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AN ELEMENTARY TREATISE ON THE  
GEOMETRY OF CONICS.



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AN ELEMENTARY TREATISE

ON THE

GEOMETRY OF CONICS.

BY

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## PREFACE.

THIS work contains elementary proofs of the principal properties of Conics, and is intended for students who proceed to the study of the subject after finishing the first six books of Euclid; the curves have not, therefore, been defined as the sections of a cone, although that method has the sanction of history and antiquity in its favour; and for the same reason, no use has been made of the method of projections.

As regards the arrangement of the subject, I have thought it best to devote separate chapters to the parabola, the ellipse, and the hyperbola. The plan of starting with a chapter on general conics, in which some fundamental propositions are proved by methods applicable to all the three curves, has no doubt the advantage of securing an appearance of brevity. But, I believe, beginners find the subject more intelligible when the properties of the three curves are discussed separately. Besides, in the other method students, and even writers of text-books, are apt to overlook the necessity of modifying an argument on account of the fundamental

difference in the figures of the several curves; see, for instance, Chap. II., Prop. x., and Chap. III., Prop. ix., which are ordinarily proved by identically the same argument. Also, as the properties of the hyperbola are proved, wherever possible, by the same methods as the corresponding properties of the ellipse, it is obvious that this arrangement does not tend to increase the work of the student.

As to the propositions included in each chapter and their sequence, I have not been able to adopt wholly the scheme of any previous writer; but I venture to hope that the book includes all the classical propositions on the subject, arranged in their proper logical order. Every attempt has been made to render the proofs simple and easily intelligible, though I have never sacrificed accuracy to brevity. Thus, for instance, I have not followed the practice of referring to a proposition when the truth of its converse is really assumed—a practice which has, in at least one instance, led to a remarkable error in the treatment of conjugate diameters in a famous text-book. Nor have I attempted to secure a fictitious appearance of conciseness by adding to each proposition a list of corollaries by no means less important than the proposition itself, and freely using them for the purpose of deducing subsequent propositions.

The exercises, of which there are about eight hundred, have been selected with great care; more than six hundred of these are placed under the different propositions from which they may be deduced; they are for the most

part of an elementary character, and have been carefully graduated. Hints and solutions have been liberally added, and these, it is hoped, will prove materially helpful to the student, and render the subject attractive. The attention of the student has also been directed to various methods of graphically describing the curves, including those used in practice by draughtsmen, and some very neat problems have been added from Newton, Book I., Sections iv. and v.

At the end of the table of contents will be found a course of reading suitable for beginners.

CALCUTTA,  
19th April, 1893.



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Propositions marked with an asterisk may be omitted by the beginner. This would leave for a first course of reading—

Chap. I.—Props. i.-ii., iv.-vii., x.-xii., xiv., xvii.-xix., xxiii.-xxv., . . . . . (16)

Chap. II.—Props. i.-v., viii.-xi., xiv.-xix., xxi.-xxiii., xxv., xxvi., xxx., xxxi., xxxiii., xxxiv., . . . . . (24)

Chap. III.—Props. i.-iv., vii.-ix., xii.-xvii., xix.-xxi., xxiii., xxvii.-xxxi., xxxiii.-xxxvi., A—D., . . . . . (30)

GEOMETRY OF CONICS.





# GEOMETRY OF CONICS.

## INTRODUCTION.

A CONIC is a curve traced by a point which moves in a plane containing a fixed point and a fixed straight line, in such a way that its distance from the fixed point is in a constant ratio to its perpendicular distance from the fixed straight line.

The fixed point is called the FOCUS.

The fixed straight line is called the DIRECTRIX.

The constant ratio is called the ECCENTRICITY, and is usually represented by the letter  $e$ .

When the eccentricity is equal to unity, the Conic is called a PARABOLA ( $e = 1$ ).

When the eccentricity is less than unity, the Conic is called an ELLIPSE ( $e < 1$ ).

When the eccentricity is greater than unity, the Conic is called a HYPERBOLA ( $e > 1$ ).

The straight line drawn through the focus perpendicular to the directrix is called the AXIS of the Conic.

The point (or points) in which the axis intersects the Conic is called the VERTEX.

The Conics are so called from the circumstance that they are, and were originally studied as, the plane sections of the surface

of a right circular cone, which is a surface formed by the revolution of a right-angled triangle about one of its sides. This conception does not lead to the simplest way of investigating the properties of Conics, as it necessitates a knowledge of the geometry of solids. In order to restrict the discussion of these curves to the domain of plane geometry, they have been defined as above.

The Conics are said to have been discovered by Menaechmus, a Greek mathematician who flourished about B.C. 350, and were accordingly called after him the "*Menaechmian Triads*." They were first systematically studied by Apollonius of Perga (B.C. 247-205).

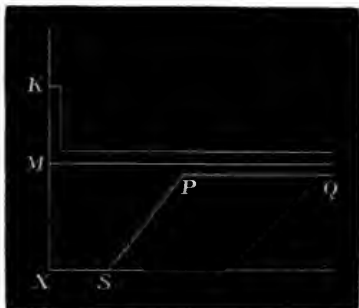
## CHAPTER I.

### THE PARABOLA.

#### DESCRIPTION OF THE CURVE.

WE have seen that the eccentricity of the parabola is unity, that is, the distance of any point on it from the focus is equal to its perpendicular distance from the directrix.

The parabola may be mechanically constructed in the following manner.



Let  $S$  be the focus and  $MX$  the directrix; and let a rigid bar  $KMQ$ , of which the portions  $KM$  and  $MQ$  are at right angles to each other, having a string of the same length as  $MQ$ , fastened at the end  $Q$ , be made to slide

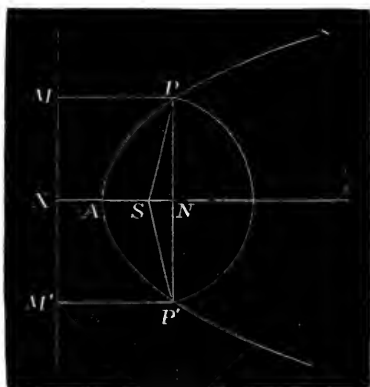
parallel to the axis  $SX$  with the end  $M$  on the directrix; then if the other end of the string be fastened at the focus  $S$ , and the string be kept stretched by means of the point of a pencil at  $P$ , in contact with the bar, it is evident that the point  $P$  will trace out a parabola, since  $SP$  is always equal to  $PM$ .

Ex. A point moves so that the sum of its distances from a fixed point and a fixed straight line is constant. Show that it describes a parabola.

In the above figure, the sum of the distances of  $P$  from  $S$  and the straight line through  $Q$  parallel to  $XK$  is evidently constant.

### PROPOSITION I.

*Given the focus and the directrix of a parabola, to determine any number of points on it.*



Let  $S$  be the focus and  $MXM'$  the directrix. Through  $S$  draw  $SX$  perpendicular to the directrix, and bisect  $SX$  in  $A$ ; then  $A$  is a point on the parabola, since  $SA = AX$ .

Take any point  $N$  in  $SX$  or  $SX$  produced. Through  $N$  draw  $PNP'$  perpendicular to  $XN$ ; with centre  $S$  and

radius equal to  $XN$ , describe a circle cutting  $PNP'$  at  $P$  and  $P'$ ; then  $P$  and  $P'$  shall be points on the parabola.

Draw  $PM$  and  $P'M'$  perpendicular to the directrix.

Then  $PS = XN$ , by construction, and  $PM = XN$ , being opposite sides of a rectangle; therefore  $PS = PM$ . Similarly it may be shown that  $P'S = P'M'$ . Therefore  $P$  and  $P'$  are points on the parabola.

In like manner, by taking any other point in  $SX$ , any number of points on the curve may be determined.

Ex. 1. The parabola is symmetrical with respect to its axis.

This follows from the fact that  $PP'$  is bisected at right angles by  $XS$ .

**Def.** A curve is said to be *symmetrical with respect to a straight line*, if, corresponding to any point on the curve, there is another point on the curve on the other side of the straight line, such that the chord joining them is bisected at right angles by the straight line.

Ex. 2. *Alternative Construction*—Join the focus  $S$  to any point  $M$  on the directrix; draw  $MP$  at right angles to the directrix, and make the angle  $MSP$  equal to the angle  $SMP$ .  $P$  is a point on the parabola.

Ex. 3. *Alternative Construction*.—Bisect  $SM$  in  $E$ , and draw  $EP$  perpendicular to  $SM$ , meeting  $MP$  in  $P$ .  $P$  is a point on the parabola.

For another construction, see Prop. X., Ex. 3.

Ex. 4. Describe a parabola of which the focus and vertex are given.

Ex. 5. Given the focus  $S$ , and two points  $P, Q$  on the parabola, construct it.

The directrix will be a common tangent to the two circles described, with centres  $P, Q$  and radii  $SP, SQ$  respectively.

Ex. 6. The distance of any point inside the parabola from the focus is less than its distance from the directrix.

Ex. 7. The distance of any point outside the parabola from the focus is greater than its distance from the directrix.

Ex. 8. A straight line parallel to the axis of a parabola meets the curve in one point only.

Ex. 9. There is no limit to the distance to which the parabola

may extend on both sides of the axis, so that the parabola is not a *closed* curve.

It is obvious that the point  $N$  may be taken *anywhere* on the axis.

Ex. 10. Any two right lines drawn from the focus to the curve on opposite sides of the axis, and equally inclined to it, are equal; and conversely.

Ex. 11. If  $SM$  meets in  $Y$  the straight line drawn through  $A$  perpendicular to the axis,  $SY = YM$ , and  $PY$  is at right angles to  $SM$  and bisects the angle  $SPM$ .

Ex. 12. If  $SZ$  is drawn at right angles to  $SP$  to meet the directrix in  $Z$ ,  $PZ$  bisects the angle  $SPM$ .

Ex. 13.  $PSp$  is a right line passing through the focus and meeting the parabola in  $P$  and  $p$ .  $PM$  and  $pm$  are perpendicular to the directrix. Show that  $MSm$  is a right angle.

Ex. 14. The locus of the centre of a circle which passes through a given point and touches a given straight line is a parabola.

Ex. 15. The locus of the centre of a circle which touches a given circle and a given straight line is a parabola.

The focus is the centre of the given circle, and the directrix a right line parallel to the given one at a distance from it equal to the radius of the given circle.

Ex. 16.  $PSp$  is a straight line through the focus  $S$ , cutting the parabola in  $P$  and  $p$ .  $PN, pn$  are drawn at right angles to the axis. Prove that  $AN \cdot An = AS^2$ .

Ex. 17. Given the directrix and two points on the curve, construct it. Show that, in general, two parabolas satisfy the conditions.

Ex. 18. If from a point  $P$  of a circle,  $PC$  be drawn to the centre  $C$ , and  $R$  be the middle point of the chord  $PQ$  drawn parallel to a fixed diameter  $ACB$ ; then the locus of the intersection of  $CP$  and  $AR$  is a parabola.

The focus will be at  $C$ , and the directrix will be the tangent to the circle at  $A$ .

### PROPERTIES OF CHORDS.

**Def.** The *chord* ( $QQ'$ ) of a conic is the finite straight line joining any two points ( $Q, Q'$ ) on the curve.

**Def.** A *focal chord* ( $PSp$ ) is any chord drawn through the focus ( $S$ ).

**Def.** The *latus rectum* ( $LL'$ ) of a conic is the focal chord drawn at right angles to the axis.

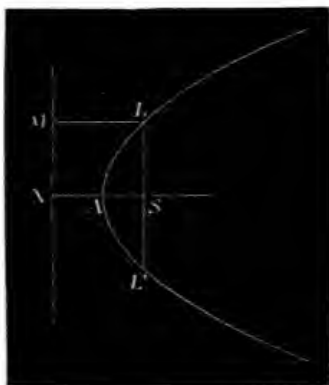
**Def.** The *focal distance* ( $SP$ ) of a point ( $P$ ) on a conic is its distance from the focus.

**Def.** The *ordinate* ( $PN$ ) of a point ( $P$ ) on a conic is the perpendicular from the point on the axis.

**Def.** The *abscissa* ( $AN$ ) of a point ( $P$ ) on a parabola, with respect to the axis, is the portion of the axis between the vertex and the ordinate of the point.

PROPOSITION II.

*The latus rectum of a parabola is equal to four times the distance of the focus from the vertex ( $LL' = 4AS$ ).*



Let  $LSL'$  be the latus rectum. Draw  $LM$  perpendicular to the directrix.

Since the parabola is symmetrical, with respect to the axis,  $LS = L'S$ . Therefore

$$LL' = 2LS = 2LM = 2XS = 4AS.$$

**Ex. 1.** Find a double ordinate of a parabola which shall be double the latus rectum.

Ex. 2. The radius of the circle described about the triangle  $LAL' = \frac{5}{8}$  latus rectum.

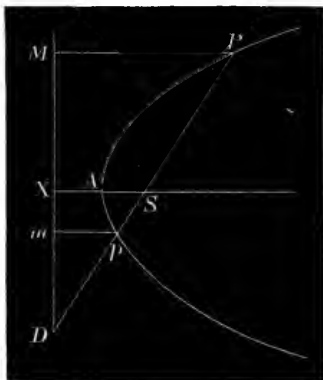
Ex. 3. Find the point  $O$  in a given ordinate  $PN$ , such that  $OR$  being drawn parallel to the axis to meet the curve in  $R$ ,  $ON + OR$  may be the greatest possible. [ $ON = 2AS$ .]

\*PROPOSITION III.

*Any focal chord of a parabola is divided harmonically by the curve, the focus, and the directrix.*

**Def.** A straight line  $AB$  is said to be divided harmonically in  $O$  and  $O'$ , if it is divided internally in  $O$  and externally in  $O'$ , in the same ratio, that is, if

$$AO : OB = AO' : O'B.$$



Produce the focal chord  $PSp$  to meet the directrix in  $D$ , and draw  $PM, pm$  from  $P, p$ , perpendicular to the directrix.

Then, from the similar triangles  $DMP, Dmp$ ,

$$PD : pD = PM : pm.$$

But  $PM = PS$ , and  $pm = pS$ .

Therefore  $PD : pD = PS : pS$ .

Hence  $Pp$  is divided harmonically in  $S$  and  $D$ .



Ex. 1. Prove that  $\frac{1}{PS} + \frac{1}{PD} = \frac{2}{Pp}$ .

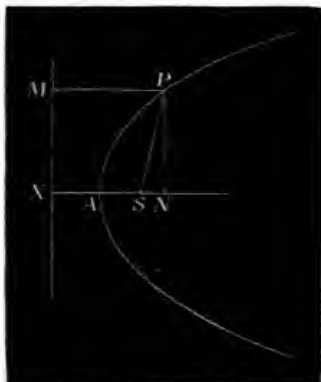
Ex. 2. Prove that  $\frac{1}{DP} + \frac{1}{Dp} = \frac{2}{DS}$ .

Ex. 3. The semi-latus rectum is a harmonic means between the two segments of any focal chord of a parabola.

Ex. 4. Focal chords of a parabola are to one another as the rectangles contained by their segments.

PROPOSITION IV.

*The square of the ordinate of any point on a parabola is equal to the rectangle contained by the latus rectum and the abscissa ( $PN^2 = 4AS \cdot AN$ ).*



Draw  $PM$  perpendicular to the directrix, and join  $SP$ . Then, because  $XS$  is bisected in  $A$  and produced to  $N$ ,

$$NX^2 = SN^2 + 4AS \cdot AN. \quad [\text{Euc. II. 8.}]$$

But  $NX = PM = SP$ .

Therefore  $NX^2 = SP^2 = SN^2 + PN^2$ . [Euc. I. 47.]

Therefore  $PN^2 = 4AS \cdot AN$ .

Ex. 1. If  $PL$  be drawn at right angles to  $AP$ , meeting the axis in  $L$ ,  $NL$  is always equal to the latus rectum.

Ex. 2. If a circle be described about the triangle  $SPN$ , the tangent to it from  $A = \frac{1}{2}PN$ .

Ex. 3. A straight line parallel to the axis bisects  $PN$ , and meets the curve in  $Q$ ;  $NQ$  meets a line through  $A$  at right angles to the axis, in  $T$ . Prove that  $3AT = 2.PN$ .

Ex. 4. If  $SQ$  be parallel to  $AP$ , and  $QM$  be the ordinate of  $Q$ , prove that  $SM^2 = AM.AN$ .

Ex. 5. If  $O$  be any point on a double ordinate  $PNP'$ , and  $OQ$  parallel to the axis meets the curve in  $Q$ , show that

$$(i.) OP.OP' = 4AS.OQ;$$

$$(ii.) PN:ON = OR:QR.$$

Ex. 6.  $PNP'$  is a double ordinate of a parabola. Through  $Q$ , another point on the curve, straight lines are drawn, one passing through the vertex, the other parallel to the axis, cutting  $PP'$  in  $l, l'$ . Prove that  $PN^2 = Nl.Nl'$ .

Ex. 7. A circle has its centre at  $A$ , and its diameter is equal to  $3AS$ . Show that the common chord of the circle and the parabola bisects  $AS$ .

Ex. 8.  $AP, BQ$  are two lines at right angles to  $AB$ ;  $A$  is joined to any point  $Q$  on  $BQ$ ; a point  $O$  is taken on  $AQ$  such that the perpendicular  $ON$  on  $AP = BQ$ . Prove that the locus of  $O$  is a parabola. [Axis,  $AP$ ; Latus rectum,  $AB$ .]

Ex. 9.  $PM, QN$  are the ordinates of the extremities of two chords  $AP, AQ$  which are at right angles to each other. Prove that  $AM.AN = (\text{Latus rectum})^2$ .

Ex. 10. The latus rectum is a mean proportional between the double ordinates of the extremities of a focal chord. (See Prop. I., Ex. 16).

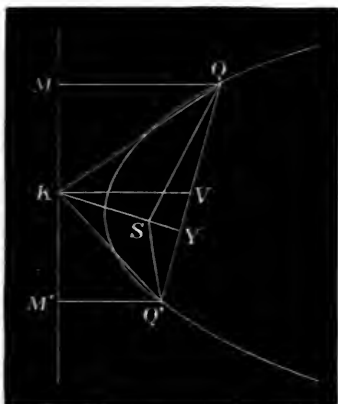
Ex. 11.  $PSp$  is a focal chord; prove that  $AP, Ap$  meet the latus rectum in points whose focal distances are equal to the ordinates of  $p$  and  $P$  respectively. (Apply Prop. I., Ex. 16.)

### PROPOSITION V.

*The locus of the middle points of any system of parallel chords of a parabola is a straight line parallel to the axis.*

Let  $QQ'$  be one of a system of parallel chords. Draw  $QM, Q'M'$  perpendicular to the directrix. Draw  $SY$

perpendicular to  $QQ'$ , produce  $YS$  to meet the directrix in  $K$ , and draw  $KV$  parallel to the axis. Then  $KV$  shall bisect  $QQ'$ . Join  $KQ$ ,  $KQ'$ ,  $SQ$ , and  $SQ'$ .



Then  $MK^2 = KQ^2 - MQ^2$  [Euc. I. 47.]  
 $= KQ^2 - QS^2.$

But  $KQ^2 = KY^2 + QY^2$  [Euc. I. 47.]

and  $QS^2 = SY^2 + QY^2.$  [Euc. I. 47.]

Therefore  $MK^2 = KY^2 - SY^2.$

Similarly  $M'K^2 = KQ'^2 - M'Q'^2$   
 $= KQ'^2 - Q'S^2$   
 $= KY^2 - SY^2.$

Therefore  $MK = M'K,$

but, since  $KV$  is parallel to  $MQ$  and  $M'Q'$ ,  $QQ'$  is bisected at  $V$ .

Now  $QQ'$  being fixed in direction and  $KSY$  being perpendicular to it,  $KSY$  is a fixed straight line and  $K$  is a fixed point. Therefore  $KV$ , which is parallel to the axis,

is a fixed straight line bisecting all chords parallel to  $QQ'$ .

**Def.** A *diameter* of any curve is the locus of the middle points of a system of parallel chords drawn in the curve.

It has just been proved that the diameters of a parabola are straight lines. It will be shown hereafter that the diameters of the other conics are also straight lines. It should be observed, however, that a diameter is not necessarily a straight line for all curves.

**Def.** The half chords ( $QV, Q'V$ ) intercepted between the diameter and the curve, are called the *ordinates* to the diameter.

**Def.** The *abscissa* of a point on a parabola with respect to any diameter is the portion of the diameter intercepted between the ordinate of the point and the parabola.

**Def.** In the parabola, the *vertex* of a diameter is the point in which it cuts the curve.

**Ex. 1.** The perpendicular from the focus upon a system of parallel chords intersects the diameter bisecting the chords upon the directrix.

**Ex. 2.** If a system of parallel chords make an angle of  $45^\circ$  with the axis, their diameter passes through an extremity of the latus rectum (see Prop. IV.).

**Ex. 3.** A parabola being traced on paper, find its focus and directrix.

The direction of the axis is given by the straight line joining the middle points of a pair of parallel chords. The position of the axis is found by observing that the middle point of any chord at right angles to its direction lies on it. At any point  $N$  on the axis, draw a perpendicular to it  $NK=2AN$ . Join  $KA$ , cutting the curve in  $L$ , which will be an extremity of the latus rectum.

**Ex. 4.** The difference between the segments of any focal chord is equal to the parallel chord through the vertex.

**Ex. 5.**  $QSQ'$  is a focal chord;  $QM, Q'M'$  are perpendicular to the axis. Show that  $MM'$  is equal to the parallel chord through the vertex.

Ex. 6.  $AP$  is any chord through the vertex, and  $PE$  is drawn at right angles to  $AP$ , meeting the axis in  $E$ .  $AE$  is equal to the focal chord parallel to  $AP$ .

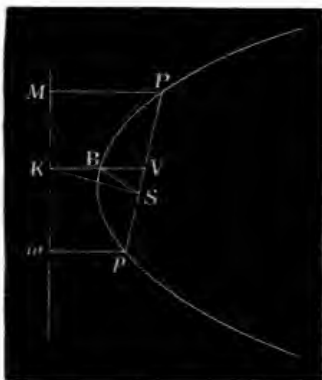
Ex. 7. The middle points of any two chords of a parabola equally inclined to the axis, are equidistant from the axis.

Ex. 8. If a parabola drawn through the middle points of the sides of a triangle  $ABC$  meets the sides again in  $a, \beta, \gamma$ , the lines  $Aa, B\beta, C\gamma$  will be parallel to each other. [Each is parallel to the axis.]

PROPOSITION VI.

*The parameter of any diameter of a parabola is four times the line joining the focus with the vertex of the diameter.*

**Def.** The *parameter* of a diameter is the length of the focal chord bisected by the diameter.



Draw  $SK$  at right angles to the focal chord  $PSp$ , to meet the directrix in  $K$ ; draw  $PM, pm$  at right angles to the directrix, and  $KBV$  parallel to them. Then  $KBV$  is the diameter bisecting the chord  $PSp$  (Prop. V.). Join  $SB$ .

Then, since  $KSV$  is a right angle, and  $KB=BS$ , we have

$$KB=BS=BV,$$

or

$$KV=2BS.$$

Now, because  $Pp$  is bisected in  $V$ ,

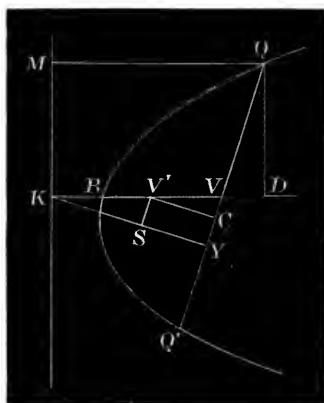
$$\begin{aligned} Pp &= PS + Sp = PM + pm \\ &= 2KV = 4BS. \end{aligned}$$

Ex. 1. Given the length of a focal chord, find its position.

Ex. 2. Draw a focal chord  $PSp$ , such that  $SP=3Sp$ .

### PROPOSITION VII.

*The ordinate to any diameter of a parabola at any point is a mean proportional to its parameter and the abscissa of the point with respect to the diameter ( $QV^2 = 4BS \cdot BV$ ).*



Let  $QQ'$  be any chord. Draw  $SY$  at right angles to it, and produce  $YS$  to meet the directrix in  $K$ . Draw  $KBV$

parallel to the axis, so that  $BV$  is the diameter bisecting  $QQ'$  in  $V$ ,  $QV$  being the ordinate and  $BV$  the abscissa.

[Prop. V.]

Draw  $SV'$  parallel to  $QQ'$ , and  $QM, QD, V'C$  at right angles to the directrix,  $KV$  and  $QQ'$  respectively.

Then 
$$QD^2 = MK^2 = KY^2 - SY^2; \quad \text{[Prop. V.]}$$

and, from the similar triangles  $QVD, KVV,$  and  $V'VC,$

$$\begin{aligned} QD : QV &= KY : KV \\ &= V'C : V'V \\ &= SY : V'V. \end{aligned}$$

Therefore 
$$QV^2 = KV^2 - V'V^2.$$

But as  $KV'$  is bisected in  $B,$  [Prop. VI.]

$$KV^2 = V'V^2 + 4BV \cdot BV'. \quad \text{[Euc. II. 8.]}$$

Therefore 
$$\begin{aligned} QV^2 &= 4BV \cdot BV' \\ &= 4BS \cdot BV. \end{aligned} \quad \text{[Prop. VI.]}$$

Ex. 1. If any chord  $BR$  meets  $QM$  and  $QQ'$  in  $L$  and  $N,$  prove that  $BL^2 = BN \cdot BR.$

Ex. 2. If  $QQ'$  meets any chord  $BR$  in  $N,$  and the diameter through  $R$  in  $N',$  prove that  $QV^2 = VN \cdot VN'.$

Ex. 3. If  $QQ'$  be any chord meeting the diameter  $BV$  in  $O,$  and  $QV, Q'V'$  ordinates to the diameter, then  $BO^2 = BV \cdot BV'.$

Let  $QB$  produced meet the diameter through  $Q'$  in  $E,$  and draw  $ER$  parallel to the ordinate meeting  $BV$  produced in  $R.$

Then 
$$QV^2 : Q'V'^2 = BV^2 : BV \cdot BV'.$$

But 
$$QV^2 : BV^2 = Q'V'^2 : BR^2;$$

$\therefore BV \cdot BV' = BR^2;$

$\therefore BV : BR = BR : BV';$

or 
$$BV : RV = BR : RV'.$$

But 
$$BV : RV = QB : QE$$

$$= BO : RV';$$

$\therefore BO = BR.$

Ex. 4. If  $POP'$  be the chord bisected by the diameter  $BOV$  at  $O,$   $PO^2 = QV \cdot Q'V'.$

Ex. 5. Through a given point, to draw a chord of a parabola which will be divided in a given ratio at the point.

Through the given point  $O,$  draw the diameter  $BO.$  Then if  $V, V'$  be the feet of the ordinates drawn through the extremities of the chord sought, it is clear that  $BV' : BV$  is as the square of the

given ratio. Also,  $BV \cdot BV' = BO^2$ , whence the points  $V, V'$  are known.

Ex. 6. If any diameter intersect two parallel chords, the rectangles under the segments of these chords are proportional to the segments of the diameter intercepted between the chords and the curve.

If  $QQ'$  be one of the chords meeting the diameter  $BV$  in  $V$ , and if  $O$  be its middle point,

$$QV \cdot Q'V = QO^2 - OV^2 = 4BS \cdot BV.$$

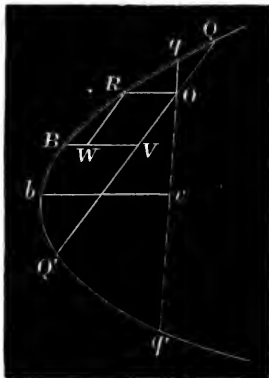
Ex. 7.  $QQ'$  is a fixed straight line, and from any point  $V$  in it,  $VB$  is drawn in a fixed direction such that  $BV$  is proportional to  $QV \cdot Q'V$ . Show that the locus of  $B$  is a parabola passing through  $Q, Q'$  and having its axis parallel to  $BV$ .

Ex. 8. Given the base and area of a triangle, the locus of its orthocentre is a parabola.

Ex. 9.  $BO, B'O'$  are any two diameters. A line is drawn parallel to the ordinate to  $BO$ , cutting the curve in  $D$ , and  $BO, BB', B'O$  in  $O, C, E$  respectively. Prove that  $OD^2 = OC \cdot OE$ . (Through  $B'$  draw a parallel to  $EO$ .)

### \*PROPOSITION VIII.

*If two chords of a parabola intersect each other, the rectangles contained by their segments are in the ratio of the parallel focal chords.*



Let the chords  $QQ'$  and  $qq'$  intersect in a point  $O$



within the parabola. Bisect  $QQ'$  in  $V$ , and draw the diameters  $OR$ ,  $VB$ . Draw  $RW$  parallel to  $QQ'$ .

Then, because  $QQ'$  is bisected in  $V$ ,

$$\begin{aligned} QO \cdot Q'O &= QV^2 - OV^2 && [\text{Euc. II. 5.}] \\ &= QV^2 - RW^2 \\ &= \frac{1}{2}BV \cdot BS - \frac{1}{2}BW \cdot BS && [\text{Prop. VII.}] \\ &= \frac{1}{2}BS \cdot WV \\ &= \frac{1}{2}BS \cdot OR. \end{aligned}$$

Similarly, if  $bv$  be the diameter bisecting  $qq'$ ,

$$qO \cdot q'O = \frac{1}{2}BS \cdot OR.$$

Therefore  $QO \cdot Q'O : qO \cdot q'O = \frac{1}{2}BS : \frac{1}{2}BS$ ;

that is, as the focal chords parallel to  $QQ'$  and  $qq'$  respectively. [Prop. VI.]

The proposition may be similarly proved when the chords intersect outside the curve.

Ex. 1. If two intersecting chords be parallel to two others, the rectangles contained by the segments of the one pair are proportional to the rectangles contained by the segments of the other pair.

Ex. 2. Deduce Prop. III.

Ex. 3. Given three points on a parabola and the direction of the axis, construct the curve.

Ex. 4. Inscribe in a given parabola a triangle having its sides parallel to three given straight lines.

### \* PROPOSITION IX.

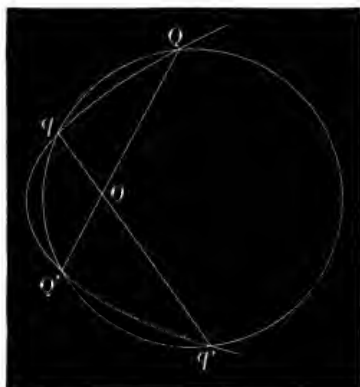
*If a circle intersect a parabola in four points their common chords will be equally inclined, two and two, to the axis.*

Let  $Q$ ,  $Q'$ ,  $q$ ,  $q'$  be the four points of intersection.

Then  $QO \cdot Q'O = qO \cdot q'O$ . [Euc. III. 35.]

Therefore, the focal chords parallel to  $QQ'$  and  $qq'$  are equal to each other. [Prop. VIII.]

And they are therefore equally inclined to the axis, from the symmetry of the figure. (See also Prop. I., Ex. 10.)



Therefore the chords  $QQ'$ ,  $qq'$  are equally inclined to the axis.

In like manner, it may be shown that the chords  $Qq$  and  $q'Q'$ , as well as the chords  $Qq'$  and  $qQ'$ , are equally inclined to the axis.

Ex. 1. If a circle cut a parabola in four points, two on one side of the axis and two on the other, the sum of the ordinates of the first two is equal to the sum of the ordinates of the other two points. (See Prop. V., Ex. 7.)

Ex. 2. If three of the points are on the same side of the axis, the sum of their ordinates is equal to the ordinate of the fourth point.

### PROPOSITION X.

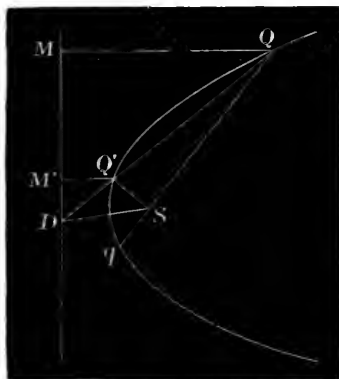
*If any chord  $QQ'$  of a parabola intersects the directrix in  $D$ ,  $SD$  bisects the exterior angle between  $SQ$  and  $SQ'$ .*

Draw  $QM$ ,  $Q'M'$  perpendicular to the directrix.

Then, by similar triangles,

$$\begin{aligned} QD : Q'D &= QM : Q'M \\ &= SQ : SQ'. \end{aligned}$$

Therefore  $SD$  bisects the exterior angle  $Q'Sq$ . [Euc. VI. A.



Ex. 1. Given the focus and two points on a parabola, find the directrix.

The point  $D$ , being the intersection of the chord  $QQ'$  and the bisector of the angle  $Q'Sq$ , is on the directrix, which touches the circle described with  $Q$  as centre and radius  $QS$ .

Ex. 2.  $PQ, pq$  are focal chords. Show that  $Pp, Qq$ , as also  $Pq, pQ$ , meet on the directrix.

If they meet the directrix in  $K, K'$ ,  $KSK'$  is a right angle.

Ex. 3. Given the focus and the directrix, trace the parabola by means of this proposition. (For other constructions, see Prop. 1., and Ex. 2, Ex. 3.)

Determine the vertex  $A$  as the middle point of  $SY$ . Take any point  $D$  on the directrix; make the angle  $DSp$  equal to the angle  $DSA$ , and let  $pS$  and  $DA$  produced meet in  $P$ .  $P$  is a point on the parabola.

Ex. 4.  $Q$  is a point on the parabola. If  $QA$  produced meet the directrix in  $D$ ,  $MSD$  is a right angle.

Ex. 5.  $PQ$  is a double ordinate, and  $PX$  cuts the curve in  $P'$ : show that the focus lies on  $P'Q$ .

Ex. 6. If two fixed points  $Q, Q'$  on a parabola be joined with a third variable point  $O$  on the curve, the segment  $qq'$  intercepted on the directrix by the chords  $QO, Q'O$  produced, subtends a constant angle at the focus.

The angle  $qSq'$  may be proved to be equal to half of the angle  $QSQ'$ .

Ex. 7. If  $QQ'$  be a focal chord, the angle  $qSq'$  is a right angle, and  $qX \cdot q'X = (\text{semi-latus rectum})^2$ .

Ex. 8. Show that a straight line which meets a parabola will, in general, meet it in two points, except when the line is parallel to the axis, in which case it meets the curve in one point only; and no straight line can meet the curve in more points than two.

Let  $DQ'$  be any straight line which meets the directrix in  $D$  and the curve in  $Q'$ . Make the angle  $DSq$  equal to the angle  $DSQ'$ , and let  $qS, DQ'$  intersect in  $Q$ . Then since

$$SQ : SQ' = QD : Q'D = QM : Q'M,$$

$Q$  is a point on the curve. If, however,  $DQ'$  be parallel to the axis,  $qS$  will coincide with the axis, and  $D'Q'$  will meet the parabola in the point  $Q'$  only (the other point of intersection in this case being really at infinity). Again  $SQ, SQ'$ , being equally inclined to  $DS$ , if there be a third point of intersection  $Q''$ ,  $SQ, SQ''$  will make the same angle with  $DS$ , which is impossible.

#### PROPERTIES OF TANGENTS.

**Def.** A *tangent* to a conic is the limiting position of a chord whose two points of intersection with the curve have become coincident.



Thus, if  $P$  and  $P'$  be two points on a conic, and if the chord  $PP'$  be so turned about  $P$  that  $P'$  may approach  $P$ , then in the limiting position when  $P'$  moves up to  $P$  and coincides with it, the chord becomes the *tangent* to the conic at  $P$ .

Again, if a chord  $PP'$  moves parallel to itself until  $P$  and  $P'$  coincide at a point  $B$  on the conic,  $PP'$  becomes in its limiting position the tangent to the curve at the point  $B$ .

Hence, a tangent may be said to be a straight line which passes through two *consecutive* or *coincident* points on the curve.

It will be seen that, generally, to a chord-property of a conic, there corresponds a tangent-property.

Thus, in Prop. V., if the chord  $QQ'$  moves parallel to itself until  $Q'$  coincides with  $Q$  at the point  $B$  on the curve, the chord in this its limiting position becomes the tangent to the parabola at  $B$ , which is thus seen to be parallel to the system of chords bisected by the diameter  $BV$ . (See Prop. XI.)

\* Again, in Prop. VIII., let the chords  $QQ'$ ,  $qq'$  intersect at a point  $O$  *outside* the parabola. Let the chord  $OQQ'$  be made to turn about the point  $O$ , until  $Q'$  coincides with  $Q$  at a point  $R$  on the curve, so that  $OR$  becomes the tangent to the curve at the point  $R$ , and  $OQ$ ,  $OQ'$  become each equal to  $OR$ . In like manner, let  $Oqq'$  be made to turn about the point  $O$ , until  $q'$  coincides with  $q$  at a point  $r$  on the curve, so that  $Or$  becomes the tangent to the curve at the point  $r$ , and  $Oq$ ,  $Oq'$ , become each equal to  $Or$ . Hence, we have the following proposition:—

The squares of any two intersecting tangents to a parabola are in the ratio of the parallel focal chords.

Ex. 1. If  $OTO'$  be the tangent to a parabola at  $T$ , and if  $OPQ$ ,  $OP'Q'$  be a pair of parallel chords,

$$OT^2 : O'T^2 = OP \cdot OQ : O'P' \cdot O'Q'.$$

Ex. 2. If  $TOO'$  be the tangent to a parabola at  $T$ ,  $O'P'$  a tangent from  $O'$ , and  $OPQ$  a chord parallel to  $O'P'$ , cutting the chord of contact  $P'Q'$  in  $R$ , prove that  $OP \cdot OQ = OR^2$ .

From Ex. 1,

$$OP \cdot OQ : OT^2 = O'P'^2 : O'T^2 = OR^2 : OT^2.$$

Cf. Prop. XXI., Ex. 8.

\* Next, in Prop. IX., suppose  $q$  to coincide with  $Q$ , and

therefore also with  $O$ ; then the circle and the parabola will touch each other at  $O$ , the chords  $OQ'$ ,  $oq'$  being equally inclined to the axis. Hence

If two chords  $OP$ ,  $OQ$  of a parabola are equally inclined to the axis, the circle round  $OPQ$  touches the parabola at  $O$ .

Ex. If one of the chords  $OP$  be at right angles to the tangent to the curve at  $O$ , the angle  $OQP$  is a right angle.

Similarly, if a circle touches a parabola at  $O$  and cuts it again in  $P$  and  $Q$ , the tangent at  $O$  and  $PQ$  are equally inclined to the axis.

Ex. If a circle touches a parabola at  $O$  and cuts it in  $P$  and  $Q$ , and  $PU$ ,  $QV$  parallel to the axis meet the circle in  $U$ ,  $V$ , show that  $UV$  is parallel to the tangent at  $O$ .

Again, consider Prop. X. Let the chord  $QQ'$  be made to turn about  $Q$ , until  $Q'$  coincides with  $Q$ , so that the chord becomes the tangent to the parabola at the point  $Q$ . The angle  $QSQ'$  vanishes, and, therefore, the exterior angle  $Q'Sq$  becomes equal to two right angles. But since  $SD$  always bisects the angle  $Q'Sq$ ,  $SD$  will, in this limiting position, be at right angles to  $SQ$ . Hence the following proposition:—

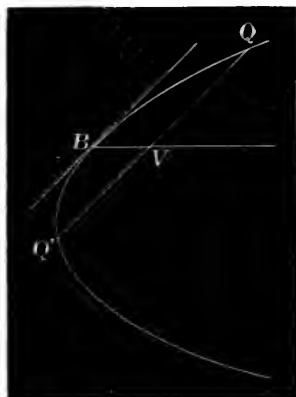
The tangent to a parabola from any point on the directrix, subtends a right angle at the focus. (See Prop. XII.)

**Def.** A circle or a conic is said to *touch* a conic at a point  $P$  when they have a *common tangent* at that point.

#### PROPOSITION XI.

*The tangent to a parabola at its point of intersection*

*with a diameter is parallel to the system of chords bisected by the diameter.*



Let  $BV$  be the diameter bisecting a system of chords parallel to  $QQ'$ .

Let  $QQ'$  be made to move parallel to itself, so that  $Q$  may coincide with  $V$ . Since  $QV$  is always equal to  $Q'V$  (Prop. V.), it is clear that  $Q'$  will also coincide with  $B$ , or, the chord in this, its limiting position, will be the tangent to the parabola at  $B$ .

Ex. Draw a tangent to a parabola making a given angle with the axis.

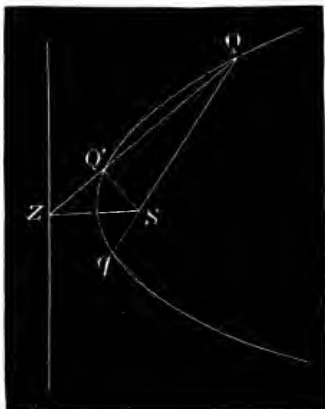
### PROPOSITION XII.

*The portion of the tangent to a parabola at any point, intercepted between that point and the directrix, subtends a right angle at the focus.*

Let any chord  $QQ'$  of the parabola intersect the directrix in  $Z$ .

Then  $SZ$  bisects the exterior angle  $Q'Sq$ . [Prop. X.

Now, let the chord  $QQ'$  be made to turn about  $Q$  until the point  $Q'$  moves up to and coincides with  $Q$ , so that



the chord becomes the tangent to the parabola at  $Q$ . In this limiting position of the chord  $QQ'$ , since  $Q$  and  $Q'$



coincide, the angle  $QSQ'$  vanishes, and therefore the angle  $Qsq$  becomes equal to two right angles. But since



$SZ$  always bisects the angle  $Q'Sq$ , in this case the angle  $QSZ$  is a right angle.

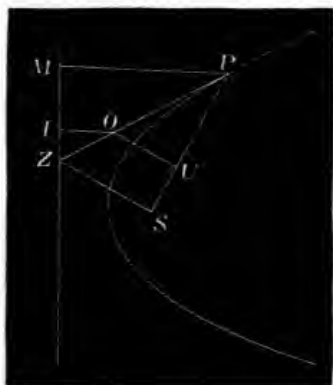
Ex. 1. If a line  $QZ$  meeting the curve in  $Q$  and the directrix in  $Z$ , subtend a right angle at the focus, it will be the tangent to the curve at  $Q$ .

Ex. 2. The tangents at the extremities of the latus rectum meet the directrix on the axis produced.

\* PROPOSITION XIII.

If from any point  $O$  on the tangent at  $P$  of a parabola perpendiculars  $OU$  and  $OI$  be drawn to  $SP$  and the directrix respectively, then

$$SU = OI.$$



Join  $SZ$ , and draw  $PM$  perpendicular to the directrix.

Because  $ZSP$  is a right angle,

[Prop. XII.

$ZS$  is parallel to  $OU$ .

