Y, Z be taken on the sides of a triangle ABC, so that

$$\frac{\sin BAX}{\sin CAX}\cdot\frac{\sin CBY}{\sin ABY}\cdot\frac{\sin ACZ}{\sin BCZ}=1,$$

it follows from § 105, that the points X, Y, Z must be collinear.

108. Ex. 1. Show that the external bisectors of the angles of a triangle meet the opposite sides in collinear points.

Ex. 2. The tangents to the circumcircle of a triangle at the angular points cut the opposite sides of the triangle in three collinear points.

Ex. 3. The lines drawn through any point O perpendicular to the lines OA, OB, OC, meet the sides of the triangle ABC in three collinear points.

Ex. 4. The tangents from the vertices of a triangle to any circle meet the opposite sides in the points X, X'; Y, Y'; and Z, Z'; respectively. Prove that if X, Y, Z are collinear, so also are X', Y', Z'.

Ex. 5. If any line cut the sides of the triangle ABC in the points X, Y, Z; the isogonal conjugates of AX, BY, CZ, with respect to the angles of the triangle will meet the opposite sides in collinear points.

Ex. 6. If a straight line cut the sides of the triangle ABC in the points X, Y, Z; the isotomic points with respect to the sides will be collinear.

Ex. 7. If D, E, F are the middle points of the sides of a triangle, and X, Y, Z the feet of the perpendiculars drawn from the vertices to the opposite sides, and if YZ, ZX, XY meet EF, FD, DE in the points P, Q, R respectively, show that DP, EQ, FR are concurrent, and also that XP, YQ, ZR are concurrent.

Ex. 8. Points X, Y, Z are taken on the sides of a triangle ABC, so that

$$BX : XC = CY : YA = AZ : ZB.$$

If AX, BY, CZ intersect in the points P, Q, R, show that

$$AQ : AR = BR : BP = CP : CQ.$$

Ex. 9. The sides AB, AC of a triangle are produced to D and E, and DE is joined. If a point F be taken on BC so that

$$BF : FC = AB \,.\, AE : AC \,.\, AD,$$

show that AF will bisect DE.

[St John's Coll. 1887.]

Ex. 10. The sides BC, CA, AB of a triangle cut a straight line in D, E, F; through D, E, F three straight lines $DLOG$, $EHOM$, $FKON$ having the common point O are drawn, cutting the sides CA, AB in L, G; AB, BC in M, H; BC, CA in N, K. Prove that

$$\frac{AK.BG.CH}{AM.BN.CL}=\frac{AG.BH.CK}{AL.BM.CN}=-\frac{GD.HE.KF}{LD.ME.NF}=-\frac{HD.KE.GF}{ND.LE.MF}.$$

[Math. Tripos, 1878.]

Ex. 11. Through the vertices of a triangle ABC, three straight lines AD, BE, CF are drawn to cut the opposite sides in the points D, E, F. The lines BE, CF intersect in A'; CF, AD intersect in B': and AD, BE in C'. Show that

$$\frac{DB'}{DC'}\cdot\frac{EC'}{EA'}\cdot\frac{FA'}{FB'}=\left(\frac{CD}{BD}\cdot\frac{AE}{CE}\cdot\frac{BF}{AF}\right)^2=\left(\frac{AC'}{AB'}\cdot\frac{BA'}{BC'}\cdot\frac{CB'}{CA'}\right)^2.$$

[De Rocquigny. Mathesis IX.]

Ex. 12. If XYZ, $X'Y'Z'$ be any two transversals of the triangle ABC, show that the lines YZ', ZX', XY' will cut the sides BC, CA, AB in three collinear points.

109. Ex. 1. If the lines joining the vertices of a triangle ABC to any point cut the opposite sides in the points X, Y, Z, and if O be any arbitrary point, show that

$$\frac{\sin BOX}{\sin XOC}\cdot\frac{\sin COY}{\sin YOA}\cdot\frac{\sin AOZ}{\sin ZOB}=1.$$

We have $\quad BX.OC : XC.OB=\sin BOX : \sin XOC.$

Hence the theorem follows from § 94.

Ex. 2. If any straight line cut the sides of a triangle ABC in the points X, Y, Z, and if O be any arbitrary point, show that

$$\frac{\sin BOX}{\sin COX}\cdot\frac{\sin COY}{\sin AOY}\cdot\frac{\sin AOZ}{\sin BOZ}=1.$$

Ex. 3. If X, Y, Z be the points in which any straight line cuts the sides of the triangle ABC, and if O be any point, show that the pencil

$$O\{AX, BY, CZ\}$$

is in involution.

Ex. 4. The sides of the triangle ABC cut any straight line in the points P, Q, R; and X, Y, Z are three points on this straight line. If AX, BY, CZ are concurrent, show that

$$\frac{QX}{XR}\cdot\frac{RY}{YP}\cdot\frac{PZ}{ZQ}=1.$$

Show also that $\{PX, QY, RZ\}$ is a range in involution.

Ex. 5. If in the last example, AX, BY, CZ cut the sides BC, CA, AB in collinear points, show that

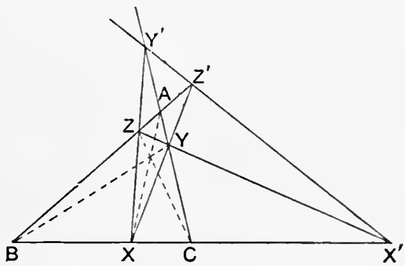
$$\frac{QX}{RX}\cdot\frac{RY}{PY}\cdot\frac{PZ}{QZ}=1.$$

Ex. 6. Prove the converse theorems of those in examples 1—5.

Ex. 7. If XYZ, $X'Y'Z'$ be any two transversals of the triangle ABC, and if YZ', $Y'Z$ meet in P; ZX', $Z'X$ in Q; and XY', $X'Y$ in R; show that AP, BQ, CR cut the sides BC, CA, AB in three collinear points.

Pole and Polar with respect to a triangle.

110. *If X, Y, Z be points on the sides of the triangle ABC, such that AX, BY, CZ are concurrent, the sides of the triangle XYZ will meet the sides of the triangle ABC in collinear points.*



Let YZ, ZX, XY meet BC, CA, AB respectively in the points X', Y', Z'.

Since X', Y, Z are collinear we have by § 104,

$$\frac{BX'}{CX'} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1.$$

But since AX, BY, CZ are concurrent, we have by § 94,

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

Therefore $$BX : XC = BX' : CX'.$$

Similarly, we shall have

$$CY : YA = CY' : AY',$$

and $$AZ : ZB = AZ' : BZ'.$$

Consequently,

$$\frac{BX'}{CX'} \cdot \frac{CY'}{AY'} \cdot \frac{AZ'}{BZ'} = \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

Hence, X', Y', Z' are collinear points.

Ex. 1. If the lines AO, BO, CO cut the sides of the triangle ABC in the points X, Y, Z; and if the points X', Y', Z' be the harmonic conjugate points of X, Y, Z with respect to B, C; C, A; and A, B; respectively, prove that :—

(i) The points X', Y', Z' are collinear.

(ii) The points X', Y, Z are collinear.

(iii) The lines AX, BY', CZ' are concurrent.

Ex. 2. If the inscribed circle of the triangle ABC touch the sides in the points X, Y, Z, show that the lines YZ, ZX, XY cut the sides BC, CA, AB in collinear points.

Ex. 3. If XYZ be any transversal of the triangle ABC, and if the lines AX, BY, CZ form the triangle PQR, show that the lines AP, BQ, CR are concurrent.

111. If the lines AX, BY, CZ meet in the point O, the line $X'Y'Z'$ (see figure § 110) is called the *polar* of the point O with respect to the triangle ABC; and the point O is called the *pole* of the line $X'Y'Z'$ with respect to the triangle.

Given any point O we can find its polar by joining AO, BO, CO, and then joining the points X, Y, Z in which these lines cut the sides of the triangle ABC. The lines YZ, ZX, XY will cut the corresponding sides of the triangle in the points X', Y', Z', which lie on the polar of O.

Given any straight line $X'Y'Z'$ to find its pole with respect to a triangle ABC; let P, Q, R be the vertices of the triangle formed by the lines AX', BY', CZ'. Then AP, BQ, CR will meet in a point (§ 110, Ex. 3) which will be the pole of the line $X'Y'Z'$.

Ex. If x denote the polar of the point O with respect to the triangle ABC, show that

$$(OBC) \,.\, Ax = (OCA) \,.\, Bx = (OAB) \,.\, Cx.$$

Special points connected with a triangle.

112. The lines drawn through the vertices of a triangle to bisect the opposite sides are called the *medians* of the triangle.

The medians of a triangle are concurrent (§ 96, Ex. 1). The point in which they intersect is called the *median point* of the triangle.

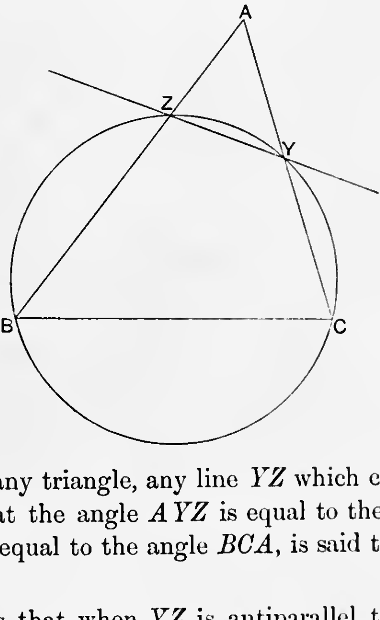
The isogonal conjugates of the medians with respect to the angles of a triangle are called the *symmedians* of the triangle. The point in which they intersect (§ 101) is called the *symmedian point.*

The median point of a triangle is also called the *centroid* of the triangle; but the name *median point* is preferred in geometry from the important connection of the point with its isogonal conjugate, the symmedian point.

The median point of a triangle is usually denoted by G, and the symmedian point by K.

Triangles which have the same median lines are called *co-median triangles;* and triangles which have the same symmedian lines are said to be *co-symmedian.*

113. In connection with the symmedian point it is convenient to define here what is meant by a line *antiparallel* to a side of a triangle.



If ABC be any triangle, any line YZ which cuts AC in Y and AB in Z, so that the angle AYZ is equal to the angle CBA, and the angle AZY equal to the angle BCA, is said to be *antiparallel* to the side BC.

It is obvious that when YZ is antiparallel to BC, the points Y, Z, B, C are concyclic; and that the line through A antiparallel to BC is the tangent at A to the circumcircle of ABC.

114. Ex. 1. If G be the median point of the triangle ABC, show that the areas (BGC), (CGA), (AGB) are equal.

Ex. 2. If K be the symmedian point of the triangle ABC, show that the areas (BKC), (CKA), (AKB) are in the ratio of the squares on BC, CA, and AB.

Ex. 3. If any circle be drawn through B and C cutting the sides AC, AB in the points M and N, show that AK will bisect MN.

Ex. 4. If the inscribed circle touch the sides of the triangle ABC in the points X, Y, Z, show that AX, BY, CZ will meet in the symmedian point of the triangle XYZ.

Ex. 5. If D, E, F be the feet of the perpendiculars from A, B, C to the opposite sides of the triangle ABC, show that the lines drawn from A, B, C to the middle points of EF, FD, DE are concurrent.

The point of concurrence is the symmedian point.

Ex. 6. Show that if lines be drawn through the symmedian point of a triangle antiparallel to the sides, the segments intercepted on them are equal.

Ex. 7. The perpendiculars from K on the sides of the triangle are proportional to the sides.

Ex. 8. If KX, KY, KZ be drawn perpendicular to the sides of the triangle, show that K is the median point of the triangle XYZ.

Ex. 9. If AD be drawn perpendicular to the side BC of the triangle ABC, show that the line joining the middle point of AD to the middle point of BC passes through the symmedian point of the triangle ABC.

Ex. 10. If from the symmedian (or median) point of a triangle, perpendiculars be drawn to the sides, the lines joining their feet are perpendicular to the medians (or symmedians) of the triangle.

Ex. 11. Show that if G be the median point, and K the symmedian point of the triangle ABC,

$$GA\,.\,KA\,.\,BC+GB\,.\,KB\,.\,CA+GC\,.\,KC\,.\,AB=BC\,.\,CA\,.\,AB.$$

[St John's Coll., 1886.]

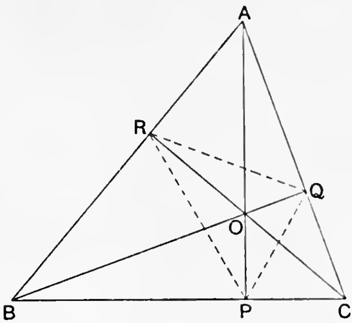
Ex. 12. Through a point P the lines XPY, $X'PZ$ are drawn parallel to the sides AB, AC of the triangle ABC, cutting the side BC in the points X, X' and the sides AC, AB in Y and Z. If the points X, X', Y, Z are concyclic show that the locus of the point P is a straight line.

Ex. 13. Any point P is taken on the line which bisects the angle BAC of a triangle internally, and PA', PB', PC' are drawn perpendicular to the sides of the triangle. Show that $A'P$ intersects $B'C'$ in a point on the median line which passes through A.

Ex. 14. The lines AA', BB', CC' connecting the vertices of two triangles ABC, $A'B'C'$ are divided in the points P, Q, R in the same ratio, $m : n$. Show that the median point of the triangle PQR divides the line joining the median points of the triangles ABC, $A'B'C'$ in the ratio $m : n$.

115. The perpendiculars from the vertices of a triangle on the opposite sides meet in a point (§ 96, Ex. 2), which is called the *orthocentre* of the triangle.

If ABC be the triangle, and O the orthocentre, it is evident from the figure that each of the four points A, B, C, O is the orthocentre of the triangle formed by the other three.



Ex. 1. Show that if AP, BQ, CR be the perpendiculars on the sides of the triangle ABC, QR will be antiparallel to BC.

Ex. 2. Show that the circles circumscribing the triangles BOC, COA, AOB, ABC are equal.

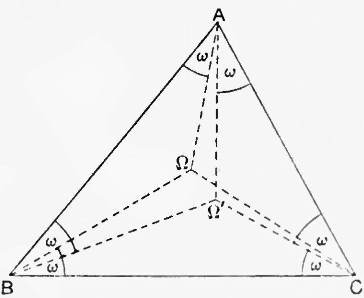
Ex. 3. Show that the triangles AQR, PBR, PQC are each similar to the triangle ABC.

Ex. 4. Show that AP bisects the angle QPR.

Ex. 5. If A, B, C, D be any four points on a circle, and if A', B', C', D' be the orthocentres of the triangles BCD, CDA, DAB, ABC, show that AA', BB', CC', DD' will be concurrent.

116. If ABC be any triangle, the lines AX, BY, CZ, drawn so as to make the angles BAX, ACY, CBZ equal, are concurrent (§ 100, Ex. 7). The point in which these lines intersect is called a *Brocard point* of the triangle ABC, and is usually denoted by Ω.

If Ω' be the point such that the angles $CA\Omega'$, $AB\Omega'$, $BC\Omega'$ are equal, Ω' is also called a Brocard point of the triangle ABC.



By § 100, Ex. 7, we see that each of the angles $BA\Omega$, $\Omega'AC$ is equal to ω, where

$$\cot\omega = \cot A + \cot B + \cot C.$$

The angle ω is called the *Brocard angle* of the triangle.

From § 101, it follows that the Brocard points Ω, Ω' are isogonal conjugate points with respect to the triangle.

117. Ex. 1. Show that the circle circumscribing the triangle $B\Omega C$ touches AC at C, and that the circle circumscribing the triangle $B\Omega' C$ touches AB at B.

This theorem gives a simple construction for finding Ω and Ω'.

Ex. 2. Show that the triangles $A\Omega' B$, $A\Omega C$ are similar.

Ex. 3. Show that the areas of the triangles $A\Omega B$, $C\Omega' A$ are equal.

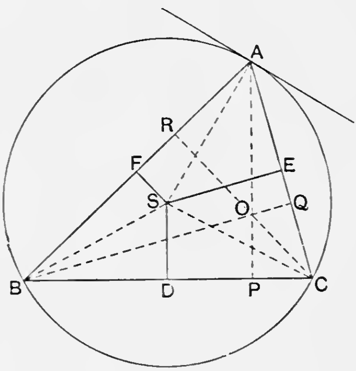
Ex. 4. Show that

$$A\Omega \,.\, B\Omega \,.\, C\Omega = A\Omega' \,.\, B\Omega' \,.\, C\Omega'.$$

The Circumcircle.

118. The circle which passes through the vertices of a triangle is called the *circumcircle* of the triangle; and the centre of this circle is called the *circumcentre*.

If ABC be the triangle, and D, E, F the middle points of the sides, the lines drawn through D, E, F perpendicular to the sides meet in the circumcentre (Euclid IV., Prop. 5).



Since the tangent at A makes the same angles with the lines AB, AC, as the side BC makes with AC, AB respectively (Euclid III., Prop. 32), it follows that the tangent at A is antiparallel to the side BC.

Since SA is perpendicular to the tangent at A, we see that SA is perpendicular to any line which is antiparallel to the side BC.

The angle ASB is double the angle ACB (Euclid III., Prop. 20), therefore the angle BAS is the complement of the angle ACB. Hence if AP be perpendicular to BC, the angle BAS is equal to the angle PAC. Thus AS and AP are isogonal conjugates with respect to the angle BAC. Hence the circumcentre and the orthocentre are isogonal conjugate points with respect to the triangle.

The circumcentre of a triangle is usually denoted by S, and the orthocentre by O.

119. Ex. 1. If D be the middle point of BC, show that $AO=2SD$.

Ex. 2. If AO meet the circumcircle in P', show that OP' is bisected by BC.

Ex. 3. Show that the line joining the circumcentre to the orthocentre passes through the median point of the triangle.

Ex. 4. Show that the circle which passes through the middle points of the sides of a triangle passes through the feet of the perpendiculars from the opposite vertices on the sides.

This follows from the fact that S and O are isogonal conjugate points (§ 102, Ex. 2).

Ex. 5. Show that if P, Q, R be the feet of the perpendiculars from A, B, C on the opposite sides of the triangle, then the perpendiculars from A, B, C to QR, RP, PQ respectively are concurrent.

Ex. 6. If from any point P on the circumcircle of the triangle ABC, PL, PM, PN be drawn perpendicular to PA, PB, PC respectively to meet BC, CA, AB in L, M, N; show that L, M, N lie on a straight line which passes through the circumcentre of the triangle. [St John's Coll., 1889.]

Ex. 7. If P be any point on the circumcircle of the triangle ABC, show that the isogonal conjugate point will be on the line at infinity.

Ex. 8. Perpendiculars are drawn to the symmedians of a triangle, at its angular points, forming another triangle. Show that the circumcentre of the former is the median point of the latter.

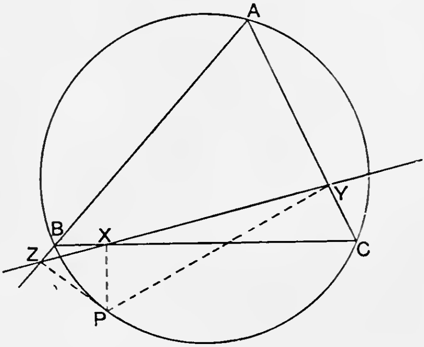
Ex. 9. If P be any point on the circumcircle of a triangle whose symmedian point is K, show that PK will cut the sides of the triangle in the points X, Y, Z so that

$$\frac{3}{PK}=\frac{1}{PX}+\frac{1}{PY}+\frac{1}{PZ}.$$

[d'Ocagne, *E. T.* Reprint, Vol. XLII., p. 26.]

120. *If from any point P on the circumcircle of the triangle ABC, PX, PY, PZ be drawn perpendicular to the sides, the points X, Y, Z will be collinear.*

Join ZX, YX. Then since the points P, X, Z, B are concyclic, the angle PXZ is the supplement of the angle ABP. And since P, Y, C, X are concyclic, the angle YXP is the supplement of the



angle YCP, and is equal to the angle ABP, because P, C, A, B are concyclic.

Hence the angles PXZ, YXP are supplementary; and therefore ZX, XY are in the same straight line.

The line XYZ is called the *Simson line* or the *pedal line* of the point P with respect to the triangle ABC.

121. Ex. 1. Show that if the feet of the perpendiculars drawn from a point P on the sides of a triangle be collinear, the locus of P is the circumcircle of the triangle.

Ex. 2. If O be the orthocentre of the triangle ABC, show that the Simson line of any point P on the circumcircle bisects the line OP.

Ex. 3. Show that if PQ be any diameter of the circumcircle, the Simson lines of P and Q are perpendicular. [Trinity Coll., 1889.]

Ex. 4. Show that the Simson line of any point P is perpendicular to the isogonal conjugate line to AP with respect to the angle BAC.

Ex. 5. If PL, PM, PN be the perpendiculars drawn from a point P on a circle to the sides BC, CA, AB of an inscribed triangle, and if straight lines Pl, Pm, Pn be drawn meeting the sides in l, m, n and making the angles LPl, MPm, NPn equal, when measured in the same sense, then the points l, m, n will be collinear. [Trinity Coll., 1890.]

Ex. 6. A triangle ABC is inscribed in a circle and the perpendiculars from A, B, C to the opposite sides meet the circle in A', B', C'; $B'E$, $C'F$ are drawn perpendicular to $C'A'$, $A'B'$ respectively, meeting AC', AB' in E and F. Show that the pedal line of the point A with respect to the triangle $A'B'C'$ bisects EF. [St John's Coll., 1890.]

Ex. 7. If P, Q be opposite extremities of a diameter of the circumcircle of a triangle, the lines drawn from P and Q perpendicular to their pedal lines respectively will intersect in a point R on the circle.

Show also that the pedal line of the point R will be parallel to PQ.

[Clare Coll., 1889.]

Ex. 8. If A, B, C, D be four points on a circle, prove that the pedal lines of each point with respect to the triangle formed by the other three meet in a point O.

If a fifth point E be taken on the circle, prove that the five points O belonging to the five groups of four points formed from A, B, C, D, E lie on a circle of half the linear dimensions. [Math. Tripos, 1886.]

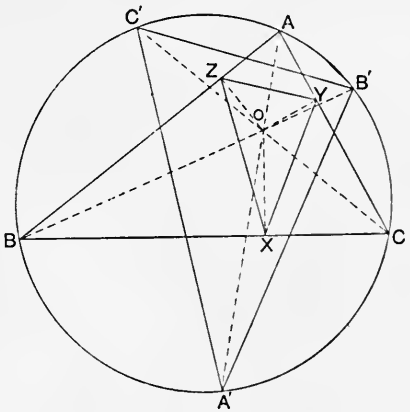
Ex. 9. If A, B, C, D be any four points on a circle, show that the projections of any point O on the circle, on the Simson lines of the point O with respect to the triangles BCD, CDA, DAB, ABC, lie on a straight line.

If this line be called the Simson line of the point O with respect to the tetrastigm $ABCD$, and if any fifth point E be taken on the circle, show that the projections of O on the Simson lines of the tetrastigms $BCDE$, $CDEA$, $DEAB$, $EABC$, $ABCD$ also lie on a straight line.

Show that the theorem may be extended.

[E. M. Langley, *E. T.* Reprint, Vol. LI., p. 77.]

122. Ex. 1. If the lines connecting the vertices of a triangle ABC to any point O cut the circumcircle in the points A', B', C', and if OX, OY, OZ be the perpendiculars on the sides of the triangle; then the triangles $A'B'C'$, XYZ are similar.



It is easy to prove that the angles $B'A'C'$, YXZ are each equal to the difference of the angles BOC, BAC. Hence the theorem follows at once.

Since the triangles $B'OC'$, COB are similar, we have

$$B'C' : OB' = BC : CO.$$

Therefore $$\frac{AO \cdot BC}{B'C'} = \frac{AO \cdot CO \cdot BO}{BO \cdot OB'}.$$

Hence, if S be the centre of the circumcircle, and R its radius, and if we denote the angles BOC, COA, AOB by α, β, γ, we have

$$\frac{AO \cdot BC}{\sin(\alpha - A)} = \frac{BO \cdot CA}{\sin(\beta - B)} = \frac{CO \cdot AB}{\sin(\gamma - C)} = \frac{2R \cdot AO \cdot BO \cdot CO}{R^2 - OS^2} \quad \text{.........(1).}$$

Again, since Y, Z, O, A are concyclic we have $YZ = OA \sin A$. Hence, if ρ be the radius of the circle XYZ, we have

$$2\rho = \frac{YZ}{\sin(\alpha - A)} = \frac{OA \cdot \sin A}{\sin(\alpha - A)}.$$

Therefore $$4\rho R = \frac{OA \cdot BC}{\sin(\alpha - A)} = \frac{OB \cdot CA}{\sin(\beta - B)} = \frac{OC \cdot AB}{\sin(\gamma - C)} \quad \text{..............(2).}$$

From (1) and (2) we have

$$2\rho = \frac{AO \cdot BO \cdot CO}{R^2 - OS^2} \quad \text{.............................. (3).}$$

Ex. 2. Show that the point O for the triangle XYZ corresponds to the isogonal conjugate point of O for the triangle $A'B'C'$.

Ex. 3. If O and O' be isogonal conjugate points with respect to the triangle ABC, and if S be the circumcentre, show that

$$\frac{AO \cdot BO \cdot CO}{AO' \cdot BO' \cdot CO'} = \frac{R^2 - OS^2}{R^2 - O'S^2}.$$

By § 102, Ex. 2, we know that if perpendiculars be drawn from O and O' to the sides of the triangle ABC, their feet lie on the same circle. Hence this result follows from Ex. 1, (3).

Ex. 4. Show that the Brocard points Ω, Ω' are equidistant from S.

See § 117, Ex. 3.

Ex. 5. If O and O' be a pair of isogonal conjugate points with respect to a triangle ABC, show that

$$AO \cdot AO' \cdot BC + BO \cdot BO' \cdot CA + CO \cdot CO' \cdot AB = BC \cdot CA \cdot AB.$$

Ex. 6. If in Ex. 1 the point O be the orthocentre of the triangle ABC, show that $B'C'$ is antiparallel to the side BC.

Ex. 7. If K be the symmedian point of the triangle ABC, and if AK, BK, CK meet the circumcircle of the triangle in A', B', C', show that the triangles ABC, $A'B'C'$ are co-symmedian.

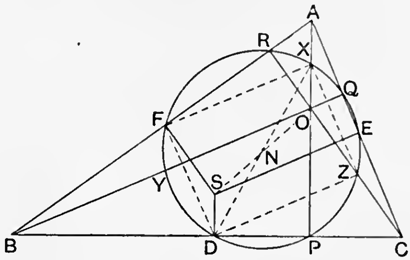
Let KX, KY, KZ be drawn perpendicular to the sides of ABC. Then K is the median point of the triangle XYZ (§ 114, Ex. 8). Therefore (Ex. 2) K is the symmedian point of the triangle $A'B'C'$.

It is evident that the medians of the triangle $A'B'C'$ are proportional to KX, KY, KZ; and therefore they are proportional to the sides of the triangle ABC.

The Nine-Point circle.

123. If S be the circumcentre, and O the orthocentre, of the triangle ABC; D, E, F the middle points of the sides; P, Q, R

the feet of the perpendiculars from the vertices on the opposite sides; and X, Y, Z the middle points of AO, BO, CO; the nine points D, E, F, P, Q, R, X, Y, Z lie on the same circle, which is called the *nine-point* circle of the triangle.



Since S and O are isogonal conjugate points with respect to the triangle ABC, it follows that a circle can be drawn through the points P, Q, R, D, E, F (§ 102, Ex. 2). Again, since A is the orthocentre of the triangle BOC (§ 114), it follows that the points P, Q, R, Y, Z, D lie on the same circle. Similarly, since B is the orthocentre of the triangle AOC, it follows that X lies on the circle PQR.

Since S and O are isogonal conjugate points, the centre of the nine-point circle N will be the middle point of SO (§ 102, Ex. 2).

124. The theorem of the last article may be proved in a more elementary manner as follows. It is easy to show that $XZDF$ and $XEDY$ are rectangles, having the common diagonal DX. And since XPD, YQE, ZRF are right angles, it follows at once that the nine points X, Y, Z, D, E, F, P, Q, R lie on a circle, whose centre is the middle point of OS.

125. Ex. 1. Show that the diameter of the nine-point circle is equal to the radius of the circumcircle.

Ex. 2. The nine-point circle of the triangle ABC is also the nine-point circle of each of the triangles BCO, CAO, ABO.

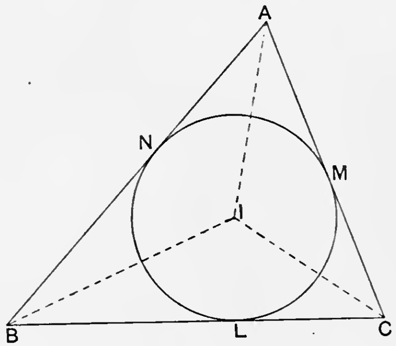
Ex. 3. Show that if P be any point on the circumcircle of a triangle, OP is bisected by the nine-point circle.

Ex. 4. Show that the Simson lines of the extremities of any diameter of the circumcircle of a triangle intersect at right angles on the nine-point circle of the triangle. [Trin. Coll., 1889.]

Ex. 5. If D, E, F be the middle points of the sides of the triangle ABC, show that the nine-point circles of the triangles AEF, BFD, CDE touch the nine-point circle of the triangle DEF at the middle points of EF, FD, DE respectively.

The inscribed and escribed circles.

126. The internal bisectors of the angles of a triangle are concurrent (§ 100, Ex. 1). It is evident that the point in which they meet is equidistant from the sides of the triangle. Therefore the circle which has this point for centre and which touches one side will touch the other sides (Euclid IV., Prop. 4). This circle is called the *inscribed circle,* or briefly the *in-circle.* Its centre is often called the *in-centre.*



If L, M, N be the points of contact of the sides, we have $AM = AN$, $BL = BN$, $CL = CM$.

Hence, denoting the lengths of the sides by a, b, c, and the perimeter by $2s$, we have

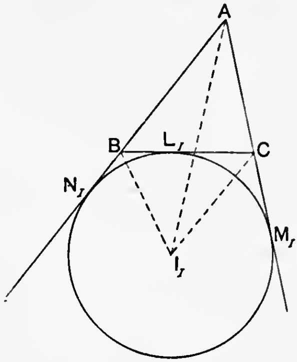
$$\begin{aligned} AM &= AN = s - a, \\ BL &= BN = s - b, \\ CL &= CM = s - c. \end{aligned}$$

127. The internal bisector of the angle BAC, and the external bisectors of the angles ABC, ACB, are concurrent. Let the point in which they meet be denoted by I_1. This point is the centre of a circle which can be drawn to touch the sides of the triangle, but it is on the side of BC remote to A. This circle is called an *escribed circle.* To distinguish it from the other escribed circles it is often called the *A-escribed circle.*

If L_1, M_1, N_1 be the points of contact of the sides with this circle, it follows at once that

$$\begin{aligned} AM_1 &= AN_1 = s, \\ BL_1 &= BN_1 = s - c, \\ CL_1 &= CM_1 = s - b. \end{aligned}$$

Similarly, if the internal bisector of the angle ABC meet the

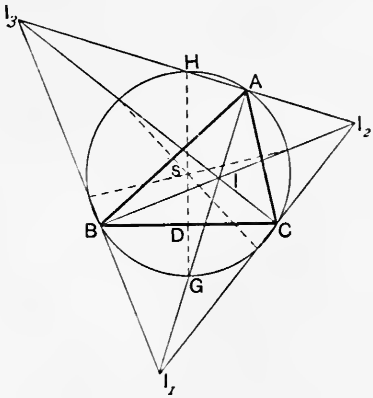


external bisectors of the other angles in the point I_2, I_2 will be the centre of the B-escribed circle. And, if the internal bisector of the angle ACB meet the external bisectors of the other angles in I_3, I_3 will be the centre of the C-escribed circle.

128. Ex. 1. Show that the circumcircle of the triangle ABC is the nine-point circle of the triangle $I_1I_2I_3$.

Ex. 2. Show that if r, r_1, r_2, r_3 be the radii of the inscribed and escribed circles, and R the radius of the circumcircle,

$$r_1+r_2+r_3-r=4R.$$



Let D be the middle point of BC, and let SD meet the circumcircle in G and H. It follows from Ex. 1, that G is the middle point of II_1, and H the middle point of I_2I_3. Hence $2HD=r_2+r_3$, $2DG=r_1-r$. Therefore

$$4R=2HG=r_1+r_2+r_3-r.$$

Ex. 3. Show that $$SI^2 = R^2 - 2Rr.$$

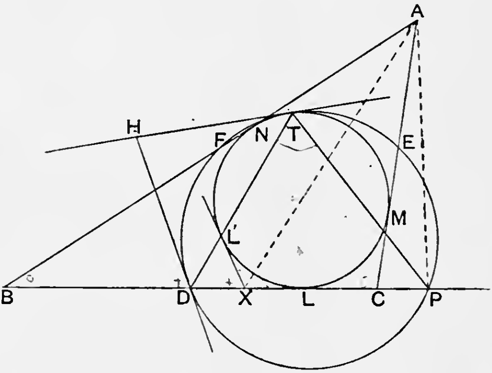
If IM be drawn perpendicular to AC, it is easy to show that the triangles AIM, HCG are similar. Therefore $IM : AI = GC : GH$. But $GC = GI$. Hence $AI . IG = IM . GH$, that is $R^2 - SI^2 = 2Rr$.

Ex. 4. Show that $$SI_1^2 = R^2 + 2Rr_1.$$

Ex. 5. If I be the in-centre of the triangle ABC, and if AI cut BC in X and the circumcircle in G, show that
$$GI^2 = GX . GA.$$

Ex. 6. If D be the middle point of BC, P the foot of the perpendicular from A, and L the point of contact of the inscribed circle with BC, show that
$$DL^2 = DX . DP.$$

Ex. 7. Show that the nine-point circle of a triangle touches the inscribed circle.



Let L, M, N be the points of contact of the inscribed circle with the sides of the triangle. Let D be the middle point of BC, and P the foot of the perpendicular from A. Let the line joining A to the centre of the inscribed circle cut BC in X, and let XL' be the other tangent drawn from X to this circle. Join DL', and let it cut the inscribed circle in T. Then T is a point on the nine-point circle, and the two circles will touch at T.

The tangent to the nine-point circle at D, DH suppose, is parallel to XL', since each of the angles HDB, $L'XB$ is equal to the difference of the angles CBA, ACB.

By Ex. 6, we have $DX . DP = DL^2 = DL' . DT$. Hence the points P, X, L', T are concyclic, and therefore the angle DTP is equal to the angle $L'XD$, that is to the angle HDB. Therefore T is a point on the nine-point circle.

Also a line through T, making with TD an angle HTD equal to TDH, is a tangent to both circles, proving that the circles touch at T.

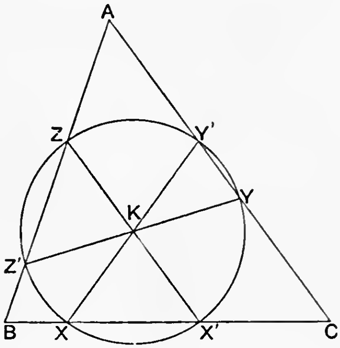
This proof was given by Mr J. Young in the *Educational Times* (see *E. T. Reprint*, Vol. LI., p. 58).

Ex. 8. Show that the nine-point circle of a triangle touches each of the escribed circles.

The Cosine circle.

129. If through the symmedian point of a triangle lines be drawn antiparallel to the sides, the six points in which they intersect the sides lie on a circle, which is called the *cosine circle* of the triangle.

The centre of this circle is the symmedian point.



Let K be the symmedian point of the triangle ABC, and let YKZ', ZKX', XKY' be drawn antiparallel to the sides BC, CA, AB respectively.

The angles KXX', $KX'X$ are each equal to the angle BAC; therefore $KX = KX'$. Similarly we have $KY = KY'$, and

$$KZ = KZ'.$$

But AK bisects all lines antiparallel to the side BC, therefore $KY = KZ'$. Similarly, $KZ = KX'$, and $KX = KY'$.

Hence the six points X, X', Y, Y', Z, Z' lie on a circle whose centre is K.

It is evident that the segments XX', YY', ZZ' are proportional to the cosines of the opposite angles: hence the name *cosine circle*. The cosine circle is the only circle which possesses the property of cutting the sides of the triangle at the extremities of three diameters.

130. Ex. 1. Show that the triangles YZX, $Z'X'Y'$ are each similar to the triangle ABC.

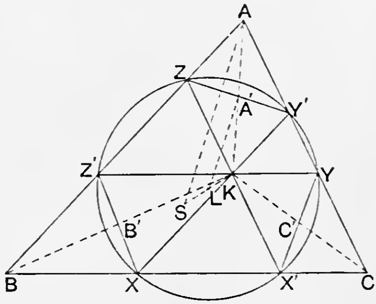
Ex. 2. If YZ', ZX', XY' be any three diameters of a circle, show that the circle is the cosine circle of the triangle formed by the lines XX', YY', ZZ'.

Ex. 3. If the tangents at B and C to the circumcircle of the triangle ABC intersect in K_1, show that the circle whose centre is K_1 and which passes through B and C will cut AB, AC in two points which are extremities of a diameter.

This circle has been called an *ex-cosine circle*.

The Lemoine circle.

131. If through the symmedian point of a triangle, lines be drawn parallel to the sides, the six points in which they intersect the sides lie on a circle, which is called the *Lemoine circle* of the triangle.



Let K be the symmedian point of the triangle ABC, and let YKZ', ZKX', XKY' be drawn parallel to the sides BC, CA, AB respectively. Let S be the circumcentre of the triangle, and L the middle point of SK.

Let AK meet $Y'Z$ in A'. Then since $KY'AZ$ is a parallelogram, A' is the middle point of AK.

Hence $$SA = 2LA'.$$

Again, AK bisects ZY'; therefore ZY' is antiparallel to the side BC, and therefore (§ 118) SA is perpendicular to ZY'. Hence LA', which is parallel to SA, is perpendicular to ZY'.

Again, since ZK is parallel to AC, and ZY' is antiparallel to BC, it follows that ZY' is equal to the radius of the cosine circle.

Hence we have $$4LY'^2 = R^2 + \rho^2,$$

where R is the radius of the circumcircle, and ρ the radius of the cosine circle.

It follows by symmetry that X, X', Y, Y', Z, Z' lie on a circle whose centre is L, the middle point of SK.

132. Ex. 1. In the figure, show that the chords $Y'Z$, $Z'X$, $X'Y$ are equal.

Ex. 2. If the Lemoine circle cut BC in X and X', show that

$$BX : XX' : X'C = BA^2 : BC^2 : CA^2.$$

Ex. 3. Show that

$$XX' : YY' : ZZ' = BC^3 : CA^3 : AB^3.$$

On account of this property the circle has been called the *triplicate ratio* circle by Mr Tucker.

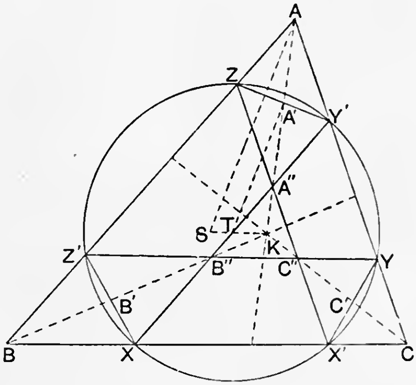
Ex. 4. Show that the triangles ZXY, $Y'Z'X'$ are each similar to the triangle ABC.

Ex. 5. If SD be drawn perpendicular to YZ', show that $Z'D$ is equal to KY.

133. If on the line SK joining the circumcentre of a triangle ABC to its symmedian point any point T be taken, and if points A', B', C' be taken on the lines KA, KB, KC respectively, so that

$$KA' : KB' : KC' : KT = KA : KB : KC : KS,$$

then lines drawn through A', B', C' antiparallel to BC, CA, AB,



will meet the sides of the triangle in six points which lie on a circle.

The system of circles obtained by taking different points T on the line KS is known as *Tucker's system of circles.*

The proof that the six points lie on a circle is very similar to that given in § 131. It is easy to see that TA' is perpendicular to ZY', and that TA', TB', TC' are proportional to SA, SB, SC respectively. Also, the chords $Y'Z$, $Z'X$, $X'Y$ are evidently equal, and are proportional to the radius of the cosine circle. Hence, the six points lie on a circle, whose centre is T.

Tucker's circles include as particular cases:—

(i) The circumcircle, when T coincides with S.

(ii) The cosine circle, when T coincides with K.

(iii) The Lemoine circle, when T is the middle point of SK.

Ex. 1. Show that the lines YZ', ZX', XY' are parallel to the sides of the triangle ABC.

Ex. 2. Show that the vertices of the triangle formed by the sides YZ', ZX', XY' lie on the symmedian lines AK, BK, CK.

Ex. 3. Show that the vertices of the triangle formed by the lines $Y'Z$, $Z'X$, $X'Y$ lie on the symmedian lines AK, BK, CK.

Ex. 4. If through any point A' on the symmedian AK, lines be drawn parallel to the sides AB, AC, meeting the symmedians BK, CK in the points B', C'; show that $B'C'$ will be parallel to BC, and that the sides of the triangle $A'B'C'$ will meet the sides of the triangle ABC in six points which lie on a Tucker circle.

Ex. 5. If through any point A'' on the symmedian AK, lines be drawn antiparallel to the sides AB, AC, meeting the symmedians BK, CK in the points B'', C''; show that $B''C''$ will be antiparallel to BC, and that the sides of the triangle $A''B''C''$ will meet the non-corresponding sides of the triangle ABC in six points which lie on a Tucker circle.

Ex. 6. From the vertices of the triangle ABC, perpendiculars AD, BE, CF are drawn to the opposite sides; and EX, FX' are drawn perpendicular to BC; FY, DY' perpendicular to CA; and DZ, EZ' perpendicular to AB. Show that the six points X, X', Y, Y', Z, Z' are concyclic.

It is easy to show that $Y'Z$ passes through the middle points of the sides DE, DF of the triangle DEF. These points obviously lie on the symmedians BK, CK. Hence, by Ex. 5, the points X, X', Y, Y', Z, Z' lie on a Tucker circle.

This particular Tucker circle is usually called *Taylor's circle*. It was first mentioned in a paper by Mr H. M. Taylor (*Proceedings of the London Mathematical Society*, Vol. xv.).

Ex. 7. Show that the centre of Taylor's circle is the in-centre of the triangle formed by the middle points of the triangle DEF.

The Brocard circle.

134. The circle whose diameter is the line joining the circumcentre of a triangle to the symmedian point is called the *Brocard circle* of the triangle.

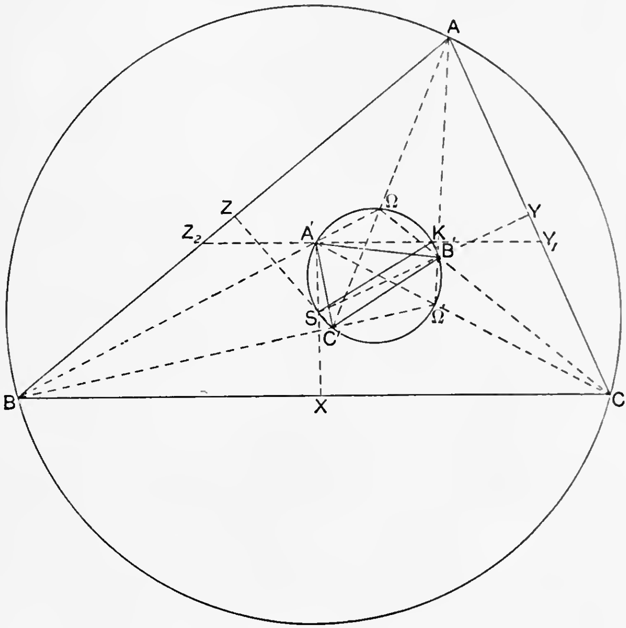
Let S be the circumcentre, and K the symmedian point of the triangle ABC. Draw SX, SY, SZ perpendicular to the sides BC, CA, AB, and let them meet the circle described on SK as diameter in the points A', B', C'.

The triangle $A'B'C'$ is called *Brocard's first triangle*.

Let BA', CB' meet in Ω. We shall find that Ω is one of the Brocard points (§ 116) and lies on the Brocard circle.

The perpendiculars from K on the sides of the triangle ABC are proportional to those sides (§ 114, Ex. 7); that is

$$A'X : B'Y = BC : CA.$$



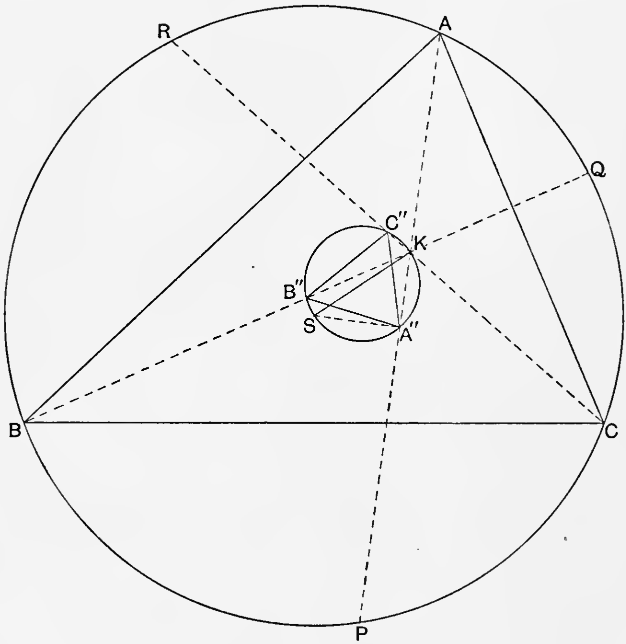
Therefore, since the angles BXA', CYB' are right angles, the triangles BXA', CYB' are similar. Therefore the angle $BA'X$ is equal to the angle $CB'Y$, that is the angle $\Omega B'S$. Therefore the point Ω lies on the circle circumscribing the triangle $A'SB'$. Similarly we can show that BA' meets AC' on the Brocard circle, that is in the point Ω.

Thus the lines AC', BA', CB' are concurrent. And since the triangles XBA', YCB', ZAC' are similar, the angles ΩAB, ΩBC, ΩCA are equal. Hence Ω is one of the Brocard points (§ 116).

Similarly we may show that AB', BC', CA' intersect on the Brocard circle in the other Brocard point.

Hence *if Ω and Ω' be the Brocard points, defined as in* § 116, *and if $A\Omega$, $B\Omega$, $C\Omega$ cut $B\Omega'$, $C\Omega'$, $A\Omega'$ respectively in the points C', A', B', the five points Ω, Ω', A', B', C' lie on a circle whose diameter is SK.*

If AK, BK, CK meet the Brocard circle in the points A'', B'', C'', the triangle $A''B''C''$ is called *Brocard's second triangle.*



Let the symmedian lines AK, BK, CK be produced to meet the circumcircle of the triangle ABC in the points P, Q, R. Then since SA'' is perpendicular to AK, it follows that A'' is the middle point of AP.

135. Ex. 1. Show that Brocard's first triangle is similar to the triangle ABC.

Ex. 2. Show that if KA', KB', KC' meet the sides of the triangle ABC in the points X_1, X_2; Y_1, Y_2; Z_1, Z_2; the sides of the triangle $Z_1X_1Y_1$ are parallel to $A\Omega$, $B\Omega$, $C\Omega$; and the sides of the triangle $Y_2Z_2X_2$ are parallel to $A\Omega'$, $B\Omega'$, $C\Omega'$.

Ex. 3. Show that the lines AA', BB', CC' are concurrent.

Since the Lemoine circle which passes through X_1, X_2, Y_1, Y_2, Z_1, Z_2, is concentric with the Brocard circle, it follows that A' and K are isotomic conjugates with respect to Y_1 and Z_2. Hence it follows that AA', BB', CC' will meet in the point which is the isotomic conjugate of K with respect to the triangle ABC.

Ex. 4. Show that Ω and K are the Brocard points of the triangle $Z_1X_1Y_1$; and that Ω' and K are the Brocard points of the triangle $Y_2Z_2X_2$.

Ex. 5. Show that

$$X_1X_2 : X_2Y_1 = \sin(A-\omega) : \sin\omega.$$

Ex. 6. Show that the line $\Omega\Omega'$ is perpendicular to SK.

Ex. 7. Show that the perpendiculars from the vertices of the triangle ABC on the corresponding sides of Brocard's first triangle are concurrent, and that their point of concurrence lies on the circumcircle of ABC.

The point in which these perpendiculars meet is called Tarry's point.

Ex. 8. Show that the Simson-line corresponding to Tarry's point is perpendicular to SK.

Ex. 9. Show that the lines drawn through the vertices of a triangle ABC parallel to the corresponding sides of the Brocard's first triangle intersect in a point on the circumcircle of ABC.

Ex. 10. Show that the point of concurrence in the last case is the opposite extremity of the diameter of the circumcircle which passes through Tarry's point.

Ex. 11. If the symmedian lines of the triangle ABC cut the circumcircle in the points P, Q, R, show that the triangles ABC, PQR have the same symmedian point, and the same Brocard circle.

Ex. 12. If $A'B'C'$ be the first Brocard triangle, and K the symmedian point, of the triangle ABC, show that the areas $(A'BC)$, $(AC'C)$, (ABB'), are each equal to the area (KBC).

Ex. 13. Show that the median point of the triangle $A'B'C'$ coincides with the median point of the triangle ABC.

If G' denote the median point of the triangle $A'B'C'$, we have (§ 36, Ex. 4),

$$3(G'BC) = (A'BC) + (B'BC) + (C'BC).$$

Therefore by the theorem of Ex. 12,

$$3(G'BC) = (KBC) + (ABK) + (AKC)$$
$$= (ABC).$$

Therefore G' coincides with the median point of the triangle ABC.

CHAPTER VII.

RECTILINEAR FIGURES.

Definitions.

136. In Euclid, a plane rectilinear figure is defined to be a figure bounded by straight lines; that is to say, a rectilinear figure is regarded as an area. Such a figure has as many sides as vertices. But in modern geometry, figures are regarded as 'systems of points' or as 'systems of straight lines.' In the present chapter we propose to consider the properties of figures consisting of finite groups of points, or of finite groups of lines. And such figures we shall call *rectilinear* figures.

The simplest rectilinear figure is that defined by three points, or by three straight lines. It is easy to see that three points may be connected by three lines, so that to have given a system of three points is equivalent to having given a system of three lines. We may therefore use the name *triangle* for either figure without ambiguity. Now let us consider the case of a figure consisting of four points. Four points may evidently be connected by six straight lines. And similarly, in the case of a figure consisting of four lines, we shall have six points of intersection. It is obvious that although four lines may be considered as a special case of a figure consisting of six points, six points will in general be connected by fifteen straight lines.

It is evident from these considerations that it will be convenient to use names for rectilinear figures which will distinguish figures consisting of points from figures consisting of straight lines. Thus, a system of four points is often called a *quadrangle*, and a system of four lines a *quadrilateral.* The latter name however is objectionable from the fact that it is commonly used to mean an area, and to avoid confusion it is customary to speak of a *complete quadrilateral* when the geometrical figure consisting of four lines is meant. But instead of these names it is preferable to use the terms *tetrastigm* and *tetragram* for the two kinds of figures, as these names are more concise. For figures consisting of any number of points we shall use the name *polystigm;* and for figures consisting of any number of straight lines, the name *polygram.*

137. In the case of a *polystigm,* the primary points are called *vertices;* and the lines joining them are called *connectors.* The connectors of a polystigm will in general intersect in certain points other than the vertices. Such points are called *centres.*

If a polystigm consist of n points, a set of n connectors may be selected in several ways so that two and not more than two pass through each of the n vertices: such a set of connectors will be called a *complete set of connectors.* For instance in the case of a tetrastigm, if A, B, C, D be the vertices, we shall have three complete sets of connectors, viz. AB, BC, CD, DA; AB, BD, DC, CA; and AC, CB, BD, DA.

In the case of a tetrastigm, it is often convenient to use the word *opposite.* Thus, in the tetrastigm $ABCD$ the connector CD is said to be *opposite* to the connector AB; and AB, CD are called a *pair of opposite connectors.* It is evident that the six connectors of a tetrastigm consist of three pairs of opposite connectors.

In the case of a polystigm, consisting of more than four vertices, the word *opposite* as applied to a pair of connectors can only be used in reference to a *complete set of connectors*, and then only when the number of vertices is even. If the vertices of the polystigm be $A_1, A_2, A_3, \dots A_{2n}$, the pair of connectors A_1A_2, $A_{n+1}A_{n+2}$ may be called *opposite connectors* of the complete set, A_1A_2, $A_2A_3, \dots A_rA_{r+1}, \dots A_nA_1$. In the case of the tetrastigm $ABCD$, it is obvious that AB and CD are opposite connectors in each of the two complete sets in which they occur; but in the case of the hexastigm $ABCDEF$, AB will occur as a member of twenty-four complete sets of connectors, and in only four of these sets is AB opposite to DE.

Again, in the case of a polystigm of $2n$ points, it is sometimes necessary to consider a group of n connectors which are such that one, and only one, passes through each of the vertices. Such a group of connectors may be called a *set of connectors.* If two sets of connectors together make up a complete set of connectors, the two sets may be called *complementary* sets. It is obvious that any particular set will have several complementary sets. For instance in the case of a hexastigm $ABCDEF$, the set of connectors AB, CD, EF will be complementary to eight sets.

Ex. 1. Show that a polystigm of n points has $\frac{1}{2}n(n-1)$ connectors, and $\frac{1}{8}n(n-1)(n-2)(n-3)$ centres.

Ex. 2. Show that a complete set of connectors of a polystigm of n points may be selected in $\frac{1}{2}(n-1)\,!$ ways.

Ex. 3. Show that a set of connectors of a polystigm of $2n$ points may be selected in $1 . 3 . 5 \ldots (2n-1)$ ways.

Ex. 4. Show that any set of connectors of a polystigm of $2n$ points has $2^n(n-1)\,!$ complementary sets.

138. In the case of a *polygram,* the points of intersection of the primary lines are called *vertices* of the figure. The vertices may be connected by certain lines other than those which determine the figure. These lines are called *diagonals.*

A group of vertices of a polygram which are such that two and not more than two lie on each of the lines of the figure, is called a *complete set of vertices.* And when the polygram consists of an even number of lines, the word *opposite* may be applied to a pair of vertices in the same way as in the case of a pair of connectors of a polystigm. Thus a tetragram will have three pairs of opposite vertices.

In the case of a polygram of $2n$ lines, a group of n vertices such that one, and only one, vertex lies on each line of the figure, is called a *set of vertices.* And any two sets which together make up a complete set may be called *complementary sets.*

Ex. 1. A polygram of n lines has $\frac{1}{2}n(n-1)$ vertices, and

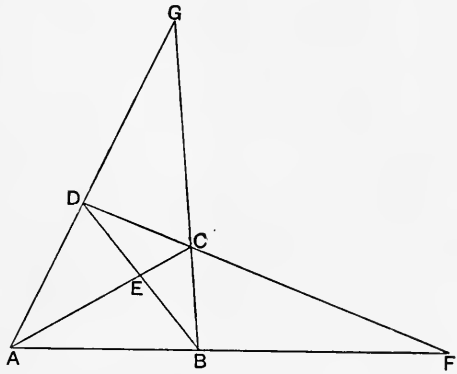
$$\tfrac{1}{8}n(n-1)(n-2)(n-3) \text{ diagonals.}$$

Ex. 2. Show that a complete set of vertices of a polygram of n lines may be selected in $\frac{1}{2}(n-1)\,!$ ways.

Ex. 3. Show that a set of vertices of a polygram of $2n$ lines may be selected in $1 . 3 . 5 \ldots (2n-1)$ ways.

Properties of a Tetrastigm.

139. A system of four points, no three of which are collinear, is called a *tetrastigm*. If these points are joined we have six connectors, or rather three pairs of opposite connectors. Each pair of opposite connectors intersect in a centre, so that there are three centres.



Let A, B, C, D be any four points, and let the connectors AC, BD meet in E, the connectors AB, CD in F, and the connectors AD, BC in G. Then E, F, G are the centres of the tetrastigm $ABCD$.

The triangle EFG is called the *central triangle* of the tetrastigm.

140. Ex. 1. If X, X' be the middle points of AC, BD; Y, Y' the middle points of AB, CD; and Z, Z' the middle points of AD, BC; show that the lines XX', YY', ZZ' are concurrent, and bisect each other.

Ex. 2. If $ABCD$ be a tetrastigm, and if AB cut CD in F, and AD cut BC in G, show that

$$FA \,.\, FC : FB \,.\, FD = GA \,.\, GC : GB \,.\, GD.$$

This result follows at once by considering GCB as a transversal of the triangle FAD, and GDA as a transversal of the triangle FBC.

Ex. 3. If A, B, C, D be any four points in a plane, show that

$$AC^2 \,.\, BD^2 = AB^2 \,.\, CD^2 + AD^2 \,.\, BC^2 - 2AB \,.\, BC \,.\, CD \,.\, DA \cos\omega,$$

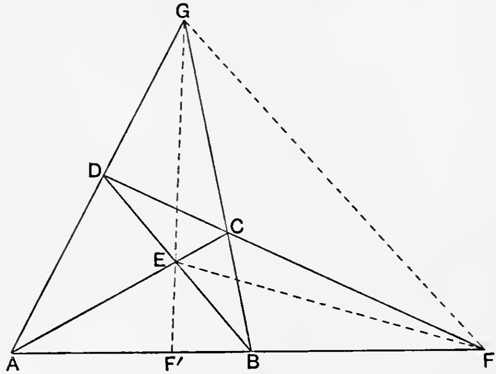
where ω is the difference of the angles BAD, BCD.

