Nathan
Bownes
Algebra year
1861

AN ELEMENTARY TREATISE ON ALGEBRA.

DESIGNED AS

FIRST LESSONS IN THAT SCIENCE.

BY

H. N. ROBINSON, A. M.,

AUTHOR OF AN UNIVERSITY EDITION OF ALGEBRA—AN ELEMENTARY TREATISE ON NATURAL PHILOSOPHY—A WORK ON GEOMETRY, CONTAINING PLANE AND SPHERICAL TRIGONOMETRY; ALSO, AUTHOR OF A TEXT BOOK ON ASTRONOMY, AND SEVERAL OTHER MATHEMATICAL WORKS.

NINTH STANDARD EDITION.

CINCINNATI:
PUBLISHED BY JACOB ERNST,
No. 112 MAIN STREET.
1856.

STEREOTYPED BY JAMES & CO.,
CINCINNATI.

PREFACE.

Every teacher is desirous of having as few textbooks in his school as is consistent with efficient and sound instruction, and in accordance with this object, great efforts have been made by several authors to produce a work on Algebra that would be a proper text book for all grades of pupils. But in this all have failed, and given up the point in despair. The student of adult age, and possessing a passably disciplined mind, requires a different book from the mere lad, who is just commencing the science. If we put a child's book into the hands of a young man, he will, probably, become displeased with the book, and possibly imbibe prejudice and distaste for the science itself; and if we put a logical and philosophical work into the hands of a child, he is sure not to comprehend it, however well and fluently he may be made to repeat the contents of its pages. But, nevertheless, as Algebra is the groundwork of all the mathematical sciences, and is of itself a system of pure logic, it is important that it should be commenced at an early age—eleven or twelve, or if otherwise well employed, thirteen or fourteen is a more suitable age.

It is a prevalent impression that Algebra should not be commenced until the pupil has acquired a good knowledge of Arithmetic, but this is a great error. The impression would be well founded, provided Arithmetic was the most elementary science, and Algebra was founded on Arithmetic; but the reverse is the fact—Algebra is elementary Arithmetic, and no one can acquire a knowledge of Arithmetic in an enlarged and scientific sense, without a previous knowledge of Algebra. Beyond notation, numeration, and the four simple rules, Arithmetic is not a science, but a sequel to all sciences, it is numerical computation applied to anything and to everything. Proportion, as a *science*, is the comparison of magnitudes, and belongs, properly speaking, to Algebra and Geometry; and the rule of three, in Arithmetic, is but little more than some of its forms of application. Problems in mensuration are very properly to be found in books called Arithmetics, but mensuration is no part of the science of Arithmetic, it is a part of Geometry, and for a good understanding of it, geometrical science must be directly consulted.

So it is with many other parts of Arithmetic, the science is elsewhere; and to have a scientific comprehension of many parts of common Arithmetic, we must go to general Arithmetic, which is emphatically Algebra; and in preparing this work, we have given constant attention to this branch of the subject, as may be seen in our treatment of fractions, proportion, progression, the roots, fellowship, and interest.

All these subjects can be better illustrated by symbols than by numbers; for numbers apply to everything, and, of course, can be made to show no particular thing; but not so with symbols, at every step the particular elements are all visible, and the logic and the reason is as distinct in every part of an operation as is the result. For these reasons, Arithmetic should be studied by symbols, as it is in many parts of Europe; many of their books, entitled Arithmetics, are as full of signs and symbols as any Algebra that ever appeared.

The prominent design of the author has been to adapt this treatise to the wants of young beginners in Algebra, and at the same time not to produce a mere childish book, but one more dignified and permanent, and to secure this end, he has kept up the same tone and spirit as though he were addressing mature and disciplined minds.

Great care has been taken in the selection of problems, and all very severe ones have been excluded, and all such as might be difficult when detached and alone, are rendered simple and easy by their connection with other leading problems of kindred character.

To bring out the original thoughts of the pupil has been another object which he designed to accomplish, and the illustrations are given in such a way as to command the constant attention of the learner, and if he learns at all, it will be naturally and easily, and what he learns will become a part of himself.

In this work, great importance is attached to equations, not merely in solving problems, but they are used as an instrument of illustrating principles, and their application is carried further in this book than in any other known to the author.

For instance, we have illustrated the nature of an equation by the aid of simple problems in subtraction and division; and conversely, the simple principle of equality is used to deduce rules for subtraction, division, the reduction of fractions to a common denominator, the multiplication of quantities affected by different fractional exponents, &c.

Notwithstanding that this book is designed to be practical, it contains more illustrations, and is more theoretical and scientific as far as it goes, than any other book designed for the same class of pupils.

We have not given demonstrations of the binomial theorem, nor made any investigations of logarithms, or the higher equations, for these subjects belong exclusively to the higher order of Algebra, and will be found very clear and full in the University Edition of Algebra by the same author.

In relation to great generalities, all books on the same science, are, in substance, much alike, yet, in the clearness and distinctness with which they present principles, they may be very different; and to arrive at perfection in this particular, is, and should be, the highest ambition of an author.

For peculiarities in this work, the teacher is respectfully referred to abbreviations generally in solving equations, to the philosophical uses made of equations in demonstrating principles—the formation of problems, and the manner of arriving at arithmetical rules, which may be found in various parts of this work.

CONTENTS.

INTRODUCTION.

ALGEBRA is the science of computation by means of symbols. Letters of the alphabet are generally used to represent quantities or numbers, and conventional signs are employed to represent operations, and to abridge and generalize the reasoning in relation to propositions or problems.

We sometimes meet with persons who can readily solve quite difficult problems, and yet are not able to explain the steps in the process: they call their operations working in the head—and, indeed, their reasoning, properly written out, is *Algebra;* but not having a knowledge of the signs, and possessing no skill in writing out the thoughts of the mind, they do not know that it is Algebra.

This natural adaptation of the mind, to solve problems without the aid of writing down the operation, is very essential to success in this science. But the mind can only go a very short distance, unaided by the pen; nor is it important that it should, for the aid given by that instrument is efficient and complete, secures the ground gone over, and leaves the mind free to advance indefinitely.

In a purely mental process, the mind must retain all the results thus far attained, and continue the reasoning onward at the same time. And this, *carried to excess,* breaks down the mind rather than strengthens it; and for this reason, a mere mental Algebra must be regarded as one of the ephemeral efforts of the times. But let no reader construe these sentiments into a disapproval of mental Algebra. Every Algebra, properly understood, is mental Algebra; for the mental process—the reasoning power—must precede every operation.

To compare the common operations of the mind with the brief and refined language of science, we propose the following problems. But before we use algebraical language, we must explain some of its symbols, and here we insert only those intended for immediate use.

THE SIGNS.

1. The perpendicular cross, thus $+$, called *plus*, denotes addition.

2. The horizontal dash, thus $-$, called *minus*, denotes subtraction.* These signs are written before the quantities to which they are affixed.

3. The diamond cross, thus $\times$, or a point between two quantities, denotes multiplication. For example, 5×4, or $5 \cdot 4$, shows that 4 and 5 must be multiplied together.

4. A horizontal line with a point above and below, thus $\div$, denotes division; also two quantities, one above another, as numerator and denominator, as $\frac{3}{7}$ or $\frac{a}{b}$, also indicates division, and shows that 3 must be divided by 7, and a must be divided by b.

5. Double horizontal lines, thus $=$, represent equality, and show that the quantities between which it is placed are equal.

6. A number or letter before any quantity shows how many times the quantity is taken, and is called the *coefficient* of the quantity, thus $3x$, shows that the quantity x is taken 3 times, and nx shows that the quantity represented by x is taken as many times as there are units in n.

7. A *vinculum* or bar ———, or parenthesis ( ), is used

---

* The signs *plus* and *minus*, in general science, have a far more comprehensive meaning than is here expressed. Here they denote simply what is to be done with the quantities to which they are attached; but in philosophical problems, they may denote the essential value of the quantities, as *credit* and *debt*; and in geometry they may represent positions, as *north* and *south*, or to the *right* or *left* of a *zero line*, &c.

to connect several quantities together. Thus, $\overline{a+b}$ or $(a+b)$ shows that a and b are there to be considered as connected, or making but one quantity.

We now turn our attention to the problems—not for the purpose of finding the answers to them, as the mere arithmetical student might suppose—but for the purpose of teaching the manner of solving them by *the science* of Algebra.

1. *A father divided* 120 *cts. among his three sons. He gave the youngest a certain number, the second* 10 *cents more, and the eldest* 10 *cents more than the second. What sum did each receive?*

By the use of *common language*, this question may be solved thus:

The youngest son had a share of the money; the second son had a like share and 10 cents more; and the eldest had also a like share as the youngest, and 20 cents more. Therefore the three boys had 3 shares and 30 cents; but the three boys had 120 cents, hence, 3 shares and 30 cents are equal to 120 cents, or the three shares are worth 90 cents, and one share is worth 30 cents, which is the sum given to the youngest.

By *algebraical language*, thus:

Let x represent the share of the youngest. Then by the conditions

$$\begin{aligned} x &= \text{3d son's share,} \\ x+10 &= \text{2d} \quad \text{"} \quad \text{"} \\ x+20 &= \text{1st} \quad \text{"} \quad \text{"} \\ \hline 3x+30 &= 120 \text{ by add.} \end{aligned}$$

This expression is called an equation, and the quantities on each side of the sign of equality are called members, or sides of the equation.

It is an axiom, that equals from equals the remainders must be equal; and in this equation, if we take 30 from both members, we have

$$3x=120-30=90$$

Dividing both members by 3 gives $x=30$, the share of the youngest.

2. *If 75 dollars be added to a share in a certain bridge company, the sum will be the value of 4 shares. What is the value of a share?*

BY COMMON LANGUAGE.

One share and 75 dollars is the same as 4 shares, therefore 75 dollars is 3 shares, and one share is 25 dollars, the third part of 3 shares.

ALGEBRAICALLY.

Let x represent the value of a share.

Then $x+75=4x$.

Taking x an equal quantity from both members and

$$75=3x.$$

Dividing by 3 gives $25=x$.

3. *A gentleman purchased a horse, a chaise and a harness, for* \$230. *The harness cost a certain sum, the chaise 3 times as much as the harness, and the horse* \$20 *more than the chaise. Required the price of each.*

BY COMMON LANGUAGE.

The harness cost a certain share of the money, and the chaise cost 3 such shares, and the horse cost 3 such shares and 20 dollars more; therefore the whole cost 7 shares and 20 dollars, which must make 230 dollars. Take the 20 dollars away, and the 7 shares is the same as 210 dollars. Therefore 1 share is 30 dollars, the value of the harness, and \$90 is the value of the chaise, and \$110 the value of the horse.

ALGEBRAICALLY.

Let $x=$ the value of the harness,

Then $3x=$ the value of the chaise,

And $3x+20=$ the value of the horse.

Sum $7x+20=230$.

Taking equals from both members $7x=210$

Or $x=30$ by division.

4. *In a certain school* $\frac{1}{3}$ *of the pupils are learning geometry,* $\frac{1}{4}$ *are learning Latin, and* 10 *more, which comprise all in the school, are learning to read. What was the whole number?*

BY COMMON LANGUAGE.	ALGEBRAICALLY.
One-third and one-fourth added together make $\frac{7}{12}$. Therefore $\frac{5}{12}$ is the number learning to read, which, by the problem, is 10; hence $\frac{1}{12}$ of the number in the school is 2, and the whole number is 24, the number required.	Let $x=$ the number in school. Then by the conditions $\frac{x}{3}+\frac{x}{4}+10=x.$

This equation may be troublesome on account of the fractions; but in due time we shall give rules to clear equations of fractions; however, fractions here, are just the same as fractions elsewhere. *One-third* and *one-fourth* of *anything* is $\frac{7}{12}$ of that thing; therefore, $\frac{7x}{12}+10=\frac{12x}{12}$. Now from the two equals take $\frac{7x}{12}$, and $10=\frac{5x}{12}$; dividing by 5, $2=\frac{x}{12}$; multiplying by 12 gives $24=x$, the final result.

From these examples it will be perceived that Algebra is but an artificial method of briefly writing out our mental operations when we solve mathematical problems, and as such, it may be extended and applied to almost every branch of the mathematics; and, therefore, the value of this science cannot be over estimated.

The three following problems are extremely simple when algebraic language is applied, but would be rather difficult by common language.

5. *On a certain day, a merchant paid out* \$2500 *to three men, A, B, and C; he paid to A a certain sum, to B* \$500 *less than the sum paid A, and to C he paid* \$900 *more than to A. Required the sum paid to each.*

Let	$x=$	the sum paid to A.
Then	$x-500=$	" " " B.
And	$x+900=$	" " " C.
By addition, . . .	$3x+400=2500$,	the whole sum paid.

Subtracting 400 from both members—that is, equals from equals—and we have $3x=2100$.

Take $\frac{1}{3}$ of each member, or divide both sides by 3 and

$x=700$, the sum paid to A.

From this we determine that \$200 was paid to B, and \$1600 to C.

In this example the sign before 500 shows what is to be done with that quantity; it shows that it is to be taken out, and accordingly it was taken out, diminishing the 900 to 400.

This operation is called *addition*, but it is *algebraic addition*, that is, *writing* the quantities according to their *signs*, and finding the result.

6. *It is required to divide the number* 99 *into five such parts that the first may exceed the second by* 3, *be less than the third by* 10, *greater than the fourth by* 9, *and less than the fifth by* 16.

Let . . . . . $x=$ the first part,

then . . . . . $x-3=$ the second,

$x+10=$ the third,

$x-9=$ the fourth,

$x+16=$ the fifth,

Sum is . . . $5x+14$, but the sum of all the parts is 99; therefore, $5x+14=99$.

By subtracting 14 from both members, and dividing the remainders by 5, we have $x=17$, the first part; and, therefore the parts are 17, 14, 27, 8, and 33.

7. *Divide* \$1000 *among four men, giving the second twice as much as the first, minus* \$200; *the third double the sum of the second, plus* \$400; *the fourth three times as much as the first, plus* \$100. *Required the share of each.*

Let . . . . $x=$ the sum paid to the 1st,

then . . . . $2x-200=$ " " " 2d,

$4x-400+400=$ " " " 3d,

$3x+100=$ " " " 4th,

The sum is $10x-100=1000$.

By adding 100 to both members we have

$$10x=1100$$

By adding 100 to —100 in the first member of the equation, makes 0, and then $10x$ only is left in that member, which must be equal to 1100, or $x=110$, the sum paid to the first, and the several sums are \$110, \$20, \$440, and \$430.

The preceding remarks and problems serve to show, *only in some small degree*, the advantage of Algebra over common language, and the learner should examine every problem, and the reason of every step in the process of its solution, until all is thoroughly understood; then he will have no difficulty in solving the examples that follow in this introduction. But before we give additional problems, let us call the student's mind to the precise idea of an

EQUATION.

An equation is simply what the word implies; equality as to value, weight, or measure; and can be best understood by comparing it to a pair of scales delicately balanced.

The balance can be preserved by adding equal weights to both sides; by taking equal weights from both sides; by multiplying both sides by the same number, or by dividing both sides by the same number, or by taking like roots or like powers of the weights in both sides.

The object of working an equation is to bring the unknown quantity to stand alone as one member of the equation, equal to known quantities in the other member. The unknown quantity thus becomes known; and *we may do anything* to accomplish this end, that the *nature of the case* may seem to require, only taking *scrupulous care* to preserve equality through every change.

It is usual to represent known quantities by their numerical values, or by the first letters of the alphabet, as a, b, c, d, &c.; and unknown quantities by the last letters, as u, t, x, y, &c.

AXIOMS.

Axioms are self-evident truths, and of course are above demonstration; no explanation can render them more clear. The following are those applicable to Algebra, and are the principles on which the truth of all algebraical operations *finally* rests.

Axiom 1. If the same quantity or equal quantities be *added* to equal quantities, their *sums* will be equal.

2. If the same quantity or equal quantities be *subtracted* from equal quantities, the *remainders* will be equal.

3. If equal quantities be multiplied into the same, or equal quantities, the *products* will be equal.

4. If equal quantities be *divided* by the same, or by equal quantities, the *quotients* will be equal.

5. If the same quantity be both *added* to and *subtracted from* another, the value of the latter will not be altered.

6. If a quantity be both *multiplied* and *divided* by another, the value of the former will not be altered.

7. Quantities which are respectively equal to any other quantity are equal to each other.

8. Like roots of equal quantities are equal.

9. Like powers of the same or equal quantities are equal.

Now suppose we have the following equation

$$x+a=b$$

in which x is the unknown quantity, and a and b known quantities. Before x can become known, a must be disengaged from it, that is, a must be subtracted from both members. It must be subtracted from the first member, because it is our object to have x stand alone, and we must subtract it from the other member, to *preserve equality*. The equation then stands $x=b-a$

Here we find the quantity a, *whatever it may be, on the other side of the equation, with the contrary sign.*

Now let us suppose we have an equation like

$$x-a=b$$

In this equation we perceive that x is diminished by a; therefore to have the single value of x we must add a to the first member; and, of course, to preserve equality, we must add a to the second member, then we shall have

$$x-a+a=b+a$$

But $-a$ and $+a$ destroy each other, and the equation is in brief $$x=b+a$$

Here, also, we find a on the opposite side of the equation, with its sign changed; and from these investigations we draw the following *rule of operation.*

RULE.—*We may change any quantity from one member of an equation to the other, if we change its sign.*

The operation itself is called transposition.

For examples, transpose the terms so that the unknown quantity x shall stand alone in the first member of the following equations:

$x+c-d=4g$ . . . . *Ans.* $x=4g+d-c$.

$x+3-a+m=30$ . . . . *Ans.* $x=30-m+a-3$.

SIMPLE PROBLEMS FOR EXERCISES.

1. A man bought a saddle and bridle for 45 dollars; the saddle cost four times as much as the bridle. What was the cost of each? *Ans.* Bridle \$9; saddle \$36.

2. Three boys had 66 cents among them; the second had twice as many as the first, and the third three times as many as the first. How many had each?

Ans. 1st boy had 11; 2d, 22; 3d, 33 cents.

3. Two men had 100 dollars between them, and one had 3 times as many as the other. How many had each?

Ans. One had \$25, the other \$75.

N. B. This last problem may be enunciated thus:

Two men had 100 dollars between them; the first had one-third as many as the other. How many had each?

4. Three men had 880 dollars among them; the first had $\frac{1}{3}$, the second had $\frac{1}{2}$ as many as the third. How many had each? Let $6x=$ what the third had.

Ans. 1st had 160; 2d, 240, and the 3d, 480 dollars.

5. There are three numbers which together make 72, the second is twice as much as the first, and the third is as much as both the others. What are the numbers?

Ans. 1st is 12; 2d, 24; 3d, 36.

6. Two men built 90 rods of fence in 3 days. The second built twice as many rods in a day as the first. How many rods did each build per day? *Ans.* 1st built 10 rods, 2d, 20.

7. A man bought 3 oxen, 4 cows, and 6 calves, for 260 dollars. He paid twice as much for an ox as he did for a cow, and twice as much for a cow as for a calf. How much did he give for each?

Ans. For a calf, $10; cow, $20; and for an ox, $40.

8. A man bought a boat load of flour for 132 dollars, one-half at 5 dollars per barrel, the other half at 6 dollars per barrel. How many barrels did the boat contain?

Let $x=$ half the number of barrels. *Ans.* 24.

9. A boy bought an equal number of apples, oranges and pears, for 96 cents: the apples at 3 cents apiece, the oranges at 4, and the pears at 5. How many of each kind did he buy? *Ans.* 8.

10. Two men bought a carriage for 86 dollars; one paid five times as much as the other, and 26 dollars more. What did each pay? *Ans.* One paid 10, the other 76 dollars.

11. If from 5 times a certain number we subtract 24, the remainder will be 196. What is the number? *Ans.* 44.

12. To the double of a certain number, if we add 18, the sum will be 96. What is the number? *Ans.* 39.

13. What number is that whose double exceeds its half by 78? Let $2x=$ the number. *Ans.* 52.

14. A man had six sons, to whom he gave 120 dollars, giving

to each one 4 dollars more than to his next younger brother. How many dollars did he give to the youngest? *Ans.* $10.

15. Three men received 65 dollars; the second received 5 dollars more than the first, and the third 10 dollars more than the second. What sum did the first receive?

Ans. $15.

16. A man paid a debt of 29 dollars, in three different payments; the second time he paid 3 dollars more than at first, and the third time he paid twice as much as at the second time. What was the amount of his first payment?

Ans. $5.

17. A man bought 6 pounds of coffee, and 10 pounds of tea, for 360 cents, giving 20 cents a pound more for the tea than for the coffee. What was the price of the coffee?

Ans. 10 cents.

18. A man bought 6 barrels of flour, and 4 firkins of butter, for 68 dollars. He gave 2 dollars more for a barrel of flour than for a firkin of butter. What was the price of flour?

Ans. $7.60.

19. A pound of coffee cost 5 cents more than a pound of sugar, and for 3 pounds of sugar or for 2 pounds of coffee you must pay the same sum. What is the price of sugar?

Ans. 10 cents.

20. A person in market selling apples, peaches, and oranges, asked 1 cent more for a peach than for an apple, and 2 cents more for an orange than for a peach, and the prices were such that 10 apples and 5 peaches cost as much as 5 oranges. What was the cost of an apple? *Ans.* 1 cent.

21. One-half of a post stands in the mud, one-third in the water, and the remainder, which is 3 feet, is above the water. What is the whole length of the post? *Ans.* 18 feet.

22. One-third and one-half of a sum of money, and ten dollars more, make the whole sum. What is the sum?

Ans. 60 dollars.

23. Divide 25 cents between two boys, and give one four times as much as the other. Required the share of each.

Ans. 5 and 20 cents.

24. Divide 15 cents between two boys, and give one double of the other. Required the share of each.

Ans. 5 and 10 cents.

Similar problems to the preceding might be framed indefinitely, but it would be improper to propose any that involve any difficulty until the pupil is better prepared to meet difficulties. We only give the preceding to convince the learner that he can find real utility in the science; but before he can go into the subject to advantage, he must learn the nature of algebraic expressions, and *acquire* the art of *adding*, *subtracting*, *multiplying*, and *dividing* algebraic quantities, both whole and fractional.

We now assure the young beginner that we will conduct him through the elements of this science with as little delay and trouble as possible; and neither remarks nor examples will be given which are not, in the judgment of the author, essential to the progress of the pupil.

With this assurance we close this introduction, and commence ALGEBRA, by giving more extended definitions of terms.

ALGEBRA

DEFINITION OF TERMS.

The signs for addition, subtraction, multiplication, division, and equality, have already been explained. We have also explained coefficient and vinculum. The word coefficient can hardly be understood by a mere definition. It means any factor connected with another, and may be simple or compound, thus, ax; a is the coefficient of x, and in the term $3ax$, $3a$ is the coefficient of x, and 3 is the coefficient of ax.

1. When a letter stands alone, as b, y, or any other letter, *one* or *unity* may be considered its coefficient. In the expression $(3a+2b-c)x$, $(3a+2b-c)$ is a compound coefficient to x. It is also a factor, and x is another factor. The word factor has the same signification as in Arithmetic.

2. When we wish to note that two quantities are *unequal*, we write this sign $>$ between them. The opening of the sign is always put toward the greater quantity, thus, $a>b$, signifies that a is greater than b, and $a<b$ shows that b is greater than a.

3. When we indicate the multiplication of numbers, without actually performing the multiplication, we must write the sign $\times$ or $\cdot$ between them, as 5×5, or $3\cdot 4$, because the multiplication could not be understood without the sign; but when we have letters in place of numbers, we may *omit the sign* and write ab, in place of $a\times b$; abx in place of $a\cdot b\cdot x$, &c.

4. When factors are equal, and each equal to a, the product of four such factors is $aaaa$; but in place of this, we may write a^4, which signifies that a is taken four times as a factor; a^7 indicates a product which is composed of a taken seven times as a factor; x^n indicates a product in which x is taken as many times as a factor as there are *units* in n.

5. The *small* number thus written to the right of a letter, (or to any quantity) and a little above, is called its EXPONENT. *Exponents* may be either whole numbers or fractions; but this will be explained hereafter.

When an exponent is a whole number, it indicates the power of the factors, or quantity to which it is attached. When it is a fraction, it indicates a root of the quantity, thus, $a^{\frac{1}{2}}$ indicates the square or second root of a; $a^{\frac{1}{3}}$ indicates the third root of a, and $a^{\frac{1}{4}}$ indicates the fourth root of a, &c.

Formerly, the sign of a root was indicated by the *radical sign* $\sqrt{}$; and, for some purposes, this sign is still in use. Whenever it is used, it is placed to the left of the quantity; thus, $\sqrt{a}$. The *number* of the root is denoted by a little figure placed over the radical sign; unless it is the second root, when the figure 2 is omitted. Thus,

$\sqrt{a}$ is the second or square root of a.

$\sqrt[3]{a}$ is the third or cube root of a.

$\sqrt[n]{a}$ is the nth root of a.

$\sqrt{a+x}$ is the second root of $(a+x)$.

It must be remembered, that if the radical sign is to affect more than one factor, the vinculum must be used with it; thus, $\sqrt{5a}$, $\sqrt{ab}$.

6. The number of literal factors which enter into any term, is the *degree* of that term; ab is of the second *degree*, a^2b of the third, ab^2c^2 of the fifth. In general, the degree of an algebraic term is found by taking the sum of the exponents of the letters which enter into that term. An algebraic quantity which has all its terms of the same degree, is said to be

homogeneous: $4a^5+2a^2b^3-3abc^3+b^4c$, is therefore homogeneous, and of the fifth degree.

Similar terms, are those which contain the same letters in the same powers, $2a^2b$ and $5a^2b$ are similar. But, $3ab^2$ and $3ab$ are not similar terms, for the letters, although the same, are not in the same power.

7. Simple Quantities, are those which consist of one term only. As $3a$, or $5ab$, or $6abc^2$.

8. Compound Quantities, are those which consist of two or more terms. As $a+b$, or $2a-3c$, or $a+2b-3c$.

9. And when the compound quantity consists of two terms it is called a Binomial, as $a+b$; when of three terms, it is a Trinomial, as $a+2b-3c$; when of four terms, a Quadrinomial, as $2a-3b+c-4d$; and so on. Also, a Multinomial or Polynomial, consists of many terms.

10. A Residual Quantity, is a binomial having one of the terms negative. As $a-2b$.

11. Positive or Affirmative Quantities, are those which are to be added, or have the sign $+$. As a or $+a$, or ab: for when a quantity is found without a sign, it is understood to be positive, or as having the sign $+$ prefixed.

12. Negative Quantities, are those which are to be subtracted. As $-a$, $-2ab$, or $-8c^2$.

13. Like Signs, are either all positive $(+)$, or all negative $(-)$.

14. Unlike Signs, are when some are positive $(+)$, and others negative $(-)$.

15. The *reciprocal* of any quantity is unity divided by that quantity. Thus, $\frac{1}{4}$ is the reciprocal of 4, $\frac{1}{a}$ is the reciprocal of a, and so on, of any other quantity.

16. The same letter, accented, is often used to denote quantities which occupy similar positions in different equations or investigations. Thus, a, a', a'', a''', represent four

different quantities; of which a' is read a prime; a'' is read a second; a''' is read a third, and so on.

That the pupil may imbibe or catch the true spirit of an algebraic expression, we give the following exercises in converting common arithmetical operations into algebraic expressions, and finding the value of each under different suppositions.

Express in algebraic language the product of three times a *into* x, *diminished by* c, *and the remainder divided by* b.

Ans. $\frac{3ax-c}{b}$.

What is the value of this expression when $a=2$, $x=3$, $c=4$, and $b=2$? *Ans.* 7.

What is its value when $a=3$, $x=5$, $c=9$, and $b=3$? *Ans.* 12.

Express in algebraic language 3 *times the square of* a, *diminished by* 2b, *and the difference divided by* c. *Ans.* $\frac{3a^2-2b}{c}$.

What is the numerical value of this expression, when $a=5$, $b=10$, and $c=5$? *Ans.* 11.

What when $a=10$, $b=7$, and $c=20$? . . *Ans.* $14\frac{3}{10}$.

What when $a=9$, $b=0$, and $c=1$? . . *Ans.* 243.

What when $a=1$, $b=1$, and $c=\frac{1}{2}$? . . *Ans.* 2.

Write the following: $6a$ diminished by x, the difference increased by the square root of c, and the whole multiplied by b. *Ans.* $(6a-x+c^{\frac{1}{2}})b$.

What is the value of this expression, when $a=3$, $x=18$, $c=16$, and $b=2$? *Ans.* 8.

What when $a=6$, $x=9$, $c=9$, and $b=7$? *Ans.* 210.

What is the value of the expression $a^2+3ab-c^2$, when $a=6$, $b=5$, and $c=4$? *Ans.* 110.

With the same value to a, b, and c, what is the value of the expression, $2a^3-3a^2b+c^3$? *Ans.* —44.

What of the expression $a^2(a+b)-2abc$? *Ans.* 156.

What is the value of $\frac{a^3}{a+3c}+c^2$? . . . . . . *Ans.* 28.

What is the value of $a^2-\sqrt{b^2-ac}$? . . . . . *Ans.* 35.

SECTION I.

ADDITION.

(ART. 1.) Addition in Algebra is connecting quantities together by their proper signs.

Here the pupil should call to mind the *fact* that unlike quantities cannot be added together. For instance, it would be an absurdity to add dollars to yards of cloth, and so of any other unlike quantities; but dollars can be added to dollars, yards to yards, &c.; so in Algebra, a may be added to a, making $2a$, or any number of a's may be added to any other number of a's by uniting their coefficients; but a cannot be added to b or to any other dissimilar quantity: we can write $a+b$, indicating the addition by the sign making a *compound quantity.*

(ART. 2.) Addition in Algebra may be divided into three cases: the first, when the quantities are alike and their signs alike; a second, when the quantities are alike and the signs unlike; and the third, when the quantities are unlike.

To discover a rule for case 1st, we propose the following problem:

On Monday a merchant sent to a steamboat 17 barrels of flour and 9 barrels of pork; on Tuesday he sent 7 barrels of flour and 10 of pork; on Wednesday, 20 barrels of flour and 6 of pork; on Thursday, 10 barrels of flour and 10 of pork. How many barrels of each has he sent?

We may write it thus:

17	barrels	of	flour	+	9	barrels	of	pork
7	"	"	"	+	10	"	"	"
20	"	"	"	+	6	"	"	"
10	"	"	"	+	10	"	"	"
54	barrels	of	flour	+	35	barrels	of	pork

Now let b represent a barrel of flour, and p represent a barrel of pork, then in place of writing out the words, we write

$$\begin{array}{r} 17b+\ 9p \\ 7b+10p \\ 20b+\ 6p \\ 10b+10p \\ \hline 54b+35p \end{array}$$

Sum is . . . . . . $54b+35p$

From this example we perceive that to add together similar quantities, we have only to add their numeral coefficients, like simple numbers in Arithmetic.

Hence, the following rule will meet

CASE 1. *When the quantities are similar and the signs alike, add the coefficients together, and set down the sum; after which set the common letter or letters of the like quantities, and prefix the common sign + or —.*

EXAMPLES.

(1)	(2)	(3)	(4)	(5)
$3a$	$-3bx$	bxy	$3a+2b-5c$	$4ab-2cd$
$9a$	$-5bx$	$2bxy$	$5a+6b-c$	$7ab-cd$
$5a$	$-4bx$	$5bxy$	$7a+11b-8c$	$15ab-2cd$
$12a$	$-2bx$	bxy	$a+b-3c$	$ab-12cd$
a	$-7bx$	$3bxy$	$16a+20b-17c$	$27ab-17cd$
$2a$	$-bx$	$6bxy$		
$32a$	$-22bx$	$18bxy$		

	(6)	(7)	(8)
	$3a+3ax+c$	$4ab+3x-2b$	$10y-x+h$
	$7a+5ax+5c$	$7ab+x-3b$	$7y-x+3h$
	$10a+7ax+2c$	$121ab+2x-b$	$3y-2x+7h$
	$2a+10ax+4c$	$99ab+x-b$	$y-10x+10h$
Sum			

(ART. 3.) Like quantities, of whatever kind, whether of powers or roots, may be added together the same as more simple quantities.

Thus $3a^2$ and $8a^2$ are $11a^2$, and $7b^3+3b^3=10b^3$. No matter what the quantities may be, if they are only alike in kind. Let the reader observe that $2(a+b)+3(a+b)$ must be together $5(a+b)$, that is, 2 times any quantity whatever added to 3 times the same quantity, must be five times that quantity. Therefore, $4\sqrt{x+y}+3\sqrt{x+y}=7\sqrt{x+y}$, for $\sqrt{x+y}$, which represents the square root of $x+y$, may be considered a single quantity.

To illustrate these remarks we give the following

EXAMPLES.

	$4(a-x)$	$(x+y)$	$\sqrt{a+x}$	$5(a^2-c)$
	$7(a-x)$	$3(x+y)$	$6\sqrt{a+x}$	(a^2-c)
	$10(a-x)$	$20(x+y)$	$13\sqrt{a+x}$	$7(a^2-c)$
Sum	$21(a-x)$	$24(x+y)$	$20\sqrt{a+x}$	

(ART. 4.)—CASE 2. *When the quantities are similar and the signs unlike, we have the following rule for* ADDITION.*

---

* In this rule, the word *addition* is not very properly used; being much too limited to express the operation here performed. The business of this operation is to incorporate into one mass, or algebraic expression, different algebraic quantities, as far as an actual incorpora-

RULE.—*Add the affirmative coefficients into one sum and the negative ones into another, and take their difference with the sign of the greater, to which affix the common literal quantity.*

EXAMPLES FOR PRACTICE.

	(1)	(2)	(3)
	$-5a$	$+3ax^2$	$+\ 8x^3+3y$
	$+4a$	$+4ax^2$	$-\ 5x^3+4y$
	$+6a$	$-8ax^2$	$-16x^3+5y$
	$-3a$	$-6ax^2$	$+\ 3x^3-7y$
	$+\ a$	$+5ax^2$	$+\ 2x^3-2y$
Sum	$+3a$	$-2ax^2$	$-\ 8x^3+3y$

(4)	(5)	(6)
$-\ 3a^2$	$+\ 3b^2y^3$	$+4ab+\ 4$
$-\ 5a^2$	$+\ 9b^2y^3$	$-4ab+12$
$-10a^2$	$-10b^2y^3$	$+7ab-14$
$+10a^2$	$-19b^2y^3$	$+\ ab+\ 3$
$+14a^2$	$-\ 2b^2y^3$	$-5ab-10$

7. Add $2xy-2a^2$, $3a^2+xy$, a^2+xy, $4a^2-3xy$, $2xy-2a^2$. *Ans.* $4a^2+3xy$.

8. Add $8a^2x^2-3xy$, $5ax-5xy$, $9xy-5ax$, $2a^2x^2+xy$, $5ax-3xy$. *Ans.* $10a^2x^2+5ax-xy$.

9. Add $3m^2-1$, $6am-2m^2+4$, $7-8am+2m^2$, and $6m^2+2am+1$. *Ans.* $9m^2+11$.

---

tion or union is possible; and to retain the algebraic marks for doing it, in cases where the former is not possible.

By using the word *united* in place of the word *added*, the reason of the rule will become obvious.

Thus $3a$ *united* to $-a$ makes $2a$
$7x$ *united* to $-2x$ makes $5x$

10. Add $12a-13ab+16ax$, $8-4m+2y$, $-6a+7ab^2+12y-24$, and $7ab-16ax+4m$.

Ans. $6a-6ab+14y+7ab^2-16$.

11. *Unite* $4a^2b-8a^2b-9a^2b+11a^2b$ into one term if possible.

Ans. $-2a^2b$.

12. *Unite* $7abc^2-abc^2-7abc^2-8abc^2+9abc^2$ into one term.

Ans. 0.

13. Add $3a(a+b)$, $7a(a+b)$, $-5a(a+b)$, and $3a(a+b)$.

Ans. $8a(a+b)$.

14. Add $7(6x+y-z)^2$, $-8(6x+y-z)^2$, $(6x+y-z)^2$, and $3(6x+y-z)^2$. *Ans.* $3(6x+y-z)^2$.

15. Add $3ab+4a(6y+b)$, $-8ab-9a(6y+b)$, $12ab+13a(6y+b)$, $ab+a(6y+b)$, and $7ab+6a(6y+b)$.

Ans. $15ab+15a(6y+b)$.

CASE 3.—*When the quantities are* UNLIKE *and the signs* ALIKE *or* UNLIKE, *we have the following rule to unite, or rather to reduce and condense the quantities.*

RULE.—*Collect together all those terms that are similar, by uniting their coefficients, as in the former cases: then write the different sums, one after another, with their proper signs.*

N. B. It is immaterial what quantity, in an aggregate sum, stands first, for the whole of a thing is equal to the sum of all its parts, whatever part may be first written. Thus, $ax+by+c$ is the same sum, whichever term stands first.

EXAMPLES.

1. Add $3ay^2$, $-2xy^2$, $-3y^2x$, $-8x^2y$, and $2xy^2$.

These terms may be arranged thus:

$$\begin{cases} 3ay^2-2xy^2 \\ \qquad -3xy^2-8x^2y \\ \qquad +2xy^2 \end{cases}$$

Sum . . . . $3ay-3xy-8x^2y$

2. Add together $15a^2-8b^2c+32a^2c^3-12bc$, $19b^2c-4a^2+11a^2c^3+2bc$, $a^2-29a^2c^3-12b^2c+5bc$, and $9a^2c^3-14bc+b^2c$.

$$\begin{array}{r} 15a^2-\ 8b^2c+32a^2c^3-12bc \\ -\ 4a^2+19b^2c+11a^2c^3+\ 2bc \\ a^2-12b^2c-29a^2c^3+\ 5bc \\ +\ \ b^2c+\ 9a^2c^3-14bc \\ \hline 12a^2\ \ *\ \ *\ +23a^2c^3-19bc \end{array}$$ the ans.

3. What is the sum of $6ab+12bc-8cd$, $3cd-7ab-9bc$, and $12cd-2ab-5bc$? *Ans.* $7cd-3ab-2bc$.

4. What is the sum of $5\sqrt{ab}-7\sqrt{bc}+8d$, $3\sqrt{ab}+8\sqrt{bc}-12d$ and $7\sqrt{ab}+3\sqrt{bc}+9d$? *Ans.* $15\sqrt{ab}+4\sqrt{bc}+5d$.

5. Add $72ax^4-8ay^3$, $-38ax^4-3ay^4+7ay^3$, $8+12ay^4$, $-6ay^3+12-34ax^4+5ay^3-9ay^4$. *Ans.* $-2ay^3+20$.

Add $a+b$ and $3a-5b$ together.

Add $6x-5b+a+8$ to $-5a-4x+4b-3$.

Add $a+2b-3c-10$ to $3b-4a+5c+10$ and $5b-c$.

Add $3a+b-10$ to $c-d-a$ and $-4c+2a-3b-7$.

Add $3a^2+2b^2-c$ to $2ab-3a^2+bc-b$.

(Art. 5.) Let it be strictly observed, that when we add similar quantities together, as $3x$, $4x$, and $10x$, we perform it by writing the coefficients in one sum, 17, and writing the x, or the quantity, whatever it may be, afterward, making in this example $17x$. As *principles never change*, we must do the same thing when the coefficients are *literal;* thus, the sum of ax, bx, and cx must be $(a+b+c)x$, and $ax-x$ may be written $(a-1)x$.

EXAMPLES.

	(1)	(2)
Add	$ax+by^2$	$ay+cx$
	$2cx+3ay^2$	$3ay+2cx$
	$4dx+7y^2$	$4y\ +6x$
Sum	$(a+2c+4d)x+(b+3a+7)y^2$	$(4a+4)y+(3c+6)x$

	(3)	(4)
Add . . .	$3x+2xy$	$ax+7y$
	$bx+cxy$	$7ax-3y$
	$(a+b)x+2cdxy$	$-2x+4y$
Sum	$(a+2b+3)x+(2cd+c+2)xy$	$(8a-2)x+8y$

5. Add $8ax+2(x+a)+3b$, $9ax+6(x+a)-9b$, and $11x+6b-7ax-8(x+a)$. *Ans.* $10ax+11x$.

6. Add $(a+b)\sqrt{x}$ and $(c+2a-b)\sqrt{x}$ together.

Ans. $(c+3a)\sqrt{x}$.

7. *Unite* $3ax+7ax-4ax-bx+3bx+4x$, as far as possible, and find the sum total of the coefficient of x.

Ans. $6ax+2bx+4x$.

The sum total of the coefficient of x is $(6a+2b+4)$, and the sum total of the whole expression may be written $(6a+2b+4)x$.

SUBTRACTION.

(Art. 6.) Subtraction in Algebra is not, in all cases, taking one quantity from another: *it is finding the difference between two quantities.*

What is the difference between 12 and 20 degrees of north latitude? This is subtraction. But when we demand the difference of latitude between 6 degrees north and 3 degrees south, the result *appeurs* like addition, for the difference is really 9 degrees, the sum of 6 and 3. This example serves to explain the true nature of the *sign minus.* It is merely an opposition to the *sign plus;* it is counting in *another direction*, and if we call the degrees north of the equator *plus,* we must call those south of it *minus*, taking the equator as the zero line.

So it is on the thermometer scale, the divisions above zero are called *plus*, those below *minus*. Money due to us may be called *plus*, money that we owe should then be called *minus*, —the one circumstance is directly opposite in effect to the other. *Indeed, we can conceive of no quantity less than nothing*, as we sometimes express ourselves. It is quantity in opposite circumstances or counted in an opposite direction; *hence the difference or space between a positive and a negative quantity is their apparent sum.*

As a further illustration of finding differences, let us take the following examples, which all can understand:

From .	16	16	16	16	16	16
Take .	12	8	2	0	—2	—4
Differ. .	4	8	14	16	18	20

Here the reader should strictly observe that the smaller the number we take away, the greater the remainder, and when the subtrahend becomes minus, its numeral value must be *added*.

(Art. 7.) We cannot take a greater quantity from a less; but we can, in all cases, *find the difference* between any two quantities, and if we *conceive* a greater quantity taken from a less, the difference cannot be *positive*, but must be *negative*, i. e. *minus*.

EXAMPLES.

From .	12	12	12	12	12	12
Take .	30	20	16	12	10	6
Differ. .	—18	—8	—4	0	2	6

(Art. 8.) *When we take any quantity from* zero, *the difference will be the same quantity with its sign changed*, as will be obvious from the following examples:

From . .	$10a$	$5a$	0	0	$-5a$
Take . .	$11a$	$6a$	$6a$	$-6a$	$-6a$
Differ. . .	$-a$	$-a$	$-6a$	$+6a$	a

(Art. 9.) Unlike quantities cannot be written in one sum, (Art. 1,) but must be taken one after another with their proper signs: therefore, the difference of unlike quantities can only be expressed by signs. Thus, the difference between a and b is $a-b$, a positive quantity if a is greater than b, otherwise it is negative. From a take $b-c$, (observe that they are unlike quantities.)

OPERATION.

From	$a+0+0$
Take	$0+b-c$
Remainder, or difference,	$a-b+c$

This formal manner of operation may be dispensed with; the ciphers need not be written, and the signs of the subtrahend need only be changed.

From the preceding observation, we draw the following

GENERAL RULE FOR FINDING THE DIFFERENCE BETWEEN ALGEBRAIC QUANTITIES.

Rule.—*Write the terms of the subtrahend, one after another, with their signs changed; and then unite terms, as far as possible, by the rules of addition.*

Or we may give the rule in the following words:

Conceive the signs in the subtrahend to be changed, and then proceed as in addition.

EXAMPLES.

	(1)	(2)	(3)
From . . .	$4a+2x-3c$	$3ax+2y$	$a+b$
Take . . .	$a+4x-6c$	$xy-2y$	$a-b$
Remainder, .	$3a-2x+3c$	$3ax-xy+4y$	$2b$

	(4)	(5)	(6)
From . .	$2x^2-3x+y^2$	$7a+2-5c$	$\frac{1}{2}x+\frac{1}{2}y$
Take . .	$-x^2-4x+a$	$-a+2+c$	$\frac{1}{2}x-\frac{1}{2}y$
Rem. . .	$3x^2+x+y^2-a$	$8a$ * $-6c$	y

	(7)	(8)
From . . .	$8x^2-3xy+2y^2+c$	$ax+bx+cx$
Take . . .	$x^2-6xy+3y^2-2c$	$x+ax+bx$
Difference, .	$7x^2+3xy-y^2+3c$	$(c-1)x$

9. Find the difference between $8xy-20$ and $-xy+12$. *Ans.* $9xy-32$.

10. Find the difference between $7a^2x+a$ and $3a^2x-2a$. *Ans.* $4a^2x+3a$.

11. Find the difference between $-8x-2y+3$ and $10x-3y+4$. *Ans.* $-18x+y-1$.

12. Find the difference between $6y^2-2y-5$ and $-8y^2-5y+12$. *Ans.* $14y^2+3y-17$.

13. From $13a^2b^3+11a-5a^2+6b$,
Take $7a-5a^2+6b-10a^2b^3$. Remainder, $23a^2b^3+4a$.

14. From $3a+b+c-d-10$,
Take $c+2a-d$. Rem. $a+b-10$.

15. From $3a+b+c-d-10$,
Take $b-19+3a$. Rem. $c-d+9$.

16. From $2ab+b^2-4c+bc-b$,
Take $3a^2-c+b^2$. Rem. $2ab-3c+bc-3a^2-b$.

17. From $a^3+3b^2c+ab^2-abc$,
Take b^2+ab^2-abc. Rem. $a^3+3b^2c-b^2$.

(Art. 10.) From a take b. The result is $a-b$. The minus sign here shows that the operation has been performed: b was positive before the subtraction; *changing the sign performed the subtraction;* so changing the sign of any other quantity would subtract it.