Therefore, by similar triangles,

$$\begin{aligned} SU : SP &= ZO : ZP \\ &= OI : PM. \end{aligned}$$

But

$$SP = PM;$$

therefore

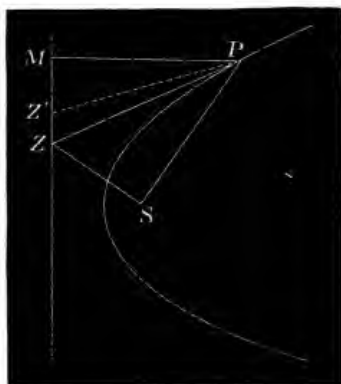
$$SU = OI.$$

This property of the parabola is the particular case of a general property of all conics discovered by Adams.

Ex. If a line  $OP$  meet the parabola at  $P$ , and  $OI$ ,  $OU$  being drawn at right angles to the directrix and  $SP$  respectively,  $SU=OI$ , then  $OP$  will be the tangent to the curve at  $P$ .

#### PROPOSITION XIV.

*The tangent at any point of a parabola bisects the angle which the focal distance of the point makes with the perpendicular drawn from the point on the directrix, and conversely.*



Let the tangent at the point  $P$  meet the directrix in  $Z$ . Draw  $PM$  perpendicular to the directrix, and join  $SP$ ,  $SZ$ .

Then, since the angle  $PSZ$  is a right angle, [Prop. XII.

$$SP^2 + SZ^2 = PZ^2. \quad [\text{Euc. I. 47.}]$$

Also  $PM^2 + MZ^2 = PZ^2$ ; [Euc. I. 47.

therefore  $SP^2 + SZ^2 = PM^2 + MZ^2$ .

But  $SP = PM$ ;

therefore  $SZ = MZ$ .

Now, in the two triangles  $ZPM$ ,  $ZPS$ , the two sides  $PM$ ,  $MZ$  are respectively equal to the two sides  $SP$ ,  $SZ$ ,

and the side  $PZ$  is common; therefore the two triangles are equal, and the angle  $SPZ$  is equal to the angle  $MPZ$ , that is,  $PZ$  bisects the angle  $SPM$ .

Conversely, if  $PZ$  bisects the angle  $SPM$ ,  $PZ$  is the tangent at  $P$ . For, if not, and if possible, let any other line  $PZ'$  be the tangent at  $P$ , then by what has been proved  $PZ'$  will bisect the angle  $SPM$ , which is impossible; therefore  $PZ$  is the tangent at  $P$ .

*Note.*—It may be shown from the definition of the parabola that the straight line which bisects the angle between  $SP$  and  $PM$  cannot meet the curve again in any other point; hence  $PZ$  would also be the tangent to the parabola at  $P$ , according to Euclid's definition of a tangent.

*Corollary.*—The tangent at the vertex of a parabola is at right angles to the axis.

Ex. 1. Show how to draw the tangent at a given point of a parabola.

Ex. 2. Draw a tangent to a parabola making a given angle with the axis.

Ex. 3. If the tangent at  $P$  meets the axis in  $T$ ,  $SP = ST$ .

Ex. 4. Two parabolas have the same focus, and their axes in the same straight line, but in opposite directions. Prove that they intersect at right angles.

*Note.*—Two curves are said to intersect at right angles when their tangents at a common point are at right angles.

Ex. 5. Given the vertex of a diameter of a parabola and a corresponding double ordinate, construct the curve. (Apply Prop. VII.)

Ex. 6. If  $ZP$  be produced to  $R$ , the angles  $SPR$  and  $MPR$  are equal.

Ex. 7.  $PZ$  bisects  $SM$  at right angles.

Ex. 8. Any point  $O$  on the tangent at  $P$  is equidistant from  $M$  and  $S$ .

Ex. 9. If the tangents to the parabola at  $Q$  and  $Q'$  meet in  $O$ , and  $QM$ ,  $Q'M'$  be the perpendiculars on the directrix from  $Q$  and  $Q'$ ,  $OM$ ,  $OS$ ,  $OM'$  are all equal.

Hence deduce, by analysis, the construction for Prop. XVII., namely, to draw two tangents to a parabola from an external point  $O$ .

Ex. 10. The tangent at any point of a parabola meets the directrix and the latus rectum in two points equidistant from the focus.

Ex. 11. The focal distance of any point on a parabola is equal to the length of the ordinate of that point produced to meet the tangent at the end of the latus rectum. (See Prop. XII., Ex. 2.)

Ex. 12.  $O$  is a point on the tangent at  $P$ , such that the perpendicular from  $O$  on  $SP$  is equal to  $2AS$ ; find the locus of  $O$ . (A parabola of which the vertex is on the directrix of the given one. Apply Prop. VII., Ex. 7.)

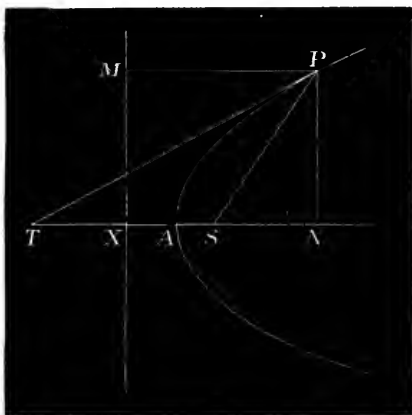
Ex. 13. If a leaf of a book be folded so that one corner moves along an opposite side, the line of the crease touches a parabola.

Let the leaf  $BCXS$  be so folded that  $S$  coincides with a point  $M$  on  $CX$ ; let the crease  $TT'$  meet  $XS$ ,  $BS$  in  $T$ ,  $T'$  respectively. Draw  $MP$  at right angles to  $CX$ , meeting  $TT'$  in  $P$ ; join  $SP$ . Then  $SP=PM$ ,  $\angle SPT=\angle MPT$ ;  $TT'$ , therefore, touches at  $P$  a parabola, of which the focus is  $S$  and directrix  $C$ .

**Def.** The portion of the axis intercepted between the tangent at any point of a conic and the ordinate of that point is called the *subtangent*.

\* PROPOSITION XV.

*The subtangent of any point of a parabola is bisected at the vertex, that is, is equal to double the abscissa of the point with respect to the axis.*



Let the tangent  $PT$  at  $P$  meet the axis in  $T$ . Draw

$PN$ ,  $PM$  perpendicular to the axis and directrix respectively.

Then, the angle  $STP =$  the angle  $TPM$   
 $=$  the angle  $TPS$ . [Prop. XIV.

Therefore  $ST = SP = PM = AN$ .

But  $AS = AX$ .

Therefore  $AT = AX$ ,

or  $NT = 2AN$ .

Ex. 1. If  $T$  is the middle point of  $AX$ , prove that  $N$  is the middle point of  $AS$ .

Ex. 2. The radius of the circle described round the triangle  $TPN$  is  $\sqrt{(SP \cdot AN)}$ .

Ex. 3. The locus of the middle points of the focal chords of a parabola is another parabola having the same axis and passing through the focus. (Apply Prop. VII., Ex. 7.)

Ex. 4. The diameter through  $P$  meets at  $E$ , a right line through  $S$  parallel to the tangent at  $P$ . Prove that the locus of  $E$  is a parabola.

If  $En$  be perpendicular to the axis,  $nS = NT = 2AN$ . If  $S'$  be taken on the axis, such that  $2SS' = AS$ , the relation  $PN^2 = 4AS \cdot AX$  gives  $En^2 = 4SS' \cdot Sn$ , showing the locus to be a parabola whose axis coincides with that of the original one, whose vertex is at  $S$ , and latus rectum half that of the original parabola.

Ex. 5. If  $SM$  meets  $PT$  in  $Y$ ,  $NY = TY$ .

Ex. 6. If the tangent at  $P$  meets the tangent at the vertex in  $Y$ ,  $AY^2 = AS \cdot AN$ .

Ex. 7. If  $SE$  be the perpendicular from  $S$  on the line through  $P$  at right angles to  $PT$ , show that  $SE^2 = AN \cdot SP$ . ( $2SE = PT$ . Apply Prop. IV.)

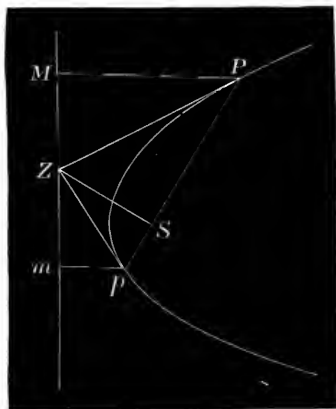
Ex. 8. Given the vertex, a tangent and its point of contact, construct the curve.

Produce  $PA$  to  $P'$ , such that  $AP' = AP$ ; if the circle on  $AP'$  as diameter meets the tangent at  $P$  in  $T$ ,  $TA$  is the axis. Then apply Prop. XIV.

Ex. 9. Find the locus of the intersection of the perpendicular from the vertex on the tangent at any point with the diameter through that point. (A right line parallel to the directrix. Apply Prop. IV.)

## \* PROPOSITION XVI.

*The tangents at the extremities of a focal chord of a parabola intersect at right angles on the directrix.*



Draw  $SZ$  at right angles to the focal chord  $PSp$ , meeting the directrix in  $Z$ . Join  $PZ$ ,  $pZ$ , and draw  $PM$ ,  $pm$  perpendiculars to the directrix.

$$\begin{aligned} \text{Then} \quad ZP^2 &= ZS^2 + SP^2 \\ &= ZM^2 + PM^2. \end{aligned} \quad [\text{Euc. I. 47.}]$$

$$\text{But} \quad SP = PM.$$

$$\text{Therefore} \quad ZS = ZM.$$

Therefore from the triangles  $ZSP$  and  $ZMP$ , the angle  $SPZ =$  the angle  $MPZ$ , and the angle  $SZP =$  the angle  $MZP$ .

[Euc. I. 8.]

Similarly,

$$\begin{aligned} \text{the angle } SpZ &= \text{the angle } mpZ, \\ \text{and the angle } SZp &= \text{the angle } mZp. \end{aligned}$$

Therefore,  $PZ$  and  $pZ$  are the tangents at  $P$  and  $p$ .

[Prop. XIV.]

Also,

the angle  $PZp = \frac{1}{2}$  the angle  $MZS + \frac{1}{2}$  the angle  $mZS$   
 $=$  one right angle

Ex. 1. Show that  $Mm$  is bisected in  $Z$ .

Ex. 2. If two tangents be drawn to a parabola from any point on the directrix, they shall be at right angles.

Ex. 3. If perpendiculars through  $P, p$ , to  $ZP, Zp$  respectively, meet in  $O$ , the distance of  $O$  from the directrix varies as  $PS.pS$ . (Apply Prop. III., Ex. 4.)

Ex. 4. Find the locus of  $O$  in Ex. 3. [A parabola having the same axis as the given one.]

Ex. 5. Show that the circle described on the focal chord  $Pp$  as diameter touches the directrix at  $Z$ .

Ex. 6. If a circle described upon a chord of a parabola as diameter meets the directrix, it also touches it; and all chords for which this is possible, intersect in a fixed point. [The focus.]

The distance of the middle point of the chord from the directrix is always greater than half the chord, unless the chord passes through the focus.

Ex. 7. Tangents at the extremities of a focal chord cut off equal intercepts on the latus rectum. (Apply Prop. XIV., Ex. 10.)

Ex. 8. Prove that  $SM, Sm$  are respectively parallel to  $Zp, ZP$ .

Ex. 9. The locus of the intersection of any two tangents to a parabola at right angles to each other, is the directrix.

Ex. 10. Given two tangents at right angles, and their points of contact, construct the curve.

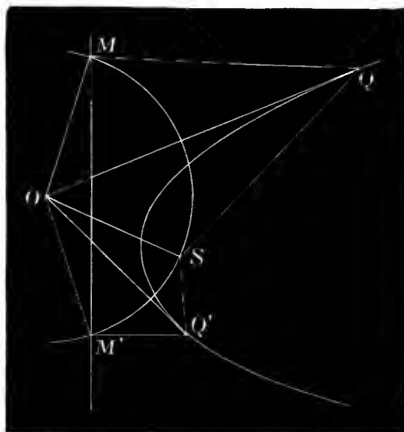
### PROPOSITION XVII.

*To draw two tangents to a parabola from an external point.*

Let  $O$  be the external point. With centre  $O$  and radius  $OS$ , describe a circle cutting the directrix in  $M$  and  $M'$ . Draw  $MQ, M'Q'$  at right angles to the directrix to meet the parabola in  $Q$  and  $Q'$ . Join  $OQ$  and  $OQ'$ ; these shall be the tangents required.

Join  $OS, OM, OM', SQ$  and  $SQ'$

Then, in the triangles  $OQM$ ,  $OQS$ , the sides  $MQ$ ,  $QO$  are equal to the sides  $SQ$ ,  $QO$  respectively, and  $OM$  is equal



to  $OS$ . Therefore the angles  $OQM$ ,  $OQS$  are equal. Therefore  $OQ$  is the tangent to the parabola at  $Q$ .

[Prop. XIV.

Similarly,  $OQ'$  is the tangent at  $Q'$ .

*Note.*—For an analysis of the construction, see Prop. XIV., Ex. 9.

It should be observed that in order that the construction may be possible, the circle described with  $O$  as centre and with radius  $OS$  must meet the directrix, that is, the distance of  $O$  from  $S$  must be either greater than or equal to its distance from the directrix. The former is the case when the point is outside the parabola (Prop. I., Ex. 7); and as in this case the circle must intersect the directrix in two points only, it follows that two tangents, and no more, can be drawn to a parabola from an *external* point. In the second case the point  $O$  is evidently on the parabola, and the circle touches the directrix, that is, meets it in two coincident points; the two tangents in this case coincide, that is, only one tangent can be drawn to a parabola at a given point on it. The distance of any point inside the parabola being less than its distance from the directrix (Prop. I., Ex. 6), no tangent can be drawn to a parabola from any point within it.



Ex. 1. If the point  $O$  be on the directrix, show from the construction that the tangents intersect at right angles.

Ex. 2. If  $O$  be on the axis produced, at a distance from the vertex  $A = \frac{1}{3}AS$ , the figure  $OQSQ'$  will be a rhombus.

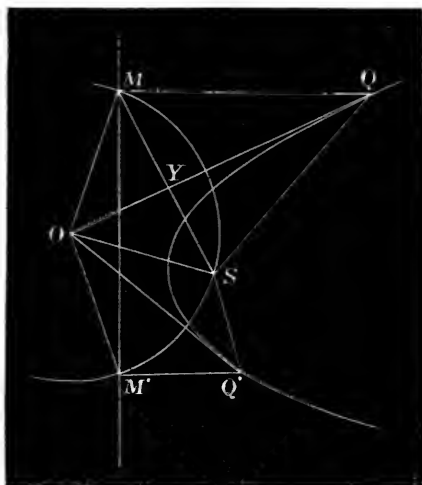
Ex. 3. *Alternative Construction.*—With the given point  $O$  as centre and radius  $OS$ , describe a circle cutting the directrix in  $M$  and  $M'$ . The perpendiculars from  $O$  upon  $SM$  and  $SM'$  will, when produced, touch the curve. (See Prop. I., Ex. 3.)

Ex. 4. *Alternative Construction.*—In the figure of Prop. XIII., taking  $O$  as the given point, draw  $OI$  at right angles to the directrix. With centre  $S$  and radius equal to  $OI$ , describe a circle; and from  $O$  draw  $OU$  and  $OU'$  tangents to this circle.  $SU, SU'$  produced will meet the parabola in the points of contact of the tangents from  $O$ . (See Prop. XIII., Ex.)

For another alternative construction, see Prop. XXIII., Ex. 13.

PROPOSITION XVIII.

*The two tangents  $OQ, OQ'$  of a parabola subtend equal angles at the focus; and the triangles  $SOQ, SOQ'$  are similar.*



With centre  $O$  and radius  $OS$ , describe a circle cutting

the directrix in  $M$  and  $M'$ ; draw  $MQ, M'Q'$  at right angles to the directrix to meet the curve in  $Q, Q'$ . Then  $OQ$  and  $OQ'$  are the tangents to the curve from  $O$ . [Prop. XVII.]

Join  $OM, OM', OS, SQ, SQ'$ , and  $SM$ , cutting  $OQ$  in  $Y$ .

In the two triangles  $MQY$  and  $SQY$ , the sides  $MQ, QY$  are equal to the sides  $SQ, QY$ , and the angles  $MQY, SQY$  are equal; [Prop. XIV.]

therefore the two triangles are equal in every respect; and the angles  $MYQ, SYQ$  are equal, each being thus equal to a right angle. [Euc. I. 4.]

Now, the angle  $SQO =$  the angle  $MQO$ ,  
and the angle  $MQO =$  the angle  $SMM'$ ,  
each being the complement of the angle  $QMY$ .

Therefore

the angle  $SQO =$  the angle  $SMM'$ .

But the angle  $SMM' = \frac{1}{2}$  the angle  $SOM'$ , [Euc. III. 20.]  
and from the equality of the triangles  $SOQ, M'OQ'$ ,  
[Prop. XVII.]

the angle  $SOQ' =$  the angle  $M'OQ'$ ,

or, the angle  $SOQ' = \frac{1}{2}$  the angle  $SOM'$ .

Therefore the angle  $SQO =$  the angle  $SOQ'$ .

Similarly, the angles  $QOS$  and  $OQ'S$  are equal, as also the remaining angles  $QSO, Q'SO$ .

Therefore the two triangles  $SOQ, SOQ'$  are similar.

Ex. 1. Prove that

$$(i.) SQ \cdot SQ' = SO^2; \quad (ii.) OQ^2 : OQ'^2 = SQ : SQ'.$$

Ex. 2. If two tangents drawn from any point on the axis be cut by any third tangent, the points of intersection are equidistant from the focus.

Ex. 3. The angle subtended at the focus by the segment intercepted on a variable tangent by two fixed tangents, is constant.

Ex. 4.  $OS$  and a line through  $O$  parallel to the axis make equal angles with the tangents.

Ex. 5. The straight line bisecting the angle  $QOQ'$  meets the axis in  $R$ ; prove that  $SO = SR$ .

Ex. 6. If two tangents drawn from any point on the axis be cut by a third tangent, their alternate segments are equal. (Cf. Prop. XXI, Ex. 10.)

Ex. 7. If the tangent and normal at any point  $P$  of a parabola meet the tangent at the vertex in  $K$  and  $L$  respectively, prove that

$$KL^2 : SP^2 = SP - AS : AS.$$

Ex. 8. If from any point on a given tangent to a parabola, tangents be drawn to the curve, the angles which these tangents make with the focal distances of the points from which they are drawn, are all equal.

Each angle is equal to the angle between the given tangent and the focal distance of the point of contact.

Ex. 9. Of the two tangents drawn to a parabola from any point, one makes with the axis the same angle as the other makes with the focal distance of the point.

Ex. 10. Two parabolas have the same focus and axis, with their vertices on the same side of their common focus. Tangents are drawn from any point  $P$  on the outer parabola to the inner one. Show that they are equally inclined to the tangent at  $P$  to the outer curve. (Apply Ex. 9, and Prop. XIV.)

Ex. 11. If the tangent at any point  $R$  meets  $OQ, OQ'$  in  $q, q'$ , show that  $Qq : qO = Oq' : q'Q' = qR : Rq'$ .  
[The triangles  $OqS, Rq'S$  are similar.]

Ex. 12. If tangents be drawn from any point on the latus rectum, show that the semi-latus-rectum is a geometric mean between the ordinates of the points of contact. (Apply Prop. I., Ex. 16, and Prop. IV.)

Ex. 13. If  $PV, P'V'$  be two diameters, and  $P'V, PV'$  ordinates to these diameters, show that  $PV = P'V'$ . (Apply Prop. VII. and Ex. 1.)

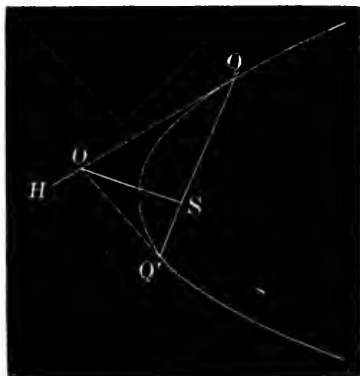
Ex. 14. If one side of a triangle be parallel to the axis of a parabola, the other sides will be in the ratio of the tangents parallel to them.

### PROPOSITION XIX.

*The exterior angle between any two tangents to a parabola is equal to the angle which either of them subtends at the focus.*

Let  $OQ$  and  $OQ'$  be the two tangents, and  $S$  the focus. Join  $SO$ ,  $SQ$ , and  $SQ'$ .

The angle  $SOQ' =$  the angle  $SQO$ . [Prop. XVIII.  
To each of these equals add the angle  $SOQ$ ; therefore the angles  $SOQ$  and  $SQO$  are together equal to the angle  $QOQ'$ . But the exterior angle  $HOQ'$  is the supplement of



the angle  $QOQ'$  (Euc. I. 13), and the angle  $OSQ$  is the supplement of the angles  $SOQ$  and  $SQO$  (Euc. I. 32). Therefore

$$\begin{aligned} \text{the angle } HOQ' &= \text{the angle } OSQ \\ &= \text{the angle } OSQ'. \end{aligned} \quad [\text{Prop. XVIII.}]$$

Ex. 1. Two tangents to a parabola, and the points of contact of one of them being given, prove that the locus of the focus is a circle.

The circle may be shown to pass through the given point of contact and the intersection of the tangents, and to touch one of them.

Ex. 2. If a parabola touch the sides of an equilateral triangle, the focal distance of any vertex of the triangle passes through the point of contact of the opposite side.

Ex. 3. Given the base  $AB$  and the vertical angle  $C$  of a triangle  $ACB$ , find the locus of the focus of a parabola touching  $CA$ ,  $CB$  in  $A$  and  $B$ .

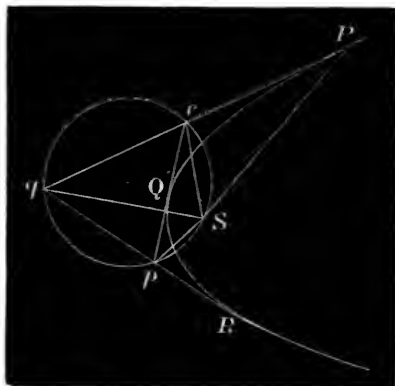
Ex. 4.  $E$  is the centre of the circle described about the triangle

$QQ'$ ; prove that the circle described about the triangle  $QEQ'$  passes through the focus.

Ex. 5. A circle passing through the focus cuts the parabola in two points. Prove that the exterior angle between the tangents to the circle at those points is four times the complement of the exterior angle between the tangents to the parabola at the same points.

\* PROPOSITION XX.

*The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.*



Let the three tangents at the points  $P, Q, R$  form the triangle  $pqr$ .

Join  $SP, Sp, Sq, Sr$ .

The angle  $Srp =$  the angle  $SPr$ , [Prop. XVIII.

and the angle  $Sqp =$  the angle  $SPQ$ ; [Prop. XVIII.

therefore the angle  $Srp =$  the angle  $Sqp$ .

Therefore the points  $p, q, r, S$  lie on a circle, or the circle round the triangle  $pqr$  passes through the focus.

Ex. 1. What is the locus of the focus of a parabola which touches three given straight lines?

Ex. 2. A parabola touches each of four straight lines given in position. Determine its focus.

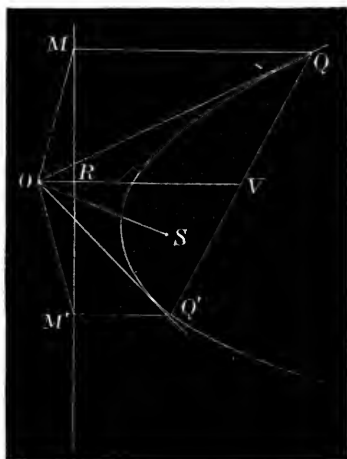
The four circles circumscribing the four triangles formed by the given straight lines, will intersect in the same point, namely, the focus required. Hence, the curve may be described. (See Prop. XXIII., Ex. 5.)

Ex. 3. If through  $p, q, r$  lines be drawn at right angles to  $Sp, Sq, Sr$  respectively, they will meet in a point.

Ex. 4. Prove that the orthocentre of the triangle  $pqr$  lies on the directrix. (Apply Prop. XII.)

### \* PROPOSITION XXI.

*If through the point of intersection of two tangents to a parabola a straight line be drawn parallel to the axis, it will bisect the chord of contact.*



Let  $OQ$  and  $OQ'$  be the two tangents, and let  $OV$  drawn parallel to the axis meet  $QQ'$  in  $V$  and the directrix in  $R$ . Draw  $QM$  and  $Q'M'$  perpendicular to the directrix, and join  $OS, OM, OM'$ .

Then  $OM = OS = OM'$ , [Prop. XVII.  
and  $OR$ , which is drawn at right angles to the base of the isosceles triangle  $OMM'$ , bisects it.

Therefore  $MR = M'R$ .  
But since  $MQ, RV, M'Q'$  are parallel to one another,

$$QV : Q'V = MR : M'R;$$

therefore  $QV = Q'V$ ,

or,  $QQ'$  is bisected in  $V$ .

Ex. 1. The tangents at the extremities of any chord of a parabola meet on the diameter bisecting that chord.

Ex. 2. The circle on any focal chord as diameter touches the directrix.

Ex. 3. The straight lines drawn through the extremities of a focal chord at right angles to the tangents at those points, meet on the diameter bisecting the chord.

Ex. 4. Given two tangents and their points of contact, find the focus and directrix.

Ex. 5. Given two points  $P, Q$  on a parabola, the tangent at one of the points  $P$ , and the direction of the axis, construct the curve.

If the tangent at  $P$  meets the diameter bisecting  $PQ$  in  $T$ ,  $TQ$  is the tangent at  $Q$ . Hence the focus by Prop. XIV.

Ex. 6. If a line be drawn parallel to the chord of contact of two tangents, the parts intercepted on it between the curve and the tangents are equal.

Ex. 7.  $OP, OQ$  are two tangents to a parabola, and  $V$  is the middle point of  $PQ$ . Prove that  $OP \cdot OQ = 2OS \cdot OV$ .

On  $QO$  produced take  $OQ' = OQ$ ; then apply Prop. XVIII. to show that the triangles  $POQ'$  and  $OSQ$  are similar.

Ex. 8. If from any point  $O$  a tangent  $OT$  and a chord  $OPQ$  be drawn, and if the diameter  $TR$  meet the chord in  $R$ , prove that  $OP \cdot OQ = OR^2$ . (Cf. *Tangent Properties*, Ex. 1, 2.)

Draw the tangent  $KO'P'$  parallel to the chord, meeting  $RT$  in  $K$ ,  $OT$  produced in  $O$ , and the curves in  $P'$ . Draw the diameter  $O'H$  bisecting  $TP'$ , so that  $O'P' = KO'$ . Then

$$OP \cdot OQ : OT^2 = O'P'^2 : O'T^2 = OK^2 : O'T^2 = OR^2 : OT^2.$$

Ex. 9. Given a chord  $PQ$  of a parabola in magnitude and position, and the point  $R$  in which the axis cuts the chord, the locus of the vertex is a circle.

If the tangent at the vertex meets  $PQ$  in  $O$ ,  $OP \cdot OQ = OR^2$ .  
∴  $O$  is a fixed point;  $OR = PR \cdot RQ / (PR - RQ)$ .

Ex. 10. The tangents from an external point are divided by any third into segments having the same ratio.

In fig. Prop. XX., draw the diameters  $rr'$ ,  $QQ'$ ,  $qq'$ ,  $pp'$ , meeting  $PR$  in  $r'$ ,  $Q'$ ,  $q'$ ,  $p'$ . Then

$$Pr : rq = rQ : Qp = qp : pR.$$

(Cf. Prop. XVIII., Ex. 11.)

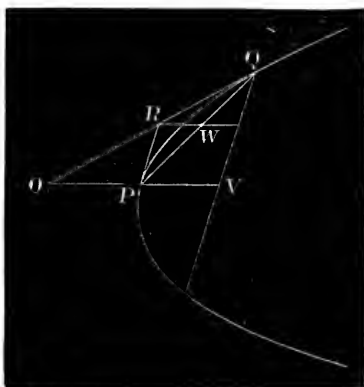
Ex. 11. The tangent parallel to  $QQ'$  bisects  $OQ$ ,  $OQ'$ .

Ex. 12. If  $E$  be the centre of the circle through  $O$ ,  $Q$ ,  $Q'$ ,  $OE$  subtends a right angle at  $S$ . (Apply Prop. XX., and Ex. 11.)

Ex. 13. If  $OQQ'$  be a right angle and  $QN$  the ordinate of  $Q$ , prove that  $QQ' : OQ = QN : AN$ .  
(Cf. Prop. XVI.)

### \* PROPOSITION XXII.

*If  $QV$  is the ordinate of a diameter  $PV$  of a parabola, and the tangent at  $Q$  meets  $VP$  produced in  $O$ , then  $OP$  shall be equal to  $PV$ .*



Let the tangent at  $P$  meet  $OQ$  in  $R$ ; through  $R$  draw the diameter  $RW$ , meeting  $PQ$  in  $W$ .

Then, since  $RP$ ,  $RQ$  are a pair of tangents,

$$QW = PW.$$

[Prop. XXI.

Also,  $RP$  is parallel to  $QV$ ;

[Prop. XI.



therefore  $OP : PV = OR : RQ$   
 $= PW : WQ.$

But  $PW = WQ;$

therefore  $OP = PV.$

Ex. 1. Tangents at the extremities of all parallel chords meet on the same straight line. (Cf. Prop. XXI., Ex. 1.)

Ex. 2. Given a tangent and a point on the curve, find the locus of the foot of the ordinate of the point of contact of the tangent, with respect to the diameter through the given point. [A right line parallel to the tangent.]

Ex. 3. If  $OV = QV$ ,  $O$  is on the directrix.

Ex. 4. If the diameter  $PV$  meets the directrix in  $O$ , and the chord drawn through the focus parallel to the tangent at  $P$  in  $V$ , prove that  $VP = OP$ .

Ex. 5. If  $OQ, OQ'$  be a pair of tangents to a parabola, and  $OQO'$  be a right angle,  $OQ$  will be bisected by the directrix.

Draw the diameter  $OPV$  and the tangent at  $P$ . (See Prop. XVI., Ex. 9.)

Ex. 6. If  $QV$  be an ordinate to the diameter  $PV$ , and  $pv$  meeting  $PQ$  in  $v$  be the diameter bisecting  $PQ$ , prove that  $PV = 4pv$ .

Ex. 7.  $PQ, PR$  are any two chords; they meet the diameters through  $R$  and  $Q$  in  $F$  and  $E$ . Show that  $EF$  is parallel to the tangent at  $P$ .

Ex. 8. If from the point of contact of a tangent a chord be drawn, and any line parallel to the axis be drawn meeting the tangent, curve, and chord, this line will be divided by them in the same ratio as it divides the chord.

Let the diameter  $RBV$  bisecting the chord  $QQ'$  in  $V$  meet the tangent at  $Q$  in  $R$ . Draw the line  $rbv$  parallel to the axis, cutting the curve and chord in  $b$  and  $v$ . Then

$$Qv : vr = QV : VR$$

$$= QV : 2VB.$$

But  $QV^2 = 4BS \cdot BV;$  (Prop. VII.)

$\therefore QV : 2BV = 2SB : QV;$

$\therefore Qv \cdot QV = 2SB \cdot vr.$

Also  $Qv \cdot Q'v = 4SB \cdot vb;$  (Prop. VIII.)

$\therefore QQ' : Q'v = rv : vb;$

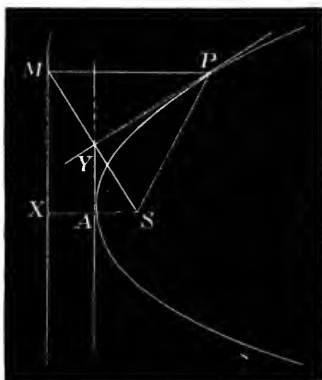
$\therefore Qv : Q'v = rb : bv.$

This is a generalisation of Prop. XXII.

Ex. 9. Through a given point within a parabola, draw a chord which shall be divided in a given ratio at that point.

## PROPOSITION XXIII.

*The locus of the foot of the perpendicular from the focus upon any tangent to a parabola is the tangent at the vertex.*



Draw  $SY$  perpendicular to the tangent at  $P$ , meeting it in  $Y$ . It is required to show that  $Y$  lies on the tangent to the parabola at the vertex.

Draw  $PM$  perpendicular to the directrix, and join  $MY, AY$ .

Now, in the two triangles  $MPY, SPY$ , the sides  $MP, PY$  are equal to the sides  $SP, PY$  respectively, and the angle  $MPY =$  the angle  $SPY$ . [Prop. XIV.

Therefore the angle  $PYM =$  the angle  $PYS$   
 $=$  one right angle; [Euc. I. 4.

therefore  $MY$  and  $YS$  are in the same straight line.

[Euc. I. 14.

Now, since  $SY = YM,$

and  $SA = AX,$

$AY$  is parallel to  $MX,$

[Euc. VI. 2.

and is, therefore, the tangent to the parabola at the vertex. [Prop. XIV., Cor.]

Ex. 1. Show that  $SY^2 = AS \cdot SP$ . [The triangles  $SYP$ ,  $SYA$  are similar. See Prop. XVIII., Ex. 1.]

Ex. 2. Show that  $SM$  is bisected at right angles by the tangent at  $P$ .

Ex. 3. If the tangent at  $P$  meet the axis in  $T$ , and  $PV$  be the ordinate of  $P$ , prove that  $PT \cdot TY = NT \cdot TS$ .

Ex. 4. If the vertex of a right angle, one leg of which always passes through a fixed point, moves along a fixed right line, the other leg will always touch a parabola.

The fixed point will be the focus, and the fixed right line the tangent at the vertex, whence the directrix is known.

Ex. 5. Given two tangents and the focus of a parabola, find the directrix.

The line joining the feet of the perpendiculars from the focus on the given tangents, is clearly the tangent at the vertex.

Ex. 6. Prove that straight lines perpendicular to the tangents of a parabola through the points where they meet a given fixed line parallel to the directrix, touch a confocal parabola.

Ex. 7. The focus and a tangent being given, the locus of the vertex is a circle.

Ex. 8. Given a tangent and the vertex, find the locus of the focus. [A parabola, of which  $A$  is the vertex and the axis the perpendicular through  $A$  on the tangent. Apply Prop. VII., Ex. 7.]

Ex. 9. The circle described on any focal distance as diameter, touches the tangent at the vertex.

Ex. 10.  $PSp$  is a focal chord; prove that the length of the common tangent of the circles described on  $Sp$ ,  $SP$  as diameters, is  $\sqrt{(AS \cdot Pp)}$ .

Ex. 11. Prove that

$$\begin{aligned} \text{(i.) } & PY \cdot PZ = PS^2; \\ \text{(ii.) } & PY \cdot YZ = AS \cdot SP. \end{aligned}$$

Ex. 12. A circle is described on the latus rectum as diameter;  $PQ$  touches the parabola at  $P$  and the circle at  $Q$ ; show that  $SP$ ,  $SQ$  are each inclined to the latus rectum at an angle of  $30^\circ$ .

Ex. 13. *Alternative Construction for Prop. XVII.*

Let  $O$  be the external point; on  $OS$  as diameter describe a circle; the lines joining  $O$  with the points of intersection of this circle with the tangent at the vertex, will be the required tangents.

Ex. 14. In the figure of Prop. VII., prove that  $QD^2 = 4AS \cdot BV$ . Let the tangent at  $B$  meet the axis in  $T$ , and the tangent at  $A$  in

Y. Then  $SYZ$  is a right angle, and the triangles  $QDV$ ,  $YAT$  are similar (Prop. XI.)

$$\therefore QD^2 : QV^2 = YA^2 : YT^2 = AS : TS = AS : BS.$$

But

$$QD^2 = 4BS \cdot BV. \quad (\text{Prop. VII.})$$

$$\therefore QV^2 = 4AS \cdot BV.$$

Ex. 15. Given the focus and two tangents, construct the curve. [Ex. 5].

Ex. 16. Given the focus, axis and a tangent, construct the parabola.

Ex. 17. Given the focus, a point  $P$  on the parabola, and the length of the perpendicular from the focus on the tangent at  $P$ , construct the curve.

Ex. 18. Given the focus, a tangent, and the length of the latus rectum, construct the curve.

Ex. 19. If a parabola roll upon another equal parabola, the vertices originally coinciding, the focus of the one traces out the directrix of the other. [The line joining the foci in any position cuts at right angles the common tangent.]

### PROPERTIES OF NORMALS.

**Def.** The straight line which is drawn through any point on a conic at right angles to the tangent at that point is called the *normal* at that point.

**Def.** The portion of the axis intercepted between the normal at any point of a conic and the ordinate of that point is called the *subnormal*.

### PROPOSITION XXIV.

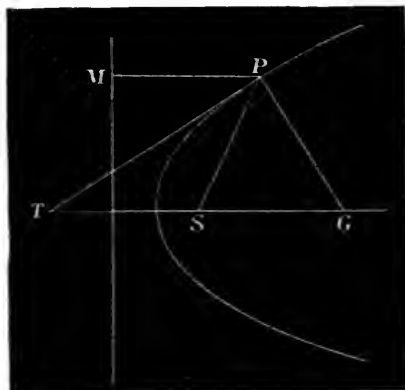
*The normal at any point of a parabola makes equal angles with the focal distance and the axis.*

Let the normal  $PG$  and the tangent  $PT$  at any point  $P$  on the parabola meet the axis in  $G$  and  $T$  respectively. Join  $SP$  and draw  $PM$  perpendicular to the directrix.

Then the angle  $SPT =$  the angle  $TPM$  [Prop. XIV.  
 $=$  the angle  $STP$ . [Euc. I. 29.]

But the angle  $TPG$  being a right angle is equal to the sum of the angle  $STP$  and  $SGP$ . [Euc. I. 32.

Therefore the angle  $SPG =$  the angle  $SGP$ .



Ex. 1. Prove that  $ST = SP = SG$ .

Ex. 2. The normal at any point bisects the interior angle between the focal distance and the diameter through that point.

Ex. 3. The focus is equidistant from  $PT$  and the straight line through  $G$  parallel to  $PT$ .

Ex. 4. From the points where the normals to a parabola meet the axis, lines are drawn at right angles to the normals; show that these lines touch an equal confocal parabola.

Ex. 5. A chord  $PQ$  of a parabola is normal to the curve at  $P$ , and subtends a right angle at  $S$ ; show that  $SQ = 2SP$ .

Ex. 6. Prove that  $SM$  and  $PT$  bisect each other at right angles.

Ex. 7. If the triangle  $SPG$  is equilateral,  $TG$  subtends a right angle at  $M$ .

Ex. 8. Prove that the points  $S, P, M, Z$  lie on a circle which touches  $PG$  at  $P$ .

Ex. 9. If in Ex. 8 the radius of the circle is equal to  $MZ$ , the triangle  $SPG$  is equilateral.

Ex. 10.  $PSp$  is a focal chord;  $pG$  is the normal at  $p$ ;  $GH$  is perpendicular on the tangent at  $P$ . Prove that  $H$  lies on the latus rectum. (Cf. Prop. XIV., Ex. 10.)

Ex. 11. If  $PF, PH$  be drawn to the axis, making equal angles

with the normal  $PG$ , prove that  $SG^2 = SF \cdot SH$ . [The triangles  $SPF$ ,  $SHP$  are similar.]

Ex. 12. If  $SY$ ,  $SZ$  be perpendicular to the tangent and normal at  $P$  respectively, prove that  $YZ$  is a diameter.

### PROPOSITION XXV.

*The subnormal of any point of a parabola is equal to half the latus rectum.*



Let the normal  $PG$  at  $P$  meet the axis in  $G$ . Draw  $PM$ ,  $PN$  perpendicular to the directrix and axis respectively. Join  $SP$ .

Then, the angle  $SPG =$  the angle  $SGP$ . [Prop. XXIV.

Therefore  $SG = SP = PM = NX$ .

Therefore  $NG = XS = 2AS = \frac{1}{2}$  latus rectum. [Prop. II.

The subnormal is therefore of constant length.

Ex. 1. If the triangle  $SPG$  is equilateral,  $SP$  is equal to the latus rectum.

Ex. 2. Show how to draw the normal at any given point without drawing the tangent.

Ex. 3. If the ordinate of a point  $Q$  bisect the subnormal of a

point  $P$ , the ordinate of  $Q$  is equal to the normal at  $P$ . (Apply Prop. IV.)

Ex. 4. Prove that  $PG^2 = 4AS \cdot SP$ .

Ex. 5. If  $C$  be the middle point of  $SG$ , prove that  
 $CG^2 - CP^2 = 4AS^2$ .

Ex. 6. If  $PL$  perpendicular to  $AP$  meets the axis in  $L$ , prove that  
 $GL = 2AS$ .

Ex. 7.  $TP, TQ$  are tangents to a given circle at  $P$  and  $Q$ . Construct a parabola which shall touch  $TP$  in  $P$  and have  $TQ$  for axis.

Ex. 8. The locus of the foot of the perpendicular from the focus on the normal is a parabola.

[Apply Prop. IV.  $SG$  is the axis, the vertex is at  $S$ , the latus rectum  $= AS$ .]

Ex. 9. If  $GK$  be drawn perpendicular to  $SP$ , prove that  
 $PK = 2AS$ .

Ex. 10.  $Pp$  is a chord perpendicular to the axis; the perpendicular from  $p$  on the tangent at  $P$  meets the diameter through  $P$  in  $R$ ; prove that  $RP = 4AS$ , and find the locus of  $R$ .

[The triangles  $PNG, RPp$  are similar. The locus of  $R$  is an equal parabola, having its vertex  $A'$  on the opposite side of  $X$ , such that  $AA' = 4AS$ .]

Ex. 11. A circle described on a given chord of a parabola as diameter cuts the curve again in two points; if these points be joined, the portion of the axis intercepted by the two chords is equal to the latus rectum.

Show also that, if the given chord is fixed in direction, the length of the line joining the middle points of the chords is constant.

[Apply Prop. VIII. The middle points of the chords are equidistant from the axis.]

### MISCELLANEOUS EXAMPLES ON THE PARABOLA.

1. Find the locus of the point of intersection of any tangent to a parabola, with the line drawn from the focus, making a constant angle with the tangent.

2.  $OQ, OQ'$  are tangents to a parabola;  $V$  is the middle point of  $QQ'$ ;  $OV$  meets the directrix in  $K$ , and  $QQ'$  meets the axis in  $N$ . Prove that  $OKNS$  is a parallelogram.

3. Inscribe in a given parabola a triangle having its sides parallel to those of a given triangle.

4. Inscribe a circle in the segment of a parabola cut off by a double ordinate.

5.  $PGQ$  is a normal chord of a parabola, meeting the axis in  $G$ . Prove that the distance of  $G$  from the vertex, the ordinates of  $P$  and  $Q$ , and the latus rectum are four proportionals.