Ex. 4. If any straight line cut the connectors AB, BC, CD, DA in the points X, Y, X', Y' respectively, show that

$$\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CX'}{X'D} \cdot \frac{DY'}{Y'A} = 1.$$

Ex. 5. Show that the bisectors of the angles of a triangle are the six connectors of a tetrastigm.

141. *Any pair of opposite connectors of a tetrastigm are harmonically conjugate with respect to the sides of the central triangle which meet at their point of intersection.*



Let $ABCD$ be the tetrastigm, EFG its central triangle; and let GE meet AB in F'.

Since AC, BD, GE are concurrent, we have (§ 94)

$$\frac{AF'}{F'B} \cdot \frac{BC}{CG} \cdot \frac{GD}{DA} = 1.$$

Also since FCD is a transversal of the triangle GAB (§ 104),

$$\frac{AF}{BF} \cdot \frac{BC}{GC} \cdot \frac{GD}{AD} = 1.$$

Therefore $$AF' : F'B = AF : BF;$$

that is, $\{FF', AB\}$ is a harmonic range.

Therefore $G\{EF, AB\}$ is a harmonic pencil, and AD, BC are harmonic conjugate rays with respect to GE, GF.

The theorem may also be stated thus: *The line joining any two vertices of a tetrastigm is divided harmonically in the centre through which it passes, and in the point of intersection with the line joining the other two centres.*

If we suppose the line FG to be at infinity, then the four points A, B, C, D are the vertices of a parallelogram; and since E is the harmonic conjugate of the point in which AC intersects FG, with respect to the points A, C, it follows that E is the middle point of AC. Thus the theorem of this article is a generalisation of the theorem:—

The diagonals of a parallelogram bisect each other.

142. Ex. 1. If AB, CD meet in F, and if through F a line be drawn cutting AC, BD in P and P', AD, BC in Q and Q', and EG in F', show that $\{FF', PQ, P'Q'\}$ will be a range in involution.

Ex. 2. The centres of the tetrastigm $ABCD$ are E, F, G; FG meets AC in X and BD in X'; GE meets AB in Y and CD in Y'; and EF meets AD in Z and BC in Z'. Show that YZ' and ZY' pass through X, and YZ, $Y'Z'$ through X'.

Ex. 3. In the same figure show that

$$\frac{AY}{YB}\cdot\frac{BZ'}{Z'C}\cdot\frac{CY'}{Y'D}\cdot\frac{DZ}{ZA}=1.$$

Ex. 4. If $ABCD$ be any tetrastigm, and if from any point in AC two straight lines be drawn, one meeting AB, BC in X and Y respectively, and the other meeting CD, AD in X' and Y' respectively; show that

$$\frac{AX}{XB}\cdot\frac{BY}{YC}\cdot\frac{CX'}{X'D}\cdot\frac{DY'}{Y'A}=1.$$

Ex. 5. If four points X, Y, X', Y' be taken on the connectors AB, BC, CD, DA respectively of a tetrastigm, such that

$$\frac{AX}{XB}\cdot\frac{BY}{YC}\cdot\frac{CX'}{X'D}\cdot\frac{DY'}{Y'A}=1,$$

show that XY, $X'Y'$ will intersect on AC, and that XY', $X'Y$ will intersect on BD.

Ex. 6. The connectors AB, CD of the tetrastigm $ABCD$ meet in F, and the connectors AD, BC meet in G. Through F a straight line is drawn meeting AD and BC in Y and Y', and through G a line is drawn meeting AB and CD in X and X'. Show that XY, $X'Y'$, and BD are concurrent; and that XY', $X'Y$, and AC are concurrent.

Ex. 7. The mid-points of the perpendiculars drawn from A, B, C to the opposite sides of the triangle ABC are P, Q, R; and D, E, F are the mid-points of BC, CA, AB.

If the sides of the triangle PQR intersect the corresponding sides of the triangle DEF in the points L, M, N, show that the pencils $A\{BC, PL\}$, $B\{CA, QM\}$, $C\{AB, RN\}$, are harmonic, and that the points L, M, N are collinear. [Sarah Marks, *E. T.* Reprint, Vol. XLVIII., p. 121.]

143. Let $ABCD$ be any tetrastigm, and let any straight line be drawn, cutting AC, BD in X and X', AB, CD in Y and Y', and AD, BC in Z and Z'.

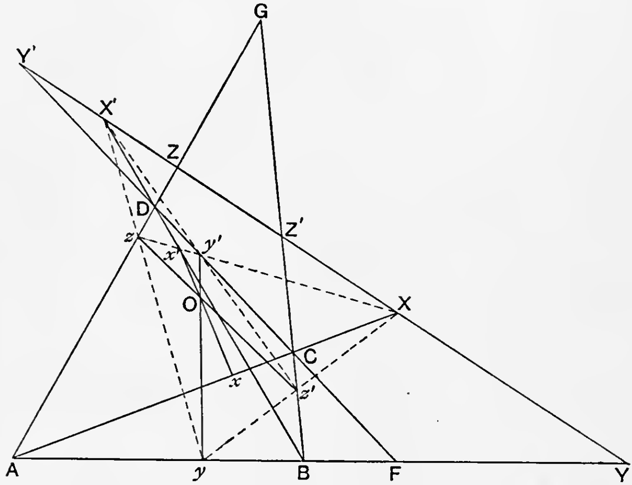
Let x, x', y, ... be the harmonic conjugate points of X, X', Y, ... with respect to the point-pairs A, C; B, D; A, B; ... respectively.

Then by § 60, since $\{AB, Yy\}$ and $\{AD, Zz\}$ are harmonic ranges, it follows that yz, BD, YZ are concurrent; that is, yz passes through X'.

Similarly, we may show that $y'z'$ passes through X', and that yz', $y'z$ intersect in the point X.

Hence, X and X' are two of the centres of the tetrastigm $yy'zz'$; and therefore by § 141, the segment yy' is divided harmonically by zz' and XX', and likewise the segment zz' is divided harmonically by yy' and XX'.

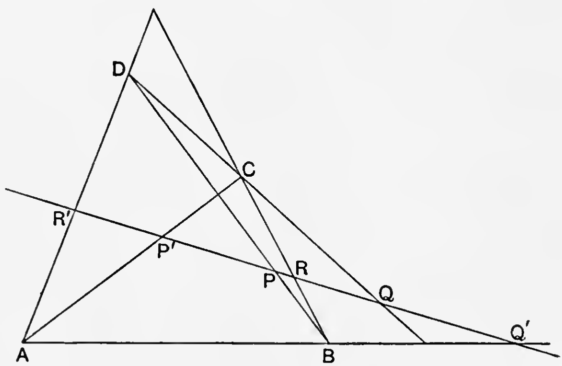
Similarly we can show that Y, Y' are two centres of the tetrastigm $xx'zz'$. Therefore, if xx', zz' intersect in O, each of the segments xx', zz' is divided harmonically in the point O, and in the point where it cuts YY'.



It follows that the lines xx', yy', zz' are concurrent, and that if O be the point in which they intersect, each segment such as xx' is divided harmonically in the point O and the point where it cuts the line XX'.

Ex. Deduce the theorem given in § 140, Ex. 1, by considering the line XX' to be the line at infinity.

144. *Any straight line is cut in involution by the three pairs of opposite connectors of any tetrastigm.*



Let $ABCD$ be any tetrastigm, and let any straight line cut the connectors BD, AC in P, P'; the connectors CD, AB in Q, Q'; and the connectors BC, AD in R, R'. Then the range $\{PP', QQ', RR'\}$ will be in involution.

Since the line BD cuts the sides of the triangle $AQ'R'$ in the points P, D, B, we have by § 104,

$$\frac{AB}{Q'B} \cdot \frac{Q'P}{R'P} \cdot \frac{R'D}{AD} = 1;$$

that is
$$\frac{Q'P}{R'P} = \frac{Q'B}{AB} \cdot \frac{AD}{R'D}.$$

Similarly, since DC cuts the sides of the triangle $AR'P'$ in the points Q, C, D, we have

$$\frac{R'Q}{P'Q} = \frac{R'D}{AD} \cdot \frac{AC}{P'C}.$$

And since BC cuts the sides of the triangle $AP'Q'$ in the points R, B, C, we have

$$\frac{P'R}{Q'R} = \frac{P'C}{AC} \cdot \frac{AB}{Q'B}.$$

Hence
$$\frac{Q'P}{R'P} \cdot \frac{R'Q}{P'Q} \cdot \frac{P'R}{Q'R} = 1;$$

that is
$$PQ' . QR' . RP' + P'Q . Q'R . R'P = 0.$$

Therefore by § 75, the range $\{PP', QQ', RR'\}$ is in involution.

145. Ex. 1. The straight lines drawn through any point parallel to the pairs of opposite connectors of a tetrastigm form a pencil in involution.

This follows by considering the range formed by the intersection of the six connectors with the line at infinity.

Ex. 2. If E, F, G be the centres of the tetrastigm $ABCD$, and O any point, the rays conjugate to EO, FO, GO with respect to the pairs of connectors which intersect in E, F, G respectively, are concurrent.

Ex. 3. If O' be the point of concurrence in the last case, show that O and O' are the double points of the range in involution formed by the points of intersection of OO' with the connectors of the tetrastigm.

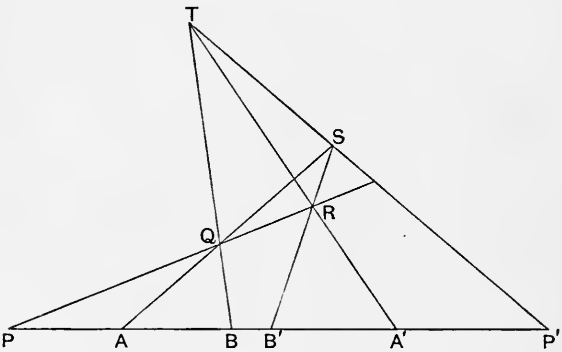
Ex. 4. If a straight line be drawn through one of the centres of a tetrastigm, show that the locus of the centre of the range in involution determined by the connectors of the tetrastigm, will be a straight line.

Ex. 5. Given any point, find a straight line passing through it, so that the given point shall be a double point of the range in involution in which it is cut by the connectors of a given tetrastigm.

Ex. 6. Any point O is taken on a transversal XYZ of a given triangle ABC. If P be the harmonic conjugate point with respect to B and C, of the point in which OA cuts BC, show that OC will intersect PY, and that OB will intersect PZ in points which lie on a fixed straight line passing through A.

If points Q, R be taken on CA, AB respectively, such that the pencils $O\{CA, BQ\}$, and $O\{AB, CR\}$ are harmonic, show that the corresponding lines passing through B and C will intersect on the straight line which passes through A.

146. The theorem of § 144 suggests a simple construction for determining the corresponding point P' of the point P in a range in involution, when two conjugate couples A, A'; and B, B'; are known.



Let any straight line PQR be drawn through P, and let Q, R be any two points on it. Let AQ meet $B'R$ in S, and let BQ meet $A'R$ in T; then TS will meet AB in P'.

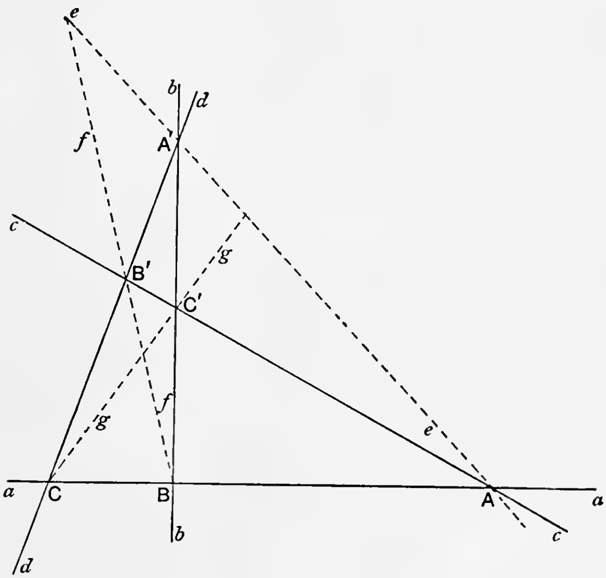
For in the tetrastigm $QRST$, the three pairs of opposite connectors are AQS, $A'RT$; BQT, $B'RS$; and PQR, $P'ST$. Therefore by § 144, the range $\{AA', BB', PP'\}$ is in involution.

Ex. If $\{AA', BB', CC'\}$ be any range in involution show how to determine three points P, Q, R such that each of the ranges $\{AA', QR\}$, $\{BB', RP\}$, $\{CC', PQ\}$ shall be harmonic.

Properties of a Tetragram.

147. A system of four lines, no three of which are concurrent, is called a *tetragram*. These four lines intersect in six points, so that we have three pairs of opposite vertices. The lines connecting each pair of opposite vertices are the diagonals, so that there are three diagonals. The triangle formed by the diagonals is called the *diagonal triangle*.

Sometimes it is convenient to denote the lines forming a tetragram by single letters, and the vertices by double letters; and sometimes it is more convenient to use letters to denote the vertices. Thus if a, b, c, d be the four lines forming the tetragram,



the points ac, bd are a pair of opposite vertices, and the line joining them, denoted by e in the figure, is a diagonal. If A, A'; B, B'; C, C'; be the three pairs of opposite vertices, the lines of the tetragram are ABC, $AB'C'$, $A'BC'$, and $A'B'C$.

148. Ex. 1. If A, A'; B, B'; C, C' be the pairs of opposite vertices of a tetragram, show that

$$AC \,.\, AC' : AB \,.\, AB' = A'C \,.\, A'C' : A'B \,.\, A'B'.$$

Ex. 2. Show that the circumcircles of the four triangles ABC', ACB', $A'B'C'$, $A'BC$ meet in a point.

Let the circumcircles of ABC', $AB'C$ meet in the point O, and then by § 120, it follows that the feet of the perpendiculars from O on the four lines constituting the tetragram are collinear. Hence the circumcircles of the triangles $A'B'C'$, $A'BC$ must also pass through the point O.

Ex. 3. Show also that the orthocentres of the four triangles are collinear.

They lie on a line which is parallel to the line which passes through the feet of the perpendiculars drawn from O.

Ex. 4. Prove that, if, for each of the four triangles formed by four lines, a line be drawn bisecting perpendicularly the distance between the circumcentre and the orthocentre, the four bisecting lines will be concurrent.

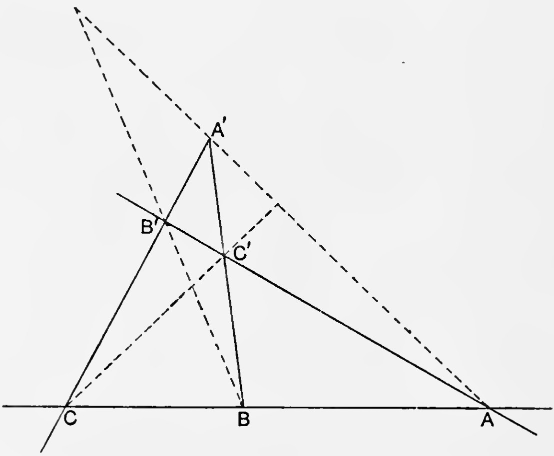
[Hervey, *E. T.* Reprint, Vol. LIV.]

Ex. 5. In every tetrastigm, the three pairs of opposite connectors intersect the opposite sides of the central triangle in six points which lie three by three on four straight lines, thus determining the three pairs of opposite vertices of a tetragram.

See § 142, Ex. 2.

Ex. 6. If *abcd* be any tetragram, and if the line joining the points *ab*, *cd* intersect the line joining the points *ad*, *bc* in the point O; show that the lines drawn through O parallel to the lines a, b, c, d will meet the lines c, d, a, b respectively in four collinear points. [Trin. Coll., 1890.]

149. *The points in which any diagonal of a tetragram cuts the other two diagonals are harmonic conjugate points with respect to the pair of opposite vertices which it connects.*



Let A, A'; B, B'; C, C'; be the three pairs of opposite vertices of the tetragram. Then evidently A and A' are a pair of centres of the tetrastigm $BC, B'C'$. Hence, by § 141, it follows that BB', and CC', cut AA' in two points which are harmonic conjugates with respect to A and A'.

If the lines of the tetragram be denoted by a, b, c, d, and the diagonals by e, f, g, we see (fig. § 147) that each of the ranges $e\{ad, gf\}, f\{ac, ge\}, g\{ab, fe\}$ is harmonic.

150. Ex. 1. Prove the theorem of § 149 directly by means of §§ 98, 106.

Ex. 2. If BB', CC' intersect in E, and if O be any point on AA', show that $O\{AE, BC, B'C'\}$ is a pencil in involution.

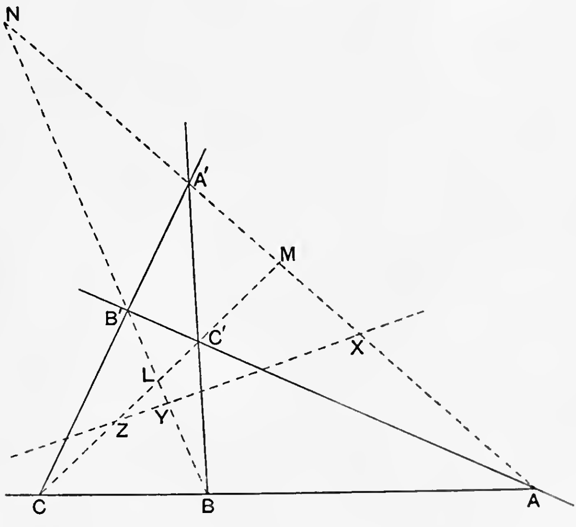
Ex. 3. Show that the three pairs of opposite vertices of a tetragram connect with the opposite vertices of the diagonal triangle, by six lines which pass three by three through four points, thus determining the three pairs of opposite connectors of a tetrastigm.

Ex. 4. If $abcd$ be any tetragram, and if the diagonal which connects the points ab, cd, meet the diagonal which connects the points ad, bc, in the point L; show that the lines which join L to the points in which any transversal cuts the lines a, b, c, d, will cut the lines c, d, a, b respectively in four collinear points.

Ex. 5. The points A, A'; B, B'; C, C'; are the opposite vertices of a tetragram. From any point P in AA', the lines PB, PB' are drawn to meet $B'C$ and BC' in H and K respectively. Show that HC' and KC intersect on the line AA'.

151. *The middle points of the diagonals of a tetragram are collinear.*

If A, A'; B, B'; C, C'; be the pairs of opposite vertices of a tetragram, then the middle points of AA', BB', CC' are collinear. (See § 38, Ex. 2.)



Let L, M, N be the vertices of the diagonal triangle; and X, Y, Z the middle points of AA', BB', CC'.

Since $\{MN, AA'\}$ is a harmonic range, and X the middle point of AA', we have by § 54, Ex. 3,

$$MX : NX = AM^2 : AN^2.$$

Similarly we shall have

$$NY : LY = BN^2 : BL^2,$$
$$LZ : MZ = CL^2 : CM^2.$$

But since ABC is a transversal of the triangle LMN, we have by § 104,

$$\frac{BL}{CL} \cdot \frac{CM}{AM} \cdot \frac{AN}{BN} = 1.$$

Hence
$$\frac{MX}{NX} \cdot \frac{NY}{LY} \cdot \frac{LZ}{MZ} = 1.$$

Therefore by § 105, X, Y, Z are collinear.

152. Ex. 1. The orthocentres of the four triangles formed by four straight lines lie on a straight line which is perpendicular to the line which bisects the diagonals of the tetragram formed by the given straight lines.

Ex. 2. If five tetragrams be formed by excluding in succession each of five given lines, show that the five lines which bisect the diagonals of these tetragrams respectively, are concurrent.

Ex. 3. If Ω, Ω' be the Brocard points of the triangle ABC, and if $A'B'C'$ be Brocard's first triangle, show that the lines joining the middle points of corresponding sides of the two triangles intersect in the point which bisects $\Omega\Omega'$.

Ex. 4. Show that the middle points of any pair of opposite sides of a tetrastigm are collinear with the middle point of one of the sides of the triangle formed by the centres of the tetrastigm.

153. The theorem of § 151 may be thus generalised :—*If any straight line cut the diagonals AA', BB', CC' of a tetragram in the points X, Y, Z; and if X', Y', Z' be the harmonic conjugate points with respect to the corresponding pairs of opposite vertices, the points X', Y', Z' will be collinear.*

Let L, M, N be the vertices of the diagonal triangle; then by § 149, the range $\{AA', MN\}$ is harmonic.

Since the range $\{AA', XX'\}$ is harmonic, and also the range $\{AA', MN\}$, we have by § 56, Ex. 4,

$$MX \,.\, MX' : NX \,.\, NX' = MA^2 : NA^2.$$

In the same way we may show that

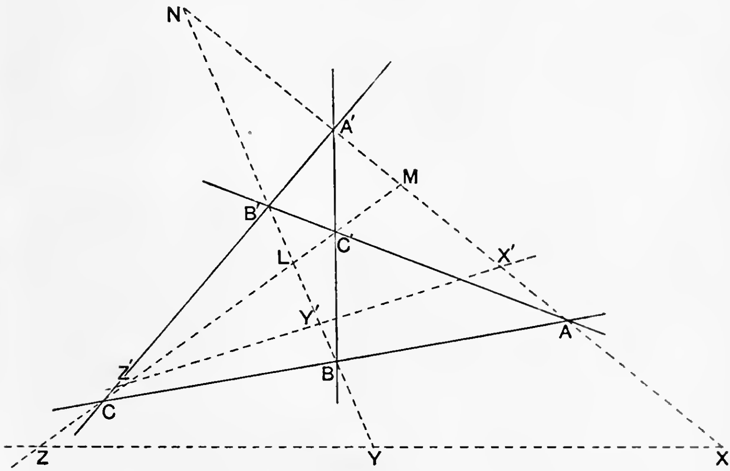
$$NY \,.\, NY' : LY \,.\, LY' = NB^2 : LB^2;$$

and
$$LZ \,.\, LZ' : MZ \,.\, MZ' = LC^2 : MC^2.$$

But since ABC is a transversal of the triangle LMN, we have by § 104,

$$\frac{MA}{NA}\cdot\frac{NB}{LB}\cdot\frac{LC}{MC}=1.$$

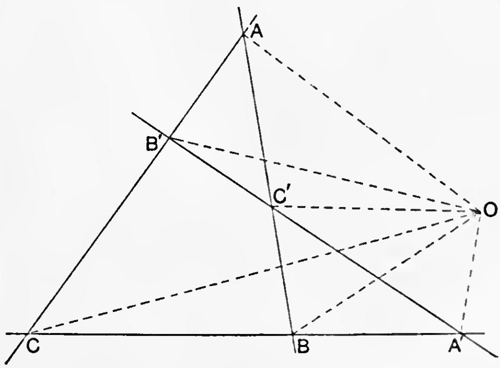
Hence
$$\frac{MX \,.\, NY \,.\, LZ}{NX \,.\, LY \,.\, MZ}\cdot\frac{MX' \,.\, NY' \,.\, LZ'}{NX' \,.\, LY' \,.\, MZ'}=1.$$



Therefore by § 105, since the points X, Y, Z are collinear, so also are the points X', Y', Z'.

If we take the line at infinity instead of the line XYZ, the points X', Y', Z' become respectively the middle points of AA', BB', CC' and thus we have the theorem of § 151.

154. *The lines connecting any point with the vertices of a tetragram form a pencil in involution.*



Let A, A'; B, B'; C, C'; be the pairs of opposite vertices of a tetragram, and let O be any point. Then the pencil

$$O\{AA', BB', CC'\}$$

will be in involution.

Since $A'B'C'$ is a transversal of the triangle ABC, we have

$$\frac{BA'}{CA'} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = 1.$$

Hence as in § 109, Ex. 2,

$$\frac{\sin BOA'}{\sin COA'} \cdot \frac{\sin COB'}{\sin AOB'} \cdot \frac{\sin AOC'}{\sin BOC'} = 1.$$

Therefore by § 91, the pencil $O\{AA', BB', CC'\}$ is in involution.

155. Ex. 1. Show that if through any point O, lines OA', OB', OC' be drawn parallel to the sides of a triangle ABC, the pencil $O\{AA', BB', CC'\}$ will be in involution.

This is proved by considering the tetragram formed by the three sides of a triangle and the line at infinity.

Ex. 2. Deduce the theorem of § 153, from the theorem of the last article.

Ex. 3. If in § 153 the line XYZ meet the line $X'Y'Z'$ in the point O, show that these lines will be the double lines of the pencil in involution $O\{AA', BB', CC'\}$.

Ex. 4. Show that the circles described on the diagonals AA', BB', CC', of a tetragram, as diameters, have two common points.

Let the circle described on BB' cut the circle described on CC' in P and P'. Then BPB', CPC' are right angles. Therefore since $P\{AA', BB', CC'\}$ is a pencil in involution, APA' is a right angle by § 87.

Ex. 5. If A, A'; B, B'; C, C'; be the opposite vertices of a tetragram, and X, Y, Z the middle points of AA', BB', CC'; show that

$$YZ.AA'^2 + ZX.BB'^2 + XY.CC'^2 = -4YZ.ZX.XY.$$

[Jesus Coll. 1890.]

Ex. 6. Apply the theorem in § 154 to obtain a construction for finding a ray which shall be the conjugate of a given ray in a pencil in involution.

Ex. 7. Through a fixed point O any straight line is drawn intersecting the sides of a triangle ABC in the points X, Y, Z. If X' be the harmonic conjugate of the point X with respect to B, C, show that the line joining Y to the point of intersection of OC and AX', and the line joining Z to the point of intersection of OB and AX', will pass through the same fixed point on BC.

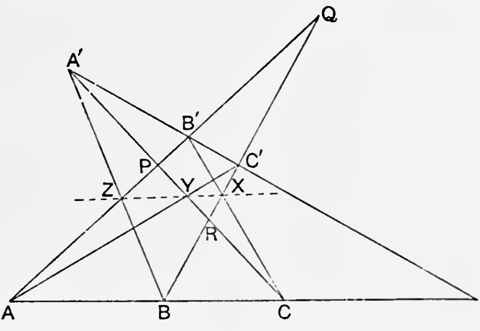
If Y', Z' be the harmonic conjugate points of Y and Z with respect to C, A and A, B respectively; if P be the point in which the line joining Y to the point of intersection of OC and AX' cuts BC; Q the point in which the line joining Z to the point of intersection of OA and BY' cuts CA; and R the point in which the line joining X to the point of intersection of OB and CZ'; show that P, Q, R are collinear.

The line PQR is the polar of the point O with respect to the triangle ABC. (§ 111.)

Special cases of polystigms and polygrams.

156. The properties of figures consisting of more than four points or straight lines have not been systematically investigated. Consequently we shall merely discuss the few special cases of interest which have been discovered. The most important of these is the case of the hexastigm in which three of the points lie on one straight line, and the remaining three on another straight line; and the correlative case of the hexagram which consists of two pencils of three rays.

157. *If* $\{ABC\}$ $\{A'B'C'\}$ *be any two ranges, the straight lines* AB', BC', CA' *intersect the three lines* $A'B$, $B'C$, $C'A$ *respectively in three points which are collinear.*



Let BC', $B'C$ intersect in X, CA', $C'A$ in Y, and AB', $A'B$ in Z; and let AB', BC', CA' form the triangle PQR.

Then since XCB', $C'YA$, $BA'Z$ are transversals of the triangle PQR, we have by § 104,

$$\frac{QX}{RX} \cdot \frac{RC}{PC} \cdot \frac{PB'}{QB'} = 1,$$

$$\frac{RY}{PY} \cdot \frac{PA}{QA} \cdot \frac{QC'}{RC'} = 1,$$

and

$$\frac{PZ}{QZ} \cdot \frac{QB}{RB} \cdot \frac{RA'}{PA'} = 1.$$

But since BCA, $C'A'B'$ are also transversals of the triangle PQR, we have by § 104,

$$\frac{QB}{RB} \cdot \frac{RC}{PC} \cdot \frac{PA}{QA} = 1,$$

and

$$\frac{QC'}{RC'} \cdot \frac{RA'}{PA'} \cdot \frac{PB'}{QB'} = 1.$$

Hence, we have

$$\frac{QX}{RX} \cdot \frac{RY}{PY} \cdot \frac{PZ}{QZ} = 1.$$

Therefore by § 105, the points X, Y, Z are collinear.

158. In the same way we may show that the lines AC', BB', CA' intersect the lines $A'B$, $C'C$, $B'A$ respectively in three collinear points. In fact, we may interchange the order of the letters $A'B'C'$ in every possible way. Thus we shall have six sets of collinear points. If we use the notation $\binom{AB'}{BA'}$ to represent the point of intersection of the lines AB', BA', we may exhibit these six sets of collinear points in the tabular form:

$$\binom{AB'}{A'B}, \quad \binom{BC'}{B'C}, \quad \binom{CA'}{C'A};$$

$$\binom{AC'}{B'B}, \quad \binom{BA'}{C'C}, \quad \binom{CB'}{A'A};$$

$$\binom{AA'}{C'B}, \quad \binom{BB'}{A'C}, \quad \binom{CC'}{B'A};$$

$$\binom{AC'}{A'B}, \quad \binom{BB'}{C'C}, \quad \binom{CA'}{B'A};$$

$$\binom{AA'}{B'B}, \quad \binom{BC'}{A'C}, \quad \binom{CB'}{C'A};$$

$$\binom{AB'}{C'B}, \quad \binom{BA'}{B'C}, \quad \binom{CC'}{A'A}.$$

Each of these triads of points are collinear.

Thus we have the theorem: *The nine lines which connect two triads of collinear points intersect in eighteen other points which lie in threes on six straight lines.*

It should be noticed that each of the collinear triads of points are the points of intersection of the three pairs of opposite connectors in a complete set of connectors of the hexastigm $ABCA'B'C'$. Hence the theorem may be stated in the form: *The* *three pairs of opposite connectors, in each of the six complete sets of a hexastigm consisting of two triads of collinear points, intersect in three collinear points.*

159. Ex. 1. Show that the nine points in which any pencil of three rays intersects any other pencil of three rays may be connected by eighteen lines which pass three by three through six points.

If a, b, c denote the rays of one pencil, and a', b', c' the rays of the other pencil, we may show by a very similar method to that used in §§ 157, 158, that the following triads of lines are concurrent :

$$\begin{pmatrix} ab' \\ a'b \end{pmatrix}, \quad \begin{pmatrix} bc' \\ b'c \end{pmatrix}, \quad \begin{pmatrix} ca' \\ c'a \end{pmatrix};$$

$$\begin{pmatrix} ac' \\ b'b \end{pmatrix}, \quad \begin{pmatrix} ba' \\ c'c \end{pmatrix}, \quad \begin{pmatrix} cb' \\ a'a \end{pmatrix};$$

$$\begin{pmatrix} aa' \\ c'b \end{pmatrix}, \quad \begin{pmatrix} bb' \\ a'c \end{pmatrix}, \quad \begin{pmatrix} cc' \\ b'a \end{pmatrix};$$

$$\begin{pmatrix} ac' \\ a'b \end{pmatrix}, \quad \begin{pmatrix} bb' \\ c'c \end{pmatrix}, \quad \begin{pmatrix} ca' \\ b'a \end{pmatrix};$$

$$\begin{pmatrix} aa' \\ b'b \end{pmatrix}, \quad \begin{pmatrix} bc' \\ a'c \end{pmatrix}, \quad \begin{pmatrix} cb' \\ c'a \end{pmatrix};$$

$$\begin{pmatrix} ab' \\ c'b \end{pmatrix}, \quad \begin{pmatrix} ba' \\ b'c \end{pmatrix}, \quad \begin{pmatrix} cc' \\ a'a \end{pmatrix}.$$

Ex. 2. If in a hexagon two pairs of opposite sides intersect on the corresponding diagonals, then the remaining pair of opposite sides will intersect on the diagonal corresponding to this pair. [Math. Tripos, 1890.]

Ex. 3. The six points A, B, C, A', B', C' are such that the lines AA', BB', CC' meet in the point O. Show that they may be connected by ten other lines which intersect in six points which are the vertices of a tetragram.

Ex. 4. The six lines a, b, c, a', b', c' are such that the points aa', bb', cc' are collinear. Show that they intersect in ten other points which lie on six lines which are the connectors of a tetrastigm.

Ex. 5. If A, B, C, A', B', C' be any six points such that the lines AB', BC', CA' are concurrent, and also the lines AC', BA', CB', show that the lines AA', BB', CC' are concurrent.

This theorem, which is contained in Ex. 1, affords a proof of § 135, Ex. 3.

Ex. 6. A pair of opposite vertices of a tetragram are given, and of the four remaining vertices, three lie on three given straight lines. Show that the sixth vertex lies on one or other of six straight lines.

CHAPTER VIII.

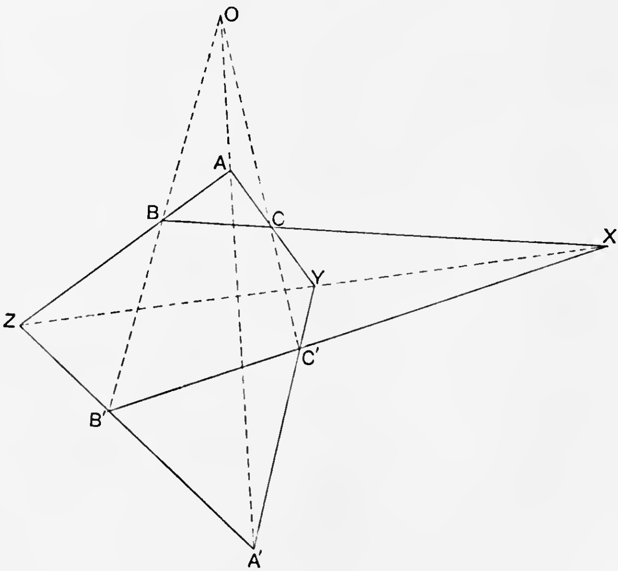
THE THEORY OF PERSPECTIVE.

Triangles in perspective.

160. Two triangles are said to be *in perspective* when the lines connecting the vertices of one triangle to the corresponding vertices of the other triangle are concurrent.

If ABC, $A'B'C'$ be two triangles in perspective, such that the lines AA', BB', CC' meet in the point O, the vertices A and A' are called *corresponding* vertices, and the sides BC, $B'C'$ are called *corresponding* sides. The point O is called the *centre of perspective* of the two triangles.

161. *When two triangles are in perspective, the corresponding sides intersect in three collinear points.*



Let ABC, $A'B'C'$ be two triangles in perspective, so that AA', BB', CC' intersect in the point O.

Let BC and $B'C'$ intersect in X; CA, $C'A'$ in Y; and AB, $A'B'$ in Z. Then X, Y, Z will be collinear.

Since $B'C'X$ is a transversal of the triangle CBO, we have by § 104,

$$\frac{BX}{CX}\cdot\frac{CC'}{OC'}\cdot\frac{OB'}{BB'}=1.$$

Similarly, since $A'C'Y$ is a transversal of the triangle CAO,

$$\frac{CY}{AY}\cdot\frac{AA'}{OA'}\cdot\frac{OC'}{CC'}=1;$$

and since $A'B'Z$ is a transversal of the triangle BAO,

$$\frac{AZ}{BZ}\cdot\frac{BB'}{OB'}\cdot\frac{OA'}{AA'}=1.$$

Hence we have, $$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}=1.$$

Therefore by § 105, the points X, Y, Z are collinear.

162. The line XYZ which passes through the points of intersection of the corresponding sides of two triangles in perspective, is called the *axis of perspective.*

Triangles in perspective are sometimes called *homologous* triangles, the centre of perspective being called the *centre of homology,* and the axis of perspective the *axis of homology.*

Triangles in perspective are also said to be *copolar,* the centre of perspective being called the *pole.*

163. *If corresponding sides of two triangles intersect in collinear points, the triangles are in perspective.*

Let YCC', ZBB' be any two such triangles; and let CC', YC, YC' meet BB', ZB, ZB' in the points O, A, A' respectively (see fig. § 161).

Then it may be proved, as in § 161, that BC, $B'C'$ intersect YZ in the same point X.

Therefore the triangles YCC', ZBB' are in perspective, the point X being their centre of perspective.

164. The theorem in § 161 may also be proved as follows: Let ABC, $A'B'C'$ be any two triangles in the same plane so situated that AA', BB', CC' meet in the point O. Let O' be any point on the normal to the plane at O, and let the normals at A', B', C' meet $O'A$, $O'B$, $O'C$ in the points A'', B'', C'' respectively.

The two planes ABC, $A''B''C''$ will intersect in a line (L say).

Also the lines BC, $B''C''$ being in the same plane $O'BC$ will meet in a point, which being common to each of the planes ABC, $A''B''C''$, must lie in the line of intersection of these planes; that is, BC and $B''C''$ will intersect on the line L.

But $B'C'$ is evidently the orthogonal projection of $B''C''$, and therefore will intersect $B''C''$ in the point in which the latter cuts the plane ABC. Consequently $B'C'$ will intersect BC in a point on the line L.

Similarly CA, AB will intersect $C'A'$, $A'B'$ respectively in points which lie on L.

Hence the corresponding sides of the triangles ABC, $A'B'C'$ intersect in collinear points.

165. Ex. 1. If the symmedian lines AK, BK, CK meet the circumcircle of the triangle ABC in the points A', B', C', show that the tangents to the circle at A', B', C' will form a triangle in perspective with the triangle ABC.

Ex. 2. If the lines joining the vertices of two triangles, which have a common median point, be parallel, their axis of perspective passes through the median point.

Ex. 3. Show that if $A'B'C'$ be the first Brocard triangle of the triangle ABC, then ABC is in perspective with the triangles $A'B'C'$, $B'C'A'$ and $C'A'B'$. See § 135, Ex. 3.

Ex. 4. Show that the triangle formed by the middle points of the sides of Brocard's first triangle is in perspective with the original triangle.

Ex. 5. If the triangle ABC be in perspective with the triangle $B'C'A'$, and also with the triangle $C'A'B'$, show that it is in perspective with the triangle $A'B'C'$. See § 159, Ex. 5.

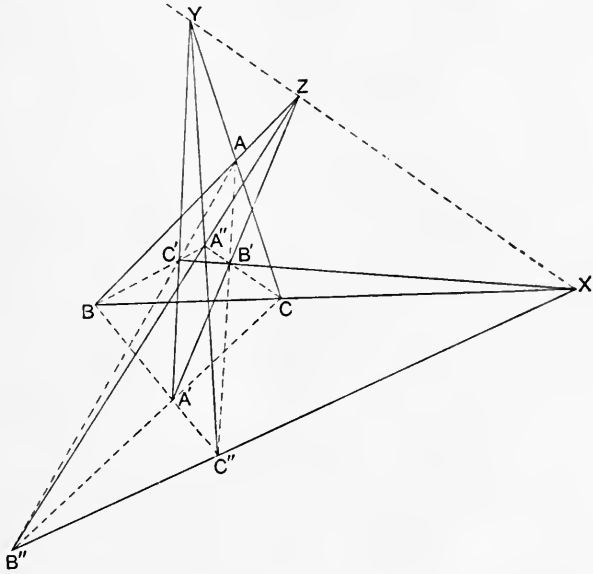
Ex. 6. Two triangles having the same median point G, are in perspective. If the centre of perspective be on the line at infinity, the axis of perspective passes through G.

Ex. 7. Two sides of a triangle pass through fixed points, and the three vertices lie on three fixed straight lines, which are concurrent; show that the third side will always pass through a fixed point.

Ex. 8. Two vertices of a triangle move on fixed straight lines, and the three sides pass through three fixed points, which are collinear; find the locus of the third vertex.

Ex. 9. Inscribe a triangle in a given triangle, so that its three sides may pass through three given points which are collinear.

166. *If ABC, $A'B'C'$ be two triangles in perspective, and if BC', $B'C$ intersect in A''; CA', $C'A$ in B''; and AB', $A'B$ in C''; the triangle $A''B''C''$ will be in perspective with each of the given triangles, and the three triangles will have the same axis of perspective.*



Let XYZ be the axis of perspective of the given triangles ABC, $A'B'C'$.

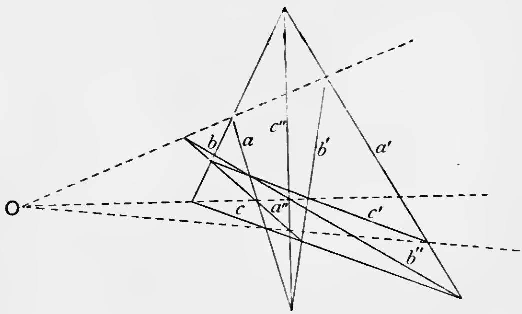
Since the given triangles are in perspective, AA', BB', CC' are concurrent. Hence the triangles $AB'C'$, $A'BC$ are in perspective; and therefore the lines $B'C'$, $C'A$, AB' will intersect BC, CA', $A'B$ respectively in collinear points (§ 161); that is, the points X, B'', C'' are collinear.

Thus $B''C''$ intersects BC in the point X.

Similarly we may show that $C''A''$ intersects CA in Y, and that $A''B''$ intersects AB in Z.

Therefore the triangle $A''B''C''$ is in perspective with each of the given triangles, and the three triangles have a common axis of perspective.

167. *If abc, $a'b'c'$ be any two triangles in perspective, the lines joining the points bc', ca', ab' to the points $b'c$, $c'a$, $a'b$ respectively form a triangle which is in perspective with each of the given* *triangles, and the three triangles have the same centre of perspective.*



Let O be the centre of perspective of the given triangles, then the lines joining the points bc, ca, ab to the points $b'c'$, $c'a'$, $a'b'$ intersect in O.

Let a'' denote the line joining the points bc', $b'c$; b'' the line joining the points ca', $c'a$; and c'' the line joining the points ab', $a'b$.

Since the triangles abc, $a'b'c'$ are in perspective, the points aa', bb', cc' are collinear. Hence by § 163, the triangles $ab'c'$, $a'bc$ are in perspective, and therefore the lines joining the points $b'c'$, $c'a$, ab' to the points bc, ca', $a'b$ are concurrent. That is, the lines b'', c'', and the line joining the points bc, $b'c'$, are concurrent.

Hence, the points bc, $b'c'$, $b''c''$ are collinear, and they lie on a line which passes through O.

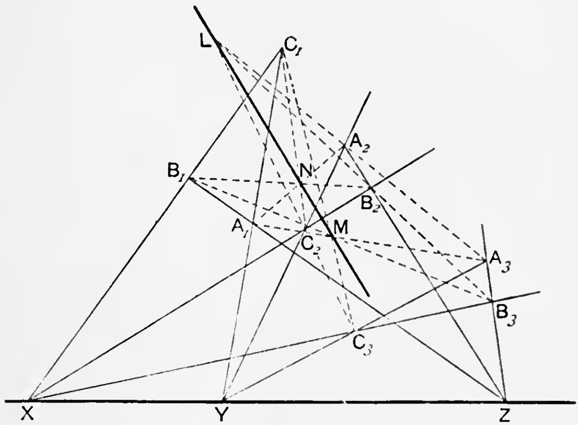
Similarly, we may show that the points ca, $c'a'$, $c''a''$ are collinear, and that the points ab, $a'b'$, $a''b''$ are collinear.

Therefore, the triangle $a''b''c''$ is in perspective with each of the triangles abc, $a'b'c'$; and the three triangles have a common centre of perspective.

168. *When three triangles are in perspective two by two, and have the same axis of perspective, their three centres of perspective are collinear.*

Let $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ be three triangles in perspective two and two, such that the sides B_1C_1, B_2C_2, B_3C_3 meet in the point X, the sides C_1A_1, C_2A_2, C_3A_3 in the point Y, and the sides A_1B_1, A_2B_2, A_3B_3 in the point Z; X, Y, Z being collinear points.

Then the triangles $B_1B_2B_3$, $C_1C_2C_3$ are in perspective, X being the centre of perspective. Therefore the lines B_2B_3, B_3B_1, B_1B_2



intersect the lines C_2C_3, C_3C_1, C_1C_2 respectively in three points L, M, N, which are collinear.

But these points are the centres of perspective of the given triangles taken two at a time. Hence, the centres of perspective of the three triangles are collinear.

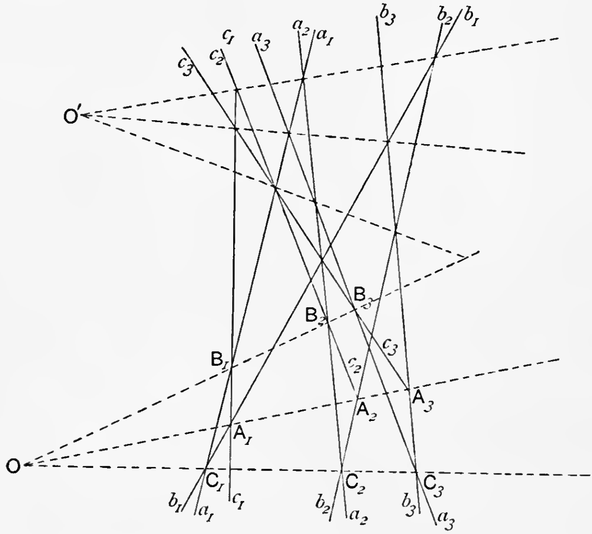
169. It is evident that the triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ are in perspective two by two, and have the same axis of perspective, namely the line of collinearity of the centres of perspective of the triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$.

Thus we have the theorem: *When three triangles are in perspective two by two, and have the same axis of perspective, the triangles formed by the corresponding vertices of the triangles are also in perspective two by two and have the same axis of perspective; and the axis of perspective of either set of triangles passes through the centres of perspective of the other set.*

170. *When three triangles are in perspective two by two, and have the same centre of perspective, their three axes of perspective are concurrent.*

Let $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ be the three triangles having the common centre of perspective O. Let a_1, b_1,... denote the sides of the triangles opposite to the vertices A_1, B_1,....

Then it is evident that the triangles $b_1b_2b_3$, $c_1c_2c_3$ are in perspective, having the line $OA_1A_2A_3$ as their axis of perspective.



Therefore the lines joining their vertices are concurrent; that is, the lines joining the points b_2b_3, b_3b_1, b_1b_2 respectively to the points c_2c_3, c_3c_1, c_1c_2 are concurrent.

But the line joining the point b_2b_3 to the point c_2c_3 is the axis of perspective of the triangles $A_2B_2C_2$, $A_3B_3C_3$.

Hence, the axes of perspective of the three triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ are concurrent.

171. It follows from the above proof, that if the triangles $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$ are in perspective and have a common centre of perspective O, their three axes of perspective will intersect in a point O', which is the common centre of perspective of the triangles $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$ whose three axes of perspective meet in O.

172. These theorems may also be easily proved by the same method that was used in § 164. Thus let $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ be any three coplanar triangles having a common centre of perspective O. Let O' be any point in the normal to the plane at O, and let the normals to the plane at A_2, B_2, C_2, A_3, B_3, C_3 meet $O'A_2$, $O'B_2$, &c. in the points A_2', B_2', &c., respectively. Then the lines of intersection of the planes $A_1B_1C_1$, $A_2'B_2'C_2'$, $A_3'B_3'C_3'$ obviously meet in the point of intersection of the three planes. But the axes of perspective of the triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ are the orthogonal projections of the lines of intersection of the planes. Consequently, since they lie in the plane $A_1B_1C_1$, they must be concurrent.

173. Ex. 1. If (ABC), $(A'B'C')$ be two ranges on different straight lines, show that the triangle formed by the lines AA', BB', CC' is in perspective with the triangle formed by the lines BC', CA', AB', and also with the triangle formed by the lines CB', AC', BA'.

This theorem follows from §§ 157, 158.

Ex. 2. Show that the three triangles in the last theorem have a common centre of perspective.

This follows from § 167.

174. We are now in a position to complete the discussion of the properties of the figure which was discussed in § 157.

We will use the same notation as in that article, namely: let (AB') represent the line joining the points A and B'; $\begin{pmatrix} AB' \\ BC' \end{pmatrix}$ the point of intersection of the lines (AB'), (BC').

In § 158 we showed that the eighteen points $\begin{pmatrix} AB' \\ A'B \end{pmatrix}$, $\begin{pmatrix} AC' \\ A'C \end{pmatrix}$, &c. lie on six lines; that is to say, each of the following triads are collinear:

$$\begin{pmatrix} AB' \\ A'B \end{pmatrix}, \quad \begin{pmatrix} BC' \\ B'C \end{pmatrix}, \quad \begin{pmatrix} CA' \\ C'A \end{pmatrix};$$

$$\begin{pmatrix} AC' \\ B'B \end{pmatrix}, \quad \begin{pmatrix} BA' \\ C'C \end{pmatrix}, \quad \begin{pmatrix} CB' \\ A'A \end{pmatrix};$$

$$\begin{pmatrix} AA' \\ C'B \end{pmatrix}, \quad \begin{pmatrix} BB' \\ A'C \end{pmatrix}, \quad \begin{pmatrix} CC' \\ B'A \end{pmatrix};$$

$$\begin{pmatrix} AC' \\ A'B \end{pmatrix}, \quad \begin{pmatrix} BB' \\ C'C \end{pmatrix}, \quad \begin{pmatrix} CA' \\ B'A \end{pmatrix};$$

$$\begin{pmatrix} AA' \\ B'B \end{pmatrix}, \quad \begin{pmatrix} BC' \\ A'C \end{pmatrix}, \quad \begin{pmatrix} CB' \\ C'A \end{pmatrix};$$

$$\begin{pmatrix} AB' \\ C'B \end{pmatrix}, \quad \begin{pmatrix} BA' \\ B'C \end{pmatrix}, \quad \begin{pmatrix} CC' \\ A'A \end{pmatrix}.$$

Let us represent the line joining the points

$$\begin{pmatrix} AB' \\ A'B \end{pmatrix}, \quad \begin{pmatrix} BC' \\ B'C \end{pmatrix}, \quad \begin{pmatrix} CA' \\ C'A \end{pmatrix}$$

by the expression
$$\begin{pmatrix} ABC \\ A'B'C' \end{pmatrix}.$$

Then the six lines will be represented by

$$\begin{pmatrix} ABC \\ A'B'C' \end{pmatrix}, \quad \begin{pmatrix} ABC \\ B'C'A' \end{pmatrix}, \quad \begin{pmatrix} ABC \\ C'A'B' \end{pmatrix}.$$

$$\begin{pmatrix} ABC \\ A'C'B' \end{pmatrix}, \quad \begin{pmatrix} ABC \\ B'A'C' \end{pmatrix}, \quad \begin{pmatrix} ABC \\ C'B'A' \end{pmatrix}.$$

We shall show that the first three are concurrent, and likewise the second three.

The first three are the axes of perspective of the triangles (AA', BB', CC'), (BC', CA', AB'), (CB', AC', BA') which have a common centre of perspective (§ 173, Ex. 2).

Therefore by § 170, these axes of perspective are concurrent. Let O be the centre of perspective, and O' the point of concurrence of the axes of perspective.

By § 171, it follows that O' will be the common centre of perspective of the triangles (AA', BC', CB'), (BB', CA', AC'), and (CC', AB', BA'); and the axes of perspective of these triangles will meet in O.

That is, the three lines

$$\begin{pmatrix} ABC \\ A'C'B' \end{pmatrix}, \quad \begin{pmatrix} ABC \\ C'B'A' \end{pmatrix}, \quad \begin{pmatrix} ABC \\ B'A'C' \end{pmatrix}$$

are concurrent, their point of intersection being the point O.

Hence we have the theorem: *The nine lines which connect two triads of collinear points intersect in eighteen points which lie in threes on six lines, three of which pass through one point, and the remaining three through another point.*

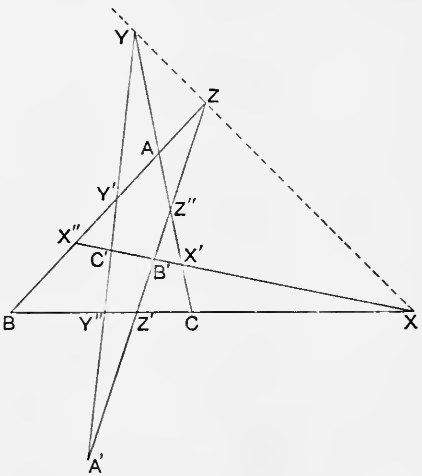
This theorem is a particular case of a more general theorem known as Pascal's theorem.

175. Ex. Show that the nine points in which any three concurrent lines intersect three other concurrent lines may be connected by eighteen lines which pass three by three through six points, which lie three by three on two other straight lines.

This theorem is a particular case of a more general theorem known as Brianchon's theorem. It may be proved in a similar way to the theorem in § 174.

Relations between two triangles in perspective.

176. *If ABC, $A'B'C'$ be two triangles in perspective, and if*



$B'C'$ cuts AC, AB in the points X', X'' respectively; if $C'A'$ cuts BA, BC in the points Y', Y'' respectively; and if $A'B'$ cuts CB, CA in the points Z', Z'' respectively; then

$$\frac{AX'}{CX'}\cdot\frac{BY'}{AY'}\cdot\frac{CZ'}{BZ'}=\frac{AX''}{BX''}\cdot\frac{BY''}{CY''}\cdot\frac{CZ''}{AZ''}.$$

Let the axis of perspective of the two triangles cut BC, CA, AB in the points X, Y, Z, respectively.

Then because $XX'X''$ is a transversal of the triangle ABC (§ 104),

$$\frac{AX''}{BX''}\cdot\frac{BX}{CX}\cdot\frac{CX'}{AX'}=1.$$

And since $YY'Y''$, $ZZ'Z''$ are also transversals of the triangle ABC,

$$\frac{AY'}{BY'}\cdot\frac{BY''}{CY''}\cdot\frac{CY}{AY}=1;$$

$$\frac{BZ'}{CZ'}\cdot\frac{CZ''}{AZ''}\cdot\frac{AZ}{BZ}=1.$$

But XYZ is also a transversal of the triangle ABC therefore

$$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}=1.$$

Hence $$\frac{AY'.AX''}{BY'.BX''}\cdot\frac{BY''.BZ'}{CY''.CZ'}\cdot\frac{CX'.CZ''}{AX'.AZ''}=1 \quad\text{.........}\quad \text{(i)}.$$

In a similar manner by considering the lines BCX, CAY, ABZ, and XYZ, as transversals of the triangle $A'B'C'$, we may deduce the relation,

$$\frac{B'X'.B'X''}{C'X'.C'X''}\cdot\frac{C'Y'.C'Y''}{A'Y'.A'Y''}\cdot\frac{A'Z'.A'Z''}{B'Z'.B'Z''}=1 \quad\text{......}\quad \text{(ii)}.$$

177. Conversely, if either of the relations (i), (ii) hold, it may be shown that the triangles are in perspective.

Let the sides $B'C'$, $C'A'$, $A'B'$ intersect the sides BC, CA, AB in the points X, Y, Z; and let us assume that relation (i) holds.

Then since $XX'X''$, $Y''YY'$, and $Z'Z''Z$ are transversals of the triangle ABC, we have

$$\frac{BX}{CX}\cdot\frac{CX'}{AX'}\cdot\frac{AX''}{BX''}=1,$$

$$\frac{BY''}{CY''}\cdot\frac{CY}{AY}\cdot\frac{AY'}{BY'}=1,$$

and
$$\frac{BZ'}{CZ'}\cdot\frac{CZ''}{AZ''}\cdot\frac{AZ}{BZ}=1.$$

But
$$\frac{AY'.AX''}{AX'.AZ''}\cdot\frac{BY''.BZ'}{BY'.BX''}\cdot\frac{CX'.CZ''}{CZ'.CY''}=1.$$

Therefore
$$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}=1.$$

Therefore X, Y, Z are collinear.

Hence by § 163, the triangles ABC, $A'B'C'$ are in perspective.

178. Two similar relations may be proved by using the theorem of § 98.

Since A', B', C' are any points in the plane of the triangle ABC, we have

$$\frac{\sin BAA'}{\sin A'AC}\cdot\frac{\sin CBA'}{\sin A'BA}\cdot\frac{\sin ACA'}{\sin A'CB}=1,$$

$$\frac{\sin BAB'}{\sin B'AC}\cdot\frac{\sin CBB'}{\sin B'BA}\cdot\frac{\sin ACB'}{\sin B'CB}=1,$$

$$\frac{\sin BAC'}{\sin C'AC}\cdot\frac{\sin CBC'}{\sin C'BA}\cdot\frac{\sin ACC'}{\sin C'CB}=1.$$

But since the triangles are in perspective, AA', BB', CC' are concurrent, therefore by § 98,

$$\frac{\sin BAA'}{\sin A'AC}\cdot\frac{\sin CBB'}{\sin B'BA}\cdot\frac{\sin ACC'}{\sin C'CB}=1.$$

Hence, we have,

$$\frac{\sin BAC'.\sin BAB'}{\sin CAC'.\sin CAB'}\cdot\frac{\sin CBA'.\sin CBC'}{\sin ABA'.\sin ABC'}\cdot\frac{\sin ACB'.\sin ACA'}{\sin BCB'.\sin BCA'}=1.$$

Similarly, we may prove the relation,

$$\frac{\sin B'A'C.\sin B'A'B}{\sin C'A'C.\sin C'A'B}\cdot\frac{\sin C'B'A.\sin C'B'C}{\sin A'B'A.\sin A'B'C}\cdot\frac{\sin A'C'B.\sin A'C'A}{\sin B'C'B.\sin B'C'A}=1.$$

Conversely, if either of these relations hold it may be proved that the lines AA', BB', CC' are concurrent; that is, the triangles ABC, $A'B'C'$ are in perspective.

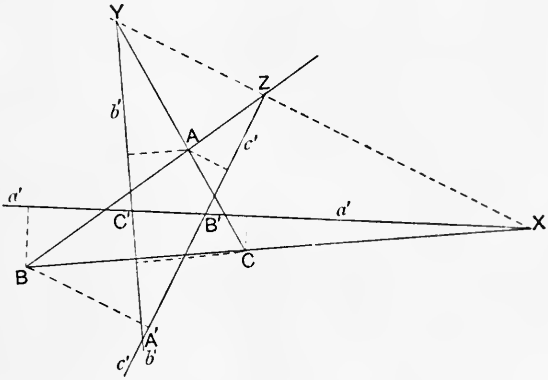
179. *When two triangles ABC, $A'B'C'$ are in perspective, the product of the ratios*

$$(Ab' : Ac'),\ (Bc' : Ba'),\ (Ca' : Cb'),$$

is equal to unity, where a', b', c' denote the sides of the triangle $A'B'C'$, and Ab' represents the perpendicular from A on b'.