18. From $3a$ take $(ab+x-c-y)$, considering the terms in the vinculum as *one term*, the difference must be $3a-(ab+x-c-y)$, but if we subtract this quantity, not as a whole, but term by term, the remainder must be $3a-ab-x+c+y$.

That is, when the vinculum is taken away, all the signs within the vinculum must be changed.

EXAMPLES.

1. From $30xy$, take $(40xy-2b^2+3c-4d)$.

Rem. $2b^2-10xy-3c+4d$.

From $3a^2$, take $(3a-x+b)$. Rem. $3a^2-3a+x-b$.

From a^2-a, take $4a-y-3a^2-1$.

Rem. $a^2-a-(4a-y-3a^2-1)$.

Or . . . . $4a^2-5a+y+1$.

From $a+b$, take $a-b$.

From $4a+4b$, take $b+a$.

From $4a-4b$, take $3a+5b$.

From $8a-12x$, take $4a-3x$.

(Art. 11.) It will be a useful exercise for the mind to look at the principle of subtraction in Algebra, through the medium of *equations*.

If we subtract 12 from 18, the remainder will be 6. Here are *three quantities*.

1. The minuend	18
2. The subtrahend	12
3. The remainder	6

In all cases, the remainder and the subtrahend, added together, must equal the minuend. Now let us suppose that we do not know the value of the remainder, and, therefore, represent it by the letter R. Then by the nature of the case we have $R+12=18$, an equation.

Taking equals from equal quantities, that is, 12 from both members of the equation, we have

$$R=6$$

Now let us take the third example under the last rule, and call its remainder R.

Then we have $R+a-b=a+b$

Rejecting a from both members, and *adding* b, or (what is the same thing), transposing $-b$, (see page 16), and we find

$$R=2b$$

Take example 9, and we have

$$R-xy+12=8xy-20$$

By transposition, $R=9xy-32$

In this manner we *may* perform all the examples in subtraction; and in this manner perform the following examples:

From $2a+2b$, take $-a-b$.

From $ax+bx$, take $ax-bx$.

From $a+c+b$, take $a+c-b$.

From $3x+2y+2$, take $5x+3y+b$.

From $6a+2x+c$, take $5a+6x-3c$.

MULTIPLICATION.

(Art. 12.) The nature of multiplication is the same in Arithmetic and Algebra. It is repeating one quantity as many times as there are units in another; the two quantities may be called factors, and in abstract quantities either may be called the multiplicand; the other of course will be the multiplier.

Thus, 4×5. It is indifferent whether we consider 4 repeated 5 *times*, or 5 repeated 4 *times;* that is, it is indifferent which we call the multiplier. Let a represent 4, and b represent 5, then the product is $a\times b$; or with letters we may omit the sign, and the product will be simply ab.

The product of any number of letters, as a, b, c, d, is $abcd$.

The product of x, y, z, is xyz.

In the product it is no matter in what order the letters are placed, xy and yx is the same product.

The product of $ax\times by$ is $axby$ or $abxy$. Now suppose $a=6$ and $b=8$, then $ab=48$, and the product of $ax\times by$ would be the same as the product of $6x\times 8y$ or $48xy$. From this we draw the following rule for multiplying simple quantities, which may be called

CASE 1. *Multiply the coefficients together, and annex the letters, one after another, to the product.*

EXAMPLES.

1. Multiply $3x$ by $7a$. . . . . . . *Product* $21ax$.
2. Multiply $4y$ by $3ab$. . . . . . . *Product* $12aby$.
3. Multiply $3b$ by $5c$, and that product by $10x$. *Ans.* $150bcx$.
4. Multiply $6ax$ by $12by$ by $7ad$. *Ans.* $504aaxydb$.
5. Multiply $3ax$ by $7b$ by $3y$. *Ans.* $63abxy$.
6. Multiply $100axy$ by $10abcy$ by 2. *Ans.* $2000aabcxyy$.

In the preceding examples no signs were expressed, and of course plus was understood as belonging to every factor; and a positive quantity, taken any number of times, must of course be *positive.*

(ART. 13.) As algebraic quantities are liable to be affected by negative signs, we must investigate the products arising from them. Let it be required to multiply —4 by 3, that is, repeat the negative quantity 3 times, the whole must be negative, *because a negative quantity taken any number of times must be negative.* Hence *minus multiplied by plus gives* *minus*, $-a \times b$ gives $-ab$; also a multiplied by $-b$ must give $-ab$, as we may conceive the minus b repeated a times.

Now let us require the product of -4 into -3.

In all cases the multiplier shows how many times the multiplicand must be taken;—when the multiplier is *plus*, it shows that the multiplicand must be *added to zero* as many times as there are units in the multiplier;—when the multiplier is *minus*, it shows that the multiplicand must be *subtracted from zero* as many times as there are units in the multiplier.

But to subtract -4 from zero *once*, gives $+4$, (Art. 8,) and to subtract it 3 times as the -3 indicates, gives $+12$. THAT IS, MINUS MULTIPLIED INTO MINUS, GIVES PLUS.

This principle is so important that we give another mode of illustrating it, making use of the following example.*

Required the product of $a-b$ by $a-c$.

Here $a-b$ must be repeated $a-c$ times.

---

* There is also another method of showing that *minus* multiplied into *minus*, must give plus; and it rests on the principle that a times 0 gives 0 for a product, or 0 times any quantity must give 0. In short, the product of two factors must be zero, if either one of them is zero.

Suppose we multiply . .	$a-a$
By	b
The product is	$ab-ab$

Here $a-a$ is in value 0. So in the product $ab-ab$ is 0, as it should be, and the whole subject is, thus far, very clear.

Now supppose we take .	$a-a$
And multiply by	$-b$
The product is	$-ab+ab$

The first part of the product is clearly $-ab$, and the whole must be zero; therefore we must take the second part, $+ab$, to destroy the first, that is, $-b$ multiplied by $-a$, gives $+ab$.

The objection to this method is, that the reasoning at the last point is rather *mechanical* than *intellectual*; we are *forced* to take ab as *plus* to make a definite sum, giving no decided *metaphysical* reason that it must be so.

If we take $a-b$, a times, we shall have too large a product, as the multiplier a is to be diminished by c.

That is $a-b$
Multiplied by a

Gives . . . $aa-ab$, which is too great by $a-b$ repeated c times, or by $ac-cb$, which must be subtracted from the former product; but to subtract we change signs, (Art. 5,) therefore the true product must be $aa-ab-ac+cb$.

That is, the product of *minus* b, by *minus* c, gives plus bc, *and, in general, minus multiplied by minus gives plus.*

But *plus* quantities multiplied by plus give plus, and minus by plus, or plus by minus, give *minus;* therefore we may say, in short,

That quantities affected by like signs, when multiplied together, give plus, and when affected by unlike signs, give minus.

(Art. 14.) The product of a into b can only be expressed by ab or ba. The product of a, b, c, d, &c., is $abcd$; but if b, c, and d are each equal to a, the product would be $aaaa$.

The product of aa into aaa is $aaaaa$; but for the sake of *brevity* and *convenience*, in place of writing aaa, we write a^3. The figure on the right of the letter shows how many times the letter is taken *as a factor*, and is called an *exponent.* The product of a^3 into a^4 is a repeated 3 times as a factor, and 4 times as a factor, in all 7 times; that is, *write the letter and add the exponents.*

EXAMPLES.

What is the product of a^3 by a^5? . . . . *Ans.* a^8.
What is the product of x^4 by x^6? . . . . *Ans.* x^{10}.
What is the product of y^2 by y^3 by y^5? . . *Ans.* y^{10}.
What is the product of a^n by a^m? . . . . *Ans.* a^{n+m}.
What is the product of b^7x^3 by bx? . . . . *Ans.* b^8x^4.
What is the product of ac by ac^2 by a^3c^2? . *Ans.* a^5c^5.

What is the product of x^3 by x^3 by x^3? . . *Ans.* x^9.

What is the product of x^2 by x^n by x^m? . *Ans.* x^{2+n+m}.

What is the product of $3x^3$ by $2x^2$ by 2? . *Ans.* $12x^5$.

Find the product in each of the following examples:

	$4ac$	$9a^2c$	$-3xy$	$-2xy$
	$-3ab$	$-4ay$	$+9xy$	$-6xy$
Product				

$-7ay$	$210xy$	$40rt$	$-21p$
$3xy$	$-3xy$	$20pq$	$-3r$

(Art. 15.) The preceding examples are sufficient to illustrate the multiplication of simple factors—we now proceed to

Case 2. *When one of the factors is a compound quantity, we have the following*

Rule.—Multiply *every term of the multiplicand, or compound quantity, separately, by the multiplier, as in the former case; placing the products one after another, with the proper signs; and the result will be the whole product required.*

The reason of this rule is obvious from Case 1.

EXAMPLES.

(1)	(2)	(3)
$5a-3c$	$3ac-4b$	$2a^2-3c+5$
$2a$	$3a$	bc
$10a^2-6ac$	$9a^2c-12ab$	$2a^2bc-3bc^2+5bc$

(4)	(5)	(6)
$12x-2ac$	$25c-7b$	$4x-b+3ab$
$4a$	$-2a$	$2ab$

(7)	(8)	(9)
$3c^2+x$	$10x^2-3y^2$	$3a^2-2x^2-6b$
$4xy$	$-4x^2$	$2ax^2$

10. Multiply $3b-2c$ by $5b$. . . . . *Ans.* $15b^2-10bc$.

11. Multiply $4xy-9$ by $6x$. . . . *Ans.* $24x^2y-54x$.

12. Multiply a^2-2x+1 by $4x^2$. *Ans.* $4a^2x^2-8x^3+4x^2$.

13. Multiply $11a^3bc^2-13xy$ by $3ax$.
Ans. $33a^4bc^2x-39ax^2y$.

14. Multiply $42c^2-1$ by -4. *Ans.* $-168c^2+4$.

15. Multiply $-30a^2bx^2y+13$ by $-5a^3$.
Ans. $+150a^5bx^2y-65a^3$.

16. Multiply $2b-7a-3$ by $4ab$.
Ans. $8ab^2-28a^2b-12ab$.

17. Multiply $a+3b-2c$ by $-3ab$.
Ans. $-3a^2b-9ab^2+6abc$.

18. Multiply $13a^2-b^2c$ by $-4c$. *Ans.* $-52a^2c+4b^2c^2$.

19. Multiply $13xy-3b$ by $-25x^2$.
Ans. $-325x^3y-75bx^2$.

CASE 3. *When both the factors are compound quantities, we have the following*

RULE.—MULTIPLY *every term of the multiplicand by every term of the multiplier, separately; setting down the products one after or under another, with their proper signs; and add the several lines of products all together for the whole product required.*

EXAMPLES.

	(1)	(2)
Multiply . . .	$2a+3b$	$6xy-2z$
By	$a+b$	$3ax-5d$
Product by a	$2a^2+3ab$	$18ax^2y-6axz-30dxy+10dz$
Product by b	$2ab+3b^2$	
Entire product	$2a^2+5ab+3b^2$	

3. Multiply $a+b+c$ by $x+y+x$, that is, repeat $a+b+c$, x times, then y times, then z times, and the operation stands thus:

	$a+b+c$
	$x+y+z$
Product by x	$ax+bx+cx$
Product by y	$ay+by+cy$
Product by z	$az+bz+cz$
Entire product	$ax+bx+cx+ay+by+cy+az+bz+cz.$

4. Multiply	$2x^2+xy-2y^2$
By	$3x-3y$
Partial product	$6x^3+3x^2y-6xy^2$
2d partial product	$-6x^2y-3xy^2+6y^3$
Whole product	$6x^3-3x^2y-9xy^2+6y^3$

5. Multiply $3a^2-2ab-b^2$ by $2a-4b$.
Prod. $6a^3-16a^2b+6ab^2+4b^3$.

6. Multiply x^2-xy+y^2 by $x+y$. . . . *Prod.* x^3+y^3.

7. Multiply $3a+4c$ by $2a-5c$. *Ans.* $6a^2-7ac-20c^2$.

8. Multiply a^2+ay-y^2 by $a-y$. *Ans.* $a^3-2ay^2+y^3$.

9. Multiply a^2+ay+y^2 by $a-y$. . . . *Ans.* a^3-y^3.

10. Multiply a^2-ay+y^2 by $a+y$. . . . *Ans.* a^3+y^3.

11. Multiply $a^3+a^2y+ay^2+y^3$ by $a-y$. . *Ans.* a^4-y^4.

12. Multiply y^2-y+1 by $y+1$. . . . *Ans.* y^3+1.

13. Multiply x^2+y^2 by x^2-y^2. . . . . *Ans.* x^4-y^4.

14. Multiply a^2-3a+8 by $a+3$. . . *Ans.* a^3-a+24.

15. Multiply $b^4+b^2x^2+x^4$ by b^2-x^2. . . *Ans.* b^6-x^6..

16. Multiply a^m+b^m by $a+b$.
Ans. $a^{m+1}+ab^m+a^mb+b^{m+1}$.

17. Multiply $x^6+x^4+x^2$ by x^2-1. . *Ans.* x^8-x^2.

18. Multiply $m+n$ by $9m-9n$. . . *Ans.* $9m^2-9n^2$.

19. Multiply y^2-20 by y^2+20. . . *Ans.* y^4-400.
20. Multiply $a+b$ by $a+b$. . . . *Ans.* $a^2+2ab+b^2$.
21. Multiply $x+y$ by $x+y$. . . . *Ans.* $x^2+2xy+y^2$.
22. Multiply $a-b$ by $a-b$. . . . *Ans.* $a^2-2ab+b^2$.
23. Multiply $x-y$ by $x-y$. . . . *Ans.* $x^2-2xy+y^2$.

(ART. 16.) When a number is multiplied by itself, the product is called its *square*, the square of one of the factors, and by inspecting the last four examples, we perceive that the square of any *binomial quantity*, (that is, the square of any two terms connected together by the sign *plus* or *minus*), the result must be THE SQUARES OF THE TWO PARTS, AND TWICE THE PRODUCT OF THE TWO PARTS.

N. B. The product of the two parts will be *plus* or *minus*, according to the sign between the terms of the binomial.

By this summary process perform the following examples:

1. Square $(3a+b)$ or multiply this quantity, by itself considered as a numeral quantity.* *Ans.* $9a^2+6ab+b^2$.

2. Square $2x-y$. *Ans.* $4x^2-4xy+y^2$.

We write the product, in the second place, in the answer, because it naturally falls there when the multiplication is formally made; but this is not essential.

Write out the following squares as indicated *by the exponent.*

$$(a-3c)^2=a^2-6ac+9c^2$$
$$(3a-c)^2=9a^2-6ac+c^2$$
$$(2x+3y)^2=4x^2+12xy+9y^2$$
$$(20x+y)^2=400x^2+40xy+y^2$$

* We make this last remark because things, arithmetically, cannot be multiplied by things. For instance, dollars cannot be multiplied by dollars, &c. In fact, every multiplier is always a number; and when we demand the square or any other power of a quantity, it always means the power of its *numeral* value considered *abstractly.*

(Art. 17.) The product of the sum and difference of two quantities is equal to the difference of their squares, *as will be seen by inspecting the following products:*

The first example should be multiplied in full to establish the principle.

What is the product of $(a+b)$ by $(a-b)$?

Ans. a^2-b^2.

What is the product of $2m+2n$ by $2m-2n$?

Ans. $4m^2-4n^2$.

What is the product of $x+y$ by $x-y$? *Ans.* x^2-y^2.

What is the product of $3x+3y$ by $3x-3y$?

Ans. $9x^2-9y^2$.

What is the product of $7a+b$ by $7a-b$?

Ans. $49a^2-b^2$.

What is the product of $1+10a$ by $1-10a$?

Ans. $1-100a^2$.

Observation.—By attention to principles much labor may be saved in the common operations of Algebra. For instance, if the product of three equal binomial factors were required, as $(x+3)(x+3)(x+3)$, we may first write out the product of two of those factors by (Art. 16); then multiply that product by the other factor.

$$\begin{array}{lr} \text{Thus,} \quad \ldots\ldots & x^2+6x+9 \\ & x+3 \\ \hline & x^3+6x^2+9x \\ & 3x^2+18x+27 \\ \hline \text{Product} \quad \ldots & x^3+9x^2+27x+27 \end{array}$$

If the product of the four factors, $(x-4)(x-5)(x+4)(x+5)$, were required, we would take the product of the *first* and *third* factors, then of the *second* and *fourth*, by (Art. 17), then the product of those two products would be the final product required.

Thus, the required product is the product of (x^2-16) by $(x^2-25)=x^4-41x^2+400$.

What is the product of $(a+c)(a+d)(a-c)(a-d)$?

Ans. $a^4-a^2c^2-a^2d^2+c^2d^2$.

DIVISION.

(Art. 18.) Division is the converse of multiplication, the product being called a dividend, and one of the factors a divisor. If a multiplied by b give the product ab, then ab divided by a must give b for a quotient, and if divided by b, give a. In short, if one simple quantity is to be divided by another simple quantity, the quotient must be found, by *inspection*, as in division of numbers.

EXAMPLES.

1. Divide $16ab$ by $4a$. . . . . **Ans.* $4b$.
2. Divide $21acd$ by $7c$. . . . . *Ans.* $3ad$.
3. Divide ab^2c by ac. . . . . *Ans.* b^2.
4. Divide $6abc$ by $2c$. . . . . *Ans.* $3ab$.
5. Divide ax^3 by ax^2. . . . . *Ans.* x.
6. Divide $3mx^6$ by mx. . . . . *Ans.* $3x^5$.
7. Divide $210c^3b$ by $7cb$. . . . . . . *Ans.* $30c^2$.
8. Divide $42xy$ by xy. . . . . *Ans.* 42.
9. Divide $3xy$ by ax. . . . . *Ans.* $\frac{3y}{a}$.

---

* The term quotient would be more exact and technical here; but, in results hereafter, we shall invariably use the term *Ans.*, as more brief and elegant, and it is equally well understood.

Remark.—In this last example we cast out the equal factor x from both the dividend and divisor, and set the other factor a of the divisor under the dividend as a denominator.

(Art. 19.) When the dividend and divisor have no factors in common, we can only *indicate the division* by setting the divisor under the dividend for a denominator, as in the following example:

Divide $3abc$ by $2xy$. . . . . *Ans.* $\frac{3abc}{2xy}$.

Divide $4axy$ by $3ay$. . . . . *Ans.* $\frac{4x}{3}$.

Divide $36aby$ by $4aby$. . . . . *Ans.* 9.

Divide $27aby$ by $11abx$. . . . . *Ans.* $\frac{27y}{11x}$.

Divide $72b^2x$ by $8abx$. . . . . *Ans.* $\frac{9b}{a}$.

(Art. 20.) In the preceding examples no signs were expressed, and, of course, every term and every factor is understood to be positive; but as algebraic quantities may have negative signs, and unlike signs, we must investigate and decide upon the sign to prefix to the quotient. This can be done by merely observing what sign must be put to the quotient so that the product of the divisor and quotient will give the same sign as in the dividend, according to the principles laid down in multiplication, (Art. 13).

For example, divide $-9y$ by $3y$, the quotient must be -3; so that $3y$ multiplied by -3 will give $-9y$, the dividend.

Divide $-9y$ by $-3y$, the quotient must be $+3$.
Divide $+9y$ by $-3y$, the quotient must be -3.

From these examples we draw the following rule for the signs:

Rule.—*When the dividend and divisor have* like signs, *both + or both —, then the quotient must be* plus.

When the dividend and divisor have unlike signs, the quotient must be MINUS.*

EXAMPLES.

Divide $-21ac$ by $-7a$. . . . . *Ans.* $+3c$.
Divide $-12xy$ by $+3y$. . . . . *Ans.* $-4x$.
Divide $72abc$ by $-8c$. . . . . *Ans.* $-9ab$.
Divide $144mn$ by $+8ac$. . . . . *Ans.* $\frac{18mn}{ac}$.

---

*Note.—We address this note to those only who are fond of the *metaphysique* of science. Division, considered in its most elementary sense, is not merely the *converse* of multiplication; it is a short process of finding how many times one quantity can be subtracted from another of the same kind. When the *subraction* is possible, and diminishes the numeral value of the minuend, and brings it nearer to zero, the operation is *real* and must be marked *plus.* When the *subtraction* is not possible without going farther from zero, we must take the converse operation, and the converse operation we must mark *minus.*

Thus, divide $18a$ by $6a$. Here, it is proposed to find how many times $6a$ can be *subtracted* from $18a$; and as we can *actually* subtract it 3 times, the quotient must be $+3$.

Divide $-18a$ by $-6a$. Here, again, the subtraction can *actually be performed,* and the number of times is 3, and, of course, the quotient is $+3$.

Divide $-18a$ by $6a$. Here, subtraction will not reduce the dividend to zero; but *addition* will, and must be performed 3 times; but the operation is the converse of the one proposed, and therefore must be marked by the converse sign to *plus,* that is -3.

Again, divide $18a$ by $-6a$. Here, if we *sub.* $-6a$ it will not reduce $18a$; but the converse operation will, and therefore the quotient must be minus, that is, -3.

Now let us inspect the common operation of division, by the help of the following example: Divide 24 by 8. Let the operation stand thus:

(ART. 21.) The product of a^3 into a^2 is a^5, (Art. 14), that is, in multiplication we add the exponents; and as division is the converse of multiplication, to divide powers of the *same letter*, we must *subtract the exponent of the divisor from that of the dividend.*

1. Divide $2a^6$ by a^4. . . . . *Ans.* $2a^2$.
2. Divide $—a^7$ by a^6. . . . . *Ans.* $—a$.
3. Divide $16x^3$ by $4x$. . . . . *Ans.* $4x^2$.
4. Divide $15axy^3$ by $—3ay$. . . . . . . *Ans.* $—5xy^2$.
5. Divide $63a^m$ by $7a^n$. . . . . . . *Ans.* $9a^{m-n}$.
6. Divide $12ax^n$ by $—3ax$. . . . . *Ans.* $—4x^{n-1}$.
7. Divide $28a^3y^4$ by $4acy^3$. . . . . . . *Ans.* $\frac{7a^2y}{c}$.

Divisor.		Divi.		Quotient.
8	)	2 4	(	3

The product of the divisor and quotient, in all cases, equals the dividend. Let d represent any divisor, D any dividend, and q the corresponding quotient, then

$$dq=D$$

Or . . . . $d=\frac{D}{q}$

In the above *numeral* example, let us suppose the divisor 8 to be —8, and the quotient —3. Then the dividend must be the product of $(—8)\times(—3)$; but suppose that we do not know whether this is *plus* or *minus*, we will therefore represent it by D.

Then . . . . . . $(—8)(—3)=D$

By dividing both members by either factor, as (—3), we have

$$—8=\frac{D}{—3} \text{ or } \frac{D}{—3}= —8$$

Here *D cannot be minus*, for *minus* divided by *minus* must give *plus* in the quotient; (as we have just determined in this note), but the quotient is *actually* (—8), therefore *D must be* plus. That is, the product of *minus* into *minus* gives *plus*; corresponding to (Art. 13).

8. Divide $-18a^3x$ by $-6ax$. . . . . *Ans.* $3a^2$.
9. Divide $6acdxy^2$ by $2adxy^2$. . . . *Ans.* $3c$.
10. Divide $45(a-x)^3$ by $15(a-x)^2$. . *Ans.* $3(a-x)$.

In this last example consider $(a-x)$ as one quantity.

11. Divide $45y^3$ by $15y^2$. . . . . . . *Ans.* $3y$.

Examples 10 and 11 are exactly alike, if we conceive $(a-x)$ equal to y.

12. Divide $12a^2x^2$ by $-3a^2x$. . . . . . *Ans.* $-4x$.
13. Divide $15ay^2$ by $3ay$. . . . . . . *Ans.* $-5y$.
14. Divide $-18ax^2y$ by $-8axz$. . . . *Ans.* $\frac{9xy}{4z}$.
15. Divide $7a^2b$ by $21x^3b^2$. . . . *Ans.* $\frac{7a^2b}{21a^3b^2}=\frac{1}{3ab}$.
16. Divide $-5a^2x^2$ by $-7a^4x^2$. . . . *Ans.* $\frac{5}{7a^2}$.
17. Divide $117a^5b^4c^3$ by $78a^5bc^4$. . . . *Ans.* $\frac{3b^3}{2c}$.
18. Divide z^5 by z^3. . . . . *Ans.* z^2.
19. Divide $(x-y)^5$ by $(x-y)^3$. . . . *Ans.* $(x-y)^2$.

Observe, that example 18 and 19 are essentially alike.

20. Divide $(a+b)^4$ by $(a+b)$. . . . *Ans.* $(a+b)^3$.
21. Divide $(a-c)^m$ by $(a-c)^n$. . . *Ans.* $(a-c)^{m-n}$.

To perform example 21 we adhere to the principle of performing 18, 19, and 20.

(Art. 22.) In the process of division, exponents may become negative, and it is the object of this article to explain their import.

To explain this, take a^4 and divide successively by a, forming the following series of quotients:

$$a^3, \quad a^2, \quad a, \quad 1, \quad \frac{1}{a}, \quad \frac{1}{a^2}, \quad \frac{1}{a^3}, \quad \&c.$$

5

Divide a^4 successively by a again, *rigidly adhering* to the principle that to divide any power of a by a, the exponent becomes *one less*, and we have

$$a^3, \quad a^2, \quad a^1, \quad a^0, \quad a^{-1}, \quad a^{-2}, \quad a^{-3}, \quad \&c.$$

Now these quotients must be equal, that is, a^3 in one series equals a^3 in another, and

$$a^2=a^2, \quad a=a^1, \quad 1=a^0, \quad \frac{1}{a}=a^{-1}, \quad \frac{1}{a^2}=a^{-2}, \quad \frac{1}{a^3}=a^{-3}.$$

Another illustration. We divide exponential quantities by subtracting the exponent of the divisor from the exponent of the dividend. Thus, a^5 divided by a^2 gives a quotient of $a^{5-2}=a^3$. a^5 divided by $a^7= a^{5-7}=a^{-2}$. We can also divide by taking the dividend for a numerator and the divisor for a denominator, thus, $\frac{a^5}{a^7}=\frac{1}{a^2}$, therefore, $\frac{1}{a^2}=a^{-2}$, (Axiom 7).

From this we learn, *that exponential terms may be changed from a numerator to a denominator, and the reverse, by changing the signs of the exponents.*

Thus, $\quad \frac{a}{x^2}=ax^{-2} \quad \frac{a^{-3}}{3y^{-1}}=\frac{y}{3a^3} \quad \frac{x^m}{x^n}=x^{m-n}$

Divide a^5bc by $a^6b^2c^{-1}$. . . . . *Ans.* $a^{-1}b^{-1}c^2$.

Observe, that to divide is to subtract the exponents.

Divide a^2x^3 by $a^4x^4y^2$. . . . *Ans.* $\frac{1}{a^2xy^2}$ or $a^{-2}x^{-1}y^{-2}$.

Divide $3ay^2$ by $5a^4x^2y^2$. . . . . *Ans.* $\frac{3}{5a^3x^2}$ or $\frac{3}{5}a^{-3}x^{-2}$.

(Art. 23.) When the dividend is a compound quantity, and the divisor a simple (or single) quantity, we have the following rule, the reason of which will be obvious if the preceding part of division has been comprehended.

Rule.—*Divide each term of the dividend by the divisor, and the several results connected together by their proper signs will be the quotient sought.*

EXAMPLES.

1. Divide $15ab-12ax$ by $3a$. . . . . . *Ans.* $5b-4x$.
2. Divide $-25a^2x+15ax^2$ by $-5ax$. . . *Ans.* $5a-3x$.
3. Divide $10ab+15ac$ by $5a$. . . . . . *Ans.* $2b+3c$.
4. Divide $30ax-54x$ by $6x$. . . . . . *Ans.* $5a-9$.
5. Divide $8x^3+12x^2$ by $4x^2$. . . . . . *Ans.* $2x+3$.
6. Divide $3bcd+12bcx-9b^2c$ by $3bc$. . *Ans.* $d+4x-3b$.
7. Divide $7ax+3ay-7bd$ by $-7ad$.

   Ans. $-\frac{x}{d}-\frac{3y}{7d}+\frac{b}{a}$.
8. Divide $3ax^3+6x^2+3ax-15x$ by $3x$.

   Ans. $ax^2+2x+a-5$.
9. Divide $3abc+12abx-3a^2b$ by $3ab$. *Ans.* $c+4x-a$.
10. Divide $25a^2bx-15a^2cx^2+5abc$ by $-5ax$.

    Ans. $-5ab+3acx-bcx^{-1}$.
11. Divide $20ab^3+15ab^2+10ab+5a$ by $5a$.

    Ans. $4b^3+3b^2+2b+1$.

(Art. 24.) We now come to the last and most important operation in division, the division of one compound quantity by another compound quantity.

The dividend may be considered a *product* of the divisor into the yet unknown factor, the quotient; and the highest power of any letter in the product, or the now called dividend, must be conceived to have been formed by the highest power of the same letter in the divisor into the highest power of that letter in the quotient. *Therefore, both the divisor and the dividend must be arranged according to the regular powers of some letter.*

After this, the truth of the following rule will become obvious by its great similarity to division in numbers.

Rule.—*Divide the first term of the dividend by the first term of the divisor, and set the result in the quotient.**

---

* Divide the *first term* of the dividend and of the remainders by the *first term* of the divisor; be not troubled about other terms.

Multiply the whole divisor by the quotient thus found, and subtract the product from the dividend.

The remainder will form a new dividend, with which proceed as before, till the first term of the divisor is no longer contained in the first term of the remainder.

The divisor and remainder, if there be a remainder, are then to be written in the form of a fraction, as in division of numbers.

EXAMPLES.

Divide $a^2+2ab+b^2$ by $a+b$.

Here, a is the leading letter, standing first in both dividend and divisor: hence no change of place is necessary.

OPERATION.

$$\begin{array}{r} a+b)a^2+2ab+b^2(a+b \\ \underline{a^2+\ ab\quad\quad} \\ ab+b^2 \\ \underline{ab+b^2} \\ . \end{array}$$

That the pupil may perceive the close connection between *multiplication* and *division*, we

$$\begin{array}{lll} \text{Multiply} & a^2+2ab+4b^2 & \\ \text{By} & \underline{2a^2-2ab+b^2} & \\ & 2a^4+4a^3b+8a^2b^2 & (1) \\ & \quad -2a^3b-4a^2b^2-8ab^3 & (2) \\ & \quad\quad \underline{+a^2b^2+2ab^3+4b^4} & (3) \\ \text{Prod. is} & 2a^4+2a^3b+5a^2b^2-6ab^3+4b^4 & \end{array}$$

Now take this product for a dividend, and one of the factors, $(a^2+2ab+4b^2)$, for a divisor, and of course the other factor, $(2a^2-2ab+b^2)$, will be the quotient, and the operation will stand thus:

$$
\begin{array}{l}
a^2+2ab+4b^2)2a^4+2a^3b+5a^2b^2-6ab^3+4b^4(2a^2-2ab+b^2 \\
\qquad\qquad \underline{2a^4+4a^3b+8a^2b^2} \qquad\qquad (1) \\
\qquad\qquad\quad -2a^3b-3a^2b^2-6ab^3 \\
\qquad\qquad\quad \underline{-2a^3b-4a^2b^2-8ab^3} \qquad (2) \\
\qquad\qquad\qquad\qquad a^2b^2+2ab^3+4b^4 \\
\qquad\qquad\qquad\qquad \underline{a^2b^2+2ab^3+4b^4} \qquad (3)
\end{array}
$$

The several partial products which make up the dividend, and marked (1), (2), (3), are again found in the operation of division, and there marked (1), (2), (3), the same as in Arithmetic.

Some operators put the divisor on the right* of the dividend, as in the following example:

Divide a^3-b^3 by $a-b$.

$$
\begin{array}{l}
a^3-b^3 \quad \Big\} \; \underline{a-b} \\
\underline{a^3-a^2b} \quad \; a^2+ab+b^2, \text{ Quo.} \\
\quad a^2b-b^3 \\
\quad \underline{a^2b-ab^2} \\
\qquad ab^2-b^3 \\
\qquad \underline{ab^2-b^3}
\end{array}
$$

GENERAL EXAMPLES.

1. Divide $a^2+2ax+x^2$ by $a+x$. *Ans.* $a+x$.
2. Divide $a^3-3a^2y+3ay^2-y^3$ by $a-y$.
   Ans. $a^2-2ay+y^2$.
3. Divide $24a^2b-12a^3cb^2-6ab$ by $-6ab$.
   Ans. $-4a+2a^2cb+1$.
4. Divide $a^3+5a^2b+5ab^2+b^3$ by $a+b$.
   Ans. $a^2+4ab+b^2$.

*** NOTE.—This is in imitation of the French, and being a mere matter of taste, involving *no principle*, we have no right to find fault with those who adopt it; and others must not complain of us because we prefer the English custom.**

5. Divide $a^3+2a^2b+2ab^2+b^3$ by a^2+ab+b^2. *Ans.* $a+b$.

6. Divide $x^3-9x^2+27x-27$ by $x-3$. *Ans.* x^2-6x+9.

7. Divide $6x^4-96$ by $6x-12$. *Ans.* x^3+2x^2+4x+8.

(ART. 25.) When a factor appears in every term of both dividend and divisor, it may be cast out of every term without affecting the quotient; thus, in the last example, the factor 6 may be cast out by division; and x^4-16 divided by $x-2$ will give the same quotient as before.

8. Divide $6a^4+9a^2-15a$ by $3a^2-3a$.

Ans. $2a^2+2a+5$.

(Observe Art. 25).

9. Divide $25x^5-x^3-2x^2-8x$ by $5x^2-4x$.

Ans. $5x^3+4x^2+3x+2$.

10. Divide $18a^2-8b^2$ by $6a+4b$. *Ans.* $3a-2b$.

11. Divide $2x^3-19x^2+26x-16$ by $x-8$.

Ans. $2x^2-3x+2$.

12. Divide y^5+1 by $y+1$. *Ans.* $y^4-y^3+y^2-y+1$.

13. Divide y^6-1 by $y-1$. *Ans.* $y^5+y^4+y^3+y^2+y+1$.

14. Divide x^2-a^2 by $x-a$. *Ans.* $x+a$.

15. Divide $6a^3-3a^2b-2a+b$ by $3a^2-1$. *Ans.* $2a-b$.

16. Divide $y^6-3y^4x^2+3y^2x^4-x^6$ by $y^3-3y^2x+3yx^2-x^3$.

Ans. $y^3+3y^2x+3yx^2+x^3$.

17. Divide $64a^4b^6-25a^2b^8$ by $8a^2b^3+5ab^4$.

Ans. $8a^2b^3-5ab^4$

18. Divide $2a^4-2x^4$ by $a-x$.

Ans. $2a^3+2a^2x+2ax^2+2x^3$.

19. Divide $(a-x)^5$ by $(a-x)^2$. *Ans.* $(a-x)^3$.

20. Divide $a^3-3a^2x+3ax^2-x^3$ by $a-x$.

Ans. $a^2-2ax+x^2$.

21. Divide a^5+1 by $a+1$. *Ans.* $a^4-a^3+a^2-a+1$.

22. Divide b^6-1 by $b-1$. *Ans.* $b^5+b^4+b^3+b^2+b+1$.

23. Divide $48a^3-92a^2x-40ax^2+100x^3$ by $3a-5x$.

Ans. $16a^2-4ax-20x^2$.

24. Divide $4d^4-9d^2+6d-1$ by $2d^2+3d-1$.

Ans. $2d^2-3d+1$.

25. Divide $10ab+15ac$ by $2b+3c$.

26. Divide $30ax-54x$ by $5a-9$.

27. Divide $8x^2+12x^2$ by $2x+3$.

28. Divide $-25a^2x+15ax^2$ by $5a-3x$.

Observe that these last four examples are the same as some in (Art. 23.)

If more examples are desired for practice, the examples in multiplication may be taken. The product or answer may be taken for a dividend, and either one of the factors for a divisor; the other will be a quotient.

Also, the examples in division may be changed to examples in multiplication; and these changes will serve to impress on the mind of the pupil the close connection between these two operations.

(Art. 26.) The operation of division is the art of finding one of two factors of a product, when the product itself and one factor is given. When the product only is presented, and its factors required, the operation is properly called

FACTORING.

Factors of a number are such numbers as may be multiplied together to produce the number; and *factors* of an algebraic expression are such quantities as being multiplied together will produce the expression. Thus, 2 and 3 are the factors of 6, because $2\times3=6$, and 3, a, and c, are factors of $3ac$, because by their multiplication they form that product.

But some numbers *have no factors*, (except 1 and the number these should not be considered factors), and such numbers are called *prime numbers.*

Also, some algebraic expressions *have no factors,* and such expressions are called *prime quantities.* Thus, $5a+c$ is a prime quantity.

The following is a list of the prime numbers up to 100:

1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

All the other intermediate numbers are *composite* numbers. Any number whatever is either a prime number, or *composed* of the product of *prime factors.*

Knowing this fact enables us to decompose any number into its prime factors by the following rule:

RULE.—*Divide the given number by any prime number that will divide it without a remainder, and divide that quotient again by any prime number that will exactly divide it, and so continue until the last quotient is a prime number. The divisors and last quotient are the factors required.*

N. B. Use the smallest prime divisors first.

EXAMPLES.

1. Required the factors composing 102. *Ans.* 2, 3, 17.

OPERATION.

$$\begin{array}{r} 2\,)\,102 \\ \hline 3\,)\,51 \\ \hline 17 \end{array}$$

2. Find the prime factors in the number 112.
Ans. 2, 2, 2, 2, 7.

3. Find the prime factors in the number 126.
Ans. 2, 3, 3, 7.

4. Find the prime factors in the number 12769.
Ans. 113, 113.

5. Find the prime factors in the number 1156.

Ans. 2, 2, 17, 17.

6. Find the prime factors in the number 1014.

Ans. 2, 3, 13, 13.

(Art. 27.) As the combination of numbers is endless, it is impossible to give any definite rule that will decide in each and every case whether a number is a prime or a composite number; and as the practical utility of factoring is limited, it is proper to confine our investigations to small numbers, and from observations on numbers, we deduce the following principles for finding these factors:

1st. That any number ending with an even number, or a cipher, can be divided by 2.

2d. Any number ending with 5 or 0, is divisible by 5.

3d. If the right hand place of any number be 0, the whole is divisible by 10; if there be two ciphers, it is divisible by 100; if three ciphers, by 1000, and so on, which is only cutting off those ciphers.

4th. If the two right hand figures of any number be divisible by 4, the whole is divisible by 4; and if the three right hand figures be divisible by 8, the whole is divisible by 8, and so on.

5th. If the sum of the digits in any number be divisible by 3 or by 9, the whole is divisible by 3 or by 9.

6th. If the right hand digit be even, and the sum of all the digits be divisible by 6, then the whole is divisible by 6.

7th. A number is divisible by 11, when the sum of the 1st, 3d, 5th, &c., or all the odd places, is equal to the sum of the 2d, 4th, 6th, &c., or of all the even places of digits.

8th. If a number cannot be divided by some quantity less than the square root of the same, that number is a prime, or cannot be divided by any number whatever.

9th. All prime numbers, except 2 and 5, have either 1, 3, 7, or 9, in the place of units; and all other numbers are composite numbers, and can be divided.

(Art. 28.) The *multiple* of a number is some *exact* number of times that number. Thus, 6 is a multiple of 2, 3; 12 is also a *multiple* of 2, 3; but not so small a *multiple* as 6 is, therefore 6 is the *least common multiple* of 2 and 3.

The least common multiple of several numbers is the least number that is divisible by these numbers without a remainder.

A COMMON MULTIPLE IS FOUND BY MEANS OF PRIME FACTORS.

For example, find the least common multiple of the numbers 24, 20, and 15.

That is, find the least number which is divisible by 24, 20, and 15. First find the *prime* factors to these numbers, (Art. 27), (2, 2, 2, 3,) (2, 2, 5,) (3, 5).

That the number required may be divisible by the first number 24, it must have all the factors in that number; that is, 2, 2, 2, 3; and to be divisible by the second number, 20, it must contain the factor 5; putting in this factor we have 2, 2, 2, 3, 5. This number is divisible also by 15, because it contains the factors 3, 5. The least common multiple required is, therefore, 120.

The least common multiple of the numbers 3, 7, 19, is their product, because the numbers are prime, and there is no common *factor* that can be cast out.

On these principles the following rule for finding the common multiple will be easily comprehended:

RULE.—*Write the numbers one after the other, and draw a line beneath them; then, take any prime number which will divide two or more of them without remainder, and divide all the numbers that will so divide—writing the quotients beneath, and all the numbers that are not divisible by it. Find a prime number that will divide two or more numbers in this second line, and proceed as before. Continue the operation until there are no two* *numbers left having a common divisor: then, multiply all the divisors and remaining numbers together, and their product will be the least common multiple sought.*

EXAMPLES.

1. Let it be required to find the least common multiple of 12, 15, 7, 18, 3, 5, and 35.

7	12,	15,	7,	18,	3,	5,	35,
5	12,	15,	1,	18,	3,	5,	5,
3	12,	3,	1,	18,	3,	1,	1,
2	4,	1,	1,	6,	1,	1,	1,
	2,	1,	1,	3,	1,	1,	1,

$$7\times5\times3\times3\times2\times2=1260.$$

2. Find the least number that can be divided by 9, 12, 16, 24, 36, without remainders. *Ans.* 144.

3. Find the least number that is divisible by each of the nine digits. *Ans.* 2520.

4. Find the least number divisible by 75, 50, 15, 20, 30, and 45. *Ans.* 900.

(ART. 29.) A *prime quantity* in Algebra, like a *prime number*, is divisible only by itself and unity. Thus, a, b, $a+b$, are prime quantities; and ab, and $ab+ac$, are composite quantities, the first is composed of the factors a, b, and the other of the factors a, and $(b+c)$.

The prime factors of a purely algebraic quantity consisting of a single term, *are visible to the eye*, and this is one of the principal advantages of an algebraic expression.

Thus, in the expression $abcx$, we perceive at once the *prime* factors, a b, c, and x; the expression a^3b^2x has *three* prime factors, each equal to a, *two* prime factors equal to b, and one equal to x.

(ART. 30.) When the algebraic expression is a *polynomial*, and has prime factors that are *monomials*, such monomial factors are visible, as in the following expressions:

		Factors.
1.	$x+ax$	$(1+a)x$
2.	$am+an+ax$	$(m+n+x)a$
3.	$bc^2+bcx+bcy$	$(c+x+y)bc$
4.	$4x^2+6xy$	$(2x+3y)2x$

Thus in the first expression, x is visible in every term, it is, therefore, a common factor to every term, and $(1+a)$ is the other factor, and the product of these two factors makes the expression; and so for the other expressions.

The examples in division (Art. 23), are analagous to these, except that in that article the divisor *is given*, and may not be contained in every term, as in example 7, (Art. 23).

(ART. 31.) When all the *prime* factors composing any algebraic expression consist of *binomials* or *polynomials*, they are not visible in the expression like a *monomial*, and we can find them only from our general knowledge of algebraic expressions.

For instance, the prime factors in the expression $(a^2+2ab+b^2)$ we know to be $(a+b)$ and $(a+b)$ by (Art. 16), and all other expressions that correspond to a binomial squared, is immediately recognized after a little experience in algebraic operations.

Also, any expression which is the difference of two squares, as (a^2-b^2) is instantly recognized as the product of the two prime factors, $(a+b)$, and $(a-b)$, (Art. 17).

The expression $ax+ay+bx+by$ can be resolved into two prime factors, by inspection, thus, $a(x+y)+b(x+y)$ is merely a change in the form of the expression. Now put $(x+y)=S$. Then the next change is $aS+bS$; the next is $(a+b)S$. Restoring the value of S, we have $(a+b)(x+y)$ for the prime factors in the original expression.

(Art. 32.) Any trinomial expression in the form of ax^2+bx+c, can be resolved into two binomial factors; but the art of finding the factors is neither more nor less than resolving an equation of the second degree, a subject of great importance and some difficulty, which will be examined very closely in a subsequent part of this work; therefore it is improper to treat upon this subject at present. (See Art. 95).

(Art. 33.) *Common multiple,* and least common multiple, have the same signification in Algebra as in Arithmetic, and are found by the same rule, except changing the words *number* and *numbers* in the rule for *quantity* and *quantities.*

Or, we may take the following rule to find the *least common multiple* in algebraic quantities.

Rule.—1. *Resolve the numbers into their prime factors.*

2. *Select all the different factors which occur, observing, when the same factor has different powers, to take the highest power.*

3. *Multiply together the factors thus selected, and their product will be the least common multiple.*

EXAMPLES.

1. Find the least common multiple of $8a^2x^2y$, and $12a^3b^3x$.

Resolving them into their prime factors,

$$8a^2x^2y=2^3\times a^2\times x^2\times y$$
$$12a^3b^3x=2^2\times a^3\times x\times b^3\times 3$$

The *different* factors are 2^3, a^3, x^2, y, b^3, 3, and their product is $24a^3b^3x^2y$, which is the least common multiple required.

2. Required the least common multiple of $27a$, $15b$, $9ab$, and $3a^2$. *Ans.* $135a^2b$.

3. Find the least common multiple of (a^2-x^2), $4(a-x)$, $(a+x)$. *Ans.* $4(a^2-x^2)$.

4. Required the least common multiple of $a^2(a-x)$, and $ax^4(a^2-x^2)$. *Ans.* $a^2x^4(a^2-x^2)$.

5. Required the least common multiple of $x^2(x-y)$, a^4x^2, and $12axy^2$. *Ans.* $12a^4x^2y^2(x-y)$.

6. Required the least common multiple of $10a^2x^2(a-b)$, $15x^5(a+b)$, and $12(a^2-b^2)$. *Ans.* $60a^2x^5(a^2-b^2)$.

ALGEBRAIC FRACTIONS.