6. If  $AR$ ,  $SY$  are perpendiculars from the vertex and focus upon any tangent, prove that

$$SY^2 = SY \cdot AR + SA^2.$$

7. Describe a parabola touching three given straight lines and having its focus on another given line.

8.  $OP$ ,  $OQ$  are tangents to a parabola at the points  $P$ ,  $Q$ . If  $SP + SQ$  is constant, prove that the locus of  $O$  is a parabola, and find its latus rectum.

9. Through any point on a parabola two chords are drawn, equally inclined to the tangent there; show that their lengths are proportional to the portions of their diameters intercepted between them and the curve.

10. The focal chord  $PSp$  is bisected at right angles by a line which meets the axis in  $O$ ; show that  $Pp = 2 \cdot SO$ .

11. On a tangent are taken two points equidistant from the focus; prove that the other tangents drawn from these points will intersect on the axis.

12. The locus of the centre of the circle circumscribing the triangle formed by two fixed tangents and any third tangent is a right line.

13. A chord  $PQ$  is normal to the parabola at  $P$ , and subtends a right angle at the vertex; prove that  $SQ = 3 \cdot SP$ .



14. Given the vertex, a tangent, and the latus rectum, construct the parabola.

15.  $P, Q$  are variable points on the sides  $AC, AB$  of a given triangle, such that  $AP:PC=BQ:QA$ . Prove that  $PQ$  touches a parabola.

16. Apply properties of the parabola to prove that—

(i.) In any triangle the feet of the three perpendiculars from any point of the circumscribing circle on the sides lie on the same straight line.

(ii.) If four intersecting straight lines be taken three together, so as to form four triangles, the orthocentres of these triangles lie on a right line.

17. Describe a parabola through four given points.

18. A parabola rolls on an equal parabola, the vertices originally coinciding. Prove that the tangent at the vertex of the rolling parabola always touches a fixed circle.

19. If two intersecting parabolas have a common focus, the angle between their axes is equal to that which their common tangent subtends at the focus.

20.  $AP, AQ$ , are two fixed straight lines, and  $B$  a fixed point. Circles described through  $A$  and  $B$  cut the fixed lines in  $P$  and  $Q$ . Prove that  $PQ$  always touches a parabola with its focus at  $B$ .

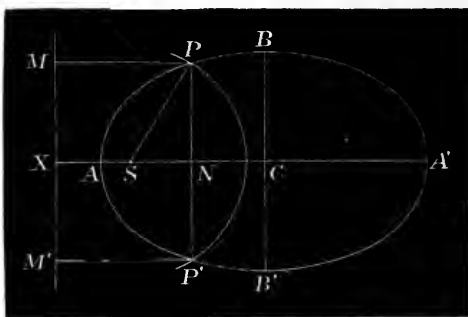
## CHAPTER II.

### THE ELLIPSE.

#### DESCRIPTION OF THE CURVE.

##### PROPOSITION I.

*Given the focus, directrix, and eccentricity of an ellipse to determine any number of points on it.*



Let  $S$  be the focus,  $MXM'$  the directrix, and  $e$  the eccentricity.

Through  $S$  draw  $SX$  perpendicular to the directrix. Divide  $SX$  in  $A$ , so that

$$SA = eAX.$$

Also in  $XS'$  produced, take  $A'$  so that

$$SA' = eA'X.*$$

Then  $A$  and  $A'$  are points on the ellipse and are its vertices.

Take any point  $N$  on  $AA'$ ; through  $N$  draw  $PNP'$  perpendicular to  $AA'$ ; with centre  $S$  and radius equal to  $e \cdot XN$ , describe a circle, cutting  $PNP'$  in  $P$  and  $P'$ . Then  $P$  and  $P'$  shall be points on the ellipse. Draw  $PM, P'M'$  perpendicular to the directrix.

Then  $SP = e \cdot XN$  [Const.  
 $= e \cdot PM,$

and  $SP' = e \cdot XN$   
 $= e \cdot P'M'.$

Therefore  $P$  and  $P'$  are points on the ellipse.

In like manner, by taking any other point on  $AA'$ , any number of points on the curve may be determined.

**Def.** The length of the axis intercepted between the vertices ( $A$  and  $A'$ ) of the ellipse is called the *major axis*.

**Def.** The middle point ( $C$ ) of the major axis is called the *centre* of the ellipse.

**Def.** The double ordinate ( $BCB'$ ) through the centre ( $c$ ) is called the *minor axis* of the ellipse.

**Ex. 1.** The ellipse is symmetrical with respect to its axis.

Corresponding to any point  $N$  on the line  $AA'$  we get two points  $P$  and  $P'$ , such that the chord  $PP'$  is bisected at right angles by the axis  $AA'$ .

**Ex. 2.** Any two right lines drawn from any point on the axis to the curve, on opposite sides of the axis and equally inclined to it, are equal, and conversely.

**Ex. 3.** If two equal and similar ellipses have a common centre, the points of intersection are at the extremities of central chords at right angles to each other.

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\* Since  $e$  is less than unity it is clear that  $A$  will lie between  $X$  and  $S$  and  $A'$  without  $XS$  on the same side as  $S$ .

Ex. 4. Prove that the ellipse lies entirely between the lines drawn through  $A$  and  $A'$  at right angles to the axis.

In order that the circle may intersect  $PNP'$  the point  $N$  must be so situated that  $SN$  may not be greater than the radius of the circle  $SP$ , that is,  $eNX$ . It may easily be shown that this is the case only when  $N$  lies between  $A$  and  $A'$ .

Ex. 5. Show that as  $P$  moves from  $A$  to  $A'$ , its focal distance ( $SP$ ) increases from  $SA$  to  $SA'$ .

For  $SP=e.NX$ , and  $NX$  has  $AX$  and  $A'X$  for its least and greatest values respectively.

Ex. 6. Hence prove that the ellipse is a closed curve.

Ex. 7. If a parabola and an ellipse have the same focus and directrix, the parabola lies entirely outside the ellipse.

Ex. 8. A chord  $QQ'$  of an ellipse meets the directrix in  $D$ . Prove that

$$SQ : QD = SQ' : Q'D.$$

Ex. 9. A straight line meets the ellipse at  $P$  and the directrix in  $D$ . From any point  $K$  in  $PD$ ,  $KU$  is drawn parallel to  $DS$  to meet  $SP$  in  $U$ , and  $KI$  is drawn perpendicular to the directrix. Prove that  $SU=e.KI$ . (Cf. Prop. XVI., which is a particular case of this.)

Ex. 10. A point  $P$  lies within, on or without the ellipse, according as the ratio  $SP : PM$  is less than, equal to, or greater than the eccentricity,  $PM$  being the perpendicular on the directrix.

### PROPOSITION II.

*The ellipse is symmetrical with respect to the minor axis and has a second focus ( $S'$ ) and directrix.*

Let  $S$  be the given focus and  $MX$  the given directrix.

Take any point  $M$  on the directrix, and through the vertices  $A$  and  $A'$  draw  $AH$  and  $A'H'$  at right angles to  $AA'$ , meeting the straight line through  $M$  and  $S$  at  $H$  and  $H'$  respectively. Describe a circle on  $HH'$  as diameter and through  $M$  draw  $MPP'$ , parallel to  $AA'$ , to meet the circle in  $P$  and  $P'$ . Then  $P$  and  $P'$  shall be points on the ellipse.

For

$$MH : HS = XA : AS = 1 : e,$$

and

$$MH' : H'S = XA' : A'S = 1 : e.$$

Therefore  $MH : HS = MH' : H'S$ ,

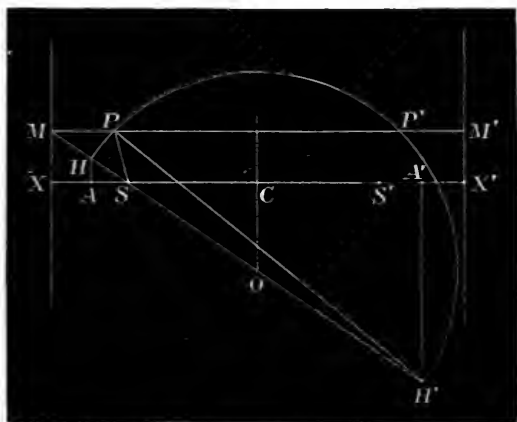
and the angle  $HPH'$  is a right angle.

[Euc. III. 31.]

Therefore,  $PH$  bisects the angle  $SPM$ .

Therefore  $SP : PM = SH : HM$   
 $= AS : AX$   
 $= e$ .

Therefore,  $P$  is a point on the ellipse. Similarly, it may be shown that  $P'$  is a point on the ellipse.



Again, the straight line drawn through  $O$ , the centre of the circle, at right angles to  $AA'$  will bisect both  $AA'$  and  $PP'$  at right angles, and will therefore coincide with the minor axis in position.

The ellipse is therefore symmetrical with respect to the minor axis. [Def.

As the minor axis divides the curve into two parts such that each is the exact reflexion of the other, if  $A'S'$  be measured off equal to  $AS$  and  $A'X' = AX$ , and  $X'M'$  be

drawn at right angles to  $X'X$ , the curve could be equally well described with  $S'$  as focus and  $X'M'$  as directrix.

The ellipse therefore has a second focus ( $S'$ ) and a second directrix ( $X'M'$ ).

Ex. Every chord drawn through the centre  $C$  is bisected at that point. (From the symmetry of the figure.)

From this property the point  $C$  is called the centre of the curve.

### PROPERTIES OF CHORDS AND SEGMENTS OF CHORDS.

#### PROPOSITION III.

*In the ellipse*  $CA = e \cdot CX$ .....(1)

$CS = e \cdot CA$ .....(2)

$CS \cdot CX = CA^2$ .....(3)



We have, from the definition,

$$SA = e \cdot AX,$$

$$SA' = e \cdot A'X = e \cdot AX'.$$

Therefore, by addition,

$$\begin{aligned} AA' &= e(AX + AX') \\ &= eXX'. \end{aligned}$$

Therefore  $CA = e \cdot CX$ .....(1)

By subtraction,  $SS' = e(A'X - AX)$   
 $= e \cdot AA'$ .

Therefore  $CS = e \cdot CA$ .....(2)

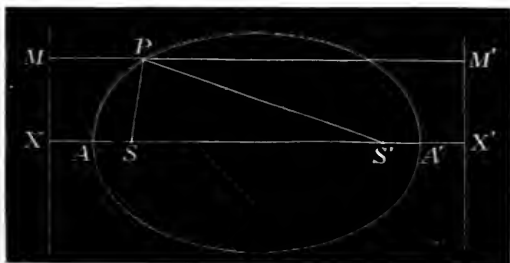
Therefore  $CS \cdot CX = CA^2$ .....(3)

**Ex.** Given the ellipse and one focus, find the centre and the eccentricity.

Describe a circle with  $S$  as centre, cutting the curve in  $P, P'$ . The axis bisects  $PP'$  at right angles.

PROPOSITION IV.

*The sum of the focal distances of any point on an ellipse is constant and equal to the major axis.*



Let  $P$  be any point on the ellipse. Join  $PS, PS'$ , and through  $P$  draw  $MPM'$  perpendicular to the directrices.

Then  $SP = e \cdot PM,$   
 $S'P = e \cdot PM'.$

Therefore  $SP + S'P = e \cdot (PM + PM')$   
 $= e \cdot MM'$   
 $= e \cdot XX'$   
 $= AA'.$

[Prop. III.]

**Ex. 1.** Show how to construct the ellipse mechanically.

*First Method.*—Fasten the ends of a string to two drawing pins fixed at  $S$  and  $S'$  on a board, and trace a curve on the board with a pencil pressed against the string, so as to keep it always

stretched. The curve traced out will be an ellipse, with foci at  $S$  and  $S'$ , and major axis equal to the length of the string.

*Second Method.*—Suppose two equal thin circular discs  $A$  and  $B$ , attached to each other, to rotate in opposite directions round an axis through their common centre; and, suppose one end of a fine string (which is wrapped round the discs, and passing through small rings at  $C$  and  $D$  in the plane of the discs, is kept stretched by the point of a pencil at  $P$ ) to be wound on to its disc, while the other is wound off. The curve traced by  $P$  will have the property

$$CP + DP = \text{constant},$$

and will, therefore, be an ellipse.

Ex. 2. The sum of the focal distances of any point is greater than, equal to, or less than the major axis, according as the point is without, upon, or within the ellipse, and conversely.

Ex. 3. The distance of either extremity of the minor axis from either focus is equal to the semi-axis-major.

Ex. 4. A circle is drawn entirely within another circle. Prove that the locus of a point equidistant from the circumferences of the two circles, is an ellipse. [The centres will be the foci.]

Ex. 5. Two ellipses have a common focus, and their major axes equal. Show that they cannot intersect in more than two points.

The common points may be shown to lie on the line bisecting at right angles the line joining the second foci.

Ex. 6. Prove that the external bisector of the angle  $SPS'$  cannot meet the ellipse again, and is, therefore, the tangent to the ellipse at  $P$ , according to Euclid's conception of a tangent. (Cf. Prop. XVII.)

Prove also that every other line through  $P$  will meet the curve again. [Apply Ex. 2.]

Ex. 7. The major axis is the longest chord that can be drawn in the ellipse.

Joining the foci with the extremities of any chord, it may be shown that twice the chord is less than the sum of the four focal distances, that is, less than twice the major axis.

Ex. 8. In what position of  $P$  is the angle  $SPS'$  greatest? [When  $P$  is at either extremity of the minor axis.]

Ex. 9. If  $r$  and  $R$  be the radii of the circles inscribed in and described about the triangle  $SPS'$ , prove that  $Rr$  varies as  $SP \cdot S'P$ .

#### PROPOSITION V.

*In the ellipse*

$$CB^2 = CA^2 - CS^2 = SA \cdot SA'.$$



Let  $B$  be an extremity of the minor axis. Join  $BS$ ,  $BS'$ .

Then  $SB + S'B = AA'$ . [Prop. IV.]

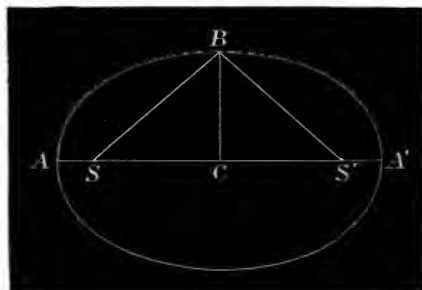
But  $SB = S'B$ .

Therefore  $SB = CA$ .

Therefore  $CB^2 = SB^2 - CS^2$  [Euc. I. 47.]

$$= CA^2 - CS^2$$

$$= SA \cdot S'A. \quad [\text{Euc. II. 5.}]$$



Ex. 1. Prove that  $e^2 = 1 - \frac{CB^2}{CA^2}$ .

Ex. 2. Prove that  $S'S^2 = A'A^2 - B'B^2$ .

Ex. 3. If the angle  $SBS'$  be a right angle, show that  $CA = \sqrt{2} \cdot CB$ .

Ex. 4. A circle is described passing through  $B$  and touching the major axis in  $S$ ; if  $SK$  be its diameter, prove that  $SK \cdot BC = AC^2$ .

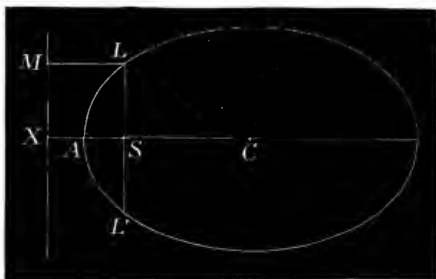
Ex. 5. Circles are described on the major and minor axes as diameters.  $PP'$  is a chord of the outer circle cutting the inner in  $Q, Q'$ . Prove that  $PQ \cdot P'Q = CS^2$ .

Ex. 6. Given a focus  $S$  and a point  $P$  on an ellipse, and the lengths of the major and minor axes, find the centre.

On  $SP$  produced, take  $SK$  equal to the major axis;  $S'$  lies on the circle with centre  $P$  and radius  $PK$ . On  $SK$  as diameter describe a circle, and place in it  $KH$  equal to the minor axis;  $S'$  lies on the circle with centre  $S$  and radius  $SH$ .

## \* PROPOSITION VI.

*The latus rectum of an ellipse is a third proportional to the major and minor axes ( $SL = CB^2/CA$ ).*



Let  $LSL'$  be the latus rectum. Draw  $LM$  perpendicular to the directrix.

$$\begin{aligned} \text{Then} \quad CS &= eCA, & [\text{Prop. III.}] \\ SL &= eLM & [\text{Def.}] \\ &= eSX; \end{aligned}$$

$$\begin{aligned} \text{therefore} \quad SL \cdot CA &= CS \cdot SX \\ &= CS(CX - CS) \\ &= CS \cdot CX - CS^2 \\ &= CA^2 - CS^2 & [\text{Prop. III.}] \\ &= CB^2. & [\text{Prop. V.}] \end{aligned}$$

Ex. 1. Construct on the minor axis as base a rectangle which shall be to the triangle  $SLS'$  in the duplicate ratio of the major axis to the minor axis.

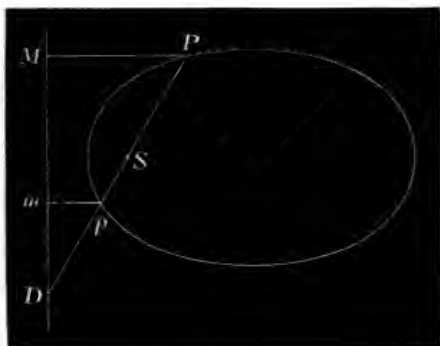
Draw  $BK$  parallel to  $LS'$ , meeting the major axis in  $K$ ; the other side of the rectangle  $= \frac{1}{2}CK$ .

Ex. 2. The extremities of the latera recta of all ellipses which have a common major axis, lie on two parabolas.

If  $LN$  be perpendicular to  $CB$ ,  $LN^2 = AC(AC - CN)$ ; hence,  $L$  lies on a parabola of which  $CB$  is the axis, and the vertex is at a distance from  $C = CA$

\* PROPOSITION VII.

*Any focal chord of an ellipse is divided harmonically by the focus and the directrix.*



Produce the focal chord  $PSp$  to meet the directrix in  $D$ , and draw  $PM$ ,  $pm$  perpendicular to the directrix.

Then  $PD : pD = PM : pm$ ,

but  $PS = e \cdot PM$ ,

and  $pS = e \cdot pm$ ;

therefore  $PD : pD = PS : pS$ .

Hence  $Pp$  is divided harmonically in  $S$  and  $D$ .

Ex. 1. The semi-latus rectum is a harmonic mean between the segments of any focal chord.

Ex. 2. Focal chords are to one another as the rectangles contained by their segments.

PROPOSITION VIII.

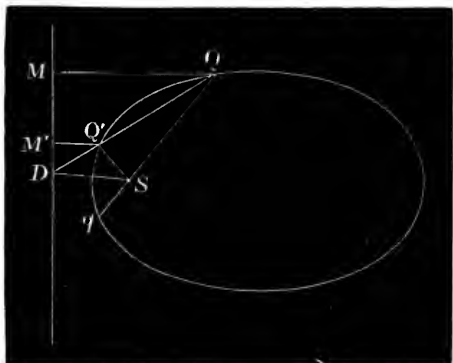
*If any chord  $QQ'$  of an ellipse intersects the directrix in  $D$ ,  $SD$  bisects the exterior angle between  $SQ$  and  $SQ'$ .*

Draw  $QM$ ,  $Q'M'$  perpendiculars on the directrix, and produce  $QS$  to meet the ellipse in  $q$ .

Then, by similar triangles,

$$\begin{aligned} QD : Q'D &= QM : Q'M' \\ &= SQ : SQ'; \end{aligned}$$

therefore  $SD$  bisects the exterior angle  $Q'Sq$ . [Euc. VI. A.]



Ex. 1.  $PSp$  is a focal chord. Prove that  $XP$  and  $Xp$  are equally inclined to the axis.

Ex. 2. Given the focus and three points on an ellipse, find the directrix and the axis.

Ex. 3. If  $P$  be any point on an ellipse, and  $PA, PA'$  when produced meet the directrix in  $E$  and  $F$ , show that  $EF'$  subtends a right angle at the focus.

Ex. 4. If  $A'S'$  be measured off along  $A'A$  equal to  $AS$ , and  $A'X'$  be measured off along  $AA'$  equal to  $AX$ , and if  $PA$  and  $PA'$  when produced meet the straight line through  $X'$  at right angles to the axis in  $E', F'$ , show that  $E'X'.F'X' = EX.FX$ , and that  $E'F'$  subtends a right angle at  $S'$ . (This is to be proved without assuming the existence of the second focus and directrix of the curve.)

Ex. 5. Hence, show that if  $PK$  be the perpendicular on  $E'F'$ ,  $S'P = e.PK$ ; and deduce the existence of a second focus and directrix corresponding to the vertex  $A'$ .

Ex. 6. If two fixed points  $Q, Q'$  on an ellipse be joined with a third variable point  $O$  on the curve, the segment  $qq'$  intercepted on either directrix by the chords  $QO$  and  $Q'O$  produced, subtends a constant angle at the corresponding focus.

The angle  $qSq'$  may be proved to be equal to half of the angle  $QSQ'$ .

Ex. 7.  $PSp$  is a focal chord;  $O$  is any point on the curve;  $PO$ ,  $pO$  produced meet the directrix in  $D, d$ . Prove that  $Dd$  subtends a right angle at the focus.

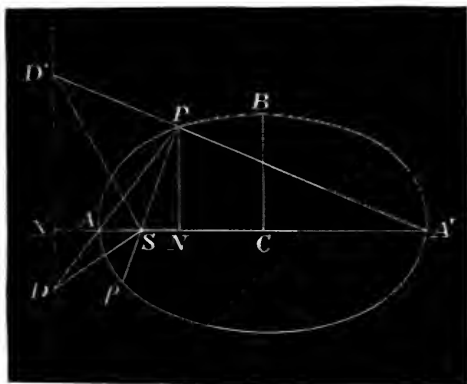
Ex. 8. Given the focus of an ellipse and two points on the curve, prove that the directrix will pass through a fixed point.

Ex. 9. A straight line which meets an ellipse will, in general, meet it in two points, and no straight line can meet it in more points than two.

The first part follows at once from the fact that the ellipse is a closed curve. (Prop. I., Ex. 6. Cf. also Ch. I., Prop. X., Ex. 8.) Then, if the line meets the curve in  $Q$  and  $Q'$ , and the directrix in  $D, S$  and  $SQ'$  will be equally inclined to  $DS$ . Hence, if there be a third point of intersection  $Q'', SQ'$  and  $SQ''$  will make the same angle with  $DS$ , which is impossible.

PROPOSITION IX.

*The square of the ordinate of any point on an ellipse varies as the rectangle under the segments of the axis made by the ordinate ( $PN^2 : AN \ A'N = CB^2 : CA^2$ ).*



Let  $PN$  be the ordinate of any point  $P$  on the ellipse. Let  $PA$  and  $A'P$  produced meet the directrix in  $D$  and  $D'$ . Join  $SD, SD'$ , and  $SP$ , and produce  $PS$  to meet the curve in  $p$ .

Then, from the similar triangles  $PAN$  and  $DAX$ ,  
 $PN : AN = DX : AX$ .

Also, from the similar triangles  $PA'N$  and  $D'A'X$ ,  
 $PN : A'N = D'X : A'X$ ;

therefore  $PN^2 : AN \cdot A'N = DX \cdot D'X : AX \cdot A'X$ .

Again,  $SD$  and  $SD'$  bisect the angles  $pSX$  and  $PSX$  respectively; [Prop. VIII.

therefore the angle  $DSD'$  is a right angle, and

$$DX \cdot D'X = SX^2; \quad [\text{Euc. VI. 8.}]$$

therefore  $PN^2 : AN \cdot A'N = SX^2 : AX \cdot A'X$ .

But the ratio  $SX^2 : AX \cdot A'X$  is constant; therefore the ratio  $PN^2 : AN \cdot A'N$  has the same value for all positions of  $P$ .

In the particular case when  $P$  coincides with the extremity  $B$  of the minor axis, the ratio  $PN^2 : AN \cdot A'N$  becomes  $CB^2 : CA^2$ ; therefore

$$PN^2 : AN \cdot A'N = CB^2 : CA^2,$$

$P$  being any point on the ellipse.

Ex. 1. Prove that  $PN^2 : CA^2 - CN^2 = CB^2 : CA^2$ .

Ex. 2. Prove that  $\frac{CN^2}{CA^2} + \frac{PN^2}{CB^2} = 1$ ,

Ex. 3. Prove that  $CP^2 = CB^2 + e^2 \cdot CN^2$ ; and hence deduce that of all lines drawn from the centre to the curve  $CA$  is the greatest and  $CB$  the least. (See Prop. V., Ex. 1.)

Ex. 4. Show that  $PN$  increases as  $N$  moves from  $A$  to  $C$ .

Ex. 5. If  $PM$  be drawn perpendicular to the minor axis, deduce that  $PM^2 : BM \cdot B'M = CA^2 : CB^2$ .

Ex. 6.  $P, Q$  are two points on an ellipse.  $AQ, A'Q$  cut  $PN$  in  $L$  and  $M$  respectively. Prove that  $PN^2 = LN \cdot MN$ .

Ex. 7. Deduce Prop. VI.

Ex. 8. If  $NQ$  be drawn parallel to  $AB$ , meeting the minor axis in  $Q$ , show that  $PN^2 = BQ \cdot B'Q$ .

Ex. 9. If a point  $P$  moves such that  $PN^2 : AN \cdot A'N$  in a constant ratio,  $PN$  being the distance of  $P$  from the line joining two fixed

points  $A, A',$  and  $N$  being between  $A$  and  $A'$ , the locus of  $P$  is an ellipse of which  $AA'$  is an axis.

Ex. 10. The locus of the intersection of lines drawn through  $A, A'$  at right angles to  $AP, A'P$ , is an ellipse. [ $AA'$  will be the minor axis. See Ex. 5, 9.]

Ex. 11. The tangent at any point  $P$  of a circle meets the tangent at the extremity  $A$  of a fixed diameter  $AB$  in  $T$ . Find the locus of the point of intersection ( $Q$ ) of  $AP$  and  $BT$ .

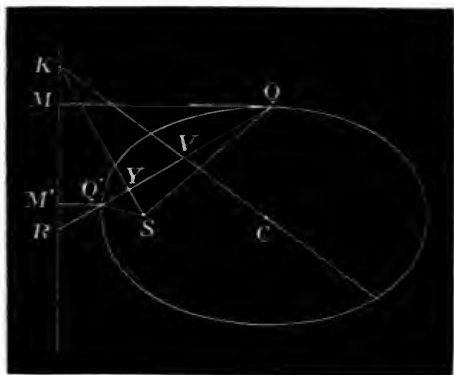
$QM$  being perpendicular to  $AB$ , the triangles  $QMA, APB$ , and  $ATC$  are similar; so are the triangles  $QMB$  and  $TAB$ . Hence  $QM^2 : AM \cdot BM = AC : AB$ .

Ex. 12. The ordinates of all points on an ellipse being produced in the same ratio, the locus of their extremities is another ellipse.

Ex. 13.  $P$  is any point on an ellipse;  $AQO$  is drawn parallel to  $CP$  meeting the curve in  $Q$  and  $CB$  produced in  $O$ . Prove that  $AO \cdot AQ = 2CB^2$ .

PROPOSITION X.

*The locus of the middle points of any system of parallel chords of an ellipse is a straight line passing through the centre.*



Let  $QQ'$  be one of a system of parallel chords and  $V$  its middle point.

Draw  $QM, Q'M'$  perpendicular to the directrix. Draw

$SY$  perpendicular to  $QQ'$  and produce it to meet the directrix in  $K$ . Produce  $QQ'$  to meet the directrix in  $R$ . Join  $SQ, SQ'$ .

$$\begin{aligned} \text{Then} \quad SQ : SQ' &= QM \cdot QM' \\ &= QR : Q'R. \end{aligned}$$

$$\text{Therefore} \quad SQ^2 - SQ'^2 : QR^2 - Q'R^2 = SQ^2 : QR^2.$$

$$\begin{aligned} \text{But} \quad SQ^2 - SQ'^2 &= QY^2 - Q'Y^2 \\ &= (QY + Q'Y)(QY - Q'Y) \\ &= 2QQ' \cdot YV. \end{aligned}$$

$$\text{Similarly} \quad QR^2 - Q'R^2 = 2QQ' \cdot RV,$$

$$\text{Therefore} \quad YV : RV = SQ^2 : QR^2.$$

Now the ratio  $SQ : QM$  is constant, also the ratio  $QM : QR$  is constant, since  $QQ'$  is drawn in a fixed direction. Therefore  $SQ : QR$  is a constant ratio.

Therefore also  $YV : RV$  is a constant ratio for all chords of the system.

But as  $R$  always lies on a fixed straight line (the directrix) and  $Y$  on another fixed straight line (the focal perpendicular on the parallel chords) intersecting the former in  $K$ ,  $V$  must also lie on a third fixed straight line passing through the same point  $K$ .

Also  $C$ , the centre of the ellipse, is evidently a point on this line, since the parallel chord through  $C$  is, from the symmetry of the figure, bisected at that point.

Hence, the diameter bisecting any system of parallel chords of an ellipse is a chord passing through its centre.

**Ex.** The diameter bisecting any system of parallel chords, meets the directrix on the focal perpendicular on the chords.

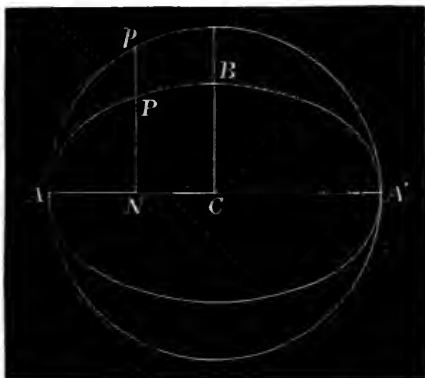
*Note.*—See Prop. XI., Ex. 10.

**Def.** The circle described on the major axis ( $AA'$ ) as diameter is called the *auxiliary circle*.



PROPOSITION XI.

*Ordinates drawn from the same point on the axis to the ellipse and the auxiliary circle are in the ratio of the minor to the major axis.*



Let  $ApA'$  be the auxiliary circle and let  $NPp$  be a common ordinate to the ellipse and the circle.

Then  $PN^2 : AN \cdot A'N = CB^2 : CA^2$ , [Prop. IX.

and  $pN^2 = AN \cdot A'N$ . [Euc. III. 3 & 35.

Therefore  $PN^2 : pN^2 = CB^2 : CA^2$ .

Therefore  $PN : pN = CB : CA$ .

*Note.*—By the help of this important property of the circle upon the major axis as diameter, many propositions concerning the ellipse may be easily proved, as will be shown hereafter. Hence the name *auxiliary circle*.

**Def.** The points  $P$  and  $p$  lying on a common ordinate  $pPN$  of the ellipse and its auxiliary circle are called *corresponding points*.

**Ex. 1.** A straight line cannot meet the ellipse in more than two points. (Cf. Prop. VIII., Ex. 9.)

Ex. 2.  $PM$  drawn perpendicular to  $BB'$  meets the circle on the minor axis as diameter in  $p'$ . Prove that

$$PM : p'M = CA : CB.$$

(See Prop. IX., Ex. 5.)

Ex. 3.  $PN, PM$  are perpendiculars on the axes, meeting the circles on the axes as diameters in  $p, p'$  respectively.

Prove that  $p$  and  $p'$  being properly selected,  $pp'$  passes through the centre.

Ex. 4. Through  $P, KPL$  is drawn making the same angle with the axes as  $pC$ , and cutting them in  $K$  and  $L$ . Show that  $KL$  is of constant length. ( $KL = CA + CB$ .)

Ex. 5. If the two extremities of a straight line move along two fixed straight lines at right angles to each other, any given point on the moving line describes an ellipse.

Let the fixed straight lines intersect in  $O$ , and let  $P$  be the given point on the moving line  $AB$  of which  $C$  is the middle point. Let  $QPN$  drawn at right angles to  $OB$ , meet  $OC, OB$  in  $Q$  and  $N$  respectively. Then, since  $OQ = AP$ , the locus of  $Q$  is a circle; also, as  $PN : QN = PB : PA$ , the locus of  $P$  is an ellipse.

Ex. 6. Given the semi-axes in magnitude and position, construct the curve mechanically.

Mark off on the straight edge of a slip of paper two lengths  $PA$  and  $PB$  in the same direction and equal to the semi-axes respectively. If the paper be now made to move so that  $A$  and  $B$  may always be on the lines representing the axes in position,  $P$  will trace out the ellipse. (See Ex. 5.)

Ex. 7. If a circle roll within another circle of double its radius, any point in the area of the rolling circle traces out an ellipse.

*First Method.*—Let  $C$  be the centre of the rolling circle, and  $O$  that of the other. If the given point  $P$  be on the radius  $CM, M$  will describe the diameter  $A'OA$  of the outer circle. Draw  $RPN$  perpendicular to  $OA'$ , meeting  $OC$  in  $R$  and  $OM$  in  $N$ . Then since  $CR = CP$ , the locus of  $R$  is a circle; and, as  $PN : RN = PM : OR$ , the locus of  $P$  is an ellipse.

*Second Method.*—The point  $M$  coincided with  $A'$  at the beginning of the motion; if in any position, the circles touch at  $Q$ ,

$$\text{arc } MQ = \text{arc } A'Q, \quad \text{angle } QCM = 2 \text{ angle } QOM,$$

$\therefore OCQ$  is always a straight line, so also is  $MCN, N$  being the intersection of the inner circle with that radius of the outer which is at right angles to  $OA$ . It is clear, therefore, that the motion of a point  $P$  in  $MN$  is exactly the same as that of a point in the moving rod in Ex. 5.

Ex. 8. From the centre of two concentric circles, a straight line is drawn to cut them in  $P$  and  $Q$ ; through  $P$  and  $Q$  straight lines are drawn parallel to two given lines at right angles. Prove that the locus of their point of intersection is an ellipse, of which the outer circle is the auxiliary circle.

Ex. 9.  $NPp$ ,  $N'P'p'$  are ordinates of the ellipse and its auxiliary circle. Show that  $PP'$ ,  $pp'$  produced meet on the axis in the same point  $T$ .

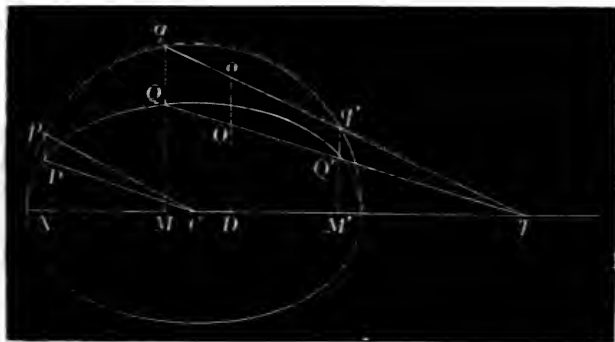
Ex. 10. Deduce from Ex. 9 a proof of Prop. X.

Let  $V, v$  be the middle points of  $PP'$ ,  $pp'$ .  $Vv$  produced bisects  $NN'$  at right angles in  $M$ . Now as long as  $PP'$  remains parallel to itself,  $pp'$  must remain parallel to itself, and, therefore, its middle point  $v$  lies on a fixed straight line, the diameter at right angles to  $pp'$ .  $V$ , therefore, lies on a fixed straight line through  $C$ , since

$$vM : VM = CB : CA.$$

\* PROPOSITION XII.

*If a system of chords of an ellipse be drawn through a fixed point the rectangles contained by their segments are as the squares of the parallel semi-diameters.*



Let  $QQQ'$  be one of the system of chords drawn through the fixed point  $O$  and  $CP$  the semi-diameter parallel to  $QQ'$ . Then  $QO \cdot OQ' : CP^2$  shall be a constant ratio.

Describe the auxiliary circle, and let  $p, q, q'$  be the corresponding points to  $P, Q, Q'$ . Join  $Cp$  and  $qq'$  and draw through  $O$  a line perpendicular to the major axis, meeting it in  $D$  and  $qq'$  in  $o$ .

Then, since  $QM : qM = Q'M' : q'M'$   
 $= CB : CA,$  [Prop. XI.

the straight lines  $QQ'$  and  $qq'$  produced meet the axis produced in the same point  $T$ .

Again, the triangles  $PNC$  and  $QMT$  being similar

$$NC : MT = PN : QM \\ = pN : qM. \quad \text{[Prop. XI.]}$$

Therefore the triangles  $pNC$  and  $qMT$  are similar.

[Euc. VI. 6.]

Therefore  $pC$  is parallel to  $qT$ .

Therefore the triangles  $pPC$  and  $qQT$  are also similar.

Now  $QO : qo = QT : qT,$

also  $OQ' : oq' = QT : qT.$

Therefore  $QO \cdot OQ' : qo \cdot oq' = QT^2 : qT^2 \\ = CP^2 : Cp^2,$

or  $QO \cdot OQ' : CP^2 = qo \cdot oq' : Cp^2.$

Now, since  $OD : oD = CB : CA,$

and the point  $O$  is fixed, the point  $o$  is also fixed; hence  $qo \cdot oq'$  is constant. [Euc. III. 35.]

Also  $Cp = CA = \text{constant}.$

Therefore  $QO \cdot OQ' : CP^2$

is a constant ratio.

Ex. 1. The ratio of the rectangles under the segments of any two intersecting chords of an ellipse, is equal to that of the rectangles under the segments of any other two chords parallel to the former, each to each.

Ex. 2. If two chords of an ellipse intersect, the rectangles under their segments are as the parallel focal chords. (Apply Prop. VII., Ex. 2.)

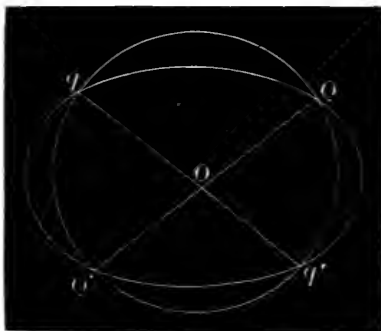
Ex. 3. Ordinates to any diameter at equal distances from the centre are equal.

Ex. 4.  $QCq$  is the central chord parallel to the focal chord  $PSp$ . Prove that

$$SP \cdot Sp : CQ \cdot Cq = CB^2 : CA^2.$$

\* PROPOSITION XIII.

*If a circle intersect an ellipse in four points their common chords will be equally inclined, two and two, to the axis.*



Let  $Q, Q', q, q'$ , be the four points of intersection.

Join  $QQ', qq'$ , intersecting in  $O$ .

Then  $QO \cdot OQ' = qO \cdot Oq'$ , [Euc. III. 35.]

Therefore the semi-diameters parallel to  $QQ'$  and  $qq'$  respectively, are equal to each other, [Prop. XII.]

and they are, therefore, equally inclined to the axis from the symmetry of the figure. (See also Prop. I., Ex. 2.)

Therefore, the chords  $QQ'$  and  $qq'$  are equally inclined to the axis.

In like manner it may be shown that the chords  $Qq$  and  $Q'q'$  as well as the chords  $Qq'$  and  $qQ'$  are equally inclined to the axis.

Ex. 1. If two chords, not parallel, be equally inclined to the axis of an ellipse, their extremities lie on a circle.

Ex. 2. If  $P$  be a fixed point on an ellipse and  $QQ'$  any ordinate to  $CP$ , show that the circle  $QPQ'$  will intersect the curve in another fixed point.

### PROPERTIES OF TANGENTS.

It has been already observed in Chapter I. that, generally, from a chord property of a conic a corresponding tangent property may be deduced. The student should work out the following exercises as illustrating the method in the case of the ellipse.

\* Deduce from Prop. XII. :—

Ex. 1. The tangents to an ellipse from an external point are proportional to the parallel semi-diameters.

Ex. 2. If the tangents at three points  $P, Q, R$  on an ellipse, intersect in  $r, q, p$ , show that

$$Pr \cdot pQ \cdot qR = Pq \cdot rQ \cdot pR.$$

Ex. 3. If two parallel tangents  $OP, O'P'$  be met by any third tangent  $OQO'$ , then  $OP : O'P' = OQ \cdot O'Q$ .

Ex. 4. If from any point without an ellipse a secant and also a tangent be drawn, the rectangle under the whole secant and the external segment is to the square of the tangent as the squares of the parallel semi-diameters.

Ex. 5. If two tangents be drawn to an ellipse, any line drawn parallel to either will be cut in geometric progression by the other tangent, the curve and the chord of contact.

Ex. 6. Any two intersecting tangents to an ellipse are to one another in the sub-duplicate ratio of the parallel focal chords.

Ex. 7. If two parallel tangents  $AQ$  and  $OR$  be cut by any third tangent  $APO$ , and  $RP$  meets  $QA$  in  $B$ , show that  $AQ = AB$ .

\* Deduce from Prop. XIII. :—

Ex. 1.  $PQ, P'Q'$  are chords of an ellipse equally inclined to the axis. Prove that the circle  $PQ'Q'$  touches the ellipse at  $P$ .

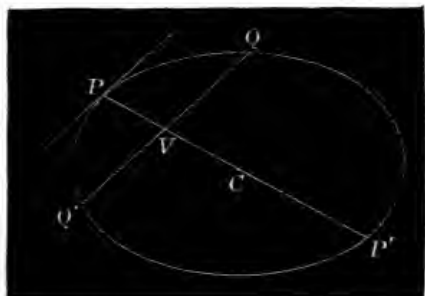
Ex. 2.  $PP'$  is a chord of an ellipse parallel to the major axis;  $PQ, P'Q'$  are chords equally inclined to that axis. Show that  $QQ'$  is parallel to the tangent at  $P$ .

Ex. 3. If a circle touch an ellipse at the points  $P$  and  $Q$ , prove that  $PQ$  is parallel to one of the axes.

See also Props. XIV. and XV.

PROPOSITION XIV.

*The tangent to an ellipse at either end of a diameter is parallel to the system of chords bisected by the diameter.*



Let  $PVCP'$  be the diameter bisecting a system of chords parallel to  $QQ'$ . Let  $QQ'$  be made to move parallel to itself so that  $Q$  may coincide with  $V$ . Since  $QV$  is always equal to  $Q'V$ , [Prop. X. it is clear that  $Q'$  will also coincide with  $V$ , and the chord in this its limiting position will be the tangent to the ellipse at  $P$ .

Ex. 1. The tangent at the vertex is at right angles to the major axis. [From symmetry, the chords at right angles to the major axis are bisected by it.]

Ex. 2. The line joining the points of contact of two parallel tangents is a diameter.