Let XYZ be the axis of perspective of the two triangles. Then we have

$$Ba' : Ca' = BX : CX,$$
$$Cb' : Ab' = CY : AY,$$
$$Ac' : Bc' = AZ : BZ.$$



But since X, Y, Z are collinear,

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1.$$

Hence,
$$\frac{Ba'}{Ca'} \cdot \frac{Cb'}{Ab'} \cdot \frac{Ac'}{Bc'} = 1;$$

that is,
$$\frac{Ab'}{Ac'} \cdot \frac{Bc'}{Ba'} \cdot \frac{Ca'}{Cb'} = 1.$$

Conversely, when this relation holds, it follows that X, Y, Z are collinear, and therefore that the triangles are in perspective.

180. Ex. 1. If any circle be drawn cutting the sides of a triangle ABC in the points X, X'; Y, Y'; Z, Z', respectively, show that the triangle formed by the lines YZ', ZX', XY' is in perspective with the triangle ABC.

This follows at once from § 177.

Ex. 2. If a circle cut the sides of the triangle ABC in the points X, X'; Y, Y'; Z, Z'; show that the triangles formed by the lines $Y'Z$, $Z'X$, $X'Y$, and the triangle formed by the lines YZ', ZX', XY', are in perspective with the triangle ABC; and that the three triangles have a common centre of perspective.

Ex. 3. If from the vertices of the triangle abc, tangents x, x'; y, y'; z, z'; be drawn to a circle, show that the triangles formed by the points yz', zx', xy', and the triangle formed by the points $y'z$, $z'x$, $x'y$, are in perspective with the triangle abc; and that the three triangles have a common axis of perspective.

If ABC be the given triangle, $A'B'C'$ the triangle formed by the points yz', zx', xy'; it is easy to prove that

$$\frac{\sin BAB' \cdot \sin BAC'}{\sin CAB' \cdot \sin CAC'} = \frac{Oc^2 - R^2}{Ob^2 - R^2},$$

where O is the centre of the circle, and R its radius.

The second part of the theorem follows from § 166.

Ex. 4. If D, E, F be the middle points of the sides of the triangle ABC, and P, Q, R the feet of the perpendiculars from the vertices on the opposite sides, show that QR, RP, and PQ will intersect EF, FD, DF in the points X, Y, Z, such that the triangle XYZ is in perspective with each of the triangles ABC, PQR, DEF.

Ex. 5. Through the vertices of the triangle ABC, parallels are drawn to the opposite sides to meet the circumcircle in the points A', B', C'. If $B'C'$, $C'A'$, $A'B'$ meet BC, CA, AB in P, Q, R respectively, show that AP, BQ, CR are concurrent. [St John's Coll. 1890.]

Ex. 6. In the last case, show that $A'P$, $B'Q$, $C'R$ are also concurrent.

Ex. 7. Through K the symmedian point of the triangle ABC, are drawn the lines YKZ', ZKX', XKY', parallel respectively to the sides BC, CA, AB, and cutting the other sides in the points Y, Z', Z, X', X, Y'. Show that the lines $Y'Z$, $Z'X$, $X'Y$ will form a triangle in perspective with the triangle ABC, and having K for centre of perspective.

See the figure of § 131.

Ex. 8. In the same figure, show that the triangle formed by the lines $Y'Z$, $Z'X'$, XY and the triangle formed by the lines YZ', ZX, $X'Y'$, will be in perspective with the triangle ABC; and have a common centre of perspective.

Ex. 9. If XYZ be any transversal of the triangle ABC, and if $XY'Z''$, $X''YY'$, $X'Y''Z$ be three other transversals passing through the point O; show that the triangles formed by the lines $Y''Z'$, $Z''X'$, $X''Y'$ will form a triangle in perspective with the triangle ABC, and having the point O for centre of perspective.

Ex. 10. Two triangles $A'B'C'$, $A''B''C''$ are inscribed in the triangle ABC, so that AA', BB', CC' are concurrent, and likewise AA'', BB'', CC''. If $B'C'$, $B''C''$ intersect in X; $C'A'$, $C''A''$ in Y; and $A'B'$, $A''B''$ in Z; show that the triangle XYZ will be in perspective with each of the triangles ABC, $A'B'C'$, $A''B''C''$.

Ex. 11. If the points of intersection of corresponding sides of two given triangles form a triangle in perspective with each of them, show that the lines joining the corresponding vertices of the given triangles will form a triangle which is in perspective with each of the given triangles, and also with the triangle formed by the points of intersection of their corresponding sides.

Ex. 12. On the sides BC, CA, AB of a triangle are taken the points X, Y, Z; and the circumcircle of the triangle XYZ is drawn cutting the sides of the triangle ABC in X', Y', Z'. The lines YZ', ZX', XY' form a triangle $A'B'C'$, and the lines $Y'Z$, $Z'X$, $X'Y$ form a triangle $A''B''C''$. Show that the triangles ABC, $A'B'C'$, $A''B''C''$ are copolar, and that when the triangle XYZ is of constant shape the common pole of these triangles is a fixed point.

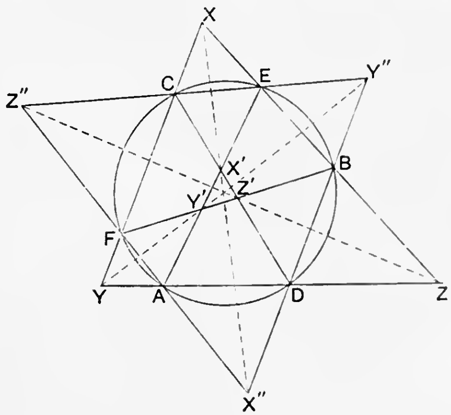
[H. M. Taylor, *L.M.S. Proc.* Vol. xv.]

Pascal's theorem.

181. To illustrate the use of the preceding theorems relating to triangles in perspective, we propose to discuss briefly the chief properties of a hexastigm inscribed in a circle. The simplest property is due to Pascal, and is called Pascal's theorem. It is usually quoted in the form: *The opposite sides of any hexagon inscribed in a circle intersect in three collinear points.* The more precise statement of the theorem would be: *The three pairs of opposite connectors in every complete set of connectors of a hexastigm inscribed in a circle intersect in three collinear points*; which is equivalent to the following: *The fifteen connectors of a hexastigm inscribed in a circle intersect in forty-five points which lie three by three on sixty lines.*

A hexastigm evidently has fifteen connectors. To find the number of points in which these intersect, apart from the vertices of the hexastigm, let us group the vertices in sets of four. This may be done in $6.5.4.3/24$, i.e. 15 ways. Now each group of four points forms a tetrastigm, which has three centres. Hence, the connectors of a hexastigm will intersect in 3.15, i.e. 45 points or centres.

182. Let A, B, C, D, E, F be any six points on a circle. Let AD, BE, CF form the triangle XYZ; BF, CD, AE the triangle $X'Y'Z'$; and CE, AF, BD the triangle $X''Y''Z''$. We shall prove



that the triangles XYZ, $X'Y'Z'$, $X''Y''Z''$ are copolar, that is, are in perspective two and two, and have the same centre of perspective.

Since the points A, B, C, D, E, F are concyclic, we have by Euclid, Bk. III., Prop. 35,

$$XE \cdot XB = XC \cdot XF,$$
$$YC \cdot YF = YA \cdot YD,$$
$$ZA \cdot ZD = ZB \cdot ZE.$$

Therefore $$\frac{XE \cdot XB}{XC \cdot XF} \cdot \frac{YC \cdot YF}{YA \cdot YD} \cdot \frac{ZA \cdot ZD}{ZB \cdot ZE} = 1.$$

Therefore by § 177, the triangle XYZ is in perspective with each of the triangles $X'Y'Z'$ and $X''Y''Z''$.

By § 167, we infer that these three triangles have the same centre of perspective.

Hence, by § 170, the axes of perspective of the three triangles are concurrent.

Let O be the common centre of perspective of the triangles, and O' the point of intersection of their axes of perspective. Then by § 171, we see that the triangles formed by the lines AD, BF, CE; BE, DC, AF; CF, AE, BD are also copolar, having O' for their common centre of perspective, and O for the point of concurrence of their axes of perspective.

183. Let us use the notation $\begin{pmatrix} AD \\ BF \end{pmatrix}$ to represent the point of intersection of the lines AD and BF. Then, since the triangles XYZ, $X'Y'Z'$ are in perspective, the points

$$\begin{pmatrix} AD \\ BF \end{pmatrix}, \begin{pmatrix} BE \\ CD \end{pmatrix}, \begin{pmatrix} CF \\ AE \end{pmatrix}$$

are collinear.

In the same way we could show that the pairs of opposite connectors in any other complete set of connectors of the hexastigm intersect in three collinear points.

The line of collinearity of three such points is called a *Pascal line.*

Since there are sixty complete sets of connectors (§ 137, Ex. 2), it follows that there are sixty Pascal lines.

Again, since the triangles XYZ, $X'Y'Z'$, $X''Y''Z''$ are copolar, it follows that the Pascal lines

$$\begin{pmatrix} AD \\ BF \end{pmatrix}, \begin{pmatrix} BE \\ CD \end{pmatrix}, \begin{pmatrix} CF \\ AE \end{pmatrix};$$

$$\binom{AD}{CE}, \binom{BE}{AF}, \binom{CF}{BD};$$
$$\binom{BF}{CE}, \binom{CD}{AF}, \binom{AE}{BD}$$

are concurrent.

The point of concurrence of three such Pascal lines is called a *Steiner point;* it may conveniently be represented by the notation

$$\binom{ABC}{DEF} \text{ or } \begin{pmatrix} AD, & BE, & CF \\ BF, & CD, & AE \\ CE, & AF, & BD \end{pmatrix}.$$

There is evidently one Steiner point on each Pascal line.

Again, from § 182, we see that the common pole of the three triangles corresponding to this Steiner point, is the Steiner point

$$\binom{ABC}{DFE}.$$

Now from six points A, B, C, D, E, F, we can select three such as A, B, C, in twenty ways, and when we combine this group with the complementary triad D, E, F, we have only ten different arrangements; but we see above that we can take one group such as (DEF) in either of two cyclic orders. Hence we infer that there are in all twenty Steiner points belonging to the figure. And since there are three Pascal lines passing through every Steiner point, we infer that there are sixty Pascal lines.

It is easy to see that a point such as $\binom{AD}{BF}$ will occur on four different Pascal lines, namely the lines

$$\binom{AD}{BF}, \binom{BE}{CD}, \binom{CF}{AE};$$
$$\binom{AD}{BF}, \binom{BC}{ED}, \binom{EF}{AC};$$
$$\binom{AD}{BF}, \binom{FE}{CD}, \binom{CB}{AE};$$
$$\binom{AD}{BF}, \binom{BE}{AC}, \binom{CF}{ED}.$$

Hence, since three of the forty-five points of intersection of the connectors of the hexastigm lie on each Pascal line, we infer that there are $4 \times 45/3$ Pascal lines; that is sixty Pascal lines.

The sixty Pascal lines pass three by three through each Steiner point, and four by four through the forty-five points of intersection of the connectors of the hexastigm. It follows that the Pascal lines will intersect one another in points other than these. For further information on this subject, the reader is referred to a Note at the end of Salmon's *Conics*, where there is a complete discussion of the question.

Steiner was the first (*Gergonne Annales de Mathém.*, Vol. XVIII.) to draw attention to the properties of the complete figure. And the subject has been fully worked out by Kirkman and Cayley.

184. Ex. 1. Show that the sixty Pascal lines pass three by three through sixty points besides the twenty Steiner points. [Kirkman.]

Let us consider the triangle formed by the lines AB, CD, EF and the triangle formed by the three Pascal lines

$$\begin{pmatrix}AB\\DE\end{pmatrix},\quad \begin{pmatrix}CE\\BF\end{pmatrix},\quad \begin{pmatrix}DF\\AC\end{pmatrix};$$
$$\begin{pmatrix}CD\\AF\end{pmatrix},\quad \begin{pmatrix}BF\\CE\end{pmatrix},\quad \begin{pmatrix}AE\\BD\end{pmatrix};$$
$$\begin{pmatrix}EF\\BC\end{pmatrix},\quad \begin{pmatrix}BD\\AE\end{pmatrix},\quad \begin{pmatrix}AC\\DF\end{pmatrix}.$$

These triangles are in perspective, for their corresponding sides intersect on the Pascal line

$$\begin{pmatrix}AB\\DE\end{pmatrix},\ \begin{pmatrix}CD\\AF\end{pmatrix},\ \begin{pmatrix}EF\\BC\end{pmatrix}.$$

Therefore the lines which join their corresponding vertices are concurrent. But these are the three Pascal lines

$$\begin{pmatrix}AB\\CD\end{pmatrix},\quad \begin{pmatrix}CE\\BF\end{pmatrix},\quad \begin{pmatrix}DF\\AE\end{pmatrix};$$
$$\begin{pmatrix}CD\\EF\end{pmatrix},\quad \begin{pmatrix}BF\\AC\end{pmatrix},\quad \begin{pmatrix}AE\\BD\end{pmatrix};$$
$$\begin{pmatrix}EF\\AB\end{pmatrix},\quad \begin{pmatrix}AC\\DF\end{pmatrix},\quad \begin{pmatrix}BD\\CE\end{pmatrix}.$$

The point of concurrence of these lines is called a *Kirkman point.*

It is easy to prove that there are three Kirkman points on each Pascal line; and that there are in all sixty Kirkman points.

Ex. 2. Show that the twenty Steiner points lie four by four on fifteen lines, and that the sixty Kirkman points lie three by three on twenty lines other than the Pascal lines.

Ex. 3. If a hexagram be circumscribed to a circle, show that its vertices may be connected by forty-five lines (or diagonals) which pass three by three through sixty points.

This theorem is known as Brianchon's theorem. It is readily deduced from § 180, Ex. 3.

Ex. 4. Show that the sixty points mentioned in the last example lie three by three on twenty lines, which pass four by four through fifteen points.

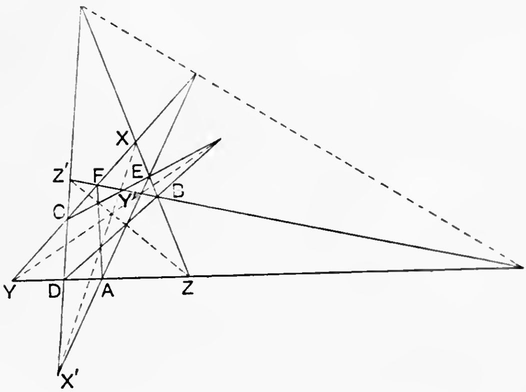
Ex. 5. Show that the sixty points mentioned in Ex. 3 also lie three by three on sixty lines, which pass three by three through twenty other points.

185. The properties which exist for a hexastigm inscribed in a circle are also true of any hexastigm formed by the points of intersection of non-corresponding sides of two triangles which are in perspective. Such a hexastigm is called a *Pascal hexastigm.*

Let XYZ, $X'Y'Z'$ be any two triangles in perspective, and let A, B, C, D, E, F be the points of intersection of non-corresponding sides of these triangles.

By § 176, we have

$$\frac{XE \,.\, XB}{XC \,.\, XF} \cdot \frac{YC \,.\, YF}{YA \,.\, YD} \cdot \frac{ZA \,.\, ZD}{ZB \,.\, ZE} = 1.$$



Hence by § 177, the triangle formed by the lines CE, AF, BD will also be in perspective with the triangles XYZ, $X'Y'Z'$. Also from § 167, it follows that these three triangles are copolar.

Again by § 177, it follows that the triangle XYZ is copolar with the triangles formed by the lines

BF, CA, DE;

CE, DF, BA.

Also, for the same reason, the triangle XYZ will be copolar with the triangles formed by the lines

EF, CD, AB;

CB, AF, ED;

and with the triangles formed by the lines

$$BC,\ FD,\ AE;$$
$$FE,\ AC,\ BD.$$

In the same way we can find three pairs of triangles copolar with the triangles $X'Y'Z'$, and the triangle formed by the lines

$$CE,\ AF,\ BD.$$

We shall thus obtain ten different triads of triangles—each triad having a common centre of perspective.

Now let us consider any one of these triads of triangles, say the triangles XYZ, $X'Y'Z'$, and the triangle formed by the lines CE, AF, BD, that is the triangles whose sides are

$$AD,\ BE,\ CF;$$
$$BF,\ CD,\ AE;$$
$$CE,\ AF,\ BD.$$

The axes of perspective of these triangles will be concurrent (§ 170); the point of concurrence being the Steiner point $\begin{pmatrix} ABC \\ DEF \end{pmatrix}$. We have evidently obtained the same arrangement by this method as we obtained in § 183, when the six points were points on a circle. Hence we may infer that if we make a list of the ten triads of triangles, as indicated above, each triangle will occur in four different triads; so that the list would be complete.

By proceeding as in § 182, we shall find by means of § 171, ten other triads of triangles, each triad producing three Pascal lines, which co-intersect in a Steiner point.

Hence we have the theorem: *If the three pairs of opposite connectors in any complete set of connectors of a hexastigm intersect in three collinear points, the three pairs of opposite connectors in every complete set will also intersect in three collinear points.*

Ex. 1. Show that any two triads of collinear points on different straight lines determine a Pascal hexastigm.

Ex. 2. Any transversal cuts the sides of the triangle ABC in the points X, Y, Z; and O is any fixed point. Show that the lines OX, OY, OZ will cut the sides of the triangle ABC in six points which determine a Pascal hexastigm.

186. The lines which join non-corresponding vertices of two triangles in perspective form a hexagram which is called a *Brianchon hexagram*.

Ex. 1. Show that every triad of opposite vertices of a Brianchon hexagram lie on three concurrent lines.

Ex. 2. Any point O is joined to the vertices of a triangle ABC, and the lines OA, OB, OC cut a given straight line in the points X, Y, Z. Show that the lines XB, XC, YC, YA, ZA, ZB determine a Brianchon hexagram.

Ex. 3. Show that if $ABCDEF$ be any Pascal hexastigm, the lines AB, BC, CA, DE, EF, FD will determine a Brianchon hexagram.

It is easy to see that a triad of diagonals of this hexagram are the lines

$$\begin{pmatrix} AB \\ DE \end{pmatrix}, \begin{pmatrix} BC \\ FD \end{pmatrix}; \quad \begin{pmatrix} BC \\ EF \end{pmatrix}, \begin{pmatrix} CA \\ DE \end{pmatrix}; \quad \begin{pmatrix} CA \\ FD \end{pmatrix}, \begin{pmatrix} AB \\ EF \end{pmatrix}.$$

But these lines are three Pascal lines of the hexastigm, which meet in a Kirkman point. (§ 184, Ex. 1.)

Hence, by applying Brianchon's theorem to this hexagram, we have at once a proof of the theorem that the sixty Kirkman points of a Pascal hexastigm lie three by three on twenty lines. (§ 184, Ex. 2.)

Ex. 4. Show that if $ABCDEF$ be any Pascal hexastigm, the lines joining the points A, D to the points $\begin{pmatrix} BC \\ DF \end{pmatrix}$, $\begin{pmatrix} AC \\ EF \end{pmatrix}$ respectively, intersect on the Pascal line

$$\begin{pmatrix} AB \\ DE \end{pmatrix}, \begin{pmatrix} CD \\ AF \end{pmatrix}, \begin{pmatrix} EF \\ BC \end{pmatrix}. \qquad \text{[Salmon.]}$$

Ex. 5. The opposite vertices of a tetragram are A, A'; B, B'; C, C'; and points X, X'; Y, Y'; Z, Z' are taken on the diagonals AA', BB', CC', so that the ranges $\{AA', XX'\}$, $\{BB', YY'\}$, $\{CC', ZZ'\}$ are harmonic. Show that X, X', Y, Y', Z, Z' are the vertices of a Pascal hexastigm.

Ex. 6. If through each centre of a tetrastigm, a pair of lines be taken, harmonically conjugate with the connectors of the tetrastigm which intersect in that centre, show that these six lines will form a Brianchon hexagram.

General theory.

187. Suppose we have any figure F consisting of any number of points A, B, C, ..., not necessarily in one plane; let these points be joined to any point O. Let any plane be drawn cutting the lines OA, OB, OC, ... in the points A', B', C', ... forming the figure F'. The figure F' is said to be the *projection* of the given figure F; the point O is called the *vertex of projection*; and the plane of F' is called the *plane of projection*.

188. Let us consider more particularly the case when the figure F is a plane figure.

i. It is evident that to any point A of F corresponds one point and only one point A' of F', and vice versa.

ii. If any three points A, B, C of F are collinear, the corresponding points A', B', C' of F' will be collinear. For since A, B, C are collinear, OA, OB, OC must lie in one plane, which can only cut the plane of projection in a straight line; that is A', B', C' must be collinear. Hence, to every straight line of F corresponds one and only one straight line of F'.

iii. If two straight lines of the figure F intersect in the point A, it is evident that the corresponding lines of F' will intersect in the corresponding point A'. Hence it follows that if any system of lines of F are concurrent, the corresponding lines of F' will be concurrent.

iv. If $\{AB, CD\}$ be any harmonic range in the figure F, then since $O\{AB, CD\}$ is a harmonic pencil, it follows that the corresponding points of F' will form a harmonic range; that is to say, $\{A'B', C'D'\}$ will be harmonic.

189. Ex. 1. Show that if $P\{AB, CD\}$ be a harmonic pencil in the figure F, $P'\{A'B', C'D'\}$ will be a harmonic pencil in the projected figure F'.

Ex. 2. Show that any range in involution will project into a range in involution.

190. Let A and B be any two points in a plane figure F, and let A', B' be the corresponding points in F' the projection of F on any plane, the vertex of projection being any point O. Let the planes of F and F' be denoted by α and α'. Then since AB, $A'B'$ are two straight lines in the same plane OAB, they must intersect. But AB lies in the plane α, and $A'B'$ in the plane α'; hence the point of intersection of AB and $A'B'$ must be a point in the line of intersection of the two planes α and α'. Similarly any straight line x of F will intersect the corresponding line x' of F' in a point lying on the line of intersection of the planes α, α'. The line of intersection of the two planes α, α' is called the *self-projected line*. It is evident that every point on it considered as belonging to the figure F, coincides with the corresponding point of F'.

191. Now suppose we have a plane figure F, and its projection F' on some plane, O being the vertex of projection. Let us take any other point P not lying on either of the planes containing F and F'; and with P as vertex let us project the whole figure on any plane, for simplicity the plane of F.

Let A, B, C, ... be any points of F; A', B', C', ... the corresponding points of F'. Let PA', PB', PC', ... cut the plane of F in the points A'', B'', C'', ... These points will form a figure F'' in the same plane as F, and A'', B'', C'', ... may be called the points of F'' which correspond to A, B, C, ... of F. Let PO cut the plane of F in O'.

It is evident that the following relations will exist between the figures F and F'':—

i. *The line joining any point of F to the corresponding point of F'' passes through a fixed point.*

For O, A, A' are collinear, therefore PO, PA, PA' lie in the same plane, and therefore O', A, A'' are collinear.

ii. *To any straight line of F corresponds a straight line of F''.*

For let A, B, C be three collinear points of F, then A', B', C' are collinear points of F', and therefore by § 188, A'', B'', C'' are collinear points of F''.

iii. *If any system of lines of F are concurrent the corresponding lines of F'' are also concurrent.*

For by § 188, the corresponding lines of F' are concurrent, and therefore the corresponding lines of F'' are concurrent.

iv. *If any points of F form a harmonic range the corresponding points of F'' will form a harmonic range.*

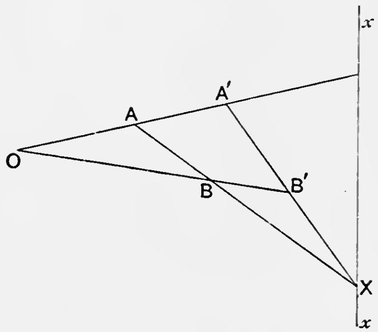
For by § 188, the corresponding points of F' form a harmonic range, therefore also do the corresponding points of F''.

v. *Every straight line of F intersects the corresponding straight line of F'' in a point lying on a fixed straight line.*

This follows at once from § 190, the straight line in which corresponding lines intersect being the line of intersection of the planes of F and F', since the plane of F'' is the same as that of F.

192. Any plane figure F being given, any other figure F'' obtained in the manner explained in the last article (viz.: by first projecting F on a plane and then with a different vertex projecting the new figure on the plane of F), is said to be in *perspective* with F. The fixed point through which pass all lines connecting corresponding points (§ 191, i.) is called the *centre of* *perspective*; and the fixed line which is the locus of the points of intersection of corresponding lines (§ 191, v.) is called the *axis of perspective.*

193. It is however unnecessary to go through the process of projection in order to construct a figure which shall be in perspective with a given figure. It is clear that if we were proceeding as in § 191, we might select the centre of perspective, and the axis of perspective. Then again, since we might have taken the plane of F' passing through the axis of perspective, and any assumed point, we may select any point A' as the point corresponding to a given point A. Hence to obtain the figure in perspective with a given figure F, let O be the centre of perspective, x the axis of perspective,



and let A' be the point corresponding to the point A. Let B be any other point of F, B' the corresponding point of F'. Then since AB, $A'B'$ are corresponding lines, they must intersect on the axis x. Let AB cut the axis x in the point X. Then $A'X$ will intersect OB in required point B'. In the same way the point corresponding to any other point may be constructed.

If F and F' be two figures in perspective; any point P may be considered as belonging to either figure. Considered as belonging to F, let P' be the corresponding point of F'; and considered as belonging to F', let Q be the corresponding point of F. Then it must be noticed that Q and P' will not coincide, unless P be a point on the axis of perspective; in which case Q and P' coincide with P.

The axis of perspective of the two figures may thus be regarded as the locus of points (other than the centre of perspective), which coincide with their corresponding points.

Likewise the centre of perspective may be regarded as the point through which pass all self-corresponding straight lines except one—the axis of perspective.

Two figures may be in perspective in more than one way. For instance, the triangles ABC, $A'B'C'$ may be so situated that AB', BC', CA' are concurrent, and also AC', BA', CB'. In this case the triangle ABC may be said to be in perspective with the triangles $B'C'A'$, $C'A'B'$. But when this is so it may be easily shown (§ 165, Ex. 5) that AA', BB', CC' must also be concurrent. Hence if two triangles are doubly in perspective, they are triply in perspective.

194. Ex. 1. If F_1, F_2, F_3 be three figures in perspective two and two in the same plane, show that if they have a common centre of perspective, their three axes of perspective are concurrent.

Let O be the common centre of perspective; $x_{2,3}$, $x_{3,1}$, $x_{1,2}$ their three axes of perspective. Let $x_{3,1}$, $x_{1,2}$ intersect in P. Then because P lies on $x_{3,1}$, P, considered as belonging to F_1, coincides with the corresponding point of F_3. Similarly because P lies on $x_{1,2}$ it coincides with the corresponding point of F_2. Hence P must lie on $x_{2,3}$, or coincide with O. In the latter case, let Q be the point of intersection of $x_{1,2}$, $x_{2,3}$; then as before it may be proved that Q must lie on $x_{1,3}$, or coincide with O. Thus, in either case the three axes of perspective $x_{2,3}$, $x_{3,1}$, $x_{1,2}$ are concurrent.

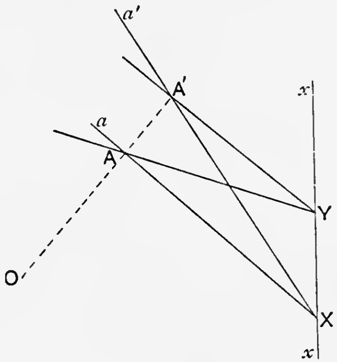
Ex. 2. Show that all triangles formed by corresponding points of F_1, F_2, F_3 in the last Ex. are in perspective, P being their common centre of perspective.

Ex. 3. If F_1, F_2, F_3 be three figures in perspective, having a common axis of perspective, show that the three centres of perspective are collinear.

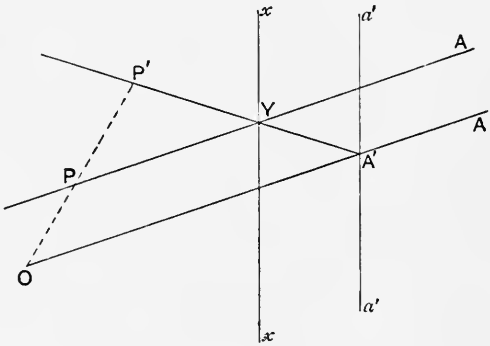
Ex. 4. If ABC, $A'B'C'$ be two triangles in perspective, and if X, Y, Z be three points on the axis of perspective, such that AX, BY, CZ are concurrent, show that $A'X$, $B'Y$, $C'Z$ will be concurrent.

195. Another method of constructing a figure F' in perspective with a given figure F, is to suppose that the line of F' which corresponds to a given line of F is known.

Thus let O be the centre of perspective, x the axis of perspective: and suppose that a, a' are a pair of corresponding lines. If any line be drawn through O cutting a, a' in A and A', it is evident that A' will be the corresponding point to A. Again, if AY be any line of F, cutting the axis of perspective in Y, and the line a in A. Then $A'Y$ will be the line of F' which corresponds to AY.



196. We may take the line at infinity in either figure as one of our given lines. Then any line in the other figure which is parallel to the axis of perspective may be taken as the corresponding



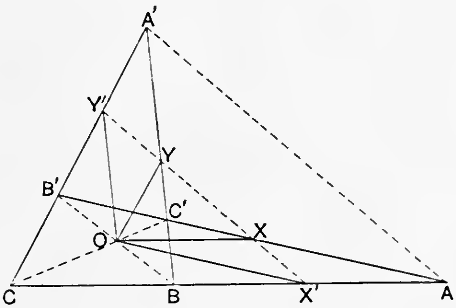
line. The construction of F' is very similar to the previous construction. Thus let a be the line at infinity, then a' is a line parallel to the axis of perspective. Draw any line through O cutting a' in A', then the corresponding point A of F is at infinity. Draw any line AY parallel to OA', cutting the axis of perspective in Y. Then YA' will be the corresponding line of F'. And if P be any point on AY, OP will cut $A'Y$ in P', so that P' is that point of F' which corresponds to P.

197. If we suppose P and P' given in the last figure, we can easily find the line of F' which corresponds to the line at infinity in F. Thus we have only to draw any line PY to meet the axis of perspective in Y; then $P'Y$ will cut the line through O parallel to PY in A', which will be the point of F' corresponding to the point at infinity on the line OA'. Therefore the line through A' parallel to the axis of perspective will be that line of F' which corresponds to the line at infinity in F.

198. Ex. 1. Through the point of intersection of two diagonals of a tetragram lines are drawn respectively parallel to the four sides and intersecting respectively the sides opposite to those to which they are parallel. Prove that these four points of intersection lie on a straight line.

[Trin. Coll., 1890.]

Let A, A'; B, B'; C, C' be the pairs of opposite vertices of the tetragram; and let BB', CC' intersect in O. Taking O for centre of perspective, and AA' for the axis of perspective, we may consider the figure $B'C'BC$ as in perspective



with the figure $BCB'C'$. If OX be drawn parallel to BC to meet $B'C'$ in X, and if OX' be drawn parallel to $B'C'$ to meet BC in X', it is evident that XX' will be that line of the figure $B'C'BC$ which corresponds to the line at infinity in the figure $BCB'C'$. Hence the theorem is proved.

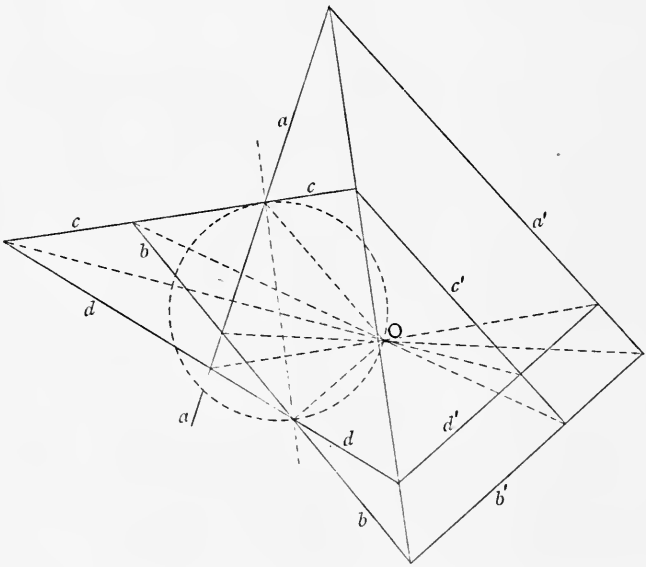
It may be noticed that XX' is parallel to AA'.

Ex. 2. A hexagon can be inscribed in one circle and circumscribed about another. Its three diagonals intersect in the point O, and lines are drawn through O parallel to the sides. Show that the points in which these lines intersect the sides opposite to those to which they are parallel, are collinear.

Ex. 3. The lines joining the vertices of the triangle ABC to any point O intersect the opposite sides in A', B', C'; and BC, CA, AB intersect $B'C'$, $C'A'$, $A'B'$ in X, Y, Z. Show that the lines drawn through O parallel to BC, CA, AB, form a triangle which is in perspective with the triangle formed by the lines AX, BY, CZ.

199. By suitably choosing the centre of perspective, and the axis of perspective, we can often form a figure F' which shall be in perspective with a given figure F, so that F' shall be a simpler figure. The advantage gained by so doing is that we are able to discover properties of the figure F by transforming known properties of the simpler figure F'.

Thus let a, b, c, d be the four sides of any tetragram, and let us take for our axis of perspective a line parallel to the diagonal joining the points ac, bd. Then if we suppose the line corresponding to this diagonal in the new figure to be at infinity, it is easy to see that the new figure will be a parallelogram. Further, if we take for our centre of perspective, a point on the circle which has the diagonal joining the points ac, bd for a diameter, the new figure becomes a rectangle.



For the lines a', c' are parallel to the line joining O to the point ac, and b', d' are parallel to the line joining O to the point bd.

200. Ex. 1. Show that the lines joining any point to the opposite vertices of a tetragram form a pencil in involution.

Ex. 2. Show that the middle points of the diagonals of a tetragram are collinear.

Ex. 3. The diagonals of a parallelogram bisect each other. Obtain the corresponding theorem for any tetragram.

Ex. 4. Any line cuts the opposite pairs of connectors of a tetrastigm in a range in involution. Prove this theorem by forming a figure in perspective, such that one connector of the given figure becomes the line at infinity in the new figure.

Ex. 5. Show that a triangle can always be constructed which shall be in perspective with one given triangle, and be similar to another given triangle.

Ex. 6. Generalise the theorem in Ex. 2.

CHAPTER IX.

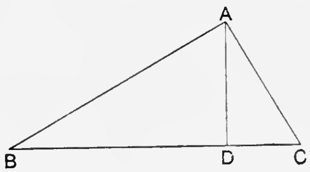
THE THEORY OF SIMILAR FIGURES.

Similar triangles.

201. Two triangles are said to be *similar* when they are equiangular. It is proved in Euclid (Bk. VI., Prop. 4) that the sides of one triangle are proportional to the homologous, or corresponding, sides of the other. It is, however, necessary to distinguish the case when the angles of the triangles are measured in the same sense, from the case when they are measured in opposite senses.

Let ABC, $A'B'C'$ be two similar triangles: then, when the angles ABC, BCA, CAB are respectively equal to the angles $A'B'C'$, $B'C'A'$, $C'A'B'$, the triangles are said to be *directly similar*; but, when the angles ABC, BCA, CAB are respectively equal to the angles $C'B'A'$, $A'C'B'$, $B'A'C'$, the triangles are said to be *inversely similar*.

As an illustration, let BAC be a right-angled triangle, and let AD be the perpendicular from the right angle on the hypotenuse. Then the triangles



BDA, ADC are directly similar, but each is inversely similar to the triangle BAC.

202. Ex. 1. If two triangles be inversely similar to the same triangle, show that they are directly similar to each other.

Ex. 2. If AA', BB', CC' be the perpendiculars from the vertices of the triangle ABC on the opposite sides, show that the triangles $AB'C'$, $A'BC'$,

$A'B'C$ are directly similar to each other, but inversely similar to the triangle ABC.

Ex. 3. If D, E, F be the middle points of the sides of the triangle ABC, show that the triangle DEF is directly similar to the triangle ABC.

Ex. 4. Two circles cut in the points A, B; and through B two lines PBQ, $P'BQ'$ are drawn, cutting one circle in P, P' and the other in Q, Q'. Show that the triangles APQ, $AP'Q'$ are directly similar.

Ex. 5. If the triangle $A'B'C'$ be inversely similar to the triangle ABC, show that the lines drawn through A', B', C' parallel respectively to BC, CA, AB will be concurrent, and that their point of intersection will lie on the circumcircle of the triangle $A'B'C'$.

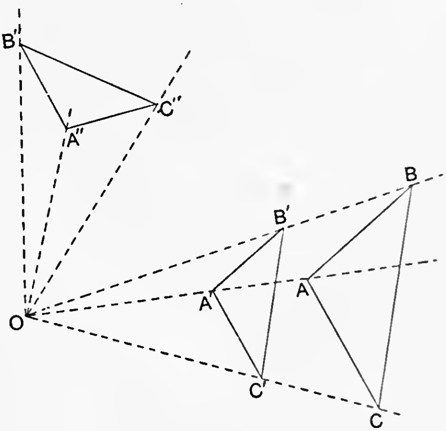
Ex. 6. If the triangles ABC, $A'B'C'$ be inversely similar, show that

$$(A'BC)+(B'CA)+(C'AB)=(ABC).$$

Ex. 7. The first Brocard triangle of any triangle is inversely similar to it.

203. When two triangles are placed so that their corresponding sides are parallel, it is evident that they are directly similar. They are also in perspective, having the line at infinity for their axis of perspective; consequently the lines joining corresponding vertices are concurrent.

Triangles so situated are said to be *homothetic*, and the centre of perspective is called their *homothetic centre*.



Let $A'B'C'$ be any triangle having its sides parallel to the corresponding sides of the triangle ABC; and let O be the centre of perspective. Since the corresponding sides are parallel, it follows at once that

$$OA' : OB' : OC' = OA : OB : OC.$$

204. Let ABC, $A'B'C'$ be two homothetic triangles, and let $A'B'C'$ be turned about the homothetic centre O, so as to come into the position $A''B''C''$.

It is obvious that the triangles $A''B''C''$, ABC are directly similar, and that

$$OA'' : OB'' : OC'' = OA : OB : OC.$$

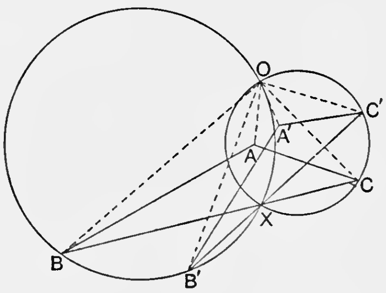
Further, it is easy to see that the angles AOA'', BOB'', COC'', and the angles at which the corresponding sides intersect are all equal.

It is evident that the triangles AOB, BOC, COA are directly similar to the triangles $A''OB''$, $B''OC''$, $C''OA''$. Hence, it appears that whatever relation the point O has to the triangle ABC, it has a similar relation to the triangle $A''B''C''$. For instance, if O were the orthocentre of the triangle ABC, it would also be the orthocentre of the triangle $A''B''C''$.

The point O is called the *centre of similitude* of the two triangles ABC, $A''B''C''$.

We shall now show that any two triangles which are directly similar, have a centre of similitude, which can be easily found. It will be perceived that when the centre of similitude is known, then, by turning one of the triangles about the centre it may be brought into such a position as to be homothetic with the other triangle.

205. *To find the centre of similitude of two triangles which are directly similar.*



Let ABC, $A'B'C'$ be any two triangles which are directly similar. Let BC, $B'C'$ intersect in the point X, and let the circumcircles of the triangles BXB', CXC' intersect in the point O.

It is evident that the triangles BOC, $B'OC'$ are directly similar. Hence the triangles AOC, AOB are directly similar to the triangles $A'OC'$, $A'OB'$.

Further, the angles COC', BOB' are each equal to the angle CXC'. Hence if the triangle $A'B'C'$ be turned about the point O through an angle equal to $C'OC$, so that the lines OC', OB' shall coincide with OC, OB, it is easy to see that the triangle $A'B'C'$ in its new position will be homothetic to the triangle ABC.

Thus O is the centre of similitude of the two triangles ABC, $A'B'C'$.

206. Ex. 1. If two directly similar triangles be inscribed in the same circle, show that the centre of the circle is their centre of similitude.

Show also that the pairs of homologous sides of the triangles intersect in points which form a triangle directly similar to each of them.

[Trinity Coll. Sch. Exam. 1885.]

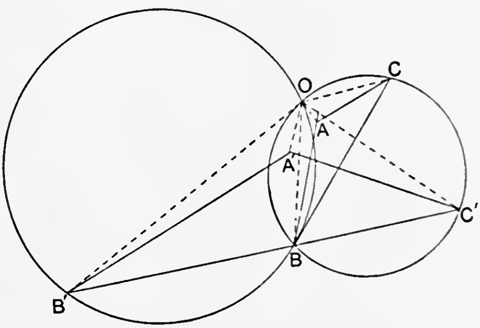
Ex. 2. If triangles directly similar to a given triangle be described on the perpendiculars of another triangle, show that their centres of similitude are the feet of the perpendiculars from the orthocentre on the medians of the triangle.

Ex. 3. If ABC be a triangle of constant shape, and if A be a fixed point, show that if the vertex B move on a fixed straight line, the vertex C will move along another straight line.

Show also that if the locus of B be a circle, then the locus of C will also be a circle.

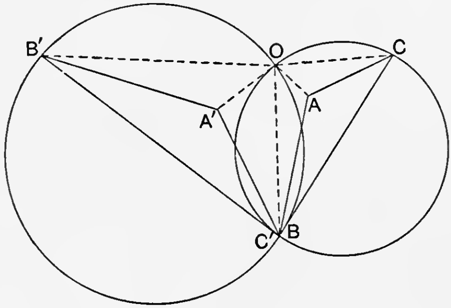
Ex. 4. If ABC, $A'B'C'$ be two triangles which are directly similar, and if the triangle $A'B'C'$ be turned about any point in its plane, show that the locus of the centre of similitude will be a circle.

207. The construction given in § 205 requires a slight modification when X, the point of intersection of BC, $B'C'$ coincides with B or C. Let us



suppose that $B'C'$ passes through B. Then, the centre of similitude O will be the point of intersection of the circle circumscribing the triangle BCC', and the circle which passes through B' and touches BC at B.

Again, if C' coincide with the point B, the centre of similitude will be the point of intersection of the circle which passes through B' and touches BC at B, and the circle which passes through C and touches BC' at B.



208. Ex. 1. If O be the centre of similitude of the directly similar triangles ABC, DAE, show that AO passes through the symmedian point of the triangle ABD.

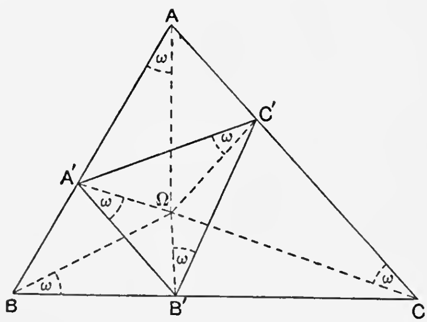
Ex. 2. In the same case, if AO meet the circumcircle of the triangle ABD in H, show that AH is bisected in the point O.

Ex. 3. If triangles be described on the sides of the triangle ABC, so as to be directly similar to each other, show that the three centres of similitude of these triangles taken two at a time, are the vertices of the second Brocard triangle of the triangle ABC. See § 134.

Ex. 4. If points A', B', C' be taken on the sides BC, CA, AB of the triangle ABC, so that the triangle $A'B'C'$ is directly similar to the triangle ABC, show that the centre of similitude of the triangle $A'B'C'$ in any two of its positions is the circumcentre of the triangle ABC.

Ex. 5. In the last case show that the circumcentre of the triangle ABC coincides with the orthocentre of the triangle $A'B'C'$.

Ex. 6. If points A', B', C' be taken on the sides AB, BC, CA of the



triangle ABC, so that the triangle $A'B'C'$ is directly similar to the triangle ABC, show that the centre of similitude is a fixed point.

Let Ω be the centre of similitude. Then, Ω will lie on the circles circumscribing the triangles $AA'C'$, $BB'A'$, $CC'B'$; and the circles $AA'\Omega$, $BB'\Omega$, $CC'\Omega$ will touch $A'B'$, $B'C'$, $C'A'$, respectively, at the points A', B', and C'. Hence, the angles ΩAB, ΩBC, ΩCA are equal, and it follows by § 116, that Ω is one of the Brocard points of the triangle ABC.

It is easily seen that the angles $\Omega A'B'$, $\Omega B'C'$, $\Omega C'A'$ are each equal to ΩAB, so that Ω is the same Brocard point of the triangle $A'B'C'$.

Ex. 7. If points A', B', C' be taken on the sides CA, AB, BC, so that the triangle $A'B'C'$ is directly similar to the triangle ABC, show that the centre of similitude is the other Brocard point.

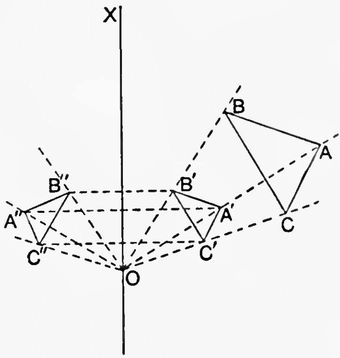
Ex. 8. If a triangle $A'B'C'$ be inscribed in a given triangle ABC, so as to be always directly similar to a given triangle, show that the centre of similitude O of the triangle $A'B'C'$, in any two of its positions is a fixed point.

[Townsend.]

Ex. 9. If a triangle $A'B'C'$ of constant shape be inscribed in a given triangle ABC, the circumcircle of the triangle $A'B'C'$ meets the sides of the triangle ABC in three points A'', B'', C'', which form another triangle of constant shape. Show that the centre of similitude O' of the triangle $A''B''C''$ in any two of its positions is a fixed point. [H. M. Taylor.]

Ex. 10. Show that the points O, O' are isogonal conjugates with respect to the triangle ABC. [Casey.]

209. Let ABC be any given triangle, and let a triangle $A'B'C'$ be constructed so as to be homothetic to the triangle ABC. Let O be the homothetic centre, and OX any line through O. Suppose the triangle $A'B'C'$ to be turned about the line OX through an angle equal to two right angles, so that its plane coincides with the plane of the triangle ABC. Let $A''B''C''$ be the new position of $A'B'C'$.



It is obvious that the triangle $A''B''C''$ is inversely similar to the triangle ABC. It is also evident from the figure that the triangles $OB''C''$, $OC''A''$, $OA''B''$ are inversely similar to the triangles OBC, OCA, OAB; that the angles AOA'', BOB'', COC'' are bisected by the line OX; and that

$$OA'' : OB'' : OC'' = OA : OB : OC.$$

Further, we see that the line OX is parallel to the internal bisector of the angles between corresponding sides of the triangles ABC, $A''B''C''$. Thus let P be any arbitrary point, and let PQ, PQ'' be drawn in the same directions as BC, $B''C''$ respectively, then OX will be parallel to the internal bisector of the angle QPQ''.

210. The point O is called the *centre of similitude* of the triangles ABC, $A''B''C''$; and the line OX the *axis of similitude* of the triangles.

Since the triangles $B''OC''$, $C''OA''$, $A''OB''$ are inversely similar to the triangles BOC, COA, AOB, the point O will have the same relative position with respect to the triangles ABC, $A''B''C''$. For instance, if O were the orthocentre of the triangle ABC, it would also be the orthocentre of the triangle $A''B''C''$.

We shall now show that any two triangles which are inversely similar, have a centre of similitude, and an axis of similitude. It is evident that when the axis of similitude is known, one of the triangles may be rotated about it so as to be brought into a position in which it is homothetic to the other triangle.

211. *To find the centre and axis of similitude of two triangles which are inversely similar.*

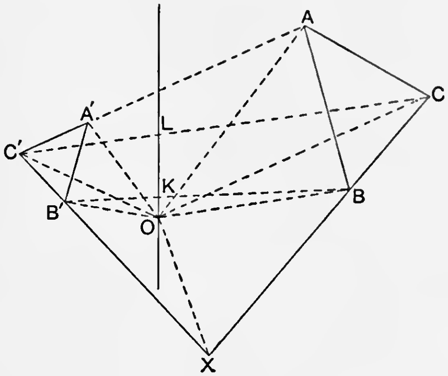
Let ABC, $A'B'C'$ be two triangles which are inversely similar. If O be the centre of similitude, it follows that the axis of similitude must bisect the angles BOB', COC'. Hence, if we divide the lines BB', CC' in the points K, L, so that

$$BK : KB' = CL : LC' = BC : B'C',$$

it is evident that KL must be the axis of similitude.

Again the triangles BOC, $B'OC'$ are inversely similar, so that the perpendiculars from O on BC, $B'C'$ must be in the same ratio as $BC : B'C'$. Consequently if BC, $B'C'$ intersect in X, the line XO will divide the angle BXB' into parts whose sines are as

$BC : B'C'$. Thus a point O can be found so that the triangles BOC, $B'OC'$ are inversely similar.



It is obvious that when O has been found in this way, the triangles $OA'B'$, $OA'C'$ are inversely similar to the triangles OAB, OAC; and that KL bisects the angle AOA'. Hence O is the centre, and KL the axis of similitude of the triangles.

212. Ex. 1. Find the centre and axis of similitude of the triangles ABC, $A'B'C'$ when $B'C'$ passes through B.

Ex. 2. Find the centre and axis of similitude of the triangles ABC, $A'B'C'$ when B and C' coincide.

Ex. 3. Show that the axis of similitude divides the lines joining corresponding points in the same ratio.

Ex. 4. If two triangles be inscribed in the same circle so as to be inversely similar, show that the triangles are in perspective.

[Trinity Coll. Sch. Exam. 1885.]

Ex. 5. In the last example, show that the axis of perspective of the triangles passes through the centre of the circle.

Ex. 6. If ABC be any triangle inscribed in a circle, and if AA', BB', CC' be drawn parallel to any given straight line meeting the circle in the points A', B', C', show that the triangles ABC, $A'B'C'$ will be inversely similar, and that their axis of perspective will pass through the centre of the circle.

Properties of two figures directly similar.

213. Let F denote any figure consisting of the system of points A, B, C,.... On the lines OA, OB, OC,... connecting these points to any point O in the same plane, let points A', B', C',... be taken so that

$$OA' : OA = OB' : OB = OC' : OC = \&c.$$

Then the figure F', consisting of the points A', B', C',..., is said to be *homothetic* to the figure F, and the point O is called the *homothetic centre.*

It is evident that if A, B, C be any three collinear points of F, the corresponding points A', B', C' of F' are also collinear; and, that the straight lines ABC, $A'B'C'$ are parallel. Hence, to every straight line of the figure F corresponds a parallel straight line of the figure F'. This also follows by considering that the two figures F and F' are in perspective, so that the theorems of § 191 hold for homothetic figures.

It is also evident that any three points A, B, C of the figure F form a triangle which is homothetic to the triangle formed by the corresponding points A', B', C' of F'.

214. If two figures be homothetic, and if one of them be turned through any angle about the homothetic centre, the two figures are said to be *directly similar.*

Let F and F' be two homothetic figures, O the homothetic centre, and let F' be turned about the point O, through an angle α. Let A, B, C,... be any points of F, and let A', B', C',... be the corresponding points of F'. Then we have

$$OA' : OA = OB' : OB = OC' : OC = \&c.$$

Also it is evident that each of the angles AOA', BOB',... is equal to α, and that each line of F, such as AB, makes with the corresponding line $A'B'$ of F' an angle equal to α.

Again, the triangles OAB, OBC,... are directly similar to the triangles $OA'B'$, $OB'C'$,...; so that the position of O with respect to one figure is exactly similar to its position with respect to the other figure.

This point O is called the *centre of similitude* of the two figures.

215. It follows, from the definition given in the last article, that two figures F and F' in the same plane will be *directly similar* when a correspondence can be established between the points of the two figures, such that: (i) To each point of F corresponds one point and only one point of F'. (ii) The distance between every pair of corresponding points subtends the same angle at a fixed point O. (iii) The distance of each point of F

from O bears a constant ratio to the distance of the corresponding point of F' from O.

Again two figures F and F' will be directly similar, when (i) each line of F makes a constant angle with the corresponding line of F', and (ii) the triangle formed by every three points of F is directly similar to the triangle formed by the corresponding points of F'. For in this case we can find the centre of similitude by proceeding as in § 205.

In applying this criterion to any two figures it is necessary to be careful as to which angle is taken as the angle between two corresponding lines. Thus, let A, B be any two points of F, A', B' the corresponding points of F'. Through any arbitrary point O draw OX parallel to and in the same direction as AB, and OX' in the same direction as $A'B'$. Then the angle between the corresponding lines AB, $A'B'$ is to be taken as equal to XOX'.

216. Directly similar figures might also have been defined to be diagrams of the same figure drawn to different scales in the same plane.

It follows at once that if two maps of the same country be placed on a table, there is one point, and only one point, which will indicate the same place on the two maps.

217. Ex. 1. The points O, A, B, C, ... of a figure F correspond to the points O', A', B', C', ... of another figure F', so that the lines OA, OB, ... are equally inclined to the lines $O'A'$, $O'B'$, ... Show that if

$$O'A' : OA = O'B' : OB = \&c.,$$

the figures F and F' will be directly similar.

Ex. 2. Hence show that any two circles are directly similar figures.

Ex. 3. Two maps of the same country, on different scales, are placed on a table, and a pin is put through both maps at a given point. If one of the maps be moved about show that the locus of the centre of similitude will be a circle.

Ex. 4. If a pair of corresponding points of two coplanar similar figures be fixed and the figures moved about in their plane, show that the locus of the centre of similitude will be a circle.

Ex. 5. Show that through any given point one and only one pair of corresponding lines of two similar figures can be drawn.

Ex. 6. If P, P' be a pair of corresponding points of two similar figures whose centre of similitude is O; show that if the locus of P be a circle passing through O, the line PP' will pass through a fixed point.

Ex. 7. If A, B, C, D be any four points on a circle, and if P, Q, R, S be the orthocentres of the triangles BCD, CDA, DAB, ABC, show that the figure $PQRS$ is directly similar to the figure $ABCD$.

218. Given any two triangles which are directly similar, it is easy to see that similar points of the two triangles will correspond. That is to say, if ABC, $A'B'C'$ be the two triangles, P and P' any similar points, (e.g. the orthocentres of the triangles), then $ABCP$ and $A'B'C'P'$ are directly similar figures. When the two triangles are homothetic, it follows that the line joining two similar points such as P and P' must pass through the centre of similitude of the two figures.

219. Ex. 1. Show that the orthocentre, the circumcentre, and the median point of any triangle are collinear.

If ABC be the triangle, D, E, F the middle points of the sides, the triangle DEF is homothetic to the triangle ABC, and the circumcentre of the latter is the orthocentre of the former.

Ex. 2. Show that if ABC be any triangle, and D, E, F be the middle points of the sides, the symmedian points of the triangles ABC, DEF are collinear with the median point of the triangle ABC.

Ex. 3. The tangents to the circumcircle of a triangle ABC form the triangle LMN, and AA', BB', CC' are the perpendiculars on the sides of the triangle ABC. Show that the lines LA', MB', NC' meet in a point which is collinear with the circumcentre and orthocentre of the triangle ABC.

Ex. 4. Show that the lines which connect the middle points of the corresponding sides of a triangle and its first Brocard triangle are concurrent.

Ex. 5. Show that, if Ω, Ω' denote the Brocard points of a given triangle, and if K' denote the isotomic conjugate point of the symmedian point of the triangle, the median point of the triangle $K'\Omega\Omega'$ coincides with the median point of the given triangle.

Properties of two figures inversely similar.

220. Let F and F' be any two homothetic figures in the same plane, and O the homothetic centre. Let F' be turned about any line OX, in its plane, through an angle equal to two right angles, so that its plane coincides with the plane of F. Then, the figure F' in its new position is said to be *inversely similar* to the figure F.

What is meant by inverse similarity is easily understood by considering F and F' to be drawings of the same map on different scales. Let us suppose F' to be drawn on transparent paper, and laid with its face downwards on the face of F, then the reverse side of F' is inversely similar to the figure F.

The point O which was originally the homothetic centre is called the *centre of similitude* of the inversely similar figures, and the line OX is called the *axis of similitude.*

221. Let $A, B, C, \ldots$ be any points of a figure F, and $A', B', C' \ldots$ the corresponding points of an inversely similar figure F'. Then if O be the centre of similitude and OX the axis of similitude, we clearly have as in § 209,

$$OA' : OA = OB' : OB = OC' : OC = \&c.$$

Also the axis OX will bisect each of the angles AOA' BOB', $COC' \ldots$; and will be parallel to the internal bisectors of the angles between the corresponding lines of the two figures.

Further, it is evident that the triangles $A'OB', A'OC', B'OC', \ldots$ will be inversely similar to the triangles AOB, AOC, $BOC, \ldots$. Hence it follows that the centre of similitude will have similar relations to the two figures.

222. Ex. 1. If A and A' be corresponding points of two figures which are inversely similar, and a, a' corresponding lines, show that the line drawn through A' parallel to a will correspond to the line through A parallel to a'.

