AN

ELEMENTARY TREATISE

ON THE

THEORY OF EQUATIONS,

WITH A COLLECTION OF EXAMPLES.

BY

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London:
MACMILLAN AND CO.
1885.

STEREOTYPED EDITION.

Cambridge:

PRINTED BY C. J. CLAY, M.A. & SON,

AT THE UNIVERSITY PRESS.

PREFACE.

THE present treatise contains all the propositions which are usually included in elementary treatises on the Theory of Equations, together with a collection of examples for exercise.

As the Theory of Equations involves a large number of interesting and important results, which can be demonstrated with simplicity and clearness, the subject may advantageously engage the attention of a student at an early period of his mathematical course. The present treatise may be read by those who are familiar with Algebra, since no higher knowledge is assumed, except in Arts. 149, 175, 268, 308...314, and Chapter XXXI., which may be postponed by those who are not acquainted with De Moivre's Theorem in Trigonometry. This work may be regarded as a sequel to that on Algebra by the present writer, and accordingly the student has occasionally been referred to the treatise on Algebra for preliminary information on some topics here discussed.

In composing the present work, the author has obtained assistance from the treatises on Algebra by Bourdon, Lefebure de Fourcy, and Mayer and Choquet; on special points he has consulted other writers, who are named in their appropriate places in the course of the work.

The examples have been selected from the College and University examination papers, and the results have been given where it appeared necessary; in most cases however, from the nature of the example, the student will be able immediately to test the correctness of his result.

In order to exhibit a comprehensive view of the subject, the present treatise includes investigations which are not to be found in all the preceding elementary treatises, and also some investigations which are not to be found in any of them. Among these may be mentioned Cauchy's proof that every equation has a root, Horner's method, the theories of elimination and expansion, Cauchy's theorem on the number of imaginary roots, the researches of Professor Sylvester respecting Newton's Rule, and the theory of determinants. The account of determinants has been principally taken from a treatise on that subject by Baltzer, which was published at Leipsic in 1857; this is an excellent work, distinguished for the completeness of its proofs of the fundamental theorems, and for the numerous applications of those theorems which it affords.

For the parts of the Theory of Equations which are beyond an elementary treatise, the advanced student may consult Serret's *Cours d'Algèbre Supérieure:* there, for example, will be found a demonstration of the theorem, that the general algebraical solution of an equation of a degree above the fourth is impossible. The article *Equation*, by Professor Cayley, in the ninth edition of the *Encyclopædia Britannica* should also be noticed. Valuable historical information, relating to the higher parts of the subject, will be found in papers on *Approximation and Numerical Solution*, by Mr James Cockle, in the *Lady's and Gentleman's Diary* for the years 1854 and 1855, and also in papers on *Equations of the Fifth Degree* by the same author in the same work, for the years 1848, 1851, 1856, 1857, 1858, and 1860.

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March, 1880.

CONTENTS.

THEORY OF EQUATIONS.

I. INTRODUCTION.

1. The reader can easily obtain a general idea of the object of the following treatise by a reference to the theory of quadratic equations which he is supposed to have already studied. The equation $ax^2 + bx + c = 0$ has two roots, namely,

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a};$$

and with respect to these roots, we know that their sum is $-\frac{b}{a}$, and their product is $\frac{c}{a}$; that is, their sum is equal to the coefficient of the second term of the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, with its sign changed, and their product is equal to the last term of this equation. (See *Algebra*, Chap. XXII.) Now it may be said that the general object of the following pages is to establish results with respect to equations of a higher degree than the second, similar to those which have been established in Algebra respecting equations of the second degree. The results obtained will be useful in other branches of mathematics, and the methods of investigation will afford valuable exercise to the student, since they are not too difficult for a person who has gained a knowledge of Algebra, and at the same time have sufficient variety to occupy his attention.

2. The words *equation* and *root* are already familiar to the student from their use in Algebra; but for distinctness we will give a definition of them.

Any Algebraical expression which contains x may be called a function of x, and may be denoted by $f(x)$. Any quantity which substituted for x in $f(x)$ makes $f(x)$ vanish, is called a root of the equation $f(x) = 0$.

An expression of the form $g.x^2 +$

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots + kx + l, \qquad g$$

where n is a positive integer, and the coefficients $a, b, c \ldots k, l$, do not involve x, is called a rational integral function of x of the n^{th} degree; and if we wish to find what value of x makes this function vanish we have to find a root of a *rational integral equation* of the n^{th} degree; this is the kind of equation which we shall consider in the present treatise. In such an equation we may if we please divide by the coefficient of the highest power of x, so as to leave that power with only unity for its coefficient; the equation then takes the form

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-2}x^2 + p_{n-1}x + p_n = 0.$$

We shall say that the equation is now in its *simplest form*; as will be seen hereafter, some of the properties of equations can be enunciated more concisely when the equation is in this form than when x^n has a coefficient which is not unity. If we do not wish to suppose the coefficient of x^n to be unity, we may conveniently denote it by p_0; then the equation takes the form

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-2}x^2 + p_{n-1}x + p_n = 0.$$

The term p_n is called the *term independent of x*.

3. It must then be remembered that by *equation* we mean *rational integral equation*; an equation which is not of this form may often be reduced to it by algebraical transformations; for example, the equation $ax^2 + bx + c\sqrt{x} = f$ may be reduced to a rational integral form by transposing $c\sqrt{x}$ and f and then squaring; it will thus become a rational integral equation of the fourth degree. Equations which involve logarithmic functions, or exponential functions, or trigonometrical functions, or irrational algebraical functions, will not be directly included in our investigations; for example, such equations as $\tan x - e^x = 0$,

or $x \log x - a = 0$, will not be included. However, the theory which will be given of rational integral equations will indirectly throw some light on these excluded equations.

And when we speak of any function $f(x)$ we shall always mean a rational integral function of x, unless the contrary is specified.

4. A remark of some importance must be made with respect to the *coefficients* $p_0, p_1, p_2, \ldots p_n$, in the equation

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-2}x^2 + p_{n-1}x + p_n = 0.$$

In the quadratic equation $ax^2 + bx + c = 0$ we are able to solve the equation without knowing what particular numbers are denoted by a, b, c; all we require to know is that a, b, c are some numbers independent of x. If we have to solve the equation $x^2 - 12x + 15 = 0$ we may either transpose the 15 and complete the square in the ordinary way, or we may take the general formulæ given in Art. 1, and put in them $a = 1$, $b = -12$, $c = 15$. If we wish to solve an equation without having the numerical values of the coefficients previously assigned, we are seeking what may be called the *algebraical solution* of the equation; and if we can effect the algebraical solution of the general equation of any degree, we may obtain a *numerical* solution of an equation of that degree, by substituting the numerical values of the coefficients in the general formula which gives the algebraical solution. As we proceed we shall find that the algebraical solution of equations up to the fourth degree inclusive has been effected; but both in equations of the third degree and of the fourth degree, when we substitute the numerical values of the coefficients in a specific equation in the general formula, the result takes a form which is sometimes practically useless. And beyond equations of the fourth degree the general algebraical solution of equations has not been carried, and it appears cannot be carried.

But with respect to what may be called the *arithmetical* solution of equations in which the coefficients are given numbers, more success has been obtained. Thus, for example, although we cannot solve algebraically the general equation of the fifth degree, we can by numerical calculation discover any root which an equation of the fifth degree with known numerical coefficients may have, or at least we can approximate as closely as we please to such a root.

5. Let us denote by $f(x)$ the expression

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-2}x^2 + p_{n-1}x + p_n;$$

then the value of this expression when $x = a$ may be denoted by $f(a)$. We will shew how the numerical value of $f(a)$ may be most easily calculated, supposing that the coefficients of $f(x)$, and also a itself, are specified numbers.

Take for example an expression of the *third* degree; then we wish to find the numerical value of

$$p_0a^3 + p_1a^2 + p_2a + p_3.$$

First obtain	$p_0a;$
add p_1, this gives	$p_0a + p_1;$
multiply by a, this gives	$p_0a^2 + p_1a;$
add p_2, this gives	$p_0a^2 + p_1a + p_2;$
multiply by a, this gives	$p_0a^3 + p_1a^2 + p_2a;$
add p_3, this gives	$p_0a^3 + p_1a^2 + p_2a + p_3.$

We may arrange the process in the following way;

$$\begin{array}{llll} p_0 & p_1 & p_2 & p_3 \\ & p_0a & p_0a^2 + p_1a & p_0a^3 + p_1a^2 + p_2a \\ \hline & p_0a + p_1 & p_0a^2 + p_1a + p_2 & p_0a^3 + p_1a^2 + p_2a + p_3 \end{array}$$

We may proceed in the same way whatever may be the degree of $f(x)$. For example, required the numerical value of $3x^4 - 2x^3 - 5x + 7$ when $x = 3$.

$$\begin{array}{rrrrr} 3 & -2 & 0 & -5 & +7 \\ & +9 & +21 & +63 & +174 \\ \hline & +7 & +21 & +58 & +181 \end{array}$$

Thus the result is 181.

6. *If any rational integral function of* x *vanishes when* x = a, *the function is divisible by* x − a.

Let $f(x)$ denote the function; then we have given that $f(a) = 0$, and we have to prove that $f(x)$ is divisible by $x - a$.

Divide $f(x)$ by $x - a$ by common algebra until the remainder no longer contains x; let Q denote the quotient and R the remainder if there be one. Then $f(x) = Q(x-a) + R$. In this identity put a for x; since Q is a rational integral function of x it cannot become infinite when $x = a$; therefore $Q(x-a)$ vanishes when $x = a$. Also $f(x)$ vanishes when $x = a$ by supposition. Thus R vanishes when $x = a$; but R does not contain x, so that if it vanishes when $x = a$ it always vanishes. That is, $R = 0$ and $x - a$ divides $f(x)$.

7. The above demonstration is important and instructive; we may however prove the theorem in another way, which will moreover have the advantage of exhibiting the form of the quotient Q. Suppose

$$f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-2}x^2 + p_{n-1}x + p_n,$$

then since $f(a) = 0$ we have $f(x) = f(x) - f(a)$

$$= p_0(x^n - a^n) + p_1(x^{n-1} - a^{n-1}) + p_2(x^{n-2} - a^{n-2}) + \dots + p_{n-1}(x - a).$$

Now the terms $x^n - a^n$, $x^{n-1} - a^{n-1}$, ... are all divisible by $x - a$ (see *Algebra*, Art. 483). By performing the division we obtain for the quotient

$$\begin{aligned}
&p_0(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1})\\
&+ p_1(x^{n-2} + ax^{n-3} + a^2x^{n-4} + \dots + a^{n-2})\\
&+ \dots\\
&+ p_{n-2}(x + a)\\
&+ p_{n-1}.
\end{aligned}$$

We may rearrange the quotient thus:

$$p_0x^{n-1} + (p_0a + p_1)x^{n-2} + (p_0a^2 + p_1a + p_2)x^{n-3} + \dots$$
$$+ p_0a^{n-1} + p_1a^{n-2} + \dots + p_{n-1},$$

and we may denote it by

$$q_0x^{n-1} + q_1x^{n-2} + q_2x^{n-3} + \dots + q_{n-2}x + q_{n-1}.$$

The new coefficients are therefore connected with each other and with the old coefficients by the formulæ

$$q_0 = p_0, \quad q_1 = aq_0 + p_1, \quad q_2 = aq_1 + p_2, \quad q_3 = aq_2 + p_3, \;\ldots\ldots;$$

that is, *each new coefficient is found by multiplying the preceding new coefficient by* a *and then adding the corresponding old coefficient.* It will be observed that these new coefficients are *successively* determined by the process of Art. 5.

8. *If* x − a *divide* f(x) *which is any rational integral function of* x, *then* a *is a root of the equation* f(x) = 0.

For let Q denote the quotient when $f(x)$ is divided by $x - a$, then $f(x) = Q(x - a)$. In this identity put a for x, then Q is not infinite, and therefore $Q(x - a)$ vanishes. Thus $f(x)$ vanishes when $x = a$, and therefore a is a root of the equation $f(x) = 0$.

9. *To find the remainder when any rational integral function of* x *is divided by* x − c, *where* c *is any constant.*

Let $f(x)$ denote any rational integral function of x, and divide $f(x)$ by $x - c$ until the remainder is independent of x; let Q denote the quotient and R the remainder. Then

$$f(x) = Q(x - c) + R.$$

In this identity put c for x, then Q is not infinite, and therefore $Q(x - c)$ vanishes; thus $f(c) = R$. That is, R is equal to $f(c)$ when $x = c$, but R does not contain x, so that R is equal to $f(c)$ always.

For example; if $3x^4 - 2x^3 - 5x + 7$ is divided by $x - 3$, the quotient is $3x^3 + 7x^2 + 21x + 58$, and the remainder is 181; see Arts. 5 and 7.

For another example let us divide the same expression by $x - 4$:

$$\begin{array}{rrrrr}
3 - & 2 & 0 - & 5 + & 7 \\
 & +12 & +40 & +160 & +620 \\
\hline
 & +10 & +40 & +155 & +627
\end{array}$$

Thus the quotient is $3x^3 + 10x^2 + 40x + 155$, and the remainder is 627.

This process is a particular case of *Synthetic Division;* see *Algebra*, Chapter LVIII.

10. Let $f(x)$ be any rational integral function of x, and suppose $x+y$ put for x; then we propose to arrange $f(x+y)$ according to powers of y, and to determine the coefficients of the different powers.

Let $f(x)=p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n$; then

$$f(x+y)=p_0(x+y)^n+p_1(x+y)^{n-1}+p_2(x+y)^{n-2}+\ldots+p_{n-1}(x+y)+p_n.$$

Expand $(x+y)^n$, $(x+y)^{n-1}$, ... by the Binomial Theorem, and arrange the whole result according to powers of y; we thus obtain for $f(x+y)$ the following series:

$$\begin{aligned}
&p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n\\
&+y\left\{np_0x^{n-1}+(n-1)p_1x^{n-2}+(n-2)p_2x^{n-3}+\ldots+p_{n-1}\right\}\\
&+\frac{y^2}{1.2}\left\{n(n-1)p_0x^{n-2}+(n-1)(n-2)p_1x^{n-3}+\ldots+2p_{n-2}\right\}\\
&+\ldots\\
&+\frac{y^r}{\lfloor r}\left\{n(n-1)\ldots(n-r+1)p_0x^{n-r}+(n-1)(n-2)\ldots(n-r)p_1x^{n-r-1}+\ldots\right\}\\
&+\ldots\\
&+\frac{y^n}{\lfloor n}\left\{\lfloor n\, p_0\right\}.
\end{aligned}$$

The first line of this series is obviously $f(x)$. We shall denote the coefficient of y by $f'(x)$, the coefficient of $\frac{y^2}{1.2}$ by $f''(x)$, the coefficient of $\frac{y^3}{\lfloor 3}$ by $f'''(x)$, and so on; this notation becomes inconvenient when the number of accents is large, and so in general the coefficient of $\frac{y^r}{\lfloor r}$ will be denoted by $f^r(x)$. Hence

$$f(x+y)=f(x)+yf'(x)+\frac{y^2}{1.2}f''(x)+\frac{y^3}{\lfloor 3}f'''(x)+\ldots$$

$$\ldots+\frac{y^r}{\lfloor r}f^r(x)+\ldots+\frac{y^n}{\lfloor n}f^n(x).$$

By inspection it will be seen that the functions $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$, ... $f^n(x)$ are connected by the following general law: in order to obtain $f^{r+1}(x)$ we multiply each term in $f^r(x)$ by the exponent of x in that term and then diminish the exponent by unity.

11. Let us suppose, for example, that $f(x)$ is of the fourth degree; let

$$f(x) = p_0x^4 + p_1x^3 + p_2x^2 + p_3x + p_4.$$

Then
$$f'(x) = 4p_0x^3 + 3p_1x^2 + 2p_2x + p_3,$$
$$f''(x) = 4 \,.\, 3p_0x^2 + 3 \,.\, 2p_1x + 2p_2,$$
$$f'''(x) = 4 \,.\, 3 \,.\, 2p_0x + 3 \,.\, 2p_1,$$
$$f''''(x) = 4 \,.\, 3 \,.\, 2 \,.\, p_0;$$

$$f(x+y) = f(x) + yf'(x) + \frac{y^2}{1 \,.\, 2} f''(x) + \frac{y^3}{\underline{|3}} f'''(x) + \frac{y^4}{\underline{|4}} f''''(x).$$

If we suppose numerical values assigned to p_0, p_1, p_2, p_3, p_4, and x, we may calculate separately $f(x)$, $f'(x)$, ... by the method of Art. 5; we shall however hereafter, in explaining Horner's method of solving equations, shew how these calculations may be most conveniently and systematically conducted.

For another example suppose that $f(x) = p(x+c)^n$.

Then $f(x) = p\left\{x^n + ncx^{n-1} + \frac{n(n-1)}{1 \,.\, 2} c^2x^{n-2} + \ldots + nc^{n-1}x + c^n\right\}$;

therefore

$$f'(x) = p\left\{nx^{n-1} + n(n-1)cx^{n-2} + \frac{n(n-1)(n-2)}{1 \,.\, 2} c^2x^{n-3} + \ldots + nc^{n-1}\right\};$$

that is
$$f'(x) = pn(x+c)^{n-1}:$$

similarly
$$f''(x) = pn(n-1)(x+c)^{n-2},$$
$$f'''(x) = pn(n-1)(n-2)(x+c)^{n-3},$$

and so on.

Suppose that $\phi(x)$ and $\psi(x)$ are two rational integral functions of x, and that $f(x) = \phi(x) + \psi(x)$; then it is easily seen that $f'(x) = \phi'(x) + \psi'(x)$, and $f''(x) = \phi''(x) + \psi''(x)$, and so on.

12. If we write the series for $f(x+y)$, beginning with the *highest* power of y, we shall have

$$f(x+y)=p_0y^n+(p_1+np_0x)y^{n-1}+\left\{p_2+(n-1)p_1x+\frac{n(n-1)}{1.2}p_0x^2\right\}y^{n-2}$$

$$+\left\{p_3+(n-2)p_2x+\frac{(n-1)(n-2)}{1.2}p_1x^2+\frac{n(n-1)(n-2)}{\lfloor 3}p_0x^3\right\}y^{n-3}$$

$$+\ldots$$

$$+\left\{p_r+(n-r+1)p_{r-1}x+\ldots+\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}p_0x^r\right\}y^{n-r}$$

$$+\ldots+f(x).$$

This may be seen from the form already given for $f(x+y)$, or by expanding separately every term in $f(x+y)$, and arranging according to descending powers of y.

13. The function $f'(x)$ is called the *first derived function* of $f(x)$, the function $f''(x)$ is called the *second derived* function of $f(x)$, and so on. The reader, when he is acquainted with the elements of the Differential Calculus, will see that each derived function is the *differential coefficient* with respect to x of the immediately preceding derived function, and that the expression for $f(x+y)$ in powers of y is an example of *Taylor's Theorem.*

Moreover, it must be observed that $f''(x)$ is deduced from $f'(x)$ in precisely the same way as $f'(x)$ is deduced from $f(x)$. Thus $f''(x)$ is the first derived function of $f'(x)$, and $f'''(x)$ is the second derived function of $f'(x)$, and so on. Hence by the preceding Article we have

$$f'(x+y)=f'(x)+yf''(x)+\frac{y^2}{1.2}f'''(x)+\frac{y^3}{\lfloor 3}f''''(x)+\ldots$$

$$\ldots+\frac{y^{r-1}}{\lfloor r-1}f^r(x)+\ldots+\frac{y^{n-1}}{\lfloor n-1}f^n(x).$$

Similarly $f''(x+y)=f''(x)+yf'''(x)+\dfrac{y^2}{1.2}f''''(x)+\ldots$

$$\ldots+\frac{y^{r-2}}{\lfloor r-2}f^r(x)+\ldots+\frac{y^{n-2}}{\lfloor n-2}f^n(x).$$

And so on.

14. *In any rational integral function of* x *arranged according to descending powers of* x, *any term which occurs may be made to contain the sum of all which follow it, as many times as we please, by taking* x *large enough, and any term may be made to contain the sum of all which precede it, as many times as we please, by taking* x *small enough.*

Let $p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-2}x^2+p_{n-1}x+p_n$ be any rational integral function of x; suppose for example that the r^{th} term $p_{r-1}x^{n-r+1}$ occurs; that is, suppose p_{r-1} not zero. Let q denote the numerical value of the greatest of the coefficients $p_r, p_{r+1}, \ldots p_n$. The sum of all the terms which follow the r^{th} term cannot exceed $q(x^{n-r}+x^{n-r-1}+\ldots+x+1)$, that is, $q\dfrac{x^{n-r+1}-1}{x-1}$. The ratio of the r^{th} term to this is $\dfrac{p_{r-1}(x-1)x^{n-r+1}}{q(x^{n-r+1}-1)}$, that is, $\dfrac{p_{r-1}(x-1)}{q-qx^{-(n-r+1)}}$. By taking x large enough, the numerator can be made as large as we please, and the denominator as near to q as we please; thus the ratio can be made as great as we please.

This proves the first part of the proposition. To prove the second part put $x=\dfrac{1}{y}$, then we obtain the series

$$y^{-n}\{p_0+p_1y+p_2y^2+\ldots+p_{n-1}y^{n-1}+p_ny^n\}.$$

We have now to prove that by taking x small enough, that is by taking y large enough, any term p_ry^r which occurs can be made to bear as great a ratio as we please to the sum of the terms $p_0+p_1y+\ldots+p_{r-1}y^{r-1}$ which precede it; this has been already proved in the first part.

15. One of the first questions which can occur in the theory of equations is whether a root must exist for every equation; and we shall now give some simple propositions which establish the existence of a root in certain cases. We shall require a theorem which is often assumed as obvious, but which may be proved in the manner shewn in the next Article.

16. Let $f(x)$ be any rational integral function of x, and $f(a)$, $f(b)$, the values of $f(x)$ corresponding to the values a and b of x;

then as x changes from a to b the function $f(x)$ will change from $f(a)$ to $f(b)$, and will pass through every intermediate value.

Let any value c be ascribed to x, and let $f(c)$ be the corresponding value of $f(x)$; let $c+h$ be another value which may be ascribed to x; then by taking h small enough $f(c+h)$ may be made to differ as little as we please from $f(c)$. For

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{1\,.\,2}f''(c)+\ldots+\frac{h^{n-1}}{\underline{|n-1}}f^{n-1}(c)+\frac{h^n}{\underline{|n}}f^n(c).$$

Then, by Art. 14, by taking h small enough, the first term of the series $hf'(c)$, $\frac{h^2}{1\,.\,2}f''(c)$, $\frac{h^3}{\underline{|3}}f'''(c)$, ... which does not vanish, can be made to contain the sum of all which follow it as often as we please, and by taking h small enough this term will itself be rendered as small as we please. Therefore $f(c+h)-f(c)$ can be made as small as we please by taking h small enough. This shews that as x changes, $f(x)$ changes *gradually*, so that if $f(x)$ takes any value for an assigned value of x, it will take another value as near as we please to the former, by taking another value of x which is sufficiently near to the assigned value. Hence as x changes from a to b, the function $f(x)$ must pass *without any interruption* from the value $f(a)$ to the value $f(b)$; for to assert that there could be *interruption* would amount to asserting that $f(x)$ could take a certain value, and could *not* take a second value as near as we please to the first value.

17. We do not assert in the preceding Article that $f(x)$ always increases from $f(a)$ to $f(b)$, or always decreases from $f(a)$ to $f(b)$; it may be sometimes increasing and sometimes decreasing. What we assert is, that it passes without any *sudden change* of value, from the value $f(a)$ to the value $f(b)$. The proposition is one of great importance, and probably will appear nearly evident to the student on reflection. It is obvious that $f(x)$ has *some* finite value for every finite value ascribed to x; also we have proved that an indefinitely small change in x can only make an indefinitely small change in $f(x)$, so that there can be no break in the succession of values which $f(x)$ assumes.

18. The student who is acquainted with Co-ordinate Geometry will find it useful and interesting to illustrate the present subject by conceiving curves drawn to represent the functions. Thus let $f(x)$ be denoted by y, so that $y = f(x)$ may be conceived to be the equation to a curve; then by supposing this curve drawn for the part lying between $x = a$ and $x = b$, a good idea is obtained of the necessary consecutiveness in the values assumed by $f(x)$ between the values $f(a)$ and $f(b)$.

It must be observed that we do not restrict a, b, $f(a)$, $f(b)$, to be *positive* quantities; and by values intermediate between $f(a)$ and $f(b)$ we mean intermediate in the *algebraical* sense; that is, any quantity z is intermediate between $f(a)$ and $f(b)$ which makes $z - f(a)$ and $f(b) - z$ of the *same sign.*

19. *If two numbers substituted for* x *in a rational integral expression* f (x) *give results with contrary signs, one root at least of the equation* f (x) = 0 *lies between those values of* x.

Let a and b denote the two numbers; then $f(a)$ and $f(b)$ have contrary signs. By Art. 16, as x changes gradually from a to b, the expression $f(x)$ passes without any interruption of value from $f(a)$ to $f(b)$; but since $f(a)$ and $f(b)$ are of contrary signs the value zero lies between them, so that $f(x)$ must be equal to zero for some value of x between a and b; that is, there is a root of the equation $f(x) = 0$ between a and b.

We do not say that there *is only* one root. And we do not say that if $f(a)$ and $f(b)$ are of the *same* sign there will be no root of the equation $f(x) = 0$ between a and b.

20. *An equation of an odd degree has at least one real root.*

Let the equation be denoted by $f(x) = 0$, where

$$f(x) = p_0x^n + p_1x^{n-1} + \dots + p_{n-1}x + p_n,$$

and n is an odd number.

When x is large enough the first term of $f(x)$, namely p_0x^n, will be larger than the sum of all the rest by Art. 14, and therefore the sign of $f(x)$ will be the same as the sign of p_0x^n. Thus, by taking x large enough, the sign of $f(x)$ can be made the *same* as the sign of p_0 when x is *positive*, and the *contrary* to that of p_0 when x is *negative*. Since then $f(x)$ changes its sign as x passes from a suitable negative value to a suitable positive value, there must be some intermediate value of x which makes $f(x)$ vanish; that is, there must be some real root of the equation $f(x) = 0$.

We may determine whether this root is positive or negative. For when we put zero for x the sign of $f(x)$ is the same as that of p_n. Thus if p_n and p_0 have the same sign, so that $\frac{p_n}{p_0}$ is positive, there will certainly be a negative root of the equation $f(x) = 0$; and if p_n and p_0 have contrary signs, so that $\frac{p_n}{p_0}$ is negative, there will certainly be a positive root of the equation $f(x) = 0$. Thus if an equation be of an odd degree, and be brought into its simplest form by dividing by the coefficient of the highest power of x, it will have a real root of the sign contrary to that of the last term.

21. *An equation of an even degree which is in its simplest form, and has its last term negative, has at least two real roots of contrary signs.* ie roots of contrary signs

Let $f(x) = 0$ be the equation; then when x is zero $f(x)$ is negative by supposition. When x is large enough $f(x)$ is positive, whether x is positive or negative. Thus there is some negative value of x which makes $f(x)$ vanish, and also some positive value of x which makes $f(x)$ vanish. That is, the equation $f(x) = 0$ has certainly one negative root and one positive root.

22. *If the rational integral expression* f (x) *consists of a set of terms in which the coefficients are all of one sign, followed by a set of terms in which the coefficients are all of the contrary sign, the equation* f $(x) = 0$ *has one positive root and only one positive root.*

By Arts. 20 and 21 the equation $f(x) = 0$ must have *one* positive root; we will shew that it has *only* one positive root.

Let $\quad f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n.$

Suppose the coefficients $p_0, p_1, \dots p_r$ all positive, and the remaining coefficients negative; let $p_{r+1} = -P_{r+1}$, $p_{r+2} = -P_{r+2}$, ... $p_n = -P_n$. Then we may write $f(x)$ thus,

$$f(x) = x^{n-r}\left\{p_0x^r + p_1x^{r-1} + p_2x^{r-2} + \ldots + p_r - \frac{P_{r+1}}{x} - \frac{P_{r+2}}{x^2} - \ldots - \frac{P_n}{x^{n-r}}\right\}.$$

The expression $p_0x^r + p_1x^{r-1} + p_2x^{r-2} + \ldots + p_r$ *increases* as x increases, unless $r = 0$, and then it remains constant; the expression $\frac{P_{r+1}}{x} + \frac{P_{r+2}}{x^2} + \ldots + \frac{P_n}{x^{n-r}}$ *diminishes* as x increases. Thus as x increases from zero onwards, the two expressions cannot be equal more than once. That is, $f(x) = 0$ has *only one* positive root.

The demonstration will be the same if we suppose the first set of coefficients *negative* and the second set positive.

23. To prevent any mistake it will be useful to draw attention to the precise results obtained in the last three Articles.

In Art. 20 it is proved that the equation considered has *at least one* real root; it is not proved that it has one only. In Art. 21 it is proved that the equation considered has *at least two* real roots; it is not proved that it has only two. In Art. 22 it is proved that the equation considered has *one* positive root and *only one positive* root: it is not proved that it has no negative root.

24. The propositions in Arts. 20, 21, and 22, as to the existence of roots in certain cases, depend upon the fact that we are able to shew that $f(x)$ undergoes a change of sign or changes of sign. We cannot infer conversely that if $f(x)$ never changes its sign within a certain range of values for x there is no root of the equation $f(x) = 0$ within that range of values for x. Take for example $x^2 - 6x + 9$; this expression never *changes* its sign, and yet it vanishes when $x = 3$: the expression is equal to $(x-3)^2$. But if the equation $f(x) = 0$ has no root within an assigned range of values for x we are sure that $f(x)$ never changes its sign within that range of values for x.

The following statements respecting the absence of roots will be seen to be obviously true:

(1) If the coefficients in $f(x)$ are all positive, the equation $f(x) = 0$ has no positive root.

(2) If all the coefficients of the *even* powers of x in $f(x)$ have one sign, and all the coefficients of the *odd* powers of x the contrary sign, the equation $f(x) = 0$ has no negative root.

(3) If $f(x)$ involves only *even* powers of x and the coefficients are all of the same sign, the equation $f(x) = 0$ has no real root.

It is supposed in this case that there is a term independent of x.

(4) If $f(x)$ involves only *odd* powers of x and the coefficients are all of the same sign, the equation $f(x) = 0$ has no real root, except $x = 0$.

It is supposed in this case, of course, that there is no term independent of x.

We say in the last two cases that the equation has no *real* root, and we do not say that the equation has no root, for we know that by virtue of some conventions an equation *may* in some cases have *imaginary* roots; see *Algebra*, Chapter XXV. And in fact we shall now proceed to shew that imaginary roots *must* exist.

II. ON THE EXISTENCE OF A ROOT.

25. We shall now prove that every rational integral equation has a root, either real or of the form $a + b\sqrt{-1}$, where a and b are real; such an expression as $a + b\sqrt{-1}$, where a and b are real, we shall call an imaginary expression. That is, when we use the term *imaginary* we shall always mean that the expression to which we apply this term is of the form $a + b\sqrt{-1}$, *where* a *and* b *are real.*

26. The student is supposed to know that by virtue of certain conventions, imaginary expressions can be used in algebraical investigations, and theorems can be established respecting them.

Thus, for example, the positive value of the square root of a^2+b^2 is called the *modulus* of each of the expressions $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$; and with this definition we can shew that the modulus of the product of two imaginary expressions is the product of the moduli of those two expressions. For the product of $a+b\sqrt{-1}$ and $a'+b'\sqrt{-1}$ is $aa'-bb'+(ab'+a'b)\sqrt{-1}$, and the modulus of this is the positive value of the square root of $(aa'-bb')^2+(ab'+a'b)^2$, that is, of $(a^2+b^2)(a'^2+b'^2)$; that is, the modulus is the product of the moduli of the two given expressions. Also an imaginary expression $a+b\sqrt{-1}$ is considered to vanish when a and b vanish; that is, an imaginary expression vanishes when its modulus vanishes. Thus, by what has just been shewn, if the product of two imaginary expressions vanishes, the modulus of one of the expressions must vanish; so that *if the product of two or more imaginary expressions vanishes, one of the expressions themselves must vanish; and if one of the expressions vanishes the product vanishes.*

27. The student who has not paid attention to the subject of imaginary expressions may consult the *Algebra*, Chap. XXV. The proof however that every equation has a root, real or imaginary, to which we shall now proceed, is somewhat difficult; the student therefore on reading this subject for the first time may assume this proposition, and reserve the remainder of the present Chapter for future consideration.

28. We shall first shew that a root, real or imaginary, exists for each of the following four equations:

$$x^n=1, \quad x^n=-1, \quad x^n=+\sqrt{-1}, \quad x^n=-\sqrt{-1}.$$

(1) $x^n=1$. It is obvious that $x=1$ is a root of this equation.

(2) $x^n=-1$. If n is an *odd* number it is obvious that $x=-1$ is a root of this equation. If n is an *even* number suppose it equal to $2m$; we have then to shew that there is a solution of $x^{2m}=-1$; this amounts to shewing that there is a solution of $x^m=\pm\sqrt{-1}$, and is therefore included in the next two cases.

(3) $x^n = +\sqrt{-1}$. If n is an *odd* number it must be of one of the two forms $4m+1$ and $4m+3$; in the former case $+\sqrt{-1}$ is a root, since $(+\sqrt{-1})^{4m+1} = +\sqrt{-1}$, and in the latter case $-\sqrt{-1}$ is a root, since $(-\sqrt{-1})^{4m+3} = +\sqrt{-1}$. If n is an *even* number suppose it equal to mp, where m is an odd number, and p is some power of 2, say 2^q. Put $y = x^p$, then the equation $x^{mp} = +\sqrt{-1}$ may be written $y^m = +\sqrt{-1}$, and by what has been already shewn $+\sqrt{-1}$ or $-\sqrt{-1}$ is a suitable value of y, according as m is of the form $4r+1$ or $4r+3$. We have then to find a value of x which will satisfy $x^p = +\sqrt{-1}$ or $x^p = -\sqrt{-1}$, where $p = 2^q$. The required value can be obtained by common Algebra. For take the square root of $+\sqrt{-1}$ or of $-\sqrt{-1}$; this will give an expression of the form $\alpha + \beta\sqrt{-1}$, where α and β are real; take the square root of $\alpha + \beta\sqrt{-1}$, which will give a similar expression; and so on: see *Algebra*, Chapter XXV. Thus after q extractions of the square root we arrive at an expression $a + b\sqrt{-1}$, such that $(a + b\sqrt{-1})^p = +\sqrt{-1}$ or $= -\sqrt{-1}$.

(4) $x^n = -\sqrt{-1}$. This case is treated like (3). If n be an *odd* number, $-\sqrt{-1}$ or $+\sqrt{-1}$ is a root, according as n is of the form $4m+1$ or $4m+3$. If n be an even number suppose it equal to mp, where m is an odd number and $p = 2^q$, and proceed as before.

29. *Every rational integral equation has a root real or imaginary.*

Let $f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-2}x^2 + p_{n-1}x + p_n$, where the coefficients $p_0, p_1, \dots p_{n-2}, p_{n-1}, p_n$ *may be either real or imaginary;* we have to shew that the equation $f(x) = 0$ has a root either real or imaginary. If any imaginary expression be substituted for x in $f(x)$, we shall obtain a result of the form $U + V\sqrt{-1}$, where U and V are real quantities, and we have to shew that an imaginary expression must exist which will make $U = 0$ and $V = 0$. This we prove in the following manner. Since $U^2 + V^2$ is always a *real positive quantity*, if it cannot be zero there must be some value which is not greater than any other value, that is, there must be some value which cannot be diminished; but we shall now prove that if U^2+V^2 have any value different from zero we *can diminish* that value by a suitable change in the expression which is substituted for x; so that it follows that U^2+V^2 must be capable of the value zero, that is, U and V must vanish simultaneously.

Suppose a particular value assigned to x, namely, $a+b\sqrt{-1}$; let $f(x)$ then become $P+Q\sqrt{-1}$, where P and Q are not both zero. Now put $a+b\sqrt{-1}+h$ for x in $f(x)$; the value which $f(x)$ then takes may be found by first expanding $f(x+h)$ in powers of h, and then putting $a+b\sqrt{-1}$ for x. Suppose then

$$f(x+h)=X+hX'+\frac{h^2}{\lfloor 2}X''+\ldots\ldots+\frac{h^n}{\lfloor n}p_0\lfloor n,$$

where X, X', X'', ... are functions of x; see Art. 10. Put $a+b\sqrt{-1}$ for x, then X becomes $P+Q\sqrt{-1}$. *Some* of the coefficients X', X'', ... may vanish for this value of x, but they cannot *all* vanish, since the last coefficient, which is that of $\frac{h^n}{\lfloor n}$, is $p_0\lfloor n$.

Suppose h^m the lowest power of h for which the coefficient does not vanish, and denote the coefficient of h^m by $R+S\sqrt{-1}$, so that R and S are not both zero. Thus when $a+b\sqrt{-1}+h$ is substituted for x the function $f(x)$ becomes

$$P+Q\sqrt{-1}+(R+S\sqrt{-1})\,h^m+\ldots,$$

where the terms not expressed can only involve powers of h higher than h^m. Denote this by $P'+Q'\sqrt{-1}$.

Let $h=\epsilon t$, where ϵ is a real positive quantity. By Art. 28 it is in our power to take t so that t^m may be $+1$ or -1; thus we can make

$$P'+Q'\sqrt{-1}=P+Q\sqrt{-1}\pm(R+S\sqrt{-1})\,\epsilon^m+\ldots,$$

so that
$$P'=P\pm R\epsilon^m+\ldots,$$
$$Q'=Q\pm S\epsilon^m+\ldots,$$

and
$$P'^2+Q'^2=P^2+Q^2\pm 2\,(PR+QS)\,\epsilon^m+\ldots,$$

where the terms not expressed can only involve powers of ϵ higher than ϵ^m.

Now ϵ may be taken so small that the sign of all the terms involving ϵ in the value of $P'^2+Q'^2$ will be the same as the sign of $\pm 2(PR+QS)\epsilon^m$, provided $PR+QS$ be not zero; see Art. 14.

We will first suppose that $PR+QS$ is not zero. Then the sign of $P'^2+Q'^2-P^2-Q^2$ is the same as the sign of $\pm 2(PR+QS)\epsilon^m$, when ϵ is taken small enough; and we can ensure that this sign shall be negative by supposing that t^m is -1 or $+1$, according as $PR+QS$ is positive or negative. We can therefore make $P'^2+Q'^2$ less than P^2+Q^2.

Next suppose that $PR+QS$ is zero. Then instead of taking $t^m=\pm 1$, take $t^m=\pm\sqrt{-1}$. Proceeding as before we shall obtain

$$P'+Q'\sqrt{-1}=P+Q\sqrt{-1}\pm(R+S\sqrt{-1})\,\epsilon^m\sqrt{-1}+\ldots,$$

so that
$$P'=P\mp S\epsilon^m+\ldots,$$
$$Q'=Q\pm R\epsilon^m+\ldots,$$
and
$$P'^2+Q'^2=P^2+Q^2\pm 2(QR-PS)\,\epsilon^m+\ldots,$$

where the terms not expressed can only involve powers of ϵ higher than ϵ^m.

Now $(PR+QS)^2+(QR-PS)^2=(P^2+Q^2)(R^2+S^2)$; and this cannot be zero, because by supposition P^2+Q^2 is not zero, and R^2+S^2 likewise is different from zero. Thus since $PR+QS$ is zero, $QR-PS$ is not zero. Therefore the sign of $P'^2+Q'^2-P^2-Q^2$ will be the same as the sign of $\pm 2(QR-PS)\,\epsilon^m$ when ϵ is taken small enough; and we can ensure that this sign shall be negative by supposing that t^m is $-\sqrt{-1}$ or $+\sqrt{-1}$, according as $QR-PS$ is positive or negative. We can therefore make $P'^2+Q'^2$ less than P^2+Q^2.

We have thus shewn that when U^2+V^2 has any value different from zero we can diminish that value by a suitable change in the expression which is substituted for x; that is, U^2+V^2 is not susceptible of any positive value which cannot be diminished;

hence, as we have already stated, it must be possible that $U=0$ and $V=0$ simultaneously.

30. It remains to be shewn that a and b in the expression $a+b\sqrt{-1}$, which is the value of x that makes $f(x)$ vanish, are *finite.*

We have $f(x)=p_0x^n\left\{1+\dfrac{p_1}{p_0x}+\dfrac{p_2}{p_0x^2}+\ldots+\dfrac{p_n}{p_0x^n}\right\}$.

Substitute $a+b\sqrt{-1}$ for x; then $f(x)$ becomes

$$p_0(a+b\sqrt{-1})^n\left\{1+\frac{p_1}{p_0(a+b\sqrt{-1})}+\frac{p_2}{p_0(a+b\sqrt{-1})^2}+\ldots+\frac{p_n}{p_0(a+b\sqrt{-1})^n}\right\}.$$

Take any term of the series within the brackets, for example, that involving p_2; we have

$$\frac{p_2}{p_0(a+b\sqrt{-1})^2}=\frac{p_2(a-b\sqrt{-1})^2}{p_0(a^2+b^2)^2}=\frac{p_2(a^2-b^2)}{p_0(a^2+b^2)^2}-\frac{2p_2ab\sqrt{-1}}{p_0(a^2+b^2)^2}$$
$$=A+B\sqrt{-1},\ \text{say}.$$

Then it is evident that A and B diminish without limit as a and b increase without limit. Thus denoting the value of $f(x)$ when $x=a+b\sqrt{-1}$ by $U+V\sqrt{-1}$, we have

$$U+V\sqrt{-1}=p_0(a+b\sqrt{-1})^n\{1+A'+B'\sqrt{-1}\},$$

where A' and B' diminish without limit as a and b increase without limit.

If we put $a-b\sqrt{-1}$ for x we shall obtain a result which can be deduced from that just given by changing the sign of $\sqrt{-1}$: thus

$$U-V\sqrt{-1}=p_0(a-b\sqrt{-1})^n\{1+A'-B'\sqrt{-1}\};$$

therefore $$U^2+V^2=p_0^2(a^2+b^2)^n\{(1+A')^2+B'^2\},$$

and this increases without limit when a and b increase without limit; for the factor $(a^2+b^2)^n$ increases without limit, and the factor $(1+A')^2+B'^2$ tends to unity as its limit. Thus U^2+V^2 cannot vanish when a and b are indefinitely great, or when either of them is indefinitely great.

31. It will be observed in the demonstration of Article 29, that the coefficients of the proposed equation may be either real or imaginary. We shall however in the subsequent part of this book always suppose the coefficients to be real unless the contrary be stated.

32. The proof given in this Chapter of the existence of a root of an equation is called Cauchy's proof. The subject has recently been again discussed by mathematicians, and two memoirs will be found on it in the Tenth Volume of the Transactions of the Cambridge Philosophical Society, one by Mr De Morgan, and the other by Mr Airy; there is a supplement to the latter. It appears from Mr De Morgan's memoir that the proof known as Cauchy's had been previously given in substance by Argand.

We may briefly notice an objection which has sometimes been urged against Cauchy's proof. It has been said that it is *conceivable, until the contrary is shewn, that* U^2+V^2 *may approach indefinitely near to some limit greater than zero without ever reaching* this limit. But this objection can be removed by the aid of Art. 30. Let z stand for U^2+V^2, that is, for

$$f(a+b\sqrt{-1}) \times f(a-b\sqrt{-1}):$$

then we know that z is finite for finite values of a and b, and infinite for infinite values of a and b. Hence the least value of z must occur when a and b have finite values; and if the least value of z were not zero the demonstration of Art. 29 would be contradicted.

The student who is acquainted with the elements of Geometry of Three Dimensions will be assisted by supposing a, b, and z to be coordinates of a point in space, and imagining the surface determined by the relation

$$z=f(a+b\sqrt{-1}) \times f(a-b\sqrt{-1}).$$

III. PROPERTIES OF EQUATIONS.

33. *Every equation has as many roots as the number which expresses its degree, and no more.*

Suppose the equation to be of the n^{th} degree, and denote it by $f(x)=0$, where $f(x)=p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n$. By Chapter II. the equation $f(x)=0$ has a root either real or imaginary; let a_1 denote that root. Therefore $f(x)$ is divisible by $x-a_1$, by Art. 6; so that $f(x)=(x-a_1)\,\phi_1(x)$, where $\phi_1(x)$ is some integral algebraical function of x of the $(n-1)^{th}$ degree. Again by Chapter II. the equation $\phi_1(x)=0$ has a root either real or imaginary; let a_2 denote that root. Therefore $\phi_1(x)$ is divisible by $x-a_2$, by Art. 6; so that $\phi_1(x)=(x-a_2)\,\phi_2(x)$, where $\phi_2(x)$ is some rational integral algebraical function of x of the $(n-2)^{th}$ degree. Therefore $f(x)=(x-a_1)(x-a_2)\,\phi_2(x)$. By proceeding in this way we shall obtain n factors of $f(x)$ denoted by $x-a_1$, $x-a_2$, ... $x-a_n$; and the only other factor must be p_0 because the coefficient of x^n in $f(x)$ is p_0. Thus

$$f(x)=p_0\,(x-a_1)\,(x-a_2)\,(x-a_3)\ldots\ldots(x-a_n).$$

Hence the equation $f(x)=0$ has n roots, because $f(x)$ vanishes when we put for x any one of the n quantities a_1, a_2, ...a_n. And the equation has no more than n roots, because if we ascribe to x a value c which is not one of the n values a_1, a_2, ...a_n, the value of $f(x)$ becomes

$$p_0\,(c-a_1)\,(c-a_2)\,(c-a_3)\ldots\ldots(c-a_n);$$

this is not zero because every factor is different from zero; and the product of factors real or imaginary will not vanish if none of the factors vanish; see Art. 26.

34. The *roots* in the preceding Article are all either real, or of the form $a+b\sqrt{-1}$, where a and b are real. And some of the roots a_1, a_2, ...a_n may be *equal* so that there are not necessarily n *different* roots of an equation of the n^{th} degree. The student may perhaps be disposed to doubt the propriety of saying that an equa-

tion of the n^{th} degree has always n roots, when these roots are not necessarily all different. It is however found convenient to consider that an equation of the n^{th} degree always has n roots, although some of the roots may be equal; just as in common algebra it is found convenient to speak of the quadratic equation $ax^2 + bx + c = 0$ as having *two equal roots* when $b^2 = 4ac$, rather than as having then only one root.

35. The only preceding Article of the book which can be at all affected by the consideration of the possibility of *equal roots*, which has just been introduced, is Article 22. In that Article it is shewn that an equation of a certain form cannot have *two different positive roots*, but the demonstration there given does not exclude the possibility of a second root or of more roots *equal* to the root which necessarily exists. After we have proved Descartes's *Rule of Signs* however it will be obvious that the equation in question can have only one positive root without any repetition.

36. If we know a root a_1 of the equation $f(x) = 0$ we know that $f(x) = (x - a_1)\phi_1(x)$ where $\phi_1(x)$ is a function of x one degree lower than $f(x)$; and the remaining roots of the equation $f(x) = 0$ can be found if we can solve the equation $\phi_1(x) = 0$ which is one degree lower than the equation $f(x) = 0$. Similarly if we know two roots a_1 and a_2 of the equation $f(x) = 0$ we know that $f(x) = (x - a_1)(x - a_2)\phi_2(x)$ where $\phi_2(x)$ is a function of x two degrees lower than $f(x)$; and the remaining roots of the equation $f(x) = 0$ can be found if we can solve the equation $\phi_2(x) = 0$, which is two degrees lower than the equation $f(x) = 0$. And so on.

37. If $f(x)$ be any rational integral algebraical function of x of the n^{th} degree, we have shewn that $f(x)$ must be capable of resolution into n factors of the first degree, so that

$$f(x) = p_0(x - a_1)(x - a_2)\ldots\ldots(x - a_n),$$

where $a_1, a_2, \ldots\ldots a_n$ are either real or imaginary. It is to be observed that there is only *one* system of factors into which $f(x)$ can be resolved; this has already appeared when the quantities $a_1, a_2, \ldots a_n$ are all unequal, but it still remains to be shewn that when some of the quantities $a_1, a_2, \ldots a_n$, are equal, $f(x)$ cannot be formed in different ways in which the same factors occur with different exponents. If possible suppose that

$$f(x) = p_0(x-a_1)^r(x-a_2)^s(x-a_3)^t \ldots\ldots$$

and also $$f(x) = p_0(x-a_1)^\rho(x-a_2)^\sigma(x-a_3)^\tau \ldots\ldots$$

Suppose r greater than ρ; then dividing by $(x-a_1)^\rho$ we have

$$p_0(x-a_1)^{r-\rho}(x-a_2)^s(x-a_3)^t \ldots\ldots = p_0(x-a_2)^\sigma(x-a_3)^\tau \ldots\ldots$$

Now the left-hand member vanishes when $x = a_1$, but the right-hand member does not; the expressions then cannot be identical, and therefore $f(x)$ cannot admit of more than one system of factors.

38. *If any rational integral function of* x *of the* n$^{\text{th}}$ *degree vanishes for more than* n *different values of* x *every coefficient in the function must be zero, so that the function must be zero for every value of* x.

For if any coefficient in the function is not zero the function will *not* vanish for more than n different values of x, so that if the function *does* vanish for more than n different values of x *every* coefficient in the function must be zero.

39. The proof in the preceding Article makes the proposition depend upon the fact that an equation of the n^{th} degree has n roots, and thus ultimately upon the investigations in Chapter II. We may however establish the proposition by an inductive proof which does not require the investigations in Chapter II.

Suppose it true that when a function of x of the n^{th} degree vanishes for more than n different values of x every coefficient in the function is zero; and that we require to shew that when a function of x of the $(n+1)^{\text{th}}$ degree vanishes for more than $n+1$ different values of x every coefficient in the function is zero.

Let $f(x) = q_0x^{n+1} + q_1x^n + q_2x^{n-1} + \ldots\ldots + q_nx + q_{n+1}$, and suppose that more than $n+1$ values of x make $f(x)$ vanish. Let a be one of these values so that $f(a) = 0$. Then $f(x) = f(x) - f(a)$

$$= q_0(x^{n+1} - a^{n+1}) + q_1(x^n - a^n) + q_2(x^{n-1} - a^{n-1}) + \ldots\ldots + q_n(x-a).$$

This may be written in the form

$$f(x) = (x - a)\,\phi(x),$$

where $\phi(x)$ is a function of x of the n^{th} degree. Since then there are more than n different values of x, exclusive of a, which make $f(x)$ vanish, there are more than n different values of x which make $\phi(x)$ vanish; therefore by supposition every coefficient in $\phi(x)$ is zero. Now by Art. 7,

$$\phi(x) = q_0 x^n + (q_0 a + q_1) x^{n-1} + (q_0 a^2 + q_1 a + q_2) x^{n-2} + \ldots\ldots;$$

thus $q_0 = 0$ because the coefficient of x^n is zero, then $q_1 = 0$ because the coefficient of x^{n-1} is also zero, then $q_2 = 0$ because the coefficient of x^{n-2} is also zero, and so on.

Thus every coefficient in $f(x)$ is zero.

This establishes the proposition, since it is known to be true for expressions of the first and second degree.

40. If $f(x)$ be any function of x of the n^{th} degree we have shewn that $f(x)$ may be resolved into n factors of the first degree. Each of these factors will divide $f(x)$ so that $f(x)$ will admit of n divisors of the first degree. Similarly as the product of *any two* of the factors of the first degree contained in $f(x)$ will be a factor of the second degree contained in $f(x)$, it follows that $f(x)$ will admit of $\frac{n(n-1)}{1 \cdot 2}$ divisors of the second degree. Proceeding thus we see that $f(x)$ will admit of as many divisors of the r^{th} degree as there are combinations of n things taken r at a time, that is, $f(x)$ will admit of $\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}$ divisors of the r^{th} degree.

But it must be remembered that the divisors of any degree, as for example the second, will not necessarily be *all different*, because the factors of the first degree in $f(x)$ are not necessarily all different. The proposition however shews that there *cannot be more* than $\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}$ different divisors of the r^{th} degree.

41. *In an equation with real coefficients imaginary roots occur in pairs.*

Let $f(x)$ be a rational integral function of x in which the coefficients are all real; then if $\alpha+\beta\sqrt{-1}$ is a root of the equation $f(x)=0$ so also is $\alpha-\beta\sqrt{-1}$ a root.

For when $\alpha+\beta\sqrt{-1}$ is put for x the function $f(x)$ takes the form $P+Q\beta\sqrt{-1}$, where P and Q involve *even* powers of β. This is obvious, because if such an expression as x^r be expanded, where $x=\alpha+\beta\sqrt{-1}$, the even powers of $\beta\sqrt{-1}$ will give rise to *real* terms, so that $\sqrt{-1}$ will occur only in connexion with *odd* powers of β. *And as the coefficients in* f (x) *are supposed real* $\sqrt{-1}$ cannot occur except with some odd power of β. If then $\alpha-\beta\sqrt{-1}$ be substituted for x in $f(x)$ the result will be obtained by changing the sign of β in the result obtained by substituting $\alpha+\beta\sqrt{-1}$ for x; the result is therefore $P-Q\beta\sqrt{-1}$.

Now suppose that $\alpha+\beta\sqrt{-1}$ is a root of $f(x)=0$; then

$$P+Q\beta\sqrt{-1}=0,$$

and, as a real quantity P cannot be equal to an imaginary quantity $-Q\beta\sqrt{-1}$, this requires

$$P=0, \text{ and } Q=0.$$

And then $\alpha-\beta\sqrt{-1}$ is also a root of $f(x)=0$.

42. Thus if $f(x)$ be a rational integral function of x with real coefficients, and have a factor $x-a_1$ where $a_1=\alpha+\beta\sqrt{-1}$, it has also a factor $x-a_2$ where $a_2=\alpha-\beta\sqrt{-1}$. The *product* of the two factors $x-\alpha-\beta\sqrt{-1}$ and $x-\alpha+\beta\sqrt{-1}$, is $(x-\alpha)^2+\beta^2$, or $x^2-2\alpha x+\alpha^2+\beta^2$; that is, the product is a real quadratic factor.

43. We have thus arrived at the result that any rational integral function of x with real coefficients may be regarded as the product of *real* factors, either simple or quadratic; and that there is only one such system of factors for any given function. Thus $f(x)$ must be of the form $(x-a)(x-b)(x-c)\ldots(x-k)\,\phi(x)$, where

$a, b, c, \ldots k$ are all the *real* roots of $f(x) = 0$, and $\phi(x)$ is a function consisting of the product of quadratic factors which cannot change its sign.

44. In the manner of Art. 41 it may be shewn that if the coefficients of any rational integral function $f(x)$ of x be themselves *rational*, and the equation $f(x) = 0$ has a root of the form $a + \sqrt{b}$ where $\sqrt{b}$ is a surd, the equation has also a root $a - \sqrt{b}$. Thus $f(x)$ has a rational quadratic factor $(x - a)^2 - b$.

45. *To investigate the relations between the coefficients of the function* f (x) *and the roots of the equation* f (x) = 0.

Let
$$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n;$$
and suppose that the roots of the equation $f(x) = 0$ are $a_1, a_2, \ldots a_n$; then
$$f(x) = (x - a_1)(x - a_2) \ldots (x - a_n).$$
Since these two expressions for $f(x)$ are identically equal, relations exist between the coefficients $p_1, p_2, \ldots p_n$, and the quantities $a_1, a_2, \ldots a_n$; these relations we shall now exhibit.

By ordinary multiplication we obtain
$$(x - a_1)(x - a_2) = x^2 - (a_1 + a_2)x + a_1 a_2,$$
$$(x - a_1)(x - a_2)(x - a_3) = x^3 - (a_1 + a_2 + a_3)x^2$$
$$+ (a_1 a_2 + a_2 a_3 + a_3 a_1)x - a_1 a_2 a_3.$$
$$(x - a_1)(x - a_2)(x - a_3)(x - a_4) = x^4 - (a_1 + a_2 + a_3 + a_4)x^3$$
$$+ (a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4)x^2$$
$$- (a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4)x + a_1 a_2 a_3 a_4.$$
Now in these results we see that the following laws hold:

I. The number of terms on the right-hand side is one more than the number of the simple factors which are multiplied together.

II. The exponent of x in the first term is the same as the number of the simple factors, and in the other terms each exponent is less than that of the preceding term by unity.

III. The coefficient of the first term is unity; the coefficient of the second term is the sum of the second terms of the simple factors; the coefficient of the third term is the sum of the products of every two of the second terms of the simple factors; the coefficient of the fourth term is the sum of the products of the second terms of the simple factors taken three at a time, and so on; the last term is the product of all the second terms of the simple factors.