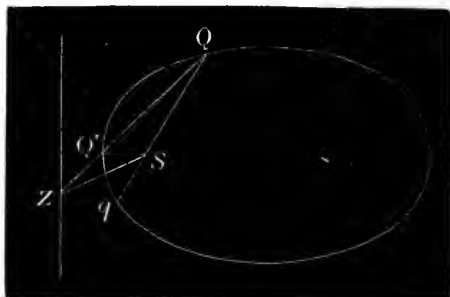
Ex. 3. Any tangent is cut harmonically by two parallel tangents and the diameter passing through their points of contact. (See note on Tangent Properties, I., Ex. 3.)

Ex. 4. An ellipse is described about the triangle  $ABC$ , having its centre at the point of intersection  $O$  of the medians.  $OA, OB, OC$  produced meet the ellipse in  $\alpha, \beta, \gamma$ . Prove that the tangents at  $\alpha, \beta, \gamma$  form a triangle similar to  $ABC$  and four times as large.

## PROPOSITION XV.

*The portion of the tangent to an ellipse at any point intercepted between that point and the directrix subtends a right angle at the focus, and conversely.*

*Also the tangents at the ends of a focal chord intersect on the directrix.*

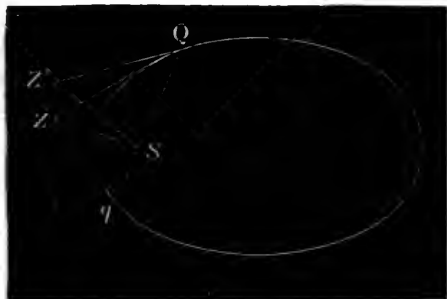


*First.*—Let any chord  $QQ'$  of the ellipse intersect the directrix in  $Z$ .

Then  $SZ$  bisects the exterior angle  $Q'Sq$ . [Prop. VIII. Now, let the chord  $QQ'$  be made to turn about  $Q$  until the point  $Q'$  moves up to and coincides with  $Q$ , so that the chord becomes the tangent to the ellipse at  $Q$ . In this limiting position of the chord  $QQ'$ , since  $Q$  and  $Q'$  coincide, the angle  $QSQ'$  vanishes and therefore the angle  $Q'Sq$  becomes equal to two right angles. But, since  $SZ$  always bisects the angle  $Q'Sq$ , in this case the angle  $Q'SZ$  is a right angle.



Again, let  $QZ$  subtend a right angle at  $S$ ; then it shall be the tangent to the ellipse at  $Q$ . For, if not, and if possible, let  $QZ'$  be the tangent at  $Q$ ; then the angle  $QSZ'$  is a right angle, which is impossible. Therefore  $QZ$  is the tangent at  $Q$ .



*Secondly.*—Let  $QSq$  be a focal chord and  $QZ$  the tangent at  $Q$ . Join  $ZS, Zq$ .

Then the angle  $QSZ$  being a right angle, the angle  $ZSq$  is also a right angle, and therefore  $qZ$  is the tangent to the ellipse at  $q$ . Therefore the tangents at  $Q$  and  $q$  intersect on the directrix.

Ex. 1. Tangents at the extremities of the latus rectum intersect in  $X$ .

Ex. 2. If through any point  $P$  of an ellipse, an ordinate  $QPN$  be drawn, meeting the tangent at  $L$  in  $Q$ , prove that  $QN=SP$ .

Ex. 3. To draw the tangent at a given point  $P$  of an ellipse.

Ex. 4. By drawing the tangent at  $B$ , prove that  $CS \cdot CX = CA^2$ .

Ex. 5. If  $ZQ$  meets the other directrix in  $Z$ ,  $ZP$  subtends a right angle at  $S$ .

Ex. 6. If  $QZ$  intersect the latus rectum in  $D$ , prove that

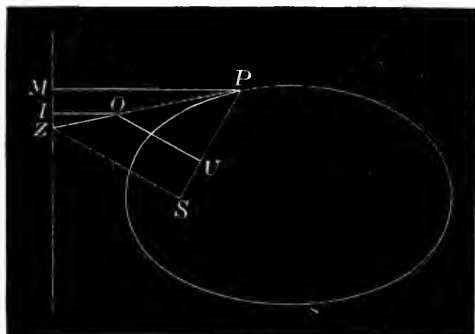
$$SD = e \cdot SZ.$$

## PROPOSITION XVI.

If from a point  $O$  on the tangent at any point  $P$  of an ellipse perpendiculars  $OU$  and  $OI$  be drawn to  $SP$  and the directrix respectively, then

$$SU = e \cdot OI,$$

and conversely.



Join  $SZ$  and draw  $PM$  perpendicular to the directrix.

Because  $ZSP$  is a right angle, [Prop. XV.]  
 $ZS$  is parallel to  $OU$ .

Therefore, by similar triangles,

$$\begin{aligned} SU : SP &= ZO : ZP \\ &= OI : PM. \end{aligned}$$

But

$$SP = e \cdot PM;$$

therefore

$$SU = e \cdot OI.$$

Again, for the converse proposition, if a line  $OP$  meets the ellipse at  $P$ , and the same construction is made as before, we have

$$SU = e \cdot OI,$$

and

$$SP = e \cdot PM;$$

therefore

$$\begin{aligned} SU : SP &= OI : PM \\ &= ZO : ZP. \end{aligned}$$

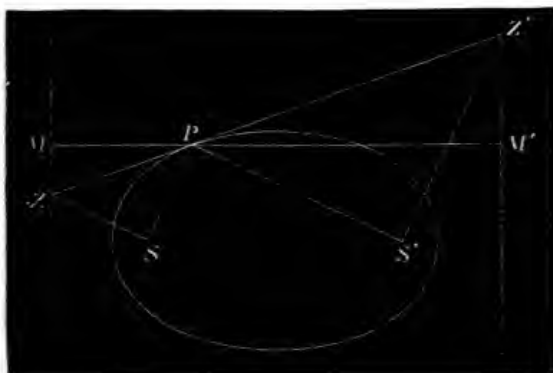
Therefore  $OU$  is parallel to  $ZS$ , [Euc. VI. 2.  
and the angle  $PSZ$  is a right angle.

$OP$  is, therefore, the tangent at  $P$ . [Prop. XV.

*Note.*—See Chap. I., Prop. XIII., also Prop. I., Ex. 9.

PROPOSITION XVII.

*The tangent at any point of an ellipse makes equal angles with the focal distances of the point.*



Let the tangent at  $P$  meet the directrices in  $Z$  and  $Z'$ .

Draw  $MPM'$  perpendicular to the directrices, meeting them in  $M$  and  $M'$  respectively. Join  $SP$ ,  $SZ$ ,  $S'P$ , and  $S'Z'$ .

Then, in the two triangles  $PSZ$  and  $PS'Z'$ , the angles  $PSZ$  and  $PS'Z'$  are equal, being right angles, [Prop. XV.  
and  $SP : S'P = PM : PM'$

$$= PZ : PZ',$$

and the angles  $PZS$  and  $PZ'S'$  are both acute angles.

Therefore the triangles are similar; [Euc. VI. 7.  
therefore the angle  $SPZ =$  the angle  $S'PZ'$ .

Ex. 1. If a line drawn through  $P$  bisect the exterior angle between  $SP$  and  $S'P$ , it will be the tangent at  $P$ .

Ex. 2. The tangent at the vertex is at right angles to the major axis.

Ex. 3. The perpendiculars from  $Z$  and  $Z'$  on  $SP$  intercept a length equal to  $AA'$ .

Ex. 4. The tangent at any point makes a greater angle with the focal distance than with the perpendicular on the directrix.

Ex. 5. If  $SY, S'Y'$  be the perpendiculars upon the tangent at  $P$ , and  $PV$  be the ordinate of  $P$ , prove that  $PV$  bisects the angle  $PYV'$ .

Ex. 6. If  $SY$ , the perpendicular on the tangent at  $P$ , meet  $S'P$  produced in  $s$ , prove that

$$(i) sY = SY, \quad (ii) SP = Ps, \quad (iii) S's = AA'.$$

On account of property (i),  $s$  is called the *image* of the focus in the tangent.

Ex. 7. Prove that the locus of the image of the focus in the tangent is a circle.

The circle, of which the centre is a focus and the radius equal to the major axis, is sometimes, though not quite properly, called the *Director Circle*, by way of analogy to the *directrix* of the parabola, which is, in the case of that curve, the locus of the image of the focus in the tangent. (See Chap. I., Prop. XIV., Ex. 7.)

Ex. 8. Given a focus and the length of the major axis, describe an ellipse touching a given straight line and passing through a given point. (Apply Prop. IV.; Newton, Book I., Prop. XVIII.)

Ex. 9. Given a focus and the length of the major axis, describe an ellipse touching two given straight lines. (Apply Prop. IV., cf. Prop. XXIII., Ex. 4; Newton, Book I., Prop. XVIII.)

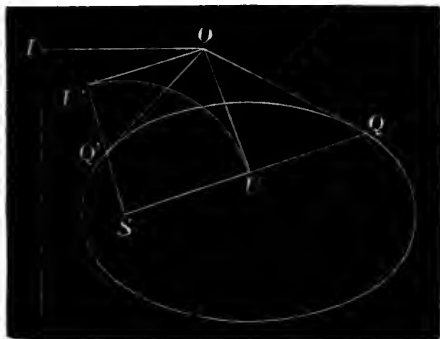
Ex. 10. If a circle be described through the foci of an ellipse, a straight line drawn from its intersection with the minor axis to its intersection with the ellipse, will touch the ellipse.

### PROPOSITION XVIII.

*To draw two tangents to an ellipse from an external point.*

Let  $O$  be the external point. Draw  $OI$  perpendicular to the directrix, and with centre  $S$  and radius equal to

$e$ .  $OI$ , describe a circle. Draw  $OU, OU'$  tangents to this circle, and let  $SU, SU'$  meet the ellipse in  $Q, Q'$ . Join  $OQ, OQ'$ . Then  $OQ, OQ'$  shall be the tangents required.



For  $OU$  is at right angles to  $SQ$ , [Euc. III. 18.  
and  $SU = e \cdot OI$ .  
Therefore  $OQ$  is the tangent to the ellipse at  $Q$ . [Prop. XVI.

Similarly  $OQ'$  is the tangent at  $Q'$ .

**Ex. 1. Alternative Construction.**—With centre  $O$  and radius  $OS$  describe a circle; with centre  $S'$  and radius equal to the major axis describe another circle intersecting the former in  $M$  and  $M'$ . Join  $S'M$  and  $S'M'$ , meeting the ellipse in  $Q$  and  $Q'$ ;  $OQ, OQ'$  are the tangents required. [The angle  $OQM =$  the angle  $OQS$ . Then apply Prop. XVII., Ex. 1. It may be shown that the construction given in Chap. I., Prop. XVI., is immediately deducible from this.]

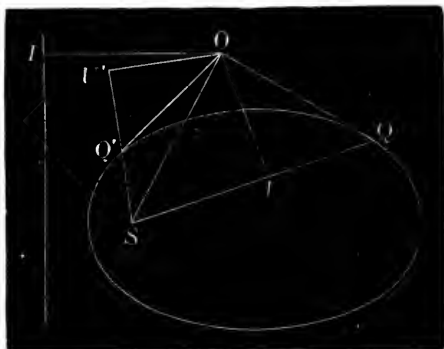
**Ex. 2.** Show that only two tangents can be drawn to an ellipse from an external point. (See Note to Chap. I., Prop. XVI.)

### PROPOSITION XIX.

*The two tangents which can be drawn to an ellipse from an external point subtend equal angles at the focus.*

Let  $OQ, OQ'$  be the two tangents from  $O$ .

Join  $SO, SQ, SQ'$ , and draw  $OI, OU, OU'$  perpendiculars upon the directrix,  $SQ, SQ'$  respectively.



Then  $SU = e \cdot OI = SU'$ . [Prop. XVI.]

Therefore  $OU = OU'$ . [Euc. I. 47.]

Therefore the angles  $OSU$  and  $OSU'$  are equal, [Euc. I. 8.]

and they are the angles which the tangents subtend at the focus  $S$ .

Ex. 1.  $OQ'$  produced meets the directrix in  $Z$ . Prove that  $OZ$  subtends a right angle at  $S$ . [Prop. XV. is a particular case of this.]

Ex. 2. If  $P$  be any point on an ellipse, the centre of the circle touching the major axis,  $SP$ , and  $S'P$  produced lies on the tangent at the vertex.

Ex. 3. The two foci and the intersections of any tangent with the tangents at the vertices, are concyclic points.

Ex. 4. A variable tangent meets a fixed tangent in  $T$ . Find the locus of the intersection with the variable tangent of the straight line through  $S$  at right angles to  $ST$ .

[The locus is the tangent at the other extremity of the focal chord through the point of contact of the fixed tangent.]

Ex. 5. The tangents at the ends of a focal chord meet the

tangents at the vertex in  $T_1$  and  $T_2$ . Prove that  $AT_1 \cdot AT_2$  is constant. ( $=AS^2$ )

Ex. 6. The angle subtended at either focus by the segment intercepted on a variable tangent by two fixed tangents is constant.

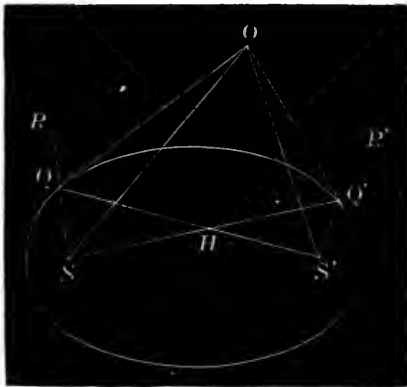
Ex. 7. If  $OS$  intersect  $QQ'$  in  $R$  and  $RK$  be drawn perpendicular to the directrix, prove that  $QK, Q'K$  are equally inclined to the axis.

Ex. 8. An ellipse is inscribed in a triangle; if one focus moves along the arc of a circle passing through two of the angular points of the triangle, find the locus of the other focus. [An arc of a circle through the same angular points.]

Ex. 9. If a quadrilateral circumscribes an ellipse, the angles subtended by opposite sides at one of the foci are together equal to two right angles.

\* PROPOSITION XX.

*The two tangents drawn to an ellipse from an external point are equally inclined to the focal distances of that point.*



Let  $OQ, OQ'$  be the two tangents from  $O$ .

Join  $SQ, SO, SQ', S'Q, S'O, S'Q'$ , and produce  $SQ$  to  $R$ .

Let  $H$  be the point of intersection of  $SQ'$  and  $S'Q$ .

Then

$$\begin{aligned} \text{the angle } SOQ &= \text{the angle } OQR - \text{the angle } OSQ && [\text{Euc. I. 32.}] \\ &= \text{half the angle } S'QR - \text{half the angle } QSQ \\ & && [\text{Props. XVII. and XIX.}] \\ &= \text{half the angle } SHQ. \end{aligned}$$

Similarly,

$$\text{the angle } S'OQ = \text{half the angle } S'HQ.$$

Therefore,

$$\text{the angle } SOQ = \text{the angle } S'OQ. \quad [\text{Euc. I. 15.}]$$

Ex. 1. Given two tangents to an ellipse and one focus, show that the locus of the centre is a right line.

Ex. 2. On  $OQ, OQ'$  take  $OK, OK'$  equal to  $OS, OS'$  respectively. Prove that  $KK'$  is equal to the major axis. [If  $SQ$  produced to  $E$  be equal to the major axis, the triangles  $SOE$  and  $KOK'$  are equal.]

Ex. 3. The straight line joining the feet of the perpendiculars from a focus on two tangents is at right angles to the line joining the intersection of the tangents with the other focus.

Ex. 4. The exterior angle between any two tangents is half the sum of the angles which the chord of contact subtends at the foci. [Cf. Chap. I., Prop. XIX.]

Ex. 5. The angle between the tangents at the extremities of a focal chord is half the supplement of the angle which the chord subtends at the other focus.

Ex. 6. Prove that

$$\angle SOS' + \angle S'QO + \angle SQ'O = 2 \text{ right angles.}$$

Ex. 7. If from any point on an ellipse tangents are drawn to a confocal ellipse, these tangents are equally inclined to the tangent at that point.

**Def.** Ellipses which have the same foci are called *confocal* ellipses.

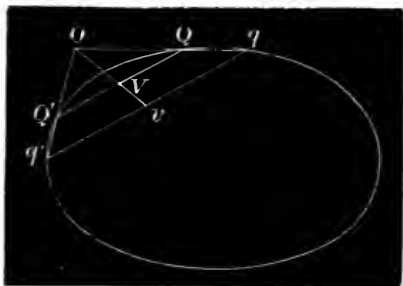
Ex. 8. If a perfectly elastic billiard ball lies on an elliptic billiard table, and is projected in any direction along the table, show that the lines in which it moves after each successive impact touch a confocal conic.

Ex. 9. Normals at the extremities of a focal chord intersect in  $O$ , and the corresponding tangents meet in  $T$ . Prove that  $OT$  passes through the other focus.



PROPOSITION XXI.

*The tangents at the extremities of any chord of an ellipse intersect on the diameter which bisects the chord.*



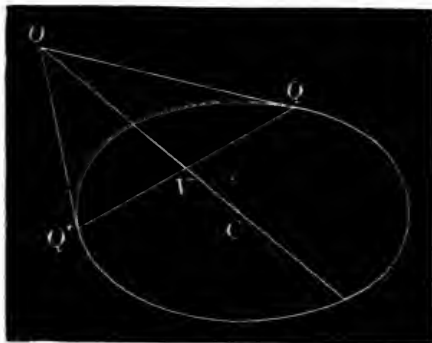
Let  $QQ'$  be the chord, and  $qq'$  any other chord parallel to it. Let  $qQ$  and  $q'Q'$  produced meet in  $O$ . Bisect  $QQ'$  in  $V$  and let  $OV$  meet  $qq'$  in  $v$ .

Then 
$$QV : qv = OV : Ov$$

$$= Q'V : q'v.$$

But 
$$QV = Q'V.$$

Therefore 
$$qv = q'v.$$



Therefore  $OVv$  is the diameter bisecting the system of chords parallel to  $QQ'$ .

[Prop. X.

If now the chord  $qq'$  be made to move parallel to itself until it coincides with  $QQ'$ ,  $qQO$  and  $q'Q'O$  will become the tangents to the curve at  $Q$  and  $Q'$  respectively, and they thus meet on the diameter bisecting  $QQ'$ .

Ex. 1. The diameter of an ellipse through an external point bisects the chord of contact of the tangents from that point.

Ex. 2. Given a diameter of an ellipse, to draw the system of chords bisected by it.

Ex. 3. The tangent at any point  $P$  of an ellipse meets the tangent at  $A$  in  $Y$ . Prove that  $CY$  is parallel to  $A'P$ .

Ex. 4. If  $OPCP'$  be a diameter through  $O$ ,  $OQ$  a tangent from  $O$ , and  $QV$  be drawn parallel to the tangent at  $P$ , then

$$OP \cdot OP' = OC \cdot OV.$$

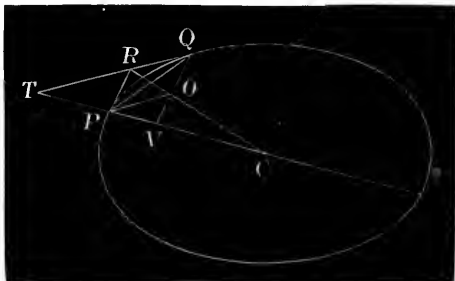
Hence show that  $OP : OP' = PV : P'V$ . [This shows that  $PP'$  is divided harmonically in  $V$  and  $O$ .]

Ex. 5. If any line drawn parallel to the chord of contact of two tangents to an ellipse meets the curve, the segments intercepted between the curve and the tangents are equal.

### PROPOSITION XXII.

*If the tangent at any point  $Q$  of an ellipse meets any diameter  $CP$  produced in  $T$ , and if  $QV$  be the ordinate to that diameter,*

$$CV \cdot CT = CP^2.$$



Draw the tangent  $PR$  at  $P$ , meeting  $QT$  in  $R$ , and draw  $PO$  parallel to  $QT$  meeting  $QV$  in  $O$ .

Then since  $POQR$  is a parallelogram,  $RO$  bisects  $PQ$ , and therefore passes through the centre  $C$ .

[Prop. XIV. and XXI.]

By similar triangles

$$CV : CP = CO : CR = CP : CT.$$

Therefore

$$CV \cdot CT = CP^2.$$

*Note.*—When the diameter coincides with the major axis, the result is stated thus:—

*If the tangent at  $Q$  meets the major axis produced in  $T$ , and  $QN$  be the perpendicular on the major axis,*

$$CN \cdot CT = CA^2.$$

When the diameter coincides with the minor axis, the result is stated thus:—

*If the tangent at  $Q$  meets the minor axis produced in  $t$ , and  $Qn$  be the perpendicular on the minor axis,*

$$Cn \cdot Ct = CB^2.$$

These two particular cases are important, and should be carefully noted by the student.

**Ex. 1.**  $VH$  drawn parallel to  $PQ$  meets  $CQ$  in  $R$ . Prove that  $PH$  is parallel to the tangent at  $Q$ .

**Ex. 2.** If a series of ellipses have the same major axis, the tangents at the extremities of their latera recta meet at the same point on the minor axis.

**Ex. 3.** If  $PT$  be a tangent to an ellipse meeting the axis in  $T$ , and  $AP, A'P$  be produced to meet the perpendicular to the major axis through  $T$  in  $Q$  and  $Q'$ , then  $QT = Q'T$ . [If  $PN$  be the ordinate of  $P$ , the relation  $CT : CA = CA : CN$  gives  $A'T : A'N = AT : AN$ .]

**Ex. 4.** If  $PN$  be perpendicular to the major axis, and the tangent at  $P$  meet the major axis produced in  $T$ , any circle through  $N$  and  $T$  cuts the auxiliary circle at right angles. [If  $E$  be the centre of the circle, show that  $EN^2 + CA^2 = EC^2$ .]

**Ex. 5.** The locus of the middle points of all focal chords of an ellipse is a similar ellipse.

Let  $O$  be the middle point of a focal chord  $PSp$ , and let the tangent at  $Q$  where  $CO$  produced meets the curve, meet the major

axis in  $T$ . If  $OM$  and  $QN$  be the ordinates to the major axis, it readily follows that

$$\frac{OM^2}{CM \cdot SM} = \frac{QN^2}{CN \cdot TN} = \frac{QM^2}{AN \cdot A'N}$$

Then apply Prop. IX., Ex. 9.

Ex. 6. If  $CY$ ,  $AZ$  be the perpendiculars from the centre and an extremity of the major axis on the tangent at any point  $P$ , show that

$$CA \cdot AZ = CY \cdot AN.$$

Ex. 7. If a variable tangent to an ellipse meet two fixed parallel tangents, it will intercept segments on them whose rectangle is constant.

Let the tangent at  $Q$  meet the two parallel tangents  $PR$  and  $pr$  in  $R$  and  $r$ .  $Pp$  is a diameter (Prop. XIV., Ex. 2). Let  $CD$  be the semi-diameter parallel to  $PR$  meeting  $Rr$  in  $t$ . Let  $QV$  and  $Qv$  be ordinates to  $CP$ ,  $CD$ ; and let  $rR$ ,  $pP$  meet in  $T$ . Then apply the proposition with respect to the diameters  $CD$ ,  $CP$ .

Ex. 8. In Ex. 7 prove that the rectangle under the segments of the variable tangent is equal to the square of the semi-diameter drawn parallel to it. (See Note on *Tangent-Properties* Ex. 1, 2. Newton, Book I., Lemma XXIV.)

Ex. 9. If  $P$  is any point on the ellipse, find the locus of the centre of the circle inscribed in the triangle  $SPS'$ . [An ellipse. If  $ON$  be the perpendicular from the centre  $O$  on  $AA'$ , it may be shown that

$$ON^2 : NS \cdot NS' = SA^2 : CB^2.$$

Then apply Prop. IX., Ex. 9.]

Ex. 10.  $CD$ ,  $CP$  are two semi-diameters of an ellipse. Tangents at  $D$  and  $P$  meet  $CP$  and  $CD$  in  $K$  and  $T$  respectively. Prove that the triangles  $CDK$  and  $CPT$  are equal in area.

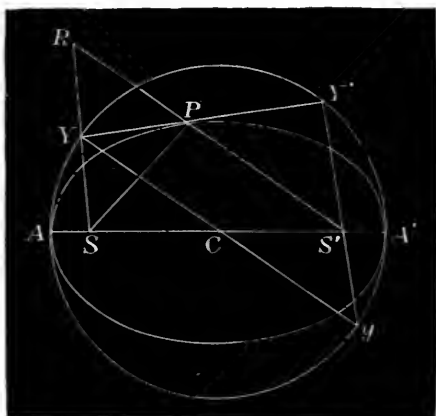
### PROPOSITION XXIII.

*The locus of the foot of the perpendicular drawn from either focus upon any tangent to an ellipse is the auxiliary circle; and the rectangle under the focal perpendiculars on the tangent is equal to the square of the semi-axis minor. ( $SY \cdot S'Y' = CB^2$ .)*

Let  $SY$ ,  $S'Y'$  be the focal perpendiculars upon the tangent at any point  $P$ .

Join  $SP$  and  $S'P$ . Produce  $S'P$  to meet  $SY$  in  $R$ . Join  $CY$ .

Then in the triangles  $SPY$ ,  $RPY$ ,  
 the angle  $SPY =$  the angle  $S'PY'$  [Prop. XVII.  
 $=$  the angle  $RPY$ , [Euc. I. 15.



and the angles  $SYP$ ,  $RYP$  are equal, each being a right angle, and  $YP$  is common,

therefore  $SP = PR$ .

and  $SY = YR$ . [Euc. I. 26.

Also  $SC = CS'$ ,

therefore  $CY$  is parallel to  $S'R$ , [Euc. VI. 2.

therefore  $CY = \frac{1}{2} \cdot S'R$  [Euc. VI. 4.

$$= \frac{1}{2}(S'P + PR)$$

$$= \frac{1}{2}(S'P + SP)$$

$$= \frac{1}{2}AA'$$

[Prop. IV.

$$= CA.$$

Therefore the locus of  $Y$  is the auxiliary circle.

Similarly it may be shown that the locus of  $Y'$  is the same circle.

Again, produce  $YC$  and  $Y'S''$  to meet in  $y$ , then  $y$  will be on the auxiliary circle.

For, since  $CS = CS'$  and  $SY$  is parallel to  $S''y$  the triangles  $SCY$  and  $S''Cy$  are equal. [Euc. I. 26.]

Therefore  $Cy = CY = CA$ , showing that  $y$  is on the auxiliary circle.

Also  $SY = S''y$   
 Therefore  $SY \cdot S''Y' = S''y \cdot S''Y'$   
 $= S''A' \cdot S''A$  [Euc. III. 35.]  
 $= SA \cdot SA'$   
 $= CB^2$ . [Prop. V.]

Ex. 1.  $CE$  parallel to the tangent at  $P$  meets  $SP, S'P$  in  $E, E'$ .  
 Prove that

$$(i) \quad PE = PE' = CA,$$

$$(ii) \quad SE = SE'.$$

(iii) the circle circumscribing the triangles  $CSE$  and  $CS'E'$  are equal.

Ex. 2. The central perpendicular on the tangent at  $P$  meets  $SP$  produced in  $Q$ . Prove that the locus of  $Q$  is a circle. [Centre  $S$ . Radius =  $CA$ .]

Ex. 3. If from the centre of an ellipse lines be drawn parallel and perpendicular to the tangent at any point, they enclose a part of one of the focal distances of that point equal to the other.

Ex. 4. Given a focus and the length of the major axis, describe an ellipse touching two given straight lines.

Ex. 5. Given a focus, a tangent, and the eccentricity, the locus of the other focus is a circle. [Since  $CS = e \cdot CY$ , the locus of the centre is obviously a circle.]

Ex. 6. Prove that the perimeter of the quadrilateral  $SY'Y'S'$  is the greatest possible when  $YY'$  subtends a right angle at the centre.

Ex. 7. A line is drawn through  $S'$  parallel to  $SP$  meeting  $YS$  in  $O$ . Prove that the locus of  $Y'$  is a circle.

Ex. 8. The right line drawn from either focus to the adjacent point of intersection of any tangent with the auxiliary circle is perpendicular to the tangent.

Ex. 9. If through any point  $Y'$  on the auxiliary circle  $Y'P$  be drawn at right angles to  $SY$ ,  $Y'P$  will be a tangent to the ellipse.

Ex. 10. If the vertex of a right angle moves on a fixed circle,

and one leg passes through a fixed point, the other leg will always touch an ellipse. (Cf. Chap. I., Prop. XXIII., Ex. 4.)

Ex. 11. Given the major axis and a tangent, show that the directrix passes through a fixed point.

Ex. 12. The circle described on  $SP$  as diameter touches the auxiliary circle.

Ex. 13. Given a focus, a tangent, and the length of the major axis, the locus of the centre is a circle.

Ex. 14. Given the foci and a tangent, construct the ellipse.

Ex. 15. *Alternative Construction for Prop. XVIII.*

Let  $O$  be the external point. On  $OS$  as diameter describe a circle intersecting the auxiliary circle in  $Y$  and  $Y'$ . Then  $OY$  and  $OY'$  produced will be the tangents required.

Ex. 16. The right line drawn from the centre parallel to either focal radius vector of any point on an ellipse to meet the tangent at that point, is equal to the semi-axis major.

Ex. 17. Draw a tangent to an ellipse parallel to a given straight line.

Ex. 18. Two ellipses, whose axes are equal, each to each, are placed in the same plane, with their centres coincident and axes inclined to each other. Draw their common tangents. [The common tangents pass through the points in which the lines joining the foci of the curves meet the common auxiliary circle.]

Ex. 19. Given a focus, a tangent, and the length of the minor axis, the locus of the other focus is a straight line.

Ex. 20. If the rectangle under the perpendiculars from the fixed points on a right line be constant ( $=k^2$ ), the line always touches an ellipse of which the fixed points are the foci, and the minor axis  $=2k$ .

Ex. 21. A chord of a circle, centre  $C$  and radius  $r$ , subtends a right angle at a fixed point  $O$ . Prove that it always touches an ellipse, of which  $C$  and  $O$  are the foci, and the square of the semi-axis minor  $=r^2 \approx CO^2$ .

Ex. 22. If a second tangent to the ellipse intersect  $FPI'$  at right angles in  $O$ , prove that  $OY \cdot OY' = CB^2$ .

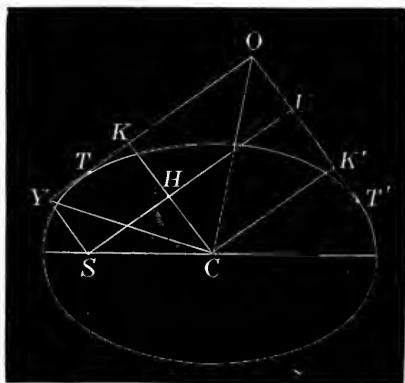
Hence, prove that  $CO^2 = CA^2 + CB^2$ .  
(Cf. Prop. XXIV.)

\* PROPOSITION XXIV.

*The locus of the intersection of tangents to an ellipse which cut at right angles is a circle.*

Let the tangents  $OT, OT'$  cut at right angles at  $O$ .

Draw  $SY, CK$  perpendicular to  $OT$  and  $SU, CK'$  perpendicular to  $OT'$ . Join  $CY, CU, CO$ . Let  $CK, SU$  intersect in  $H$ .



Now  $Y$  and  $U$  are on the auxiliary circle, [Prop. XXIII.

therefore

$$CY = CU = CA.$$

Then

$$CO^2 = CK^2 + CK'^2$$

[Euc. I. 47.

and

$$CY^2 = CK^2 + YK^2,$$

therefore

$$CA^2 = CK^2 + SH^2;$$

also

$$CU^2 = CK'^2 + UK'^2,$$

therefore

$$CA^2 = CK'^2 + CH^2,$$

therefore

$$2CA^2 = CK^2 + CK'^2 + SH^2 + HC^2$$

$$= CO^2 + CS^2;$$

[Euc. I. 47.

but

$$CS^2 = CA^2 - CB^2,$$

[Prop. V.

therefore

$$CO^2 = CA^2 + CB^2.$$

Hence the locus of  $O$  is a circle described with the centre  $C$  and radius equal to  $AB$ .

*Note.*—This circle is called the *Director Circle* of the ellipse.

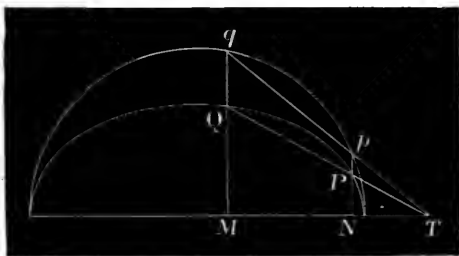


Ex. 1. An ellipse slides between two fixed lines at right angles to each other; prove that the locus of its centre is an arc of a circle.

Ex. 2. Any rectangle circumscribing an ellipse is inscribed in the director circle.

PROPOSITION XXV.

*Tangents at corresponding points of an ellipse and its auxiliary circle intersect on the major axis.*



Let the ordinate  $pPN$  meet the ellipse in  $P$  and the auxiliary circle in the corresponding point  $p$ . Let  $qQM$  be any other ordinate.

Then, because

$$\begin{aligned} QM : qM &= CB : CA \\ &= PN : pN, \end{aligned} \quad [\text{Prop. XI.}]$$

the straight line  $QP, qp$  produced meet the major axis in the same point  $T$ .

Now, if  $qQM$  be made to move parallel to itself so as to coincide with  $pPN$ , the points  $Q, P$  and  $q, p$  will coalesce, and the chords  $QPT$  and  $qpT$  will become tangents to the ellipse and the circle at  $P$  and  $p$  respectively.

Ex. 1. Deduce this proposition from the property  $CN \cdot CT = CA^2$ . (Prop. XXII.)

Ex. 2. The tangent at  $p$  meets  $CB$  produced in  $K$ . Prove that  $CK \cdot PN = CA \cdot CB$ .

Ex. 3. Show that the locus of the intersection of the normals at  $P$  and  $p$  is a circle of which the radius is  $CA + CB$ . [If the normals intersect in  $O$ , and if  $PR$  be drawn parallel to the major axis to meet  $CO$  in  $R$ , then, by similar triangles, it may easily be shown that  $OR = CA$ ,  $CR = CB$ .]

Ex. 4.  $OQ, OQ'$  are tangents to an ellipse;  $ON$  is drawn perpendicular to the axis. Prove that the tangents to the auxiliary circle at the corresponding points  $q, q'$  meet on  $ON$ .

If  $QQ'$  produced meet the major axis in  $T$ , prove also that

$$CN \cdot CT = CA^2.$$

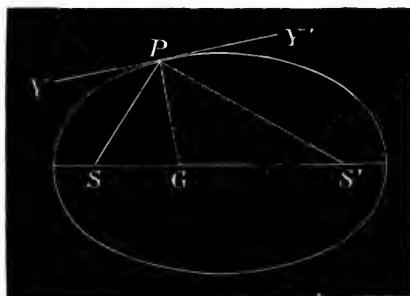
[For the second part, note that if  $ON$  meet the auxiliary circle in  $R$ , the tangent at  $R$  meets the major axis at the point where  $QQ', qq'$  meet it. Cf. also Prop. XXII., note, which is a limiting case.]

Ex. 5. In Ex. 4, if  $ON$  meets the ellipse in  $r$ , the tangent at  $r$  intersects the major axis in  $T$ .

## PROPERTIES OF NORMALS.

### PROPOSITION XXVI.

*The normal at any point of an ellipse bisects the angle between the focal distances of the point.*



Let the normal at the point  $P$  meet the major axis in  $G$ . Let  $YPY'$  be the tangent at  $P$ .

Then the angle  $SPY =$  the angle  $S'PY'$ . [Prop. XVII.]

But the angles  $GPY$ ,  $GPY'$  are equal, being right angles ;  
 [Def.

therefore the angle  $SPG =$  the angle  $S'PG$ .

Ex. 1. If the tangent and normal at any point  $P$  meet the minor axis in  $t$  and  $g$ , then  $P$ ,  $t$ ,  $g$ ,  $S$ , and  $S'$  lie on the same circle.

Ex. 2. Prove that the triangles  $SPG$  and  $gPS'$  are similar.

Ex. 3. If from  $g$  a perpendicular  $gK$  be drawn on  $SP$  or  $S'P$ , show that  $PK = CA$ .

Ex. 4. Prove that  $SP \cdot S'P = PG \cdot Pg$ . [The triangles  $PSg$ ,  $PS'G$  are similar. Ex. 1.]

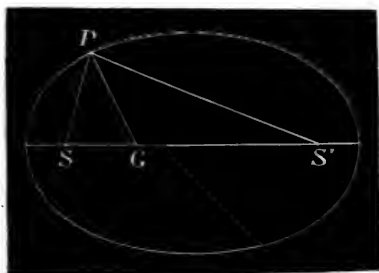
Ex. 5. No normal can pass through the centre, except it be at an end of one of the axes.

Ex. 6. The normal  $PG$  and the focal perpendiculars on the tangent at  $P$  are in harmonic progression.

Ex. 7. The circle described on  $PG$  as diameter cuts  $SP$ ,  $S'P$  in  $K$  and  $L$ . Prove that  $PG$  bisects  $KL$  at right angles.

\* PROPOSITION XXVII.

If the normal at any point  $P$  of an ellipse meets the major axis in  $G$ ,  $SG = e \cdot SP$ .



Join  $S'P$ .

Then, since  $PG$  bisects the angle  $SPS'$ , [Prop. XXVI.

$$SG : S'G = SP : S'P ; \quad [\text{Euc. VI. 3.}$$

therefore  $SG : SG + S'G = SP : SP + S'P$ ,

or  $SG : SP = SG + S'G : SP + S'P$ .

But  $SG + S'G = SS' = e \cdot AA'$ , [Prop. III.  
 and  $SP + S'P = AA'$ ; [Prop. IV.  
 therefore  $SG = e \cdot SP$ .

Ex. 1. Show how to draw the normal at any point without drawing the tangent.

Ex. 2. If  $PM$  be drawn perpendicular to the directrix, and  $MS$  meet the minor axis in  $g$ , show that  $Pg$  is the normal at  $P$ .

Ex. 3. A perpendicular is drawn from a fixed point  $M$  on the major axis of an ellipse, on the tangent at any point  $P$ . The locus of the intersection of this perpendicular with  $SP$  is a circle.

Ex. 4. If  $GE$  be perpendicular to  $SP$ , prove that  $PE$  is equal to half the latus rectum. [ $PSN$  and  $SEG$  are similar triangles; therefore  $SE = e \cdot SN$ ,  $SP = e \cdot NX$ , so that  $PE = e \cdot SX$ .]

Ex. 5. In Ex. 4, show that  $GE = e \cdot PN$ .

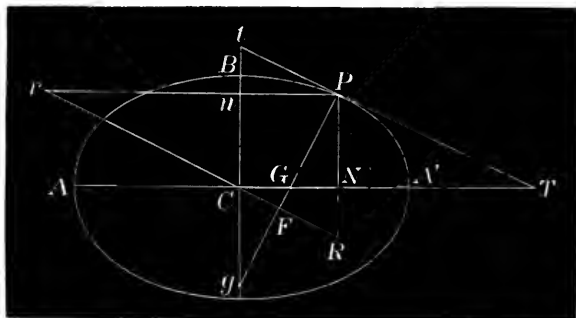
Ex. 6. Show that

$$PG^2 : SP \cdot S'P = CB^2 : CA^2.$$

(Cf. Prop. XXVI., Ex. 4, and Prop. XXVIII.)

### PROPOSITION XXVIII.

*The normal at any point of an ellipse, terminated by either axis, varies inversely as the central perpendicular on the tangent. ( $PG \cdot PF = CB^2$   $Pg \cdot PF = CA^2$ .)*



Let the normal at  $P$  meet the major axis in  $G$  and the minor axis in  $g$ ; let the tangent at  $P$  meet them in  $T$ .

and  $t$  respectively. Draw  $PN$ ,  $Pn$  perpendicular to the major and minor axis, and let a straight line through the centre, drawn parallel to the tangent at  $P$ , meet  $PN$ ,  $PG$ , and  $Pn$  produced, in  $R$ ,  $F$ , and  $r$  respectively.

Then, since the angles at  $N$  and  $F$  are right angles,  $G$ ,  $F$ ,  $R$ ,  $N$  lie on a circle; therefore

$$\begin{aligned} PG \cdot PF &= PN \cdot PR && [\text{Euc. III. 36.}] \\ &= Cn \cdot Ct && [\text{Euc. I. 34.}] \\ &= CB^2. && [\text{Prop. XXII., Note.}] \end{aligned}$$

Again, since the angles at  $n$  and  $F$  are right angles,  $g$ ,  $F$ ,  $n$ ,  $r$  lie on a circle; therefore

$$\begin{aligned} Pg \cdot PF &= Pn \cdot Pr && [\text{Euc. III. 36.}] \\ &= CN \cdot CT && [\text{Euc. I. 34.}] \\ &= CA^2. && [\text{Prop. XXII., Note.}] \end{aligned}$$

Therefore both  $PG$  and  $Pg$  vary inversely as  $PF$ , which is equal to the central perpendicular upon the tangent at  $P$ .

Ex. 1. If  $CF$  meet the focal distances of  $P$  in  $E$  and  $E'$ , prove that  $Pg$  subtends a right angle at  $E$  and  $E'$ . (See Prop. XXIII., Ex. 1.)

Ex. 2. If the circle through  $S$ ,  $P$ ,  $S'$  meets the minor axis in  $g$  on the side opposite to  $P$ , prove that  $Sg$  varies as  $PG$ .

Ex. 3.  $PQ$  is drawn at right angles to  $SP$ , meeting the diameter parallel to the tangent at  $P$  in  $Q$ . Prove that  $PQ$  varies inversely as  $PN$ .

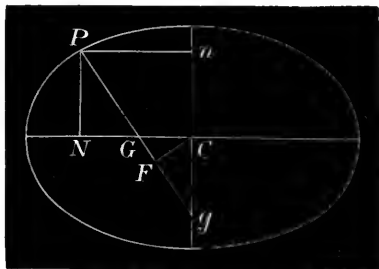
### PROPOSITION XXIX.

*If the normal at any point  $P$  on an ellipse meets the major axis in  $G$ , and  $PN$  be the ordinate to that axis,*

(i)  $GN : CN = CB^2 : CA^2$ .

(ii)  $CG = e^2 \cdot CN$ .

Let the normal meet the minor axis in  $g$ . Draw  $Pn$  perpendicular to the minor axis, and  $CF$  parallel to the tangent at  $P$ .



Then, because the triangles  $PNG$  and  $Png$  are similar,

$$GN : CN = PG : Pg \quad [\text{Enc. VI. 2.}]$$

$$= PG \cdot PF : Pg \cdot PF$$

$$= CB^2 : CA^2; \quad [\text{Prop. XXVIII.}]$$

therefore  $ON - GN : CN = CA^2 - CB^2 : CA^2$ ,

or  $CG : CN = CS^2 : CA^2$ . [Prop. V.]

But  $CS = e \cdot CA$ ; [Prop. III.]

therefore  $CG = e^2 \cdot CN$ .