Ex. 2. If ABC, $A'B'C'$ be two triangles which are inversely similar, the lines through the vertices of each parallel to the sides of the other are concurrent.

If P, P' be the points of concurrence, show that P and P' are corresponding points.

Ex. 3. If $A'B'C'$ be the first Brocard triangle of the triangle ABC, and if the perpendiculars from A, B, C on the sides of the triangle $A'B'C'$ intersect in T, show that the circumcentre of ABC is that point of $A'B'C'$ which corresponds to the point T.

The point T is called Tarry's point (§ 135, Ex. 7) of the triangle ABC.

Ex. 4. Find the axis of similitude and centre of similitude of any triangle and its first Brocard triangle.

The centre of similitude is the median point of the triangles (§ 135, Ex. 13).

Ex. 5. If K' be the symmedian point of the first Brocard triangle of the triangle ABC, and if S be the circumcentre, L the Lemoine centre, and T Tarry's point of the triangle ABC, show that LK' is parallel to TS.

Properties of three figures directly similar.

223. Let F_1, F_2, F_3 be any three figures which are directly similar; let S_1 be the centre of similitude of F_2 and F_3; S_2 that of F_3 and F_1; and S_3 that of F_1 and F_2.

The triangle formed by the three centres of similitude S_1, S_2, S_3, is called the *triangle of similitude* of the figures F_1, F_2, F_3; and the circumcircle of this triangle is called the *circle of similitude.*

It will be convenient to explain here the notation which will be used in the following articles. The scales on which the figures are drawn will be denoted by k_1, k_2, k_3; the constant angles at which corresponding lines of the figures intersect will be denoted by a_1, a_2, a_3; corresponding points will be denoted by P_1, P_2, P_3; and corresponding lines by x_1, x_2, x_3. The perpendicular distance of any point P from a line x will be denoted by Px.

224. *In every system of three directly similar figures, the triangle formed by three corresponding lines is in perspective with the triangle of similitude, and the locus of the centre of perspective is the circle of similitude.*

Let x_1, x_2, x_3 be any three corresponding lines, forming the triangle X_1, X_2, X_3. Then we have,

$$S_1x_2 : S_1x_3 = k_2 : k_3,$$

$$S_2x_3 : S_2x_1 = k_3 : k_1,$$

$$S_3x_1 : S_3x_2 = k_1 : k_2.$$

Therefore
$$\frac{S_1x_2}{S_1x_3} \cdot \frac{S_2x_3}{S_2x_1} \cdot \frac{S_3x_1}{S_3x_2} = 1.$$

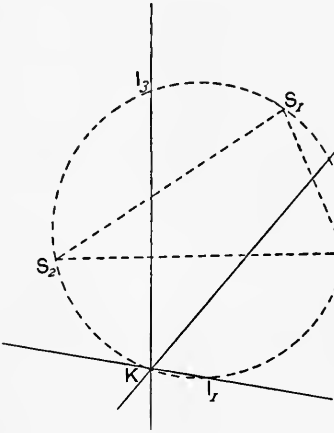
Hence by § 179, the triangle formed by the lines x_1, x_2, x_3, is in perspective with the triangle of similitude $S_1S_2S_3$.

If K be the centre of perspective of the triangles $S_1S_2S_3$, $X_1X_2X_3$, it is evident that

$$Kx_1 : Kx_2 : Kx_3 = k_1 : k_2 : k_3.$$

Now since x_1, x_2, x_3 are corresponding lines, they intersect each other at angles equal to α_1, α_2, α_3. Hence the angles of the triangle $X_1X_2X_3$ are known, and therefore the angles X_2KX_3, X_3KX_1, X_1KX_2 are constant. That is, the angles S_2KS_3, S_3KS_1, S_1KS_2 are constant; and therefore the point K must lie on the circle $S_1S_2S_3$.

225. Since three corresponding lines form a triangle in perspective with the triangle of similitude, so that the centre of perspective is a point on the circle of similitude, it follows that if three corresponding lines are concurrent their point of intersection is a point on the circle of similitude.



Let x_1, x_2, x_3 be any three corresponding lines which are concurrent, and let K be the point of intersection. Then we have

$$S_1x_2 : S_1x_3 = k_2 : k_3.$$

Hence S_1K divides the angle between x_2 and x_3 into parts whose sines are in a constant ratio.

Let x_1, x_2, x_3 cut the circle of similitude in the points I_1, I_2, I_3. Since x_2, x_3 are corresponding lines, it follows that the angle I_2KI_3 is equal to $\pi - \alpha_1$.

Hence it follows that the angles I_2KS_1, I_3KS_1 are constant. And similarly we can show that the angles I_3KS_2, I_1KS_2, I_1KS_3, and I_1KS_1 are constant.

Therefore I_1, I_2, I_3 are fixed points on the circle of similitude.

Thus we have the theorem: *Every triad of corresponding lines which are concurrent pass through three fixed points on the circle of similitude.*

These fixed points on the circle of similitude are called the *invariable points*, and the triangle formed by them is called the *invariable triangle*.

226. Ex. 1. Show that the invariable points of a system of three similar figures are corresponding points.

Ex. 2. Show that the triangle formed by any three corresponding lines is inversely similar to the invariable triangle.

Ex. 3. If K be any point on the circle of similitude, show that KI_1, KI_2, KI_3 are corresponding lines of the figures F_1, F_2, F_3.

Ex. 4. Show that the invariable triangle is in perspective with the triangle of similitude. If O be the centre of perspective, show that the distances of O from the sides of the invariable triangle are inversely proportional to k_1, k_2, k_3.

Ex. 5. If K be the centre of perspective of the triangle formed by three corresponding lines x_1, x_2, x_3 and the triangle of similitude, show that KI_1, KI_2, KI_3 are parallel to x_1, x_2, x_3 respectively.

Ex. 6. If x_1, x_2, x_3 and x_1', x_2', x_3' be two triads of corresponding lines, and if K, K' be the centres of perspective of the triangles $x_1x_2x_3$, $x_1'x_2'x_3'$ and the triangle of similitude, show that K and K' are corresponding points of the directly similar triangles $x_1x_2x_3$, $x_1'x_2'x_3'$.

Ex. 7. Show that the centre of similitude of the triangles formed by two triads of corresponding lines, is a point on the circle of similitude.

227. *The triangle formed by any three corresponding points of three directly similar figures, is in perspective with the invariable triangle, and the centre of perspective is a point on the circle of similitude.*

Let P_1, P_2, P_3 be any three corresponding points. Then if I_1, I_2, I_3 be the invariable points, the lines I_1P_1, I_2P_2, I_3P_3 are corresponding lines. But these lines intersect on the circle of similitude, since they pass through the invariable points. Hence the triangles $P_1P_2P_3$, $I_1I_2I_3$ are in perspective.

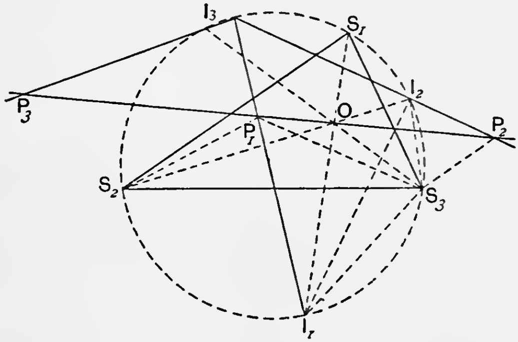
228. Ex. 1. If S_1' be that point of F_1 which corresponds to S_1 considered as a point of F_2 or F_3, show that S_1, S_1' and I_1 are collinear.

Ex. 2. If S_2', S_3' be similar points corresponding to S_2 and S_3, show that the triangles $S_1S_2S_3$, $S_1'S_2'S_3'$, and $I_1I_2I_3$ are copolar.

Ex. 3. If two triangles, formed by two triads of corresponding points, be in perspective, the locus of their centre of perspective is the circle of similitude. [Tarry.]

229. *If three corresponding points be collinear, their line of collinearity will pass through the centre of perspective of the triangle of similitude and the invariable triangle.*

Let P_1, P_2, P_3 be three corresponding points which are collinear, and let I_1, I_2, I_3 be the invariable points. Since I_1, P_1 are points of F_1 and I_2, P_2 the corresponding points of F_2, it follows that the triangles $S_3I_1I_2$, $S_3P_1P_2$ are directly similar. Therefore, the angle $S_3P_1P_2$ is equal to the angle $S_3I_1I_2$, and therefore to the angle S_3S_2O. Similarly, we can show that the angle $S_2P_1P_3$ is equal to the angle S_2S_3O. Hence the angle $S_2P_1S_3$ is equal to the angle S_2OS_3. Therefore P_1 must lie on the circumcircle of S_2OS_3.



Hence, the angle S_3P_1O is equal to the angle S_3S_2O, and therefore to the angle $S_3P_1P_2$. Therefore the line $P_1P_2P_3$ must pass through the point O.

230. It is evident from the last article that when three corresponding points P_1, P_2, P_3 are collinear, each of them lies on a fixed circle. That is, P_1 lies on the circumcircle of the triangle S_2OS_3, P_2 on the circumcircle of the triangle S_3OS_1, and P_3 on the circle S_1OS_2.

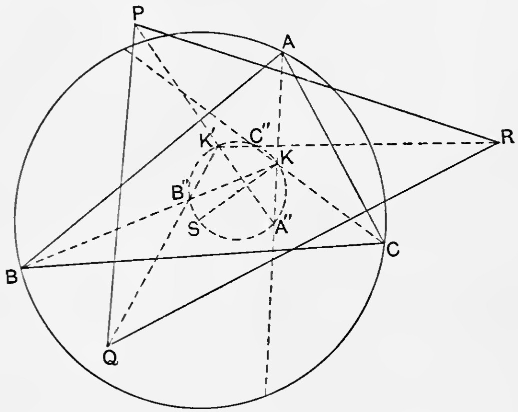
If P_2 and P_3 coincide with S_1, P_1 will coincide with the point S_1' of the figure F_1 which corresponds to the point S_1 considered as a point of F_2 or F_3. It is evident then that S_1' must lie on the circumcircle of the triangle S_2OS_3.

Special cases of three directly similar figures.

231. If three figures be described on the sides of the triangle ABC so as to be directly similar to each other, the triangle of similitude of the figures will be the second Brocard triangle of the triangle ABC (§ 208, Ex. 3); and the circle of similitude will be the Brocard circle of the triangle.

The sides of the triangle ABC will be corresponding lines, and the centre of perspective of this triangle and the triangle of similitude will be the symmedian point of ABC.

Let K be the symmedian point of the triangle ABC, and let A', B', C' be the first Brocard triangle. Then KA', KB', KC' are parallel to BC, CA, AB respectively. Hence A', B', C' are the invariable points of the system (§ 226, Ex. 5).



If PQR be the triangle formed by any three corresponding lines, and if K' be the centre of perspective of the triangles PQR, and $A''B''C''$, the triangle of similitude, it follows from § 226, Ex. 6, that the triangle PQR will be directly similar to ABC, and that K' will be the symmedian point of the triangle PQR.

Thus: *If three directly similar figures be described on the sides of a triangle, any three corresponding lines form a triangle whose symmedian point lies on the Brocard circle of the given triangle.*

232. If $A'B'C'$ be the first Brocard triangle, and $A''B''C''$ the second Brocard triangle of the triangle ABC, the lines $A'A''$, $B'B''$, $C'C''$ are concurrent. For A'', B'', C'' are the centres of similitude, and A', B', C' the invariable points of three directly similar figures described on the sides of the triangle ABC. Hence by § 226, Ex. 4, the triangles $A'B'C'$, $A''B''C''$ are in perspective.

Let $A'A''$, $B'B''$, $C'C''$ intersect in G, then by § 226, Ex. 4, it follows that the distances of G from the sides of the triangle ABC are inversely proportional to K_1, K_2, K_3, and therefore are inversely proportional to BC, CA, AB. But the triangles $A'B'C'$, ABC are inversely similar, so that

$$B'C' : C'A' : A'B' = BC : CA : AB.$$

Hence G is the median point of $A'B'C'$. This point is also the median point of ABC (§ 222, Ex. 4).

233. Ex. 1. Show that if $A'B'C'$ be the first Brocard triangle of the triangle ABC, the lines BA', CB', AC' are concurrent, and intersect on the Brocard circle.

The points B, C, A are corresponding points of three directly similar figures described on the sides of ABC, hence BA', CB', AC' are corresponding lines, and the theorem follows from § 225.

Ex. 2. Triangles are described on the sides of a triangle ABC, so as to be directly similar to each other; show that their vertices form a triangle in perspective with the first Brocard triangle of the triangle ABC.

Ex. 3. If in the last case the vertices be collinear, show that their line of collinearity passes through the median point of the triangle ABC.

Show also that each vertex lies on a circle.

If G be the median point; A'', B'', C'' the vertices of the second Brocard circle; the vertices lie on the circumcircles of $B''C''G$, $C''A''G$, $A''B''G$. These three circles are called McCay's circles.

Ex. 4. If P, Q, R be corresponding points of three directly similar figures described on the sides of the triangle ABC, and if two of the lines AP, BQ, CR be parallel, show that the three are parallel.

Ex. 5. Similar isosceles triangles BPC, CQA, ARB are described on the sides of a triangle ABC. If the triangle ABC, the triangle whose sides are AB', BC', CA', and the triangle whose sides are $A'B$, $B'C$, $C'A$, be denoted by F_1, F_2, F_3 respectively, show that the triangle of similitude of F_1, F_2, F_3 is the triangle $S\Omega\Omega'$ formed by the circumcentre and the Brocard points of the triangle ABC.

Show also that the symmedian points of the triangles are the invariable points of the system. [Neuberg.]

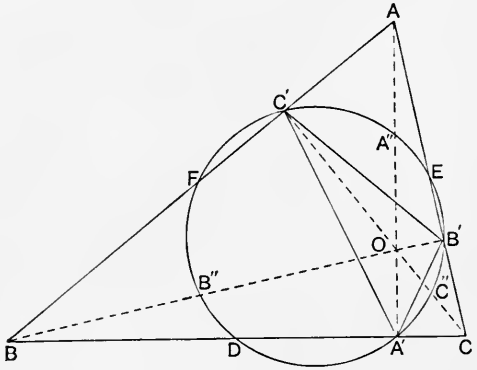
234. Let ABC be any triangle, and let AA', BB', CC' be drawn perpendicular to the sides. Then the triangles $AB'C'$, $A'BC'$, $A'B'C$ are inversely similar to the triangle ABC and therefore directly similar to each other. The centres of similitude of these triangles are evidently the points A', B', C'.

Let D, E, F be the middle points of the sides of the triangle ABC; and A'', B'', C'' the middle points of AO, BO, CO, where O is the orthocentre.

Then the perpendiculars at the middle points of AB', $A'B$, $A'B'$ are corresponding lines. But these lines meet in the point F.

Similarly, the perpendiculars at the middle points of AC', $A'C'$,

$A'C$ which meet in E are corresponding lines; and the perpendiculars at the middle points of $B'C'$, BC', $B'C$ which meet in D are corresponding lines.



Hence, by § 225, D, E, F are points on the circle of similitude; that is, the circle $A'B'C'$.

Again, the perpendiculars at the middle points of $B'C'$, AC', AB' meet in A''. Therefore A'' is one of the invariable points of the system. Similarly B'', C'' are the other invariable points. Hence A'', B'', C'' lie on the circle of similitude.

Hence the nine points A', B', C', A'', B'', C'', D, E, F, lie on a circle.

235. Ex. 1. Show that three corresponding lines of the triangles $AB'C'$, $A'BC'$, $A'B'C$ form a triangle in perspective with the triangle $A'B'C'$.

Ex. 2. Show that the circumcentre of the triangle formed by three corresponding lines lies on the nine-point circle.

Ex. 3. The three lines joining A'', B'', C'' to corresponding points of the three triangles cointersect on the nine-point circle of ABC.

Ex. 4. Every line which passes through the orthocentre of the triangle ABC meets the circumcircles of the triangles $AB'C'$, $A'BC'$, $A'B'C$ in points which are corresponding points for the three triangles.

Ex. 5. If P, P_1, P_2, P_3 be corresponding points of the triangles ABC, $AB'C'$, $A'BC'$, $A'B'C$, show that $A''P_1$, $B''P_2$, $C''P_3$ meet the nine-point circle of ABC in the point which is the isogonal conjugate, with respect to the triangle $A''B''C''$, of the point at infinity on the line joining P to the circumcentre of ABC.

Ex. 6. Show that the lines joining A'', B'', C'' to the in-centres of the triangles $AB'C'$, $BC'A'$, $CA'B'$ respectively cointersect in the point of contact of the nine-point circle of the triangle ABC with its inscribed circle.

236. Ex. 1. If directly similar figures be described on the perpendiculars of a triangle ABC, show that the circle of similitude will be the circle whose diameter is the line joining the median point of the triangle to the orthocentre.

Ex. 2. If L, M, N be the invariable points of these figures, show that the triangle LMN is inversely similar to the triangle ABC, and that the centre of similitude of these triangles is the symmedian point of each.

Ex. 3. Show that any three corresponding lines form a triangle whose median point lies on the circle of similitude.

Ex. 4. If directly similar triangles be described on the perpendiculars of a given triangle, so that their three vertices are collinear, show that the line of collinearity will pass through the symmedian point of the given triangle.

Ex. 5. If G be the median point, and O the orthocentre of the triangle ABC, show that the line joining the feet of the perpendiculars from O and G on AG, AO respectively, passes through the symmedian point of the triangle.

CHAPTER X.

THE CIRCLE.

Introduction.

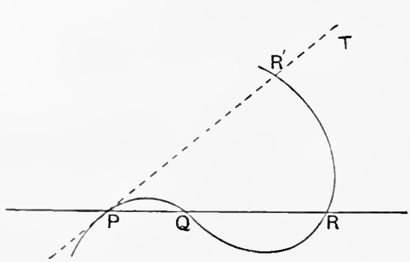
237. A CIRCLE is defined to be the locus of a point which moves in one plane so as to be always at a constant distance from a fixed point.

A circle is a curve of the second order; for, every straight line which cuts a circle meets it in two points, and no straight line can be drawn to cut a circle in more than two points. When a straight line does not cut a circle in real points, it is said to cut it in two imaginary points.

A straight line may meet a circle in apparently only one point. In this case, the line is said to cut the circle in two coincident points, and is called a *tangent* to the circle.

238. This definition of a tangent may be extended to include the case of any curve:

The chord joining two consecutive (i.e. indefinitely near) points on a curve is said to touch the curve.



Let P be any point on a curve, and let Q be a near point, at a finite distance from P. Join PQ. Now let the point Q move along the curve towards the point P. Then the line PQ turns about the point P, until Q coincides with P, when PQ will have the position PT. Thus PT is the limiting position of the chord PQ, that is PT is the tangent to the given curve at the point P.

In the case of a circle, or any curve of the second order, the tangent at any point cannot cut the curve again; but in the case of curves of order greater than the second, the tangent at any point will in general cut the curve again.

It is left to the reader to show that the definition of a tangent to a circle, as given in Euclid, is equivalent to the definition given above.

239. If we consider the assemblage of lines formed by drawing the tangents at every point of a circle, it is easy to see that two of these lines will pass through any given point. Hence a circle is a curve of the second class.

From a point within a circle, no real tangents can be drawn to the circle; that is, the tangent lines which pass through such a point are imaginary. If the given point be on the circle, only one tangent can be drawn through it; that is to say, the two tangents are coincident.

It follows that when a circle is treated as a curve of the second class, any point on it is to be regarded as the point of intersection of two consecutive tangents. More generally, we see that, in the case of a curve of any class, the point of contact of any tangent line is the limiting position of the point in which it intersects a near tangent, when the latter is turned about so as to coincide with the given line.

240. The simplest definition of a circle regarded as a curve of the second class is the following:

The envelope of a straight line which moves in one plane so as to be always at a constant distance from a fixed point is a circle.

Ex. 1. A triangle given in species and magnitude is turned about in a plane, so that two of its sides pass through two fixed points. Show that the envelope of the third side is a circle.

Ex. 2. Two sides of a given triangle touch two fixed circles. Show that the envelope of the third side is a circle.

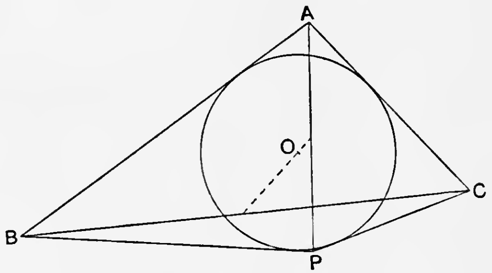
Ex. 3. Two circles intersect in the points A and B, and from a point P on one of them PA, PB are drawn cutting the other circle in the points Q and R. Show that the envelope of QR is a circle.

Ex. 4. If two sides of a triangle and its inscribed circle be given in position, the envelope of its circumcircle is a circle.

Ex. 5. If two sides of a triangle be given in position, and if its perimeter be given in magnitude, find the envelope of its circumcircle.

241. It is very often instructive to consider how the enunciation of a particular theorem requires modification when two or more points, or lines, of a figure coincide. On the other hand a theorem may sometimes be easily recognised as a special case of a general theorem by taking a slightly more complicated figure.

Ex. The inscribed circle of the triangle ABC touches the side BC in the point P, show that the line joining the middle points of BC and AP passes through the centre of the circle.



Consider any circle touching the sides AB, AC of the triangle; and let the other tangents which can be drawn from B and C meet in P. Then we know that the line joining the middle point of BC to the middle point of AP passes through the centre of this circle (§ 38, Ex. 4). If now we suppose the circle to be drawn smaller and smaller until it touches BC, P will become the point of contact of the circle with BC. Hence the theorem is proved.

242. Ex. 1. A circle touches the sides of the triangle ABC in the points P, Q, R; show that the lines AP, BQ, CR are concurrent.

This may be deduced from Pascal's theorem (§ 181).

Ex. 2. Any point D is taken on the side BC of the triangle ABC, and circles are drawn passing through D and touching AB, AC respectively at B and C. Show that these circles meet in a point P, which lies on the circumcircle of the triangle ABC; and that the Simson line of P with respect to the triangle ABC is perpendicular to the line which joins the middle points of BC, and AD.

See § 148, Ex. 2.

Ex. 3. If A, B, C, D be four points on a circle, such that the pencil $P\{AB, CD\}$ is harmonic, where P is any other point on the circle; show that the tangents at A and B intersect on CD.

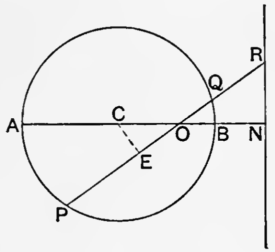
Let the tangent at A meet CD in T, and let AB cut CD in V. By § 48, Ex. 4, the pencil $A\{AB, CD\}$, that is the pencil $A\{TB, CD\}$, is harmonic. Therefore the range $\{TV, CD\}$ is harmonic. If the tangent at B meet CD in T', we can prove in the same way that $\{T'V, CD\}$ is a harmonic range. Hence T and T' coincide.

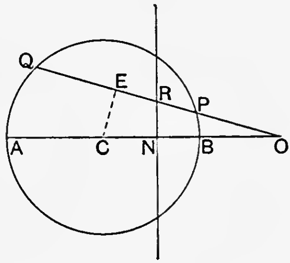
Ex. 4. If the pairs of tangents drawn to a circle from two points, A and B, cut any fifth tangent harmonically, show that the chord of contact of the tangents from A will pass through B.

See § 48, Ex. 5.

Poles and Polars.

243. *If a straight line be drawn through a fixed point O, and if the point R be taken on it, which is the harmonic conjugate of O, with respect to the two points in which the line cuts a given circle, the locus of the point R will be a straight line.*





Let P, Q be the points in which the straight line cuts the circle, and let E be the middle point of PQ. Then we have (§ 54, Ex. 1)

$$OP.OQ = OE.OR.$$

Also if AOB be the diameter of the circle which passes through O, and N the harmonic conjugate of O with respect to A and B, we shall also have

$$OA.OB = OC.ON,$$

where C is the centre.

But $$OA.OB = OP.OQ;$$

therefore $$OE.OR = OC.ON.$$

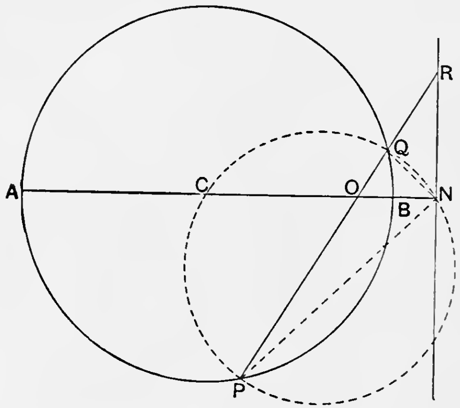
Therefore the points C, E, R, N are concyclic. Hence it follows that the angle ONR is a right angle.

Consequently, the locus of the point R is the straight line which passes through N and is at right angles to CO.

This straight line is called the *polar* of the point O with respect to the circle; and the point O is said to be the *pole* of the line.

It should be noticed that if the point O is without the circle, the straight line OR may not intersect the circle in real points. But in this case the foot of the perpendicular from C may still be regarded as the middle point of PQ, and the proof given above applies.

244. The theorem of the last article may also be proved otherwise thus:



Let PQ be any chord of a given circle which passes through the given point O, and let R be the harmonic conjugate of O with respect to the points P, Q.

Let C be the centre of the circle, and let a circle be drawn through the points C, P, Q, cutting CO in the point N.

Then since $\quad OC \,.\, ON = OP \,.\, OQ = OA \,.\, OB,$

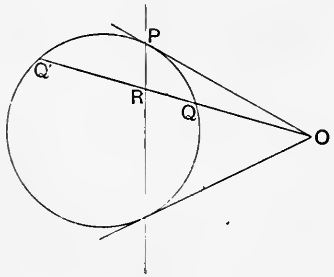
it follows that N is a fixed point.

Now C is the middle point of the arc PCQ, therefore CN bisects the angle PNQ.

But $N\{OR, PQ\}$ is a harmonic pencil, by hypothesis. Therefore NR must be the other bisector of the angle PNQ; that is, RNC must be a right angle.

Therefore the point R always lies on the straight line which cuts OC at right angles in the point N.

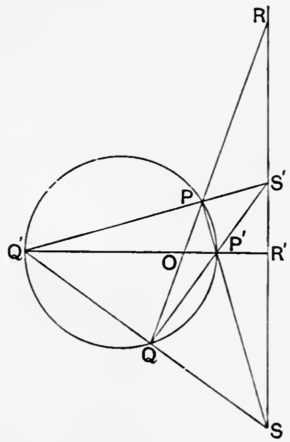
245. It is evident that the polar of a point within a circle cuts the circle in imaginary points; and that the polar of an external point cuts the circle in real points. Further, if O be an external point, it is easy to see that its polar will pass through the points of contact of the two tangents which can be drawn from O to the circle. Let any chord be drawn through the point O cutting the



circle in Q and Q', and the polar of O in the point R. Then if this line be turned about the point O, so as to make the points Q and Q' approach one another, the point R, which lies between them, will ultimately coincide with them. Hence, if P be the point of contact of one of the tangents from O, when Q and Q' coincide with the point P, so also will the point R. That is to say, P is a point on the polar of O.

246. *To construct the polar of a point with respect to a given circle.*

Let O be the given point, and let any two chords POQ, $P'OQ'$



be drawn. Let PP' intersect QQ' in S; and let PQ', $P'Q$ intersect in S'. Then SS' is the polar of O.

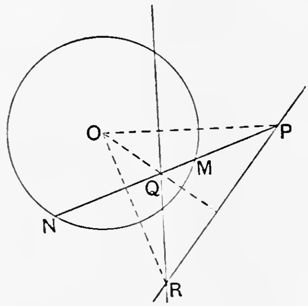
For O, S, S' are the centres of the tetrastigm $PP'QQ'$; and therefore POQ meets SS' in a point R, which is the harmonic conjugate of O with respect to P and Q.

Thus R is a point on the polar of O.

Similarly, if $P'Q'$ meet SS' in R', it follows that R' is a point on the polar of O.

Hence SS' is the polar of O.

247. *If the polar of a point P with respect to a circle pass through the point Q, the polar of Q will pass through P.*



Let PQ cut the circle in M and N. Then because PMN cuts the polar of P in Q, $\{PQ, MN\}$ is a harmonic range. Therefore P must lie on the polar of Q.

248. We infer that the polars of every point on any straight line, with respect to a circle, pass through the same point, namely the pole of the straight line.

Suppose now that the polars of two points P and Q, intersect in the point R. Then since R is on the polar of P, P is on the polar of R. Similarly Q is on the polar of R. Hence, PQ is the polar of R.

Thus, the line joining any two points is the polar of the point of intersection of the polars of the points; or, what is the same thing, the point of intersection of any two lines is the pole of the line joining the poles of the two lines.

249. This theorem furnishes us with a simple method for constructing the pole of a given straight line.

For take any two points on the line, and draw their polars; the point in which they intersect will be the pole of the given line.

250. It follows from § 247 that the polar of any point on a circle is the tangent to the circle, and that the pole of any tangent to the circle is its point of contact.

Let R be any point on the circle, and let P and Q be any two points on the tangent at R. The polars of P and Q each pass through R; hence R is the pole of PQ. That is, R is the pole with respect to the circle of the tangent at R to the circle.

251. Ex. 1. If a chord of a circle pass through a fixed point, the locus of the point of intersection of the tangents at its extremities is the polar of the point with respect to the circle.

Ex. 2. If P be any point on the polar of O, show that the line PO will be the harmonic conjugate of the polar of O with respect to the tangents from P to the circle.

Ex. 3. If any three points be collinear, show that their polars with respect to a circle will be concurrent.

Ex. 4. Show that the poles with respect to a circle of three concurrent lines are collinear.

Ex. 5. If from any two points on a given straight line, pairs of tangents be drawn to a circle, show that the diagonals of the tetragram formed by them will intersect in the pole of the given line.

Ex. 6. The tangents at the points B and C on a circle intersect in the point A; and the tangent at any point P cuts the sides of the triangle ABC in the points X, Y, Z. Show that $\{PX, YZ\}$ is a harmonic range.

Ex. 7. Any two points P and Q are taken on a chord AB of a circle, and the polars of P and Q cut AB in the points P', Q' respectively. Show that the range $\{AB, PP', QQ'\}$ is in involution.

Ex. 8. If PM, QN be drawn perpendicular to the polars of Q and P, with respect to a circle whose centre is O; show that

$$PM : QN = OP : OQ.$$ [Salmon.]

Ex. 9. The tangents at three points A, B, C on a circle form the triangle $A'B'C'$. Show that the centre of perspective of the triangles ABC, $A'B'C'$, is the pole with respect to the circle of the axis of perspective of the triangles.

Ex. 10. Show that the poles of the symmedian lines of a triangle, with respect to the circumcircle, lie on the corresponding sides of the triangle.

Hence show that if the symmedian lines of the triangle ABC cut the circumcircle in the points A', B', C', the two triangles ABC, $A'B'C'$ are co-symmedian.

Ex. 11. Show that the lines drawn from the circumcentre of a triangle perpendicular to the symmedian lines intersect the corresponding sides of the triangle in three points which are collinear.

Ex. 12. Through the middle point O of a chord AOB of a circle, are drawn any other chords POQ, and ROS. If PR, QS cut AB in H and K, show that O will be the middle point of HK.

Ex. 13. Given the base and the sum or difference of the sides of a triangle, show that the polar of the vertex with respect to a circle, whose centre is one extremity of the base, will always touch a fixed circle.

252. Since every diameter of a circle is bisected at the centre, it follows that the harmonic conjugate of the centre of any circle with respect to the extremities of any diameter is the point at infinity on that diameter. Hence, we infer that *the centre of any circle is the pole of the line at infinity.*

253. It also follows that the pole of any diameter is the point at infinity on the diameter which is perpendicular to the given diameter.

Let O be the centre of a circle, and let P, P' be the points in which two diameters at right angles cut the line at infinity. Then P is the pole of OP', and therefore the points P, P' are harmonic conjugates with respect to the two imaginary points in which the circle cuts the line at infinity; or, what is the same thing, the two imaginary points in which the circle cuts the line at infinity are harmonic conjugates with respect to P and P'. Again, if another pair of diameters at right angles be drawn cutting the line at infinity in the points Q and Q', it follows in the same way that the imaginary points in which the circle cuts the line at infinity are also harmonic conjugates with respect to Q and Q'.

Hence, if we draw a series of pairs of diameters at right angles, the points in which they meet the line at infinity will form a range $\{PP', QQ', \ldots\}$ in involution, having for double points the points in which the circle cuts the line at infinity.

If these points be joined to any point A, we clearly have a pencil $A\{PP', QQ', \ldots\}$, such that the conjugate rays intersect at right angles, and the lines joining A to the points in which the circle cuts the line at infinity are the double rays of this pencil.

Hence, we infer that *every circle passes through the same two imaginary points on the line at infinity.*

These two imaginary points have many important properties. They are called the *circular points.*

254. Since the centre of a circle is the pole of the line at infinity, it follows that the lines joining the centre of a circle to the circular points touch the circle at these points. Hence, concentric circles have the same tangents at the circular points, and therefore may be said to touch each other at the circular points.

Conjugate points and lines.

255. Any two points are said to be *conjugate points* with respect to a circle, when the polar of either passes through the other.

Any two straight lines are said to be *conjugate lines* with respect to a circle, when the pole of either lies on the other.

It is evident that the polars of a pair of conjugate points are conjugate lines; and that the poles of a pair of conjugate lines are conjugate points.

256. It is easy to see that there is in general only one point on a given straight line which is conjugate to a given point; namely, the point in which the given straight line cuts the polar of the given point. Similarly, through a given point we can draw but one line which shall be conjugate to a given straight line, unless the given point be the pole of the given line.

257. Ex. 1. Show that perpendicular diameters of a circle are conjugate lines with respect to the circle.

Hence, perpendicular diameters are called *conjugate diameters.*

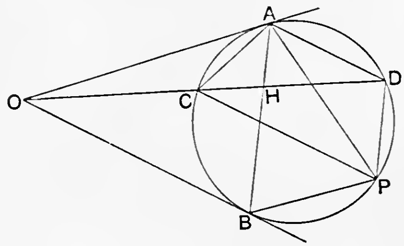
Ex. 2. Show that the line joining any pair of conjugate points is cut harmonically by the circle.

Ex. 3. Show that the tangents drawn to a circle from the point of intersection of two conjugate lines with respect to the circle, form with these lines a harmonic pencil.

Ex. 4. If A, A'; B, B'; C, C'; be pairs of conjugate points with respect to a circle, on the same straight line, show that $\{AA', BB', CC'\}$ is a range in involution.

Ex. 5. Through a point O two conjugate lines are drawn, and any tangent meets them in the points P and Q. Show that the other tangents from P and Q to the circle intersect on the polar of O.

258. *Any two conjugate lines with respect to a circle, cut the circle in the points A, B and C, D respectively; if P be any other point on the circle, the pencil $P\{AB, CD\}$ is harmonic.*



Let O be the pole of AB, and let AB intersect CD in H.

Then, $\{OH, CD\}$ is a harmonic range; and therefore the pencil $A\{OB, CD\}$ is harmonic.

Therefore $$\frac{\sin OAC}{\sin CAB} = \frac{\sin OAD}{\sin BAD}.$$

But the angle OAC is equal to the angle APC, and the angle CAB to the angle CPB.

Therefore $$\frac{\sin OAC}{\sin CAB} = \frac{\sin APC}{\sin CPB}.$$

Similarly we can show that

$$\frac{\sin OAD}{\sin BAD} = \frac{\sin APD}{\sin BPD}.$$

Hence, $$\frac{\sin APC}{\sin CPB} = \frac{\sin APD}{\sin BPD}.$$

Therefore the pencil $P\{AB, CD\}$ is harmonic.

259. Ex. 1. If AB be any chord of a circle, and if the conjugate line to AB cut the circle in C and D, show that

$$AC : CB = AD : BD.$$

Ex. 2. If P, A, B, C, D be five points on a circle, such that the pencil $P\{AB, CD\}$ is harmonic, show that the lines AB, CD are conjugate with respect to the circle.

Ex. 3. If A and B be a pair of conjugate points with respect to a circle, show that the tangents drawn from them to the circle will cut any fifth tangent in a harmonic range.

Ex. 4. Deduce from § 258, that if AA', BB', CC' be concurrent chords of a circle, and if P be any other point on the circle, the pencil $P\{AA', BB', CC'\}$ will be in involution.

Ex. 5. If P be any point on the polar of the point A with respect to the inscribed (or an escribed) circle of the triangle ABC, show that PB and PC will be conjugate lines with respect to the circle.

Ex. 6. Any straight line is drawn through the pole of the line BC, with respect to the circumcircle of the triangle ABC, cutting AC, AB in the points Q and R. Show that Q and R are conjugate points with respect to the circle.

Ex. 7. The centre O of a circle ABC lies on another circle ABP, any chord of which OP cuts AB in Q. Show that P and Q are conjugate points with respect to the circle ABC.

Ex. 8. If I be the centre of the inscribed circle of a triangle, and if BP, CQ be drawn perpendicular to CI, BI respectively; show that P and Q lie on the polar of A with respect to the circle.

Ex. 9. Through a fixed point of a circle chords are drawn equally inclined to a fixed direction; show that the line joining their extremities passes through a fixed point.

Ex. 10. The tangents to a circle at the points A, B, C, form the triangle $A'B'C'$; and AA' cuts the circle in P. If from any point Q on the tangent at P, the other tangent QR be drawn, show that the pencil $Q\{RA', B'C'\}$ is harmonic.

Ex. 11. Three fixed tangents to a circle form a triangle ABC, and on the tangent at any point P is taken a point Q such that the pencil $Q\{PA, BC\}$ is harmonic. Show that the locus of the point Q is a straight line which touches the circle.

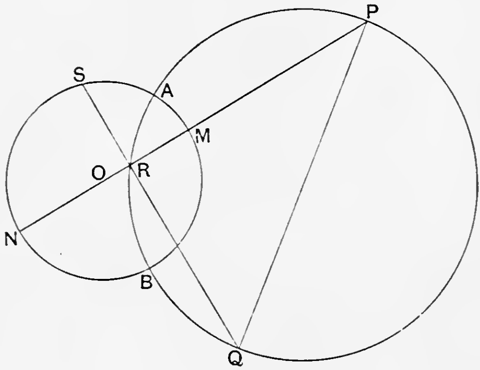
Ex. 12. Two conjugate lines with respect to a circle cut the circle in the points A, B; and C, D; respectively. Through any point P on AB are drawn the lines CP, DP cutting the circle in C' and D': show that $C'D'$ passes through a fixed point on AB.

Ex. 13. Through a point O on a circle are drawn any two chords OA, OB. If a chord PQ be drawn conjugate to OA and cutting OB in R, show that the pencil $A\{BR, PQ\}$ is harmonic.

Ex. 14. A fixed straight line meets a circle in A and B, and through a fixed point C on the line AB is drawn a straight line meeting the tangents at A and B in P and Q; show that the other tangents to the circle from P and Q intersect in a point whose locus is a straight line.

Ex. 15. The tangent at the point A to the circumcircle of the triangle ABC meets the tangents at B and C in C' and B'. If the lines OB', OC' connecting B and C to any point O, meet BC in P and Q, show that AB, AC intersect $B'Q$, $C'P$, respectively, in points which lie on the polar of the point O.

260. *The circle described on the line joining a pair of conjugate points with respect to a given circle, as diameter, will cut the given circle orthogonally.*



Let P, Q be a pair of conjugate points with respect to the circle SAB, and let the circle whose diameter is PQ cut the circle SAB in the points A and B.

Let O be the centre of the circle SAB; and let OP cut this circle in M and N, and the circle PAQ in R.

Then, since PRQ is a right angle, it follows that QR must be the polar of P with respect to the circle SAB.

Therefore $\{PR, MN\}$ is a harmonic range, and therefore

$$OR \, . \, OP = OM^2 = OA^2.$$

Hence, OA touches the circle PAQ at the point A; and the circle PAQ cuts the given circle orthogonally.

261. Ex. 1. If two circles cut orthogonally, show that the extremities of any diameter of either are conjugate points with respect to the other.

Ex. 2. If a system of circles be drawn to cut a given circle orthogonally, show that the polars with respect to them, of a point on the given circle, are concurrent.

Ex. 3. Show that any straight line which cuts one circle in a pair of points conjugate with respect to another circle, cuts the latter in points which are conjugate with respect to the former.

Ex. 4. Show how to draw a straight line which shall cut two of three given circles in pairs of conjugate points with respect to the third.

Ex. 5. Show that the circles described on the diagonals of a tetragram as diameters, cut the circumcircle of the triangle formed by the diagonals orthogonally.

Ex. 6. Any pair of conjugate points with respect to a given circle are taken as centres of two circles which cut the given circle orthogonally. Show that these circles will cut each other orthogonally.

Conjugate triangles.

262. The triangle formed by the polars of the vertices of a given triangle with respect to a circle, is called the *conjugate triangle* of the given triangle.

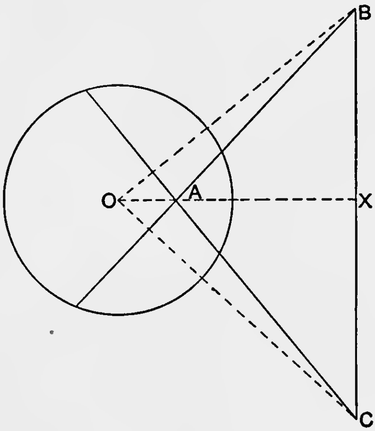
If ABC be the given triangle, and $A'B'C'$ the conjugate triangle, so that $B'C'$, $C'A'$, $A'B'$ are the polars of A, B, C, respectively, it follows by § 247, that A', B', C' will be the polars of BC, CA, AB, respectively. Thus the triangle ABC is the conjugate triangle of $A'B'C'$.

263. In the particular case when a triangle coincides with its conjugate, that is when each vertex is the pole of the opposite side, the triangle is said to be *self-conjugate.*

Given any point A, we can always construct a triangle having one vertex at A, which shall be self-conjugate with respect to a given circle. Let any point B be taken on the polar of A, and let the polar of B cut the polar of A in the point C. Then the triangle ABC is self-conjugate with respect to the circle.

For, since B lies on the polar of A, the polar of B passes through A. Therefore AC is the polar of B. Also by § 248, C must be the polar of AB.

264. If ABC be any self-conjugate triangle with respect to a



circle whose centre is O, it is easy to see that O must be the orthocentre of the triangle. For, since A is the pole of BC, OA is perpendicular to BC. Similarly, OB, OC are perpendicular to CA and AB respectively.

Let OA meet BC in X, and let r denote the radius of the circle, then we shall have

$$r^2 = OA \,.\, OX.$$

Hence it follows that, given the triangle ABC, only one circle can be drawn such that the triangle is self-conjugate with respect to it. The centre of the circle will be the orthocentre of the triangle, and its radius will be determined by the above formula.

265. This circle is called the *polar circle* of the triangle. It is evident that it is real, only when the orthocentre lies outside the triangle; that is, when one angle of the triangle is greater than a right angle. If one angle of a triangle be a right angle, the radius of its polar circle is evanescent.

266. Ex. 1. Show that the polar circle of a triangle cuts orthogonally the circles described on the sides of the triangle as diameters.

Ex. 2. If ABC be any triangle and O its orthocentre, show that the polar circles of the four triangles ABC, BOC, COA, AOB are mutually orthotomic.

One of these circles is imaginary.

Ex. 3. The polar circles of the four triangles formed by four straight lines, taken three at a time, cut orthogonally the circles described on the diagonals of the tetragram formed by the lines, as diameters.

Ex. 4. If ABC be any self-conjugate triangle with respect to a circle, and if B and C be joined to any point P on the circle, show that BP, CP will cut the circle in two points Q and R, such that QR will pass through A.

Ex. 5. If ABC be any self-conjugate triangle with respect to a circle, and if QAR be any chord of this circle; show that BQ, CR will intersect on the circle.

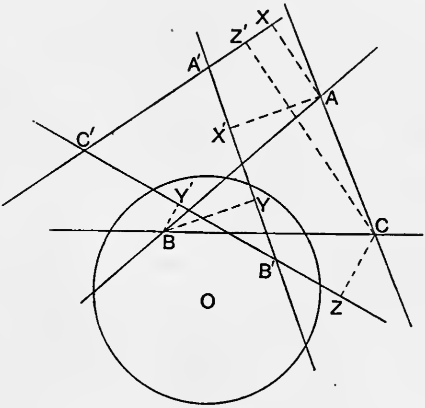
Also if BQ intersect CR in P, and if BR intersect CQ in P', show that PP' will pass through A.

Ex. 6. Show that each side of a triangle cuts the polar circle in two points which are conjugate with respect to the circumcircle.

Ex. 7. Two triangles are self-conjugate with respect to a circle; show that their six vertices form a Pascal hexastigm, and that their six sides form a Brianchon hexagram.

267. *Any triangle and its conjugate triangle with respect to a given circle are in perspective.*

Let ABC be any triangle, $A'B'C'$ the conjugate triangle with respect to a circle whose centre is O. Let AX, AX' be drawn perpendicular to $C'A'$ and $A'B'$; BY, BY' perpendicular to $A'B'$ and $B'C'$; and CZ, CZ' perpendicular to $B'C'$ and $C'A'$.



Then since $A'B'$ is the polar of C, and $A'C'$ the polar of B, by § 251, Ex. 8,

$$BY : CZ' = OB : OC;$$

similarly we shall have,

$$CZ : AX' = OC : OA;$$

and

$$AX : BY' = OA : OB.$$

Therefore

$$\frac{BY}{CZ'} \cdot \frac{CZ}{AX'} \cdot \frac{AX}{BY'} = 1.$$

Hence, by § 179, the triangle ABC, $A'B'C'$ are in perspective.

268. Let the sides of the triangle ABC cut the corresponding sides of the triangle $A'B'C'$ in the points P, Q, R. Then, since A is the pole of $B'C'$ and A' the pole of BC, it follows that P is the pole of AA'. Similarly, Q and R are the poles of BB', and CC respectively.

But, AA', BB', CC' meet in the centre of perspective of the two triangles; and P, Q, R lie on the axis of perspective.

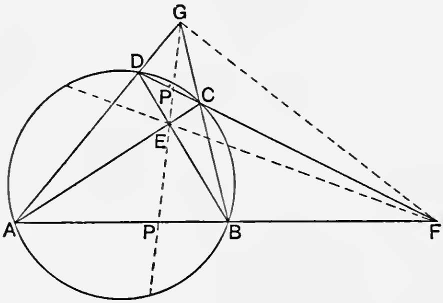
Hence, *the axis of perspective of any triangle and its conjugate is the polar of the centre of perspective of the triangles.*

269. Ex. 1. Show that any triangle inscribed in a circle is in perspective with the triangle formed by the tangents at its vertices.

Ex. 2. If ABC and $A'B'C'$ be a pair of conjugate triangles with respect to a circle whose centre O is the circumcentre of the triangle ABC; show that O will be the in-centre of the triangle $A'B'C'$.

Tetrastigm inscribed in a circle.

270. *The centres of any tetrastigm inscribed in a circle form a self-conjugate triangle.*



Let $ABCD$ be any tetrastigm inscribed in a circle, and let E, F, G be its centres.

Then, if AB, CD cut GE in P and P', it follows by § 141, that the ranges

$$\{AB, PF\} \text{ and } \{CD, P'F\}$$

are harmonic.

Therefore GE is the polar of the point F.

Similarly, EF, FG are the polars of G and E respectively.

Therefore EFG is a self-conjugate triangle with respect to the circle.

271. Ex. 1. Show that the orthocentre of the triangle formed by the centres of a tetrastigm inscribed in a circle coincides with the centre of the circle.

Ex. 2. Show that the circles described on the sides of the triangle formed by the centres of any tetrastigm inscribed in a given circle, as diameters, cut the given circle orthogonally.

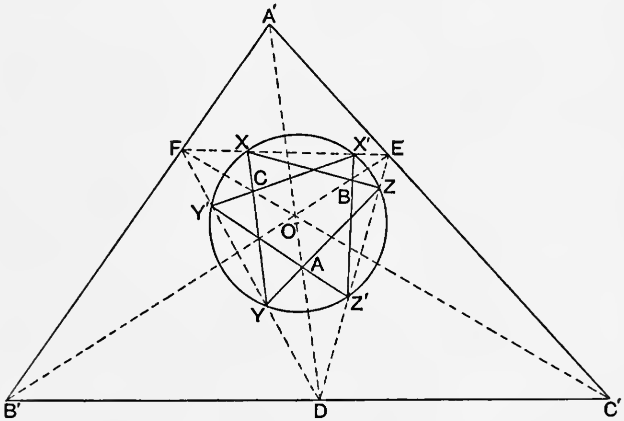
Ex. 3. If A and B be two fixed points on a circle and PQ any diameter, show that the locus of the point of intersection of AP and BQ is a circle which cuts the given circle orthogonally in the points A and B.

Ex. 4. Two circles intersect orthogonally in the points A and B, and from any point P on one of them PA, PB are drawn cutting the other in the points Q and R. Show that AR and QB intersect in a point which lies on the circle PAB.

Ex. 5. Through the vertex A of the triangle ABC, which is self-conjugate to a given circle, are drawn two straight lines cutting the circle in the points P, P' and Q, Q' respectively: show that if the pencil $A\{PQ, BC\}$ be harmonic, then B and C will be the other centres of the tetrastigm $PP'QQ'$.

Ex. 6. Show how to inscribe a triangle in a given circle, so that its sides shall pass respectively through three given points.

Let A, B, C be the given points; and let $A'B'C'$ be the conjugate triangle to the triangle ABC, with respect to the given circle. Let AA', BB', CC' cut $B'C'$, $C'A'$, and $A'B'$, in the points D, E, F respectively; and let EF, FD,



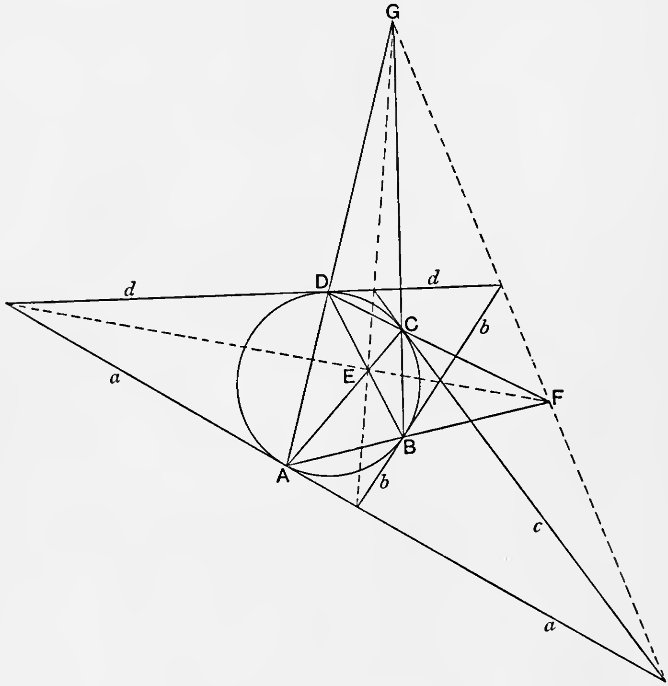
DE cut the circle in the points X, X'; Y, Y'; Z, Z'; respectively. Then these points determine two triangles XYZ, $X'Y'Z'$ which satisfy the given conditions.

For, since $A'D$, $B'E$, $C'F$ are concurrent (§ 267), it follows (§ 96, Ex. 11) that $D\{C'A, EF\}$ is harmonic. Therefore A is one of the centres of the tetrastigm $YZY'Z'$, by the theorem in Ex. 5.

272. Let $ABCD$ be any tetrastigm inscribed in a circle, and let E, F, G be its centres. Then since AC and BD pass through E, which is the pole of FG, it follows that the poles of AC and BD must lie on FG.

Similarly the poles of AB and DC will lie on EG, and the poles of BC and AD on FE.

Hence, the tangents to the circle at the vertices of the tetrastigm $ABCD$ form a tetragram, whose vertices lie in pairs on the lines EF, FG, GE; that is, the diagonals of the tetragram are the lines joining the centres of the tetrastigm.



273. *If a tetrastigm be inscribed in a circle, any straight line will be cut in involution by the circle and the three pairs of opposite connectors of the tetrastigm.*

Let $ABCD$ be a tetrastigm inscribed in a circle, and let any straight line be drawn cutting the connectors AC, BD in P and P'; the connectors CD, AB in Q and Q'; the connectors AD, BC in R and R'; and the circle in S and S'.

Then the range $\{PP', QQ', RR', SS'\}$ will be in involution.

Let AC and BD intersect in E. Then since the angles PAR, $R'BP'$ are equal,

$$\frac{RP}{AR} \sin RPA = \frac{P'R'}{BR'} \sin BP'R'.$$

Therefore
$$\frac{AR}{RP} \cdot \frac{P'R'}{BR'} \cdot \frac{PE}{EP'} = 1.$$

Similarly, since the angles RDP', PCR' are equal,

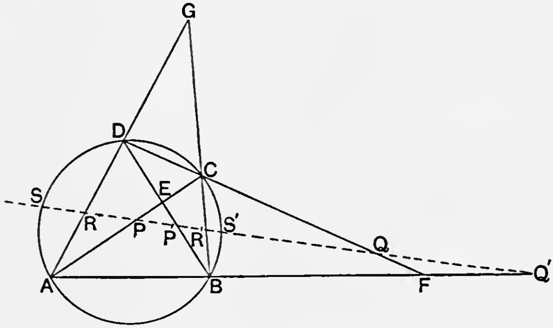
$$\frac{RD}{RP'} \cdot \frac{PR'}{R'C} \cdot \frac{EP'}{EP} = 1.$$

Hence, $AR.RD : BR'.R'C = RP.RP' : PR'.P'R'$.

But since ARD, SRS' are chords of a circle,

$$AR.RD = SR.RS'.$$

And similarly $$BR'.R'C = SR'.R'S'.$$



Therefore $SR.RS' : SR'.R'S' = RP.RP' : PR'.P'R'$.

Hence, by § 76, the range $\{SS', PP', RR'\}$ is in involution.

Similarly it may be proved that the range $\{SS', QQ', RR'\}$ is in involution.

Consequently the range $\{SS', PP', QQ', RR'\}$ is in involution.

274. Ex. 1. If E, F, G be the centres of any tetrastigm inscribed in a circle, and P any given point, show that the conjugate rays of EP, FP, GP with respect to the connectors of the tetrastigm which intersect in E, F, G, respectively, will intersect in a point which lies on the polar of P with respect to the circle.

If the point P be on the circle, the lines will intersect on the tangent at P.

Ex. 2. If in the last example, P' be the point of intersection of the rays conjugate to EP, FP, and GP, show that P and P' are the double points of the range in involution in which PP' is cut by the circle and the connectors of the tetrastigm.

Ex. 3. If E, F, G be the centres of a tetrastigm inscribed in a circle whose centres is O, the conjugate rays of EO, FO, GO with respect to the connectors of the tetrastigm which pass through E, F, and G will be parallel.

Ex. 4. If through any point P, straight lines be drawn parallel to the connectors of a tetrastigm inscribed in a circle, they will form a pencil in involution, the double rays of which are perpendicular.

Hence, the bisectors of the angles formed by the pairs of opposite connectors of a tetrastigm inscribed in a circle are parallel.

275. Since every circle passes through the same pair of imaginary points on the line at infinity, it follows that a system of circles which have two finite points common may be considered as circumscribing the same tetrastigm. Consequently we have the theorem:

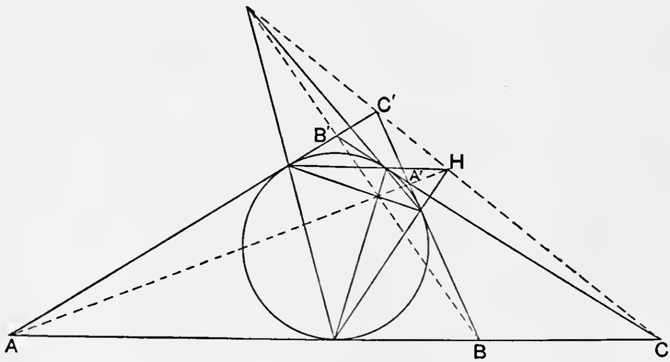
A system of circles having two common points, cuts any straight line in a range in involution.

Ex. 1. Two circles intersect in A and B, and a common tangent touches them in P and Q. Show that if a system of circles be drawn through the points A and B, they will cut the line PQ in a range in involution, the double points of which are P and Q.

Ex. 2. Show that the polar of a given point with respect to any circle which passes through two fixed points, passes through a fixed point.

Tetragram circumscribed to a circle.

276. *The diagonals of any tetragram circumscribed to a circle form a self-conjugate triangle with respect to the circle.*



Let A, A'; B, B'; C, C' be the three pairs of opposite vertices of a tetragram circumscribed to a circle.

Let AA' cut CC' in H, then the pencil $B\{AA',\ HB'\}$ is harmonic.

Therefore H is the pole of BB'.

That is, the point of intersection of the diagonals AA', CC', is the pole of the diagonal BB'.

Similarly it may be proved that BB', CC' intersect in the pole of AA'; and that AA', BB' intersect in the pole of CC'.

Hence, the lines AA', BB', CC' form a self-conjugate triangle.

277. Since H is the pole of BB', it follows that the polars of B and B' must pass through H. That is, the lines joining the points of contact of BA, BA' and the line joining the points of contact of $B'A$, $B'A'$ meet in the point of intersection of AA', CC'.

Hence, the centres of the tetrastigm formed by the points of contact of the tetragram are the points of intersection of the diagonals of the tetragram.

It should be noticed that these theorems might have been inferred from § 272.

278. Ex. 1. If a tetrastigm be inscribed in a circle, show that the diagonals of the tetragram formed by the tangents at its vertices, intersect the three pairs of opposite connectors of the tetrastigm in six points which are the vertices of a tetragram.