We shall now prove that these laws always hold whatever be the number of simple factors. Suppose these laws to hold when $n-1$ factors are multiplied together; that is, suppose

$$(x-a_1)(x-a_2)\dots(x-a_{n-1}) = x^{n-1} + q_1x^{n-2} + q_2x^{n-3} + \dots + q_{n-2}x + q_{n-1},$$

where $q_1 =$ the sum of the terms $-a_1, -a_2, \dots -a_{n-1}$,

$q_2 =$ the sum of the products of these terms taken two at a time,

$q_3 =$ the sum of the products of these terms taken three at a time,

....

$q_{n-1} =$ the product of all these terms.

Multiply both sides of this identity by another factor $x - a_n$; thus

$$(x-a_1)(x-a_2)\dots(x-a_n) = x^n + (q_1 - a_n)x^{n-1} + (q_2 - q_1a_n)x^{n-2} + (q_3 - q_2a_n)x^{n-3} + \dots\dots - q_{n-1}a_n.$$

Now $q_1 - a_n = -a_1 - a_2 - \dots - a_{n-1} - a_n$

$=$ the sum of all the terms $-a_1, -a_2, \dots -a_n$;

$q_2 - q_1a_n = q_2 + a_n(a_1 + a_2 + \dots + a_{n-1})$

$=$ the sum of the products taken two and two of all the terms $-a_1, -a_2, \dots - a_n$;

$q_3 - q_2a_n = q_3 - a_n(a_1a_2 + a_2a_3 + \dots)$

$=$ the sum of the products taken three and three of all the terms $-a_1, -a_2, \dots - a_n$;

....

$-q_{n-1}a_n =$ the product of all the terms $-a_1, -a_2, \dots - a_n$.

Hence if the laws hold when $n-1$ factors are multiplied together they hold when n factors are multiplied together; but they have been proved to hold when four factors are multiplied together, therefore they hold when five factors are multiplied together, and so on; thus they hold universally.

We have used the inductive method in establishing these laws; but they may also be obtained in another way: see *Algebra*, Art. 506.

Since if $a_1, a_2, \ldots a_n$ are the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0,$$

the left-hand member is equivalent to the product of the factors $x - a_1$, $x - a_2$, ... $x - a_n$, we have the following results. In any equation in its simplest form the coefficient of the second term is equal to the sum of the roots with their signs changed; the coefficient of the third term is equal to the sum of the products of every two of the roots with their signs changed; the coefficient of the fourth term is equal to the sum of the products of every three of the roots with their signs changed;......the last term is the product of all the roots with their signs changed.

Or we may enunciate the laws thus: the coefficient of the second term *with its sign changed* is equal to the sum of the roots; the coefficient of the third term is equal to the sum of the products of every two of the roots; the coefficient of the fourth term *with its sign changed* is equal to the sum of the products of every three of the roots; and so on. Thus generally if p_r denote as usual the coefficient of x^{n-r} in the equation, $(-1)^r p_r =$ the sum of the products of every r of the roots.

46. It might appear perhaps that the relations given in the preceding Article would enable us to find the roots of any proposed equation; for they supply equations involving the roots, and the number of these equations is the same as the number of the roots, so that it might be supposed practicable to eliminate all the roots but one and thus to determine that root. But on attempting this elimination we *merely reproduce the proposed equation itself.* Take, for example, the equation of the third degree

$$x^3+p_1x^2+p_2x+p_3=0;$$

suppose the roots to be a, b, c; then

$$-a-b-c=p_1,$$
$$ab+bc+ca=p_2$$
$$-abc=p_3.$$

In order to eliminate b and c and so to obtain an equation which contains only a, the simplest method is to multiply the first of the above three equations by a^2, and the second by a, and add the results to the third. Thus

$$-a^3-a^2b-a^2c+a^2b+abc+ca^2-abc=p_1a^2+p_2a+p_3;$$

that is,
$$a^3+p_1a^2+p_2a+p_3=0;$$

we have thus the proposed equation with a instead of x to represent the unknown quantity. And it is not difficult to see that we ought to expect a cubic equation in a, if we eliminate b and c from the relations we are considering. For the letters a, b, c represent the roots without any distinction of one root from the others; thus any equation which we deduce for determining a ought to allow of three values for a, since a may stand for any one of the three roots of the proposed equation. Thus we may feel certain that we shall only reproduce the original form of the proposed equation by performing any algebraical operations on the relations which connect the known coefficients of the equation with its unknown roots, with the view of eliminating all the roots but one.

47. Although the relations given in Art. 45 will not determine the roots of any proposed equation, we shall find that they will enable us to deduce various important results with respect to equations. For example, if a_1, a_2, ... a_n are the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0,$$

we have
$$-p_1=a_1+a_2+a_3+\ldots+a_n,$$
$$p_2=a_1a_2+a_1a_3+\ldots+a_2a_3+\ldots;$$

thus
$$p_1^2-2p_2=a_1^2+a_2^2+a_3^2+\ldots+a_n^2,$$

that is $p_1^2 - 2p_2$ is equal to the sum of the squares of the roots of the proposed equation. If then in any equation $p_1^2 - 2p_2$ is *negative*, the roots of the equation cannot be all real.

48. In the same manner as in the preceding Article we may deduce other relations involving the roots. Thus for example

$(-1)^{n-1}p_{n-1}$ = the sum of the products of the roots $n-1$ at a time,

$(-1)^n p_n$ = the product of all the roots;

therefore by division

$$-\frac{p_{n-1}}{p_n} = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}$$

= the sum of the reciprocals of the roots.

Also
$$p_1 \frac{p_{n-1}}{p_n} = (a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right)$$
$$= n + \frac{a_1}{a_2} + \frac{a_1}{a_3} + \ldots + \frac{a_2}{a_1} + \frac{a_2}{a_3} + \ldots;$$

therefore
$$\frac{a_1}{a_2} + \frac{a_1}{a_3} + \ldots + \frac{a_2}{a_1} + \frac{a_2}{a_3} + \ldots = \frac{p_1 p_{n-1}}{p_n} - n.$$

IV. TRANSFORMATION OF EQUATIONS.

49. The general object of the present Chapter is to deduce from a given equation another equation the roots of which shall have an assigned relation to those of the given equation. It will be seen as we proceed that various transformations of this kind can be effected without knowing the roots of the given equation; and hereafter examples will occur shewing that such transformations may be of use in the solution of equations.

50. *To transform an equation into another the roots of which are those of the proposed equation with contrary signs.*

Let $f(x) = 0$ denote the proposed equation; assume $y = -x$, so that when x has any particular value, y has numerically the same value but with the contrary sign; thus $x = -y$, and the required equation is $f(-y) = 0$.

If $\quad f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n,$

the equation $f(-y) = 0$ is

$$p_0(-y)^n + p_1(-y)^{n-1} + p_2(-y)^{n-2} + \dots - p_{n-1}y + p_n = 0,$$

that is, $$p_0y^n - p_1y^{n-1} + p_2y^{n-2} - \dots \pm p_{n-1}y \mp p_n = 0;$$

thus the transformed equation may be obtained from the proposed equation by *changing the sign of the coefficient of every other term beginning with the second.*

51. The rule at the end of the preceding Article assumes that the proposed equation has all the terms which can occur in an equation of its degree, that is, it is assumed that no coefficient is zero. But suppose we take an example in which this is not the case; thus let it be required to transform the equation

$$x^6 + 3x^5 - 4x^3 - 4x + 7 = 0,$$

into another in which the roots shall be numerically the same but with contrary signs. Put $x = -y$, and we get

$$y^6 - 3y^5 + 4y^3 + 4y + 7 = 0.$$

We may if we please write the original equation thus,

$$x^6 + 3x^5 + 0x^4 - 4x^3 + 0x^2 - 4x + 7 = 0;$$

then the transformed equation according to the rule in Art. 50, is

$$y^6 - 3y^5 + 0y^4 + 4y^3 + 0y^2 + 4y + 7 = 0,$$

that is, $$y^6 - 3y^5 + 4y^3 + 4y + 7 = 0,$$

as before.

An equation is said to be *complete* when it has all the terms which can occur in an equation of its degree, that is, when no coefficient is zero. And we shall sometimes find it useful to render an equation complete by the artifice used above, that is, by introducing the missing terms with zero for the coefficient of each of them.

52. *To transform an equation into another the roots of which are equal to those of the proposed equation multiplied by a given quantity.*

Let $f(x)=0$ denote the proposed equation; and let it be required to transform it into another the roots of which are k times as large. Assume $y=kx$, so that when x has any particular value, the value of y is k times as large; thus $x=\frac{y}{k}$, and the required equation is $f\left(\frac{y}{k}\right)=0$.

53. For example, transform the equation

$$x^3-\frac{3x^2}{2}+\frac{5x}{4}-\frac{2}{9}=0$$

into another the roots of which are k times as large. Put $x=\frac{y}{k}$ and then multiply throughout by k^3; thus we obtain

$$y^3-\frac{3ky^2}{2}+\frac{5k^2y}{4}-\frac{2k^3}{9}=0.$$

This example will shew us an application which may be made of the present transformation. The coefficients of the proposed equation are not all integers; by properly assuming k we may make the coefficients of the transformed equation all integers. For instance, if $k=6$, the transformed equation is

$$y^3-9y^2+45y-48=0.$$

Generally, suppose the proposed equation to be

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0,$$

then if we put $x=\frac{y}{k}$, and multiply throughout by k^n, all that is necessary to ensure that the coefficients of the transformed equation shall be integers is, that for each term of the transformed equation $p_rk^ry^{n-r}$, every prime factor which occurs in the denominator of p_r shall occur to at least as high a power in k^r.

54. *To transform an equation into another the roots of which shall be less than those of the proposed equation by a constant difference.*

Let $f(x)=0$ denote the proposed equation; and let it be required to transform this equation into another the roots of which shall be less than the roots of the proposed equation by a constant difference k. Assume $y=x-k$, so that when x has any particular value, the value of y is less by k; thus $x=k+y$, and the required equation is $f(k+y)=0$.

By Art. 10 the expanded form of the equation $f(k+y)=0$ is

$$f(k)+yf'(k)+\frac{y^2}{1\,.\,2}f''(k)+\frac{y^3}{\lfloor 3}f'''(k)+\ldots+y^n\frac{f^n(k)}{\lfloor n}=0.$$

Thus if $f(x)=p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n$ the equation $f(k+y)=0$ when arranged according to *descending* powers of y is by Art. 12

$$p_0y^n+(p_1+np_0k)\,y^{n-1}+\left\{p_2+(n-1)\,p_1k+\frac{n\,(n-1)}{1\,.\,2}\,p_0k^2\right\}y^{n-2}$$
$$+\ldots$$
$$+\left\{p_r+(n-r+1)\,p_{r-1}k+\ldots\ldots+\frac{n\,(n-1)\ldots(n-r+1)}{\lfloor r}\,p_0k^r\right\}y^{n-r}$$
$$+\ldots+f(k)=0.$$

A good practical mode of conducting the operation will be found in Chapter XVIII.

55. If an equation is to be transformed into another the roots of which *exceed* those of the proposed equation by the constant quantity h, we use the method of the preceding Article. Let the proposed equation be denoted by $f(x)=0$, and suppose $y=x+h$; then $x=y-h$, and the required equation is $f(y-h)=0$. Thus we have only to put $-h$ for k in the result of the preceding Article, and we obtain the required equation. But in fact this is included in the preceding Article; for that Article does not require k to be necessarily a *positive* quantity.

56. The principal use of the transformation in Art. 54 is to obtain from a proposed equation another which *wants an as-*

signed term. Thus if we wish the transformed equation in y to be without its *second* term, we take k such that $p_1 + np_0 k = 0$, that is, $k = -\frac{p_1}{np_0}$. If we wish the transformed equation in y to be without its *third* term, we must find k from the quadratic equation

$$p_2 + (n-1)p_1 k + \frac{n(n-1)}{1.2} p_0 k^2 = 0.$$

And generally, if we wish the transformed equation in y to be without its $(r+1)^{\text{th}}$ term, we must find k from an equation of the r^{th} degree, namely

$$p_0 k^r + \frac{r}{n} p_1 k^{r-1} + \frac{r(r-1)}{n(n-1)} p_2 k^{r-2} + \ldots + \frac{\underline{|r}\,\underline{|n-r}}{\underline{|n}} p_r = 0.$$

We shall see hereafter that the solution of an equation is sometimes facilitated by first removing some assigned term.

57. For example, transform the equation $x^3 - 6x^2 + 4x + 5 = 0$ into another without its second term. Here $p_0 = 1$, $p_1 = -6$; thus $k = 2$, and the required equation is

$$(y+2)^3 - 6(y+2)^2 + 4(y+2) + 5 = 0,$$

that is, $$y^3 - 8y - 3 = 0.$$

Again, transform the equation $x^3 - 2x^2 - 4x + 9 = 0$ into another without its third term. Put $y + k$ for x; the transformed equation is

$$(y+k)^3 - 2(y+k)^2 - 4(y+k) + 9 = 0,$$

that is, $$y^3 + y^2(3k-2) + y(3k^2 - 4k - 4) + k^3 - 2k^2 - 4k + 9 = 0.$$

If the third term is to disappear k must be found from the equation $3k^2 - 4k - 4 = 0$; this gives $k = 2$ or $-\frac{2}{3}$. With the value $k = 2$ the transformed equation is

$$y^3 + 4y^2 + 1 = 0.$$

With the value $k = -\frac{2}{3}$ the transformed equation is

$$y^3 - 4y^2 + \frac{283}{27} = 0.$$

58. *To transform an equation into another the roots of which are the reciprocals of the roots of the proposed equation.*

Let $f(x)=0$ denote the proposed equation. Assume $y=\frac{1}{x}$, so that when x has any particular value, the value of y is the reciprocal of that value; thus $x=\frac{1}{y}$ and the required equation is $f\left(\frac{1}{y}\right)=0$.

Thus if $f(x)=p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n$ the equation $f\left(\frac{1}{y}\right)=0$ is

$$\frac{p_0}{y^n}+\frac{p_1}{y^{n-1}}+\frac{p_2}{y^{n-2}}+\ldots+\frac{p_{n-1}}{y}+p_n=0,$$

that is, $$p_ny^n+p_{n-1}y^{n-1}+p_{n-2}y^{n-2}+\ldots+p_1y+p_0=0.$$

59. *To transform an equation into another the roots of which are the squares of the roots of the proposed equation.*

Let $f(x)=0$ denote the proposed equation. Assume $y=x^2$, so that when x has any particular value the value of y is the square of that value: thus $x=\sqrt{y}$ and the required equation is $f(\sqrt{y})=0$.

Thus if $f(x)=p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n$ the equation $f(\sqrt{y})=0$ is

$$p_0y^{\frac{n}{2}}+p_1y^{\frac{n-1}{2}}+p_2y^{\frac{n-2}{2}}+\ldots+p_{n-1}y^{\frac{1}{2}}+p_n=0.$$

By transposing and squaring we have

$$\left(p_0y^{\frac{n}{2}}+p_2y^{\frac{n-2}{2}}+p_4y^{\frac{n-4}{2}}+\ldots\right)^2=\left(p_1y^{\frac{n-1}{2}}+p_3y^{\frac{n-3}{2}}+\ldots\right)^2.$$

The equation will be in a rational form when both sides are developed, and by bringing all the terms to one side we obtain

$$p_0^2y^n+(2p_0p_2-p_1^2)\,y^{n-1}+(2p_0p_4+p_2^2-2p_1p_3)\,y^{n-2}+\ldots=0.$$

60. These cases of transformation of equations might be increased, but we have given sufficient to explain this part of the subject. We will conclude with three examples which will illustrate the use of some of the relations established in Art. 45.

(1) If the roots of the equation $x^3 + px^2 + qx + r = 0$ be a, b, c, form the equation of which the roots are

$$\frac{a}{b+c}, \quad \frac{b}{c+a}, \quad \frac{c}{a+b}.$$

Denote the required equation by

$$y^3 + Py^2 + Qy + R = 0.$$

Then we have, by Art. 45,

$$-P = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b},$$

$$Q = \frac{ab}{(b+c)(c+a)} + \frac{bc}{(c+a)(a+b)} + \frac{ca}{(a+b)(b+c)},$$

$$-R = \frac{abc}{(b+c)(c+a)(a+b)};$$

and $\qquad a+b+c=-p, \quad ab+bc+ca=q, \quad abc=-r.$

Thus we may now proceed to express the values of P, Q, and R in terms of p, q, and r. For example

$$R = \frac{r}{(b+c)(c+a)(a+b)};$$

now by actual multiplication we find

$$(b+c)(c+a)(a+b) = (a+b+c)(ab+bc+ca) - abc$$
$$= -pq + r;$$

therefore $$R = \frac{r}{r-pq}.$$

Similarly we can express P and Q.

But we may evade the trouble of this process by an algebraical artifice. We have

$$\frac{a}{b+c} = \frac{a}{a+b+c-a} = \frac{a}{-p-a}.$$

Thus if $y = -\frac{x}{p+x}$, when x takes the value a the value of y is $\frac{a}{b+c}$; and similarly when x takes the values b and c the values of y are respectively $\frac{b}{c+a}$ and $\frac{c}{a+b}$.

Thus the required equation will be obtained by eliminating x between the proposed equation and $y = -\frac{x}{p+x}$.

Hence $x = -\frac{py}{1+y}$; and by substituting this value in the proposed equation we obtain

$$-\frac{p^3y^3}{(1+y)^3} + \frac{p^3y^2}{(1+y)^2} - \frac{pqy}{1+y} + r = 0,$$

or $$r(1+y)^3 + p^3y^2(1+y) - pqy(1+y)^2 - p^3y^3 = 0,$$

that is $$(r-pq)y^3 + (3r+p^3-2pq)y^2 + (3r-pq)y + r = 0.$$

Hence by this method we arrive indirectly at the values of P, Q, and R: we see that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = -\frac{3r+p^3-2pq}{r-pq},$$

$$\frac{ab}{(b+c)(c+a)} + \frac{bc}{(c+a)(a+b)} + \frac{ca}{(a+b)(b+c)} = \frac{3r-pq}{r-pq},$$

$$\frac{abc}{(b+c)(c+a)(a+b)} = -\frac{r}{r-pq}.$$

The last result has already been obtained by direct investigation.

(2) Required to transform the equation $x^3 + qx + r = 0$ into another the roots of which are the squares of the differences of the roots of the proposed equation.

Let a, b, c denote the roots of the proposed equation; then, by Art. 45,

$$a + b + c = 0, \qquad ab + bc + ca = q, \qquad abc = -r;$$

therefore $$a^2 + b^2 + c^2 = -2q.$$

The roots of the transformed equation are to be $(a-b)^2$, $(b-c)^2$, and $(a-c)^2$; now

$$(a-b)^2 = a^2 - 2ab + b^2 = a^2 + b^2 + c^2 - 2ab - c^2 = a^2 + b^2 + c^2 - \frac{2abc}{c} - c^2$$

$$= -2q + \frac{2r}{c} - c^2;$$

thus if $y = -2q + \frac{2r}{x} - x^2$, when x takes the value c the value of y is $(a-b)^2$; and similarly when x takes the values a and b, the values of y are respectively $(b-c)^2$ and $(c-a)^2$. Thus the transformed equation will be obtained by eliminating x between the proposed equation and $y = -2q + \frac{2r}{x} - x^2$.

Thus $$x^3 + qx + r = 0,$$

and $$x^3 + (2q + y)x - 2r = 0;$$

therefore $$(q + y)x - 3r = 0.$$

Hence $x = \frac{3r}{q+y}$; substituting this value in the proposed equation and reducing, we have finally

$$y^3 + 6qy^2 + 9q^2y + 27r^2 + 4q^3 = 0.$$

Thus if $27r^2 + 4q^3$ is positive the transformed equation has a real negative root by Art. 20; and therefore the proposed equation must have two imaginary roots, since it is only such a pair of roots which can produce a *negative* root in the transformed equation.

If $27r^2 + 4q^3$ is zero the transformed equation has one root equal to zero, and therefore the proposed equation must have two equal roots.

(3) Required to transform the equation $x^3 + px^2 + qx + r = 0$ into another the roots of which are the squares of the differences of the roots of the proposed equation.

Put $x = x' - \frac{p}{3}$; thus the proposed equation becomes

$$\left(x' - \frac{p}{3}\right)^3 + p\left(x' - \frac{p}{3}\right)^2 + q\left(x' - \frac{p}{3}\right) + r = 0,$$

that is, $$x'^3 + q'x' + r' = 0,$$

where $$q' = q - \frac{p^2}{3}, \quad r' = \frac{2p^3}{27} - \frac{pq}{3} + r.$$

Each root of the last equation exceeds the corresponding root of the proposed equation by $\frac{p}{3}$; and thus the squares of the differences of the roots of the last equation are the same as the squares of the differences of the roots of the proposed equation. Therefore by the former example the required equation is

$$y^3 + 6q'y^2 + 9q'^2y + 27r'^2 + 4q'^3 = 0;$$

that is,

$$y^3 + 2(3q - p^2)y^2 + (3q - p^2)^2 y + \frac{(2p^3 - 9pq + 27r)^2 + 4(3q - p^2)^3}{27} = 0.$$

Hence if a, b, c are the roots of $x^3 + px^2 + qx + r = 0$, we see that

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = -2(3q - p^2),$$

$$(a-b)^2(b-c)^2 + (b-c)^2(c-a)^2 + (c-a)^2(a-b)^2 = (3q - p^2)^2,$$

$$(a-b)^2(b-c)^2(c-a)^2 = -\frac{1}{27}\left\{(2p^3 - 9pq + 27r)^2 + 4(3q - p^2)^3\right\}.$$

V. DESCARTES'S RULE OF SIGNS.

61. We have already in Arts. 21...24 given instances of the connexion which exists between the signs of the coefficients in $f(x)$ and the nature of the roots of the equation $f(x) = 0$, and we now proceed to investigate a general theorem on the subject after some preliminary definitions.

62. When each term of a set of terms has one of the signs + and − before it, then in considering the terms in order, a *continuation* is said to occur when a sign is the same as the immediately preceding sign, and a *change* is said to occur when a sign is the contrary to the immediately preceding sign. Thus in the expression $x^8 - 3x^7 - 4x^6 + 7x^5 + 3x^4 + 2x^3 - x^2 - x + 1$, there are four continuations and four changes; the first continuation occurs at $-4x^6$, the second at $+3x^4$, the third at $+2x^3$, the fourth at $-x$; the first change occurs at $-3x^7$, the second at $+7x^5$, the third at $-x^2$, the fourth at $+1$.

It is obvious that in any *complete* equation the number of continuations together with the number of changes is equal to the number which expresses the degree of the equation; see Art. 51. And if in any complete equation we put $-x$ for x, the continuations and changes in the original equation become respectively changes and continuations in the new equation. In an equation $f(x) = 0$ which is not complete, the sum of the numbers of the changes of $f(x)$ and $f(-x)$ cannot be *greater* than the degree of the equation; because if terms are missing in $f(x)$, although it may happen that the number of changes in $f(x)$ or in $f(-x)$ is thus diminished, it cannot be increased.

We shall now enunciate and prove a theorem which is called Descartes's *Rule of Signs*.

63. *In any equation, complete or incomplete, the number of positive roots cannot exceed the number of changes in the signs of the coefficients, and in any complete equation the number of negative roots cannot exceed the number of continuations in the signs of the coefficients.*

We shall first shew that if any polynomial be multiplied by a factor $x - a$ there will be at least one more *change* in the product than in the original polynomial.

Suppose for example that the signs of the terms in the original polynomial are $+ + - - - + - + - - +$. We have to multiply the polynomial by a binomial in which the signs of the terms are $+ -$.

Then writing down only the *signs* which occur in the process and in the result we have

$$
\begin{array}{l}
+\,+\,-\,-\,-\,+\,-\,+\,-\,-\,+ \\
+\,- \\
\hline
+\,+\,-\,-\,-\,+\,-\,+\,-\,-\,+ \\
\quad -\,-\,+\,+\,+\,-\,+\,-\,+\,+\,- \\
\hline
+\,\pm\,-\,\mp\,\mp\,+\,-\,+\,-\,\mp\,+\,-
\end{array}
$$

A double sign is placed where the sign of any term in the product is ambiguous. The following laws will be seen by inspection to hold.

(1) Every group of continuations in the original polynomial has a group of the same number of ambiguities corresponding to it in the new polynomial.

(2) In the new polynomial the signs before and after an ambiguity or a group of ambiguities are contrary.

(3) In the new polynomial a change of sign is introduced at the end.

Now in the new polynomial take the most unfavourable case and suppose all the ambiguities to be replaced by continuations; by the second law we may then without influencing the number of continuations adopt the upper sign for the ambiguities; and thus the signs of the original polynomial will be repeated in the new polynomial, except that by the third law there is an additional change of sign introduced at the end of the new polynomial. Thus in the most unfavourable case there is one more change of sign in the new polynomial than in the original polynomial.

If then we suppose the product of all the factors corresponding to the negative and imaginary roots of an equation already formed, by multiplying by the factor corresponding to each positive root we introduce at least *one change of sign.* Therefore no equation can have more positive roots than it has changes of sign.

To prove the second part of Descartes's rule of signs we suppose the equation *complete*, and put $-y$ for x; then the original *conti-*

nuations of sign become *changes* of sign. And the transformed equation cannot have more positive roots than it has changes; and thus there cannot be more negative roots of the original equation than the number of continuations of sign in that original equation.

64. Whether the equation $f(x)=0$ be complete or not its roots are equal in magnitude but contrary in sign to the roots of $f(-x)=0$, that is, the negative roots of $f(x)=0$ are the positive roots of $f(-x)=0$; and whether the equation be complete or not the number of the positive roots of $f(-x)=0$ cannot exceed the number of changes of sign in $f(-x)$. Thus the whole rule of signs may be enunciated in the following manner: an equation $f(x)=0$ cannot have more positive roots than $f(x)$ has changes of sign, and cannot have more negative roots than $f(-x)$ has changes of sign.

65. For example, take the equation $x^4+3x^2+5x-7=0$. Here there is *one* change of sign, and therefore there cannot be more than one positive root. And by writing $-x$ for x we obtain the equation $x^4+3x^2-5x-7=0$; here there is *one* change of sign, and therefore there cannot be more than one positive root, so that the original equation cannot have more than one negative root. Thus the original equation cannot have more than two real roots.

In this example we know by Art. 21 that there *is* one positive root, and that there *is* one negative root; and we have just ascertained that there cannot be more than one of each.

Again, consider the equation $x^3+qx+r=0$, where q and r are both positive. Here there is no change of sign, and therefore no positive root; this also appears from Art. 24. If we write $-x$ for x, we obtain an equation with *one* change of sign, so that the original equation cannot have more than one negative root, and therefore the original equation must have two imaginary roots.

Again, consider the equation $x^3-qx+r=0$, where q and r are both positive. Here there are *two* changes of sign, and therefore there cannot be more than two positive roots. If we write

$-x$ for x, we obtain an equation with *one* change of sign, so that the original equation cannot have more than one negative root.

In this example we know by Art. 20 that there *is* one negative root, and we have just ascertained that there cannot be more than one; whether the other two roots are real positive quantities or imaginary, we cannot infer from Descartes's rule of signs. But from Art. 60 it follows that the equation which has for its roots the squares of the differences of the roots of the proposed equation is $y^3 - 6qy^2 + 9q^2y + 27r^2 - 4q^3 = 0$; and by Descartes's rule of signs, or by Art. 24, if $27r^2 - 4q^3$ is negative, the last equation has no negative root, and therefore the original equation no imaginary roots; also if $27r^2 - 4q^3$ is positive, the last equation has a negative root by Art. 20, and therefore the original equation must have two imaginary roots.

66. The student should observe that the results given in Art. 24, are all consistent with Descartes's rule of signs, and may all be deduced from it. Also the proposition in Art. 22 is included in Descartes's rule of signs; and we learn from this rule that such an equation as that considered in Art. 22 can have only one positive root, without repetition; see Art. 35.

67. It is shewn in the proof of Descartes's rule of signs, that on multiplying a polynomial by the factor which corresponds to a real positive root, one change of sign *at least* is introduced; it may be observed, that the number of the changes of sign introduced must be an *odd* number. For suppose in the first place that the last sign in the original polynomial is +; then since the first sign is +, the whole number of changes of sign in the original polynomial must be an *even* number or zero; and the sign of the last term of the new polynomial is −, so that the number of changes of sign in the new polynomial is an *odd* number. Therefore an *odd* number of changes of sign must have been introduced. Next suppose that the last sign in the original polynomial is −, so that the last sign in the new polynomial is +; then there must be an odd number of changes of sign in the original polynomial, and an even number of changes of sign in

the new polynomial. Therefore an *odd* number of changes of sign must have been introduced.

68. *When all the roots of an equation* f (x) = 0 *are real, the number of positive roots is equal to the number of changes of sign in* f (x), *and the number of negative roots is equal to the number of changes of sign in* f (– x).

Let n denote the degree of the equation, m the number of positive roots, and m' the number of negative roots, μ the number of changes of sign in $f(x)$, and μ' the number of changes of sign in $f(-x)$. Since all the roots of the equation are real $m+m'=n$. Also m cannot be greater than μ, and m' cannot be greater than μ', by Art. 63. Therefore $\mu+\mu'=n$, for the sum of μ and μ' cannot exceed n. Thus $m+m'=\mu+\mu'$. And m cannot be greater than μ; nor can m be less than μ, for then m' would be greater than μ', which is impossible. Thus $m=\mu$, and $m'=\mu'$.

In this proposition we assume that $f(x)$ has a term independent of x, so that the equation $f(x)=0$ is not satisfied by $x=0$. A root *zero* cannot properly be considered either positive or negative.

If we wish to introduce the consideration of zero roots we may proceed thus: suppose the equation to have m positive roots, m' negative roots, and the root zero repeated r times. Then we have $m+m'+r=n$, so that $m+m'=n-r$. And we can shew that $\mu+\mu'$ can be neither less nor greater than $n-r$; so that $\mu+\mu'=n-r$. Then as before $m=\mu$ and $m'=\mu'$.

69. Suppose μ the number of changes of sign in $f(x)$, and μ' the number of changes of sign in $f(-x)$. Then the equation $f(x)=0$ cannot have more than μ positive roots, and cannot have more than μ' negative roots, and therefore cannot have more than $\mu+\mu'$ real roots. Hence if n is greater than $\mu+\mu'$ the equation $f(x)=0$ must have at least $n-\mu-\mu'$ imaginary roots. In the next two Articles we shall shew more definitely what inferences we can draw as to the number of imaginary roots of an equation when that equation is not complete.

70. *If any group consisting of an even number of terms is deficient in any equation there are at least as many imaginary roots of the equation.*

Suppose the $2r$ terms which might occur in $f(x)$ between x^m and x^{m-2r-1} to be deficient; then the equation $f(x)=0$ will have at least $2r$ imaginary roots. Let A and B denote the coefficients of x^m and x^{m-2r-1} respectively in $f(x)$, and suppose the deficient terms introduced with coefficients $q_1, q_2, q_3, \ldots$; and denote the new function by $F(x)$. Then in the expression

$$Ax^m + q_1x^{m-1} + q_2x^{m-2} + \ldots + q_{2r}x^{m-2r} + Bx^{m-2r-1}$$

the number of changes of sign together with the number of continuations of sign is $2r+1$; in other words the number of changes of sign in this expression, together with the number of changes of sign which it would present if the sign of x were changed, is $2r+1$. But now let the hypothetical terms be removed; then if A and B are of contrary signs there will be one change of sign for $f(x)$, and no change of sign for $f(-x)$; and if A and B are of the same sign there will be one change of sign for $f(-x)$ and no change of sign for $f(x)$. Therefore in both cases the loss of $2r$ terms ensures the loss of $2r$ from the sum of the number of changes of sign in $F(x)$ and in $F(-x)$.

And this result holds for every deficient group consisting of an even number of terms. Thus there are at least as many imaginary roots of the equation $f(x)=0$ as the sum of the numbers of terms in such deficient groups.

71. *If any group consisting of an odd number of terms is deficient in any equation, the equation has at least one more than that number of imaginary roots if the deficient group is between two terms of the same sign, and the equation has at least one less than that number of imaginary roots if the deficient group is between two terms of contrary signs.*

Suppose the $2r+1$ terms which might occur in $f(x)$ between x^m and x^{m-2r-2} to be deficient. Let A and B denote the coefficients of x^m and x^{m-2r-2} in $f(x)$ respectively; then if A and B are of the same sign the equation $f(x)=0$ has at least $2r+2$ imaginary roots; if A and B are of contrary signs the equation $f(x)=0$ has at least $2r$ imaginary roots.

Suppose the deficient terms introduced with coefficients q_1, q_2, q_3,...; and denote the new function by $F(x)$. Then in the expression

$$Ax^m + q_1x^{m-1} + q_2x^{m-2} + \ldots + q_{2r+1}x^{m-2r-1} + Bx^{m-2r-2}$$

the number of changes of sign together with the number of continuations of sign is $2r+2$; or in other words the number of changes of sign in this expression, together with the number of changes of sign which it would present if the sign of x were changed, is $2r+2$. But when the hypothetical terms are removed there will be no change of sign either for $f(x)$ or $f(-x)$ if A and B have the same sign, and there will be one change of sign for $f(x)$ and one change of sign for $f(-x)$ if A and B have contrary signs. Therefore the loss of $2r+1$ terms from $F(x)$ ensures the loss of $2r+2$, or of $2r$, from the sum of the number of changes of sign in $F(x)$ and in $F(-x)$, according as the deficient group is between two terms of the same sign, or of contrary signs.

And this result holds for every deficient group consisting of an odd number of terms; therefore there will be at least as many imaginary roots of the equation $f(x)=0$ as the sum furnished by considering the deficient groups.

72. Thus as an example of Art. 71 we see that if a single term is deficient any where in $f(x)$ between two terms of the same sign, there must be at least two imaginary roots; if a single term is deficient between two terms of contrary signs we cannot deduce from this fact any inference as to the number of imaginary roots.

It will be observed that when in consequence of the deficiency of terms the sum of the number of changes of sign in $f(x)$ and $f(-x)$ falls short of the number which expresses the degree of the equation $f(x)=0$, the difference is always an even number. This appears from the examination of the two possible cases in Arts. 70 and 71. That is, with the notation of Art. 69, the number $n-\mu-\mu'$ is always an even number. This might have been anticipated from Art. 41.

VI. ON EQUAL ROOTS.

73. It is sometimes convenient or necessary to know whether a proposed equation has *equal roots*, as we shall see in the course of the work. We shall therefore now explain how we can determine whether an equation has equal roots, and how we can remove factors which correspond to the equal roots when they exist, and thus reduce the equation to one which has only unequal roots. We have first to prove a property concerning the *first derived function* of a given function.

74. *Let* f(x) *be any rational integral function of* x *and* f′(x) *the first derived function; then will*

$$f'(x) = \frac{f(x)}{x-a} + \frac{f(x)}{x-b} + \frac{f(x)}{x-c} + \ldots + \frac{f(x)}{x-k},$$

where a, b, c,...k, *are the roots real or imaginary of the equation* f(x) = 0.

For let p_0 be the coefficient of the highest power of x in $f(x)$, then we have identically by Art. 33,

$$f(x) = p_0(x-a)(x-b)(x-c)\ldots(x-k). \qquad (1)$$

Put $y+z$ for x; thus

$$f(y+z) = p_0(y+z-a)(y+z-b)(y+z-c)\ldots(y+z-k);$$

expand each side in a series proceeding according to ascending powers of z; then the left-hand side becomes by Art. 10,

$$f(y) + f'(y)z + f''(y)\frac{z^2}{1.2} + \ldots$$

Thus the coefficient of z is $f'(y)$, and therefore $f'(y)$ must be equal to the coefficient of z on the right-hand side, that is, to

$$p_0(y-b)(y-c)\ldots(y-k) + p_0(y-a)(y-c)\ldots(y-k) + \ldots,$$

that is, to

$$\frac{f(y)}{y-a} + \frac{f(y)}{y-b} + \frac{f(y)}{y-c} + \ldots + \frac{f(y)}{y-k}.$$

And as it is immaterial what symbol we use for a variable which may have any value, we may change y into x; thus we have

$$f'(x) = \frac{f(x)}{x-a} + \frac{f(x)}{x-b} + \frac{f(x)}{x-c} + \ldots + \frac{f(x)}{x-k}. \qquad (2)$$

The result here obtained is true if among the quantities $a, b, c, \ldots k$, there should occur one or more equal to a, or equal to $b, \ldots$ and so on. Suppose that on the whole a occurs exactly r times, b exactly s times, c exactly t times,...; then (1) may be written

$$f(x) = p_0(x-a)^r(x-b)^s(x-c)^t \ldots,$$

and (2) may be written

$$f'(x) = \frac{rf(x)}{x-a} + \frac{sf(x)}{x-b} + \frac{tf(x)}{x-c} + \ldots$$

75. *The equation* $f(x) = 0$ *has or has not equal roots according as* $f(x)$ *and* $f'(x)$ *have or have not a common measure which involves* x.

Suppose $a, b, c, \ldots k$ the roots real or imaginary of the equation $f(x) = 0$, so that $f(x) = p_0(x-a)(x-b)(x-c)\ldots(x-k)$; then

$$f'(x) = p_0(x-b)(x-c)\ldots(x-k) + p_0(x-a)(x-c)\ldots(x-k) + \ldots$$

If $a, b, c, \ldots k$ are all unequal, none of the factors $x-a$, $x-b$, $x-c, \ldots x-k$ will divide $f'(x)$, for $(x-a)$ for example divides every term in $f'(x)$, *except the first;* and no product of any number of them will divide $f'(x)$. Thus if $f(x)$ has no equal factors $f(x)$ and $f'(x)$ have no common measure. Hence if $f(x)$ and $f'(x)$ have a common measure the factors of $f(x)$ cannot be all unequal.

Next suppose that the equation $f(x) = 0$ has equal roots; suppose that a occurs r times, that b occurs s times, that c occurs t times, and so on. Then

$$f'(x) = p_0(x-a)^r(x-b)^s(x-c)^t \ldots \left\{\frac{r}{x-a} + \frac{s}{x-b} + \frac{t}{x-c} + \ldots\right\}.$$

In this case the factor $(x-a)^{r-1}(x-b)^{s-1}(x-c)^{t-1}\ldots$ occurs in every term of $f'(x)$. Thus if $f(x)$ has equal factors, $f(x)$ and $f'(x)$ have a common measure. Hence if $f(x)$ and $f'(x)$ have no common measure $f(x)$ has no equal factors.

76. For example, consider the equation

$$f(x) = x^4 - 11x^3 + 44x^2 - 76x + 48 = 0.$$

Here $$f'(x) = 4x^3 - 33x^2 + 88x - 76.$$

It will be found that $f(x)$ and $f'(x)$ have the common measure $x - 2$; this shews that $(x-2)^2$ is a factor of $f(x)$. It will be found that

$$f(x) = (x-2)^2(x^2 - 7x + 12) = (x-2)^2(x-3)(x-4);$$

thus the roots of the equation $f(x) = 0$ are 2, 2, 3, 4.

Again, consider the equation

$$f(x) = 2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0.$$

Here $f(x)$ and $f'(x)$ will be found to have the common measure $x - 3$; and $f(x) = (x-3)^2(2x^2+1)$. Thus the roots of the equation $f(x) = 0$ are 3, 3, $+\sqrt{\left(-\frac{1}{2}\right)}$, $-\sqrt{\left(-\frac{1}{2}\right)}$.

77. In the enunciation of Art. 75, the words "*which involves x*" occur at the end. We mean to indicate by these words that we do not regard the factor p_0, although that may in a certain sense be considered as a common measure of $f(x)$ and $f'(x)$.

As we are here for the first time making an important use of *common measures* of expressions it will be convenient to introduce a remark on the subject. It is usual to consider the theory of common measures and of the greatest common measure in works on Algebra; but the theory is not necessary at an early stage of mathematical study, and becomes more intelligible after the result has been obtained which we have given in Art. 33. Let $f(x)$ and $\phi(x)$ denote two rational integral functions of x; then $f(x)$ and $\phi(x)$ may be resolved into factors, so that

$$f(x) = p_0(x-a_1)(x-a_2)(x-a_3)\ldots,$$

$$\phi(x) = q_0(x-b_1)(x-b_2)(x-b_3)\ldots;$$

and each of the functions can be thus resolved in only one way. Hence the function of x of the highest degree which will divide both $f(x)$ and $\phi(x)$ is the product of all the common factors of the first degree in x; and this we may call the greatest common measure of $f(x)$ and $\phi(x)$.

Here we have taken no notice of p_0 and q_0; but we may if we please find their greatest arithmetical common measure if they are numbers, or if they are both functions of another quantity, as y, we may find the greatest common measure of these functions of y.

78. Suppose $f(x) = p_0(x-a)^r(x-b)^s(x-c)^t \dots$; then we have found in Art. 75 that $f(x)$ and $f'(x)$ have the common measure $(x-a)^{r-1}(x-b)^{s-1}(x-c)^{t-1}\dots$. Thus the common measure involves all the equal factors which occur in $f(x)$, but the exponent in each case is less than the corresponding exponent in $f(x)$ by unity. If we divide $f(x)$ by the common measure of $f(x)$ and $f'(x)$, the quotient involves all the factors which occur in $f(x)$, each factor occurring singly. Thus the equation obtained by putting this quotient equal to zero contains *without repetition* all the roots which the equation $f(x) = 0$ has.

79. We see that if the factor $(x-a)^r$ occurs in $f(x)$ the factor $(x-a)^{r-1}$ occurs in $f'(x)$; so that the equation $f'(x) = 0$ has $r-1$ roots each equal to a. Now $f''(x)$ is the first derived function of $f'(x)$; thus if $r-1$ be greater than unity $f'(x)$ and $f''(x)$ will have a common measure, and the equation $f''(x) = 0$ will have $r-2$ roots equal to a. Thus in this way we can shew that if $(x-a)^r$ is a factor of $f(x)$ then the derived functions $f'(x), f''(x), \dots f^{r-1}(x)$, all vanish when $x = a$.

This may also be proved in the following way.

Let $f(x) = (x-a)^r \phi(x)$, where $\phi(x)$ is a rational integral function of x which is supposed not to contain the factor $x - a$; put $x = a + z$; thus

$$z^r \phi(a+z) = f(a+z)$$
$$= f(a) + f'(a)z + \dots + f^r(a)\frac{z^r}{\lfloor r} + \dots + f^n(a)\frac{z^n}{\lfloor n}.$$

As the left-hand member of this identity is divisible by z^r the right-hand member must be so too. Therefore we must have

$$f(a)=0, \qquad f'(a)=0, \ldots\ldots f^{r-1}(a)=0.$$

And as the left-hand member is not divisible by a power of z higher than z^r the right-hand member cannot be, and therefore $f^r(a)$ is not zero. Thus the number of terms in the series $f(x)$, $f'(x)$, $f''(x)$,...which vanish when $x=a$, is the same as the exponent of $x-a$ in $f(x)$.

For example, suppose

$$f(x)=x^5+2x^4+3x^3+7x^2+8x+3;$$

here it will be found that $f'''(x)$ is the first of the series $f(x)$, $f'(x)$,... which does not vanish when $x=-1$; thus the factor $(x+1)^3$ occurs in $f(x)$. It will be found that $f(x)=(x+1)^3(x^2-x+3)$.

For another example we will investigate the conditions which must hold in order that the equation

$$x^4+qx^2+rx+s=0$$

may have three equal roots.

Here
$$f(x)=x^4+qx^2+rx+s,$$
$$f'(x)=4x^3+2qx+r,$$
$$f''(x)=12x^2+2q.$$

Hence from $f''(x)=0$ we obtain

$$x^2=-\frac{q}{6} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Substitute this value in $f(x)=0$ and $f'(x)=0$: thus

$$-\frac{5q^2}{36}+rx+s=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$x\left(-\frac{2q}{3}+2q\right)+r=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (3) we obtain

$$x=-\frac{3r}{4q} \quad\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and substituting this in (2) we have

$$s - \frac{3r^2}{4q} - \frac{5q^2}{36} = 0 \text{.........................} (5).$$

And from (1) and (4) $r^2 = -\frac{8q^3}{27}$ ....(6).

Hence (5) becomes $s = -\frac{q^2}{12}$ ....(7).

Thus (6) and (7) express the required conditions.

Conversely if (6) and (7) be satisfied, it will be found that $f(x)$, $f'(x)$ and $f''(x)$ all vanish when $x = -\frac{3r}{4q}$.

80. We will briefly indicate another way in which the test for equal roots may be investigated. If the equation $f(x) = 0$ has more than one root equal to a, then it follows that if $f(x)$ be divided by $x - a$ the quotient will vanish when $x = a$. Hence by taking the form of the quotient given in Art. 7, we must have

$$np_0a^{n-1} + (n-1)p_1a^{n-2} + \dots + 2ap_{n-2} + p_{n-1} = 0;$$

that is, $f'(x)$ vanishes when $x = a$.

81. It appears then that when we wish to determine the equal roots of an equation $f(x) = 0$, we may begin by finding the greatest common measure of $f(x)$ and $f'(x)$; then we equate this greatest common measure to zero, and we have an equation to solve which has for its roots those roots of the equation $f(x) = 0$ which are repeated. As this greatest common measure may be itself a complex expression, involving repeated factors, it is useful to have a systematic process by which the roots may be obtained with as little trouble as possible. This we shall now give.

82. Suppose $f(x) = 0$ to be an equation which has equal roots; and let

$$f(x) = X_1X_2^2X_3^3X_4^4 \dots X_m^m,$$

where the product of all the factors which occur singly in $f(x)$ is denoted by X_1, the product of all the factors which occur just twice is denoted by X_2^2, the product of all the factors which occur just three times is denoted by X_3^3, and so on. Any one or more of the quantities X_1, X_2, X_3,... will be unity, if there is no factor in $f(x)$ which is repeated just the corresponding number of times.

Now form the first derived function $f'(x)$ of $f(x)$, and then obtain the greatest common measure of $f(x)$ and $f'(x)$. We will denote this greatest common measure by $f_1(x)$, so that

$$f_1(x) = X_2 X_3^2 X_4^3 \dots X_m^{m-1}.$$

Next obtain the greatest common measure of $f_1(x)$ and its first derived function $f_1'(x)$, and denote it by $f_2(x)$, so that

$$f_2(x) = X_3 X_4^2 \dots X_m^{m-2}.$$

Proceed in this way and form in succession

$$\begin{aligned} f_3(x) &= X_4 X_5^2 \dots X_m^{m-3}, \\ f_4(x) &= \quad X_5 \dots X_m^{m-4}, \\ &\dots\dots\dots\dots\dots \\ f_{m-1}(x) &= \qquad\quad X_m, \\ f_m(x) &= \qquad\quad 1. \end{aligned}$$

Now form a new series of functions by dividing each term of the series $f(x), f_1(x), f_2(x), \dots f_m(x)$ down to $f_{m-1}(x)$ by the immediately succeeding term. Thus we get

$$\begin{aligned} \frac{f(x)}{f_1(x)} &= X_1 X_2 \dots X_m, = \phi_1(x) \text{ say}, \\ \frac{f_1(x)}{f_2(x)} &= \quad X_2 \dots X_m, = \phi_2(x) \text{ say}, \\ &\dots\dots\dots\dots\dots\dots \\ \frac{f_{m-2}(x)}{f_{m-1}(x)} &= X_{m-1} X_m, = \phi_{m-1}(x) \text{ say}, \\ \frac{f_{m-1}(x)}{f_m(x)} &= \qquad X_m, = \phi_m(x) \text{ say}. \end{aligned}$$

Then finally

$$\frac{\phi_1(x)}{\phi_2(x)} = X_1, \ \frac{\phi_2(x)}{\phi_3(x)} = X_2, \dots \frac{\phi_{m-1}(x)}{\phi_m(x)} = X_{m-1}, \ \phi_m(x) = X_m.$$

Thus the factors $X_1, X_2, \dots X_m$ are now separated, and by solving the equations $X_1 = 0$, $X_2 = 0, \dots X_m = 0$, we obtain all the roots of the proposed equation $f(x) = 0$; and any root found from $X_r = 0$ occurs r times in the equation $f(x) = 0$.

83. For an example of the process of the preceding Article suppose that

$$f(x) = x^8 + x^7 - 8x^6 - 6x^5 + 21x^4 + 9x^3 - 22x^2 - 4x + 8.$$

Then retaining the notation of the preceding Article we shall find that

$$\begin{aligned}
f_1(x) &= x^4 + x^3 - 3x^2 - x + 2,\\
f_2(x) &= x - 1,\\
f_3(x) &= 1,\\
\phi_1(x) &= x^4 - 5x^2 + 4,\\
\phi_2(x) &= x^3 + 2x^2 - x - 2,\\
\phi_3(x) &= x - 1,\\
X_1 &= x - 2,\\
X_2 &= x^2 + 3x + 2,\\
X_3 &= x - 1.
\end{aligned}$$

Therefore $f(x) = (x-2)(x^2+3x+2)^2(x-1)^3$

$$= (x-2)(x+1)^2(x+2)^2(x-1)^3.$$

Thus the roots of the equation $f(x) = 0$ are $2, -1, -1, -2, -2, 1, 1, 1$.

84. When the coefficients of an equation are all commensurable quantities the expressions $X_1, X_2, \ldots$ of Art. 82 have likewise all their coefficients commensurable. Hence if one and only one of the roots of an equation, with commensurable quantities for coefficients, is repeated r times, that root *must be a commensurable* quantity; for it will be determined by an equation $X_r = 0$ which involves no incommensurable quantities.

Hence we can deduce the following results:

If an equation of the third degree with commensurable quantities for coefficients have no commensurable roots it has no equal roots. For if an equation of the third degree have equal roots, there must be either one root occurring three times, or one root occurring twice and another root occurring once; and in either case, as we have just seen, if the coefficients are commensurable quantities so also are the roots.

If an equation of the fourth degree with commensurable quantities for coefficients have no commensurable roots it cannot have either one root occurring four times, or one root occurring three times and another root occurring once. If then such an equation have equal roots it must have two incommensurable roots each repeated twice. Thus if $f(x)=0$ be the equation $f(x)$ must be a perfect square.

If an equation of the fifth degree with commensurable quantities for coefficients have no commensurable roots it has no equal roots. For it will be found on examining every case which can exist that if there be equal roots there must be one or more commensurable roots. Suppose, for example, that the equation has two roots each occurring twice and another root occurring once; then if the coefficients are commensurable quantities the unrepeated root must be a commensurable quantity.

VII. LIMITS OF THE ROOTS OF AN EQUATION. SEPARATION OF THE ROOTS.

85. In the present Chapter we shall first investigate some theorems which will shew between what limits *all* the real roots of any proposed equation must lie; and we shall then consider to some extent the possibility of discovering limits between which the real roots separately lie. The advantage of such a Chapter arises from the fact that the *algebraical* solution of the general equation of any degree above the fourth has not been obtained; and as we shall see hereafter, the *numerical* solution of equations is a systematic process based on the supposition that we have some knowledge of the approximate values of particular roots.

It is to be observed that unless anything to the contrary is specially stated, the whole of the present Chapter relates to the *real* roots of equations.

86. When we say that a certain quantity is a *superior limit* of the positive roots of an equation, we mean that no positive root can be greater than that quantity.

87. *The numerically greatest negative coefficient increased by unity is a superior limit of the positive roots of an equation which is in its simplest form.*

Let $f(x)=0$ be the equation; suppose it of the n^{th} degree. Let p be the numerically greatest negative coefficient which occurs in $f(x)$. Then if such a value be found for x that $f(x)$ is positive for that value of x and for all greater values, that value is a superior limit of the positive roots of the equation $f(x)=0$; now if any positive value of x make

$$x^n - p(x^{n-1}+x^{n-2}+x^{n-3}+\ldots+x+1)$$

positive, it will *a fortiori* make $f(x)$ positive. That is, $f(x)$ is positive for a positive value of x if $x^n - p\dfrac{x^n-1}{x-1}$ is positive, and therefore *a fortiori* if $x^n - 1 - p\dfrac{x^n-1}{x-1}$ is positive, that is if $(x^n-1)\left(1-\dfrac{p}{x-1}\right)$ is positive; and the last expression is positive if $x-1$ is greater than p. Thus $f(x)$ is positive if x is equal to $p+1$ or greater than $p+1$; that is, $p+1$ is a superior limit of the positive roots of the equation $f(x)=0$.

88. In the equation $f(x)=0$ put $-y$ for x, and if n is an odd number change the sign of every term so that the coefficient of y^n may be $+1$. Let q be the numerically greatest negative coefficient of the equation in this form; then $q+1$ is a limit of the positive values of y, and therefore $-(q+1)$ is a limit of the negative values of x.

Hence all the roots of the equation $f(x)=0$ must lie between $p+1$ and $-(q+1)$.

Hence *a fortiori* if m be the numerical value of the greatest coefficient in an equation without regard to sign, all the roots of the equation lie between $m+1$ and $-(m+1)$.

89. *In an equation of the* n^th^ *degree in its simplest form if* p *be the numerical value of the greatest negative coefficient, and* x^{n-r} *the highest power of* x *which has a negative coefficient,* $1+\sqrt[r]{p}$ *is a superior limit of the positive roots.*

Let $f(x)=0$ be the proposed equation; since all the terms which precede x^{n-r} have positive coefficients $f(x)$ will certainly be positive for a positive value of x if

$$x^n - p\,(x^{n-r} + x^{n-r-1} + \ldots + x^2 + x + 1)$$

be positive, that is, if $x^n - p\,\dfrac{x^{n-r+1}-1}{x-1}$ be positive. Hence, supposing x greater than unity, $f(x)$ will be positive *a fortiori* if $x^n - p\,\dfrac{x^{n-r+1}}{x-1}$ is positive, that is if $x^n(x-1) - px^{n-r+1}$ is positive, that is if $x^{r-1}(x-1)-p$ is positive, that is *a fortiori* if $(x-1)^r$ is equal to or greater than p. Hence if $x = 1+\sqrt[r]{p}$ or any greater value, $f(x)$ is positive, that is $1+\sqrt[r]{p}$ is a superior limit of the positive roots of the equation $f(x)=0$.

90. *If each negative coefficient be taken positively and divided by the sum of all the positive coefficients which precede it, the greatest of all the fractions thus formed increased by unity, is a superior limit of the positive roots.*

Let the equation be $f(x)=0$, where $f(x)$ denotes

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} - p_3x^{n-3} + p_4x^{n-4} + \ldots - p_rx^{n-r} + \ldots + p_n.$$

Now we have

$$x^m = (x-1)(x^{m-1} + x^{m-2} + \ldots + x + 1) + 1;$$

let all the terms of the equation with positive coefficients be transformed by means of this formula, and let the others remain unchanged. Thus $f(x)$ becomes

$x^m - 1 = (-$

$$
\begin{aligned}
&p_0(x-1)x^{n-1}+p_0(x-1)x^{n-2}+p_0(x-1)x^{n-3}+\ldots+p_0(x-1)+p_0\\
&\qquad\qquad\quad +p_1(x-1)x^{n-2}+p_1(x-1)x^{n-3}+\ldots+p_1(x-1)+p_1\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad +p_2(x-1)x^{n-3}+\ldots+p_2(x-1)+p_2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -p_3x^{n-3}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad +\ldots
\end{aligned}
$$

Consider now the successive vertical columns of this expression. Where there is no negative coefficient the value of the column is positive if x is greater than unity. To ensure a positive value of the columns in which a negative coefficient occurs we must have

$$(p_0+p_1+p_2)(x-1) \text{ greater than } p_3,$$

$$\ldots\ldots\ldots$$

$$(p_0+p_1+p_2+\ldots+p_{r-1})(x-1) \text{ greater than } p_r,$$

$$\ldots\ldots\ldots$$

Therefore x must be greater than $\dfrac{p_3}{p_0+p_1+p_2}+1,\ldots$ and greater than $\dfrac{p_r}{p_0+p_1+p_2+\ldots+p_{r-1}}+1,\ldots$ Therefore if x be taken equal to the greatest of the expressions thus obtained, that value of x, or any greater value, will make $f(x)$ positive; that is, the greatest of the expressions is a superior limit of the positive roots of the equation $f(x)=0$.

91. We will now illustrate the rules by two examples. First, take the equation

$$x^5+8x^4-14x^3-53x^2+56x-18=0.$$

By Art. 87 we have $53+1$, that is 54, as a superior limit of the positive roots.

By Art. 89, since $n=5$ and $r=2$, we have $1+\sqrt{53}$ as a limit, so that 9 is a limit.