Ex. 1. In the figure of Prop. XXVIII., prove that:—

(i)  $CG \cdot CT = CS^2$ .

(ii)  $Cg \cdot Ct = CS^2$ .

(iii)  $NG \cdot CT = CB^2$ .

(iv)  $Tg, tG$  intersect at right angles.

Ex. 2. Find a point  $P$  on the ellipse such that  $PG$  may bisect the angle between  $PC$  and  $PN$ .

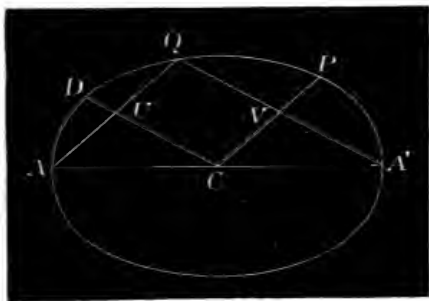
Ex. 3. In the figure of Prop. XXVIII., prove that the rectangle under the focal perpendiculars on  $PG = CF \cdot PT$ .

## PROPERTIES OF CONJUGATE DIAMETERS.

### PROPOSITION XXX.

*If one diameter of an ellipse bisects chords parallel to*

a second, the second diameter bisects chords parallel to the first.



Let  $CP$  bisect chords parallel to  $CD$ ; then  $CD$  bisects chords parallel to  $CP$ .

Draw  $A'Q$  parallel to  $CD$ , meeting  $CP$  in  $V$ ; join  $AQ$ , meeting  $CD$  in  $U$ .

Then  $A'Q$  is bisected in  $V$  and  $AA'$  in  $C$ ; therefore  $CV$  is parallel to  $AQ$ . [Euc. VI. 2.]

Again, since  $AA'$  is bisected in  $C$ , and  $CD$  is parallel to  $A'Q$ ,  $AQ$  is bisected by  $CD$ . [Euc. VI. 2.]

Therefore  $CD$  bisects all chords parallel to  $AQ$ , [Prop. X.] and therefore all chords parallel to  $CP$ .

**Def.** Two diameters so related that each bisects chords parallel to the other are called *Conjugate Diameters*.

Thus  $CP$  and  $CD$  are conjugate to each other; so also are the major and minor axes.

Ex. 1. If one diameter is conjugate to another, the first is parallel to the tangent at an extremity of the second. (Prop. XIV.)

Ex. 2. Given an ellipse and two conjugate diameters, show how to draw the tangent at any point.

If  $CP, CD$  be conjugate diameters, and  $QV$  is drawn parallel to  $CD$ ,  $QV$  is the ordinate to  $CP$ . In  $CP$  produced take  $T$ , such that  $CV \cdot CT = CP^2$ .  $QT$  is the tangent at  $Q$ . (Prop. XXII.)

Ex. 3. If  $CQ$  be conjugate to the normal at  $P$ , then  $CP$  is conjugate to the normal at  $Q$ .

Ex. 4. The focal perpendiculars upon  $CP$  and  $CD$ , when produced backwards, will intersect  $CD$  and  $CP$  on the directrix. (Apply Prop. XXIX., Ex. 2.)

Ex. 5. The focus is the orthocentre of the triangle formed by any two conjugate diameters and the directrix. (See Prop. X., Ex. 1.)

Ex. 6. Any diameter is a mean proportional between the focal chord parallel to it and the major axis. [The conjugate diameter  $CD$  will bisect the focal chord. Then apply Prop. XXII., and Prop. XXIII., Ex. 16.]

Ex. 7. The rectangle under the intercepts on any tangent between the curve and any two conjugate diameters, is equal to the square of the semi-diameter parallel to the tangent, and conversely.

Let the tangent at  $Q$  meet the conjugate semi-diameters  $CP$ ,  $CD$  in  $T$ ,  $T'$ , and let  $CR$  be the semi-diameter parallel to  $TT'$ . Let the tangent at  $R$  parallel to  $CQ$  meet  $CD$  in  $t$ . Draw the ordinates  $QV$ ,  $Rv$  with respect to  $CD$ , parallel to  $CP$ . Then

$$CV \cdot CT' = Cv \cdot Ct = CD^2. \quad [\text{Prop. XXII.}]$$

By similar triangles,

$$QT : CR = CV : Cv = Ct : CT' = CR : QT'.$$

Therefore

$$QT \cdot QT' = CR^2.$$

Ex. 8. Given in magnitude and position any two conjugate semi-diameters  $CP$ ,  $CD$  of an ellipse, find the major and minor axes.

Produce  $CP$  to  $K$ , such that  $CP \cdot PK = CD^2$ . Bisect  $CK$  in  $O$ , and let the line through  $O$  at right angles to  $CK$  meet the line through  $P$  parallel to  $CD$  in  $H$ . With centre  $H$  and radius  $HC$ , describe a circle cutting  $PH$  in  $T$ ,  $T'$ ; the circle will also pass through  $K$ . Then  $CT$ ,  $CT'$  will coincide with the directions of the major and minor axes respectively.

For  $PT \cdot PT' = CP \cdot PK = CD^2$ ; therefore  $CT$ ,  $CT'$  are conjugate diameters (Ex. 7), and as they are at right angles, they must coincide with the directions of the major and minor axes. (Cf. Prop. XXXIII., Ex. 3; see also Miscellaneous Examples, 13, 14, 15, 16.)

To determine the magnitudes of the axes, observe that  $TPPT'$  is the tangent at  $P$ , and apply Prop. XXII., note.

Ex. 9.  $PP'$  is a fixed line. Find the locus of a point  $Q$  which so moves that  $QV$  being drawn in a fixed direction to meet  $PP'$  in  $V$ ,  $QV^2$  is to  $PV \cdot P'V$  in a given ratio.

Bisect  $PP'$  in  $C$ , and through  $C$  draw  $CD$  in the fixed direction, such that  $CD^2$  is to  $CP^2$  in the given ratio. Then the locus of  $Q$



will be the ellipse described with  $CP$  and  $CD$  as conjugate semi-diameters (Ex. 8). Apply Prop. XII., and cf. Prop. XXXII.

*Note.*—If  $QV^2 = PV \cdot P'V$ , the semi-diameters  $CP, CD$  will be *equi-conjugate*. In this case the position of the major and minor axes may be at once determined, as they bisect the angles between the equiconjugate diameters. (See Prop. XXXI., Ex. 3)

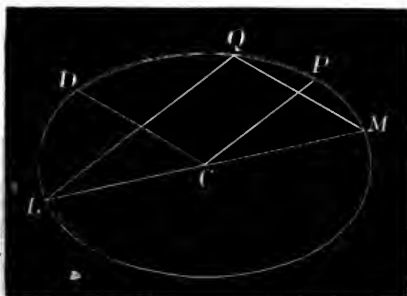
**Ex. 10.** A series of ellipses have their equiconjugate diameters of the same magnitude. One of these diameters is fixed and common, while the other varies. The tangents drawn from any point on the fixed diameter produced will touch the ellipses in points situated on a circle. (Apply Prop. XXII.)

**Ex. 11.** If  $CN, CP$  are the abscissa and ordinate of a point  $P$  on a circle whose centre is  $C$ , and  $NQ$  be taken equal to  $NP$ , and be inclined to it at a constant angle, the locus of  $Q$  is an ellipse.

**Def.** Chords which join any point on an ellipse to the extremities of a diameter are called *supplemental chords*.

PROPOSITION XXXI.

*Supplemental chords of an ellipse are parallel to conjugate diameters.*



Join any point  $Q$  on the ellipse to the extremities of a diameter  $LCM$ . Then  $QL$  and  $QM$  are supplemental chords.

Draw  $CP, CD$  parallel to  $QL, QM$  respectively; then they shall be conjugate diameters.

Because  $LM$  is bisected in  $C$  and  $CP$  is parallel to  $LQ$ ,  
 $CP$  bisects  $MQ$ , [Euc. VI. 2.  
 and, therefore, all chords parallel to  $CD$ . [Prop. X.  
 Therefore  $CD$  bisects all chords parallel to  $CP$ , [Prop. XXX.  
 and is therefore conjugate to  $CP$ .

Ex. 1. Prove that for any assumed pair of conjugate diameters there can be drawn a pair of supplemental chords parallel to them.

Ex. 2. The diagonals of any parallelogram circumscribed to an ellipse are conjugate diameters. [The diagonals pass through the centre of the ellipse. Then see Note on *Tangent-Properties*, Ex. 1, 3.]

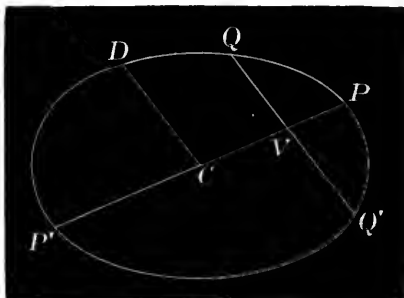
Ex. 3. The diagonals of the rectangle formed by the tangents at the extremities of the major and minor axes of an ellipse are equi-conjugate diameters.

Ex. 4. The tangent at any point  $Q$  on an ellipse meets the equi-conjugate diameters in  $T$  and  $T'$ . Prove that the triangles  $QCT$  and  $QCT'$  are as  $CT^2 : CT'^2$ . [Apply Prop. XXII.]

### \* PROPOSITION XXXII.

*The square of the ordinate of any point on an ellipse with respect to any diameter varies as the rectangle under the segments of the diameter made by the ordinate.*

$$(QV^2 : PV \cdot P'V = CD^2 : CP^2.)$$



Let  $QVQ'$  be a double ordinate with respect to the

diameter  $PCP'$ , meeting it in  $V$ . Let  $CD$  be the semi-diameter parallel to  $QV$ .

Now  $CP$  bisects  $QQ'$  and therefore all chords parallel to  $QV$  or  $CD$ . [Def. and Prop. X.

Therefore  $CD$  is conjugate to  $CP$ . [Def.

But  $QV \cdot Q'V : PV \cdot P'V = CD^2 : CP^2$ . [Prop. XII.

and  $QV = Q'V$ .

Therefore  $QV^2 : PV \cdot P'V = CD^2 : CP^2$ .

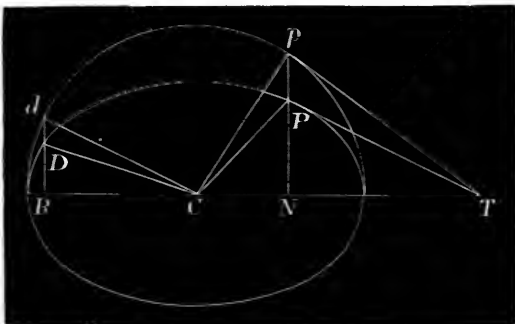
Ex. If  $QP, QP'$  meet  $CD, CP$  in  $M, N$  respectively, prove that  $CM \cdot CN = CD^2$ .

PROPOSITION XXXIII.

If  $CP, CD$  be two conjugate semi-diameters of an ellipse and ordinates  $PN, DR$  be drawn to the major axis, then

(i)  $PN : CR = DR : CN = CB : CA$ .

(ii)  $CN^2 + CR^2 = CA^2$ .



Let  $NP$  and  $RD$  produced meet the auxiliary circle in  $p$  and  $d$ . Join  $Cp, Cd$ , and let the tangents at  $P$  and  $p$  meet the major axis produced in  $T$ . [Prop. XXV.

Then, because  $PT$  is parallel to  $CD$ , [Props. X. and XIV. the triangles  $NPT$  and  $RDC$  are similar.

Therefore  $NT:RC=PN:DR$ ; [Euc. VI. 4.]

but  $PN:DR=pN:dR$ , [Prop. XI.]

therefore  $NT:RC=pN:dR$ ,

and the angles  $pNT$  and  $dRC$  are equal, being right angles. Therefore, the triangles  $NpT$  and  $RdC$  are similar. [Euc. VI. 6.]

Therefore the angles  $pTN$  and  $dCR$  are equal.

Therefore  $pT$  is parallel to  $dC$  and the angle  $dCP$  = the angle  $CpT$  = a right angle.

Therefore the angle  $pCN$  = the angle  $CdR$ , each being the complement of the angle  $dCR$ .

Therefore the two triangles  $pCN$  and  $dCR$  are equal in every respect. [Euc. I. 26.]

Therefore  $CR=pN$

and  $PN:CR=PN:pN$   
 $=CB:CA$ . [Prop. XI.]

Similarly  $DR:CN=CB:CA$ .

Again,  $CN^2+CR^2=CN^2+pN^2$   
 $=Cp^2=CA^2$ .

Ex. 1. If  $CQ$  be perpendicular to  $PT$ , prove that  
 $CQ \cdot QT:CT^2=CN \cdot PN:CD^2$ .

Ex. 2. If the normal at  $P$  meets the major and minor axes in  $G$  and  $g$  respectively, prove that

$$(i) \quad PG:CD=CB:CA,$$

$$(ii) \quad Pg:CD=CA:CB,$$

$$(iii) \quad PG \cdot Pg=CD^2.$$

Ex. 3. Prove that if two conjugate diameters be at right angles to each other, they must be the major and minor axes of the ellipse.

Ex. 4. Prove that

$$(SP-CA)^2+(SD-CA)^2=CS^2.$$

Ex. 5. If the tangent at the vertex  $A$  cut any two conjugate diameters in  $T$  and  $t$ , show that  $AT \cdot At=CB^2$ .

Ex. 6. Apply Prop. XXII. to prove this proposition.

If the tangents at  $P$  and  $D$  meet the major axis in  $T$  and  $t$ , it may easily be shown from the relation

$$CR : CN = CT : Ct,$$

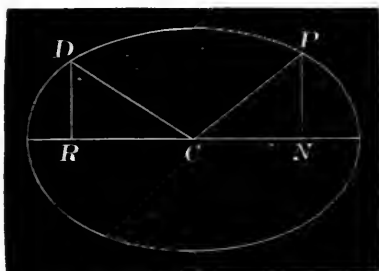
that

$$CN^2 = CR \cdot Rt = AR \cdot A'R.$$

Then apply Prop. IX.

PROPOSITION XXXIV.

*The sum of the squares of any two conjugate semi-diameters is constant. ( $CP^2 + CD^2 = CA^2 + CB^2$ .)*



Let  $CP, CD$  be the conjugate semi-diameters, and let  $PN, DR$  be the ordinates to the major axis.

Then  $PN : CR = CB : CA$ . [Prop. XXXIII.]

Therefore  $PN^2 : CR^2 = CB^2 : CA^2$ .

Similarly  $DR^2 : CN^2 = CB^2 : CA^2$ ,

therefore  $PN^2 + DR^2 : CN^2 + CR^2 = CB^2 : CA^2$ ;

but  $CN^2 + CR^2 = CA^2$ ; [Prop. XXXIII.]

therefore  $PN^2 + DR^2 = CB^2$ ,

therefore  $CP^2 + CD^2 = CA^2 + CB^2$ . [Euc. I. 47.]

Therefore, in the ellipse, the sum of the squares of any conjugate semi-diameters is constant, being equal to the sum of the squares of the semi-axis major and semi-axis minor.

Ex. 1. Find the greatest value of the sum of a pair of conjugate diameters. [The diameters must be equiconjugate.]

Ex. 2. If  $PG$ ,  $DH$  be the normals at  $P$  and  $D$ , prove that  $PG^2 + DH^2$  is constant.

Ex. 3. Prove that  $SP \cdot S'P = CD^2$ . [ $SP + S'P = 2CA$ . Then square and substitute.]

Ex. 4.  $OP$ ,  $OQ$  are tangents to an ellipse, and  $SQ$  is produced to meet the directrices in  $R$ ,  $R'$ . Prove that

$$PR \cdot PR' : QR \cdot QR' = OP^2 : OQ^2.$$

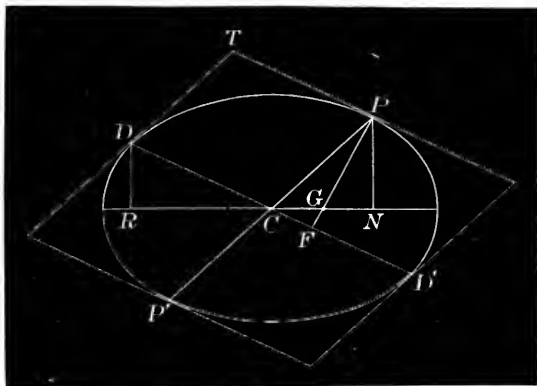
[If  $PM$  and  $QN$  be the ordinates, it can easily be shown that

$$\frac{PR \cdot PR'}{QR \cdot QR'} = \frac{MX \cdot MX'}{NX \cdot NX'} = \frac{SP \cdot S'P}{SQ \cdot S'Q}$$

Then apply Ex. 3 and Note on *Tangent-Properties*, Ex. I., 1.]

\* PROPOSITION XXXV.

The area of the parallelogram formed by the tangents at the extremities of a pair of conjugate diameters is constant. ( $CD \cdot PF = CA \cdot CB$ .)



The tangents at the extremities of two conjugate diameters  $PCP'$  and  $DCD'$  will evidently form a parallelogram, [Prop. XIV. the area of which is four times that of the parallelogram  $CDTP$ , where  $T$  is the intersection of the tangents at  $P$  and  $D$ .

Let the normal at  $P$  meet the major axis in  $G$  and  $DCD'$  in  $F$ . Draw the ordinates  $PN$  and  $DR$  to the major axis.

Then, since the angles at  $N$  and  $F$  are right angles, the angle  $GPN =$  the angle  $GCF =$  the angle  $DCR$ .

[Euc. I. 15 and I. 32.

Therefore the two right-angled triangles  $GPN$  and  $DCR$  are similar.

Therefore  $PG : CD = PN : CR$   
 $= CB : CA,$  [Prop. XXXIII.

therefore  $PG . PF : CD . PF = CB^2 : CA . CB ;$

but  $PG . PF = CB^2,$  [Prop. XXVIII.

therefore  $CD . PF = CA . CB.$

Again, the area of the parallelogram  $CDTP$   
 $= CD . PF = CA . CB =$  constant,

which proves the proposition.

Ex. 1. Find the least value of the sum of a pair of conjugate diameters. [The diameters are the major and the minor axis. Cf. Prop. XXXIV., Ex. 1.]

Ex. 2. Prove that the parallelogram formed by the tangents at the extremities of a pair of conjugate diameters is the least that can be circumscribed about the ellipse.

Ex. 3. If  $PG$  meets the minor axis in  $g$ , prove that

$$PG . Pg = CD^2.$$

(Prop. XXVIII. Cf. Prop. XXXIII., Ex. 2.)

Ex. 4. If  $SY$  be the perpendicular upon the tangent at  $P$ , prove that

$$SP : SY = CD : CB.$$

[In the figure of Prop. XXIII.,

$$\frac{SP}{SY} = \frac{S'P}{S'Y'} = \frac{SP + S'P}{SY + S'Y'} = \frac{CA}{CK},$$

where  $CK$  is the central perpendicular upon the tangent at  $P$ .

Therefore  $\left[ \frac{SP}{SY} = \frac{CA}{PF} = \frac{CD}{CB} \right]$

Ex. 5. Prove that  $SP . S'P = CD^2$ . [From Ex. 4

$$\frac{SP . S'P}{SY . S'Y'} = \frac{CD^2}{CB^2}$$

Then apply Prop. XXIII. Cf. also Prop. XXXIV., Ex. 3, and Prop. XXXIII., Ex. 2, along with Prop. XXVI., Ex. 4.]

Ex. 6. If the tangent at  $P$  meet the minor axis in  $T$ , prove that the areas of the triangles  $SPS'$ ,  $STS'$  are as  $CD^2 : ST^2$ . [Cf. Prop. XXVI., Ex. 1.]

Ex. 7. If  $DQ$  be drawn parallel to  $SP$  and  $CQ$  perpendicular to  $DQ$ , prove that  $CQ = CB$ . (See Ex. 4.)

Ex. 8. The tangents drawn from  $D$  to the circle on the minor axis as diameter are parallel to the focal distances of  $P$ . (See Ex. 4.)

Ex. 9. If on the normal at  $P$ ,  $PQ$  be taken equal to the semi-conjugate diameter  $CD$ , the locus of  $Q$  is a circle whose centre is  $C$  and radius equal to  $CA - CB$ . [Apply Prop. XXXIV.]

#### MISCELLANEOUS EXAMPLES ON THE ELLIPSE.

1. Find the locus of the point of intersection of any tangent to an ellipse, with the line drawn from the focus making a constant angle with the tangent.

[A circle. Cf. Prop. XXIII. Observe that if the vertex of a triangle of a given species be fixed, while one base angle moves along a fixed circle, the locus of the other base angle is a circle.]

2. The line drawn parallel to the axis through the intersection of normals at the extremities of a focal chord, bisects the chord.

3.  $S, S'$  are the foci of an ellipse;  $S'R$  is drawn equal to  $AA'$ ; the line bisecting  $RS$  at right angles touches the ellipse. (Newton, Book I., Prop. XVII.)

4. Given a focus, the length of the major axis and two points on the curve, to construct it. (Apply Prop. IV. Newton, Book I., Prop. XVIII.)

5. Given a focus, the eccentricity, and two tangents, to construct the curve. (Apply Prop. XXIII., Ex. 5. Newton, Book I., Prop. XX.)

6. Given a focus, the eccentricity and two points



on the curve, to construct it. (Newton, Book I., Prop. XX.)

[The directrix touches the two circles having their centres at the given points, and radii equal to  $e$  times their focal distances.]

7. Given a focus and the eccentricity, to describe an ellipse touching a given line at a given point. (Newton, Book I., Prop. XX.)

[Let  $S$  be the given focus, and  $P$  the given point on the tangent  $YPP'$ . (Fig. Prop. XXIII.) Draw  $SY$  at right angles to  $PP'$ , and produce it to  $R$ , so that  $YR = YS$ . Divide  $SR$  internally and externally at the points  $K, L$  in the ratio  $SA : AY$ ; the circle on  $KL$  as diameter meets  $RP$  in  $S'$ .]

8. The rectangle under the perpendiculars let fall from any point on an ellipse on two opposite sides of an inscribed quadrilateral is in a constant ratio to the rectangle under the perpendiculars let fall on the other two sides.

[The proposition holds if instead of perpendiculars on the sides, lines are drawn making a constant angle with them. Newton, Book I., Lemmas XVII.-XIX.]

9. The rectangle under the perpendiculars let fall from any point on an ellipse on two fixed tangents is in a constant ratio to the square of the perpendicular on their chord of contact.

10. If two fixed tangents to an ellipse be cut by a diameter parallel to their chord of contact and by a third variable tangent, the rectangle under the segments of the two fixed tangents, intercepted between the diameter and the variable tangent, is constant.

11. The right line joining the middle points of the diagonals of a quadrilateral circumscribing an ellipse will pass through the centre. (Apply Ex. 10 and Prop. XXI., Ex. 5.)

12. If a quadrilateral be circumscribed to an ellipse the diagonals will intersect on the chord of contact of the sides.

13. Given two conjugate diameters in magnitude and position to construct the ellipse.

[Through the extremities  $P, P', D, D'$  of the given conjugate diameters  $PCP', DCD'$ , draw lines parallel to them, forming the parallelogram  $EFGH$ . Divide the half side  $DE$  into any number of equal parts at  $R', R''$ , etc. Divide  $DC$  into the same number of equal parts at  $r', r''$ , etc. The intersection of  $PR'$  and  $P'r''$  determines a point on the ellipse.]

14. Given two conjugate semi-diameters in magnitude and position, determine the axes.

[Let  $CP, CD$  be the conjugate semi-diameters. Draw  $PR$  perpendicular to  $CD$ , and on  $PR$  take  $PQ, P'Q'$  on opposite sides of  $P$ , each equal to  $CD$ ; then the axes are in direction the bisections of the angle  $QCQ'$ , while their lengths are the sum and difference of  $CQ, CQ'$ .]

15. Given two conjugate semi-diameters in magnitude and position, determine the axes.

[Let  $CP, CD$  be the conjugate semi-diameters. Draw  $PR$  perpendicular to  $CD$ , and on it take  $PQ=CD$ . On  $CQ$  as diameter, describe a circle, and let  $O$  be its centre. Join  $OP$ , cutting the circle in  $E$  and  $F$ ; join  $CE, CF$ , and take on  $CE, CF, CA=FP, CB=EP$ . Then  $CA, CB$  are the semi-axes sought.]

16. Given two conjugate semi-diameters  $CP, CD$ , with centre  $C$  and radius  $CP$  describe a circle, and let  $KK'$  be its diameter at right angles to  $CP$ ; then will the axes of the ellipse be equal to  $KD \pm K'D$ , and parallel to the bisectors of the angle  $KDK'$ .

17. Any diameter of an ellipse varies inversely as the perpendicular focal chord of its auxiliary circle.

18. If any rectangle circumscribe an ellipse the perimeter of the parallelogram formed by joining the points

of contact is twice the diameter of the director circle.  
(Prop. XXIV.)

19. Given a focus, the length of the major axis, and that the second focus lies on a fixed straight line, prove that the ellipse touches two fixed parabolas having the given focus for focus.

20. Two given ellipses in the same plane have a common focus, and one revolves about the common focus while the other remains fixed; the locus of the point of intersection of their common tangents is a circle.

[If  $H$  be the second focus of the fixed ellipse,  $K$  of the revolving ellipse, and  $b_1, b_2$  their semi-minor axes,

$$HT : KT = b_1^2 : b_2^2,$$

where  $T$  is the point whose locus is sought.]

21.  $TQ, TQ'$  are tangents to an ellipse;  $CQ, CQ', QQ', CT$  are joined;  $QQ'$  and  $CT$  intersect in  $V$ . Prove that the area of the triangle  $QQC'$  varies inversely as

$$\left(\frac{CV}{TV}\right)^{\frac{1}{2}} + \left(\frac{TV}{CV}\right)^{\frac{1}{2}}.$$

22.  $SY, S'Y'$  are perpendiculars on the tangent at  $P$ . Perpendiculars from  $Y, Y'$  on the major axis cut the circles of which  $SP, S'P$  are diameters in  $I, J$  respectively. Prove that  $IS, JS'$ , and  $BC$  produced meet in the same point.

23. An ellipse touches two given lines  $OP, OQ$  in  $P$  and  $Q$ , and has one focus on the line  $PQ$ . Find the other focus and the directrices.

24.  $S, S'$  are the foci of an ellipse;  $SY$  is perpendicular on the tangent at  $P$ . Prove that  $S'Y$  bisects the normal at  $P$ .

25.  $CP, CD$  are two conjugate semi-diameters of an

ellipse;  $Rr$  is a tangent parallel to  $PD$ ; a straight line  $CIJ$  cuts at a given angle  $PD$ ,  $Rr$  in  $I, J$ . Prove that the loci of  $I$  and  $J$  are similar curves. [It can easily be shown that  $CI^2 : CJ^2 = 1 : 2$ .]

26. A system of parallelograms is inscribed in an ellipse whose sides are parallel to the equiconjugate diameters. Prove that the sum of the squares on the sides is constant.

27.  $OP, OQ$  are tangents to an ellipse;  $CU, CV$  are the parallel semi-diameters. Prove that

$$OP \cdot OQ + CU \cdot CV = OS \cdot OS'.$$

28.  $P, Q$  are points on two confocal ellipses at which the line joining the common foci subtends equal angles. Prove that the tangents at  $P, Q$  include an angle equal to that subtended by  $PQ$  at either focus.

29. The foci of a given ellipse  $A$  lie on an ellipse  $B$  the extremities of a diameter of  $A$  being the foci of  $B$ . Prove that the eccentricity of  $B$  varies as the diameter of  $A$ .

30. Normals at the extremities  $P$  and  $D$  of two conjugate semi-diameters meet in  $K$ . Prove that  $CK$  is perpendicular to  $PD$ .

31. If  $CP, CP'$  be semi-diameters of an ellipse at right angles to each other, prove that

$$\frac{1}{CP^2} + \frac{1}{CP'^2}$$

is constant.

32. Having given the auxiliary circle of an ellipse and a tangent to the ellipse touching the ellipse at a given point, find the foci.

33. Find the locus of the centres of circles cutting a given ellipse orthogonally.

34. An ellipse is inscribed in a given triangle. If one of the foci is known, show how to find the ellipse and its points of contact with the sides of the triangle.

35. Two fixed points  $Q$ ,  $R$  and a variable point  $P$  are taken on an ellipse; the locus of the orthocentre of the triangle  $PQR$  is an ellipse.

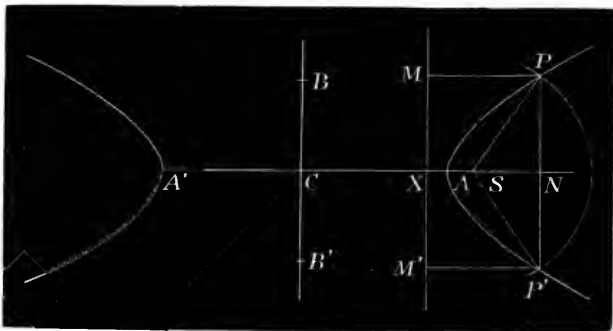
## CHAPTER III.

### THE HYPERBOLA.

#### DESCRIPTION OF THE CURVE.

#### PROPOSITION I.

*Given the focus, directrix, and eccentricity of a hyperbola to determine any number of points on it.*



Let  $S$  be the focus,  $MXM'$  the directrix, and  $e$  the eccentricity.

Through  $S$  draw  $SX$  perpendicular to the directrix. Divide  $SX$  in  $A$  so that

$$SA = e \cdot AX.$$

Also, in  $SX$  produced,\* take  $A'$  so that

$$SA' = e \cdot AX.$$

Then  $A$  and  $A'$  are points on the hyperbola and are its vertices.

Take any point  $N$  on  $A'A$  produced. Through  $N$  draw  $PNP'$  perpendicular to  $AA'$ . With centre  $S$  and radius equal to  $e \cdot XN$ , describe a circle cutting  $PNP'$  in  $P$  and  $P'$ . Then  $P$  and  $P'$  shall be points on the hyperbola.

Draw  $PM, P'M'$  perpendicular to the directrix.

$$\begin{aligned} \text{Then} \quad SP &= e \cdot XN && [\text{Const.}] \\ &= e \cdot PN, \end{aligned}$$

$$\begin{aligned} \text{and} \quad SP' &= e \cdot XN \\ &= e \cdot P'M'. \end{aligned}$$

Therefore  $P$  and  $P'$  are points on the hyperbola.

In like manner, by taking any other point on  $A'A$  produced, a series of points on the curve may be determined lying on the right hand side of the directrix.

Again, if  $N$  be taken on  $AA'$  produced, another series of points on the curve may be determined lying on the left hand side of the directrix.

**Def.** The length of the axis intercepted between the vertices ( $A, A'$ ) of the hyperbola is called the *transverse axis*.

**Def.** The middle point ( $C$ ) of the transverse axis is called the *centre* of the hyperbola.

**Def.** A straight line  $BCB'$  passing through the centre and perpendicular to the transverse axis, such that

$$CB^2 = CB'^2 = CS^2 - CA^2 = SA \cdot SA'$$

is called the *conjugate axis*.

\* Since  $e$  is greater than unity, it is clear that  $A$  will lie between  $S$  and  $X$ , and  $A'$  without  $SX$  on the side remote from  $S$ .

The conjugate axis, unlike the minor axis of the ellipse, does not meet the curve at all. (See Ex. 3 below.) Its utility in establishing properties of the hyperbola will appear later on.

Ex. 1. The hyperbola is symmetrical with respect to its axis.

Corresponding to any point  $N$  on the line  $A'A$  produced, we get two points  $P$  and  $P'$  such that the chord  $PP'$  is bisected at right angles by the axis  $A'A$ . [Def.]

Ex. 2. Any two right lines drawn from any point on the axis to the curve on opposite sides of the axis, and equally inclined to it, are equal, and conversely.

Ex. 3. Show that the hyperbola lies wholly outside the lines drawn through  $A$  and  $A'$  at right angles to the axis.

In order that the circle may intersect the line  $P'NP$ , the point  $N$  must be so situated that  $SN$  may not be greater than the radius of the circle  $SP$ , that is,  $e \cdot AN$ . It may be shown that this is the case only when  $N$  does not lie between  $A$  and  $A'$ .

Ex. 4. Hence, the hyperbola consists of *two distinct branches* lying on opposite sides of the lines drawn through the vertices at right angles to the axis.

Ex. 5. There is no limit to the distance to which each branch of the hyperbola may extend on both sides of the axis, so that the hyperbola consists of two *infinite* branches.

It is obvious that the point  $N$  may be taken anywhere on the axis beyond  $A$  and  $A'$ .

*Note.*—It will be remembered that the parabola consists of one infinite branch (Chap. I., Prop. I., Ex. 9) and that the ellipse is a closed oval (Chap. II., Prop. I., Ex. 6).

Ex. 6. In any conic, if  $PR$  be drawn to the directrix parallel to a fixed straight line, the ratio  $SP : PR$  is constant.

Ex. 7. If an ellipse, a parabola, and a hyperbola have the same focus and directrix, the parabola will lie entirely outside the ellipse and inside the hyperbola. (Cf. Chap. I., Prop. I., Ex. 6 and 7.)

Ex. 8. Prove that the locus of a point of trisection of an arc of a circle described on a given base is a hyperbola.

Ex. 9. If a circle touches the transverse axis at the focus, and passes through one end of the conjugate axis, the portion of the conjugate axis intercepted  $= CA^2/CB$ .

Ex. 10. Prove that the locus of the point of intersection of two tangents to a parabola which cut at a constant angle is a hyperbola.

Let  $OP, OQ$  be two tangents to a parabola, cutting at a constant angle  $\alpha$ . Draw  $OI, OU$  perpendicular to the directrix and  $SP$ ; then  $OI = SU$  (Chap. I., Prop. XIII.), and

$$OS : OI = OS' : SU,$$



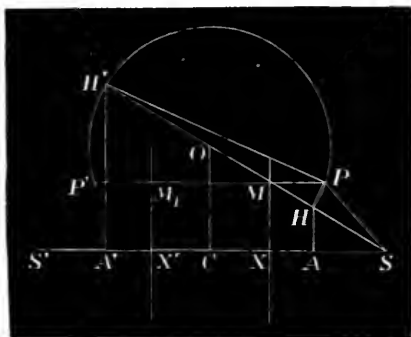
which is a constant ratio greater than unity since  $\angle OSP = \pi - a$ . (Chap. I., Prop. XIX.) The locus of  $O$  is, therefore, a hyperbola having the same focus and directrix as the parabola.

Ex. 11.  $P$  is any point on a given hyperbola ( $e=2$ ).  $D$  is taken on the axis such that  $SD=SA'$ . If  $A'P$  meets the latus rectum in  $K$ , find the locus of the intersection of  $DK$  and  $SP$ . [The circle on  $A'D$  as diameter.]

Ex. 12. The angular point  $A$  of a triangle  $ABC$  is fixed, and the angle  $A$  is given, while the points  $B$  and  $C$  move on a fixed right line. Find the locus of the centre of the circumscribing circle of the triangle. [A hyperbola of which  $A$  is the focus and  $BC$  the directrix.]

PROPOSITION II.

*The hyperbola is symmetrical with respect to the conjugate axis and has a second focus ( $S'$ ) and directrix.*



Let  $S$  be the given focus and  $MX$  the given directrix.

Take any point  $M$  on the directrix and through the vertices  $A$  and  $A'$  draw  $AH, A'H'$  at right angles to  $AA'$ , meeting the straight line through  $M$  and  $S$  at  $H$  and  $H'$  respectively.

Describe a circle on  $HH'$  as diameter, and through  $M$  draw  $PMP'$  parallel to  $AA'$ , to meet the circle in  $P$  and  $P'$ . Then  $P$  and  $P'$  shall be points on the hyperbola.

$$\text{For} \quad SH : HM = SA : AX \\ = e,$$

$$\text{and} \quad SH' : MH' = SA' : XA' \\ = e,$$

$$\text{therefore} \quad SH : HM = SH' : MH',$$

and the angle  $HPH'$  is a right angle; therefore  $PH$  bisects the angle  $SPM$ .

$$\text{Therefore} \quad SP : PM = SH : HM \\ = SA : AX \\ = e.$$

Therefore  $P$  is a point on the hyperbola.

Similarly it may be shown that  $P'$  is a point on the hyperbola.

Again, the straight line drawn through  $O$ , the centre of the circle, at right angles to  $AA'$ , will bisect both  $AA'$  and  $PP'$  at right angles, and will therefore coincide with the conjugate axis in position.

The hyperbola is therefore symmetrical with respect to the conjugate axis.

Hence the two branches of the hyperbola, lying on opposite sides of the conjugate axis, are such that each is the exact reflexion of the other. Therefore, if  $A'S'$  be measured off  $= AS$  and  $A'X' = AX$ , and  $X'M_1$  be drawn at right angles to  $X'S$ , the curve could be equally well described with  $S'$  as focus and  $X'M_1$  as directrix. The hyperbola has therefore a second focus  $S'$  and a second directrix  $X'M_1$ .

**Ex.** Every chord drawn through the centre  $C$  and terminated by the two branches is bisected at that point. [From the symmetry of the figure.]

From this property the point  $C$  is called the *centre* of the curve.

PROPERTIES OF CHORDS AND SEGMENTS OF CHORDS.

PROPOSITION III.

- In the hyperbola*       $CA = e \cdot CX$ .....(i.)  
                                   $CS = e \cdot CA$ .....(ii.)  
                                   $CS \cdot CX = CA^2$ .....(iii.)



We have from the definition

$$SA = e \cdot AX,$$

$$SA' = e \cdot A'X = e \cdot AX'.$$

Therefore, by subtraction,

$$AA' = e(AX' - AX)$$

$$= e \cdot XX'.$$

Therefore       $CA = e \cdot CX$ .....(i.)

By addition       $SS' = e \cdot (AX + A'X)$

$$= e \cdot AA'.$$

Therefore       $CS = e \cdot CA$ .....(ii.)

Therefore       $CS \cdot CX = CA^2$ .....(iii.)

Ex. 1. Given the transverse and the conjugate axis, find the focus and the directrix.

Ex. 2. Prove that 
$$e^2 = 1 + \frac{CB^2}{CA^2}.$$

Ex. 3. If the line through  $B$  parallel to the transverse axis meet the latus rectum in  $D$ , then will the triangles  $SCD$ ,  $SXD$  be similar.

Ex. 4. Prove that

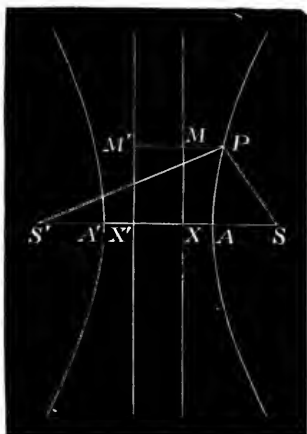
$$SX^2 : AX \cdot A'X = CB^2 : CA^2.$$

Ex. 5. If any line through the centre meet the perpendicular through  $A$  to the transverse axis in  $O$  and the directrix in  $E$ , then  $AE$  is parallel to  $SO$ .

Ex. 6. In Prop. I., Ex. 8, find the distance between the centres of the two hyperbolas which are the loci of the points of trisection of an arc of a circle described on a given base. [One-third of the given base.]

#### PROPOSITION IV.

*The difference of the focal distances of any point on a hyperbola is constant and equal to the transverse axis.*

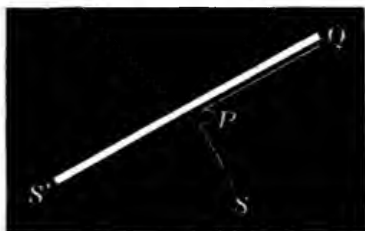


Let  $P$  be any point on the hyperbola. Join  $PS$ ,  $PS'$ , and through  $P$  draw  $PMM'$  perpendicular to the directrices.

Then  $SP = e \cdot PM$ ,  
 and  $S'P = e \cdot PM'$ .  
 Therefore  $S'P - SP = e(PM' - PM)$   
 $= e \cdot MM'$   
 $= e \cdot XX'$   
 $= AA'$ .

[Prop. III.

Ex. 1. Show how to construct the hyperbola mechanically.



*First Method.*—Suppose a bar  $SQ$ , length  $r$ , to revolve round its extremity  $S'$  which is fixed. Then if a string of given length  $l$ , attached to the bar at  $Q$  and also to a fixed point  $S$ , be always kept stretched by means of a pencil at  $P$  pressed against it (the part  $QP$  of the string being in contact with the rod), the pencil will trace out a hyperbola with foci at  $S$  and  $S'$ , and the transverse axis equal to  $(r - l)$ . For

and  $S'P + PQ = r$   
 $SP + PQ = l$ ,  
 $\therefore S'P - SP = r - l = \text{constant}$ .

It should be observed that  $l$  must be less than  $r$  and greater than  $r - SS'$ .

In the same manner, by making the bar revolve round  $S$  as centre, the other branch of the hyperbola may be described. The other branch may also be described by taking the string longer than the rod by the length  $(r - l)$ .

*Second Method.*—Suppose two equal thin circular discs  $A$  and  $B$  attached to each other, to rotate in the same direction round an axis through their common centre; and suppose the two ends of a fine string (which is wrapped round the discs and passing through small rings at  $C$  and  $D$  in the plane of the discs, is kept stretched by the point of a pencil at  $P$ ) to be wound off from the two discs. The curve traced by  $P$  will have the property  $CP - DP = \text{constant}$ , and will, therefore, be a hyperbola.

Ex. 2. Given the foci and the transverse axis to determine any number of points on the curve.

Describe a circle with centre  $S$  and any radius  $r$ ; describe a circle with centre  $S'$  and radius  $=r+AA'$ . The two circles intersect in points on the curve.

Ex. 3. Given a focus, a tangent, and a point on an ellipse, prove that the locus of the other focus is a hyperbola. [The foci will be the given point and the image of the focus in the tangent. Chap. II., Prop. XXIII.]

Ex. 4. Given a focus, a tangent, and two points on an ellipse to construct the curve. (Newton, Book I., Prop. XXI.)

Ex. 5. Given a focus, two tangents, and a point on an ellipse to construct the curve. (Newton, Book I., Prop. XXI.)

Ex. 6. Given a focus, the eccentricity, a tangent, and a point on an ellipse to construct the curve. (Apply Chap. II., Prop. XXIII., Ex. 5. Newton, Book I., Prop. XX.)

Ex. 7. The difference of the focal distances of any point is greater than, equal to, or less than the transverse axis, according as the point is within, upon, or without the hyperbola, and conversely.

Ex. 8. The locus of the centre of a circle which touches two fixed circles is an ellipse or a hyperbola. (Cf. Chap. II., Prop. IV., Ex. 4.)

Ex. 9. Given one focus of an ellipse and two points on the curve, the locus of the other focus is a hyperbola.

Ex. 10. A parabola passes through two fixed points, and has its axis parallel to a given line; prove that the locus of its focus is a hyperbola.