Ex. 2. Show also that the three centres of the tetrastigm connect with the vertices of the tetragram by six lines which constitute the connectors of a tetrastigm.

Ex. 3. If P be any point on the side BC of a triangle ABC, self-conjugate with respect to a given circle, and if Q be the harmonic conjugate of P with respect to B and C; show that the tangents drawn from P and Q to the circle will form a tetragram whose diagonals are the sides of the triangle ABC.

Ex. 4. Construct a triangle whose sides shall touch a fixed circle, and whose vertices shall lie on three given straight lines.

Ex. 5. The tangents drawn from the vertices of a triangle ABC, to touch a given circle, meet the opposite sides in the points X, X'; Y, Y'; Z, Z'; respectively. If P be the point of intersection of the other tangents which can be drawn from X and X'; Q the point of intersection of the tangents from Y and Y'; and R the point of intersection of the tangents from Z and Z'; show that the triangles PQR, ABC are in perspective.

Ex. 6. If $ABCD$ be any tetrastigm inscribed in a circle, so that the connectors AB, BC, CD, DA touch another circle in the points P, Q, R, S respectively, show that:—

(i) The lines AC, BD, PR, QS are concurrent.

(ii) PR, QS bisect the angles between AC and BD.

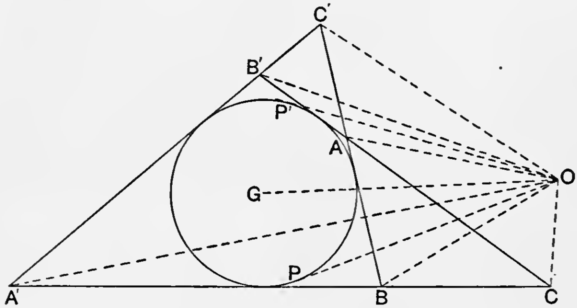
(iii) The polars of the point of intersection of AC and BD with respect to the two circles are coincident.

279. *The tangents drawn from any point to a circle, and the pairs of straight lines connecting the point to the three pairs of opposite vertices of a tetragram circumscribed to the circle, form a pencil in involution.*

If OP, OP' be the tangents from O to the circle, and if A, A'; B, B'; C, C' be the pairs of opposite vertices of a circumscribing tetragram, then the pencil $O\{PP', AA', BB', CC'\}$ will be in involution.

Let G be the centre of the circle; then, since GO bisects the angle POP', we have

$$\sin AOP . \sin AOP' = \sin^2 AOG - \sin^2 POG.$$



If r denote the radius of the circle, and a the perpendicular from G on AO, this result may be written,

$$GO^2 . \sin AOP . \sin AOP' = a^2 - r^2.$$

Let a' denote the perpendicular from G on OA', and p the perpendicular on AA', then we shall have:

$$\sin AOP \ . \sin AOP' : \sin A'OP . \sin A'OP' = a^2 - r^2 : a'^2 - r^2,$$

$$\sin A'AB' . \sin A'AB : \sin OAB' . \sin OAB = p^2 - r^2 : a^2 - r^2,$$

$$\sin OA'B' . \sin OA'B : \sin AA'B' . \sin AA'B = a'^2 - r^2 : p^2 - r^2.$$

Therefore,

$$\frac{\sin AOP . \sin AOP'}{\sin A'OP . \sin A'OP'} = \frac{\sin OAB' . \sin OAB . \sin AA'B' . \sin AA'B}{\sin OA'B' . \sin OA'B . \sin A'AB' . \sin A'AB}.$$

But since the lines $B'A$, $B'O$, $B'A'$ are concurrent (§ 98),

$$\frac{\sin B'A'A . \sin B'AO . \sin B'OA}{\sin B'A'O . \sin B'AA' . \sin B'OA} = -1;$$

and since the lines BA, BO, BA' are concurrent,

$$\frac{\sin BA'A . \sin BAO . \sin BOA'}{\sin BA'O . \sin BAA' . \sin BOA} = -1.$$

Hence $$\frac{\sin AOP . \sin AOP'}{\sin A'OP . \sin A'OP'} = \frac{\sin AOB' . \sin AOB}{\sin A'OB' . \sin A'OB}.$$

Therefore the pencil $O\{PP', AA', BB'\}$ is in involution (§ 89). In the same way it may be shown that the pencil $O\{PP', AA', CC'\}$ is in involution.

Hence, the pencil $O\{PP', AA', BB', CC'\}$ is in involution.

280. Ex. 1. If any line be drawn to intersect the diagonals AA', BB', CC' of a tetragram circumscribed to a circle, in the points X, Y, Z, show that the harmonic conjugates of these points with respect to the pairs of opposite vertices of the tetragram lie on a straight line which is conjugate to the given line with respect to the circle.

Ex. 2. Show that the line which bisects the diagonals of a tetragram circumscribed to a circle passes through the centre.

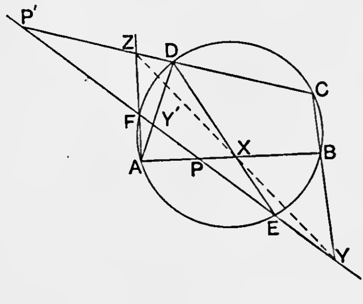
Ex. 3. If any tetragram be circumscribed to a given circle, show that the circles described on the diagonals of the tetragram will intersect on a fixed circle concentric with the given circle.

Ex. 4. Given any straight line, find the point on it, such that the pencil in involution determined by a given circle and a circumscribed tetragram will have the given line as a double line.

Pascal's and Brianchon's theorems.

281. Pascal's theorem, which relates to a hexastigm inscribed in a circle, has already been proved in Chapter VIII. (§ 181), where some further properties of such a hexastigm were investigated. Pascal's theorem asserts that *the opposite connectors of a hexastigm inscribed in a circle intersect in three collinear points*; that is to say, if A, B, C, D, E, F be any six points on a circle, then AB, BC, CD will intersect DE, EF, FA, respectively, in three collinear points.

The theorem may be readily deduced as a consequence of the



theorem proved in § 273, viz., that any circle and the pairs of opposite connectors of any inscribed tetrastigm determine a range in involution on any straight line.

Let A, B, C, D, E, F be any six points on a circle; let EF cut AB, CD, BC, AD in P, P', Y, Y' respectively; let AF cut CD in Z; and let AB cut DE in X.

Since $ABCD$ is a tetrastigm inscribed in the circle, therefore by § 273, $\{EF, YY', PP'\}$ is a range in involution.

But the connectors of the tetrastigm $AXDZ$ will cut the line EF in a range in involution (§ 144). Therefore if XZ cut EF in W, the range $\{EF, Y'W, PP'\}$ will be in involution.

It follows that W must coincide with Y. Hence the points X, Y, Z must be collinear, which is Pascal's theorem.

282. Brianchon's theorem asserts that if a hexagram be circumscribed to a circle, the three diagonals which connect the pairs of opposite vertices will be concurrent. That is to say, if a, b, c, d, e, f be any six tangents to a circle, then lines joining the points ab, bc, cd respectively to the points de, ef, fa, will be concurrent. The theorem follows at once from § 180, Ex. 3, and it may also be deduced from the theorem of § 279.

283. Let us now consider the hexagram formed by drawing the tangents to a circle at the six points A, B, C, D, E, F; and let us denote these tangents by a, b, c, d, e, f.

It follows from § 272, that the line connecting the points ab, de is the polar of the point of intersection of the lines AB, DE. And similarly, every diagonal of the hexagram will be the polar of the corresponding centre of the inscribed hexastigm.

Hence we may deduce properties of a hexagram circumscribed to a circle from the properties of a hexastigm inscribed in a circle.

Thus from the theorem: *The fifteen connectors of a hexastigm inscribed in a circle intersect in forty-five points which lie three by three on sixty lines, which pass three by three through twenty points;* we have the theorem: *The fifteen vertices of any hexagram circumscribed to a circle, connect by forty-five lines which pass three by three through sixty points, which lie three by three on twenty lines.*

When the points of contact of the hexagram are the vertices of the hexastigm, it is easy to see that the sixty *Brianchon* points of the former are respectively the poles of the sixty *Pascal* lines of the latter.

Ex. Show that if the Lemoine circle of the triangle ABC, cut the sides in the points X, X'; Y, Y'; Z, Z', respectively, the axis of perspective of the triangle ABC, and the triangle formed by the lines $Y'Z$, $Z'X$, $X'Y$, is the polar of the symmedian point of the triangle ABC with respect to the Lemoine circle.

CHAPTER XI.

THE THEORY OF RECIPROCATION.

The Principle of Duality.

284. LET us suppose that we have given any geometrical figure consisting of an assemblage of points. The polars of each point of the figure with respect to a fixed circle constitute another figure consisting of an assemblage of lines. These figures are said to be *reciprocal figures* with respect to the fixed circle.

Let F and F' be two such reciprocal figures; we propose to show that to every descriptive proposition concerning the figure F corresponds a proposition concerning the figure F'. That is to say, that when a proposition concerning any figure, regarded as an assemblage of points, has been proved, a corresponding proposition may be inferred for the reciprocal figure, regarded as an assemblage of lines; and *vice versa*. In fact it will be seen that the proofs of two such propositions will correspond step for step.

A proposition relating to any geometrical figure and the corresponding proposition relating to the reciprocal figure are called *reciprocal* propositions. The method by which the truth of a theorem is inferred from the reciprocal theorem, is known as the "principle of duality."

285. Firstly, let us consider the composition of two reciprocal figures. Let us suppose that F consists of certain points, lines, and curves. It is obvious that to each point of F will correspond a line of F'; and to each point on any line of F will correspond a line of F' passing through the pole of the line (§ 247). Consequently, to each line of F regarded as an assemblage of points will correspond an assemblage of lines of F' passing through a point. Or, we may say that to every line of F corresponds a point of F' (§ 4).

Now let us consider a curve of the nth order as belonging to F. An arbitrary line will cut this curve in n points; and the lines of F' corresponding to these points will be concurrent. Hence, corresponding to an assemblage of points of the nth order belonging to F, there will be an assemblage of lines of the nth class belonging to F'; that is, corresponding to a curve of the nth order belonging to F there will be a curve of the nth class belonging to F'.

In the same manner we may show that if there be a curve of the nth class belonging to the figure F, there will correspond a curve of the nth degree belonging to the figure F'.

It is evident that if the same process by which F' was obtained from F, be applied to the figure F' we shall obtain the original figure F. Hence the name "reciprocal figures."

286. Secondly, let us consider what relations will subsist between the several parts of a figure F' corresponding to given relations between the corresponding parts of a given figure F, of which F' is the reciprocal figure.

i. If certain points of F lie on a straight line, it follows from § 247, that the corresponding lines of F' will pass through a point.

Hence, corresponding to the line joining any two points of F, we shall have the point of intersection of the corresponding lines of F'.

ii. If two lines of F intersect in the point P, the corresponding points of F' will lie on the line which corresponds to P.

Hence, if several lines of F be concurrent, the corresponding points of F' will be collinear.

iii. If certain points of F lie on a curve of the nth order, the corresponding lines of F' will be tangents to a curve of the nth class.

Hence, corresponding to the tangent at a point P on a curve belonging to F, we shall have the point of contact with the corresponding curve of the line of F' which corresponds to the point P. For a tangent to a curve, considered as an assemblage of points, is the line joining two consecutive points of the system, and a point on a curve, considered as an assemblage of lines, is the point of intersection of two consecutive lines of the system.

iv. If two tangents to a curve belonging to F intersect in a point P, the corresponding points on the curve belonging to F' will lie on the line which corresponds to the point P.

v. Corresponding to a point of intersection of two curves of F, we shall have a common tangent to the corresponding curves of F'.

Hence, if two curves of F touch, the corresponding curves of F' will touch each other.

Thus, it appears that to every descriptive proposition concerning any geometrical figure, a corresponding proposition may be inferred for the reciprocal figure.

287. We propose now to give in parallel columns some examples of descriptive theorems with their reciprocals. The reader however is recommended to attempt to form the reciprocal theorem for himself, before looking at the reciprocal theorem as given.

Ex. 1. If the lines connecting the corresponding vertices of two triangles be concurrent, the corresponding sides of the triangles will intersect in collinear points. (§ 161.)

If the points of intersection of the corresponding sides of two triangles be collinear, the corresponding vertices of the triangles will lie on concurrent lines. (§ 163.)

Ex. 2. When three triangles are in perspective, and have a common centre of perspective, their three axes of perspective will be concurrent. (§ 170.)

When three triangles are in perspective, and have a common axis of perspective, their three centres of perspective will be collinear. (§ 168.)

Ex. 3. The nine lines which connect two triads of collinear points intersect in eighteen points which lie three by three on six lines, which pass three by three through two points. (§ 174.)

The nine points of intersection of two triads of concurrent lines may be connected by eighteen lines which pass three by three through six points, which lie three by three on two other lines. (§ 175.)

Ex. 4. In every tetrastigm the three pairs of opposite connectors intersect the opposite sides of the triangle formed by the centres of the tetrastigm, in six points which are the pairs of opposite vertices of a tetragram. (§ 148, Ex. 4.)

In every tetragram the three pairs of opposite vertices connect with the opposite vertices of the triangle formed by the diagonals of the tetragram, by six lines which are the pairs of opposite connectors of a tetrastigm. (§ 150, Ex. 2.)

Harmonic Properties.

288. Let us now consider what properties will subsist for a figure, reciprocal to a given figure, corresponding to harmonic properties of the given figure.

Let A, B, C, D be four collinear points of a figure F, and let a, b, c, d be the corresponding lines of the reciprocal figure F'. Let O be the centre of the circle of reciprocation: then a, b, c, d are the polars with respect to this circle of the points A, B, C, D respectively. Therefore a, b, c, d are respectively perpendicular to OA, OB, OC, OD.

Suppose now that $\{AB, CD\}$ is a harmonic range. Then $O\{AB, CD\}$ is a harmonic pencil, and consequently $\{ab, cd\}$ is a harmonic pencil. If, however, the line $ABCD$ passes through O, the lines a, b, c, d cut this line in points which are conjugate to A, B, C, D respectively; and therefore (§ 257, Ex. 2) the pencil $\{ab, cd\}$ is harmonic.

Hence, *if four points of a figure form a harmonic range, the corresponding lines of the reciprocal figure form a harmonic pencil.*

289. In the same way we can show that if any system of points of one figure form a range in involution, the corresponding system of lines of the reciprocal figure will form a pencil in involution.

290. The following are reciprocal theorems.

Ex. 1. The lines joining any centre of a given tetrastigm to the other centres are harmonic conjugate lines with respect to the connectors of the tetrastigm which pass through that centre. (§ 141.)

The points in which any diagonal of a given tetragram cuts the other diagonals are harmonic conjugate points with respect to the vertices of the tetragram which lie on that diagonal. (§ 149.)

Ex. 2. Any straight line is cut in involution by the pairs of opposite connectors of any tetrastigm. (§ 144.)

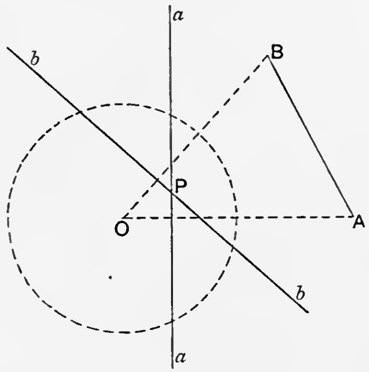
The lines connecting any point to the pairs of opposite vertices of a tetragram form a pencil in involution. (§ 154.)

Ex. 3. On each diagonal of a tetragram are taken a pair of points harmonically conjugate to the vertices of the tetragram which lie on that diagonal. If three of these points be collinear, so also will be the other three points. (§ 153.)

Through each centre of a tetrastigm are drawn a pair of lines harmonically conjugate to the connectors of the tetrastigm which intersect in that centre. If three of these lines be concurrent, so also will be the other three lines.

Reciprocation applied to metrical propositions.

291. Let A, B be any two points of a figure F, and let a, b be the corresponding lines of the reciprocal figure F'. Let O be the centre of the circle of reciprocation, and let p denote the perpendicular distance from O on the line AB.



Then $$p \, . \, AB = OA \, . \, OB \, . \sin AOB.$$

But since a, b are the polars of A and B,
$$Oa \, . \, OA = Ob \, . \, OB = r^2;$$
and $$\sin AOB = \sin ab.$$

Therefore if a and b intersect in P,
$$AB = \frac{r^2 \, . \, OP}{Oa \, . \, Ob} \sin ab;$$
and $$\sin ab = \frac{p}{OA \, . \, OB} \, . \, AB.$$

292. Let A be any point, and x any line of a figure; and let a be the corresponding line, and X the corresponding point of the reciprocal figure.

Then, O being the centre of the circle of reciprocation, we have (§ 251, Ex. 8)
$$Ax : Xa = OA : OX.$$
Therefore $$Ax = \frac{r^2}{OX \, . \, Oa} \, . \, Xa.$$

293. By means of these formulae we are able to transform any metrical theorem so as to obtain the reciprocal theorem. In a great many instances it will be found that although the formulae are apparently complicated, the reciprocal theorem is as simple as the original theorem.

Ex. 1. If $\{ABCD\}$ be any range,

$$AB \,.\, CD + BC \,.\, AD + CA \,.\, BD = 0.$$

If $\{abcd\}$ be any pencil,

$$\sin ab \,.\, \sin cd + \sin bc \,.\, \sin ad + \sin ca \,.\, \sin bd = 0.$$

Ex. 2. If the straight lines which connect the vertices A, B, C of a triangle to a point O, cut the opposite sides in X, Y, Z,

$$\frac{BX \,.\, CY \,.\, AZ}{XC \,.\, YA \,.\, ZB} = 1.$$

(§ 94.)

If any straight line cut the sides of a triangle ABC in the points X, Y, Z,

$$\frac{\sin BAX}{\sin CAX} \cdot \frac{\sin CBY}{\sin ABY} \cdot \frac{\sin ACZ}{\sin BCZ} = 1.$$

(§ 106.)

Ex. 3. If a straight line move so as to be divided in a constant ratio by the sides of a triangle, the locus of a point which divides one of the segments in a given ratio will be a straight line.

If a point move so that the sines of the angles subtended at it by the sides of a triangle are in constant ratio, the straight line which divides one of these angles into parts whose sines have a given ratio, will pass through a fixed point.

294. *To find the curve which is reciprocal to a circle.* A circle being a curve of the second order and second class, the reciprocal curve will be of the second class and second order. It will not in general be a circle. For if A be the centre of the given circle, P any point on it, we have

$$AP = \frac{OA \,.\, OX}{Ox} \,.\, \sin ax,$$

where x, and a, are the lines corresponding to the points P and A; and X denotes the point ax. Hence, denoting the line OX by z, we see that the ratio $\sin ax : \sin zx$ will be constant.

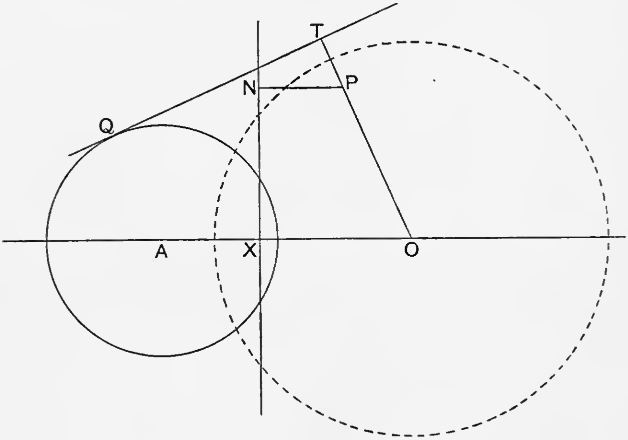
It follows that the figure reciprocal to a circle may be defined as the envelope of a line x which divides the angle between a fixed line a and a variable line z passing through a fixed point O, into parts whose sines are in a constant ratio.

If we wish to obtain a definition for such a curve as a locus we must proceed otherwise.

Let TQ be any tangent to the given circle, and let A be its centre. Let XN be the polar of A, and P the pole of QT with respect to the circle of reciprocation. Let OX, PN be perpendiculars on NX, and let OT, AQ be perpendiculars on TQ.

Then we have (§ 251, Ex. 8),

$$AQ : PN = OA : OP.$$



That is $$OP : PN = OA : AQ.$$

Thus the reciprocal curve to the given circle is the locus of a point which moves so that its distance from a fixed point varies as its distance from a fixed straight line.

295. If however the circle of reciprocation be concentric with the given circle, let QT be a tangent to the given circle, and let P be its pole with respect to the circle of reciprocation; then we have $OQ \,.\, OP$ constant, and therefore the locus of P will be a concentric circle.

When we wish to reciprocate theorems concerning a circle, it is usual to take the circle itself as the circle of reciprocation; for this circle evidently reciprocates into itself.

296. The following are examples of reciprocal theorems.

Ex. 1. If a tetrastigm be inscribed in a circle, its three centres form a self-conjugate triangle with respect to the circle. (§ 270.)

If a tetragram be circumscribed to a circle, its three diagonals form a self-conjugate triangle with respect to the circle. (§ 276.)

Ex. 2. If a hexastigm be inscribed in a circle, its opposite connectors intersect in three collinear points. (Pascal's theorem.)

If a hexagram be circumscribed to a circle, the lines which connect the three pairs of opposite vertices will be concurrent. (Brianchon's theorem.)

Ex. 3. If any system of chords of a circle be drawn through a fixed

If pairs of tangents be drawn to a given circle from points on a fixed point, the lines which join their extremities to any point on the circle will form a pencil in involution. (§ 259, Ex. 4.)

line, they will cut any other tangent to the circle in a range in involution.

Ex. 4. If any straight line be drawn through the pole of BC, with respect to the circumcircle of the triangle ABC, cutting AB and AC in Q and R, Q and R will be conjugate points with respect to the circle. (§ 259, Ex. 6.)

If any point P be taken on the polar of the point A with respect to the inscribed (or escribed) circle of the triangle ABC, the lines PB, PC will be conjugate lines with respect to the circle. (§ 259, Ex. 5.)

The Reciprocal of a circle.

297. It was proved in § 294 that the reciprocal curve of a given circle is the locus of a point which moves so that its distance from the centre of reciprocation varies as its distance from the line which is the reciprocal of the centre of the given circle. Thus the reciprocal of a given circle is a conic section, whose focus is the centre of reciprocation and directrix the line which corresponds to the centre of reciprocation. Referring to § 264, we see that this conic will be an ellipse, hyperbola, or parabola, according as the centre of reciprocation lies within, without, or on the given circle.

298. We propose to derive a few of the properties of conic sections from the corresponding properties of the circle.

Ex. 1. A circle is a curve of the second order and second class.

A conic is a curve of the second class and second order. (§ 285.)

Ex. 2. Any tangent to a circle is perpendicular to the line joining its point of contact to the centre.

Any point on a conic, and the point where its tangent meets the directrix subtend a right angle at the focus.

Ex. 3. The line joining the points of contact of two parallel tangents to a circle passes through the centre of the circle.

The point of intersection of the tangents at the extremities of any focal chord of a conic intersect on the directrix.

Ex. 4. Every chord of a circle which subtends a right angle at a fixed point on the circle passes through the centre.

The locus of the point of intersection of tangents to a parabola which cut at right angles is the directrix.

Ex. 5. The difference of the perpendiculars let fall from a fixed point on any pair of parallel tangents to a circle is constant.

The difference of the reciprocals of the segments of any focal chord of a conic is constant.

Ex. 6. The rectangle contained by the segments of any chord of a circle which passes through a fixed point is constant.

The rectangle contained by the perpendiculars drawn from the focus of a conic to a pair of parallel tangents is constant.

299. If any point P be taken on a given straight line x, and a pair of tangents be drawn to a given circle, we know that the straight line which is the harmonic conjugate of the line x with respect to the pair of tangents will pass through a fixed point, the pole of the line x with respect to the circle. Reciprocating with respect to any point we have the theorem: *If a chord of a conic be drawn through a fixed point, the locus of the harmonic conjugate of the fixed point with respect to the extremities of the chord is a straight line.*

This straight line is called the *polar* of the fixed point with respect to the conic. Thus the definition of the polar of a point with respect to a conic is exactly similar to the definition (§ 243) for a circle.

If we use the words 'pole,' 'conjugate,' and 'self-conjugate' in the same sense for a conic as in the case of a circle, we see that in enunciating the reciprocal of a given theorem concerning a circle, we shall have to interchange the words 'pole' and 'polar;' but the words 'conjugate' and 'self-conjugate' will be unchanged.

Ex. 1. The line joining any point to the centre of a circle is perpendicular to the polar of the point.

The line joining the point of intersection of any line with the directrix of a conic to the pole of the line subtends a right angle at the focus.

Ex. 2. Any triangle and its conjugate with respect to a circle are in perspective. (§ 267.)

Any triangle and its conjugate with respect to a conic are in perspective.

Ex. 3. If a chord of a circle subtend a right angle at a fixed point, the locus of its pole is another circle.

If two tangents to a conic intersect at right angles, the polar of the point of intersection envelopes a conic confocal with the given conic.

Ex. 4. The centres of any tetrastigm inscribed in a circle form a triangle which is self-conjugate with respect to the circle. (§ 270.)

The diagonals of any tetragram circumscribed to a conic form a triangle which is self-conjugate with respect to the conic.

Ex. 5. The diagonals of a tetragram circumscribed to a circle form a triangle which is self-conjugate with respect to the circle. (§ 276.)

The centres of any tetrastigm inscribed in a conic form a triangle which is self-conjugate with respect to the conic.

CHAPTER XII.

PROPERTIES OF TWO CIRCLES.

Power of a point with respect to a circle.

300. If through a fixed point O, any straight line be drawn cutting a given circle in the points P and Q, the rectangle $OP.OQ$ has the same value for all positions of the line OP (Euclid, Bk. III., Props. 35, 36). The value of this rectangle is called the *power* of the point O with respect to the circle.

If C be the centre of the circle, and R its radius, the power of the point O is equal to $OC^2 - R^2$, which is equal to the square of the tangent drawn from O to the circle.

For convenience we propose to call the square on the distance between two points, the *power* of one point with respect to the other; and the perpendicular from a point on a straight line, the *power* of the point with respect to the line.

301. Ex. 1. If two circles intersect in the points A and B, the powers of any point on the line AB with respect to the circles are equal.

Ex. 2. The locus of a point whose power with respect to a given circle is constant is a concentric circle.

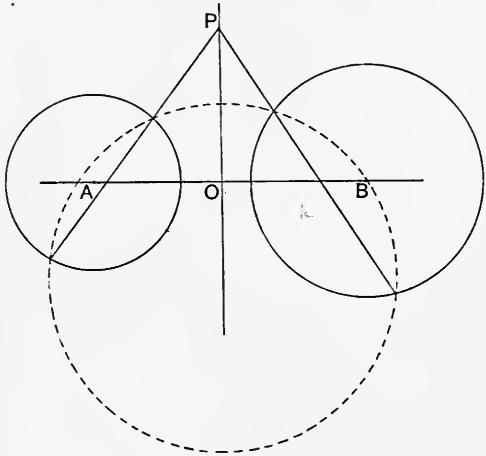
Ex. 3. If the sum of the powers of a point with respect to two given circles (or a point and a circle) be constant, the locus of the point is a circle.

Ex. 4. Find a point O on the line joining the centres of two circles, such that its powers with respect to the two circles shall be equal.

Let A and B be the centres of the circles; a and b their radii. Then $OA^2 - a^2 = OB^2 - b^2$. But if E be the middle point of AB, $OA^2 - OB^2 = 2OE.AB$. Therefore $2OE.AB = a^2 - b^2$. This determines the position of the point O uniquely, so that there is only one such point on the line AB.

It should be noticed however, that the point at infinity on the line AB may also be considered as a point whose powers with respect to the two circles are equal.

302. *The locus of a point whose powers with respect to two given circles are equal, is a straight line.*



Let A and B be the centres of the circles; and let a, b be their radii.

Let any circle be drawn cutting each of the circles in real points; and let the common chords of this circle and the given circles cut in the point P. Then evidently P is a point whose powers with respect to the given circles are equal.

Draw PO perpendicular to AB.

Then since $$PA^2 - a^2 = PB^2 - b^2,$$

therefore $$OP^2 + OA^2 - a^2 = OP^2 + OB^2 - b^2,$$

or $$OA^2 - a^2 = OB^2 - b^2.$$

Thus O is a point on AB whose powers with respect to the two circles are equal. But there is only one such point (§ 301, Ex. 4).

Hence, the locus of P is the straight line through the point O which is at right angles to AB.

This straight line is called the *radical axis* of the two circles.

303. Ex. 1. Show that the locus of point, whose power with respect to a circle is equal to its power with respect to a point, is a straight line.

Ex. 2. If the power of a point with respect to a circle be proportional to its power with respect to a straight line, show that the locus of the point will be a circle.

Ex. 3. If the powers of a point with respect to two given circles (or points) be in a constant ratio, show that the locus of the point will be a circle.

Ex. 4. Show that the power of any point on the line at infinity with respect to any circle is constant.

The Radical axis of two Circles.

304. The *radical axis* of two circles is the straight line which is the locus of a point whose powers with respect to two given circles are equal.

When the circles intersect in real points, the radical axis passes through these points (§ 301, Ex. 1). Hence the polars with respect to the circles of any point on their radical axis will intersect on the radical axis.

But whether the circles intersect in real points or not, the tangents to the circles from any point on the radical axis are equal. Therefore any circle which has its centre on the radical axis of two given circles, and which cuts one of them orthogonally will also cut the other orthogonally. Let such a circle cut the radical axis of the given circles in the points P and P'. Then P and P' will be conjugate points with respect to each of the given circles (§ 261, Ex. 1). Therefore the polars of any point on the radical axis intersect on the radical axis.

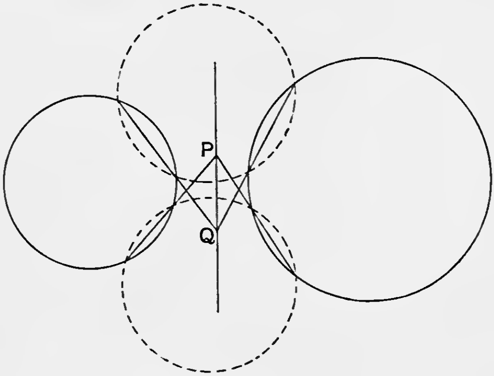
Now let $P, Q, R, \ldots$ be any points on the radical axis of two circles; and let the polars of these points with respect to the circles intersect in $P', Q', R', \ldots$, respectively. Then $\{PP', QQ', RR', \ldots\}$ is a range in involution. And the double points of this range must be the points in which the radical axis cuts either circle. It follows that the radical axis of two circles passes through their points of intersection, whether these points be real or imaginary.

305. *The radical axes of any three circles taken two at a time are concurrent.*

Let two of the radical axes meet in the point P. Then evidently the powers of the point P with respect to the circles are equal. Therefore P is a point on the third radical axis.

The point of concurrence of the radical axes of three circles is called the *radical centre* of the circles.

306. Hence, we can construct the radical axis of two circles which do not intersect in real points.



Draw any circle cutting the given circles in real points, and let the radical axes, that is the common chords, of these circles intersect in the point P. Then P is a point on the radical axis of the given circles.

Similarly, by drawing another circle we can find another point Q on the radical axis.

The line PQ will then be the radical axis of the circles.

307. Ex. 1. Show that the six radical axes of the inscribed and escribed circles of any triangle are the six connectors of a tetrastigm, each vertex of which is the orthocentre of the triangle formed by the other three.

Ex. 2. If AD, BE, CF be the perpendiculars on the sides of the triangle ABC, show that the axis of perspective of the triangles ABC, DEF, is the radical axis of the circumcircles of the triangles.

Ex. 3. Show that the radical axis of the circumcircle of a triangle and the Lemoine circle of the triangle, is the polar of the symmedian point with respect to the Lemoine circle.

Ex. 4. Show that the circumcircle of a triangle, its polar circle, and its nine-point circle have a common radical axis.

Ex. 5. Three circles are described with their centres on the sides BC, CA, AB of the triangle ABC, and cutting the circumcircle at right angles in A, B, C, respectively. Prove that these circles have a common radical axis.

[St John's Coll., 1886.]

Ex. 6. Any four points A, B, C, D are taken in a circle; AC, BD intersect in E; AB, CD in F; and AD, BC in G. Show that the circles circumscribing the triangles EAB, ECD intersect the lines AD, BC, in four points lying on a fourth circle; and that if these four circles be taken three at a time, the radical centres of the systems so formed will be the vertices of a parallelogram whose diagonals are the line EF and a line parallel to FG.

[Math. Tripos, 1887.]

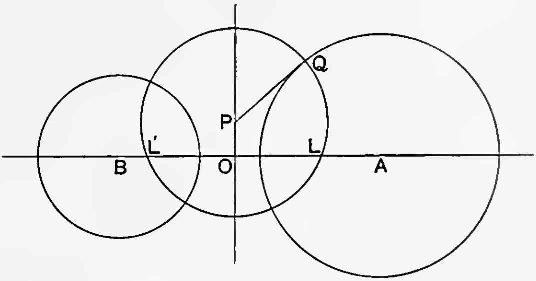
Ex. 7. The locus of a point the difference of whose powers with respect to two given circles is constant, is a straight line parallel to the radical axes of the circles.

308. The radical axis of two circles might have been defined as the locus of the centre of a circle which cuts each of them orthogonally.

For if P be the centre of a circle which cuts two given circles orthogonally, the radius of the circle is equal to the tangent drawn from P to either of the given circles. Hence P must be a point on the radical axis of the circles.

Hence we infer that only one circle can be drawn to cut three given circles orthogonally. The centre of this circle is clearly the radical centre of the given circles.

309. *Every circle which cuts two given circles orthogonally, passes through two fixed points on the line joining the centres of the given circles.*



Let A and B be the centres of the given circles; and let OP be their radical axis, cutting AB in the point O. Let any circle which cuts the circles orthogonally meet AB in L and L'; and let P be the centre of this circle.

Then $$PL^2 = PQ^2 = PA^2 - AQ^2.$$

Therefore $$OL^2 = OA^2 - AQ^2.$$

Hence the circle whose centre is O, and radius OL, will cut the given circles orthogonally.

It follows that every circle which cuts the given circles orthogonally will pass through the points L, L'.

It is easy to see that these points are real or imaginary according as the given circles intersect in imaginary or real points.

310. Ex. 1. If two circles cut two other circles orthogonally, show that the radical axis of either pair is the line joining the centres of the other pair.

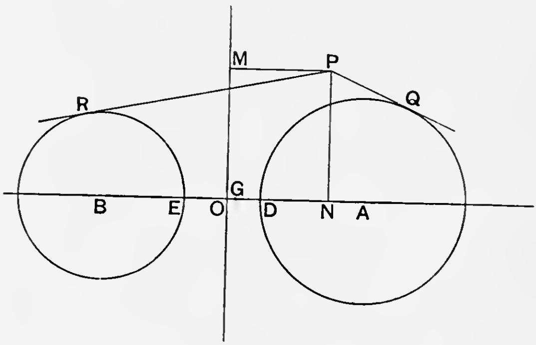
Ex. 2. If four circles be mutually orthotomic, show that the centre of any one of them is the orthocentre of the triangle formed by the centres of the other three.

Ex. 3. Show that the points L and L' (§ 309) are conjugate points with respect to each of the given circles.

Ex. 4. If A, A'; B, B'; C, C' be the pairs of opposite vertices of a tetragram, show that the circles described on AA', BB', CC' have a common radical axis, which passes through the centre of the circumcircle of the triangle formed by the lines AA', BB', CC'.

Ex. 5. If four circles cut a fifth circle orthogonally, show that their six radical axes form a pencil in involution.

311. *The difference of the powers of any point with respect to two given circles is proportional to the power of the point with respect to the radical axis of the circles.*



Let A and B be the centres of the circles; OM their radical axis; and let G be the middle point of AB.

Let P be any point; and let PM, PN be the perpendiculars from P on OM and AB.

Then the difference of the powers of P with respect to the circles is equal to

$$PR^2 - PQ^2,$$

that is, $$PB^2 - PA^2 + AD^2 - BE^2,$$

or, $$NB^2 - NA^2 + AD^2 - BE^2.$$

But $$NB^2 - NA^2 = 2NG \,.\, AB,$$

and (§ 301, Ex. 4), $$AD^2 - BE^2 = 2OG \,.\, BA.$$

Therefore $$PR^2 - PQ^2 = 2GN \,.\, BA + 2OG \,.\, BA$$
$$= 2ON \,.\, BA = 2PM \,.\, AB.$$

Thus the difference of the powers of the point P with respect to the given circles is equal to $2PM \,.\, AB$.

312. Ex. 1. Show that the power, with respect to a circle, of a point on another circle, is proportional to the power of the point with respect to the radical axis of the circles.

Ex. 2. Given any three circles having a common radical axis, show that the powers with respect to two of them of any point on the third circle are in a constant ratio.

Ex. 3. If the powers of any point with respect to two given circles be in a constant ratio, show that the locus of the point is a circle which has a common radical axis with the given circles.

Ex. 4. The radius of a circle which touches two given circles bears a constant ratio to the distance of its centre from the radical axis of the given circles.

Power of two circles.

313. The square on the distance between the centres of two circles less the squares on their radii is called the *power* of the two circles, or the power of one circle with respect to the other.

It will be convenient to consider the angle of intersection of two circles to be the angle subtended at either point of intersection by the line which joins the centres of the circles; so that in the case of two equal circles, the angle of intersection is the angle through which one of them must be turned about its point of intersection with the other, so that the two may coincide.

If d denote the distance between the centres of two circles; r, r' their radii; and θ their angle of intersection; the power of the circles is equal to $d^2 - r^2 - r'^2$, or $-2rr' \cos\theta$.

The power of two circles is always a real magnitude, even when the circles are imaginary, provided their centres are real points; but it may be either positive or negative. When the circles cut orthogonally the power vanishes; when they touch the power is equal to $\pm 2rr'$ according as the contact is external or internal.

The power of two coincident circles, that is the power of a circle with respect to itself, is equal to $-2r^2$.

If any two circles be denoted by X, Y, the power of the circles is usually denoted by (X, Y).

314. It is often convenient to consider a point as a circle whose radius is indefinitely small, and a straight line as a circle whose radius is infinitely great. When a point is treated as a circle, it is usually referred to as a *point-circle.*

If in the definition of the power of two circles, either of the circles be a point-circle, its power with respect to the other is clearly equal to the square on the tangent which can be drawn from the point to the circle. So that the definition given in § 300 is included in that given in § 313. Similarly, the power of two point-circles will be the square of the distance between the points.

In the case of a circle and a straight line, considered as a circle whose radius is infinite, the power is clearly proportional to $r \cos \theta$, where r is the radius of the circle, and θ the angle at which the circle cuts the line. Hence we may take as the power of a straight line and a circle the perpendicular distance from the centre of the circle on the straight line. Similarly we may take as the power of two straight lines the cosine of their angle of intersection.

Considering the case of the line at infinity, it is easy to see that the powers of any two circles with respect to the line at infinity will be in a ratio of equality, but the power of a straight line with respect to it will be zero.

315. The definitions given in the last article are seldom required, but it will generally be found that if any theorem relating to points, lines, and circles, can be expressed as a *power-theorem* (that is a metrical theorem in which the only metrical magnitudes involved are powers), a corresponding theorem may be enunciated for a more general figure in which the points and lines are replaced by circles.

Ex. 1. If the power of a variable circle with respect to a given circle be constant, the variable circle will cut orthogonally a fixed circle concentric with the given circle. (Cf. § 301, Ex. 2.)

Let X denote the fixed circle, and Z the variable circle; let A, C denote their centres; and let a, c denote their radii. Then we have $AC^2 - a^2 - c^2 =$ constant $= k^2$. Hence, if a circle X' be described with A for centre, and radius a', given by $a'^2 = a^2 + k^2$, it is clear that the power of the circles Z and X' will be zero; that is, the circle Z will cut X' orthogonally.

Ex. 2. If the sum of the powers of a variable circle and two given circles be constant, the variable circle will cut orthogonally a fixed circle. (Cf. § 301, Ex. 3.)

Ex. 3. The difference of the powers of a circle with respect to two given circles is proportional to the power of that circle with respect to the radical axis of the given circles. (Cf. § 311.)

Ex. 4. If a circle be drawn cutting orthogonally one of two given circles, its power with respect to the other given circle is proportional to its power with respect to the radical axis of the given circles. (Cf. § 312, Ex. 1.)

Ex. 5. If the powers of a variable circle with respect to two given circles be in a constant ratio, the variable circle will cut orthogonally a fixed circle which has a common radical axis with the given circles. (Cf. § 312, Ex. 3.)

Ex. 6. If a circle touch two given circles it must cut orthogonally one or other of two fixed circles.

Ex. 7. The locus of the centre of a circle which bisects two given circles, that is cuts them in points which are opposite ends of diameters, is a straight line parallel to the radical axis of the circles.

Ex. 8. Show that one circle can be drawn which shall bisect three given circles, and construct it.

Ex. 9. Show that one circle can be drawn which shall be bisected by three given circles.

This circle is concentric with, and cuts orthogonally the circle which cuts the given circles orthogonally. Hence, the former is a real circle only when the latter is imaginary.

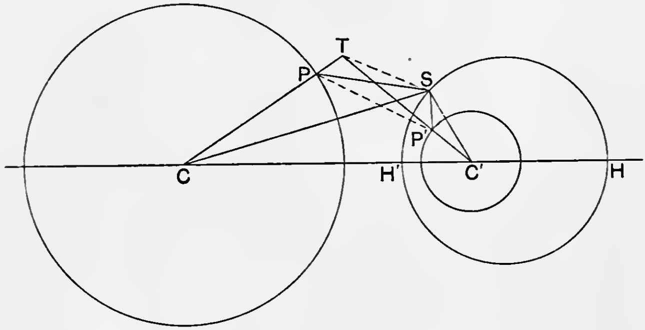
Centres of similitude of two circles.

316. Any two circles may evidently be regarded as diagrams of the same figure drawn to different scales. Hence two circles may be considered as directly similar figures (§ 216).

Let P be any point on one circle: then we may obviously take any point P' on the other circle as the point which corresponds to P. The correspondence will then be determined. For, if the points Q, Q' be any other pair of corresponding points, the arcs PQ, $P'Q'$ must subtend at the centres of the circles equal angles measured in the same sense. It follows that there will be an infinite number of positions for the centre of similitude.

Let us suppose that we have given a pair of corresponding points on the two circles.

To find the centre of similitude we must proceed as in § 205. Thus let P and P' be the given points which correspond, and let C and C' be the centres of the circles. Then if CP meet $C'P'$ in T, the circles which circumscribe the triangles TPP', TCC' will intersect in the centre of similitude.



Let S be the centre of similitude, then it follows from § 214, that

$$SC : SC' = SP : SP' = CP : C'P'.$$

Hence the locus of the centre of similitude of two circles is a circle which has a common radical axis with the point-circles C and C' (§ 319, Ex. 3).

This circle is called the *circle of similitude* of the given circles.

317. Ex. 1. Show that the circle of similitude of two given circles has with them a common radical axis.

Let S be any point on the circle of similitude. Then $SC : SC' = r : r'$; therefore the powers of the point S with respect to the given circles are in the ratio of the squares on their radii. Hence, the theorem follows from § 312, Ex. 3.

Ex. 2. Show that if from any point on the circle of similitude of two given circles, pairs of tangents be drawn to both circles, the angle between one pair will be equal to the angle between the other pair.

Ex. 3. Show that the three circles of similitude of three circles taken in pairs have a common radical axis.

Ex. 4. Show that the three circles of similitude of three given circles cut orthogonally the circumcircle of the triangle formed by the centres of the given circles.

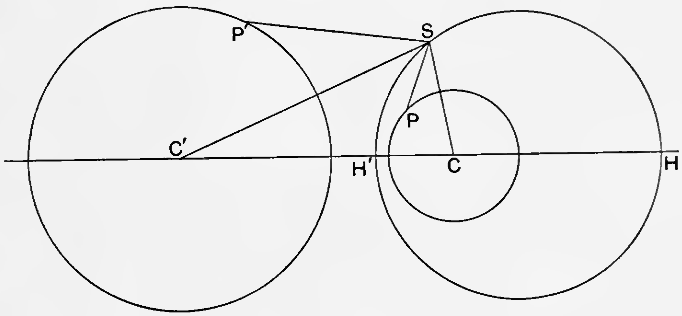
Ex. 5. Prove that there are two points, each of which has the property that its distances from the angular points of a triangle are proportional to the opposite sides; and that the line joining them passes through the centre of the circumcircle. [Math. Tripos, 1888.]

This theorem is also true when the distances from the angular points are in any given ratio.

Ex. 6. If A, B, C, D be any four points on a circle; and if AB, CD intersect in E: AC, BD in F; and AD, CB in G; show that the circle described on FG as diameter is the circle of similitude of the circles described on AB and CD.

Ex. 7. If O be the orthocentre, and G the median point of the triangle ABC, show that the circle described on OG as diameter is the circle of similitude of the circumcircle and nine-point circle of the triangle.

318. *Given a centre of similitude of two given circles to find the corresponding points on the circles.*



Let C, C' be the centres of the given circles; S the given centre of similitude, on the circle of similitude.

If P be any point on the circle whose centre is C, the corresponding point P' on the other circle will be such that the angles CSP, $C'SP'$ are equal and measured in the same sense (§ 214).

Also the angle PSP' will be equal to the angle CSC'. Hence if S coincide with either of the points in which the circle of similitude cuts the line CC', the points P and P' will be collinear with S. That is, the circles will have these points for *homothetic* centres (§ 213).

319. Let the circle of similitude cut the line joining the centres C, C' in the points H and H'. Then when it is necessary to distinguish these points, we shall call that point which does not lie between the centres the *homothetic centre* of the circles, and the point which lies between the centres the *anti-homothetic centre* of the circles.

These points are often called the *external and internal centres of similitude,* but these names are clearly inappropriate, since any point on the circle of similitude may be considered as a centre of similitude.

320. Ex. 1. Show that two of the common tangents of two circles pass through each homothetic centre.

Ex. 2. If H and H' be the homothetic centres of two circles whose centres are C and C', show that $\{HH', CC'\}$ is a harmonic range.

Ex. 3. If K and K' be the poles of the radical axis of two circles, and H and H' their homothetic centres, show that $\{KK', HH'\}$ is a harmonic range.

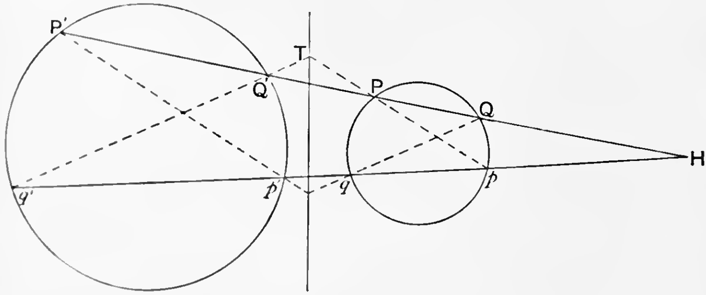
Ex. 4. If through either homothetic centre of two circles, a line be drawn to cut the circles in the points P, Q; P', Q'; respectively, so that P and P' are corresponding points; show that

$$HP \,.\, HQ' = HQ \,.\, HP';$$

and that these rectangles have a constant value for all positions of the line HP.

Ex. 5. If the line joining the homothetic centres of two circles, cut them in A, B and A', B' respectively, show that $\{HH', AB, A'B'\}$ is a range in involution.

Ex. 6. Through either homothetic centre of two given circles are drawn two lines HP, Hp cutting the circles respectively in the points P, Q, p, q; P', Q', p', q'; show that any pair of non-corresponding chords such as Pp, $Q'q'$ will intersect on the radical axis of the given circles.

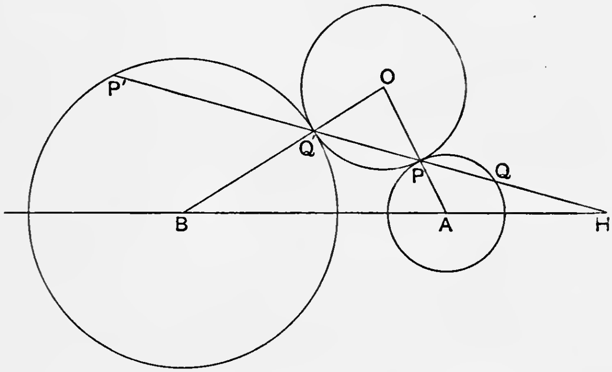


Since $HP \,.\, HQ' = Hp \,.\, Hq'$ (Ex. 4), the points P, Q', p, q' are concyclic. Therefore if Pp meet $Q'q'$ in T, $TP \,.\, Tp = TQ' \,.\, Tq'$. Hence T is a point on the radical axis of the circles.

Ex. 7. If from any point T, on the radical axis of two circles, tangents be drawn to the circles; show that the homothetic centres of the circles will be two of the centres of the tetrastigm formed by the points of contact.

Ex. 8. The line joining the centres of two circles meets one of the circles in the point A, and the other in the point B; and P is any point on the radical axis of the circles. If PA, PB cut the circles in Q and R, show that the tangents at Q and R meet on the radical axis.

Ex. 9. If any circle be drawn to touch two given circles, show that the line joining the points of contact will pass through one of the homothetic centres.



Let a circle be drawn touching two given circles in the points P and Q'. Then, if O be its centre, and A and B the centres of the given circles, it is evident that AP, BQ' are equally inclined to PQ'. Therefore, if PQ' cut the given circles in Q and P', AQ and BQ' are parallel. Therefore PQ must pass through H, one of the homothetic centres of the given circles.

If the variable circle touch the given circles both internally, or both externally, the line joining the points of contact will pass through the homothetic centre of the given circles; but if the circle touch one of the given circles internally and one externally, the line joining the points of contact will pass through the anti-homothetic centre.

Ex. 10. Show that if a variable circle touch two given circles it will cut orthogonally one or other of two fixed circles, whose centres are the homothetic centres of the given circles, and which have a common radical axis with the given circles.

Ex. 11. If two circles be drawn to touch two given circles, show that the radical axis of either pair will pass through a homothetic centre of the other pair, provided that: if one of the circles touches the given circles both externally or both internally, so also does the other; or, if one of the circles touch one of the given circles internally and the other externally, so also does the other.

Ex. 12. Two circles are drawn through a fixed point O to touch two fixed straight lines AB, AC in the points D, E and F, G respectively. Show that the circles circumscribing the triangles ODE, OFG touch one another in the point O. [St John's Coll. 1887.]

Ex. 13. Two circles APB, AQB touch a third in the points P and Q. Show that

$$AP : AQ = BP : BQ.$$

Ex. 14. If the inscribed circle of the triangle ABC touch the side BC in the point P, and if D, R be the middle points of BC and AP, show that DR passes through the centre of the inscribed circle.

Let the escribed circle which is on the opposite side of BC touch BC in Q, and let AQ cut the inscribed circle in P'. Then, since A is the homothetic centre of the two circles, if O, O' be the centres of the circles, OP' and $O'Q$ are parallel. Hence P, O, P' are collinear. But D is the middle point of PQ. Therefore D, O, R are collinear.

The theorem is also true of any one of the circles which touch the sides of the triangle.

For another proof of this theorem see § 241.

Ex. 15. If O, O' be the centres of any two circles which touch two given circles in the same sense, at the points P, Q and P', Q' respectively, show that

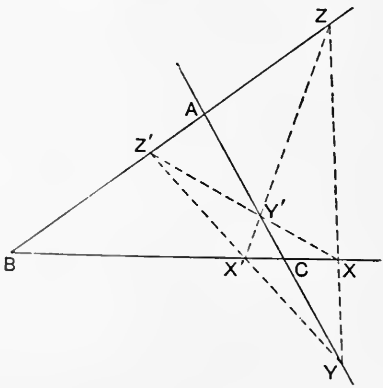
$$PQ^2 : P'Q'^2 = AO \, . \, BO : AO' \, . \, BO',$$

where A and B are the centres of the given circles.

Ex. 16. A circle whose centre is O touches two given circles, whose centres are A and B, at P and Q, and FG is the common tangent of the given circles which passes through the point of intersection of AB and PQ. Show that

$$PQ^2 : FG^2 = OP^2 : AO \, . \, BO.$$

321. *The six homothetic centres of three circles taken in pairs, are the six vertices of a tetragram.*



Let A, B, C be the centres of the given circles; and let their radii be denoted by a, b, c. Let X, Y, Z be the homothetic centres and X', Y', Z' the anti-homothetic centres of the three pairs of circles.

Then since
$$BX : CX = b : c,$$
$$CY : AY = c : a,$$
$$AZ : BZ = a : b;$$
therefore
$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1.$$
Therefore (§ 105) the points X, Y, Z are collinear.

Again, since
$$CY' : Y'A = c : a,$$
$$AZ' : Z'B = a : b;$$
therefore
$$\frac{BX}{CX} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ'}{Z'B} = 1;$$
that is,
$$\frac{BX}{CX} \cdot \frac{CY'}{AY'} \cdot \frac{AZ'}{BZ'} = 1.$$
Therefore the points X, Y', Z' are collinear.

In the same way we may show that the points X', Y, Z' are collinear; and that the points X', Y', Z are collinear.

Hence X, X'; Y, Y'; Z, Z' are the opposite pairs of vertices of a tetragram.

These four lines are called the *axes of similitude* or *the homothetic axes* of the given circles.

322. Ex. 1. Show that the lines AX, AX'; BY, BY'; CZ, CZ' are the three pairs of opposite connectors of a tetrastigm.

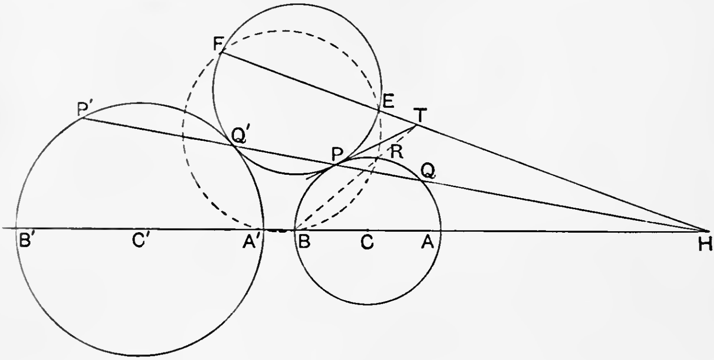
Ex. 2. If a variable circle touch two fixed circles, the line joining the points of contact passes through one of the homothetic centres of the given circles.

Let A, B denote the given circles, and let X denote a circle touching A and B in the points P and Q. Then P and Q are homothetic centres of the pairs of circles A, X; and B, X; respectively.

Ex. 3. If the nine-point circle of the triangle ABC touch the inscribed circle at the point P, and the escribed circles at the points P_1, P_2, P_3; show that PP_1 and P_2P_3 cut BC in the same points as the internal and external bisectors of the angle BAC.

Ex. 4. Describe a circle which shall touch two given circles and pass through a given point.

Let E be the given point. Let it be required to draw a circle passing through E, which shall touch each of the given circles *externally*. Then the line joining the points of contact P and Q' must pass through H, and $HP \cdot HQ' = HA' \cdot HB$.



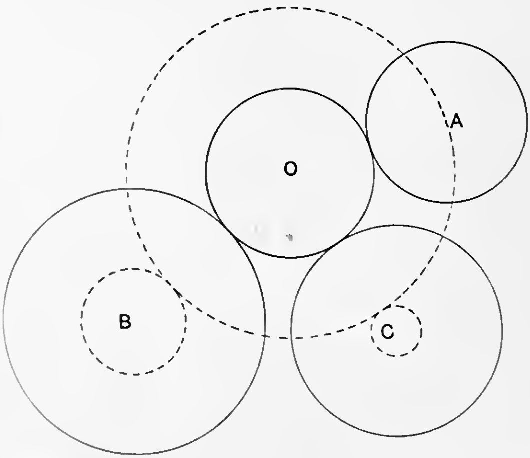
Draw the circle $A'BE$, and let it cut HE in F, and one of the given circles in R. Let BR cut EF in T, and from T draw a line touching the circle BRA in P. Then the circle circumscribing the triangle EFP will touch the given circles.

Since two tangents may be drawn from the point T to the circle ABR, it follows that two circles can be drawn to touch the given circles, so that the line joining the points of contact shall pass through the homothetic centre H.

Similarly, it is evident that two circles can be drawn to pass through E and touch the given circles, so that the line joining the point of contact shall pass through the anti-homothetic centre H'.

Thus, four circles can be drawn satisfying the given conditions.

Ex. 5. Show how to describe a circle which shall touch three given circles.



There will generally be eight circles which can be drawn to touch three given circles; that is, two circles touching the given circles each in the same sense, and three pairs of circles which touch one of the given circles in the opposite sense to that in which it touches the other two.

Let us suppose that O is the centre of the circle which touches each of the three given circles externally. Let A, B, C be the centres of the given circles; and let a, b, c denote their radii, and let us suppose that a is not greater than b or c. Then, if r denote the radius of the circle which touches them, it is evident that a circle described with O for centre, and with radius equal to $r+a$, will pass through the point A and touch externally two circles whose centres are B and C, and radii $b-a$, $c-a$, respectively.

Now this circle may be easily constructed as in Ex. 4; and thus we shall be able to find the point O.

In the same manner the centres of the other seven circles can be found.

Ex. 6. If two circles X, X' be drawn to touch three given circles A, B, C, so that each touches all of the given circles externally, or all internally, show that the radical axis of X and X' passes through the three homothetic centres of A, B, and C; and that the radical centre of A, B, and C is the anti-homothetic centre of X and X'.

Ex. 7. Describe a circle which shall touch two given circles and cut a given circle orthogonally.

Show that four circles can be drawn satisfying these conditions.