THE nature of fractions is the same, whether in Arithmetic or Algebra, and of course those who understand fractions in Arithmetic, can have no difficulty with the same subject in Algebra.

(ART. 33.) *A fraction is one quantity divided by another when the division is indicated and not actually performed.*

Hence every fraction consists of two parts, the *dividend* and *divisor*, which take the name of *numerator* and *denominator*.

The numerator is written above a line, and the denominator below it, thus, $\frac{a}{b}$, and is read, a divided by b.

For illustration, we may consider any simple fraction as $\frac{3}{5}$; here we consider *one* or unity divided into 5 parts, and 3 of these parts are taken. The 5 *denotes* the parts that the *unit* is divided into, hence it is properly named *denominator*, and the 3, numbers the parts taken, and is, therefore, properly called the *numerator*. So in the fraction $\frac{a}{b}$, b denotes the parts into which unity is divided, and a shows the number of parts taken.

In a numeral fraction, as $\frac{2}{3}$, it is evident that if we double both numerator and denominator, we do not change the value of the fraction; thus, $\frac{4}{6}$ is the same part of the whole unit as

$\frac{2}{3}$, and thus it would be if we multiplied by any other number; and conversely, we may divide both numerator and denominator by the same number, without changing the value of the fraction. Hence, if any fraction contains any factors common to both numerator and denominator, we may suppress them by division, and thus reduce the terms of the fractions to smaller quantities.

Hence, to reduce fractions to lower terms when possible, we have the following rule:

RULE.—*Divide both terms by their greatest common divisor.* Or, *resolve the numerator and denominator into their prime factors, and then cancel those factors common to both terms.*

EXAMPLES.

1. Reduce $\frac{14a^2b^3c}{21abc^2}$ to its lowest terms.

Here $7abc$ is the common divisor, and dividing according to the rule, gives $\frac{2ab^2}{3c}$, the fraction reduced.

2. Reduce $\frac{33a^2x^2}{55a^2x^3}$. . . . . *Ans.* $\frac{3}{5x}$.

3. Reduce $\frac{12ax}{18ab}$ to its lowest terms. . . . . *Ans.* $\frac{2x}{3b}$.

4. Reduce $\frac{14a^2x^2y}{21ax^2}$ to its lowest terms. . . . *Ans.* $\frac{2ay}{3}$.

5. Reduce $\frac{116a^5x^2y}{68a^3xy^2}$ to its lowest terms. . *Ans.* $\frac{29a^2x}{17y}$.

6. Reduce $\frac{51a^3b-63a^2b^2}{36a^4b^2-9ab}$ to its lowest terms.

Ans. $\frac{17a^2-21ab}{12a^3b-3}$.

7. Reduce $\frac{4a^2-4x^2}{3(a+x)}$ to its lowest terms. *Ans.* $\frac{4(a-x)}{3}$.

8. Reduce $\frac{x^5-b^2x^3}{x^4-b^4}$ to its lowest terms. . *Ans.* $\frac{x^3}{x^2+b^2}$.

9. Reduce $\frac{x^2-1}{xy+y}$ to its lowest terms. . . *Ans.* $\frac{x-1}{y}$.

10. Reduce $\frac{cx+cx^2}{acx+abx}$ to its lowest terms. *Ans.* $\frac{c+cx}{ac+ab}$.

11. Divide $x^3y^2+x^2y^3$ by ax^2y+axy^2. . . . *Ans.* $\frac{xy}{a}$.

12. Divide $4a+4b$ by $2a^2-2b^2$. . . . . *Ans.* $\frac{2}{a-b}$.

13. Divide n^3-2n^2 by n^2-4n+4. . . . *Ans.* $\frac{n^2}{n-2}$.

(Art. 34.) Fractions in Algebra, as in Arithmetic, may be simple or complex, proper or improper, and the same definitions to these terms should be given, as well as the same rules of operation; for in fact this part of Algebra is but a generalization of Arithmetic, and in some cases we give arithmetical and algebraical examples side by side.

A mixed quantity in Algebra is an integer quantity and a fraction; and to reduce these to improper fractions, we have the following rule:

Rule.—*Multiply the integer by the denominator of the fraction, and to the product add the numerator, or connect it with its proper sign + or —; then the denominator being set under this sum, will give the improper fraction required.*

EXAMPLES.

1. Reduce $2\frac{3}{8}$ and $a+\frac{x}{b}$ to improper fractions.

Ans. $\frac{19}{8}$ and $\frac{ab+x}{b}$.

These two operations, and the principle that governs them, are exactly alike.

2. Reduce $5\frac{7}{8}$ and $a+\frac{a^2}{b}$ to improper fractions.

Ans. $\frac{47}{8}$ and $\frac{ab+a^2}{b}$.

3. Reduce $7\frac{1}{7}$ and $ax+\frac{b}{c}$ to improper fractions.

Ans. $\frac{50}{7}$ and $\frac{acx+b}{c}$.

4. Reduce $3-\frac{1}{2}$ and $x^2-\frac{x}{y}$ to improper fractions.

Ans. $\frac{5}{2}$ and $\frac{x^2y-x}{y}$.

5. Reduce $y-1+\frac{1-y}{1+y}$ to a fractional form. *Ans.* $\frac{y^2-y}{y+1}$.

6. Reduce $x+y+\frac{a}{x+y}$ to the form of a fraction.

Ans. $\frac{x^2+2xy+y^2+a}{x+y}$.

7. Reduce $4+2x+\frac{b}{c}$ to an improper fraction.

8. Reduce $5x-\frac{2x+5}{3}$ to an improper fraction.

9. Reduce $3a-9-\frac{3a^2-30}{a+3}$ to a simple fraction.

Ans. $\left(\frac{3}{a+3}.\right)$

The converse of this operation must be true, and, therefore, to reduce an improper fraction to a mixed quantity, we have the following

RULE.—*Divide the numerator by the denominator, as far as possible, and set the remainder, (if any), over the denominator for the fractional part; the two joined together with their proper sign, will be the mixed quantity sought.*

EXAMPLES.

1. Reduce $\frac{47}{8}$ and $\frac{ab+x}{b}$ to mixed quantities.

Ans. $5\frac{7}{8}$ and $a+\frac{x}{b}$.

2. Reduce $\frac{19}{8}$ and $\frac{a^2+bx}{a}$ to mixed quantities.

Ans. $2\frac{3}{8}$ and $a+\frac{bx}{a}$.

3. Reduce $\frac{5ay+ab+x}{y}$ to mixed quantities.

Ans. $5a+\frac{ab+x}{y}$.

4. Reduce $\frac{2a^2-2b^2}{a-b}$ to a whole or mixed quantity.

Ans. $2a+2b$.

5. Reduce $\frac{15a^3+2x}{5a^2}$ to a mixed quantity. *Ans.* $3a+\frac{2x}{5a^2}$.

6. Reduce $\frac{a^2+ab+b^2}{a}$ to a mixed quantity.

Ans. $a+b+\frac{b^2}{a}$.

7. Reduce $\frac{12a^2+4a-3c}{4a}$ to a mixed quantity.

Ans. $3a+1-\frac{3c}{4a}$.

(Art. 35.) A fraction is an expression for *unperformed* division. Thus, 2 divided by 5, is written $\frac{2}{5}$. The double of this is $\frac{4}{5}$, 3 times $\frac{2}{5}$ is $\frac{6}{5}$, &c. That is, to multiply a fraction by any number, we *multiply the numerator of the fraction by the number, without changing the denominator.*

The nature of division is the same, whatever numbers represent the dividend and divisor. Hence, for the sake of simplicity, let us consider the result of dividing 24 by 6. Here 24 is the dividend and 6 the divisor, and the division expressed and unperformed, must be written $\frac{24}{6}$, and the *value* of this expression, *or quotient,* is 4. Now observe, that we can double the quotient by doubling 24, or by taking the half of 6. We can find 3 times the value of this quotient, by multiplying the numerator 24 by 3, *or by dividing* the denominator 6 by 3.

Hence, to multiply a fraction by a whole number, we have the following rule:

RULE.—*Multiply the numerator by the whole number; or, when you can, divide the denominator by the whole number.*

EXAMPLES.

1. Multiply $\frac{3}{7}$ by 5. . . . . *Ans.* $\frac{15}{7}=2\frac{1}{7}$.
2. Multiply $\frac{4}{9}$ by 3. . . . . *Ans.* $\frac{4}{3}=1\frac{1}{3}$.
3. Multiply $\frac{17}{19}$ by 4. . . . . *Ans.* $\frac{68}{19}=3\frac{11}{19}$.
4. Multiply $\frac{9}{14}$ by 100. . . . . . . *Ans.* $\frac{900}{14}=64\frac{2}{7}$.
5. Multiply $\frac{11}{27}$ by 18. . . . . *Ans.* $7\frac{1}{3}$.
6. Multiply $\frac{13}{47}$ by 19. . . . . *Ans.* $5\frac{12}{47}$.
7. Multiply $\frac{7}{3}$ by 24. . . . . *Ans.* 56.
8. Multiply $\frac{17}{21}$ by 105. . . . . *Ans.* 85.
9. Multiply $\frac{3}{7}$ by 63. . . . . *Ans.* 27.
10. Multiply $\frac{a}{b}$ by c. . . . . *Ans.* $\frac{ac}{b}$.

(ART. 36.) When we multiply a fraction by its denominator, we merely suppress the denominator. Thus, multiply $\frac{1}{3}$ by 3, the result is 1, the numerator of the fraction; multiply $\frac{2}{5}$ by 5, and we have 2, the *numerator* for the product.

EXAMPLES.

1. Multiply $\frac{3}{7}$ by 7. . . . . *Ans.* 3.
2. Multiply $\frac{a}{b}$ by b. . . . . *Ans.* a.
3. Multiply $\frac{4}{11}$ by 11. . . . . *Ans.* 4.
4. Multiply $\frac{x}{11}$ by 11. . . . . *Ans.* x.
5. Multiply $\frac{3ax}{5b}$ by $5b$. . . . . *Ans.* $3ax$.
6. Multiply $\frac{3x}{7}$ by 7. . . . . *Ans.* $3x$.
7. Multiply $\frac{3a-x}{10}$ by 20. . . . . . *Ans.* $6a-2x$.

8. Multiply $\frac{21ax-b}{3bx}$ by $6bx$. . . . . *Ans.* $42ax-2b$.

9. Multiply $\frac{x}{2}+\frac{x}{3}$ by 6. . . . . *Ans.* $3x+2x$.

10. Multiply $\frac{2x}{3}+\frac{3x}{2}$ by 3. . . . . . . *Ans.* $2x+\frac{9x}{2}$.

11. Multiply $3\frac{1}{3}$ by 3. . . . . *Ans.* 10.

(ART. 37.) As a fraction is an expression for unperformed division, we may express the division of $3\frac{1}{2}$ by $5\frac{2}{3}$, in the following form:

$$\frac{3\frac{1}{2}}{5\frac{2}{3}}$$

But this is certainly a *complex fraction*; so are $\frac{3\frac{1}{2}}{7}$ and $\frac{5}{8\frac{1}{4}}$ complex fractions; hence complex fractions may be defined thus:

A complex fraction is one in which the numerator or denominator, or both, are fractions or mixed quantities.

To simplify a complex fraction, we multiply both numerator and denominator by the denominators of the fractional parts: or by their product, or by their least common multiple.

For example, let us simplify the fraction $\frac{3\frac{1}{2}}{5\frac{2}{3}}$. If we multiply both numerator and denominator by 2, the numerator will contain no fraction, and the result will be $\frac{7}{10\frac{4}{3}}$. Multiply numerator and denominator of this fraction by 3, and the denominator will contain no fraction; and the final result will be $\frac{21}{34}$, a simple fraction, equal in value to the complex fraction.

But we could have arrived at this result at once, by multiplying both terms by 6, the product of $2 \cdot 3$. Hence, the rule just given.

EXAMPLES.

1. Reduce $\dfrac{2\frac{1}{2}}{4\frac{1}{7}}$ to a simple fraction. . . . . . *Ans.* $\frac{35}{58}$.

2. Reduce $\dfrac{5}{3\frac{1}{8}}$ to a simple fraction. . . . . *Ans.* $\frac{8}{5}=1\frac{3}{5}$.

3. Reduce $\dfrac{a+\frac{m}{n}}{b-\frac{c}{d}}$ to a simple fraction. . *Ans.* $\dfrac{nad+md}{nbd-cn}$.

4. Reduce $\dfrac{a+\frac{a}{3}}{b}$ to a simple fraction. . . . *Ans.* $\dfrac{4a}{3b}$.

5. Reduce $\dfrac{\frac{1}{4}a+c}{x+\frac{y}{2}}$ to a simple fraction. . . *Ans.* $\dfrac{a+4c}{4x+2y}$.

6. Divide $\dfrac{a}{b}$ by $\dfrac{c}{d}$, that is, simplify the complex fraction $\dfrac{\frac{a}{b}}{\frac{c}{d}}$.

Here the division is expressed, but *unperformed,* and by the rule to simplify the fraction, we find its value to be $\dfrac{ad}{bc}$.

From this result we can draw a rule for dividing one fraction by another; and the rule here indicated, when expressed in words, is the rule commonly found in Arithmetic.

7. Simplify the fraction $\dfrac{m}{a+\frac{1}{c}}$ . . . . . . *Ans.* $\dfrac{cm}{ac+1}$.

8. Simplify the fraction $\dfrac{a}{1+\frac{m}{n}}$. . . . . . *Ans.* $\dfrac{na}{n+m}$.

MULTIPLICATION OF FRACTIONS.

(Art. 38.) We have already given a rule to multiply a fraction by a whole number; (Art. 35); but when two fractions are multiplied together, the result is the same, whichever we consider the multiplier. That is, $\frac{3}{5}$ multiplied by 4, and 4 multiplied by $\frac{3}{5}$, is the same product. Also, $\frac{a}{b}$ multiplied by x is $\frac{ax}{b}$, therefore, x multiplied by $\frac{a}{b}$ is also $\frac{ax}{b}$. Hence to multiply a quantity by a fraction, observe the following rule:

Rule.—*Multiply the quantity by the numerator of the fraction, and set the denominator under the result.*

EXAMPLES.

1. Multiply 7 by $\frac{2}{3}$. . . . . *Ans.* $\frac{14}{3}$.
2. Multiply a by $\frac{x}{y}$. . . . . *Ans.* $\frac{ax}{y}$.
3. Multiply 5 by $\frac{4}{7}$. . . . . *Ans.* $\frac{20}{7}$.

Now, in this last example write *one* under the 5, which will give it a fractional form without changing its value. Then it will be $\frac{5}{1}\times\frac{4}{7}$; and if we multiply the *numerators* together, and the *denominators* together, we have $\frac{20}{7}$, as before. Again, we may take $\frac{5}{1}$ and multiply both numerator and denominator by any number, say 3, and we have $\frac{15}{3}$, which is really 5 as at first. We have now to multiply $\frac{15}{3}$ by $\frac{4}{7}$, and if we multiply *numerators* and *denominators* as before, we shall have $\frac{60}{21}$ for the product, which is in value $\frac{20}{7}$, as it ought to be.

4. Multiply a by $\frac{c}{d}$. . . . . *Prod.* $\frac{ac}{d}$.

5. Multiply $\frac{a}{1}$ by $\frac{c}{d}$. . . . . *Prod.* $\frac{ac}{d}$.

As $a=\frac{na}{n}$ we can thus change the form of the first factor without changing its value, then the example will be to

Multiply $\frac{na}{n}$ by $\frac{c}{d}$. . . . . *Prod.* $\frac{nac}{nd}=\frac{ac}{d}$.

From these examples we have the following rule for multiplying fractions together:

RULE.—*Multiply the numerators together for a new numerator, and the denominators, for a new denominator.*

N. B. Equal factors in numerators and denominators may be canceled out, which will save the reduction of the product to lower terms.

To find such equal factors, separate the quantities into their prime factors (Art. 27), before multiplication.

EXAMPLES.

1. Multiply $\frac{3a}{4x}$ by $\frac{5x}{8}$. . . . . *Ans.* $\frac{15a}{32}$.

2. Multiply $\frac{3a}{5y}$ by $\frac{3y}{9x}$. . . . . *Ans.* $\frac{a}{5x}$.

3. Multiply $\frac{3x^2}{10y}$ by $\frac{5y}{9x}$. . . . . *Ans.* $\frac{x}{6}$.

4. Multiply $\frac{a-b}{5}$ by $\frac{25x-25}{a^2-b^2}$ by $\frac{1}{x-1}$. . . . *Ans.* $\frac{5}{a+b}$.

In this example we separate the second fraction into its prime factors, (Art. 27), and the operation stands thus:

$$\frac{a-b}{5}\times\frac{25(x-1)}{(a+b)(a-b)}\times\frac{1}{x-1}$$

Suppressing all the factors which are found common in the numerator and denominator, and the result is $\frac{5}{a+b}$, ans.

5. Multiply $\frac{x}{a+x}$, $\frac{a^2-x^2}{x^2}$, and $\frac{a}{a-x}$ together. *Ans.* $\frac{a}{x}$.

6. Multiply $\frac{3x}{2}$ by $\frac{3a}{b}$. . . . . *Ans.* $\frac{9ax}{2b}$.

7. Multiply $\frac{4a^2x}{3}$ by $\frac{3a}{4}$. . . . . *Ans.* a^3x.

8. Multiply $\frac{2x}{a}$ by $\frac{3ab}{c}\times\frac{3ac}{2b}$. . . . . . . *Ans.* $9ax$.

9. Multiply $-\frac{a}{x}$ into $-\frac{y}{z}$. . . . . *Ans.* $\frac{ay}{xz}$.

10. Multiply $\frac{a}{x}$, $\frac{3x}{y}$, $\frac{4y}{3z}$ together. . . . . . *Ans.* $\frac{4a}{z}$.

11. Multiply $\frac{(a+x)}{30}$ by $\frac{5a}{3(a+x)}$. . . . . . *Ans.* $\frac{a}{18}$.

12. Multiply $\frac{2x+3y}{2a}$ by $\frac{2a}{5x}$. . . . . . *Ans.* $\frac{2x+3y}{5x}$.

13. What is the product of $\frac{4ax}{y}$, $\frac{3xy}{2a}$ and $\frac{2}{x}$? *Ans.* $12x$.

14. What is the product of $\frac{2a}{3b+c}$ into $\frac{2ac-bc}{5ab}$?

Ans. $\frac{4ac-2bc}{15b^2+5bc}$.

15. Multiply $b+\frac{bx}{a}$ by $\frac{a}{x}$. . . . . . . *Ans.* $\frac{ab+bx}{x}$.

16. Multiply $\frac{x^2-b^2}{bc}$ by $\frac{x^2+b^2}{b+c}$. . . . . . *Ans.* $\frac{x^4-b^4}{b^2c+bc^2}$.

17. Multiply $\frac{a^2-x^2}{2y}$ by $\frac{2a}{a+x}$. . . . . . *Ans.* $\frac{(a-x)a}{y}$.

18. Multiply $\frac{x^2-y^2}{x}$, $\frac{x}{x+y}$ and $\frac{a}{x-y}$. . . . . . *Ans.* a.

19. Multiply $3a$, $\frac{x+1}{2a}$, and $\frac{x-1}{a+b}$ together. *Ans.* $\frac{3(x^2-1)}{2(a+b)}$.

20. Multiply $\frac{3x^2-5x}{14}$ by $\frac{7a}{2x^3-3x}$. . . *Ans.* $\frac{3ax-5a}{4x^2-6}$.

21. Multiply $\frac{3x^2}{5x-10}$ by $\frac{15x-30}{2x}$. . . . . . *Ans.* $\frac{9x}{2}$.

22. Multiply $\frac{8ab}{3}$ by $\frac{3}{8ab}$. . . . . *Ans.* 1.

DIVISION IN FRACTIONS.

(Art. 39.) When we multiply a fraction by a whole number, we multiply the numerator by that number, or divide the denominator, (Art. 35); and as division is the converse of multiplication, therefore, conversely, when we divide a fraction by a whole number, we *divide* the numerator (when possible), or *multiply* the denominator by that number. Thus, $\frac{3}{7}$ divided by 3, would be $\frac{1}{7}$, and divided by 4, would be $\frac{3}{28}$; in the first case the division is actually performed, in the second it is only expressed.

EXAMPLES.

1. Divide $\frac{3}{8}$ by 3. . . . . *Ans.* $\frac{1}{8}$.
2. Divide $\frac{2}{7}$ by 9. . . . . *Ans.* $\frac{2}{63}$.
3. Divide $\frac{5}{9}$ by 5. . . . . *Ans.* $\frac{1}{9}$.
4. Divide $\frac{13}{19}$ by 13. . . . . *Ans.* $\frac{1}{19}$.
5. Divide $\frac{9}{10}$ by 8. . . . . *Ans.* $\frac{9}{80}$.

6. Divide $\frac{3a}{b}$ by $3c$. . . . . *Ans.* $\frac{a}{cb}$.

In this example we divide first by 3, and that quotient by c.

7. Divide $\frac{3a+x}{2y}$ by n. . . . . *Ans.* $\frac{3a+x}{2ny}$.

(ART. 40.) Let us now consider division when the divisor is a fraction. We must now go back to the elementary principle of division. *It is the art of discovering how many times a number or quantity, called the divisor, can be subtracted from another number or quantity of the same kind, called the dividend.*

For example, we require the division of 6 by $\frac{1}{3}$. The undisciplined and inconsiderate often understand this as demanding the *third* of 6; but it is not so, it is demanding how many times $\frac{1}{3}$ is contained in 6, or how many times $\frac{1}{3}$ can be *subtracted from* 6.

To arrive at the true result, we consider that $\frac{1}{3}$ is contained in 1 three times; therefore, it must be contained in 6, 18 times.

Now, $\frac{2}{3}$ must be contained in 6, 9 times; and we may arrive at this result, thus, $\frac{6 \cdot 3}{2}$.

Again, suppose we divide the number a by $\frac{1}{7}$.

The divisor $\frac{1}{7}$ is contained in *one unit* 7 times, therefore, it is contained in a units, $7a$ times.

To make this more general, we will suppose the denominator of the divisor to be any other number as well as 7; therefore, suppose it n, the quotient will then be na. To make the example still more general, let us suppose a to be divided by $\frac{m}{n}$, m being a whole number.

The divisor can be resolved into *two factors*, $\frac{1}{n}$ and m. Dividing a by the factor $\frac{1}{n}$, we have already shown the quotient to be na; dividing this by the whole number m, (Art. 39), the result must be $\frac{na}{m}$.

This shows that when the divisor is a fraction, the quotient is found by the following rule:

Rule.—*Multiply the dividend (whatever it may be) by the denominator of the divisor, and divide that product by the numerator.*

In the result last given, let the dividend a be a fraction $\frac{c}{d}$, and in the place of a write $\frac{c}{d}$.

Then the problem will be to divide $\frac{c}{d}$ by $\frac{m}{n}$, that is, one fraction by another, and the result must be

$$\frac{n\frac{c}{d}}{m}$$

This is a *complex* fraction, and simplifying it (by Art. 37), we have $\frac{nc}{md}$.

From this result we draw the following rule for dividing one fraction by another:

Rule.—*Invert the terms of the divisor, and proceed as in multiplication.*

(Art. 41.) For the purpose of illustrating the nature of an equation, and showing the power and simplicity of algebraic operations, we will arrive at this rule by another course of reasoning.

Let us again consider the nature of division, and for this purpose, divide 32 by 8.

Divisor.	Dividend.	Quotient.
8)	32	(4

Here it is visible that the product of the divisor and quotient is equal to the dividend; and this is a general principle, true in every possible case.

Now let us divide $\frac{a}{b}$ by $\frac{c}{d}$. There will be a certain quotient which we will represent by Q. Then the product of the divisor and quotient will be equal to the dividend; that is, we shall have the following equation:

$$\frac{cQ}{d}=\frac{a}{b}$$

Both members of this equation are fractional, and if we multiply the first member by d, the denominator, the product will be the numerator, (Art. 35); but if we take d times one member, we must take d times the other member, to preserve equality. (Ax. 3).

Therefore, multiplying by d, we have

$$cQ=\frac{ad}{b}$$

Dividing both members by c, then Q will *stand alone.*

And . . . . $Q=\frac{ad}{bc}$

This equation shows that when we divide one fraction by another, the value of the quotient is found by inverting the terms of the divisor, and then multiplying the numerators together for a new numerator, and the denominators together for a new denominator; or more briefly, we say

Invert the terms of the divisor, and proceed as in multiplication.

EXAMPLES.

1. Divide $\frac{3}{7}$ by $\frac{2}{5}$. . . . . *Ans.* $\frac{15}{14}=1\frac{1}{14}$.

$$\frac{2Q}{5}=\frac{3}{7} \text{ or } \tfrac{3}{7}\times\tfrac{5}{2}=\tfrac{15}{14}$$

$$2Q=\tfrac{15}{7}$$

$$Q=\tfrac{15}{14}$$

2. Divide $\frac{a}{1-a}$ by $\frac{a}{5}$. . . . . Ans. $\frac{5}{1-a}$.

3. Divide $\frac{2x}{ab}$ by $\frac{3xy}{ab}$. . . . . Ans. $\frac{2}{3y}$.

4. Divide $\frac{3a-b}{ab}$ by $\frac{2a^2}{ab}$. . . . . Ans. $\frac{3a-b}{2a^2}$.

5. Divide $\frac{7rx+a^2}{5ast}$ by $\frac{4b+ax}{5ast}$. . . . . Ans. $\frac{7rx+a^2}{4b+ax}$.

6. Divide $\frac{a+b}{c}$ by $\frac{c}{a+b}$. . . . . . . Ans. $\frac{(a+b)^2}{c^2}$.

7. Divide $\frac{5x}{a}$ by $\frac{b}{c}$.

8. Divide $\frac{15ab}{a-x}$ by $\frac{10ac}{a^2-x^2}$.*

Operation, $\frac{15ab}{a-x}\times\frac{(a+x)(a-x)}{10ac}$. . . . Ans. $\frac{3b(a+x)}{2c}$.

9. Divide $\frac{2ax+x^2}{a^3-x^3}$ by $\frac{x}{a-x}$. . . . Ans. $\frac{2a+x}{a^2+ax+x^2}$.

10. Divide $\frac{14x-3}{5}$ by $\frac{10x-4}{25}$. . . . . Ans. $\frac{70x-15}{10x-4}$.

11. Divide $\frac{9x^2-3x}{5}$ by $\frac{x^2}{5}$. . . . . Ans. $\frac{9x-3}{x}$.

12. Divide $\frac{6x-7}{x+1}$ by $\frac{x-1}{3}$. . . . . . Ans. $\frac{18x-21}{x^2-1}$.

13. Divide $\frac{16ax}{5}$ by $\frac{4x}{15}$. . . . . Ans. $12a$.

14. Divide $\frac{6z+4}{5}$ by $\frac{3z+2}{4y}$. . . . . Ans. $\frac{8y}{5}$.

15. Divide $\frac{7x}{3}$ by $\frac{4x^2}{6}$. . . . . Ans. $\frac{21}{6x}$.

---

* Separate into factors wherever separation is obvious.

16. Divide $\frac{a+1}{6}$ by $\frac{2a}{3}$. . . . . *Ans.* $\frac{a+1}{4a}$.

17. Divide $\frac{x}{x-1}$ by $\frac{x}{2}$. . . . . *Ans.* $\frac{2}{x-1}$.

18. Divide $\frac{x^2-2xy+y^2}{ab}$ by $\frac{x-y}{bc}$. . . . . *Ans.* $\frac{cx-cy}{a}$.

19. Divide $\frac{m^2-n^2}{3}$ by $\frac{m+n}{6}$. . . . . . *Ans.* $2m-2n$.

20. Divide $\frac{5x}{3}$ by $\frac{2a}{3b}$. . . . . *Ans.* $\frac{5bx}{2a}$.

21. Divide $\frac{x-b}{8cd}$ by $\frac{3cx}{4d}$. . . . . *Ans.* $\frac{x-b}{6c^2x}$.

22. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$. . . . *Ans.* $x+\frac{b^2}{x}$.

Operation, $\frac{(x^2+b^2)(x^2-b^2)}{(x-b)(x-b)}\times\frac{x-b}{(x+b)x}=\frac{x^2+b^2}{x}$.

ADDITION OF FRACTIONS.

(Art. 42.) When fractions have a common denominator, they can be readily added together by adding their numerators, because $\frac{2}{7}$ and $\frac{3}{7}$ is obviously $\frac{5}{7}$, and $\frac{4}{n}$ and $\frac{7}{n}$ is obviously $\frac{11}{n}$, or $\frac{a}{n}$ and $\frac{b}{n}$ is $\frac{a+b}{n}$, &c.

But when the denominators are unlike, we cannot directly add the fractions together, because we cannot add unlike things, as dollars and cents, or units and tens, &c.

In all such cases we can only indicate the addition by signs, unless we first reduce the quantities to *like* denominations, or (as applied to fractions), to *common* denominators.

We shall investigate a rule for the addition of fractions through the medium of *equations*.

For example, we require the sum of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{5}$. By the *summary* process of Algebra, we pronounce the sum to be S. Then we have the following equation:

$$S=\tfrac{1}{2}+\tfrac{2}{3}+\tfrac{3}{5} \qquad (1)$$

The first member of this equation is a symbol merely; and in the second member the addition is only *indicated*, not performed; and to perform it, the fractional form of the equation must be changed to whole numbers, or the denominators made common.

If we multiply every term of both members by 2, *the first fraction will be removed*, (Art. 35), *and the equation will stand thus:*

$$2S=1+\tfrac{4}{3}+\tfrac{6}{5} \qquad (2)$$

If we multiply every term by 3, *the second fraction will be removed*, and the equation will stand thus:

$$6S=3+4+\tfrac{18}{5} \qquad (3)$$

In the same manner we can *remove* the third fraction by multiplying by 5; then we have

$$30S=15+20+18 \qquad (4)$$

Now, if we divide every term of equation (4) by 30, we shall have

$$S=\tfrac{15}{30}+\tfrac{20}{30}+\tfrac{18}{30} \qquad (5)$$

Here we have the sum S equal to fractions having a *common denominator*, and that common denominator is the product of the denominators of the given fractions 2, 3, and 5.

In equation (4), we may add the numbers 15, 20. and 18 directly, making 53, and the equation will be

$$30S=53$$

Dividing by 30, and $S=\frac{53}{30}$ or $1\frac{23}{30}$.

Also, the sum of the fractions in equation (5) is $S=\frac{53}{30}$.

That the operation may be more distinct, we will require the sum of the literal fractions $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{g}{h}$.

Assume S to be their sum as before.

Then . . . . . $S=\frac{a}{b}+\frac{c}{d}+\frac{g}{h}$ (1)

Remove the fractions first, by multiplying by b, then by d, then by h; *or by multiplying the whole at once by the product bdh.*

Multiplying by b, gives $bS=a+\frac{cb}{d}+\frac{gb}{h}$ (2)

Again by d, gives $dbS=ad+cb+\frac{gbd}{h}$ (3)

Again by h, gives $hdbS=adh+cbh+gbd$ (4)

Dividing both members of equation (4) by hdb, and we have

$$S=\frac{adh}{hdb}+\frac{cbh}{hdb}+\frac{gbd}{hdb} \qquad (5)$$

But these fractions in the second member of equation (5), have a *common denominator*, and, therefore, it need not be written under every numerator, if it be written under their sum.

Thus . . . $S=\frac{(adh+cbh+gbd)}{hdb}$ (6)

Here, then, we have the sum of the fractions in *one quantity.*

By inspecting the second member of equation (5), and comparing it with the original fractions to be added, we perceive that the numerator of the first fraction, a, is multiplied by the denominators of the other fractions; and the numerator of the second fraction, c, is also multiplied by the denominators of the other fractions: and the same is true of the third fraction, and so on.

The common denominator is made up—or *is the product—of all the denominators.*

Hence, we derive the following rule for reducing fractions to a common denominator:

RULE.—*Multiply each numerator into all the denominators except its own, for the new numerators; and all the denominators together, for a common denominator.*

And to add fractions, we have the following rule:

RULE.—*Reduce the fractions to a common denominator; and the sum of the numerators, written over the common denominator, will be the sum of the fractions.*

EXAMPLES.

1. Add $\frac{3x}{5}$, $\frac{2x}{7}$ and $\frac{x}{3}$ together.

Ans. $\frac{63x+30x+35x}{105}=\frac{128x}{105}$.

2. Add $\frac{a}{b}$ and $\frac{a+b}{c}$. . . . . *Ans.* $\frac{ac+ab+b^2}{bc}$.

3. Add $\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$ together. . . . . *Ans.* $x+\frac{x}{12}$.

4. Add $\frac{x-2}{3}$ and $\frac{4x}{7}$ together. . . . . *Ans.* $\frac{19x-14}{21}$.

5. Add $\frac{1}{a+b}$ and $\frac{1}{a-b}$ together. . . . *Ans.* $\frac{2a}{a^2-b^2}$.

6. Add $\frac{x}{x+y}$ and $\frac{y}{x-y}$ together. . . . *Ans.* $\frac{x^2+y^2}{x^2-y^2}$.

7. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and d, to fractions having a common denominator. *Ans.* $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, and $\frac{6acd}{6ac}$.

8. Reduce $\frac{3}{4}$, $\frac{2x}{3}$, and $a+\frac{2x}{a}$, to fractions having a common denominator. *Ans.* $\frac{9a}{12a}$, $\frac{8ax}{12a}$, and $\frac{12a^2+24x}{12a}$.

(Art. 43.) The preceding rules are general, and correspond to quantities that are prime to each other; but in cases of multiple denominators, the *general* rule would carry the operator through a much longer process than necessary.

We will, therefore, investigate a more convenient practical rule, which will apply to fractions having multiple denominators. For example, we require the sum of the fractions

$$\frac{a}{b},\ \frac{c}{nb},\ \frac{d}{mb}$$

As before, we designate the sum by S, which gives the equation

$$S=\frac{a}{b}+\frac{c}{nb}+\frac{d}{mb} \qquad (1)$$

Multiplying every term by b, then we have

$$bS=a+\frac{c}{n}+\frac{d}{m} \qquad (2)$$

Multiplying by n, and then by m, or multiply at once by nm, then we have

$$nmbS=anm+cm+dn \qquad (3)$$

Dividing equation (3) by nmb, and we have

$$S=\frac{anm+cm+dn}{nmb}$$

The product nmb is composed of all the different factors in the denominators, and no more; it is, therefore, the *least common multiple* of the denominators, (Art. 32).

To find the numerators, we divide this product by the denominator of any one of the fractions, and multiply the quotient by the numerator. For instance, take the first fraction, $\frac{a}{b}$. Divide nmb by b, and we have nm; multiply this by a, and we have anm, the new numerator for the first fraction; and by the same operation we find the numerators for the other fractions.

Hence, we have the following rule for reducing fractions to equivalent fractions having a least common denominator, and thence finding their sum.

RULE 1.—*Find the least common multiple of all the denominators, which will be the least common denominator.*

2. *Divide the common denominator by the denominator of the first given fraction, and multiply the quotient by the numerator, the product will be the first of the required numerators.*

3. *Proceed in like manner to find each of the required numerators.*

4. *The sum of the fractions will be the algebraic sum of these numerators, with the common denominator under them.*

NOTE.—The fractions should be reduced to their lowest terms before this or the preceding rules are applied.

OTHER EXAMPLES.

1. Add $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$, and $\frac{7}{12}$ together. . . . *Ans.* $\frac{31}{12}=2\frac{7}{12}$.

$$S=\frac{1}{2}+\frac{2}{3}+\frac{5}{6}+\frac{7}{12} \qquad (1)$$

The least common multiple of these denominators is obviously 12; therefore, multiply both members of the equation by 12, and we have

$$12S=6+8+10+7 \qquad (2)$$

Dividing again by 12, and we have

$$S=\frac{6}{12}+\frac{8}{12}+\frac{10}{12}+\frac{7}{12} \qquad (3)$$

The second member of equation (3) is composed of equivalent fractions to those in equation (1), as may be seen by reducing these fractions to their lowest terms. In equation (3), the fractions have a common denominator, composed of the least common multiple of the original denominators.

The sum of these fractions is, of course, the sum of the numerators with the common denominator under it; thus, $\frac{31}{12}$, and might have been taken for equation (2), thus:

$$12S=31$$

Or . . . . . . $S=\frac{31}{12}$

2. Add $\frac{12b-a}{35c}$, and $\frac{3a-b}{7c}$ together. . . *Ans.* $\frac{2a+b}{5c}$.

3. Add $\frac{1}{1+a}$, $\frac{a}{1-a}$, and $\frac{a}{1+a}$ together. . *Ans.* $\frac{1}{1-a}$.

4. Add $\frac{a}{b}$, $\frac{2a}{3b}$, and $\frac{5b}{4a}$ together. . . *Ans.* $\frac{20a^2+15b^2}{12ab}$.

5. Add $\frac{6ab-3b^2-12ac+16bc}{12bc}$, and $\frac{3a-4b}{3b}$ together.

Ans. $\frac{2a-b}{4c}$.

NOTE.—Examples 2, 3, and 5, and all others like them, had better be performed by solving an equation. If not so performed, multiply the numerator and denominator of the second fraction in example 2, by 5, and in example 5, by $4c$, and thus make the denominators common *by inspection*. Then *unite* the numerators, and reduce to lowest terms.

In examples like the following, consisting of entire quantities and fractions, make two examples of the operation, by first uniting the entire quantities, and then the fractions, and lastly uniting the two sums together by their proper signs.

6. Add $2x$, $3x+\frac{3a}{5}$ and $x+\frac{2a}{9}$ together. *Ans.* $6x+\frac{37a}{45}$.

7. Add $5x+\frac{x-2}{3}$ and $4x-\frac{2x-3}{5x}$ together.

Ans. $9x+\frac{5x^2-16x+9}{15x}$.

8. Add $\frac{2b}{(a-b)(a+b)}$ and $\frac{1}{a+b}$ together. *Ans.* $\frac{1}{a-b}$.

9. Add $\frac{a-b}{ab}$, $\frac{b-c}{bc}$ and $\frac{c-a}{ac}$ together. . . *Ans.* 0.

10. Add $\frac{a^2-x^2}{ax}$ and $\frac{x-a}{x}$ together. . . . *Ans.* $\frac{a-x}{a}$.

11. Add $\frac{5+x}{y}$, $\frac{3-ax}{ay}$ and $\frac{b}{3a}$ together.

Ans. $\frac{15a+by+9}{3ay}$.

12. Add $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$ together. . . *Ans.* $\frac{2(a^2+b^2)}{a^2-b^2}$.

13. Add $\frac{a}{b}$, $\frac{a-3b}{cd}$ and $\frac{a^2-b^2-ab}{bcd}$ together.

Ans. $\frac{acd-4b^2+a^2}{bcd}$.

14. Add $\frac{a}{a+b}$ and $\frac{b}{a-b}$ together. . . . *Ans.* $\frac{a^2+b^2}{a^2-b^2}$.

15. Add $\frac{1}{x+y}$ and $\frac{y}{x^2-y^2}$ together. . . . *Ans.* $\frac{x}{x^2-y^2}$.

16. Add $\frac{4a^2}{1-a^4}$ and $\frac{1-a^2}{1+a^2}$ together. . . . *Ans.* $\frac{1+a^2}{1-a^2}$.

17. Add $\frac{1}{a}+\frac{1}{b}$ and $1-\left(\frac{a+b}{ab}\right)$ together. . . *Ans.* 1.

SUBTRACTION OF FRACTIONS.

(Art. 44.) We would remind the pupil that, in addition, we took the sum of the numerators, after the fractions were reduced to a common denominator. Hence, the difference of the two fractions must be found by taking the difference of their numerators, when the denominators are alike. For example, the difference between $\frac{7}{12}$ and $\frac{5}{12}$, must be $\frac{2}{12}=\frac{1}{6}$, and the difference between $\frac{7}{8}$ and $\frac{2}{8}$, must be $\frac{5}{8}$, &c. These observations must give us the following

RULE 1.—*Reduce the fractions to a common denominator.*

2. *Subtract the numerator of the subtrahend from the numerator of the minuend, and place the difference over the common denominator.*

(ART. 45.) We may also find the result of any proposed example by means of equations, as in addition.

For example, from $\frac{3}{4}$ take $\frac{5}{7}$. The remainder is some number, which we may represent by R.

Then . . . . . . $R=\frac{3}{4}-\frac{5}{7}$ (1)

Or, we may consider that in every possible example, the remainder and subtrahend added together, must equal the minuend; that is*

$$R+\frac{5}{7}=\frac{3}{4} \quad (2)$$

Equation (2) is the same as equation (1), except the fraction $\frac{5}{7}$ is transposed, according to the rule of transposition on page 17.

Multiply equation (1) by 7, and we have

$$7R=\frac{21}{4}-5$$

Multiplying by 4, and $28R=21-20=1$

By division . . . . $R=\frac{1}{28}$

EXAMPLES.

1. From $\frac{7x}{2}$ take $\frac{2x-1}{3}$. *Ans.* $\frac{21x-4x+2}{6}=\frac{17x+2}{6}$.

2. From $\frac{1}{x-y}$ take $\frac{1}{x+y}$. Eq. fractions $\frac{x+y}{x^2-y^2}$, $\frac{x-y}{x^2-y^2}$.

Difference or *Ans.* $\frac{2y}{x^2-y^2}$.

---

* We take this view of the subject to show the pupil the nature of equations; not that it is, or is not, a better method of solving the problems.

3. From $\frac{x}{3}$ take $\frac{2x}{7}$. . . . . Diff. $\frac{x}{21}$

4. From $\frac{2ax}{3}$ take $\frac{5ax}{2}$. . . . . *Ans.* $-\frac{11ax}{6}$.

5. From $\frac{1}{a+1}$ take $\frac{a-2}{a^2-a+1}$. . . . . . . *Ans.* $\frac{3}{1+a^3}$.

6. From $\frac{3}{4a}$ take $\frac{5}{2x}$. . . . . *Ans.* $\frac{3x-10a}{4ax}$.

7. From $\frac{3a}{4x}$ take $\frac{4x}{3a}$. . . . . *Ans.* $\frac{9a^2-16x^2}{12ax}$.

8. From $\frac{2a^3b^2}{4a^2-b^4}$ take $\frac{a^2b^2}{2a+b^2}$. . . . . . *Ans.* $\frac{a^2b^4}{4a^2-b^2}$.

OPERATION.

$$R+\frac{a^2b^2}{2a+b^2}=\frac{2a^3b^2}{(2a+b^2)(2a-b^2)}$$

$$(4a^2-b^4)R+2a^3b^2-a^2b^4=2a^3b^2$$

Dropping from both members $(2a^3b)$, and transposing a^2b^4, and we have

$$(4a^2-b^4)R=a^2b^4$$

By division, ... $R=\frac{a^2b^4}{4a^2-b^4}$

9. From $\frac{1}{x-1}$ take $\frac{2}{x+1}$. . . . . *Ans.* $\frac{3-x}{x^2-1}$.

10. From $2a-2x+\frac{a-x}{a}$ take $2a-4x+\frac{x-a}{x}$.

Ans. $2x+\frac{a^2-x^2}{ax}$.

11. From $\frac{2a+b}{5c}$ take $\frac{3a-b}{7c}$. . . . . . *Ans.* $\frac{12b-a}{35c}$.

12. From $\frac{5x+1}{7}$ take $\frac{21x+3}{4}$. . *Ans.* $\left(-\frac{127x-17}{28}\right)$

13. From $\frac{x-y}{2a}$ take $\frac{x+y}{3a}$. . . . *Ans.* $\left(\frac{ax-5ay}{6a^2}\right)$

14. From $\frac{1+a^2}{1-a^2}$ take $\frac{1-a^2}{1+a^2}$. . . . . . *Ans.* $\frac{4a^2}{1-a^4}$.

15. From $x+\frac{x-y}{x^2+xy}$ take $\frac{x+y}{x^2-xy}$. . . *Ans.* $x-\frac{4y}{x^2-y^2}$.

16. From $\frac{a-b}{2c}$ take $\frac{2b-4a}{5d}$. *Ans.* $\frac{5ad-5bd-4bc+8ac}{10cd}$.

17. From $\frac{2(a^2+b^2)}{a^2-b^2}$ take $\frac{a-b}{a+b}$. . . . . *Ans.* $\frac{a+b}{a-b}$.

18. From $\frac{x}{x-3}$ take $\frac{x+3}{x}$. . . . . *Ans.* $\frac{9}{x^2-3x}$.

19. From $6a+\frac{14a-13}{20}$ take $4a+\frac{2a-5}{4}$.

Ans. $2a+\frac{a+3}{5}$.

SECTION II.

EQUATIONS.

The most interesting and the most essential part of Algebra is comprised in equations; and nearly all of our previous preparations have been with a view to a more ready understanding of equations.

The use of equations is the solving of problems in almost every branch of mathematical science, and also in the investigation of scientific truths.

For instance, we have already investigated rules for the addition, subtraction, and division of fractional quantities, by means of algebraical equations, and we have thus, incidentally, given some explanations concerning the nature of equations; but now coming to the subject in order, we shall disregard all this, and commence on the supposition that the pupil must yet learn every particular.