By Art. 90 we have to take the greatest of the following expressions; $\dfrac{14}{1+8}+1$, $\dfrac{53}{1+8}+1$, $\dfrac{18}{1+8+56}+1$, that is, we must take $\dfrac{53}{9}+1$; so that 7 is a limit.

Again, take the equation

$$x^5 - 5x^4 - 13x^3 + 2x^2 + x - 70 = 0.$$

Here Arts. 87 and 89 give $70 + 1$ as a limit; and Art. 90 gives $\frac{70}{4} + 1$, so that 19 is a limit.

Thus, in both these examples, Art. 90 supplies us with the smallest superior limit. It is easy to see that Art. 89 always gives a smaller limit than Art. 87, except when $r = 1$, and then the two limits coincide. Art. 89 is advantageous in general when several positive coefficients occur before the first negative coefficient, so that r is large. Art. 90 always gives a smaller limit than Art. 87, except when the greatest negative coefficient is preceded by only one positive coefficient, namely that of the first term, and then the two limits coincide. Art. 90 is advantageous in general when large positive coefficients occur before the first large negative coefficient.

92. By particular artifices we may frequently obtain a smaller superior limit than the general rules supply.

Consider the first example of the preceding Article. Here we have to find a superior limit of the positive roots of $f(x) = 0$, where $f(x)$ may be written thus,

$$x^2(x^3 - 53) + 8x^3\left(x - \frac{14}{8}\right) + 56\left(x - \frac{9}{28}\right);$$

now if x be equal to 4, or to any greater number, the expressions within the brackets are all positive, and so $f(x)$ is positive. Thus 4 is a superior limit of the positive roots of the equation $f(x) = 0$.

Again, consider the second example of the preceding Article. Here we may write $f(x)$ thus,

$$x^3(x^2 - 5x - 13) + 2x^2 + x - 70;$$

now by the aid of Art. 87 we see that $x^2 - 5x - 13$ is positive if $x = 13 + 1$ or any greater number, and obviously $2x^2 + x - 70$ is positive when $x = 14$ or any greater number. Thus 14 is a superior limit of the positive roots of the equation $f(x) = 0$.

93. We may now easily find an *inferior* limit of the positive roots of an equation, that is a number which is not greater than any of the positive roots. For transform the proposed equation into one whose roots are the reciprocals of the roots of the proposed equation, and then the reciprocal of the superior limit of the positive roots of the transformed equation will be an inferior limit of the positive roots of the proposed equation. Thus suppose the proposed equation to be

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_{n-1} x + p_n = 0;$$

put $\frac{1}{y}$ for x, and multiply by y^n and divide by p_n, so that the transformed equation is

$$y^n + \frac{p_{n-1}}{p_n} y^{n-1} + \dots + \frac{p_2}{p_n} y^2 + \frac{p_1}{p_n} y + \frac{1}{p_n} = 0.$$

Let a superior limit of the positive roots of this equation be found by one of the preceding Articles, and denote it by L; then $\frac{1}{L}$ is an inferior limit of the positive roots of the proposed equation. Suppose that we use Art. 87; let $\frac{p_r}{p_n}$ denote that coefficient which is numerically the greatest of the negative coefficients of the transformed equation; then $1 - \frac{p_r}{p_n}$ is a superior limit of the positive roots of the transformed equation, and therefore $\frac{p_n}{p_n - p_r}$ is an inferior limit of the positive roots of the proposed equation. Here p_r is in fact the numerically greatest among those coefficients of the proposed equation which have the contrary sign to the sign of p_n.

For example, in the first equation of Art. 91 we have $p_n = -18$ and $p_r = 56$; thus $\frac{-18}{-18-56}$, that is $\frac{18}{74}$, is an inferior limit of the positive roots.

94. We will now explain another method of determining a superior limit to the positive roots of an equation; this method is called *Newton's Method.*

Let $f(x)=0$ denote the equation which is to be considered; put $h+y$ for x and expand $f(h+y)$ by Art. 10. Thus the equation becomes

$$f(h)+yf'(h)+\frac{y^2}{2}f''(h)+\ldots+\frac{y^n}{\underline{|n}}f^n(h)=0.$$

Now suppose h positive and of such a value that $f(h)$, $f'(h)$, $f''(h)$,.....$f^n(h)$ are all positive; then no *positive* value of y can satisfy the above equation. But $y=x-h$, and as y cannot be positive, x cannot be greater than h; thus h is a superior limit of the positive roots of the equation $f(x)=0$. We may observe that if the proposed equation is in its simplest form $f^n(h)$ is necessarily positive, being equal to $\underline{|n}$.

95. For example, take the equation

$$x^5+x^4-4x^3-6x^2-700x+500=0.$$

Here
$$f(h)=h^5+h^4-4h^3-6h^2-700h+500,$$
$$f'(h)=5h^4+4h^3-12h^2-12h-700,$$
$$\frac{1}{2}f''(h)=10h^3+6h^2-12h-6,$$
$$\frac{1}{\underline{|3}}f'''(h)=10h^2+4h-4,$$
$$\frac{1}{\underline{|4}}f''''(h)=5h+1.$$

It is convenient to begin with the last function of h and ascend regularly. Any positive value of h makes $f''''(h)$ positive; $h=1$ makes $f'''(h)$ positive; $h=2$ makes $f''(h)$ positive; $h=4$ makes $f'(h)$ positive; $h=5$ makes $f(h)$ positive. Then it will be found that $h=5$ makes all the functions of h positive; and therefore 5 is a superior limit of the positive roots of the proposed equation.

It must be observed, that when according to the method here given we begin with the last function and *increase* the value of h suitably as we ascend to the other functions, we shall not require ever to re-examine the sign of those functions of h which we have passed. For suppose, for example, we have ascertained that a certain value a when put for h renders all the functions of h positive up to $f''(h)$. Then put a greater value for h, say $a+b$; and since

$$f''(a+b)=f''(a)+bf'''(a)+\frac{b^2}{1\,.\,2}f''''(a)+\dots$$

and all the terms on the right-hand side are positive by supposition, $f''(a+b)$ is positive also. Hence in the preceding example, when it was found that $h=5$ rendered $f(h)$ positive, it was unnecessary to try whether this value of h rendered the other functions of h positive, because the method of proceeding ensured this result.

96. To find the limits of the *negative* roots of an equation $f(x)=0$ we put $-y$ for x, and then find the limits of the positive roots of the transformed equation in y; then these limits, with their signs changed, will be limits of the negative roots of the proposed equation.

Take, for example, the equation

$$x^5-7x^4-15x^3+3x^2+4x\div 48=0;$$

put $-y$ for x and we obtain

$$y^5+7y^4-15y^3-3y^2+4y-48=0.$$

By Art. 90 we have $\dfrac{48}{1+7+4}+1$, that is 5, as a superior limit of the positive roots, and by Art. 93 we have $\dfrac{48}{48+7}$ as an inferior limit of the positive roots. Thus the negative roots of the proposed equation must lie between -5 and $-\dfrac{48}{55}$.

97. Having thus shewn how limits may be found between which *all* the real positive roots of an equation must lie, and limits between which *all* the real negative roots of an equation must lie, we proceed to give some theorems with respect to the

situation of the roots taken singly or in groups. It will be seen hereafter that the complete investigation of this part of the subject is involved in *Sturm's Theorem.*

98. *If we substitute successively for* x *in* f(x) *two quantities which include between them an odd number of roots of the equation* f(x) = 0, *we shall obtain results with contrary signs; if we substitute successively two quantities which include between them no root or an even number of roots we shall obtain results with the same sign.*

Suppose λ and μ two quantities of which λ is the greater; let $a, b, c, \ldots, k$, be all the real roots of the equation $f(x) = 0$ which lie between λ and μ; by Art. 43 we have

$$f(x) = (x-a)(x-b)(x-c)\ldots(x-k)\psi(x),$$

where $\psi(x)$ is a function formed of the product of quadratic factors which can never change their sign, and of real factors which cannot change their sign while x lies between λ and μ.

Substitute successively λ and μ for x; thus

$$f(\lambda) = (\lambda-a)(\lambda-b)(\lambda-c)\ldots(\lambda-k)\psi(\lambda),$$

$$f(\mu) = (\mu-a)(\mu-b)(\mu-c)\ldots(\mu-k)\psi(\mu).$$

Now all the factors $\lambda-a$, $\lambda-b$, $\lambda-c, \ldots \lambda-k$, are positive, and all the factors $\mu-a$, $\mu-b$, $\mu-c, \ldots \mu-k$, are negative; and $\psi(\lambda)$ and $\psi(\mu)$ have the same sign. Therefore $f(\lambda)$ and $f(\mu)$ have the *same* sign or *contrary* signs, according as the number of the roots $a, b, c, \ldots, k$, is *even* or *odd.*

99. Hence conversely, if two quantities when substituted for x in $f(x)$ give results with *contrary* signs an *odd* number of the roots of the equation $f(x) = 0$ must lie between the two quantities; if they give results with the *same* sign either no root or an even number of roots must lie between the two quantities.

This result includes that of Art. 19 as a particular case.

100. It is to be observed that the demonstration in Art. 98 does not require the roots $a, b, c, \ldots, k$, to be all *unequal;* only

it must be remembered that a root repeated m times is to be counted as m roots.

We see that if $f(\lambda)$ and $f(\mu)$ be of the same sign, either no root of the equation $f(x)=0$ lies between λ and μ, or else an even number of roots. Now in the preceding Articles of the present Chapter an argument of the following kind has been sometimes used; the value μ or any greater value of x makes $f(x)$ positive, therefore μ is a superior limit of the positive roots of the equation $f(x)=0$. It must be observed that by the words *makes* $f(x)$ *positive*, we mean makes $f(x)$ a *positive quantity and not zero*. For example, if $f(x)=(x-4)^2(x-1)$, then if x is greater than unity $f(x)$ cannot become negative; but we must not infer that unity is a superior limit of the positive roots, for 4 is a root.

If then we only know that $f(x)$ cannot become negative for any value of x greater than μ, we cannot infer that there is no root greater than μ; but we may infer that there is either *no* root or else a root or roots each repeated an *even* number of times.

101. We shall now investigate an important theorem which furnishes relations between the roots of the equation $f(x)=0$ and the roots of the equation $f'(x)=0$, where $f'(x)$ is the first derived function of $f(x)$. The theorem is sometimes called by the name of *Rolle*, who first used it.

102. *A real root of the equation* $\mathrm{f}'(\mathrm{x})=0$ *lies between every adjacent two of the real roots of the equation* $\mathrm{f}(\mathrm{x})=0$.

Let the real roots of the equation $f(x)=0$ arranged in descending order of algebraical magnitude be denoted by $a, b, c, \ldots k$. Let $\phi(x)$ be the product of the quadratic factors corresponding to the imaginary roots of the equation $f(x)=0$, so that $\phi(x)$ cannot change its sign. Then by Art. 43

$$f(x)=(x-a)(x-b)(x-c)\ldots(x-k)\phi(x).$$

In this identity put $y+z$ for x; thus

$$f(y+z)=(y+z-a)(y+z-b)(y+z-c)\ldots(y+z-k)\,\phi(y+z).$$

Suppose each member of this identity expanded in a series proceeding according to ascending powers of z. The coefficient of z on the left-hand side will be $f'(y)$; see Art. 10. The coefficient of z on the right-hand side will be

$$\left\{(y-b)(y-c)\dots(y-k)+(y-a)(y-c)\dots(y-k)+\dots\right\}\phi(y)$$
$$+(y-a)(y-b)(y-c)\dots(y-k)\phi'(y).$$

By equating these coefficients of z, and changing y into x in the resulting identity, we have

$$f'(x)=\left\{(x-b)(x-c)\dots(x-k)+(x-a)(x-c)\dots(x-k)+\dots\right\}\phi(x)$$
$$+(x-a)(x-b)(x-c)\dots(x-k)\phi'(x).$$

Now put successively $a, b, c, \dots, k$, for x; the last term on the right-hand side of the identity vanishes in every case, and therefore the sign of $f'(a)$ is the same as the sign of $(a-b)(a-c)\dots(a-k)$, the sign of $f'(b)$ is the same as the sign of $(b-a)(b-c)\dots(b-k)$, the sign of $f'(c)$ is the same as the sign of $(c-a)(c-b)\dots(c-k)$, and so on; and these signs are alternately positive and negative, for the first expression has *no* negative factor, the second expression has *one* negative factor, the third expression has *two* negative factors, and so on. Hence by Art. 99 an odd number of the roots of the equation $f'(x)=0$ lies between every adjacent two of the roots of the equation $f(x)=0$.

103. The demonstration of the preceding Article implies that the roots $a, b, c, \dots k$, are all unequal. Suppose however that the root a is repeated r times, that the root b is repeated s times, that the root c is repeated t times, and so on. We shall have

$$f(x)=(x-a)^r(x-b)^s(x-c)^t\dots\phi(x),$$

$$f'(x)=\phi(x)\left\{r(x-a)^{r-1}(x-b)^s(x-c)^t\dots+s(x-a)^r(x-b)^{s-1}(x-c)^t\dots+\dots\right\}$$
$$+(x-a)^r(x-b)^s(x-c)^t\dots\phi'(x).$$

Let $f_1(x)$ denote the greatest common measure of $f(x)$ and $f'(x)$, that is, let $f_1(x) = (x-a)^{r-1}(x-b)^{s-1}(x-c)^{t-1}\dots$ Then

$$\frac{f'(x)}{f_1(x)} = \phi(x)\left\{r(x-b)(x-c)\dots + s(x-a)(x-c)\dots + \dots\right\}$$
$$+ (x-a)(x-b)(x-c)\dots\phi'(x).$$

Call this expression $F(x)$; then as before we see that the equation $F(x)=0$ has an odd number of roots between a and b, an odd number between b and c, and so on. And since we have $f'(x) = f_1(x)F(x)$, whenever $F(x)$ vanishes so also does $f'(x)$. Thus an odd number of the roots of the equation $f'(x)=0$ lies between every adjacent two *unequal* roots of the equation $f(x)=0$.

With respect to the *equal* roots of the equation $f(x)=0$, we know that the root a which is repeated r times in the equation $f(x)=0$ is repeated $r-1$ times in the equation $f'(x)=0$; similarly the root b which is repeated s times in the equation $f(x)=0$ is repeated $s-1$ times in the equation $f'(x)=0$: and so on.

It will be convenient for us to imagine that the r roots equal to a of the equation $f(x)=0$ include $r-1$ intervals, in each of which a root a occurs of the equation $f'(x)=0$; and similarly for the other repeated roots. With this conception we may regard the enunciation of Art. 102 as holding universally, whether the roots of the equation $f(x)=0$ are all unequal or not.

104. No more than one root of the equation $f(x)=0$ can lie between any adjacent two of the roots of the equation $f'(x)=0$. For if there could be more than one there would be a root or roots of the equation $f'(x)=0$ comprised between them, and so the two roots of the equation $f'(x)=0$ which were by supposition adjacent would not be adjacent.

And similarly the equation $f(x)=0$ cannot have more than one root greater than the greatest root of the equation $f'(x)=0$, or more than one root less than the least root of the equation $f'(x)=0$.

If the equation $f(x)=0$ has all its roots real, so also has the equation $f'(x)=0$; for the latter equation is of a degree lower than the former by unity, and a root of the latter equation exists between each adjacent two of the roots of the former equation. And generally if the equation $f(x)=0$ has m real roots the equation $f'(x)=0$ has certainly $m-1$ real roots, and may have more.

105. Since $f''(x)$ is the first derived function of $f'(x)$, the equation $f''(x)=0$ has an odd number of roots between every two adjacent roots of the equation $f'(x)=0$. Thus if the equation $f(x)=0$ has m real roots, the equation $f'(x)=0$ has at least $m-1$ real roots, and the equation $f''(x)=0$ has at least $m-2$ real roots. Proceeding in this way we arrive at the result that if the equation $f(x)=0$ has m real roots, the equation $f^r(x)=0$ has at least $m-r$ real roots.

Hence if the equation $f^r(x)=0$ has μ imaginary roots, the equation $f(x)=0$ has at least μ imaginary roots. For if the equation $f(x)=0$ had less than μ imaginary roots it would have more than $n-\mu$ real roots, supposing n the degree of the equation; thus the equation $f^r(x)=0$ would have more than $n-\mu-r$ real roots, and as this equation is of the degree $n-r$ it could not have so many as μ imaginary roots, which is contrary to the supposition.

For example, let $f(x)=x^n(1-x)^n$.

The equation $f(x)=0$ has all its roots real, namely, n equal to zero, and n equal to unity. Hence the equation $f^n(x)=0$ will have all its n roots real and all lying between 0 and 1; this equation is

$$0=1-\frac{n}{1}\frac{n+1}{1}x+\frac{n(n-1)}{1.2}\frac{(n+1)(n+2)}{1.2}x^2-\ldots\ldots$$

106. From Art. 105 we may deduce the following simple test, which will often indicate the existence of imaginary roots in an equation.

Let p_{r-1}, p_r, *and* p_{r+1} *be the coefficients of three consecutive terms in* $f(x)$, *then if* p_r^2 *is less than* $p_{r-1}p_{r+1}$ *there must be a pair of imaginary roots in the equation* $f(x)=0$.

Take the $(n-r-1)^{\text{th}}$ derived function of $f(x)$ and equate it to zero; thus

$$\frac{p_0 x^{r+1}\,|\underline{n}}{|\underline{r+1}} + \ldots + \frac{p_{r-1}x^2\,|\underline{n-r+1}}{1\,.\,2} + p_r x\,|\underline{n-r} + p_{r+1}\,|\underline{n-r-1} = 0.$$

Put $\dfrac{1}{y}$ for x, and multiply by y^{r+1}, and divide by $p_{r+1}\,|\underline{n-r-1}$; thus

$$y^{r+1} + \frac{(n-r)p_r}{p_{r+1}}\,y^r + \frac{(n-r+1)(n-r)}{1\,.\,2}\,\frac{p_{r-1}}{p_{r+1}}\,y^{r-1} + \ldots = 0.$$

If the roots of this equation are all real the sum of their squares is positive; and therefore, by Art. 47,

$$\frac{(n-r)^2 p_r^2}{p_{r+1}^2} - \frac{(n-r+1)(n-r)\,p_{r-1}}{p_{r+1}}$$

is positive. Therefore

$$p_r^2 \text{ is greater than } \frac{n-r+1}{n-r}\,p_{r-1}\,p_{r+1},$$

and *a fortiori*

$$p_r^2 \text{ is greater than } p_{r-1}\,p_{r+1}.$$

If then this condition does not hold there must be a pair of imaginary roots in the derived equation, and therefore also in the original equation. See also Art. 331.

107. If we know all the real roots of the equation $f'(x) = 0$ we can determine how many real roots the equation $f(x) = 0$ has. For let the roots of the equation $f'(x) = 0$ be α, β, $\gamma, \ldots, \kappa$, arranged in descending order of algebraical magnitude. Substitute for x in $f(x)$ successively α, β, $\gamma, \ldots, \kappa$, and observe the signs of the results. Then *one* root or *no* root of the equation $f(x) = 0$ lies between any adjacent two substituted values, according as the corresponding results have *contrary* signs or the *same* sign. This follows from Arts. 98 and 104.

The equation $f(x)=0$ has one root algebraically greater than α, or none, according as $f(\alpha)$ is negative or positive; and it has one root algebraically less than κ if the equation be of an even degree and $f(\kappa)$ be negative, or if the equation be of an odd degree and $f(\kappa)$ be positive, otherwise not. See Arts. 98 and 104.

Hence the number of real roots of the equation $f(x)=0$ will be the same as the number of *changes of sign* in the series obtained by substituting $+\infty$, α, β, γ,... κ, $-\infty$, for x in $f(x)$ successively. If however $f(x)$ *vanishes* when any of the substitutions are made, it indicates that the equation $f(x)=0$ has *equal* roots, and the number of these may be discovered by Chap. VI.

108. As an example we will investigate the conditions that the equation $x^3-qx+r=0$ may have all its roots possible, supposing q a positive quantity. Here $f'(x)=3x^2-q$, so that the roots of the equation $f'(x)=0$ are $\pm\sqrt{\left(\frac{q}{3}\right)}$; let $\alpha=+\sqrt{\left(\frac{q}{3}\right)}$ and $\beta=-\sqrt{\left(\frac{q}{3}\right)}$.

$$\text{Then}\qquad f(\alpha)=+\left(\frac{q}{3}\right)^{\frac{3}{2}}-q\left(\frac{q}{3}\right)^{\frac{1}{2}}+r=-2\left(\frac{q}{3}\right)^{\frac{3}{2}}+r,$$

$$f(\beta)=-\left(\frac{q}{3}\right)^{\frac{3}{2}}+q\left(\frac{q}{3}\right)^{\frac{1}{2}}+r=2\left(\frac{q}{3}\right)^{\frac{3}{2}}+r.$$

First suppose $\left(\frac{r}{2}\right)^2$ greater than $\left(\frac{q}{3}\right)^3$; then if r be positive $f(\alpha)$ and $f(\beta)$ are both positive, and the equation $f(x)=0$ has only one real root, which is algebraically less than β; if r be negative $f(\alpha)$ and $f(\beta)$ are both negative, and the equation $f(x)=0$ has only one real root, which is greater than α.

Next suppose $\left(\frac{r}{2}\right)^2$ less than $\left(\frac{q}{3}\right)^3$; then $f(\alpha)$ is negative and $f(\beta)$ is positive, and the equation $f(x)=0$ has three real roots, namely one greater than α, one between α and β, and one algebraically less than β.

109. A method of discovering the situation of the real roots of an equation was indicated by Waring, and reproduced by Lagrange, which we shall now explain; it is called *Waring's Method of separating the Roots.*

Let us suppose that the equal roots of an equation, if it has any, have been discovered and the corresponding factors removed, so that we have to deal with an equation which has only unequal roots. Let $f(x)=0$ represent this equation. Suppose k to be a quantity which is less than the difference of any two roots, and let s be a superior limit to the positive roots. Substitute for x in $f(x)$ successively s, $s-k$, $s-2k$, $s-3k$,... and so on down to a quantity which is algebraically less than the least root which the equation can have; and observe the series of the signs of the results. Then when a *change* of sign occurs one root exists between the two corresponding substituted values, and when there is a *continuation* of sign no root exists in that interval. For since k is less than the difference of any two of the roots we are sure that more than one root cannot occur in each interval.

We have then to consider how the quantity k may be determined. Suppose that the equation has been formed which has for its roots the squares of the differences of the roots of the proposed equation, and that an inferior limit of the positive roots of this equation has been found; denote this by δ. Then $\sqrt{\delta}$ is a suitable value for k.

We have already in Art. 60 given an example of the construction of an equation which has for its roots the squares of the differences of the roots of a proposed equation, and we shall hereafter consider the question generally: see Chapter XX. It will then be found that on account of the complexity of the result obtained, Waring's method of separating the roots of a proposed equation is generally useless in practice for equations of a degree higher than the third, although theoretically it attains its proposed object.

110. As an example of Waring's method take the equation

$$x^3 - 3x^2 - 4x + 13 = 0.$$

By Art. 60 the equation which has for its roots the squares of the differences of the roots of the proposed equation is

$$y^3 - 42y^2 + 441y - 49 = 0.$$

Put $y = \frac{1}{z}$; thus $49z^3 - 441z^2 + 42z - 1 = 0$,

that is, $$49z^2(z-9) + 42\left(z - \frac{1}{42}\right) = 0;$$

thus 9 is a superior limit to the values of z, and therefore $\frac{1}{9}$ is an inferior limit to the values of y. Hence $\sqrt{\frac{1}{9}}$, that is, $\frac{1}{3}$, is less than the difference of any two roots of the proposed equation.

Now $4+1$, that is 5, is a superior limit of the positive roots of the proposed equation, by Art. 87. And $-(1+\sqrt{13})$ is numerically a superior limit to the negative roots, by Arts. 96 and 89. Thus all the roots of the proposed equation lie between 5 and -5. By substituting in succession for x the values 5, $5-\frac{1}{3}$, $5-\frac{2}{3}$,... it will be found that one root lies between 3 and $2\frac{2}{3}$, one root between $2\frac{2}{3}$ and $2\frac{1}{3}$, and one root between -2 and $-2\frac{1}{3}$.

111. We will conclude this Chapter with a proposition which may serve as an example of some of the principles already established. In the equation $f(x) = 0$,

where $$f(x) = p_0 x^n + p_1 x^{n-1} + \ldots + x - r,$$

if q is the numerical value of the numerically greatest coefficient, and r is positive and less than $\frac{1}{2+4q}$, there is a real positive root less than $2r$.

When x is zero $f(x)$ is negative. Now a positive value of x will make $f(x)$ positive, *a fortiori*, if it make

$$x - r - q(x^n + x^{n-1} + \ldots + x^3 + x^2)$$

positive, that is, if it make $x - r - qx^2 \frac{1-x^{n-1}}{1-x}$ positive.

Hence *a fortiori* $f(x)$ is positive if x is less than unity and $(1-x)(x-r)-qx^2$ is positive. Now put $2r$ for x in the last expression and it becomes $r\{1-2r-4qr\}$, and this is positive because by supposition $r(2+4q)$ is less than unity. Thus $f(x)$ is positive when $x=2r$; and $f(x)$ is negative when $x=0$; therefore a root of the equation $f(x)=0$ lies between 0 and $2r$.

In like manner if the last term in $f(x)$ is r instead of $-r$ and r is positive and less than $\frac{1}{2+4q}$ the equation $f(x)=0$ has a root between 0 and $-2r$.

VIII. COMMENSURABLE ROOTS.

112. By a *commensurable root* is meant a root which can be expressed exactly in a finite form, whole or fractional; so that it involves no irrational quantities. We shall now shew that when the coefficients of an equation are rational numbers, whole or fractional, the commensurable roots of the equation can easily be found.

We have seen in Art. 53 that if the coefficients of an equation are rational but not all integers, we can transform the equation into another which has all its coefficients integers and the coefficient of its first term unity. We may therefore confine ourselves to equations of the latter form; and we shall first shew that equations of that form cannot have rational fractional roots.

113. *If the coefficients of an equation are whole numbers, and the coefficient of its first term unity, the equation cannot have a rational fractional root.*

Let the equation be

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-2}x^2+p_{n-1}x+p_n=0,$$

and if possible suppose it to have a rational fractional root which in its lowest terms is expressed by $\frac{a}{b}$. Substitute this value for x, and multiply all through by b^{n-1}; thus

$$\frac{a^n}{b}+p_1a^{n-1}+p_2a^{n-2}b+\ldots+p_{n-2}a^2b^{n-3}+p_{n-1}ab^{n-2}+p_nb^{n-1}=0,$$

and therefore

$$-\frac{a^n}{b}=p_1a^{n-1}+p_2a^{n-2}b+\ldots+p_{n-2}a^2b^{n-3}+p_{n-1}ab^{n-2}+p_nb^{n-1}.$$

The last result is impossible because the right-hand member of the equation is an *integer*, and the left-hand member is *not an integer*. Therefore $\frac{a}{b}$ cannot be a root of the proposed equation.

114. Thus we are only concerned with the investigation of *integral* commensurable roots, and we shall now explain the method by which they may be found. The method is sometimes called the *Method of divisors*, and sometimes *Newton's Method.*

Let the equation be

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-2}x^2+p_{n-1}x+p_n=0,$$

and suppose a an integral root. Then substituting and writing the terms in the reverse order we have

$$p_n+p_{n-1}a+p_{n-2}a^2+\ldots+p_2a^{n-2}+p_1a^{n-1}+a^n=0,$$

and therefore by division by a

$$\frac{p_n}{a}+p_{n-1}+p_{n-2}a+\ldots+p_2a^{n-3}+p_1a^{n-2}+a^{n-1}=0.$$

Hence $\frac{p_n}{a}$ must be an integer; denote it by q_1 and divide again by a; thus

$$\frac{q_1+p_{n-1}}{a}+p_{n-2}+\ldots+p_2a^{n-4}+p_1a^{n-3}+a^{n-2}=0.$$

Hence $\frac{q_1+p_{n-1}}{a}$ must be an integer; denote it by q_2 and divide again by a, and we shall find that $\frac{q_2+p_{n-2}}{a}$ must be an integer. Proceeding in this way after dividing n times by a we shall arrive at a result denoted by $\frac{q_{n-1}+p_1}{a}+1=0.$

Hence the following conditions are *necessary* in order that the integer a may be a root of the equation $f(x) = 0$.

The last term of the equation must be divisible by a. Add to the quotient thus obtained the coefficient of x in the equation; the sum must be divisible by a. Add to the quotient thus obtained the coefficient of x^2 in the equation; the sum must be divisible by a. Proceed in this way until $n-1$ divisions have been effected, add to the quotient the coefficient of x^{n-1}; the sum must be divisible by a and the quotient must be -1.

If at any step the required condition is not satisfied the integer a is not a root.

115. We have in the preceding Article found the conditions which are *necessary* in order that the integer a may be a root of the equation $f(x) = 0$; it is easy to see that if the last of these conditions is satisfied the integer a is a root. For that last condition may be expressed thus;

$$\frac{p_n}{a^n} + \frac{p_{n-1}}{a^{n-1}} + \frac{p_{n-2}}{a^{n-2}} + \dots + \frac{p_2}{a^2} + \frac{p_1}{a} = -1,$$

and if this is true we see by multiplying by a^n that a is a root of $f(x) = 0$.

In order then to find all the commensurable roots of an equation we have only to determine all the divisors of the last term, and try whether they satisfy the conditions of Art. 114. The labour will often be lessened by first finding positive and negative limits of the roots, because of course no integer need be tried which does not fall within these limits.

116. For an example take the equation

$$x^3 - 3x^2 - 8x - 10 = 0.$$

Here $1+10$ is a superior limit of the positive roots, by Art. 87; and by writing $-y$ for x we obtain the equation

$$y^3 + 3y\left(y - \frac{8}{3}\right) + 10 = 0,$$

for which 3 is a superior limit of the positive roots. Hence all the roots of the proposed equation lie between 11 and -3. The divisors of -10 which fall between these limits are 10, 5, 2, 1, -1, -2; and we proceed to try if any of these numbers are roots.

$+10$	$+5$	$+2$	$+1$	-1	-2
-1	-2	-5	-10	10	5
-9	-10	-13	-18	2	-3
	-2		-18	-2	
	-5		-21	-5	
	-1		-21	$+5$	

In the first line all the divisors of the last term are written which it is necessary to try, and beneath each divisor the results are placed which arise from carrying on the trial with that divisor. Thus taking the divisor 10, we first divide the last term -10 by it, and set down the quotient -1; then we add this to the coefficient of x which is -8, and set down the sum -9; *this is not divisible* by 10, so that 10 is not a root. With respect to 5 all the conditions are fulfilled, so that 5 is a root. With respect to $+2$ and -2 we arrive at points where exact division is not possible, so that these numbers are not roots. With respect to $+1$ and -1 the final condition is not satisfied, so that these numbers are not roots.

Thus the only commensurable root is 5; and denoting the equation by $f(x)=0$, we know that $x-5$ is a factor of $f(x)$. The other factor will be found to be x^2+2x+2.

For another example take the equation

$$x^5+5x^4+x^3-16x^2-20x-16=0.$$

It will be found that the commensurable roots are 2, -2, and -4.

117. It is usual to omit $+1$ and -1 from the divisors to be tried, as it is simpler to test whether these values are roots by substituting them for x in the given equation.

If any powers of x are missing from the proposed equation they should be supposed to be introduced with zero coefficients; see Art. 51.

When we have ascertained by the method here exemplified that certain numbers a, b, c,..., are the only commensurable roots of an equation $f(x)=0$, it still remains to determine whether any of these roots are repeated. We may divide $f(x)$ by the product $(x-a)(x-b)(x-c)\ldots$ and denoting the quotient by $\phi(x)$ we may apply the method to the equation $\phi(x)=0$, and thus determine whether any of the quantities a, b, c,... are roots of this equation. Proceeding in this way we shall determine the repeated roots of the equation $f(x)=0$, and how often each root is repeated.

Or we may apply the test of equal roots found in Chapter VI. to the equation $f(x)=0$.

118. Suppose that instead of taking an equation, with *unity* for the coefficient of the first term, as in Art. 114, we take an equation with *any integer* p_0 for the coefficient of the first term. The only difference in the resulting conditions is that the last quotient must be $-p_0$ and not -1. Suppose for example

$$2x^3-12x^2+13x-15=0.$$

Here $\dfrac{15}{2}+1$ is a superior limit of the positive roots by Art. 87, and there is no negative root by Art. 24, and by trial we see that 1 is not a root; thus the only divisors of the last term to be used are 5 and 3. The process being arranged as before we have

$$\begin{array}{rr} 5 & 3 \\ -3 & -5 \\ 10 & 8 \\ 2 & \\ -10 & \\ -2 & \end{array}$$

Thus 5 is a root, for all the conditions are satisfied, the last quotient being -2; and 3 is not a root, because 8 is not divisible by 3.

It must be remembered that if the coefficient of the first term is not unity the equation may have a commensurable fractional root; see Art. 113.

119. The number of divisors of the last term which it is necessary to try may sometimes be diminished by the following principle. Suppose a a root of the equation $f(x)=0$; for x put $m+y$, then $a-m$ is a value of y which satisfies the equation $f(m+y)=0$. The term independent of y in this equation is $f(m)$, and all the coefficients of y are integers, if the coefficients in $f(x)$ are integers and m also an integer; see Art. 10. Thus if a be an integer $a-m$ is an integer and must therefore divide $f(m)$ by Art. 114. Thus any integer a which divides the last term of $f(x)$ is to be rejected if $a-m$ does not divide $f(m)$.

Here m may be any integer positive or negative; the values $+1$ and -1 are advantageous from the ease with which $f(m)$ can then be calculated.

Take for example the second equation given in Art. 116; here 4 divides the last term, but $4+1$ does not divide $f(-1)$ which is -9; thus 4 cannot be a root of the proposed equation.

Again, take the example $x^3-20x^2+164x-400=0$. This equation has no negative root by Art. 24; and by writing it in the form $x^2(x-20)+164\left(x-\frac{100}{41}\right)$, we see that 20 is a superior limit of the positive roots. The positive divisors of the last term which are less than 20 are 2, 4, 5, 8, 10, and 16. Of these 5, 8, and 10 are not roots; for $f(1)=-255$, and this is not divisible by $5-1$, or by $8-1$, or by $10-1$. Thus the only divisors of the last term which remain for trial are 2, 4, and 16; it will be found that 4 is a root.

120. As an example of a rational fractional root, consider the equation $4x^4-11x^2+7x-6=0$, that is,

$$x^4-\frac{11}{4}x^2+\frac{7}{4}x-\frac{3}{2}=0.$$

First, put $x=\frac{y}{2}$, in order to transform the equation into one with integral coefficients; see Art. 53. Thus

$$y^4-11y^2+14y-24=0,$$

that is, $$y^4+0y^3-11y^2+14y-24=0.$$

By Arts. 89 and 96 all the roots of this equation must lie between $1+\sqrt{24}$ and $-(1+\sqrt{24})$; and we see by trial that $+1$ and -1 are not roots. Thus the only divisors of the last term to be tried are $4, 3, 2, -2, -3, -4$. Also $f(1)=-20$, and this is not divisible by $4-1$ or by $-2-1$; thus the numbers 4 and -2 may be rejected. The process being arranged as before we have

3	2	−3	−4
−8	−12	8	6
6	2	22	20
2	1		−5
−9	−10		−16
−3	−5		4
−3			4
−1			−1

Thus 3 and -4 are roots; and since $x=\frac{y}{2}$, we have $\frac{3}{2}$ and -2 as roots of the original equation.

IX. OF THE DEPRESSION OF EQUATIONS.

121. In the present Chapter we shall shew how the solution of an equation may be made to depend upon the solution of an equation of lower degree, in certain cases where known relations subsist among the roots; this process is called the *depression of equations.*

122. *When two equations have a root or roots in common, it is required to determine the root or roots.*

Suppose the equations $f(x)=0$ and $F(x)=0$ to have a common root a; then $f(x)$ and $F(x)$ have the common factor $x-a$. Hence the greatest common measure of $f(x)$ and $F(x)$ must have $x-a$ as a factor. Similarly every factor common to $f(x)$ and $F(x)$ will be a factor of their greatest common measure, and no other factors will occur in the greatest common measure.

Hence, if we find the greatest common measure of $f(x)$ and $F(x)$, and equate it to zero, the roots of this equation will coincide with the required roots which are common to the equations $f(x)=0$ and $F(x)=0$.

If any factor is repeated in $f(x)$ and $F(x)$ it will also be repeated in their greatest common measure.

123. Suppose, for example, we have the two equations

$$x^4+3x^3-5x^2-6x-8=0$$

$$\text{and } x^4+x^3-9x^2+10x-8=0.$$

The greatest common measure of the expressions which form the left-hand members of these equations is x^2+2x-8; and if this be put equal to zero we obtain $x=-4$, or $x=2$. Thus 2 and -4 are the roots common to the two equations.

124. Suppose we know that there exists between a and b, two roots of the equation $f(x)=0$, the relation $pa+qb=r$; it is required to determine these roots.

Since a and b are roots of the equation $f(x)=0$, we have $f(a)=0$, and $f(b)=0$; but $b=\dfrac{r-pa}{q}$, therefore $f\left(\dfrac{r-pa}{q}\right)=0$. Thus a is a common root of the equations $f(x)=0$ and $f\left(\dfrac{r-px}{q}\right)=0$. Hence a may be found by the preceding Article. Thus a is known and then b from the relation $pa+qb=r$. Hence $f(x)$ may be divided by the product of the factors $x-a$ and $x-b$; and if the quotient be equated to zero we obtain an equation for determining the remaining roots of the equation $f(x)=0$.

125. Suppose, for example, that we have the equation

$$x^4 - 7x^3 + 11x^2 - 7x + 10 = 0 \quad \text{.................... (1)},$$

and that it is known that two of its roots a and b are connected by the relation $b = 2a + 1$.

Substitute $2x + 1$ for x in (1); thus

$$(2x+1)^4 - 7(2x+1)^3 + 11(2x+1)^2 - 7(2x+1) + 10 = 0,$$

that is $$16x^4 - 24x^3 - 16x^2 - 4x + 8 = 0,$$

or $$4x^4 - 6x^3 - 4x^2 - x + 2 = 0 \quad \text{.......................... (2)}.$$

The greatest common measure of the left-hand members of (1) and (2) will be found to be $x - 2$. Thus $a = 2$, and therefore $b = 5$; that is, 2 and 5 are two of the roots of the proposed equation. Then it will be found that

$$x^4 - 7x^3 + 11x^2 - 7x + 10 = (x-2)(x-5)(x^2+1),$$

so that the other roots are $\pm\sqrt{(-1)}$.

126. It may happen that another pair of roots α and β is subject to the same condition $p\alpha + q\beta = r$. In this case the expressions $f(x)$ and $f\left(\frac{r - px}{q}\right)$ will have for their greatest common measure an expression of the second degree in x which will involve the factors $x - a$ and $x - \alpha$.

If the roots a and b are both repeated in the equation $f(x) = 0$, the factor $x - a$ will be repeated in the greatest common measure of $f(x)$ and $f\left(\frac{r - px}{q}\right)$.

127. Generally suppose that two roots a and b of the equation $f(x) = 0$ are connected by the relation $b = \phi(a)$. Then the equations $f(x) = 0$ and $f\{\phi(x)\} = 0$ have a common root, namely a, and we may determine this common root by Art. 122.

128. There is a case in which the method of Arts. 124 and 126 does not assist us in solving a proposed equation. Suppose,

for example, we have an equation $f(x)=0$, and it is known that the roots of this equation occur in pairs, and that *each* pair of roots a and b satisfies the relation $a+b=2r$. Then according to Art. 124 we should proceed to investigate the common roots of the equations $f(x)=0$ and $f(2r-x)=0$. But these equations will be found to coincide completely; for by supposition $f(a)=0$, that is, $f(2r-b)=0$, and $f(b)=0$, that is, $f(2r-a)=0$, so that the roots a and b are common to the two equations. Similarly every other pair of roots is common to the two equations, and so the two equations must coincide.

129. There are various ways in which we may *depress* the equation in the case considered in the preceding Article; we will explain two of them as they furnish exercises on the subject of the present Chapter.

I. We may proceed thus. Assume $a-b=2z$, so that we have simultaneously

$$f(a)=0, \quad a+b=2r, \quad a-b=2z.$$

From the second and third of these equations $a=z+r$. Substitute in the first equation, so that $f(z+r)=0$. From this equation values of z must be found, and then corresponding values of a and b. It is easy to shew that the equation $f(r+z)=0$ only involves *even* powers of z, and so if we regard z^2 as the unknown quantity the degree of this equation will be half the degree of the proposed equation. For let a and b be one pair of roots of the proposed equation, α and β another pair, and so on; then

$$f(x)=(x-a)(x-b)(x-\alpha)(x-\beta)\ldots$$

$$f(z+r)=(z+r-a)(z+r-b)(z+r-\alpha)(z+r-\beta)\ldots$$

$$=\left(z+\frac{a+b}{2}-a\right)\left(z+\frac{a+b}{2}-b\right)\left(z+\frac{\alpha+\beta}{2}-\alpha\right)\left(z+\frac{\alpha+\beta}{2}-\beta\right)\ldots$$

$$=\left\{z^2-\left(\frac{a-b}{2}\right)^2\right\}\left\{z^2-\left(\frac{\alpha-\beta}{2}\right)^2\right\}\ldots;$$

that is, $f(z+r)$ involves only even powers of z

In fact, as no distinction in theory exists between the roots a and b, it might have been expected that an equation which should be constructed to have $\frac{a-b}{2}$ for a root would also have $\frac{b-a}{2}$ as a root; and such is the case.

II. We may also proceed thus. Assume $z = ab$. Then

$$(x-a)(x-b) = x^2 - (a+b)x + ab = x^2 - 2rx + z.$$

Hence if z be suitably determined, $x^2 - 2rx + z$ will be a factor of $f(x)$. Perform the process of dividing $f(x)$ by $x^2 - 2rx + z$ until the remainder takes the form $Px + Q$, where P and Q are functions of z, but do not contain x. Hence the necessary and sufficient conditions for $x^2 - 2rx + z$ being a factor of $f(x)$ are $P = 0$ and $Q = 0$. Find by Art. 122 a value of z which will satisfy both these equations; then find a and b from

$$a + b = 2r \text{ and } z = ab.$$

130. Suppose we know that between three roots a, b, c of the equation $f(x) = 0$, the relation $pa + qb + rc = s$ exists; it is required to determine these roots.

Since a, b, and c are roots of the equation $f(x) = 0$, we have

$$f(a) = 0, \; f(b) = 0, \; f(c) = 0. \quad \text{Thus}$$

$$f(a) = 0, \; f(b) = 0, \; f\left(\frac{s - pa - qb}{r}\right) = 0.$$

Suppose b eliminated between the last two equations; we thus obtain an equation which we may denote by $\phi(a) = 0$. Thus the equations $f(x) = 0$, and $\phi(x) = 0$ have a common root a, and this may be found by Art. 122.

131. We will here give a few miscellaneous examples connected with the subject of the present Chapter.

(1) It is required to determine the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0,$$

which are all in arithmetical progression.

Denote them by a, $a+b$, $a+2b$, ...

By Art. 47,

$$-p_1 = a + (a+b) + (a+2b) + \ldots + (a + \overline{n-1}b),$$

$$p_1^2 - 2p_2 = a^2 + (a+b)^2 + (a+2b)^2 + \ldots (a + \overline{n-1}b)^2.$$

That is, $$-p_1 = na + \frac{n(n-1)}{2} b,$$

$$p_1^2 - 2p_2 = na^2 + n(n-1)ab + \frac{n(n-1)(2n-1)}{6} b^2;$$

see *Algebra*, Chapter XXX.

By squaring the first result and subtracting it from n times the second we obtain

$$(n-1)p_1^2 - 2np_2 = \frac{n^2(n^2-1)b^2}{12};$$

thus b is known, and then a can be found.

(2) The equation $x^4 + 3x^3 - 12x^2 - 48x - 64 = 0$ has two roots which are equal in magnitude and of opposite signs; find them.

Here the equation obtained by changing the sign of x will have a root in common with the proposed equation. That is, the proposed equation has a root in common with the equation

$$x^4 - 3x^3 - 12x^2 + 48x - 64 = 0.$$

Then by Art. 122 we may proceed to find the greatest common measure of the left-hand members of these equations. Or thus; by subtraction,

$$6x^3 - 96x = 0;$$

therefore either $x = 0$, or else $x^2 = 16$.

The former does not give a root; the latter gives $x = \pm 4$; and $+4$ and -4 are roots of the proposed equation.

(3) The equation $3x^4 - 19x^3 + 9x^2 - 19x + 6 = 0$ has two roots the product of which is 2; find them.

Suppose y to denote one root; then $\frac{2}{y}$ is another; hence

$$3y^4 - 19y^3 + 9y^2 - 19y + 6 = 0 \ldots\ldots\ldots\ldots\ldots(1);$$

$$\text{and } 3\left(\frac{2}{y}\right)^4 - 19\left(\frac{2}{y}\right)^3 + 9\left(\frac{2}{y}\right)^2 - 19\left(\frac{2}{y}\right) + 6 = 0,$$

$$\text{that is, } 6y^4 - 38y^3 + 36y^2 - 152y + 48 = 0,$$

$$\text{or } 3y^4 - 19y^3 + 18y^2 - 76y + 24 = 0 \ldots\ldots\ldots(2).$$

The greatest common measure of the left-hand members of (1) and (2) is $3y^2 - 19y + 6$; and putting this equal to zero we obtain $y = \frac{1}{3}$, or $y = 6$. Thus $\frac{1}{3}$ and 6 are the required roots.

X. RECIPROCAL EQUATIONS.

132. A *reciprocal* equation is one which is not changed when the unknown quantity is changed into its reciprocal. Hence if a be a root of such an equation, the reciprocal of a, that is, $\frac{1}{a}$, is also a root. We shall see that the solution of a reciprocal equation may be made to depend on the solution of an equation of not higher than half the degree of the proposed equation. We shall first determine the relations which must hold among the coefficients of an equation in order that it may be a reciprocal equation, and shall then shew how the equation may be *depressed* and so rendered easier of solution.

133. *To find the conditions that a proposed equation may be a reciprocal equation.*

Let the equation be

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-2} x^2 + p_{n-1} x + p_n = 0 \ldots(1).$$

Change x into $\frac{1}{x}$, then multiply by x^n and divide by p_n, and re-arrange the terms; thus we have

$$x^n + \frac{p_{n-1}}{p_n} x^{n-1} + \frac{p_{n-2}}{p_n} x^{n-2} + \dots + \frac{p_2}{p_n} x^2 + \frac{p_1}{p_n} x + \frac{1}{p_n} = 0 \dots (2).$$

In order that (2) may coincide with (1), the coefficients of the same powers of x must be coincident; thus

$$p_1 = \frac{p_{n-1}}{p_n},\ p_2 = \frac{p_{n-2}}{p_n},\ \dots p_{n-2} = \frac{p_2}{p_n},\quad p_{n-1} = \frac{p_1}{p_n},\quad p_n = \frac{1}{p_n};$$

from the last equation we have $p_n^2 = 1$, therefore $p_n = +1$, or -1, and this gives rise to two classes of reciprocal equations.

I. Suppose $p_n = 1$; then we obtain

$$p_1 = p_{n-1},\quad p_2 = p_{n-2},\ \dots p_{n-2} = p_2,\quad p_{n-1} = p_1.$$

Thus an equation is a reciprocal equation when the coefficients of the terms equidistant from the first and last are equal.

II. Suppose $p_n = -1$; then we obtain

$$p_1 = -p_{n-1},\quad p_2 = -p_{n-2},\ \dots p_{n-2} = -p_2,\quad p_{n-1} = -p_1.$$

In this case if the equation is of an *even* degree, we have among the above series of conditions $p_m = -p_m$, where $m = \frac{1}{2}n$, and this is impossible unless $p_m = 0$. Thus an equation is a reciprocal equation when the coefficients of terms equidistant from the beginning and end are equal in magnitude and of contrary signs; with the condition that if the equation is of an even degree the coefficient of the middle term is zero.

134. A reciprocal equation of the first class of an *odd* degree has a root -1, as is obvious by inspection. Thus if $f(x) = 0$ denote the equation, $f(x)$ is divisible by $x + 1$; see Art. 6. Let $\phi(x)$ be the quotient, then $\phi(x) = 0$ will be a reciprocal equation of an even degree with its last term positive.

A reciprocal equation of the second class of an *odd* degree has a root $+1$, as is obvious by inspection. Thus if $f(x) = 0$ denote the equation, $f(x)$ is divisible by $x - 1$; see Art. 6. Let $\phi(x)$ be the quotient, then $\phi(x) = 0$ will be a reciprocal equation of an even degree with its last term positive.

A reciprocal equation of the second class of an *even* degree has a root $+1$, and a root -1, as is obvious by inspection. Thus, if $f(x)=0$ denote the equation, $f(x)$ is divisible by x^2-1; see Art. 36. Let $\phi(x)$ be the quotient, then $\phi(x)=0$ will be a reciprocal equation of an even degree with its last term positive.

135. The statements made in the preceding Article respecting the results of certain divisions will probably be admitted as obvious. But it is easy to give formal proofs. Consider the last case, that of a reciprocal equation of the second class of an even degree. Suppose $f(x)=0$ to represent the equation; then we know that $f(x)$ is such that $f(x)=-x^n f\left(\frac{1}{x}\right)$, and we know that $f(x)$ is divisible by x^2-1; we wish to prove that the quotient is a function which has the coefficients of the terms equidistant from the first and last equal.

We have $f(x)=-x^n f\left(\frac{1}{x}\right)$;

$$\text{therefore } \frac{f(x)}{x^2-1}=-\frac{x^n}{x^2-1}f\left(\frac{1}{x}\right)=x^{n-2}\frac{f\left(\frac{1}{x}\right)}{1-\frac{1}{x^2}}.$$

And this shews the truth of the statement, since $\dfrac{f\left(\frac{1}{x}\right)}{1-\frac{1}{x^2}}$ is what we obtain when we change x into $\frac{1}{x}$ in $-\dfrac{f(x)}{x^2-1}$.

136. It follows from Art. 134 that any reciprocal equation is either of an even degree with its last term positive, or may be depressed to this form. We may then consider this as the standard form of a reciprocal equation, and we shall now shew that such an equation may be depressed to one of half its degree.

The fact that a reciprocal equation could be thus depressed was noticed by De Moivre in 1718: see his *Doctrine of Chances*, first edition, page 113.

137. *It is required to depress a reciprocal equation which is of an even degree with its last term positive.*

Let the equation be $x^{2m}+p_1x^{2m-1}+p_2x^{2m-2}+\ldots+p_2x^2+p_1x+1=0$. Divide by x^m and collect the terms in pairs which are equidistant from the beginning and end; thus

$$x^m+\frac{1}{x^m}+p_1\left(x^{m-1}+\frac{1}{x^{m-1}}\right)+p_2\left(x^{m-2}+\frac{1}{x^{m-2}}\right)+\ldots=0.$$

Now assume $x+\frac{1}{x}=y$; then

$$x^2+\frac{1}{x^2}=y^2-2,$$

$$x^3+\frac{1}{x^3}=\left(x^2+\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)-y=y^3-3y,$$

....

and generally, $x^{p+1}+\frac{1}{x^{p+1}}=\left(x^p+\frac{1}{x^p}\right)\left(x+\frac{1}{x}\right)-\left(x^{p-1}+\frac{1}{x^{p-1}}\right)$,

so that we can express $x^{p+1}+\frac{1}{x^{p+1}}$ as a rational function of y of the degree $p+1$. Hence by substitution in the above equation we obtain an equation in y of the degree m. Then from each value of y we deduce two corresponding values of x from the equation $x^2-yx+1=0$.

138. The general relation in the preceding Article may be thus expressed;

$$x^{p+1}+\frac{1}{x^{p+1}}=\left(x^p+\frac{1}{x^p}\right)y-\left(x^{p-1}+\frac{1}{x^{p-1}}\right).$$

This shews that we may regard the quantities

$$x+\frac{1}{x},\ x^2+\frac{1}{x^2},\ x^3+\frac{1}{x^3},\ \ldots$$

as forming a recurring series in which the *scale of relation* is $1-y+1$; see *Algebra*, Chapter XLIX. We shall hereafter give in Chapter XXI. a general expression for $x^p+\frac{1}{x^p}$ in terms of y.

139. For an example of a reciprocal equation take the equation

$$2x^6+x^5-13x^4+13x^2-x-2=0.$$

Here $+1$ and -1 are roots by inspection; and we can therefore divide the left-hand member by x^2-1. Thus we obtain

$$2x^4+x^3-11x^2+x+2=0;$$

$$\text{therefore } x^2+\frac{1}{x^2}+\frac{1}{2}\left(x+\frac{1}{x}\right)-\frac{11}{2}=0.$$

$$\text{Put } x+\frac{1}{x}=y;\text{ thus}$$

$$y^2-2+\frac{y}{2}-\frac{11}{2}=0,$$

$$\text{or}\quad y^2+\frac{y}{2}-\frac{15}{2}=0;$$

$$\text{therefore}\quad y=\frac{5}{2}\text{ or }-3.$$

$$\text{Hence}\quad x+\frac{1}{x}=\frac{5}{2},\quad\text{or } x+\frac{1}{x}=-3;$$

$$\text{therefore } x=2\text{ or }\frac{1}{2}\text{ or }\frac{1}{2}(-3\pm\sqrt{5}).$$

140. The following equation may be transformed into a reciprocal equation:

$$x^{2m}+p_1x^{2m-1}+p_2x^{2m-2}+\dots+p_mx^m+p_{m-1}cx^{m-1}+p_{m-2}c^2x^{m-2}$$
$$+\dots+p_2c^{m-2}x^2+p_1c^{m-1}x+c^m=0.$$

For assume $x=z\sqrt{c}$, and divide by c^m; we thus obtain a reciprocal equation in z of the standard form.

XI. BINOMIAL EQUATIONS.

141. An equation of the form $x^n-A=0$ where A is a known quantity is called a *binomial equation.*

The roots of this equation are all different because the first derived function of $x^n - A$ is nx^{n-1}, and no value of x will make $x^n - A$ and nx^{n-1} vanish simultaneously; see Art. 75.

142. If $x^n - A = 0$ we have $x = \sqrt[n]{A}$; that is, x is equal to an n^{th} root of A. But the equation $x^n - A = 0$ has n roots by Art. 33, and these roots are all different by Art. 141. Hence we obtain the following important result, *any algebraical quantity has* n *different* nth *roots.* By an algebraical quantity here we mean either a real quantity, or an imaginary quantity of the form $p + q\sqrt{-1}$.

143. Let a denote one of the n^{th} roots of any quantity A, so that $a^n = A$. Then in the equation $x^n - A = 0$ assume $x = ay$, so that $a^n y^n - A = 0$; therefore $y^n - 1 = 0$. Hence $y = \sqrt[n]{1}$, that is, y is equal to an n^{th} root of unity. And $x = ay = a\sqrt[n]{1}$; but $x = \sqrt[n]{A}$; therefore $\sqrt[n]{A} = a\sqrt[n]{1}$. Thus *all the* nth *roots of any algebraical quantity may be found by multiplying any one of them in succession by the values of the* nth *roots of unity.*

144. Let us now suppose that A is a real positive quantity, and that we have to solve the equation $x^n - A = 0$ and the equation $x^n + A = 0$. Let a be the arithmetical value of the n^{th} root of A, which may always be obtained, at least approximately, by the aid of the Binomial Theorem; see *Algebra*, Chapter XXXVI. Assume $x = ay$, then the proposed equations become respectively $y^n - 1 = 0$, and $y^n + 1 = 0$. These equations can both be solved by the aid of Trigonometry; see *Trigonometry*, Chapter XXIII. We shall however now consider these equations without using the Trigonometrical expressions; and although we are not able to solve them generally by means of algebraical expressions, we shall be able to prove important results respecting them.

145. *If* α *be any root of the equation* $x^n - 1 = 0$, *then* α^m *is also a root, where* m *is any integer, positive or negative.*

For $(\alpha^m)^n = \alpha^{mn} = (\alpha^n)^m = 1^m = 1$.

146. *If* α *be any root of the equation* $x^n+1=0$, *then* α^m *is also a root, where* m *is any odd integer positive or negative.*

For $(\alpha^m)^n=\alpha^{mn}=(\alpha^n)^m=(-1)^m=-1$, if m be odd.