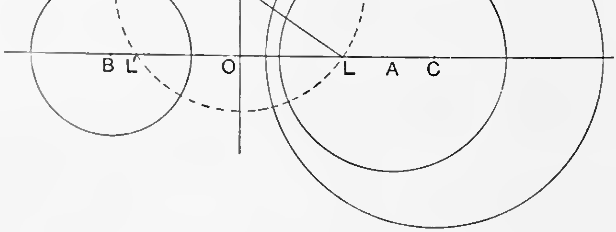
CHAPTER XIII.

COAXAL CIRCLES.

The Limiting Points.

323. If any system of circles have a common radical axis, the circles are said to be *coaxal.*

It was proved in § 308, that if with any point P, on the radical axis of two circles, as centre, a circle be described cutting either circle orthogonally, it will also cut the other circle orthogonally. Hence, if the centre of a circle which cuts one circle of a coaxal system orthogonally, lie on the radical axis, the circle will cut all the circles of the system orthogonally. From § 309, it follows that any such circle will cut the line of centres of the circles of the system in two fixed points.



Let these fixed points be L and L'. Then it is evident that the power of the point P with respect to the point-circle L is equal to the power of P with respect to any circle of the system.

Hence the point L, and similarly the point L', may be considered as a point-circle belonging to the coaxal system.

Hence, in every coaxal system there are two circles whose radii are indefinitely small.

These point-circles are called the *limiting points* of the system. They are evidently real only when the circles do not intersect in real points.

324. Ex. 1. If the circles of a coaxal system touch at the point O, show that the limiting points coincide in the point O.

Ex. 2. If any circle of a coaxal system pass through a limiting point of the system, show that the two limiting points must coincide, and that the circles of the system will touch each other at this point.

Ex. 3. Show that the polars of a fixed point with respect to the circles of a coaxal system are concurrent.

Let Q be the given point, and let the limiting points of the system be L, and L'. Let Q' be the opposite extremity of the diameter of the circle QLL' which passes through Q. Then since this circle cuts each circle of the given coaxal system orthogonally, it follows from § 261, Ex. 1, that Q and Q' will be conjugate points with respect to every circle of the system. Therefore the polars of the point Q will intersect concurrently in the point Q'.

If, however, Q be a point on the line LL', this proof fails. But in this case it is evident that the polars of the point Q will be perpendicular to the line LL', and will therefore meet in a point at infinity.

Ex. 4. If that circle of the system which passes through the point Q, in the last example, be drawn, show that it will touch QQ'.

Ex. 5. Show that the polar of a limiting point with respect to any circle of the system is the line which passes through the other limiting point and is parallel to the radical axis.

Ex. 6. If Q, Q' be a pair of points which are conjugate with respect to every circle of a coaxal system, show that QQ' subtends a right angle at each limiting point.

Ex. 7. Show that the radical axes of the circles of a coaxal system and any given circle are concurrent.

Ex. 8. If two systems of coaxal circles have one circle (or a limiting point) common, they have a common orthogonal circle.

Ex. 9. Three circles have their centres collinear and cut orthogonally a given circle, show that they are coaxal.

Ex. 10. Show that any line is cut in involution by the circles of a coaxal system.

Ex. 11. If a straight line cut any two circles of a coaxal system in the points P, Q; P', Q': respectively, and the radical axis in the point O, show that PP' and QQ' will subtend equal or supplementary angles at any point of the circle, whose centre is O, which cuts the given circles orthogonally.

Ex. 12. On two straight lines are taken the points P, Q, R, S,... ; and P', Q', R', S',... ; respectively, so that

$$PQ : P'Q' = PR : P'R' = PS : P'S' = \&c...$$

If the straight lines intersect in the point O, show that the circles OPP', OQQ', ORR', &c., are coaxal.

325. Ex. 1. Construct a circle which shall be coaxal with a given system of coaxal circles, and cut a given circle orthogonally.

Let Z denote the given circle, and let Y, Y' be any circles which cut each circle of the given system orthogonally. Then the circle which cuts Z, Y, Y' orthogonally will clearly satisfy the conditions of the question.

There is only one solution to the problem.

Ex. 2. Construct a circle which shall be coaxal with a given system, and touch a given circle.

Let Z denote the given circle, and let X, X' denote any circles of the coaxal system. Let the circle, Y say, which cuts Z, X, X' orthogonally, cut Z in the points P and Q. Then if the tangents to Y at P and Q cut the line joining the centres of X and X' and C and C', it is easy to see that the circles whose centres are C and C' and radii CP, $C'Q$ respectively, will touch the circle Z and be coaxal with X and X'.

Ex. 3. Show that two circles of a coaxal system can be drawn which shall touch a given straight line.

Ex. 4. If the two circles of a coaxal system which touch a given straight line, touch it in P and Q, show that PQ subtends a right angle at each of the limiting points of the system.

Orthogonal coaxal systems.

326. Every circle which cuts two circles of a coaxal system orthogonally, cuts every circle of the system orthogonally, and every such circle passes through the limiting points of the system (§ 323). Hence, given any system of coaxal circles, another system of coaxal circles may be constructed such that every circle of either system cuts orthogonally every circle of the other system.

Two such systems are called *orthogonal systems* of coaxal circles.

It is evident from § 309, that if the limiting points of a given system be real, the limiting points of the orthogonal system will be imaginary. The limiting points of either system are sometimes called the *antipoints* of the limiting points of the other system.

327. Ex. 1. Show that the polar circles of the four triangles formed by four straight lines, taken three at a time, and the circles described on the diagonals of the tetragram formed by the lines as diameters, are orthogonal systems of coaxal circles.

Hence, the orthocentres of the four triangles formed by four lines lie on a straight line which is perpendicular to the line which bisects the diagonals of the tetragram formed by the lines.

Ex. 2. If X, Y, Z be collinear homothetic centres of three circles, show that the circles described with these points for centres and coaxal with the three pairs of circles, will be coaxal.

Ex. 3. Show that the antipoints of four concyclic points lie four by four on three circles orthotomic with each other and the original circle.

Relations between the powers of coaxal circles.

328. *The difference of the powers of a variable circle with respect to two given circles is equal to twice the rectangle contained by the power of the variable circle with respect to the radical axis of the given circles, and the distance between their centres.*

Let X, Y denote the given circles, and let Z denote the variable circle. Then if A, B, C be the centres of these circles, we have by § 311,

$$\begin{aligned}(ZX) - (ZY) &= (CX) - (CY)\\ &= 2AB \,.\, NC,\end{aligned}$$

where CN is the perpendicular from C on the radical axis of the system.

329. Let X_1, X_2, X_3,..., denote any circles of a given coaxal system, and let X be that circle of the system which cuts orthogonally a given circle Z. Then if A, A_1, A_2, A_3,... be the centres of the circles X, X_1, X_2, X_3,..., we have since $(ZX) = 0$,

$$(ZX_1) : (ZX_2) : (ZX_3) = AA_1 : AA_2 : AA_3.$$

Thus: *If a variable circle be drawn cutting a fixed circle of a coaxal system orthogonally, its powers with respect to any fixed circles of the system are in a constant ratio.*

The converse of this theorem is also true. For, let Z denote any circle whose powers with respect to two circles X_1, X_2, of a coaxal system are in a constant ratio. And let X denote that circle of the system which cuts Z orthogonally. Then if A, A_1, A_2, be the centres of the circles X, X_1, X_2, we have

$$(ZX_1):(ZX_2)=AA_1:AA_2.$$

Therefore $AA_1:AA_2$ is a constant ratio; and therefore the point A is fixed, that is to say, the circle Z will always cut the same circle, X, orthogonally. Thus we have the theorem: *If the powers of a variable circle with respect to two given circles be in a constant ratio, the variable circle cuts orthogonally a fixed circle coaxal with the given circles.*

330. Let us consider the case of a variable circle which cuts two given circles at constant angles.

Let Z be a variable circle which cuts the given circles X_1, X_2, at angles α_1, α_2. Then if ρ, r_1, r_2 denote the radii of these circles, we have

$$(ZX_1)=-2\rho r_1\cos\alpha_1,\ (ZX_2)=-2\rho r_2\cos\alpha_2.$$

Therefore

$$(ZX_1):(ZX_2)=r_1\cos\alpha_1:r_2\cos\alpha_2.$$

Hence by § 329, Z cuts orthogonally a fixed circle coaxal with the circles X_1, X_2.

Again, let X_3 denote any other circle coaxal with X_1 and X_2, and let X_3 cut Z at the angle α_3. Then by the last article, the ratio $(ZX_1):(ZX_3)$ is constant; that is, the ratio $r_1\cos\alpha_1:r_3\cos\alpha_3$ is constant.

Therefore α_3 is a constant angle.

Hence we have the theorem: *A variable circle which cuts two fixed circles of a coaxal system at constant angles, cuts every circle of the same system at a constant angle.*

Now two circles of a coaxal system can always be drawn to touch a given circle. We infer from the last theorem that if X, X' be the two circles coaxal with X_1, X_2, which touch the variable circle Z in any position, then X, and X' will touch Z in all its positions. Thus: *A variable circle which cuts two fixed circles of a coaxal system at constant angles, will always touch two fixed circles of the same system.*

331. Ex. 1. Show that if the powers of a variable circle with respect to three given circles be in constant ratio, the variable circle will be coaxal with the circle which cuts the given circles orthogonally.

Ex. 2. If X, Y, Z be any three given circles, and if circles X', Y', Z' be drawn cutting a fourth given circle orthogonally, and coaxal respectively with the pairs of circles Y, Z; Z, X; X, Y; show that the circles X', Y', Z' are coaxal.

Ex. 3. Show that all circles which cut three given circles at the same angle form a coaxal system.

Ex. 4. Show that all circles which cut three given circles at the same or supplementary angles form four coaxal systems, whose radical axes are the axes of similitude of the given circles.

Ex. 5. If the product of the tangents, from a variable point P to two given circles, has a given ratio to the square of the tangent from P to a third given circle coaxal with the former, the locus of P is a circle of the same system.

Ex. 6. If the product of the powers of a variable circle with respect to two given circles, has a constant ratio to the square of the power of the circle with respect to a third circle coaxal with the former, the variable circle will cut orthogonally a circle of the same system.

Ex. 7. A straight line cuts two given circles in the points P, P'; Q, Q'; respectively. Show that the tangents at P and P' will intersect the tangents at Q and Q' in four points which lie on a circle coaxal with the given circles.

Ex. 8. If ABC be a triangle inscribed in a circle of a coaxal system; and if P, P' be the points of contact of BC with the two circles of the system which it touches; Q, Q' the similar points on CA; and R, R' the similar points on AB; show that:—

i. The point-pairs P, P'; Q, Q'; R, R' are the pairs of opposite vertices of a tetragram.

ii. The line-pairs AP, AP'; BQ, BQ'; CR, CR' are the pairs of opposite connectors of a tetrastigm.

Ex. 9. The sides of the triangle ABC touch three circles of a coaxal system in the points X, Y, Z. If AX, BY, CZ be concurrent, or if X, Y, Z be collinear, show that the centres of the circles will form with the centres of those circles of the system which pass through the points A, B, C, a range in involution.

Ex. 10. If A, B, C be the centres of any three coaxal circles, and if a, b, c denote their radii, show that

$$BC\,.\,a^2+CA\,.\,b^2+AB\,.\,c^2=-BC\,.\,CA\,.\,AB.$$

Ex. 11. If A, B, C be the centres of any three coaxal circles, and if p, q, r denote their powers with respect to any other circle, show that

$$BC\,.\,p+CA\,.\,q+AB\,.\,r=0.$$

332. In the theorems given in §§ 328, 329, any circle of the coaxal system may be replaced by one of the limiting points of the system. Hence we have the following theorems:

(i) *If P be any point on a fixed circle of a coaxal system, the square on the distance from P to a limiting point of the system is proportional to the perpendicular from P on the radical axis.*

(ii) *If P be any point on a fixed circle of a coaxal system, the tangent drawn from P to any other circle of the system is proportional to the distance of P from either limiting point of the system.*

(iii) *If the tangent drawn from a point to a circle be proportional to its distance from a fixed point, the locus of the point will be a circle coaxal with the fixed point and the given circle.*

333. Ex. 1. If through either limiting point of a system of coaxal circles, a straight line be drawn intersecting a circle of the system, show that the rectangle contained by the perpendiculars from the points of intersection on the radical axis is constant.

Ex. 2. Two circles are drawn, one lying within the other. From L, the limiting point which lies outside them, are drawn tangents to the circles, touching the outer circle in A and the inner in B. If LB cut the outer circle in C, and D, prove that

$$LA^2 = LB^2 + CB \cdot BD.$$ [St John's Coll. 1886.]

Ex. 3. Two circles touch each other internally at the point O, and a straight line is drawn cutting the circles in the points A, B; and C, D; respectively. The tangent at A intersects the tangents at B and C in E and F, and the tangent at B intersects the tangents at B and C in G and H. Prove that E, F, G, and H lie on a circle which touches each of the given circles at O.

Ex. 4. If a variable circle touch two circles of a coaxal system, the tangents drawn to it from the limiting points have a constant ratio.

Ex. 5. If a variable circle touch two circles of a coaxal system, its radius varies as the square of the tangent drawn to it from either limiting point.

Ex. 6. If a variable circle cut two circles of a coaxal system at given angles, the tangents drawn to it from the limiting points have a constant ratio.

Ex. 7. From the vertices of the triangle ABC, AP, BQ, CR are drawn to touch a given circle. Show that if the sum of two of the rectangles

$$BC \cdot AP, \quad CA \cdot BQ, \quad AB \cdot CR,$$

be equal to the third, then the circle will touch the circumcircle of the triangle ABC. [Purser.]

Suppose we have given $BC \cdot AP = CA \cdot BQ + AB \cdot CR$.

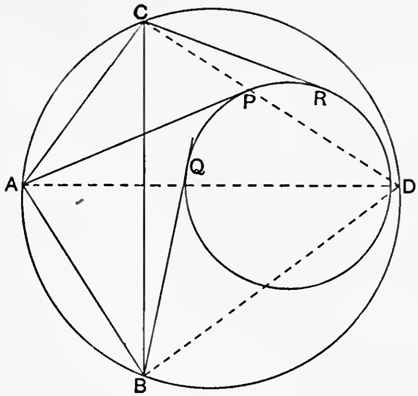
On the arc BC find a point D such that

$$BD : CD = BQ : CR.$$

Then $$BC \cdot AD = CA \cdot BD + AB \cdot CD.$$

Hence $$AP : BQ : CR = AD : BD : CD.$$

It follows from § 322 (iii), that D must be one of the limiting points of the circles ABC, PQR.



Hence the circles ABC, PQR must touch each other*.

Ex. 8. Show that the nine-point circle of a triangle touches the inscribed and escribed circles of the triangle.

Let D, E, F be the middle points of the sides of the triangle; P, Q, R, the points of contact of the sides with the inscribed circle. Then, if a, b, c denote the sides of the triangle, it is easy to see that

$$DP=\tfrac{1}{2}(b\sim c),\quad EQ=\tfrac{1}{2}(c\sim a),\quad FR=\tfrac{1}{2}(a\sim b),$$

and therefore $$EF\,.\,DP\pm FD\,.\,EQ\pm DE\,.\,FR=0.$$

Hence by the last theorem the nine-point circle touches the inscribed circle of the triangle.

Ex. 9. A chord of a circle subtends a right angle at a fixed point O. Show that the locus of the middle point of the chord is a circle coaxal with the given circle and the point-circle O.

Ex. 10. Show that the locus of the foot of the perpendicular from a fixed point O on any chord of a given circle which subtends a right angle at O, is a circle coaxal with the given circle and the point-circle O.

Ex. 11. If a chord of a circle subtend a right angle at a fixed point O, show that the locus of the pole of the chord will be a circle coaxal with the given circle and the point-circle O.

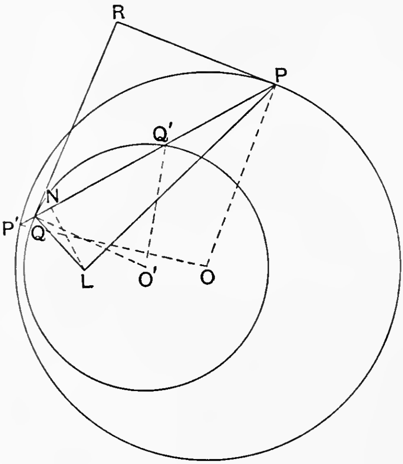
Ex. 12. If P and Q be points on two circles of a coaxal system such that PQ subtends a right angle at a limiting point of the system, show that the tangents at P and Q will intersect in a point, the locus of which is a circle of the same system.

Let L be the limiting point; and let O, O' be the centres of the circles. Let PQ cut the circles again in P' and Q'. Then if the tangents at P and Q intersect in R, we have

$$RP:RQ=\sin RQP:\sin RPQ=\cos O'QQ':\cos OPP'.$$

* This proof is due to Mr A. Larmor.

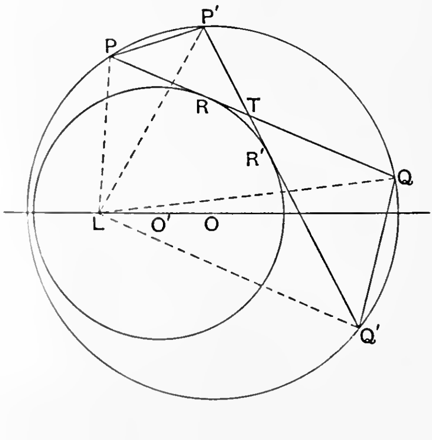
Therefore $$RP : RQ = QQ' . OP : PP' . O'Q.$$
But $$PL^2 : PQ . PQ' = LO : O'O.$$



Let LN be drawn perpendicular to PQ; then
$$PL^2 = PN . PQ.$$
Therefore $$PN : PQ' = LO : O'O;$$
and therefore $$PN : Q'N = LO : LO'.$$
Similarly $$QN : P'N = LO' : LO.$$
Hence $$PP' : QQ' = LO : LO'.$$
Therefore $$RP : RQ = LO' . OP : LO . O'Q.$$
Hence the locus of R is a circle coaxal with the given circles.

Ex. 13. Show that the locus of the point N (see figure Ex. 12) is a circle coaxal with the given circles.

Ex. 14. One circle lies within the other, and the tangents at any two



points of the former cut the latter in the points P, Q; P', Q'; respectively. If L be a limiting point of the system, show that

$$PP' : QQ' = PL + P'L : QL + Q'L.$$

By § 332, (ii), we have, if R, R' be the points of contact of PQ, $P'Q'$,

$$PR : PL = P'R' : P'L = QR : QL = Q'R' : Q'L.$$

Let PQ cut $P'Q'$ in T, then it is evident that

$$PR + P'R' = PT + P'T,$$

$$QR + Q'R' = QT + Q'T.$$

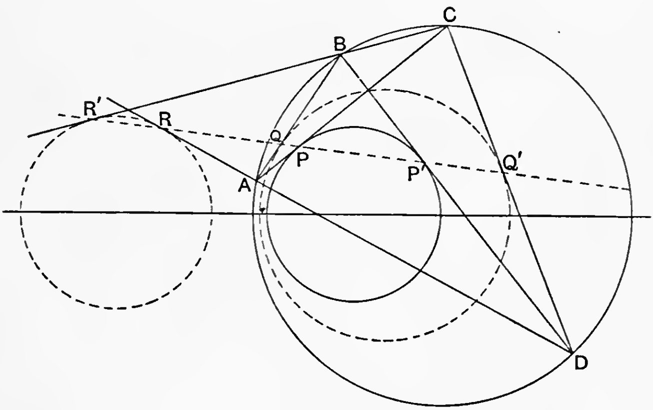
But, since the triangles TPP', $TQ'Q$ are equiangular,

$$PP' : QQ' = PT + P'T : TQ' + TQ.$$

Hence $$PP' : QQ' = PL + P'L : QL + Q'L.$$

Poncelet's theorem.

334. *If a tetrastigm be inscribed in a circle of a given coaxal system so that one pair of opposite connectors touches another circle of the system, then each pair of opposite connectors will touch a circle of the system, and the six points of contact will be collinear.*



Let A, B, C, D be any four points on a circle, and let AC, BD touch another circle at the points P, P'. Let PP' cut AB, CD in Q and Q'; and AD, BC in R and R'.

The triangles AQP, $DQ'P'$ are obviously similar; therefore

$$AQ : AP = DQ' : DP'.$$

Again, $$AP : AQ = \sin AQP : \sin APQ,$$

and $$BP' : BQ = \sin BQP' : \sin BP'Q.$$

But the angles APQ, $QP'B$ are equal.

Therefore $$AP : AQ = BP' : BQ.$$

Hence $$AP : AQ = BP' : BQ = DP' : DQ' = CP : CQ'.$$

Let Z_1, Z_2, Z_3, Z_4 denote the circles whose centres are A, B, C, D, and whose radii are AQ, BQ, CQ', DQ', respectively. Now only one circle can be drawn coaxal with the given circles, which will cut Z_1 orthogonally (§ 325, ex. 1). Let this circle be denoted by X.

By § 329 we have

$$(AX) : (BX) : (CX) : (DX) = AP : BP' : CP : DP'.$$

Therefore

$$(AX) : (BX) : (CX) : (DX) = AQ : BQ : CQ' : DQ'.$$

But $$(AX) = AQ;$$

therefore $$(BX) = BQ, \; (CX) = CQ', \; (DX) = DQ'.$$

Since $(BX) = BQ$, it follows that X must cut Z_2 orthogonally. Therefore X must pass through the limiting points of the circles Z_1, Z_2. But these circles touch at the point Q. Hence the circle X must touch AB at the point Q.

Similarly, the circle X will cut orthogonally the circles Z_3, Z_4. Therefore, since these circles touch at the point Q', the circle X must touch CD at Q'.

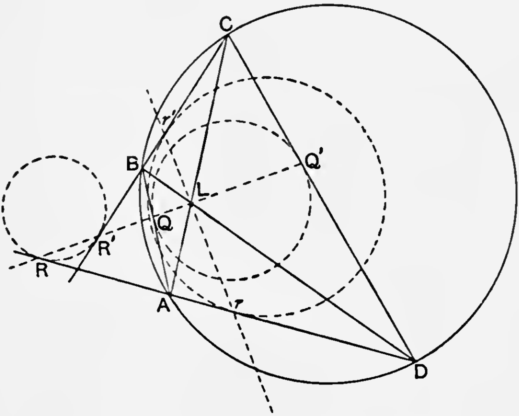
Thus the pair of connectors AB, CD touch the same circle of the coaxal system at the points Q and Q'.

In a similar manner it may be proved that the pair of connectors AD, BC will touch a circle coaxal with the given circles at the points R and R'.

It should be noticed that when the points, in which the line PP' cuts a pair of opposite connectors of the tetrastigm $ABCD$, are internal to the circle $ABCD$, the circle which touches this pair of connectors will have its centre on the same side of the radical axis as the centre of the circle $ABCD$. But when the points are external to the circle $ABCD$, the centre of the corresponding circle will be on the side of the radical axis opposite to the centre of the circle $ABCD$. Thus of the three circles which touch the pairs of connectors, two of the centres will lie on the same side of the radical axis as the centre of the circle $ABCD$.

335. Let us consider the case when the connectors AC, BD of the inscribed tetrastigm intersect in a limiting point.

Let L be the point of intersection of AC and BD; and let the bisectors of the angles between these lines be drawn, cutting the



pair of connectors AB, CD in Q, Q' and q, q' respectively, and the connectors AD, BC in R, R' and r, r' respectively.

Then it is easy to show that

$$AR : AL = DR : DL = BR' : BL = CR' : CL,$$

and $$AQ : AL = DQ' : DL = BQ : BL = CQ' : CL.$$

Hence it follows, as in § 334, that AD and BC will touch a circle of the system in the point R and R'; and that AB, CD will touch another circle of the system in Q and Q'.

In the same way it may be shown that AB, CD will touch a circle of the system in the points q, q'; and that AD, BC will touch another circle of the system in r and r'.

Hence we have the theorem: *If any four points be taken on a circle of a given coaxal system, so that one pair of opposite connectors of the tetrastigm formed by them intersect in a limiting point of the system, the other pairs of opposite connectors will each touch two circles of the system.*

It should be noticed that although each pair of connectors touches two circles, they do not constitute a *pair* of common tangents of the two circles.

336. Ex. 1. If a tetrastigm be inscribed in a circle, and if one pair of opposite connectors touch two circles coaxal with the former, show that one of the centres of the tetrastigm coincides with a limiting point of the system.

Let $ABCD$ be the given tetrastigm, and let AD, BC touch one circle in R and R', and another circle in r and r'. Then it is easy to see that RR' will cut rr' at right angles.

Let L be the point of intersection, then since the circles whose diameters are Rr and $R'r'$ intersect in the limiting points of the given circles, it follows that L must be one of these limiting points. Again, the range $\{Rr, AD\}$ is harmonic, therefore LR and Lr must bisect the angles ALD. Let AL, DL meet the circle $ABCD$ in C' and B' respectively. Then from § 335, we see that $B'C'$ must touch the same circles as AD at the points in which it is cut by rr' and RR'. Therefore $B'C'$ must coincide with BC, that is, B' coincides with either B or C. Hence L must be one of the centres of the tetrastigm $ABCD$.

Ex. 2. If a tetrastigm be inscribed in a circle of a coaxal system, so that two pairs of its opposite connectors touch another circle of the system, show that the remaining pair of connectors will intersect in a limiting point of the system.

Let $ABCD$ be the given tetrastigm, and let AB, BC, CD, DA touch a circle of the system at the points Q, R, Q', R', respectively. It follows from § 334, that AB, CD will touch another circle of the system at the points in which these lines cut the line RR'. Hence this theorem follows from that in Ex. 1.

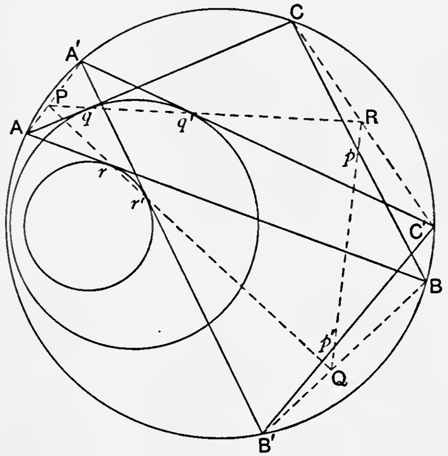
Ex. 3. If $ABCD$ be a tetrastigm inscribed in a circle, and if AB, CD touch respectively at Q and Q', a coaxal circle, show that if QQ' pass through a limiting point of the system, this point will be a centre of the tetrastigm.

Ex. 4. If $ABCD$ be a tetrastigm inscribed in a circle of a coaxal system, and if AB, CD touch one circle of the system at the points Q, Q', and AD, BC another circle of the system at R, R', show that the connectors AC, BD will intersect in a limiting point of the system, provided that Q, Q', R and R' are not collinear.

Ex. 4. The sides of the triangle ABC touch the inscribed circle in the points P, Q, R. If the lines QR, RP, PQ cut the lines BC, CA, AB in the points X, Y, Z, and the tangents to the circumcircle of the triangle ABC at its vertices, in the points X', Y', Z'; show that the three circles which touch respectively BC, AX' at X and X'; CA, BY' at Y and Y'; and AB, CZ' at Z and Z'; will be coaxal with the circumcircle and the inscribed circle of the triangle ABC.

337. *If the vertices of a triangle move continuously, and in the same direction, on the circumference of a circle of a given coaxal system, so that two of its sides touch two fixed circles of the system, the third side will touch another fixed circle of the system.*

Let A, B, C be any positions of the vertices of the triangle on the circle X; and let X_1, X_2 denote the circles which are the envelopes respectively of AB, AC, as the points A, B, C describe continuously the circle X. Let q, r be the points of contact of AC, AB with the circles X_2, X_1; and let q', r' be the new positions



of q and r when the points A, B, C have moved to the positions A', B', C'.

Since the points A, B, C move in the same direction, it is obvious that the centres of the circles X_1, X_2 must lie on the same side of the radical axis as the centre of the circle X. Also it is evident that qq' and rr' will intersect AA' between A and A'. Similarly if qq', rr' intersect CC', BB' in R and Q, R will lie between C and C', and Q between B and B'.

Now since the four points A, A', C, C' lie on a circle X, and the lines AC, $A'C'$ touch a circle X_2 in the points q, q', it follows from § 334, that AA' and CC' will touch a circle, coaxal with X and X_2, at the points in which qq' cuts them. Similarly it may be proved that AA' and BB' will touch a circle, coaxal with X and X_1, at the points in which rr' cuts them. But since AA' can only touch one circle of the given coaxal system at a point between A and A', it follows that qq' and rr' must intersect AA' in the same point, and that AA', BB', CC' must touch the same circle of the system.

Let us denote the circle which touches AA', BB', CC' by X'. Then since BB', CC' is a tetrastigm inscribed in a circle X, and a pair of connectors BB', CC' touch another circle X' at the points Q and R, it follows from § 334 that BC and $B'C'$ must touch a circle, X_3 say, coaxal with X and X', at the points in which QR cuts these lines.

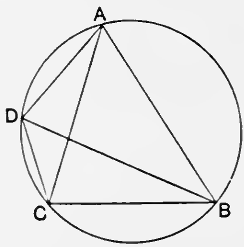
Let QR meet the lines BC, $B'C'$ in p and p'. Then it is obvious that p must lie between B and C, and p' between B' and C'. Hence the circle X_3 which is touched by BC and $B'C'$ will have its centre on the same side of the radical axis as the circle X.

Thus as A, B, C describe the circle X, the side BC will envelope a fixed circle X_3 coaxal with X, X_1, and X_2.

The proof given above requires but slight modification when the restriction that A, B, C should move in the same direction is removed. Thus if A move in the opposite direction to B and C, it is easy to see that the circles X_1, X_2 must have their centres on the side of the radical axis opposite to the circle X, and in this case it may be proved that X_3 which is the envelope of BC has its centre on the same side as the circle X. Again, if A and B move in one direction and C in the other, then X_1 must have its centre on the same side of the radical axis as X, and X_2 must have its centre on the opposite side of the radical axis, and then it may be proved that X_3 the envelope of BC will have its centre on the same side of the radical axis as X_2.

Hence we may state the theorem in the form: *If a triangle be inscribed in a circle so that two sides touch two given circles coaxal with the former, the third side will touch a fixed circle of the same coaxal system.*

338. Let us suppose that AB, BC touch respectively the circles X_1 and X_2, and let X_3 denote the circle which is always touched by CA, as the vertices of the triangle move in the same



direction round the circle ABC. Let us take the point D between A and C, so that AD touches the circle X_2. Then by § 334, since

AD and BC touch the same circle X_2, therefore BD will touch the same circle as AC, that is the circle X_3.

Thus BAD is a triangle, the vertices of which occur in the opposite order to those of the triangle ABC, and the sides of which touch the same circles as the corresponding sides of the triangle ABC.

But if we consider the triangle ABD with its vertices occurring in the same order as the vertices of ABC, we see that the sides AB, BD, DA touch the circles X_1, X_3, X_2.

Hence, we infer that if we take the vertices ABC always in the same order, it is immaterial in which order the sides touch the circles X_1, X_2, X_3.

339. If ABC be a triangle inscribed in a circle X, such that when two sides touch two circles X_1, X_2 coaxal with X, the envelope of the third side is the circle X_3, the circles X_1, X_2, X_3 are said to form a *poristic system with respect to the circle* X.

Suppose that we have given any three circles X_1, X_2, X_3 coaxal with a given circle X, it is evident that the problem "to inscribe a triangle in the circle X so that its sides shall touch respectively the three circles X_1, X_2, X_3," is indeterminate when the circles X_1, X_2, X_3 form a poristic system with respect to the circle X. But when this is not the case, let Y_1 be the circle of the coaxal system (X, X_1, X_2, X_3) which forms with X_2 and X_3 a poristic system with respect to X, then since the circles Y_1, X_1 will have four common tangents, we shall find four solutions to the problem. Similarly if Y_2, Y_3 be the circles which form with X_3, X_1; and X_1, X_2; respectively, poristic systems, we may obtain eight other solutions by drawing the common tangents of the pairs of circles Y_2, X_2; Y_3, X_3. Thus, when X_1, X_2, X_3 do not constitute a poristic system of circles, twelve triangles may in general be inscribed in X so that their sides touch respectively the circles X_1, X_2, X_3. But of these twelve solutions some or all may be imaginary.

340. Let A, B, C, D be any four points on a circle of a given coaxal system, and let AB, BC, CD touch respectively three fixed circles of the system. Then if A', B', C', D' be four other points on the circle ABC (taken in the same order as the points A, B, C, D), so that $A'B'$, $B'C'$, and $C'D'$ touch respectively the same circles as

AB, BC and CD, it may be proved in the same manner as in § 337, that AA', BB', CC' and DD' will touch a circle of the system, and that $A'D'$, $A'C'$ and $B'D'$ will touch respectively the same circles of the system as AD, AC and BD.

The second part of this theorem may be deduced from the theorem in § 337. For since AB and BC always touch fixed circles of the system, therefore AC must always touch a fixed circle of the system. And since AC and CD always touch fixed circles of the system, therefore AD must touch a fixed circle of the system. Similarly, it may be proved that BD must always touch a fixed circle of the system.

Now let us suppose that AB, BC and CD touch respectively the circles X_1, X_2 and X_3. Then AD must touch a circle, X_4 say. Let CE be drawn to touch the circle X_4, then it may be proved that EA must touch the circle X_3.

For CA will always touch a fixed circle, X_5 say. Therefore by § 338, since CA, CD and DA touch the circles X_5, X_3, X_4 respectively, if CE touch X_4, EA must touch X_3.

Hence, we infer that: *If A, B, C, D be four points taken in the same order, on a fixed circle belonging to a given coaxal system, so that AB, BC, CD touch, respectively, the fixed circles X_1, X_2, X_3 of the system, then DA must touch a fixed circle, X_4, of the system; and, further, if AB, BC, CD touch respectively any three of the four circles X_1, X_2, X_3, X_4, then DA must touch the remaining circle.*

341. In exactly the same way, we may prove Poncelet's celebrated theorem: *If $A_1, A_2, \ldots A_n$ be any number of points taken in order on a circle of a given coaxal system, so that A_1A_2, $A_2A_3, \ldots A_{n-1}A_n$ touch respectively $(n-1)$ fixed circles $X_1, X_2, \ldots X_{n-1}$ of the system, then A_nA_1 must touch a fixed circle, X_n, of the system; and, further, if A_1A_2, $A_2A_3, \ldots A_{n-1}A_n$ touch respectively any $n-1$ of the circles X_1, $X_2, \ldots X_n$, then A_nA_1 must touch the remaining circle.*

The theorem may also be stated in the form: *If a polystigm can be inscribed in a circle of a given coaxal system, so that each one of a complete set of connectors* (§ 137) *touches respectively a fixed circle of the system, then an infinite number of such polystigms can be inscribed.*

342. Ex. 1. If $A_1, A_2, \ldots A_n$ be n points on a circle X of a coaxal system, so that A_1A_2, $A_2A_3, \ldots A_{n-1}A_n$, A_nA_1 touch respectively the circles of the system $X_1, X_2, \ldots X_n$, which form with respect to the circle X a poristic system; and if $A_1', A_2', \ldots A_n'$ be n other points taken in the same order on the circle X, so that $A_1'A_2'$, $A_2'A_3'$, &c., touch respectively the same circles as A_1A_2, A_2A_3, &c.; show that A_1A_1', $A_2A_2', \ldots A_nA_n'$ will touch a circle of the coaxal system.

Ex. 2. If A_1, A_2, A_3, A_4, A_5 be five points on a circle, such that the connectors A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_1 touch another circle, show that the connectors A_1A_3, A_3A_5, A_5A_2, A_2A_4, A_4A_1 will touch another circle coaxal with the given circles.

Ex. 3. If A_1, A_2, A_3, A_4, A_5, A_6 be six points on a circle, and if the connectors A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_1 touch another circle, show that the connectors A_1A_4, A_2A_5, A_3A_6 will intersect in a limiting point of the given circles, and that the connectors A_1A_3, A_2A_4, A_3A_5, A_4A_6, A_5A_1, A_6A_2 will touch a circle belonging to the same coaxal system.

Ex. 4. Show that if $2n$ points $A_1, A_2, \ldots A_{2n}$ be taken on a circle such that a complete set of connectors touch another circle, there exists a set of n connectors which intersect in a limiting point of the circles, and that there are $(n-2)$ other complete sets of connectors which touch respectively $(n-2)$ circles coaxal with the given circles.

The $2n(n-1)$ connectors which do not intersect in the limiting point may be arranged in $n(n-1)$ pairs, each pair being common tangents of two of the circles of the system.

CHAPTER XIV.

THE THEORY OF INVERSION.

Inverse points.

343. If on the line joining a point P to the centre O of a given circle, a point Q be taken so that the rectangle $OP \, . \, OQ$ is equal to the square on the radius of the circle; the point Q is said to be the *inverse point* with respect to the circle of the point P.

If Q be the inverse point of P, it is evident that P is the inverse of Q. Hence P and Q are called a pair of inverse points with respect to the circle.

The inverse of any point with respect to a circle might also be defined as the conjugate point with respect to the circle which lies on the diameter which passes through the given point. Thus, if P, Q be a pair of inverse points with respect to a circle, P, Q are a pair of conjugate points, and therefore every circle which passes through P and Q will cut the given circle orthogonally.

344. If we have any geometrical figure consisting of an assemblage of points, the inverse points with respect to a fixed circle will form another figure, which is called the *inverse figure* with respect to the circle of the given figure.

It will be shown that when certain relations exist between the parts of any figure, other relations may be inferred concerning the corresponding parts of the inverse figure. And as the inverse figure may be of a more complicated character we are thus able to obtain properties of such figures from known properties of simpler figures.

The fixed circle is called the *circle of inversion,* and the process by which properties of inverse figures are derived is known as 'inversion.' It will be seen that as a rule, the nature of the inverse figure is independent of the magnitude of the circle of inversion, but depends on the position of the centre of this circle. Consequently it is usual to designate the process briefly by the phrase 'inverting with respect to a point;' but it must be remembered that when this phrase is used, the inversion is really taken with respect to a circle whose centre is the point.

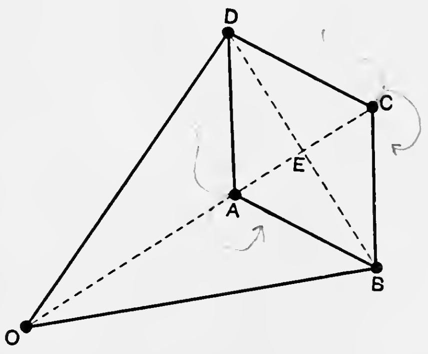
It is often convenient to invert a figure with respect to an imaginary circle, having a real centre. In this case, if O be the centre of inversion, and P, Q a pair of inverse points, P and Q will lie on opposite sides of O, and the rectangle $PO \,.\, OQ$ will be constant.

345. Ex. 1. Show that the limiting points of a system of coaxal circles are inverse points with respect to every circle of the system.

Ex. 2. If a pair of points be inverse points with respect to two circles, they must be the limiting points of the circles.

Ex. 3. Show that the extremities of any chord of a circle, the centre, and the inverse of any point on the chord, are concyclic.

346. We may mention here a method by which the inverse of any given figure may be drawn with the aid of a simple mechanical instrument. Let $ABCD$ be a rhombus formed by four rigid bars of equal lengths hinged together; and let the joints B, D be connected with a fixed point O, by means of two equal rigid bars hinged at O. Then the points A and C will be inverse points with respect to a circle whose centre is O.



It is evident that the points O, A, C will be collinear. Let E be the point of intersection of BD and AC. Then we have

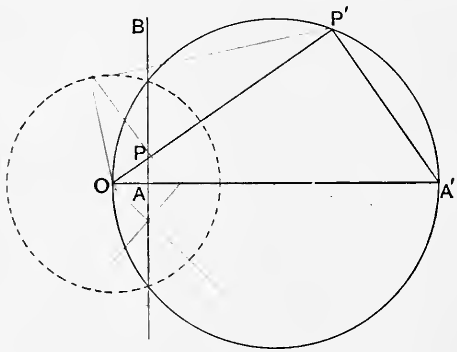
$$OA \cdot OC = OE^2 - AE^2 = OD^2 - DA^2.$$

Hence A and C are inverse points with respect to a circle whose centre is O. Consequently if the point A be made to describe any curve the point C will describe the inverse curve.

This arrangement of bars is called *Peaucellier's cell.*

The inverse of a straight line.

347. *The inverse of a straight line with respect to any circle is a circle which passes through the centre of the circle of inversion.*



Let P be any point on the straight line AB, and let P' be the inverse point with respect to a circle whose centre is O.

Let OA be the perpendicular from O on the straight line, and let A' be the inverse point of A.

Then we have

$$OP \cdot OP' = OA \cdot OA'.$$

Therefore the points A, P, A', P' are concyclic; and therefore the angle $OP'A'$ is equal to the angle OAP, which is a right angle.

Hence P' is a point on the circle whose diameter is OA'.

Thus, the inverse of a straight line is a circle which passes through the centre of inversion.

Conversely, it is evident that the inverse of a circle which passes through the centre of inversion is a straight line. In other words: *The inverse of a circle with respect to any point on it is a straight line.*

348. Ex. 1. Show that the inverse of the line at infinity is a point-circle coincident with the centre of inversion.

Ex. 2. If C be the centre of a circle which passes through the centre of inversion, and if C' be the inverse of the point C, show that the straight line which is the inverse of the given circle bisects CC'.

Ex. 3. Show that the inverse circles of a system of parallel straight lines touch each other at the centre of inversion.

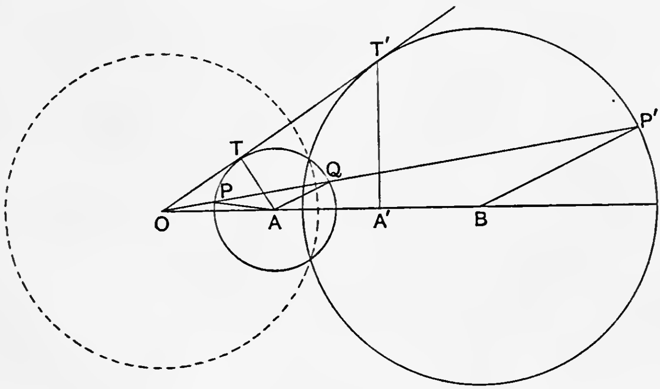
Ex. 4. If a system of lines be concurrent, show that the circles which are inverse to them are coaxal.

Ex. 5. The inverse circles of two straight lines intersect at the same angle as the lines.

The radii of the circles drawn to the centre of inversion are perpendicular respectively to the lines.

Inverse circles.

349. *The inverse of a circle with respect to any circle is a circle.*



Let A be the centre of the given circle, and O the centre of the circle of inversion. Let P be any point on the given circle, and let P' be the inverse point.

Let OP cut the given circle in Q, and let $P'B$ be drawn parallel to QA, and meeting OA in B.

Then since the rectangles $OP \,.\, OP'$, $OP \,.\, OQ$ are constant, the ratio $OP' : OQ$ is constant.

But since AQ, BP' are parallel,

$$BP' : AQ = OB : OA = OP' : OQ.$$

Therefore B is a fixed point, and BP' a constant length. Hence the locus of the point P' is a circle whose centre is B.

350. If X denote any circle, and X' its inverse with respect to any circle of inversion, it is easy to see that X will be the inverse of X', so that X and X' may be called a *pair of inverse circles* with respect to the circle of inversion.

If the circle X cut the circle of inversion orthogonally, the point P' will coincide with the point Q, and the circle X' will coincide with X. Thus, *the inverse of a given circle with respect to any circle which cuts it orthogonally coincides with the given circle.*

351. Ex. 1. Show that the circle of inversion may be so chosen that the inverse circles of three given circles shall be coincident with themselves.

Ex. 2. If three circles intersect two and two in the points A, A'; B, B'; C, C'; and if through any point O the circles OAA', OBB', OCC' be described, prove that these three circles will be coaxal.

Ex. 3. Show that the nine-point circle of a triangle is the inverse of the circumcircle with respect to the polar circle of the triangle.

Ex. 4. If two circles X and Y be so related that a triangle can be inscribed in X, so that its sides touch Y, show that the nine-point circle of the triangle formed by the points of contact with Y is the inverse of X with respect to Y.

Ex. 5. Show that the nine-point circle of a triangle ABC is the inverse of the fourth common tangent of the two escribed circles, which are opposite to B and C, with respect to the circle whose centre is the middle point of BC, and which cuts these escribed circles orthogonally.

Ex. 6. Show that McCay's circles (§ 233, Ex. 3) are the inverses of the sides of the first Brocard triangle of a given triangle, with respect to the circle whose centre is the median point of the triangle, and which cuts the Brocard circle orthogonally.

352. It is evident that the centre of inversion is a homothetic centre of the given circle and its inverse. When the circle of inversion is real, its centre is the homothetic centre; and when the circle of inversion is imaginary, its centre is the antihomothetic centre of the pair of inverse circles.

Let X, X' denote a pair of inverse circles with respect to any circle of inversion, S. Then these circles are coaxal.

For, referring to the figure in § 349, we have,

$$\begin{aligned}(P'S):(P'X) &= P'O^2 - OP\,.\,OP' : P'Q\,.\,PP'\\ &= OP' : QP'\\ &= OB : AB.\end{aligned}$$

Thus the powers with respect to the circles S, X of any point P' on the circle X', are in a constant ratio. Therefore, by § 330, the circle X' is coaxal with the circles S and X.

Hence we may infer that, given a pair of circles X and X', two circles can be found, which will be such that X and X' are a pair of inverse circles with respect to either. For these two circles of inversion will be the circles whose centres are the homothetic centres of X and X', and which are coaxal with X and X'.

353. Ex. 1. If X and X' be inverse circles with respect to each of the circles S and S', show that S and S' cut each other orthogonally.

Ex. 2. Show that the circumcircle of a triangle, the nine-point circle, and the polar circle are coaxal.

354. *To find the radius of the inverse of a circle.*

Let R denote the radius of the circle of inversion, and let r, r' denote the radii of the given circle and its inverse. Then from the figure in § 349, we have

$$r : r' = OQ : OP' = OP \,.\, OQ : OP \,.\, OP'.$$

Therefore $$r : r' = (OX) : R^2,$$

where (OX) denotes the power of the point O with respect to the given circle.

Thus $$r' = \frac{rR^2}{(OX)}.$$

355. Ex. 1. Show that if the centre of inversion lie on a certain circle, the inverse circles of two given circles will be equal.

Let X_1, X_2 denote the given circles, and let r_1, r_2 denote their radii. Then we must have $(OX_1) : (OX_2) = r_1 : r_2$. Hence O must lie on a fixed circle coaxal with the circles X_1, X_2.

Ex. 2. Show that there are two points with respect to which three given circles may be inverted into three equal circles.

356. Let A be the centre of a given circle, and let A' be the inverse point of A with respect to a given circle of inversion whose centre is O.

Let OTT' be the common tangent to the given circle and its inverse. Then we shall have (see fig. in § 349),

$$OA \,.\, OA' = OP \,.\, OP' = OT \,.\, OT'.$$

Therefore the points A, A', T, T' are concyclic, and therefore the angle $OA'T'$ will be equal to the angle OTA, which is a right angle.

Hence $A'T'$ is the polar of O with respect to the circle $P'T'Q'$; that is, A' is the inverse point of O with respect to the circle $P'T'Q'$.

Thus, *the inverse of the centre of a given circle is the inverse with respect to the inverse circle of the centre of inversion.*

Hence it follows that the inverse circles of a system of concentric circles will be a coaxal system of circles having the centre of inversion for a limiting point. For the polars with respect to the inverse circles of the centre of inversion will evidently be coincident and the result follows from § 345, Ex. 2.

Corresponding properties of inverse figures.

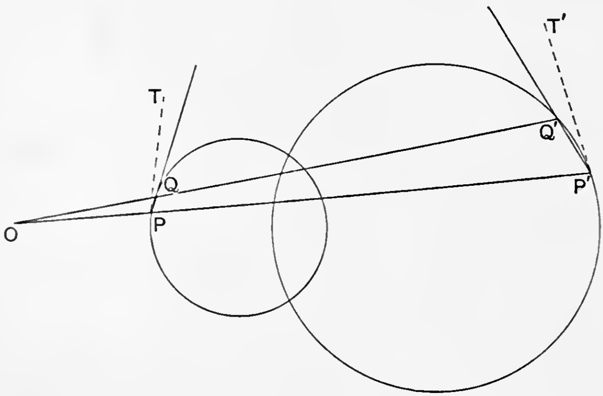
357. *If two circles touch each other, the inverse circles will also touch each other.*

If two circles touch they intersect in two coincident points. It follows that the inverse circles will intersect in two coincident points, and therefore will touch each other.

It should be noticed however that the nature of the contact will not necessarily be the same.

A similar theorem is evidently true for any two curves.

358. *If two circles intersect, their angle of intersection is equal or supplementary to the angle at which the inverse circles intersect.*



Let P, Q be two near points on any circle, and let P', Q' be the inverse points on the inverse circle.

Then since $$OP \cdot OP' = OQ \cdot OQ',$$
the points P, P', Q, Q' are concyclic. Therefore the angle OPQ is equal to the angle $OQ'P'$. It may happen however that the point O falls within the circle which can be drawn through the points P, P', Q, Q'; in which case the angles OPQ, $OQ'P'$, will be supplementary.

Now let the point Q approach indefinitely near to the point P, so that the line PQ becomes the tangent at P. Then at the same time $Q'P'$ will become the tangent at P'.

Hence if PT, $P'T'$ be the tangents at P and P', the angles $P'PT$, $T'P'P$ will be equal or supplementary.

It follows that if any two circles intersect in the point P, the angle between the tangents to the circles at this point will be equal or supplementary to the angle between the tangents to the inverse circles at the point P'.

If the two circles cut orthogonally the inverse circles will also cut orthogonally.

359. Ex. 1. If X and Y denote any two circles, and if X', Y' denote the inverse circles with respect to any point O; show that X' and Y' will intersect at the same angle as X and Y, when the point O is either external to both the circles X and Y, or is internal to both; but when the point is internal to one circle and external to the other, the angle of intersection of X and Y will be supplementary to the angle of intersection of X' and Y'.

Ex. 2. Show that the nine-point circle of a triangle touches the inscribed and escribed circles.

This may be deduced from the theorem in § 351, Ex. 5.

Ex. 3. Show that four circles can be drawn which shall touch two given circles and their inverse circles with respect to any circle of inversion.

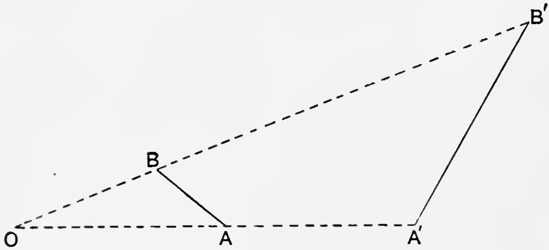
Discuss the case when one of the given circles cuts the circle of inversion orthogonally.

360. *If P and Q be a pair of inverse points with respect to any circle S, and if P', Q' be the inverse points of P and Q, and S' the inverse of S, with respect to any circle, then P' and Q' will be inverse points with respect to the circle S'.*

Since P and Q are inverse points with respect to S, therefore any circle which passes through P and Q will cut S orthogonally. Consequently P' and Q' will be two points such that any circle which passes through them will cut S' orthogonally.

It follows at once from this theorem, that if two figures F_1, F_2 be inverse figures with respect to a circle S, and if F_1', F_2', S' be the inverse figures of F_1, F_2, S with respect to any circle of inversion, then F_1', F_2' will be inverse figures with respect to the circle S'.

361. *Given the distance between any two points to find the distance between the inverse points with respect to any circle of inversion.*



Let A, B be any two points, and let A', B' be the inverse points with respect to any circle of inversion whose centre is O.

Then since $OA \,.\, OA' = OB \,.\, OB'$, the points A, A', B, B' are concyclic. Therefore the triangles OAB, $OB'A'$ are similar; and therefore

$$AB : A'B' = OA : OB' = OB : OA'.$$

Therefore

$$A'B' : AB = OA \,.\, OA' : OA \,.\, OB.$$

Also $$A'B'^2 : AB^2 = OA' \,.\, OB' : OA \,.\, OB.$$

Again if p, p' denote the perpendiculars from O on the lines AB, $A'B'$, we shall have

$$A'B' : AB = p' : p.$$

In the case when the points A, B are collinear with the point O, we shall have

$$OA : OB' = OB : OA' = AB : B'A',$$

whence $$B'A' : AB = OA \,.\, OA' : OA \,.\, OB;$$

and $$A'B'^2 : AB^2 = OA' \,.\, OB' : OA \,.\, OB.$$

362. Ex. 1. If A, B, C, D be any four points on a straight line, show that

$$AB \,.\, CD + AC \,.\, DB + AD \,.\, BC = 0.$$

If B', C', D' be the inverse points, with respect to the point A of the points B, C, D, we shall have

$$B'C' + C'D' + D'B' = 0.$$

Hence the above relation may be deduced by § 361.

Ex. 2. If A, B, C, D be any four points taken in order on a circle, show that:

i. $AC.BD=AB.CD+AD.BC.$

ii. $BD.CD.BC+AD.BD.AB=BC.AC.AB+CD.AD.AC.$

Ex. 3. If A, B, C, D be four points on a circle such that the pencil $O\{AC, BD\}$ is harmonic, where O is any variable point on the circle, show that

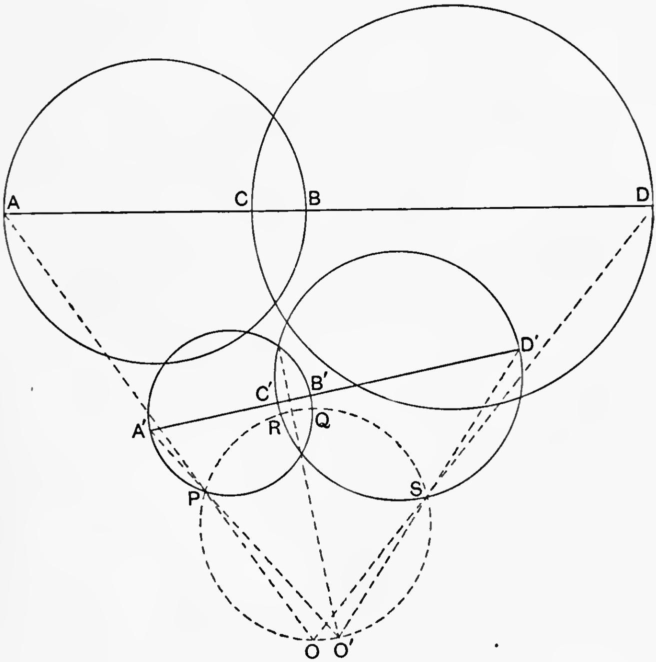
$$AB.CD=AD.BC.$$

Ex. 4. If three straight lines be drawn through a point O, making equal angles with each other, and if any other straight line cut them in the points L, M, N, show that

$$OM.ON+ON.OL+OL.OM=0.$$

Ex. 5. Show that if four points A, B, C, D on a circle be such that AB and CD are conjugate lines with respect to the circle, the inverse points A', B', C', D' with respect to any circle will be such that $A'B'$, $C'D'$ are conjugate lines with respect to the inverse circle.

Ex. 6. If the line joining the centres of any two circles cut them in the



points A, B and C, D, respectively; and if the line joining the centres of the inverse circles cut them in the points A', B'; and $C'D'$; show that

$$AC.BD : AB.CD=A'C'.B'D' : A'B'.C'D',$$

where the points A', B', C', D' are supposed to occur in the same order as the points A, B, C, D respectively.

Let X, Y denote the given circles, and let X', Y' denote the inverse circles with respect to a circle of inversion whose centre is O. Then if P, Q, R, S be the inverse points of the points A, B, C, D respectively, these points will lie on a circle which will pass through O and cut the circles X', Y' orthogonally. Also we shall have from § 361,

$$AC.BD : AB.CD = PR.QS : PQ.RS.$$

Now the radical axis of the circles X', Y' will cut the circle $PQRS$ in two points. Let O' be one of these points. Then if we take for circle of inversion the circle whose centre is O' and which cuts X' and Y' orthogonally, the common diameter $A'B'C'D'$ of the circles X' and Y' will clearly be the inverse of the circle $PQRS$. It follows that the lines $A'P$, $B'Q$, $C'R$, $D'S$ will intersect in one of the points in which the radical axis of X', Y' cuts the circle $PQRS$. Hence, if we take this point as O', we shall have by § 361,

$$A'C'.B'D' : A'B'.C'D' = PR.QS : PQ.RS.$$

Therefore we shall have

$$AC.BD : AB.CD = A'C'.B'D' : A'B'.C'D'.$$

In the same way it may be proved that

$$AD.BC : AB.CD = A'D'.B'C' : A'B'.C'D'.$$

Now it is easy to prove that the rectangles $AC.BD$, and $AD.BC$ are equal to the squares on the common tangents of the circles X and Y. Hence, if T, t denote the common tangents, and r_1, r_2 the radii of the circles X, Y, and if T', t' denote the common tangents and r_1', r_2' the radii of the inverse circles X', Y', we shall have

$$T^2 : T'^2 = t^2 : t'^2 = r_1 r_2 : r_1' r_2'.$$

Ex. 7. If A', B', C' be the inverse points of three given points A, B, C, with respect to any centre of inversion O, show that the triangle $A'B'C'$ will be similar to the triangle PQR, where P, Q, R are the points in which the lines AO, BO, CO cut the circumcircle of the triangle ABC.

Ex. 8. If the inverse points of three given points A, B, C form a triangle which is similar to a given triangle, show that the centre of inversion must coincide with one or other of two fixed points which are inverse points with respect to the circumcircle of the triangle ABC.

Power relations connecting inverse circles.

363. Let X and X' denote a pair of inverse circles with respect to any circle. Let S denote the circle of inversion, and let S' denote the circle which cuts S orthogonally and is coaxal with X and X'. Then X and X' are also a pair of inverse circles with respect to the circle S'.