(Art. 46.) An equation is an algebraical expression, meaning that certain quantities are equal to certain other quantities. Thus, $3+4=7$; $a+b=c$; $x+4=10$, are equations, and express that 3 added to 4 is equal to 7, and in the second equation, that a added to b is equal to c, &c. The signs are only abbreviations for words.

The quantities on each side of the sign of equality are called *members*. Those on the left of the sign form the *first* member, those on the right, the *second* member.

(Art. 47.) As unlike things can neither be added to, nor subtracted from each other, it follows that a member of an equation must consist of the same *kind* of quantities; and as it is absurd to suppose one kind of quantity equal to another in any other sense than a numerical one, it also follows that the members of an equation must be equal in *kind* as well as in number. That is, Dollars = Dollars,

Or . . . . Pounds = Pounds, &c., &c.

It is true we may say that a farmer has as many dollars in his purse as he has sheep and cows on his farm.

Here we cannot say that his sheep and cows are *equal* to his dollars; but the *number* of his sheep added to the *number* of his cows, are equal to the *number* of his dollars.

That is, . . Number = Number.

Indeed, when dollars equal dollars, or yards equal yards, it is but really a number of dollars equal to a number of dollars, &c.; that is, universally, number equal to number.

(Art. 48.) In the solution of problems, every equation is supposed to contain at least one *unknown quantity;* and the solution of an equation is the art of changing and operating on the terms by means of addition, subtraction, multiplication, or division, or by all these combined, so that the unknown quantity may stand alone as one member of the equation, equal to known quantities in the other member, by which it then becomes known.

Every equation is to be regarded as the statement, in algebraic language, of a particular question.

Thus, $x-3=4$, may be regarded as the statement of the following question: To find a number from which, if 3 be subtracted, the remainder will be equal to 4.

An equation is said to be *verified*, when the value of the unknown quantity being substituted for it, the two members are rendered equal to each other.

Thus, in the equation $x-3=4$, if 7, the value of x, be substituted instead of it, we have $7-3=4$,

Or . . . . $4=4$.

(ART. 49.) Equations are of the *first, second, third* and higher degrees, according to the highest power of the unknown quantity involved.

Thus, $\left.\begin{array}{r} x=a \\ x+bx=c \end{array}\right\}$ are equations of the first degree.

$\left.\begin{array}{r} x^2+ax=c \\ ax^2+bx=h \end{array}\right\}$ are equations of the second degree.

$\left.\begin{array}{r} x^3=a \\ x^3+bx=c \end{array}\right\}$ are equations of the third degree, &c.

Equations of the first degree are also called *simple* equations, and equations of the second degree are called quadratic equations; but *quadratic equations* may include many other equations of any *even* degree, according to certain relations that may exist between the several parts of the equation, which will be explained hereafter. At present we shall confine our investigations to simple equations.

(ART. 50.) Equations are either *numeral* or *literal*. Numeral equations contain numbers only, excepting the unknown quantity. In literal equations, the given quantities are represented by letters, in whole or in part.

An *identical equation*, is one in which the two members are identical; or, one in which one of the members is the result of the operations indicated in the other.

Thus, $\left.\begin{array}{r} 2x-1=2x-1 \\ 5x+3x=8x \end{array}\right\}$ are identical equations.

(ART. 51.) The unknown quantity of an equation may be united to known quantities, in *four* different ways; by addition, by subtraction, by multiplication, and by division, and further by various combinations of *these four* ways, as shown by the following equations, both numeral and literal:

	NUMERAL.	LITERAL.
1st. By addition, ...	$x+6=10$	$x+a=b$
2d. By subtraction, . . .	$x-8=12$	$x-c=d$
3d. By multiplication, . . .	$20x=80$	$ax=e$
4th. By division, ...	$\frac{x}{4}=16$	$\frac{x}{d}=g+a$

5th. $x+6-8+4=10+2-3$, $x+a-b+c=d+c$, &c., are equations in which the *unknown* is connected with known quantities, both by addition and subtraction.

$2x+\frac{x}{3}=21$, $ax+\frac{x}{b}=c$, are equations in which the *unknown* is connected with known quantities, by both multiplication and division.

Equations often occur, in solving problems, in which all of these operations are combined.

(ART. 52.) Let us now examine and discover, if possible, how the *unknown* quantity can be separated from known quantities, and be made to *stand alone* as one member of the equation. For this purpose, let us take the equation

$$x+a=b$$

Take equals from equals, $a=a$

Remainders are equal, $x=b-a$ (Ax. 2).

Here the quantity a, connected to x, appears on the other side of the equation, with its *opposite* sign.

Again, suppose we have the equation

$$x-8=10$$

Add equals to equals, $8=8$

Sums will be equal, $x=10+8$ (Ax. 1).

Here, again, the quantity connected with x appears on the opposite side of the equation, with its opposite sign.

From this we derive the following operation, which operation is called

TRANSPOSITION.

RULE.—*Any quantity may be changed from one member of an equation to the other, if, in so doing, we change its sign.*

Now, suppose we have an equation in the form of

$$ax=c$$

Here, x is united to a by multiplication; it can be *disunited* by division. Dividing by a, gives

$$x=\frac{c}{a}$$

Again, suppose an equation appears in the form of

$$\frac{x}{a}=g$$

Here, x is *united* to a known quantity by division, and it can be *disunited* by multiplication; that is, multiply by a, and we have $x=ag$

From these observations, we deduce this general principle:

That to separate the unknown quantity from additional terms, we must use subtraction; from subtracted terms, we must use addition; from multiplied terms, we must use division; from division, we must use multiplication.

In all cases take the opposite operation.

(ART. 53.) In many practical problems, the unknown quantity is often combined with the known quantities, not merely in a simple manner, but under various fractional and compound forms. Hence, rules can only embody general principles, and skill and tact must be acquired by close attention and practical application; but from the foregoing principles, we derive the following

GENERAL RULE.—*Connect and unite, as much as possible, all the terms of a similar kind on both sides of the equation.*

Then, to clear of fractions, multiply both sides by the denominators, one after another, in succession. Or, multiply by their continued product, or by their least common multiple, (when such a number is obvious), *and the equation will be free of fractions.*

Then transpose the unknown terms to the first member of the equation, and the known terms to the other. Then unite the similar terms, and divide by the coefficient of the unknown term, and the equation is solved.

EXAMPLES.

1. Given $3x-2+5=2x+12$, to find x. . *Ans.* $x=9$.

By transposition, $3x-2x=12+2-5$

Uniting terms, $x=9$

In place of transposing, we may *drop* equals from both sides, or add equals to both sides, as the circumstances may require.

In the present example, we *drop* $2x$ from both sides, and conceive $-2+5$ united, then we have

$$x+3=12$$

Drop 3 from both sides, and we have

$$x=9, \text{ as before.}$$

Dropping and transposing is one and the same operation, differing only in form.

2. Given $6-2x+10=20-3x-2$, to find x. *Ans.* $x=2$.

Uniting similar terms in both members, we have

$$-2x+16=18-3x$$

Adding $3x$ to both sides, and dropping 16 from both, we have $x=2$

3. Given $\frac{x}{2}-\frac{x}{4}+x=2x-3$, to find x. . . *Ans.* $x=4$.

Drop x from both members, then we have

$$\frac{x}{2}-\frac{x}{4}=x-3.$$

Multiply every term by 4, and we have

$$2x-x=4x-12$$

Transpose $4x$, and unite,

Then . . . . . $-3x=-12$

Divide both members by -3, and $x=4$.

4. Given $5x+22-2x=31$, to find x. . . *Ans.* $x=3$.
5. Given $4x+20-6=34$, to find x. . . *Ans.* $x=5$.
6. Given $3x+12+7x=102$, to find x. . *Ans.* $x=9$.
7. Given $10x-6x+14=62$, to find x. . *Ans.* $x=12$.
8. Given $ax+bx=ma+mb$, to find x. . . *Ans.* $x=m$.

Separate both members into their prime factors,

Thus, ... $(a+b)x=(a+b)m$.

Dividing both members by $(a+b)$, gives $x=m$.

9. Given $ax+dx=a-c$, to find x. . . *Ans.* $x=\frac{a-c}{a+d}$.
10. Given $3(x+1)+4(x+2)=6(x+3)$, to find x.
Ans. $x=7$.

Perform the multiplication indicated, then reduce.

11. Given $\frac{3x}{4}+16=\frac{x}{2}+\frac{x}{8}+17$, to find x. . . *Ans.* $x=8$.

In the first place, drop 16 from both members, according to the general rule. Then

$$\frac{3x}{4}=\frac{x}{2}+\frac{x}{8}+1$$

Multiply both members by 8, the least common multiple of the denominators, and we have

$$6x=4x+x+8, \text{ or, } x=8$$

12. Given $\frac{x}{2}-3+\frac{x}{3}=5-3$, to find x. . . *Ans.* $x=6$.
13. Given $\frac{x}{3}-\frac{x}{4}+2=3$, to find x. . . . *Ans.* $x=12$.

14. Given $\frac{x}{4}+\frac{x}{8}-\frac{x}{6}=\frac{5}{12}$, to find x. . . . *Ans.* $x=2$.

15. Given $\frac{5x}{8}+\frac{1}{4}=\frac{11}{6}+\frac{7x}{12}$, to find x. . . . *Ans.* $x=38$.

16. Given $\frac{x}{a}+\frac{x-5}{2}+2b=3b$, to find x. *Ans.* $x=\frac{2ab+5a}{2+a}$.

17. Given $\frac{3x}{5}+2\frac{1}{2}+11=\frac{x}{4}+17$, to find x. *Ans.* $x=10$.

18. Given $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x=39$, to find the value of x.

Here are no scattering terms to collect, and clearing of fractions is the first operation.

By examination of the denominators, 12 is obviously their least common multiple, therefore, multiply by 12.

Hence, . . . $6x+4x+3x=39\times 12$

Collect the terms, $13x=39\times 12$

Divide by 13, and $x=3\times 12=36$, *Ans.*

19. Given $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x=a$, to find x.

This example is essentially the same as the last. It is identical if we suppose $a=39$.

Solution, . . . $6x+4x+3x=12a$

Or, ... $13x=12a$

Divide and . . . . . . $x=\frac{12a}{13}$

Now if a be any multiple of 13, the problem is easy and brief in numerals.

20. Given $\frac{1}{3}x-5+\frac{1}{4}x+8+\frac{1}{5}x-10=100-6-7$ to find the value of x.

Collecting and uniting the numeral quantities, we have

$$\tfrac{1}{3}x+\tfrac{1}{4}x+\tfrac{1}{5}x=94$$

Multiply every term by 60, and we have

$$20x+15x+12x=94\cdot60$$

Collecting terms, $47x=94\cdot60$

Divide both sides by 47, and $x=2\cdot60=120$, *Ans.*

21. Given $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x+\frac{1}{5}x=77$, to find x. *Ans.* $x=60$.

22. Given $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x=130$, to find x. . *Ans.* $x=120$.

23. Given $\frac{1}{2}x+\frac{1}{6}x+\frac{1}{12}x=90$, to find x. . *Ans.* $x=120$.

24. Given $\frac{1}{2}y+\frac{1}{3}y+\frac{1}{7}y=82$, to find y. . *Ans.* $y=84$.

25. Given $5x+\frac{1}{6}x+\frac{1}{2}x=34$, to find x. . . *Ans.* $x=6$.

N. B. In solving 21, 22, 23, 24 and 25, take 19 for a model, and write a to represent the second members of the equations, to save numeral multiplications.

26. Given $\frac{3x}{4}-\frac{x-1}{2}=6x-\frac{20x+13}{4}$ to find x.

Multiply by 4, to clear of fractions, and

$$3x-2x+2=24x-20x-13. \quad \text{Reduced, } x=5.$$

(Art. 54.) When a *minus* sign stands before a compound quantity, it indicates that the whole is to be subtracted; but we subtract by changing signs, (Art. 5). The minus sign before $\frac{x-1}{2}$ in the last example, does not indicate that the x is minus, but that this term must be subtracted. When the term is multiplied by 4, the numerator becomes $2x-2$, and subtracting it, we have $-2x+2$.

27. Given $x-\frac{x-3}{2}=\frac{9}{2}-\frac{x+4}{3}$, to find x. . *Ans.* $x=2$.

28. Given $\frac{x+2}{3}-\frac{x-3}{4}+2=x-\frac{x-1}{2}$, to find x.

Ans. $x=7$.

29. Given $\frac{4x-2}{11}-\frac{3x-5}{13}=1$, to find x. . *Ans.* $x=6$.

30. Given $\frac{x}{5}-\frac{x-2}{3}=-\frac{x}{2}+\frac{13}{3}$, to find x. . *Ans.* $x=10$.

PROPORTION.

(ART. 55.) Sometimes an equation may arise, or a problem must be solved through the aid of proportion.

Proportion is nothing more than an assumption that the same *relation*, or the same *ratio* exists between two quantities as exists between two other quantities.

Quantities can only be compared when they are alike in kind, and one of them must be the *unit* of measure for the other.

Thus, if we compare A and B, we find *how many times*, or part of a time, A is contained in B, by dividing B by A, thus,

$$\frac{B}{A}=r, \quad \text{or} \quad B=rA$$

That is, a certain number of times A is equal to B.

Now if we have two other quantities, C and D, having the same *relation or ratio* as A to B, that is, if $D=rC$,

Then A is to B as C is to D.

But in place of writing the words between the letters, we write the signs that indicate them.

Thus, ... $A:B::C:D$

But in place of B and D, write their values rA, and rC.

Then, ... $A:rA::C:rC$

Multiply the extreme terms, and we have rCA.

Multiply the mean terms, and we have rAC.

Obviously the same product, whatever quantities may be represented by either A, or r, or C.

Hence, to convert a proportion into an equation, we have the following

RULE.—*Place the product of the extremes equal to the product of the means.*

(Art. 56.) The relation between two quantities is not changed by multiplying or dividing both of them by the same quantity. Thus, $a:b::2a:2b$, or more generally, $a:b::na:nb$, for the product of the extremes is obviously equal to the product of the means.

That is, a is to b as any number of times a is to the same number of times b.

We shall take up proportion again, but Articles 55 and 56 are sufficient for our present purpose.

EXAMPLES.

1. *If 3 pounds of coffee cost 25 cents, what will a bag of 60 pounds cost?* *Ans.* 500 cents.

Ans. It will cost a certain number of cents, which I designate by x, and the numerical value of x can be deduced from the following proportion: Pounds compare with pounds, as cents compare with cents. That is, these different kinds of quantities must have the same *numerical ratio*.

Thus, ... $3:60::25:x$

Without the x, this is the rule of *three* in Arithmetic, because there are three terms given to find the fourth; and in Algebra we designate the fourth term by a symbol before we know its numerical value, which makes the proportion complete.

By the rule (Art. 55), $3x=60\cdot25$

Or, ... $x=\frac{60\cdot25}{3}$

Hence, when the first three terms of a proportion are given to find the fourth, *multiply the second and third together, and divide by the first.*

In Arithmetic it requires more care to state a question than it does in Algebra, because in the former science we have not so much capital at command as in the latter.

In Algebra it is immaterial what position the unknown term has in the proportion, if the comparison is properly made. Thus, in the foregoing question the demand is money, and money must be compared with money; and the statement may be made thus, . . $25 : x :: 3 : 60$

Or thus, . . . $x : 25 :: 60 : 3$

From either one of these proportions the value of x is found by multiplying and dividing by the same numbers.

2. If 2 cords of wood cost 5 dollars, what will 48 cords cost? *Ans.* \$120.

Given $5 : x :: 2 : 48$, to find x.

3. Given $2 : x :: 6 : 5x-4$, to find x. . . *Ans.* $x=2$.

The equation, . . $10x-8=6x$

4. Given $\frac{(x-1)(x+1)}{3a} : \frac{x+1}{3a} :: 2x : 1$, to find x.

Divide the first two terms by $(x+1)$, (Art. 56). Also multiply by $3a$.

5. Given $x+2 : a :: b : c$, to find the value of x.

Ans. $x=\frac{ab}{c}-2$.

6. Given $2x-3 : x-1 :: 2x : x+1$, to find the value of x. *Ans.* $x=3$.

7. Given $x+6 : 38-x :: 9 : 2$, to find x. *Ans.* $x=30$.

8. Given $x+4 : x-11 :: 100 : 40$, to find x. *Ans.* $x=21$.

9. Given $x+a : x-a :: c : d$, to find x. *Ans.* $x=\frac{a(c+d)}{c-d}$.

10. Given $x : 2x-a :: a : b$, to find x. . *Ans.* $x=\frac{a^2}{2a-b}$.

11. Given $a : b :: 2y : d$, to find y. . . . *Ans.* $y=\frac{ad}{2b}$.

12. Given $a^2-ac : ax :: 1 : (d-b)$, to find x.

Ans. $x=(d-b)(a-c)$.

13. Given $x : 75-x :: 3 : 2$, to find x. . . *Ans.* $x=45$.

QUESTIONS PRODUCING SIMPLE EQUATIONS.

(ART. 57.) We now suppose the pupil can readily reduce a simple equation containing but one unknown quantity, and he is, therefore, prepared to solve the following questions. The only difficulty he can experience, is the want of tact to reason briefly and powerfully with algebraic symbols; but this tact can only be acquired by practice and strict attention to the solution of questions. We can only give the following general direction:

Represent the unknown quantity by some symbol or letter, and really consider it as definite and known, and go over the same operations as to verify the answer when known.

EXAMPLES.

1. A merchant paid \$480 to two men, A and B, and he paid three times as much to B as to A. How many dollars did he pay to each? *Ans.* To A, \$120, to B, \$360.

Let . . . . $x=$ *the sum to* A,
Then . . . . $3x=$ *the sum to* B,
Sum . . . . $4x=$ *the sum paid to both.*
But . . . . . . $480=$ *the sum paid to both.*

Thus, when any question has been clearly and fully stated, it will be found that some condition has been represented in two ways; one having the *unknown* quantity in it, and the other having a *known* quantity. These two expressions must be put in the same line, with the sign $=$ between them, so as to form an *equation*. And then, by reducing the equation, the required result will be found.

Thus, . . . $4x=480$, therefore, $x=120$.

Observe that the problem would be essentially the same, whatever number of dollars were paid out. It is not necessary that the number should have been 480, any more than

48, or any other number. Therefore, to make the problem *more* general, we may represent the number of dollars paid out by a, and the equation will then be $4x=a$. And $x=\frac{a}{4}$.

Again, the problem would have been the same in character, and equally as simple, had the merchant paid 4 times, or 5 times, or n times as much to B as to A.

We may therefore make it general by stating it in the following words:

A merchant paid a *dollars to two men,* A *and* B, *and he paid* n *times as many dollars to* B *as to* A. *What did he pay to each?*

Let . . . . . $x=$ the sum to A,

Then . . . . $nx=$ the sum to B,

By add. . . . $x+nx$

Or . . . $(1+n)x=$ the sum to both.

Also, ... $a=$ the sum to both.

Therefore, . $(1+n)x=a$, or $x=\frac{a}{1+n}$.

This shows that the sum paid to A was $\frac{a}{1+n}$ dollars, and as B had n times as many, the sum to B was $\frac{na}{1+n}$.

For proof, $\frac{a}{1+n}+\frac{na}{1+n}$ must equal a. As the denominators are common, the sum of the two is $\frac{a+na}{1+n}$ or $\frac{(1+n)a}{1+n}$ or a, by suppressing the common factors in numerator and denominator

2. My horse and saddle are worth \$100, and my horse is worth 7 times my saddle. What is the value of each?

Ans. Saddle, \$12½; horse, \$87½.

3. My horse and saddle are worth a dollars, and my horse is worth n times my saddle. What is the value of each?

Ans. Saddle, $\frac{a}{1+n}$; horse, $\frac{na}{1+n}$.

4. A farmer said he had 4 times as many cows as horses, and 5 times as many sheep as cows; and the number of them all was 100. How many horses had he? *Ans.* 4.

5. A farmer said he had n times as many cows as horses, and m times as many sheep as cows; and the number of them all was a. How many horses had he?

Ans. $\frac{a}{1+n+mn}$ horses.

6. A school-girl said that she had 120 pins and needles; and that she had seven times as many pins as needles. How many had she of each sort? *Ans.* 15 needles, and 105 pins.

7. A teacher said that her school consisted of 64 scholars; and that there were three times as many in Arithmetic as in Algebra, and four times as many in Grammar as in Arithmetic. How many were there in each study?

Ans. 4 in Algebra; 12 in Arithmetic; and 48 in Grammar.

8. A certain school consisted of a number of scholars; a certain portion of them studied Algebra; n times as many studied Arithmetic, and there were m times as many in Grammar as in Arithmetic. How many were in Algebra?

Ans. $\frac{a}{1+n+mn}$.

9. A person said that he was \$450 in debt. That he owed A a certain sum, B twice as much, and C twice as much as to A and B. How much did he owe each?

Ans. To A, \$50; to B, \$100; to C, \$300.

10. A person said that he was owing to A a certain sum; to B four times as much; and to C eight times as much; and to D six times as much; so that \$570 would make him even with the world. What was his debt to A? *Ans.* \$30.

11. A person said that he was in debt to four individuals, A, B, C, and D, to the amount of a dollars; and that he was indebted to B, n times as many dollars as to A; to C, m times as many dollars as to A; and to D, p times as many dollars as to A. What was his debt to A?

Ans. $\frac{a}{1+n+m+p}$ dollars.

12. If \$75 be divided between two men in the proportion of 3 to 2, what will be the respective shares?

Ans. \$45 and \$30.

Let . . . . . $x=$ the greater share,

Then . . . $75-x=$ the other.

To answer the demands of the problem, we must have

$$x : 75-x :: 3 : 2$$

see example 13, (Art. 56). Observe the following method of solution:

Let $3x=$ the greater share, and $2x$ the smaller share,

Then $5x$ the two shares, must equal the whole sum.

That is, $5x=75$ or $x=15$. Therefore, $3x=45$, the greater share.

13. Divide \$150 into two parts, so that the smaller may be to the greater as 7 to 8. *Ans.* \$70; and \$80.

14. Divide \$1235 between A and B, so that A's share may be to B's as 3 to 2. *Ans.* A's share, \$741; B's, \$494.

N. B. When proportional numbers are required, it is generally most convenient to represent them by one unknown term, with coefficients of the given relation. Thus, numbers in proportion of 3 to 4, may be expressed by $3x$ and $4x$, and the proportion of a to b may be expressed by ax and bx.

15. Divide d dollars between A and B so that A's share may be to B's as m is to n.

Ans. A's share, $\frac{md}{m+n}$; B's, $\frac{nd}{m+n}$.

16. A gentleman is now 25 years old, and his youngest

brother is 15. How many years must elapse before their ages will be in the proportion of 5 to 4? *Ans.* 25 years.

$$25+x : 15+x :: 5 : 4.$$

17. Two men commenced trade together; the first put in \$40 more than the second; and the stock of the first was to that of the second as 5 to 4. What was the stock of each? *Ans.* \$200; and \$160.

18. A man was hired for a year for \$100, and a suit of clothes; but at the end of 8 months he left, and received his clothes and \$60 in money, as full compensation for the time expired. What was the value of the suit of clothes? *Ans.* \$20.

19. Three men trading in company gained \$780, which must be divided in proportion to their stock. A's stock was to B's as 2 to 3, and A's to C's was in the proportion of 2 to 5. What part of the gain must each receive? *Ans.* A, \$156; B, \$234; C, \$390.

Let . . . . $2x=$ A's share of the gain,
Then . . . . . . $3x=$ B's " " "
And . . . . . . $5x=$ C's " " "

Therefore, . . . $10x=780$, or $x=78$.

20. A field of 864 acres is to be divided among three farmers, A, B, and C; so that A's part shall be to B's as 5 to 11, and C may receive as much as A and B together. How much must each receive? *Ans.* A, 135; B, 297; C, 432 acres.

21. Three men trading in company, put in money in the following proportion; the first 3 dollars as often as the second 7, and the third 5. They gain \$960. What is each man's share of the gain? *Ans.* \$192; \$448; \$320.

22. A man has two flocks of sheep, each containing the same number; from one he sells 80, from the other 20; then the number remaining in the former is to that in the lat-

ter as 2 to 3. How many sheep did each flock originally contain? *Ans.* 200.

23. There are two numbers in proportion of 3 to 4; but if 24 be added to each of them, the two sums will be in the proportion of 4 to 5. What are the nnmbers?

Ans. 72 and 96.

24. A man's age when he was married was to that of his wife as 3 to 2; and when they had lived together 4 years, his age was to hers as 7 to 5. What were their ages when they were married? *Ans.* His age, 24; hers, 16 years.

25. A certain sum of money was put at simple interest, and in 8 months it amounted to \$1488, and in 15 months it amounted to \$1530. What was the sum? *Ans.* \$1440.

Let $x=$ the sum. The sum or principle subtracted from the amount will give the interest: therefore $1488-x$ represents the interest for 8 months, and $1530-x$ is the interest for 15 months.

Now, whatever be the rate per cent. double time will give double interest, &c. Hence, $8:15::1488-x:1530-x$.

N. B. To acquire true delicacy in algebraical operations, it is often expedient not to use large numerals, but let them be represented by letters. In the present example, let $a=1488$. Then $a+42=1530$, and the proportion becomes $8:15::a-x:a+42-x$.

Multiply extremes and means, then

$$8a+8\cdot42-8x=15a-15x$$

Drop $8a$ and $-8x$ from both members, and we have

$$8\cdot42=7a-7x$$

Dividing by 7, and transposing, we find

$$x=a-48=1440, \textit{ Ans.}$$

26. A certain sum of money was put at simple interest for $2\frac{1}{2}$ years, and in that time it amounted to \$3526, and in 38 months it amounted to \$3606. What was the sum put at interest? *Ans.* \$3226.

(Art. 58.) The object of solving problems should be to acquire a knowledge of the utility and the *power* of the science, and this knowledge cannot be attained to the fullest extent by merely solving problems; we must also learn how to propose them, and to propose such as are convenient and proper for instruction.

Problem 25 is extracted from an English work; and let the reader observe that the two amounts, \$1488, \$1530, and \$1440, the sum put at interest, are all *whole numbers,* no fraction of a dollar in any of them, which makes the problem a neat and convenient one.

The question now is, how the proposer discovered these numbers? Did he happen upon them? Did he find them by repeated trials? or did he deduce them naturally and easily from a scientific process?

We can best answer these questions by showing how we found the numbers to form problem 26.

Wanting another example of the same kind as 25, but of different *data,* I wrote on a slip of paper thus:

A sum of money was put at interest for $2\frac{1}{2}$ years, and the amount for that time was a dollars; and for 38 months the amount was $a+d$ dollars. What was the sum?

The amount for 38 months must be greater than the amount for 30 months, therefore d is a positive number.

Let x represent the sum lent. Then $a-x=$ the interest for 30 months, and $a+d-x=$ the interest for 38 months.

Hence, . . $30:38::a-x:a+d-x$.

Product of extremes and means gives

$$30a+30d-30x=38a-38x$$

Dropping $30a$ and $-30x$ from both members, we have

$$30d=8a-8x$$

Dividing by 8, and transposing, gives

$$x=a-\frac{30d}{8}$$

Here a stands alone, and any whole number greater than $\frac{30d}{8}$ can be written in its place; and if we take d of such a value as to render it divisible by 8, the fraction $\frac{30d}{8}$ will be a whole number, and cause x to be a whole number also.

In preparing the example, I took d equal 80, then $\frac{30d}{8}$ is in value 300; and I took a, *hap-hazard* at \$3526; therefore, $a+d=3606$, and $x=3226$, the numbers given in the problem. By taking different values to a and d, we may form as many numeral problems as we please like problem 25 or 26; and if, in every instance, we take care to take d of such a value as to render it divisible by 8, no fractions will appear in the problems.

Again, observe the expression $x=a-\frac{30d}{8}$. The numerator of the fraction has 30 for a coefficient, and that is the number of months that the sum of money was out at interest before the first amount was rendered; and 8, the denominator, is the number of months between the times of rendering the two amounts.

Observing these facts, we may solve another problem of the like kind without going through the steps of the process. For example.

27. *A certain sum of money was put at simple interest, and in* 13 *months it amounted to* a *dollars, and in* 20 *months it amounted to* a+d *dollars. What was the sum?* *Ans.* x.

And . . . . . $x=a-\frac{13d}{7}$ (1)

To form a numerical problem from equation (1), such as shall contain only whole numbers, and correspond to the times here mentioned, we must take $d=7$, or *some multiple of* 7.

Suppose we take $d=14$; then

$$x=a-26 \qquad (2)$$

Now it is my object to form another *numerical* problem of this kind, corresponding to the times mentioned in 27, having such numbers that the answer—the sum put at interest, shall be just 100 dollars.

Take equation (2), and in place of x write 100, transpose -26, and we have $a=126$, the first amount; and as $d=14$, $a+d$, the second amount, must be 140.

Hence, we may write the problem thus:

28. A certain sum of money was put at interest, and in 13 months the amount due was $126, and if continued at interest for 20 months, the amount due would have been $140. What was the sum put at interest? *Ans.* $100.

In equation (1), the fraction $\frac{13d}{7}$ is the interest on the sum for 13 months, because it is the sum, which, if added to the principal, will give the amount.

Here in these problems, d and a are perfectly arbitrary; we pay no attention to the rate of interest; and if we take d of any great value, there will be an unreasonable quantity of interest; and if d is taken very small in relation to a, the rate of interest will be small; but the algebraist can adjust the rate by putting $\frac{13d}{7}$, equal to any given rate of interest; but in a work like this, it is not proper to carry these investigations any further.

EQUATIONS CONTINUED.

(Art. 59.) *Problems in which fractions mostly occur.*

1. The number 12 is $\frac{3}{4}$ of what number?

Ans. It is $\frac{3}{4}$ of the number x.

To determine the numerical value of x, we solve the following equation, $\frac{3x}{4}=12$. Hence, 16 is the number.

2. The number a is $\frac{3}{4}$ of what number? *Ans.* $\frac{4a}{3}$.

3. The number 21 is $\frac{3}{7}$ of what number? *Ans.* 49.

4. The number 21 is the $\frac{m}{n}$th part of what number?

Ans. $\frac{21n}{m}$.

5. The number a is the $\frac{m}{n}$th part of what number?

Ans. $\frac{an}{m}$.

6. If you add together $\frac{1}{6}$ and $\frac{1}{7}$ of a certain number, the sum will be 130. What is the number? *Ans.* 420.

The following solution is taken from another book, and it is a fair specimen of the manner of teaching Algebra, both in this country and in England; but in this particular we insist on improvement.

Let . . . . $x=$ the number,

Then . . . . . . $\frac{x}{7}+\frac{x}{6}=130$

Multiplying both members by 7 and 6, or by 42, we have

$$6x+7x=5460$$
$$13x=5460$$
$$x=420$$

This is but *half Algebra.* An algebraist *never multiplies numbers together*, except in *final results*, or in some rare cases where it is impossible to do otherwise.

To avoid this, let numbers be represented by letters; and in place of 130 in the equation, write a to represent it, as taught in (Art. 53.)

Then . . . . . . $\frac{x}{7}+\frac{x}{6}=a$

Clearing of fractions, $6x+7x=42a$

Or . . . . $13x=42a$

Now, as a is divisible by 13, and the quotient 10, dividing both members by 13, gives $x=420$, without the least effort at numerical computation.

It is not, in fact, necessary to write a; we may retain the number as a factor, or what is better, take its *obvious factors*.

Thus, ... $6x+7x=42 \cdot 13 \cdot 10$

Uniting and suppressing the factors common to both members, and . . . . . . $x=420$

We extract a solution to the following problem:

7. A farmer wishes to mix 116 bushels of provender, consisting of rye, barley, and oats, so that it may contain $\frac{5}{7}$ as much barley as oats, and $\frac{1}{2}$ as much rye as barley. How much of each must there be in the mixture?

Stating the question, $x=$ oats; and $\frac{5x}{7}=$ barley.

Then, ... $\frac{1}{2}$ of $\frac{5x}{7}$ is $\frac{5x}{14}=$ rye.

Forming the equation, $x+\frac{5x}{7}+\frac{5x}{14}=116$

Multiplying by 14, . . $14x+10x+5x=1624$

Uniting terms, ... $29x=1624$

Dividing by 29, ... $x=56$ the *Ans.*

If we keep the factors separate, we have

$$29x=116 \cdot 14$$

Dividing by 29, gives $x=4 \cdot 14=56$.

Here we find a reason why the farmer wished to mix 116 bushels—not 100, or 115, or 117—it must be some multiple of 29 to have the different kinds of grain come out in whole numbers. Indeed, the numbers in all numeral problems are so chosen that the final coefficient of the unknown quantity *shall be* some factor in the other member; therefore it is *worse than useless* to hide the factors (as is often done), by laborious multiplication.

8. Divide 48 into two such parts, that if the less be divided by 4, and the greater by 6, the sum of the ouotients will be 9.

Ans. 12 and 36.

9. A clerk spends $\frac{2}{3}$ of his salary for his board, and $\frac{2}{3}$ of the remainder in clothes, and yet saves \$150 a year. What is his yearly salary? *Ans.* \$1350.

10. An estate is to be divided among 4 children, in the following manner:

The first is to have \$200 more than $\frac{1}{4}$ of the whole.
The second is to have \$340 more than $\frac{1}{5}$ of the whole.
The third is to have \$300 more than $\frac{1}{6}$ of the whole.
And the fourth is to have \$400 more than $\frac{1}{8}$ of the whole.
What is the value of the estate? *Ans.* \$4800.

11. Of a detachment of soldiers, $\frac{2}{3}$ are on actual duty, $\frac{1}{8}$ of them sick, $\frac{1}{5}$ of the remainder absent on leave, and the rest, which is 380, have deserted. What was the number of men in the detachment? *Ans.* 2280 men.

12. A man has a lease for 99 years, and being asked how much of it was already expired, answered that $\frac{2}{3}$ of the time past was equal to $\frac{4}{5}$ of the time to come. Required the time past and the time to come.

Assume $a=99$. *Ans.* Time past, 54 years.

13. It is required to divide the number 204 into two such parts, that $\frac{2}{5}$ of the less being taken from the greater, the remainder will be equal to $\frac{3}{7}$ of the greater subtracted from 4 times the less. *Ans.* The numbers are 154 and 50.

Put $a=204$, and resubstitute in the result.

14. In the composition of a quantity of gunpowder
The nitre was 10 pounds more than $\frac{2}{3}$ of the whole,
The sulphur $4\frac{1}{2}$ pounds less than $\frac{1}{6}$ of the whole,
The charcoal 2 pounds less than $\frac{1}{7}$ of the nitre.
What was the amount of gunpowder? *Ans.* 69 pounds.

Let . . . . $x=$ the whole,

Then . . . . . $\frac{2x}{3}+10=$ the nitre,

$\frac{x}{6}-4\frac{1}{2}=$ the sulphur,

$\frac{2x}{21}+\frac{10}{7}-2=$ the charcoal.

By addition, $\frac{2x}{3}+\frac{x}{6}+\frac{2x}{21}+\frac{10}{7}+3\frac{1}{2}=x.$

Multiply both members by 6, and

$$4x+x+\frac{4x}{7}+\frac{60}{7}+21=6x$$

Drop $5x$ from both members, then

$$\frac{4x}{7}+\frac{60}{7}+21=x$$

Multiply by 7, and drop $4x$ from both members,

And . . . . $60+21\cdot7=3x$

Dividing by 3, $20+7\cdot7=x$

Or, ... $69=x$

15. Divide \$44 between three men, A, B, and C, so that the share of A may be $\frac{3}{5}$ that of B, and the share of B, $\frac{3}{4}$ that of C. *Ans.* A, \$9; B, \$15, C, \$20.

Will the student find the reason why the problem requires the division of the number 44; why not 45, 47, or any other number, as well as 44?

Let $3x=$A's share, $5x=$B's, and $y=$C's share,

Then . . . . $\frac{3y}{4}=5x$

Or . . . . . . $y=\frac{20x}{3}=$ C's share.

Hence, . . . $3x+5x+\frac{20x}{3}=44$

Clearing of fractions, and uniting terms, we have

$$44x=44\cdot 3, \text{ or } x=3$$

If the problem had required the division of any other number of dollars, for instance, a dollars, the value of x would have been $\frac{3a}{44}$ dollars. Taking a equal 44, or any number of times 44, gives *whole numbers* for the respective shares.

16. What number is that, to which, if we add its $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, the sum will be 50? *Ans.* 24.

17. What number is that, to which, if we add its $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, the sum will be a? *Ans.* $\frac{24a}{50}$.

18. If A can build a certain wall in 10 days, and B can do the same in 14 days, what number of days will be required to build the wall, if they both work together?

Ans. $5\frac{5}{6}$ days.

Let x represent the days required. If A can do the work in 10 days, in one day he will do $\frac{1}{10}$ of it, and in x days he will do $\frac{x}{10}$ of the whole work. By the same reasoning, $\frac{x}{14}$ is the part of the work done by B.

Therefore, $\frac{x}{10}+\frac{x}{14}=1$. (1 is the whole work).

19. If A can do a piece of work in a days, and B can do the same in b days, how long will it take them, if they both work together? *Ans.* $\frac{ab}{a+b}$ days.

I now wish to propose a numerical problem in all respects like problem 18, except that the number of days shall be a whole number, and the answer shall be 8.

The answer to 19 is a *general* answer; and now if we require a particular answer, 8, we simply require the verification of the following equation.

$$\frac{ab}{a+b}=8$$

Or . . . . $ab=8a+8b$

In this equation, if we assume a, the equation will give b, or if we assume b, the equation will give a corresponding value to a. But whichever letter we assume, it must be assumed greater than 8; because it requires either man more than 8 days to do the work, for they together do it in 8 days.

Now, assume $a=12$, then the equation becomes

$$12b=8\cdot12+8b$$
$$4b=8\cdot12$$
$$b=24$$

We can now write out our numerical problem thus:

20. A can do a piece of work in 12 days; B can do the same in 24 days. How many days will be required, if they both work together? *Ans.* 8.

21. A young man, who had just received a fortune, spent $\frac{3}{8}$ of it the first year, and $\frac{4}{5}$ of the remainder the next year; when he had \$1420 left. What was his fortune?

Ans. \$11360.

22. If from $\frac{1}{3}$ of my hight in inches, 12 be subtracted, $\frac{1}{5}$ of the remainder will be 2. What is my hight?

Ans. 5 feet 6 inches.

23. A laborer, A, can perform a piece of work in 5 days, B can do the same in 6 days, and C in 8 days; in what time can the three together perform the same work?

Ans. $2\frac{2}{59}$ days.

Let $x=$ the number of days in which all three can do it.

24. After paying out $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had remaining 66 guineas. How many guineas had I at first?

Ans. 120.

25. In a certain orchard, $\frac{1}{2}$ are apple trees, $\frac{1}{4}$ peach trees, $\frac{1}{6}$ plum trees, 100 cherry trees, 100 pear trees. How many trees in the orchard? *Ans.* 2400.

26. A farmer has his sheep in five different fields, viz: $\frac{1}{4}$ in the first field, $\frac{1}{6}$ in the second, $\frac{1}{8}$ in the third, $\frac{1}{12}$ in the fourth, and 45 in the fifth field. How many sheep in the flock? *Ans.* 120.

27. A person at play, lost $\frac{1}{4}$ of his money, and then won 3 shillings; after which he lost $\frac{1}{3}$ of what he then had; and, on counting, found that he had 12 shillings remaining. What had he at first? *Ans.* 20 shillings.

(Art. 60). When equations contain *compound* fractions, and simple ones, clear them of the simple fractions first, and unite, as far as possible, all the simple terms.

We give a few examples to show the advantage of observing this expedient.

1. Given $\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}$ to find the value of x.

Multiply all the terms by the smallest denominator, 3. That is, divide all the *denominators* by 3, and

$$\frac{6x+7}{3}+\frac{7x-13}{2x+1}=2x+4$$

Multiplying again by 3, and dropping $6x+7$ from both members, we have

$$\frac{21x-39}{2x+1}=5$$

Clearing of fractions, transposing, &c., we find $x=4$.

2. Given $\frac{7x+16}{21}=\frac{x+8}{4x-11}+\frac{x}{3}$ to find x.

Multiply by 21, and from both members drop $7x$, then

$$16=\frac{21x+21\cdot 8}{4x-11}$$

Clearing of fractions

And . . . $64x-11\cdot 16=21x+21\cdot 8$

For the purpose of showing something of the spirit of Algebra, we will put $a=8$: after dropping $21x$ from both members,

Then . . $43x-11\cdot 2a=21a$

Or . . . $43x-22a=21a$

Or . . . . . $43x=43a$ or $x=a=8$.

3. Given $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$, to find x.

Multiplying by 36, and dropping $9x$ from both members,

Then . . . . . $20=\frac{36(4x-12)}{5x-4}$

4. Given $\frac{a-b}{x}+\frac{2a^2-b^2}{ax+bx}=\frac{(3a^2-2b^2)(3a^2+2b^2)}{a+b}$ to find x.

Multiply by x,

Then . . $a-b+\frac{2a^2-b^2}{a+b}=\frac{(3a^2-2b^2)(3a^2+2b^2)x}{a+b}$

Multiply by $(a+b)$, and unite known quantities,

Then . . . $3a^2-2b^2=(3a^2-2b^2)(3a^2+2b^2)x$

$$x=\frac{1}{3a^2+2b^2}.$$

5. Given $\frac{5x+5}{x+2}+\frac{9}{4}=\frac{6x-12}{x-2}$ to find x. . *Ans.* $x=2$.

6. Given $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$ to find x.

Ans. $x=72$.

7. Given $\frac{18x-19}{28}+\frac{11x+21}{6x+14}=\frac{9x+15}{14}$ to find x.

Ans. $x=7$.

8. Given $\frac{20x+36}{25}+\frac{5x+20}{9x-16}=\frac{4x}{5}+3\frac{11}{25}$ to find x.

Ans. $x=4$.

9. Given $\frac{6}{6-5}-\frac{3x}{2}+\frac{2}{3}=\frac{2x+5}{3}-\frac{3}{2}$ to find x.

Ans. $x=3$.

10. Given $\frac{1}{x-5}+5+x=\frac{3x+18}{3}$ to find x. . *Ans.* $x=6$.

11. Divide the number 48 into two such parts, that 7 divided by one part shall be equal to 5 divided by the other part. Required the parts. *Ans.* 28 and 20.

12. Divide the number 48 into two such parts, that one may be to the other as 7 to 5. Required the parts. *Ans.* 28 and 20.

13. A person in play, lost a fourth of his money, and then won back 3 shillings; after which he lost a third of what he now had, and then won back 2 shillings; lastly, he lost a seventh of what he then had, and then found he had but 12 shillings remaining. What had he at first?

Ans. 20 shillings.

14. A shepherd was met by a band of robbers, who plundered him of half of his flock and half a sheep over. Afterward a second party met him, and took half of what he had left, and half a sheep over; and soon after this, a third party met him and treated him in like manner; and then he had 5 sheep left? How many sheep had he at first?

Ans. 47 sheep.

15. A man bought a horse and chaise for 341 (a) dollars. Now, if $\frac{3}{8}$ of the price of the horse be subtracted from twice the price of the chaise, the remainder will be the same as if $\frac{5}{7}$ of the price of the chaise be subtracted from 3 times the price of the horse. Required the price of each.

Ans. Horse, \$152; chaise, \$189.

N. B. Let $8x=$ the price of the horse.

Or let . $7x=$ the price of the chaise.

Solve this question by both of these notations.

16. A laborer engaged to serve for 60 days, on these conditions: That for every day he worked he should have 75 cents and his board, and for every day he was idle he should forfeit 25 cents for damage and board. At the end of the time a settlement was made, and he received \$25. How many days did he work, and how many days was he idle?

The common way of solving such questions is to let $x=$ the days he worked; then $60-x$ represents the days he was idle. Then sum up the account and put it equal to $25.

Another method is, to consider that if he worked the whole 60 days, at 75 cents per day, he must receive $45. But for every day he was idle, he not only lost his wages, 75 cents, but 25 cents in addition. That is, he lost $1 every day he was idle.

Now, let $x=$ the days he was idle. Then, $x=$ the dollars he lost. And $45-x=25$ or $x=20$, the days he was idle.

17. A person engaged to work a days on these conditions: For each day he worked he was to receive b cents; for each day he was idle he was to forfeit c cents. At the end of a days he received d cents. How many days was he idle?

Ans. $\frac{ab-d}{b+c}$ days.

Let $x=$ the number of days he was idle.

Had he worked every day he must have received ab cents. But for every idle day we must diminish this sum by $(b+c)$ cents; and for x days, the diminution must be $(b+c)x$ cents. That is, $ab-(b+c)x=d$ by the question.

Hence, ... $x=\frac{ab-d}{b+c}$

18. A boy engaged to convey 30 glass vessels to a certain place, on condition of receiving 5 cents for every one he delivered safe, and forfeiting 12 cents for every one he broke. On settlement, he received 99 cents. How many did he break? *Ans.* 3.

19. A boy engaged to carry n glass vessels to a certain place, on condition of receiving a cents for every one he delivered, and to forfeit b cents for every one he broke. On settlement he received d cents. How many did he break?

Ans. The number represented by $\frac{an-d}{a+b}$.

SIMPLE EQUATIONS

CONTAINING TWO UNKNOWN QUANTITIES.

(Art. 61.) We have thus far considered such equations only as contained but one unknown quantity; but we now suppose the pupil sufficiently advanced to comprehend equations containing two or more unknown quantities.

There are many simple problems which one may meet with in Algebra, which cannot be solved by the use of a single *unknown* quantity, and there are also some which *may be solved* by a single letter, that may become much more simple by using two or more unknown quantities.