147. *If* m *be prime to* n, *the equations* $x^m-1=0$ *and* $x^n-1=0$ *have no common root except unity.*

Let p and q be two integers which satisfy the relation $pm-qn=1$; such integers can always be found by Algebra; see *Algebra*, Chapter XLVI. And suppose that a is a common root of the two equations. Then $a^m=1$, therefore $a^{pm}=1$; and $a^n=1$, therefore $a^{qn}=1$. Hence, by division, $a^{pm-qn}=1$; that is $a=1$.

148. *If* n *is a prime number, and* α *any root of the equation* $x^n-1=0$, *except unity, then all the roots of the equation will be furnished by the series* $\alpha, \alpha^2, \alpha^3, \ldots \alpha^n$.

For these quantities are all roots by Art. 145. We have therefore only to shew that no two of them are equal. If possible, suppose $\alpha^r=\alpha^s$; then $\alpha^{r-s}=1$; and thus the equations $x^n-1=0$ and $x^{r-s}-1=0$ have a common root which is not unity. But this is impossible by Art. 147, since $r-s$ is less than n and therefore prime to it.

149. If n is not a prime number, and α is any root of the equation $x^n-1=0$, it is true by Art. 145 that any power of α is also a root; but it is not necessarily true that the successive powers of α will furnish all the roots. Suppose for example that $n=pq$; and let α be a root of the equation $x^p-1=0$; then α is also a root of the equation $x^n-1=0$, and so is any power of α. But we cannot obtain more than p different values by taking powers of α; for $\alpha^{p+1}=\alpha^p\times\alpha=\alpha$, $\alpha^{p+2}=\alpha^p\times\alpha^2=\alpha^2$, and so on. Thus the powers of α will not furnish *all* the roots of the equation $x^n-1=0$.

If n be not a prime number it is still true that *some* of the roots of the equation $x^n-1=0$ have the property of furnishing all the roots by their successive powers. This we shall shew from the Trigonometrical expressions for the roots.

For let r be any integer; then

$$\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}$$

is a root; denote it by α. Suppose r prime to n, then the successive powers of α will furnish all the roots.

For let s and t be two integers, neither of which exceeds n; then α^s and α^t will not be equal. For

$$\alpha^s=\cos\frac{2sr\pi}{n}+\sqrt{-1}\sin\frac{2sr\pi}{n},$$

$$\alpha^t=\cos\frac{2tr\pi}{n}+\sqrt{-1}\sin\frac{2tr\pi}{n};$$

and in order that these should be equal $\frac{2sr\pi}{n}$ and $\frac{2tr\pi}{n}$ must either be equal or differ by a multiple of four right angles. See *Plane Trigonometry*, Art. 93. Thus

$$\frac{(s\sim t)\,r}{n}\text{ must be an integer;}$$

but this is impossible since r is prime to n and $s \sim t$ is less than n.

150. *The solution of the equation* $x^n-1=0$ *where* n *is the product of different prime numbers can be made to depend upon the solution of equations of a similar form having for the index of* x *the different prime factors of* n.

Suppose, for example, that n is the product of three prime factors m, p, q. Let α be a root of the equation $x^m-1=0$, let β be a root of the equation $x^p-1=0$, let γ be a root of the equation $x^q-1=0$; these roots being all supposed different from unity. Then the roots of the equation $x^n-1=0$ will be the terms of the product

$$(1+\alpha+\alpha^2+\ldots+\alpha^{m-1})(1+\beta+\beta^2+\ldots+\beta^{p-1})(1+\gamma+\gamma^2+\ldots+\gamma^{q-1}).$$

First, any term of this product is a root. For suppose $\alpha^r\beta^s\gamma^t$ to denote such a term; then $(\alpha^r\beta^s\gamma^t)^n=1$, since $\alpha^{rn}=1$, $\beta^{sn}=1$, and

$\gamma^{tn}=1$. Secondly, no two terms of this product are equal. For, if possible, suppose $\alpha^r\beta^s\gamma^t=\alpha^\rho\beta^\sigma\gamma^\tau$; then $\alpha^{r-\rho}=\beta^{\sigma-s}\gamma^{\tau-t}$. The quantity on the left-hand side is a root of the equation $x^m-1=0$, and the quantity on the right-hand side is a root of the equation $x^{pq}-1=0$; but since m is prime to pq it is impossible that these equations can have any common root except unity.

Similarly we may proceed when n has more than three prime factors.

151. Next suppose that the prime factors of n occur more than once in n; for example, let $n=\mu\,.\,\pi\,.\,\kappa$, where μ, π, κ are respectively any powers of the prime numbers m, p, and q. Then it will still be true that if we obtain the μ roots of the equation $x^\mu-1=0$, the π roots of the equation $x^\pi-1=0$, and the κ roots of the equation $x^\kappa-1=0$, and take every possible product of these roots, one from each system, we shall obtain all the roots of the equation $x^n-1=0$. But, by Art. 149, the roots of each system cannot necessarily be represented by the powers of one root taken arbitrarily.

Similarly we may proceed when n involves more than three different primes.

3 152. It is usual to add one more proposition respecting the equation $x^n-1=0$ when n is a power of a prime; and we will give it here although it is of little practical importance. Suppose, for example, that $n=m^3$ where m is a prime number. Let α be a root of the equation $x^m-1=0$, let β be a root of the equation $x^m-\alpha=0$, and let γ be a root of the equation $x^m-\beta=0$. Then the roots of the equation $x^n-1=0$ will be the terms of the product

$$(1+\alpha+\alpha^2+\ldots+\alpha^{m-1})(1+\beta+\beta^2+\ldots+\beta^{m-1})(1+\gamma+\gamma^2+\ldots+\gamma^{m-1}).$$

First, any term of the product is a root. For suppose $\alpha^r\beta^s\gamma^t$ to denote such a term; then $(\alpha^r\beta^s\gamma^t)^n=\alpha^{rn}\beta^{sn}\gamma^{tn}=1$. Secondly, no two terms of this product are equal. For, if possible, suppose $\alpha^r\beta^s\gamma^t=\alpha^\rho\beta^\sigma\gamma^\tau$; thus $\alpha^l=\alpha^\lambda$, where

$$l=r+\frac{s}{m}+\frac{t}{m^2}\text{ and }\lambda=\rho+\frac{\sigma}{m}+\frac{\tau}{m^2}.$$

Therefore $\alpha^{l-\lambda}=1$, therefore $\alpha^{\nu}-1=0$, where $\nu=m^2(l-\lambda)$. But $m^2(l-\lambda)=m^2(r-\rho)+m(s-\sigma)+t-\tau$, and this is prime to m, and therefore to m^3; and therefore the equations $x^n-1=0$ and $x^{\nu}-1=0$ cannot have a common root different from unity.

153. The preceding Article is of little practical importance, because the operations which it involves cannot be generally effected. Suppose that we can solve the equation $x^m-1=0$, and so find α; then all the quantities $1, \alpha, \alpha^2, \ldots \alpha^{m-1}$, are roots of the equation $x^n-1=0$; so that we thus obtain m roots. But to find β we have to solve the equation $x^m-\alpha=0$, that is, we have to find $\sqrt[m]{\alpha}$ where $\alpha=\sqrt[m]{1}$; and there is no algebraical method of effecting this generally.

Thus, for example, when we have solved the equations $x^3-1=0$ and $x^5-1=0$ we can *immediately* form all the solutions of the equation $x^{15}-1=0$ by Art. 150. But we cannot practically solve the equations $x^9-1=0$ or $x^{25}-1=0$ by the method of Art. 152; we can only obtain three roots of the former equation and five roots of the latter equation.

154. We will now indicate the methods by which we can practically solve the equations $x^n-1=0$ and $x^n+1=0$, when n is not too great.

We may observe however that if n be any power of 2 these equations may be solved by the process given in Algebra for extracting the square root of a binomial surd, repeated as often as is necessary; see Art. 28. If $n=pm$, where $p=2^r$, assume $x^p=y$, thus the equations $x^n-1=0$ and $x^n+1=0$ become respectively $y^m-1=0$ and $y^m+1=0$. Then if y can be found we can deduce x by the process of extracting the square root repeated r times.

155. In the equation $x^n-1=0$ suppose that n is an odd number, and let $n=2m+1$. The equation $x^{2m+1}-1=0$ has only one real root, namely $+1$; for it has no negative root, and if x be made equal to any other quantity than unity x^{2m+1} will not be equal to unity; thus the equation has only one real root. Divide $x^{2m+1}-1$

by $x-1$; thus we reduce the equation to be solved to the following,

$$x^{2m}+x^{2m-1}+x^{2m-2}+\ldots+x^2+x+1=0.$$

This is a *reciprocal* equation, and its solution can be made to depend upon the solution of an equation of the degree m.

156. In the equation $x^n-1=0$ suppose that n is an even number, and let $n=2m$. The only real roots of the equation are $+1$ and -1; and we may divide $x^{2m}-1$ by the product of $x-1$ and $x+1$, that is, by x^2-1. Thus we reduce the equation to be solved to the following,

$$x^{2m-2}+x^{2m-4}+\ldots+x^2+1=0.$$

This is a *reciprocal* equation, and its solution can be made to depend upon the solution of an equation of the degree $m-1$.

The equation $x^{2m}-1=0$ may also be conveniently treated by writing it thus, $(x^m-1)(x^m+1)=0$, and so resolving it into the equations $x^m-1=0$ and $x^m+1=0$. Or we may adopt the method given in Art. 154.

157. In the equation $x^n+1=0$, suppose that n is an odd number, and let $n=2m+1$. The equation $x^{2m+1}+1=0$ has only one real root, namely -1; and we may divide $x^{2m+1}+1$ by $x+1$, and thus reduce the equation to be solved to the following,

$$x^{2m}-x^{2m-1}+x^{2m-2}-\ldots+x^2-x+1=0;$$

this is a *reciprocal* equation, and its solution can be made to depend upon the solution of an equation of the degree m.

If n is an odd number in the equation $x^n+1=0$, and we change x into $-x$, we obtain $x^n-1=0$; so we may if we please solve the latter equation, and then change the signs of the roots, and thus obtain the solution of the former equation.

158. In the equation $x^n+1=0$, suppose that n is an even number; then the equation has no real root. The equation is a *reciprocal* equation, and its solution may be made to depend

upon the solution of an equation of half the degree. Or the equation may be treated by the method given in Art. 154.

159. Thus in the four preceding Articles we have shewn how the solution of the proposed equations can be made to depend upon the solution of other equations which are not of higher degrees than half the degrees of the proposed equations. In each case we remove the factors which correspond to the real roots and then put $x + \frac{1}{x} = z$, and obtain an equation in z. Now it may be observed that this equation in z will have all its roots real. For suppose that $\alpha + \beta\sqrt{-1}$ denotes one of the imaginary values of x; then the corresponding value of z is

$$\alpha + \beta\sqrt{-1} + \frac{1}{\alpha + \beta\sqrt{-1}}, \text{ that is, } \alpha + \beta\sqrt{-1} + \frac{\alpha - \beta\sqrt{-1}}{\alpha^2 + \beta^2},$$

and this is a real quantity, namely, 2α, provided that $\alpha^2 + \beta^2 = 1$. We shall shew that $\alpha^2 + \beta^2$ is $= 1$.

Since $\alpha + \beta\sqrt{-1}$ is a root of the proposed equation $x^n \mp 1 = 0$, by Art. 41, $\alpha - \beta\sqrt{-1}$ is also a root. Thus

$$(\alpha + \beta\sqrt{-1})^n = \pm 1, \text{ and } (\alpha - \beta\sqrt{-1})^n = \pm 1;$$

hence by multiplication $(\alpha^2 + \beta^2)^n = 1$; therefore $\alpha^2 + \beta^2 = \pm 1$, and since $\alpha^2 + \beta^2$ is necessarily positive it must be equal to $+1$.

160. We will now consider some examples of the equations

$$x^n + 1 = 0 \text{ and } x^n - 1 = 0.$$

(1) $x^3 - 1 = 0$; this gives $(x-1)(x^2 + x + 1) = 0$.

Hence the roots are 1 and $\frac{-1 \pm \sqrt{-3}}{2}$; these values are then the three cube roots of $+1$. By changing their signs we shall obtain the three cube roots of -1, or in other words the roots of the equation $x^3 + 1 = 0$.

(2) $x^4+1=0$. Put $x+\dfrac{1}{x}=z$; we get $z^2-2=0$.

Thus $$z=\pm\sqrt{2}.$$

Therefore $x^4+1=(x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1)$;

and the solution can be completed by finding the roots of two quadratic equations.

(3) $x^5-1=0$. This gives $(x-1)(x^4+x^3+x^2+x+1)=0$.

Hence we have to solve $x^2+\dfrac{1}{x^2}+x+\dfrac{1}{x}+1=0$, that is

$$z^2+z-1=0. \quad \text{Thus } z=\frac{-1\pm\sqrt{5}}{2}.$$

Therefore

$$x^5-1=(x-1)\left(x^2+x\frac{1-\sqrt{5}}{2}+1\right)\left(x^2+x\frac{1+\sqrt{5}}{2}+1\right);$$

and the solution can be completed by finding the roots of two quadratic equations. The roots with their signs changed will be roots of the equation $x^5+1=0$.

161. If we attempt to solve the equation $x^7-1=0$, we obtain an equation of the *third* degree in z; and if we attempt to solve the equation $x^9-1=0$ we obtain an equation of the *fourth* degree in z. We shall in the next two Chapters shew how to solve equations of the third and fourth degrees; it will however be found that the methods of solution are of little practical value when the equations to be solved have all their roots real, which is the case we have here to consider, by Art. 159.

162. In an equation of the form $x^{2n}+px^n+q=0$, we can by the solution of a quadratic equation find the values of x^n, and then the method of the present Chapter may be applied to find the values of x.

We will close this Chapter by a proposition respecting the number of values of the product of two surd quantities.

163. Suppose A and B any two algebraical quantities, and m and n any positive integers. Then $\sqrt[m]{A}$ has m different values, and $\sqrt[n]{B}$ has n different values by Art. 142. Hence the product of $\sqrt[m]{A}$ and $\sqrt[n]{B}$ cannot have *more* than mn different values; and we shall shew that it cannot have so many values unless m and n are prime to each other. This we shall shew by proving the following proposition; *the number of different values of the product of $\sqrt[m]{A}$ and $\sqrt[n]{B}$ is equal to the least common multiple of m and n.*

Let a be one value of $\sqrt[m]{A}$; then all the values of $\sqrt[m]{A}$ are included in $a\sqrt[m]{1}$. Let b be one of the values of $\sqrt[n]{B}$; then all the values of $\sqrt[n]{B}$ are included in $b\sqrt[n]{1}$. Hence all the values of the product are included in $ab \times \sqrt[m]{1} \times \sqrt[n]{1}$; and therefore the number of the different values of the product is the same as the number of the different values of $\sqrt[m]{1} \times \sqrt[n]{1}$. Let r be the least common multiple of m and n; then $(\sqrt[m]{1} \times \sqrt[n]{1})^r = 1$. Thus $\sqrt[m]{1} \times \sqrt[n]{1}$ is equal to an r^{th} root of unity, and therefore cannot have *more* than r different values.

We have however still to shew that $\sqrt[m]{1} \times \sqrt[n]{1}$ *really has* r different values. Let p be the greatest common measure of m and n, and let $m = p\mu$, and $n = p\nu$. Let α denote a value of $\sqrt[\mu]{1}$, and β a value of $\sqrt[\nu]{1}$; then $\sqrt[m]{1} \times \sqrt[n]{1}$ may be written thus $\sqrt[p]{\alpha} \times \sqrt[p]{\beta}$, or $\sqrt[p]{\alpha\beta}$. Now $\alpha\beta$ has $\mu \times \nu$ values, and as each p^{th} root of $\alpha\beta$ has p values we have in all $p\mu\nu$ values, that is r values. And these values are all different. For let α' denote another of the μ values, and β' another of the ν values, and suppose if possible that $\sqrt[p]{\alpha'\beta'} = \sqrt[p]{\alpha\beta}$; raise both sides to the p^{th} power, then $\alpha'\beta' = \alpha\beta$; therefore $\frac{\alpha'}{\alpha} = \frac{\beta}{\beta'}$. The left-hand member is a root of the equation $x^\mu - 1 = 0$, and the right-hand member is a root of the equation $x^\nu - 1 = 0$; and these equations can have no common root except unity by Art. 147. Thus there are $\mu\nu$ different values of $\alpha\beta$, and r different values of $\sqrt[m]{1} \times \sqrt[n]{1}$.

164. The essential part of the preceding Article is sometimes treated thus. We have $\sqrt[m]{1} \times \sqrt[n]{1} = 1^{\frac{m+n}{mn}}$, and if $\frac{m+n}{mn}$ be reduced

to its lowest terms, the numerator will be an integer and the denominator will be r; thus $1^{\frac{m+n}{mn}} = 1^{\frac{1}{r}}$ which has r different values. This method however is unsatisfactory, because the ordinary theory of surds in Algebra is only proved there for the arithmetical values of the surds, and thus does not furnish the relation $1^{\frac{1}{m}} \times 1^{\frac{1}{n}} = 1^{\frac{m+n}{mn}}$, in the sense in which this relation is here required.

XII. CUBIC EQUATIONS.

From Here

165. It is unnecessary to say anything on the solution of quadratic equations because that subject is fully considered in treatises on Algebra. We propose in the present Chapter to give the solution of equations of the third degree which are also called cubic equations.

It appears from Art. 56, that any proposed equation can always be transformed into another equation without the second term. As the roots of a cubic equation without the second term are more simple expressions than the roots of a complete cubic equation, we shall suppose that the cubic equation which we have to solve is without the second term. The process which we shall now give is usually called *Cardan's solution of a cubic equation.*

166. *To solve the equation* $x^3 + qx + r = 0$.

Assume $x = y + z$, so that y and z are two quantities which are at present unknown. Substitute for x in the given equation; thus

$$(y+z)^3 + q(y+z) + r = 0,$$

that is,
$$y^3 + z^3 + (3yz + q)(y+z) + r = 0.$$

Now we have made only one assumption with respect to the two quantities y and z, namely that their sum is the value of a root of the proposed equation. We are therefore at liberty to make another assumption; suppose then that $3yz+q=0$. Thus we have

$$y^3+z^3+r=0.$$

Substitute for z in terms of y; thus

$$y^3+\left(-\frac{q}{3y}\right)^3+r=0,$$

that is

$$y^6+ry^3-\frac{q^3}{27}=0.$$

Hence

$$y^3=-\frac{r}{2}\pm\sqrt{\left(\frac{r^2}{4}+\frac{q^3}{27}\right)},$$

and

$$z^3=-r-y^3=-\frac{r}{2}\mp\sqrt{\left(\frac{r^2}{4}+\frac{q^3}{27}\right)}.$$

Also $x=y+z$; it will lead to the same result in the value of x whether we adopt the upper sign or lower sign in the values of y^3 and z^3; for distinctness suppose the upper sign taken. Therefore

$$x=\left\{-\frac{r}{2}+\sqrt{\left(\frac{r^2}{4}+\frac{q^3}{27}\right)}\right\}^{\frac{1}{3}}+\left\{-\frac{r}{2}-\sqrt{\left(\frac{r^2}{4}+\frac{q^3}{27}\right)}\right\}^{\frac{1}{3}}.$$

Thus the expression for x is the sum of two cube roots, and as every quantity has three cube roots, we must examine which cube roots are to be used in the present case. Let

$$\alpha=\frac{1}{2}(-1+\sqrt{-3}),$$

then by Art. 160, the three cube roots of 1 are 1, α, and α^2. Let m denote one of the cube roots of $-\frac{r}{2}+\sqrt{\left(\frac{r^2}{4}+\frac{q^3}{27}\right)}$, then the other cube roots are $m\alpha$ and $m\alpha^2$; let n denote one of the cube roots of $-\frac{r}{2}-\sqrt{\left(\frac{r^2}{4}+\frac{q^3}{27}\right)}$; then the other cube roots are $n\alpha$ and $n\alpha^2$. If we could ascribe to each of the cube roots which occur in the expression for x any one of its three values, we should obtain on the whole nine values of x. But a cubic equation can only have three roots, so that we are led to conclude that only three values will be admissible for x. And in fact the process of solution requires that $yz = -\frac{q}{3}$, and it is this condition which determines the admissible values of the cube roots. Suppose that m and n are so taken as to satisfy the condition $mn = -\frac{q}{3}$; thus we can have $y = m$ and $z = n$ as admissible values. Then we can also have $y = \alpha m$ and $z = \alpha^2 n$; and we can also have $y = \alpha^2 m$ and $z = \alpha n$; for in these two cases we have the relation $yz = -\frac{q}{3}$ satisfied. No other pair of values however is admissible; for instance, if we suppose $y = m$ and $z = \alpha n$, we get $yz = -\frac{\alpha q}{3}$ and not $-\frac{q}{3}$, and any other pair of values except those which we have admitted will make $yz = -\frac{\alpha q}{3}$ or $= -\frac{\alpha^2 q}{3}$ instead of $-\frac{q}{3}$.

167. For example, suppose $x^3 + 6x - 20 = 0$. Here $q = 6$ and $r = -20$; thus

$$x = (10 + \sqrt{108})^{\frac{1}{3}} + (10 - \sqrt{108})^{\frac{1}{3}}. \qquad = 2$$

By numerical work it may be ascertained that

$$(10 + \sqrt{108})^{\frac{1}{3}} = 2{\cdot}732 \ldots, \text{ and } (10 - \sqrt{108})^{\frac{1}{3}} = -{\cdot}732 \ldots,$$

so that we may presume that $x = 2$ is a root, and this will be found the case on trial. Instead of expressing the other two roots by the method of the preceding Article it will be preferable to depress the equation to a quadratic. Since 2 is a root of the proposed equation we know that $x^3 + 6x - 20$ is divisible by $x - 2$, and we find that

$$x^3 + 6x - 20 = (x - 2)(x^2 + 2x + 10);$$

therefore the other two roots of the proposed equation may be found by solving the equation

$$x^2 + 2x + 10 = 0;$$

thus these roots are

$$-1 \pm \sqrt{-9}, \text{ that is } -1 \pm 3\sqrt{-1}.$$

In the preceding example we may verify *by trial* that

$$(10+\sqrt{108})^{\frac{1}{3}}=1+\sqrt{3} \text{ and } (10-\sqrt{108})^{\frac{1}{3}}=1-\sqrt{3},$$

and so find the root 2 without any numerical extraction of roots. There is however no algebraical process by which we can universally obtain the cube root of an expression of the form $a+\sqrt{b}$ in a finite form; see *Algebra*, Art. 310. We may apply the binomial theorem to find the value of $(a+\sqrt{b})^{\frac{1}{3}}$ in an infinite series; in this case in order to obtain a *convergent* series, we must expand in ascending powers of $\sqrt{b}$ or of a, according as $\sqrt{b}$ is less or greater than a; see *Algebra*, Chapters XXXVI. and XL.

168. We have seen in Art. 166, that although apparently nine values are furnished for x only three are really admissible. We may see a reason for the occurrence of the nine values. For the relation $yz=-\frac{q}{3}$ was assumed, but this was transformed into $y^3z^3=-\frac{q^3}{27}$ in the process; and the latter relation would not be changed if q were changed into $q\alpha$ or into $q\alpha^2$. Thus, in solving the equation $x^3+qx+r=0$, we really found nine solutions, three belonging to this equation, three to the equation $x^3+q\alpha x+r=0$, and three to the equation $x^3+q\alpha^2x+r=0$.

169. Let us now consider more particularly the form of the roots of the proposed cubic equation. We will assume that q and r denote real quantities. The expressions for y^3 and z^3 may be either real or imaginary.

First suppose that these expressions are real. We may then suppose that m and n denote respectively the arithmetical values of the cube roots of y^3 and z^3. The proposed cubic equation has in this case one root which is certainly real, namely $m+n$; the other two roots are $m\alpha+n\alpha^2$ and $m\alpha^2+n\alpha$. By substituting for α its value these roots become respectively

$$-\frac{1}{2}(m+n)+\frac{1}{2}(m-n)\sqrt{-3},$$

and $$-\frac{1}{2}(m+n)-\frac{1}{2}(m-n)\sqrt{-3},$$

and these roots are imaginary unless $m=n$. When $m=n$ the cubic equation has two equal roots each being equal to $-m$ or $-n$. The condition which is necessary and sufficient to ensure $m=n$, that is, $y^3=z^3$, is that $\frac{r^2}{4}+\frac{q^3}{27}=0$.

Conversely, if the roots of the cubic equation are all real and unequal the expressions for y^3 and z^3 must be imaginary.

Next suppose that the expressions for y^3 and z^3 are imaginary; that is, suppose that $\frac{r^2}{4}+\frac{q^3}{27}$ is a negative quantity. We know from Art. 142 that y^3 and z^3 will each have cube roots of a certain form. We may therefore suppose that $m=\mu+\nu\sqrt{-1}$, and as z^3 only differs from y^3 in the sign of the radical, we can take $n=\mu-\nu\sqrt{-1}$. In this case the roots of the proposed cubic equation are all real, namely,

$$\mu+\nu\sqrt{-1}+\mu-\nu\sqrt{-1}, \text{ that is } 2\mu,$$

$$(\mu+\nu\sqrt{-1})\,\alpha+(\mu-\nu\sqrt{-1})\,\alpha^2, \text{ that is } -\mu-\nu\sqrt{3},$$

and $$(\mu+\nu\sqrt{-1})\,\alpha^2+(\mu-\nu\sqrt{-1})\,\alpha, \text{ that is } -\mu+\nu\sqrt{3}.$$

170. It will now be seen that Cardan's solution of a cubic equation is of little practical use when the roots of the proposed equation are real and unequal. For in this case the expressions for y^3 and z^3 are imaginary; and although we know that cube roots of these expressions exist, there is no arithmetical method of obtaining them, and no algebraical method of obtaining them exactly. We have the roots in this case exhibited in a form which is alge-

braically correct, but arithmetically of little value. For example, take the equation

$$x^3 - 15x - 4 = 0.$$

Here $r = -4$ and $q = -15$. Hence we obtain

$$x = (2 + \sqrt{-121})^{\frac{1}{3}} + (2 - \sqrt{-121})^{\frac{1}{3}};$$

that is, $$x = (2 + 11\sqrt{-1})^{\frac{1}{3}} + (2 - 11\sqrt{-1})^{\frac{1}{3}}.$$

Now here we have no obvious mode of extracting the cube roots. It may be verified by trial that

$$(2 + 11\sqrt{-1})^{\frac{1}{3}} = 2 + \sqrt{-1},$$

and $$(2 - 11\sqrt{-1})^{\frac{1}{3}} = 2 - \sqrt{-1}.$$

Thus $$x = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4.$$

Hence 4 is a root. The other roots can then be found by the method of Art. 169; or we may proceed thus,

$$x^3 - 15x - 4 = (x - 4)(x^2 + 4x + 1).$$

We have therefore to solve the equation $x^2 + 4x + 1 = 0$; the roots are $-2 \pm \sqrt{3}$.

Again, consider the equation $x^3 - 3\sqrt[3]{2}x - 2 = 0$.

Here $r = -2$ and $q = -3\sqrt[3]{2}$. Thus

$$x = (1 + \sqrt{-1})^{\frac{1}{3}} + (1 - \sqrt{-1})^{\frac{1}{3}}.$$

It may be verified by trial that

$$(1 + \sqrt{-1})^{\frac{1}{3}} = \frac{\sqrt{3} + 1}{2\sqrt[3]{2}} + \frac{\sqrt{3} - 1}{2\sqrt[3]{2}}\sqrt{-1},$$

$$(1 - \sqrt{-1})^{\frac{1}{3}} = \frac{\sqrt{3} + 1}{2\sqrt[3]{2}} - \frac{\sqrt{3} - 1}{2\sqrt[3]{2}}\sqrt{-1}.$$

Thus

$$x = \left(\frac{\sqrt{3} + 1}{2\sqrt[3]{2}} + \frac{\sqrt{3} - 1}{2\sqrt[3]{2}}\sqrt{-1}\right) + \left(\frac{\sqrt{3} + 1}{2\sqrt[3]{2}} - \frac{\sqrt{3} - 1}{2\sqrt[3]{2}}\sqrt{-1}\right) = \frac{\sqrt{3} + 1}{\sqrt[3]{2}}.$$

The other roots may then be found; they are

$$\frac{1 - \sqrt{3}}{\sqrt[3]{2}} \text{ and } -\frac{2}{\sqrt[3]{2}}.$$

171. The case in which the three roots of a cubic equation are real and unequal is sometimes called the *irreducible case*, and sometimes it is said that Cardan's solution *fails* in this case; these expressions are used to indicate the fact that the roots are in this case presented to us in a form which is very inconvenient for arithmetical purposes.

We may however use the binomial theorem in order to approximate to the cube root of an expression of the form $p+q\sqrt{-1}$. For if q be numerically less than p we can expand $(p+q\sqrt{-1})^{\frac{1}{3}}$ in a *converging* series proceeding according to ascending powers of $q\sqrt{-1}$; see *Algebra*, Chapter XXXVI. We can thus obtain approximately $(p+q\sqrt{-1})^{\frac{1}{3}}$ in the form $P+Q\sqrt{-1}$; and then $(p-q\sqrt{-1})^{\frac{1}{3}}$ will have an approximate value $P-Q\sqrt{-1}$; and the sum of the two cube roots will be $2P$. But if q be numerically greater than p we may proceed thus;

$$p+q\sqrt{-1}=\sqrt{-1}\,(q-p\sqrt{-1});$$

hence $$(p+q\sqrt{-1})^{\frac{1}{3}}=(\sqrt{-1})^{\frac{1}{3}}(q-p\sqrt{-1})^{\frac{1}{3}}.$$

Now $-\sqrt{-1}$ is a cube root of $\sqrt{-1}$ as we find by trial, so that we have $$(p+q\sqrt{-1})^{\frac{1}{3}}=-\sqrt{-1}\,(q-p\sqrt{-1})^{\frac{1}{3}}.$$

And we can expand $(q-p\sqrt{-1})^{\frac{1}{3}}$ in a converging series proceeding according to ascending powers of $p\sqrt{-1}$; and thus we may find as before the sum of the cube roots of $p+q\sqrt{-1}$ and $p-q\sqrt{-1}$.

The case in which $p=q$ is really involved in the second example of the preceding Article.

It may be observed that by means of De Moivre's theorem, we can express the cube root of any quantity $p+q\sqrt{-1}$ in a form involving Trigonometrical functions.

172. It appears from the preceding Articles that the cubic equation $x^3+qx+r=0$ may always be solved by Cardan's process

without any difficulty when q is a *positive* quantity, and also when q is a *negative* quantity provided q^3 is numerically less than $\frac{27r^2}{4}$; and in these cases two of the roots are imaginary. If q^3 is a negative quantity and numerically greater than $\frac{27r^2}{4}$, Cardan's solution is inconvenient, and in this case all the roots are real.

If q^3 be negative and numerically equal to $\frac{27r^2}{4}$, so that $\frac{r^2}{4}+\frac{q^3}{27}=0$, the proposed cubic equation has two of its roots equal by Art. 60. We have by Art. 166 in this case $m=n=\sqrt[3]{-\frac{r}{2}}$; and the three roots are $2m$, $-m$, and $-m$.

In every case where one root of a cubic equation has been found we can, if we please, depress the equation to a quadratic, and so find the other two roots, instead of finding the other two roots by the process of the preceding Articles.

173. We will briefly indicate the results which are obtained in the solution of a *complete* cubic equation. Let the equation be

$$ax^3+3bx^2+3cx+d=0;$$

assume $x=z-\frac{b}{a}$, then we obtain

$$z^3+qz+r=0,$$

where $$q=3\frac{c}{a}-\frac{3b^2}{a^2},\quad r=\frac{d}{a}-\frac{3bc}{a^2}+\frac{2b^3}{a^3}.$$

Hence by Cardan's method

$$z=\left(-\frac{r}{2}+\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right)^{\frac{1}{3}}+\left(-\frac{r}{2}-\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right)^{\frac{1}{3}}.$$

The condition which must hold if there are equal roots is

$$\frac{r^2}{4}+\frac{q^3}{27}=0;$$

that is

$$(2b^3 - 3abc + a^2d)^2 + 4(ac - b^2)^3 = 0.$$

It will be found by common Algebraical work that this can be put in the form

$$(ad - bc)^2 - 4(b^2 - ac)(c^2 - bd) = 0.$$

174. Some cubic equations in which the coefficients have special values may be solved without using Cardan's method. For example, suppose

$$x^3 + 3x = a^3 - a^{-3}.$$

This may be written

$$x^3 + 3x = \left(a - \frac{1}{a}\right)^3 + 3\left(a - \frac{1}{a}\right),$$

that is, $$x^3 - \left(a - \frac{1}{a}\right)^3 + 3\left\{x - \left(a - \frac{1}{a}\right)\right\} = 0;$$

and now we see that one root is given by $x = a - \dfrac{1}{a}$.

Again, suppose we have the complete cubic equation

$$x^3 + ax^2 + bx + c = 0,$$

and that the relation $3ac = b^2$ holds among the coefficients. The proposed equation may be written

$$-x^3 = ax^2 + bx + c,$$

therefore $$-3abx^3 = 3ba^2x^2 + 3b^2ax + b^3,$$

therefore $$(a^3 - 3ab)x^3 = a^3x^3 + 3ba^2x^2 + 3b^2ax + b^3 = (ax + b)^3,$$

therefore $$x\sqrt[3]{a^3 - 3ab} = ax + b,$$

therefore $$x = \frac{b}{\sqrt[3]{a^3 - 3ab} - a}.$$

175. A process is given in the *Trigonometry*, Chapter XVII. by which we may obtain the roots of a cubic equation in the *irreducible case*, by the aid of the Trigonometrical Tables. This

is a matter of very little practical value, but we will shew how the Trigonometrical Tables may also be used for examples which do not belong to the *irreducible case.*

Suppose $x^3 + qx + r = 0$; then

$$x = \left(-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{\frac{1}{3}}.$$

If q is positive, assume $\frac{q^3}{27} = \frac{r^2}{4}\tan^2\theta$; then we get

$$x = \left(-\frac{r}{2} + \frac{r}{2}\sec\theta\right)^{\frac{1}{3}} + \left(-\frac{r}{2} - \frac{r}{2}\sec\theta\right)^{\frac{1}{3}}$$

$$= \left(-\frac{r}{\cos\theta}\right)^{\frac{1}{3}}\left\{\left(\cos\frac{\theta}{2}\right)^{\frac{2}{3}} - \left(\sin\frac{\theta}{2}\right)^{\frac{2}{3}}\right\}.$$

If q is negative, and $4q^3$ numerically less than $27r^2$, assume $\frac{q^3}{27} = -\frac{r^2}{4}\sin^2\theta$; then we get

$$x = \left(-\frac{r}{2} + \frac{r}{2}\cos\theta\right)^{\frac{1}{3}} + \left(-\frac{r}{2} - \frac{r}{2}\cos\theta\right)^{\frac{1}{3}}$$

$$= (-r)^{\frac{1}{3}}\left\{\left(\cos\frac{\theta}{2}\right)^{\frac{2}{3}} + \left(\sin\frac{\theta}{2}\right)^{\frac{2}{3}}\right\}.$$

176. An important cubic equation occurs in many mathematical investigations, and it may be noticed here although not connected with the special subject of this Chapter.

We propose to shew that the roots of the equation $f(x) = 0$ are all real, where $f(x)$ denotes

$$(x-a)(x-b)(x-c) - a'^2(x-a) - b'^2(x-b) - c'^2(x-c) - 2a'b'c'.$$

The equation may be written thus,

$$(x-a)\left\{(x-b)(x-c) - a'^2\right\} - \left\{b'^2(x-b) + c'^2(x-c) + 2a'b'c'\right\} = 0.$$

Let h and k denote the roots of the quadratic equation

$$(x-b)(x-c)-a'^2=0,$$

and suppose h not less than k. Then by solving the quadratic equation it will be seen that h is greater than b or c, and that k is less than b or c. Substitute successively $+\infty$, h, k, $-\infty$ for x in $f(x)$; the results will be respectively

$$+\infty, -\left\{b'\sqrt{(h-b)}+c'\sqrt{(h-c)}\right\}^2, \left\{b'\sqrt{(b-k)}-c'\sqrt{(c-k)}\right\}^2, -\infty.$$

Thus the equation $f(x)=0$ has three real roots, one greater than h, one between h and k, and one less than k.

177. There are two cases which require further examination as they are not provided for by this demonstration, (1) that in which $h=k$, (2) that in which h or k is a root of the cubic equation.

(1) Suppose $h=k$. Since the roots of the quadratic equation are equal we shall obtain the condition $(b-c)^2+4a'^2=0$; therefore $b=c$ and $a'=0$. Hence it will be found that c is a root of the cubic equation; and on dividing $f(x)$ by $x-c$ and equating the quotient to zero we obtain a quadratic equation which has real roots.

(2) Suppose that h or k is a root of the cubic equation; for example, suppose that h is. Then the process of Art. 176 shews that the cubic equation has also a real root less than k; thus it has two real roots, and the third root must therefore also be real. Similarly if k be a root of the cubic equation, it has a real root greater than h; and thus the third root must also be real.

178. We may investigate the condition which must hold in order that h or k may be a root of the cubic equation. Suppose that λ is a root of the quadratic equation and also of the cubic equation.

Since λ is a root of the quadratic equation, we have

$$(\lambda-b)(\lambda-c)-a'^2=0\ldots\ldots\ldots\ldots\ldots(1);$$

and since λ is also supposed to be a root of the cubic equation, we obtain

$$b'^2(\lambda - b) + c'^2(\lambda - c) + 2a'b'c' = 0 \ldots\ldots (2).$$

From (1) and (2) we deduce

$$b'^2(\lambda - b) + c'^2(\lambda - c) + 2b'c'\sqrt{(\lambda - b)(\lambda - c)} = 0,$$

that is, $$\left\{b'\sqrt{(\lambda - b)} + c'\sqrt{(\lambda - c)}\right\}^2 = 0;$$

therefore $$b'^2(\lambda - b) = c'^2(\lambda - c) \ldots\ldots\ldots\ldots (3).$$

From (2) and (3) we obtain

$$\lambda - b = -\frac{a'c'}{b'}, \quad \lambda - c = -\frac{a'b'}{c'} \ldots\ldots\ldots\ldots (4);$$

and therefore $$b - \frac{a'c'}{b'} = c - \frac{a'b'}{c'} \ldots\ldots\ldots\ldots (5).$$

Hence the relation (5) must hold among the coefficients of the cubic equation in order that one of the roots of the quadratic equation may also be a root of the cubic equation.

Conversely, if (5) holds we may give to λ the single value determined by (4), and then both (1) and (2) will be satisfied; and thus the quadratic equation and the cubic equation will have a common root.

In obtaining (4) and (5) we assume that neither b' nor c' vanishes.

Suppose that b' vanishes; then from (3) either c' vanishes or $\lambda = c$. If $\lambda = c$ then from (1) it follows that a' must vanish.

179. Let us now investigate the conditions in order that the cubic equation may have equal roots.

If neither h nor k is a root of the cubic equation, the demonstration in Art. 176 shews that the roots of the cubic equation are unequal. But the process of Art. 176 may be conducted so as to use either of the quadratic equations

$$(x - c)(x - a) - b'^2 = 0, \text{ or } (x - a)(x - b) - c'^2 = 0,$$

instead of the quadratic equation

$$(x - b)(x - c) - a'^2 = 0.$$

Hence the cubic equation cannot have equal roots unless it has a root in common with any one of these quadratic equations. Hence from equation (5) we obtain the following as necessary conditions for the existence of equal roots of the cubic equation,

$$a-\frac{b'c'}{a'}=b-\frac{c'a'}{b'}=c-\frac{a'b'}{c'}.$$

Conversely; if these conditions hold the cubic equation has equal roots. For denote these equal quantities by r, so that

$$a=r+\frac{b'c'}{a'},\quad b=r+\frac{c'a'}{b'},\quad c=r+\frac{a'b'}{c'};$$

substitute for a, b, c in the cubic equation, and it becomes

$$(x-r)^3-(x-r)^2\left(\frac{b'c'}{a'}+\frac{c'a'}{b'}+\frac{a'b'}{c'}\right)=0;\ \therefore$$

so that the root r occurs twice, and the other root is

$$r+\frac{b'c'}{a'}+\frac{c'a'}{b'}+\frac{a'b'}{c'}.$$

This assumes that a', b', and c' are all different from zero.

Suppose now that one of these quantities vanishes, say a'. Then from the quadratic equation

$$(x-b)(x-c)-a'^2=0$$

it follows that x must be equal to c or b. Suppose $x=c$; then from the other quadratic equations we see that

$$b'=0 \text{ and } (c-a)(c-b)-c'^2=0.$$

If a', b' and c' all vanish then in order that there may be equal roots, two of the three a, b, c must be equal; if they are all equal the cubic equation reduces to $(x-a)^3=0$.

XIII. BIQUADRATIC EQUATIONS.

180. We shall now proceed to explain some methods for the solution of equations of the fourth degree, which are also called biquadratic equations. We suppose the biquadratic equation which is to be solved to be deprived of its second term, for a reason already given; see Art. 165. The first solution which we shall give is called *Descartes's Solution.*

181. *To solve the equation*

$$x^4 + qx^2 + rx + s = 0.$$

Assume $$x^4 + qx^2 + rx + s = (x^2 + ex + f)(x^2 - ex + g);$$

we have then to shew that the quantities e, f, and g can be found. Multiply together the factors on the right-hand side, and equate the coefficients of the several powers of x to those on the left-hand side; thus

$$g + f - e^2 = q, \quad e(g - f) = r, \quad gf = s;$$

that is, $$g + f = q + e^2, \quad g - f = \frac{r}{e}, \quad gf = s.$$

Find g and f in terms of e from the first two of these equations, and substitute in the third; thus

$$\left(q + e^2 + \frac{r}{e}\right)\left(q + e^2 - \frac{r}{e}\right) = 4s.$$

From this equation by reduction we obtain

$$e^6 + 2qe^4 + (q^2 - 4s)e^2 - r^2 = 0.$$

This may be considered as a cubic equation for finding e^2, and it will certainly have one real positive root by Art. 20. When e^2 is known we can find e, and then g and f become known. Thus the expression $x^4 + qx^2 + rx + s$ is resolved into the product of two real quadratic factors, and we can obtain the four roots of the proposed biquadratic equation by solving the two quadratic equations

$$x^2 + ex + f = 0, \quad x^2 - ex + g = 0.$$

182. It will be observed that in one of the two assumed quadratic factors we introduced the term ex, and in the other quadratic factor the term $-ex$; and the reason for this is that there is no term involving x^3 in the expression which we wish to resolve into quadratic factors. Now e is equal to the sum of the two roots of the second quadratic equation given at the end of the preceding Article, so that e is equal to the sum of two of the roots of the proposed biquadratic equation. Out of the *four* roots of a biquadratic equation *two* roots can be selected in $\frac{4 \cdot 3}{1 \cdot 2}$ ways, that is, in 6 ways; and thus we see the reason why the equation in e should be of the *sixth* degree. But as the sum of the four roots of the biquadratic equation is zero by Art. 45, the sum of any two roots is equal in magnitude and opposite in sign to the sum of the remaining two roots; and thus we see the reason why the equation in e only involves *even* powers of e, so that the values of e^2 can be found by the solution of a cubic equation.

We may observe that when we have found e^2 we can give either sign to the value of e, which we obtain by extracting the square root; for by changing the sign of e we merely interchange the values of f and g, and this has no influence on the results which are obtained by solving the biquadratic equation.

183. Suppose, for example, that $x^4 - 10x^2 - 20x - 16 = 0$. Here $q = -10$, $r = -20$, $s = -16$. The cubic equation in e^2 becomes $e^6 - 20e^4 + 164e^2 - 400 = 0$, and a root of this is $e^2 = 4$; see Art. 119. Thus $e = 2$; then $f = 2$, and $g = -8$; therefore

$$x^4 - 10x^2 - 20x - 16 = (x^2 + 2x + 2)(x^2 - 2x - 8).$$

The four roots of the proposed biquadratic equation will be found to be 4, -2, $-1 + \sqrt{-1}$, and $-1 - \sqrt{-1}$.

184. Thus it appears that the solution of a biquadratic equation can be effected if we can obtain one root of a certain auxiliary cubic equation. It becomes therefore a point of importance to ascertain when this cubic equation falls under the *irreducible*

case; see Art. 171. This gives occasion for the following proposition. *The auxiliary cubic equation will not fall under the irreducible case when the biquadratic equation has two real roots and two imaginary roots.*

For suppose the imaginary roots of the biquadratic equation to be denoted by $\alpha+\beta\sqrt{-1}$ and $\alpha-\beta\sqrt{-1}$; then since the sum of the four roots is zero, the two real roots will be of the forms $-\alpha+\gamma$ and $-\alpha-\gamma$. By taking the sum of every pair of these roots we obtain the expressions $\pm 2\alpha$, $\pm(\gamma+\beta\sqrt{-1})$, and $\pm(\gamma-\beta\sqrt{-1})$. Thus the three values of e^2 will be $(2\alpha)^2$, $(\gamma+\beta\sqrt{-1})^2$, and $(\gamma-\beta\sqrt{-1})^2$; if γ is not zero two of these values of e^2 are imaginary, and if γ is zero the values of e^2 are all real, but two of them are equal; thus the cubic equation in e^2 will not fall under the irreducible case.

185. If the roots of the biquadratic equation are all real the roots of the auxiliary cubic equation will be all real. If the roots of the biquadratic equation are all imaginary they will be of the forms $\alpha\pm\beta\sqrt{-1}$ and $-\alpha\pm\gamma\sqrt{-1}$. By taking the sum of every pair of these roots we obtain the expressions $\pm 2\alpha$, $\pm(\beta+\gamma)\sqrt{-1}$, and $\pm(\beta-\gamma)\sqrt{-1}$; thus the values of e^2 are $4\alpha^2$, $-(\beta+\gamma)^2$, and $-(\beta-\gamma)^2$, and so are all real.

Hence if the biquadratic equation has its roots all real or all imaginary, the auxiliary cubic equation will *in general* fall under the irreducible case; we say *in general*, because it may happen that the cubic equation has two of its roots equal, and then it does not fall under the irreducible case.

186. We have in the two preceding Articles shewn what will be the forms of the roots of the auxiliary cubic equation corresponding to the various forms of the roots of the proposed biquadratic equation. We will now state conversely what will be the forms of the roots of the proposed biquadratic equation corresponding to the various forms of the roots of the auxiliary cubic equation. Since the last term of the cubic equation is

negative, there must be *one* positive root; and as the product of the roots is positive, by Art. 45, the only cases which can occur are, (1) all the roots positive, (2) one positive root and two negative roots, (3) one positive root and two imaginary roots. The following results follow from Arts. 184 and 185.

(1) If the cubic equation has all its roots positive, the roots of the biquadratic equation are all real.

(2) If the cubic equation has one positive root and two negative roots, the biquadratic equation has two real roots and two imaginary roots, or else four imaginary roots.

(3) If the cubic equation has one positive root and two imaginary roots, the biquadratic equation has two real roots and two imaginary roots.

187. The four roots of the biquadratic equation can be expressed very simply in terms of the three roots of the auxiliary cubic equation. Let α^2, β^2, γ^2 denote the three values of e^2 obtained from the cubic equation

$$e^6 + 2qe^4 + (q^2 - 4s)\, e^2 - r^2 = 0.$$

Then by Art. 45 we have $r^2 = \alpha^2\beta^2\gamma^2$, and $-2q = \alpha^2 + \beta^2 + \gamma^2$. Thus we may put $r = \alpha\beta\gamma$, and take α as a value of e; therefore

$$x^2 + ex + f = x^2 + \alpha x + \frac{1}{2}\left(q + \alpha^2 - \frac{r}{\alpha}\right)$$

$$= x^2 + \alpha x + \frac{1}{4}(\alpha^2 - \beta^2 - \gamma^2 - 2\beta\gamma).$$

By solving the equation $x^2 + ex + f = 0$ we shall therefore obtain

$$x = \frac{1}{2}(-\alpha - \beta - \gamma), \quad \text{or} \quad x = \frac{1}{2}(-\alpha + \beta + \gamma).$$

Similarly, by putting $x^2 - ex + g = 0$ we shall obtain

$$x = \frac{1}{2}(\alpha - \beta + \gamma), \qquad \text{or} \quad x = \frac{1}{2}(\alpha + \beta - \gamma).$$

Thus the four roots of the biquadratic equation are

$$\frac{1}{2}(-\alpha - \beta - \gamma), \quad \frac{1}{2}(-\alpha + \beta + \gamma), \quad \frac{1}{2}(\alpha - \beta + \gamma), \quad \frac{1}{2}(\alpha + \beta - \gamma).$$

In order that the biquadratic equation may have equal roots the auxiliary cubic equation must have equal roots. For suppose, for example, that

$$\frac{1}{2}(-\alpha-\beta-\gamma)=\frac{1}{2}(-\alpha+\beta+\gamma),$$

then $$\beta+\gamma=0,$$

therefore $$\beta^2=\gamma^2;$$

and a similar result will follow in any other case.

Hence we can express the condition which must hold in order that the proposed biquadratic equation may have equal roots; for by Art. 173 the condition in order that the auxiliary cubic equation may have equal roots is

$$(27r^2-72qs+2q^3)^2=4(q^2+12s)^3.$$

It will be seen, by Art. 79, that the conditions which must hold in order that the proposed biquadratic equation may have three equal roots may be expressed thus:

$$27r^2-72qs+2q^3=0, \text{ and } q^2+12s=0.$$

It will be useful to note the forms of these conditions for a complete biquadratic equation.

Let the equation be

$$ax^4+4bx^3+6cx^2+4dx+e=0;$$

assume $x=z-\dfrac{b}{a}$, then we obtain

$$z^4+qz^2+rz+s=0,$$

where $$q=\frac{6c}{a}-\frac{6b^2}{a^2},$$

$$r=\frac{4d}{a}-\frac{12bc}{a^2}+\frac{8b^3}{a^3},$$

$$s=\frac{e}{a}-\frac{4bd}{a^2}+\frac{6b^2c}{a^3}-\frac{3b^4}{a^4}.$$

Hence we shall find that

$$q^2 + 12s = \frac{12}{a^2}(ae - 4bd + 3c^2),$$

and $$27r^2 - 72qs + 2q^3 = \frac{16 \times 27}{a^3}(ad^2 + eb^2 + c^3 - ace - 2bcd).$$

Thus the condition for equal roots is

$$(ae - 4bd + 3c^2)^3 = 27(ad^2 + eb^2 + c^3 - ace - 2bcd)^2;$$

and the conditions for three equal roots are

$$ae - 4bd + 3c^2 = 0,$$

and $$ad^2 + eb^2 + c^3 - ace - 2bcd = 0.$$

188. Another mode of solving a biquadratic equation has been given under slightly different forms by various mathematicians; and thus it is sometimes called *Ferrari's* method, sometimes *Waring's* method, and sometimes *Simpson's* method. We will now explain it.

Let the biquadratic equation be

$$x^4 + px^3 + qx^2 + rx + s = 0;$$

add to both sides $ax^2 + bx + c$, and then let a, b, c be so determined as to render each side a perfect square. We have then

$$x^4 + px^3 + (q + a)x^2 + (r + b)x + s + c = ax^2 + bx + c.$$

The right-hand member will be a perfect square if $b^2 = 4ac$. Suppose the left-hand member to be equal to

$$\left(x^2 + \frac{px}{2} + m\right)^2;$$

by comparing the coefficients we obtain

$$2m + \frac{p^2}{4} = q + a, \quad pm = r + b, \quad m^2 = s + c.$$

These three relations express a, b, c in terms of m; substituting the values of a, b, and c in the equation $b^2 = 4ac$ we obtain

$$(pm - r)^2 = 4\left(2m + \frac{p^2}{4} - q\right)(m^2 - s).$$

From this cubic equation m must be found, and then a, b, and c. And since we now have

$$\left(x^2+\frac{px}{2}+m\right)^2 = ax^2+bx+c = ax^2+bx+\frac{b^2}{4a},$$

we obtain $$x^2+\frac{px}{2}+m = \pm\frac{2ax+b}{2\sqrt{a}}.$$

Thus we have two quadratic equations to solve, namely,

$$x^2+\frac{px}{2}+m+\frac{2ax+b}{2\sqrt{a}}=0, \text{ and } x^2+\frac{px}{2}+m-\frac{2ax+b}{2\sqrt{a}}=0.$$

189. It may be shewn that the auxiliary cubic equation which this method requires us to solve will in general fall under the irreducible case, unless the proposed biquadratic equation has two real roots and two imaginary roots. For let α, β, γ, δ, denote the four roots of the proposed biquadratic equation; then from considering the two quadratic equations obtained in Art. 188, it follows that $m+\dfrac{b}{2\sqrt{a}}$ must be equal to the product of two of the four quantities α, β, γ, δ, and $m-\dfrac{b}{2\sqrt{a}}$ must be equal to the product of the remaining two. Suppose then

$$m+\frac{b}{2\sqrt{a}}=\alpha\beta, \text{ and } m-\frac{b}{2\sqrt{a}}=\gamma\delta;$$

thus $$m=\frac{1}{2}(\alpha\beta+\gamma\delta).$$

Hence we infer by symmetry that the other two values of m will be $\frac{1}{2}(\alpha\gamma+\beta\delta)$ and $\frac{1}{2}(\alpha\delta+\beta\gamma)$.

It is obvious that if α, β, γ, δ, are all real, these three values of m are all real; and it may be shewn that such will be the case if α, β, γ, δ, are all imaginary. If however two of the four quantities are real and two imaginary, it will be found that two of the values of m are imaginary and one real, or else they are all real and two of them equal.

190. We will now give *Euler's method* of solving a biquadratic equation. Suppose the equation to be

$$x^4 + qx^2 + rx + s = 0.$$

Assume $x = y + z + u$; thus

$$x^2 = y^2 + z^2 + u^2 + 2(yz + zu + uy),$$

that is, $$x^2 - y^2 - z^2 - u^2 = 2(yz + zu + uy).$$

Square both sides; thus

$$\begin{aligned} x^4 - 2x^2(y^2 + z^2 + u^2) + (y^2 + z^2 + u^2)^2 &= 4(yz + zu + uy)^2 \\ &= 4(y^2z^2 + z^2u^2 + u^2y^2) + 8yzu(y + z + u). \end{aligned}$$

Put x for $y + z + u$, and transpose; thus

$$x^4 - 2x^2(y^2 + z^2 + u^2) - 8xyzu + (y^2 + z^2 + u^2)^2 - 4(y^2z^2 + z^2u^2 + u^2y^2) = 0.$$

In order that this equation may coincide with the proposed biquadratic equation, we must have

$$q = -2(y^2 + z^2 + u^2), \quad r = -8yzu,$$
$$s = (y^2 + z^2 + u^2)^2 - 4(y^2z^2 + z^2u^2 + u^2y^2).$$

Thus $$y^2 + z^2 + u^2 = -\frac{q}{2},$$

$$y^2z^2 + z^2u^2 + u^2y^2 = \frac{1}{4}\left(\frac{q^2}{4} - s\right) = \frac{q^2 - 4s}{16},$$

$$y^2z^2u^2 = \frac{r^2}{64}.$$

Therefore it follows from Art. 45, that y^2, z^2, and u^2 are the values of t furnished by the following cubic equation,

$$t^3 + \frac{q}{2}t^2 + \frac{q^2 - 4s}{16}t - \frac{r^2}{64} = 0.$$

Let the roots of this equation be denoted by t_1, t_2, and t_3; then

$$y = \pm\sqrt{t_1}, \qquad z = \pm\sqrt{t_2}, \qquad u = \pm\sqrt{t_3}.$$

If we substitute these values in the expression for x, namely, $y + z + u$, we obtain *eight* different results on account of the ambiguities in sign. But these results are not all admissible; for we must have $yzu=-\frac{r}{8}$, so that the sign of the product of y, z, and u, must be the contrary to the sign of r.

If we suppose r *positive*, we have the following admissible values of x,

$$-\sqrt{t_1}-\sqrt{t_2}-\sqrt{t_3},\quad -\sqrt{t_1}+\sqrt{t_2}+\sqrt{t_3},\quad \sqrt{t_1}-\sqrt{t_2}+\sqrt{t_3},\quad \sqrt{t_1}+\sqrt{t_2}-\sqrt{t_3}.$$

If we suppose r *negative*, we have the following admissible values of x,

$$\sqrt{t_1}+\sqrt{t_2}+\sqrt{t_3},\quad \sqrt{t_1}-\sqrt{t_2}-\sqrt{t_3},\quad -\sqrt{t_1}+\sqrt{t_2}-\sqrt{t_3},\quad -\sqrt{t_1}-\sqrt{t_2}+\sqrt{t_3}.$$

191. The reason why *eight* values of x present themselves in the preceding Article is because the relation $yzu=-\frac{r}{8}$ was squared and used in the process in the form $y^2z^2u^2=\frac{r^2}{64}$; for since the relation in the latter form is not changed by changing the sign of r, the process really determines the roots of the biquadratic equation $x^4+qx^2-rx+s=0$, as well as the roots of the biquadratic equation $x^4+qx^2+rx+s=0$.