Ex. 11. Given the base of a triangle and its point of contact with the inscribed circle, show that the locus of its vertex is a hyperbola.

Ex. 12. Find the locus of the intersection of the tangents from two given points  $A$  and  $B$  to all circles touching  $AB$  at a given point  $C$ .

[An ellipse when  $C$  is outside  $A$  and  $B$ ; a hyperbola when  $C$  is between  $A$  and  $B$ , except when  $CA=CB$ , in which case the locus is a right line.]

Ex. 13. An ellipse and a hyperbola having the same foci intersect in  $P$ . If  $CA, Ca$  be their semi-axes major respectively and  $PN$  the ordinate of  $P$ , show that

$$CA : CS = CN : Ca.$$

Ex. 14.  $P$  is any point on an ellipse, of which  $CA, CB$  are the semi-axes;  $CD$  is the semi-diameter conjugate to  $CP$ ;  $Cb$  is the semi-conjugate axis of the confocal hyperbola through  $P$ . Prove that

$$CB^2 + Cb^2 = CD^2.$$

Let

$$Ca = \text{semi-transverse axis.}$$

Then

$$Cb^2 = CS^2 - Ca^2 = CS^2 - \frac{1}{4}(SP - S'P)^2$$

$$= CS^2 - \frac{1}{4}\{(SP + S'P)^2 - 4SP \cdot S'P\}$$

$$= CD^2 - Cb^2. \quad [\text{Chap. II., Prop. XXXV., Ex. 5.}]$$

Ex. 15.  $SF, S'F'$  are the focal perpendiculars on the tangent at any point  $P$  of an ellipse. Prove that  $PY.PY'$  is equal to the square on the semi-conjugate axis of the confocal hyperbola through  $P$ .

$$\left[ \frac{SP}{PY} = \frac{S'P}{PY'} = \frac{CD}{\sqrt{PY.PY'}} \right]$$

$$\frac{SF}{PY} = \frac{S'F'}{PY'} = \frac{CB}{\sqrt{PY.PY'}}$$

Apply Ex. 14. Cf. Prop. XXI., Ex. 8.]

Ex. 16. Two adjacent sides of a quadrilateral are given in magnitude and position; if a circle can be inscribed on the quadrilateral, the locus of the intersection of the other two sides is a hyperbola.

Ex. 17. Prove that the circle in Prop. I., Ex. 12, always touches a fixed circle. [Centre is second focus of the hyperbola, radius = transverse axis.]

\* PROPOSITION V.

*The latus rectum of a hyperbola is a third proportional to the transverse and conjugate axes.  $\left( SL = \frac{CB^2}{CA} \right)$*



Let  $LSL'$  be the latus rectum. Draw  $LM$  perpendicular to the directrix.

Then  $CS = e \cdot CA.$  [Prop. III.

$SL = e \cdot LM$  [Def.

$= e \cdot SX.$

Therefore  $SL \cdot CA = CS \cdot SX$

$= CS(CS - CX)$

$= CS^2 - CS \cdot CX$

$= CS^2 - CA^2$

[Prop. III.

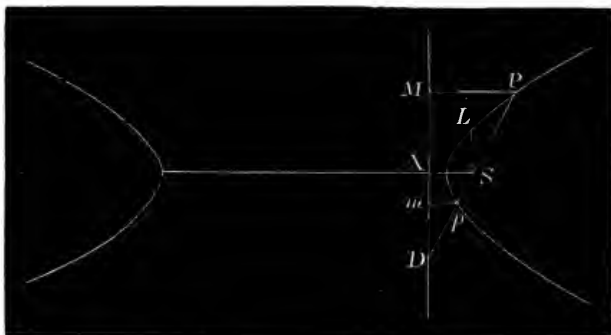
$= CB^2.$

[Def.

Ex. Prove this proposition by means of Prop. III., Ex. 4.

\* PROPOSITION VI.

*Any focal chord of a hyperbola is divided harmonically by the focus and directrix; and focal chords are to one another as the rectangles contained by their segments.*



Produce the focal chord  $PSp$  to meet the directrix in  $D$ , and draw  $PM$  and  $pm$  perpendicular to the directrix.

Then  $PD : pD = PM : pm;$

but  $PS = e \cdot PM,$

and  $ps = e \cdot pm;$

[Def.



therefore  $PD : pD = PS : pS$ .

Hence  $Pp$  is divided harmonically in  $S$  and  $D$ .

Again,  $PD$ ,  $SD$ , and  $pD$  being in harmonic progression,  $PM$ ,  $SX$ , and  $pm$  are also in harmonic progression. But

$$SP : PM = SL : SX = Sp : pm = e;$$

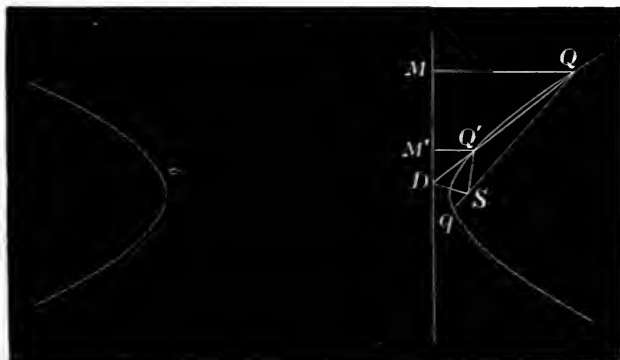
therefore  $SP$ ,  $SL$ , and  $Sp$  are also in harmonic progression. Therefore

$$SL = \frac{2SP \cdot Sp}{SP + Sp} = \frac{2SP \cdot Sp}{Pp};$$

therefore the focal chord  $Pp$  varies as  $SP \cdot Sp$ .

PROPOSITION VII.

*If any chord  $QQ'$  of a hyperbola intersects the directrix in  $D$ ,  $SD$  bisects the angle between  $SQ$  and  $SQ'$ .*



*First, let  $Q$  and  $Q'$  be on the same branch of the hyperbola*

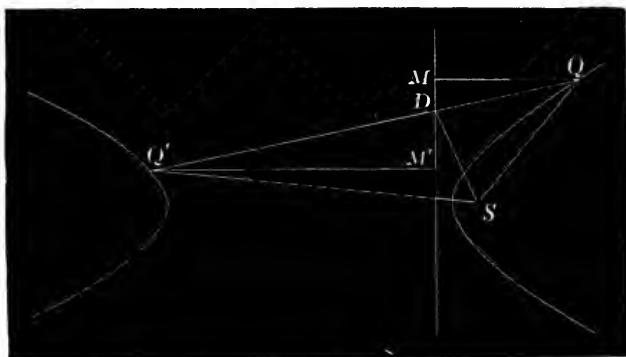
*Draw  $QM$ ,  $Q'M'$  perpendicular to the directrix.*

*Then, by similar triangles,*

$$\begin{aligned} QD : Q'D &= QM : Q'M' \\ &= SQ : SQ'. \end{aligned}$$

Therefore  $SD$  bisects the *exterior* angle  $Q'Sq$ . [Euc. VI. A.

*Secondly*, let  $Q, Q'$  be on opposite branches of the hyperbola; then it may be similarly shown that  $SD$  bisects the *interior* angle  $QSQ'$ . [Euc. VI. 3.



Ex. 1. Prove that a straight line can cut a hyperbola in two points only. (Cf. Chap. I., Prop. X., Ex. 8; Chap. II., Prop. VIII., Ex. 9.)

Ex. 2. If two points  $Q, Q'$  on a hyperbola be joined with a third variable point  $O$  on the curve, the segment  $qq'$  intercepted on either directrix by the chords  $QO$  and  $Q'O$  produced, subtends a constant angle at the corresponding focus.

Ex. 3. Given the focus and three points on a hyperbola, find the directrix and the axis.

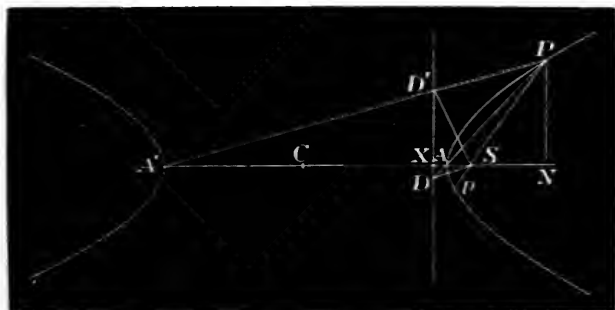
### PROPOSITION VIII.

*The square of the ordinate of any point on a hyperbola varies as the rectangle under the segments of the axis produced, made by the ordinate.*

$$(PN^2 : AN \cdot A'N = CB^2 : CA^2.)$$

Let  $PN$  be the ordinate of any point  $P$  on the hyperbola. Let  $PA, PA'$ , produced if necessary, meet the

directrix in  $D$  and  $D'$ . Join  $SP$ ,  $SD$ ,  $SD'$ , and produce  $PS$  to meet the curve in  $p$ .



Then, from the similar triangles  $PAN$  and  $DAX$ ,  
 $PN : AN = DX : AX$ .

Also from the similar triangles  $PA'N$  and  $D'A'X$ ,  
 $PN : A'N = D'X : A'X$ ;

therefore  $PN^2 : AN \cdot A'N = DX \cdot D'X : AX \cdot A'X$ .

Again,  $SD$  and  $SD'$  bisect the angles  $pSX$  and  $PSX$  respectively; [Prop. VII.

therefore the angle  $DSD'$  is a right angle, and

$$DX \cdot D'X = SX^2. \quad [\text{Euc. VI. 8.}]$$

Therefore  $PN^2 : AN \cdot A'N = SX^2 : AX \cdot A'X$ .

But the ratio  $SX^2 : AX \cdot A'X$  is constant; therefore the ratio  $PN^2 : AN \cdot A'N$  has the same value for all positions of  $P$ .

To determine this constant value we have

$$SA : AX = CS : CA; \quad [\text{Prop. III.}]$$

therefore  $SX : AX = CS + CA : CA$ .

Similarly  $SX : A'X = CS - CA : CA$ ;

therefore  $SX^2 : AX \cdot A'X = CS^2 - CA^2 : CA^2$   
 $= CB^2 : CA^2;$  [Def.]

therefore  $PN^2 : AN \cdot A'N = CB^2 : CA^2.$

Ex. 1. Prove that

$$PN^2 : CN^2 - CA^2 = CB^2 : CA^2.$$

Ex. 2. Having shown that

$$PN^2 : AN \cdot A'N = SX^2 : AX \cdot A'X,$$

apply Prop. V. to complete the proof. [Make  $P$  coincide with the extremity  $L$  of the latus rectum.]

Ex. 3. Prove that

$$\frac{CN^2}{CA^2} - \frac{PN^2}{CB^2} = 1.$$

Ex. 4.  $NQ$  parallel to  $AB$  meets the conjugate axis in  $Q$ . Show that  $QB \cdot QB' = PN^2$ .

Ex. 5.  $Q$  is a point on the curve;  $AQ, A'Q$  meet  $PN$  in  $D$  and  $E$ ; prove that  $DN \cdot EN = PN^2$ .

Ex. 6. If a point  $P$  moves such that  $PN^2 : AN \cdot A'N$  in a constant ratio,  $PN$  being the distance of  $P$  from the line joining two fixed points  $A, A'$ , and  $N$  falling outside  $AA'$ ; the locus of  $P$  is a hyperbola of which  $AA'$  is an axis.

Ex. 7.  $PNP'$  is a double ordinate of an ellipse; show that the locus of intersection of  $AP'$  and  $A'P$  is a hyperbola.

Ex. 8. A circle is described through  $A, A'$  and  $P$ . If  $NP$  meets the circle again in  $Q$ , the locus of  $Q$  is a hyperbola.

Ex. 9.  $NQ$  is a tangent to the circle on  $AA'$  as diameter;  $PM$  is drawn parallel to  $CQ$ , meeting  $AA'$  in  $M$ ; show that  $MN = CB$ . [The triangles  $PMN, QCN$  are similar.]

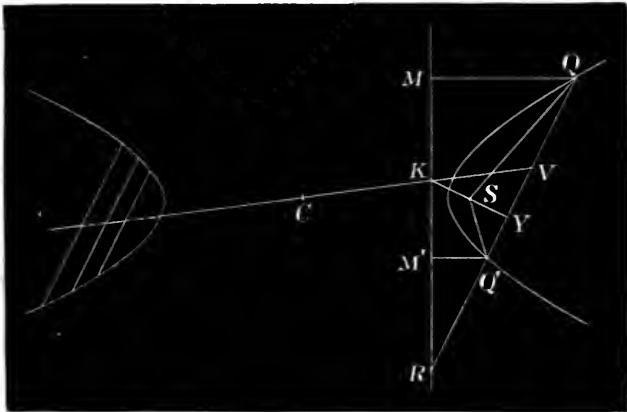
Ex. 10. A chord  $AP$  is divided in  $Q$ , so that  $AQ : QP = CA^2 : CB^2$ . Prove that the line through  $Q$  at right angles to  $QN$  is parallel to  $A'P$ .

### PROPOSITION IX.

*The locus of the middle points of any system of parallel chords of a hyperbola is a straight line passing through the centre.*

Let  $QQ'$  be one of a system of parallel chords, and  $V$  its middle point.

Draw  $QM, Q'M'$  perpendicular to the directrix; draw  $SY$  perpendicular to  $QQ'$  and produce  $YS$  to meet the directrix in  $K$ . Produce  $QQ'$  to meet the directrix in  $R$ , and join  $SQ, S'Q'$ .



Then  $SQ : QM = SQ' : Q'M'$   
 therefore  $SQ : SQ' = QM : Q'M'$   
 $= QR : Q'R;$

therefore

$$SQ^2 - SQ'^2 : QR^2 - Q'R^2 = SQ^2 : QR^2.$$

But

$$SQ^2 - SQ'^2 = QY^2 - Q'Y^2 \quad [\text{Euc. I. 47.}]$$

$$= (QY + Q'Y)(QY - Q'Y)$$

$$= 2 \cdot QQ' \cdot YV.$$

Similarly

$$QR^2 - Q'R^2 = 2 \cdot QQ' \cdot RV;$$

therefore

$$YV : RV = SQ^2 : QR^2.$$

Now, the ratio  $SQ : QM$  is constant; also, the ratio  $QM : QR$  is constant, since  $QQ'$  is drawn in a fixed direction. Therefore  $SQ : QR$  is a constant ratio; therefore also  $YV : RV$  is a constant ratio for all chords of the

system. But as  $R$  always lies on a fixed straight line (the directrix), and  $Y$  on another fixed straight line (the focal perpendicular on the parallel chords), intersecting the former in  $K$ ,  $V$  must also lie on a third fixed straight line, passing through the same point  $K$ .

Again, corresponding to a system of parallel chords in one branch of the hyperbola, there is in the other branch another system exactly similar thereto; and the middle points of the chords of both the systems must lie on  $VK$ , which therefore divides the two branches symmetrically. Hence, from the symmetry of the curve about the major and minor axes, and therefore about the centre,  $VK$  must pass through  $C$ .

Hence the diameter bisecting any system of parallel chords of a hyperbola is a straight line passing through the centre.

Ex. The diameter bisecting any system of parallel chords meets the directrix on the focal perpendicular to the chords.

\* PROPOSITION X.

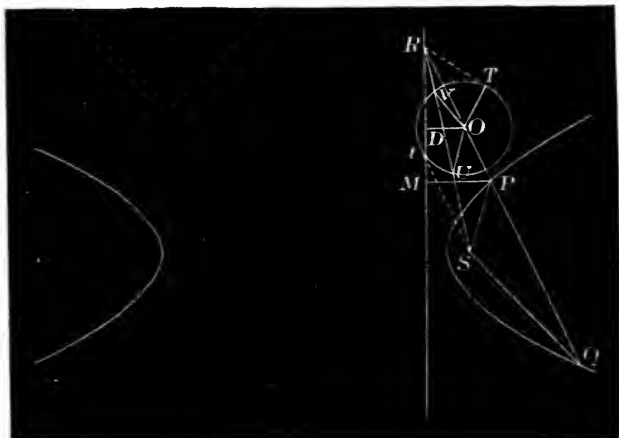
*If any two parallel chords of a hyperbola be drawn through two fixed points, the ratio of the rectangles under their segments will be constant, whatever be the directions of the chords.*

Let  $OPQ$  be a chord drawn through one of the fixed points  $O$ , outside the curve.

Produce  $QPO$  to meet the directrix in  $R$ , and join  $SR$ ,  $SP$ ,  $SQ$ . Draw  $OU$ ,  $OV$  parallel to  $SP$ ,  $SQ$  respectively; and draw  $OD$ ,  $PM$  perpendicular to the directrix.

$$\begin{aligned} \text{Then} \quad RO:RP &= OU:PS \\ &= OD:PM, \end{aligned}$$

but  $PS = e \cdot PM$ ;  
 therefore  $OU = e \cdot OD$ .  
 Similarly  $OV = e \cdot OD$ .



Describe a circle with centre  $O$  and radius equal to  $e \cdot OD$ , passing through  $U$  and  $V$ ; and draw  $RT$ ,  $St$  tangents to this circle.

Now, by similar triangles,

$$OP : OR = SU : RU,$$

and  $OQ : OR = SV : RV$ ;

therefore  $OP \cdot OQ : OR^2 = SU \cdot SV : RU \cdot RV$   
 $= St^2 : RT^2.$  [Enc. III. 36.]

Therefore  $OP \cdot OQ : St^2 = OR^2 : RT^2.$

Now, for a given direction of the chord  $OPQ$  the ratio  $OR : OD$  is constant, and, therefore, also the ratio  $OR : OT$ , since  $OT = e \cdot OD$ . Therefore, also, the ratio  $OR : RT$  is constant.

If, now, through another fixed point  $O'$  a parallel chord

$O'P'Q'$  be drawn, and similar constructions be made, we shall have  $OP \cdot OQ : St^2 = O'P' \cdot O'Q' : St'^2$ ;

therefore  $OP \cdot OQ : O'P' \cdot O'Q' = St^2 : St'^2$ .

But since the points  $O$  and  $O'$  are fixed, the two circles are fixed in magnitude and position, and, therefore,  $St$  and  $St'$  are constants.

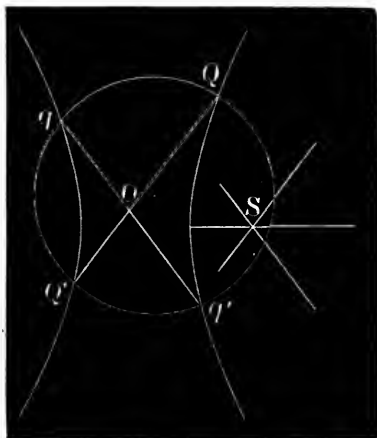
Therefore the ratio  $OP \cdot OQ : O'P' \cdot O'Q'$  is constant.

Ex. 1. If a system of chords of a hyperbola be drawn through a fixed point, the rectangles contained by their segments are as the parallel focal chords, and also as the squares of the parallel semi-diameters where they exist. (Apply Prop. VI.)

Ex. 2. The ordinates to any diameter at equal distances from the centre are equal.

#### \* PROPOSITION XI.

*If a circle intersect a hyperbola in four points, their common chords will be equally inclined, two and two, to the axis.*



Let  $Q, Q', q, q'$  be the four points of intersection.



Then  $QO \cdot OQ' = qO \cdot Oq'$ . [Euc. III. 35.]

Therefore the rectangles under the segments of the focal chords parallel to  $QQ'$  and  $qq'$  respectively are equal,

[Prop. X.]

and therefore the focal chords themselves are equal.

[Prop. VI.]

They are, therefore, equally inclined to the axis, from the symmetry of the figure. (See also Prop. I., Ex. 2.)

Therefore, the chords  $QQ'$ ,  $qq'$  are equally inclined to the axis.

In like manner it may be shown that the chords  $Qq$  and  $Q'q'$ , as well as the chords  $Qq'$  and  $qQ'$ , are equally inclined to the axis.

#### PROPERTIES OF TANGENTS.

The student should work out the following exercises as illustrating the method of deducing tangent properties from the corresponding chord-properties.

I. Deduce from Prop. X., Ex. 1 :—

1. The tangents to a hyperbola from an external point are proportional to the parallel semi-diameters where they exist, and are in the subduplicate ratio of the parallel focal chords.

2. If two parallel tangents  $OP$ ,  $O'P'$  be met by a third tangent at  $Q$ , in  $O$  and  $O'$ , prove that

$$OP : O'P' = OQ : O'Q.$$

II. Deduce from Prop. XI. :—

1.  $PQ$  and  $PQ'$  are chords of a hyperbola equally inclined to the axis; prove that the circle  $PQQ'$  touches the hyperbola at  $P$ .

2. If a circle touch a hyperbola at the points  $P$  and  $Q$ , show that  $PQ$  is parallel to one of the axes.

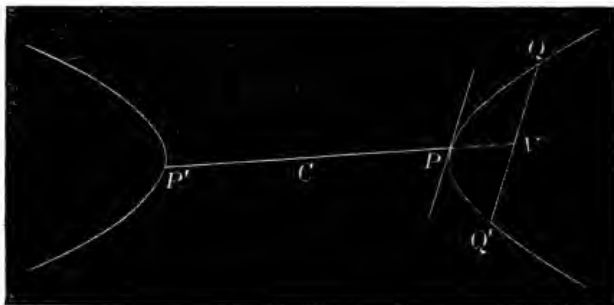
III. Deduce from Prop. VII., Ex. 1 :—

1. A tangent to one branch of a hyperbola cannot meet the other branch.

See also Prop. XII. and XIII.

## PROPOSITION XII.

*The tangent to a hyperbola at either end of a diameter is parallel to the system of chords bisected by the diameter.*



Let  $P'CPV$  be the diameter bisecting a system of chords parallel to  $QQ'$ . Let  $QQ'$  be made to move parallel to itself, so that  $Q$  may coincide with  $V$ . Since  $QV$  is always equal to  $Q'V$ , [Prop. X. it is clear that  $Q'$  will also coincide with  $V$ , and the chord in this its limiting position will be the tangent to the hyperbola at  $P$ .

Ex. 1. The tangent at the vertex is at right angles to the transverse axis.

Ex. 2. The line joining the points of contact of two parallel tangents is a diameter.

## PROPOSITION XIII.

*The portion of the tangent to a hyperbola at any point, intercepted between that point and the directrix, subtends a right angle at the focus, and conversely.*

*Also, tangents at the ends of a focal chord intersect on the directrix.*

First, let any chord  $QQ'$  of the hyperbola intersect the directrix in  $Z$ ; then  $SZ$  bisects the exterior angle  $Q'Sq$ .

[Prop. VII.]

Now, let the chord  $QQ'$  be made to turn about  $Q$  until the point  $Q'$  moves up to and coincides with  $Q$ , so that the



chord becomes the tangent to the hyperbola at  $Q$ . In this limiting position of the chord  $QQ'$ , since  $Q$  and  $Q'$  coincide, the angle  $Q'Sq$  vanishes; therefore the angle



$Q'Sq$  becomes equal to two right angles. But since  $SZ$  always bisects the angle  $Q'Sq$ , in this case the angle  $Q'SZ$  is a right angle.

*Conversely*, let  $QZ$  subtend a right angle at  $S$ , then it shall be the tangent to the hyperbola at  $Q$ . For if not and if possible let  $QZ'$  be the tangent at  $Q$ . Then the angle  $QSZ'$  is a right angle, which is impossible; therefore  $QZ$  is the tangent at  $Q$ .

*Secondly*, let  $Qsq$  be a focal chord and  $QZ$  the tangent at  $Q$ .

Join  $ZS$ ,  $Zq$ .

Then the angle  $QSZ$  being a right angle, the angle  $ZSq$  is also a right angle. Therefore  $qZ$  is the tangent to the hyperbola at  $q$ . Therefore the tangents at  $Q$ ,  $q$  intersect on the directrix.

Ex. 1. Tangents at the extremities of the latus rectum intersect in  $X$ .

Ex. 2. To draw the tangent at a given point of a hyperbola.

Ex. 3. If  $QZ$ ,  $qZ$  meet the latus rectum produced in  $D$  and  $d$ , then  $SD = Sd$ . (Cf. Chap. II., Prop. XV., Ex. 6.)

#### PROPOSITION XIV.

*If from a point  $O$  on the tangent at any point  $P$  of a hyperbola perpendiculars  $OU$ ,  $OI$  be drawn to  $SP$  and the directrix respectively, then*

$$SU = e \cdot OI,$$

*and conversely.*

Join  $SZ$  and draw  $PM$  perpendicular to the directrix.

Because  $ZSP$  is a right angle,

[Prop. XIII.

$ZS$  is parallel to  $OU$ .

Therefore, by similar triangles,

$$\begin{aligned} SU : SP &= ZO \cdot ZP \\ &= OI : PM; \end{aligned}$$

but

$$SP = e \cdot PM.$$

Therefore

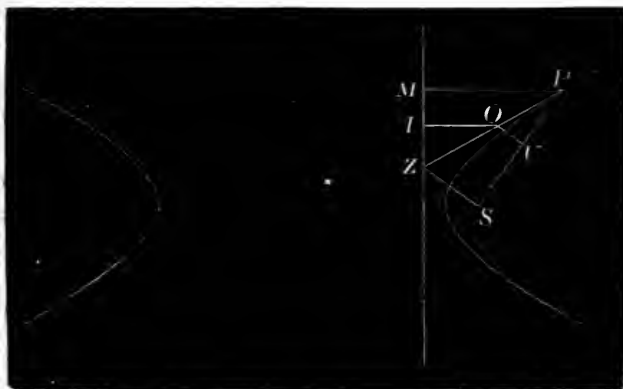
$$SU = e \cdot OI.$$

Again, for the converse proposition, we have

$$SU = e \cdot OI,$$

and

$$SP = e \cdot PM$$



Therefore  $SU : SP = OI : PM$   
 $= ZO : ZP.$

Therefore  $OU$  is parallel to  $ZS$ , [Enc. VI. 2.]  
 and the angle  $PSZ$  is a right angle.

Therefore  $PZ$  is a tangent at  $P$ . [Prop. XIII.]

Ex. If a perpendicular through  $O$  on the transverse axis meet the curve in  $Q$  and  $Q'$ , then  $SU = SQ$ , and  $OU^2 = OQ \cdot OQ'$ .

### PROPOSITION XV.

*The tangent at any point of a hyperbola makes equal angles with the focal distances of the point.*

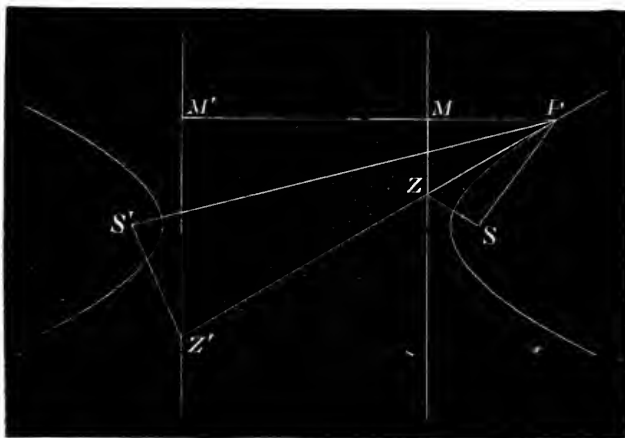
Let the tangent at  $P$  meet the directrices in  $Z$  and  $Z'$ .

Draw  $PMM'$  perpendicular to the directrices, meeting them in  $M$  and  $M'$  respectively. Join  $SP, SZ, S'P, S'Z'$ .

Then, in the two triangles  $PSZ$  and  $PS'Z'$ , the angles  $PSZ$  and  $PS'Z'$  are equal, being right angles, [Prop. XIII. and

$$\begin{aligned} SP : S'P &= PM : PM' \\ &= PZ : PZ', \end{aligned}$$

and the angles  $PZS$  and  $PZ'S'$  are both acute angles.



Therefore the triangles are similar.

[Euc. VI. 7.]

Therefore the angle  $SPZ =$  the angle  $S'PZ'$ .

Ex. 1. The tangent at the vertex is perpendicular to the axis.

Ex. 2. Given a focus, a tangent and its point of contact, find the locus of the other focus.

Ex. 3. If  $PCp$  be a diameter, and if  $Sp$  meet the tangent at  $P$  in  $T$ ,  $SP = ST$ .

Ex. 4. If an ellipse and a hyperbola have the same foci, they intersect at right angles. (See Chap. I., Prop. XIV., Ex. 4.)

Such Conics are called *Confocal Conics*.

Ex. 5. If the tangent at  $P$  meet the axes in  $T, t$ , the angles  $PSt, STP$  are supplementary. [The circle round  $SPS'$  obviously passes through  $t$ .]

Ex. 6. If the diameter parallel to the tangent at  $P$  meet  $SP$  and  $S'P$  in  $E$  and  $E'$ , the circles about the triangles  $SCE, S'CE'$  are equal.

Ex. 7. Tangents at the extremities of a focal chord  $PSQ$  meet in  $T$ . Prove that

$$2\angle PTQ - \angle PS'Q = 2 \text{ right angles.}$$

Ex. 8.  $Y, Y'$  are the feet of the focal perpendiculars on the tangent at  $P$ ; if  $PY$  be the ordinate, the angles  $PNY, PNY'$  are supplementary. [ $\angle PNY = \angle PSY = \angle PS'Y' = \pi - \angle PNY'$ .]

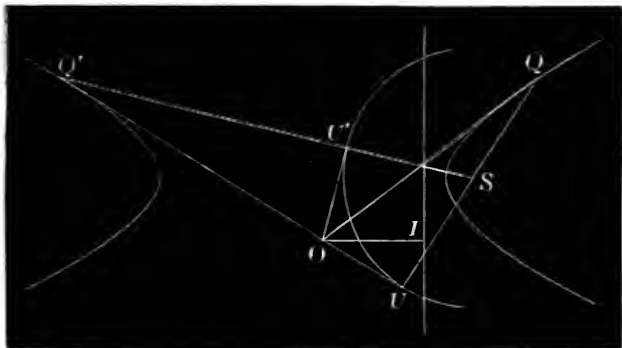
Ex. 9. A parabola and a hyperbola have a common focus  $S$ , and their axes in the same direction. A line  $SPQ$  cuts the curves in  $P$  and  $Q$ . If the tangents at  $P, Q$  meet in  $T$ , prove that  $\angle PTQ = \frac{1}{2}\angle SS'Q$ . (See Prop. I., Ex. 7.)

Ex. 10.  $P$  is a point on a hyperbola whose foci are  $S, S'$ ; another hyperbola is described whose foci are  $S, P$ , and whose transverse axis  $= SP - 2PS'$ . Prove that the hyperbolas will meet at only one point, and that they will have the same tangent at that point. [Apply Prop. IV. If  $Q$  be a point of intersection,  $QP = QS' + PS'$ ;  $Q$ , therefore, is the other extremity of the focal chord  $PS'$ .]

Ex. 11. A chord  $PRVQ$  meets the directrices in  $R$  and  $V$ ,  $P, Q$  being on different branches. Prove that  $PR$  and  $VQ$  subtend, each at the focus nearer to it, angles of which the difference is equal to the angle between the tangents at  $P$  and  $Q$ . (Apply Prop. VII.)

PROPOSITION XVI.

To draw two tangents to a hyperbola from an external point.



Let  $O$  be the external point.

Draw  $OI$  perpendicular to the directrix, and with

centre  $S$  and radius equal to  $e.OI$  describe a circle. Draw  $OU, OU'$  tangents to this circle and let  $US$  and  $SU'$  produced meet the hyperbola in  $Q, Q'$ . Then  $OQ, OQ'$  shall be the tangents required.

For  $OU$  is at right angles to  $SU$ , [Euc. III. 18.  
and  $SU = e.OI$

Therefore  $OQ$  is the tangent to the hyperbola at  $Q$ .

[Prop. XIV.

Similarly  $OQ'$  is the tangent at  $Q'$ .

*Note.*—If it had been necessary to produce *both*  $SU$  and  $SU'$  in the *same* direction, to meet the curve, the points of contact would have been on the same branch, instead of being on opposite branches, as in the figure.

Ex. 1. *Alternative Construction.*—With centre  $O$  and radius  $OS$ , describe a circle. With centre  $S'$  and radius equal to the transverse axis, describe another circle intersecting the former in  $M$  and  $M'$ . Join  $S'M$  and  $S'M'$ , and produce them to meet the curve in  $Q$  and  $Q'$ .  $OQ, OQ'$  are the tangents required. (Cf. Chap. II., Prop. XVIII., Ex. 1.)

Ex. 2. Prove that only two tangents can be drawn to a hyperbola from an external point.

### PROPOSITION XVII.

*The two tangents that can be drawn to a hyperbola from an external point subtend equal or supplementary angles at the focus according as the points of contact are on the same or opposite branches of the curve.*

*First*, let  $OQ, OQ'$  be the two tangents from  $O$ ,  $Q$  and  $Q'$  being on the same branch of the curve.

Join  $SO, SQ, SQ'$ , and draw  $OI, OU, OU'$  perpendiculars upon the directrix,  $SQ, SQ'$  respectively.

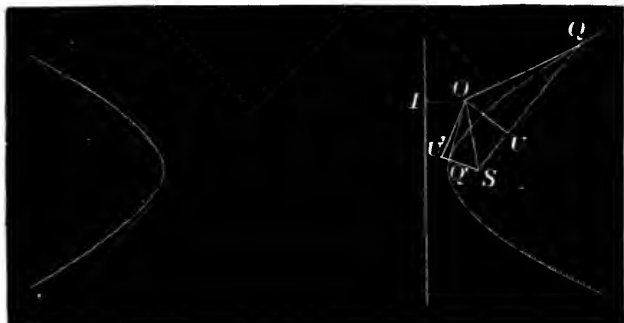
Then  $SU = e.OI = SU'$ . [Prop. XIV.

Therefore  $OU = OU'$ . [Euc. I. 47.

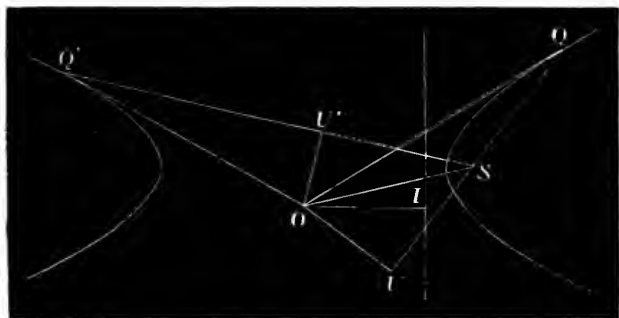
Therefore the angles  $OSU$  and  $OSU'$  are equal, [Euc. I. 8.



and they are the angles which the tangents subtend at the focus.



Secondly, let  $Q$  and  $Q'$  be on opposite branches of the curve. Then it may be similarly proved that the angles



$OSU$  and  $OSU'$  are equal; therefore the angles  $OSQ$  and  $OSQ'$  are supplementary.

Ex. 1. In Fig. 1 prove that  $OQ, OQ'$  subtend equal angles at  $S'$ .

Ex. 2. The portion of any tangent intercepted between the tangents at the vertices, subtends a right angle at either focus.

Ex. 3. Find the locus of the centre of the inscribed circle of the triangle  $SQS'$ . [The tangent at the vertex  $A$ .]

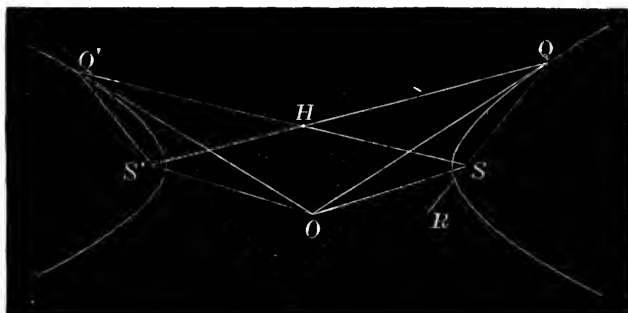
Ex. 4. Show that the chord of contact  $QQ'$  is divided harmonically by  $SO$  and the directrix.

Ex. 5. If  $PN$  be the ordinate of  $P$ , and  $PT$  the tangent, prove that  
 $SP : ST = AN : AT$ .

Ex. 6. Two points  $P$  and  $Q$  are taken on the same branch of the curve and on the same side of the axis; prove that a circle can be drawn touching the four focal distances. [The centre is the point of intersection of the tangents at  $P$  and  $Q$ . Apply Prop. XV.]

\* PROPOSITION XVIII.

*The two tangents that can be drawn to a hyperbola from an external point make equal or supplementary angles with the focal distances of the point according as the points of contact are on the opposite or same branches of the curve.*



First, let  $OQ, OQ'$  be the two tangents from  $O, Q$  and  $Q'$  being on opposite branches of the curve.

Join  $SQ, SQ', SO, S'Q, S'Q, S'O$ , and produce  $QS$  to  $R$ . Let  $H$  be the point of intersection of  $SQ'$  and  $S'Q$ .

Then the angle  $SOQ$

$$= \text{the angle } OSR - \text{the angle } OQS \quad [\text{Enc. I. 32.}]$$

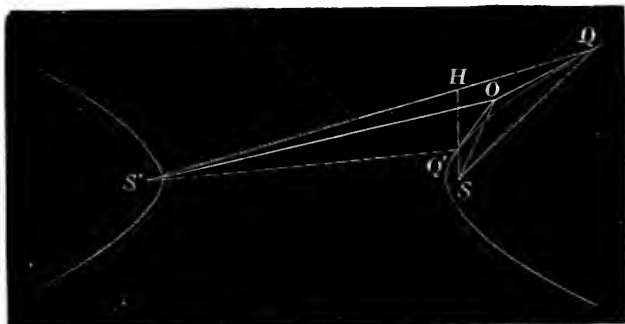
$$= \text{half the angle } Q'SR - \text{half the angle } SQS'$$

[Props. XVII. and XV.]

$$= \text{half the angle } SHQ.$$

Similarly the angle  $S'OQ'$   
 = half the angle  $S'HQ'$ .

Therefore the angle  $SOQ$  = the angle  $S'OQ'$ . [Euc. I. 15.]



Secondly, let  $Q, Q'$  be on the same branch.

Then the angle  $SOQ$   
 = two right angles – the angle  $OSQ$  – the angle  $OQS'$   
 [Euc. I. 32.]

= two right angles – half the angle  $QSQ'$  – half the  
 angle  $SQS'$  [Prop. XVII. and XV.]

= two right angles – half the angle  $SHS'$ . [Euc. I. 32.]

Again, the angle  $S'OQ'$

= two right angles – the angle  $OQ'S'$  – the angle  $OS'Q'$   
 [Euc. I. 32.]

= half the angle  $SQ'S'$  – half the angle  $QS'Q'$ .  
 [Props. XV. and XVII.]

= half the angle  $SHS'$

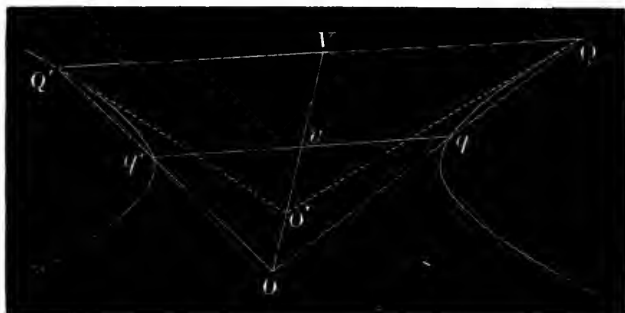
Therefore, the angles  $SOQ$  and  $S'OQ'$  are together equal  
 to two right angles.

Ex. 1. Tangents are drawn from any point on a circle through  
 the foci. Prove that the lines bisecting the angle between the  
 tangents, or between one tangent and the other produced, all  
 pass through a fixed point. [A point of intersection of the circle  
 with the conjugate axis.]

Ex. 2. A hyperbola is described, touching the four sides (produced, if necessary) of a quadrilateral  $ABCD$  which is inscribed in a circle. If one focus lies on the circle, the other also lies on it. [ $\angle S'CD = \angle SCB = \angle SAB = \angle S'AD$ .]

PROPOSITION XIX.

*The tangents at the extremities of any chord of a hyperbola intersect on the diameter which bisects the chord.*



Let  $QQ'$  be the chord and  $qq'$  any other chord parallel to it.

Let  $Qq, Q'q'$  produced meet in  $O$ . Bisect  $QQ'$  in  $V$  and let  $OV$  meet  $qq'$  in  $v$ .

$$\begin{aligned} \text{Then} \quad QV:qv &= OV:Ov \\ &= Q'V:q'v; \end{aligned}$$

$$\text{but} \quad QV = Q'V,$$

$$\text{therefore} \quad qv = q'v.$$

Thus  $OvV$  is the diameter bisecting the system of chords parallel to  $QQ'$ . [Prop. IX.

If now the chord  $qq'$  be made to move parallel to itself till it coincide with  $QQ'$ ,  $QqO$  and  $Q'q'O$  will become the tangents to the curve at  $Q$  and  $Q'$  respectively. They thus meet on the diameter bisecting  $QQ'$ .

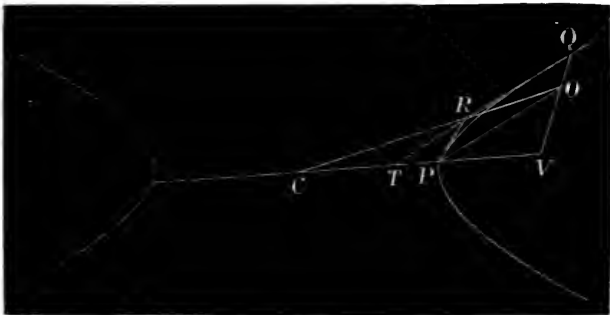
Ex. 1. Given a diameter of a hyperbola, to draw the system of chords bisected by it.

Ex. 2. If a circle passing through any point  $P$  on the curve, and having its centre on the normal at  $P$ , meets the curve again in  $Q$  and  $R$ , the tangents at  $Q$  and  $R$  intersect on a fixed straight line.

[The tangent at  $P$  and  $QR$  are equally inclined to the axis (see Prop. XI.);  $QR$  is, therefore, fixed in direction.]

PROPOSITION XX.

If the tangent at any point  $Q$  of a hyperbola meet any diameter  $CP$  in  $T$  and if  $QV$  be the ordinate to that diameter,  $CV \cdot CT = CP^2$ .



Draw the tangent  $PR$  at  $P$ , meeting  $QT$  in  $R$ , and draw  $PO$  parallel to  $QT$ , meeting  $QV$  in  $O$ .

Then since  $POQR$  is a parallelogram, [Prop. XII.  
 $RO$  bisects  $PQ$ , and therefore passes through the centre  $C$ .  
 [Prop. XIX.]

By similar triangles

$$CV : CP = CO : CR = CP : CT,$$

therefore  $CV \cdot CT = CP^2$ .