When two unknown quantities are used, two independent equations must exist, in which the value of the unknown letters must be the same in each. When three unknown quantities are used, there must exist three independent equations, in which the value of any one of the unknown letters is the same in each.

In short, there must be as many independent equations as unknown quantities used in the question.

An *independent* equation may be called a primitive or *prime* equation—one that is not derived from any other equation. Thus, $x+3y=a$, and $2x+6y=2a$, are not independent equations, because one can be derived from the other; but $x+3y=a$, and $4x+5y=b$, are independent equations, because neither one can be reduced to the other by any arithmetical operation.

The reason that two equations are required to determine two unknown quantities, will be made clear by considering the following equation:

$$x+y=20$$

This equation will be verified if we make $x=1$, and $y=19$, or $x=2$, and $y=18$, or $x=\frac{1}{2}$, and $y=19\frac{1}{2}$, &c., &c., without limit. But if we combine another equation with this, as

$x-y=4$, then we have to verify two equations with the same values to x and y, and only one value for x and one value for y will answer both conditions.

Thus, ... $x+y=20$
And . . . . . . $x-y=4$
By addition . . . $2x=24$
$x=12$, and $y=8$.

That is, we have found a value for x and another to y (12 and 8), so that their sum shall be 20, and their difference 4; and no other possible numbers will answer.

A merchant sends me a bill of 16 *dollars for* 3 *pairs of shoes and* 2 *pairs of boots; afterward he sends another bill of* 23 *dollars for* 4 *pairs of shoes and* 3 *pairs of boots, charging at the same rate. What was his price for a pair of shoes, and what for a pair of boots?*

This can be resolved by one unknown quantity, but it is far more simple to use two.

Let $x=$ the price of a pair of shoes,
And $y=$ the price of a pair of boots.
Then by the question $3x+2y=16$
And . . . . . . $4x+3y=23$

These two equations are independent; that is, one cannot be converted into the other by multiplication or division, notwithstanding the value of x and of y are the same in both equations.

Equations are *independent* when they express different conditions, and *dependent* when they express the same conditions under different forms.

To reduce equations involving two unknown quantities, it is necessary to perform some arithmetical operation upon them, which will cause one of the unknown quantities to disappear. These operations are called elimination.

There are three principal methods of *elimination.*

1. *By comparison.* 2. *By substitution.* 3. By addition or subtraction.

All the operations rest on the axioms.

FIRST METHOD.

(Art. 62.) Transpose the terms containing y to the right hand sides of the equations, and divide by the coefficients of x, and

From equation (A) we have $x=\frac{16-2y}{3}$ (C)

And from (B) we have . $x=\frac{23-3y}{4}$ (D)

Put the two expressions for x equal to each other (Ax 7),

And . . . . . . $\frac{16-2y}{3}=\frac{23-3y}{4}$

An equation which readily gives $y=5$, which, taken as the value of y in either equation (C) or (D), will give $x=2$.

SECOND METHOD.

(Art. 63.) To explain the second method of elimination, resume the equations

$$3x+2y=16 \quad (A)$$
$$4x+3y=23 \quad (B)$$

The value of x from equation (A) is $x=\frac{1}{3}(16-2y)$.

Substitute this value for x in equation (B), and we have $4\times\frac{1}{3}(16-2y)+3y=23$, an equation containing only y.

Reducing it, we find $y=5$, the same as before.

Observe, that this method consists in finding the value of one of the unknown quantities from one equation, and substituting that value in the other. Hence, it is properly called the method by *substitution*.

THIRD METHOD OF ELIMINATION.

(Art. 64.) Resume again $3x+2y=16$ (A)

$4x+3y=23$ (B)

When the coefficients of either x or y are the same in both equations, and the signs alike, that term will disappear by subtraction.

When the signs are unlike, and the coefficients equal, the term will disappear by addition.

To make the coefficients of x *equal, multiply each equation by the coefficient of* x *in the other.*

To make the coefficients of y *equal, multiply each equation by the coefficient of* y *in the other.*

Multiply equation (A) by 4 and $12x+8y=64$

Multiply equation (B) by 3 and $12x+9y=69$

Difference . . . . $y=5$, as before.

To continue this investigation, let us take the equations

$2x+3y=23$ (A)

$5x-2y=10$ (B)

Multiply equation (A) by 2, and equation (B) by 3, and we have $4x+6y=46$

$15x-6y=30$

Equations in which the coefficients of y are equal, and the signs unlike. In this case add, and the y's will destroy each other, giving . . . . $19x=76$

Or . . . . $x=4$

Of these three methods of elimination, sometimes one is preferable and sometimes another, according to the relation of the coefficients and the positions in which they stand.

No one should be prejudiced against either method; and in practice we use either one, or modifications of them, as the case may require. The forms may be disregarded when the principles are kept in view.

EXAMPLES.

1\. Given $\left\{ \begin{array}{ll} 4x+y=34 & (A) \\ x+4y=16 & (B) \end{array} \right\}$ to find x and y.

From (A) . . . . $x=\frac{34-y}{4}$

From (B) . . . . $x=16-4y$

Therefore . . $\frac{34-y}{4}=16-4y$ (Ax. 7).

Ans. $y=2$, $x=8$.

2\. Given $\left\{ \begin{array}{ll} 7x+4y=58 & (A) \\ 9x-4y=38 & (B) \end{array} \right\}$ to find x and y.

Ans. $x=6$, $y=4$.

Here, it would be very inexpedient to take the first method of elimination.

Observe that the coefficients of y are alike in number, but opposite in signs.

A skillful operator takes great advantage of circumstances, and very rarely goes through all the operations of set rules; but this skill can only be acquired by observation and practice.

Add the two equations. Why?

3\. Given $\left\{ \begin{array}{ll} 5x+6y=58 & (A) \\ 2x+6y=34 & (B) \end{array} \right\}$ to find x and y.

Ans. $x=8$; $y=3$.

Subtract (B) from (A). Why?

4\. Given $\left\{ \begin{array}{ll} 4x+3y=22 & (A) \\ -4x+2y=-12 & (B) \end{array} \right\}$ to find x and y.

Ans. $x=4$; $y=2$.

Add (A) and (B). Why?

5\. Given $\left\{ \begin{array}{ll} 6x+5y=128 & (A) \\ 3x+4y=88 & (B) \end{array} \right\}$ to find x and y.

Ans. $x=8$; $y=16$.

Multiply (B) by 2. Why?

6. Given $\begin{cases} 2x+3z=38 \\ 6x+5z=82 \end{cases}$ to find x and z.

Ans. $x=7$; $z=8$.

7. Given $\begin{cases} 4x+6y=46 \\ 5x-2y=10 \end{cases}$ to find x and y.

Ans. $x=4$; $y=5$.

8. Given $\begin{cases} 2x+3y=31 \\ 4x-3y=17 \end{cases}$ to find x and y.

Ans. $x=8$; $y=5$.

9. Given $\begin{cases} 4y+z=102 \\ y+4z=48 \end{cases}$ to find y and z.

Ans. $y=24$; $z=6$

10. Given $\begin{cases} 2x+3y=7 \\ 8x+10y=26 \end{cases}$ to find x and y.

Ans. $x=2$: $y=1$.

11. Given $\begin{cases} 5y+3x=93 \\ 3y+4x=80 \end{cases}$ to find y and x.

Ans. $y=12$; $x=11$.

12. Given $\frac{1}{2}x+\frac{1}{3}y=14$, and $\frac{1}{3}x+\frac{1}{2}y=11$, to find x and y.

Ans. $x=24$; $y=6$.

13. Given $x+\frac{1}{2}y=8$, and $\frac{1}{2}x+y=7$, to find x and y.

Ans. $x=6$; $y=4$.

14. Given $\frac{1}{7}x+7y=99$, and $\frac{1}{7}y+7x=51$, to find x and y.

Ans. $x=7$; $y=14$.

PROBLEMS PRODUCING EQUATIONS OF TWO UNKNOWN QUANTITIES.

1. A man bought 3 bushels of wheat and 5 bushels of rye for 38 shillings; and at another time, 6 bushels of wheat and 3 bushels of rye for 48 shillings. What was the price for a bushel of each?

Let $x=$ price of wheat, and $y=$ price of rye.

By the first condition, $3x+5y=38$ (*A*)

By the second, . . $6x+3y=48$ (*B*)

Ans. $x=6$; $y=4$.

2. A gentleman paid for 6 pairs of boots, and 4 pairs of shoes, \$44; and afterward, for 3 pairs of boots, and 7 pairs of shoes, \$32. What was the price of each per pair?

Ans. Boots, \$6; shoes, \$2.

3. A man spends 30 cents for apples and pears, buying his apples at the rate of 4 for a cent, and his pears at the rate of 5 for a cent. He afterward let his friend have half of his apples and one-third of his pears, for 13 cents, at the same rate. How many did he buy of each sort?

Let $x=$ the number of apples,

$y=$ " " of pears,

$\frac{1}{4}$ cent $=$ the price of 1 apple; hence, x apples are worth $\frac{x}{4}$ cents. Therefore,

By the first condition, . . $\frac{x}{4}+\frac{y}{5}=30$ (A)

By the second, . . . $\frac{x}{8}+\frac{y}{15}=13$ (B)

Multiplying (B) by 2, . $\frac{x}{4}+\frac{2y}{15}=26$ (1)

Subtracting (1) from (A), $\frac{y}{5}-\frac{2y}{15}=4$ (2)

Multiplying (2) by 15, . $3y-2y=60$, or $y=60$

This value of y put in (A), gives $\frac{x}{4}+12=30$, or $x=72$.

4. What fraction is that, to the numerator of which, if 1 be added, its value will $\frac{1}{3}$, but if 1 be added to the denominator, its value will be $\frac{1}{4}$?

Let . . . . $x=$ the numerator,

And . . . . $y=$ the denominator,

Then . . . . . . $\frac{x}{y}=$ the fraction, And we shall have the two equations,

$$\frac{x+1}{y}=\frac{1}{3} \quad (A)$$

$$\frac{x}{y+1}=\frac{1}{4} \quad (B)$$

Clearing of fractions, $3x+3=y$ (1)

And . . . . . $4x-1=y$ (2)

Taking (1) from (2), $x-4=0$, or $x=4$.

Hence, from 1 we find $y=15$, and the fraction is $\frac{4}{15}$.

5. What fraction is that, to the numerator of which, if 4 be added, the value is $\frac{1}{2}$, but if 7 be added to its denominator, its value will be $\frac{1}{5}$? *Ans.* $\frac{5}{18}$.

6. A and B have certain sums of money: says A to B, "Give me \$15 of your money, and I shall have five times as much as you have left." Says B to A, "Give me \$5 of your money, and I shall have exactly as much as you have left." How many dollars had each?

Ans. A had \$35; and B, \$25.

7. What fraction is that whose numerator being doubled, and its denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$? *Ans.* $\frac{4}{5}$.

8. If A give B \$5 of his money, B will have twice as much money as A has left; and if B give A \$5, A will have thrice as much as B has left. How much had each?

Ans. A had \$13; B, \$11.

9. A merchant has sugar at 9 cents and at 13 cents a pound, and he wishes to make a mixture of 100 pounds that shall be worth 12 cents a pound. How many pounds of each quality must he take?

Ans. 25 pounds at 9 cents, and 75 pounds at 13 cents.

10. A person has a saddle worth £50, and two horses. When he saddles the poorest horse, the horse and saddle are worth twice as much as the best horse; but when he saddles

the best, the horse and saddle are together worth three times the other. What is the value of each horse?

Ans. Best, £40; poorest, £30.

11. One day a gentleman employs 4 men and 8 boys to labor for him, and pays them 40 shillings; the next day he hires at the same rate, 7 men and 6 boys, for 50 shillings. What are the daily wages of each?

Ans. Man's, 5 shillings; boy's, 2 shillings 6 pence.

12. A merchant sold a yard of broadcloth and 3 yards of velvet, for \$25; and, at another time, 4 yards of broadcloth and 5 yards of velvet, for \$65. What was the price of each per yard? *Ans.* Broadcloth, \$10; velvet, \$5.

13. Find two numbers, such that half the first, with a third part of the second, make 9, and a fourth part of the first, with a fifth part of the second, make 5. *Ans.* 8 and 15.

14. A gentleman being asked the age of his two sons, answered, that if to the sum of their ages 18 be added, the result will be double the age of the elder; but if 6 be taken from the difference of their ages, the remainder will be equal to the age of the younger. What were their ages?

Ans. 30 and 12.

15. A says to B, "Give me 100 of your dollars, and I shall have as much as you." B replies, "Give me 100 of your dollars and I shall have twice as much as you. How many dollars has each? *Ans.* A, \$500; B, \$700.

16. Find two numbers, such that $\frac{3}{5}$ of the first and $\frac{2}{7}$ of the second added together, will make 12; and if the first be divided by 2, and the second multiplied by 3, $\frac{2}{3}$ of the sum of these results will be 26. *Ans.* 15 and $10\frac{1}{2}$.

17. Says A to B, "$\frac{1}{3}$ of the difference of our money is equal to yours; and if you give me \$2, I shall have five times as much as you." How much has each?

Ans. A, \$48; B, \$12.

18. A market-woman bought eggs, some at the rate of 2 for a cent, and some at the rate of 3 for 2 cents, to the amount

of 65 cents. She afterward sold them all for 100 cents, thereby gaining half a cent on each egg. How many of each kind did she buy? *Ans.* 50 of one; 60 of the other.

19. What two numbers are those, whose sum is a and difference b?

Let $x=$ the greater. *Ans.* The greater is $\frac{a}{2}+\frac{b}{2}$,

$y=$ the less. The less is $\frac{a}{2}-\frac{b}{2}$,

Sum is . . a

Difference is b.

(Art. 64.) From the result of this problem, we learn one important fact, which will be of use to us in solving other problems.

The fact is this: *That the half sum of any two numbers, added to the half difference, is the greater of the two numbers; and the half sum, diminished by the half difference, gives the less.*

20. There are two numbers whose sum is 100, and three times the less taken from twice the greater, gives 150 for remainder. What are the numbers? *Ans.* 90 and 10.

The half sum of the two numbers is 50. Now let $x=$ the half difference. Then $50+x=$ the greater number.

And . . . . . . $50-x=$ the less number.

Twice the greater is $100+2x$

Three times less is $150-3x$

Difference is . $-50+5x=150$.

21. What two numbers are those whose sum is 12, and whose product is 35? *Ans.* 7 and 5.

Let . . . . $6+x=$ the greater,

Then . . . . . . $6-x=$ the less.

Product, ... $36-x^2=35$. Hence, $x=1$.

22. What two numbers are those whose difference is 4, and product 96? *Ans.* 12 and 8.

Let $x=$ the half sum; $2=$ the half difference.

23. The difference of two numbers is 6, and the sum of their squares is 50. What are the numbers? *Ans.* 7 and 1.

24. The difference of two numbers is 8, and their product is 240. What are the numbers? *Ans.* 12 and 20.

(Art. 65.) To reduce equations containing three or more unknown quantities, we employ the same principles as for two unknown quantities, and no more; and from these principles we draw the following

Rule.—*Combine any one of the equations with each of the others, so as to eliminate the same unknown quantity; there will thus arise a new set of equations containing one less unknown quantity.*

In the same manner combine one of these new equations with each of the others, and thus obtain another set of equations containing one less unknown quantity than the last set; and so continue, until an equation is found containing one unknown quantity; solving this equation, and substituting the value of its unknown quantity in the other equations, the other unknown quantities are easily found.

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+y+z=9 \\ x+2y+3z=16 \\ x+3y+4z=21 \end{array}\right\}$ to find x, y, and z.

By the first method, transpose the terms containing y and z in each equation, and

$$x=9-y-z$$
$$x=16-2y-3z$$
$$x=21-3y-4z$$

Then putting the 1st and 2d values equal, and the 2d and 3d values equal, gives

$$9-y-z=16-2y-3z$$

$$16-2y-3z=21-3y-4z$$

Transposing and condensing terms,

And . . . . $y=7-2z$

Also, ... $y=5-z$

Hence, ... $5-z=7-2z$, or $z=2$.

Having $z=2$, we have $y=5-z=3$, and having the values of both z and y, by the first equation we find $x=4$.

2. Given $\left\{\begin{array}{l} 2x+4y-3z=22 \\ 4x-2y+5z=18 \\ 6x+7y-z=63 \end{array}\right\}$ to find x, y and z.

Multiplying the first equation by 2, .	$4x+8y-6z=44$
And subtracting the second, . . .	$4x-2y+5z=18$
The result is, (A)	$10y-11z=26$
Then multiply the first equation by 3,	$6x+12y-9z=66$
And subtract the third, ...	$6x+7y-z=63$
The result is, (B)	$5y-8z=3$
Multiply the new equation, (B), by 2,	$10y-16z=6$
And subtract this from equation (A),	$10y-11z=26$
The result is, ...	$5z=20$

Therefore, $z=4$.

Substituting the value of z in equation (B), and we find $y=7$.

3. Given $\left\{\begin{array}{l} 3x+9y+8z=41 \\ 5x+4y-2z=20 \\ 11x+7y-6z=37 \end{array}\right\}$ to find x, y, and z.

Ans. $x=2$; $y=3$; $z=1$.

(Art. 66.) When three, four, or more unknown quantities, with as many equations, are given, and their coefficients are all *prime* to each other, the operation is necessarily long. But when several of the coefficients are multiples, or *measures* of each other, or unity, several *expedients* may be resorted to for the purpose of facilitating calculation.

No specific rules can be given for mere expedients. Examples alone can illustrate. Some few expedients will be illustrated by the following

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+y+z=31 \\ x+y-z=25 \\ x-y-z=\ 9 \end{array}\right\}$ to find x, y, and z.

Subtract the 2d from the 1st, and $2z=6$.

Subtract the 3d from the 2d, and $2y=16$.

Add the 1st and 3d, and . . $2x=40$.

2. Given $\left\{\begin{array}{l} x+y+z=26 \\ x-y \quad\ \ =\ 4 \\ x-z \quad\ \ =\ 6 \end{array}\right\}$ to find x, y, and z.

Add all three, and $3x=36$, or $x=12$.

3. Given $\left\{\begin{array}{l} x-y-z=\ 6 \\ 3y-x-z=12 \\ 7z-y-x=24 \end{array}\right\}$ to find x, y, and z.

4. Given $x+\frac{1}{2}y=100$, $y+\frac{1}{3}z=100$, $z+\frac{1}{4}x=100$, to find x, y, and z.

Put $a=100$. *Ans.* $x=64$; $y=72$; and $z=84$.

5. Given $\left\{\begin{array}{l} u+v+x+y=10 \\ u+v+z+x=11 \\ u+v+z+x=12 \\ u+x+y+z=13 \\ v+x+y+z=14 \end{array}\right\}$ to find the value of each.

Here are *five* letters and five equations. Each letter has the *same* coefficient, *one* understood. Each equation has 4 letters, z is wanting in the 1st equation, y in the 2d, &c.

Now assume $u+v+x+y+z=s$.

Then . . . . . . $s-z=10$ (A)

$$s-y=11$$

$$s-x=12$$

$$s-v=13$$

$$s-u=14$$

Add, and . . . . $5s-s=60$ Or $s=15$.

Put this value of s in equation (A), and $z=5$, &c.

6. Given $x+y=a$, $x+z=b$, $y+z=c$.

Add the 1st and 2d, and from the sum subtract the 3d.

Ans. $x=\frac{1}{2}(a+b-c)$, $y=\frac{1}{2}(a+c-b)$, $z=\frac{1}{2}(b+c-a)$.

5. Reduce the equations $\left\{\begin{array}{l} x+y=52 \\ y+z=82 \\ z+w=68 \\ w+u=30 \\ u+x=32 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=20, \\ y=32, \\ z=50, \\ w=18, \\ u=12. \end{array}\right.$

6. Reduce the equations $\left\{\begin{array}{r} \frac{1}{3}x+3y=23 \\ x+\frac{z}{4}=8 \\ y+3z=31 \\ x+y+z+2w=39 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=6, \\ y=7, \\ z=8, \\ w=9. \end{array}\right.$

7. Reduce the equations $\left\{\begin{array}{r} 4x+2y-3z=4 \\ 3x-5y+2z=22 \\ x+y+z=12 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=5, \\ y=1, \\ z=6. \end{array}\right.$

8. Reduce the equations $\left\{\begin{array}{r} \frac{1}{2}x+\frac{1}{3}y+z=46 \\ \frac{1}{4}x-y+z\frac{1}{2}=9 \\ x+\frac{1}{4}y-\frac{1}{8}z=19 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=20, \\ y=12, \\ z=32. \end{array}\right.$

QUESTIONS PRODUCING EQUATIONS

CONTAINING THREE OR MORE UNKNOWN QUANTITIES

1. There are three persons, A, B, and C, whose ages are as follows: If B's age be subtracted from A's, the difference will be C's age; if five times B's age and twice C's, be added together, and from their sum A's age be subtracted, the remainder will be 147. The sum of all their ages is 96. What are their ages? *Ans.* A's, 48; B's, 33; C's, 15.

2. Find what each of three persons, A, B, and C, is worth, from knowing, 1st, that what A is worth added to 3 times what B and C are worth make 4700 dollars; 2d, that what B is worth added to 4 times what A and C are worth make 5800 dollars; 3d, that what C is worth added to 5 times what A and B are worth make 6300 dollars.

Ans. A is worth \$500; B, \$600; C, \$800.

3. A gentleman left a sum of money to be divided among four servants, so that the share of the first was $\frac{1}{2}$ the sum of the shares of the other three; the share of the second, $\frac{1}{3}$ of the sum of the other three; and the share of the third, $\frac{1}{4}$ the sum of the other three; and it was found that the share of the last was 14 dollars less than that of the first. What was the amount of money divided, and the shares of each respectively?

Ans. The sum was \$120; the shares, 40, 30, 24 and 26.

4. The sum of three numbers is 59; $\frac{1}{2}$ the difference of the first and second is 5, and $\frac{1}{2}$ the difference of the first and third is 9; required the numbers. *Ans.* 29, 19, and 11.

5. There is a certain number consisting of two places, a unit and a ten, which is four times the sum of its digits, and if 27 be added to it, the digits will be inverted. What is the number? *Ans.* 36.

NOTE.—Undoubtedly the reader has learned in Arithmetic that numerals have a specific and a local value, and every remove from the unit multiplies by 10. Hence, if x represents a digit in the place of tens, and y in the place of units, the number must be expressed by $10x+y$. A number consisting of three places, with x, y, and z to represent the digits, must be expressed by $100x+10y+z$.

6. A number is expressed by three figures; the sum of these figures is 11; the figure in the place of units is double that in the place of hundreds, and when 297 is added to this number, the sum obtained is expressed by the figures of this number reversed. What is the number? *Ans.* 326.

7. Divide the number 90 into three parts, so that twice the first part increased by 40, three times the second part increased by 20, and four times the third part increased by 10, may be all equal to one another.

Ans. First part, 35; second, 30; and third, 25.

If the object is merely to solve the seventh example, it would not be expedient to use three unknown symbols.

Let $\frac{1}{2}(x-40)=$ the first part, &c.

8. Find three numbers, such that the first with $\frac{1}{3}$ of the other two, the second with $\frac{1}{4}$ of the other two, and the third with $\frac{1}{5}$ of the other two, shall be equal to 25.

Ans. 13, 17, and 19.

9. A man with his wife and son, talking of their ages, said that his age, added to that of his son, was 16 years more than that of his wife; the wife said that her age, added to that of her son, made 8 years more than that of her husband; and that all their ages added together amounted to 88 years. What was the age of each?

Ans. Husband, 40, wife, 36, and son 12 years.

10. There are three numbers, such that the first, with $\frac{1}{2}$ the second, is equal to 14; the second, with $\frac{1}{3}$ part of the third, is equal to 18; and the third, with $\frac{1}{4}$ part of the first, is equal to 20; required the numbers. *Ans.* 8, 12, and 18.

11. Find three members, such that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third shall be equal to 62; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third equal to 47; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third equal to 38.

Ans. 24, 60, and 120.

12. There are two numbers, such that $\frac{1}{2}$ the greater added to $\frac{1}{3}$ the lesser, is 13; and if $\frac{1}{2}$ the lesser is taken from $\frac{1}{3}$ the greater, the remainder is nothing. Required the numbers.

Ans. 18 and 12.

12. Find three numbers of such magnitude, that the first with the $\frac{1}{2}$ sum of the other two, the second with $\frac{1}{3}$ of the other two, and the third with $\frac{1}{4}$ of the other two, may be the same, and amount to 51 in each case. *Ans.* 15, 33, and 39.

14. A said to B and C, "Give me, each of you, 4 of your sheep, and I shall have 4 more than you will have left." B said to A and C, "If each of you will give me 4 of your sheep, I shall have twice as many as you will have left." C then said to A and B, "Each of you give me 4 of your sheep, and I shall have three times as many as you will have left." How many had each? *Ans.* A, 6; B, 8; C, 10.

15. A person bought three silver cups; the price of the first, with $\frac{1}{2}$ the price of the other two, was 25 dollars; the price of the second, with $\frac{1}{3}$ of the price of the other two, was 26 dollars; and the price of the third, with $\frac{1}{2}$ the price of the other two, was 29 dollars; required the price of each.

Ans. \$8, \$18, and \$16

16. A's age is double that of B's, and B's is triple that of C's, and the sum of all their ages is 140; what is the age of each? *Ans.* A's $=84$; B's $=42$; C's $=14$.

17. A man wrought 10 days for his neighbor, his wife 4 days, and son 3 days, and received 11 dollars and 50 cents; at another time he served 9 days, his wife 8 days, and his son 6 days, at the same rates as before, and received 12 dollars; a third time he served 7 days, his wife 6 days, his son 4 days,

at the same rates as before, and received 9 dollars. What were the daily wages of each?

Ans. Husband's wages, \$1.00; wife, 0; son, 50 cts.

(Art. 67.) In this last example we put x to represent the daily wages of the husband, y the wages of the wife, and z the wages of the son, and in conclusion, y was found equal to 0; but it might have came out a *minus* quantity, and if it had, it would have shown that the presence of the wife was not a source of income, but expense; and if correct results are given at the settlements, the *signs* to the different quantities will show whether any particular individual received wages or was on expense for board, as in the following problems:

18. A man worked for a person ten days, having his wife with him 8 days, and his son 6 days, and he received 10 dollars and 30 cents as compensation for all three; at another time he wrought 12 days, his wife 10 days, and son 4 days, and he received 13 dollars and 20 cents; at another time he wrought 15 days, his wife 10 days, and his son 12 days, at the same rates as before, and he received 13 dollars 85 cents. What were the daily wages of each?

Ans. The husband 75 cts.; wife, 50 cts. The son 20 cts. *expense* per day.

Here the language of the problem is improper, as it implies that all received wages; but the solution shows that this could not be the case; for the value of the son's wages comes out *minus*, which is *opposition to plus* or to positive wages, that is, expense.

A stronger illustration of this principle will be shown by the following problem:

19. Two men, A and B, commenced trade at the same time; A had 3 times as much money as B, and continuing in trade, A gains 400 dollars, and B 150 dollars; now A has twice as much money as B. How much did each have at first?

Without any special consideration of the question, it implies that both had money, and asks how much. But on resolving the question with x to represent A's money, and y B's, we find . . . . $x=-300$

And . . . . $y=-100$ dollars.

That is, they had no money, and the minus sign in this case indicates *debt;* and the solution not only reveals the numerical values, but the true conditions of the problem, and points out the necessary corrections of language to correspond to an arithmetical sense.

That is, the problem should have been written thus:

A *is three times as much in debt as* B; *but* A *gains* 400 *dollars, and* B 150; *now* A *has twice as much money as* B. *How much were each in debt?*

As this enunciation corresponds with the real circumstances of the case, we can resolve the problem without a *minus* sign in the result. Thus:

Let $x=$ B's debt, then $3x=$ A's debt.

$150-x=$ B's money, $400-3x=$ A's money.

Per question, $400-3x=300-2x$. Or $x=100$.

20. What number is that whose fourth part exceeds its third part by 12? *Ans.* -144.

But there is no such abstract number as -144, and we cannot interpret this as *debt.* It points out error or *impossibility*, and by returning to the question we perceive that a fourth part of any number whatever cannot exceed its third part; it must be, its third part exceeds its fourth part by 12, and this enunciation gives the positive number, 144. Thus do equations rectify *subordinate* errors, and point out special conditions.

21. A man when he was married was 30 years old, and his

wife 15. How many years must elapse before his age will be three times the age of his wife?

Ans. The question is incorrectly enunciated; $7\frac{1}{2}$ years *before* the marriage, *not* after, their ages bore the specified relation.

22. What fraction is that which becomes $\frac{3}{5}$ when 1 is added to its numerator, and becomes $\frac{5}{7}$ when 1 is added to its denominator?

Ans. In an arithmetical sense, there is no such fraction. The algebraic expression, $\frac{-10}{-15}$, will give the required results.

23. Divide the number 10 into two such parts that their product shall be 50.

Let . . . . $x+y=$ the greater number,

And . . . . $x-y=$ the less.

Then . . . . $2x=10$, or $x=5$.

The product of the two numhers is x^2-y^2, and by the question must be equal to 50.

That is, ... $x^2-y^2=50$.

But, $x=5$; hence, $x^2=25$, which, drop from both members,

And . . . . $-y^2=25$

Or . . . . $y^2=-25$

That is, the question calls for two equal factors whose product is *minus* 25; but equal factors will never give a minus product; there is no 2d root of -25, and the value of the unknown quantity in such cases is said to be imaginary, which shows that the problem is impossible.

Here, $y=\pm5\sqrt{-1}$, a value that has no existence in numbers.

SECTION III.

INVOLUTION.

(Art. 68.) Equations, and the resolution of problems producing equations, do not always result in the first powers of the unknown quantities, but different powers are frequently involved, and therefore it is necessary to investigate methods of resolving equations containing higher powers than the first; and preparatory to this we must learn involution and evolution of algebraic quantities.

(Art. 69.) Involution is the method of raising any quantity to a given power. Evolution is the reverse of involution, and is the method of determining what quantity raised to a proposed power will produce a given quantity.

As in Arithmetic, involution is performed by multiplication, and evolution by the extraction of roots.

The first power is the root or quantity itself.

The second power, commonly called the *square*, is the quantity multiplied by itself.

The third power is the product of the second power by the quantity.

The fourth power is the third power multiplied into the quantity, &c.

Or we may consider the *second* power of a quantity to be the quantity taken *twice* as a *factor*.

The *third* power is the quantity taken *three times* as a *factor*.

The *fourth* power is the quantity taken *four times* as a factor.

The *tenth* power is the quantity taken ten times as a factor; and so on for any other power.

The *n*th would be the quantity taken n times as a factor.

Thus, let a represent any quantity.

Its *first* power is . . . $a=a$

Its *second* power is . . $a \cdot a=a^2$

The *third* power is . . $a \cdot a \cdot a=a^3$

The *fourth* power is . $a \cdot a \cdot a \cdot a=a^4$

The *fifth* power is $a \cdot a \cdot a \cdot a \cdot a=a^5$

In general terms a to the nth power is $a \cdot a$, &c., $=a^n$, and n may be any number whatever.

(Art. 70.) When the quantity is negative, *all the odd powers* will be *minus*, and all the *even* powers will be plus.

For, by the rules of multiplication,

$$-a \times -a = +a^2$$

And . . . . $-a \times -a \times -a = -a^3$

&c.=&c.

(Art. 71.) When we require the 5th power of a, we simply write a^5; when the 8th power we write a^8, &c., for any other power.

That is, a is the same as a^1, the exponent 1 is understood; and when we require the nth power of a, we conceive its exponent 1 understood multiplied by n, which makes a^n.

The *second* power of a^3 is $a^3 \times a^3 = a^6$.

The *third* power of a^3 is $a^3 \cdot a^3 \cdot a^3 = a^9$.

The *tenth* power of a^3 is a^{30}.

The nth power of a^3 is a^{3n}.

That is, it is the exponent of the quantity repeated as many times as there are *units* in the index of the power.

Thus, the 7th power of a^4 has the exponent of a (4) repeated 7 times, and the result is a^{28}.

From this we derive the following rule to raise a single quantity to any power.

RULE.—*Multiply the exponent of the quantity by the index of the required power.*

EXAMPLES.

Raise x^2 to the 3d power. . . . . *Ans.* x^6.

Raise y^5 to the 4th power. . . . . *Ans.* y^{20}.

Raise P^7 to the 5th power. . . . . *Ans.* P^{35}.

Raise x^3 to the 4th power. . . . . *Ans.* x^{12}.

Raise y^7 to the 3d power. . . . . *Ans.* y^{21}.

Raise x^n to the 6th power. . . . . *Ans.* x^{6n}.

Raise x^n to the mth power. . . . . *Ans.* x^{mn}.

Raise ax^2 to the 3d power. . . . . *Ans.* a^3x^6.

Raise ab^2x^4 to the 2d power. . . . . . . *Ans.* $a^2b^4x^8$.

Raise c^2y^4 to the 5th power. . . . . . . *Ans.* $c^{10}y^{20}$.

(ART. 72.) By the definition of powers, the second power is any quantity multiplied by itself; hence the second power of ax is a^2x^2, the second power of the coefficient a, as well as the other quantity x; but a may be a numeral, as $6x$, and its second power is $36x^2$. *Hence, to raise any simple quantity to any power, raise the numeral coefficient, as in Arithmetic, to the required power, and annex the powers of the given literal quantities.*

EXAMPLES.

1. Required the 3d power of $3ax^2$. . . . *Ans.* $27a^3x^6$
2. Required the 4th power of $\frac{2}{3}y^2$. . . . *Ans.* $\frac{16}{81}y^8$
3. Required the 3d power of $-2x$. . . . *Ans.* $-8x^3$
4. Required the 4th power of $-3x$ . . . *Ans.* $81x^4$
5. Required the 2d power of $8a^2b^3$. . . . *Ans.* $64a^4b^6$.
6. Required the 3d power of $5x^2z$. . . . *Ans.* $125x^6z^3$.

7. Required the 3d power of $6a^5y^2x$. . *Ans.* $216a^{15}y^6x^3$.

8. Required the 4th power of $2a^2b^3c^4$. . *Ans.* $16a^8b^{12}c^{16}$.

(ART. 73.) When the quantity to be raised to a power is a fraction, we must observe the rules for the multiplication of fractions, and multiply numerators by numerators, and denominators by denominators.*

Thus, the 2d power of $\frac{a}{b}$ is $\frac{a\times a}{b\times b}=\frac{a^2}{b^2}$

Hence, to raise fractions to powers, we have the following

RULE.—*Raise both numerator and denominator to the required power.*

EXAMPLES.

Observe, that by the rules laid down for multiplication, the *even* powers of minus quantities must be *plus*, and the odd powers *minus*.

1. Required the 2d power of $\frac{2a^2b^6}{5c}$. . . . *Ans.* $\frac{4a^4b^{12}}{25c^2}$.

2. Required the 6th power of $-\frac{2a}{3x}$. . . . *Ans.* $\frac{64a^6}{729x^6}$.

---

* Suppose we were required to raise $\frac{a}{b}$ to the fifth power, and did not know whether the denominator was to be raised or not, we could decide the point by means of an equation, as follows:

The fraction has *some value* which we represent by a symbol, say P. Then $P=\frac{a}{b}$. Now if we can find the true 5th power of P, it will be the required 5th power of the fraction.

Clearing the equation of fractions, we have

$$bP=a$$

Taking the 5th power of both members gives

$$b^5P^5=a^5$$

By division, ... $P^5=\frac{a^5}{b^5}$

This equation shows that to raise any fraction to any power, the numerator and denominator must be raised to that power

3. Required the 6th power of $\frac{a^2b}{\frac{1}{3}x}$. . . *Ans.* $\frac{729a^{12}b^6}{x^6}$.

4. Required the 6th power of $\frac{2}{3}a^2b$. . . *Ans.* $\frac{64}{729}a^{12}b^6$.

5. Required the 2d power of $\frac{3}{a^2}$. . . . . . *Ans.* $\frac{9}{a^4}$.

6. Required the 3d power of $\frac{ac}{x^2y}$. . . . *Ans.* $\frac{a^3c^3}{x^6y^3}$.

7. Required the 4th power of $-\frac{2y}{5x}$. . . . *Ans.* $\frac{16y^4}{625x^4}$.

8. Required the 3d power of $\frac{av}{3y}$. . . . *Ans.* $\frac{a^3v^3}{27y^3}$.

(Art. 74.) The powers of compound quantities are raised by the application of the rule for compound multiplication, (Art. 14).

Let $a+b$ be raised to the 2d, 3d, 4th, &c., powers.

$$
\begin{array}{ll}
 & a+b \\
 & \underline{a+b} \\
 & a+ab \\
 & \quad\quad ab+b^2 \\
\text{2d power or square,} & \overline{a^2+2ab+b^2} \\
 & \underline{a+b} \\
 & a^3+2a^2b+ab^2 \\
 & \quad\quad a^2b+2ab^2+b^3 \\
\text{3d power or cube,} & \overline{a^3+3a^2b+3ab^2+b^3} \\
 & \underline{a+b} \\
 & a^4+3a^3b+3a^2b^2+ab^3 \\
 & \quad\quad a^3b+3a^2b^2+3ab^3+b^4 \\
\text{The 4th power,} \quad . & \overline{a^4+4a^3b+6a^2b^2+4ab^3+b^4} \\
 & \underline{a+b} \\
 & a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
 & \quad\quad a^4b+4a^3b^2+6a^2b^3+4ao^4+b^5 \\
\text{The 5th power,} \quad . & \overline{a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5}
\end{array}
$$

By inspecting the result of each product, we may arrive at general principles, according to which any power of a binomial may be expressed, without the labor of actual multiplication. This theorem for abbreviating powers, and its general application to both powers and roots, first shown by Sir Isaac Newton, has given it the name of Newton's binomial, or the *binomial theorem.*

Observations.—Observe the 5th power: a, being the first, is called the leading term; and b, the second, is called the following term. The *sum* of the exponents of the two letters in each and all of the terms amount to the index of the power. In the 5th power, the sum of the exponents of a and b is 5; in the 4th power it is 4; in the 10th power it would be 10, &c. In the 2d power there are *three* terms; in the 3d power there are 4 terms; in the 4th power there are 5 terms; always *one more* term than the index of the power denotes.

The 2d letter does not appear in the first term; the 1st letter does not appear in the last term.

The highest power of the leading term is the index of the given power, and the powers of that letter decrease by *one* from term to term. The second letter appears in the 2d term, and its exponent increases by one from term to term, as the exponent of the other letter decreases.

The 8th power of $(a+b)$ is indicated thus, $(a+b)^8$. When expanded, its literal part (according to the preceding observations) must commence with a^8, and the sum of the exponents of every term amount to 8, and they will stand thus $a^8, a^7b, a^6b^2, a^5b^3, a^4b^4, a^3b^5, a^2b^6, ab^7, b^8$.

The coefficients are not so obvious. However, we observe that the coefficients of the first and last terms must be *unity* The coefficients of the terms next to the first and last are equal, and are the *same* as the index of the power. The coefficients *increase* to the middle of the series, and then *decrease* in the same manner, and it is manifested that there must be some law of connection between the exponents and the coefficients.

By inspecting the 5th power of $a+b$, we find that the 2d coefficient is 5, and the 3d is 10.

$$\frac{5\times 4}{2}=10$$

The 3d coefficient is the 2d, multiplied by the exponent of the leading letter, and *divided by the exponent of the second letter increased by unity.*

In the same manner, the fourth coefficient is the third, multiplied by the exponent of the leading letter, and divided by the exponent of the second letter *increased* by *unity*, and so on from coefficient to coefficient.

The 4th coefficient is $\frac{10\times 3}{3}=10$

The 5th is . . . . $\frac{10\times 2}{4}=5$

The last is . . . . $\frac{5\times 1}{5}=1$, understood.

Now let us expand . . $(a+b)^8$

For the 1st term write . . a^8

For the 2d term write . . $8a^7b$

For the 3d, $\frac{8\times 7}{2}=28$. $28a^6b^2$

For the 4th, $\frac{28\times 6}{3}$ . . . $56a^5b^3$

For the 5th, $\frac{56\times 5}{4}$ . . . $70a^4b^4$

Now, as the exponents of a and b are equal, we have arrived at the middle of the series, and of course to the highest coefficient. The coefficients now decrease in the reverse order in which they increased.

Hence, the expanded power is

$$a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8.$$

Let the reader observe, that the exponent of b, increased by unity, is always equal to the number of terms from the beginning or from the left of the power. Thus, b^2 is in the 3d term, &c. Therefore in finding the coefficients, we may divide by the number of terms already written, in place of the exponents of the second term increased by unity.

If the binomial $(a+b)$ becomes $(a+1)$, that is, when b becomes unity, the 8th power becomes,

$$a^8+8a^7+28a^6+56a^5+70a^4+56a^3+28a^2+8a+1.$$

Any power of 1 is 1, and 1 as a factor never appears.

If a becomes 1, then the expanded power becomes,

$$1+8b+28b^2+56b^3+70b^4+56b^5+28b^6+8b^7+b^8.$$

The manner of arriving at these results is to represent the unit by a letter, and *expand the simple literal terms*, and afterward substitute their values in the result.

(Art. 74.) If we expand $(a-b)$ in place of $(a+b)$, the exponents and coefficients will be precisely the same, but the principles of multiplication of quantities affected by different signs will give the *minus* sign to the second and to every alternate term.

Thus, the 6th power of $(a-b)$ is

$$a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6.$$

(Art. 75.) This method of readily expanding the powers of a *binomial* quantity is one application of the "*binomial theorem*," and it was thus by induction and by observations on the result of particular cases that the theorem was established. Its rigid demonstration is somewhat difficult, but its application is simple and useful.

Its most general form may arise from expanding $(a+b)^n$.

When $n=3$, we can readily expand it.

When $n=4$, we can expand it.

When $n=$ any whole positive number, we can expand it.

Now let us operate with n just as we would with a known number, and we shall have

$$(a+b)^n=a^n+na^{n-1}b+n\frac{n-1}{2}a^{n-2}b^2, \text{ \&c.}$$

We know not where the series would terminate, until we know the value of n. We are convinced of the truth of the result, when n represents any positive whole number; but let n be negative or fractional, and we are not so sure of the result.

The result would be true, however, whatever be the value of n; but this requires demonstration, and a deeper investigation than it would be proper to go into in a work like this.

When n is a fraction, the operation is *extracting a root* in place of *expanding a power.*

But for the demonstration of the binomial theorem, and its application to the extraction of roots, we refer the reader to our University Edition of Algebra.

EXAMPLES.

1. Expand $(x+y)^3$. . . . . *Ans.* $x^3+3x^2y+3xy^2+y^3$.

2. Expand $(y+z)^7$.
Ans. $y^7+7y^6z+21y^5z^2+35y^4z^3+35y^3z^4+21y^2z^5+7yz^6+z^7$.

3. Required the third power of $3x+2y$.

We cannot well expand this by the binomial theorem, because the terms are not simple *literal quantities.* But we can assume $3x=a$ and $2y=b$. Then

$$3x+2y=a+b, \text{ and } (a+b)^3=a^3+3a^2b+3ab^2+b^3$$

Now to return to the values of a and b, we have,

$$a^3 = \qquad 27x^3$$
$$3a^2b = 3\times 9x^2\times 2y = 54x^2y$$
$$3ab^2 = 3\times 3x\times 4y^2 = 36xy^2$$
$$b^3 = \qquad 8y^3$$

Hence, $(3x+2y)^3 = 27x^3+54x^2y+36xy^2+8y^3$.

4. Required the 4th power of $2a^2-3$.

Let $x=2a^2$, $y=3$. Then expand $(x-y)^4$, and return the values of x and y, and we shall find the result.

$$16a^8-96a^6+216a^4-216a^2+81.$$

5. Required the cube of $(a+b+c+d)$.

As we can operate in this summary manner *only* on *binomial* quantities, we represent $a+b$ by x, or assume $x=a+b$, and $y=c+d$.

Then $(x+y)^3=x^3+3x^2y+3xy^2+y^3$.

Returning the values of x and y, we have

$$(a+b)^3+3(a+b)^2(c+d)+3(a+b)(c+d)^2+(c+d)^3.$$

Now we can expand by the binomial, these quantities contained in parentheses.

6. Required the 4th power of $2a+3x$.

Ans. $16a^4+96a^3x+216a^2x^2+216ax^3+81x^4$.

7. Expand $(x^2+3y^2)^5$.

Ans. $x^{10}+15x^8y^2+90x^6y^4+270x^4y^6+405x^2y^8+243y^{10}$.

8. Expand $(2a^2+ax)^3$. *Ans.* $8a^6+12a^5x+6a^4x^2+a^3x^3$.

9. Expand $(x-1)^6$.

Ans. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

10. Expand $(3x-5)^3$. *Ans.* $27x^3-135x^2+225x^2-125$.

11. Expand $(2a-5b)^3$. *Ans.* $8a^3-60a^2b+150ab^2-125b^3$.

12. Expand $(4a^3b-2c^2)^4$.

Ans. $256a^{12}b^4-512a^9b^3c^2+384a^6b^2c^4-128a^3bc^6+16c^8$.

EVOLUTION.

(Art. 76.) Evolution is the converse of involution.

Involution is the expanding of roots to powers. Evolution is extracting the root when the power is given.

To find rules for operation, we must *observe* how powers are formed, and then we shall be able to *trace the operations back*. Thus, to square a, we double its exponent, which makes a^2, (Art. 71). The square of a^2 is a^4, the cube of a^2 is a^6, &c. Take the 4th power of x, and we have x^4. The nth power of x^4 is x^{4n}, &c., &c.