The auxiliary cubic equation of Art. 181 will be found to coincide with that of Art. 190 by supposing $e^2=4t$; thus the remarks made in Arts. 184...186, respecting the connexion between the roots of the auxiliary cubic equation and the biquadratic equation, and the circumstances under which the cubic equation falls under the irreducible case, apply to Euler's method of solution as well as to Descartes's.

192. It may happen that special forms of biquadratic equations admit of simpler solution than the general equation. The following is an example. The biquadratic equation

$$x^4+px^3+qx^2+rx+s=0$$

can be solved as a quadratic equation if $p^3-4pq+8r=0$. For the equation $x^4+px^3+qx^2+rx+s=0$ may be written

$$x^2\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)x\left(x+\frac{r}{q-\frac{p^2}{4}}\right)+s=0;$$

and this may be solved as a quadratic equation, if $\frac{r}{q-\frac{p^2}{4}}=\frac{p}{2}$, that is, if $p^3-4pq+8r=0$.

Some valuable remarks on Biquadratic Equations by Professor R. S. Ball will be found in the *Quarterly Journal of Mathematics*, Vol. VII. 1866.

XIV. STURM'S THEOREM.

193. In the preceding Chapters of the present work we have demonstrated various theorems respecting the roots of equations, and have given the algebraical solution of equations of the third and fourth degrees. We are now about to enter upon a different part of the subject, namely, the methods of finding approximately the numerical values of the roots of equations; the present Chapter commences this part of the subject by proving Sturm's theorem, the object of which is to determine the situation and the number of the real roots of any equation. We shall enunciate and prove the theorem in the next Article; we shall then give some remarks connected with the theorem, and finally apply it to some examples.

194. *Sturm's Theorem.* Let $f(x)=0$ be an equation cleared of equal roots, and let $f_1(x)$ be the first derived function of $f(x)$; let the operation of finding the greatest common measure of $f(x)$ and $f_1(x)$ be performed with this modification, that the sign of every remainder is changed before it is used as a divisor, and let the operation be continued until the remainder is obtained which is independent of x, and change the sign of that remainder also.

Let $f_2(x), f_3(x), \ldots f_m(x)$, be the series of modified remainders thus obtained. Let α be any quantity, and β another which is algebraically greater, then the number of real roots of the equation $f(x)=0$ between α and β is the excess of the number of changes of sign in the series $f(x), f_1(x), f_2(x), \ldots f_m(x)$, when $x=\alpha$, over the number of changes of sign when $x=\beta$.

We shall call the *whole* series $f(x), f_1(x), f_2(x), \ldots f_m(x)$, *Sturm's functions*, and we shall call the series $f_1(x), f_2(x), \ldots f_m(x)$, the *auxiliary functions*, so that the auxiliary functions consist of Sturm's functions omitting $f(x)$.

Let $q_1, q_2, \ldots q_{m-1}$, denote the successive quotients which arise in performing the operations indicated; then we have the following relations,

$$f(x)=q_1 f_1(x)-f_2(x),$$
$$f_1(x)=q_2 f_2(x)-f_3(x),$$
$$f_2(x)=q_3 f_3(x)-f_4(x),$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$f_{m-2}(x)=q_{m-1} f_{m-1}(x)-f_m(x).$$

From these relations we can draw three inferences.

(1) The last of the functions $f_m(x)$ is not zero; for by supposition it is independent of x and if it were zero $f(x)$ and $f_1(x)$ would have a common measure, and then the equation $f(x)=0$ would have equal roots by Art. 75, and this is contrary to the hypothesis.

(2) Two consecutive auxiliary functions cannot vanish simultaneously; for if they could all the succeeding auxiliary functions would vanish including $f_m(x)$; and this is impossible by (1).

(3) When any auxiliary function vanishes the two adjacent functions have contrary signs. Suppose for example that $f_3(x)=0$; then from the third of the above system of relations we have $f_2(x)=-f_4(x)$.

Now no alteration can be made in the sign of any one of Sturm's functions except when x passes through a value which makes that function vanish; and we shall now prove that when x passes through a value which makes $f(x)$ vanish one change of sign is lost by Sturm's functions, and that no change of sign is lost or gained in consequence of x passing through a value which makes one of the auxiliary functions vanish.

I. Suppose c a root of the equation $f(x) = 0$, so that $f(c) = 0$.

Let h be a positive quantity. Now $f(c-h)$ may be expanded in powers of h by Art. 10, and h may be taken so small that the sign of the whole series shall be the same as the sign of the first term that does not vanish, by Art. 14; that is, the sign of $f(c-h)$ will be the same as the sign of $-hf_1(c)$ since $f(c) = 0$. The sign of $f_1(c-h)$ will be the same as the sign of $f_1(c)$ when h is taken small enough. Thus if $x = c - h$ and h is taken small enough, $f(x)$ and $f_1(x)$ *have contrary signs.*

Similarly, it may be shewn that if $x = c + h$ and h is taken small enough, $f(x)$ and $f_1(x)$ have the *same sign.*

Thus as x increases through a root of the equation $f(x) = 0$, Sturm's functions *lose one change of sign.*

II. Let c now denote a value of x which makes one of the auxiliary functions vanish, for example, $f_r(x)$, so that $f_r(c) = 0$. Then $f_{r-1}(c)$ and $f_{r+1}(c)$ have *contrary signs*, and thus just before $x = c$ and also just after $x = c$, the three terms $f_{r-1}(x)$, $f_r(x)$, $f_{r+1}(x)$ will present one permanence of sign and one change of sign; for if $f_{r-1}(x)$ and $f_r(x)$ have the same sign, $f_r(x)$ and $f_{r+1}(x)$ have contrary signs, and *vice versa.* Thus Sturm's functions neither lose nor gain a change of sign when x passes through a value which makes one of the auxiliary functions vanish.

No value of x can make two *consecutive* functions simultaneously vanish. If two or more vanish simultaneously which are not consecutive, then, if $f(x)$ be one of them, it follows by I. that a change of sign is lost as x increases through that value, and if $f(x)$ be not one of them it follows by II. that no change of sign is lost.

Thus we have proved that as x increases, Sturm's functions never lose a change of sign except when x passes through a root of the equation $f(x) = 0$, and never gain a change of sign. Hence the number of changes of sign lost as x increases from any value α to a greater value β, is equal to the number of the roots of the equation $f(x)=0$ which lie between α and β.

195. We have shewn that no alteration occurs in the *number* of the changes of sign in Sturm's functions in consequence of x passing through a value which makes one of the auxiliary functions vanish; but alterations may take place, and in general do take place, with respect to the order in which the signs + and − are distributed among the series of functions. Suppose, for example, that a and b are two roots of the equation $f(x)=0$ and that a is less than b; then $f(x)$ and $f_1(x)$ have contrary signs *just before* $x=a$ and have the same sign *just after* $x=a$. Now *just before* $x=b$ the signs of $f(x)$ and $f_1(x)$ are again contrary. In fact the equation $f_1(x)=0$ has one root between $x=a$ and $x=b$, and so $f_1(x)$ must pass from positive to negative or *vice versa* between $x=a$ and $x=b$. This transition of $f_1(x)$ from positive to negative or *vice versa* between a and b, cannot alter the whole number of changes of sign in the series of Sturm's functions, as we have proved, but it does modify the distribution of the signs + and − among the series, and thus renders it possible after a change has been lost as x increases through a, for another change to be lost as x increases through b.

The present Article adds nothing to the proof of Sturm's theorem; but is merely intended to assist a student in the difficulty which is often felt as to how the changes of sign are lost.

196. In counting the number of changes of sign in the series of Sturm's functions, it may happen that the value of x which we are considering makes one of the auxiliary functions vanish. Then it is indifferent whether we ascribe the positive sign or the negative sign to the vanishing function, since the signs of the functions which precede and follow it are necessarily contrary.

197. In order to find the whole number of real roots of an equation $f(x)=0$, we may first put $-\infty$ for x and then $+\infty$ for x in Sturm's functions; the excess of the number of changes of sign in

the first case over the number of changes of sign in the second case is the whole number of real roots. When x is made equal to $+\infty$ or $-\infty$ the sign of any one of the functions will be the same as the sign of the highest power of x in that function.

198. Let n denote the degree of $f(x)$; then the number of the auxiliary functions $f_1(x)$, $f_2(x)$,...will in general also be n; because each remainder is generally of one degree lower than the preceding remainder. We will suppose that the number of auxiliary functions is the same as the degree of $f(x)$, and we will suppose that the highest power of x in $f(x)$ has a positive coefficient.

(1) If the first terms in all the auxiliary functions have positive coefficients all the roots of the equation $f(x)=0$ are real. For all Sturm's functions will then be positive when $x=+\infty$, and they will be alternately positive and negative when $x=-\infty$; thus n changes of sign are lost as x passes from $-\infty$ to $+\infty$.

(2) If the coefficients of the first terms are not all positive, there will be a pair of imaginary roots for every change of sign in the series formed of these coefficients. For suppose that in this series of coefficients there are m changes of sign and $n-m$ continuations of sign. Then when $x=+\infty$ there are m changes of sign and $n-m$ continuations of sign in Sturm's functions. Now change x from $+\infty$ to $-\infty$; then the changes of sign are replaced by continuations of sign and the continuations of sign by changes of sign, so that for $x=-\infty$ there are $n-m$ changes of sign. The excess of the number of changes of sign when $x=-\infty$ over the number when $x=+\infty$ is therefore $n-2m$; thus there are $n-2m$ real roots of the equation $f(x)=0$, and therefore $2m$ imaginary roots.

Hence in order that an equation may have all its roots real, it is necessary and sufficient that the coefficients of the first terms in all the auxiliary functions should be of the same sign.

199. Suppose that among the auxiliary functions we find one, as $f_r(x)$, which cannot change its sign; then we may disregard all the functions which follow it, and count only the number of changes of sign in the series $f(x)$, $f_1(x)$, $f_2(x)$,...$f_r(x)$. For in the original

demonstration of Sturm's theorem the necessary property of the last auxiliary function is that it *should not vanish*, and as $f_r(x)$ cannot vanish, the demonstration will hold for the series $f(x)$, $f_1(x)$, $f_2(x)$, ... $f_r(x)$.

This remark is of practical importance, because the labour attending the formation of Sturm's functions is considerable in examples of equations of high degrees, and thus it is useful to have a rule which sometimes relieves us from the necessity of forming the entire series of functions.

200. Suppose $\phi(x)$ to be a function which has no factor in common with $f(x)$, and suppose that $\phi(x)$ and $f_1(x)$ take the same sign when any root of the equation $f(x) = 0$ is substituted for x in them. Then we may use $\phi(x)$ instead of $f_1(x)$ and deduce the remaining auxiliary functions from $f(x)$ and $\phi(x)$ instead of from $f(x)$ and $f_1(x)$. For on recurring to the demonstration of Sturm's theorem it will be seen that with this new set of functions the two fundamental properties are still true, namely, that no change of sign is lost owing to the vanishing of any auxiliary function, and that a change of sign is lost when $f(x)$ vanishes.

201. We have hitherto supposed that the equation to be treated by Sturm's method is cleared of equal roots; we shall now shew that this limitation is unnecessary, and that the theorem will always give the number of *distinct* roots between assigned limits, no regard being had to the *repetition* of any roots.

Suppose for example that the root a occurs p times and the root b occurs q times in the equation $f(x) = 0$.

Let $$f(x) = (x-a)^p (x-b)^q (x-c)(x-d)\ldots$$

then $$f_1(x) = (x-a)^{p-1}(x-b)^{q-1}\left\{\begin{aligned} &p(x-b)(x-c)(x-d)\ldots \\ &+q(x-a)(x-c)(x-d)\ldots \\ &+\ldots \end{aligned}\right\}$$

Thus $(x-a)^{r-1}(x-b)^{r-1}$ is the greatest common measure of $f(x)$ and $f_1(x)$, and this expression will divide all the auxiliary functions $f_2(x), f_3(x), \ldots f_m(x)$ which are formed as in Art. 194.

Now let $\psi(x) = (x-a)(x-b)(x-c)(x-d)\ldots$

and
$$\begin{aligned}\phi(x) = p\,&(x-b)(x-c)(x-d)\ldots \\ &+ q(x-a)(x-c)(x-d)\ldots \\ &+ (x-a)(x-b)(x-d)\ldots \\ &+ \ldots\end{aligned}$$

Then $\phi(x)$ is not the first derived function of $\psi(x)$, for that would be what $\phi(x)$ would become if $p=1$ and $q=1$; but $\phi(x)$ has the same sign as the first derived function of $\psi(x)$, when we make $x=a$, or b, or $c, \ldots$ Hence, by Art. 200, we may determine the situation of the real roots of the equation $\psi(x)=0$ by taking $\psi(x)$ and $\phi(x)$ as the first two of Sturm's functions and forming the rest from them.

But the series of Sturm's functions formed from $f(x)$ and $f_1(x)$ only differs from the series formed from $\psi(x)$ and $\phi(x)$ by reason of the additional factor $(x-a)^{p-1}(x-b)^{q-1}$ in every term of the series. Thus when any value is ascribed to x, the signs of the terms in the former series will all be the same as those of the latter, or all contrary; and thus the number of changes of sign will be the same.

Hence by examining the series of Sturm's functions formed from $f(x)$ and $f_1(x)$ we can ascertain how many of the roots of the equation $\psi(x)=0$ lie between assigned limits, that is, how many *distinct and separate* roots of the equation $f(x)=0$ lie between those limits.

Thus we need not apply the test for equal roots before we apply Sturm's method; in fact, in calculating Sturm's functions we shall be warned of equal roots if they exist by the fact that the last remainder will be zero.

202. We may observe that in the operation by which all the auxiliary functions after the first are found, we may always multiply or divide the divisors or dividends by any *positive* number we please, as in the operation of finding the greatest common measure; for the auxiliary functions thus only become multiplied or divided by positive numbers, so that their signs remain unchanged.

We may by Sturm's theorem determine the number of real roots of any proposed equation. Then, by substituting successive integers for x in the series of Sturm's functions, we can determine between what consecutive integers the roots lie; or if it is found that more than one root lies between two assigned integers, we can substitute for x successively fractions which lie between those integers, until we at last determine intervals between which the roots lie singly.

203. We will now take some examples.

Suppose $f(x) = x^3 - 3x^2 - 4x + 13 = 0$.

Here
$$f_1(x) = 3x^2 - 6x - 4,$$
$$f_2(x) = 2x - 5,$$
$$f_3(x) = +1.$$

The roots of the equation are all real by Art. 198. The following is the series of signs corresponding to the values of x indicated.

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$
0	+	−	−	+
1	+	−	−	+
2	+	−	−	+
3	+	+	+	+

Here there are two changes of sign when $x = 2$, and none when $x = 3$; thus there are two positive roots between 2 and 3, and no other positive roots.

It will be found that when $x = -3$, the succession of signs is $-+-+$, and when $x = -2$ it is $++-+$, so that one change of sign is lost in proceeding from -3 to -2, and therefore the negative root lies between -2 and -3. To separate the two roots which lie between 2 and 3 we should substitute for x some number or numbers lying between 2 and 3. Suppose, for example, we put $x = 2\frac{1}{2}$; then the succession of signs is $--0+$, and thus we have only one change of sign, whether we consider the 0 to carry the sign $+$ or $-$. Thus a change of sign is lost in proceeding from 2 to $2\frac{1}{2}$, and therefore one root lies between 2 and $2\frac{1}{2}$; hence the other root lies between $2\frac{1}{2}$ and 3.

Again, suppose $f(x) = x^4 - 6x^3 + 5x^2 + 14x - 4 = 0$.

Here $\qquad f_1(x) = 2x^3 - 9x^2 + 5x + 7$, omitting a factor 2,

$$f_2(x) = 17x^2 - 57x - 5,$$

$$f_3(x) = 152x - 457,$$

$$f_4(x) = +.$$

In this example it will be found that the calculation of $f_4(x)$ is somewhat complicated; it is sufficient for our purpose however to know the *sign*, and thus when we ascertain that it is *positive* we need not calculate it exactly, but merely put down $f_4(x) = +$.

The roots of the equation are all real by Art. 198.

The following is the series of signs corresponding to the values of x indicated.

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$
-2	+	−	+	−	+
-1	−	−	+	−	+
0	−	+	−	−	+
1	+	+	−	−	+
2	+	−	−	−	+
3	+	−	−	−	+
4	+	+	+	+	+

There is one change of sign lost between -2 and -1, one between 0 and 1, and two between 3 and 4.

If we put $3\frac{1}{2}$ for x the succession of signs is $- 0 + + +$, and thus there is only one change of sign, so that one root of the equation lies between 3 and $3\frac{1}{2}$; therefore another root lies between $3\frac{1}{2}$ and 4.

Again, suppose $f(x) = 2x^4 - 13x^2 + 10x - 49 = 0$.

Here $\quad f_1(x) = 4x^3 - 13x + 5$, omitting a factor 2,

$$f_2(x) = 13x^2 - 15x + 98.$$

It is easy to see that the roots of the equation $f_2(x) = 0$ are imaginary, that is, $f_2(x)$ cannot vanish for any real value of x; therefore by Art. 199 we need not obtain any more of Sturm's functions in this example. When $x = -\infty$ the succession of signs is $+ - +$, and when $x = +\infty$ the succession of signs is $+ + +$; thus the equation has two real roots and two imaginary roots. One of the real roots is positive and the other negative by Art. 21.

XV. FOURIER'S THEOREM.

204. Sturm's theorem constitutes the complete solution of a problem which has engaged the attention of many of the most eminent mathematicians during the last two hundred years; this theorem was published in the volume of *Mémoires présentés...... par des Savants Étrangers*, Paris, 1835.

Among those who attempted the solution of the problem before Sturm two are deserving of especial notice, Budan and Fourier; the methods of these two mathematicians start from a theorem which English writers usually call *Fourier's theorem*, and which French writers connect with the name of Budan as well as with that of Fourier. Fourier's work on equations was published in 1831 after the death of the author; Budan published a work on the subject in 1807. There is evidence however that Fourier had given the theorem in a course of lectures delivered before the publication of Budan's work. We will now enunciate and prove the theorem.

205. *Fourier's Theorem.* Let $f(x)$ be an algebraical function of the n^{th} degree; let $f_1(x), f_2(x), \ldots f_n(x)$ be the successive derived functions of $f(x)$. Let α be any quantity and β another which is algebraically greater; then the number of the real roots of the equation $f(x)=0$ between α and β, cannot be greater than the excess of the number of the changes of sign in the series $f(x), f_1(x), f_2(x), \ldots f_n(x)$, when $x=\alpha$, over the number of the changes of sign when $x=\beta$.

We shall call the whole series $f(x), f_1(x), f_2(x), \ldots f_n(x)$, *Fourier's functions.*

No alteration can occur in the sign of any one of Fourier's functions except when x passes through a value which makes that function vanish. We shall now have four cases to consider.

I. Suppose when $x=c$ that $f(x)$ vanishes and that $f_1(x)$ does not vanish. Put $c-h$ for x where h is a positive quantity; then h may be taken so small that the sign of $f(c-h)$ is the same as that of $-hf_1(c)$, and the sign of $f_1(c-h)$ the same as that of $f_1(c)$; see Art. 14. Thus if $x=c-h$ and h is taken small enough, $f(x)$ and $f_1(x)$ have *contrary signs.*

Similarly it may be shewn that if $x=c+h$ and h is taken small enough, $f(x)$ and $f_1(x)$ have the *same sign.*

Thus as x increases through a value c, which is an unrepeated root of the equation $f(x)=0$, Fourier's functions *lose one change of sign.*

II. Suppose when $x=c$ that $f(x)$ vanishes and also the derived functions $f_1(x), f_2(x), \ldots$ up to $f_{r-1}(x)$, and that $f_r(x)$ does not vanish. Put $c-h$ for x where h is a positive quantity; then h may be taken so small that the signs of the series of terms

$$f(c-h), \quad f_1(c-h), \quad f_2(c-h), \ldots\ldots f_{r-1}(c-h), \quad f_r(c-h)$$

shall be respectively the same as the signs of the series of terms

$$(-h)^r f_r(c), \quad (-h)^{r-1} f_r(c), \quad (-h)^{r-2} f_r(c), \ldots -hf_r(c), \; f_r(c);$$

see Arts. 10 and 14. Thus if $x=c-h$ and h is taken small enough, the first $r+1$ of Fourier's functions present r changes of sign.

Similarly it may be shewn that if $x=c+h$ and h is taken small enough, the first $r+1$ of. Fourier's functions present no change of sign.

Thus as x increases through a value c which is a root of the equation $f(x)=0$ repeated r times, Fourier's functions *lose* r *changes of sign.*

III. Suppose when $x=c$ that *one* of the derived functions vanishes, but neither of the two adjacent functions; thus let $f_r(x)$ vanish when $x=c$ but neither $f_{r-1}(x)$ nor $f_{r+1}(x)$. Then if h is taken small enough, when $x=c-h$ the signs of the three terms $f_{r-1}(x)$, $f_r(x)$, $f_{r+1}(x)$, are respectively the same as the signs of $f_{r-1}(c)$, $-hf_{r+1}(c)$, $f_{r+1}(c)$, and when $x=c+h$ the signs are the same as the signs of $f_{r-1}(c)$, $hf_{r+1}(c)$, $f_{r+1}(c)$. Thus if $f_{r-1}(c)$ and $f_{r+1}(c)$ have the *same* sign, Fourier's functions lose two changes of sign as x increases through c, and if $f_{r-1}(c)$ and $f_{r+1}(c)$ have *contrary* signs Fourier's functions neither gain nor lose a change of sign.

IV. Suppose when $x=c$ that several successive derived functions vanish; for example, suppose when $x=c$ that the m functions $f_r(x)$, $f_{r+1}(x)$, $\ldots f_{r+m-1}(x)$ vanish, and that $f_{r-1}(x)$ and $f_{r+m}(x)$ do not vanish. By proceeding as before, and supposing h taken small enough and positive, we shall obtain the following results with respect to the $m+2$ terms, $f_{r-1}(x)$, $f_r(x)$, $\ldots f_{r+m-1}(x)$, $f_{r+m}(x)$.

(1) Let m be even. If $f_{r-1}(c)$ and $f_{r+m}(c)$ have the *same* sign, the terms present m changes of sign when $x=c-h$, and no change of sign when $x=c+h$. If $f_{r-1}(c)$ and $f_{r+m}(c)$ have *contrary* signs, the terms present $m+1$ changes of sign when $x=c-h$, and one change of sign when $x=c+h$. Thus in both cases Fourier's functions lose m changes of sign as x increases through c.

(2) Let m be odd. If $f_{r-1}(c)$ and $f_{r+m}(c)$ have the *same* sign the terms present $m+1$ changes of sign when $x=c-h$, and no change of sign when $x=c+h$. Thus Fourier's functions lose $m+1$

changes of sign as x increases through c. If $f_{r-1}(c)$ and $f_{r+m}(c)$ have *contrary* signs, the terms present m changes of sign when $x = c - h$, and one change of sign when $x = c + h$. Thus Fourier's functions lose $m - 1$ changes of sign as x increases through c.

Thus on the whole Fourier's functions never gain a change of sign, but they do lose one change of sign when x increases through a root of the equation $f(x) = 0$; and thus the theorem is proved.

206. It will be observed that the demonstration of Art. 205 gives us something more than the enunciation to which for simplicity we confined ourselves. For it appears that whenever an alteration occurs in the number of the changes of sign of Fourier's functions, except by reason of the variable increasing through a root of the given equation, an *even* number of changes of sign is lost. Thus on the whole we have the following result if we substitute successively a number α and a greater number β in Fourier's functions.

(1) Suppose that Fourier's functions lose no change of sign; then no root of the given equation lies between α and β.

(2) Suppose that Fourier's functions lose an *odd* number of changes of sign; then we are certain that some odd number of roots lies between α and β, but cannot tell what odd number, except when only *one* change of sign is lost, and then we are certain of *one* root.

(3) Suppose that Fourier's functions lose an *even* number of changes of sign; then we can only infer that there is either no root or else some even number of roots between α and β.

207. The advantage of Fourier's theorem is that it can be easily applied, because the successive derived functions of a given function can be immediately formed. The disadvantage of the theorem is that it may require an almost unlimited number of trials. For if two roots are very nearly equal, it would require very minute subdivision of an interval in which they were conjectured to lie, in order to distinguish them from two imaginary

roots. It would be necessary to apply the test for equal roots before beginning Fourier's process, as otherwise an even number of repeated roots might remain undiscovered.

208. Budan and Fourier both gave methods for examining a doubtful interval more closely in order to discover whether roots of the proposed equation were or were not situated in the interval. But it is unnecessary to explain these methods since Sturm's theorem attains the proposed object with simplicity and certainty.

209. It may be shewn that Descartes's *rule of signs* is included in Fourier's Theorem.

Suppose that $f(x)=0$ is a *complete* equation.

If we put $x=0$ in Fourier's functions the signs are the same as the signs in the expression $f(x)$ taken from right to left; and if we put $x=\infty$ in Fourier's functions the signs are all positive. Hence, by Fourier's theorem, the equation $f(x)=0$ cannot have more positive roots than $f(x)$ has changes of sign.

If the proposed equation be not complete, we may suppose the absent terms supplied with zero coefficients, and such signs may be ascribed to these coefficients as to make Fourier's functions have the same number of changes of sign when these terms are counted as when they are omitted.

The part of the *rule of signs* which relates to the negative roots can be deduced from that part of it which refers to the positive roots; see Art. 63.

210. Fourier's theorem also includes the rule given by Newton for finding a superior limit to the positive roots of an equation; see Art. 94. For if $f(x)=0$ be the equation, Newton's method directs us to find h such that when $x=h$ Fourier's functions are all positive; and then by Fourier's theorem no roots of the proposed equation exist between $x=h$ and $x=+\infty$.

XVI. LAGRANGE'S METHOD OF APPROXIMATION.

211. We have already shewn how the commensurable roots of an equation may be found; we shall now consider how the approximate numerical values of the real incommensurable roots may be calculated.

By Sturm's theorem we can always determine how many roots lie within a given interval, and we may then divide that interval into smaller intervals within which the roots lie singly. Suppose then that we know that an equation has one root and only one between two given quantities α and β, and we wish to approximate to the value of this root. If we substitute any quantity γ which is intermediate between α and β for x in $f(x)$, we shall know by the sign of $f(\gamma)$ whether the root lies between α and γ or between γ and β. Suppose it to lie between α and γ; then we may substitute for x a quantity δ which lies between α and γ, and we shall know by the sign of $f(\delta)$ whether the root lies between α and δ or between δ and γ. This process may be continued to any extent, and we may approximate as closely as we please to the numerical value of the root; for by each operation we can thus halve the interval within which the root must lie.

The operation here described would however be very laborious and methods have been proposed for attaining the required result, with less calculation. We shall first explain Lagrange's method.

212. Let $f(x)=0$ be an equation which is known to have one root, and only one, between two consecutive positive integers a and $a+1$. Put $x=a+\frac{1}{y}$, and substitute this value of x in the proposed equation; thus $f\left(a+\frac{1}{y}\right)=0$. If we clear this equation of fractions, we obtain an equation in y of the same degree as the original equation in x; denote it by $\phi(y)=0$. This equation in y has only one positive integral root, since the original equation in x

has only one root between a and $a+1$. We may then determine the consecutive integers between which the value of y must lie, by substituting in $\phi(y)$ successively the values 1, 2, 3,... until two consecutive results are obtained which are of contrary signs. Suppose it is thus found that y lies between b and $b+1$. Put $y=b+\frac{1}{z}$, and substitute; thus $\phi\left(b+\frac{1}{z}\right)=0$. Hence, as before, we obtain an equation in which the unknown quantity has only one positive root, and we may determine the consecutive integers between which the value of z must lie; let these be c and $c+1$. Then put $z=c+\frac{1}{u}$; and so on.

Thus we shall obtain the required value of x to any degree of approximation in the form of a continued fraction, namely,

$$x=a+\cfrac{1}{b+\cfrac{1}{c+\dots}}$$

213. Next suppose that the equation $f(x)=0$ has more than one root lying between the integers a and $a+1$. By Sturm's theorem, or by some other method of separating the roots, we may determine by what number the roots of the equation which lie between the same two consecutive integers must be multiplied in order that the products may lie between different consecutive integers. Transform the equation into another whose roots are those of the proposed equation multiplied by the number thus determined; and then the method of the preceding Article may be applied to the transformed equation.

Or we may adopt the method of the preceding Article without effecting this transformation. In this case the equation in y will have more than one positive root and we must seek the greatest integer in each root, and then proceed to the separate calculation of the several resulting values of z. It may happen that the equation in y has more than one root between certain consecutive integers; then the equation in z may be used to discriminate them, and the calculation of each root continued; and so on.

214. From the given equation $f(x)=0$ we deduce $f\left(a+\frac{1}{y}\right)=0$, that is, supposing $f(x)$ of the degree n,

$$f(a)+\frac{1}{y}f'(a)+\frac{1}{y^2}\frac{f''(a)}{2}+\frac{1}{y^3}\frac{f'''(a)}{\underline{|3}}+\ldots+\frac{1}{y^n}\frac{f^n(a)}{\underline{|n}}=0\,;$$

multiply by y^n and we obtain

$$y^nf(a)+y^{n-1}f'(a)+y^{n-2}\frac{f''(a)}{2}+\ldots+\frac{f^n(a)}{\underline{|n}}=0.$$

Thus in order to form the equation in y we must calculate the numerical values of $f(a)$, $f'(a)$, $f''(a)$,...; these calculations may be performed in the manner explained in Art. 5; but, as we have stated in Art. 11, the best method will be explained hereafter in the Chapter on Horner's method. A similar remark holds with respect to the formation of the equation in z.

By referring to Arts. 54 and 58, we see that Lagrange's method of approximation may be thus stated. Suppose a root of an assigned equation to lie between a and $a+1$, diminish the roots of the equation by a, and take the reciprocal equation. Find a root of the last equation lying between integers b and $b+1$, diminish the roots by b, and take the reciprocal equation. Find a root of the last equation lying between integers c and $c+1$, diminish the roots by c, and take the reciprocal equation. Proceed in this way. Then the continued fraction

$$a+\cfrac{1}{b+\cfrac{1}{c+\ldots}}$$

is a root of the original equation.

215. Example. $x^3-2x-5=0$.

By Art. 108, this equation has only one real root; and by Art. 20, this root must be a positive quantity; it will be found on trial to lie between 2 and 3.

Assume $x = 2 + \frac{1}{y}$; then

$$f(2) = 2^3 - 2 \,.\, 2 - 5 = -1,$$
$$f'(2) = 3 \,.\, 2^2 - 2 = 10,$$
$$\tfrac{1}{2} f''(2) = 3 \,.\, 2 = 6,$$

and the equation in y is $-y^3 + 10y^2 + 6y + 1 = 0$, that is,

$$y^3 - 10y^2 - 6y - 1 = 0, \text{ say } \phi(y) = 0.$$

Here $y = 10$ makes $\phi(y)$ negative, and $y = 11$ makes $\phi(y)$ positive; therefore the required value of y must lie between 10 and 11. Assume $y = 10 + \frac{1}{z}$; then

$$\phi(10) = 10^3 - 10 \,.\, 10^2 - 6 \,.\, 10 - 1 = -61,$$
$$\phi'(10) = 3 \,.\, 10^2 - 20 \,.\, 10 - 6 = 94,$$
$$\tfrac{1}{2}\phi''(10) = 3 \,.\, 10 - 10 = 20,$$

and the equation in z is $-61z^3 + 94z^2 + 20z + 1 = 0$, that is,

$$61z^3 - 94z^2 - 20z - 1 = 0, \text{ say } \psi(z) = 0.$$

Here $z = 2$ makes $\psi(z)$ positive, so that the required value of z must lie between 1 and 2. Assume $z = 1 + \frac{1}{u}$; then

$$\psi(1) = 61 \,.\, 1^3 - 94 \,.\, 1^2 - 20 \,.\, 1 - 1 = -54,$$
$$\psi'(1) = 183 \,.\, 1^2 - 188 \,.\, 1 - 20 = -25,$$
$$\tfrac{1}{2}\psi''(1) = 183 \,.\, 1 - 94 = 89,$$

and the equation in u is $-54u^3 - 25u^2 + 89u + 61 = 0$, that is,

$$54u^3 + 25u^2 - 89u - 61 = 0.$$

This equation shews that the value of u must lie between 1 and 2; and we may proceed as before.

Hence
$$x = 2 + \cfrac{1}{10 + \cfrac{1}{1 + \cfrac{1}{1 + \&c.}}}$$

The convergents corresponding to this continued fraction are $\frac{2}{1}, \frac{21}{10}, \frac{23}{11}, \frac{44}{21}, \ldots\ldots$ See *Algebra*, Chapter XLIV. The difference between $\frac{44}{21}$ and the real value of the root is less than $\frac{1}{21(21+11)}$, that is, less than $\frac{1}{672}$.

For another example take the equation $x^3 - 7x + 7 = 0$. By Art. 108 this equation has all its roots real; and by Sturm's theorem it may be shewn that one root lies between 1 and $1\frac{1}{2}$, and that another root lies between $1\frac{1}{2}$ and 2. Therefore if we put $x = \frac{x'}{2}$ and form an equation in x' this equation will have one root between 2 and 3, and one root between 3 and 4. The equation in x' is $\left(\frac{x'}{2}\right)^3 - 7\frac{x'}{2} + 7 = 0$, that is, $x'^3 - 28x' + 56 = 0$.

The root which lies between 2 and 3 will be found to be

$$2 + \cfrac{1}{1 + \cfrac{1}{2 + \&c.}}$$

The root which lies between 3 and 4 will be found to be

$$3 + \cfrac{1}{2 + \cfrac{1}{1 + \&c.}}$$

The roots of the original equation will be obtained by taking half of each of these values.

Or we may apply Lagrange's method to the original equation without any preliminary transformation. Assume $x = 1 + \frac{1}{y}$; thus $\left(1 + \frac{1}{y}\right)^3 - 7\left(1 + \frac{1}{y}\right) + 7 = 0$. This will give $y^3 - 4y^2 + 3y + 1 = 0$, say $\phi(y) = 0$. Here $\phi(1)$ is positive, $\phi(2)$ is negative, and $\phi(3)$ is positive; thus one value of y must lie between 1 and 2, and the other between 2 and 3. Then we may put $y = 1 + \frac{1}{z}$ in order to continue the approximation to the first root, and $y = 2 + \frac{1}{z}$ in order to continue the approximation to the second root.

The equation $x^3 - 7x + 7 = 0$ has one negative root; we may find it by changing x into $-x$ and calculating the positive root of the resulting equation, that is of the equation

$$(-x)^3 - 7(-x) + 7 = 0.$$

Or since the sum of the three roots of the equation $x^3 - 7x + 7 = 0$ is zero, when two of the roots are calculated approximately the third can be immediately found approximately.

216. If in following Lagrange's method we arrive at an equation which has an integer for a root, we obtain a finite continued fraction as a root of the original equation, that is, we obtain a *commensurable* fractional root. This of course cannot occur if we have previously determined all the commensurable roots both whole and fractional of any proposed equation, and removed the corresponding factors by division.

217. It may happen that in following Lagrange's method we arrive at an equation which is identical with one of those which preceded it; in this case the quotients of the continued fraction recur, so that the continued fraction is a periodic continued fraction and its value can be found by solving a quadratic equation; see *Algebra*, Chapter XLV. The roots of this quadratic equation will involve a quadratic surd, and *both* of the roots will be roots of the proposed equation by Art. 44.

218. We will here give the general process which has been exemplified in Art. 215 in the second method of treating the equation $x^3 - 7x + 7 = 0$. The object in view, is to apply Lagrange's method of approximation when a proposed equation has more than one root between consecutive integers. Let $f(x) = 0$ be the proposed equation; form the *auxiliary functions* $f_1(x), f_2(x), f_3(x), \ldots$

which occur in Sturm's theorem, stopping when one is obtained which is positive for all values of x; see Art. 199. Suppose that more than one root of the proposed equation lies between the consecutive integers a and $a+1$. Put $a+\frac{1}{y}$ for x in the functions $f(x), f_1(x), f_2(x), \ldots$, and denote what they become respectively by $F(y)$, $F_1(y)$, $F_2(y), \ldots$ If in the latter series of functions we substitute successively any two numbers, as b and $b+1$, the difference of the numbers of the changes of sign in the two cases will give us the number of roots of the equation $F(y)=0$ which lie between b and $b+1$. For the results which we obtain by substituting b and $b+1$ in $F(y)$, $F_1(y)$, $F_2(y), \ldots$, are the same as those we should obtain by substituting respectively $a+\frac{1}{b}$ and $a+\frac{1}{b+1}$ in the series $f(x), f_1(x), f_2(x), \ldots$; and therefore the difference of the numbers of the changes of sign must be equal to the number of the roots of the equation $f(x)=0$ which lie between $a+\frac{1}{b}$ and $a+\frac{1}{b+1}$, that is, to the number of the roots of the equation $F(y)=0$ which lie between b and $b+1$.

If then we find that more than one value of y lies between the consecutive integers b and $b+1$, we substitute $b+\frac{1}{z}$ for y in the series $F(y)$, $F_1(y)$, $F_2(y), \ldots$; then, by giving two consecutive integral values successively to z and substituting them we can determine whether more than one value of z lies between two consecutive integers.

We proceed in this way until we obtain an equation which has only one root between consecutive integers; and after that we need not pay any regard to Sturm's functions but continue the calculation for this particular root by the method of Art. 212.

Thus we are able to separate the roots and can calculate them without any omissions.

As we do not require to know the *values*, but only the *signs* of $F(y)$, $F_1(y)$, $F_2(y), \ldots$, we may in all cases multiply these functions by such powers of y as will clear them of fractions; for y is supposed to be a positive quantity, and therefore any power of y is positive. Thus, for example, instead of $F(y)$, that is, instead of $f\left(a+\frac{1}{y}\right)$, we may use

$$y^n f(a) + y^{n-1} f'(a) + \frac{y^{n-2}}{2} f''(a) + \dots + \frac{1}{\lfloor n} f^n(a),$$

supposing that $f(x)$ is of the degree n.

XVII. NEWTON'S METHOD OF APPROXIMATION WITH FOURIER'S ADDITIONS.

219. We shall now explain Newton's method of approximation to the numerical value of a root of an equation.

Let $f(x)=0$ be an equation which has a root between certain limits α and β the difference of which is a small fraction; let c be a quantity between α and β which is assumed as a first approximation to the required root, and let $c+h$ denote the exact value of the root, so that h is a small fraction which is to be determined. Thus $f(c+h)=0$, that is, by Art. 10,

$$f(c) + hf'(c) + \frac{h^2}{1 \cdot 2} f''(c) + \frac{h^3}{\lfloor 3} f'''(c) + \dots + \frac{h^n}{\lfloor n} f^n(c) = 0.$$

Now since h is supposed to be a small fraction h^2, h^3,... will be small compared with h; if we neglect the squares and higher powers of h in the above equation we obtain $f(c)+hf'(c)=0$; thus

$$h = -\frac{f(c)}{f'(c)}.$$

Supposing then that we thus obtain an approximation to the value of h, we have $c-\frac{f(c)}{f'(c)}$ as a new approximation to the root of the proposed equation. Denote this new approximation by c_1, and then proceeding as before we obtain $c_1-\frac{f(c_1)}{f'(c_1)}$ as a new approximation; and so on.

We shall presently examine more closely the conditions which must hold in order that this method may be safely applied. It is of course obvious that such examination is necessary, since the process is not universally applicable; for if $f'(c)$ is small compared with $f(c)$ the supposed approximate value of h is not a small fraction as it should be.

220. As an example of Newton's method we will take the equation which Newton himself selected, namely, $x^3-2x-5=0$, say $f(x)=0$. Here $x=2$ makes $f(x)$ negative, and $x=3$ makes $f(x)$ positive, so that a root of the equation $f(x)=0$ lies between 2 and 3. Again, $x=2\frac{1}{2}$ makes $f(x)$ positive, so that the root lies between 2 and $2\frac{1}{2}$; also $x=2{\cdot}2$ makes $f(x)$ positive; thus the root cannot differ from $2{\cdot}1$ by so much as $\cdot 1$. Suppose then $c=2{\cdot}1$; then

$$c_1=c-\frac{f(c)}{f'(c)}=c-\frac{c^3-2c-5}{3c^2-2}=2{\cdot}1-\frac{\cdot 061}{11{\cdot}23}=2{\cdot}1-\cdot 0054 \text{ nearly};$$

thus $c_1=2{\cdot}0946$ nearly.

Then for a new approximation we have

$$c_1-\frac{f(c_1)}{f'(c_1)}=c_1-\cdot 00004852 \text{ nearly}=2{\cdot}09455148 \text{ nearly}.$$

221. This process is very simple in theory and not difficult in practice; but it is not of certain success unless some precautions are taken which we shall presently explain. For suppose that c is an approximate value of the root, and that $c_1=c-\dfrac{f(c)}{f'(c)}$, we are not sure without further investigation that c_1 is nearer than c to the real value of the root. In the preceding example, after we had ascertained that there was a root between 2 and $2{\cdot}2$, we assumed $2{\cdot}1$ as a first approximation and deduced $2{\cdot}0946$ as a new approximation. But we are not sure as yet that $2{\cdot}0946$ is nearer to the root than $2{\cdot}2$; if however we put $2{\cdot}1$ for x we find that $f(x)$ is positive, and thus the required root must lie between 2 and $2{\cdot}1$, and now we know that $2{\cdot}0946$ is nearer than $2{\cdot}2$ to this root. But we do not know even now that $2{\cdot}0946$ is nearer to the root than $2{\cdot}1$. If we put $2{\cdot}0946$ for x we find

that $f(x)$ is positive, and this shews that the root lies between 2·0946 and 2; thus 2·0946 is nearer to the root than 2·1.

222. Fourier has given a rule by which we are saved the trouble of such repeated examinations as we have exemplified in the preceding Article; this rule guarantees the success of Newton's method when certain conditions are satisfied. Fourier's supplement to Newton's method depends upon a property of the first derived function of a given function, which we will now prove.

223. If a and b are any two quantities, some quantity λ intermediate between a and b exists, such that

$$f(b) - f(a) = (b-a) f'(\lambda).$$

For let $F(x)$ denote $f(x) - f(a) - \dfrac{x-a}{b-a}\left\{f(b) - f(a)\right\}$; then $F(x)$ vanishes when $x = a$ and also when $x = b$. Therefore by Art. 102 the equation $F'(x) = 0$ has a root between a and b. And, by Art. 11, $F'(x) = f'(x) - \dfrac{f(b) - f(a)}{b-a}$; hence some quantity λ intermediate between a and b must exist, such that $f'(\lambda) - \dfrac{f(b) - f(a)}{b-a} = 0$; therefore $f(b) - f(a) = (b-a) f'(\lambda)$.

224. Suppose that b is greater than a: then $f(b)$ is algebraically greater or less than $f(a)$ according as $f'(\lambda)$ is positive or negative. If $f'(x)$ is positive between $x = a$ and $x = b$, then $f'(\lambda)$ is necessarily positive, and if $f'(x)$ is negative between $x = a$ and $x = b$, then $f'(\lambda)$ is necessarily negative.

Hence we have the following result; if $f'(x)$ is constantly positive through any interval, $f(x)$ increases with x through that interval, and if $f'(x)$ is constantly negative, $f(x)$ decreases as x increases through that interval. By the increase or decrease of $f(x)$ we mean *algebraical* increase or decrease. We may however state our result thus; if $f'(x)$ retain the same sign through any interval, then as x increases through that interval $f(x)$ *increases numerically* when it has the same sign as $f'(x)$, and *decreases numerically* when it has the contrary sign.

225. We shall now enunciate and prove Fourier's rule. Let $f(x)=0$ be an equation which has one root and only one between α and β; and suppose that the equation $f'(x)=0$ has no root between α and β, and also that the equation $f''(x)=0$ has no root between α and β; then Newton's method of approximation will certainly be successful if it be begun and continued from that limit for which $f(x)$ and $f''(x)$ have the same sign.

It follows from our suppositions that $f(x)$ changes sign once and only once between α and β, and that $f'(x)$ and $f''(x)$ do not change sign between α and β. We will suppose $\beta-\alpha$ to be positive.

(1) Suppose that $f(x)$ and $f''(x)$ have the same sign when $x=\alpha$. Take α for the first approximation; then Newton's second approximation is $\alpha-\dfrac{f(\alpha)}{f'(\alpha)}$. Let $\alpha+h$ denote the true value of the root; then $f(\alpha+h)=0$. Now by Art. 223, we have $f(\alpha+h)-f(\alpha)=hf'(\lambda)$, where λ lies between α and $\alpha+h$; thus $h=-\dfrac{f(\alpha)}{f'(\lambda)}$, and the true value of the root is $\alpha-\dfrac{f(\alpha)}{f'(\lambda)}$. We have then to shew that $\alpha-\dfrac{f(\alpha)}{f'(\alpha)}$ is nearer than α to the true value of the root. Since h is necessarily a positive quantity, $f(\alpha)$ and $f'(\lambda)$ are of contrary signs, and $f(\alpha)$ is of the same sign as $f''(\alpha)$, and therefore $f'(\lambda)$ and $f''(\alpha)$ are of contrary signs. Hence $f'(x)$ decreases numerically as x increases between α and β, by Art. 224, so that $f'(\lambda)$ is numerically less than $f'(\alpha)$; therefore $-\dfrac{f(\alpha)}{f'(\alpha)}$ is a positive quantity which is numerically less than the positive quantity $-\dfrac{f(\alpha)}{f'(\lambda)}$. This shews that Newton's second approximation is nearer to the true value of the root than the first approximation.

Let $\alpha_1=\alpha-\dfrac{f(\alpha)}{f'(\alpha)}$; then $f(\alpha_1)$ and $f''(\alpha_1)$ have the same sign, and the approximation can be continued from α_1.

(2) Suppose that $f(x)$ and $f''(x)$ have the same sign when $x=\beta$. Take β for the first approximation, then Newton's second approximation is $\beta-\frac{f(\beta)}{f'(\beta)}$. Let $\beta+h$ denote the true value of the root; then $f(\beta+h)=0$. Now, by Art. 223, we have $f(\beta+h)-f(\beta)=hf'(\lambda)$, where λ lies between β and $\beta+h$; thus $h=-\frac{f(\beta)}{f'(\lambda)}$. We have then to shew that $\beta-\frac{f(\beta)}{f'(\beta)}$ is nearer than β to the true value of the root. Since h is necessarily a negative quantity, $f(\beta)$ and $f'(\lambda)$ are of the same sign, and $f(\beta)$ is of the same sign as $f''(\beta)$, and therefore $f'(\lambda)$ and $f''(\beta)$ are of the same sign. Hence $f'(x)$ increases numerically as x increases between a and β, by Art. 224, so that $f'(\lambda)$ is numerically less than $f'(\beta)$. Therefore $\frac{f(\beta)}{f'(\beta)}$ is a positive quantity which is numerically less than the positive quantity $\frac{f(\beta)}{f'(\lambda)}$. This shews that Newton's second approximation is nearer to the true value of the root than the first approximation.

Let $\beta_1=\beta-\frac{f(\beta)}{f'(\beta)}$; then $f(\beta_1)$ and $f''(\beta_1)$ have the same sign, and the approximation can be continued from β_1.

226. The preceding Article shews that the conditions given by Fourier are *sufficient* to ensure the success of Newton's method of approximation. When these conditions are satisfied, and the approximation is begun and continued from that limit for which $f(x)$ and $f''(x)$ have the same sign, we obtain a succession of values, which continuously increase up to the real value of the root or diminish down to it, according as the limit from which we start is less or greater than the true value of the root. We will now briefly shew that Fourier's conditions are *necessary.*

If we start with an assumed value c, Newton's second approximation corrects this by adding $-\frac{f(c)}{f'(c)}$, while the true value of the root would be obtained by adding $-\frac{f(c)}{f'(\lambda)}$. Hence the permanence of sign of $f'(x)$ is necessary in order that we may be sure that $f'(c)$ and $f'(\lambda)$ have the same sign; if these quantities do not have the same sign the Newtonian correction has the wrong sign, and Newton's second approximation is further from the true value of the root than the first approximation.

The permanence of sign of $f''(x)$ is necessary in order to ensure that $f'(\lambda)$ is numerically less than $f'(c)$. If this is not the case the Newtonian correction is numerically greater than the true correction, and thus, supposing the correction to be of the right sign, the true value of the root lies *between* Newton's first and second approximations. In this case Newton's second approximation *may* be nearer to the true value of the root than the first approximation, but is not necessarily so.

227. In the example of Art. 220, it may be shewn that the equation $f(x)=0$ has only one root between 2 and 2·1, and that the equations $f'(x)=0$ and $f''(x)=0$ have no roots between these limits; also $f(x)$ and $f''(x)$ are both positive when $x=2{\cdot}1$. Hence the Newtonian approximation will certainly succeed if it be begun and continued from the limit 2·1.

For another example take the equation $x^3-7x+7=0$, say $f(x)=0$. It may be shewn by trial that the equation has one root between 1·3 and 1·4; the equations $f'(x)=0$, and $f''(x)=0$, have no roots between these limits; also $f(x)$ and $f''(x)$ are both positive when $x=1{\cdot}3$. Hence the Newtonian approximation will certainly succeed if it be begun and continued from the limit 1·3.

228. We will now shew how to estimate the rapidity of the approximation. Suppose c to be the approximate value of the root which has been obtained at any stage of the process; then the true value of the root is $c-\dfrac{f(c)}{f'(\lambda)}$, so that the numerical value of the error at this stage is $\dfrac{f(c)}{f'(\lambda)}$, which we will denote by r. The next approximate value will be $c-\dfrac{f(c)}{f'(c)}$, and now the nu-

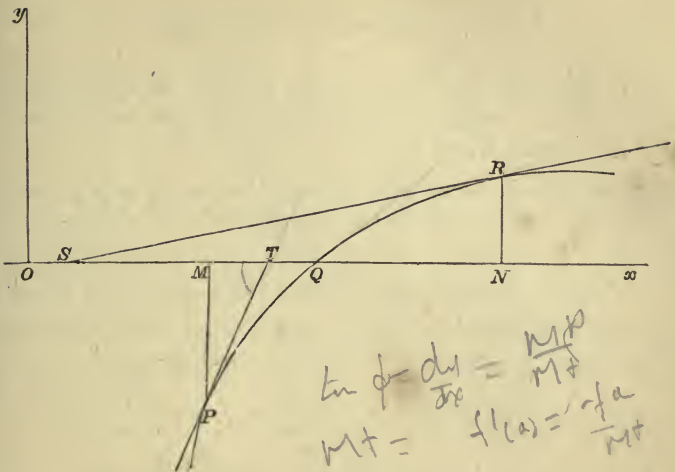
merical value of the error is $\frac{f(c)}{f'(\lambda)} - \frac{f(c)}{f'(c)}$, that is, $r\frac{f'(c)-f'(\lambda)}{f'(c)}$. And by Art. 223, we have $f'(c)-f'(\lambda)=(c-\lambda)f''(\mu)$, where μ lies between c and λ; thus the error is $\frac{r(c-\lambda)f''(\mu)}{f'(c)}$. Now λ lies between c and the real value of the root, so that $c-\lambda$ is less than r; hence the error is less than $\frac{r^2 f''(\mu)}{f'(c)}$. Let the greatest value which $f''(x)$ can take between the limits considered be divided by the least value which $f'(x)$ can take, and denote the quotient by q; then the error is *a fortiori* less than qr^2.

For example, in Art. 220, the root lies between 2 and 2·1. Thus to find q we divide the value of $6x$ when $x=2{\cdot}1$ by the value of $3x^2-2$ when $x=2$; therefore $q=1{\cdot}26$; and as q is nearly unity, the number of exact decimal places in the approximation will be nearly doubled at each step.

229. The student who is acquainted with the elements of the application of the Differential Calculus to the theory of curves, will find it easy to illustrate geometrically Fourier's rule for conducting Newton's approximation.

Suppose PQR to be a part of the curve determined by the equation $y=f(x)$. Then we may be supposed to know OM and ON, and to require the value of OQ; that is, we require to know the point where the curve cuts the axis of x.

At the point P it is obvious that $f(x)$ is negative if Oy be the positive direction of the axis of y; and $f''(x)$ is also negative at P, since the curve at P is convex to the axis of x. Draw the tangent PT; let $OM=a$, then $MT=-\frac{f(a)}{f'(a)}$, as is known by the Differential Calculus; so that, starting from M the Newtonian approximation proceeds to T. And as T falls between M and Q it is obvious that the method succeeds in this case, and that the approximation can be continued from T.



At the point R we have $f(x)$ positive and $f''(x)$ negative. Draw the tangent RS; then, starting from N the Newtonian approximation proceeds to S, and S and N are on *opposite* sides of Q. Moreover there is no security that QS is less than QN, and there is no security that the approximation can be continued from S. Thus the approximation cannot be safely begun from N.

The student may easily illustrate by figures the condition that $f'(x)$ and $f''(x)$ should retain an unchanged sign between the limits considered.

If however, in any example, we know that NS is less than NM we may start from N, as the point S will then fall between Q and M, and the approximation can be continued from S.

Let $ON=\beta$; then we may start from N if $\dfrac{f(\beta)}{f'(\beta)}$ is less than $\beta-\alpha$. *Messenger of Mathematics*, Vol. III. page 40.

XVIII. HORNER'S METHOD. 1816,

230. We shall now explain the method of approximating to the numerical value of a root of an equation which was invented by the late W. G. Horner.

For the history of this part of the subject we refer to a memoir by Professor De Morgan in the *Companion to the Almanac* for 1839.

Let $f(x)=0$ be any equation; then $f(a+x)=0$ is an equation the roots of which are less by a than the roots of the first equation. The equation $f(a+x)=0$ becomes when developed

$$f(a)+xf'(a)+x^2\frac{f''(a)}{1\,.\,2}+x^3\frac{f'''(a)}{\lfloor 3}+\ldots+x^n\frac{f^n(a)}{\lfloor n}=0.$$

Now the essential part of Horner's method consists of a process by which the coefficients of the last equation may be systematically and economically calculated; we have already observed that such a process will be useful; see Arts. 11, 54, and 214.

231. Suppose, for example, that

$$f(x)=Ax^5+Bx^4+Cx^3+Dx^2+Ex+F;$$

$$\begin{aligned}
\text{then } f(a) &= Aa^5+Ba^4+Ca^3+Da^2+Ea+F,\\
f'(a) &= 5Aa^4+4Ba^3+3Ca^2+2Da+E,\\
\frac{1}{2}f''(a) &= 10Aa^3+6Ba^2+3Ca+D,\\
\frac{1}{\lfloor 3}f'''(a) &= 10Aa^2+4Ba+C,\\
\frac{1}{\lfloor 4}f''''(a) &= 5Aa+B,\\
\frac{1}{\lfloor 5}f'''''(a) &= A.
\end{aligned}$$

(1) We may calculate $f(a)$ in the manner explained in Art. 5, thus;

$$\begin{aligned}
&A &&= A,\\
&Aa + B &&= P \text{ say},\\
&Pa + C = Aa^2 + Ba + C &&= Q \text{ say},\\
&Qa + D = Aa^3 + Ba^2 + Ca + D &&= R \text{ say},\\
&Ra + E = Aa^4 + Ba^3 + Ca^2 + Da + E &&= S \text{ say},\\
&Sa + F = Aa^5 + Ba^4 + Ca^3 + Da^2 + Ea + F &&= f(a).
\end{aligned}$$

Here each line is obtained by multiplying the preceding line by a, and adding on in succession the terms B, C, D, E, F.