Note. When the diameter coincides with the transverse axis the result is stated thus:—

If the tangent at  $Q$  meets the transverse axis in  $T$  and  $QN$  be the perpendicular on the transverse axis.

$$CN \cdot CT = CA^2.$$

From this it may be shown that

If the tangent at  $Q$  meets the conjugate axis, produced if necessary in  $t$ , and  $Qn$  be the perpendicular on the conjugate axis,

$$Cn \cdot Ct = CB^2.$$

$$\frac{QN \cdot Ct}{CN \cdot CT} = \frac{QN^2}{CN \cdot NT} = \frac{QN^2}{CN^2 - CN \cdot CT} = \frac{QN^2}{CN^2 - CA^2}.$$

$$\therefore Cn \cdot Ct = CB^2. \quad [\text{Prop. VIII.}]$$

These two results are important, and should be carefully noted by the student.

Ex. 1. If the tangent at  $Q$  meet the transverse axis in  $T$  and  $QN$  be the perpendicular on the transverse axis, show that

$$CN \cdot NT = AN \cdot NA'.$$

Ex. 2. In Ex. 1, if  $TD$  be drawn perpendicular to the axis to meet the circle described on  $AA'$  as diameter, then  $DN$  touches the circle.

Ex. 3. In Ex. 2, prove that

$$DN : QN = CA : CB.$$

Also if  $DA$  be produced to meet  $PN$  in  $K$ ,

$$QN : NK = CB : CA.$$

(Apply Prop. VIII., and see Ex. 1.)

Ex. 4. Any diameter is cut harmonically by a tangent and the ordinate of the point of contact of the tangent with respect to the diameter.

Ex. 5. Any tangent is cut harmonically by any two parallel tangents and the diameter through their points of contact. (Ex. 4.)

Ex. 6. If  $PN$  be the ordinate of a point  $P$ , and  $NQ$  be drawn parallel to  $AP$  to meet  $CP$  in  $Q$ ,  $AQ$  shall be parallel to the tangent at  $P$ .

Ex. 7. If the tangent at  $P$  intersect the tangents at the vertices and the transverse axis in  $R$ ,  $r$  and  $T$ , show that

$$(i) \quad AT \cdot A'T = CT \cdot TN.$$

$$(ii) \quad AR \cdot A'r = CB^2.$$

Ex. 8.  $P$  is any point on a hyperbola. Prove that the locus of the centre ( $Q$ ) of the circle touching  $SP$ ,  $S'P$  produced, and the transverse axis, is a hyperbola.

[Let  $QM$  be the ordinate of  $Q$ ; then, if the tangents at  $A$  and  $P$  meet in  $F$ ,  $QSF$  is a right angle, and

$$\frac{QM}{MS} = \frac{SA}{AF}, \quad \frac{QM}{MS'} = \frac{S'A'}{A'F'}$$

$\therefore QM^2 \cdot MS \cdot MS' = SA^2 \cdot CB^2$ . Ex. 7.

Then apply Prop. VIII., Ex. 6.]

Ex. 9. The tangent at  $P$  bisects any straight line perpendicular to  $AA'$ , and terminated by  $AP$  and  $A'P$ .

[Let the tangent at  $P$ ,  $AP$ ,  $A'P$  meet the conjugate axis in  $t$ ,  $E$ ,  $E'$  respectively. Then

$$\frac{CE - CE'}{PN} = \frac{CA \cdot A'N - CA' \cdot AN}{AN \cdot A'N} = \frac{2CA^2}{AN \cdot A'N} = \frac{2Ct}{PN}$$

[Prop. VIII.

$\therefore CE - CE' = 2Ct$ , or  $t$  bisects  $EE'$ .]

Ex. 10. An ellipse and a hyperbola are described, so that the foci of each are at the extremities of the transverse axis of the other; prove that the tangents at their points of intersection meet the conjugate axis in points equidistant from the centre. [The conjugate axes of the two curves are equal in length.]

PROPOSITION XXI.

*The locus of the foot of the perpendicular drawn from either focus upon any tangent to a hyperbola is the circle described on the transverse axis as diameter; and the rectangle under the focal perpendiculars on the tangent is equal to the square of the semi-conjugate axis.*

$$(SY \cdot S'Y' = CB^2)$$

Let  $SY$ ,  $S'Y'$  be the focal perpendiculars upon the tangent at any point  $P$ .

Join  $SP$ ,  $S'P$ , and produce  $SY$  to meet  $S'P$  in  $R$ . Join  $CY$ .

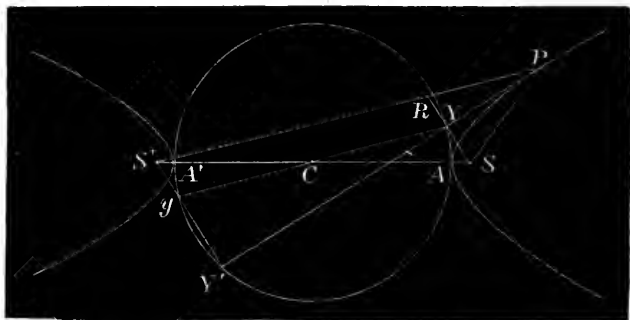
Then in the triangles  $SPY$ ,  $RPY$ , the angle  $SPY =$  the angle  $RPY$ ,

[Prop. XV.

the angles  $SYP$ ,  $RYP$  are equal, being right angles, and  $YP$  is common.

Therefore  $SP = PR, SY = YR.$  [Euc. I. 26.]  
 Also  $SC = CS';$   
 therefore  $CY$  is parallel to  $S'P.$  [Euc. VI. 2.]  
 Therefore  $CY = \frac{1}{2}S'R$  [Euc. VI. 4.]  
 $= \frac{1}{2}(S'P - PR)$   
 $= \frac{1}{2}(SP - SP)$   
 $= \frac{1}{2}AA'$  [Prop. IV.]  
 $= CA;$

therefore the locus of  $Y$  is the circle described on the transverse axis as diameter.



Similarly it may be shown that the locus of  $Y'$  is the same circle.

Again, produce  $YC$  to meet  $S'Y'$  in  $y$ . Then  $y$  will be on the circle.

For, since  $CS = CS'$ , and  $SY$  is parallel to  $S'Y'$ , the triangles  $SCY, S'Cy$  are equal. [Euc. I. 26.]

Therefore  $Cy = CY = CA$ , showing that  $y$  is on the circle.

Also  $SY = S'y,$

therefore  $SY \cdot S'Y' = S'y \cdot S'Y' = S'A' \cdot S'A$  [Euc. III. 35.]

$= SA \cdot SA' = CB^2.$  [Def.]



Ex. 1. If  $CE$  drawn parallel to the tangent at  $P$  meet  $S'P$  in  $E$ , then  $PE=CA$ .

Ex. 2. From a point on the circle on  $AA'$  as diameter lines are drawn touching the curve in  $P, P'$ . Prove that  $SP', S'P$  are parallel. [Each is parallel to  $CY$ .]

Ex. 3. If through any point  $Y$  on the circle on  $AA'$  as diameter  $YP$  be drawn at right angles to  $SY$ ,  $YP$  will be a tangent to the hyperbola.

Ex. 4. If the vertex of a right angle moves on a fixed circle, and one leg passes through a fixed point *outside* the circle, the other leg will always touch a hyperbola.

Ex. 5. Given a focus, a tangent, and a point on a hyperbola, find the locus of the other focus. [An arc of a fixed hyperbola of which the foci are the given point and the image of the focus in the tangent.]

Ex. 6. Given a focus, a tangent, and the transverse axis, find the locus of the other focus. [A circle ; centre  $R$ , radius= $AA'$ .]

Ex. 7. If  $PN$  be the ordinate of  $P$ , the points  $Y, Y', N, C$  lie on a circle.

Ex. 8. The right lines joining each focus to the foot of the perpendicular from the other focus on the tangent meet on the normal and bisect it.

Ex. 9. Alternative Construction for Prop. XVI.

Let  $O$  be the external point. On  $OS$  as diameter describe a circle, cutting the circle on  $AA'$  as diameter in  $Y$  and  $Y'$ . Then  $OY$  and  $OY'$  produced will be the tangents required.

Ex. 10. If tangents be drawn from  $P$  to a circle described with  $S'$  as centre and radius equal to  $CB$ , the chord of contact will touch the circle described on  $AA'$  as diameter. [The line through  $y$  perpendicular to  $S'P$  will be the chord of contact.]

Ex. 11. If the tangent at  $P$  cuts the transverse axis in  $T$ , prove that  $AT \cdot A'T = YT \cdot Y'T$ .

Ex. 12. Find the position of  $P$  when the area of the triangle  $YCY'$  is the greatest possible.

[ $CY=CY'=CA$  ; therefore  $YCY'$  must be a right angle. If the tangent at  $P$  meets  $CB$  in  $t$ ,  $PN \cdot Ct = CB^2$ . (Prop. XX.) Also the triangles  $CYS, CY't$  are equal ; therefore  $PN \cdot CS = CB^2$ .]

Ex. 13. If  $SY, SZ$  be perpendiculars on two tangents which meet in  $O$ ,  $OZ$  is perpendicular to  $S'O$ . [ $S'O$  is parallel to the bisector of  $YCZ$ . Apply Prop. XVII.]

Ex. 14. An ellipse and a hyperbola are confocal ; if a tangent to the one intersects at right angles a tangent to the other, the locus of the point of intersection is a circle.

Let  $SY, S'Y'$  be the focal perpendiculars upon the tangent to the ellipse, and  $SZ, S'Z'$  those upon the tangent to the hyperbola ; let the tangents meet at  $O$  ; let  $a, b$  be the semi-axes of the ellipse, and  $\alpha, \beta$  those of the hyperbola. Then if  $CV$  be perpendicular to  $YOY'$ ,

$$OY \cdot OY' = Y'V^2 - OV^2$$

and  $CO^2 + OY \cdot OY' = CY'^2 = CA^2$  ;

$\therefore CO^2 + SZ \cdot S'Z' = a^2$

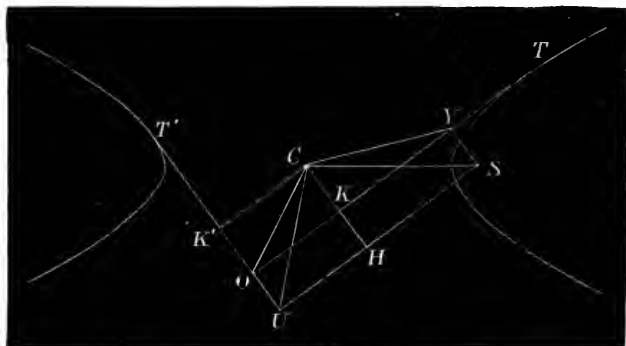
or  $CO^2 = a^2 - \beta^2$ .

See also Prop. IV., Ex. 14, 15.

Ex. 15. If an ellipse and a hyperbola are confocal, the difference of the squares of the central distances of parallel tangents is constant ( $=b^2 + \beta^2$ . Ex. 14.)

### \* PROPOSITION XXII.

*The locus of the intersection of tangents to a hyperbola which cut at right angles is a circle.*



Let the tangents  $OT, OT'$  cut at right angles at  $O$ .

Draw  $SY, CK$  perpendicular to  $OT$ , and  $SU, CK'$  perpendicular to  $OT'$ . Join  $CY, CU, CO$ , and produce  $CK$  to meet  $SU$  in  $H$ .

Now  $Y$  and  $U$  are on the circle on  $AA'$  as diameter;

[Prop. XXI.

therefore  $CY = CU = CA$ .

Now  $CO^2 = CK^2 + CK'^2$ , [Euc. I. 47.

and  $CY^2 = CK^2 + YK^2$ ;

therefore  $CA^2 = CK^2 + SH^2$ .

Also  $CU^2 = CK'^2 + UK'^2$ ,

therefore  $CA^2 = CK'^2 + HC^2$ ;

therefore  $2CA^2 = CK^2 + CK'^2 + SH^2 + HC^2$   
 $= CO^2 + CS^2$ . [Euc. I. 47.

But  $CS^2 = CA^2 + CB^2$ ; [Def.

therefore  $CO^2 = CA^2 - CB^2$ .

Hence the locus of  $O$  is a circle described with centre  $C$ .

*Note.*—This circle is called the *director circle* of the hyperbola. In the case when  $CB$  is greater than  $CA$ ,  $CA^2 - CB^2$  is negative, and, therefore, the locus does not exist, that is, when  $CB$  is greater than  $CA$  the hyperbola has no tangents cutting at right angles.

*Ex.* Four tangents to a hyperbola form a rectangle; if one side  $UV$  of the rectangle intersect a directrix in  $F$ , and  $S$  be the corresponding focus, the triangles  $FSU$ ,  $FVS$  are similar.

$$\begin{aligned}
 [SF^2 &= CF^2 - CX^2 + SX^2 \\
 &= CF^2 + CS^2 - 2CS. \quad CX = CF^2 - CA^2 + CB^2 \\
 &= \text{square of tangent from } F \text{ to the director circle} \\
 &= FU \cdot FV.]
 \end{aligned}$$

PROPERTIES OF NORMALS.

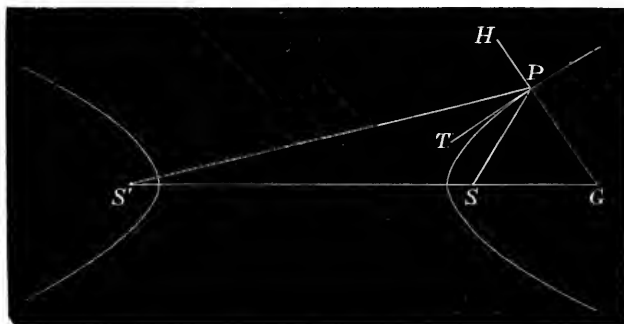
PROPOSITION XXIII.

*The normal at any point of a hyperbola makes equal angles with the focal distances of the point.*

Let the normal  $HPG$  at the point  $P$  meet the axis in  $G$ .

Let  $PT$  be the tangent at  $P$ . Then

the angle  $SPT =$  the angle  $S'PT$ . [Prop. XV.



But the angles  $TPG$  and  $TPH$  are equal, being right angles; [Def.

therefore the angle  $SPG =$  the angle  $S'PH$ .

Ex. 1. If the tangent and normal at  $P$  meet the conjugate axis in  $t$  and  $g$ ,  $P, t, g, S, S'$  lie on the same circle.

Ex. 2. If a circle through the foci meet two confocal hyperbolas in  $P$  and  $Q$ , the angle between the tangents at  $P$  and  $Q$  is equal to  $P'SQ$ .

Ex. 3. The tangent at  $P$  meets the conjugate axis in  $t$ , and  $tQ$  is perpendicular to  $SP$ . Prove that  $SQ$  is of constant length.

[If  $SY$  is perpendicular to  $Ct$ ,  $CY = CA$ . Prop. XXI. Also  $Q, S, C, t$  lie on a circle.  $\therefore \angle tQC = \angle tSS' = \angle tPS' = \angle tPS$ .  $\therefore CQ \parallel SY$ , and  $SQ = CY = CA$ .]

Ex. 4. If from  $g$  a perpendicular  $gK$  be drawn on  $SP$ , show that  $PK = CA$ . (Cf. Chap. II., Prop. XXVI., Ex. 3.)

Ex. 5. Prove that  $SP \cdot S'P = PG \cdot Pg$ . (Cf. Chap. II., Prop. XXVI., Ex. 4.)

#### \* PROPOSITION XXIV.

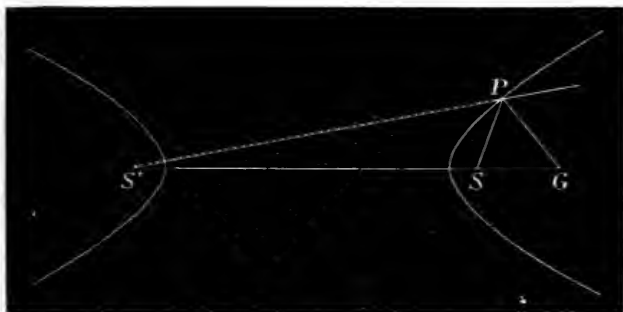
If the normal at any point  $P$  of a hyperbola meet the transverse axis in  $G$ ,

$$SG = e \cdot SP.$$

Join  $S'P$ .

Then, since  $PG$  bisects the exterior angle between  $SP$  and  $S'P$ ,

$$S'G : SG = S'P : SP; \quad [\text{Euc. VI. A.}]$$



therefore  $S'G - SG : SG = S'P - SP : SP$ ,  
 or  $SG : SP = S'G - SG : S'P - SP$ .  
 But  $S'G - SG = SS' = e \cdot AA'$ , [Prop. III.]  
 and  $S'P - SP = AA'$ ; [Prop. IV.]  
 therefore  $SG = e \cdot SP$ .

Ex. 1. The projection of the normal upon the focal distance of any point is equal to the semi-latus rectum. (Cf. Chap. II., Prop. XXVII., Ex. 4.)

Ex. 2. A circle passing through a focus, and having its centre on the transverse axis, touches the curve; prove that the focal distance of the point of contact is equal to the latus rectum.

Ex. 3. Draw the normal at any point without drawing the tangent.

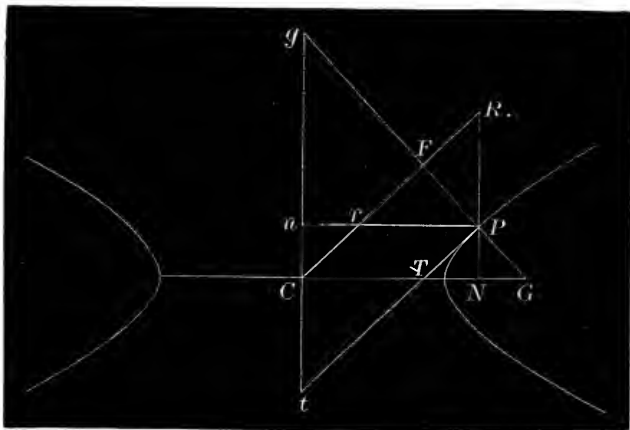
\* PROPOSITION XXV.

*The normal at any point of a hyperbola terminated by either axis varies inversely as the central perpendicular upon the tangent.*

$$(PG \cdot PF = CB^2; \quad Pg \cdot PF = CA^2.)$$

Let the normal at  $P$  meet the transverse and conjugate axis in  $G$  and  $g$  respectively, and let the tangent at  $P$  meet them in  $T$  and  $t$  respectively.

Draw  $PN$ ,  $Pn$  perpendicular to the transverse and conjugate axis, and let a straight line through the centre, drawn parallel to the tangent at  $P$ , meet  $NP$ ,  $GP$  produced and  $Pn$  in  $R$ ,  $F$ , and  $r$  respectively.



Then, since the angles at  $N$  and  $F$  are right angles,  $G, N, F, R$  lie on a circle.

$$\begin{aligned} \text{Therefore} \quad PG \cdot PF &= PN \cdot PR && [\text{Euc. III. 35.}] \\ &= Cn \cdot Ct && [\text{Euc. I. 34.}] \\ &= CB^2. && [\text{Prop. XX., Note.}] \end{aligned}$$

Again, since the angles at  $n$  and  $F$  are right angles,  $g, F, r, n$  lie on a circle.

$$\begin{aligned} \text{Therefore} \quad Pg \cdot PF &= Pn \cdot Pr && [\text{Euc. III. 36.}] \\ &= CN \cdot CT && [\text{Euc. I. 34.}] \\ &= CA^2. && [\text{Prop. XX., Note.}] \end{aligned}$$

Therefore both  $PG$  and  $Pg$  vary inversely as  $PF$ , which

is equal to the central perpendicular upon the tangent at  $P$ .

Ex. In Prop. XXIII., Ex. 1, prove that

$$Gg = e \cdot Sg$$

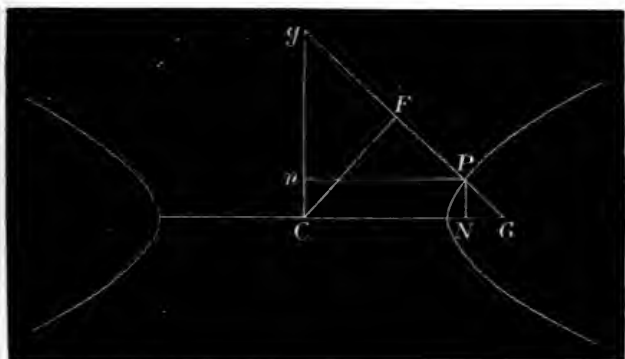
Apply Prop. III., Ex. 2.

\* PROPOSITION XXVI.

If the normal at any point  $P$  of a hyperbola meet the transverse axis in  $G$ , and  $PN$  be the ordinate to that axis.

$$(i) \quad GN : CN = CB^2 : CA^2.$$

$$(ii) \quad CG = e^2 \cdot CN$$



Let the normal meet the conjugate axis in  $g$ . Draw  $Pn$  perpendicular to the conjugate axis, and  $CF$  parallel to the tangent at  $P$ .

Then, because the triangles  $PNG$  and  $Png$  are similar,

$$GN : CN = PG : Pg \quad [\text{Euc. VI. 2.}]$$

$$= PG \cdot PF : Pg \cdot PF$$

$$= CB^2 : CA^2; \quad [\text{Prop. XXV.}]$$

therefore  $CN + GN : CN = CA^2 + CB^2 : CA^2,$

or  $CG : CN = CS^2 : CA^2. \quad [\text{Def.}]$

But  $CS = e \cdot CA$ ; [Prop. III.  
therefore  $CG = e^2 \cdot CN$ .

Ex. 1. Prove that

$$CG \cdot Cn : Cg \cdot CN = CB^2 : CA^2.$$

Ex. 2. Show that

$$Sn : Cn = CA^2 : CB^2.$$

Ex. 3. If the tangent and normal at  $P$  meet the axis in  $T$  and  $G$ , prove that

$$(i) \quad NG \cdot CT = CB^2.$$

$$(ii) \quad CG \cdot CT = CS^2.$$

[Apply Prop. XX.]

Ex. 4. Find the locus of the points of contact of tangents to a series of confocal hyperbolas from a fixed point on the axis.

[From Ex. 3 (ii),  $G$  the foot of the normal is fixed; hence  $P$  lies on the circle of which  $TG$  is diameter.]

#### PROPERTIES OF ASYMPTOTES.

**Def.** When a curve continually approaches to a fixed straight line without ever actually meeting it, but so that its distance from it, measured along any straight line, becomes ultimately less than any finite length, the fixed straight line is called an *asymptote* to the curve.

#### PROPOSITION XXVII.

*The diagonals of the rectangle formed by perpendiculars to the axes of a hyperbola, drawn through their extremities, are asymptotes to the curve.*

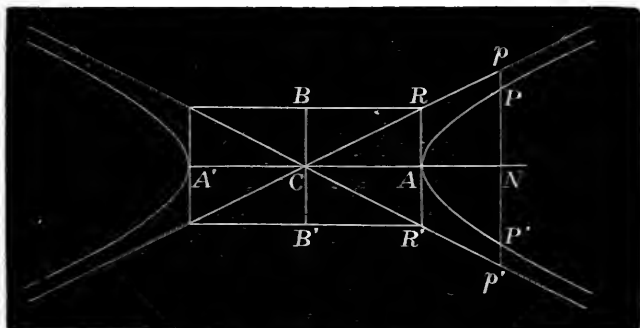
Let  $CR, CR'$  be the diagonals of the rectangle formed by perpendiculars through the extremities  $A, A', B, B'$  of the axes of the hyperbola. Through any point  $N$  on the transverse axis draw  $pPNP'p'$  perpendicular to it, meeting the curve in  $P$  and  $P'$ , and  $CR, CR'$  in  $p, p'$  respectively.

Now  $PN^2 : AN \cdot A'N = CB^2 : CA^2$ , [Prop. VIII.]

or  $PN^2 : CN^2 - CA^2 = CB^2 : CA^2$ . [Euc. II. 6.]



Again  $pN^2 : CN^2 = AR^2 : CA^2$   
 $= CB^2 : CA^2$ ;  
 therefore  $pN^2 - PN^2 : CA^2 = CB^2 : CA^2$ ,  
 or  $pN^2 - PN^2 = CB^2$



But since  $pp'$  is bisected in  $N$ ,

$$pN^2 - PN^2 = pP \cdot p'P. \quad [\text{Euc. II. 5.}]$$

Therefore  $pP \cdot p'P = CB^2$ .

Now  $p'P = NP + Np'$ ,

and  $NP^2$  varies as  $AN \cdot A'N$ , [Prop. VIII.]

and  $Np'$  varies as  $CN$ .

Hence, as  $N$  moves along  $A'A$  produced, both  $NP$  and  $Np'$ , and therefore also  $p'P$ , continually increase. But the product  $pP \cdot p'P$ , of which one factor  $p'P$  continually increases, is constant; therefore  $p'P$  continually diminishes, and becomes ultimately less than any finite length, however small.  $CR$ , therefore, is an asymptote to the hyperbola. Similarly,  $CR'$  is another asymptote.

Ex. 1. The lines joining the extremities of the axes are bisected by one asymptote and parallel to the other.

Ex. 2. Any line parallel to an asymptote cannot meet the curve in more than one point.

Ex. 3. Prove that the angle between the asymptotes of the

hyperbola in Prop. I., Ex. 10, is double the exterior angle between the tangents.

Ex. 4. The circle on  $AA'$  as diameter cuts the directrices in the same points as the asymptotes.

Ex. 5. If the directrix meets  $CR$  in  $F'$ , prove that (i)  $CF=AC$ ; (ii)  $CFS$  is a right angle.

Ex. 6. Given one asymptote, the direction of the other, and the position of one focus, find the vertices.

Ex. 7. If  $CR$  meets the directrix in  $F'$ ,  $AF'$  is parallel to  $SR$ .

Ex. 8. Given the asymptotes and a focus to find the directrix. [Apply Ex. 5 (ii).]

Ex. 9. Given the centre, an asymptote, and a directrix, to find the focus. [Apply Ex. 5 (ii).]

Ex. 10. Given an asymptote, the directrix, and a point on the hyperbola, to construct the curve. (Ex. 5.)

Ex. 11. The straight line drawn from the focus to the directrix, parallel to an asymptote, is equal to the semi-latus rectum, and is bisected by the curve. (Cf. Ex. 13.)

Ex. 12. The perpendicular from the focus on either asymptote is equal to the semi-conjugate axis.

Ex. 13. The focal distance of any point on the curve is equal to the length of the line drawn from the point parallel to an asymptote to meet the directrix. (Cf. Ex. 11.)

Ex. 14. Given the eccentricity of a hyperbola, find the angle ( $\theta$ ) between the asymptotes.  $\left( \sec \frac{\theta}{2} = e. \right)$

Ex. 15. Prove that the tangents to a hyperbola from  $C$  coincide with the asymptotes.

Apply Prop. XVI., Ex. 1, observing that the tangents are lines bisecting  $SM, SM'$  at right angles.

The asymptotes may thus be regarded as tangents to the hyperbola whose points of contact are at infinity.

Ex. 16. If the tangent at  $P$  meets an asymptote in  $T$ , prove that  $ST$  will bisect the angle between  $PS$  and the line through  $S$  parallel to the asymptote. (Apply Ex. 15 and Prop. XVII.)

Ex. 17. If the tangent at  $P$  meets an asymptote in  $T$ , prove that  $\angle STP = \angle S'TC = \angle PS'T$ . (Ex. 15.)

Ex. 18. If a tangent meet the asymptotes in  $L$  and  $M$ , the angle subtended by  $LM$  at the farther focus is half the angle between the asymptotes.

[Apply Ex. 16 and Prop. XVIII. If  $S'L, S'M'$  be drawn parallel to the asymptotes,  $LS', MS'$  bisect the angles  $PS'L, PS'M'$ .]

Ex. 19. Given an asymptote, the focus, and a point on the hyperbola to construct the curve.

[The feet of the focal perpendiculars on the asymptote and the tangent at the point (Ex. 16) will lie on the circle described on  $AA'$  as diameter (Ex. 15 and Prop. XXI.), whence the centre is determined; the directrix is found at once by Ex. 5.]

Ex. 20. The tangent and normal at any point meet the asymptotes and the axes respectively in four points lying on a circle, which passes through the centre of the hyperbola, and of which the radius varies inversely as the central perpendicular on the tangent.

Ex. 21. The radius of the circle which touches a hyperbola and its asymptotes is equal to the part of the latus rectum intercepted between the curve and an asymptote. (Apply Prop. V.)

Ex. 22. A parabola  $P$  and a hyperbola  $H$  have a common focus, and the asymptotes of  $H$  are tangents to  $P$ . Prove that the tangent at the vertex of  $P$  is a directrix of  $H$ , and that the tangent to  $P$  at its intersection with  $X$  passes through the farther vertex of  $H$ .

[The line joining the feet of the focal perpendiculars upon the asymptotes is the tangent at the vertex of  $P$  (Chap. I., Prop. XXIII.), and the directrix of  $H$  (Ex. 5). If  $P$  be a common point, and  $PM$  be perpendicular to the directrix of  $H$ , we have  $SP : PM = SC : CA$ , and  $SP = PM + SX$ .  $\therefore SP : SX = CS : AS$ .  $\therefore SP \cdot AS = SX \cdot CS = CB^2 = SA \cdot SA'$ .  $\therefore SP = SA'$  and  $A'P$  touches the parabola at  $P$ . (Chap. I., Prop. XIV.)]

Ex. 23. If an ellipse and a confocal hyperbola intersect in  $P$ , an asymptote passes through the point on the auxiliary circle corresponding to  $P$ . (Apply Prop. IV., Ex. 13.)

PROPOSITION XXVIII.

*If through any point on a hyperbola a straight line parallel to either axis be drawn meeting the asymptotes, the rectangle under its segments is equal to the square of the semi-axis to which it is parallel.*

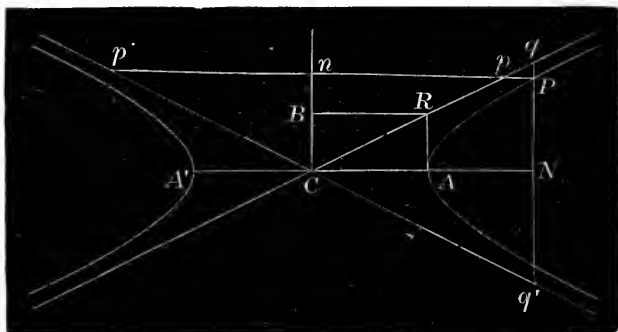
*First case.*

Through any point  $P$  on the hyperbola draw  $Ppp'$  parallel to the transverse axis, meeting the asymptotes in  $p$  and  $p'$  and the conjugate axis in  $n$ .

Then, since  $pp'$  is bisected at  $n$ ,

$$Pp \cdot Pp' = Pn^2 - pn^2. \quad [\text{Euc. II. 6.}]$$

Now  $PN^2 : AN \cdot A'N = CB^2 : CA^2$ , [Prop. VIII.  
 therefore  $PN^2 : CN^2 - CA^2 = CB^2 : CA^2$ ; [Euc. II. 6.  
 or  $Cn^2 : Pn^2 - CA^2 = CB^2 : CA^2$ ,  
 but  $Cn^2 : pn^2 = CB^2 : BR^2$   
 $= CB^2 : CA^2$ ,  
 therefore  $PN^2 - CA^2 = pn^2$ ,  
 or  $Pn^2 - pn^2 = CA^2$ ,  
 therefore  $Pp \cdot Pp' = CA^2$ .



*Second case.*

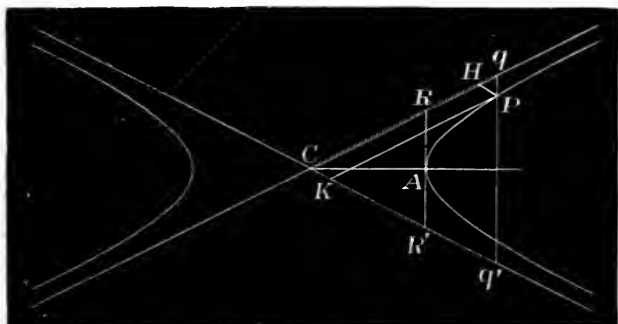
Through  $P$  draw  $qPq'$  parallel to the conjugate axis, meeting the asymptotes in  $q, q'$ .

Then, as before,

$PN^2 : CN^2 - CA^2 = CB^2 : CA^2$ ,  
 or  $PN^2 + CB^2 : CN^2 = CB^2 : CA^2$ ,  
 or  $PN^2 + CB^2 : Pn^2 = CB^2 : CA^2$ ;  
 but  $qN^2 : Pn^2 = qN^2 : CN^2$   
 $= AR^2 : CA^2$   
 $= CB^2 : CA^2$ ;  
 therefore  $qN^2 = PN^2 + CB^2$ ,  
 or  $qN^2 - PN^2 = CB^2$ ,  
 or  $Pq \cdot Pq' = CB^2$ . [Euc. II. 6.

PROPOSITION XXIX.

If through any point on a hyperbola lines be drawn parallel to the asymptotes, the rectangle under the segments intercepted between the point and the asymptotes is constant.



Through any point  $P$  on the hyperbola draw  $PH, PK$  parallel to the asymptotes, meeting them in  $H, K$ . Draw  $RAR'$  and  $qPq'$  perpendicular to  $CA$

Then, by similar triangles,

$$PH : Pq = CR' : RR',$$

and  $PK : Pq' = CR : RR',$

therefore  $PH \cdot PK : Pq \cdot Pq' = CR' \cdot CR : RR'^2,$

or  $PH \cdot PK : CB^2 = CR'^2 : 4RA^2.$  [Prop. XXVIII.  
 $= CA^2 + CB^2 : 4CB^2$   
 $= CS^2 : 4CB^2.$  [Def.

or  $PH \cdot PK = \frac{1}{4}CS^2.$

Ex. 1. Find the locus of the point of intersection of the medians of the triangle formed by a tangent with the asymptotes. [A hyperbola having the same asymptotes.]

Ex. 2.  $P, Q$  are points on a hyperbola.  $PL, QM$  are drawn parallel to each other to meet one asymptote;  $PR, QN$  are drawn also parallel to each other to meet the other asymptote. Prove that  $PL \cdot PR = QM \cdot QN.$

Ex. 3. If through  $P, P'$  on a hyperbola lines are drawn parallel to the asymptotes, forming a parallelogram, one of its diagonals will pass through the centre.

Ex. 4. If  $P$  be the middle point of a line which moves so as to form with two intersecting lines a triangle of constant area, the locus of  $P$  is a hyperbola.

Ex. 5. If through any point of a hyperbola, lines be drawn parallel to the asymptotes meeting any semi-diameter  $CQ$  in  $P$  and  $R$ , then  $CP \cdot CR = CQ^2$ .

Ex. 6. A series of hyperbolas having the same asymptotes is cut by a fixed straight line parallel to one of the asymptotes, and through the points of intersection lines are drawn parallel to the other, and equal to either axis of the corresponding hyperbola; prove that the locus of their extremities is a parabola.

Ex. 7. Given the asymptotes and a point on the curve, to construct it. (Apply Prop. XXVII., Ex. 5.)

Ex. 8. If a line through the centre meets  $PH, PK$  in  $U, V$ , and the parallelogram  $PUQV$  be completed, prove that  $Q$  is on the curve.

[If  $QU, VQ$  meet the asymptotes in  $U', V'$ , since the parallelograms  $HK, U'V'$  are equal,  $PH \cdot PK = QU' \cdot QV'$ .]

Ex. 9. The ordinate  $NP$  at any point of an ellipse is produced to  $Q$ , such that  $NQ$  is equal to the subtangent at  $P$ . Prove that the locus of  $Q$  is a hyperbola.

[If  $P$  is on the quadrant  $AB$ , the asymptotes are  $CB$  and the bisector of the angle  $ACB'$ .]

Ex. 10. If a straight line passing through a fixed point  $C$ , meets two fixed lines  $OA, OB$  in  $A, B$ , and if  $P$  be taken on  $AB$  such that  $CP^2 = CA \cdot CB$ , find the locus of  $P$ .

[Through  $C$  draw  $CD, CE$  parallel to  $OA, OB$ , to meet them. Through  $P$  draw lines parallel to  $OA, OB$  meeting  $CE$  in  $K$ , and  $DC$  in  $H$ . Then  $OD \cdot OE = PH \cdot PK$ . The locus of  $P$  is, therefore, a hyperbola of which the asymptotes are  $CH, CK$ .]

**Def.** Two hyperbolas are said to be *conjugate* when the transverse axis of each coincides with the conjugate axis of the other.

Thus, a hyperbola which has  $CB$  and  $CA$  for transverse and conjugate axes respectively, is called the *Conjugate hyperbola*, with reference to the one we have been dealing with.

The conjugate hyperbola has the same asymptotes as the original one, since they are the diagonals of the same rectangle. It is evident that a pair of conjugate hyperbolas lie on opposite sides of their common asymptotes.

It has already been pointed out that the *two* branches of a hyperbola together constitute *one* complete curve; but it must not, by analogy, be supposed that a pair of conjugate hyperbolas together constitutes *one entire* curve. They are a pair of totally distinct hyperbolas, although one is of use in deducing some properties of the other.

Ex. 1. Tangents  $TP$ ,  $TQ$  are drawn to a hyperbola from any point  $T$  on one of the branches of the conjugate. Prove that  $PQ$  touches the other branch of the conjugate.

[ $CT$  bisects  $PQ$  in  $V$ , Prop. XIX.; and  $CT \cdot CV = CT^2$ . Prop. XX.]

Ex. 2. An ordinate  $NP$  meets the conjugate hyperbola in  $Q$ ; prove that the normals at  $P$  and  $Q$  meet on the transverse axis.

[If the normal at  $Q$  meets the axes in  $G$  and  $G'$ ,

$$\frac{QG'}{QG} = \frac{CA^2}{CB^2} = \frac{CN}{NG'}$$

Apply Props. XXV., XXVI.]

PROPOSITION XXX.

*If through any point on a hyperbola or its conjugate a straight line be drawn in a given direction to meet the asymptotes, the rectangle under its segments is constant.*



Let  $P$  be the point on the given hyperbola and  $Q$  a point either on the same hyperbola or its conjugate.

Draw  $pPp'$  and  $qQq'$  in the given direction, meeting the asymptotes in  $p, p'$  and  $q, q'$  respectively. Through  $P, Q$  draw  $uPu', vQv'$  parallel to the conjugate axis, meeting the asymptotes in  $u, u'$  and  $v, v'$  respectively.

Now, by similar triangles,

$$Pp : Qq = Pu : Qv,$$

and

$$Pp' : Qq' = Pu' : Qv',$$

therefore  $Pp \cdot Pp' : Qq \cdot Qq' = Pu \cdot Pu' : Qv \cdot Qv'$ ;

but

$$Pu \cdot Pu' = CB^2 = Qv \cdot Qv', \quad [\text{Prop. XXVIII.}]$$

therefore

$$Pp \cdot Pp' = Qq \cdot Qq'.$$

Ex. 1. Prove that

$$Pp \cdot Pp' = Qq \cdot Qq' = CD^2,$$

where  $CD$  is the parallel semi-diameter terminated by the curve or its conjugate.

Ex. 2. An ordinate  $QV$  of any diameter  $CP$  is produced to meet the asymptote in  $R$ , and the conjugate hyperbola in  $Q'$ . Prove that

$$QV^2 + Q'V^2 = 2RV^2.$$

Prove also that the tangents at  $Q, Q'$  meet  $CP$  in points equidistant from  $C$ . [ $Q'V^2 - RV^2 = CD^2$ . For the second part, apply Prop. XX.]

### PROPOSITION XXXI.

*If any line cut a hyperbola the segments intercepted between the curve and its asymptotes are equal, and the portion of any tangent intercepted between the asymptotes is bisected at the point of contact.*

Let any line meet the curve and its asymptotes in  $Q, Q'$  and  $q, q'$  respectively.

Now  $Qq \cdot Qq' = Q'q \cdot Q'q'$ . [Prop. XXX.

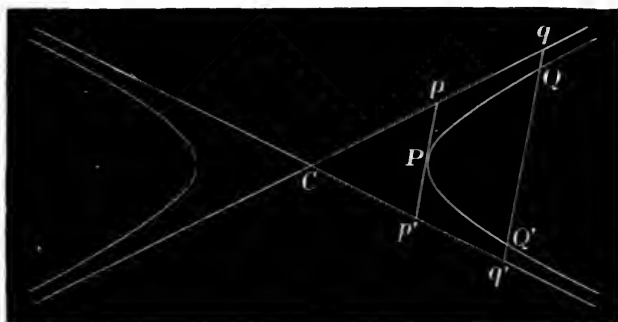
or  $Qq \cdot QQ' + Qq \cdot Q'q' = Q'q' \cdot QQ' + Qq \cdot Q'q'$ , [Euc. II. 1

or  $Qq \cdot QQ' = Q'q' \cdot QQ'$ ,

therefore  $Qq = Q'q'$ .



If now  $QQ'$  be made to move parallel to itself until the points  $Q, Q'$  coincide at a point  $P$  on the curve it becomes the tangent to the curve at  $P$  and  $Pp = Pp'$



Ex. 1. From a given point on a hyperbola, draw a straight line such that the segment intercepted between the other intersection with the hyperbola and a given asymptote, shall be equal to a given line.

When does the problem become impossible?

Ex. 2. The foot of the normal at  $P$  is equidistant from  $p, p'$ .

Ex. 3. Prove that  $Qq \cdot Qq' = Pp^2$ .

Ex. 4. If  $QK$  be drawn parallel to  $Cq'$  and  $Q'K'$  parallel to  $Cq$ , then  $Kq = K'Q'$ , and  $KQ = K'q'$ .

Ex. 5. The tangent at  $P$  meets an asymptote in  $T$ , and a line  $TQ$  drawn parallel to the other asymptote meets the curve in  $Q$ ; if  $PQ$  produced meets the asymptotes in  $R, R'$ , prove that  $RR'$  is trisected at  $P$  and  $Q$ .

Ex. 6. The diameter bisecting any chord  $QQ'$  of a hyperbola meets the curve in  $P$ ; and  $QH, PK, Q'H'$  are drawn parallel to one asymptote meeting the other in  $H, K, H'$ . Prove that  $CH \cdot CH' = CK^2$ .

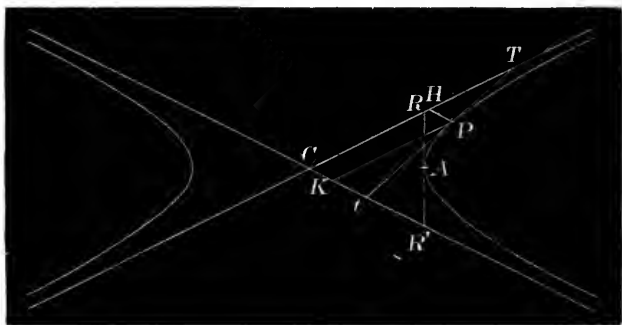
Ex. 7. A line drawn through one of the vertices of a hyperbola, and terminated by two lines drawn through the other vertex parallel to the asymptotes, will be bisected at the other point where it cuts the hyperbola.