Now, if *multiplying* exponents raises simple literal quantities to powers, *dividing* exponents must extract roots. Thus, the square root of a^4 is a^2. The cube root of a^2 must be $a^{\frac{2}{3}}$.

The cube root of a must have its exponent, (1 understood) divided by 3, which will make $a^{\frac{1}{3}}$.

Therefore, roots are properly expressed by fractional exponents.

The square root of a is $a^{\frac{1}{2}}$, and the exponents, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c., indicate the third, fourth, and fifth roots. The 6th root of x^5 is $x^{\frac{5}{6}}$; hence, we perceive that the numerators of the exponent indicate the power of the quantity, and the denominator the root of that power.

(Art. 77.) The square of ax is a^2x^2. We square both factors, and so, for any other powers, we raise *all* the factors to the required power. Conversely, then, we extract roots by taking the required roots of all the factors. Thus the cube root of $8x^3$ is $2x$.

The square root of $64a^4$ is obviously $8a^2$, and from these examples we draw the following rule for the extraction of roots of monomials.

RULE.—*Extract the root of the numeral coefficients, and divide the exponent of each letter by the index of the root.*

EXAMPLES.

1. What is the *second* root of $9a^2x^4y^6$? . . *Ans.* $3ax^2y^3$.
2. What is the *third* root of $8a^6y^3$? . . . *Ans.* $2a^2y$.
3. What is the *fourth* root of $81a^4x^{12}$? . . *Ans.* $3ax^3$.
4. What is the *fifth* root of $32a^5x^{10}y^{15}$? . *Ans.* $2ax^2y^3$.

For illustration, we will observe that this last example requires us to find *five equal factors*, which, when multiplied together, will produce 32; and *five equal factors*, which, when multiplied together, will produce a^5, *five equal factors*, which, when multiplied together, will produce x^{10}, and *five equal factors*, which, when multiplied together, will produce y^{15}.

Now, a^5 shows *five equal factors*, each equal to a; therefore, a is one of the factors required. In the same manner x^{10} shows ten equal factors, and the product of *two* of these, or x^2 is one of the *five* equal factors required. In the same manner y^3 is another of the equal factors; and there is no trouble in finding any root of any literal monomial quantity; for all we have to do is to divide its exponent (whatever it may be) by the index of the proposed root. But when the factor is a numeral, like 32, we can find the factor only by *trial*. Sometimes no such factor as the one required exists; in such cases we consider the number as a letter with 1 understood for its exponent, and then divide such exponent by the index of the root. For example, the *fifth* root of 32 is $(32)^{\frac{1}{5}}$; but this is only an indication of the root or factor, not an actual discovery of it. Take particular notice of the following examples:

5. What is the *third* root of $7a^2x^3$? . . *Ans.* $7^{\frac{1}{3}}a^{\frac{2}{3}}x$.

The number 7 is here regarded as a letter.

6. What is the *second* root of $20ax$? *Ans.* $\pm(20)^{\frac{1}{2}}a^{\frac{1}{2}}x^{\frac{1}{2}}$.
7. What is the *fourth* root of $16a^4x^8$?

Ans. $-2ax^2$, or $2ax^2$.

8. What is the *square* root of $36a^2y^4$? . . *Ans.* $\pm 6ay^2$.

(ART. 78.) The even roots of algebraic quantities may be taken with the double sign, as indicating either *plus* or *minus*, for either quantity will give the same square, and we may not know which of them produced the power, (Art 70). For example, the square root of 16 may be either $+4$ or -4, for either of them, when multiplied by itself, will produce 16.

The cube root of a *plus* quantity is always plus, and the cube root of a *minus* quantity is always minus. For $+2a$ cubed, gives $+8a^3$, and $-2a$ cubed, gives $-8a^3$, and a may represent any quantity whatever.

9. What is the fourth root of $81a^4b^8c^{12}$? *Ans.* $\pm 3ab^2c^3$.

10. What is the third root of $-27a^{12}x^3$? *Ans.* $-3a^4x$.

11. What is the *third* root of $16a^4$? *Ans.* $2a(2a)^{\frac{1}{3}}$.

In this example it is obvious that there are no *three* equal factors, which, when multiplied together, will produce 16, and there are no three equal factors expressed in entire quantities that will produce a^4. Therefore, we must write $(16)^{\frac{1}{3}}a^{\frac{4}{3}}$ for the answer. But this is only indicating the operation, *not performing* it, and we have no clearer idea of the result now than at first. However, to see what can be done, we will separate $16a^4$ into the two factors $(8a^3)(2a)$. The first of these is a complete *third* power, and the other is not; but the *third* root of the whole is the third root of the two factors written together as a product; that is, $2a(2a)^{\frac{1}{3}}$, and this is all we can do to reduce or simplify it.

12. What is the *second* root of $20a^2x^3$? *Ans.* $\pm 2ax(5x)^{\frac{1}{2}}$.

All the square factors in this are $4a^2x^2$, the other factors are $5x$. We can take the second root of the *square factors*, and of the others *we cannot*. In relation to them we can only *indicate* the root. Therefore, the whole root is $\pm 2ax(5x)^{\frac{1}{2}}$.

13. What is the *second* root of 75? . . . *Ans.* $\pm 5(3)^{\frac{1}{2}}$.

14. What is the *second* root of $98a^2x$? . *Ans.* $\pm 7a(2x)^{\frac{1}{2}}$.

15. What is the *third* root of $32a^3$? . . . *Ans.* $2a(4)^{\frac{1}{3}}$.

16. What is the *third* root of $24a^3x^2$? . *Ans.* $2a(3x^2)^{\frac{1}{3}}$.

17. What is the *third* root of $27a^3$? . . . . . *Ans.* $3a$.

18. What is the third root of $19a$? . . . *Ans.* $(19a)^{\frac{1}{3}}$.

(Art. 78.) By comparing examples 17 and 18, we perceive that some monomials have such roots (or what is the same thing), such equal factors as may be required, and some have not. When no such factors exist, all we can do is to indicate an operation, to be performed as example 18. So it is with polynomials—some may have *equal factors*, and others not. When equal factors do exist in any polynomial, they are commonly apparent to any one who has had a little experience in raising roots to powers as explained in Articles 73, 74, and 75. For instance, any one can perceive that the polynomial $a^2+2ab+b^2$ has two equal factors, each equal to $(a+b)$; and after a little more observation we can perceive that the polynomial $x^3+3x^2y+3xy^2+y^3$ has three equal factors, each equal to $(x+y)$, or perceive that $(x+y)$ is its third root.

It *is only regular polynomials* that have equal factors, and it is only by *observing* how the powers are formed by multiplication that we can determine

HOW TO EXTRACT ROOTS OF POLYNOMIALS.

On the supposition that we know that the square root of the polynomial

$$a^2+2ab+b^2, \text{ is } (a+b),$$

we propose to extract it out of the polynomial itself.

We know that a^2, the first term, must have been formed by the multiplication of a into itself, therefore, a must be part of the root sought.

The next term is $2a\times b$, that is *twice* the root of the first term into the second term of the root. Hence, if we divide the second term of the square by twice the root of the first term, we shall obtain b, the second term of the root, and as b must be multiplied into itself to form a square, we add b to $2a$, and $2a+b$ we call a divisor.

OPERATION.

$$\begin{array}{l} \qquad\quad a^2+2ab+b^2(a+b \\ \qquad\quad a^2 \\ \hline 2a+b)2ab+b^2 \\ \qquad\quad\;\; 2ab+b^2 \\ \hline \end{array}$$

We take a for the first term of the root, and subtract its square (a^2) from the whole square. We then double a and divide $2ab$ by $2a$ and we find b, which we place in both the divisor and quotient. Then we multiply $2a+b$ by b, and we have $2ab+b^2$, to subtract from the two remaining terms of the square, and in this case nothing remains.

Again, let us take $a+b+c$, and square it. We shall find its square to be

$$a^2+2ab+b^2+2ac+2bc+c^2$$

$$\begin{array}{lll} & a^2+2ab+b^2+2ac+2bc+c^2(a+b+c & \\ & a^2 & \\ \cline{2-2} 2a+b & \quad 2ab+b^2 & \\ & \quad 2ab+b^2 & \\ \cline{2-3} 2a+2b+c & & 2ac+2bc+c^2 \\ & & 2ac+2bc+c^2 \\ \cline{3-3} \end{array}$$

By operating as before, we find the first two terms of the root to be $a+b$, and a remainder of $2ac+2bc+c^2$. Double the root already found, and we have $2a+2b$ for a partial divisor. Divide the first term of the remainder $2ac$ by $2a$, and we have c for the third term of the root, which must be added to $2a+2b$ to complete the divisor. Multiply the divisor by the last term of the root, and set the product under the three terms last brought down, and we have no remainder.

Again, let us take $a+b+c$ to square; but before we square it, let the single letter $s=a+b$.

Then we shall have $s+c$ to square, which produces

$$s^2+2sc+c^2$$

To take the square root of this, we repeat the first operation, and thus the root of any quantity can be brought into a binomial, and the rule for a binomial root will answer for a root containing any number of terms by considering the root *already found*, however great, as *one* term.

Hence, the following *rule to extract the square root of a compound quantity.*

RULE.—*Arrange the terms according to the powers of some letter, beginning with the highest, and set the square root of the first term in the quotient.*

Subtract the square of the root thus found from the first term, and bring down the next two terms for a dividend.

Divide the first term of the dividend by double of the root already found, and set the result both in the root and in the divisor.

Multiply the divisor, thus completed, by the term of the root last found, and subtract the product from the dividend, and so on.

EXAMPLES.

1. What is the square root of

$$
\begin{array}{lll}
 & a^4+4a^2b+4b^2-4a^2-8b+4(& a^2+2b-2 \\
 & a^4 & \\
\hline
2a^2+2b &)4a^2b+4b^2 & \\
 & 4a^2b+4b^2 & \\
\hline
2a^2+4b-2 & \quad -4a^2-8b+4 & \\
 & \quad -4a^2-8b+4 & \\
\hline
\end{array}
$$

2. What is the square root of $1-4b+4b^2+2y-4by+y^2$?
Ans. $1-2b+y$.

3. What is the square root of $4x^4-4x^3+13x^2-6x+9$?
Ans. $2x^2-x+3$.

4. What is the square root of $4x^4-16x^3+24x^2-16x+4$?
Ans. $2x^2-4x+2$.

5. What is the square root of $16x^4+24x^3+89x^2+60x$ 100?
Ans. $4x^2+3x+10$.

6. What is the square root of $4x^4-16x^3+8x^2+16x+4$?
Ans. $2x^2-4x-2$

7. What is the square root of $x^2+2xy+y^2+6xz+6yz+9z^2$?
Ans. $x+y+3z$.

8. What is the square root of $a^2-ab+\frac{1}{4}b^2$? Ans. $a-\frac{1}{2}b$

9. What is the square root of $\frac{a^2}{b^2}-2+\frac{b^2}{a^2}$?
Ans. $\frac{a}{b}-\frac{b}{a}$ or $\frac{b}{a}-\frac{a}{b}$.

10. What is the square root of $x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$?
Ans. $x^{\frac{1}{3}}-y^{\frac{1}{3}}$ or $y^{\frac{1}{3}}-x^{\frac{1}{3}}$.

(ART. 79.) Every square root will be equally a root if we change the sign of all the terms. In the first example, for instance, the root may be taken $-a^2-2b+2$, as well as a^2+2b-2, for either one of these quantities, by squaring, will produce the given square. Also, observe that every square consisting of three terms only, has a binomial root.

Algebraic squares may be taken for formulas, corresponding to numeral squares, and their roots may be extracted in the same way, and by the *same rule.*

For example, $a+b$ squared is $a^2+2ab+b^2$, and to apply this to numerals, suppose $a=40$, and $b=7$.

$$
\begin{array}{lr}
\text{Then the square of 40 is} & a^2 = 1600 \\
& 2ab = 560 \\
& b^2 = 49 \\
\hline
\text{Therefore, } \ldots\ldots & (47)^2 = 2209
\end{array}
$$

Now the necessary divisions of this square number, 2209, are not visible, and the chief difficulty in discovering the root is to make these separations.

The first observation to make is, that the square of 10 is 100, of 100 is 10000, and so on. Hence, the square root of any square number less than 100, consists of one figure, and of any square number over 100 and less than 10000, of two figures, and so on. *Every two places in a power demanding one place in its root.*

Hence, to find the number of places or figures in a root, we must *separate the power into periods of two figures, beginning at the unit's place.* For example, let us require the square root of 22 09. Here are two periods indicating two places in the root, corresponding to tens and units. The greatest square in 22 is 16, its root is 4, or 4 tens $=40$. Hence, $a=40$.

$$
\begin{array}{rl}
& 22\ 09(40+7=47 \\
a^2 = & 16\ 00 \\
\hline
2a+b=80+7=87\ ) & 6\ 09 \\
& 6\ 09 \\
\hline
\end{array}
$$

Then $2a=80$, which we use as a divisor for 609, and find it is contained 7 times. The 7 is taken as the value of b, and $2a+b$, the complete divisor, is 87, which, multiplied by 7, gives the two last terms of the binomial square. $2ab+b^2=560+49=609$, and the entire root, $40+7=47$, is found.

Arithmetically, a may be taken as 4 in place of 40, and 1600 as 16, the place occupied by the 16 makes it 16 hundred, and the ciphers are superflous. Also, $2a$ may be considered 8 in place of 80, and 8 in 60 (not in 609) is contained 7 times, &c.

If the square consists of more than *two periods*, treat it as *two*, and obtain the two superior figures of the root, and when obtained, bring down another period to the remainder, and consider the root already obtained as one quantity, or one figure.

For another example, let the square root of 399424 be extracted.

```
         39 94 24(632
         36
  123 |  394
      |  369
 1262   |25 24
        |25 24
```

In this example, if we disregard the local value of the figures, we have $a=6$, $2a=12$, and 12 in 39, 3 times, which gives $b=3$. Afterward we suppose $a=63$, and $2a=126$, 126 in 252, 2 times, or the second value of $b=2$. In the same manner, we would repeat the formula of a binomial square as many times as we have periods.

EXERCISES FOR PRACTICE.

1. What is the square root of 8836? . . . *Ans.* 94.
2. What is the square root of 106929? . . *Ans.* 327.
3. What is the square root of 4782969? . . *Ans.* 2187.
4. What is the square root of 43046721? . . *Ans.* 6561.
5. What is the square root of 387420489? *Ans.* 19683.

When there are whole numbers and decimals, point off periods both ways from the decimal point, and make the decimal places even, by annexing ciphers when necessary, extending the decimal as far as desired. When there are decimals only, commence pointing off from the decimal point.

EXAMPLES.

1. What is the square root of 10.4976? . . *Ans.* 3.24.
2. What is the square root of 3271.4207? *Ans.* 57.19+.
3. What is the square root of 4795.25731?
Ans. 69.247+.
4. What is the square root of .0036? . . *Ans.* .06.
5. What is the square root of .00032754?
Ans. .01809+.
6. What is the square root of .00103041? *Ans.* .0321.

As the square of any quantity is the quantity multiplied by itself, and the product of $\frac{a}{b}$ by $\frac{a}{b}$ (Art. 64) is $\frac{a^2}{b^2}$; hence, *to take the square root of a fraction, we must extract the square root of both numerator and denominator.*

A fraction may be equal to a square, and the terms, as given, not square numbers; such may be reduced to square numbers.

EXAMPLES.

What is the square root of $\frac{72}{128}$?

Observe $\frac{72}{128}=\frac{36}{64}$. Hence, the square root is $\frac{6}{8}$.

1. What is the square root of $\frac{98}{128}$? . . . . . *Ans.* $\frac{7}{8}$.
2. What is the square root of $\frac{112}{175}$? . . . . . *Ans.* $\frac{4}{5}$.
3. What is the square root of $\frac{2304}{5184}$? . . . . . *Ans.* $\frac{2}{3}$.
4. What is the square root of $\frac{2704}{4225}$? . . . . . *Ans.* $\frac{4}{5}$.

When the given fractions cannot be reduced to square terms, reduce the value to a decimal, and extract the root, as in the last article.

TO EXTRACT THE CUBE ROOT OF COMPOUND QUANTITIES.

(Art. 80.) We may extract the cube root in a similar manner as the square root, by dissecting or retracing the combination of terms in the formation of a binomial cube.

The cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$ (Art. 67). Now, to extract the root, it is evident we must take the root of the first term (a^3), and the next term is $3a^2b$. *Three times the square of the first letter or term of the root multiplied by the 2d term of the root.*

Therefore, to find this second term of the root, we must divide the second term of the power ($3a^2b$) by three times the square of the root already found (a).

$$3a^2)3a^2b(b$$
$$\underline{3a^2b}$$

When we can decide the value of b, we may obtain the complete divisor for the remainder, after the cube of the first term is subtracted, thus:

The remainder is . $3a^2b+3a^2b+b^3$

Take out the factor b, and $3a^2+3ab+b^2$ is the complete divisor for the remainder. But this divisor contains b, the very term we wish to find by means of the divisor; hence, it must be found before the divisor can be completed. In distinct algebraic quantities there can be no difficulty, as the terms stand separate, and we find b by dividing simply $3a^2b$ by $3a^2$; but in numbers the terms are mingled together and b can only be found by trial.

Again, the terms $3a^2+3ab+b^2$ explain the common arithmetical rule, as $3a^2$ stands in the place of hundreds, it corresponds with the words: "Multiply the square of the quotient by 300," "and the quotient by 30," ($3a$), &c.

By inspecting the various powers of $a+b$ (Art. 73), we draw the following general rule for the extraction of roots:

RULE.—*Arrange the terms according to the powers of some letter; take the required root of the first term and place it in the quotient; subtract its corresponding power from the first term, and bring down the second term for a dividend.*

Divide this term by *twice* the root already *found for the* SQUARE *root, three times the square of it for the* CUBE *root, four times the third power for the fourth root, &c., and the quotient* *will be the next term of the root. Involve the whole of the root thus found, to its proper power, which subtract from the given quantity, and divide the first term of the remainder by the same divisor as before; proceed in this manner till the whole root is determined.*

EXAMPLES.

1. What is the cube root of $x^6+6x^5-40x^3+96x-64$?

$$x^6+6x^5-40x^3+96x-64\ (x^2+2x-4$$
$$x^6$$

Divisor $3x^4)\ 6x^5$=1st remainder.

$$x^6+6x^5+12x^4+8x^3=(x^2+2x)^3$$

Divisor $3x^4\quad)-12x^4$=2d remainder.

$$x^6+6x^5-40x^3+96x-64$$

2. What is the cube root of $27a^3+108a^2+144a+64$?

Ans. $3a+4$.

3. What is the cube root of $a^3-6a^2x+12ax^2-8x^3$?

Ans. $a-2x$.

4. What is the cube root of $x^6-3x^5+5x^3-3x-1$?

Ans. x^2-x-1.

5. What is the cube root of $a^3-6a^2b+12ab^2-8b^2$?

Ans. $a-2b$.

6. What is the cube root of $x^3+3x+\frac{3}{x}+\frac{1}{x^3}$? *Ans.* $x+\frac{1}{x}$.

7. Extract the fourth root of

$$a^4+8a^3+24a^2+32a+16(a+2$$
$$a^4$$
$$4a^3)\quad 8a^3,\ \&c.$$
$$a^4+8a^3+24a^2+32a+16$$

(Art. 81.) To apply this general rule to the extraction of the cube root of numbers, we must first observe that the cube of 10 is 1000, of 100 is 1000000, &c.; ten times the root producing 1000 times the power, or one cipher in the root producing 3 in the power; hence, any cube within 3 places of figures can have only one in its root, any cube within 6 places can have only two places in its root, &c. Therefore, we must divide off the given power into periods consisting of three places, commencing at the unit. If the power contains decimals, commence at the unit place, and count three places each way, and the number of periods will indicate the number of figures in the root.

EXAMPLES.

1. Required the cube root of 12812904.

$$
\begin{array}{lrl}
 & & 12\ 812\ 904(234 \\
a=2 & a^3= & 8 \\
\text{Divisor} & 3a^2=12 &)48 \\
 & & 12167 = (23)^3 \\
 & 3(23)^2=1587) & \quad 6459\quad (4 \\
 & & 12\ 812\ 904=(234)^3
\end{array}
$$

Here, 12 is contained in 48, 4 times; but it must be remembered that 12 is only a trial or partial divisor; when completed it will exceed 12, and of course the next figure of the root cannot exceed 3.

The first figure in the root was 2. Then we assumed $a=2$. Afterward we found the next figure must be 3. Then we assumed $a=23$. To have found a succeeding figure, had there been a remainder, we should have assumed $a=234$, &c., and from it obtained a new partial divisor.

2. What is the cube root of 148877? . . . *Ans.* 53.
3. What is the cube root of 571787? . . *Ans.* 83.
4. What is the cube root of 1367631? . . *Ans.* 111.
5. What is the cube root of 2048383? . . *Ans.* 127.
6. What is the cube root of 16581375? . . *Ans.* 255.
7. What is the cube root of 44361864? . . *Ans.* 354.
8. What is the cube root of 100544625? . . *Ans.* 465.

(Art. 82.) The methods of direct extraction of the cube root of such numbers as have surd roots, are all too tedious to be much used, and several eminent mathematicians have given more brief and practical methods of approximation.

One of the most useful methods may be investigated as follows:

Suppose a and $a+c$ two cube roots, c being *very small* in relation to a, a^3 and $a^3+3a^2c+3ac^2+c^3$ are the cubes of the supposed roots.

Now, if we double the first cube (a^3), and add it to the second, we shall have

$$3a^3+3a^2c+3ac^2+c^3$$

If we double the second cube and add it to the first, we shall have $\quad 3a^3+6a^2c+6ac^2+2c^3$

As c is a very small fraction compared to a, the terms containing c^2 and c^3 are very small in relation to the others; and the relation of these two sums will not be materially changed by rejecting those terms containing c^2 and c^3, and the sums will then be . . . . . $3a^3+3a^2c$

And . . . . . $3a^3+6a^2c$

The ratio of these terms is the same as the ratio of $a+c$ to $a+2c$.

Or the ratio is . . . . $1+\frac{c}{a+c}$.

But the ratio of the roots a to $a+c$, is $1+\frac{c}{a}$.

Observing again, that c is supposed to be very small in relation to a, the fractional parts of the ratios $\frac{c}{a+c}$ and $\frac{c}{a}$ are both small, and very near in value to each other. Hence, we have found an operation on two cubes which are near each other in magnitude, that will give a proportion very *near* in proportion to their roots; and by knowing the root of one of the cubes, by this ratio we can find the other.

For example, let it be required to find the cube root of 28, true to 4 or 5 places of decimals. As we wish to find the cube root of 28, we may assume that 28 is a *cube*. 27 is a cube near in value to 28, and the root of 27 we know to be 3.

Hence, a, in our investigation, corresponds to 3 in this example, and c is unknown; but the cube of $a+c$ is 28, and a^3 is 27.

Then . . .	27	28
	2	2
	54	56
Add . . .	28	27
Sums . . .	82 :	83 : : 3 : $a+c$ very nearly.

Or, $(a+c)=\frac{249}{82}=3.03658+$, which is the cube root of 28, true to 5 places of decimals.

By the laws of proportion, which we hope more fully to investigate in a subsequent part of his work, the above proportion, ... $82 : 83 : : a : a+c$,
may take this change, $82 : 1 : : a : c$

Hence, $c=\frac{3}{82}$; c being a correction to the known root, which, being applied, will give the unknown or sought root.

From what precedes, we may draw the following rule for finding approximate cube roots:

RULE.—*Take the nearest rational cube to the given number, or assume a root and cube it. Double this cube, and add the number to it; also double the number and add the assumed cube to it. Then, by proportion, as the first sum is to the second, so is the known root to the required root.*

EXAMPLES.

1. What is the approximate cube root of 122?

Ans. 4.95967+.

By the rule .	$125\times2=250$	244
Add . . .	122	125
	372 :	369 : : 5 : *Ans.*

2. What is the cube root of 10? . . *Ans.* 2.15443+.

Assume 2.1 for the root, then 9.261 is its cube.

3. What is the approximate cube root of 720?
Ans. 8.9628+.

4. What is the approximate cube root of 345?
Ans. 7.01357+.

5. What is the approximate cube root of 520?
Ans. 8.04145+.

6. What is the approximate cube root of 65?
Ans. 4.0207+.

7. What is the approximate cube root of 16?

The cube root of 8 is 2, and of 27 is 3; therefore the cube root of 16 is between 2 and 3. Suppose it 2.5. The cube of this root is 15.625, which shows that the cube root of 16 is a little more than 2.5, and by the rule

31.25		32
16		15.625
47.25	:	47.625 : : 2.5 : to the required root.
47.25	:	.375 : : 2.5 : .01984

Assumed root . .	2.50000
Correction . . .	.01984
Approximate root	2.51984

We give the last as an example to be followed in most cases where the root is about midway between two integral numbers.

This rule may be used with advantage to extract the root of perfect cubes, when the powers are very large.

EXAMPLE.

The number 22.069.810.125 is a cube; required its root.

Dividing this cube into periods, we find that the root must contain 4 figures, and the superior period is 22, and the cube root of 22 is near 3, and of course the whole root near 3000; but it is less than 3000. Suppose it 2800, and cube this number. The cube is 21952000000, which, being less than the given number, shows that our assumed root is not large enough.

To apply the rule, it will be sufficient to take six superior figures of the given and assumed cubes. Then by the rule,

```
 219520      220698
      2           2
 ------      ------
 439040      441396
 220698      219520
 ------      ------
 659738  :   660916::2800
             659738
             ------
 659738  :     1178::2800
                 2800
             --------
               942400
              2356
             --------
      659738)3298400(5
             3298690
             --------
```

Assumed root,	2800
Correction, .	5
True root, .	2805

The result of the last proportion is not exactly 5, as will be seen by inspecting the work; the slight imperfection arises from the rule being approximate, not perfect.

When we have cubes, however, we can always decide the unit figure by inspection, and, in the present example, the unit figure in the cube being 5, the unit figure in the root must be 5, as no other figure when cubed will give 5 in the place of units.

[For several other abbreviations and expedients in extracting cube root in numerals, see "Robinson's Arithmetic."]

To obtain the 4th root, we may extract the square root of the square root. To obtain the 6th root, we may take the square root first, and then the cube root of that quantity.

To extract odd roots of high powers in numeral quantities is very tedious, and of no practical utility; we, therefore, give no examples.

(ART. 83.) It is sometimes necessary to multiply roots together or to divide one by another, and we must, therefore, find rules for such operations.

For instance, I wish to find the product of the square root of 3 into the square root of 12, and I know not how to find it, unless I first extract the square root of 3, and then of 12, and multiply the two roots together. But this would require a great amount of labor, and even then it would not be done to accuracy, as no exact square roots of either 3 or 12 exist.

It is possible, however, that the product of these two roots is the same as the square root of the product of 3 and 12, that is, the square root of 36, which is 6; but how are we to demonstrate whether this be true or not?

In answer to this inquiry, we say let a and b represent any two numbers then $a^{\frac{1}{2}}$ and $b^{\frac{1}{2}}$ will represent their square roots, (Art. 76).

In Algebra we represent the product of any two quantities by writing the quantities as factors with or without the sign of multiplication between them. Thus, the product of x and y is $x \cdot y$ or xy, so the product of $a^{\frac{1}{2}}$ into $b^{\frac{1}{2}}$ is $a^{\frac{1}{2}} \cdot b^{\frac{1}{2}}$; but the question is whether or not that this is the same as $(ab)^{\frac{1}{2}}$.

Now the product of these two roots must be *some number*. Let that number be indicated by P. Then we shall have this equation . . . $P = a^{\frac{1}{2}} b^{\frac{1}{2}}$

Square both members of this equation (and we square by doubling the exponent of every factor, Art. 71), and we have

$$P^2=ab$$

Now, by considering ab as a single number, and extracting the square root of both members, we have

$$P=(ab)^{\frac{1}{2}}$$

This last equation *answers the question, and we learn that the product of the roots is the same as the root of the product.*

Hence, $(3)^{\frac{1}{2}}\times(12)^{\frac{1}{2}}=(36)^{\frac{1}{2}}=6.$

What is the product of $(2)^{\frac{1}{2}}$ by $(8)^{\frac{1}{2}}$. . . . . . *Ans.* 4.

What is the product of $5(5)^{\frac{1}{2}}$ by $3(8)^{\frac{1}{2}}$. . *Ans.* $15(40)^{\frac{1}{2}}$.

Hence, when we wish to multiply numbers together which contain factors under the same root, we have the following

Rule.—*Multiply the rational parts together for the rational part of the product, and the radical parts together for the radical part of the product.*

EXAMPLES.

1. Multiply $a(b)^{\frac{1}{2}}$ by $c(d)^{\frac{1}{2}}$. . . . . . . *Ans.* $ac(bd)^{\frac{1}{2}}$.
2. Multiply $3(3)^{\frac{1}{2}}$ by $2(3)^{\frac{1}{2}}$. . . . . *Ans.* 18.
3. Multiply $3(2)^{\frac{1}{2}}$ by $4(8)^{\frac{1}{2}}$. . . . . *Ans.* 48.
4. Multiply $2(14)^{\frac{1}{3}}$ by $3(4)^{\frac{1}{3}}$. . . . . . *Ans.* $6(56)^{\frac{1}{3}}$.

But in 56 there is a cube factor 8, the other factor is 7, therefore the last answer is $6(8)^{\frac{1}{3}}(7)^{\frac{1}{3}}=12(7)^{\frac{1}{3}}$.

5. Multiply $2(5)^{\frac{1}{2}}$ by $2(10)^{\frac{1}{2}}$. . . . . . *Ans.* $20(2)^{\frac{1}{2}}$.

(Art. 84.) The roots to be multiplied together may not be the same—one may be a *square* root, the *other* a *cube* or some other root. In such cases, is there any other mode of expressing the product except by a representation of the factors?

For example, what is the product of $a^{\frac{1}{2}}$ by $b^{\frac{1}{3}}$. Is there another manner of expressing it than

$$a^{\frac{1}{2}}b^{\frac{1}{3}}$$

As in the last article, the product must be some number which we can represent by P.

Then . . . . $P=a^{\frac{1}{2}}b^{\frac{1}{3}}$

By squaring, ... $P^2=ab^{\frac{2}{3}}$

By cubing, ... $P^6=a^3b^2$

Taking the 6th root, . $P=(a^3b^2)^{\frac{1}{6}}$

In this manner we can find the product of other roots; but in a work like this it is not important to carry this subject to any great length; we give, however, the following

EXAMPLES.

1. What is the product of $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$? . . . *Ans.* $a^{\frac{5}{6}}$.
2. What is the product of $(6)^{\frac{1}{2}}$ by $(150)^{\frac{1}{2}}$? . *Ans.* 30.
3. What is the product of $(\frac{1}{2})^{\frac{1}{2}}$ by $(\frac{3}{8})^{\frac{1}{2}}$? . *Ans.* $\frac{1}{4}(3)^{\frac{1}{2}}$.
4. What is the product of $(2)^{\frac{1}{2}}$ by $(2)^{\frac{1}{3}}$? . *Ans.* $(32)^{\frac{1}{6}}$.

(Art. 85.) As division is the converse of multiplication, we may infer at once from Art. 83 that the quotient arising from the division of one root by another, is the same as the root of the quotient; but for greater clearness we had better denote the quotient by a letter and use an equation.

For example, divide $(8)^{\frac{1}{2}}$ by $(2)^{\frac{1}{2}}$, the quotient is $(4)^{\frac{1}{2}}$ or 2; but to establish the principle of operation, let Q represent the required quotient. Here, as in all examples of division, the product of the divisor and quotient is equal to the dividend. Therefore, in this case we must have

$$2^{\frac{1}{2}}Q=8^{\frac{1}{2}}$$

By squaring, ... $2Q^2=8$

15

This equation shows that to obtain the true quotient, *we must divide one number by the other regardless of the root, and then write the root over the quotient.*

Thus, ... $Q=(4)^{\frac{1}{2}}$

EXAMPLES.

1. Divide $(54)^{\frac{1}{2}}$ by $(6)^{\frac{1}{2}}$. . . . . . . *Ans.* $9^{\frac{1}{2}}=3$.
2. Divide $8(72)^{\frac{1}{2}}$ by $2(6)^{\frac{1}{2}}$. . . . . . . *Ans.* $4(12)^{\frac{1}{2}}$.
3. Divide $3(10)^{\frac{1}{2}}$ by $(15)^{\frac{1}{2}}$. . . . . . . *Ans.* $(6)^{\frac{1}{2}}$.

$$(15)^{\frac{1}{2}}Q=3(10)^{\frac{1}{2}}$$
$$15Q^2=9\times 10, \text{ or } Q^2=6$$

4. Divide 18 by $2(3)^{\frac{1}{2}}$. . . . . *Ans.* $3(3)^{\frac{1}{2}}$.
5. Divide $6a$ by $3(a)^{\frac{1}{2}}$ . . . . *Ans.* $2(a)^{\frac{1}{2}}$.
6. Divide $(160)^{\frac{1}{2}}$ by $(8)^{\frac{1}{2}}$. . . . . . . *Ans.* $2(5)^{\frac{1}{2}}$.
7. Divide 9 by $(27)^{\frac{1}{2}}$. . . . . *Ans.* $(3)^{\frac{1}{2}}$.
8. Divide 1 by $(\frac{1}{3})^{\frac{1}{2}}$. . . . . *Ans.* $(3)^{\frac{1}{2}}$.
9. Divide a by $(a)^{\frac{1}{2}}$. . . . . *Ans.* $(a)^{\frac{1}{2}}$.

(Art. 86.) When the roots are different, we proceed on the same principle, which will be sufficient for every possible case.

For example, divide $7^{\frac{1}{2}}$ by $7^{\frac{1}{3}}$; the quotient must be some number which we can represent by Q, and from the equation

$$7^{\frac{1}{3}}Q=7^{\frac{1}{2}}$$

Cubing, ... $7Q^3=7^{\frac{3}{2}}$

Squaring, ... $7^2Q^6=7^3$

Dividing by 7^2 . . . $Q^6=7$. Hence, $Q=7^{\frac{1}{6}}$, *Ans.*

2. Divide $(a^2b^2d^3)^{\frac{1}{6}}$ by $d^{\frac{1}{2}}$. . . . . *Ans.* $(ab)^{\frac{1}{3}}$.

SECTION IV.

EQUATIONS.

(Art. 87.) We have thus far been able to resolve only simple equations, or equations of the first degree; but many problems and many philosophical investigations present equations of the second, third, and higher degrees, which may demand a solution, as, for instance, the first example of Art. 86 incidentally demanded the solution of the equation

$$Q^6=7$$

which is an equation of the *sixth* degree; but it appears in so simple a form that there is no mistaking the principle on which its solution depends, and thus generally, When an unknown quantity is involved to any power, we find the *first power* (that is, the quantity itself) by extracting the corresponding root of both members.

As in that equation . . $Q=(7)^{\frac{1}{6}}$

In the same manner, if $x^n=a$, then $x=a^{\frac{1}{n}}$.

The converse of these equations may often occur, that is, the unknown quantity may appear under the form of a root, as in the following equation:

$$x^{\frac{1}{3}}=a+c$$

Here, it is obvious that the value of x must be found by cubing; but if we cube the first member of the equation, we must cube the second to preserve equality, (Ax. 9). That is

$$x=(a+c)^3$$

(ART. 88.) From the foregoing observations, we draw the following general rule of operation:

RULE.—*To free a quantity from a power, extract the corresponding root. To free it from a root, involve to the corresponding power.*

When the unknown quantity is connected to a known quantity, and the whole number a power or root, the power or root, as the case may be, is removed in the same manner as before.

Thus . . . . . . $(x+a)^3=ap$

The value of x is found, by first taking the cube root of both numbers and afterward transposing a.

Again . . . . . $(2x+c)^{\frac{1}{2}}=a$

Here, after squaring both members, we have

$2x+c=a^2$, a simple equation.

(ART. 89.) The equations that appear in Articles 87 and 88, and all other equations of like kind where the unknown quantity is raised to a complete power, or is under some one particular root, are called

PURE EQUATIONS.

Thus, the equation $ax^2=b$, or, which is the same thing, $x^2=\frac{b}{a}$ is a pure equation, because the power of the unknown quantity is *complete;* but the equation $x^2+bx=c$ is not a pure equation, because it contains *different powers* of the unknown quantity.

The equation $(x+a)^{\frac{1}{3}}=c$ is a pure equation; but the equation $x^{\frac{1}{3}}+x=c$ is an impure equation, because it contains no *complete* power of the unknown quantity.

Again, $x^2+2ax+a^2=c+b$ is a pure equation, because the first member is a complete power of $(x+a)$, and $(x+a)$ may be represented by y, then $y^2=c+b$, obviously a pure equation.

There is no difficulty in resolving pure equations as we have already seen, for all we have to do is to apply the rule expressed in Art. 88; but impure equations in the higher degrees, present *serious difficulties;* and even equations of the second degree, when impure, compel us to *complete the power* before we can solve the equations. Equations of the second degree can be represented by a geometrical square; and when the equation is pure, the square corresponding to the first member is complete, and when impure, it must be *completed*, and the necessary operation is very appropriately called

COMPLETING THE SQUARE

But before we go into the investigation of completing a square, we will give some examples to exercise the learner in resolving pure equations.

EXAMPLES.

1. Given $\sqrt{4+(x-2)^{\frac{1}{2}}}=3$, to find x. . . *Ans.* $x=27$.

To remove the first radical sign, we square both members, then . . . . $4+(x-2)^{\frac{1}{2}}=9$

Dropping 4 from both members, and then squaring, we find . . . . $x-2=25$

2. Given $x-\sqrt{x^2+6}=-2$, to find x. . . *Ans.* $x=\frac{1}{2}$.

Transpose x for the purpose of having the quantity under the radical *stand alone* as one member of the equation,

Thus, ... $-\sqrt{x^2+6}=2-x$

Now, by squaring, the radical sign will disappear; but if any other quantity were joined to this by + or —, the radical could not disappear in the square.

The square is . . $x^2+6=4-4x+x^2$

3. Given $x+\sqrt{x^2-7}=7$, to find x. . . . *Ans.* $x=4$.

4. Given $\sqrt{x+12}=2+\sqrt{x}$, to find x. . . *Ans.* $x=4$.

N. B. No rules can be given that will meet every case, for the combination of quantities is too various. The pupil must depend mainly on general principles and his own practical experience.

5. Given $2+(3x)^{\frac{1}{2}}=\sqrt{5x+4}$, to find x. . *Ans.* $x=12$.

6. Given $\left(\frac{20x^2-9}{4x}\right)^{\frac{1}{2}}=x^{\frac{1}{2}}$, to find x. . *Ans.* $x=\frac{3}{4}$.

7. Given $3x^2-29=\frac{x^2}{4}+510$, to find x. . *Ans* $x=14$.

8. Given $x+2=\sqrt{4+x\sqrt{64+x^2}}$, to find x. *Ans.* $x=6$.

9. Given $x-\frac{1}{2}\sqrt{x}=\sqrt{x^2-x}$, to find x. . . *Ans.* $x=\frac{25}{16}$.

10. Given $x\sqrt{a^2+x^2}=a^2-x^2$, to find x. *Ans.* $x=a\sqrt{\frac{1}{3}}$.

11. Given $\sqrt{x-32}=16-\sqrt{x}$, to find x. . *Ans.* $x=81$.

For the sake of brevity, put $a=16$, then the last equation will be $\sqrt{x-2a}=a-\sqrt{x}$. At the conclusion resubstitute the value of a.

12. Given $\sqrt{x-16}=8-\sqrt{x}$, to find x. . . *Ans.* $x=25$.

13. Given $x^2-ax^2=x$, to find x. . . . *Ans.* $x=\frac{1}{1-a}$.

14. Given $x^{\frac{1}{2}}+\sqrt{3+x}=\frac{6}{\sqrt{3+x}}$, to find x. . *Ans.* $x=1$.

PROBLEMS PRODUCING PURE EQUATIONS.

(Art. 90.) In solving problems, it often depends on the manner or means of notation we employ, whether the equation comes out simple or complex, or whether it is a pure equation or a common quadratic. For example,

1. *Find two numbers, whose difference is* 6, *and their product* 40. *Ans.* 4 and 10.

If we represent the least number by x,
Then the greater number must be $x+6$
Their product is . . . . x^2+6x
But, by the problem this product is 40.

Therefore, $x^2+6x=40$; but as yet, we *cannot solve* this equation, because it contains two separate powers of x; we *know not* what to do with it. But in truth, the problem in itself is so simple we should be able to solve it, and must do so by some artifice or other.

After some reflection, we conclude to

Put . $x-3$ to represent the least number.

Then . $x+3$ will represent the greatest number.

Product $x^2-9=40$, a pure equation giving ±7 for the value of x. (See Art. 64).

The reason of taking the double sign is found in Art. 78.

If we take $+7$, then the least number is 4 and the greater 10. If we take -7, the numbers are -10 and -4; but as there are really no such numbers as -10 and -4, we take only $+7$ from the answer to x.

The solving of this problem shows us how to solve any equation in the form of

$$x^2+ax=b$$

Consider the first member as the product of x, and $x+a$, the difference of these two factors is a.

Put . . . . $x=y-\frac{a}{2}$ (1)

$x+a=y+\frac{a}{2}$ (2)

Product, . . . $x^2+ax=y^2-\frac{a^2}{4}=b$ (3)

Hence, ... $y^2=b+\frac{a^2}{4}$ (4)

And . . . . $y=\pm\sqrt{b+\frac{a^2}{4}}$ (5)

The value of y, as determined in equation (5), put in equation (1), gives

$$x=-\frac{a}{2}\pm\sqrt{b+\frac{a^2}{4}}$$

(Art. 91). The student will perceive that Article 90 is a digression, but one that should be pardoned. We now continue our problems in pure equations, and if any problem does not produce such an equation, it will be because the notation designed by the author has not been taken.

2. The sum of two numbers is 6, and the sum of their cubes is 72. What are the numbers? *Ans.* 2 and 4.

Let $3+x=$ the greater,
And $3-x=$ the less.

3. Divide the number 56 into two such parts, that their product shall be 640. *Ans.* 40 and 16.

Let $28+x=$ the greater,
$28-x=$ the less.

4. A and B distributed 1200 dollars each, among a certain number of persons. A relieved 40 persons more than B, and B gave to each individual 5 dollars more than A. How many were relieved by A and B? *Ans.* A, 120; B, 80.

Let $x+20=$ the number relieved by A.
And $x-20=$ the number relieved by B.

Then . . . . $\frac{1200}{x+20}+5=\frac{1200}{x-20}$

Dividing by 5, gives $\frac{240}{x+20}+1=\frac{240}{x-20}$

To avoid numeral multiplication and division, put $a=20$, and $b=240$. Then the equation becomes

$$\frac{b}{x+a}+1=\frac{b}{x-a}$$

5. Find a number, such that one-third of it multiplied by one-fourth, shall produce 108. *Ans.* 36.

6. What number is that whose square plus 18 shall be equal to half its square plus $30\frac{1}{2}$? *Ans.* 5.

7. What two numbers are those which are to each other as 5 to 6, and the difference of whose squares is 44?

Ans. 10 and 12.

Let $6x=$ the greater, and $5x=$ the less.

8. What two numbers are those which are to each other as 3 to 4, and the difference of whose squares is 28?

Ans. 6 and 8.

9. What two numbers are those whose product is 144, and the quotient of the greater by the less is 16?

Ans. 48 and 3.

10. The length of a lot of land is to its breadth as 9 to 5, and it contained 405 square feet. Required the length and breadth in feet. *Ans.* 27 and 15.

11. What two numbers are those whose difference is to the greater as 2 to 9, and the difference of whose squares is 128?

Ans. 18 and 14.

12. Find two numbers in the proportion of $\frac{1}{2}$ to $\frac{2}{3}$, the sum of whose squares shall make 225? *Ans.* 9 and 12.

The thoughtful student will not use the fractional numbers $\frac{1}{2}$ and $\frac{2}{3}$; but he will use whole numbers in the same proportion. Let this observation apply to other problems as well as to this one.

13. There is a rectangular field, whose breadth is $\frac{5}{6}$ of the length. After laying out $\frac{1}{6}$ of the whole ground for a garden, it was found that there were left 625 square rods for mowing. Required the length and breadth of the field.

Ans. Length, 30 rods; breadth, 25.

14. Two men talking of their ages, one said that he was 94 years old. Then, replied the younger, the sum of your

age and mine, multiplied by the difference between our ages, will produce 8512. What is the age of the younger?

Ans. 18 years.

15. A fisherman being asked how many fish he had caught, replied, "If you add 14 to the number, the square root of the sum, diminished by 8, will equal nothing." How many had he caught? *Ans.* 50.

16. A merchant gains in trade a sum, to which 320 dollars bears the same proportion as five times the sum does to 2500 dollars. What is the sum? *Ans.* $400.

17. What number is that, the fourth part of whose square being subtracted from 8, leaves a remainder equal to 4?

Ans. 4.

18. Find two numbers, such that the second power of the greater, multiplied by the less, produces 448, and the second power of the less, multiplied by the greater, gives 392.

Ans The numbers are 8 and 7.

Let $x=$ the greater, $y=$ the less.

Then . . . . . . $x^2y=448$ (1)

And . . . . $xy^2=392$ (2)

The product of (1) and (2) is

$$x^3y^3=(448)(392) \qquad (3)$$

This equation indicates that there may be *cube* factors in 448 and in 392. Therefore, we will try to find them by dividing by 8, the *least cube number* above unity.

Thus, ... 8)448 8)392

8)56 7)49

7 7

Hence, . . $x^3y^3=(8\cdot8\cdot7)(8\cdot7\cdot7)=8^3\cdot7^3$ (4)

Or, . . . $xy=8\cdot7$ (5)

Divide (1) by (5), and we have $x=8$.