Let A, A' denote the centres of the circles X, X'; and let

O, O' denote the centres of the circles S and S'. Then since O, O' are the homothetic centres of X and X' (§ 352), the range $\{OO', AA'\}$ is harmonic, and therefore

$$\frac{OA}{OA'} + \frac{O'A}{O'A'} = 0.$$

But since the circles X, X', S, S' are coaxal, and the circles S, S' cut orthogonally, we have by § 329,

$$(S'X) : (S'X') = OA : OA';$$

and $$(SX) : (SX') = O'A : O'A'.$$

Hence we have

$$\frac{(S'X)}{(SX)} + \frac{(S'X')}{(SX')} = 0.$$

364. Let T denote the circle which is concentric with S, and which cuts S orthogonally. The circle T will be real when S is imaginary. Then since the circles T and S' cut one circle, S, of the coaxal system $\{S, X, X'\}$, orthogonally, therefore by § 329,

$$(TX) : (TX') = (S'X) : (S'X').$$

Hence the formula of the last article may be written,

$$\frac{(TX)}{(SX)} + \frac{(TX')}{(SX')} = 0.$$

Consequently the ratios $(TX) : (SX)$, and $(TX') : (SX')$ have opposite signs.

365. We also have from § 329,

$$(TX) : (TS') = (S'X) : (S'S').$$

Therefore if R, R' denote the radii of the circles S, S', we shall have

$$(TS') = (SS') - (SS) = 2R^2,$$

and therefore $$(TX) : (S'X) = R^2 : -R'^2.$$

Hence if either of the circles S and S' be imaginary, the powers (TX), $(S'X)$ will have the same sign.

It is easy to see that the ratio $(SX) : (S'X)$ is negative or positive according as the centre of X does, or does not, lie between O and O'.

Hence, if we call that circle of the pair of inverse circles X, X', the *positive circle* for which the ratio $(TX) : (SX)$ is positive, and the other the *negative circle* of the pair, we can easily discriminate between the circles.

366. Again it is easy to see that

$$(TX) = (SX) - (SS).$$

Hence from the relation of § 364, we may deduce the relation

$$\frac{1}{(SX)} + \frac{1}{(SX')} = \frac{2}{(SS)}.$$

From this we may deduce the more general formula

$$\frac{(ZX)}{(SX)} + \frac{(ZX')}{(SX')} = 2\frac{(ZS)}{(SS)},$$

where Z denotes any circle.

To prove this, let Y denote the circle which is coaxal with X and X', and which cuts Z orthogonally. Then if B denote the centre of Y, we have by § 329,

$$(ZX) : (ZX') : (ZS) = BA : BA' : BO.$$

But we have, $$\frac{BO}{(SX)} + \frac{BO}{(SX')} = 2\frac{BO}{(SS)};$$

and (§ 363) $$\frac{OA}{(SX)} + \frac{OA'}{(SX')} = 0.$$

Therefore $$\frac{BA}{(SX)} + \frac{BA'}{(SX')} = 2\frac{BO}{(SS)}.$$

Hence $$\frac{(ZX)}{(SX)} + \frac{(ZX')}{(SX')} = 2\frac{(ZS)}{(SS)}.$$

367. In § 361 it was proved that if A and B be any two points, and A', B' the inverse points with respect to any circle of inversion, S, whose centre is O, then

$$A'B'^2 : AB^2 = OA' \,.\, OB' : OA \,.\, OB.$$

If (AB) denote as usual the power of the points A and B, this formula may be written in the form

$$(A'B') : (AB) = (A'S) \,.\, (B'S) : (AS) \,.\, (BS),$$

for, as proved in § 363, we have

$$(A'S) : (AS) = A'O : OA,$$
$$(B'S) : (BS) = B'O : OB.$$

368. We shall now show that a similar formula connects the powers of inverse circles: *If X', Y' be the inverse circles of X and Y with respect to any circle of inversion, S, then*

$$(X'Y') : (XY) = (X'S) \,.\, (Y'S) : (XS) \,.\, (YS).$$

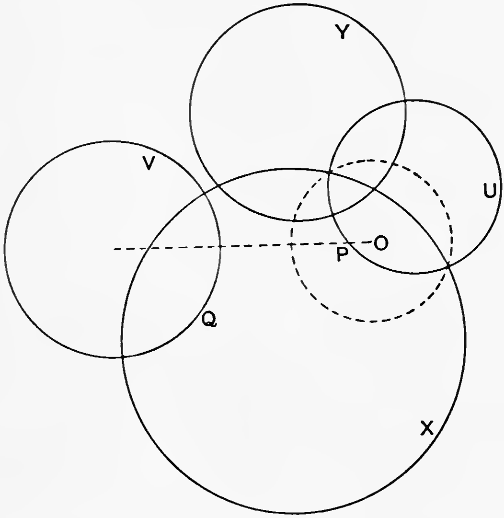
Let a circle U be described coaxal with the circles S and X, and cutting Y orthogonally. Let P be any point on U, and let a circle V be described coaxal with S and the point-circle P, and cutting X orthogonally. Then if Q be any point on the circle V, we have by § 329,

$$(XY) : (YS) = (PX) : (PS);$$

and $$(XP) : (XS) = (QP) : (QS).$$

Therefore

$$(XY) : (XS) . (YS) = (PQ) : (PS) . (QS).$$



Let U', V' be the inverse circles of U and V. Then it is evident that U' will be coaxal with S and X', and will cut Y' orthogonally. Also if P' be the inverse of P with respect to S, it is evident that P' will be a point on U'. Also V' will cut X' orthogonally and will be coaxal with P' and S.

Hence we shall have

$$(X'Y') : (X'S) . (Y'S) = (P'Q') : (P'S) . (Q'S).$$

But by § 367,

$$(P'Q') : (PQ) = (P'S) . (Q'S) : (PS) . (QS).$$

Therefore

$$(X'Y') : (XY) = (X'S) . (Y'S) : (XS) . (YS).$$

369. The proof given above requires modification when either of the given circles X, Y cuts the circle of inversion orthogonally. Let us suppose that Y cuts S orthogonally, then Y' will coincide with Y.

Now since Y is a circle which cuts orthogonally the circle S, which is coaxal with the circles X and X', therefore by § 329,

$$(XY) : (YX') = OA : OA',$$

where A, A' are the centres of the circles X, X'.

But in § 363, it was shown that

$$OA : OA' = (S'X) : (S'X') = -(SX) : (SX'),$$

where S' is the circle, coaxal with X and X', which cuts S orthogonally.

Hence $$(XY) : (X'Y') = (XS) : -(X'S).$$

This relation is easily seen to be in agreement with the relation of the last article; for if we suppose Y and Y' to be nearly coincident, (YS) and $(Y'S)$ are small quantities which are ultimately equal but have opposite signs.

370. Ex. 1. Prove that if X', Y' be the inverse circles of X and Y, with respect to any circle whose centre is O,

$$(XY) : (X'Y') = (OX) : (OY') = (OY) : (OX').$$

Ex. 2. Show that if a, b, a', b' denote the radii of the circles X, Y, X', Y';

i. $(XY) : (X'Y') = ab : a'b'$,

when O is external to both X and Y, or is internal to both circles;

ii. $(XY) : (X'Y') = ab : -a'b'$,

when O is external to one of the circles X, Y, and internal to the other.

Ex. 3. If T, t denote the common tangents of the circles X, Y, and T', t' the common tangents of X', Y', show that

$$T^2 : T'^2 = t^2 : t'^2 = ab : a'b',$$

provided O be internal to both the circles X, Y, or external to both.

We have $$T^2 = (XY) + 2ab, \quad t^2 = (XY) - 2ab.$$

Hence the result follows from Ex. 2, i.

If O be internal to the circle X, and external to Y, we shall have from Ex. 2, ii.,

$$T^2 : t'^2 = t^2 : T'^2 = ab : -a'b'.$$

Ex. 4. Deduce the theorem of § 359, Ex. 1, from Ex. 2, of this section.

Ex. 5. A series of circles X_1, X_2,...X_m,... are described, so that each circle of the system touches two given circles (one of which lies within the other), and its two neighbours in the series. If X_{m+1} coincide with X_1, so that there is a ring of circles traversing the space between the given circles n times, show that the radii of the given circles are connected with the distance between their centres by the formula,

$$(r-r')^2 - 4rr' \tan^2 \frac{n\pi}{m} = \delta^2. \qquad \text{[Steiner.]}$$

Ex. 6. Show that if the circles X_1, X_2, X_3, X_4 touch another circle each in the same sense, the direct common tangents $T_{1,2}$, $T_{1,3}$, &c., are connected by a relation of the form

$$T_{1,2} \cdot T_{3,4} \pm T_{1,3} \cdot T_{2,4} \pm T_{1,4} \cdot T_{2,3} = 0. \qquad \text{[Casey.]}$$

If we invert the figure with respect to any point on the common tangent circle, we shall have a group of four circles touching a straight line and lying on the same side of the line. If A, B, C, D be the four points of contact, it is evident that

$$AB \,.\, CD + AC \,.\, DB + AD \,.\, BC = 0.$$

Hence by the theorem of Ex. 3, the above result follows.

A similar relation holds when the given circles do not touch the circle in the same sense, provided that in the cases of two circles which touch it in opposite senses the direct common tangent is replaced by the corresponding transverse common tangent.

It should be carefully noticed that the converse of this important theorem cannot be inferred from the nature of the proof here given. In the next chapter, however, we shall give another proof of the theorem, and shall show that the converse theorem is also true.

Inversion applied to coaxal circles.

371. To illustrate the advantage of using the method of inversion to prove propositions relating to geometrical figures we shall show how the principal properties of a system of coaxal circles may be derived. When a system of coaxal circles intersect in real points, we may take either point as the centre of inversion, and thus obtain for the inverse figure a system of concurrent lines (§ 348, Ex. 4); and when the coaxal systems have real limiting points, by taking either as the centre of inversion we obtain a system of concentric circles (§ 356). Consequently the properties of a system of coaxal circles may be derived from the properties of the simpler figures consisting either of concurrent lines, or concentric circles. In either case, it will be observed that the centre of inversion will not have any particular relation to the simple figure.

372. Ex. 1. Every circle which touches two given straight lines cuts orthogonally one or other of two straight lines concurrent with the given lines.

Every circle which touches two given concentric circles cuts orthogonally one or other of two circles concentric with the given circles.

Every circle which touches two given circles cuts orthogonally one or other of two circles coaxal with the given circles.

Ex. 2. If a variable circle touch two given concentric circles, the locus of its centre is one or other of two circles concentric with the given circles.

If a variable circle touch two given circles, the locus of the inverse point with respect to it of either of the limiting points of the given circles, is one or other of two circles coaxal with the given circles.

Ex. 3. If a variable circle cut two given concentric circles at constant angles:

i. It will cut orthogonally a fixed circle concentric with the given circles.

ii. It will cut every concentric circle at a constant angle.

iii. It will touch two circles concentric with the given circles.

If a variable circle cut two given circles at constant angles:

i. It will cut orthogonally a fixed circle coaxal with the given circles.

ii. It will cut every coaxal circle at a constant angle.

iii. It will touch two circles coaxal with the given circles.

Ex. 4. If the powers of a variable circle with respect to two concentric circles are in a constant ratio, the circle will cut orthogonally a fixed circle concentric with the given circles.

If the powers of a variable circle with respect to two given circles are in a constant ratio, the circle will cut orthogonally a fixed circle with the given circle.

Ex. 5. The powers of a variable point on a fixed circle with respect to two concentric circles are in a constant ratio.

The powers of a variable point on a given circle with respect to two coaxal circles are in a constant ratio.

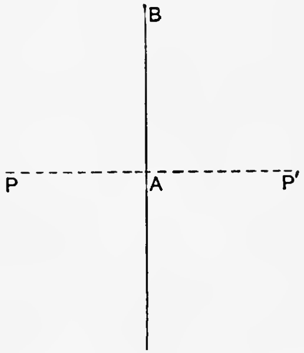
Miscellaneous Theorems.

373. Hitherto we have supposed the circle of inversion to be of finite dimensions. It remains to consider the case when the circle of inversion is a point-circle, and the case when the radius of the circle is infinitely great.

When the circle of inversion is a point-circle, O, let us enquire what will be the form of the inverse of a given figure F. If no part of the given figure pass through the point O, we may imagine a circle drawn having O for centre, and its radius small but finite, which will not cut F in real points. The inverse figure of F with respect to this circle will evidently lie entirely within the circle, and will therefore be evanescent when the radius of the circle is indefinitely diminished. Hence, when the circle of inversion is a point-circle, the inverse of any figure which does not pass through the point is evanescent.

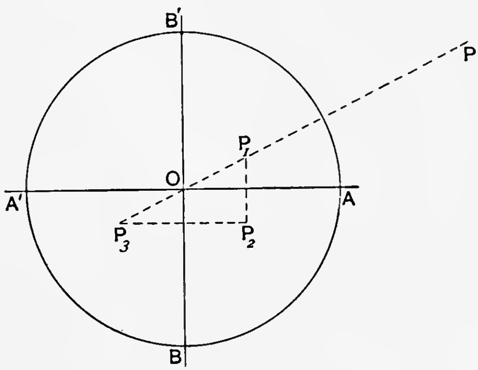
But if any part of the figure F be a straight line or a circle which passes through the centre of inversion, such line or circle may be considered as cutting the point-circle of inversion orthogonally, and will therefore coincide with the corresponding part of the inverse figure. Hence, when the circle of inversion is a point-circle, every straight line or circle which passes through the point coincides with its inverse with respect to the point.

374. When the radius of the circle of inversion is infinitely great, the circle may be considered as consisting of a finite straight line and the line at infinity.



Let AB be any straight line, and let us find the inverse point with respect to the line AB of any given point P. Let PA be drawn perpendicular to AB, then the point A and the point at infinity on the line PA may be considered as opposite extremities of a diameter of the line-circle AB, (that is the infinite circle whose finite part is the straight line AB). If P' be the inverse point of P, P and P' must be harmonically conjugate with the point A and the point at infinity on the line AP. Hence PP' is bisected in the point A.

375. If four circles be mutually orthogonal, and if any figure be inverted with respect to each of the four circles in succession, the fourth inversion will coincide with the original figure.



Let O be a point of intersection of two of the circles, then if the figure be inverted with respect to the point O, we shall have a real circle centre O, two rectangular diameters, and an imaginary concentric circle.

Let P be any point, and let P_1, P_3 be the inverse points with respect to the two circles. Then since they cut orthogonally, we shall have

$$OP \,.\, OP_1 + OP \,.\, OP_3 = 0.$$

Therefore $$OP_1 = P_3O.$$

Let P_2 be the inverse of P with respect to the diameter OA, then P_1P_2 is bisected by OA. It follows that P_2P_3 will be bisected by OB; that is, P_3 will be the inverse of P_2 with respect to OB.

Hence, if the point P be inverted with respect to the two circles, and the two diameters successively, the fourth inversion will coincide with P.

It follows from § 360, that if any point be inverted successively with respect to four mutually orthotomic circles, the fourth position will concide with the original position of the point.

Hence also if any figure be inverted successively with respect to four mutually orthotomic circles, the ultimate figure will coincide with the original figure.

376. Ex. 1. A straight line is drawn through a fixed point O cutting a given circle whose centre is C, in the points P and Q. Show that if the direction of the line PQ vary, two of the four circles which can be drawn to touch the circles OPC, OQC, CPQ belong respectively to two coaxal systems and the other two cut orthogonally the circle whose diameter is OC.

Ex. 2. If G be the median point of the triangle ABC, and if AG, BG, CG cut the circumcircle of the triangle in the points A', B', C'; show that the symmedian point of the triangle $A'B'C'$ lies on the diameter which passes through the Tarry point of the triangle ABC (§ 135, Ex. 7).

[E. Vigarié. E. T. Reprint, Vol. LII. p. 73.]

Ex. 3. Three circles are drawn through any point O. Show that four circles may be drawn to touch them, and that these four circles are touched by another circle.

If the first set of circles intersect in the points A, B, C show that the circle which touches the second set will cut the circles BOC, COA, AOB in three points P, Q, R, such that:

i. The lines AP, BQ, CR are concurrent.

ii. The groups of points B, C, Q, R; C, A, R, P; A, B, P, Q; are concyclic.

iii. The circle PQR is the inverse of the circle ABC with respect to the circle which cuts orthogonally the three circles $BCQR$, $CARP$, $ABPQ$.

CHAPTER XV.

SYSTEMS OF CIRCLES.

System of three circles.

377. THE radical axes of three given circles taken in pairs are concurrent (§ 305), the point of intersection being called the radical centre of the circles. If, with this point for centre, a circle be described cutting any one of the circles orthogonally, it will cut each of the circles orthogonally (§ 304). It follows also from the properties of the radical axis of two circles (§ 308), that this circle is the only circle which cuts each of the three given circles orthogonally.

This circle is called the *orthogonal circle*, or the *radical circle* of the given system. It has an important relation to all the groups of circles which are connected with three given circles, owing to the fact that all such groups occur in pairs, each pair being inverse circles with respect to the orthogonal circle of the system.

When the radical centre is internal to each of the three given circles, the orthogonal circle is evidently imaginary. In this case a concentric circle can be drawn so as to be bisected by each of the given circles (§ 315, Ex. 9).

378. If P and Q are opposite extremities of a diameter of the radical circle of three given circles, it follows from § 261, Ex. 1, that the points P and Q are conjugate points with respect to each of the given circles. Hence, *the radical circle is the locus of a point whose polars with respect to three given circles are concurrent.*

379. It was proved in § 321, that the homothetic centres of three circles taken in pairs are the six vertices of a tetragram. The four lines of this tetragram are called the *homothetic axes*, or *axes of similitude* of the given circles. It will be found that these axes have important relations in connection with the geometry of three circles.

Convention relating to the sign of the radius of a circle.

380. In § 358 it was proved that the angle of intersection of two circles is equal or supplementary to the angle of intersection of the inverse circles with respect to any circle of inversion. If X and Y denote two given circles, and if X', Y' denote the inverse circles with respect to a circle whose centre is O, it is easy to see that the angle of intersection of the circles X', Y' is equal to the angle of intersection of X, Y, provided that the point O is either internal to both the circles X, Y, or external to both circles; but that when the point O is external to one circle and internal to the other, the angle of intersection of X' and Y' is supplementary to the angle of intersection of X and Y (§ 339, Ex. 1).

Now the radius of a circle may be conceived either as a positive or as a negative magnitude. But, if r, r', d denote the radii and the distance between the centres of two circles, their power (§ 313)

$$= d^2 - r^2 - r'^2 = -2rr' \cos \omega.$$

Hence, if ω be regarded as the angle of intersection of the circles when r, r' are considered as of like sign, their angle of intersection must be regarded as $\pi - \omega$ when r, r' are considered as of unlike sign. It will be found that considerable advantage will accrue from the use of this idea in the case of pairs of inverse circles.

Let us consider the radii of the inverse pair of circles X, X' as having the same sign when their centres are situated on the same side of the centre of inversion, and as having different signs when their centres are situated on opposite sides of the centre of inversion. It is easy to see that, if we regard the radius of the circle X as positive, the radius of X' will be positive or negative according as the centre of inversion is external or internal to the circle X, when the circle of inversion is real; and that the radius of X' will be positive or negative according as the centre of inversion is internal or external to the circle X, when the circle of inversion is imaginary.

Hence, if we adopt the above rule of sign as a convention, we may say that the inverse circles of two given circles intersect at the *same* angle as the given circles.

When it is convenient to specify which circle of a pair of inverse circles is to be considered as having its radius positive, we may say that the radius of that circle is positive whose centre lies on the opposite side of the centre of inversion to the radical axis of the circles. Now when the circle of inversion is imaginary, and the centres of two inverse circles are situated on opposite sides of the centre of inversion, they are also situated on opposite sides of the radical axis. Therefore, when the circle of inversion is imaginary, we may say that the radius of that circle of the pair is positive, whose centre lies on the *same* side of the radical axis as the centre of inversion.

381. Let us suppose that we have three given circles X, Y, Z; and let S be the radical circle of the system. Then each of the given circles coincides with its inverse with respect to the circle S. Now let us imagine a circle U to be drawn cutting the given circles at given angles. Then if U' denote the inverse circle of U with respect to S, it follows from § 380 that U' will cut the given circles at the same angles as U.

Hence, if the problem: *To draw a circle cutting three given circles at given angles,* admits of one solution, it will admit of two solutions.

It must be noticed, however, that the two circles which can be drawn cutting the given circles at angles θ, ϕ, ψ will be coincident with the two circles which can be drawn cutting the given circles at the angles $\pi - \theta, \pi - \phi, \pi - \psi$.

Assuming then, for the present, that a circle can always be drawn cutting three given circles at given angles θ, ϕ, ψ, we infer that:—a pair of circles can be drawn cutting the given circles at angles θ, ϕ, ψ; a pair cutting them at angles $\pi - \theta, \phi, \psi$; a pair cutting them at angles $\theta, \pi - \phi, \psi$; and a pair cutting them at angles $\theta, \phi, \pi - \psi$.

Thus, every pair of circles which cut three given circles at given angles may be considered as one of four associated pairs of circles.

Four such pairs of circles are called a *group* of circles.

Circles cutting three given circles at given angles.

382. *To describe a circle which shall cut three given circles at given angles.*

Let X, Y, Z be the three given circles, and let Σ denote a circle which cuts them at the angles θ, ϕ, ψ respectively. It follows from § 330, that Σ must cut orthogonally three circles U, V, W, which are coaxal with the pairs Y, Z; Z, X; X, Y; respectively. Now these circles U, V, W are coaxal circles; for if A, B, C, be the centres of X, Y, Z, and D, E, F the centres of U, V, W, we have as in § 329,

$$BD : CD = (\Sigma Y) : (\Sigma Z);$$

$$CE : AE = (\Sigma Z) : (\Sigma X);$$

$$AF : BF = (\Sigma X) : (\Sigma Y).$$

Therefore
$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1;$$

and therefore the points D, E, F are collinear. Consequently the circles U, V, W are coaxal.

Also since

$$(\Sigma Y) : (\Sigma Z) = r_2 \cos \phi : r_3 \cos \psi,$$

where r_2, r_3 are the radii of the circles Y, Z, the point D is easily found, and likewise the points E and F. Therefore the line DEF may be constructed.

Again the circles U, V, W evidently cut orthogonally the radical circle of the system X, Y, Z. Denoting this circle by S, we see that the circles Σ, S belong to the orthogonal coaxal system of the system U, V, W.

Hence the centre of the circle Σ must lie on the straight line which passes through the radical centre of the circles X, Y, Z, and is perpendicular to the line DEF.

Again, the circle Σ must touch two circles coaxal with Y and Z (§ 330). Let these circles be U_1 and U_2. Then U_1 and U_2 are a pair of inverse circles with respect to the circle U. Hence, if a circle be drawn through the limiting points of the system (U, V, W) to touch U_1, it will also touch U_2. Now two circles may be drawn passing through two given points and touching a given circle. Hence we infer that two circles can be drawn cutting the circles U, V, W orthogonally, and touching U_1 and U_2. These circles will evidently cut the circles X, Y, Z at the given angles.

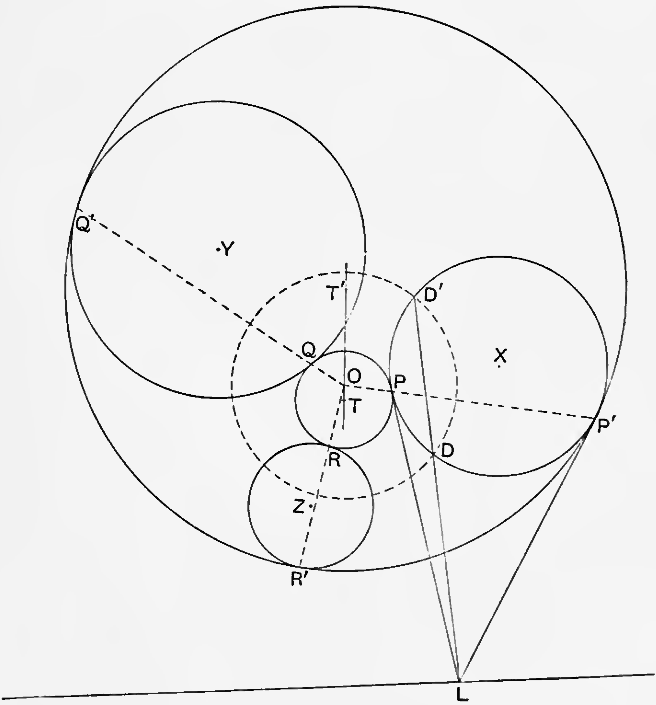
To show that the construction is practicable, we have only to show that the circles U_1, U_2 can be drawn. Now the locus of the centre of a circle which cuts a given circle at a given angle is a circle concentric with the given one. Therefore two circles can be drawn having a given radius and touching the two given circles Y, Z. If then we draw (§ 325, Ex. 2) the two circles coaxal with Y and Z which touch either of the two circles of given radius which cut Y and Z at the given angles, these circles will evidently be the circles U_1, U_2 (§ 330).

If the limiting points of the system (U, V, W) are imaginary, we can still draw two circles cutting these circles orthogonally and touching the circles U_1, U_2, as in § 325, Ex. 2.

Thus, we can in general always describe two circles which shall cut three given circles at given angles.

Circles which touch three given circles.

383. The eight circles which touch three given circles consist of four pairs of circles (§ 381); namely, a pair which touch the given circles each in the same sense, and three pairs which touch one of the given circles in one sense and the other two circles in



the opposite sense. The construction of any pair may be deduced from the general case given in the last article, or it may be done as indicated in § 322, Ex. 5. But the simplest method is to proceed as explained below.

Let us suppose the given circles to be external to each other, so that the radical circle of the system is real; and let us suppose that the two circles which touch each of the given circles in the same sense have been drawn. Let P, Q, R be the points of contact of one of the circles, and P', Q', R' the points of contact of the other.

Let us denote the given circles by X, Y, Z; the radical circle of the system by S; and the tangent circles by T, T'. Then, since the circles Y and Z touch the circles T and T' in the same sense, the radical axis of T, T' must pass through the homothetic centre of Y, Z (§ 320, Ex. 9). Similarly the radical axis of T, T' must pass through the homothetic centres of the pairs of circles Z, X; and X, Y. Hence the radical axis of the circles T, T' is a homothetic axis of the circles X, Y, Z.

Again, let the tangents to the circle X at the points P, P' meet in L. Then, since $LP = LP'$, it follows that L is a point on the radical axis of T and T'; therefore L is a point on the homothetic axis of X, Y, Z. But since the circles T, T' are coaxal with the radical circle of the system X, Y, Z, therefore the point L is the radical centre of the radical circle, and the circles X, T. Consequently, if the radical circle cut the circle X in the points D and D' the chord DD' must pass through the point L.

Hence we have the following simple construction for drawing the circles T and T'; *Draw the radical axes of the pairs of circles* S, X; S, Y; S, Z; *and from the points of intersection of these axes with that homothetic axis of the given circles, which passes through these homothetic centres, draw tangents to the given circles; then the points of contact are points on the circles which touch the given circles.*

Similarly, the other pairs of tangent circles may be constructed by finding the points in which the radical axes of the pairs of circles S, X; S, Y; and S, Z; cut the other three homothetic axes of the given circles. Corresponding to each homothetic axis there will be one pair of tangent circles.

384. Let O be the radical centre of the given circles. Then since the circles T and T' are inverse circles with respect to the radical circle, it follows that the lines PP', QQ', RR' must intersect in the point O.

Again, since the tangents at P and P' intersect on a homothetic axis of X, Y, Z, therefore PP' must pass through the pole of this line with respect to the circle X.

Hence, we have the following construction: *Draw any homothetic axis of the given circles, and find the poles of this line with respect to each of the circles; then the lines joining these poles to the radical centre of the given circles, will cut them in the six points of contact of a pair of tangent circles.*

This method is not of such easy application as the preceding one, but it is always practicable, whereas the former is impracticable when the radical circle is imaginary.

385. Let any circle U be drawn coaxal with the circles T, T', which (see fig. § 383) touch each of three given circles X, Y, Z in the same sense. It follows from § 329, that the powers (UX), (UY), (UZ) will be in the same ratio as the powers of the circles X, Y, Z with respect to the radical axis of S and S', that is the homothetic axis of the circles X, Y, Z. Therefore the powers (UX), (UY), (UZ) are in the ratio of the radii of the circles X, Y, Z. Hence every circle which is coaxal with the circles S and S' will cut the circles X, Y, Z at equal angles.

Hence, to construct a circle which shall cut three given circles at the same angle, θ say, we infer that it is sufficient to draw a circle coaxal with the circles S and S', and cutting one of the given circles X at the angle θ.

Hence it appears that a circle can always be drawn which shall cut four given circles at the same angle. Let X_1, X_2, X_3, X_4 denote the four circles, and let O_1, O_2, O_3, O_4 be the radical centres of the four triads of circles. Let the perpendiculars from O_1, O_2 on the homothetic axes of the triads of circles X_2, X_3, X_4; X_1, X_3, X_4; intersect in O. Then it follows from the above argument that a circle whose centre is O will cut each of the given circles at the same angle.

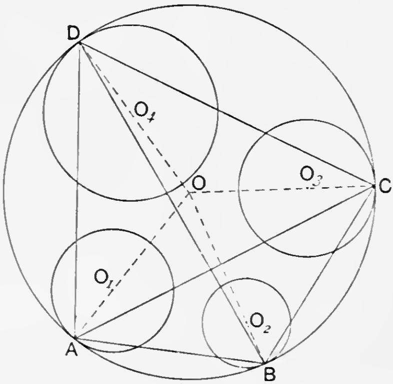
386. Ex. 1. Show that eight circles can be drawn each of which will cut four given circles at the same, or supplementary, angles.

Ex. 2. From the radical centre of each triad of four given circles, lines are drawn perpendicular to the four homothetic axes of the triad. Show that the sixteen lines, so obtained, pass four by four through eight other points.

System of four circles having a common tangent circle*.

387. It has been already proved (§ 370, Ex. 6) that, when four circles have a common tangent circle, the common tangents of the four circles are connected by a certain relation. It was pointed out, however, in that article, that the converse of the theorem does not follow from the proof there given. We propose now to give a different proof of this important theorem, and at the same time to show that the converse is true under all circumstances.

Let X_1, X_2, X_3, X_4 denote any four circles which touch a circle X, in the points A, B, C, D. Let O, O_1, O_2, O_3, O_4 be the centres of the circles X, X_1, X_2, X_3, X_4; and let r, r_1, r_2, r_3, r_4 denote their radii. Also let us denote the direct common tangents of the pairs of circles X_1, X_2; X_1, X_3; &c., by the symbols 12, 13, &c.; and the transverse common tangents of the same pairs by (12); (13); &c.



Firstly let us suppose that the circle X touches each of the circles X_1, X_2, X_3, X_4 in the same sense

By § 320, Ex. 16, we have

$$12^2 : AB^2 = OO_1 . OO_2 : OA . OB;$$

$$13^2 : AC^2 = OO_1 . OO_3 : OA . OC;$$

....

* The greater part of this section is taken from a paper by Mr A. Larmor;—*Proc. L. M. S.* vol. XXIII, p. 135. (1891.)

But since the points A, B, C, D are concyclic,

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

Hence $$12 \cdot 34 + 14 \cdot 23 - 13 \cdot 24 = 0 \qquad \text{(i)}.$$

Secondly, let us suppose that the circle X touches X_1 in the opposite sense to that in which it touches the circles X_2, X_3, X_4.

Then, by § 320, Ex. 16, we have

$$(12)^2 : AB^2 = OO_1 \cdot OO_2 : OA \cdot OB.$$

Hence $$(12) \cdot 34 + (14) \cdot 23 - (13) \cdot 24 = 0 \qquad \text{(ii)}.$$

And thirdly, if the circle X touches the circles X_1, X_2 in the opposite sense to that in which it touches the circles X_3, X_4, we shall have

$$12 \cdot 34 + (14) \cdot (23) - (13) \cdot (24) = 0 \qquad \text{(iii)}.$$

Thus, when four circles have a common tangent circle, their common tangents must be connected by a relation of the type (i), (ii) or (iii).

It is to be noticed that the product which is affected with the negative sign corresponds to the pairs of circles for which the chords of contact intersect in a point which is internal to the circle X.

388. Let us suppose that the circle X_4 is a point-circle. Then we see that, if O_4 be a point on either of the circles which touch X_1, X_2, X_3 all internally or all externally,

$$12 \cdot 34 - 14 \cdot 23 + 13 \cdot 24 = 0,$$
$$12 \cdot 34 + 14 \cdot 23 - 13 \cdot 24 = 0,$$

or $$-12 \cdot 34 + 14 \cdot 23 + 13 \cdot 24 = 0,$$

according as the point O_4 lies on the arc 23, 31, or 12, respectively.

If O_4 be a point on either circle which has contacts of similar nature with X_2, X_3, and of the opposite nature with X_1, then

$$(12) \cdot 34 - (14) \cdot 23 + (13) \cdot 24 = 0,$$
$$(12) \cdot 34 + (14) \cdot 23 - (13) \cdot 24 = 0,$$

or $$-(12) \cdot 34 + (14) \cdot 23 + (13) \cdot 24 = 0.$$

If O_4 be a point on either circle which has contacts of similar nature with X_3, X_1, and of the opposite nature with X_2, then

$$(12) \cdot 34 - 14 \cdot (23) + 13 \cdot (24) = 0,$$
$$(12) \cdot 34 + 14 \cdot (23) - 13 \cdot (24) = 0,$$

or $$-(12) \cdot 34 + 14 \cdot (23) + 13 \cdot (24) = 0.$$

If O_4 be a point on either circle which has contacts of similar nature with X_1, X_2, and of the opposite nature with X_3, then

$$12 . 34 - (14) . (23) + (13) . (24) = 0,$$

$$12 . 34 + (14) . (23) - (13) . (24) = 0,$$

or

$$-12 . 34 + (14) . (23) + (13) . (24) = 0.$$

In each case these alternatives hold according as the point O_4 lies on the arc 23, 31, or 12, respectively, X_4 being regarded as a point-circle lying on the same side of the tangent circle as the circle X_3.

389. Conversely, if any one of the relations which occur in the last article subsist between the common tangents of the circles X_1, X_2, X_3, and the point-circle X_4, that point must lie on one or other of the pair of tangent circles of X_1, X_2, X_3, for which that particular relation has here been proved to subsist.

The proof depends on the following lemma: *Given three circles X_1, X_2, X_3 and a point P there is only one other point Q for which*

$$1Q : 2Q : 3Q = 1P : 2P : 3P.$$

This theorem follows at once from § 312, Ex. 3. The point Q is in fact the other point of concourse of the three circles which can be drawn through P coaxal with the pairs of circles X_2, X_3; X_3, X_1; X_1, X_2; respectively. Also from § 345, Ex. 1, we see that P, Q are inverse points with respect to the radical circle of the system X_1, X_2, X_3.

390. Let us suppose now that the common tangents of the circles X_1, X_2, X_3, and the point-circle X_4 are connected by the relation

$$12 . 34 - 14 . 23 + 12 . 34 = 0.$$

This relation holds for any point on either of the arcs of the pair of circles (Y, Y', say) which touch each of the circles X_1, X_2, X_3 in the same sense.

Through the point O_4 describe a circle coaxal with X_2 and X_3, and let it cut either of these arcs in Q.

Then, by § 388,

$$12 . 3Q - 23 . 1Q + 13 . 2Q = 0;$$

and, by hypothesis,

$$12 . 34 - 23 . 14 + 13 . 24 = 0.$$

But, since O_4, Q lie on a circle coaxal with X_2 and X_3, by § 329,

$$24 : 34 = 2Q : 3Q.$$

Hence $$14 : 24 : 34 = 1Q : 2Q : 3Q,$$

and, by the above Lemma, since Q is on one of the circles Y, Y', which are inverse circles with respect to the radical circle of the system X_1, X_2, X_3, it follows that O_4 must be a point on the other circle.

391. Suppose now that the common tangents of four given circles X_1, X_2, X_3, X_4 are connected by a relation of the form

$$12 \,.\, 34 \pm 14 \,.\, 23 \pm 13 \,.\, 24 = 0,$$

$$(12) \,.\, 34 \pm (14) \,.\, 23 \pm (13) \,.\, 24 = 0,$$

or $$(12) \,.\, (34) \pm (14) \,.\, (23) \pm 13 \,.\, 24 = 0.$$

Then the four circles have a common tangent circle.

For, take that circle, X_4 say, whose radius is not greater than that of the three remaining circles. With the centre of each of the remaining circles as centre describe a circle, whose radius is equal to the sum or difference of its radius and that of the circle X_4, according as the common tangent of it and X_4 is transverse or direct.

These three new circles X_1', X_2', X_3', together with O_4 (the centre of X_4—a point-circle) form a group of four circles having the same common tangents as the four given circles, so that the given relation is satisfied for this system; and it follows by § 390 that the point O_4 must lie on one or other of a pair of common tangent circles of the system X_1', X_2', X_3'; and hence, that X_4 touches one or other of a pair of common tangent circles of the system X_1, X_2, X_3.

If the given circles X_1, X_2, X_3, X_4 have a common orthogonal circle, then it is easy to see that X_4 will touch both circles of the pair.

392. Ex. 1. Show that the circle which passes through the middle points of the sides of a triangle, touches the inscribed and escribed circles of the triangle.

This theorem follows at once by treating the middle points of the sides as point-circles.

Ex. 2. Show that a circle can be drawn to touch the escribed circles of a triangle in one sense, and the inscribed circle in the opposite sense.

Ex. 3. If the circles X_1, X_2 are the inverse circles of X_3, X_4, respectively, with respect to any circle, show that the common tangents of the circles are connected by the relations:

$$23.14=12.34+13.24;$$
$$(23).(14)=(12).(34)+13.24.$$

393. When four circles which touch the same circle intersect in real points, we may obtain relations connecting their angles of intersection which are equivalent to the relations given in § 387.

If two circles whose centres are O_1, O_2, touch another circle whose centre is O, at the points P and Q, it is easy to prove that, if the circles cut at the angle ω:

$$PQ^2 : 4O_1P\,.\,O_2Q\sin^2\tfrac{1}{2}\omega = OP\,.\,OQ : OO_1\,.\,OO_2,$$

when the contacts are of the same nature; and that

$$PQ^2 : 4O_1P\,.\,O_2Q\cos^2\tfrac{1}{2}\omega = OP\,.\,OQ : OO_1\,.\,OO_2,$$

when the contacts are of the opposite nature.

Hence, if X_1, X_2, X_3, X_4 be four circles which touch a fifth circle X, we shall have:

$$\sin\tfrac{1}{2}\omega_{1,2}\,.\,\sin\tfrac{1}{2}\omega_{3,4}+\sin\tfrac{1}{2}\omega_{1,4}\,.\,\sin\tfrac{1}{2}\omega_{2,3}-\sin\tfrac{1}{2}\omega_{1,3}\,.\,\sin\tfrac{1}{2}\omega_{2,4}=0\ldots\text{(i)},$$

when X touches all the circles in the same sense;

$$\sin\tfrac{1}{2}\omega_{1,2}\,.\,\cos\tfrac{1}{2}\omega_{3,4}+\sin\tfrac{1}{2}\omega_{2,3}\,.\,\cos\tfrac{1}{2}\omega_{1,4}-\sin\tfrac{1}{2}\omega_{1,3}\,.\,\cos\tfrac{1}{2}\omega_{2,4}=0\ldots\text{(ii)},$$

when X touches X_4 in one sense, and X_1, X_2, X_3 in the opposite sense;

$$\sin\tfrac{1}{2}\omega_{1,2}\,.\,\sin\tfrac{1}{2}\omega_{3,4}+\cos\tfrac{1}{2}\omega_{2,3}\,.\,\cos\tfrac{1}{2}\omega_{1,4}-\cos\tfrac{1}{2}\omega_{1,3}\,.\,\cos\tfrac{1}{2}\omega_{2,4}=0\ldots\text{(iii)},$$

when X touches X_1 and X_2 in the same sense, and X_3, X_4 in the opposite sense.

Conversely, if the angles of intersection of the circles X_1, X_2, X_3, X_4 be connected by any one of the above relations, it may be proved, as in § 391, that the circles will have a common tangent circle.

394. We propose now to give an alternative method* by which the truth of the theorem of § 391 may be inferred.

If the circles X_1, X_2. X_3 touch the same straight line, it is evident that their common tangents must be connected by the relation,

$$23 \pm 31 \pm 12=0,$$

or by a relation of the type

$$23 \pm (31) \pm (12)=0,$$

* This method was suggested by Mr Baker.

according as the circle X_1 is on the same side, or the opposite side, of the line as the circles X_2, X_3.

The converse of this theorem is not so obvious, but it is easily seen from a figure that it is true when any one of the circles is a point-circle.

When the radius of each circles is finite, let X_1 be that circle whose radius is not greater than the radii of the other two, and let circles X_2', X_3' be drawn concentric with X_2 and X_3 with radii equal to the sum or difference of the radii of these circles, respectively, and the circle X_1, according as their common tangents with X_1 are transverse or direct.

Then the circles X_2', X_3', and the point-circle O_1 (the centre of X_1) have the same common tangents as the circles X_1, X_2, X_3, so that the given relation is satisfied for this system, and therefore the point O_1 must lie on one of the common tangents of the circles X_2', X_3'. Consequently the circle X_1 must touch one of the common tangents of the circles X_2, X_3; that is, the circles X_1, X_2, X_3 touch the same line.

395. Let us suppose now that the common tangents of the circles X_1, X_2, X_3 and a point-circle O_4, are connected by a relation of the form

$$23 \,.\, 14 \pm 31 \,.\, 24 \pm 12 \,.\, 34 = 0.$$

Let X_1', X_2', X_3' denote the inverse circles of X_1, X_2, X_3, respectively, with respect to any circle whose centre is O_4, and whose radius is R; and let r_1, r_1', &c. denote the radii of the circles X_1, X_1', &c. Then we have by § 370, Ex. 3,

$$12^2 : 1'2'^2 = r_1 r_2 : r_1' r_2',$$

provided O_4 be external to both the circles X_1, X_2, or internal to both; and

$$12^2 : (1'2')^2 = (12)^2 : 1'2'^2 = r_1 r_2 : r_1' r_2',$$

when O_4 is external to one and internal to the other circle.

Also by § 354, we have,

$$14^2 : R^2 = r_1 : r_1'.$$

Hence it follows that the common tangents of the circles X_1', X_2', X_3' will be connected by a relation of the type

$$2'3' \pm 3'1' \pm 1'2' = 0,$$

or

$$(2'3') \pm 3'1' \pm 1'2' = 0.$$

Therefore, by § 394, the circles X_1', X_2', X_3' will have a common tangent line.

Hence it follows that the circles X_1, X_2, X_3 must touch a circle passing through the point O_4.

We may proceed in the same manner when the common tangents of the circles X_1, X_2, X_3 and the point-circle O_4 are connected by either of the relations (ii) or (iii) of § 387.

Finally, the general case may be deduced as in § 391.

396. Ex. 1. Show that, if the circle X_4 cut the circles X_1, X_2, X_3 at equal angles, and if

$$\sin \tfrac{1}{2}\omega_{1,2} + \sin \tfrac{1}{2}\omega_{2,3} - \sin \tfrac{1}{2}\omega_{1,3} = 0,$$

the circle X_4 and the two circles Y, Y' which touch the circles X_1, X_2, X_3, each in the same sense, will touch each other at the same point. [A. Larmor.]

Ex. 2. If three circles X_1, X_2, X_3 intersect at angles α, β, γ, and if X be the circle which intersects them at angles $\beta \sim \gamma$, $\gamma \sim \alpha$, $\alpha \sim \beta$ respectively, show that:

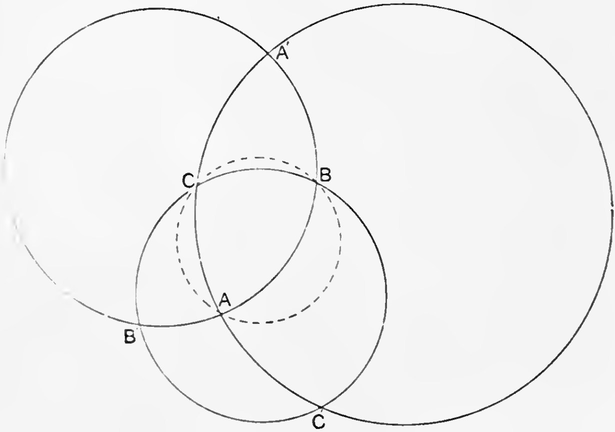
i. A circle can be drawn to touch the circles X, X_1, X_2, X_3 in the same sense.

ii. Three circles can be drawn to touch two of the circles X_1, X_2, X_3 in one sense, and the third circle and the circle X in the opposite sense.

It is easily verified that the following relations subsist connecting the angles of intersection of the four circles:

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}(\beta - \gamma) + \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}(\gamma - \alpha) + \sin \tfrac{1}{2}\gamma \sin \tfrac{1}{2}(\alpha - \beta) = 0;$$
$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}(\beta - \gamma) + \cos \tfrac{1}{2}\beta \cos \tfrac{1}{2}(\gamma - \alpha) - \cos \tfrac{1}{2}\gamma \cos \tfrac{1}{2}(\alpha - \beta) = 0;$$
$$\cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}(\beta - \gamma) - \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}(\gamma - \alpha) - \cos \tfrac{1}{2}\gamma \cos \tfrac{1}{2}(\alpha - \beta) = 0;$$
$$\cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}(\beta - \gamma) - \cos \tfrac{1}{2}\beta \cos \tfrac{1}{2}(\gamma - \alpha) + \sin \tfrac{1}{2}\gamma \sin \tfrac{1}{2}(\alpha - \beta) = 0.$$

Ex. 3. Three given circles intersect two by two in the points A, A'; B, B'; C, C'. Show that the circles ABC, $AB'C'$, $A'BC'$, $A'B'C$ are touched by four other circles. [A. Larmor.]



If the given circles intersect at angles a, β, γ, it is easy to see that the angles of intersection of the circles ABC, $AB'C'$, $A'BC'$, $A'B'C$, are given by the scheme :

	ABC	$AB'C'$	$A'BC'$	$A'B'C$
ABC		$\beta \sim \gamma$	$\gamma \sim a$	$a \sim \beta$
$AB'C'$	$\beta \sim \gamma$		$\pi - a - \beta$	$\pi - a - \gamma$
$A'BC'$	$\gamma \sim a$	$\pi - a - \beta$		$\pi - \beta - \gamma$
$A'B'C$	$a \sim \beta$	$\pi - a - \gamma$	$\pi - \beta - \gamma$	

Hence, the theorem follows from the theorem in Ex. 2.

Ex. 4. Show that the circles ABC, $A'B'C'$, $AB'C'$, $A'BC$ have four common tangent circles. [A. Larmor.]

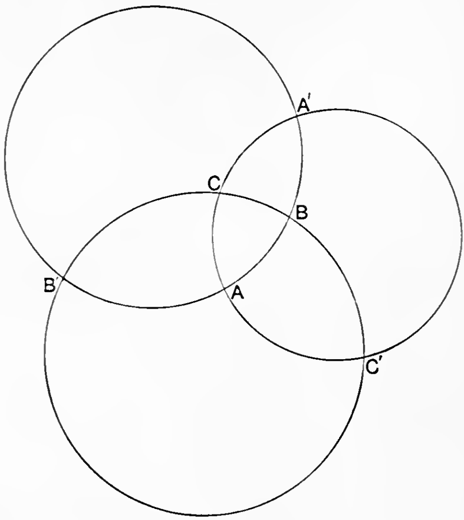
Ex. 5. Show the circumcircles of the eight circular triangles which are formed by three given circles are touched by thirty-two circles, each of which touches four of the eight circles. [A. Larmor.]

Properties of a circular triangle.

397. Let ABC be any triangle formed by three given circular arcs, and let the complete circles be drawn, intersecting again in the points A', B', C'. We thus obtain three triangles $A'BC$, $AB'C$, ABC', which may be called the *associated triangles* of the given triangle ABC; and four triangles $A'B'C'$, $AB'C'$, $A'BC'$, $A'B'C$, which are the inverse triangles, with respect to the circle which cuts the given circles orthogonally, of the given triangle and its associated triangles respectively.

Each of the above triangles has a circumcircle, and each has an inscribed circle, the eight inscribed circles being the eight circles which can be drawn to touch the three circles which form the triangles. Each of these systems of circles have some remarkable properties, in the discussion of which we shall meet with other circles which will be found to correspond to some of the circles connected with a linear triangle.

We shall find it convenient to consider the angles of a triangle as measured in the same way as the angles of a linear triangle. The angles of a triangle will not necessarily be the same as the angles of intersection of the circles which form it. Thus, if in the figure we take α, β, γ as the angles of the triangle ABC, the angles of intersection of the circles will be the supplements of these angles.

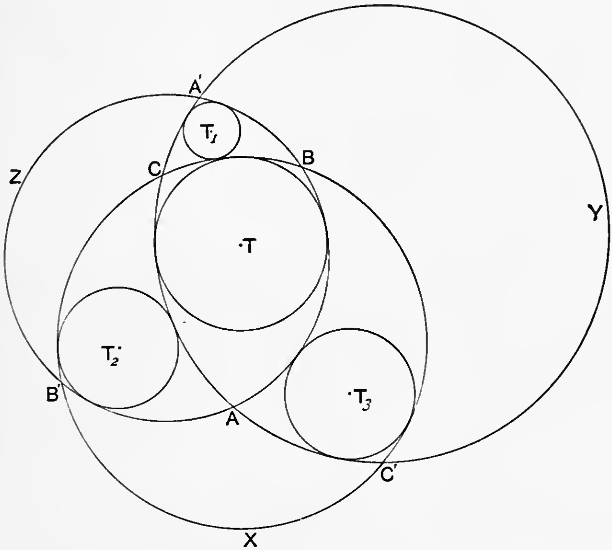


The angles of the several triangles formed by the circles BC, CA, AB are easily seen to be given by the scheme:

ABC	a	β	γ
$A'BC$	a	$\pi-\beta$	$\pi-\gamma$
$AB'C$	$\pi-a$	β	$\pi-\gamma$
ABC'	$\pi-a$	$\pi-\beta$	γ
$AB'C'$	a	$\pi-\beta$	$\pi-\gamma$
$A'BC'$	$\pi-a$	β	$\pi-\gamma$
$A'B'C$	$\pi-a$	$\pi-\beta$	γ
$A'B'C'$	$2\pi-a$	$2\pi-\beta$	$2\pi-\gamma$

398. *The inscribed circle of any triangle and the inscribed circles of the three associated triangles are touched by another circle which touches the former in one sense and the latter in the opposite sense.*

Let T, T_1, T_2, T_3 denote the inscribed circles of the triangles ABC, $A'BC$, $AB'C$, ABC'; and let 01, (01); 12, (12); &c., denote the common tangents of the pairs of circles T, T_1; T_1, T_2; &c.



Then, since the circle BC touches T_1 externally, and T, T_2, T_3 internally, we have by § 387, (ii),

$$(13).02 = (01).23 + (12).03.$$

Similarly, since the circle CA touches T_2 externally, and T, T_1, T_3 internally,

$$(12).03 = (02).13 + (23).01;$$

and, since the circle AB touches T_3 externally, and T, T_1, T_2 internally,

$$(13).02 = (23).01 + (03).12.$$

Hence, we have,

$$(03).12 = (01).23 + (02).13.$$

Therefore (§ 391) a circle can be drawn touching the circle T internally and the circles T_1, T_2, T_3 externally.

This theorem is evidently analogous to Feuerbach's theorem concerning the inscribed and escribed circles of a linear triangle. The extension of the theorem is due to Dr Hart, and the proof given above is a modification of Dr Casey's proof.

399. The circle which touches the inscribed circles of a circular triangle and its associated triangles is called the *Hart circle* of the triangle. It has several properties which are analogous to the properties of the nine-point circle of a linear triangle.

We have already seen § 396, Ex. 2, that the circle which cuts the sides of the triangle ABC at angles equal to $\beta \sim \gamma$, $\gamma \sim \alpha$, $\alpha \sim \beta$, respectively, touches the circles T, T_1, T_2, T_3. Hence, we infer that the Hart circle of the triangle ABC cuts the sides at angles equal to the differences of the angles of the triangle.

If we denote the circles BC, CA, AB by X, Y, Z, and the Hart circle of the triangle ABC by H, we see that the circles form a system touched by four other circles T, T_1, T_2, T_3, such that:—

T touches X, Y, Z, H in the same sense;

T_1 touches X, H in one sense, and Y, Z in the other sense;

T_2 touches Y, H in one sense, and Z, X in the other sense;

T_3 touches Z, H in one sense, and X, Y in the other sense.

Hence, we infer that the circles X, Y, Z, H form a system such that each is the Hart circle of one of the triangles formed by the other three circles.

400. There being a Hart circle connected with each of the eight triangles formed by three circles, we have in all a system of eight Hart circles. And since the Hart circle of any triangle touches the inscribed circles of its own triangle and the three associated triangles, we see that: *The Hart circle of any triangle and the Hart circles of the three associated triangles have a common tangent circle which touches the former in the opposite sense to that in which it touches the latter.*

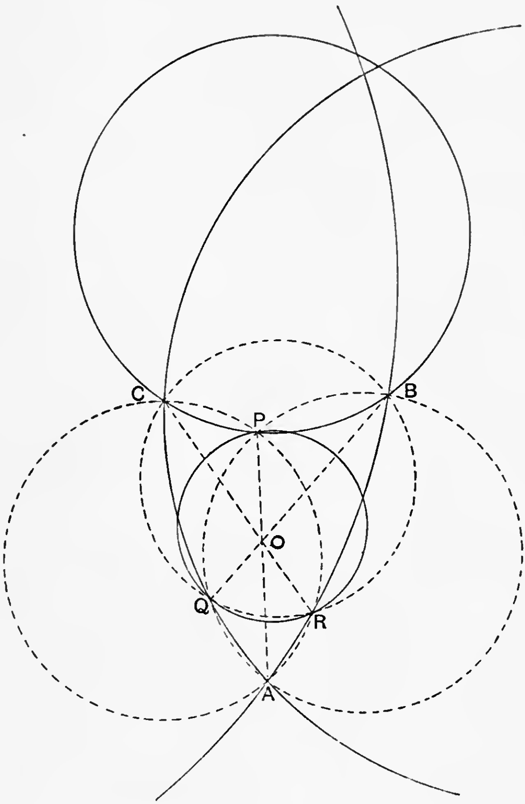
401. In § 396, Ex. 3, it was proved that the circumcircles of the triangles ABC, $AB'C'$, $A'BC'$, $A'B'C$ form a system such that one of them, ABC, for instance, cuts the others at angles equal to the differences of the angles at which they intersect.

Hence, we have the theorem: *The circumcircle of any circular triangle is the Hart circle of the triangle formed by the circumcircles of the inverse associated triangles**.

402. Several properties of the Hart circle of a triangle may be derived by considering that the circle ABC is the Hart circle of

* This theorem was first stated by Mr A. Larmor.

the triangle $A'B'C'$, formed by the circular arcs $AB'C'$, $A'BC'$, $A'B'C$.



Thus let us consider three circles $BCQR$, $CARP$, $ABPQ$, intersecting in the three pairs of points A, P; B, Q; C, R; each pair being inverse points with respect to the circle which cuts the three circles orthogonally. It follows that the circle PQR is the Hart circle of the triangle ABC formed by the circular arcs BPC, CQA, ARB.

Hence we infer that the Hart circle of a circular triangle ABC cuts the arcs BC, CA, AB in three points P, Q, R, respectively, such that the straight lines AP, BQ, CR are concurrent.

If O be the point of concurrence of the lines AP, BQ, CR, we have the theorems:

i. Each group of points: B, C, Q, R; C, A, R, P; A, B, P, Q: are concyclic.

ii. The point O is the radical centre of the circles $BCQR$, $CARP$, $ABPQ$.

iii. The Hart circle PQR is the inverse of the circumcircle ABC with respect to the circle which cuts the circles $BCQR$, $CARP$, $ABPQ$, orthogonally.

iv. The circumcircles of the triangles AQR, BRP, CPQ are the inverses of the circles BPC, CQA, ARB, with respect to the circle which cuts the circles $BCQR$, $CARP$, $ABPQ$, orthogonally.

The points P, Q, R are evidently analagous in the case of a linear triangle to the feet of the perpendiculars from the vertices on the opposite sides. The circle which cuts the circles $BCQR$. $CARP$, $ABPQ$ orthogonally, or the circle of similitude of the circumcircle and the Hart circle, is analogous to the polar circle of a linear triangle.

403. Ex. 1. If the angles of a circular triangle ABC be α, β, γ, and if circles be drawn through the pairs of points B, C; C, A; A, B; cutting the arcs BC, CA, AB, at angles equal to $\frac{1}{4}(\pi+\alpha+\beta+\gamma)$; show that these circles will cut the arcs of the triangle in three points P, Q, R respectively, such that the circumcircle of the triangle PQR is the Hart circle of the triangle ABC.

Ex. 2. If the Hart circle of the triangle ABC cut the arcs BC, CA, AB in the points P, P'; Q, Q'; R, R'; respectively, the points Q, R being concyclic with B, C; R, P with C, A; and P, Q with A, B; show that the circumcircles of the triangles $AQ'R'$, $BR'P'$, $CP'Q'$, touch the circumcircle of the triangle ABC at the points A, B, C, respectively.

404. When the given circles do not cut in real points, the Hart circles of the system are in general real circles. Their existence may be inferred in a similar manner to that adopted in § 398, by using the relations of § 387.