(2) We may now calculate $f'(a)$ in the same way as $f(a)$ was calculated, using A, P, Q, R, S in the same way as A, B, C, D, E, F were used;

$T = 2Aa + b$

$$\begin{aligned}
&A = A,\\
&Aa + P = 2Aa + B = T \text{ say},\\
&Ta + Q = 3Aa^2 + 2Ba + C = U \text{ say},\\
&Ua + R = 4Aa^3 + 3Ba^2 + 2Ca + D = V \text{ say},\\
&Va + S = 5Aa^4 + 4Ba^3 + 3Ca^2 + 2Da + E = f'(a).
\end{aligned}$$

(3) We may now calculate $\frac{1}{2}f''(a)$ in the same way as $f(a)$ and $f'(a)$ were calculated, using A, T, U, V;

$$\begin{aligned}
&A = A,\\
&Aa + T = 3Aa + B = W \text{ say},\\
&Wa + U = 6Aa^2 + 3Ba + C = X \text{ say},\\
&Xa + V = 10Aa^3 + 6Ba^2 + 3Ca + D = \tfrac{1}{2}f''(a).
\end{aligned}$$

(4) We may now calculate $\frac{1}{\lfloor 3}f'''(a)$ in the same way, using A, W, X;

$$\begin{aligned}
&A = A,\\
&Aa + W = 4Aa + B = Y \text{ say},\\
&Ya + X = 10Aa^2 + 4Ba + C = \frac{1}{\lfloor 3}f'''(a).
\end{aligned}$$

(5) We may now calculate $\frac{1}{\underline{|4}}f''''(a)$ in the same way, using A and Y;

$$A \qquad = A,$$

$$Aa + Y = 5Aa + B = \frac{1}{\underline{|4}}f''''(a).$$

(6) Lastly, $$A = \frac{1}{\underline{|5}}f'''''(a).$$

The above process may be conveniently arranged thus;

$$\begin{array}{cccccc}
A & B & C & D & E & F \\
 & \underline{Aa} & \underline{Pa} & \underline{Qa} & \underline{Ra} & \underline{Sa} \\
 & P & Q & R & S & f(a) \\
 & \underline{Aa} & \underline{Ta} & \underline{Ua} & \underline{Va} & \\
 & T & U & V & f'(a) & \\
 & \underline{Aa} & \underline{Wa} & \underline{Xa} & & \\
 & W & X & \frac{1}{2}f''(a) & & \\
 & \underline{Aa} & \underline{Ya} & & & \\
 & Y & \frac{1}{\underline{|3}}f'''(a) & & & \\
\underline{Aa} & & & & & \\
\frac{1}{\underline{|4}}f''''(a) & & & & &
\end{array}$$

The quantity under any horizontal line is obtained by adding the two quantities immediately over the line.

We have thus shewn Horner's process of forming the coefficients of the equation $f(a+x)=0$ when the equation is of the *fifth* degree; we will hereafter prove that this process is applicable whatever may be the degree of the equation. We will give a numerical illustration of the process and then explain the use of the process in approximating to the root of an equation.

For a numerical illustration suppose $a=2$ and

$$f(x) = 3x^5 - x^3 + 4x^2 + 5x - 8.$$

3	0	− 1	+ 4	+ 5	− 8
2	6	12	22	52	114
	6	11	26	57	106 $f(2)$
	6	24	70	192	
	12	35	96	249 $f'(2)$	
	6	36	142		
	18	71	238 $\frac{1}{2}f''(2)$		
	6	48			
	24	119 $\frac{1}{6}f'''(2)$			
	6				
	30 $\frac{1}{24}f^{(4)}(2)$				

Thus $f(2+x) = 3x^5 + 30x^4 + 119x^3 + 238x^2 + 249x + 106$.

232. Suppose, for example, that we have an equation with a root lying between 300 and 400; form a second equation the roots of which are less than those of the first equation by 300, so that the second equation has a root lying between 0 and 100. By trial let the greatest multiple of 10 which is contained in this root be found; suppose it to be 70; form a third equation the roots of which are less than those of the second equation by 70, so that the third equation has a root between 0 and 10. By trial let the greatest integer which is contained in this root be found; suppose it to be 2; form a fourth equation the roots of which are less than those of the third equation by 2, so that the fourth equation has a root lying between 0 and 1. By trial let the greatest number of tenths which is contained in this root be found; suppose it to be 8 tenths; form a fifth equation the roots of which are less than those of the fourth equation by ·8, so that the fifth equation has a root lying between 0 and ·1. By trial let the greatest number of hundredths which is contained in this root be found; suppose it to be 7 hundredths.

Now suppose that ·07 is exactly a root of the fifth equation; it follows that 372·87 is exactly a root of the first equation.

Next suppose that ·07 is not exactly a root of the fifth equation; then it follows that an equation exists the roots of which are less than those of the first equation by 372·87, and which has a root lying between 0 and ·01. Thus the first equation has a root which lies between 372·87 and 372·88.

Thus we see how by a series of operations of the kind given in Art. 231, we either arrive at the exact value of the root of an equation, or we may approximate to it as closely as we please.

233. In the preceding Article we have stated that certain numbers must be found *by trial;* we shall now shew that we can easily guide ourselves in these trials. Let $f(x)=0$ be the proposed equation, and suppose that by one or more operations we have derived the equation which has its roots less than those of the proposed equation by c, that is, suppose we have formed the equation $f(c+x)=0$, and suppose that this last equation has a small root. Then c is an approximate value of a root of the original equation; hence by the preceding Chapter $c-\dfrac{f(c)}{f'(c)}$ will be in general a nearer approximation to that root. Thus $-\dfrac{f(c)}{f'(c)}$ is an approximate value of the number which we want in order to continue the operation.

234. Example. Let $f(x)=2x^3-473x^2-234x-711$. It will be found by trial that $f(200)$ is negative and $f(300)$ positive, so that the equation $f(x)=0$ has a root between 200 and 300. We proceed to diminish the roots by 200.

$$\begin{array}{llll}
2 & -473 & -234 & -711\ (200 \\
 & \underline{\ \ \ 400} & \underline{-14600} & \underline{-2966800} \\
 & -73 & -14834 & -2967511 \\
 & \underline{\ \ \ 400} & \underline{\ \ \ 65400} & \\
 & \ \ \ 327 & \ \ \ 50566 & \\
 & \underline{\ \ \ 400} & & \\
 & \ \ \ 727 & &
\end{array}$$

Hence the equation which has its roots less than those of $f(x)=0$ by 200 is $2x^3+727x^2+50566x-2967511=0$; so that $f(200)=-2967511$ and $f'(200)=50566$.

Hence $-\dfrac{f(200)}{f'(200)}$ is more than 50. We then proceed to diminish the roots of the equation just given by 50.

2	727	50566	-2967511 (50
	100	41350	4595800
	827	91916	1628289

We thus find that 50 is too large a number, for we have $f(250) = 1628289$ a *positive* quantity, while $f(200)$ is *negative;* so that the root we are seeking is less than 250. In fact, in guiding ourselves in the manner explained in Art. 233 we are liable to select *too large* a number for trial, especially in the early part of the operation; a similar failure occurs sometimes in the ordinary process of extracting the square root of a number.

We then try 40.

2	727	50566	-2967511 (40
	80	32280	3313840
	807	82846	346329

Thus 40 is also too large, for $f(240)$ is positive. We then try 30.

2	727	50566	-2967511 (30
	60	23610	2225280
	787	74176	-742231
	60	25410	
	847	99586	
	60		
	907		

Thus $f(230) = -742231$ a *negative* quantity, so that 30 is the right number.

Hence the equation which has its roots less than those of $f(x) = 0$ by 230 is $2x^3 + 907x^2 + 99586x - 742231 = 0$.

Here $f'(230) = 99586$ so that $-\dfrac{f(230)}{f'(230)} = 7$ approximately.

We proceed then to diminish the roots of the equation just given by 7.

2	907	99586	−742231 (7
	14	6447	742231
	921	106033	0

This shews that $f(237)=0$; so that 237 is a root of the original equation.

The whole operation is usually exhibited thus;

2	−473	−234	−711 (237
	400	−14600	−2966800
	−73	−14834	−2967511*
	400	65400	2225280
	327	50566*	−742231† 2
	400	23610	742231
	727*	74176	
	60	25410	
	787	99586†	
	60	6447	
	847	106033	
	60		
	907†		
	14		
	921		

Here the mark * shews where the first part of the operation ends, and the mark † shews where the second part of the operation ends.

235. We will now take an example of an equation which has no commensurable root. Let $f(x)=x^3-3x^2-2x+5$. It will be found by trial that $f(3)$ is negative and $f(4)$ positive, so that the equation $f(x)=0$ has a root between 3 and 4. The following will be the operation for approximating to this root as far as three places of decimals.

1	−3	−2	5 (3·128
	3	0	−6
	0	−2	−1*
	3	9	·761
	3	7*	−·239†
	3	·61	·167128
	6*	7·61	−·071872‡
	·1	·62	·068273152
	6·1	8·23†	−·003598848
	·1	·1264	
	6·2	8·3564	
	·1	·1268	
	6·3†	8·4832‡	
	·02	·050944	
	6·32	8·534144	
	·02	·051008	
	6·34	8·585152	
	·02		
	6·36‡		
	·008		
	6·368		
	·008		
	6·376		
	·008		
	6·384		

Here to find the second figure of the root we have $-\frac{-1}{7}$, so that ·1 is the nearest number to be tried; to find the third figure of the root we have $-\frac{-\cdot 239}{8\cdot 23}$, so that ·02 is the nearest number to be tried; to find the fourth figure of the root we have

$-\dfrac{-\cdot071872}{8\cdot4832}$, so that ·008 is the nearest number to be tried. In all these cases the number suggested is found to be correct.

236. As another example take the equation given in the preceding Article, and approximate to the root which it has between 1 and 2. The operation is usually exhibited thus;

1	−3	−2	5 (1·2016
	1	−2	−4
	−2	−4	1000*
	1	−1	−992
	−1	−500*	8000000†
	1	4	−4879399
	00*	−496	3120601000‡
	2	8	−2927060904
	2	−4880000†	193540096
	2	601	
	4	−4879399	
	2	602	
	600†	−487879700‡	
	1	36216	
	601	−487843484	
	1	36252	
	602	−487807232	
	1		
	6030‡		
	6		
	6036		
	6		
	6042		
	6		
	6048		

The difference between this arrangement and that in Art. 235 arises from the fact that it is usual in practice to omit the *decimal points,* just as they are omitted in the process for extracting the square roots of numbers approximately. The following rule with respect to the decimal part of the root will be sufficient. When all the whole figures of the root have been found and the decimal part of the root is about to appear, annex one cipher to the right of the first working column, two ciphers to the right of the second working column, three ciphers to the right of the third working column, and so on if there are more than three working columns; then proceed completely through one stage of the operation as if the new figure of the root were a whole number. Then annex ciphers again as before.

It will be observed that after the 2 in the root the next figure considered as an integer would be approximately given by $-\frac{8000}{-48800}$, and this is less than unity; so a cipher is put in the root and we annex another cipher to the first working column, two more to the second working column, and three more to the third, and proceed as before. The ciphers will serve to distinguish the several stages of the operation, so that the marks *†‡ may be omitted.

It is obvious that in all the preceding examples the first working column might have been shortened by performing in the head the easy work which occurs, and putting down only the results, but we have thought it clearer to exhibit the whole for the student.

237. After a certain number of figures in the root have been found correctly, an additional number may be obtained by a contracted operation. We will exemplify this by calculating the positive root of the equation $x^3 + 3x^2 - 2x - 5 = 0$. We will first perform the operation at full until five decimal places of the root have been determined.

1	3	−2	−5 (1·33005
	1	4	2
	4	2	−3000
	1	5	2667
	5	700	−333000
	1	189	332337
	60	889	−663000000000
	3	198	564352475125
	63	108700	−98647524875
	3	2079	
	66	110779	
	3	2088	
	690	112867000000	
	3	3495025	
	693	112870495025	
	3	3495050	
	696	112873990075	
	3		
	699000		
	5		
	699005		
	5		
	699010		
	5		
	699015		

The rule for contracting the operation is the following; strike off at every step one figure from the right of the last column but one, two figures from the right of the last column but two, and so on.

We will now resume the example just considered and apply this contracted process.

1	699015	112873990075	−98647524875	(1·33005873
		55921	90299639432	
		11287454929	−8347885443*	
		55921	7901261018	
		11287510850*	−446624425†	
		489	338625624	
		1128751574	−107998801	
		489		
		1128752063†		
		2		
		112875208		
		2		
		112875210		

At the point where the full operation terminated we have 8 suggested for the next figure; we then reject 5 from the end of the last working column but one, and 15 from the end of the last working column but two. The first step in carrying on the work is to multiply 6990 by 8, and put the product in the next working column; the product is considered to be 55921, because we conceive 69901 multiplied by 8 and the last figure struck off, and so 55921 is nearer than 55920 to the true value. Then we add 55921 to 11287399007; the figure in the units' place of the sum we take to be 9 by allowing for the rejected 5. The mark * indicates where the first stage of the contracted operation finishes. Now strike off 0 from the end of the second working column and 90 from the end of the first working column, so that the first working column is reduced to 69. The next figure of the root is 7, and this stage of the operation finishes where the mark † is put. Strike off 3 from the end of the second working column and 69 from the end of the first working column. The first working column now disappears, but still exercises a slight influence because the next figure in the root is 3, and when 69 is multiplied by 3 and two figures rejected there remains a 2.

Only two working columns are now left; the remainder of the work coincides with the ordinary process of *contracted division*, and it will supply eight more figures in the root.

11287521,0	-107998801	(1·3300587395679825
	101587689	
1128752,1	-6411112	
	5643761	
112875,2	-767351	
	677251	
11287,5	-90100	
	79013	
1128,7	-11087	
	10158	
112,8	-929	
	902	
11,2	-27	
	22	
1,1	-5	

The approximation may be relied upon very nearly up to the last figure. For if the whole operation were performed at full, the last working column would present a large number of figures on the right-hand side of those here exhibited, but those which are here exhibited would retain their places without alteration except perhaps the exchange in some lines of the last figure for another differing from it by unity.

238. The root found in the preceding Article is the numerical value of the *negative* root of the equation $x^3 - 3x^2 - 2x + 5 = 0$. Hence the sum of the roots found in Arts. 235 and 236 should exceed the root found in Art. 237 by 3; because the sum of the three roots of the equation with their proper signs is 3. This will be found to hold approximately; and the student may exercise himself in carrying on the approximations to the two

positive roots to more places of decimals than we have given, in order to verify more clearly the connexion between the sum of those two roots and the root calculated in Art. 237.

239. Various suggestions have been offered with the view of saving labour in the use of Horner's method. With respect to such suggestions we may quote the following remarks which occur in connexion with one of them. "But considering that the process is one which no person will very often perform, we doubt whether to recommend even this abridgment. All such simplifications tend to make the computer lose sight of the uniformity of method which runs through the whole; and we have always found them, in rules which only occur now and then, afford greater assistance in forgetting the method than in abbreviating it." *Penny Cyclopædia*, article *Involution*.

240. In Art. 231 it was stated that it would be proved that Horner's method of forming the equation $f(a+x)=0$ is universally true. We will now consider this point.

Let $\quad f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n,$

for x put $y+a$, and suppose that $f(x)$ then becomes

$$q_0y^n + q_1y^{n-1} + q_2y^{n-2} + \ldots + q_{n-1}y + q_n;$$

we have to prove that q_n, q_{n-1}, ... q_1, q_0, are found correctly by Horner's process. It is obvious that $q_0=p_0$. Since $y=x-a$ the following expressions are identically equal,

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n,$$

and $\quad q_0(x-a)^n + q_1(x-a)^{n-1} + q_2(x-a)^{n-2} + \ldots + q_{n-1}(x-a) + q_n.$

Therefore q_n is the *remainder* that would be found on dividing $f(x)$ by $x-a$; also the quotient arising from this division must be identically equal to

$$q_0(x-a)^{n-1} + q_1(x-a)^{n-2} + q_2(x-a)^{n-3} + \ldots + q_{n-1}.$$

Then again q_{n-1} is the *remainder* that would be found on dividing the last expression by $x-a$; also the quotient arising from this division must be identically equal to

$$q_0(x-a)^{n-2}+q_1(x-a)^{n-3}+q_2(x-a)^{n-4}+\ldots+q_{n-2}.$$

Then again q_{n-2} is the *remainder* that would be found on dividing the last expression by $x-a$; also the quotient arising from this division must be identically equal to

$$q_0(x-a)^{n-3}+q_1(x-a)^{n-4}+q_2(x-a)^{n-5}+\ldots+q_{n-3};$$

and so on.

Thus q_n, q_{n-1}, q_{n-2}, q_{n-3}, ... are the successive remainders which occur in dividing, first $f(x)$ by $x-a$, then the quotient by $x-a$, then the new quotient by $x-a$; and so on. And we see by Arts. 5, 7, and 9 that Horner's process determines these successive remainders.

241. We have thus sufficiently discussed the subject of the approximate values of the *real* roots of equations. There is no easy practical method of calculating the *imaginary* roots of equations at present known; but theoretically this may be made to depend on what has been already given. For suppose $a+b\sqrt{-1}$ is an imaginary root of an equation $f(x)=0$; then since the real and imaginary parts of $f(a+b\sqrt{-1})$ must separately vanish, we obtain two results, which we may denote by $P=0$ and $Q=0$, as in Art. 41. Here P and Q will be functions of a and b, and if we eliminate a or b from the equations $P=0$ and $Q=0$, we obtain a single equation involving one unknown quantity; and we require *real values* of this unknown quantity. Hence we can determine the imaginary roots of a given equation if we can form a certain other equation and determine its real roots. We shall hereafter shew how to form the equation which results by eliminating one of two unknown quantities from two given equations.

We shall in Chapter XXI. explain another method which has been used for calculating the imaginary roots of equations. The student may also consult Dr Rutherford's essay on the *Complete Solution of Numerical Equations.*

XIX. SYMMETRICAL FUNCTIONS OF THE ROOTS.

242. A function of two or more quantities is said to be a *symmetrical* function of those quantities if the function is not altered when any two of the quantities are interchanged.

Thus, for example, $a^2+b^2+c^2$ is a symmetrical function of the three quantities a, b c; so also is $ab+bc+ca$; for each of these functions is unaltered when we interchange a and b, or a and c, or b and c.

243. *The coefficients of an equation are symmetrical functions of the roots of the equation.*

For by Art. 45, if the equation be $x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots=0$, we have

$-p_1=$ the sum of the roots,

$p_2=$ the sum of the products of the roots taken two at a time,

and so on; and it is manifest that the functions of the roots which occur here are symmetrical functions.

The object of the present Chapter is to shew that every rational symmetrical function of the roots of an equation can be expressed in terms of the coefficients of that equation. We shall begin with proving Newton's theorem for the sums of the powers of the roots of an equation.

244. Let $f(x)$ denote $x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n$, and let $a, b, c, d, \ldots$ denote the roots of the equation $f(x)=0$.

Let
$$S_1=a+b+c+d+\ldots,$$
$$S_2=a^2+b^2+c^2+d^2+\ldots,$$
$$S_3=a^3+b^3+c^3+d^3+\ldots,$$

and so on; thus S_1 is the sum of the roots, S_2 is the sum of the squares of the roots, S_3 is the sum of the cubes of the roots, and in general S_m is the sum of the m^{th} powers of the roots.

By Art. 74 we have

$$f'(x) = \frac{f(x)}{x-a} + \frac{f(x)}{x-b} + \frac{f(x)}{x-c} + \dots$$

The divisions indicated on the right-hand side of this identity can all be exactly performed by Art. 7; and we have

$$\frac{f(x)}{x-a} = x^{n-1} + (a+p_1)x^{n-2} + (a^2+p_1a+p_2)x^{n-3} + \dots$$
$$+ (a^m + p_1a^{m-1} + p_2a^{m-2} + \dots + p_m)x^{n-m-1} + \dots;$$

and similar expressions hold for $\frac{f(x)}{x-b}$, $\frac{f(x)}{x-c}$, ...

By addition then we obtain

$$f'(x) = nx^{n-1} + (S_1 + np_1)x^{n-2} + (S_2 + p_1S_1 + np_2)x^{n-3} + \dots$$
$$+ (S_m + p_1S_{m-1} + p_2S_{m-2} + \dots + np_m)x^{n-m-1} + \dots.$$

Also $f'(x) = nx^{n-1} + (n-1)p_1x^{n-2} + (n-2)p_2x^{n-3} + \dots$
$$+ (n-m)p_mx^{n-m-1} + \dots.$$

Equate the coefficients of the same powers of x in the identity; thus

$$S_1 + np_1 = (n-1)p_1 \text{ or } S_1 + p_1 = 0,$$

$$S_2 + p_1S_1 + np_2 = (n-2)p_2 \text{ or } S_2 + p_1S_1 + 2p_2 = 0,$$

and generally

$$S_m + p_1S_{m-1} + p_2S_{m-2} + \dots + np_m = (n-m)p_m,$$

or $$S_m + p_1S_{m-1} + p_2S_{m-2} + \dots + p_{m-1}S_1 + mp_m = 0.$$

In this general result m is supposed to be less than n.

By means of the general result we can express the sum of the m^{th} powers of the roots in terms of the coefficients and the sums of inferior powers of the roots; and thus by repeated operations we may express the sum of the m^{th} powers of the roots in terms of the coefficients only.

Next suppose that m is not restricted to be less than n. Multiply the given equation $f(x) = 0$ by x^{m-n}; thus $x^{m-n} f(x) = 0$,

that is, $$x^m + p_1 x^{m-1} + p_2 x^{m-2} + \dots + p_n x^{m-n} = 0.$$

Substitute for x successively $a, b, c, \dots$ and add the results; thus

$$S_m + p_1 S_{m-1} + p_2 S_{m-2} + \dots + p_n S_{m-n} = 0.$$

By this theorem we can express the sum of the m^{th} powers of the roots of an equation in terms of the coefficients and the sums of inferior powers of the roots when m is not less than n; and thus by repeated operations we may express the sum of the m^{th} powers of the roots in terms of the coefficients.

Practically the following is a very convenient method. We have

$$f'(x) = \frac{f(x)}{x-a} + \frac{f(x)}{x-b} + \frac{f(x)}{x-c} + \dots;$$

therefore

$$\frac{xf'(x)}{f(x)} = \frac{x}{x-a} + \frac{x}{x-b} + \frac{x}{x-c} + \dots$$

$$= \left(1 - \frac{a}{x}\right)^{-1} + \left(1 - \frac{b}{x}\right)^{-1} + \left(1 - \frac{c}{x}\right)^{-1} + \dots$$

$$= n + \frac{S_1}{x} + \frac{S_2}{x^2} + \frac{S_3}{x^3} + \dots.$$

Thus, if we actually divide $xf'(x)$ by $f(x)$, the coefficients of the terms in order will be $n, S_1, S_2, \dots$. The division may be advantageously performed in the manner explained in the *Algebra*, Chapter LVIII.

245. To find the sum of the *negative* powers of the roots of the equation $f(x) = 0$, we may put $\frac{1}{y}$ for x and find the sum of the corresponding *positive* powers of the roots of the transformed equation in y.

Or we may make m successively equal to $n-1$, $n-2$, $n-3$,... in the result of the preceding Article; and thus obtain successively S_{-1}, S_{-2}, S_{-3},....

246. The general problem of finding the value of any rational symmetrical function of certain quantities may be reduced to the problem of finding the value of certain simple functions, as we shall now shew.

Any rational symmetrical function which is not integral will be the quotient of one rational symmetrical integral function by another; so that only integral functions need be considered. Any rational symmetrical integral function which is not homogeneous will be the sum of two or more rational symmetrical integral functions which are homogeneous; so that only homogeneous functions need be considered. A homogeneous function may consist of different parts in which although the *sum* of the exponents remains the same, the exponents themselves are different; in such a case the homogeneous function is the sum of two or more homogeneous functions of the same degree in which the exponents are the same for all the terms.

Hence it follows that we need only consider such rational symmetrical functions as are integral and homogeneous, and in which the exponents are the same for all the terms.

247. Let a, b, c, d,... denote the roots of a given equation.

By Art. 244 we can express in terms of the coefficients the value of

$$a^m + b^m + c^m + d^m + \dots$$

This function may be said to be of the *first order*, since each term involves only *one* of the roots.

A function may be said to be of the *second order* when each term involves *two* of the roots, as

$$a^m b^p + a^m c^p + b^m c^p + \dots$$

Here every permutation is to be formed of the roots taken two at a time, and the exponent m placed over the first root and p over the second. We shall denote this function by $\Sigma a^m b^p$, as it is the sum of all the terms which can be formed like $a^m b^p$.

A function may be said to be of the *third order* when each term involves *three* of the roots, as

$$a^m b^p c^q + a^m c^p d^q + a^m b^p d^q + \dots$$

Here every permutation is to be formed of the roots taken three at a time, and the exponent m placed over the first root, p over the second, and q over the third. We shall denote this function by $\Sigma a^m b^p c^q$, as it is the sum of all the terms which can be formed like $a^m b^p c^q$.

Similarly we may have functions of the fourth and higher orders, and may use a similar notation to represent them.

Since we have shewn how to express the function denoted by S_m in terms of the coefficients of the equation it will be sufficient to shew that any of the functions we have to consider can be expressed in terms of such functions as S_m.

248. *To find the value of the symmetrical function of the second order* $\Sigma a^m b^p$.

We have
$$S_m = a^m + b^m + c^m + \dots,$$
$$S_p = a^p + b^p + c^p + \dots.$$

By multiplication we obtain

$$S_m S_p = a^{m+p} + b^{m+p} + c^{m+p} + \dots$$
$$+ a^m b^p + a^m c^p + b^m a^p + \dots;$$

that is,
$$S_m S_p = S_{m+p} + \Sigma a^m b^p,$$
and therefore $\Sigma a^m b^p = S_m S_p - S_{m+p}$.

This supposes that m and p are unequal. If we suppose p equal to m the terms in $\Sigma a^m b^p$ become equal two and two, so that this sum may be expressed thus, $2\Sigma (ab)^m$; and therefore

$$2\Sigma (ab)^m = S_m^{\,2} - S_{2m}.$$

249. *To find the value of the symmetrical function of the third order* $\Sigma a^m b^p c^q$.

We have $$\Sigma a^m b^p = a^m b^p + b^m c^p + a^m c^p + \ldots,$$
$$S_q = a^q + b^q + c^q + \ldots$$

By multiplication we obtain
$$\begin{aligned} S_q \Sigma a^m b^p = a^{m+q} b^p &+ b^{m+q} c^p + c^{m+q} a^p + \ldots \\ &+ a^m b^{p+q} + b^m c^{p+q} + c^m a^{p+q} + \ldots \\ &+ a^m b^p c^q + \ldots \end{aligned}$$

The terms on the right-hand side form three sets, which in our notation are denoted by $\Sigma a^{m+q} b^p$, $\Sigma a^{p+q} b^m$, $\Sigma a^m b^p c^q$; thus
$$S_q \Sigma a^m b^p = \Sigma a^{m+q} b^p + \Sigma a^{p+q} b^m + \Sigma a^m b^p c^q.$$

Substitute for $\Sigma a^m b^p$, $\Sigma a^{m+q} b^p$, and $\Sigma a^{p+q} b^m$ their values from Art. 248, and we obtain
$$\Sigma a^m b^p c^q = S_m S_p S_q - S_{m+p} S_q - S_{m+q} S_p - S_{p+q} S_m + 2S_{m+p+q}.$$

We have supposed m, p, q all unequal. Suppose, however, that $m = p$; then, as in Art. 248, we have
$$2\Sigma (ab)^m c^q = S_m^2 S_q - S_{2m} S_q - 2S_{m+q} S_m + 2S_{2m+q}.$$

If $m = p = q$, the sum $\Sigma a^m b^p c^q$ reduces to $2 \,.\, 3\Sigma (abc)^m$; thus
$$6\Sigma (abc)^m = S_m^3 - 3S_{2m} S_m + 2S_{3m}.$$

The method of this and the preceding Article may be continued to any extent, and thus a function of any order like $\Sigma a^m b^p$ and $\Sigma a^m b^p c^q$ may be expressed in terms of the coefficients. Hence by Art. 246, the object proposed in the present Chapter can be attained.

250. We have shewn how the function denoted by S_m can be expressed in terms of the coefficients; and thus of course the sum of any number of such functions as S_m can be so expressed. The following method will, however, be generally more advantageous in such a case. If $\phi(x)$ denote any rational integral function of x, it is required to express in terms of the coefficients the sum $\phi(a) + \phi(b) + \phi(c) + \ldots$

We have $$\frac{f'(x)}{f(x)}=\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}+\dots;$$

therefore $$\frac{\phi(x)f'(x)}{f(x)}=\frac{\phi(x)}{x-a}+\frac{\phi(x)}{x-b}+\frac{\phi(x)}{x-c}+\dots$$

$$=\frac{\phi(x)-\phi(a)}{x-a}+\frac{\phi(x)-\phi(b)}{x-b}+\frac{\phi(x)-\phi(c)}{x-c}+\dots$$

$$+\frac{\phi(a)}{x-a}+\frac{\phi(b)}{x-b}+\frac{\phi(c)}{x-c}+\dots$$

In this identity the integral parts and the fractional parts will be separately equal; also such expressions as $\frac{\phi(x)-\phi(a)}{x-a}$ are integral by Art. 7. Let $\phi(x)f'(x)$ be divided by $f(x)$, the process being carried on until the remainder is an integral function of x of lower degree than $f(x)$; let R be this remainder. Then by considering the fractional parts of the identity we have

$$\frac{R}{f(x)}=\frac{\phi(a)}{x-a}+\frac{\phi(b)}{x-b}+\frac{\phi(c)}{x-c}+\dots$$

Multiply up; then

$$R=x^{n-1}\left\{\phi(a)+\phi(b)+\phi(c)+\dots\right\}$$

$$+\text{ terms involving lower powers of } x \text{ than } x^{n-1}.$$

Thus $\phi(a)+\phi(b)+\phi(c)+\dots$ is equal to the coefficient of x^{n-1} in R.

251. As an example of the formulæ of this Chapter suppose it required to find the sums of the powers of the roots of the equation $$x^4-x^3-7x^2+x+6=0.$$

$$S_1=-p_1=1,$$
$$S_2=-p_1S_1-2p_2=1+14=15,$$
$$S_3=-p_1S_2-p_2S_1-3p_3=15+7-3=19,$$
$$S_4=-p_1S_3-p_2S_2-p_3S_1-4p_4=19+105-1-24=99,$$
$$S_5=-p_1S_4-p_2S_3-p_3S_2-p_4S_1=99+133-15-6=211,$$
$$S_6=-p_1S_5-p_2S_4-p_3S_3-p_4S_2=211+693-19-90=795,$$
and so on.

Put $\frac{1}{y}$ for x in the given equation; then

$$y^4 + \frac{1}{6}y^3 - \frac{7}{6}y^2 - \frac{1}{6}y + \frac{1}{6} = 0.$$

Thus for the sums of negative powers of the roots of the original equation we have

$$S_{-1} = -\frac{1}{6},$$

$$S_{-2} = -\frac{1}{6}S_{-1} - 2\left(-\frac{7}{6}\right) = \frac{14}{6} + \frac{1}{36} = \frac{85}{36},$$

and so on.

These results may be easily verified, as the original equation has been constructed so as to have for its roots -2, -1, 1, 3.

Again, suppose we require the values of S_1, S_2, S_3 and S_4 in the biquadratic equation

$$x^4 + px^3 + qx^2 + rx + s = 0.$$

$S_1 + p = 0$, therefore $S_1 = -p$,

$S_2 + pS_1 + 2q = 0$, therefore $S_2 = p^2 - 2q$,

$S_3 + pS_2 + qS_1 + 3r = 0$, therefore $S_2 = -p(p^2 - 2q) + pq - 3r$
$= -p^3 + 3pq - 3r,$

$S_4 + pS_3 + qS_2 + rS_1 + 4s = 0,$

therefore $S_4 = -p(-p^3 + 3pq - 3r) - q(p^2 - 2q) + rp - 4s$
$= p^4 - 4p^2q + 4rp + 2q^2 - 4s.$

As another example, let α, β, γ, δ denote the four roots of the biquadratic equation $x^4 + px^3 + qx^2 + rx + s = 0$;

$$\text{let } A = \frac{1}{2}(\alpha\beta + \gamma\delta), \quad B = \frac{1}{2}(\alpha\gamma + \beta\delta), \quad C = \frac{1}{2}(\alpha\delta + \beta\gamma);$$

and let it be required to find the value of the following symmetrical functions of the roots of the biquadratic equation,

(1) $A + B + C$,

(2) $AB + BC + CA$,

(3) ABC.

(1) $A+B+C=\frac{1}{2}(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)=\frac{q}{2}$,

(2) $AB+BC+CA=\frac{1}{4}(\alpha^2\beta\gamma+\alpha^2\gamma\delta+\dots)=\frac{1}{4}\Sigma\alpha^2\beta\gamma$

$=\frac{1}{8}(S_1^2S_2-S_2^2-2S_1S_3+2S_4)$; by the method of Art. 249.

Then the values of S_1, S_2, S_3 and S_4 may be substituted which have already been obtained, and the value of $\frac{1}{4}\Sigma\alpha^2\beta\gamma$ will be known. Or we may proceed thus,

$$\Sigma\alpha^2\beta\gamma=\Sigma\frac{\alpha^2\beta\gamma\delta}{\delta}=\alpha\beta\gamma\delta\Sigma\frac{\alpha}{\delta}.$$

And $\alpha\beta\gamma\delta=s$, and $\Sigma\frac{\alpha}{\delta}=\frac{pr}{s}-4$, by Art. 48;

therefore $AB+BC+CA=\frac{1}{4}(pr-4s)$.

(3) $ABC=\frac{1}{8}(\alpha^3\beta\gamma\delta+\dots+\alpha^2\beta^2\gamma^2+\dots)=\frac{1}{8}\Sigma\alpha^3\beta\gamma\delta+\frac{1}{8}\Sigma\alpha^2\beta^2\gamma^2$.

The values of these two symmetrical functions may be found by the methods of the present Chapter directly; or we may abbreviate those methods thus,

$$\Sigma\alpha^3\beta\gamma\delta=\alpha\beta\gamma\delta\Sigma\alpha^2=s(p^2-2q),$$

$$\Sigma\alpha^2\beta^2\gamma^2=\alpha^2\beta^2\gamma^2\delta^2\Sigma\frac{1}{\alpha^2}=s^2\left(\frac{r^2}{s^2}-\frac{2q}{s}\right),$$

for to find $\Sigma\frac{1}{\alpha^2}$ we have only to obtain the sum of the squares of the roots of the equation in y which is formed by writing $\frac{1}{y}$ for x.

Thus $ABC=\frac{1}{8}(r^2+p^2s-4qs)$.

The values of the functions of A, B, C which have been found may be verified; for A, B, C, by Art. 189, are the roots of the cubic equation in m in Art. 188.

XX. APPLICATIONS OF SYMMETRICAL FUNCTIONS.

252. In the present Chapter we shall give two applications of the theory of symmetrical functions of the roots of an equation; the first application will consist in forming the equation which has for its roots the squares of the differences of the roots of a given equation, and the second application will be to prove an important theorem in elimination.

253. *To form the equation which has for its roots the squares of the differences of the roots of a given equation.*

Suppose the given equation to be of the n^{th} degree, and denote its roots by a, b, c,.... Then the roots of the required equation will be $(a-b)^2$, $(a-c)^2$,...$(b-c)^2$,...; the number of these is the same as the number of combinations of n things taken 2 at a time, that is, $\frac{1}{2}n(n-1)$; and this number will therefore denote the degree of the required equation. Put m for $\frac{1}{2}n(n-1)$, and suppose that the required equation is denoted by

$$x^m + q_1 x^{m-1} + q_2 x^{m-2} + \dots + q_m = 0.$$

Also let s_r denote the sum of the r^{th} powers of the roots of this equation. We have only to determine s_1, s_2,... s_m, and then the coefficients of the required equation will be found in succession by the formulæ of Art. 244, namely, $s_1 + q_1 = 0$, $s_2 + q_1 s_1 + 2q_2 = 0$, and so on.

Let $$\phi(x) = (x-a)^{2r} + (x-b)^{2r} + (x-c)^{2r} + \dots,$$

then $$2s_r = \phi(a) + \phi(b) + \phi(c) + \dots$$

Now let S_1, S_2, S_3,... denote the sums of the powers of the roots of the given equation; thus

$$\phi(x) = nx^{2r} - 2rS_1 x^{2r-1} + \frac{2r(2r-1)}{1.2} S_2 x^{2r-2} - \dots + S_{2r}.$$

Put for x in succession $a, b, c, \ldots$ and add; thus

$$2s_r = nS_{2r} - 2rS_1S_{2r-1} + \frac{2r(2r-1)}{1.2} S_2S_{2r-2} - \ldots + nS_{2r}.$$

The terms on the right-hand side which are equidistant from the beginning and the end are equal; therefore by rearranging and dividing by 2 we obtain

$$s_r = nS_{2r} - 2rS_1S_{2r-1} + \frac{2r(2r-1)}{1.2} S_2S_{2r-2} - \ldots$$

$$\ldots + \frac{1}{2}(-1)^r \frac{2r(2r-1)\ldots(r+1)}{\lfloor r} S_r^2.$$

Now $S_1, S_2, \ldots$ can be expressed in terms of the coefficients of the given equation; thus s_r can be found, and then finally the coefficients of the required equation.

254. The last term of the required equation, namely that denoted by q_m in the preceding Article, may be calculated in another way. Let the given equation be denoted by $f(x) = 0$, so that

$$f(x) = (x-a)(x-b)(x-c)\ldots$$

Then $\quad f'(x) = (x-b)(x-c)\ldots + (x-a)(x-c)\ldots + \ldots;$

thus
$$f'(a) = (a-b)(a-c)\ldots,$$
$$f'(b) = (b-a)(b-c)\ldots$$

Hence $q_m = f'(a)f'(b)f'(c)\ldots.$

Now let $\alpha, \beta, \gamma, \ldots$ be the roots of the equation $f'(x) = 0$; then

$$f'(x) = n(x-\alpha)(x-\beta)(x-\gamma)\ldots;$$

therefore
$$f'(a)f'(b)f'(c)\ldots$$
$$= n^n(a-\alpha)(a-\beta)(a-\gamma)\ldots(b-\alpha)(b-\beta)\ldots(c-\alpha)\ldots.$$

But
$$(a-\alpha)(b-\alpha)(c-\alpha)\ldots = (-1)^n f(\alpha)\ldots,$$
$$(a-\beta)(b-\beta)(c-\beta)\ldots = (-1)^n f(\beta)\ldots,$$

and so on;

thus $$f'(a)f'(b)f'(c)\ldots = n^n(-1)^{n(n-1)} f(\alpha)f(\beta)f(\gamma)\ldots$$
$$= n^n f(\alpha)f(\beta)f(\gamma)\ldots,$$
for $(-1)^{n(n-1)} = 1$.

Now $f(\alpha)f(\beta)f(\gamma)\ldots$ is a *symmetrical* function of the roots of the derived equation $f'(x) = 0$, and may therefore be calculated.

255. In Art. 109 we have explained one use which we may make of the equation whose roots are the squares of the differences of the roots of a proposed equation; namely, we may thus determine the situation of the real roots of the proposed equation. But Sturm's theorem now answers this purpose more readily. However the equation which has for its roots the squares of the differences of the roots of a proposed equation will sometimes on inspection give information respecting the number of *imaginary* roots in the proposed equation; for it is obvious that if this new equation can have *negative* roots the proposed equation must have *imaginary* roots; and if the new equation has no negative roots the proposed equation has no imaginary roots. Also if the new equation has imaginary roots the proposed equation must have imaginary roots; it will not however follow that if the new equation has no imaginary roots the proposed equation has none. For example, the proposed equation might be a biquadratic equation with roots $\pm\lambda\sqrt{-1}$ and $\pm\mu\sqrt{-1}$; in this case the new equation will only have real negative roots.

It will be convenient to give the product of the squares of the differences of the roots in algebraical equations of the second, third, and fourth degrees.

(1) $$ax^2 + 2bx + c = 0.$$

The product is $$\frac{4(b^2 - ac)}{a^2}.$$

(2) $$x^3 + px^2 + qx + r = 0.$$

By Art. 60 the product is
$$-\frac{1}{27}\left\{(2p^3 - 9pq + 27r)^2 + 4(3q - p^2)^3\right\}.$$

If the equation be

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

this becomes $$-\frac{27}{a^6}\left\{(2b^3 - 3abc + a^2d)^2 + 4(ac - b^2)^3\right\},$$

or more symmetrically

$$-\frac{27}{a^4}\left\{(ad - bc)^2 - 4(b^2 - ac)(c^2 - bd)\right\}.$$

(3) $$x^4 + qx^2 + rx + s = 0.$$

By Art. 187 the product is

$$(\alpha^2 - \beta^2)^2 (\beta^2 - \gamma^2)^2 (\gamma^2 - \alpha^2)^2,$$

where α^2, β^2, γ^2 are the roots of a certain cubic.

Hence the product is

$$-\frac{1}{27}\left\{(27r^2 - 72qs + 2q^3)^2 - 4(q^2 + 12s)^3\right\}.$$

If the equation be

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0,$$

this becomes by Art. 187,

$$\frac{256}{a^6}\left\{(ae - 4bd + 3c^2)^3 - 27(ad^2 + eb^2 + c^3 - ace - 2bcd)^2\right\}.$$

256. We shall now shew how to eliminate one of the unknown quantities from two equations containing two unknown quantities, by the theory of symmetrical functions.

Let the equations be

$$p_0x^m + p_1x^{m-1} + p_2x^{m-2} + \ldots + p_m = 0,$$

and $$q_0x^n + q_1x^{n-1} + q_2x^{n-2} + \ldots + q_n = 0.$$

The coefficients $p_0, p_1, p_2, \ldots, q_0, q_1, q_2, \ldots$ are supposed rational integral functions of a quantity y, and x is to be eliminated.

Suppose that from the first of these equations the values of x could be found in terms of y; let these values be denoted by $a, b, c, \ldots$ Substitute them in the second equation, and we obtain m equations for determining y, namely

$$q_0a^n + q_1a^{n-1} + q_2a^{n-2} + \ldots + q_n = 0,$$
$$q_0b^n + q_1b^{n-1} + q_2b^{n-2} + \ldots + q_n = 0,$$
$$q_0c^n + q_1c^{n-1} + q_2c^{n-2} + \ldots + q_n = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

so that all admissible values of y are contained among the roots of these equations. And conversely any root of any one of these equations is an admissible value of y. For suppose, for example, that the first of these equations has a root β, and suppose, when β is put for y in a, that the value is α; then $x = \alpha$, $y = \beta$ will satisfy the two original equations. For these values obviously satisfy the second equation; and the first equation is satisfied by $x = a$, *whatever* y *may be*, and is therefore satisfied when we take $x = a$ and give to y in a the value β. Hence it follows that by multiplying together the left-hand members of the above equations in y and equating the product to zero we obtain the final equation in y. Now in this product no alteration is made by interchanging any two of the quantities $a, b, c, \ldots$, so that the product is a *symmetrical function* of these quantities, and the value of it can therefore be expressed in terms of the coefficients $p_0, p_1, p_2, \ldots$ of the first equation. Thus we shall finally obtain a rational integral equation in y, and this equation has for its roots all the admissible values of y and no others.

257. For a particular example, suppose that the first equation is a cubic in x, and the second a quadratic in x, so that we have to eliminate x from the equations

$$p_0x^3 + p_1x^2 + p_2x + p_3 = 0, \text{ and } q_0x^2 + q_1x + q_2 = 0,$$

where the coefficients are supposed functions of y. Here with the notation of the preceding Article we have

$(q_0a^2 + q_1a + q_2)(q_0b^2 + q_1b + q_2)(q_0c^2 + q_1c + q_2) = 0$, that is,

$$q_2^3 + q_1^3abc + q_0^3a^2b^2c^2 + q_0^2q_2\Sigma a^2b^2 + q_0^2q_1\Sigma a^2b^2c + q_1^2q_2\Sigma ab$$

$$+ q_1q_2^2\Sigma a + q_0q_2^2\Sigma a^2 + q_0q_1^2\Sigma a^2bc + q_0q_1q_2\Sigma a^2b = 0.$$

Also $abc = -\dfrac{p_3}{p_0}$, $\quad a^2b^2c^2 = \dfrac{p_3^2}{p_0^2}$,

$$\Sigma a^2b^2 = a^2b^2c^2\Sigma\frac{1}{a^2} = \frac{p_3^2}{p_0^2}\left(\frac{p_2^2}{p_3^2} - \frac{2p_1}{p_3}\right),$$

$$\Sigma a^2b^2c = abc\Sigma ab = -\frac{p_3}{p_0}\Sigma ab = -\frac{p_2p_3}{p_0^2},$$

$$\Sigma a = -\frac{p_1}{p_0}, \qquad \Sigma a^2 = \frac{p_1^2}{p_0^2} - \frac{2p_2}{p_0},$$

$$\Sigma a^2bc = abc\Sigma a = \frac{p_1p_3}{p_0^2},$$

$$\Sigma a^2b = abc\Sigma\frac{a}{b} = -\frac{p_3}{p_0}\left(\frac{p_1p_2}{p_0p_3} - 3\right), \text{ by Art. 48.}$$

And by substituting these values we shall obtain the equation which results from the elimination of x.

258. *If we eliminate one unknown quantity between two equations of the degrees* m *and* n *respectively, the degree of the resulting equation will not exceed* mn.

Let the equations be

$$p_0x^m + p_1x^{m-1} + p_2x^{m-2} + \ldots\ldots + p_m = 0,$$

$$q_0x^n + q_1x^{n-1} + q_2x^{n-2} + \ldots\ldots + q_n = 0;$$

the coefficients in these equations are supposed to be functions of y. Moreover it is now supposed that the sum of the exponents of x and y in the same term is never greater than m in the first equation, and never greater than n in the second equation; so that p_ρ and q_ρ may be of the degree ρ in y, but not higher.

Now suppose that x is eliminated by the method of Art. 257; the first member of the final equation in y then consists of a series of terms, each of which is the product of m factors, and is of the form $q_r a^{n-r} \times q_s b^{n-s} \times q_t c^{n-t} \times \ldots$ And as we know that the series of terms forms a *symmetrical* function of $a, b, c, \ldots$, the aggregate of the terms with the exponents just indicated will be

$$q_r q_s q_t \ldots \Sigma a^{n-r} b^{n-s} c^{n-t} \ldots$$

Now the degree of $q_r q_s q_t \ldots$ is not higher than $r+s+t+\ldots$, so that we have only to shew that the degree of $\Sigma a^{n-r} b^{n-s} c^{n-t} \ldots$ is not higher than $n-r+n-s+n-t+\ldots$, and then it will follow that the degree of the product is not higher than mn. The required result follows from two observations. (1) From the formulæ of Art. 244, it can be shewn that S_ρ does not involve higher powers of y than y^ρ. (2) From the process of Arts. 248 and 249, it will follow that the value of $\Sigma a^\lambda b^\mu c^\nu \ldots$ will involve powers and products of $S_1, S_2, S_3, \ldots S_{\lambda+\mu+\nu+\ldots}$; and in each term the sum of the subscript letters attached to the symbol S is $\lambda+\mu+\nu+\ldots$

Hence we conclude that in the final equation in y no power of y higher than y^{mn} will occur.

259. The preceding Article gives the limit which the degree of the final equation in y cannot surpass; it may however in particular cases fall short of this limit.

The theorem may be extended and the following general result obtained; if between any number of equations involving the same number of unknown quantities all those quantities are eliminated except one, the degree of the final equation cannot exceed the product of the degrees of the original equations. See Serret's *Cours d'Algèbre Supérieure*.

XXI. SUMS OF THE POWERS OF THE ROOTS.

260. By Newton's method, which is explained in Art. 244, the sums of the powers of the roots of an equation may be found *successively;* we shall now explain a method by which the sum for any assigned integral power of the roots of an equation may be obtained *independently.*

Let a, b, c,... denote the roots of an equation $f(x) = 0$, so that we have $f(x) = (x-a)(x-b)(x-c)\ldots$; and suppose the equation of the n^{th} degree. Then

$$\frac{f(x)}{x^n} = \left(1-\frac{a}{x}\right)\left(1-\frac{b}{x}\right)\left(1-\frac{c}{x}\right)\ldots$$

Take the logarithm of both sides, and then expand the logarithms on the right-hand side; thus

$$\begin{aligned}\log\frac{f(x)}{x^n} = &-\frac{1}{x}(a+b+c+\ldots)\\ &-\frac{1}{2x^2}(a^2+b^2+c^2+\ldots)\\ &-\frac{1}{3x^3}(a^3+b^3+c^3+\ldots)\\ &-\ldots\ldots\end{aligned}$$

Thus on the right-hand side the coefficient of $\frac{1}{x^m}$ is $-\frac{S_m}{m}$; hence we have $\frac{S_m}{m}$ = the coefficient of $\frac{1}{x^m}$ in the expansion of $-\log\frac{f(x)}{x^n}$ in descending powers of x.

This supposes m positive; if the sum for any negative integral power is required we can change x into $\frac{1}{y}$ and find the sum for the corresponding positive power of the roots of the equation in y.

261. For example, find the sum of the m^{th} powers of the roots of the equation $x^2 - px + q = 0$.

Here
$$\frac{f(x)}{x^2} = 1 - \left(\frac{p}{x} - \frac{q}{x^2}\right);$$

$$-\log \frac{f(x)}{x^2} = -\log\left\{1 - \left(\frac{p}{x} - \frac{q}{x^2}\right)\right\}$$

$$= \frac{p}{x} - \frac{q}{x^2} + \frac{1}{2}\left(\frac{p}{x} - \frac{q}{x^2}\right)^2 + \frac{1}{3}\left(\frac{p}{x} - \frac{q}{x^2}\right)^3 + \ldots + \frac{1}{m}\left(\frac{p}{x} - \frac{q}{x^2}\right)^m + \ldots$$

The complete coefficient of $\frac{1}{x^m}$ may be obtained by selection from the various terms in the value of $-\log\frac{f(x)}{x^2}$ in which this power of x can occur; these terms written in the reverse order are

$$\frac{1}{m}\left(\frac{p}{x} - \frac{q}{x^2}\right)^m + \frac{1}{m-1}\left(\frac{p}{x} - \frac{q}{x^2}\right)^{m-1} + \frac{1}{m-2}\left(\frac{p}{x} - \frac{q}{x^2}\right)^{m-2} + \ldots$$

The coefficient of $\frac{1}{x^m}$ is therefore

$$\frac{1}{m}p^m - \frac{1}{m-1}\,\frac{m-1}{1}\,p^{m-2}q + \frac{1}{m-2}\,\frac{(m-2)(m-3)}{1\,.\,2}\,p^{m-4}q^2 - \ldots$$

Thus $S_m = p^m - mp^{m-2}q + \frac{m(m-3)}{1\,.\,2}p^{m-4}q^2 - \ldots$

$$+ (-1)^r \frac{m(m-r-1)\ldots\ldots(m-2r+1)}{\lfloor r}p^{m-2r}q^r + \ldots$$

Suppose $q = 1$, then the quadratic equation is a reciprocal equation, and its roots are of the form a and $\frac{1}{a}$; see Art. 133.

Thus we have $a + \frac{1}{a} = p$, and also

$$a^m + \frac{1}{a^m} = p^m - mp^{m-2} + \frac{m(m-3)}{1 \cdot 2} p^{m-4} - \ldots$$
$$+ (-1)^r \frac{m(m-r-1)\ldots(m-2r+1)}{\underline{|r}} p^{m-2r} + \ldots$$

We have thus obtained a general expression for $a^m + \dfrac{1}{a^m}$ in terms of powers of $a + \dfrac{1}{a}$; see Art. 138.

Again, suppose $q = -1$; then the roots of the quadratic are of the form a and $-\dfrac{1}{a}$: thus we get an expression for $a^m + \left(-\dfrac{1}{a}\right)^m$ in terms of powers of $a - \dfrac{1}{a}$.

262. Again, let it be required to find the sum of the m^{th} powers of the roots of the equation $x^n - 1 = 0$.

Here $\dfrac{f(x)}{x^n} = 1 - \dfrac{1}{x^n}$, $\quad -\log \dfrac{f(x)}{x^n} = \dfrac{1}{x^n} + \dfrac{1}{2x^{2n}} + \dfrac{1}{3x^{3n}} + \dfrac{1}{4x^{4n}} + \ldots$

Here the coefficient of $\dfrac{1}{x^m}$ is zero unless m is a multiple of n, and then the coefficient is $\dfrac{n}{m}$; so that $S_m = 0$ unless m is a multiple of n, and then $S_m = n$.

This result is often useful, and we will give three applications of it in the following three Articles.

263. We will shew how to find the sum of selected terms of a given series.

Suppose that the sum of $a_0 + a_1 x + a_2 x^2 + \ldots$ *ad infinitum* is known, and denote it by $\phi(x)$: and let it be required to find the sum of the series

$$a_m x^m + a_{m+n} x^{m+n} + a_{m+2n} x^{m+2n} + \ldots\ldots \textit{ ad infinitum.}$$

Let $\alpha, \beta, \gamma, \ldots$ denote the n^{th} roots of unity, that is, the n roots of the equation $x^n - 1 = 0$. Multiply both sides of the given identity by α^{n-m}, and then change x into αx; thus

$$\alpha^{n-m}\phi(\alpha x) = a_0\alpha^{n-m} + a_1\alpha^{n-m+1}x + a_2\alpha^{n-m+2}x^2 + \ldots$$

Similarly,

$$\beta^{n-m}\phi(\beta x) = a_0\beta^{n-m} + a_1\beta^{n-m+1}x + a_2\beta^{n-m+2}x^2 + \ldots,$$

$$\gamma^{n-m}\phi(\gamma x) = a_0\gamma^{n-m} + a_1\gamma^{n-m+1}x + a_2\gamma^{n-m+2}x^2 + \ldots,$$

and so on.

Add together the n identities which can thus be formed; then on the right-hand side we obtain n times the required series, by Art. 262; thus

$$a_m x^m + a_{m+n}x^{m+n} + a_{m+2n}x^{m+2n} + \ldots$$

$$= \frac{1}{n}\left\{\alpha^{n-m}\phi(\alpha x) + \beta^{n-m}\phi(\beta x) + \gamma^{n-m}\phi(\gamma x) + \ldots\right\}.$$

As an example we will find the sum of

$$x + \frac{x^4}{\underline{|4}} + \frac{x^7}{\underline{|7}} + \ldots \textit{ ad infinitum.}$$

Here $\qquad m = 1, \quad n = 3, \quad \phi(x) = e^x.$

Thus the required sum

$$= \frac{1}{3}\left\{\alpha^2\phi(\alpha x) + \beta^2\phi(\beta x) + \gamma^2\phi(\gamma x)\right\}.$$

Now $\quad \alpha = 1, \quad \beta = \dfrac{-1+\sqrt{-3}}{2}, \quad \gamma = \dfrac{-1-\sqrt{-3}}{2}.$

Hence
$$\phi(\beta x) = e^{-\frac{x}{2} + \frac{x\sqrt{-3}}{2}}$$

$$= e^{-\frac{x}{2}}\left(\cos\frac{x\sqrt{3}}{2} + \sqrt{-1}\sin\frac{x\sqrt{3}}{2}\right);$$

$$\phi(\gamma x) = e^{-\frac{x}{2}}\left(\cos\frac{x\sqrt{3}}{2} - \sqrt{-1}\sin\frac{x\sqrt{3}}{2}\right).$$

And finally the required sum is

$$\frac{1}{3}e^x - \frac{1}{3}e^{-\frac{x}{2}}\left(\cos\frac{x\sqrt{3}}{2} - \sqrt{3}\sin\frac{x\sqrt{3}}{2}\right).$$

264. Again, by means of Art. 262 we can prove the following theorem; the expression $(x+y)^n - x^n - y^n$ is divisible by x^2+xy+y^2 if n be an odd positive integer not divisible by 3, and it is divisible by $(x^2+xy+y^2)^2$ if n be a positive integer of the form $6m+1$.

Let 1, α, β, be the three cube roots of unity, that is, the three roots of the equation $x^3-1=0$. Then the product of these roots is 1, that is, $\alpha\beta=1$, by Art. 45; and $1+\alpha^m+\beta^m=0$, provided m be not a multiple of 3, by Art. 262.

Thus $$x^2+xy+y^2=(x-\alpha y)(x-\beta y).$$

Hence $(x+y)^n - x^n - y^n$ is divisible by x^2+xy+y^2 provided it vanishes when $x=\alpha y$, and when $x=\beta y$; and it is divisible by $(x^2+xy+y^2)^2$, provided its derived function *also* vanishes when $x=\alpha y$, and when $x=\beta y$: this derived function, by Arts. 11, 13, is

$$n(x+y)^{n-1} - nx^{n-1}.$$

When $x=\alpha y$ we have

$$(x+y)^n - x^n - y^n = y^n\left\{(1+\alpha)^n - \alpha^n - 1\right\} = y^n\left\{(-\beta)^n - \alpha^n - 1\right\},$$

and this vanishes when n is an odd integer which is not divisible by 3.