Ex. 8. If  $qT$  be the tangent from  $q$ , and  $QH, TK, Q'H'$  be drawn parallel to  $Cq$  meeting  $Cq'$  in  $H, K, H'$ , prove that  $QH + Q'H' = 2TK$ .

Ex. 9. Through any point  $P$  on a hyperbola lines are drawn parallel to the asymptotes, meeting them in  $M$  and  $N$ ; and any ellipse is constructed having  $CM, CN$  for semi-diameters. If  $CP$  cut the ellipse in  $Q$ , show that the tangent to the ellipse at  $Q$  is parallel to the tangent to the hyperbola at  $P$ . [Each is parallel to  $MN$ .]

\* PROPOSITION XXXII.

*The area of the triangle formed by the asymptotes and any tangent to a hyperbola is constant.*



Let the tangents at the vertex  $A$  and at any point  $P$  meet the asymptotes in  $R, R'$  and  $T, t$  respectively.

Draw  $PH, PK$  parallel to the asymptotes, meeting them in  $H$  and  $K$ .

Then, since  $Tt$  is bisected at  $P$ ,

$$CT = 2 \cdot CH, \quad [\text{Prop. XXXI.}]$$

and  $Ct = 2 \cdot CK, \quad [\text{Euc. VI. 2.}]$

therefore  $CT \cdot Ct = 4 \cdot CK \cdot CH$   
 $= 4 \cdot PH \cdot PK$

$$= CS^2 \quad [\text{Prop. XXIX.}]$$

$$= CR \cdot CR'. \quad [\text{Def.}]$$

Therefore the triangle  $CTt$  is equal to the triangle  $CRR'$ ,

[Euc. VI. 15.]

and is, therefore, constant.

Ex. 1. If any two tangents be drawn to a hyperbola, the lines joining the points where they met the asymptotes will be parallel.

Ex. 2. If  $TOt$ ,  $T'Ot'$  be two tangents meeting one asymptote in  $T$ ,  $T'$ , and the other in  $t$ ,  $t'$ , prove that

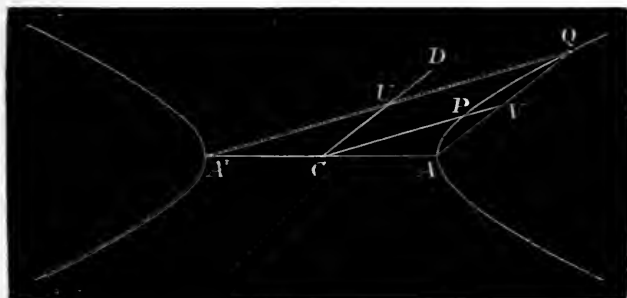
$$TO : Ot = t'O : T'O.$$

Ex. 3. Tangents are drawn to a hyperbola, and the portion of each tangent intercepted between the asymptotes is divided in a constant ratio. Prove that the locus of the points of section is a hyperbola. (Apply Prop. XXIX.)

PROPERTIES OF CONJUGATE DIAMETERS.

PROPOSITION XXXIII.

*If one diameter of a hyperbola bisects chords parallel to a second the second diameter bisects chords parallel to the first.*



Let  $CP$  bisect chords parallel to  $CD$ , then  $CD$  bisects chords parallel to  $CP$ .

Draw  $AQ$  parallel to  $CD$  meeting  $CP$  produced in  $V$ . Join  $A'Q$ , intersecting  $CD$  in  $U$ .

Then, because  $AQ$  is bisected in  $V$  and  $AA'$  in  $C$ ,  $CV$  is parallel to  $A'Q$ . [Enc. VI. 2.]

Again, since  $AA'$  is bisected in  $C$  and  $CD$  is parallel to  $AQ$ ,  $A'Q$  is bisected by  $CD$ . [Enc. VI. 2.]

Therefore  $CD$  bisects all chords parallel to  $A'Q$ ,

[Prop. IX.

and, therefore, all chords parallel to  $CP$ .

**Def.** Two diameters so related that each bisects chords parallel to the other are called *conjugate diameters*.

Thus  $CP$  and  $CD$  are conjugate to each other; so also are the transverse and the conjugate axes.

It is clear that of two conjugate diameters, one (as  $CP$ ) will meet the hyperbola, and the other (as  $CD$ ) the conjugate hyperbola.

The portion  $CD$  terminated by the conjugate hyperbola is usually called the semi-diameter conjugate to  $CP$ .

**Ex. 1.** If any tangent to a hyperbola meet any two conjugate diameters, the rectangle under its segments is equal to the square of the parallel semi-diameter. (Cf. Chap. II., Prop. XXX., Ex. 7.)

**Ex. 2.** Given in magnitude and position any two conjugate semi-diameters of a hyperbola, find the transverse and conjugate axes. (Cf. Chap. II., Prop. XXX., Ex. 8.)

**Ex. 3.** Draw a tangent to a hyperbola parallel to a given straight line.

[The point of contact ( $P$ ) of the required tangent is obtained by drawing  $CD$  parallel to the given straight line, and  $CP$  parallel to the tangent to the conjugate hyperbola at  $D$ .]

**Ex. 4.** If  $CQ$  be conjugate to the normal at  $P$ ,  $CP$  is conjugate to the normal at  $Q$ .

**Ex. 5.**  $OP$ ,  $OQ$  are tangents to a hyperbola from  $O$ . Prove that  $CO$ ,  $PQ$  are parallel to a pair of conjugate diameters. (Prop. XIX.)

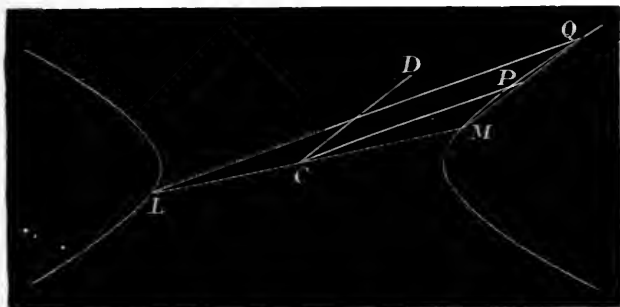
**Ex. 6.** An ellipse or a hyperbola is drawn touching the asymptotes of a given hyperbola. Prove that two of the chords of intersection of the curves are parallel to the chord of contact of the conic with the asymptotes.

[If  $PP'$  be the chord of contact and  $CV$  bisect  $PP'$ , then  $CV$ ,  $PP'$  are parallel to a pair of conjugate diameters in both conics.]

**Def.** Chords which join any point on a hyperbola to the extremities of a diameter are called *supplemental chords*.

PROPOSITION XXXIV.

*Supplemental chords of a hyperbola are parallel to conjugate diameters.*



Join any point  $Q$  on the hyperbola to the extremities of a diameter  $LM$ . Then  $QL$  and  $QM$  are supplemental chords.

Draw  $CP$ ,  $CD$  parallel to  $QL$  and  $QM$  respectively, then they shall be conjugate diameters.

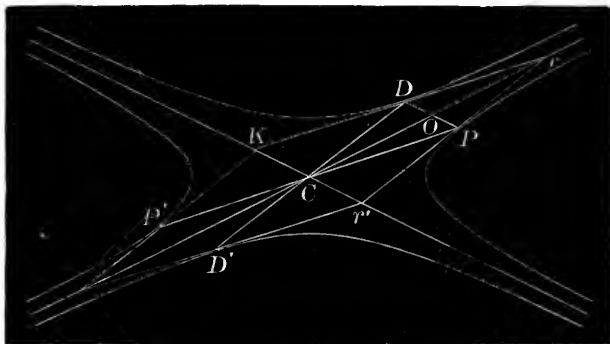
Because  $LM$  is bisected in  $C$ , and  $CP$  is parallel to  $LQ$ ,  
 $CP$  produced bisects  $MQ$ , [Euc. VI. 2.  
 and, therefore, all chords parallel to  $CD$ . [Prop. IX.

Therefore  $CD$  bisects all chords parallel to  $CP$ ,  
 [Prop. XXXIII.  
 and is, therefore, conjugate to it.

PROPOSITION XXXV.

*The tangents at the extremities of any pair of conjugate diameters meet on the asymptotes, and the line joining the extremities is parallel to one asymptote and bisected by the other.*

Let  $CP$ ,  $CD$  be a pair of conjugate semi-diameters. Draw  $rPr'$  the tangent at  $P$ , meeting the asymptotes in  $r$  and  $r'$ . Join  $Dr$  and produce  $rD$  to meet the other asymptote in  $K$ .



Now, since  $P$  is a point on the curve and  $D$  on its conjugate, and  $DC$  meets both the asymptotes in  $C$  and is parallel to  $Pr$ ,

$$DC^2 = Pr \cdot Pr' \quad [\text{Props. XII. and XXXIII.}]$$

$$= Pr^2; \quad [\text{Prop. XXX.}]$$

$$[\text{Prop. XXXI.}]$$

therefore

$$CD = Pr.$$

Therefore  $Dr$  is parallel to  $CP$ , and  $Cr$ ,  $PD$  bisect each other at  $O$ .

Again, since  $Pr = Pr'$ , [Prop. XXXI.]

and  $Or = Oc$ ,

therefore  $PD$  is parallel to  $r'K$ . [Euc. VI. 2.]

Therefore  $Dr = DK$ , [Euc. VI. 2.]

and  $KDr$  is the tangent at  $D$ . [Prop. XXXI.]

Ex. 1. If  $PD$  be drawn parallel to an asymptote to meet the conjugate hyperbola in  $D$ ,  $CP$ ,  $CD$  are conjugate diameters.

Ex. 2. Conjugate diameters of a hyperbola are also conjugate diameters of the conjugate hyperbola.

Ex. 3.  $CP, CD$  are conjugate diameters of a hyperbola.  $PV, DM$  are ordinates to the transverse axis. Prove that

- (i)  $CM : PV = CA : CB.$
- (ii)  $DM : CN = CB : CA.$

Let the tangent to the hyperbola at  $P$  and to the conjugate at  $D$ , meet the transverse axis in  $T, t$  respectively. Then  $CP, PT$  are parallel to  $Dt, DC$ . Now

$$CT \cdot CN = CA^2 = Ct \cdot CM. \quad (\text{Prop. XX.})$$

$$\therefore CM : CN = CT : Ct = PT : CD = PV : DM = CN : Mt ;$$

$$\therefore CN^2 = CM \cdot Mt = CA^2 + CM^2. \quad (\text{Prop. XX.})$$

$$\therefore CM^2 = CN^2 - CA^2.$$

But  $PV^2 : CN^2 - CA^2 = CB^2 : CA^2. \quad (\text{Prop. VIII.})$

$\therefore$  (i) follows immediately.

Ex. 4. If the normal at  $P$  meet the axes in  $G, g$ , prove that

- (i)  $PG : CD = CB : CA.$
- (ii)  $Pg : CD = CA : CB.$
- (iii)  $PG \cdot Pg = CD^2.$

[The triangles  $DCM$  and  $PGN$  are similar, as also the triangles  $DCM$  and  $Pgn$ .]

Ex. 5. A circle is drawn touching the transverse axis at  $C$ , and also touching the curve. Prove that the diameter conjugate to the diameter through either point of contact, is equal to  $SS'$ .

[If the normal at  $P$  meets the axes in  $G, g$ , and the tangent at  $P$  meets  $CB$  in  $t$ ,  $Ct = PG$ , and  $CF^2 = PG \cdot Pg = Ct \cdot Cg = CS^2$ . Prop. XXIII., Ex. 1.]

Ex. 6. The area of the parallelogram formed by the tangents at the extremities of any pair of conjugate diameters, is constant and equal to  $4 \cdot CA \cdot CB$ . (Apply Prop. XXXII.)

Ex. 7. The tangent at a point  $P$  of an ellipse (centre  $O$ ) meets the hyperbola having the same axes as the ellipse, in  $C$  and  $D$ . If  $Q$  be the middle point of  $CD$ , prove that  $OQ, OP$  are equally inclined to the axes.

[Draw  $OrR$  parallel to  $PQ$ , meeting the ellipse and hyperbola in  $r$  and  $R$ ; then  $OP, Or$  are conjugate in the ellipse, and  $OQ, OR$  in the hyperbola. If  $PV, QM, rl, RL$  be the ordinates, we have, for the ellipse,

$$\frac{PV}{Ol} = \frac{OB}{OA} = \frac{rl}{ON}. \quad (\text{Chap. II., Prop. XXXIII.})$$

or, 
$$\frac{PV}{ON} = \frac{OB^2}{OA^2} \cdot \frac{Ol}{rl}.$$

Similarly, for the hyperbola,

$$\frac{QM}{OM} = \frac{OB^2}{OA^2} \cdot \frac{Ol}{rl}. \quad (\text{Ex. 3.})$$

$\therefore PV : ON = QM : OM$ .]

Ex. 8. With two conjugate diameters of an ellipse as asymptotes, a pair of conjugate hyperbolas is described. Prove that if the ellipse touch one hyperbola, it will also touch the other.

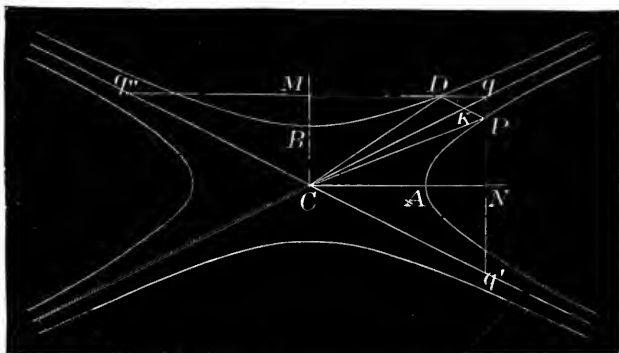
[The diameters drawn through the points of contact are conjugate to each other.]

Ex. 9. Apply this proposition to prove Prop. X.

### PROPOSITION XXXVI.

*The difference of the squares of any two conjugate semi-diameters of a hyperbola is constant*

$$(CP^2 \sim CD^2 = CA^2 \sim CB^2).$$



Let  $CP, CD$  be a pair of conjugate semi-diameters.

Draw the ordinate  $qPNq'$ , meeting the asymptotes in  $q, q'$ , and join  $PD$ ; let  $PD$  meet the asymptote in  $K$ . Join  $Dq$ .

Then, since the asymptotes are equally inclined to the ordinate  $qPNq'$ ,

[Const.]

and  $PK$  is parallel to the asymptote  $Cq'$ , [Prop. XXXV.]  
the angles  $KqP$  and  $KPq$  are equal.

Therefore  $Kq = KP = KD$ . [Prop. XXXV.]

Therefore the circle described on  $PD$  as diameter passes through  $q$ , and the angle  $PqD$  is a right angle. [Euc. III. 31.]



If, therefore,  $qD$  produced meet the conjugate axis in  $M$  and the asymptote  $Cq'$  in  $q''$ ,  $qMq''$  will be at right angles to  $CB$ .

Now  $Cq^2 - CP^2 = qN^2 - PN^2$  [Euc. I. 47.]  
 $= Pq \cdot Pq'$  [Euc. II. 5.]  
 $= CB^2,$  [Prop. XXVIII.]

and  $Cq^2 - CD^2 = qM^2 - DM^2$  [Euc. I. 47.]  
 $= Dq \cdot Dq''$  [Euc. II. 5.]  
 $= CA^2;$  [Prop. XXVIII.]

therefore  $CP^2 - CD^2 = CA^2 - CB^2.$

Ex. 1. If from any point on an asymptote of a hyperbola, ordinates be drawn to the curve and its conjugate, meeting them in  $P$  and  $D$  respectively, show that  $CP$  and  $CD$  will be conjugate semi-diameters, and conversely.

Ex. 2. Apply Prop. XXXV., Ex. 3, to prove this proposition. We have  $CN^2 - CM^2 = CA^2.$

Similarly, if  $Pn, Dm$  be ordinates to  $CB,$

$$Cm^2 - Cn^2 = CB^2,$$

or  $DM^2 - PN^2 = CB^2.$

Subtracting,  $CP^2 - CD^2 = CA^2 - CB^2.$

Ex. 3. The difference between the sum of the squares of the distances of any point on the curve from the ends of any diameter, and the sum of the squares of its distances from the ends of the conjugate, is constant. [=  $2(CA^2 - CB^2).$ ]

Ex. 4.  $\sigma$  is the focus of the conjugate hyperbola lying on  $CB.$  Prove that  $\sigma D - SP = CA - CB.$

(Apply Ex. 1, and Prop. XXVII., Ex. 5 and 13.)

Ex. 5. Prove that  $SP \cdot S'P = CD^2.$

[ $SP \cdot S'P = 2 \cdot CA.$  Then square and substitute. Cf. also Prop. XXIII., Ex. 5, and Prop. XXXV., Ex. 3.]

Ex. 6. In Prop. XXIII., Ex. 1, prove that

$$St : tg = CB : CD,$$

$CD$  being conjugate to  $CP.$  [Apply Ex. 5 and Prop. XXI.]

Ex. 7. If the tangent at  $P$  meet any conjugate diameters in  $T$  and  $t,$  the triangles  $SPT, S'Pt$  are similar.

[ $SP : PT = Pt : S'P.$  Apply Ex. 5 and Prop. XXXIII., Ex. 1.]

Ex. 8. If the tangent at  $P$  meet the conjugate axis in  $t,$  the areas of the triangles  $SPS', StS'$  are the ratio of  $CD^2 : St^2.$  (Apply Prop. XXIII., Ex. 1.)

Ex. 9. Through  $C$  a line is drawn parallel to either focal distance of  $P$ ; if  $DE$  is drawn perpendicular to this line, prove that  $DE=CB$ .

[If  $SY$  is perpendicular to the tangent at  $P$ , the triangles  $SYP$ ,  $CDE$  are similar. Then

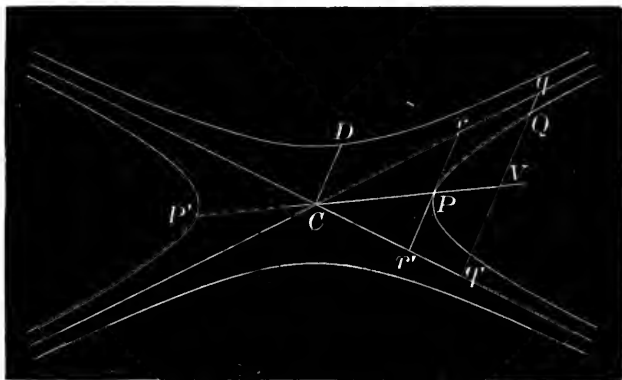
$$DE : CD = SY : SP = S'Y' : S'P ;$$

$$\therefore \frac{DE^2}{CD^2} = \frac{SY \cdot S'Y'}{SP \cdot S'P} = \frac{BC^2}{CD^2} \quad \text{Prop. XXI. and Ex. 5.]}$$

\* PROPOSITION XXXVII.

*The square of the ordinate of any point of a hyperbola with respect to any diameter varies as the rectangle under the segments of the diameter made by the ordinate.*

$$(QV^2 : PV \cdot P'V = CD^2 : CP^2.)$$



Let  $QV$  be an ordinate to the diameter  $PCP'$ , meeting the asymptotes in  $q, q'$ .

Draw the tangent at  $P$  meeting the asymptotes in  $r, r'$ .

Then  $Pr$  is parallel to  $QV$ .

[Prop. XII.]

Therefore, by similar triangles,

$$qV^2 : Pr^2 = CV^2 : CP^2,$$

therefore  $qV^2 - Pr^2 : Pr^2 = CV^2 - CP^2 : CP^2,$

but  $Pr \cdot Pr' = Qq \cdot Qq'$ , [Prop. XXX.

or  $Pr^2 = qV^2 - QV^2$ ,  
[Prop. XXXI. and Euc. II. 5.

therefore  $qV^2 - Pr^2 = QV^2$ .

Also  $CV^2 - CP^2 = PV \cdot P'V$ , [Euc. II. 5.

therefore  $QV^2 : Pr^2 = PV \cdot P'V : CP^2$ ,

or  $QV^2 : PV \cdot P'V = Pr^2 : CP^2$ ,

which is constant.

Since  $CD^2 = Pr \cdot Pr'$  [Prop. XXX.

$= Pr^2$ , [Prop. XXXI.

this result may also be expressed as

$$QV^2 : PV \cdot P'V = CD^2 : CP^2.$$

Ex. If the tangent at  $D$  to the conjugate hyperbola meet an asymptote in  $r'$  and the hyperbola in  $q'$ , and the ordinate  $vq'$  parallel to the tangent at  $P$  be produced to meet the same asymptote in  $R$ , show that  $\triangle CPr' = \frac{1}{2} \triangle CrR$ .

### THE EQUILATERAL HYPERBOLA.

The rectangle contained by the transverse axis of a central conic and its latus rectum has been called by Apollonius the "figure of the conic upon its axis." It is evident that the "minor" or "conjugate" axis of a central conic, according as it is an ellipse or a hyperbola, is equal to the side of a square equivalent in area to the "figure." (Chap. II., Prop. VI., and Chap. III., Prop. V.)

A hyperbola which has the sides of its "figure" equal is called an *equilateral* hyperbola. The latus rectum being thus equal to the transverse axis it is clear that the conjugate axis is equal to the transverse axis (Chap. III., Prop. V.); in other words the two axes of an *equilateral* hyperbola are equal.

From Prop. XXVII. it is clear that the asymptotes of an *equilateral* hyperbola are at right angles to each other. From this property the curve is also called a *rectangular* hyperbola.

Ex. Prove that the locus of the intersection of tangents to a parabola including half a right angle, is a rectangular hyperbola. (Prop. I., Ex. 10, and Prop. XXVII., Ex. 3.)

The properties of the hyperbola proved in the preceding propositions are, of course, true for the equilateral hyperbola as well. In some cases, however, the results assume forms which are deserving of notice.

Thus, for the equilateral hyperbola, we have

$$\begin{aligned} \text{Prop. III.} \quad e &= \sqrt{2}, & (\text{See Ex. 2.}) \\ CS^2 &= 2CA^2, \\ CS &= 2CX. \end{aligned}$$

Ex. If a circle be described on  $SS'$  as diameter, the tangents at the vertices will intersect the asymptotes in the circumference.

$$\begin{aligned} \text{Prop. V.} \quad SL &= CA, \\ \text{or,} \quad \text{Latus rectum} &= AA'. \end{aligned}$$

$$\text{Prop. VIII.} \quad PN^2 = AN \cdot A'N.$$

Ex. 1. If  $PNP'$  be a double ordinate, the angles  $PAP'$  and  $P'A'P'$  are supplementary.

Ex. 2. The triangle formed by the tangent at any point and its intercepts on the axes, is similar to the triangle formed by the central radius to that point and the abscissa and ordinate of the point. (See Prop. XX., Ex. 1.)

Ex. 3. If  $M$  be a point on the conjugate axis, and  $MP$  be drawn parallel to the transverse axis meeting the curve in  $P$ , then  $PM = AM$ .

Ex. 4. The tangent at any point  $P$  of a circle meets a fixed diameter  $AB$  produced in  $T$ , show that the straight line through  $T$  perpendicular to  $AB$  meets  $AP$   $BP$  produced in points which lie on an equilateral hyperbola.

Ex. 5. If  $AB$  be any diameter of a circle and  $PNQ$  an ordinate to it, the locus of intersection of  $AP$ ,  $BQ$  is an equilateral hyperbola.

Ex. 6. The locus of the point of intersection of tangents to an ellipse which make equal angles with the major and minor axis respectively, and are not at right angles, is a rectangular hyperbola. (The foci of the ellipse will be the vertices.)

Prop. XXVI.  $CN = NG,$   
 $PG = Pg = CP.$

Prop. XXXI.  $CP = Pr = Pr'.$

Ex. 1. A circle whose centre is any point  $P$  and radius  $CP$ , intersects the normal on the axes and the tangent on the asymptotes.

Ex. 2. If the tangents at two points  $Q$  and  $Q'$  meet in  $T$ , and if  $CQ, CQ'$  meet these tangents in  $R$  and  $R'$ , the circle circumscribing  $RTR'$  passes through  $C$ .

Ex. 3. The angle subtended by any chord at the centre is the supplement of the angle between the tangents at the ends of the chord.

#### PROPOSITION A.

*Conjugate diameters are equal in the equilateral hyperbola and the asymptotes bisect the angle between them.*

Let  $CP, CD$  be any two conjugate semi-diameters. Then  $CP^2 \sim CD^2 = CA^2 \sim CB^2 = O$ , [Prop. XXXVI. since the axes are equal.

Therefore  $CP = CD.$

Again, since the asymptote  $Cr$  (Fig., Prop. XXXV.) bisects  $PD$  it must bisect the angle  $PCD$ .

Similarly, it may be shown that the asymptote  $C'r'$  bisects the angle  $PCD'$ .

Ex. 1. A circle is described on the transverse axis as diameter. Prove that if any tangent be drawn to the hyperbola, the straight lines joining the centre of the hyperbola with the point of contact and with the middle point of the chord of intersection of the tangent with the circle, are inclined to the asymptotes at complementary angles.

Ex. 2. The lines drawn from any point on the curve to the extremities of any diameter make equal angles with the asymptotes. (Prop. XXXIV.)

Ex. 3. The focal chords drawn parallel to conjugate diameters are equal. (Props. VI. and X.)

Ex. 4. If two concentric rectangular hyperbolas be described, the axes of one being the asymptotes of the other, they will cut at right angles.

Ex. 5. The normals at the ends of two conjugate diameters intersect on the asymptote and are parallel to another pair of conjugate diameters. (Prop. XXXV.)

Ex. 6. If  $QV$  be an ordinate of a diameter  $PCp$ ,  
 $QV^2 = PV \cdot pV$ . (Prop. XXXVII.)

Ex. 7. If tangents parallel to a given direction are drawn to a system of circles passing through two fixed points, the points of contact lie on a rectangular hyperbola. (Apply Ex. 6.)

Ex. 8. Given the base of a triangle and the difference of the angles at the base, prove that the locus of the vertex is a rectangular hyperbola. (Apply Ex. 6.)

Ex. 9.  $PCp$  is a diameter and  $QV$  an ordinate, prove that  $QV$  is the tangent at  $Q$  to the circle round the triangle  $PQp$ . (Apply Ex. 6.)

Ex. 10. If  $P$  be a point on an equilateral hyperbola and if the tangent at  $Q$  meet  $CP$  in  $T$ , the circle circumscribing  $CTQ$  touches the ordinate  $QV$  conjugate to  $CP$ . (Apply Ex. 6 and Prop. XX.)

Ex. 11. The angle between a chord  $PQ$  and the tangent at  $P$ , is equal to the angle subtended by  $PQ$  at the other extremity of the diameter through  $P$ .

Ex. 12. The distance of any point on the curve from the centre is a geometric mean between its distances from the foci. (Apply Prop. XXXVI., Ex. 5.)

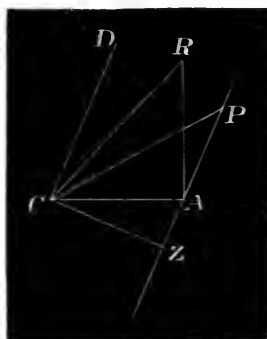
Ex. 13. The points of intersection of an ellipse and a confocal rectangular hyperbola are the extremities of the equi-conjugate diameters of the ellipse. (Apply Prop. XXXVI., Ex. 5, and Chap. II., Prop. XXXV., Ex. 5.)

Ex. 14. If two focal chords be parallel to conjugate diameters, the lines joining their extremities intersect on the asymptotes.

[If  $PSp$ ,  $QSp$  be the chords, it may be shown that  $pq$ ,  $PQ$  and an asymptote will meet on the directrix at the same point. Prop. VII. and Prop. XXVII., Ex. 5.]

PROPOSITION B.

*In the equilateral hyperbola the transverse axis bisects the angle between the central radius vector of any point and the central perpendicular on the tangent at that point.*



Let  $P$  be any point on an equilateral hyperbola and  $CD$  the semi-diameter conjugate to  $CP$ ; let  $CZ$  be the perpendicular on the tangent at  $P$ .

If  $CR$  be the asymptote, because

$$CA = AR, \quad [\text{Prop. XXVII.}]$$

the angle  $ACR$  is half a right angle, that is, half of the angle  $DCZ$ , since  $CD$  is parallel to  $PZ$ .

[Props. XII. and XXXIII.]

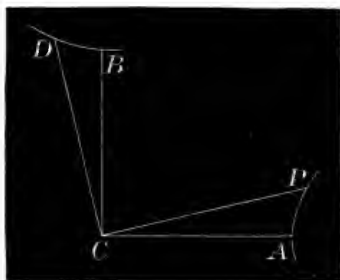
But the angle  $PCR$  is half of the angle  $PCD$ ; [Prop. A. therefore the remaining angle  $PCA$  is half of the remaining angle  $PCZ$ , that is,  $CA$  bisects the angle  $PCZ$ .

Ex. 1. Prove that  $CZ \cdot CP = CA^2$ . (Apply Prop. XX.)

Ex. 2. Prove that the angles  $CPA$  and  $CAZ$  are equal.

## PROPOSITION C.

*In the equilateral hyperbola diameters at right angles to each other are equal.*



Let there be two semi-diameters  $CP$ ,  $CD$  at right angles to each other, meeting the curve and its conjugate in  $P$  and  $D$  respectively.

Then the angle  $ACB =$  the angle  $PCD$ ,  
each being a right angle. Taking away the common angle  $PCB$ ,

the angle  $ACP =$  the angle  $BCD$ .

Hence from symmetry, since the curve and its conjugate are equal and similarly placed with respect to the axes,

$$CP = CD.$$

Ex. 1. Prove that focal chords at right angles to each other are equal.

Ex. 2. If a right-angled triangle be inscribed in the curve, the normal at the right angle is parallel to the hypotenuse. (See Prop. X.)

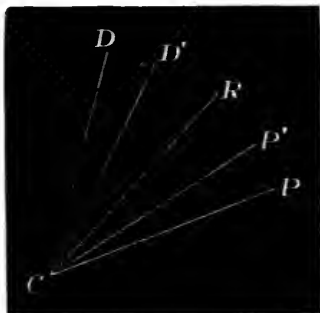
Ex. 3. Chords which subtend a right angle at a point  $P$  of the curve, are all parallel to the normal at  $P$ .

## PROPOSITION D.

*The angle between any two diameters of an equilateral hyperbola is equal to the angle between their conjugates.*



Let  $CP, CP'$  be any two semi-diameters, and  $CD, CD'$  the semi-diameters conjugate to them respectively.



Then, if  $CR$  be the asymptote,  
                     the angle  $PCR =$  the angle  $DCR$ ,     [Prop. A.  
 and                 the angle  $P'CR =$  the angle  $D'CR$ ;     [Prop. A.  
 therefore, by subtraction,  
                     the angle  $PCP' =$  the angle  $DCD'$ .

Ex. 1. Conjugate diameters are inclined to either axes at angles which are complementary.

Ex. 2. If  $CP, CD$  be conjugate semi-diameters and  $PN, DM$  ordinates, the triangles  $PCN, DCM$  are equal in all respects.

Ex. 3. The difference between the angles which the lines joining any point on the curve to the extremities of a diameter make with the diameter, is equal to the angle which the diameter makes with its conjugate.

Ex. 4. The angles subtended by any chord at the extremities of a diameter are equal or supplementary. (Apply Prop. XXXIV.)

Ex. 5.  $AB$  is a chord of a circle and a diameter of a rectangular hyperbola,  $P$  is any point on the circle,  $AP, BP$ , produced if necessary, meet the hyperbola in  $Q, Q'$  respectively. Prove that  $BQ$  and  $AQ'$  intersect on the circle. (Apply Ex. 4.)

Ex. 6. A circle and a rectangular hyperbola intersect in four points and one of their common chords is a diameter of the hyperbola. Show that the other common chord is a diameter of the circle. (Apply Ex. 4.)

Ex. 7.  $QN$  is drawn perpendicular from any point  $Q$  on the curve to the tangent at  $P$ . Prove that the circle round  $CNP$  bisects  $PQ$ . (Apply Ex. 4.)

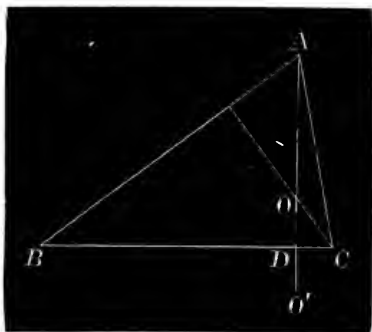
Ex. 8. If a rectangular hyperbola circumscribe a triangle, the locus of its centre is the nine-point circle.

[The diameters to the middle points of the sides are conjugate to the sides respectively.]

Ex. 9. The tangent at a point  $P$  of a rectangular hyperbola meets a diameter  $QCQ'$  in  $T$ . Prove that  $CQ$  and  $TQ'$  subtend equal angles at  $P$ .

\* PROPOSITION E.

*If a rectangular hyperbola circumscribe a triangle it passes through the orthocentre.*



Let a rectangular hyperbola circumscribing a triangle  $ABC$  meet  $AD$ , drawn perpendicular to  $BC$ , in  $O$ .

Then the rectangles  $AD \cdot OD$ ,  $BD \cdot CD$  are as the squares of the semi-diameters parallel to  $AD$ ,  $BC$ . [Prop. X. But the semi-diameters being at right angles to each other, are equal : [Prop. C.

therefore  $AD \cdot OD = BD \cdot CD$ .

Therefore, as is well known, the point  $O$  must coincide either with the orthocentre or with the point  $O'$  where  $AD$  meets the circle circumscribing the triangle  $ABC$ .

But the latter case is impossible; for then the lines  $AD, BC$ , which are at right angles to each other, will be equally inclined to the axis, [Prop. XI.

and will, therefore, be parallel to the asymptotes, which are also at right angles to each other and equally inclined to the axis. [Prop. XXVII.

Hence  $BC$ , being parallel to an asymptote, cannot meet the curve in two points (see Prop. XXVII., Ex. 2), which is contrary to the hypothesis.

Hence the curve must pass through the orthocentre.

Ex. 1. Every conic passing through the centres of the four circles which touch the sides of a triangle is a rectangular hyperbola.

Ex. 2. Any conic passing through the four points of intersection of two rectangular hyperbolas, is itself a rectangular hyperbola.

Ex. 3. If two rectangular hyperbolas intersect in  $A, B, C, D$ , the circles described on  $AB, CD$  as diameters intersect each other orthogonally.

[ $D$  is the orthocentre of the triangle  $ABC$ . Observe that the distance between the middle points of  $AB$  and  $CD$  is equal to the radius of the circumscribing circle.]

#### MISCELLANEOUS EXAMPLES ON THE HYPERBOLA.

1. Given the two asymptotes and a point on the curve, show how to construct the curve and find the position of the foci.

2.  $CP, CD$  are conjugate semi-diameters and the tangent at  $P$  meets an asymptote in  $r$ . If  $rn$  be the perpendicular from  $r$  on the transverse axis  $DPn$  is a right line.

3.  $P$  is any point on a hyperbola whose foci are  $S, S'$ ; if the tangent at  $P$  meet an asymptote in  $T$  the angle between that asymptote and  $S'P$  is double the angle  $STP$ .

4. Given four points on an equilateral hyperbola which are at the extremities of two chords at right angles and also the tangent at one of the points, find the centre of the curve.

5. The tangents at the extremities  $P, P'$  of a chord of a conic parallel to the transverse axis meet in  $T$ . If two circles be drawn through  $S$ , touching the conic at  $P$  and  $P'$  respectively, prove that  $F$ , the second point of intersection of the circles, will be at the intersection of  $PP'$  and  $ST$ .

Prove also that the locus of  $F$  from different positions of  $PP'$  will be a parabola with its vertex at  $S$  and passing through the ends of the conjugate axis.

6. Given a pair of conjugate diameters  $PCP', DCD'$ , find the position of the axis.

[Join  $PD, PD'$ , bisect them in  $E$  and  $F$ ; join  $CE, CF$ ; bisect the angle  $ECF$  by the line  $A'CA$ , and through  $C$  draw  $BCB'$  perpendicular to  $ACA'$ ; these are the axes sought.]

7. If the focal radii vectors, the ordinate and the tangent at any point  $P$  of a hyperbola meet an asymptote in  $Q, R, E, T$  respectively, and  $M$  be the middle point of  $QR$ , prove that  $PQ \simeq PR = 2(CM \simeq ET)$ .

8. If  $P$  and  $Q$  be the points of contact of orthogonal tangents from  $O$  to two confocal conics, the normals at  $P$  and  $Q$  to the two conics will intersect on the line joining  $O$  to their common centre.

9. Describe the hyperbolas which have a common focus, pass through a given point and have their asymptotes parallel to two given straight lines.

10. From each of two points on a rectangular hyperbola a perpendicular is drawn on the tangent at the

other; prove that these perpendiculars subtend equal angles at the centre.

11. If the focal distances of a point  $P$  on a hyperbola meet an asymptote in  $U$  and  $V$ , the perimeter of the triangle  $PUV$  is constant for all positions of  $P$ .

12. If a hyperbola be described touching the three sides of a triangle, one focus lies within one of the three outer segments of the circumscribing circle made by the sides of the triangle.

13. Two fixed points  $P, Q$  are taken in the plane of a given circle and a chord  $RS$  of a circle is drawn parallel to  $PQ$ ; prove that the locus of intersection of  $RP$  and  $SQ$  is a conic.

14. Tangents are drawn to a rectangular hyperbola from a point  $T$  on the transverse axis, meeting the tangents at the vertices in  $Q, Q'$ . Prove that  $QQ'$  touches the auxiliary circle at  $R$ , such that  $RT$  bisects the angle  $QTQ'$ .

15. If the tangents at the ends of a chord of a hyperbola meet in  $T$  and  $TM, TM'$  be drawn parallel to the asymptotes to meet them in  $M, M'$ , then  $MM'$  is parallel to the chord.

16. The locus of the intersection of two equal circles which are described on two sides  $AB, AC$  of a triangle as chords is a rectangular hyperbola whose centre is the middle point of  $BC$  and which passes through  $A, B, C$ .

17. Through a fixed point  $O$  a chord  $POQ$  of a hyperbola is drawn,  $PL, QL$  are drawn parallel to the asymptotes; show that the locus of  $L$  is a similar and similarly situated hyperbola.

18. A circle and a rectangular hyperbola circumscribe a triangle  $ABC$ , right angled at  $C$ . If the tangent to the circle at  $C$  meets the hyperbola again in  $C'$ , the tangents to the hyperbola at  $C, C'$  intersect on  $AB$ .

19. Find the locus of the middle points of a system of chords of a hyperbola passing through a fixed point on one of the asymptotes.

20.  $CP, CD$  are conjugate semi-diameters; if

$$CD = 2\sqrt{2} \cdot CB,$$

prove that the tangent at  $P$  passes through a focus of the conjugate hyperbola.

21. Given a focus and three points on a conic, find the directrix. Show that three at least of the four possible conics must be hyperbolas.

22. The normal at any point  $P$  of a hyperbola meets the asymptotes in  $g_1, g_2$  and the conjugate diameter in  $f$ ; prove that  $Pf$  is the harmonic mean between  $Pg_1, Pg_2$ .

23. The sum of the squares of the perpendiculars drawn from the foci of a hyperbola on any tangent to the conjugate hyperbola is constant ( $= 2 \cdot CB^2$ )

24. The tangent at  $P$  meets the asymptotes in  $T, t$ , and the normal at  $P$  meets the transverse axis in  $G$ ; prove that the triangle  $TGt$  remains similar to itself as  $P$  varies.

25. The intercept on any tangent to a hyperbola made by the asymptotes subtends a constant angle at either focus.

26. Given two tangents to a rectangular hyperbola and their points of contact, to find the asymptotes.

27. A circle touches a conic at a fixed point and cuts it

in  $P$  and  $Q$ ; the locus of the middle point of  $PQ$  is a right line.

28. If two conics with a common directrix meet in four points, these four points lie on a circle whose centre is on the straight line joining the corresponding foci.

29. The locus of the middle point of a line which moves so as to cut off a constant area from the corner of a rectangle is an equilateral hyperbola. (Prop. XXIX., Ex. 4.)

30. If between a rectangular hyperbola and its asymptotes a concentric elliptic quadrant be inscribed, the rectangle contained by its axes is constant. (Apply Chap. II., Prop. XXII., and Chap. III., Prop. XXIX.)

31. Given an asymptote, a tangent and its point of contact, to construct a rectangular hyperbola.

[Let the tangent at  $P$  meet the asymptote in  $L$ . Make  $PM = LP$  and draw  $MC$  at right angles to  $LC$ .  $C$  is the centre and the focus  $S$ , which lies on the bisector of the angle  $LCM$ , is determined by the relation  $CS^2 = CL \cdot CM$ . Prop. XXXII. The directrix bisects  $CS$ .]

32. Straight lines, passing through a given point, are bounded by two fixed lines at right angles to each other. Find the locus of their middle points.

[Let  $OX, OY$  be the fixed straight lines and  $P$  the given point. If  $C$  be the middle point of  $OP$ , the locus will be a rectangular hyperbola of which the lines through  $C$  parallel to  $OX$  and  $OY$  are the asymptotes. Apply Prop. XXIX.]

33. Given a point  $Q$  and a straight line  $AB$ , if a line  $QCP$  be drawn cutting  $AB$  in  $C$ , and  $P$  be taken in it, so that  $PD$  being perpendicular upon  $AB$ ,  $CD$  may be of constant magnitude, the locus of  $P$  is a rectangular hyperbola (Prop. XXIX.)

34. Parallel tangents are drawn to a series of confocal

ellipses. Prove that the locus of the points of contact is a rectangular hyperbola.

[See figure, Chap. II., Prop. XXVIII.  $CF \propto CG$  and  $PF \propto PR \propto Ct \propto CT$ . Therefore  $PF \cdot CF \propto CG \cdot CT = CS^2 = \text{constant}$ .]

35. From the point of intersection of the directrix with one of the asymptotes of a rectangular hyperbola a tangent is drawn to the curve, meeting the other asymptote in  $T$ . Prove that  $CT$  is equal to the transverse axis. (Apply Prop. XXXII. and Prop. XXVII., Ex. 5.)

36. If a rectangular hyperbola, having its asymptotes coincident with the axes of an ellipse, touch the ellipse, the axis of the hyperbola is a mean proportional between the axes of the ellipse. (Apply Props. XXXI., XXXII., and XX.)

37. Ellipses are inscribed in a given parallelogram; prove that their foci lie on a rectangular hyperbola.

38. Given the centre, a tangent, and a point on a rectangular hyperbola, find the asymptotes.

39. Prove that the parallel focal chords of conjugate hyperbolas are to one another as the eccentricities of the hyperbolas.

40. With each pair of three given points as foci a hyperbola is drawn passing through the third point. Prove that the three hyperbolas thus drawn intersect in a point.



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