If we denote the pairs of tangent circles by T, T'; T_1, T_1'; T_2, T_2': T_3, T_3'; and the pairs of Hart circles by H, H'; H_1, H_1'; H_2, H_2'; H_3, H_3'; and if we consider the radii of the circles $T, T_1, T_2, T_3, H, H_1, H_2, H_3$, as positive, we see that, for the figure of § 398, the radii of T' and H' will be positive, and the radii of the circles $T_1', T_2', T_3', H_1', H_2', H_3'$ negative, in accordance with the convention of § 380. Hence the nature of the contacts of the several circles will be those given in the scheme:—

	T	T'	T_1	T_1'	T_2	T_2'	T_3	T_3'
H	in		ex		ex		ex	
H'		in		ex		ex		ex
H_1	ex		in			in		in
H_1'		ex		in	in		in	
H_2	ex			in	in			in
H_2'		ex	in			in	in	
H_3	ex			in		in	in	
H_3'		ex	in		in			in

It will be found that for any other figure the nature of the contacts will be the same as in this scheme, provided we choose the signs of the radii of any four of the circles T, T_1, T_2, T_3, so that the contacts of them with the circle H are as here indicated. For instance, let us consider the case of three given circles external to each other. Let T be the circle which touches each internally, and let T_1, T_2, T_3 be the circles which touch one of the given circles internally and the other two externally,—here the words internally and external have their ordinary meanings. Then it is easy to see that the circle H will touch each of the circles T, T_1, T_2, T_3, internally. But if we consider the radii of T and H as positive, and the radii of T_1, T_2, T_3 as negative, the contacts, in the generalised sense, will be the same as given by the scheme; and the nature of the contacts of any other group of circles may be inferred.

Circular reciprocation.

405. We propose now to explain a method analogous to the method of polar reciprocation (Ch. XI.), by which we may derive from known properties of figures consisting of circles, other properties. It will be seen, however, that the reciprocal figure will in general be a more complicated figure than the original; consequently the method is not so powerful as polar reciprocation, when used as an instrument of research.

Let S denote a fixed circle, and let P, P' be a pair of inverse points with respect to S. Then there can be found (§ 325, Ex. 1) *one* circle, which cuts S orthogonally and is coaxal with the system $\{S, P, P'\}$. This circle we shall call the *reciprocal* with regard to S of the point-pair P, P'; or simply the *reciprocal* of the point P. The circle S will be called the *circle of reciprocation.*

The reciprocal circle of a point will evidently be a real circle only when the circle of reciprocation is imaginary. Consequently we shall assume, unless the contrary is stated, that the circle of reciprocation is an imaginary circle having a real centre.

We shall presently prove that when the locus of a point P is a circle, the reciprocal of the point will envelope two circles, constituting a pair of inverse circles with respect to the circle of reciprocation. These circles will be called the *reciprocal* of the circle which is the locus of P.

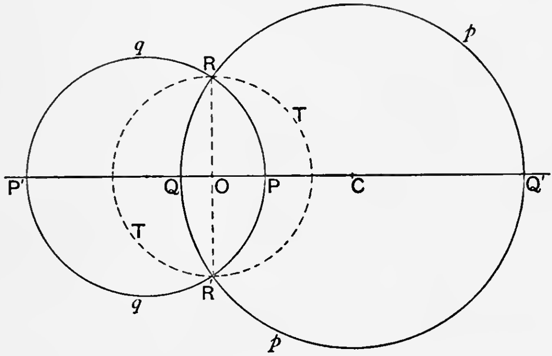
Further, if x, x' denote the pair of circles reciprocal to a circle X, and y, y' the pair of circles reciprocal to Y, we shall show that when the circles X and Y touch, the circles x, x' will each touch one of the circles y, y'.

406. To be able to apply the last theorem, it is necessary to distinguish between two circles which are inverse circles with respect to a given circle of inversion. Let X, X' be a pair of circles inverse with respect to S, and let T be the circle concentric with S and cutting it orthogonally; then we have shown in § 364, that the ratios $(XT):(XS)$, $(X'T):(X'S)$ have opposite signs. We shall call that circle of the pair for which this ratio is positive, the *positive* circle of the pair, and the other circle the *negative* circle of the pair. In § 365 it was shown that when the circle of inversion is imaginary, the centre of the negative circle of the pair X, X' must lie between the centres of the circles S, S', where S' is that circle coaxal with X and X' which cuts S orthogonally.

It will be necessary to use the convention as to the sign of the radius of a circle, which was given in § 380; and we shall suppose the radius of either of a pair of inverse circles to be *positive,* when its centre is situated on the same side of the radical axis as the centre of inversion. It is to be noticed that the positive circle of a given pair of inverse circles may have a negative radius, and that the radii of both circles of a pair may have the same sign.

Assuming the convention as here stated to be always understood, and using the definitions given above, we shall find that the theorem stated in the last article may be stated in the form: *When two circles X, Y touch internally, the positive and negative circles of the reciprocal pairs x, x'; y, y' touch respectively; and when X, Y touch externally, the positive and negative circles of the pair x, x' touch respectively the negative and positive circles of the pair y, y'.*

407. *To construct the reciprocal of a point.*



Let P be any given point, and let P' be the inverse point with respect to an imaginary circle S whose centre is O. Let T denote the circle whose centre is O which cuts S orthogonally. Then if p denote the reciprocal of the point P with respect to S, p will have its centre on the line OP, and will bisect the circle T.

Let C be the centre of p; and let p cut OP in Q and Q', and the circle T in the points R, R'. Let q denote the circle whose diameter is PP'. Then, since P, P' are by definition the limiting points of the circles S and p, the circle q must cut these circles orthogonally. Therefore the circle q will pass through the points R, R'; and the pencil $R\{PP', QQ'\}$ will be harmonic. Hence RP will bisect the angle QRQ'. But the angles QRQ', ORC evidently have the same bisectors: therefore RP, RP' bisect the angle ORC, and therefore the range $\{OC, PP'\}$ is harmonic.

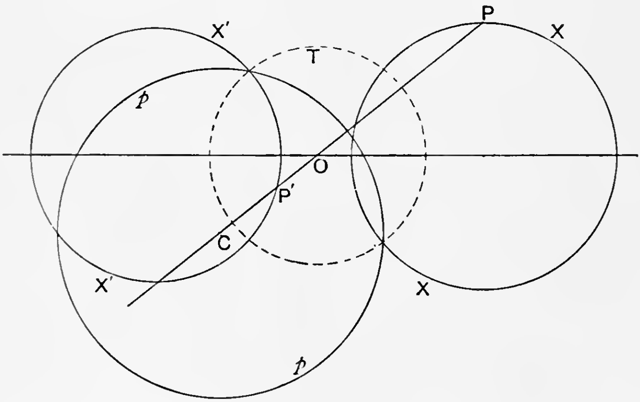
Hence we have the following construction for the circle p: *Find the harmonic conjugate of the point O with respect to the points P, P', and with this point for centre draw a circle cutting the circle T in the same points as the diameter perpendicular to the line OP.*

408. When the point P coincides with the point O, P' is at infinity, and therefore the point C must coincide with O. We infer then that the circle T is the reciprocal of the point O, and also of the line at infinity.

Again if P and P' be points on the circle T, it is obvious that C the centre of p will be the point at infinity on the line OP. That is to say the reciprocal of any point P on the circle T is the diameter of this circle which is perpendicular to OP.

409. *To find the reciprocal of a given circle.*

Let P be any point on a given circle X, and let P' be the inverse point with respect to S, the circle of reciprocation, on the inverse circle X'. Then the circle p which is the reciprocal of P with respect to S will touch two circles coaxal with X and X'; we shall prove that these circles are fixed for all positions of P.



Since the circle X passes through the point P (so that the power (XP) is zero) which is a limiting point of the system (p, S, P, P'), it follows by § 329, that

$$(Xp) : (XS) = PC : PO.$$

But (§ 407) $\qquad CP : PO = \rho : k,$

where ρ, k denote the radii of the circles p, T.

Hence, if r denote the radius of X, and θ the angle of intersection of the circles X, p, we shall have

$$\cos\theta = \frac{(XS)}{2rk}.$$

Similarly, if r' denote the radius of X', and θ' the angle of intersection of X', p, we shall have

$$\cos\theta' = -\frac{(X'S)}{2r'k}.$$

Hence the circle p belongs to a system of circles which cut the circles X, X' at constant angles. Therefore (§ 330) the circle p will touch two fixed circles coaxal with X and X'.

The circles enveloped by p are called the *reciprocal* pair of circles corresponding to the pair X, X'.

It is evident that these circles are a pair of inverse circles with respect to the circle of reciprocation.

410. Let x, x' denote the reciprocal circles of the circles X, X'. Then it is evident that if either of the latter is a straight line, each of the circles x, x' will touch the circle T. Also if the circles X, X' are point-circles, the circles x, x' will evidently coincide with the circle which is the reciprocal of the points.

411. *To construct the reciprocal pair of circles of a given pair of circles which are inverse with respect to the circle of reciprocation.*

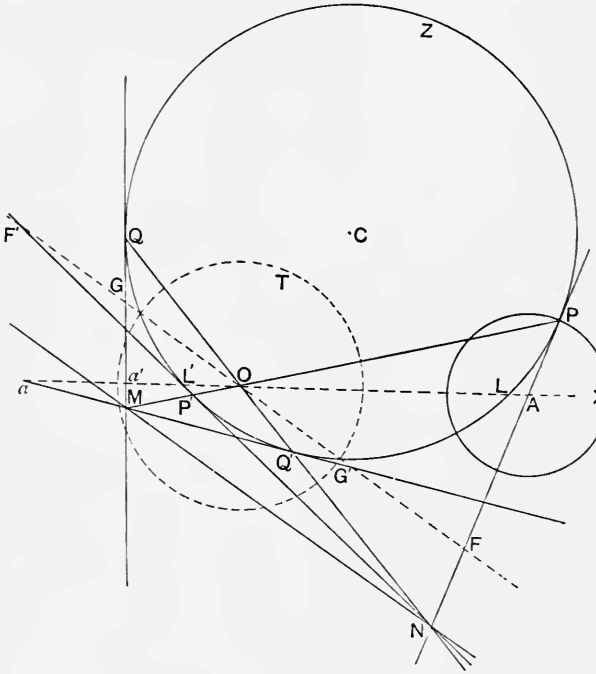
Let X be any given circle, X' the inverse circle with respect to S; let L, L' be the limiting points of X and X'; and let P, P' be a pair of inverse points on them. The points P, P', L, L' are concyclic: let Z denote the circle which passes through them. The circle Z evidently cuts orthogonally the circles S, X, and p, the reciprocal of the point P. Hence, if Q, Q' be the points in which Z cuts p, Q and Q' will be the points in which p touches the circles x, x', which are the reciprocal pair of X and X'.

Let M be the centre of the circle p. Then, since $\{MO, PP'\}$ is harmonic (§ 407), it follows that O and M are conjugate points with respect to Z. Also, since p cuts Z orthogonally, M is the pole of QQ' with respect to Z.

It follows that, if N be the pole of PP' with respect to Z, QQ' must pass through N; and that N will be the centre of the circle q which is the reciprocal of the point-pair Q, Q'.

Hence we have the following construction for drawing the circles x, x', the reciprocal pair of X, X':

Take any point P on X, and draw the circle Z cutting X orthogonally in P and X' orthogonally in P'. Let PP' meet the polar of O with respect to Z in the point M, and let MQ, MQ' be the tangents



from M to Z. Then, if the diameter of X which passes through O cut MQ, MQ' in the points a, a', the circles whose centres are a, a', and whose radii are aQ, $a'Q'$ will be the circles reciprocal to X and X'.

412. Let us suppose that Z is a given circle, and let X be any variable circle cutting Z orthogonally in the point P. Let A be the centre of X, and let O' be the centre of the circle S', which cuts S orthogonally and is coaxal with S and X. The circle S' will also cut Z orthogonally, and therefore O' must lie on the radical axis of S and Z, that is to say the locus of O', for different positions of X, is the polar of O with respect to Z.

Now X will be the *positive* circle of the pair X, X', when its centre A does *not* lie between O and O' (§ 406). Hence, if a line FF' (see fig. § 411) be drawn through the point O parallel to MN, the polar of O, cutting the lines NP, NP' in F and F', we see that the circle X will be the positive circle of the pair X, X', provided its centre does *not* lie between N and F.

Let x denote the positive, and x' the negative circle of the pair x, x'. Then x and x' cut Z orthogonally in the points Q, Q'. If Q denote the positive point of the pair Q, Q', we see from the figure (§ 411) that:

i. When A lies between P and F, x will cut Z in Q';

ii. When A lies between F and N, x will cut Z in Q;

iii. When A has any other position on PN, x will cut Z in Q.

Again, let us enquire which of the circles X, X', x, x' have negative radii, when A has different positions on the line PN. The radical axis of the system (X, X', x, x') evidently cuts the line OA in a point which lies on the circle whose diameter is OC, where C is the centre of the circle Z. The tangent to this circle at O is the line OF. Therefore, when A lies on the same side of F as the point P, the radius of X is negative.

Also we see that the radius of X' will be negative when A' lies on the opposite side of F' to P'; and that the radius of x or x' will be negative when a or a' lies on the same side of G as Q, or on the opposite side of G' to Q'.

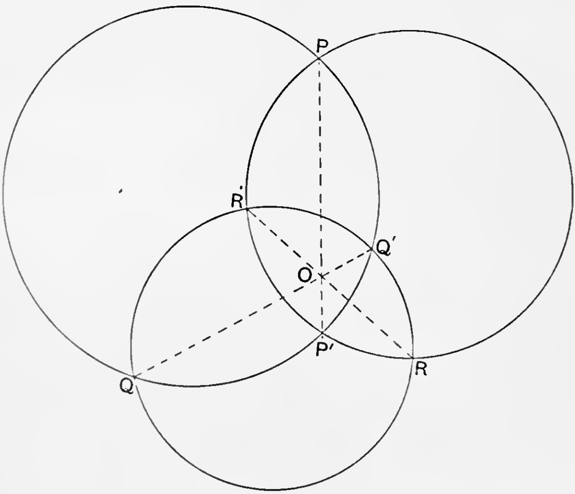
413. Let X, Y be any two circles touching at the point P (fig. § 411); let x and x' be the positive and negative circles of the pair reciprocal to X, and let y and y' be the positive and negative circles of the pair reciprocal to Y. Let A, B be the centres of X, Y. Then we see that the positive circles x, y will touch (i) at the point Q' when A and B both lie between P and F; (ii) at the point Q provided that neither A nor B lies between P and F. In either case the circles X and Y must touch *internally* at the point P, where the word *internally* has a generalised meaning in accordance with the convention stated in § 406.

Again if A lie between P and F, and if B do not lie between P and F, that is to say if the circles X, Y touch externally, it follows that the positive circle x will touch the negative circle y' at the point Q', and that y will touch x' at the point Q.

Hence we have the theorem: *When two circles touch internally, the positive reciprocal circles touch each other, and likewise the negative reciprocal circles touch; but when the circles touch* *externally, the positive reciprocal circle of either touches the negative reciprocal circle of the other.*

We can evidently determine the nature of the contacts of the reciprocal circles by considering whether the given circles are positive or negative circles. Thus, when the given circles X, Y are both positive, or both negative circles, the reciprocal circles must touch internally; and when one of the circles X, Y is a positive circle and the other a negative circle the reciprocal circles must touch externally.

414. To illustrate the use of the method of circular reciprocation, let us consider the case of three given circles intersecting in the three pairs of points A, A'; B, B'; C, C'; and having an imaginary radical circle. Then, if we take the radical circle of the system as the circle of reciprocation, the reciprocals of the point-pairs A, A'; B, B'; C, C'; will be three circles having the circle of reciprocation for their radical circle, and intersecting in the point-pairs P, P'; Q, Q'; R, R'; which will evidently be the reciprocals of the given circles. Again the reciprocals of the group of tangent circles of the given circles will obviously be the group of circumcircles of the reciprocal system. Hence, the properties of the group of tangent circles of a given system of three circles must correspond reciprocally to the properties of the group of circumcircles of such a system.



Let T, T'; T_1, T_1'; T_2, T_2'; T_3, T_3' denote the pairs of tangent circles of the given system; and let H, H'; H_1, H_1'; H_2, H_2'; H_3, H_3' denote the pairs of Hart circles of the system. Then, if P, Q, R be the positive points of the pairs of points in which the reciprocal circles intersect, it is easy to see that the circumcircles PQR, $P'QR$, $PQ'R$, PQR' will be the positive reciprocal circles of the pairs T, T'; T_1, T_1'; T_2, T_2'; T_3, T_3'; respectively. Let K, K_1, K_2, K_3 denote respectively the positive reciprocal circles, and K', K_1', K_2', K_3' the negative reciprocal circles of the pairs of Hart circles of the given system. Then, since H touches T internally, and T_1, T_2, T_3 externally (§ 404), it follows by the last article that K must touch the circumcircles PQR, $PQ'R'$, $P'QR'$, $P'Q'R$. Also, since H_1 touches T externally and T_1, T_2', T_3' internally, it follows that K_1' must touch the circles PQR, $PQ'R'$, $P'QR'$, and $P'Q'R$. Similarly it follows that the circles K_2', K_3' must touch the same four circles.

Hence the four circumcircles PQR, $PQ'R'$, $P'QR'$, $P'Q'R$, have four common tangent circles; that is to say any one may be considered as a Hart circle of the system formed by the other three.

Similarly we may show that the circles $P'Q'R'$, $P'QR$, $PQ'R$, PQR' have four common tangent circles (cf. § 401).

Mr A. Larmor was the first, I believe, to state the theorem in § 401, and to point out the reciprocal relation which exists between the circumcircles and the tangent circles of a system of three circles, in a paper communicated to the British Association in 1887. The theorem stated above in § 413, although arrived at independently, is merely the equivalent in plane geometry of Lemmas (a) and (β) given in his paper on 'Contacts of systems of circles,' *London Math. Soc. Proc.* Vol. XXIII., pp. 136—157. In this paper the subject is treated at greater length than in this treatise.

CHAPTER XVI.

THEORY OF CROSS RATIO.

Cross Ratios of ranges and pencils.

415. If P be any point on the line AB, the ratio $AP : BP$ is called the *ratio* of the point P with respect to the points A and B.

The ratio of the ratios of two points P and Q with respect to the points A and B is called the *cross ratio* of the points P, Q with respect to A and B; or briefly the cross ratio of the range $\{AB, PQ\}$.

It will be convenient to use the notation $\{AB, PQ\}$ to mean the cross ratio of the range $\{AB, PQ\}$, so that we have

$$\{AB, PQ\} = AP \, . \, BQ : AQ \, . \, BP.$$

In this definition it is necessary to observe the order in which the points are taken.

Now four points may be taken in twenty-four different orders; that is to say, four collinear points determine twenty-four ranges. Thus the points A, B, C, D determine the ranges:

$$\begin{array}{llll}
\{AB, CD\}, & \{BA, DC\}, & \{CD, AB\}, & \{DC, BA\}, \\
\{AB, DC\}, & \{BA, CD\}, & \{DC, AB\}, & \{CD, BA\}, \\
\{AC, BD\}, & \{CA, DB\}, & \{BD, AC\}, & \{DB, CA\}, \\
\{AC, DB\}, & \{CA, BD\}, & \{DB, AC\}, & \{BD, CA\}, \\
\{AD, BC\}, & \{DA, CB\}, & \{BC, AD\}, & \{CB, DA\}, \\
\{AD, CB\}, & \{DA, BC\}, & \{CB, AD\}, & \{BC, DA\}.
\end{array}$$

From the definition it is evident that the four ranges in each row of this scheme have the same cross ratio. That is to say: *If any two points of a range be interchanged, the cross ratio of the* *range is unaltered, provided that the other two points are also interchanged.*

Again, we have from the definition,

$$\{AB, CD\} \cdot \{AB, DC\} = 1;$$
$$\{AC, BD\} \cdot \{AC, DB\} = 1;$$
$$\{AD, BC\} \cdot \{AD, CB\} = 1.$$

And, since A, B, C, D are four collinear points, so that

$$AB \cdot CD + AC \cdot DB + AD \cdot BC = 0,$$

we have
$$\{AB, CD\} + \{AC, BD\} = 1;$$
$$\{AB, DC\} + \{AD, BC\} = 1;$$
$$\{AC, DB\} + \{AD, CB\} = 1.$$

Hence, if $\{AB, CD\} = \kappa$, we have

$$\{AB, CD\} = \kappa, \qquad \{AB, DC\} = \frac{1}{\kappa},$$
$$\{AC, BD\} = 1 - \kappa, \quad \{AC, DB\} = \frac{1}{1-\kappa},$$
$$\{AD, BC\} = \frac{\kappa - 1}{\kappa}, \quad \{AD, CB\} = \frac{\kappa}{\kappa - 1}.$$

416. If the two points A and B coincide, it is obvious that

$$AC \cdot BD = BC \cdot AD;$$

and therefore that
$$\{AB, CD\} = 1.$$

In this case we have

$$\{AC, BD\} = \{AD, BC\} = 0,$$

and
$$\{AC, DB\} = \{AD, CB\} = \infty.$$

Conversely, if $\kappa = 0$, we have

$$\{AC, BD\} = \{AD, BC\} = 0;$$

and therefore
$$AB \cdot CD = AB \cdot DC = 0.$$

Therefore either A and B coincide, or else C and D coincide.

Hence, if the cross ratio of the range $\{AB, CD\}$ have the value 1, 0, or ∞, two of the points must coincide.

417. If $\{AB, CD\} = -1$, the range $\{AB, CD\}$ is harmonic. In this case we have,

$$\{AC, BD\} = \{AD, BC\} = 2;$$

and
$$\{AC, DB\} = \{AD, CB\} = \tfrac{1}{2}.$$

Conversely, if the cross ratio of the range $\{AB, CD\}$ have the value -1, 2, or $\frac{1}{2}$; the points A, B, C, D, taken in some order, form a harmonic range.

In fact we have the following theorems:

(i) When $\{AB, CD\} = \{BA, CD\}$, the range $\{AB, CD\}$ is harmonic.

(ii) When $\{AB, CD\} = \{AC, BD\}$, the range $\{AD, BC\}$ is harmonic.

(iii) When $\{AB, CD\} = \{AD, CB\}$, the range $\{AC, BD\}$ is harmonic.

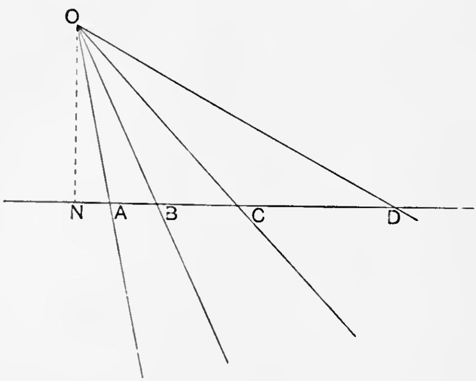
418. There is another special case of some importance. If the cross ratio of $\{AB, CD\}$, that is κ, satisfy the equation $\kappa^2 - \kappa + 1 = 0$, we have

$$\{AB, CD\} = \{AC, DB\} = \{AD, BC\} = \kappa,$$
$$\{AC, BD\} = \{AD, BC\} = \{AB, DC\} = -\kappa^2.$$

In this case the points may be said to form a *bivalent* range.

419. If OA, OB, OC, OD be any four rays of a pencil, the ratio of $\sin AOC \cdot \sin BOD : \sin BOC \cdot \sin AOD$ is called the *cross ratio* of the pencil $O\{AB, CD\}$.

If A, B, C, D be any four points on the same straight line, the pencil $O\{AB, CD\}$, formed by joining these points to any point O, will have the same cross ratio as the range $\{AB, CD\}$.



For, if ON be perpendicular to the line AB, we have,

$$ON \cdot AB = OA \cdot OB \sin AOB.$$

Therefore

$$AC \cdot BD : BC \cdot AD = \sin AOC \cdot \sin BOD : \sin BOC \cdot \sin AOD.$$

420. Ex. 1. If $\{ABCD\}$ be any range, and if the circles described on AB, CD, as diameters intersect at the angles 2θ, shew that

$$\{AB, CD\} = -\cot^2\theta, \quad \{AB, DC\} = -\tan^2\theta,$$
$$\{AC, BD\} = \operatorname{cosec}^2\theta, \quad \{AC, DB\} = \sin^2\theta,$$
$$\{AD, BC\} = \cos^2\theta, \quad \{AD, CB\} = \sec^2\theta.$$

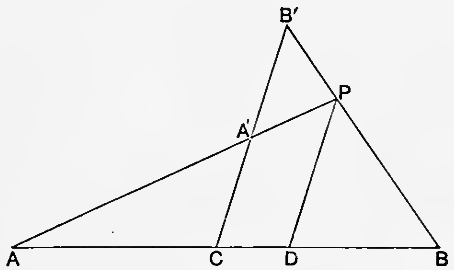
[Casey.]

Ex. 2. If $\{ABXYZ\}$ be any range, show that

$$\{YZ, AB\} . \{ZX, AB\} . \{XY, AB\} = 1.$$

Ex. 3. If A, B, C, I, J be any five coplanar points, show that the product of the cross ratios of the pencils $A\{BC, IJ\}$, $B\{CA, IJ\}$, $C\{AB, IJ\}$, is equal to unity.

421. *Given any three collinear points A, B, C: to find a point D on the same line, such that the range $\{AB, CD\}$ may have a given cross ratio.*



Draw any straight line through the point C, and take on it two points A', B', so that the ratio of $CA' : CB'$ is equal to the given cross ratio. Let the lines AA', BB' meet in P, and let PD be drawn parallel to $A'C$ meeting AB in D. Then D is a point such that the range $\{AB, CD\}$ has the given cross ratio.

For, $\quad AC : AD = CA' : DP$;

and $\quad BC : BD = CB' : DP$.

Therefore $\quad AC . BD : BC . AD = CA' : CB'$.

It is evident that there is only one solution to the problem. Hence it follows that, if

$$\{AB, CD\} = \{AB, CD'\},$$

the points D and D' must coincide.

Also from § 419 we infer that, if $O\{AB, CD\} = O\{AB, CD'\}$, the rays OD, OD' must be coincident.

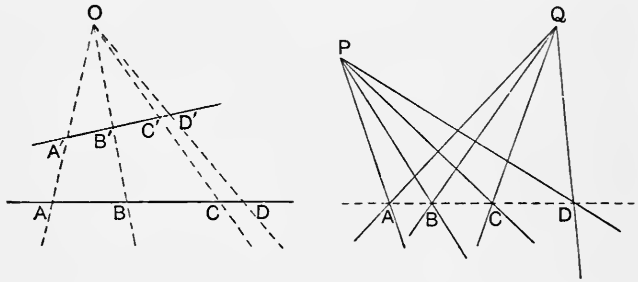
422. Ranges and pencils which have equal cross ratios are said to be *equicross*.

It is often convenient to express the fact that two ranges or pencils are equicross by an equation such as

$$\{AB, CD\} = \{A'B', C'D'\} = O\,\{PQ, RS\}.$$

But when this notation is used, it is necessary to observe the order of the points, or rays.

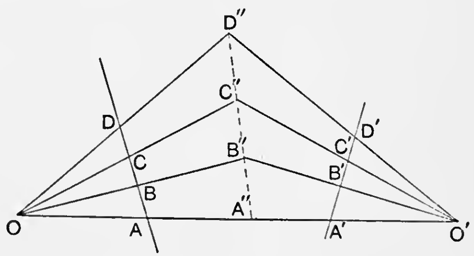
423. In § 419 we proved that, when the points of a range $\{AB, CD\}$ are joined to any point O, the range $\{AB, CD\}$ and the pencil $O\,\{AB, CD\}$ are equicross.



Hence, if the rays of a pencil be cut by two transversals in the points A, A'; B, B'; C, C'; D, D'; the ranges $\{AB, CD\}$, $\{A'B', C'D'\}$ are equicross.

It is also evident that, if $\{ABCD\}$ be any range, and if P and Q be any two points, the pencils $P\{AB, CD\}$, $P'\,\{A'B', C'D'\}$ are equicross.

424. Let $\{AB, CD\}$, $\{A'B', C'D'\}$ by any two equicross ranges, and let O, O' be any two points on the line AA', then the lines OB, OC, OD will intersect the lines $O'B'$, $O'C'$, $O'D'$, respectively, in collinear points.



Let OB, OC meet $O'B'$, $O'C'$, in B'', C''; and let $B''C''$ meet AA' in A''. Let OD, $O'D'$ meet $B''C''$ in D'', D''', respectively.

Then we have $\{A''B'', C''D''\} = \{AB, CD\}$,

and $\{A''B'', C''D'''\} = \{A'B', C'D'\}$.

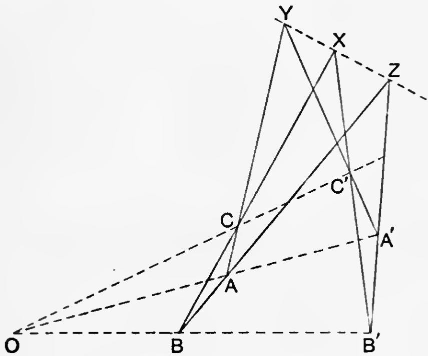
But, by hypothesis, $\{AB, CD\} = \{A'B', C'D'\}$.

Therefore $\{A''B'', C''D''\} = \{A''B'', C''D'''\}$.

Hence, by § 421, the points D'', D''' must coincide; that is to say, the lines OD, OD' intersect in a point on the line $A''B''C''$.

The theorem of this article may also be stated in the form: *If two equicross pencils have a common ray, they will also have a common transversal.*

425. Ex. 1. If ABC, $A'B'C'$ be two triangles such that AA', BB', CC' are concurrent, the corresponding sides of the triangles will intersect in collinear points (§ 161).

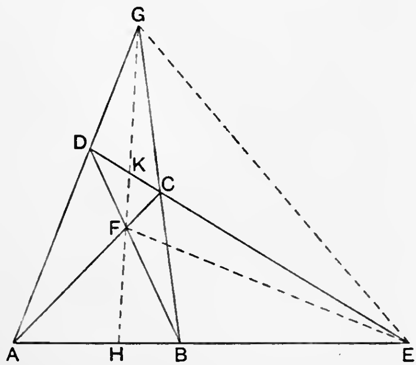


Let AA', BB', CC' meet in O; and let BC, $B'C'$ intersect in the point X. Then we have

$$A\{BC, OX\} = 0\{BC, AX\} = 0\{B'C', A'X\} = A'\{B'C', OX\}.$$

Hence by § 424, AB, AC will intersect $A'B'$, $A'C'$, in points which are collinear with X.

Ex. 2. If $ABCD$ be any tetrastigm, and if E, F, G be respectively the



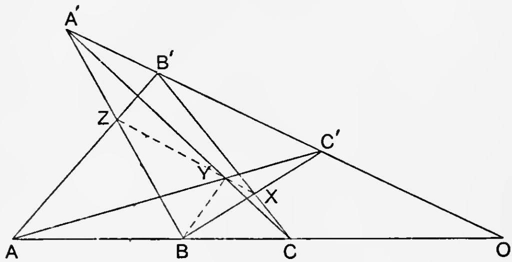
points of intersection of AB, CD; AC, BD; AD, BC; show that the pencil $G\{EF, AB\}$ is harmonic (§ 141).

Let GF cut AB in H, and CD in K. Then we have,

$$G\{EF, AB\}=F\{EH, AB\}=F\{EK, CD\}=G\{EF, BA\}.$$

Therefore, by § 417, the pencil $G\ \{EF, AB\}$ is harmonic.

Ex. 3. If $\{ABC\}$, $\{A'B'C'\}$ be two ranges on different lines, show that the points of intersection of the pairs of lines BC', $B'C$; CA', $C'A$; AB', $A'B$; will be collinear. (§ 157.)



Let BC', $B'C$ intersect in X; CA', $C'A$ in Y; and AB', $A'B$ in Z. Join YX, YZ, YB.

Then it is easy to see that,

$$Y\{AA', BZ\}=A\{C'A', BB'\},$$

since these pencils have the common transversal $A'B$.

Similarly, $$Y\{C'C, BX\}=C\{C'A', BB'\}.$$

But it is evident that

$$A\{C'A', BB'\}=C\{C'A', BB'\}.$$

Hence, $$Y\{AA', BZ\}=Y\{C'C, BX\}.$$

Therefore the points X, Y, Z must be collinear.

Ex. 4. If $\{abc\}$, $\{a'b'c'\}$ be any two pencils, show that the lines joining the pairs of points bc', $b'c$; ca', $c'a$; ab', $a'b$; will be concurrent.

Ex. 5. The sides of a triangle PQR pass respectively through the fixed points A, B, C; and two of the vertices, Q and R, move on fixed straight lines which intersect in the point O. If the points O, B, C be collinear, show that the locus of the point P will be a straight line.

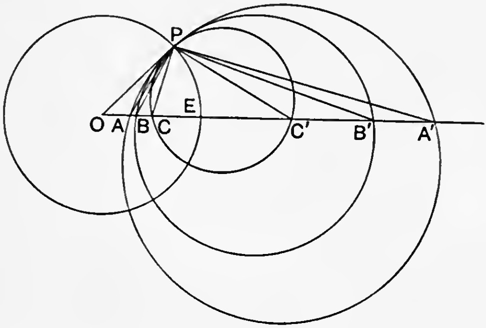
Involution.

426. *If $\{AA', BB', CC'\}$ be a range in involution the ranges $\{AA', BC\}$, $\{A'A, B'C'\}$ are equicross and conversely.*

Let O be the centre of the involution, then by definition (§ 66) we have

$$OA \cdot OA'=OB \cdot OB'=OC \cdot OC'.$$

(i) Let us suppose that each point lies on the same side of he centre as its conjugate. Then the double points of the range re real.

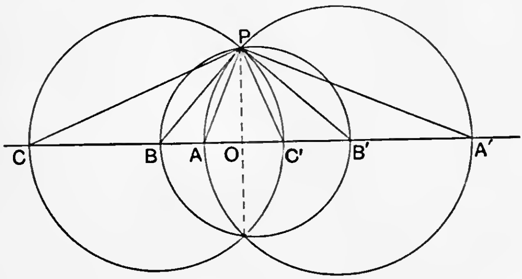


Let E be one of the double points, and let P be any point on he circle whose centre is O and radius OE. The circle which cuts his circle orthogonally and passes through A will pass through A', ince $OA \,.\, OA' = OE^2 = OP^2$. It is evident therefore that the ircles PAA', PBB', PCC' will touch each other at P.

Hence the angles OPA, OPB, OPC, &c., are equal to the ngles $PA'O$, $PB'O$, $PC'O$, &c.; and therefore the angles APA', BPB', &c. have a common bisector, the tangent to the circles t P.

Hence the pencils $P\{AA', BC\}$, $P\{A'A, B'C'\}$ are equicross.

(ii) Let us suppose that each point of the range lies on the pposite side of the centre to its conjugate. Then the double oints are imaginary.



Let the circles described on AA', BB' as diameters intersect in P. Then the angle CPC' will also be a right angle (§ 80, Ex. 8).

It is evident that any segment such as AB subtends an angle at P equal or supplementary to that subtended by the conjugate segment $A'B'$.

Hence the pencils $P\{AA', BC\}$, $P\{A'A, B'C'\}$ are equicross.

Conversely, if the ranges $\{AA', BC\}$, $\{A'A, B'C'\}$ be equicross, the range $\{AA', BB', CC'\}$ will be in involution.

For if not, let us find the point C'', the conjugate point of C, in the involution determined by the point pairs A, A'; B, B' (§ 68). Thus we have, because $\{AA', BB', CC''\}$ is a range in involution,

$$\{AA', BC\} = \{A'A, B'C''\}.$$

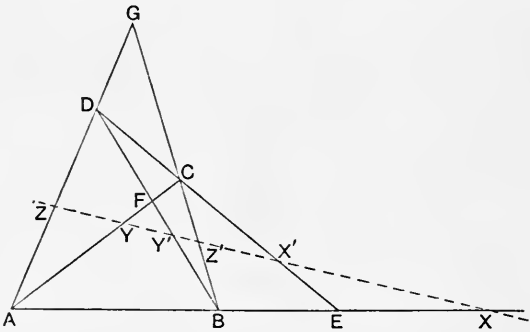
Therefore $$\{A'A, B'C'\} = \{A'A, B'C''\}.$$

Hence the points C', C'' must coincide.

427. Ex. 1. If $\{AA', BB', CC', DD'\}$ be a range in involution, show that the ranges $\{AB, CD\}$, $\{A'B', C'D'\}$ will be equicross.

Show that the converse of this theorem is not true.

Ex. 2. Show that any straight line is cut by the pairs of opposite connectors of a tetrastigm in a system of points which form a range in involution.



We have, $$A\{XX', YZ\} = \{EX', CD\};$$

and $$B\{X'X, Y'Z'\} = \{X'E, DC\} = \{EX', CD\}.$$

Therefore $$\{XX', YZ\} = \{X'X, Y'Z'\},$$

and therefore $\{XX', YY', ZZ'\}$ is a range in involution.

Ex. 3. Show that if A, A'; B, B'; C, C'; be the pairs of opposite vertices of a tetragram, and if O be any other point, the pencil $O\{AA', BB', CC'\}$ will be in involution.

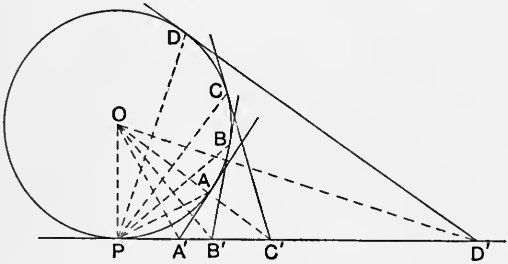
Ex. 4. The middle points of the diagonals of a tetragram lie on a line called the diameter of the tetragram. Show that the diameters of the five tetragrams formed by five straight lines are concurrent.

Cross ratio properties of a circle.

428. *Four fixed points on a circle subtend a pencil, whose cross ratio is constant, at all points on the circle.*

If A, B, C, D be four fixed points on a circle, the pencil $P\{AB, CD\}$ has a constant cross ratio for all positions of the point P on the circle since the angles APB, APC, &c. are of constant magnitude.

429. *If the tangents at four fixed points A, B, C, D on a circle, intersect the tangent at a variable point P, in the points A', B', C', D', the range $\{A'B', C'D'\}$ and the pencil $P\{AB, CD\}$ are equicross.*

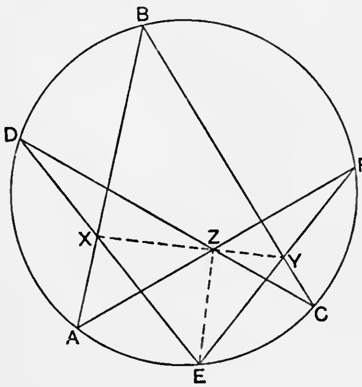


Let O be the centre of the circle. Then, since the angles $PA'O$, $PB'O$ are respectively complementary to half the angles AOP, BOP, the angle $A'OB'$ is equal to half the angle AOB, and is therefore equal to the angle APB.

Hence we have, $\quad P\{AB, CD\} = O\{A'B', C'D'\}$.

Hence, since the cross ratio of the pencil $P\{AB, CD\}$ is constant for all positions of P, it follows that: *Four fixed tangents to a circle determine on a variable tangent a range whose cross ratio is constant.*

430. Ex. 1. If A, B, C, D, E, F be any six points on a circle, show that



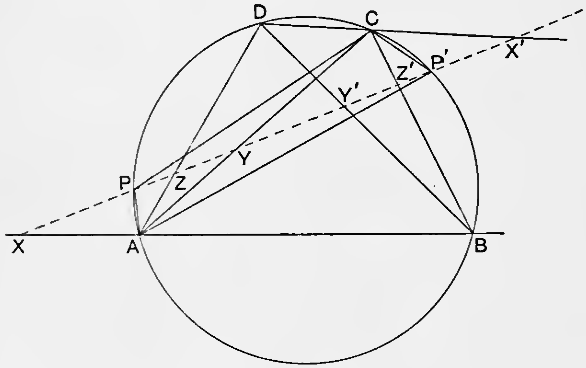
the points of intersection of the pairs of lines AB, DE; BC, EF; CD, FA; are collinear. (Pascal's theorem. Cf. § 181.)

Let AB, DE intersect in X; BC, EF in Y; and CD, FA in Z.

Then the pencils $Z\{FY, CE\}$, $C\{FB, DE\}$ are equicross because they have the common transversal EF; and the pencils $Z\{AX, DE\}$, $A\{FB, DE\}$ are equicross because they have the common transversal ED. But the pencils $A\{FB, DE\}$, $C\{FB, DE\}$ are equicross by § 428. Therefore $Z\{FY, CE\}=Z\{AX, DE\}$. Hence the points X, Y, Z are collinear.

Ex. 2. If a, b, c, d, e, f be six tangents to a circle, prove that the lines joining the pairs of points ab, de; bc, ef; cd, fa; are concurrent. (Brianchon's theorem.)

Ex. 3. Any straight line is cut in involution by a circle, and the opposite connectors of an inscribed tetrastigm (§ 273).



Let $ABCD$ be a tetrastigm inscribed in a circle, and let a straight line be drawn, cutting the circle in P, P', and the pairs of connectors of the tetrastigm in the points X, X'; Y, Y'; Z, Z'. Join AP, AP', CP, CP'.

Then we have $$A\{PP', XZ\}=A\{PP', BD\};$$

and $$C\{P'P, X'Z'\}=C\{P'P, DB\}=C\{PP', BD\}.$$

But by § 428, $$A\{PP', BD\}=C\{PP', BD\}.$$

Therefore $$A\{PP', XZ\}=C\{P'P, X'Z'\}.$$

Hence the range $\{PP', XX', ZZ'\}$ is in involution.

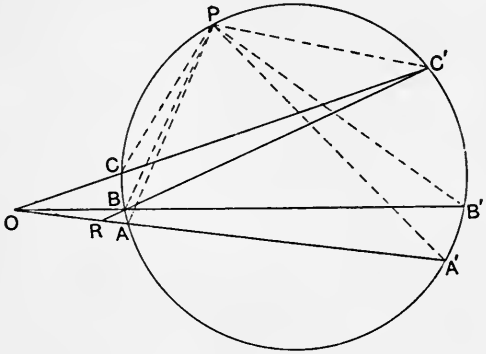
Ex. 4. Show that if A, A'; B, B'; C, C' be the pairs of opposite vertices of a tetragram circumscribed to a circle, and if the tangents at the points P, P' intersect in the point O, the pencil $O\{PP', AA', BB', CC'\}$ is in involution (§ 279).

Ex. 5. If through any point O three straight lines be drawn cutting a circle in the points A, A'; B, B'; C, C'; and if P be any other point on the circle, show that the pencil $P\{AA', BB', CC'\}$ will be in involution.

Let BC' cut AA' in the point R. Then we have,

$$P\{AA', BC\} = C'\{AA', BC\} = \{AA', RO\};$$

and
$$P\{A'A, B'C'\} = B\{A'A, B'C'\} = \{A'A, OR\}.$$



But,
$$\{AA', RO\} = \{A'A, OR\}.$$

Therefore
$$P\{AA', BC\} = P\{A'A, B'C'\},$$

and therefore the pencil $P\{AA', BB', CC'\}$ is in involution.

Ex. 6. If any straight line drawn through a fixed point O on a circle, cut the sides of an inscribed triangle ABC in the points A', B', C', and the circle in the point P, show that the range $\{PA', B'C'\}$ will have a constant cross ratio.

Ex. 7. Two points P, Q are taken on the circumcircle of the triangle ABC, so that the cross ratios of the pencils $Q\{PA, BC\}$, $P\{QA, CB\}$ are equal. Show that the lines BC, PQ intersect in a point on the tangent at the point A.

Ex. 8. A chord PQ of the circumcircle of the triangle ABC cuts the sides of the triangle in the points X, Y, Z. Show that if the range $\{QX, YZ\}$ have a constant cross ratio, the point P will be a fixed point.

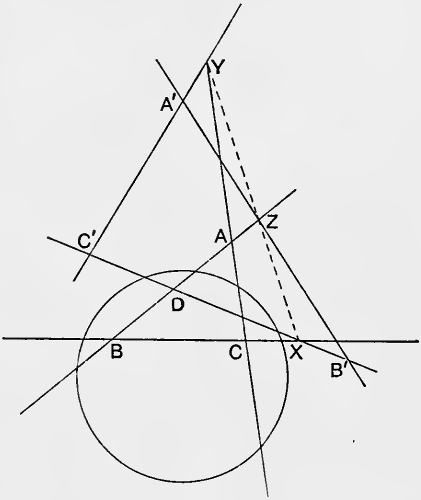
Ex. 9. Four fixed tangents to a circle form a tetragram whose pairs of opposite vertices are A, A'; B, B'; C, C'. If the tangent at any point P meet AA' in p, and if PB, PB', PC, PC' meet AA' in the points b, b', c, c', respectively, show that

$$Ap^2 : A'p^2 = Ab \,.\, Ab' : A'b \,.\, A'b' = Ac \,.\, Ac' : A'c \,.\, A'c'.$$

431. *If four points be collinear the range formed by them is equicross with the pencil formed by the polars of the points with respect to a circle.*

Let A, B, C, D be any four collinear points, and let PA', PB', PC', PD' be the polars of A, B, C, D, with respect to a circle whose centre is O. Then, since the lines PA', PB', &c., are perpendicular to the lines OA, OB, &c., it follows that the pencils $O\{AB, CD\}$, $P\{A'B', C'D'\}$ are equicross.

432. Ex. 1. Any triangle and its conjugate with respect to a circle are in perspective (§ 267).



Let $A'B'C'$ be the conjugate triangle of ABC; and let the corresponding sides intersect in the points X, Y, Z. Then, if $B'C'$ cut AB in the point D, the point D will be the pole of AC'. Also X is the pole of AA'. Therefore by § 431, the range $\{B'X, DC'\}$ is equicross with the pencil $A\{YA', C'B\}$. But, $\{B'X, DC'\}=Z\{A'X, AC'\}$; and $A\{YA', C'B\}=Z\{YA', C'A\}$. Therefore $Z\{A'X, AC'\}=Z\{YA', C'A\}=Z\{A'Y, AC'\}$. Hence the points X, Y, Z are collinear, and therefore the triangles ABC, $A'B'C'$ are in perspective.

Ex. 2. The tangents to a circle at the points A, B, C, form the triangle $A'B'C'$, and the tangent at any point P meets the sides of the triangle ABC in the points a, b, c, and the sides of the triangle $A'B'C'$ in the points a', b', c': show that $\{Pa, bc\}=\{Pa', b'c'\}$.

Ex. 3. The tangent at any point P on a circle which touches the sides of the triangle ABC, meets a fixed tangent in T. Show that the pencil $T\{PA, BC\}$ has a constant cross ratio.

Ex. 4. On the tangent at any point P on the inscribed circle of the triangle ABC, a point Q is taken such that the pencil $Q\{PA, BC\}$ has a constant cross ratio. Show that the locus of Q is a straight line which touches the circle.

Ex. 5. If ABC, $A'B'C'$ be any two triangles self conjugate with respect to a circle, show that the pencils $A\{BC, B'C'\}$, $A'\{BC, B'C'\}$ will be equicross.

Homographic ranges and pencils.

433. Any two ranges $\{ABC...\}$, $\{A'B'C'...\}$, situated on the same, or on different lines, are said to be *homographic*, when the cross ratio of any four points of one range is equal to the cross ratio of the corresponding points of the other range.

Similarly, two pencils are said to be *homographic* when the cross ratio of any four rays of one pencil is equal to that of the corresponding rays of the other pencil; and a pencil is said to be *homographic* with a range under similar circumstances.

434. Any two ranges which have a *one to one correspondence* (that is, when to each point of one range corresponds one, and only one, point of the other), are homographic.

For, if A and B be two fixed points of one of the ranges, and A', B' the corresponding points of the other range, any two corresponding points P, P' of the ranges must be such that the ratios $AP : BP$, $A'P' : B'P'$, have a constant ratio.

Hence, if P, Q be any two points of the range $\{AB...\}$, and P', Q' the corresponding points of the range $\{A'B'...\}$, we shall have

$$\{AB, PQ\} = \{A'B', P'Q'\}.$$

That is to say, the ranges will be homographic.

Similarly, if two pencils, or if a pencil and a range, have a one to one correspondence, they will be homographic.

435. Ex. 1. Show that a variable tangent to a circle determines two homographic ranges on any two fixed tangents.

Ex. 2. Show that a range of points on any straight line and their polars with respect to a circle form two homographic systems.

Ex. 3. Show that the polars with respect to a fixed triangle of a range of points on any straight line cut any other straight line in a range which is homographic with the former.

Ex. 4. Show that if two homographic pencils have a common ray they will also have a common transversal.

436. Let $\{ABC...\}$, $\{A'B'C'...\}$ be any two homographic ranges on different lines; and let O, O' be the points of each range which correspond respectively to the point at infinity on the other. Then we shall have

$$\{AB, O\infty\} = \{A'B', \infty' O'\},$$

where ∞, ∞' denote the points at infinity on the lines AB, $A'B'$.

That is, $$\frac{AO \, . \, B\infty}{BO \, . \, A\infty} = \frac{A'\infty' \, . \, B'O'}{B\infty' \, . \, A'O'}.$$

Therefore $$\frac{AO}{BO} = \frac{B'O'}{A'O'};$$

that is $$AO \, . \, A'O' = BO \, . \, B'O'.$$

Hence, if P, P' be any pair of corresponding points we shall have,

$$OP \, . \, O'P' = \text{constant}.$$

The points O, O' are called the *centres* of the ranges.

It is evident that if the lines be superposed, so that the points O and O' coincide, the pairs of corresponding points will be conjugate couples of a range in involution.

437. When two homographic ranges $\{ABC...\}$, $\{A'B'C'...\}$ are situated on the same straight line, there will be two points of one range which coincide with the corresponding points of the other range. For, if O, O' be the centres of the ranges, and S a point of the range $\{ABC...\}$ which coincides with the corresponding point of the range $\{A'B'C'...\}$, we shall have by the last article,

$$OS \, . \, O'S = OA \, . \, O'A' = OB \, . \, O'B' = \&c.$$

Thus S will be a point whose power, with respect to the circle described on OO' as diameter, is constant. But the locus of such a point is a circle whose centre is the middle point of OO'. Hence there are two points S, S' which are coincident corresponding points.

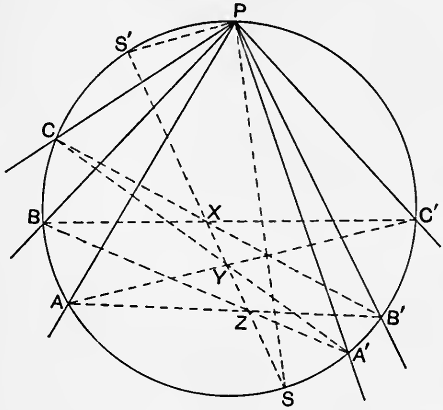
These points are called the *double points* of the ranges.

By joining the points of the ranges to any point not on the line it follows that any two homographic pencils having a common vertex will have two *double rays*, that is to say rays which, considered as belonging to one pencil, coincide with the corresponding rays of the other pencil.

438. *To find the double rays of a pair of homographic pencils which have a common vertex.*

Let $P\{ABC...\}$, $P\{A'B'C'...\}$, be any two homographic pencils. Let a circle be described passing through P and cutting the rays of the pencils in the points A, B, C...; and A', B', C',...; respectively.

Then, if X, Y, Z be the points of intersection of the pairs of ines BC', $B'C$; CA', $C'A$; AB', $A'B$; we know that the points X, Y, Z will be collinear. (Pascal's theorem.)



Let XYZ cut the circle in S and S'. Then we shall have

$$P\{A'B', C'S\} = A\{A'B', C'S\} = A\{A'Z, YS\}$$
$$= A'\{AZ, YS\} = A'\{AB, CS\} = P\{AB, CS\}.$$

Therefore PS will be one double ray of the pencils $P\{ABC\ldots\}$, $P\{A'B'C'\ldots\}$; and similarly PS' will be the other double ray.

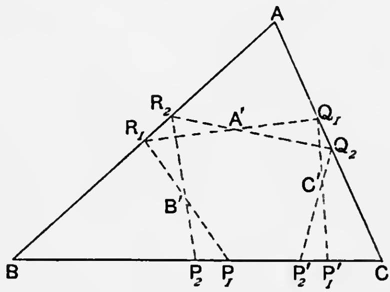
439. Ex. 1. Show that, if S, S' be the double points of the homographic ranges $\{ABC\ldots\}$, $\{A'B'C'\ldots\}$, the circle whose diameter is SS' will be coaxal with the circles whose diameters are AB' and $A'B$.

Ex. 2. If PS, PS' be the double rays of two homographic pencils $P\{ABC\ldots\}$, $P\{A'B'C'\ldots\}$, show that the pencil $P\{SS', AB', A'B\}$ will be in involution.

Ex. 3. Show how to find a point on each of two given straight lines such that the line joining them shall subtend given angles at two given points.

Ex. 4. Show how to inscribe a triangle in a given triangle, such that the sides of the triangle shall pass through three given points.

Let ABC be the given triangle; A', B', C' the given points. Through A'



draw any line cutting CA in Q and AB in R; and let QC', RB' meet BC in P' and P. Then, as the line QR turns about the point A', the points P, P' will form two homographic ranges. If the double points of these ranges be S and S', it is evident that the lines SB', SC' will cut AB, AC in points collinear with A'. Thus the problem admits of two solutions.

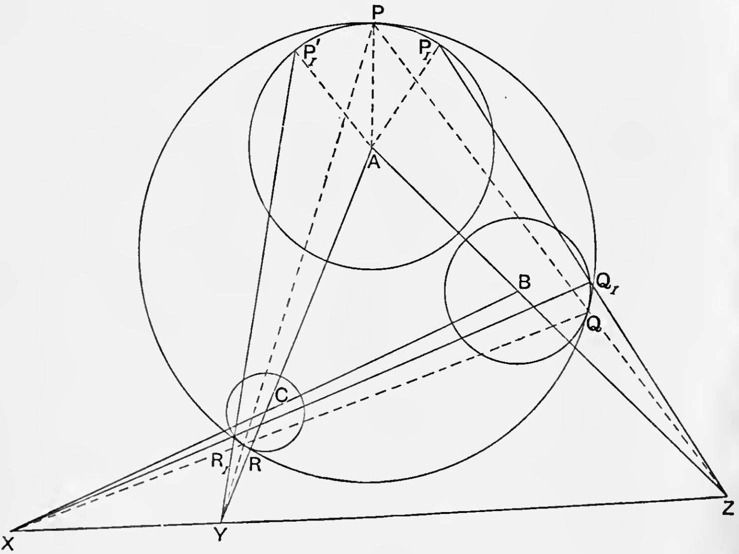
Ex. 5. Inscribe in a circle a triangle whose sides shall pass through three given points.

Ex. 6. Inscribe in a circle a triangle whose sides shall touch three given circles.

Ex. 7. Show how to find two corresponding pairs of points P, Q; P', Q'; on the homographic ranges $\{ABCPQ...\}$, $\{A'B'C'P'Q'...\}$, such that PP', QQ' shall pass through a given point O.

Ex. 8. Describe a circle which shall touch three given circles*.

Let A, B, C be the centres of the given circles; and suppose that a circle can be drawn touching them at the points P, Q, R, respectively. Then the triangles PQR, ABC are in perspective and have one of the homothetic axes of the given circles for their axis of perspective. Let X, Y, Z be the homothetic



centres of the given circles on this axis of perspective. Through X draw any straight line cutting the circles whose centres are B and C in the points Q_1 and R_1, and let ZQ_1, YR_1 cut the circle whose centre is A in P_1 and P'_1. Then, if a pencil of lines be drawn through X, it is clear that the pencils $A\{P_1\}$, $A\{P'_1\}$ will be homographic. Hence if AP and AP' be the double rays of these pencils, P and P' will be the points of contact with the circle, whose centre is A, of a pair of circles which touch the given circles.

* This method is due to Casey.

NOTES.

Page 78, § 134. In connection with the Brocardian geometry of the riangle, McClelland's treatise "On the geometry of the circle" (1891) may e consulted. He deduces several theorems from the theorem that, if P, Q, R e any points on the sides BC, CA, AB of a triangle, the circles AQR, BRP, $'PQ$ will have a common point.

Page 113, § 180, ex. 12. In conection with this subject a paper by Ir Jenkins "On some geometrical proofs of theorems connected with the ıscription of a triangle of constant form in a given triangle," *Quarterly 'ournal*, Vol. XXI., p. 84, (1886) may be consulted.

Page 140, § 223. The theory of similar figures is chiefly due to Neuberg nd Tarry, whose papers will be found in Mathesis, Vol. II.

Page 145, § 232, ex. 3. See a paper by McCay in the *Trans. Royal Irish cademy*, Vol. XXVIII.

Page 189, § 313. The definition of the *power* of a point with respect to a ircle was first given by Steiner, *Crelle*, Vol. I., p. 164 (1826). Darboux gave ıe definition of the *power of two circles* in a paper published in *Annales de École Normale superieure*, Vol. I. (1872). Clifford also used the same definition in a paper said to have been written in 1866, but published for the rst time in his Collected Mathematical Papers (1882).

Page 206, § 333, ex. 7. The theorem in this example which is afterwards sed to prove Feuerbach's theorem was taken from Nixon, *Euclid Revised*, nd edit. p. 350 (1888). The theorem together with the proof are said to be ue to Prof. Purser, but the proof given by Nixon is invalid. I am informed ıat another proof has been inserted in a new edition of this treatise which to appear shortly. It may be mentioned that an elegant proof by McCay f Feuerbach's theorem is to be found in McClelland's *Geometry of the circle*, 183 (1891). McCay's proof depends on the theorem that the Simson lines f two diametrically opposite points on the circumcircle of a triangle intersect ı a point on the nine-point circle.

Page 235, § 375. This theorem is taken from Casey, *Sequel to Euclid*, 112. It was first stated by Casey (*Phil. Trans.*, Vol. CLXVII.), and the proof ven is attributed by him to McCay.

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