19. A man purchased a field, whose length was to its breadth as 8 to 5. The number of dollars paid per acre was

equal to the number of rods in the length of the field: and the number of dollars given for the whole, was equal to 13 times the number of rods round the field. Required the length and breadth of the field.

Ans. Length, 104; breadth, 65 rods.

Let $8x=$ the length, and $5x=$ the breadth,

Then . $\frac{40x^2}{160}=\frac{x^2}{4}=$number of acres.

And . $\frac{x^2}{4}\times 8x=2x^3=$ the whole number of dollars.

Again, $8x+5x=13x=$ half round the field.

And $13x\cdot 2\cdot 13=$ thirteen times round, which is equal to the dollars paid.

20. There is a stack of hay, whose length is to its breadth as 5 to 4, and whose highth is to its breadth as 7 to 8. It is worth as many cents per cubic foot as it is feet in breadth; and the whole is worth at that rate 224 times as many cents as there are square feet in the bottom. Required the dimensions of the stack.

Ans. Length, 20; breadth, 16; and highth 14 feet.

Let $5x=$ the length, $4x=$ the breadth, and $\frac{7x}{2}=$ highth.

Then $\left(5x\cdot 4x\cdot \frac{7x}{2}\right)4x=$ cost in cents.

Again, $5x\cdot 4x=$ square feet on the bottom;

Hence, . $224\cdot 5x\cdot 4x=$ cost in cents;

Therefore, $5x\cdot 4x\cdot \frac{7x}{2}\cdot 4x=224\cdot 5x\cdot 4x$

By striking out equal factors, we have

$$7x\cdot 2x=224$$

21. It is required to divide the number of 14 into two such parts, that the quotient of the greater divided by the less,

may be to the quotient of the less divided by the greater, as 16 : 9. *Ans.* The parts are 8 and 6.

Let $x=$ the greater part. Then $14-x=$ the less.

Per question, $\frac{x}{14-x}:\frac{14-x}{x}::16:9$.

Multiply extremes and means, and $\frac{9x}{14-x}=\frac{16(4-x)}{x}$

Clearing of fractions, we have $9x^2=16(14-x)^2$.

By evolution, . $3x=4(14-x)=4\cdot14-.4x$.

By transposition, $7x=4\cdot14$.

By division . . $x=4\cdot2=8$, the greater part.

We solved the last four examples for the purpose of strongly recommending the factor system, which has not been practiced or appreciated half as much as its merits deserve.

(ART. 92.) We will now return to equations in the form of $x^2+ax=b$, for if we know how to solve them, we need not be so particular in our notation as we have been in the last Article.

As we have before remarked, all these equations can be resolved by considering the first member as the product of two factors, one of which is x, the other $(x+a)$, and their difference is a. Then if we put $y-\frac{a}{2}$ for one factor, and $y+\frac{a}{2}$ for the other, we shall have $y^2-\frac{a^2}{4}=b$, and $y^2=b+\frac{a^2}{4}$

That is, if to b, or to its equal, x^2+ax, we add $\frac{a^2}{4}$, the square of $\frac{a}{2}$ (*the square of half the coefficient of the first power of x*), we shall have y^2, that is, *some square.* Hence, $x^2+ax+\frac{a^2}{4}$ is a square, and we have the equation

$$x^2+ax+\frac{a^2}{4}=b+\frac{a^2}{4}$$

Now, the first member is a square in *form;* but whether it is a *numerical square* or not, depends on $(b+\frac{a^2}{4})$ being a numerical square; but whatever it is, we have, by extracting the square root of both members

$$x+\frac{a}{2}=\pm\sqrt{b+\frac{a^2}{4}}$$

$$x=-\frac{a}{2}\pm\sqrt{b+\frac{a^2}{4}}$$

We have found this formula before, in Art. 90.

(Art. 93.) An equation in the form of $x^2-ax=b$, its first member may be considered the product of the two factors, x and $(x-a)$, and these two factors differ by a.

Let . . . . . . $y+\frac{a}{2}=x$ (1)

And . . . . . . $y-\frac{a}{2}=x-a$ (2)

Multiply (1) and (2), and we have

$$y^2-\frac{a^2}{4}=x(x-a)=b$$

Hence, ... $y^2=b+\frac{a^2}{4}$ (3)

Equation (3) shows that whether we have an equation in the form of $x^2+ax=b$, or of $x^2-ax=b$; that is, whether ax is either *plus* or *minus*, we make the first member a *square* by the addition of precisely the same quantity $\frac{a^2}{4}$, *which is the square of half* the coefficient of x.

In other words, . . . $x^2+ax+\frac{a^2}{4}$ and . . . . $x^2-ax+\frac{a^2}{4}$ } are both squares.

The square root of the first is $x+\frac{a}{2}$, the square root of the second is $x-\frac{a}{2}$ } See Art. 78.

Hence, to complete the square of the first member of any equation in the form of $x^2+ax=b$, or of $x^2-ax=b$, or more generally when the exponent of the unknown quantity in the first term *is double* of that in the other, we have the following

RULE.—*Add the square of half the coefficient of the lowest power of the unknown quantity.*

EXAMPLES.

Complete the squares in the following equations:

N. B. We add a quantity to the first member to complete its square, we add the same quantity to the second member *to preserve equality.*

1. $x^2+4x=96$. . . . *Ans.* $x^2+4x+4=96+4$.
2. $x^2-4x=45$. . . . *Ans.* $x^2-4x+4=49$.
3. $x^2-7x=8$. . . . . . *Ans.* $x^2-7x+\frac{49}{4}=8+\frac{49}{4}$.
4. $x^4-2x^2=24$. . . . *Ans.* $x^4-2x^2+1=24+1$.
5. $x^{2n}-4x^n=a$. . . . *Ans.* $x^{2n}-4x^n+4=a+4$.
6. $x^2+6x=16$. . . . . *Ans.* $x^2+6x+9=25$.
7. $x^2-15x=-54$. . . *Ans.* $x^2-15x+\frac{225}{4}=\frac{225}{4}-54$.
8. $x^2-\frac{2}{3}x=\frac{133}{3}$. . . . *Ans.* $x^2-\frac{2}{3}x+\frac{1}{9}=\frac{133}{3}+\frac{1}{9}$.
9. $x^2-\frac{5}{6}x=\frac{1}{6}$. . . . . . *Ans.* $x^2-\frac{5x}{6}+\frac{25}{144}=\frac{1}{6}+\frac{25}{144}$.
10. $x-\frac{a}{b}x=\frac{c}{d}$. . . . . . *Ans.* $x^2-\frac{a}{b}x+\frac{a^2}{4b^2}=\frac{c}{d}+\frac{a^2}{4b^2}$.

Find the values of x in each of the ten preceding equations. First by extracting the square root of both members.

1. $x+2=\pm10$. . . . . . . Therefore, $x=8$ or -12.
2. $x-2=\pm7$. . . . . . . " $x=5$ or -9.

3. $x-\frac{7}{2}=\pm\frac{9}{2}$. . . . . Therefore, $x=8$ or -1.

4. $x^2-1=\pm 5$. . . . . . . " $x=(6)^{\frac{1}{2}}$ or $\sqrt{-4}$.

5. $x^n-2=\pm(a+4)^{\frac{1}{2}}$.

Therefore, . . . $x=\sqrt[n]{2+\sqrt{a+4}}$ or $(2-\sqrt{a+4})^{\frac{1}{n}}$

6. $x+3=\pm 5$. . . . . . . Therefore, $x=2$ or -8.

7. $x-\frac{15}{2}=\pm\frac{3}{2}$. . . . . . " $x=9$ or 6.

8. $x-\frac{1}{3}=\pm\frac{20}{3}$. . . . . " $x=7$ or $-\frac{19}{3}$.

9. $x-\frac{5}{12}=\pm\frac{7}{12}$. . . . . " $x=1$ or $-\frac{1}{6}$.

10. $x-\dfrac{a}{2b}=\pm\left(\dfrac{c}{d}+\dfrac{a^2}{4b^2}\right)^{\frac{1}{2}}$. . " $x=\dfrac{a}{2b}\pm\left(\dfrac{c}{d}+\dfrac{a^2}{4b^2}\right)^{\frac{1}{2}}$.

The preceding ten equations are all prepared for completing the square; that is, the highest power of the unknown quantity stands first, and is positive.

It is necessary that it should be positive, because we must take its square root, and there is no *square root* to a negative quantity. Therefore, in reducing an equation preparatory to completing its square, if the highest power comes out *minus*, make it *plus*, by changing all the signs to both members of the equation.

Example. Find the values of x from the following equation

$$1-\frac{x}{2}=5-\frac{36}{x+2}$$

Multiplying by 2, and afterward dropping 2 from both members, we have . . $-x=8-\dfrac{72}{x+2}$

Clearing of fractions,

$$-x^2-2x=8x+16-72$$

Transposing $8x$, uniting 16 and -72, and afterward changing all the signs, we have

$x^2+10x=56$. Hence, $x=4$, or -14

2. Find the values of x from the equation $3x^2+2x-9=76$.
Ans. $x=5$ or $-5\frac{2}{3}$.

3. Find x from $x^2+\frac{x}{2}=\frac{2x^2}{5}-\frac{x}{5}+1\frac{3}{10}$. *Ans.* $x=1$ or $-2\frac{1}{6}$.

4. Find x from $\frac{x^2}{4}-30+x=2x-22$.
Ans. $x=8$ or -4.

5. Find x from $\frac{x^2}{2}-\frac{x}{3}+7\frac{1}{2}=8\frac{1}{8}$. . *Ans.* $x=1\frac{1}{2}$ or $-\frac{5}{6}$.

6. Find x from $\frac{x^2}{4}-15=\frac{2x}{3}-14\frac{3}{4}$. *Ans.* $x=3$ or $-\frac{1}{3}$.

7. Find x from $\frac{x}{x+8}=\frac{x+3}{2x+1}$. . *Ans.* $x=12$ or -2.

8. Find x from $x-1+\frac{2}{x-4}=0$. . *Ans.* $x=3$ or 2.

9. Find x from $\frac{22-x}{20}=\frac{15-x}{x-6}$. . *Ans.* $x=36$ or 12.

10. Find x from $\frac{2x-7}{x-1}=\frac{x+1}{2x+7}$. . *Ans.* $x=4$ or -4.

(Art. 94.) For a more definite understanding of quadratics, we will solve and strictly examine the following equation:

$$x^2+4x=60$$

Completing the square, then

$$x^2+4x+4=64$$

Extracting the square root of both members,

And . . . . $x+2=\pm 8$

The reason of taking the double sign to 8 has been several times explained.

If we take $+8$, then . $x=6$

If . . -8, then . $x=-10$

That is, either $+6$, or -10 will verify the equation.

For . . . . . $6^2+4\cdot 6=60$

Also, . . $(-10)^2-4\cdot 10=60$

If $x=6$, ... $x-6=0$ (1)

If $x=-10$ . . . . $x+10=0$ (2)

If we multiply equations (1) and (2) together, we shall have as follows:

$$\begin{array}{r} x-6 \\ x+10 \\ \hline x^2-6x \qquad \\ 10x-60 \\ \hline x^2+4x-60=0 \end{array}$$

As the two factors are in value equal to 0, the product of the two must, of course, equal 0, and we have the equation as above. Transpose -60, and we have $x+4x=60$, *the original equation.*

Thus we perceive, that a quadratic equation may be considered as the product of two simple equations, and the values of x in the simple equations are said to be *roots* of the quadratic, and this view of the subject gives the *rationale* of the unknown quantity having *two* values.

This example shows us how to form an equation when we have the *two* roots; that is, gives us the following

RULE:—*Connect each root with a contrary sign to an unknown quantity. Take the product of the two binomial factors thus formed for the first member of the equation sought, and* 0 *for the other member.*

EXAMPLES.

1. Find the equation which has 3 and -2 for its roots.
Ans. $x^2-x-6=0$.

2. Find the equation which has 5 and -9 for its roots.
Ans. $x^2+4x-45=0$.

3. Find the equation which has 7 and -7 for its roots.
Ans. $x^2-49=0$.

4. Find the equation which has 8 and -12 for its roots.
Ans. $x^2+4x-96=0$.

Let the pupil observe that this last equation is equation 1 in Art. 93, and he can take the roots of those ten equations and deduce the equations again if desirable.

5. Find the equation which has a and b for its roots.

Ans. $x^2-(a+b)x+ab=0$.

If one of the roots is negative, suppose $-a$, the equation is then . . . . . $x^2+(a-b)x-ab=0$

If b is negative and a positive, the equation is

$$x^2+(b-a)x-ab=0$$

If both roots are negative, then the equation is

$$x^2+(a+b)x+ab=0$$

Now, let the pupil observe that the exponent of the highest power of the unknown quantity is 2; *and there are two roots. The coefficient of the first power of the unknown quantity is the algebraic sum of the two roots, with their signs changed; and the absolute term, independent of the unknown quantity, is the product of the roots (the sign conforming to the rules of multiplication).*

From these observations we can instantly form the equation when the two roots are given, without the formality of going through the multiplication; for example,

Find the equation which has 7 and -9 for its roots.

$$7-9=-2\text{; changed } +2,\quad 7(-9)=-63$$

Hence, $x^2+2x-63=0$ is the equation.

(Art. 95.) When a quadratic equation is formed, or found, or *given*, we may consider it as the product of two *binomial factors*, and those factors may be obvious to one who fully understands the subject, or any one can find them who can resolve the equation.

In giving examples in FACTORING (Art. 26), we omitted trinomial quantities of the second degree. The reason of that omission must now be perfectly comprehended by the careful student. We now return to that subject, and require the factors composing the expression

$$x^2+5x+6$$

Put the expression equal to zero, and resolve the quadratic, and we shall find the roots to be -2 and -3. Therefore, the sought factors are $(x+2)$ and $(x+3)$.

Find the factors composing each of the following expressions. Each expression must be taken as a quadratic equation presented for solution:

1. $x^2-x-20=0$. . . . . *Ans.* $(x-5)(x+4)$.
2. $a^2-7a+12$. . . . . *Ans.* $(a-3)(a-4)$.
3. a^2-7a-8. . . . . *Ans.* $(a-8)(a+1)$.
4. x^2-x-30. . . . . *Ans.* $(x-6)(x+5)$.
5. $x^2+7x-18$. . . . . *Ans.* $(x+9)(x-2)$.
6. $x^2+2ax+a^2$. . . . . *Ans.* $(x+a)(x+a)$.
7. $x^2-2ax+a^2$. . . . . *Ans.* $(x-a)(x-a)$.
8. x^2-a^2. . . . . *Ans.* $(x+a)(x-a)$.
9. x^2-2x+4. . . . . *Ans.* $(x-r)(x-r)$.

In this last expression $r=1\pm\sqrt{-3}$, and it not being a numerical quantity, the roots are said to be *imaginary*.

(Art. 96). When a quadratic equation is reduced to the form of $x^2+ax=b$, to complete the square of the first member we take the half of the coefficient of x to square, therefore, it will be more convenient to represent that coefficient by $2a$ in place of a, and as it may have the negative as well as the positive sign, and as b can be negative as well as positive; therefore, for a representation of every variety of quadratic equations, we have the four general forms.

$$x^2+2ax=b \quad (1)$$
$$x^2-2ax=b \quad (2)$$
$$x^2-2ax=-b \quad (3)$$
$$x^2+2ax=-b \quad (4)$$

A solution of each of these equations gives for the values of x as follows:

$$x=-a\pm\sqrt{b+a^2} \quad (1)$$
$$x=+a\pm\sqrt{b+a^2} \quad (2)$$
$$x=+a\pm\sqrt{a^2-b} \quad (3)$$
$$x=-a\pm\sqrt{a^2-b} \quad (4)$$

The quantity x has no conceivable value in equations (3) and (4) when applied to any problem in which b has a greater numerical value than a^2, for the solution requires the square root of $\sqrt{a^2-b}$, a negative quantity; and there being no square roots to such quantities, we have no conception of any value to x, and, of course, we call the value *imaginary*.

After we reduce an equation to one of the preceding forms, the solution is only substituting particular values for a and b; but in many cases it is more easy to resolve the equation as an original one, than to refer and substitute from the formula.

(Art. 97.) We may meet with many quadratic equations that would be very inconvenient to reduce to the form of $x^2+2ax=b$; for when reduced to that form, $2a$ and b may both be troublesome fractions.

Such equations may better be left in the form of

$$ax^2+bx=c$$

An equation in which the known quantities, a, b, and c, are all *whole numbers*, and prime to each other.

We now desire to find some method of making the first member of this equation a square, without making fractions. We therefore cannot divide by a, because b is not divisible by a, the two letters being prime to each other by hypothesis. But the first term of a binomial square is always a square; therefore, if we desire the first member of our equation to be converted into a binomial square, we must render the first term a square, and we can accomplish this by multiplying every term by a.

The equation then becomes

$$a^2x^2+bax=ca$$

Put . . . . $y=ax$

Then . . . . . . $y^2+by=ca$

Complete the square, by the preceding rule, and we have

$$y^2+by+\frac{b^2}{4}=ca+\frac{b^2}{4}$$

We are sure the first member is a square; but one of the terms is fractional, a condition we wished to avoid; but the denominator of the fraction is 4, a *square*, and a square multiplied by a square produces a square.

Therefore, multiply by 4, and we have the equation

$$4y^2+4by+b^2=4ca+b^2$$

An equation in which the first member is a binomial square, and not fractional.

If we return the values of y and y^2, this last equation becomes $4a^2x^2+4abx+b^2=4ac+b^2$

Compare this with the primitive equation

$$ax^2+bx=c$$

We multiplied this equation first by a, then by 4, and in addition to this, we find b^2 on both sides of the *rectified* equation, b being the coefficient of the first power of the unknown quantity. From this it is obvious, that to convert the expression ax^2+bx into a binomial square, we may use the following

RULE.—*Multiply by four times the coefficient of* x^2, *and add the square of the coefficient of* x.

To preserve equality, both sides of an equation must be multiplied by the same quantity, and the same addition must be made to both sides. We operate on the first member of an equation to make it a square; we operate on the second member to preserve equality.

EXAMPLES.

1. Given $5x^2+4x=204$, to find the values of x.

By the rule, we multiply 4 times 5, and add to both members 4^2. That is,

$$4\cdot5^2x^2+80x+16=4080+16$$

By extracting square root, we have

$$2\cdot5x+4=\pm64,\ x=6 \text{ or } -6\tfrac{4}{5}$$

By extracting the square root of the first member, the second term always *disappears;* it is, therefore, *not necessary to compute it,* and for that reason we may simply represent it by a letter, as in the following example:

2. Given $7x^2-20x=32$, to find the values of x.

Multiply by 4 times 7 and add 20^2.

Then . . $4\cdot7^2x^2-A+400=896+400$

Square root, . $2\cdot7x-20=\pm36$; hence, $x=2$ or $-\frac{2}{7}$.

3. Given $2x^2-5x=117$, to find the values of x.

Ans. $x=9$ or $-6\frac{1}{2}$.

4. Given $3x^2-5x=28$, to find the values of x.

Ans. $x=4$ or $-\frac{7}{3}$.

5. Given $3x^2-x=70$, to find the values of x.

Ans. $x=5$ or $-\frac{14}{3}$.

6. Given $5x^2+4x=273$, to find the values of x.

Ans. $x=7$ or $-7\frac{8}{5}$.

7. Given $2x^2+3x=65$, to find the values of x.

Ans. $x=5$ or $-6\frac{1}{2}$.

8. Given $3x^2+5x=42$, to find the values of x.

Ans. $x=3$ or $-4\frac{2}{3}$.

9. Given $8x^2-7x+16=181$. to find x.

Ans. $x=5$ or $-4\frac{1}{8}$.

10. Given $10x^2-8x+8=320$, to find x.

Ans. $x=6$ or $-5\frac{1}{5}$.

11. Given $3x^2+2x=4$, to find x. . *Ans.* $x=-\frac{1}{3}\pm\frac{1}{3}\sqrt{13}$.

12. Given $5x^2+7x=7$, to find x. *Ans.* $x=-\frac{7}{10}\pm\frac{3}{10}\sqrt{21}$.

13. Given $\frac{240}{x}+\frac{4}{10}=\frac{216}{x-15}$, to find x. . *Ans.* $x=75$.

QUESTIONS

GIVING RISE TO QUADRATIC EQUATIONS.

1. If four times the square of a certain number be diminished by twice the number it will leave a remainder of 30. What is the number? *Ans.* 3.

N. B. The number 3 is the only number that will answer the required conditions—the algebraic expression $-\frac{5}{2}$ will also answer the conditions; but the expression is not a number in any *arithmetical sense.*

2. A person purchased a number of horses for 240 dollars. If he had obtained 3 more for the same money, each horse would have cost him 4 dollars less. Required the number of horses. *Ans.* 12.

3. A grazier bought as many sheep as cost him 240 dollars, after reserving 15 out of the number, he sold the remainder for 216 dollars, and gained 40 cents a head on the number sold. How many sheep did he purchase? *Ans.* 75.

(See equation 13 just passed over).

4. A company dining at a house of entertainment, had to pay 3 dollars and 50 cents; but before the bill was presented two of them went away; in consequence of which, those who remained, had to pay each 20 cents more than if all had been present. How many persons dined? *Ans.* 7.

5. There is a certain number, which being subtracted from 22, and the remainder multiplied by the number, the product will be 117. What is the number? *Ans.* 13 or 9.

6. In a certain number of hours a man traveled 36 miles, but if he had traveled one mile more per hour, he would have taken 3 hours less than he did to perform his journey. How many miles did he travel per hour? *Ans.* 3 miles.

7. A man being asked how much money he had in his purse, answered, that the square root of the number taken from half the number would give a remainder of 180 dollars. How much money had he? *Ans.* \$400.

8. Divide 100 into two such parts, that the sum of their square roots may be 14. *Ans.* 64 and 36.

9. Divide the number 14 into two such parts, that the sum of the squares of those parts shall be 100. *Ans.* 8 and 6.

10. Divide the number a into two such parts, that the sum of the squares of those parts shall be b.

Ans. $\frac{1}{2}(a\pm\sqrt{2b-a^2})$.

11. It is required to divide the number 24 into two such parts, that their product may be equal to 35 times their difference. *Ans.* 10 and 14.

12. The sum of two numbers is 8, and the sum of their cubes 152. What are the numbers? *Ans.* 3 and 5.

Let $4-x=$ the less number.

And $4+x=$ the greater number.

Then put $a=4$, cube, &c.

(Art. 98.) In the preceding examples we have only considered the *resulting equation* after all the other unknown quantities have been eliminated.

In solving a problem, however, the operator may use *one, two, three,* or *more* unknown quantities, and operate as in simple equations, and in eliminating one quantity after another, there will result a final equation, which may be of the *first, second, third,* or higher degree, according to the conditions of the problem and the *tact* of the operator in taking hold of the matter.

(Art. 99.) When two independent equations are drawn from a problem, if one of them is quadratic, the other must be simple, or the resulting equation cannot be brought down to the second degree, except in rare cases, where the two equations are homogeneous or are symmetrical.

(Art. 100.) Two equations essentially quadratic, involving two unknown quantities, depend for their solution on a resulting equation of the fourth degree.

(Art. 101.) No words will cover every case of similar or symmetrical equations; but as a general thing, in similar equations we may change x to y, and y to x without changing the form of the equations or falsifying them.

Thus, ... $x+y=a$

And . . . . $xy=b$

Are similar equations, and $x+y=a$

$$x^2+y^2=b$$

Are both similar and symmetrical.

When equations are similar, we can reduce them without completing the square, as the learner will discover by the following examples:

1. Given $\left\{\begin{array}{r} x+y=a \\ xy=b \end{array}\right\}$ to find x and y.

Squaring the first, . . $x^2+2xy+y^2=a^2$ (1)
Four times the second, . . $4xy \quad =4b$ (2)

Subtracting (2) from (1) $x^2-2xy+y^2=a^2-4b$ (3)

Square root of (3), ... $x-y=\pm\sqrt{a^2-4b}$ (4)
But . . . . $x+y=a$ (5)

Add (4) to (5), ... $2x=a\pm\sqrt{a^2-4b}$ (6)

Take (4) from (5) . . . . . $2y=a\mp\sqrt{a^2-4b}$ (7)

Dividing (6) and (7) by 2 and we have the values of x and y, and by reason of the double sign each letter has two values. When the quantity $\sqrt{a^2-4b}$ is a complete second power, the values of x and y will be rational, otherwise they will contain surds.

2. Given $\left\{\begin{array}{ll} x+y=a & (1) \\ x^2+y^2=b & (2) \end{array}\right\}$ to find x and y.

Squaring (1), . . $x^2+2xy+y^2=a^2$ (3)

Subtracting (2), ... $2xy=a^2-b$ (4)

Taking (4) from (2), $x^2-2xy+y^2=2b-a^2$ (5)

Square root of (5), . . $x-y=\pm\sqrt{2b-a^2}$ (6)

Adding (6) and (1), . . . $2x=a\pm\sqrt{2b-a^2}$

Taking (6) from (1), . . . $2y=a\mp\sqrt{2b-a^2}$

3. Given $\left\{\begin{array}{ll} x+y=a & (1) \\ x^2-y^2=b & (2) \end{array}\right\}$ to find x and y.

Divide (2) by (1), and $x-y=\frac{b}{a}$. (3)

Equation (1) and (3) will give the values sought.

4. Given $\left\{\begin{array}{ll} x+y=a & (1) \\ x^3+y^3=b & (2) \end{array}\right\}$ to find x and y.

Cube (1), . $x^3+3x^2y+3xy^2+y^3=a^3$ (3)

From (3) take (2), and $3x^2y+3xy=a^3-b$ (4)

Or, ... $3xy(x+y)=a^3-b$

Divide (4) by (1), and . . $3xy=a^2-\frac{b}{a}$ (5)

Equations (1) and (5) combined, will make an example the same in form as example 1, and may be solved in the same manner.

5. Given $\left\{\begin{array}{ll} x+y=a & (1) \\ x^3-y^3=b & (2) \end{array}\right\}$ to find x and y.

The equation resulting from these cannot be reduced to the second degree, and we mention the fact to save the time of the operator from making useless trials.

6. Given $\left\{\begin{array}{ll} x-y=a & (1) \\ x^3-y^3=b & (2) \end{array}\right\}$ to find x and y.

Divide (2) by (1), then we have

$$x^2+xy+y^2=\frac{b}{a} \qquad (3)$$

Squaring (1), . . $x^2-2xy+y^2=a^2$ (4)

Taking (4) from (3), . . . $3xy=\frac{b}{a}-a^2$ (5)

Dividing (5) by (3), . . $xy=\frac{b}{3a}-\frac{a^2}{3}$ (6)

Adding (6) and (3), $x^2+2xy+y^2=\frac{4b}{3a}-\frac{a^2}{3}$ (7)

Square root of (7), . . $x+y=\pm\sqrt{\frac{4b}{3a}-\frac{a^2}{3}}$ (8)

Combine (1) and (8) for the values of x and y.

To exercise in these general principles, we give the following numeral examples:

7. Given $\begin{cases} x+y=12 \\ xy=35 \end{cases}$ to find x and y. *Ans.* $\begin{cases} x=5 \text{ or } 7, \\ y=7 \text{ or } 5. \end{cases}$

8. Given $\begin{cases} x-y=-1 \\ xy=42 \end{cases}$ to find x and y. *Ans.* $\begin{cases} x=6 \text{ or } -7, \\ y=7 \text{ or } -6. \end{cases}$

9. Given $\begin{cases} x+y=1125 \\ x^2-y^2=1125 \end{cases}$ to find x and y. *Ans.* $\begin{cases} x=563, \\ y=562. \end{cases}$

10. Given $\begin{cases} x-y=4 \\ x^3-y^3=124 \end{cases}$ to find x and y. *Ans.* $\begin{cases} x=5 \text{ or } -1, \\ y=1 \text{ or } -5. \end{cases}$

11. Given $\begin{cases} x^3+y^3=19(x+y) \\ x-y=3 \end{cases}$ to find x and y.

Ans. $\begin{cases} x=5 \text{ or } -2, \\ y=2 \text{ or } -5. \end{cases}$

12. Given $\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{5}{6} \\ \frac{1}{x^2}+\frac{1}{y^2}=\frac{31}{36} \end{cases}$ to find x and y. *Ans.* $\begin{cases} x=2 \text{ or } 3, \\ y=3 \text{ or } 2. \end{cases}$

To solve this, put, $\frac{1}{x}=P$, and $\frac{1}{y}=Q$, then we have an example in the form of example 2.

13. Given $\begin{cases} x^3+y^3=152 \\ x+y=8 \end{cases}$ to find x and y. *Ans.* $\begin{cases} x=5 \text{ or } 3, \\ y=3 \text{ or } 5. \end{cases}$

(ART. 102.) Equations are homogeneous, when the sum of the exponents of the unknown quantities is the same in every term.

Thus, $\left.\begin{matrix} 2x^2-xy=6 \\ 2y^2+3xy=8 \end{matrix}\right\}$ are homogeneous equations, because the sum of the exponents of x and y is the same in every term; that is, 2.

Such equations may always be resolved by putting one unknown quantity equal to the other multiplied by a new unknown factor.

To solve these equations, put $x=vy$.

Substituting this value of x in the two equations, and they become . . . . $2v^2y^2-vy^2=6$ (1)

And . . . . . . $2y^2+3vy^2=8$ (2)

From (1) . . . . $y^2=\dfrac{6}{2v^2-v}$ (3)

From (2) . . . . $y^2=\dfrac{8}{2+3v}$ (4)

Equating (3) and (4), clearing of fractions and reducing, we have . . . . $8v^2-13v=6$ (5)

Thus, from every pair of homogeneous equations, we may have a resulting equation of the second degree in reference to the new factor introduced.

Solving equation (5), we find $v=2$ or $-\frac{3}{8}$.

Taking 2 for the value, $x=2y$, and from equation (4) $y^2=\dfrac{8}{2+6}=1$. Hence, $y=\pm 1$, which gives $x=2$, or -2.

The equations . $x^2+y^2-x-y=78$

$xy+x+y=39$, are both quadratic, and, therefore, by Art. 100, they will produce a resulting equation of the *fourth* degree; but they are also similar and symmetrical, and for this reason, it is *possible* to bring out a resulting quadratic, but no general rule of operation can be laid down, and the operator must depend mainly on his own acquired tact and skill.

To resolve these equations, we double the second, and add it to the first, we then have

$$x^2+2xy+y^2+x+y=156$$

The first member of this equation is obviously the same as

$$(x+y)^2+(x+y)=156$$

For the purpose of simplification, put $(x+y)=s$.

Then, $s^2+s=156$, a quadratic equation in relation to s, and a solution gives s, or $x+y=12$. This equation taken from the second of the primitive equations gives $xy=27$, and from these last two equations, $x=9$ or 3, and $y=3$ or 9.

There are a great variety of circumstances that may come in aid, or deter the solution of equations; but it is not proper to notice them in an elementary work like this. For a more full development of these particulars in equations, see Robinson's Algebra, University Edition.

(ART. 103.) It is not essential that the unknown quantity should be in involved literally to its first and second powers; it is only essential that the index of one power should be double that of the other.* In such cases, the equations can be resolved as quadratics. For example, $x^6-4x^3=621$ is an impure equation of the sixth degree, yet with a view to its solution, it may be called a quadratic. For we can assume

* From this and the following article we perceive that the term *quadratic* equations, is far more proper and comprehensive than equations of the second degree.

We speak of this because it has been suggested to us, that the modern rules of science required the systematic use of the term equations of the first, *second*, *third*, &c., degrees. The author of this work is modern in all his views, and is an advocate for modern improvements; but it must be improvements, not merely varieties, or changes in technicalities.

$y=x^3$; then $y^2=x^6$, and the equation becomes $y^2-4y=621$, a quadratic in relation to y, giving $y=27$, or -23.

Therefore, . . . $x^3=27$ or -23

And . . . . . $x=3$ or $\sqrt[3]{-23}$

There are other values of x; but it would be improper to seek for them now; such inquiries belong to the higher order of equations.*

For another example, take $x^3-x^{\frac{3}{2}}=56$, to find the values of x.

Here we perceive one exponent of x is *double* that of the other; it is therefore essentially a quadratic.

Such cases can be made clear by assuming the lowest power of the unknown quantity equal to any single letter. In the present case, assume $y=x^{\frac{3}{2}}$; then $y^2=x^3$, and the equation becomes . . . $y^2-y=56$

A solution gives $y=8$, or -7, and by returning to the assumption, $y=x^{\frac{3}{2}}$, we find $x^{\frac{3}{2}}=8$, or $x^{\frac{1}{2}}=2$, or $x=4$.

(Art. 104.) When a compound quantity appears under different powers or fractional exponents, one exponent being double of the other, we may put the quantity equal to a single letter, and make its quadratic form apparent and simple. For example, suppose the values of x were required in the equation

$$2x^2+3x+9-5\sqrt{2x^2+3x+9}=6$$

Assume . . . . . . $\sqrt{2x^2+3x+9}=y$

Then by involution, . . $2x^2+3x+9=y^2$ (A)

And the equation becomes . $y^2-5y=6$ (B)

Which equation gives $y=6$ or -1. These values of y, substituted for y in equation (A), give

$$2x^2+3x+9=36$$

Or, ... $2x^2+3x+9=1$

From the first of these we find . . $x=3$ or $-4\frac{1}{2}$.

---

* See Algebra, University Edition.

From the last, we find $x=\frac{1}{4}(-3\pm\sqrt{-55})$, imaginary quantities.

We give a few examples to fix the principles explained in Articles 103 and 104.

1. Given $x+3+2(x+3)^{\frac{1}{2}}=35$, to find one value of x.

Ans. $x=22$.

2. Given $(y^2+2y)^2+4(y^2+2y)=96$, to find one value of y.

Ans. $y=2$.

3. Given $10+x-(10+x)^{\frac{1}{2}}=12$, to find one value of x.

Ans. $x=6$.

4. Given $\left(\frac{6}{y}+y\right)^2+\left(\frac{6}{y}+y\right)=30$ to find y.

Ans. $y=3$ or 2, or $-3\pm\sqrt{3}$.

5. Given $(x+12)^{\frac{1}{2}}+(x+12)^{\frac{1}{4}}=6$, to find the values of x.

Ans. $x=4$ or 69.

6. Given $(x+a)^{\frac{1}{2}}+2b(x+a)^{\frac{1}{4}}=3b^2$, to find the values of x.

Ans. $x=b^4-a$ or $81b^4-a$.

It is very seldom that problems produce such compound equations as the last six, or indeed never will unless expressly designed so to do. The following is one:

1. A poulterer going to market to buy turkeys, met with four flocks. In the second, were 6 more than 3 times the square root of double the number in the first. The third contained 3 times as many as in the first and second; and the fourth contained 6 more than the square of one-third the number in the third; and the whole number was 1938. How many were in each flock? *Ans.* 18, 24, 126, 1770.

Let . . . . $2x^2=$ the number in the first,

Then . . $6x+6=$ the number in the second,

$3(2x^2+6x+6)=$ the number in the third,

$(2x^2+6x+6)^2+6=$ the number in the fourth.

Assume $2x^2+6x+6=y$. Then the whole sum is

$$y^2+4y+6=1938$$

Subtracting 2 from both members, and extracting square root, we have . . . $y+2=44$

We do not take the minus sign, for minus cannot apply to this problem.

From the assumed equation, we have

$$2x^2+6x+6=42$$

2. If a certain number be increased by 3, and the square root of the sum taken and added to the number, the sum will be 17. What is the number? *Ans.* 13.

3. The square of a certain number, and 11 times the number makes 80. What is the number? *Ans.* 5.

4. Find two numbers, such that the less may be to the greater as the greater is to 12, and that the sum of their squares may be 45. *Ans.* 3 and 6.

5. What two numbers are those, whose difference is 3, and the difference of their cubes 189? *Ans.* 3 and 6.

6. What two numbers are those, whose sum is 5, and the sum of their cubes 35? *Ans.* 2 and 3.

7. A merchant has a piece of broadcloth and a piece of silk. The number of yards in both is 110; and if the square of the number of yards of silk be subtracted from 80 times the number of yards of broadcloth, the difference will be 400. How many yards are there in each piece?

Ans. 60 of silk; 50 of broadcloth.

8. A is 4 years older than B; and the sum of the squares of their ages is 976. What are their ages?

Ans. A's age, 24 years; B's, 20 years.

9. Divide the number 10 into two such parts, that the square of 4 times the less part, may be 112 more than the square of 2 times the greater. *Ans.* 4 and 6.

10. Find two numbers, such that the sum of their squares may be 89, and their sum multiplied by the greater, may produce 104. *Ans.* 5 and 8

11. What number is that, which, being divided by the product of its two digits, the quotient is $5\frac{1}{3}$; but when 9 is subtracted from it, there remains a number having the same digits inverted? *Ans.* 32.

12. Divide 20 into three parts, such that the continual product of all three may be 270, and that the difference of the first and second may be 2 less than the difference of the second and third. *Ans.* 5, 6, and 9.

13. A regiment of soldiers, consisting of 1066, formed into two squares, one of which has four men more in a side than the other. What number of men are in a side of each of the squares? *Ans.* 21 and 25.

14. The plate of a lookingglass is 18 inches by 12, and is to be framed with a frame of equal width, whose area is to be equal to that of the glass. Required the width of the frame. *Ans.* 3 inches.

15. A square courtyard has a rectangular gravel walk round it. The side of the court wants two yards of being six times the width of the gravel walk, and the number of square yards in the walk exceeds the number of yards in the periphery of the court by 164. Required the area of the court. *Ans.* 256 yards.

16. A and B start at the same time to travel 150 miles; A travels 3 miles an hour faster than B, and finishes his journey $8\frac{1}{3}$ hours before him; at what rate per hour did each travel? *Ans.* 9 and 6 miles per hour.

17. A company at a tavern had 1 dollar and 75 cents to pay; but before the bill was paid two of them went away, when those who remained had each 10 cents more to pay; how many were in the company at first? *Ans.* 7.

18. A set out from C, toward D, and traveled 7 miles a day. After he had gone 32 miles, B set out from D toward C, and went every day $\frac{1}{19}$ of the whole journey; and after he

had traveled as many days as he went miles in a day, he met A. Required the distance from C to D.

Ans. 76 or 152 miles; both numbers will answer the condition.

19. A farmer received 24 dollars for a certain quantity of wheat, and an equal sum at a price 25 cents less by the bushel for a quantity of barley, which exceeded the quantity of wheat by 16 bushels. How many bushels were there of each? *Ans.* 32 bushels of wheat, and 48 of barley.

20. A laborer dug two trenches, one of which was 6 yards longer than the other, for 17 pounds, 16 shillings, and the digging of each of them cost as many shillings per yard as there were yards in its length. What was the length of each?

Ans. 10 and 16 yards.

21. A and B set out from two towns which were distant from each other 247 miles, and traveled the direct road till they met. A went 9 miles a day, and the number of days at the end of which they met, was greater, by 3, than the number of miles which B went in a day. How many miles did each travel? *Ans.* A, 117, and B 130 miles.

22. The fore wheels of a carriage make 6 revolutions more than the hind wheels, in going 120 yards; but if the circumference of each wheel be increased 1 yard, they will make only 4 revolutions more than the hind wheels, in the same distance; required the circumference of each wheel.

Ans. 4 and 5 yards.

23. There are two numbers whose product is 120. If 2 be added to the lesser, and 3 subtracted from the greater, the product of the sum and remainder will also be 120. What are the numbers? *Ans.* 15 and 8.

24. There are two numbers, the sum of whose squares exceeds twice their product, by 4, and the difference of their squares exceeds half their product, by 4; required the numbers. *Ans.* 6 and 8.

25. What two numbers are those, which being both multiplied by 27, the first product is a square, and the second the root of that square; but being both multiplied by 3, the first product is a cube, and the second the root of that cube?

Ans. 243 and 3.

26. A man bought a horse, which he sold, after some time, for 24 dollars. At this sale he loses as much per cent. upon the price of his purchase as the horse cost him. What did he pay for the horse?

Ans. He paid $60 or $40; the problem does not decide which sum.

27. What two numbers are those whose product is equal to the difference of their squares; and the greater number is to the less as 3 to 2? *Ans.* No such numbers exist.

28. What two numbers are those, the double of whose product is less than the sum of their squares by 9, and half their product is less than the difference of their squares by 9?

Ans. The numbers are 9 and 12.

Will the student show that examples 24 and 28 are essentially the same.

SECTION V.

ARITHMETICAL PROGRESSION.

(Art. 105.) A series of numbers or quantities, increasing or decreasing by the same difference, from term to term, is called arithmetical progression.

Thus, 2, 4, 6, 8, 10, 12, &c., is an increasing or ascending arithmetical series, having a common difference of 2; and 20, 17, 14, 11, 8, &c., is a decreasing series, whose common difference is 3.

We can more readily investigate the properties of an arithmetical series from literal than from numeral terms. Thus, let a represent the first term of a series, and d the common difference. Then

$$a,\ (a+d),\ (a+2d),\ (a+3d),\ (a+4d),\ \&c.,$$

represents an ascending series; and

$$a,\ (a-d),\ (a-2d),\ (a-3d),\ (a-4d),\ \&c.,$$

represents a descending series.

Observe that the coefficient of d in any term, is equal to the number of the preceding term.

The first term exists without the common difference. All other terms consist of the first term and the common difference multiplied by *one* less than the number of terms.

Thus, if the first term of an arithmetical series is a, and d the common difference, the tenth term would be expressed by

$$a+9d$$

The 17th term by . $a+16d$

The 53d term by . $a+52d$

The nth term by . $a+(n-1)d$

When the series is decreasing, the sign to the term containing d will be minus, the 20th term, for example, would be

$$a-19d$$

The nth term . . . . $a-(n-1)d$

We add a few examples to exercise the pupil in finding any term of a series, when the first term, a, and the common difference, d, are given.

1. When $a=2$ and $d=3$, what is the 10th term? *Ans.* 29.
2. When $a=3$ and $d=2$, what is the 12th term? *Ans.* 25.
3. When $a=7$ and $d=10$, what is the 21st term? *Ans.* 207.
4. When $a=1$ and $d=\frac{1}{2}$, what is the 100th term? *Ans.* $50\frac{1}{2}$.
5. When $a=3$ and $d=\frac{1}{3}$, what is the 100th term? *Ans.* 36.
6. When $a=0$ and $d=\frac{1}{8}$, what is the 89th term? *Ans.* 11.
7. When $a=6$ and $d=-\frac{1}{2}$, what is the 20th term? *Ans.* $-3\frac{1}{2}$.
8. When $a=30$ and $d=-3$, what is the 31st term? *Ans.* -60.

Wherever the series is supposed to terminate, is the last term, and if such term be designated by L, and the number of terms by n, the last term must be $a+(n-1)d$, or

$a-(n-1)d$, according as the series may be ascending or descending, which we draw from inspection.

Hence, ... $L=a\pm(n-1)d$ (A)

(Art. 106.) It is manifest, that the sum of the terms will be the same, in whatever order they are written.

Take, for instance, the series .	3,	5,	7,	9,	11,
And the same inverted, . . .	11,	9,	7,	5,	3.
The sums of the terms will be	14,	14,	14,	14,	14.
Take . .	a	$a+d$,	$a+2d$,	$a+3d$,	$a+4d$,
Inverted, . . .	$a+4d$,	$a+3d$,	$a+2d$,	$a+d$,	a
Sums, . .	$2a+4d$,	$2a+4d$,	$2a+4d$,	$2a+4d$,	$2a+4d$.

Here we discover the important property, that, in arithmetical progression, *the sum of the extremes is equal to the sum of any other two terms equally distant from the extremes. Also, that twice the sum of any series is equal to the extremes, or first and last term repeated as many times as the series contains terms.*

Hence, if S represents the sum of a series, and n the number of terms, a the first term, and L the last term, we shall have . . . . . . $2S=n(a+L)$

Or, ... $S=\frac{n}{2}(a+L)$ (B)

The two equations (A) and (B) contain *five* quantities, a, d, L, n, and S; any three of them being given, the other *two* can be determined.

Two independent equations are sufficient to determine two unknown quantities (Art. 45), and it is immaterial which two are unknown, if the other three are given.

By examining the two equations they will become familiar.

$$L=a+(n-1)d \qquad (A)$$

$$S=\frac{n}{2}(a+L) \qquad (B)$$

Equation (A) and (B) furnish all the rules given in Arithmetics in relation to arithmetical progression.

For instance, the rule to find the last term of any arithmetical series, is equation (A) put in words, thus:

RULE.—*Multiply the common difference by the number of terms less one, and to the product add the first term.*

A rule for finding the sum of any series, we draw from equation (B), thus:

RULE.—*Multiply the sum of the extremes by half the number of terms.*

EXAMPLES.

1. The first term of an arithmetical series is 5, the last term 92, and the number of terms 30. What is the sum of the terms? *Ans.* 1455.

2. The first term of an arithmetical series is 2, the number of terms 10, and the last term 30. What is the sum of the terms? *Ans.* 160.

3. The first term of an arithmetical series is 5, the common difference 3, and the number of terms 30. What is the last term? *Ans.* 92.

4. The first term of an arithmetical series is 7, the last term 207, and the number of terms 21. What is the sum of the terms? *Ans.* 2247.

5. The first term of an arithmetical series is 6, the last term $-3\frac{1}{2}$, and the number of terms 20. What is the sum of the terms? *Ans.* 25.

The two equations (A) and (B) cover the whole subject of arithmetical progression, when any three of the five quantities are given; for there would be *two unknown quantities*, and we have *two equations*, which are sufficient to find them; we, therefore, give the following miscellaneous examples:

Use the equations *without modification* or change, by putting in the given values just as they stand, and afterward reduce them as numeral equations.