Also, when $x=\alpha y$,

$$n(x+y)^{n-1} - nx^{n-1} = ny^{n-1}\left\{(1+\alpha)^{n-1} - \alpha^{n-1}\right\} = ny^{n-1}\left\{(-\beta)^{n-1} - \alpha^{n-1}\right\};$$

this vanishes if $n-1$ is an even integer and a multiple of 3, because $\alpha^3=1$, and $\beta^3=1$. And if $n-1$ is an even integer and a multiple of 3, it follows that n is an odd integer and not divisible by 3, so that $(x+y)^n - x^n - y^n$ also vanishes.

The same results would be obtained by putting βy for x.

Comptes Rendus......Vol. IX. p. 360.

265. The last application we shall make of Art. 262 is to prove the following theorem.

Let S denote the sum of the series

$$1-\frac{n-3}{2}+\frac{(n-4)(n-5)}{\lfloor 3}-\frac{(n-5)(n-6)(n-7)}{\lfloor 4}+\ldots$$
$$+(-1)^{r-1}\frac{(n-r-1)(n-r-2)\ldots(n-2r+1)}{\lfloor r}+\ldots$$

Then $S=\frac{3}{n}$ if n is an odd positive integer divisible by 3,

$S=0$ if n is an odd positive integer not divisible by 3,

$S=-\frac{1}{n}$ if n is an even positive integer divisible by 3,

$S=\frac{2}{n}$ if n is an even positive integer not divisible by 3.

In Art. 261 put xy for q and $x+y$ for p, so that $S_n=x^n+y^n$; thus, if n is a positive integer,

$$(x+y)^n-x^n-y^n=nxy(x+y)\left\{(x+y)^{n-3}-\frac{n-3}{2}xy(x+y)^{n-5}\right.$$
$$\left.+\frac{(n-4)(n-5)}{\lfloor 3}(xy)^2(x+y)^{n-7}-\ldots\right\}\ldots(1).$$

Let 1, α, β, denote the three cube roots of unity; put $x=\alpha y$, then the right-hand member of (1) becomes

$$n\alpha(1+\alpha)y^n\left\{(1+\alpha)^{n-3}-\frac{n-3}{2}\alpha(1+\alpha)^{n-5}+\frac{(n-4)(n-5)}{\lfloor 3}\alpha^2(1+\alpha)^{n-7}-\ldots\right\}.$$

But $\alpha\beta=1$, and therefore $\beta^2=\alpha\beta^3=\alpha$; also $\alpha+\beta+1=0$, so that $-\beta=\alpha+1$; thus $\alpha=(\alpha+1)^2$. Hence the right-hand member of (1) reduces to

$$n(1+\alpha)^n y^n\left\{1-\frac{n-3}{2}+\frac{(n-4)(n-5)}{\lfloor 3}-\ldots\right\},$$

that is $\qquad n(-\beta)^n y^n S.$

Also when $x = \alpha y$ the left-hand member of (1) becomes

$$y^n \left\{(1+\alpha)^n - \alpha^n - 1\right\}, \text{ that is, } y^n \left\{(-\beta)^n - \alpha^n - 1\right\}.$$

Therefore $\qquad (-\beta)^n - \alpha^n - 1 = n(-\beta)^n S \dots\dots\dots\dots\dots\dots(2).$

If n is an odd positive integer divisible by 3, the left-hand member of (2) is equal to -3 by Art. 262; therefore $-3 = -n\beta^n S = -nS$; therefore $S = \frac{3}{n}$.

If n is an odd positive integer not divisible by 3, the left-hand member of (2) is zero by Art. 262; therefore $S = 0$.

If n is an even positive integer divisible by 3, the left-hand member of (2) is -1, and the right-hand member is nS; therefore $S = -\frac{1}{n}$.

If n is an even positive integer not divisible by 3, the left-hand member of (2) is $\beta^n - \alpha^n - 1$, that is $2\beta^n$, since $\alpha^n + \beta^n + 1 = 0$; thus $2\beta^n = n\beta^n S$, and therefore $S = \frac{2}{n}$.

It is to be observed that the series denoted by S consists of a *finite* number of terms; in fact if $n = 2m$ or $2m+1$ there are m terms in the series.

Crelle's *Mathematical Journal*, Vol. xx. p. 321.

This Article serves to illustrate the present subject: but we may observe that the result can be obtained more simply by another method.

It is known, see *Plane Trigonometry*, Chapter xx, that

$$2\cos n\theta = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{n(n-3)}{1 \cdot 2}(2\cos\theta)^{n-4} -$$

$$\dots + (-1)^r \frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{\lfloor r}(2\cos\theta)^{n-2r} + \dots$$

Put $\theta = \frac{\pi}{3}$; hence, transposing and dividing by n, we obtain

$$S = \frac{1}{n}\left(1 - 2\cos\frac{n\pi}{3}\right).$$

266. As another example of the theorem of Art. 260 we will shew how to express $x^n + y^n + (-x-y)^n$ in terms of $x^2 + xy + y^2$, and $xy(x+y)$.

Let $$a = x^2 + xy + y^2, \quad b = xy(x+y),$$

and put z for $-x-y$.

Then $$x + y + z = 0,$$

$$xy + yz + zx = xy - (x+y)^2 = -a,$$

$$xyz = -b;$$

thus x, y, and z are the roots of the cubic equation

$$t^3 - at + b = 0;$$

and therefore $\frac{1}{n}(x^n + y^n + z^n)$ is equal to the coefficient of $\frac{1}{t^n}$ in the expansion of $-\log\left(1 - \frac{a}{t^2} + \frac{b}{t^3}\right)$.

Now $-\log\left(1 - \frac{a}{t^2} + \frac{b}{t^3}\right)$

$$= \frac{1}{t^2}\left(a - \frac{b}{t}\right) + \frac{1}{2t^4}\left(a - \frac{b}{t}\right)^2 + \frac{1}{3t^6}\left(a - \frac{b}{t}\right)^3 + \dots$$

We can then expand $\left(a - \frac{b}{t}\right)^2$, $\left(a - \frac{b}{t}\right)^3$, ... and collect the coefficient of any assigned power of $\frac{1}{t}$.

If n be an even number we thus obtain a formula for

$$(x+y)^n + x^n + y^n;$$

and if n be an odd number for

$$(x+y)^n - x^n - y^n.$$

The following are special cases:

$$(x+y)^7 - x^7 - y^7 = 7a^2b = 7(x^2 + xy + y^2)^2 xy(x+y),$$

$$(x+y)^8 + x^8 + y^8 = 2a^4 + 8ab^2$$

$$= 2(x^2 + xy + y^2)\{(x^2 + xy + y^2)^3 + 4x^2y^2(x+y)^2\}.$$

The general formulæ may be easily obtained by putting $2m$ and $2m+1$ for n. Thus it will be found that

$$\frac{(x+y)^{2m}+x^{2m}+y^{2m}}{2m}=\frac{a^m}{m}+\frac{m-2}{1.2}a^{m-3}b^2+\frac{(m-3)(m-4)(m-5)}{\underline{|4}}a^{m-6}b^4$$
$$+\ldots+\frac{(m-r-1)(m-r-2)\ldots(m-3r+1)}{\underline{|2r}}a^{m-3r}b^{2r}+\ldots;$$

and that

$$\frac{(x+y)^{2m+1}-x^{2m+1}-y^{2m+1}}{2m+1}=a^{m-1}b+\frac{(m-2)(m-3)}{\underline{|3}}a^{m-4}b^3+\ldots$$
$$+\frac{(m-r-1)(m-r-2)\ldots(m-3r)}{\underline{|2r+1}}a^{m-3r-1}b^{2r+1}+\ldots$$

267. It has been proposed to make use of the values of the sums of the powers of the roots of an equation in order to approximate to a root of the equation; we will give an account of this method drawn from Murphy's *Treatise on the Theory of Algebraical Equations.*

Let a, b, c,... denote the roots of an equation; suppose them all real and a numerically the greatest. We have

$$\frac{S_{m+1}}{S_m}=\frac{a^{m+1}+b^{m+1}+c^{m+1}+\ldots}{a^m+b^m+c^m+\ldots}$$
$$=a\frac{1+\left(\frac{b}{a}\right)^{m+1}+\left(\frac{c}{a}\right)^{m+1}+\ldots}{1+\left(\frac{b}{a}\right)^{m}+\left(\frac{c}{a}\right)^{m}+\ldots}$$

Thus if m be taken large enough the right-hand member can be made to approach as near as we please to a, that is, to the value of the numerically greatest root.

268. We may now examine how far the result of the preceding Article is modified by the presence of imaginary roots. Let $\beta+\gamma\sqrt{-1}$ and $\beta-\gamma\sqrt{-1}$ be a pair of conjugate imaginary roots; their sum is 2β and their product is $\beta^2+\gamma^2$, which is the square of their *modulus;* see *Algebra,* Chap. XXV.

Now $$\beta \pm \gamma\sqrt{-1} = \mu\left(\frac{\beta}{\mu} \pm \frac{\gamma}{\mu}\sqrt{-1}\right).$$

Assume $$\frac{\beta}{\mu} = \cos\theta, \text{ and } \frac{\gamma}{\mu} = \sin\theta,$$

so that $$\tan\theta = \frac{\gamma}{\beta} \text{ and } \mu^2 = \beta^2 + \gamma^2;$$

thus μ is the modulus. Then the conjugate roots may be put in the form $\mu(\cos\theta \pm \sqrt{-1}\sin\theta)$; and by De Moivre's theorem the sum of the m^{th} powers of the two roots is $2\mu^m \cos m\theta$.

Thus if the numerical value of the greatest real root be greater than the greatest modulus of the imaginary roots, $\frac{S_{m+1}}{S_m}$ will tend to a limit as m is indefinitely increased, namely, to the numerically greatest root; but if there is a modulus of the imaginary roots greater than the numerically greatest root, there will be no limiting value of $\frac{S_{m+1}}{S_m}$.

Example. $x^3 - 2x - 5 = 0$. Here the series $S_1, S_2, S_3, \ldots\ldots$ is 0, 4, 15, 8, 50, 91, 140, 432, 735, 1564, 3630, 6803, 15080, 31756, 64175, 138912, 287130, 598699,...... By dividing each term by the preceding, we observe a tendency to a limit a little greater than 2, so that we may conclude that there is a real root a little greater than 2. The example however is not a very favourable one for the method; for since the product of all the roots is 5, and the real root is rather greater than 2, the product of the other two roots is nearly 2·5. These two roots are imaginary by Art. 172, and as their modulus is the square root of their product, the modulus is greater than 1·5; thus the modulus is not very small compared with the real root, and so the expression $\frac{S_{m+1}}{S_m}$ approaches slowly towards its limit.

269. We may obtain the product of the two numerically greatest roots in certain cases, by a method similar to that in Art. 267.

For
$$S_m = a^m + b^m + c^m + \dots,$$
$$S_{m+1} = a^{m+1} + b^{m+1} + c^{m+1} + \dots,$$
$$S_{m+2} = a^{m+2} + b^{m+2} + c^{m+2} + \dots .$$

Therefore
$$S_m S_{m+2} - S^2_{m+1} = a^m b^m (a-b)^2 + a^m c^m (a-c)^2 + b^m c^m (b-c)^2 + \dots$$

We will denote this by u_m, so that
$$u_m = a^m b^m (a-b)^2 \left\{1 + \frac{c^m}{b^m}\left(\frac{a-c}{a-b}\right)^2 + \frac{c^m}{a^m}\left(\frac{b-c}{a-b}\right)^2 + \dots\right\}.$$

Hence by proceeding as in Arts. 267 and 268 we may obtain the following results.

(1) If all the roots are real $\frac{u_{m+1}}{u_m}$ can be brought as near as we please to the product of the two numerically greatest roots by increasing m sufficiently.

(2) If there are real roots numerically greater than the modulus of any imaginary root, there is a limiting value of $\frac{u_{m+1}}{u_m}$, namely, the product of the two greatest of these roots.

(3) If there be one or more moduli greater than the numerically greatest real root there is a limiting value of $\frac{u_{m+1}}{u_m}$, namely, the square of the greatest of these moduli, that is, the product of the corresponding imaginary roots.

(4) Thus the only case in which $\frac{u_{m+1}}{u_m}$ can fail to have a limit is when there is one real root, and only one, numerically greater than the greatest modulus of the imaginary roots. In this case that real root can be found by Art. 267.

270. We may also obtain in certain cases the sum of two roots of an equation by a similar method.

From the values of S_m, S_{m+1}, S_{m+2}, and S_{m+3}, we shall obtain

$$S_m S_{m+3} - S_{m+1} S_{m+2} = a^m b^m (a+b)(a-b)^2 + a^m c^m (a+c)(a-c)^2 + b^m c^m (b+c)(b-c)^2 + \dots;$$

we will denote this by v_m. Then u_m having the meaning assigned in the preceding Article, we shall find that there is a limit of $\frac{v_m}{u_m}$ in the cases named in the preceding Article, and that this limit is the sum of the numerically greatest roots, or the sum of the two imaginary roots with the greatest modulus.

271. Thus in cases (1), (2), and (3) of Art. 269 we can get the product of two roots by Art. 269 and their sum by Art. 270; and in cases (1) and (2) we can get the sum of two roots by Art. 270 and the greater of them by Art. 267.

272. Example. $x^4 + x^3 + 4x^2 - 4x + 1 = 0$.

Here we obtain the following values:

for $S_1, S_2, \dots -1, -7, 23, -3, -116, 227, 202, -1571, \dots$;

for $u_1, u_2, \dots -72, -508, -2677, -14137, -74961, -397421, \dots$;

for $v_1, v_2, \dots 164, 881, 4873, 25726, 136382, \dots$

Here no definite limit is obtained by dividing each term in the series $S_1, S_2, \dots$ by its predecessor; we are therefore sure of the existence of imaginary roots. By dividing each term of the series $u_1, u_2, \dots$ by its predecessor, we obtain quotients which indicate $5{\cdot}301\dots$ as the value of the product of two roots. By dividing each term of the series $v_1, v_2, \dots$ by the corresponding term of the series $u_1, u_2, \dots$ we obtain quotients which indicate $-1{\cdot}819\dots$ as the sum of these two roots. From these values we can obtain approximate values of two imaginary roots.

Since the sum of all the four roots of the equation is -1, and their product is 1, the sum of the remaining two roots is $\cdot 819\dots$ and their product $\frac{1}{5{\cdot}301\dots}$; these two roots are therefore also imaginary.

Thus we shall find in this example that the modulus of the first pair of imaginary roots is about five times as great as the modulus of the other pair. Hence with the notation of Art. 269 we shall find that in taking $u_m = a^m b^m (a-b)^2$ and neglecting the other terms, the error is about $\frac{1}{5^m}$ of the whole quantity; and hence we can judge of the accuracy of our result. For example; we have given above the values of u_m as far as u_5 and u_6, so that we can depend upon having found the product of the roots with an error of only about $\left(\frac{1}{5^5}\right)^{\text{th}}$ part of the whole.

XXII. ELIMINATION.

273. Suppose that we have to solve two simultaneous equations involving two unknown quantities; there are certain cases in which the solution can be readily effected. Suppose that x and y denote the unknown quantities; then if one of the equations involves x^m and no other power of x, we can immediately find x^m from this equation in terms of y and substitute it in the other equation; we shall thus obtain an equation involving y only, and the roots of this equation may be found exactly or approximately by methods already explained.

Again suppose that the equations are represented by $A=0$ and $B=0$, and that A and B can be readily decomposed into factors; suppose for example that $A = UU'U''$ and $B = VV'$. Then all the solutions of the proposed equations are obtained by solving the simultaneous equations $U=0$ and $V=0$, $U=0$ and $V'=0$, $U'=0$ and $V=0$, $U'=0$ and $V'=0$, $U''=0$ and $V=0$, $U''=0$ and $V'=0$. Thus the solution of the proposed equations is made to depend upon the solution of other equations of lower degrees.

It may happen that one of the factors of A is identical with one of the factors of B; for example, suppose that U and V are identical. Then any values of x and y which satisfy the equation $U=0$ will satisfy the simultaneous equations $A=0$ and $B=0$. Thus if U involves both x and y, we can assign any value we please to one of the unknown quantities and determine the corresponding value of the other, and so obtain as many solutions as we please. If U involves only one of the unknown quantities we can satisfy the equations $A=0$ and $B=0$, by giving to that unknown quantity a value deduced from the equation $U=0$, and any value we please to the other unknown quantity.

274. We have already shewn how by the aid of the theory of symmetrical functions we can eliminate one of the unknown quantities from two equations, and so obtain a final equation which involves only the other unknown quantity. We are now about to explain another method of performing the elimination, which depends on the process of finding the greatest common measure of two algebraical expressions.

275. Let the two simultaneous equations be denoted by $f_1(x, y)=0$ and $f_2(x, y)=0$. Suppose that $x=\alpha$ and $y=\beta$ are values which satisfy these equations; then the equations $f_1(x, \beta)=0$ and $f_2(x, \beta)=0$ are satisfied by the value $x=\alpha$. Hence $f_1(x, \beta)$ and $f_2(x, \beta)$ must have a common measure; this common measure must be such that when equated to zero it furnishes the value α, and also any other value or values by which in conjunction with $y=\beta$ the proposed equations are satisfied.

Suppose then that we arrange $f_1(x, y)$ and $f_2(x, y)$ according to descending powers of x, and proceed in the usual way to find their greatest common measure, carrying on the operation until we arrive at a remainder which is a function of y only, say $\phi(y)$. Then no values of y will be admissible except such as make $\phi(y)=0$; for unless $\phi(y)$ vanishes $f_1(x, y)$ and $f_2(x, y)$ have no common measure and therefore do not vanish simultaneously. It is not however true conversely that every value of y which makes $\phi(y)$ vanish is necessarily admissible. For it may happen that in the process the coefficients of some of the powers of x are fractions involving y *in their denominators;* and a value of y which satisfies the equation $\phi(y)=0$ may make some of these denominators vanish, and thus introduce infinite or indeterminate quantities. Suppose, for example, that we have

$$f_1(x, y)=qf_2(x, y)+\phi(y).$$

Then if q is an integral expression it will not be rendered infinite by any finite value of y, and any value of y which makes $\phi(y)$ vanish, combined with the corresponding value of x deduced from the equation $f_2(x, y)=0$, will make $f_1(x, y)$ vanish. But if q is a fraction, involving y in its denominator, q may be infinite when $\phi(y)$ vanishes, and $f_1(x, y)$ will not necessarily vanish when $\phi(y)=0$ and $f_2(x, y)=0$. The same exception may occur when we carry on the process in the usual way, and introduce factors which are not functions of x in order to avoid fractional coefficients. Suppose, for example, that we multiply $f_1(x, y)$ by a quantity c in order to avoid the fractional coefficients which are functions of y; and suppose we now have

$$cf_1(x, y)=qf_2(x, y)+\phi(y).$$

If we find y from the equation $\phi(y)=0$, and then x from the equation $f_2(x, y)=0$, the values so obtained must necessarily make $cf_1(x, y)$ vanish; but it does not follow that $f_1(x, y)$ vanishes, for it may be that the value of y which has been taken makes c vanish.

Hence we require a rule which shall point out the admissible solutions, and to this rule we shall now proceed. We shall suppose that in finding the greatest common measure the usual precautions are taken to avoid fractional coefficients. We may assume that in the equations which we shall denote by $A=0$ and $B=0$, neither A nor B contains any factor which is a function of y *only;* for such a factor can be separately considered and all the solutions found which depend on it. The method we are about to explain is due to MM. Labatie and Sarrus; we shall give it from the *Algebra* of MM. Mayer and Choquet.

276. Let the two simultaneous equations be denoted by $A=0$ and $B=0$; we will suppose that neither A nor B has a

factor which is a function of y only, and that B is not of a higher degree in x than A. Let c denote the factor by which A must be multiplied in order that it may be divisible by B; let q be the quotient and rR the remainder, where r is a function of y only. Let c_1 denote the factor by which B must be multiplied in order that it may be divisible by R; let q_1 be the quotient and r_1R_1 the remainder, where r_1 is a function of y only. Proceed in this way, and suppose, for example, that at the fourth division we have a remainder which does not contain x, and which we may denote by r_3. Thus we shall have the following identities:

$$\left.\begin{aligned} cA &= qB + rR, \\ c_1B &= q_1R + r_1R_1, \\ c_2R &= q_2R_1 + r_2R_2, \\ c_3R_1 &= q_3R_2 + r_3 \ . \end{aligned}\right\} \quad \text{.....................} (1)$$

Let d be the greatest common measure of c and r, let d_1 be the greatest common measure of $\frac{cc_1}{d}$ and r_1, let d_2 be the greatest common measure of $\frac{cc_1c_2}{dd_1}$ and r_2, let d_3 be the greatest common measure of $\frac{cc_1c_2c_3}{dd_1d_2}$ and r_3. We shall now prove that the solutions of the equations $A=0$ and $B=0$ will be obtained by solving the following systems:

$$\left.\begin{aligned} \frac{r}{d} &= 0 \text{ and } B = 0, \\ \frac{r_1}{d_1} &= 0 \text{ and } R = 0, \\ \frac{r_2}{d_2} &= 0 \text{ and } R_1 = 0, \\ \frac{r_3}{d_3} &= 0 \text{ and } R_2 = 0; \end{aligned}\right\} \quad \text{.....................} (2)$$

that is, we shall shew in the first place that all the solutions obtained from (2) do satisfy the equations $A=0$ and $B=0$, and in the second place that all the values of x and y which satisfy the equations $A=0$ and $B=0$ are included among the solutions obtained from the system (2).

Divide both members of the first identity (1) by d; thus

$$\frac{c}{d}A=\frac{q}{d}B+\frac{r}{d}R \text{..........................} (3).$$

Now, by hypothesis, $\frac{c}{d}$ and $\frac{r}{d}$ are both integral functions of y; thus $\frac{qB}{d}$ is also an integral function; but by hypothesis B has no factor which is a function of y only, and therefore d must divide q.

The identity (3) shews that the values of x and y which satisfy the equations $\frac{r}{d}=0$ and $B=0$ make $\frac{c}{d}A$ vanish; but $\frac{c}{d}$ and $\frac{r}{d}$ by hypothesis have no common factor, and therefore these values make A vanish. Hence all the solutions of the equations $\frac{r}{d}=0$ and $B=0$ satisfy the equations $A=0$ and $B=0$.

Again, multiply both members of the identity (3) by c_1, and substitute for c_1B its equivalent obtained from the second of the identities (1); thus

$$\frac{cc_1}{d}A=\frac{c_1r+qq_1}{d}R+\frac{q}{d}r_1R_1.$$

The expression $\frac{c_1r+qq_1}{d}$ is integral, for r and q are divisible by d; moreover this expression is divisible by d_1, for d_1 divides $\frac{cc_1}{d}$ and r_1 and does not divide R. Divide by d_1; then, for shortness, putting M for $\frac{q}{d}$ and M_1 for $\frac{c_1r+qq_1}{dd_1}$, we have

$$\frac{cc_1}{dd_1}A=M_1R+\frac{r_1}{d_1}MR_1 \text{..........................} (4).$$

Multiply both members of the second of the identities (1) by $\frac{c}{d}$; thus

$$\frac{cc_1}{d}B = \frac{cq_1}{d}R + \frac{c}{d}r_1R_1.$$

Since d_1 will divide $\frac{cc_1}{d}$ and r_1, it will divide $\frac{cq_1}{d}R$; but R is not divisible by d_1 and therefore $\frac{cq_1}{d}$ must be. Divide by d_1; then, for shortness, putting N for $\frac{c}{d}$ and N_1 for $\frac{cq_1}{dd_1}$, we have

$$\frac{cc_1}{dd_1}B = N_1R + \frac{r_1}{d_1}NR_1 \text{.........................} (5).$$

The identities (4) and (5) shew that all the values of x and y which make $\frac{r_1}{d_1}$ and R vanish, make $\frac{cc_1}{dd_1}A$ and $\frac{cc_1}{dd_1}B$ vanish; but $\frac{cc_1}{dd_1}$ and $\frac{r_1}{d_1}$ have no common factor, and therefore all the solutions of the equations $\frac{r_1}{d_1} = 0$ and $R = 0$ satisfy the equations $A = 0$ and $B = 0$.

Again, multiply both members of the identity (4) by c_2, and substitute for c_2R its equivalent from the third of the identities (1); thus

$$\frac{cc_1c_2}{dd_1}A = \left(q_2M_1 + \frac{c_2r_1}{d_1}M\right)R_1 + r_2M_1R_2.$$

By hypothesis d_2 divides the first member of this identity, and also divides r_2; it must therefore divide $\left(q_2M_1 + \frac{c_2r_1}{d_1}M\right)R_1$, but R_1 is not divisible by d_2; therefore $q_2M_1 + \frac{c_2r_1}{d_1}M$ is divisible by d_2. Denote the quotient by M_2; thus

$$\frac{cc_1c_2}{dd_1d_2}A = M_2R_1 + \frac{r_2}{d_2}M_1R_2 \text{...................} (6).$$

Multiply both members of the identity (5) by c_2, and substitute for c_2R its equivalent from the third of the identities (1); thus

$$\frac{cc_1c_2}{dd_1} B = \left(q_2N_1 + \frac{c_2r_1}{d_1} N\right) R_1 + r_2N_1R_2.$$

We may prove as before that the coefficient of R_1 is divisible by d_2, and denoting the quotient by N_2 we have

$$\frac{cc_1c_2}{dd_1d_2} B = N_2R_1 + \frac{r_2}{d_2} N_1R_2 \text{..................} (7).$$

The identities (6) and (7) shew that all the values of x and y which make $\frac{r_2}{d_2}$ and R_1 vanish, make the first members of these identities vanish; but $\frac{cc_1c_2}{dd_1d_2}$ and $\frac{r_2}{d_2}$ have no common factor, and therefore all the solutions of the equations $\frac{r_2}{d_2} = 0$ and $R_1 = 0$ satisfy the equations $A = 0$ and $B = 0$.

In the same way as before if we multiply both members of the identities (6) and (7) by c_3, and substitute for c_3R_1 its equivalent from the fourth of the identities (1), we obtain

$$\frac{cc_1c_2c_3}{dd_1d_2d_3} A = M_3R_2 + \frac{r_3}{d_3} M_2 \text{.....................} (8),$$

$$\frac{cc_1c_2c_3}{dd_1d_2d_3} B = N_3R_2 + \frac{r_3}{d_3} N_2 \text{.....................} (9),$$

where M_3 and N_3 are integral functions of x and y. The identities (8) and (9) shew that all the solutions of the equations $\frac{r_3}{d_3} = 0$ and $R_2 = 0$ satisfy the equations $A = 0$ and $B = 0$.

We have thus proved the first part of the proposition, namely, that all the solutions obtained from the system of equations (2) do satisfy the equations $A = 0$ and $B = 0$; we have now to shew that all the values of x and y which satisfy the equations $A = 0$ and $B = 0$ are included among the solutions obtained from the system (2).

The identity (3) may be written

$$NA - MB = \frac{r}{d} R \text{ ....(10).}$$

Multiply (4) by B and (5) by A and subtract; thus

$$(M_1B - N_1A) R + (MB - NA) \frac{r_1}{d_1} R_1 = 0,$$

and therefore by (10)

$$(M_1B - N_1A) R - \frac{rr_1}{dd_1} RR_1 = 0,$$

and therefore

$$M_1B - N_1A = \frac{rr_1}{dd_1} R_1 \text{ ....(11).}$$

Multiply (6) by B and (7) by A and subtract; thus

$$(M_2B - N_2A) R_1 + (M_1B - N_1A) \frac{r_2}{d_2} R_2 = 0,$$

and therefore by (11)

$$(M_2B - N_2A) R_1 + \frac{rr_1r_2}{dd_1d_2} R_1R_2 = 0,$$

and therefore

$$M_2B - N_2A = -\frac{rr_1r_2}{dd_1d_2} R_2 \text{........................(12).}$$

Similarly from (8) and (9) we deduce

$$M_3B - N_3A = \frac{rr_1r_2r_3}{dd_1d_2d_3} \text{ ....(13).}$$

The identity (13) shews that all the values of x and y which make A and B vanish make $\frac{r}{d}\frac{r_1}{d_1}\frac{r_2}{d_2}\frac{r_3}{d_3}$ vanish; so that one of the factors $\frac{r}{d}$, $\frac{r_1}{d_1}$, $\frac{r_2}{d_2}$, and $\frac{r_3}{d_3}$ must vanish. Hence the equations

$$\frac{r}{d} = 0, \quad \frac{r_1}{d_1} = 0, \quad \frac{r_2}{d_2} = 0, \text{ and } \frac{r_3}{d_3} = 0,$$

supply all the admissible values of y.

Suppose then that $x = \alpha$ and $y = \beta$ are values which satisfy the equations $A = 0$ and $B = 0$.

First suppose that β is a root of the equation $\frac{r}{d} = 0$; then it is manifest that the values $x = \alpha$ and $y = \beta$ satisfy the equations $\frac{r}{d} = 0$ and $B = 0$.

Next suppose that β is not a root of the equation $\frac{r}{d} = 0$, but is a root of the equation $\frac{r_1}{d_1} = 0$; since $\frac{r}{d}$ does not vanish when $y = \beta$, it follows from (10) that the values $x = \alpha$ and $y = \beta$ make R vanish, and so they satisfy the equations $\frac{r_1}{d_1} = 0$ and $R = 0$.

Next suppose that β is not a root of the equation $\frac{r}{d} = 0$, nor of the equation $\frac{r_1}{d_1} = 0$, but is a root of the equation $\frac{r_2}{d_2} = 0$; since $\frac{r}{d}\frac{r_1}{d_1}$ does not vanish when $y = \beta$, it follows from (11) that the values $x = \alpha$ and $y = \beta$ make R_1 vanish, and so they satisfy the equations $\frac{r_2}{d_2} = 0$ and $R_1 = 0$.

Next suppose that β is not a root of any of the equations $\frac{r}{d} = 0$, $\frac{r_1}{d_1} = 0$, $\frac{r_2}{d_2} = 0$, but is a root of the equation $\frac{r_3}{d_3} = 0$; since $\frac{r}{d}\frac{r_1}{d_1}\frac{r_2}{d_2}$ does not vanish when $y = \beta$, it follows from (12) that the values $x = \alpha$ and $y = \beta$ make R_2 vanish, and so they satisfy the equations $\frac{r_3}{d_3} = 0$ and $R_2 = 0$.

This proves the second part of the proposition.

The equation $\frac{r}{d}\frac{r_1}{d_1}\frac{r_2}{d_2}\frac{r_3}{d_3} = 0$ which gives all the admissible values of y may be called the *final equation* in y.

277. Examples.

(1) $$x^3 + 3yx^2 + (3y^2 - y + 1)x + y^3 - y^2 + 2y = 0,$$
$$x^2 + 2yx + y^2 - y = 0.$$

Here we have $x + 2y$ for the first remainder, so that $r = 1$, and $y^2 - y$ for the second remainder, which is independent of x. The only solutions are those furnished by $\frac{r_1}{d_1} = 0$ and $R = 0$, that is, by $y^2 - y = 0$ and $x + 2y = 0$.

(2) $$x^3 + 2yx^2 + 2y(y-2)x + y^2 - 4 = 0,$$
$$x^2 + 2yx + 2y^2 - 5y + 2 = 0.$$

The first remainder here is $(y-2)(x+y+2)$; so that $r = y - 2$ and $R = x + y + 2$; the second remainder is $y^2 - 5y + 6$, which is independent of x. The solutions are those furnished by $\frac{r}{d} = 0$ and $B = 0$, that is, by $y - 2 = 0$ and $x^2 + 2yx + 2y^2 - 5y + 2 = 0$; and those furnished by $\frac{r_1}{d_1} = 0$ and $R = 0$, that is, by $y^2 - 5y + 6 = 0$ and $x + y + 2 = 0$.

The *final equation* in y is $(y-2)(y^2 - 5y + 6) = 0$.

(3) $$x^3 + 3yx^2 - 3x^2 + 3y^2x - 6yx - x + y^3 - 3y^2 - y + 3 = 0,$$
$$x^3 - 3yx^2 + 3x^2 + 3y^2x - 6yx - x - y^3 + 3y^2 + y - 3 = 0.$$

The first remainder is $2(y-1)(3x^2 + y^2 - 2y - 3)$; the second remainder is $8(y^2 - 2y)x$; the third remainder is $y^2 - 2y - 3$. The solutions are those furnished by

$y - 1 = 0$, and $x^3 - 3yx^2 + 3x^2 + 3y^2x - 6yx - x - y^3 + 3y^2 + y - 3 = 0$,

by $\qquad y^2 - 2y = 0$, and $3x^2 + y^2 - 2y - 3 = 0$,

and by $\qquad y^2 - 2y - 3 = 0$, and $x = 0$.

The *final equation* in y is $(y-1)(y^2 - 2y)(y^2 - 2y - 3) = 0$.

(4) $$(y-2)x^2-2x+5y-2=0,$$
$$yx^2-5x+4y=0.$$

Here we multiply the left-hand member of the first expression by y to render the division possible without introducing fractional coefficients. Thus $c=y$. The first remainder is $(3y-10)x+y^2+6y$. In order to carry on the division we now multiply $yx^2-5x+4y$ by $3y-10$, and perform the following operation:

$$\begin{array}{l} (3y-10)\,x+y^2+6y\left.\vphantom{x}\right\}\;(3y-10)\,yx^2-(3y-10)\,5x+(3y-10)\,4y\left\{\vphantom{x}\right.yx \\ \qquad\qquad\qquad\qquad\;\; \underline{(3y-10)yx^2+(y^2+6y)yx\qquad\qquad\qquad\qquad} \\ \qquad\qquad\qquad\qquad\qquad -(y^3+6y^2+15y-50)\,x+12y^2-40y \end{array}$$

We may either regard the terms in the last line as forming the second remainder, or we may continue the operation of division as the remainder is not of a lower degree in x than the divisor; if we adopt the latter plan we must again multiply by $3y-10$, which will give rise to the same remainder as if we had originally multiplied by $(3y-10)^2$. Thus we continue the operation as follows:

$$\begin{array}{l} -(y^3+6y^2+15y-50)(3y-10)x+(12y^2-40y)(3y-10)\left\{\vphantom{x}\right.-(y^3+6y^2+15y-50) \\ \underline{-(y^3+6y^2+15y-50)(3y-10)x-(y^3+6y^2+15y-50)(y^2+6y)} \\ \qquad\qquad\qquad\qquad y^5+12y^4+87y^3-200y^2+100y \end{array}$$

We have here a remainder independent of x, which is the value of r_1; and d_1 here $=y$; so that the solutions are those furnished by

$$y^4+12y^3+87y^2-200y+100=0, \text{ and } (3y-10)x+y^2+6y=0.$$

278. The following remarks may be made on the process of Art. 276.

I. We may always take c such that c and r have no common factor. For if d be the greatest common measure of c and r the division of $\frac{c}{d}A$ by B can be effected without introducing fractional coefficients, as appears from the identity (3); thus c is not the most simple factor which can be used as a multiplier of A before dividing by B. Hence by choosing the most simple factor we can make $d = 1$.

Similarly we may take c_1, c_2,..., such that c_1 and r_1 shall have no common factor, and that c_2 and r_2 shall have no common factor, and so on.

Hence on the whole we may take c, c_1, c_2, c_3,... so that $d = 1$, that d_1 is the greatest common measure of c and r_1, that d_2 is the greatest common measure of $\frac{cc_1}{d_1}$ and r_2, that d_3 is the greatest common measure of $\frac{cc_1c_2}{d_1d_2}$ and r_3, and so on.

II. Suppose that the remainder independent of x which has been denoted by r_3 is zero; then R_2 is a common measure of A and B. Hence the solutions of the equations $A = 0$ and $B = 0$ consist, (1) of an infinite number of values of x and y which may be deduced from the single equation $R_2 = 0$, (2) of the finite number of values of x and y which may be obtained by solving the equations $\frac{A}{R_2} = 0$ and $\frac{B}{R_2} = 0$. But since $r_3 = 0$ it follows from the identities (1) of Art. 276 that R_2 divides R and R_1. Divide the identities (3), (4), (5), (6), (7), (10), (11), (12) of Art. 276 by R_2; we thus obtain new identities in which A, B, R, R_1 and R_2 are replaced by $\frac{A}{R_2}$, $\frac{B}{R_2}$, $\frac{R}{R_2}$, $\frac{R_1}{R_2}$ and $\frac{R_2}{R_2}$. By means of these identities we can prove, as in Art. 276, that all the solutions of the equations $\frac{A}{R_2} = 0$ and $\frac{B}{R_2} = 0$ will be obtained by solving the following systems:

$$\frac{r}{d} = 0 \text{ and } \frac{B}{R_2} = 0,$$

$$\frac{r_1}{d_1} = 0 \text{ and } \frac{R}{R_2} = 0,$$

$$\frac{r_2}{d_2} = 0 \text{ and } \frac{R_1}{R_2} = 0.$$

For example, suppose

$$x^3 + yx^2 - (y^2+1)x + y - y^3 = 0,$$

and
$$x^3 - yx^2 - (y^2 + 6y + 9)x + y^3 + 6y^2 + 9y = 0.$$

Here the first division gives $2\left\{yx^2 + (3y+4)x - (y^3 + 3y^2 + 4y)\right\}$ for the remainder, so that we may take

$$R = yx^2 + (3y + 4)x - (y^3 + 3y^2 + 4y).$$

To perform the second division multiply the dividend by y, and after one step in the division multiply again by y in order to continue the division. We then obtain $8(y^2 + 3y + 2)(x - y)$ for the remainder r_1R_1. Divide R by $x - y$ and the quotient is $yx + y^2 + 3y + 4$, and there is no remainder.

Thus the solutions of the proposed equations consist, (1) of an infinite number of values of x and y which may be deduced from the single equation $x - y = 0$, (2) of the finite number of values of x and y which may be obtained by solving the equations

$$y^2 + 3y + 2 = 0 \text{ and } yx + y^2 + 3y + 4 = 0.$$

III. The demonstration in Art. 276 implicitly supposes that the values of x and y are *finite;* it is however possible to have infinite solutions of an equation. Suppose for example that $(y-1)x^2 - 2x + y^2 = 0$; then so long as y is not equal to unity the two values of x furnished by this quadratic equation are finite. If y approaches indefinitely near to unity one value of x increases indefinitely; see *Algebra*, Chapter XXII. Thus when $y = 1$ we may say that x has an infinite value.

We have not included such infinite values of x and y in our investigations in Art. 276; these can be easily discovered independently. If, for example, we wish to ascertain whether an infinite value of x is admissible, we may put $\frac{1}{x'}$ for x, then clear of fractions, and suppose $x' = 0$; we have now *two* equations in y, and if they have a common root or roots, such root or roots combined with an infinite value of x may be said to satisfy the proposed equations.

XXIII. EXPANSION OF A FUNCTION IN SERIES.

279. Suppose we have an equation connecting two unknown quantities x and y. If we could solve the equation so as to obtain the values of y in terms of x, we might expand each value of y in a series proceeding according to powers of x. We are now about to explain a method for effecting these expansions of the values of y in series, without having previously obtained the values of y in finite terms.

The method in its complete form is due to Lagrange; it was suggested by a process given by Newton which is called *Newton's Parallelogram*. For the history of the method, and for full information respecting it, the student may refer to Memoirs by Professor De Morgan in the first volume of the *Quarterly Journal of Mathematics* and in the ninth volume of the *Cambridge Philosophical Transactions;* from these memoirs the brief account of the method which we shall give has been derived. An account of *Newton's Parallelogram* will also be found in the translation of Newton's work on *Lines of the Third Order* by C. R. M. Talbot, published in 1861.

280. Let the equation be denoted by

$$Ay^{\alpha} + By^{\beta} \ldots + Ky^{\kappa} + \ldots + Sy^{\sigma} = 0,$$

where $A, B, \ldots K, \ldots S$, are all functions of x. We suppose $\alpha, \beta, \ldots \kappa, \ldots \sigma$ to be arranged in descending order of algebraical magnitude; and throughout the investigation such words as *greater* and *less*, *greatest* and *least*, are to have their algebraical meaning.

Let A be of the degree a, that is, suppose x^a the greatest power of x which occurs in A; let B be of the degree b, ..., K of the degree k, ..., S of the degree s. Our object now requires the solution of the problem given in the next Article.

281. It is required to determine all the ways in which t can be taken so that two or more out of the following series of terms may be equal and greater than any of the rest:

$$a + \alpha t, \;\; b + \beta t, \;\ldots\ldots k + \kappa t, \;\; \ldots\ldots s + \sigma t.$$

Begin by supposing that t is $+\infty$; the first term is then greater than any of the others. As t diminishes each term diminishes, but each term diminishes more slowly than any of the terms which precede it. Let t have that value for which $a + \alpha t$ *first* becomes equal to one or more of the subsequent terms. This is found by taking the greatest value of t which can be obtained from the equations

$$a + \alpha t = b + \beta t,\ \ a + \alpha t = c + \gamma t,\ \ \ldots a + \alpha t = k + \kappa t,\ \ \ldots a + \alpha t = s + \sigma t,$$

that is, the greatest value of t must be found from the set

$$\frac{b-a}{\alpha-\beta},\ \frac{c-a}{\alpha-\gamma},\ \ldots\ldots \frac{k-a}{\alpha-\kappa},\ \ldots\ldots \frac{s-a}{\alpha-\sigma}.$$

Let $\dfrac{k-a}{\alpha-\kappa}$ be the greatest of these values, if one is greater than any of the others; or if several are equal and greater than any of the rest, let $\dfrac{k-a}{\alpha-\kappa}$ be the last of them; denote $\dfrac{k-a}{\alpha-\kappa}$ by τ.

Let t continue to diminish from the value τ until $k + \kappa t$ *first* becomes equal to one or more of the similar subsequent terms. This value of t is found, as before, by taking the greatest value of t which can be obtained from the equations

$$k + \kappa t = l + \lambda t,\ \ k + \kappa t = m + \mu t,\ \ \ldots\ldots k + \kappa t = s + \sigma t,$$

that is, the greatest value must be taken from the set

$$\frac{l-k}{\kappa-\lambda},\ \frac{m-k}{\kappa-\mu},\ \ldots\ldots \frac{s-k}{\kappa-\sigma}.$$

Let the greatest of these be selected, if one is greater than any of the others; or if several are equal and greater than any of the rest let the last of them be selected; let τ' denote the value of the selected term, which we will suppose to be $\dfrac{n-k}{\kappa-\nu}$.

Let t continue to diminish from the value τ'; and proceed as before to find another value τ'' from the equations

$$n + \nu t = p + \varpi t, \ldots\ldots n + \nu t = s + \sigma t.$$

This process must be continued until the term $s+\sigma t$ is used in obtaining a value of t.

Thus we see how all the suitable values of t may be found.

282. Suppose now that $A=x^a(a_1+A_1)$, where a_1 is independent of x, and A_1 vanishes when x is infinite; similarly let $B=x^b(b_1+B_1)$; and so on. Assume $y=x^t(u+U)$, where u is independent of x, and U vanishes when x is infinite. Substitute these values in the proposed equation involving x and y; thus

$$x^{a+\alpha t}(a_1+A_1)(u+U)^\alpha+x^{b+\beta t}(b_1+B_1)(u+U)^\beta+\ldots$$
$$\ldots+x^{k+\kappa t}(k_1+K_1)(u+U)^\kappa+\ldots+x^{s+\sigma t}(s_1+S_1)(u+U)^\sigma=0.$$

Since this is to hold for all values of x it must hold when x is infinite; and this will not be the case if the highest power of x occurs in only one term. In other words, the sum of the coefficients of the highest power of x must vanish. At this point the investigation of the preceding Article finds its application.

By supposition τ is the greatest admissible value of t, and we obtain for the part of the expression on the left-hand side of the above equation involving the highest power of x,

$$x^{a+\alpha\tau}\left\{(a_1+A_1)(u+U)^\alpha+\ldots+(k_1+K_1)(u+U)^\kappa\right\}.$$

When x is infinite the coefficient of $x^{a+\alpha\tau}$ must vanish; this gives the following equation for finding u,

$$a_1u^\alpha+\ldots\ldots+k_1u^\kappa=0.$$

From this equation values of u must be obtained, and to each value of u corresponds a value of y in which the term involving the highest power of x is ux^τ.

In a similar way by considering the value τ' we arrive at the following equation for determining u,

$$k_1u^\kappa+\ldots\ldots+n_1u^\nu=0.$$

From this equation values of u must be found, and to each value of u corresponds a value of y in which the term involving the highest power of x is $ux^{\tau'}$.

By proceeding in this way, we shall obtain the highest power of x in each value of y.

Next use one of the pairs of corresponding values of t and u which have been determined; put $y = x^t(u + U)$, and substitute this value of y in the original equation involving x and y. We thus obtain an equation connecting x and U and known quantities. We then apply the method to determine the highest power of x in the values of U, and thus we obtain the *second* terms in the expansions of the several values of y in series proceeding according to descending powers of x. And this process may be continued to any extent we please.

283. There is nothing in the preceding method which requires the given exponents $\alpha, \beta, \ldots \sigma, a, b, \ldots s$, to be *integers;* they will however be such when we apply the method to determine the *first* terms in the case of equations of the kind considered in the present Treatise.

We will now apply the method to an example.

Suppose we have the equation

$$y^4(x^2 - 3x) + y^2(x^3 + 2x^2) - y(4x^5 + 3) + 3x^6 = 0.$$

The set of terms $\frac{b-a}{\alpha-\beta}, \frac{c-a}{\alpha-\gamma}, \ldots$ is, in the present case, $\frac{3-2}{4-2}, \frac{5-2}{4-1}, \frac{6-2}{4-0}$. The second and third of these are equal to 1, which is greater than $\frac{1}{2}$, which is the value of the first term. Thus $\tau = 1$. Hence we put $y = x(u + U)$, and substitute in the proposed equation. The highest power of x is then x^6, and the term involving it is

$$x^6\left\{(u+U)^4 - 4(u+U) + 3\right\}.$$

The coefficient must vanish when x is infinite; this gives

$$u^4 - 4u + 3 = 0.$$

It is obvious that $u = 1$ is a solution, and as the derived function $4u^3 - 4$ also vanishes when $u = 1$, the root 1 is repeated.

Divide u^4-4u+3 by $(u-1)^2$; the quotient is u^2+2u+3. Thus the other values of u are furnished by the equation $u^2+2u+3=0$, and they are $-1\pm\sqrt{-2}$. We infer then that the proposed equation will only furnish two real values of y in terms of x, and that x is the first term in each of these values when they are expanded in series according to descending powers of x.

We may now put $x(1+U)$ for y in the proposed equation, and proceed to find the values of U; we will resume this example presently.

284. The following inferences may be drawn from Arts. 281 and 282.

(1) If $a+\alpha$, $b+\beta$, ..., $k+\kappa$, ..., $s+\sigma$ are all equal, the quantities $\tau, \tau', \tau'', \dots$ are all equal to unity.

(2) If of the quantities $a+\alpha$, $b+\beta$, ..., $k+\kappa$, ..., $s+\sigma$, two or more are equal and greater than all the rest, then unity occurs among the set $\tau, \tau', \tau'', \dots$ For it is obvious that $t=1$ is a suitable value in the investigation of Art. 281, since this value makes two or more of the terms there given equal, and greater than all the rest.

These two inferences involve the theory of the rectilinear asymptotes of algebraical curves.

In the remainder of this Article we suppose that $\alpha, \beta, \gamma, \dots$ are all integers, and that σ is zero.

(3) The first equation for u in Art. 282 will have $\alpha-\kappa$ roots, the second will have $\kappa-\nu$ roots, and so on; thus on the whole we get α values for the first term of y, as should be the case, since the proposed equation is of the degree α in y.

(4) Suppose that the degrees of all the functions of x from K to N inclusive are equal and higher than any of the others. Then out of the values of y there will be $\alpha-\kappa$ which begin with a positive power of x, and $\kappa-\nu$ which begin with the zero power of x, and ν which begin with a negative power of x. For the $\kappa-\nu$ values of y which begin with the zero power of x arise

from the fact that by hypothesis the value $t=0$ makes all the following terms equal and greater than any which follow them, $k+\kappa t$, $l+\lambda t$, ... $n+\nu t$. The $\alpha-\kappa$ values of y which begin with a positive power of x arise from positive values of t, and the corresponding values of u obtained relative to the exponents $\alpha, \beta, \ldots \kappa$. The ν values of y which begin with a negative power of x arise from negative values of t, and the corresponding values of u obtained relative to the exponents $\nu, \ldots \sigma$, where $\sigma=0$.

(5) If $A, B, \ldots S$, are all of the same degree except M, and M is of a higher degree than the rest, there are $\alpha-\mu$ values of y in which the highest power of x has the positive index $\dfrac{m-a}{\alpha-\mu}$, and μ values of y in which the highest power of x has the negative index $-\dfrac{m-a}{\mu}$.

285. A remark should be made respecting the equation in U which is obtained when the second terms in the values of y are required; see Art. 282. Suppose we assume $y=x^t(u+U)$, where u and t are known, and substitute this value of y in the proposed equation. We thus obtain an equation in U of the same degree as the original equation in y. However in general only some of the values of U will be admissible. For, by supposition, U vanishes when x is infinite, and so we must reject any value of U which commences with a positive power of x or with the zero power of x. These rejected values of U must belong to the other values of y with which we are not at the moment concerned, since by supposition we are seeking only that particular value of y which commences with ux^t, or those particular values which so commence if there are more than one, where u and t have known values.

286. Let us now resume the example in Art. 283. We have to substitute $x(u+U)$ for y, and make $u=1$. We shall thus obtain the following result after dividing by x,

$$U^4(x^5\ldots)+U^3(4x^5\ldots)+U^2(6x^5\ldots)-U(10x^4\ldots)-2x^4\ldots=0.$$

Here in the coefficients of the powers of U we have only expressed the highest powers of x. Form the fractions according to Art. 282; thus we obtain

$$\frac{5-5}{4-3}, \frac{5-5}{4-2}, \frac{4-5}{4-1}, \frac{4-5}{4-0}.$$

Here the first two terms are zero, and are algebraically greater than the others; but a zero value is to be rejected as explained in the preceding Article. We therefore proceed in the manner of Art. 281, supposing that $\tau=0$, and that we have to find τ'. Thus we form the fractions

$$\frac{4-5}{2-1}, \frac{4-5}{2-0}.$$

Of these the second, which is $-\frac{1}{2}$, is algebraically the greater. Accordingly we put $U=ux^{-\frac{1}{2}}$, and to find u we obtain the equation $6u^2-2=0$, so that $u=\frac{1}{\pm\sqrt{3}}$. Thus the first term of U is $\frac{1}{\sqrt{3x}}$ or $-\frac{1}{\sqrt{3x}}$. Therefore, as far as we have gone, we have

$$y=x\left(1+\frac{1}{\sqrt{3x}}+\dots\right) \text{ or } y=x\left(1-\frac{1}{\sqrt{3x}}+\dots\right).$$

287. The nature of the values of U may be seen by examining the formation of the general equation in U. Let us first put $x^t u$ for y and then change u into $u+U$. When we put $x^t u$ for y the left-hand member of the proposed equation will take the form

$$\chi_1(u)x^{n_1}+\chi_2(u)x^{n_2}+\chi_3(u)x^{n_3}+\dots$$

where $n_1, n_2, n_3, \dots$ are supposed in descending order of magnitude. Denote this expression by $\phi(u)$; then the equation in U will be $\phi(u+U)=0$. We will suppose the exponents of y in the proposed equation positive integers. The equation in U may be written

$$\phi_\alpha U^\alpha+\phi_{\alpha-1}U^{\alpha-1}+\phi_{\alpha-2}U^{\alpha-2}+\dots+\phi_1 U+\phi=0,$$

where ϕ_α stands for $\frac{1}{\lfloor\alpha}\phi^\alpha(u)$, and similar meanings belong to

$\phi_{\alpha-1}$, $\phi_{\alpha-2}$, ... Now if no special value were assigned to u, the coefficients of the several powers of U in the above equation would be functions of x, all of the same degree, namely n_1. Thus by Art. 284 the values of U would all commence with the zero power of x. But if u be such that $\chi_1(u)=0$, the function ϕ is of a lower degree in x than the function ϕ_1; hence one of the values of U begins with a negative power of x, namely, with $x^{-(n_1-n_2)}$. And this is the value of U which we are seeking, because $\chi_1(u)=0$ is the equation from which u is to be found according to our process. If however the equation $\chi_1(u)=0$ has *equal* roots, we obtain more than one suitable value of U. Suppose, for example, that the particular root which we have selected occurs *four* times; then ϕ_4 will be of the degree n_1 in x, while ϕ_3, ϕ_2, ϕ_1, ϕ, will only be of the degree n_2. Hence, by Art. 284, there will be four suitable values of U, each commencing with x raised to the negative power $-\frac{1}{4}(n_1-n_2)$.

We have here supposed that $\chi_2(u)$ and its derived functions do not vanish for the value of u which is considered.

288. In what we have hitherto given we have investigated values of y proceeding according to *descending* powers of x. Thus if we illustrate our results by geometry, and suppose curves traced corresponding to the values of y in terms of x, the first term of the series which we have found for a value of y will exhibit the nature of the curve at a great distance from the origin.

But the method may also be applied to find the values of y proceeding according to *ascending* powers of x, so that the first term in a value of y will exhibit the nature of the curve close to the origin, when the curve passes through the origin.

In order to apply the method to find the values of y proceeding according to *ascending* powers of x we need only make the following changes. We must suppose α, β, ... σ arranged in *ascending* order of algebraical magnitude; and A_1 must vanish when x vanishes and not when x is infinite, so that x^a must be the *lowest* power of x in A and not, as before, the *highest* power; a similar change of meaning must be made in B_1 and b, and in the remaining similar quantities.

Then when t is $+\infty$ the following quantities are in *ascending* order of magnitude, $a+\alpha t,\ b+\beta t,\ \dots\ k+\kappa t,\ \dots\ s+\sigma t.$

As before, the greatest value of t is to be found from the equations

$$a+\alpha t=b+\beta t,\ a+\alpha t=c+\gamma t,\ \dots\ a+\alpha t=k+\kappa t,\ \dots\ a+\alpha t=s+\sigma t.$$

XXIV. MISCELLANEOUS THEOREMS.

289. In the present Chapter we shall collect some miscellaneous theorems of interest and importance, which will exemplify many of the principles established in the preceding pages.

To prove that the following equation has no imaginary roots,

$$\frac{A^2}{x-a}+\frac{B^2}{x-b}+\frac{C^2}{x-c}+\dots+\frac{K^2}{x-k}-\lambda=0.$$

If possible suppose that $p+q\sqrt{-1}$ is a root; then $p-q\sqrt{-1}$ is also a root. Substitute successively these values for x and subtract one result from the other; thus

$$q\left\{\frac{A^2}{(p-a)^2+q^2}+\frac{B^2}{(p-b)^2+q^2}+\frac{C^2}{(p-c)^2+q^2}+\dots+\frac{K^2}{(p-k)^2+q^2}\right\}=0,$$

and this is impossible unless $q=0$.

Or we may prove the theorem thus. Denote the left-hand member of the proposed equation by $\phi(x)$, and suppose $a, b, c, \dots k$, in ascending order of algebraical magnitude. When x is a little greater than a the first term of $\phi(x)$ is very large and positive, and by taking x sufficiently near to a we may ensure a *positive* value for $\phi(x)$. When x is a little less than b the second term of $\phi(x)$ is very large and negative, and by taking x sufficiently near to b we may ensure a *negative* value for $\phi(x)$. Thus $\phi(x)$ changes sign for some value of x between a and b. Similarly, $\phi(x)$ changes sign for some value of x between b and c; and so on. In this way we may shew that the roots of the equation $\phi(x)=0$ are all real and unequal.