EXAMPLES.

1. The sum of an arithmetical series is 1455, the first term 5, and the number of terms 30. What is the common difference? *Ans.* 3.

Here, $S=1455$, $a=5$, $n=30$. L and d are sought.

Equation (B), $1455=(5+L)15$. Reduced, $L=92$.

Equation (A), $92=5+29d$. Reduced, $d=3$, *Ans.*

2. The sum of an arithmetical series is 567, the first term 7, and the common difference 2. What is the number of terms? *Ans.* 21.

Here, $S=567$, $a=7$, $d=2$. L and n are sought.

Equation (A), $L=7+2n-2=5+2n$

Equation (B), $567=(7+5+2n)\frac{n}{2}=6n+n^2$

Or, ... $n^2+6n+9=576$

$n+3=24$, or $n=21$, *Ans.*

3. Find *seven* arithmetical means between 1 and 49.

Observe that the series must consist of 9 terms.

Hence, $a=1$, $L=49$, $n=9$.

Ans. 7, 13, 19, 25, 31, 37, 43.

4. The first term of an arithmetical series is 1, the sum of the terms 280, the number of terms 32. What is the common difference, and the last term? *Ans.* $d=\frac{1}{2}$, $L=16\frac{1}{2}$.

5. Insert three arithmetical means between $\frac{1}{3}$ and $\frac{1}{2}$.

Ans. The means are $\frac{3}{8}$, $\frac{5}{12}$, $\frac{11}{24}$.

6. Insert five arithmetical means between 5 and 15.

Ans. The means ars $6\frac{2}{3}$, $8\frac{1}{3}$, 10, $11\frac{2}{3}$, $13\frac{1}{3}$.

7. Suppose 100 balls be placed in a straight line, at the distance of a yard from each other; how far must a person travel to bring them one by one to a box placed at the distance of a yard from the first ball?

Ans. 5 miles and 1300 yards.

8. A speculator bought 47 house lots in a certain village, giving 10 dollars for the first, 30 dollars for the second, 50 dollars for the third, and so on. What did he pay for the whole 47? *Ans.* \$22090.

9. In gathering up a certain number of balls, placed on the ground in a straight line, at the distance of 2 yards from each other, the first being placed 2 yards from the box in which they were deposited, a man, starting from the box, traveled 11 miles and 840 yards. How many balls were there?

Ans. 100.

10. How many strokes do the clocks of Venice, which go on to 24 o'clock, strike in a day? *Ans.* 300.

11. In a descending arithmetical series the first term is 730, the common difference 2, and the last term 2. What is the number of terms? *Ans.* 365.

12. The sum of the terms of an arithmetical series is 280, the first term 1, and the number of terms 32. What is the common difference? *Ans.* $\frac{1}{2}$.

13. The sum of the terms of an arithmetical series is 950, the common difference 3, and the number of terms 25. What is the first term? *Ans.* 2.

14. What is the sum of n terms of the series 1, 2, 3, 4, 5, &c.? *Ans.* $S=\frac{n}{2}(1+n)$

15. Suppose a man owes 1000 dollars, what sum shall he pay daily so as to cancel the debt, principal and interest, at the end of a year, reckoning it at 6 per cent, simple interest?

Divide 1000 dollars by 365, and call the quotient a. This would be the sum he must pay daily, provided there were no interest to be paid.

Cast the interest on a for *one day*, at 6 per cent, and call this interest i.

18

Then the first day he must pay $a+i$

The second day, ... $a+2i$

The third day, ... $a+3i$; and so, on in arithmetical progression.

The last day he must pay . $a+365i$

Altogether, he must pay . $\left(\frac{2a+366i}{2}\right)365.$

Or, he must pay daily, . . $a+183i=$ the answer.

(Art. 107.) Bodies falling near the surface of the earth, and unresisted by the atmosphere, fall in the first second of time $16\frac{1}{12}$ feet, and increase the distance which they fall $2(16\frac{1}{12})$ feet every second. Hence, $16\frac{1}{12}$ feet may be considered the first term of an arithmetical series, and $2(16\frac{1}{12})$ the common difference. We call $16\frac{1}{12}$ feet g, the symbol for gravity. Then g is the first term of an arithmetical series, and $2g$ the common difference. Hence, g, $3g$, $5g$, $7g$, $9g$, &c., are the spaces corresponding to 1, 2, 3, 4, &c., seconds.

These facts being admitted, show a formula for the fall of a body in 10 seconds, and for its fall the last second of the ten,

From (A) . . . $L=g+9\cdot2g$

From (B) . . . $S=5(2g+18g)$

Hence, its fall during the last second of the ten is $19g$, and the whole space fallen through is $100g$, which is the *square of the seconds* multiplied by *the force of gravity*, and this is the general rule in Astronomy.

But *this manner* of arriving at the result is *not* recommended, except as an exercise in progression.

PROBLEMS IN ARITHMETICAL PROGRESSION

TO WHICH THE PRECEDING FORMULAS, (A) AND (B), DO NOT IMMEDIATELY APPLY.

(Art. 108.) When three quantities are in arithmetical progression, it is evident that the middle one must be the exact *mean* of the three, otherwise, it would not be arithmetical progression; therefore the sum of the extremes must be double that of the mean.

Take, for example, any three consecutive terms of a series, as . . . . . . $a+2d,\quad a+3d,\quad a+4d$

and we perceive, by inspection, that the sum of the extremes is double that of the mean.

When there are four terms, the sum of the extremes is equal to the sum of the means, by (Art. 106).

To facilitate the solution of problems, when three terms are in question, let them be represented by $(x-y)$, x, $(x+y)$, y being the common difference.

When four numbers are in question, let them be represented by $(x-3y)$, $(x-y)$, $(x+y)$, $(x+3y)$, $2y$ being the common difference.

So in general for any other number, assume such terms *that the common difference will disappear by addition.*

EXAMPLES.

1. Three numbers are in arithmetical progression, the product of the first and second is 15, and of the first and third is 21. What are the numbers? *Ans.* 3, 5, and 7.

2. There are four numbers in arithmetical progression, the sum of the two means is 25, and the second, multiplied by the common difference is 50. What are the numbers? *Ans.* 5, 10, 15, and 20.

3. There are four numbers in arithmetical progression, the product of the first and third is 5, and of the second and fourth is 21. What are the numbers? *Ans.* 1, 3, 5, and 7.

4. There are five numbers in arithmetical progression, the sum of these numbers is 65, and the sum of their squares 1005. What are the numbers? *Ans.* 5, 9, 13, 17, and 21.

Let $x=$ the middle term, and y the common difference.

Then $x-2y$, $x-y$, x, $x+y$, $x+2y$, will represent the numbers, and their sum will be $5x=65$, or $x=13$. Also, the sum of their squares will be

$$5x^2+10y^2=1005, \text{ or } x^2+2y^2=201.$$

5. The sum of three numbers in arithmetical progression is 15, and their continued product is 105. What are the numbers? *Ans.* 3, 5, and 7.

6. There are three numbers in arithmetical progression, their sum is 18, and the sum of their squares 158. What are those numbers? *Ans.* 1, 6, and 11.

7. Find three numbers in arithmetical progression, such that the sum of their squares may be 56, and the sum arising by adding together once the first and twice the second, and thrice the third, may amount to 28. *Ans.* 2, 4, 6.

8. Find three numbers having equal differences, so that their sum may be 12, and the sum of their fourth powers 962. *Ans.* 3, 4, 5.

9. Find three numbers having equal differences, and such that the square of the least added to the product of the two greater, may make 28, but the square of the greatest added to the product of the two less, may make 44. *Ans.* 2, 4, 6.

10. Find three numbers in arithmetical progression, such that their sum shall be 15, and the sum of their squares 93. *Ans.* 2, 5, and 8.

11. Find three numbers in arithmetical progression, such that the sum of the first and third shall be 8, and the sum of the squares of the second and third shall be 52. *Ans.* 2, 4, and 6.

12. Find four numbers in arithmetical progression, such that the sum of the first and fourth shall be 13, and the difference of the squares of the two means shall be 39. *Ans.* 2, 5, 8, and 11.

13. Find seven numbers in arithmetical progression, such

that the sum of the first and sixth shall be 14, and the product of the third and fifth shall be 60.

Ans. 2, 4, 6, 8, 10, 12, and 14.

15. Find five numbers in arithmetical progression, such that their sum shall be 25, and their continued product 945.

Ans. 1, 3, 5, 7, and 9.

16. Find four numbers in arithmetical progression, such that the difference of the squares of the first and second shall be 12, and the difference of the squares of the third and fourth shall be 28. *Ans.* 2, 4, 6, and 8.

GEOMETRICAL PROGRESSION.

(ART. 109.) When a series of numbers or quantities increase or decrease by a constant multiplier from term to term, the numbers or quantities are said to be in *geometrical progression*, and the constant multiplier is called the *ratio.*

Thus, let a be the first term of the progression, and r the ratio, then a, ar, ar^2, ar^3, ar^4, &c., will represent the series.

If r is greater than 1, the series will be *ascending;* if less than 1, the series will be *descending*, and if $r=1$, every term of the series will be the same in value.

For example, 2, 6, 18, 54, 162, &c., is a geometrical series in which the first term a is 2, and the ratio is 3.

The series 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c., is also a *geometrical series* in which the first term a is 9, and the *multiplier*, the *ratio*, is $\frac{1}{3}$.

The series 3, 3, 3, 3, &c., is also a *geometrical series* in which the first term a, is 3, and the *multiplier*, the *ratio*, is 1.

Geometry compares magnitudes, and inquires how many times one magnitude is greater than another, and thus, in the series, 2, 4, 8, 16, &c., 4 is two times 2, 8 is 2 times 4, &c.; hence, numbers so compared, and a regular series thus obtained, is called a *geometrical* series.

(ART. 110.) *In any given series we may find the ratio, by dividing any term by its preceding term.*

(ART. 111.) Taking the general series a, ar, ar^2, ar^3, ar^4, &c., under inspection, we find that the *first* power of r is a factor in the second term, the second power of r in the third term, the third power of r in the fourth term, and thus, universally, *the power of the ratio* in any term, is one less than the number of the term.

The first term is a factor in every term. Hence, the 10th term of this general series is ar^9. The 17th term would be ar^{16}. The 25th term would be ar^{24}, and, in general, the nth term would be ar^{n-1}.

Therefore, if n represent the number of terms in any series, and L the last term, then

$$L=ar^{n-1} \qquad (1)$$

Wherever a series is supposed to terminate, is the last term, and equation (1) is a general representation of it; and if we multiply that equation by r, we shall have

$$rL=ar^n$$

(ART. 112.) Let S represent the sum of any geometrical series, then we have

$$S=a+ar+ar^2+ar^3, \text{ \&c., to } ar^{n-1}$$

Multiply this equation by r, and we have

$$rS=ar+ar^2+ar^3, \text{ \&c., to } ar^{n-1}+ar^n$$

Subtracting the upper equation from the lower, and observing that . . . . . . $rL=ar^n$

Then . . . $(r-1)S=rL-a$

Therefore, . . . $S=\dfrac{rL-a}{r-1} \qquad (2)$

As the equations (1) and (2) are fundamental, and cover the whole subject of geometrical progression, let them be brought together for critical inspection.

Here we perceive five quantities, a, r, n, L, and S, and any three of them being given in any problem, the other two can be determined from the equations,

$$L=ar^{n-1} \qquad (1)$$

$$S=\frac{Lr-a}{r-1} \qquad (2)$$

These two equations furnish the rules given for the operations in common arithmetic.

Thus, in almost every Arithmetic, the rules for finding the last term of any arithmetical series is expressed in the following words:

RULE.—*Raise the ratio to a power one less than the number of terms, and multiply that number by the first term.*

This rule is simply equation (1) put in words.

Equation (2) gives the following rule for the sum of a series.

RULE.—*Multiply the last term by the ratio, and from the product subtract the first term, and divide the remainder by the ratio less one.*

GENERAL EXAMPLES IN GEOMETRICAL PROGRESSION.

1. What is the ratio of the series 2, 6, 18, 54, &c.? *Ans.* 3.
2. What is the ratio of the series 5, 20, 80, &c.? *Ans.* 4.
3. What is the ratio of the series $\frac{2}{3}$, $\frac{4}{9}$, $\frac{8}{27}$, &c.? *Ans.* $\frac{2}{3}$.
4. What is the ratio of the series $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$, &c. *Ans.* $\frac{1}{10}$.
5. What is the ratio of the series $\frac{5}{3}$, 1, $\frac{3}{5}$, &c.? *Ans.* $\frac{3}{5}$.

6. What is the ratio of the series 8, 20, 50, &c.? *Ans.* $2\frac{1}{2}$.

7. What is the ratio of the series $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}$, &c.? *Ans.* $\frac{1}{x}$.

8. What is the ratio of the series $a, -b, +\frac{b^2}{a}$ &c.?

Ans. $-\frac{b}{a}$.

9. What is the 11th term of the series 1, 2, 4, &c.?

Ans. 1024.

10. What is the 9th term of the series 5, 20, 80, &c.?

Ans. 327680.

11. What is the 8th term of the series 2, 6, 18, &c.?

Ans. 4374.

12. What is the 6th term of the series 1, $\frac{3}{4}$, $\frac{9}{16}$, &c.

Ans. $\frac{243}{1024}$.

13. What is the sum of 8 terms of the series 2, 6, 18, &c.?

Ans. 6560.

$$(2) \qquad S=\frac{rL-a}{r-1}=\frac{3\cdot 4374-2}{2}=6560$$

14. What is the sum of 10 terms of the series 4, 12, 36, &c.? *Ans.* 118096.

15. What is the sum of 9 terms of the series 5, 20, 80, &c.? *Ans.* 436905.

16. What is the sum of 5 terms of the series 3, $4\frac{1}{2}$, $6\frac{3}{4}$, &c.? *Ans.* $39\frac{9}{16}$.

17. What is the sum of 10 terms of the series 1, $\frac{2}{3}$, $\frac{4}{9}$, &c.? *Ans.* $\frac{174075}{59049}$.

18. A man purchased a house, giving 1 dollar for the first door, 2 dollars for the second, 4 dollars for the third, and so on, there being 10 doors. What did the house cost him?

Ans. $1023.

(Art. 113.) By equation (2), and the rule subsequently given, we perceive that the sum of a series depends on the first and last terms and the ratio, *and not on the number of terms;* and whether the terms be many or few, there is no variation in the rule. Hence, we may require the sum of any descending series, as $1, \frac{1}{3}, \frac{1}{4}, \frac{1}{8}$, &c., to infinity, *provided we determine the* LAST *term.* Now, we perceive the magnitude of the terms decrease as the series advances; the hundredth term would be extremely small, the thousandth term would be very much less, and the *infinite term* nothing; not too small to be noted, as some tell us, but absolutely *nothing.*

Hence, in any decreasing series, when the number of terms is *conceived* to be infinite, the last term, L, becomes 0, and equation (2) becomes

$$s=\frac{-a}{r-1}$$

By change of signs . $s=\frac{a}{1-r}$

This gives the following rule for the sum of a decreasing infinite series:

RULE.—*Divide the first term by the difference between unity and the ratio.*

EXAMPLES.

1. Find the value of $1, \frac{3}{4}, \frac{9}{16}$, &c., to infinity.

$a=1,\ r=\frac{3}{4}$ *Ans.* 4.

2. Find the exact value of the series 2, 1, $\frac{1}{2}$, &c., to infinity. *Ans.* 4.

3. Find the exact value of the series 6, 4, &c., to infinity. *Ans.* 18.

4. Find the exact value of the decimal .3333, &c., to infinity. *Ans.* $\frac{1}{3}$.

This may be expressed thus: $\frac{3}{10}+\frac{3}{100}$, &c. Hence, $a=\frac{3}{10}$, $r=\frac{1}{10}$.

5. Find the value of .323232, &c., to infinity.

$a=\frac{32}{100}$, $ar=\frac{32}{100000}$; therefore, $r=\frac{1}{100}$. *Ans.* $\frac{32}{99}$.

6. Find the value of .777, &c., to infinity. . . *Ans.* $\frac{7}{9}$.

7. Find the sum of the infinite series $1+\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+$, &c.

Ans. $\frac{x^2}{x^2-1}$.

8. Find the sum of the infinite geometrical progression $a-b+\frac{b^2}{a}-\frac{b^3}{a^2}+\frac{b^4}{a^3}-$, &c., in which the ratio is $-\frac{b}{a}$.

Ans. $\frac{a^2}{a+b}$

(Art. 114.) When three numbers are in geometrical progression, the product of the extremes is equal to the square of the mean.

This principle is obvious from the general series

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ \&c.$$

Taking any three consecutive terms anywhere along the series, we observe, *that the product of the extremes is equal to the square of the mean.*

That is, if the three terms taken, are a, ar, ar^2,

$$a^2r^2=(ar)^2$$

If ar^2, ar^3, ar^4 are the three terms,

$$ar^2\times ar^4=(ar^3)^2$$

Hence, to find a geometrical mean between two numbers, we must multiply them together, and take the square root. *If we take four consecutive terms, the product of the extremes will be equal to the product of the means.*

(Art. 115.) This last property belongs equally to geometrical proportion, as well as to a geometrical series, and the learner must be careful not to confound *proportion* with a *series.*

$a : ar :: b : br$, is a geometrical proportion, *not* a continued series. The ratio is the same in the two couplets, but the magnitudes, a and b, to which the ratio is applied, may be very different.

We may suppose $a : ar$ two consecutive terms of one series, and $b : br$ any two consecutive terms of another series having the same ratio as the first series, and, being brought together, they form a geometrical proportion. *Hence, the equality of the ratio constitutes proportion.*

EXAMPLES.

1. Find the geometrical mean between 2 and 8. *Ans.* 4.

(Art. 114) $\sqrt{2\times 8}=4$

2. Find the geometrical mean between 3 and 12. *Ans.* 6.

3. Find the geometrical mean between 5 and 80. *Ans.* 20.

4. Find the geometrical mean between a and b. *Ans.* $(ab)^{\frac{1}{2}}$.

5. Find the geometrical mean between $\frac{1}{4}$ and 9. *Ans.* $\frac{3}{2}$.

6. Find the geometrical mean between $3a$ and $27a$. *Ans.* $9a$.

7. Find the geometrical mean between 1 and 9. *Ans.* 3.

8. Find the geometrical mean between 2 and 3. *Ans.* $\sqrt{6}$.

9. Find two geometrical means between 4 and 256.

N. B. When the two means are found, the series will consist of *four* terms, and 4 will be the first term and 256 will be the last term.

Comparing this with the general series,

$$a,\ ar,\ ar^2,\ ar^3, \text{ we have}$$

$$a=4 \text{ and } ar^3=256$$

Hence, . . . $r^3=64$ or $r=4$

Therefore, 16 and 64 are the means required.

10. Find three geometrical means between 1 and 16.

Here, the first term of the series is 1, the last term 16, and the number of terms 5, because three terms are required, and two are already given.

Now, by equation (1), $L=ar^{n-1}$

That is, ... $ar^{n-1}=16$

But as $a=1$, and $n=5$, this equation is

$$r^4=16$$

Hence, ... $r=2$

Therefore, the means required are 2, 4, and 8.

We may obtain the ratio when the first and last terms are given, by the following *formula:*

$$r=\left(\frac{L}{a}\right)^{\frac{1}{n-1}}$$

11. The first and last terms of a geometrical series are 2 and 162, and the number of terms 5; required the ratio.

Ans. 3

12. The first term of a geometrical series is 28, the last term 17500, and the number of terms 5; what is the ratio?

Ans. 5.

PROBLEMS THAT INVOLVE THE PRINCIPLES OF GEOMETRICAL PROGRESSION.

(ART. 116.) When we wish to express three unknown quantities in geometrical progression, we may represent them by x, $\sqrt{xy}$, y, or by x^2, xy, y^2, or by x, xy, xy^2, for either of these correspond with Art. 114; that is, the product of the extremes is equal to the square of the mean.

When we wish to express four unknown quantities in geometrical progression, we may express them by x, xy, xy^2, xy^3, or by P, x, y, Q.

The object of this last notation, is to reduce P and Q to terms expressed by x and y, thus:

Taking the first three terms only, we shall have

$$Py=x^2$$

Or, ... $P=\dfrac{x^2}{y}$

Taking the last three terms only, we shall have

$$Q=\frac{y^2}{x}$$

Therefore, four quantities in geometrical progression may be expressed by x and y only, and the terms stand symmetrically thus:

$$\frac{x^2}{y},\ x,\ y,\ \frac{y^2}{x}$$

In a similar manner, we might express more terms by x and y only, and have them stand symmetrically, if it were proper to extend this subject in a work as elementary as this.

1. Three numbers are in geometrical progression, the sum of the first and second is 90, and the sum of the second and third is 180. What are the numbers?

Ans. 30, 60, and 120.

Represent the numbers by x, xy, and xy^2.

2. The sum of three numbers in geometrical progression is 7, and the sum of their squares is 21. What are the numbers? *Ans.* 1, 2, 4.

This problem furnishes the following equations:

$$x+\sqrt{xy}+y=7 \qquad (1)$$

$$x^2+xy+y^2=21 \qquad (2)$$

From (1) . . . . . $x+y=a-\sqrt{xy}$ (3)

From (2) . . . . . $x^2+y^2=3a-xy$ (4)

Squaring (3), . $x^2+2xy+y^2=a^2-2a\sqrt{xy}+xy$ (5)

Subtracting (4) from (5), $2xy=a^2-3a-2a\sqrt{xy}+2xy$ (6)

Dropping $2xy$ from both members, dividing by a, and transposing, we have . . $2\sqrt{xy}=a-3$

That is, ... $\sqrt{xy}=2$ (7)

This value of $\sqrt{xy}$ put in equation (3), gives

$$x+y=5 \quad (8)$$

From equations (7) and (8), we find x and y, as taught in Art. 101.

3. The sum of the first and third of four numbers in geometrical progression is 20, and the sum of the second and fourth is 60. What are the numbers? *Ans.* 2, 6, 18, 54.

4. Divide the number 210 into three parts, so that the last shall exceed the first by 90, and the parts be in geometrical progression. *Ans.* 30, 60, and 120.

5. The sum of four numbers in geometrical progression is 30; and the last term divided by the sum of the mean terms is $1\frac{1}{3}$. What are the numbers? *Ans.* 2, 4, 8, and 16.

6. The sum of the first and third of four numbers in geometrical progression is 148, and the sum of the second and fourth is 888. What are the numbers?

Ans. 4, 24, 144, and 864.

7. The continued product of three numbers in geometrical progression is 216, and the sum of the squares of the extremes is 328. What are the numbers? *Ans.* 2, 6, 18.

8. The sum of three numbers in geometrical progression is 13, and the sum of the extremes being multiplied by the mean, the product is 30. What are the numbers?

Ans. 1, 3, and 9.

9. There are three numbers in geometrical progression whose product is 64, and the sum of their cubes is 584. What are the numbers? *Ans.* 2, 4, and 8.

Let x^2, xy, y^2 represent the three numbers.

Then, ... $x^3y^3=64$ (1)

Also, ... $x^6+x^3y^3+y^6=584$ (2)

Add, ... x^3y^3 . . $=64$

And . . . $x^6+2x^2y^3+y^3=648=324\cdot2$ (3)

Square root, . . . $x^3+y^3=18\sqrt{2}$ (4)

From (2) subtract three times (1), and we have

$$x^6-2x^3y^3+y^6=392=196\cdot2 \quad (5)$$

Square root, . . . $y^3-x^3=14\sqrt{2}$ (6)

We give the *minus* sign to x^3, because y must be greater than x from the position it occupies in our notation, and x^3-y^3 or y^3-x^3, when squared, will produce the same power.

Subtracting (6) from (4), and

$$2x^3=4\sqrt{2}$$

Or, ... $x^3=2\sqrt{2}$

Squaring, ... $x^6=8$

Cube root, ... $x^2=2$, *Ans.*

10. There are three numbers in geometrical progression, the sum of the first and last is 52, and the square of the mean is 100. What are the numbers? *Ans.* 2, 10, 50.

11. There are three numbers in geometrical progression, their sum is 31, and the sum of the squares of the first and last is 626. What are the numbers? *Ans.* 1, 5, 25.

12. It is required to find three numbers in geometrical progression, such that their sum shall be 14, and the sum of their squares 84. *Ans.* 2, 4, and 8.

13. There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means, as 7 to 3. What are the numbers? *Ans.* 1, 3, 9, and 27.

14. The sum of four numbers in geometrical progression is equal to the common ratio +1, and the first term is $\frac{1}{10}$. What are the numbers? *Ans.* $\frac{1}{10}$, $\frac{3}{10}$, $\frac{9}{10}$, $\frac{27}{10}$.

PROPORTION.

(ART. 117.) Two magnitudes of the same kind can be compared with each other, and the numerical relation between them determined. The manner of determining this relation, is to divide one by the other, and the quotient is called the ratio between the two magnitudes. When two quantities have the same *ratio* as two other quantities, the four quantities may constitute a proportion.

Therefore, proportion is the equality of ratios.

Proportion is written in two ways,

Thus, ... $a:b::c:d$

Or thus, ... $a:b=c:d$

The last is the modern method, and means that the ratio of a to b is equal to the ratio of c to d.

If a is taken as the unit of measure between a and b, then $\frac{b}{a}$ is the numerical ratio between these two magnitudes.

If c is taken for the unit of measure between c and d, then $\frac{d}{c}$ is the numerical ratio between these two magnitudes.

The magnitudes a and b may be very different in kind from those of c and d; for instance, a and b may be bushels of wheat, and c and d sums of money.

This manner of comparing magnitudes, by taking one of them as a whole (regardless of other units) is called *geometrical proportion,* and if there are more than two magnitudes having the same ratio, the magnitudes are said to be in geometrical progression.

Two magnitudes compared by *ratio* are called a couplet. Thus, $a:b$ is a couplet, and $c:d$ is another couplet.

The first magnitude of a couplet is called the *antecedent,* the second the *consequent.*

A ratio can exist between two magnitudes; but a proportion requires *four—two antecedents* and *two consequents* having the same ratio.

Thus, if . . . . . $a:b=c:d$

Then . . . . $\frac{b}{a}=\frac{d}{c}$ by the def. of proportion.

All operations in proportion rest on this *fundamental* equation; and to prove a principle or an operation true, we directly, or remotely compare the principle or the operation to this equation, and if we find a correspondence, the principle or the operation is true—otherwise, false.

PROPOSITION I.

In every proportion, the product of the extremes is equal to the product of the means.

Let . . . . $a:b=c:d$ represent any proportion,

Then, ... $\frac{b}{a}=\frac{d}{c}$ must be a true equation.

Multiply both members of this equation by ac, and as the product of equal factors are equal (Ax. 3),

Therefore, ... $cb=ad$

That is, the product of c and b, the means, is equal to the product of a and d, the extremes.

SCHOLIUM.—Divide both members of this equation by a,

Then . . . . . . $\frac{cb}{a}=d$

This equation shows, that the *fourth* term of any proportion may be found from the first *three*, by the following

RULE.—*Multiply the second and third terms of the proportion together, and divide that product by the first term.*

This is a *part of the well known rule of three*, in Arithmetic.

PROPOSITION II.

Conversely. *If the product of two quantities is equal to the product of two others, then two of them may be taken for the means, and the other two for the extremes of a proportion.*

Let . . . . $cb=ad$

Divide both members of this equation by any one of the four factors, say c, then we have

$$b=\frac{ad}{c}$$

Divide this last equation by another of the factors, say a,

Then . . . . $\frac{b}{a}=\frac{d}{c}$

This is the fundamental equation for proportion, and gives

$$a:b=c:d$$

Now, as the principle is established, we may proceed more summarily, and take the two factors in one member for the extremes,

Thus, ... $a: \quad = \quad :d$

To fill up the means, we must take the factor which has the same name as a to stand before the equality, and the other factor to stand after the equality will be of the same name as d, and the proportion will be complete.

If the quantities are all numerals, it is immaterial which factor stands first in the means.

Thus, $a:b=c:d$ } are proportions equally true in numeri-
Or, . $a:c=b:d$ } cal values.

SCHOLIUM.—A proportion and an equation may be regarded as but a different form for the same expression, and every equation may be put into a proportion. For example,

What proportion is equivalent to the following equation?

$$xy=a(a+b)$$

Ans. . . . . . $x:a=a+b:y,$

Or, ... $xy:a=a+b:1$

Or, ... $a:x=y:(a+b)$

What proportion is equivalent to the equation

$$x=cd+a$$

Ans. . . . $x:\left(\frac{cd}{a}+1\right)=a:1,$

Thus, we might give examples without end.

PROPOSITION III.

If three quantities are in continued proportion, the product of the extremes is equal to the square of the mean.

If . . . . . . $a:b=b:c$

From proposition 1, $ac=bb=b^2$

That is, if proposition 1 is true, the truth of this proposition follows as an inevitable consequence.

If $ac=b^2$, then $b=\sqrt{ac}$, which shows that

The mean proportional between two quantities is found by extracting the square root of their product.

PROPOSITION IV.

If four quantities are in proportion, they will be in proportion by INVERSION, *that is, the second will be to the first, as the fourth to the third.*

Let . . . . . . $a:b=c:d$

Then, by the definition of ratio and proportion, we have

$$\frac{b}{a}=\frac{d}{c}$$

Divide 1 by $\frac{b}{a}$ and the quotient is $\frac{a}{b}$.

Divide 1 by $\frac{d}{c}$ and the quotient is $\frac{c}{d}$.

But equals divided by equals must produce equal quotients (Ax. 4).

Therefore, ... $\frac{a}{b}=\frac{c}{d}$

Or, ... $b:a=d:c$

In numbers if . . $3:5=12:20$

Then . . . . . . $5:3=20:12$

PROPOSITION V.

Magnitudes which are proportional to the same proportionals, are proportional to each other.

If $a:b=P:Q$ }
And $c:d=P:Q$ } Then $a:b=c:d$.

From the first proportion, $\frac{b}{a}=\frac{Q}{P}$

By the second, . . . $\frac{d}{c}=\frac{Q}{P}$

Therefore, ... $\frac{b}{a}=\frac{d}{c}$ (Ax. 1).

Or, ... $a:b=c:d$

PROPOSITION VI.

If four magnitudes be in proportion, they must be in proportion by COMPOSITION; *that is, the first will be to the sum of the first and second, as the third will be to the sum of the third and fourth; and the first is to the difference between the first and second, as the third is to the difference between the third and fourth.*

On the supposition that . $a:b=c:d$

We are to prove that $a:a+b=c:c+d$

From the supposition, . $\frac{b}{a}=\frac{d}{c}$

Add each member of this equation to unity, and then we have . . . . $1+\frac{b}{a}=1+\frac{d}{c}$

Reducing these mixed quantities to improper fractions,

And . . . . $\frac{a+b}{a}=\frac{c+d}{c}$

That is, ... $a:a+b=c:c+d$

Subtracting each member of the original equation from unity, and we have

$$1-\frac{b}{a}=1-\frac{d}{c}$$

Or, ... $\frac{a-b}{a}=\frac{c-d}{c}$

Therefore, . . . $a:a-b=c:c-d$

SCHOLIUM.—This *composition* may be carried to almost any extent, as we see by the following investigation:

Take the equation, . . $\frac{b}{a}=\frac{d}{c}$

Multiply both members by m, then

$$\frac{mb}{a}=\frac{md}{c}$$

Add each member of this equation to n,

Then, ... $n+\frac{mb}{a}=n+\frac{md}{c}$

By reduction, . . $\frac{na+mb}{a}=\frac{nc+md}{c}$

Hence, . . . $a:na+mb=c:nc+md$

PROPOSITION VII.

If four quantities be in proportion, the sum of the two quantities which form the first couplet is to their difference, as the sum of the two quantities which form the second couplet is to their difference.

On the supposition that . . $a:b=c:d$

We are to prove that $a+b:a-b=c+d:c-d$

From proposition 6, . . $a:a+b=c:c+d$ (1)

Also, ... $a:a-b=c:c-d$ (2)

From (1), ... $\frac{a+b}{a}=\frac{c+d}{c}$

Dividing both members of this equation by $(a+b)$, and multiplying both members by c, we have

$$\frac{c}{a}=\frac{c+d}{a+b}$$

Operating in the same manner with (2), we shall find

$$\frac{c}{a}=\frac{c-d}{a-b}$$

Therefore, ... $\frac{c+d}{a+b}=\frac{c-d}{a-b}$ (Ax. 1).

Whence, ... $a+b:c+d=a-b:c-d$

Or, ... $a-b:c-d=a+b:c+d$

PROPOSITION VIII.

If four quantities be in proportion, either couplet may be multiplied or divided by any number whatever, and the quantities will still be in proportion.

Let . . . . $a:b=c:d$

Then, ... $\frac{b}{a}=\frac{d}{c}$

Multiplying both numerator and denominator of either of these fractions by any number, n,

Then, ... $\frac{nb}{na}=\frac{d}{c}$

Also, ... $\frac{b}{a}=\frac{nd}{nc}$

That is, ... $na:nb=c:d$

Also, ... $a:b=nc:nd$

Here, n may represent any number whatever; and if it represents a whole number, as 3, 7, 8, &c., then the couplet is multiplied. If n represent a fraction, as $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{8}$, &c., then the couplet is divided.

PROPOSITION IX.

If four quantities be in proportion, the antecedents may be multiplied by any number, and they will still be in proportion; also, the consequents may be multiplied by any number, and the four quantities will still be in proportion.

Let . . . . $a:b=c:d$

Then, ... $\frac{b}{a}=\frac{d}{c}$

Multiplying this equation by m, then

$$\frac{mb}{a}=\frac{md}{c}$$

Therefore, ... $a:mb=c:md$

Divide both members of the original equation by m,

Then, ... $\frac{b}{ma}=\frac{d}{mc}$

Hence, ... $ma:b=mc:d$

PROPOSITION X.

If four magnitudes be in proportion, like powers or roots of the same will be in proportion.

Let . . . . $a:b=c:d$

Then, ... $\frac{b}{a}=\frac{d}{c}$

Raise both members of this equation to any power denoted by n,

Then, ... $\frac{b^n}{a^n}=\frac{d^n}{c^n}$

Hence, ... $a^n:b^n=c^n:d^n$

By extracting any root of the primitive equations, which may be designated by $\frac{1}{n}$, we have

$$\frac{b^{\frac{1}{n}}}{a^{\frac{1}{n}}}=\frac{d^{\frac{1}{n}}}{c^{\frac{1}{n}}}$$

Hence, ... $a^{\frac{1}{n}}:b^{\frac{1}{n}}=c^{\frac{1}{n}}:d^{\frac{1}{n}}$

PROPOSITION XI.

If four quantities be in proportion, also four others, the product or quotient of the two, term by term, will still form proportions.

If . . . . $a:b=c:d$

And, ... $x:y=m:n$

Then we are to prove that

$$ax:by=cm:dn$$

And, ... $\frac{a}{x}:\frac{b}{y}=\frac{c}{m}:\frac{d}{n}$

From the first proportion we have

$$\frac{b}{a}=\frac{d}{c} \qquad (1)$$

From the second, . . . $\frac{y}{x}=\frac{n}{m} \qquad (2)$

By multiplying these two equations together, term by term, we find

$$\frac{by}{ax}=\frac{nd}{mc}$$

That is, ... $ax : by = mc : nd$

Apply proposition 1 to the two given proportions, and we have . . . . $ad = bc$

And, ... $nx = my$

Dividing one of these equations by the other, we have

$$\left(\frac{a}{x}\right)\left(\frac{d}{n}\right)=\left(\frac{b}{y}\right)\left(\frac{c}{m}\right)$$

By the reverse application of proposition 1, we have

$$\frac{a}{x}:\frac{b}{y}=\frac{c}{m}:\frac{d}{n}$$

PROPOSITION XII.

If any number of proportionals have the same ratio, any one of the antecedents will be its consequent, and as the sum of all the antecedents to the sum of all the consequents.

Let . . . . $a : b = a : b$

Also, ... $a : b = c : d$

$a : b = m : n$

&c.=&c.

Then we are to prove that

$$a : b = (a+c+m) : (b+d+n)$$

From the first prop., $ab = ab$

From the second, . $ad = cb$

From the third, . . $an = nb$

By addition, $a(b+d+n) = b(a+c+m)$

By prop. 1, ... $a : b = (a+c+m) : (b+d+n)$

The following examples are intended to illustrate the practical utility of the foregoing propositions:

EXAMPLES.

1. Find two numbers, the greater of which is to the less as their sum to 42, and the greater to the less as their difference is to 6.

Let . $x=$ the greater and $y=$ the less.

Then, by conditions $\begin{cases} x:y=x+y:42 \\ x:y=x-y:6 \end{cases}$

(Prop. 5), . . $x+y:42=x-y:6$

Changing means, $x+y:x-y=42:6$

(Prop. 9), . . . $2x:2y=48:36$

(Prop. 8), . . . $x:\ y=\ 4:3$

With these proportions of x and y, we return to the original conditions; applying proposition 5, and we have

$$4:3=x+y:42$$
$$4:3=x-y:6$$

From the first, . . $x+y=56$

From the second, . $x-y=8$

Hence, ... $x=32,\ y=24$

2. Divide the number 14 into two such parts, that the quotient of the greater, divided by the less, shall be to the less, divided by the greater, as 100 to 16.

Let . $x=$ the greater, and $y=$ the less part,

Then, ... $\frac{x}{y}:\frac{y}{x}=100:16$

And, ... $x+y=14$

(Prop. 8), . . . $x^2:y^2=100:16$

(Prop. 10), . . . $x:y=10:4$

Hence, ... $2x=5y$

But, ... $x+y=14$

Therefore, ... $x=10,\ y=4$

3. Find three numbers in geometrical progression whose sum is 13, and the sum of the extremes is to the double of the mean as 10 to 6.

Let x, xy, and xy^2 represent the numbers,

Then, by the conditions, $x+xy+xy^2=13$
And, ... $xy^2+x:2xy=10:6$
(Prop. 8), ... $y^2+1:2y=10:6$
(Prop. 7), $(y^2+2y+1):(y^2-2y+1)=16:4$
(Prop. 10), ... $y+1:y-1=4:2$
(Prop. 8), ... $2y:2=6:2$
Hence, ... $y=3,\ x=1$

4. The product of two numbers is 35, and the difference of their cubes is to the cube of their difference as 109 to 4. What are the numbers? *Ans.* 7 and 5.

Let x and y represent the numbers. Then, by the given conditions, ... $xy=35$

And, ... $x^3-y^3:(x-y)^3=109:4$

Divide the first couplet by $(x-y)$ (Prop. 8). Then we have . . . . $x^2+xy+y^2:(x-y)^2=109:4$

Expanding $(x-y)^2$, and then making application of (Prop. 7), we have . . . . $3xy:(x-y)^2=105:4$

But, ... $3xy=105$
Therefore, ... $(x-y)^2=4$
And, ... $x-y=2$

5. What two numbers are those, whose difference is to their sum as 2 to 9, and whose sum is to their product as 18 to 77? *Ans.* 11 and 7.

6. Two numbers have such a relation to each other, that if 4 be added to each, they will be in proportion as 3 to 4; and if 4 be subtracted from each, they will be to each other as 1 to 4. What are the numbers? *Ans.* 5 and 8.

7. Divide the number 16 into two such parts that their product shall be to the sum of their squares as 15 to 34. *Ans.* 10 and 6.

8. There are two numbers whose product is 320; and the difference of their cubes, is to the cube of their difference, as 61 to 1. What are the numbers? *Ans.* 20 and 16.

CONCLUSION.

We conclude this volume by giving a general investigation of the rules in Arithmetic for the computation of interest, and the adjustment of accounts in fellowship.

Interest is a percentage paid for the use of money for a specified time.

On this single definition, all the rules of computation are founded. The unit for time is commonly one year.

Let r represent the interest corresponding to unity of principal remaining at interest for unity of time.

Then, as a double capital would demand double interest for the same time, a treble capital, treble interest, and so on. Therefore, if P represents any principal or capital, we have the following proportion:

Prin. Int. Prin. Int.

$$1 : r = P : rP$$

The last term of this proportion shows that to find the interest of any principal for one year, we must multiply that principal by the decimal rate per cent.*

For double the length of time, the interest must be double, for treble the length of time, the interest must be treble; and so on.

Now, let t represent the length of time that any principal, P, remains at interest, r being the rate per cent., and I the aggregate interest, then we shall have this general equation.

$$Prt = I \qquad (1)$$

This gives the following universal rule for computing interest.

RULE.—*Multiply the principal by the decimal rate per cent., and that product by the time.*

---

* *Rate per cent is but another definition for the interest of unity of* principal for unity of time.

If we consider that the principal and interest added together must give the amount, and if we put A to represent the amount, then we shall have

$$Prt+P=A \qquad (2)$$

Equations (1) and (2) embrace all the conditions in relation to interest, and furnish all the rules for computations.

For instance, equation (1) gives

$$P=\frac{I}{rt}$$

Equation (2) gives . $P=\frac{A}{rt+1}$

That is, when any problem requires the finding of the principal, observe the following rules:

RULE 1.—*Divide the interest by the product of the rate and time.*

RULE 2.—*Divide the amount by the product of the rate and time, increased by unity.*

Equation (1) gives . $t=\frac{I}{Pr}$

That is, to find the time, we have the following rule:

RULE.—*Divide the whole interest by the interest for one year.*

Equation (1) gives . $r=\frac{I}{Pt}$

To find the rate per cent., take the following rule:

RULE.—*Divide the interest by the product of the principal and time.*

FELLOWSHIP.

Two men united capital to engage in a certain enterprise, the first put in a dollars, the second b dollars, and they gained g dollars. Give a rule for the equitable division of this gain.

Let x represent the portion belonging to that one which paid in a dollars, and y the portion of the other.

Then, ... $x+y=g$

But their portions of the gain should be in just the same proportion as their capital paid in,

That is, ... $x:y=a:b$, or $bx=ay$

Multiply the first equation by a, then

$$ax+ay=ag, \text{ or } ax+bx=ag$$

Or, ... $x=\frac{ag}{a+b}$, $y=\frac{bg}{a+b}$

Hence, we have the following rule to find each man's share.

RULE.—*Multiply the gain by each man's stock, and divide the product by the whole capital invested.*

Again, suppose three persons, A, B, and C, enter into partnership, and furnish capital in proportion to a, b, and c, and they gain a sum, g, what is each man's share of it?

Let . $x=$ A's share, $y=$ B's share, and $z=$C's share,

Then, . . . $x+y+z=g$ (1)

And, ... $x:y=a:b$, also, $y:z=a:c$

From the first propor., $y=\frac{bx}{a}$ (2)

From the second, . . $z=\frac{cx}{a}$ (3)

These values of y and z, put in equation (1), give

$$x+\frac{bx}{a}+\frac{cx}{a}=g, \text{ or } ax+bx+cx=ag$$

Or, ... $x=\frac{ag}{a+b+c}$ (4)

This value of x put in (2) gives y, and in (3) gives z.

$$y=\frac{bg}{a+b+c} \quad (5)$$

$$z=\frac{cg}{a+b+c} \quad (6)$$

Here, again, we find that each man's share of the gain *is equal to the whole gain multiplied by his particular portion of the stock, and that product divided by the whole stock invested.*

The same results would be obtained in relation to any number of partners. Observe, that g can be of any value, positive, negative, or zero. When it is zero, each numerator is zero; and, thus, x, y, and z become zero, as they ought in that case. When g is negative, it denotes loss, and losses must be shared in the same proportion.

It is not necessary that a, b, and c should designate the actual stock of each partner if they represent their due proportional parts.

In taking up a book on common Arithmetic, we find the following rule for fellowship:

As the whole amount of stock or labor
Is to each man's portion,
So is the whole property, loss or gain,
To each man's share of it.

These four lines express either one of the equations, (4), (5), or (6); for, by resolving (4), for example, into a proportion, we have . . $(a+b+c):a=g:x$

Thus, we perceive that this, like most other arithmetical rules, is the result of algebraic investigation.

Let us now consider the case in which time is an element, and for the sake of clearness we will suppose an example.

Two men, A *and* B, *hired a pasture, for which they agreed to pay* g *dollars.* A *put in* a *cows* 3 *weeks,* B *put in* b *cows for* 5 *weeks; what shall each pay?*

Consider that, a cows for three weeks would consume as much as $3a$ cows for one week. Also, b cows for five weeks would consume as much as $5b$ cows for one week. *Thus, we reduce all action to some unit of time.* To be more general, we will consider 3 and 5 as t and t', any number of weeks or days whatever, then the action will be at and bt', and it is evident that the partners must pay in proportion to this action, or in this case, to the consumption of the cows.

Now, let $x=$ what A must pay, and $y=$ what B must pay,

Then, ... $x+y=g$ (1)

And, ... $x:y=at:bt'$

Hence, ... $y=\left(\frac{bt'}{at}\right)x$ (2)

This value of y put in equation (1), gives

$$x+\frac{bt'}{at}x=g, \text{ or } (at+bt')x=(at)g \quad (3)$$

Hence, ... $x=\frac{(at)g}{at+bt'}$ (4)

$$y=\frac{(bt')g}{at+bt} \quad (5)$$

Equation (4) will furnish the following proportion:

$$(at+bt'):at=g:x$$

Equation (5), $at+bt':bt'=g:y$

Taking up a work on Arithmetic, I found the following rule for computing results in compound fellowship.

RULE.—*Multiply the active agents by the time each was in action.* Then by proportion.

As the sum of the products
Is to each particular product,
So is the whole gain or loss
To each man's share of it.

Now, it is evident that the words of this rule were dictated by the preceding proportions.

THE END.