The form in which the equation $\phi(x) = 0$ is presented, enables us to recognise more easily the property we had to prove. But our result will not be affected if we clear the equation of fractions, so as to bring it to the standard form; that is, in fact, if instead of $\phi(x) = 0$ we consider the equation

$$\phi(x)(x-a)(x-b)(x-c)\ldots(x-k) = 0.$$

290. Required the values of the n quantities $x_1, x_2, x_3, \ldots x_n$ from the following n equations,

$$x_1 + x_2 + x_3 + \ldots + x_n = 0,$$

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = 0,$$

$$a_1^2x_1 + a_2^2x_2 + a_3^2x_3 + \ldots + a_n^2x_n = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_1^{n-2}x_1 + a_2^{n-2}x_2 + a_3^{n-2}x_3 + \ldots + a_n^{n-2}x_n = 0,$$

$$a_1^{n-1}x_1 + a_2^{n-1}x_2 + a_3^{n-1}x_3 + \ldots + a_n^{n-1}x_n = b.$$

Multiply these equations respectively by $c_{n-1}, c_{n-2}, \ldots c_2, c_1, 1$, where $c_{n-1}, c_{n-2}, \ldots c_2, c_1$, are at present undetermined, and add the results. Assume $c_{n-1}, c_{n-2}, \ldots c_2, c_1$, such that the coefficients of $x_2, x_3, \ldots x_n$, vanish; then

$$x_1(a_1^{n-1} + c_1a_1^{n-2} + c_2a_1^{n-3} + \ldots + c_{n-2}a_1 + c_{n-1}) = b.$$

From the assumption with respect to $c_{n-1}, c_{n-2}, \ldots c_2, c_1$, it follows that $a_2, a_3, \ldots a_n$ are the roots of the equation

$$z^{n-1} + c_1z^{n-2} + c_2z^{n-3} + \ldots + c_{n-2}z + c_{n-1} = 0.$$

Therefore the left-hand side of this equation is identically equal to

$$(z-a_2)(z-a_3)\ldots(z-a_n).$$

Hence substituting a_1 for z the equation which determines x may be put in the form

$$x_1(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n) = b.$$

Thus x_1 is known; and the values of $x_2, x_3, \ldots x_n$, can be deduced by symmetry.

291. Required the values of the n quantities $x, y, z, \ldots$ from the following n equations,

$$\frac{x}{k_1-a}+\frac{y}{k_1-b}+\frac{z}{k_1-c}+\ldots=1,$$

$$\frac{x}{k_2-a}+\frac{y}{k_2-b}+\frac{z}{k_2-c}+\ldots=1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{x}{k_n-a}+\frac{y}{k_n-b}+\frac{z}{k_n-c}+\ldots=1.$$

We may regard the n quantities $k_1, k_2, \ldots k_n$ as the roots of the single equation

$$\frac{x}{k-a}+\frac{y}{k-b}+\frac{z}{k-c}+\ldots=1,$$

which is of the n^{th} degree with respect to k. Assume $k=a-t$; it will follow that $a-k_1$, $a-k_2$, $a-k_3, \ldots$ are the values of the roots of the following equation in t,

$$1+\frac{x}{t}+\frac{y}{t+b-a}+\frac{z}{t+c-a}+\ldots=0.$$

Multiply by the product of the denominators so as to put this equation in the usual form; thus

$$t^n+A_1t^{n-1}+A_2t^{n-2}+\ldots+A_n=0,$$

where the term independent of t, that is A_n, is $x(b-a)(c-a)\ldots$

Therefore, by Art. 45,

$$(a-k_1)(a-k_2)(a-k_3)\ldots=(-1)^n x(b-a)(c-a)\ldots,$$

that is,
$$x=-\frac{(a-k_1)(a-k_2)(a-k_3)\ldots}{(a-b)(a-c)\ldots}$$

From this expression the values of $y, z, \ldots$ may be deduced by symmetrical changes in the letters $a, b, c\ldots$

Grunert's *Archiv der Mathematik und Physik*, Vol. XXIII. p. 235.

292. To prove that the sum of the products of the n quantities $c, c^2, c^3, \ldots c^n$, taken m at a time is

$$\frac{(c^n-1)(c^{n-1}-1)\ldots(c^{n-m+1}-1)}{(c-1)(c^2-1)\ldots(c^m-1)}c^{\frac{m(m+1)}{2}}.$$

Assume

$$(x+c)(x+c^2)\ldots(x+c^n)=x^n+p_1x^{n-1}+\ldots+p_{n-1}x+p_n\ldots\ldots(1).$$

Then by Art. 45 we have to find the value of p_m. In (1) change x into $\frac{x}{c}$ and multiply by c^n; thus

$$(x+c^2)(x+c^3)\ldots(x+c^{n+1})=x^n+p_1cx^{n-1}+\ldots+p_{n-1}c^{n-1}x+p_nc^n\ldots(2).$$

From (1) and (2) we obtain

$$(x+c^{n+1})(x^n+p_1x^{n-1}+\ldots+p_{n-1}x+p_n)$$
$$=(x+c)(x^n+p_1cx^{n-1}+\ldots+p_{n-1}c^{n-1}x+p_nc^n).$$

Equate the coefficients of x^{n-m+1} in the two members of this identity; thus

$$p_m+c^{n+1}p_{m-1}=p_mc^m+p_{m-1}c^m;$$

therefore
$$p_m=p_{m-1}\frac{c^m(c^{n-m+1}-1)}{c^m-1}\ldots\ldots\ldots(3).$$

And $p_1=c+c^2+\ldots+c^n=\dfrac{c(c^n-1)}{c-1}$; then by means of (3) we can determine successively $p_2, p_3, p_4, \ldots$; and thus we shall arrive at the required value for p_m.

293. Let there be n quantities $a, b, c, \ldots$; let s_n denote their sum, s_{n-1} the sum of any $n-1$ of them, and so on; and let S denote

$$(s_n)^r-\Sigma(s_{n-1})^r+\Sigma(s_{n-2})^r-\ldots+(-1)^{n-1}\Sigma(s_1)^r.$$

Here $\Sigma(s_m)^r$ denotes the sum of such terms as $(s_m)^r$ formed by taking all possible selections of m quantities out of the n quantities $a, b, c, \ldots$ Then we shall shew that $S=0$ if r is less than n, and that S is divisible by $abc\ldots$ if r is equal to n or greater than n; and in particular that

$$S = \lfloor n \, abc\ldots, \text{ when } r = n,$$

$$\text{and } S = \frac{\lfloor n+1}{2}(a+b+c+\ldots)\,abc\ldots, \text{ when } r = n+1.$$

We may separate S into two parts, one part in which a occurs in every term and another part in which a does not occur at all. We may write the former part thus,

$$(s_n)^r - \Sigma_1 (s_{n-1})^r + \Sigma_1 (s_{n-2})^r - \ldots + (-1)^{n-1} a^r,$$

and the latter part thus,

$$-\Sigma_2 (s_{n-1})^r + \Sigma_2 (s_{n-2})^r - \ldots + (-1)^{n-1} \Sigma_2 (s_1)^r,$$

where Σ_1 indicates certain of the terms formerly included under Σ, and Σ_2 indicates the remainder. Now suppose $a = 0$, then S vanishes; for we have in this case

$$(s_n)^r - \Sigma_2 (s_{n-1})^r = 0,$$

$$\Sigma_1 (s_{n-1})^r - \Sigma_2 (s_{n-2})^r = 0,$$

$$\Sigma_1 (s_{n-2})^r - \Sigma_2 (s_{n-3})^r = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Similarly, we may prove that S vanishes when $b = 0$, and when $c = 0$, and so on. Thus we conclude that S is in general divisible by each of the quantities $a, b, c, \ldots$ and therefore by their product. But the product will be of n dimensions, and therefore if S be of less than n dimensions it must be identically zero. And as S is of r dimensions it follows that S vanishes when r is less than n, and is divisible by $abc\ldots$ when r is not less than n.

When $r = n$ we have therefore $S = \lambda abc\ldots$, where λ is some *numerical* quantity which is to be determined. To determine λ suppose that $a, b, c, \ldots$ are all equal to unity; then S becomes

$$n^n - n(n-1)^n + \frac{n(n-1)}{1 \cdot 2}(n-2)^n - \ldots,$$

that is $\lfloor n$, by *Algebra*, Chapter XXXIX.

Next, suppose $r = n + 1$. Then S is divisible by $abc...$; and as S is of $n+1$ dimensions, it must have a factor which is of one dimension and symmetrical with respect to $a, b, c...$; this factor must therefore be $a + b + c + ...$

Hence $S = \mu abc...(a + b + c + ...)$, where μ is a numerical quantity which is to be determined. To determine μ suppose that $a, b, c, ...$ are all equal to unity; then S becomes

$$n^{n+1} - n(n-1)^{n+1} + \frac{n(n-1)}{1.2}(n-2)^{n+1} - ...,$$

and this must equal μn. Hence by *Algebra*, Chapter XXXIX. we have $\mu = \frac{\lfloor n+1}{2}$.

294. Let $[c]_r$ denote $c(c-1)(c-2)...(c-r+1)$, whatever c may be; then will

$$[a+b]_n = [a]_n + n[a]_{n-1}b + \frac{n(n-1)}{1.2}[a]_{n-2}[b]_2 + ... + [b]_n.$$

For suppose that a is a positive integer; then we know that this theorem is true for *any* positive integral value of b, for it follows by equating the coefficients of x^n in $(1+x)^{a+b}$ and in $(1+x)^a \times (1+x)^b$. Hence since this is true for more than n values of b it is identically true by Art. 39; that is, when a is a positive integer the theorem is true for *all* values of b. Then since it is true for *any* positive integral value of a, it is true for more than n values of a, and therefore by Art. 39 it is true for *all* values of a.

Thus we are able to prove the proposed theorem, by assuming the Binomial Theorem for a positive integral index and also the Theorem of Art. 39. The theorem is sometimes called by the name of Vandermonde. The theorem is required in Euler's proof of the Binomial Theorem for any index, and as is well known, is there established by an appeal to the principle of the *permanence of equivalent forms.*

295. Let $\phi(x)=0$ be an equation which has a root a, so that we may suppose $\phi(x)=(x-a)\psi(x)$; then

$$\frac{\phi(x)}{x}=\left(1-\frac{a}{x}\right)\psi(x),$$

$$\log\frac{\phi(x)}{x}=\log\left(1-\frac{a}{x}\right)+\log\psi(x)$$

$$=-\left(\frac{a}{x}+\frac{1}{2}\frac{a^2}{x^2}+\ldots\right)+\log\psi(x).$$

Suppose that $\log\dfrac{\phi(x)}{x}$ can be expanded in a series involving positive and negative powers of x, and that $\log\psi(x)$ can be expanded in a series involving only positive powers of x; then assuming the identity of the two members of the equation we obtain this result,

$$-a=\text{the coefficient of } \frac{1}{x} \text{ in the expansion of } \log\frac{\phi(x)}{x}.$$

296. The theorem of the preceding Article is given by Murphy in his *Theory of Equations* and illustrated by examples; see his pages 77...82. The demonstration of the theorem is imperfect, since the infinite series may be divergent; but the theorem is of some importance. It had been noticed before Murphy drew attention to it; see De Morgan's *Differential and Integral Calculus*, pages 328 and 644, and also the *Philosophical Magazine* for June 1848, page 421; according to the latter work the theorem was given by Lagrange in 1768.

It appears that the process furnishes the numerically *least* root of the equation to which it is applied; and some reason may be assigned for this, at least when all the roots are real.

For suppose that the roots of the equation $\phi(x)=0$ are $a, b, c,\ldots$ in ascending order of magnitude. Then

$$\phi(x)=A(x-a)(x-b)(x-c)\ldots\ldots,$$

where A is a constant.

Thus $$\frac{\phi(x)}{x}=B\left(1-\frac{a}{x}\right)\left(1-\frac{x}{b}\right)\left(1-\frac{x}{c}\right)\ldots\ldots,$$

where B is a constant.

Since a is the numerically least root of the equation $\phi(x)=0$, if x lies between a and b the expansions of

$$\log\left(1-\frac{a}{x}\right), \quad \log\left(1-\frac{x}{b}\right), \quad \log\left(1-\frac{x}{c}\right), \ldots\ldots$$

will all give *convergent* series; and hence we see that $\log\dfrac{\phi(x)}{x}$ can be developed in the required form in a manner which is arithmetically intelligible and true. Then as x can have *any* value between a and b we may, by a natural extension of the theory of indeterminate coefficients, equate the coefficient of $\dfrac{1}{x}$ in the expansion of $\log\dfrac{\phi(x)}{x}$ to $-a$.

In the same way as the coefficient of $\dfrac{1}{x}$ in the expansion of $\log\dfrac{\phi(x)}{x}$ is seen to be $-a$, we see that the coefficient of $\dfrac{1}{x^n}$ is $-\dfrac{a^n}{n}$; thus we can determine the value of any assigned integral power of the numerically least root of the equation $\phi(x)=0$.

297. For example, required a root of the equation

$$x^n+cx-b=0.$$

$$\text{Here } \frac{\phi(x)}{x}=c-\frac{b}{x}+x^{n-1},$$

$$\log\frac{\phi(x)}{x}=\log c+\log\left(1-\frac{b}{cx}+\frac{x^{n-1}}{c}\right)$$

$$=\log c-z-\frac{1}{2}z^2-\frac{1}{3}z^3-\ldots,$$

$$\text{where } z=\frac{b}{cx}-\frac{x^{n-1}}{c}=\frac{b}{cx}\left(1-\frac{x^n}{b}\right).$$

We have now to pick out the terms involving $\dfrac{1}{x}$; we shall obtain such a term from z, from z^{n+1}, from z^{2n+1}, and so on. Hence we shall find for the root the series

$$\frac{b}{c}-\frac{b^n}{c^{n+1}}+\frac{2n}{2}\frac{b^{2n-1}}{c^{2n+1}}-\frac{3n(3n-1)}{2\,.\,3}\frac{b^{3n-2}}{c^{3n+1}}+\ldots$$

298. Let $\phi(x)=0$ be an equation of which $a_1, a_2, \ldots a_m$, are roots, so that we may suppose

$$\phi(x)=(x-a_1)(x-a_2)\ldots(x-a_m)\psi(x);$$

$$\text{then } \frac{\phi(x)}{x^m}=\left(1-\frac{a_1}{x}\right)\left(1-\frac{a_2}{x}\right)\ldots\left(1-\frac{a_m}{x}\right)\psi(x).$$

Take the logarithms of both sides; then, as in Art. 295, we infer that $-(a_1+a_2+\ldots+a_m)$ is equal to the coefficient of $\frac{1}{x}$ in the expansion of $\log\frac{\phi(x)}{x^m}$. See Murphy's *Theory of Equations*, pages 82 and 83.

As in Art. 296 we may conclude that the process will give the sum of the numerically *least m* roots.

299. We shall now give some theorems relating to the decomposition of a rational fraction into other fractions, which relatively to the original fraction are called *partial fractions*.

Suppose that $\phi(x)$ is a function of x of the n^{th} degree; let the roots of the equation $\phi(x)=0$ be all unequal and let them be denoted by $a, b, c, \ldots k$. Let $\psi(x)$ be a function of x which is of the $(n-1)^{\text{th}}$ degree or of a lower degree. Then the following relation will be identically true,

$$\frac{\psi(x)}{\phi(x)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}+\ldots\ldots+\frac{K}{x-k},$$

provided proper constant values be assigned to $A, B, C, \ldots K$.

For in order that this relation may be identically true it is necessary and sufficient that the following should be identically true:

$$\psi(x)=A\frac{\phi(x)}{x-a}+B\frac{\phi(x)}{x-b}+C\frac{\phi(x)}{x-c}+\ldots\ldots+K\frac{\phi(x)}{x-k}.$$

The members of this equation are not of a higher degree than that expressed by $n-1$, hence the relation will be identically true if n values of x can be found for which it is true; see Art. 39. And by properly choosing $A, B, C, \ldots K$ the relation can be made true for the n values $a, b, c, \ldots k$, of x. For suppose $x=a$, then all the terms on the right-hand side vanish, except that which involves A; and we obtain

$$\psi(a) = A\left\{\frac{\phi(x)}{x-a}\right\}_{x=a},$$

that is, by Art. 74,

$$\psi(a) = A\phi'(a).$$

This determines A; and similar values will be found for $B, C, \ldots K$.

300. Next suppose that $\psi(x)$ is not of lower degree than $\phi(x)$. By common division we may obtain

$$\frac{\psi(x)}{\phi(x)} = F(x) + \frac{f(x)}{\phi(x)},$$

where $F(x)$ and $f(x)$ are integral functions of x, and $f(x)$ is of a lower degree than $\phi(x)$. We may then decompose $\frac{f(x)}{\phi(x)}$ into partial fractions in the manner shewn in the preceding Article.

Since we have

$$\psi(x) = \phi(x)\,F(x) + f(x);$$

it follows that $\psi(x)$ and $f(x)$ *have the same value* when $\phi(x)$ vanishes. Hence the *partial fractions* corresponding to $\frac{\psi(x)}{\phi(x)}$, when determined by the method of Art. 299, can be found without previously dividing $\psi(x)$ by $\phi(x)$; we must however not omit the part $F(x)$ if we wish to obtain the complete value of $\frac{\psi(x)}{\phi(x)}$.

301. Various Algebraical identities may be established by means of the principles of the preceding two Articles.

For example, if n be any positive integer

$$\frac{\lfloor n}{(x+1)(x+2)\ldots(x+n+1)} = \frac{1}{x+1} - \frac{n}{1}\frac{1}{x+2} + \frac{n(n-1)}{1.2}\frac{1}{x+3} - \ldots\ldots + \frac{(-1)^n}{x+n+1}.$$

For we may assume that the left-hand member can be put in the form

$$\frac{A_1}{x+1} + \frac{A_2}{x+2} + \frac{A_3}{x+3} + \ldots\ldots + \frac{A_{n+1}}{x+n+1};$$

and then we may determine $A_1, A_2, \ldots A_{n+1}$: this is effected by multiplying both sides by

$$(x+1)(x+2)\ldots(x+n+1),$$

and then substituting for x in succession the values $-1, -2, \ldots$

Again, if n be any positive integer

$$\frac{1}{x+1} - \frac{n}{(x+1)(x+2)} + \frac{n(n-1)}{(x+1)(x+2)(x+3)} - \ldots + \frac{(-1)^n \lfloor n}{(x+1)(x+2)\ldots(x+n+1)} = \frac{1}{x+n+1}.$$

For we may assume that the left-hand member can be put in the form

$$\frac{A_1}{x+1} + \frac{A_2}{x+2} + \frac{A_3}{x+3} + \ldots\ldots + \frac{A_{n+1}}{x+n+1};$$

multiply both sides by $(x+1)(x+2)\ldots(x+n+1)$ and then substitute for x in succession the values $-1, -2, \ldots$. Thus we shall obtain

$$A_1 = (1-1)^n = 0,$$
$$A_2 = n(1-1)^{n-1} = 0,$$
$$A_3 = \frac{n(n-1)}{1.2}(1-1)^{n-2} = 0,$$

and by proceeding thus we find that $A_1, A_2, \ldots A_n$ are all zero, and that $A_{n+1} = 1$.

Again, if m be any positive integer

$$(1-y)^m \left\{ \frac{1}{x+1} + \frac{m}{(x+1)(x+2)} \frac{y}{1-y} \right.$$
$$+ \frac{m(m-1)}{(x+1)(x+2)(x+3)} \left(\frac{y}{1-y}\right)^2 + \ldots$$
$$\left. + \frac{\lfloor m}{(x+1)(x+2)\ldots(x+m+1)} \left(\frac{y}{1-y}\right)^m \right\}$$
$$= \frac{1}{x+1} - \frac{my}{x+2} + \frac{m(m-1)}{1.2} \frac{y^2}{x+3} + \ldots\ldots + \frac{(-1)^m y^m}{x+m+1}.$$

This may be demonstrated in the way already exemplified by assuming that the left-hand member can be put in the form

$$\frac{A_1}{x+1} + \frac{A_2}{x+2} + \frac{A_3}{x+3} + \ldots\ldots + \frac{A_{m+1}}{x+m+1}:$$

then we deduce

$$A_1 = (1-y+y)^m = 1,$$
$$A_2 = -my(1-y+y)^{m-1} = -my,$$

and so on.

Or we may establish this result by the aid of the second example. For if we expand the left-hand member in powers of y, and compare the coefficients of y^r in the two sides, we find them equal by the second example.

302. We have in Articles 299 and 300 given separately the decomposition of a rational fraction when its denominator has *no repeated factors*, on account of the simplicity of the result; it

is however only a particular case of the general investigation to which we now proceed.

Suppose that $\phi(x)$ is a function of x which involves repeated factors; for example, let

$$\phi(x) = p_0(x-a)^r(x-b)^s(x-c)^t \ldots (x-k),$$

and let $\psi(x)$ be any other function of x. Then the expression $\dfrac{\psi(x)}{\phi(x)}$ may be resolved into the following parts.

(1) Any factor $x-k$ which is not repeated will give rise to a single term $\dfrac{K}{x-k}$.

(2) The factor $(x-a)^r$ will give rise to the series of terms

$$\frac{A}{(x-a)^r} + \frac{A_1}{(x-a)^{r-1}} + \frac{A_2}{(x-a)^{r-2}} + \ldots\ldots + \frac{A_{r-1}}{x-a}.$$

A similar series of terms will arise from each of the other repeated factors.

(3) There will also be an integral expression if $\psi(x)$ be not of a lower degree than $\phi(x)$.

For suppose $\phi(x) = (x-a)^r \chi(x)$; then we have identically, whatever A may be,

$$\frac{\psi(x)}{\phi(x)} = \frac{A}{(x-a)^r} + \frac{\psi(x) - A\chi(x)}{\phi(x)}.$$

Now let A be determined by the equation $\psi(a) - A\chi(a) = 0$; then $\psi(x) - A\chi(x)$ vanishes when $x=a$, and is therefore divisible by $x-a$. Therefore with this value of A we may put

$$\psi(x) - A\chi(x) = (x-a)\psi_1(x),$$

and therefore

$$\frac{\psi(x)}{\phi(x)} = \frac{A}{(x-a)^r} + \frac{\psi_1(x)}{(x-a)^{r-1}\chi(x)}.$$

In the same way we may decompose the last fraction and obtain

$$\frac{\psi_1(x)}{(x-a)^{r-1}\chi(x)}=\frac{A_1}{(x-a)^{r-1}}+\frac{\psi_2(x)}{(x-a)^{r-2}\chi(x)}.$$

By proceeding in this way the required result is established.

303. It is easy to shew after the manner of Art. 37 that there is only one mode of decomposing $\frac{\psi(x)}{\phi(x)}$ into an integral function, and a series of partial fractions each of which involves only one distinct factor in its denominator. Hence we infer that the *result* obtained must be the same in whatever order the operations are conducted, that is, whatever factor we first consider.

Practically the best way to determine the numerators of the partial fractions will often be the following. Put $x=a+h$; thus

$$\frac{\psi(x)}{\phi(x)}=\frac{\psi(x)}{(x-a)^r\chi(x)}=\frac{\psi(a+h)}{h^r\chi(a+h)};$$

now expand by some algebraical method $\frac{\psi(a+h)}{\chi(a+h)}$ in powers of h, and according to the notation already used the result must be

$$\frac{\psi(a+h)}{\chi(a+h)}=A+A_1h+A_2h^2+A_3h^3+\ldots\ldots$$

That is, A_m must be the coefficient of h^m in the expansion of $\frac{\psi(a+h)}{\chi(a+h)}$ according to ascending powers of h.

Similarly, the numerators of the other partial fractions may be determined.

304. In the next two Articles we shall give some theorems relative to limits of the roots of an equation; they were communicated to the writer by the late Professor de Morgan, in a letter dated Feb. 6, 1858.

305. The following theorem relative to limits of the roots of an equation will be found to include two of those which are given in Chapter VII., and to add something to them.

Let $f(x) = p_0x^n + p_1x^{n-1} + \ldots + p_{n-1}x + p_n$; then we proceed to investigate a superior limit to the positive roots of the equation $f(x) = 0$.

Let a be equal to the coefficient of the first term, or to anything less; let b be equal to the least of the positive coefficients which immediately follow, and precede any negative coefficient, or to anything less; let c be equal to the numerical value of the numerically greatest negative coefficient, or to anything greater. Suppose that x^{n-k-1} is the first term with a negative coefficient. Then $f(x)$ is certainly positive when the following expression is positive,

$$ax^n + b(x^{n-1} + \ldots + x^{n-k}) - c(x^{n-k-1} + \ldots + x + 1),$$

that is, when the following expression is positive,

$$ax^n + b\frac{x^n - x^{n-k}}{x-1} - c\frac{x^{n-k} - 1}{x-1};$$

that is, supposing x greater than unity, when

$$\left\{a(x-1) + b\right\}x^n - (b+c)x^{n-k} + c$$

is positive, that is, *a fortiori*, when

$$\left\{a(x-1) + b\right\}x^k - (b+c)$$

is zero or positive.

(1) Take $b = 0$, and let c be the numerically greatest negative coefficient; then $f(x)$ is positive if $a(x-1) - c$ is zero or positive, that is, if $x = 1 + \frac{c}{a}$ or anything greater. See Art. 87.

(2) Take $b=0$, and let c be the numerically greatest negative coefficient; then $f(x)$ is positive if $a(x-1)x^k - c$ is zero or positive, and therefore *a fortiori* if $a(x-1)^{k+1} - c$ is so; that is, if $x = 1 + \left(\frac{c}{a}\right)^{\frac{1}{k+1}}$ or anything greater. See Art. 89.

(3) Put zero for a; then $f(x)$ is positive if $bx^k - (b+c)$ is zero or positive, that is, if $x = \left(1 + \frac{c}{b}\right)^{\frac{1}{k}}$ or anything greater. This is a new limit, which may be less than that in (2) when b can be taken greater than p_0.

(4) If a is not greater than b we have $f(x)$ positive if

$$\left\{a(x-1) + a\right\} x^k - (a+c)$$

is zero or positive, that is, if $x = \left(1 + \frac{c}{a}\right)^{\frac{1}{k+1}}$ or anything greater. This furnishes a less limit than that in (3) whenever b cannot be taken so great as p_0.

(5) Suppose that a is not less than c; then from (2) we obtain $1 + 1^{\frac{1}{k+1}}$, that is 2, as a superior limit.

(6) Suppose that b is not less than c; then from (3) we obtain $2^{\frac{1}{k}}$ as a superior limit.

(7) Suppose that neither a nor b is less than c; then from (4) we obtain $2^{\frac{1}{k+1}}$ as a superior limit.

306. We shall now give another theorem on the limits of the roots of equations. It depends on the mode of calculating the value of an expression of the form $ax^n + bx^{n-1} + cx^{n-2} + \dots$ for an assigned value of x, which we have explained in Art. 5. If θ denote that assigned value the calculation determines successively

$$a\theta, \quad a\theta + b, \quad (a\theta + b)\theta, \quad (a\theta + b)\theta + c, \dots\dots$$

Let $f(x)=0$ be the equation. Arrange $f(x)$ in groups, each group consisting of all the positive terms which come together followed by all the negative terms which come together before the next positive term. Thus, writing only the signs, supposing we have the succession,

$$++---+-++----+--+,$$

then they will be arranged in groups thus,

$$(++---),\ (+-),\ (++----),\ (+--),\ +.$$

Let the first group involve the powers of x from x^n to x^{n-k} both inclusive. Suppose the factor x^{n-k} removed by division. Take θ on trial as a value of x, and calculate the value when $x=\theta$ of the quotient after division by x^{n-k}. If the result is positive denote it by A_1, and put A_1x^{n-k} at the head of the next group. Suppose this group to extend to the term involving x^{n-l}. After A_1x^{n-k} has been prefixed to the second group divide by x^{n-l}, and find the value of the quotient when $x=\theta$. If the result be positive denote it by A_2, and put A_2x^{n-l} at the head of the next group; and so on. If all the results be positive up to the last, θ is a superior limit of the positive roots. The number θ to be tried may be selected by one of the easier rules, remembering that it is not likely a number will be required much higher than the superior limit found from considering only the first group.

For example, take an equation of the 18th degree. We will write down coefficients only, in groups,

$$(7+4+3-80-100)+(20-100)+(3+2+1-40-1000-1000)$$
$$+(70-8000-2000)+(1000-400-4000).$$

Here from considering only the first group we see that 2 is too small; we will try 3. We proceed to calculate the value when $x=3$ of

$$7x^4+4x^3+3x^2-80x-100$$

7	4	3	-80	-100
7	25	78	154	362

Thus $A_1=362$.

We proceed to calculate the value when $x=3$ of

$$362x^2+20x-100$$

$$\begin{array}{ccc} 362 & 20 & -100 \\ 362 & 1106 & 3218 \end{array}$$

Thus $A_2=3218$.

We have next to calculate the value when $x=3$ of

$$3218x^6+3x^5+2x^4+x^3-40x^2-1000x-1000.$$

It is however sufficiently obvious now that we shall obtain positive results to be denoted by A_3, A_4, and A_5; so that 3 is a superior limit of the positive roots.

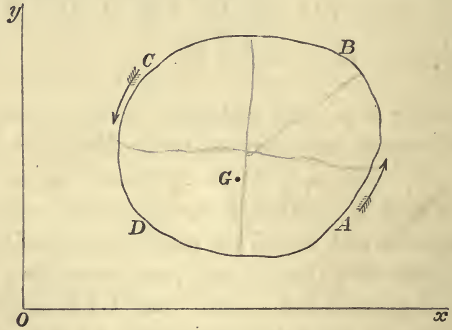
In this example the rule of Art. 90 would give $1+\dfrac{8000}{110}$, which is more than 70; and the rule of Art. 89 would give $1+\sqrt[3]{\dfrac{8000}{7}}$, which is more than 11.

The following is a brief statement of the theorem. Divide the whole expression into successive positive and integer lots, $A_p-B_q+C_r-D_s+\dots$; p, q, r, s,... representing the last exponent of x in each lot. Divide A_p-B_q by x^q, and ascertain a value of x, say λ, which makes the quotient positive; let l be this quotient. Divide $lx^q+C_r-D_s$ by x^s, and ascertain a value of x, say μ, which is perhaps not greater than λ but must not be less than λ, which makes the quotient positive; let m be this quotient. Continue the process with $mx^s+E_t-F_u$, and so on to the end. The last value of x used is greater than any root of the equation; and the first value of x, namely λ, is very often the last also.

XXV. CAUCHY'S THEOREM.

307. We shall devote the present Chapter to the demonstration of a remarkable theorem given by Cauchy, the object of which is to ascertain how many roots real or imaginary lie within assigned limits; in fact, the theorem proposes to effect with respect to the roots in general what Sturm's theorem effects with respect to the *real* roots.

308. Take any rectangular axes, and let x, y be the co-ordinates of any point. Let $\phi(z)$ be any rational function of z; then $\phi(x + y\sqrt{-1})$ can be expressed in the form $p + q\sqrt{-1}$. A point whose co-ordinates are such that p and q simultaneously vanish, will be called a *radical point*. Describe any contour $ABCD$; then the number of radical points which lie within this contour will be given by the following rule. Let a point move round this contour in the positive direction, and note how often $\frac{p}{q}$ passes through the value 0 and changes its sign; suppose it to change k times from + to −, and l times from − to +; then the number of radical points which lie within the contour is $\frac{1}{2}(k - l)$.



It is to be observed that the contour is supposed to be so taken that no radical point lies *on* it; also if any imaginary root of the equation $\phi(z) = 0$ is repeated two, or three, or more times, we consider that we have two, or three, or more radical points, although these points coincide. By movement in the *positive* direction we imply that a radius vector drawn from a fixed point within the contour to the moving point passes over a positive angle equal to four right angles, while the moving point passes round the contour.

The theorem is proved by first considering the case of an infinitesimal contour, and then the case of a finite contour.

309. Take any point G, which is not a radical point, within the contour, and describe an infinitesimal contour including G. Suppose that the moving point passes in the positive direction round this infinitesimal contour; we have then four cases to consider.

(1) Suppose that neither p nor q vanishes within or on the contour. Here $\frac{p}{q}$ does not change sign at all during the circuit; so that the rule asserts that there is no radical point within the contour, and this is true because p and q do not vanish.

(2) Suppose that q does not vanish within or on the contour, but that p does. In this case $\frac{p}{q}$ may change sign as the moving point passes through a position for which p vanishes. But at the end of the circuit p has resumed its original sign, and thus there must have been the same number of changes from + to − as from − to +. Hence k and l are equal, and the rule asserts that there is no radical point within the contour, and this is true because q does not vanish.

(3) Suppose that p does not vanish within or on the contour, but that q does. In this case $\frac{p}{q}$ never vanishes, so that the rule asserts that there is no radical point within the contour, and this is true because p does not vanish.

(4) Suppose that both p and q vanish within or on the contour. If they do not vanish simultaneously we may divide the space bounded by the contour into other spaces, for some of which p alone vanishes, and for others q alone vanishes; thus we obtain two or more contours instead of one, and these fall under the cases (2) and (3). We have then only to consider the case in which p and q vanish simultaneously, so that there is a radical point within or on the contour. And we may suppose the contour so small that there is only one distinct radical point *within* it, and none *on* it.

Let a, b be the co-ordinates of this radical point; and put $x=a+r\cos\theta$, and $y=b+r\sin\theta$; thus

$$x+y\sqrt{-1}=a+b\sqrt{-1}+r(\cos\theta+\sqrt{-1}\sin\theta),$$
$$=a+b\sqrt{-1}+v, \text{ say.}$$

Suppose now that the equation $\phi(z)=0$ has the root $a+b\sqrt{-1}$ repeated m times; then $\phi(a+b\sqrt{-1}+v)$ takes the form $cv^m+c_1v^{m+1}+c_2v^{m+2}+\ldots$, where c, c_1, c_2,... are certain imaginary expressions of the standard form; so that we may suppose

$$c=h(\cos\alpha+\sqrt{-1}\sin\alpha),\quad c_1=h_1(\cos\alpha_1+\sqrt{-1}\sin\alpha_1),\ldots$$

Hence, by De Moivre's theorem we shall obtain

$$\frac{p}{q}=\frac{h\cos(m\theta+\alpha)+h_1r\cos\{(m+1)\theta+\alpha_1\}+h_2r^2\cos\{(m+2)\theta+\alpha_2\}+\ldots}{h\sin(m\theta+\alpha)+h_1r\sin\{(m+1)\theta+\alpha_1\}+h_2r^2\sin\{(m+2)\theta+\alpha_2\}+\ldots}$$

We may suppose r so small that the number of changes of sign shall be unaffected by r; that is, we may proceed as if $\frac{p}{q}=\cot(m\theta+\alpha)$. And as $m\theta$ *increases* from one multiple of π to the next multiple of π, there is always one passage through zero accompanied by a change of sign from + to −. Thus we have $k=2m$, and $l=0$; so that $\frac{1}{2}(k-l)=m$, according to the rule.

310. The theorem is thus proved for an infinitesimal contour; and we shall now consider the finite contour $ABCD$. Let the contour be divided into an indefinitely large number of infinitesimal contours, these contours being so taken that no radical point falls *on* any of them. Then the number of radical points within $ABCD$ can be found by making a point describe all these infinitesimal contours, and adding together the numbers furnished by the rule, which we may denote by $\frac{1}{2}\Sigma(k-l)$. But the same result will be obtained if we omit all the interior lines of division, and retain only the boundary $ABCD$. For each point on any interior line of division belongs to *two* contours, and is therefore

traversed by the describing point twice and in *contrary* directions; so that, if in one case there is a change in $\frac{p}{q}$ from + to −, there is a change in the other case from − to +, and on the whole the number $\frac{1}{2}\Sigma(k-l)$ is unaffected. Hence the interior lines of division may be omitted, and the moving point constrained to describe the contour $ABCD$ alone.

Thus the theorem is proved.

311. We can now immediately deduce the theorem that an equation of the n^{th} degree must have n roots. Suppose the contour $ABCD$ to be a circle with the origin as centre and an indefinitely *large* radius. The value of $\frac{p}{q}$ will now depend only on the term involving the highest power of z in $\phi(z)$; and if we suppose that term to be $h(\cos\alpha + \sqrt{-1}\sin\alpha)z^n$, we shall have $\frac{p}{q} = \cot(n\theta + \alpha)$. Thus we shall obtain $k = 2n$, and $l = 0$; so that $\frac{1}{2}(k-l) = n$.

312. We have drawn the figure in Art. 308 so that if from any point within the contour a radius vector is drawn in one direction it meets the contour in only one point. The figure however need not be so restricted; it may be such that a radius vector drawn in one direction may meet the contour any *odd* number of times. Hence as a point moves round the contour the radius vector drawn to the moving point from any fixed origin within the contour will not always revolve in the same direction. By the positive direction of movement of the describing point we must understand that for which, although the vectorial angle may not be *always* increasing, yet on the whole the *positive* angle 2π is gained in the circuit.

The demonstration will not be affected by the admission of the kind of figure here contemplated; for the *infinitesimal* contours may still be supposed, if we please, ovals which have only one radius vector drawn in any definite direction from a fixed origin. Or if we do not adopt this restriction we must observe that at the end of Art. 309, as θ now does not always increase, there may be more values of θ for which $\frac{p}{q}$ vanishes, than we contemplated; but if so, there will be exactly as many more changes from + to − as from − to +.

313. We have supposed throughout that there is no radical point *on* a contour considered. If there be, no change is made in our investigations except at the end of Art. 309; and here instead of having the range 2π for θ we have only π, so that m occurs instead of $2m$ as the number of changes of sign.

314. Cauchy's Theorem is given in the *Penny Cyclopædia*, Article *Theory of Equations*, in Mr De Morgan's *Trigonometry and Double Algebra*, and in Mr De Morgan's Memoir to which we have referred in Art. 32; from these sources the present account of it has been derived.

XXVI. NEWTON'S RULE AND SYLVESTER'S THEOREM.

315. Newton enunciated a rule respecting the number of positive, of negative, and of imaginary roots in an equation, which remained without demonstration until the recent researches of Professor Sylvester, who has established a remarkable general theorem which includes Newton's rule as a particular case. The original sources of information on the subject are the *Philosophical Transactions* for 1864, the publications of the *London Mathematical Society*, No. II., and the *Philosophical Magazine* for March, 1866; from these sources the exposition which we shall now give has been essentially derived.

316. We begin by enunciating in substance Newton's rule.

Let $f(x)=0$ be an algebraical equation of the n^{th} degree; and suppose

$$f(x)=a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{1.2}a_2x^{n-2}+\ldots+na_{n-1}x+a_n;$$

then a_0, a_1, a_2,...a_n may be termed the *simple elements* of $f(x)$.

Let a new series of quantities A_0, A_1, A_2,...A_n be formed in the following way:

$$A_0=a_0^2,\quad A_1=a_1^2-a_0a_2,\quad A_2=a_2^2-a_1a_3,\ldots\ldots$$
$$\ldots A_{n-1}=a_{n-1}^2-a_{n-2}a_n,\quad A_n=a_n^2;$$

then A_0, A_1, A_2,...A_n may be termed the *quadratic elements* of $f(x)$.

We shall call a_r, a_{r+1} a *succession of simple elements*, and A_r, A_{r+1} a *succession of quadratic elements*; and we shall call

$$\begin{matrix} a_r, & a_{r+1}, \\ A_r, & A_{r+1}, \end{matrix}$$

an *associated couple of successions.*

Now a succession may present either a permanence or a variation of sign; and this will be termed for brevity a permanence or a variation. Thus in an associated couple of successions we shall have one of four cases; two permanences, or two variations, or a superior permanence with an inferior variation, or a superior variation with an inferior permanence: these may be called respectively a double permanence, a double variation, a permanence-variation and a variation-permanence.

The following is equivalent to Newton's complete rule:

Write the whole series of quadratic elements of $f(x)$ under the whole series of simple elements in their natural order; then:

The number of double permanences in the associated series is a superior limit of the number of negative roots of the equation $f(x)=0$.

The number of variation-permanences is a superior limit of the number of positive roots.

From either of these two statements the other follows by changing the sign of x in $f(x)$.

It follows from these two statements that the whole number of real roots cannot exceed the number of permanences in the series of quadratic elements; and therefore the number of imaginary roots cannot be less than the number of variations in the series of quadratic elements.

It should be noticed that writers who have quoted Newton's rule seem always to have restricted themselves to that part which relates to the number of imaginary roots.

317. We will illustrate Newton's rule by some examples.

Suppose $$2x^4 - 13x^2 + 10x - 49 = 0.$$

Here the series of simple and quadratic elements are

$$2, \quad 0, \; -\frac{13}{6}, \quad \frac{10}{4}, \quad -49,$$

$$4, \quad \frac{13}{3}, \quad \frac{169}{36}, \quad -\frac{1199}{12}, \quad 2401.$$

Thus whether we suppose the zero which forms the second of the simple elements to be positive or negative, we find that there is one double permanence, and one variation-permanence; so there cannot be more than one positive root, and there cannot be more than one negative root: there are then certainly two imaginary roots.

These results agree with those in Art. 203. In this example Descartes's rule would indicate that there cannot be more than *three* positive roots; so that Newton's rule gives us fuller information than Descartes's.

Next suppose $$x^6 + x^5 - x^4 - x^3 + x^2 - x + 1 = 0.$$

We will write down the series of simple elements, and the signs of the quadratic elements:

$$1, \quad \frac{1}{6}, \quad -\frac{1}{15}, \quad -\frac{1}{20}, \quad \frac{1}{15}, \quad -\frac{1}{6}, \quad 1,$$

$$+, \quad +, \quad +, \quad +, \quad -, \quad -, \quad +.$$

Here there are two double permanences, and two variation-permanences; so that by Newton's rule there cannot be more than two positive roots, and there cannot be more than two negative roots.

Descartes's rule would indicate that there cannot be more than *four* positive roots.

In this example it may be shewn by Sturm's theorem that all the roots are imaginary; or we may obtain the same result thus: It is obvious that there can be no positive root greater than unity; and the equation may be written in the forms

$$x^6 + (1-x)(1+x^2-x^4) = 0, \quad x^3(x+1)^2(x-1) + x^2 - x + 1 = 0,$$

which shew respectively that there is no positive root less than unity, and no negative root.

Next suppose $x^6 - 12x^5 + 60x^4 + 123x^2 + 4567x - 89012 = 0$.

We will write down the series of simple elements, and the signs of the quadratic elements·

$$1, \ -2, \ 4, \ 0, \ \frac{41}{5}, \ \frac{4567}{6}, \ -89012,$$

$$+, \quad 0, \ +, \ -, \ +, \quad +, \quad +.$$

There is one double permanence whether we suppose the zero in the upper series to be positive or negative, and one variation-permanence if we suppose the zero in the lower series to be negative; so that by Newton's rule there cannot be more than one positive root, and there cannot be more than one negative root.

Descartes's rule would indicate that there cannot be more than *three* positive roots.

In this example we know by Art. 21 that there is certainly one positive root and one negative root; it will be found on trial that the former lies between 1 and 10, and the latter between -1 and -10.

318. The preceding examples shew that Newton's rule may often be applied with facility. It is obvious that it always tells us as much as Descartes's rule, and often tells us more. For with respect to positive roots, for example, Descartes's rule takes the number of variations in the series of simple elements, while Newton's rejects those variations which are unaccompanied by a permanence in the series of quadratic elements.

319. The following is Professor Sylvester's Theorem :

Let $f(x+\lambda)$ be arranged according to powers of x; let the series of simple elements and the series of quadratic elements be formed, and let the number of double permanences be called the number of double permanences due to λ, and be denoted by $\varpi(\lambda)$. In like manner let the number of double permanences for $f(x+\mu)$ be called the number of double permanences due to μ, and be denoted by $\varpi(\mu)$. Suppose μ greater than λ; then $\varpi(\mu)-\varpi(\lambda)$ is either equal to the number of roots of the equation $f(x)=0$ between λ and μ, or surpasses that number by some even integer.

320. Before demonstrating this theorem we will shew that it includes Newton's rule.

Put 0 for μ and $-\infty$ for λ. We have $\varpi(-\infty)=0$; for when λ is $-\infty$, the simple elements of $f(x+\lambda)$ are alternately positive and negative, so that there can be no double permanences.

Thus $\varpi(0)=\varpi(0)-\varpi(-\infty)$.

Therefore, by the above theorem, $\varpi(0)$ is either equal to the number of roots of the equation $f(x)=0$ between $-\infty$ and 0, or surpasses that number by some even integer. This establishes the first part of Newton's rule, from which the other parts follow.

321. The simple elements of $f(x+\lambda)$ are

$$\frac{f^n(\lambda)}{\lfloor n}, \ \frac{1}{n}\frac{f^{n-1}(\lambda)}{\lfloor n-1}, \ \frac{1\,.\,2}{n(n-1)}\frac{f^{n-2}(\lambda)}{\lfloor n-2}, \ \ldots\ldots \frac{1}{n}f'(\lambda), \ f(\lambda).$$

It will make no change in sign if we multiply every element by $\lfloor n$; thus the series becomes

$$f^n(\lambda), \ f^{n-1}(\lambda), \ 1\,.\,2f^{n-2}(\lambda), \ \lfloor 3 f^{n-3}(\lambda), \ldots\ldots \lfloor n-1 f'(\lambda), \ \lfloor n f(\lambda).$$

In like manner by omitting the square of $\lfloor r-1$ from the r^{th} quadratic element we obtain the series

$$G_n(\lambda), \ G_{n-1}(\lambda), \ G_{n-2}(\lambda), \ldots\ldots \ G_1(\lambda), \ G(\lambda),$$

where $G_r(\lambda)$ stands for $\{f^r(\lambda)\}^2-\gamma_r f^{r-1}(\lambda) f^{r+1}(\lambda)$,

$$\gamma_r \text{ denoting } \frac{n-r+1}{n-r}.$$

We have then to determine the laws of the change in the number of double permanences in the associated series

$$f^n(t),\ f^{n-1}(t),\ f^{n-2}(t),\ \ldots\ldots f'(t),\ f(t),$$

$$G_n(t),\ G_{n-1}(t),\ G_{n-2}(t),\ \ldots\ldots G_1(t),\ G(t),$$

as t increases.

No change can take place except when t passes through a value which makes one or more terms vanish in either or both of the series of elements.

322. It will be necessary to investigate the value of the derived function of a quadratic element; let $G_m(t)$ denote the quadratic element, and $G'_m(t)$ its derived function.

To obtain $G'_m(t)$ we must suppose $G_m(t+h)$ to be expanded in powers of h, and take the coefficient of h.

$$\begin{aligned} G_m(t+h) &= \{f^m(t+h)\}^2 - \gamma_m f^{m-1}(t+h) f^{m+1}(t+h) \\ &= \{f^m(t) + hf^{m+1}(t) + \ldots\}^2 \\ &\quad - \gamma_m\{f^{m-1}(t) + hf^m(t) + \ldots\}\{f^{m+1}(t) + hf^{m+2}(t) + \ldots\}. \end{aligned}$$

The coefficient of h is

$$(2-\gamma_m) f^m(t) f^{m+1}(t) - \gamma_m f^{m-1}(t) f^{m+2}(t).$$

Now it is easily seen that

$$2-\gamma_m = \frac{1}{\gamma_{m+1}};$$

thus $$\begin{aligned} G'_m(t) &= \frac{1}{\gamma_{m+1}} f^m(t) f^{m+1}(t) - \gamma_m f^{m-1}(t) f^{m+2}(t) \\ &= \frac{1}{\gamma_{m+1}} f^m(t) f^{m+1}(t) - \frac{\{f^m(t)\}^2}{f^{m+1}(t)} f^{m+2}(t) + \frac{f^{m+2}(t)}{f^{m+1}(t)} G_m(t) \\ &= \frac{f^m(t)}{\gamma_{m+1} f^{m+1}(t)} G_{m+1}(t) + \frac{f^{m+2}(t)}{f^{m+1}(t)} G_m(t). \end{aligned}$$

323. Suppose that a single term in the series of simple elements intermediate between the first and the last vanishes when $t=c$, say $f^r(c)=0$.

Let h be an indefinitely small quantity; this will be the meaning of h throughout the investigation: then $f^r(c+h)$ has the sign of $hf^{r+1}(c)$. Thus the associated terms

$$f^{r+1}(c+h), \quad f^r(c+h), \quad f^{r-1}(c+h),$$
$$G_{r+1}(c+h), \quad G_r(c+h), \quad G_{r-1}(c+h),$$

have the same signs as

$$f^{r+1}(c), \quad hf^{r+1}(c), \quad f^{r-1}(c),$$
$$\left\{f^{r+1}(c)\right\}^2, \ -f^{r+1}(c)f^{r-1}(c), \ \left\{f^{r-1}(c)\right\}^2.$$

If $f^{r+1}(c)$ and $f^{r-1}(c)$ have the same sign, the terms here considered have no double permanence. If $f^{r+1}(c)$ and $f^{r-1}(c)$ have contrary signs, there is one double permanence whether we suppose h negative or positive.

Thus no change is made in the number of double permanences when t increases through the value c.

324. Suppose that a single term in the series of quadratic elements intermediate between the first and last vanishes when $t=c$, say $G_r(c)=0$.

Since $G_r(c)=0$ it follows that $f^{r-1}(c)$ and $f^{r+1}(c)$ have the same sign. Thus the associated terms

$$f^{r+1}(c+h), \quad f^r(c+h), \quad f^{r-1}(c+h),$$
$$G_{r+1}(c+h), \quad G_r(c+h), \quad G_{r-1}(c+h),$$

have the same signs as

$$f^{r+1}(c), \quad f^r(c), \quad f^{r-1}(c),$$
$$G_{r+1}(c), \ hG'_r(c), \ G_{r-1}(c),$$

and by Art. 322 the sign of $G'_r(c)$ is the same as that of

$$\frac{f^r(c)}{f^{r+1}(c)}\,G_{r+1}(c).$$

If $f^{r+1}(c)$ and $f^r(c)$ have contrary signs, the terms here considered have no double permanence.

If $f^{r+1}(c)$ and $f^r(c)$ have the same sign, and $G_{r+1}(c)$ and $G_{r-1}(c)$ have contrary signs, there is one double permanence whether we suppose h negative or positive.

If $f^{r+1}(c)$ has the same sign as $f^r(c)$, and $G_{r+1}(c)$ the same sign as $G_{r-1}(c)$, there is no double permanence when h is negative, and there are two when h is positive: thus in this case two double permanences are gained when t increases through the value c.

325. Suppose that several consecutive terms of the series of simple elements vanish when $t=c$, say

$$f^{r+s-1}(c)=0, \quad f^{r+s-2}(c)=0, \ldots f^r(c)=0.$$

Thus we suppose s consecutive terms to vanish, and as $f^n(c)$ is a constant which cannot vanish, $r+s$ cannot be greater than n: we suppose that r is not zero.

We have to consider the changes in the signs of

$$f^{r+s}(t), \quad f^{r+s-1}(t), \quad f^{r+s-2}(t), \ldots f^r(t), \quad f^{r-1}(t),$$

$$G_{r+s}(t), \quad G_{r+s-1}(t), \quad G_{r+s-2}(t), \ldots G_r(t), \quad G_{r-1}(t),$$

produced when t increases through the value c. Let $\phi(c)$ stand for $f^{r+s}(c)$; then when $t=c+h$, the signs of the simple elements here considered are the same as the signs of

$$\phi(c), \ h\phi(c), \ h^2\phi(c), \ldots h^s\phi(c), \ f^{r-1}(c).$$

We proceed to investigate the signs of the quadratic elements:

$$G_{r+s}(c)=\left\{f^{r+s}(c)\right\}^2, \text{ which is positive,}$$

$$G_{r-1}(c)=\left\{f^{r-1}(c)\right\}^2, \text{ which is positive,}$$

$$G_r(c+h) = \left\{f^r(c+h)\right\}^2 - \gamma_r f^{r-1}(c+h) f^{r+1}(c+h);$$

expand in powers of h and take the term which involves the lowest power of h: thus we obtain

$$-\gamma_r f^{r-1}(c)\,\phi(c)\,\frac{h^{s-1}}{\underline{|s-1}},$$

so that the sign is the same as that of

$$-f^{r-1}(c)\,\phi(c)\,h^{s-1}.$$

We shall now shew that the other quadratic elements which we have to consider are positive. For

$$G_m(c+h) = \left\{f^m(c+h)\right\}^2 - \gamma_m f^{m-1}(c+h) f^{m+1}(c+h);$$

let μ stand for $r+s-m$; then by expanding and picking out the term which involves the lowest power of h we obtain

$$\left\{\frac{\phi(c)\,h^\mu}{\underline{|\mu}}\right\}^2 - \gamma_m \phi(c)\,\frac{h^{\mu+1}}{\underline{|\mu+1}}\,\phi(c)\,\frac{h^{\mu-1}}{\underline{|\mu-1}};$$

the sign of this is the same as the sign of

$$\frac{\mu+1}{\mu} - \gamma_m,$$

that is as the sign of

$$\frac{r+s-m+1}{r+s-m} - \frac{n-m+1}{n-m},$$

that is as the sign of

$$\frac{1}{r+s-m} - \frac{1}{n-m}.$$

Now $r+s$ is not greater than n so that the sign is never negative; the case in which $r+s=n$ will require further examination.

In this case

$$f^r(c) = 0,\ f^{r+1}(c) = 0, \ldots f^{n-1}(c) = 0;$$

and as $f^r(t)$ is of $n-r$ dimensions in t it follows that all the roots of $f^r(t) = 0$ are equal to c. Thus $f^r(t)$ is of the form $C(t-c)^{n-r}$ where C is a constant.

Then $$f^{r+1}(t) = C(n-r)(t-c)^{n-r-1},$$
$$f^{r+2}(t) = C(n-r)(n-r-1)(t-c)^{n-r-2};$$

and thus it will be found that $G_{r+1}(t)$ is identically zero. And in like manner it will be found that $G_{r+2}(t)$, $G_{r+3}(t)$, ... $G_{n-1}(t)$ are all identically zero.

We will adopt the convention that these quadratic elements which are identically zero shall be supposed to have the *positive* sign; and thus the case in which $r+s=n$ will lead to the same results as that in which $r+s$ is less than n.

Thus finally the signs of the terms of the associated series which we have to consider are the same as the signs of

$$\phi(c),\ h\phi(c),\ h^2\phi(c), \ldots\ldots\ldots\ldots h^s\phi(c),\ f^{r-1}(c),$$
$$+, \quad +, \quad + \ldots\ldots +,\ -h^{s-1}\phi(c)f^{r-1}(c),\ +.$$

We can now ascertain the number of double permanences; the following results will be easily obtained:

Suppose s even, and $\phi(c)$ and $f^{r-1}(c)$ of the same sign; when h is negative there is one double permanence, and when h is positive there are $s-1$: thus $s-2$ double permanences are gained when t increases through the value c.

Suppose s even, and $\phi(c)$ and $f^{r-1}(c)$ of contrary signs; when h is negative there is no double permanence, and when h is positive there are s: thus s are gained.

Suppose s odd, and $\phi(c)$ and $f^{r-1}(c)$ of the same sign; when h is negative there is no double permanence, and when h is positive there are $s-1$: thus $s-1$ are gained.

Suppose s odd, and $\phi(c)$ and $f^{r-1}(c)$ of contrary signs; when h is negative there is one double permanence, and when h is positive there are s: thus $s-1$ are gained.

Hence an *even* number of double permanences is gained when t increases through the value c.

326. Suppose that several consecutive terms of the series of quadratic elements vanish when $t=c$, say

$$G_{r+s-1}(c)=0, \quad G_{r+s-2}(c)=0, \ldots\ldots G_r(c)=0.$$

Thus we suppose s consecutive terms to vanish, and as $G_n(c)$ is a constant which cannot vanish, $r+s$ cannot be greater than n: we suppose that r is not zero.

In consequence of the vanishing of the s consecutive quadratic elements, we have the following conditions holding among the simple elements comprised between $f_{r+s}(c)$ and $f_{r-1}(c)$, both inclusive:

$f_{r+s}(c)$, $f_{r+s-2}(c)$, $f_{r+s-4}(c), \ldots$ are all of the same sign;

$f_{r+s-1}(c)$, $f_{r+s-3}(c)$, $f_{r+s-5}(c), \ldots$ are all of the same sign.

If the terms in the second set have the contrary sign to those in the first set there is no double permanence when $t=c+h$, whether we suppose h positive or negative.

We have then only to consider the case in which the terms in the two sets have all the same sign.

Let $G_{m+1}(t)$ and $G_m(t)$ be any two consecutive quadratic elements comprised between $G_{r+s}(t)$ and $G_r(t)$, both inclusive: then $G_{m+1}(c+h)$ and $G_m(c+h)$ shall have contrary signs when h is negative and the same sign when h is positive.

For by Art. 322,

$$G'_m(t)=\frac{f^m(t)}{\gamma_{m+1}f^{m+1}(t)}G_{m+1}(t)+\frac{f^{m+2}(t)}{f^{m+1}(t)}G_m(t).$$

Put $c+h$ for t, and expand in powers of h.

Suppose that in $G_m(c+h)$ the term which involves the lowest power of h is $\frac{R}{\lfloor\rho}h^\rho$, so that R is the value of the ρ^{th} differential coefficient of $G_m(c)$ with respect to c. Then the term which involves the lowest power of h in $G'_m(c+h)$ will be $\frac{R}{\lfloor\rho-1}h^{\rho-1}$.

Hence from the above equation the term which involves the lowest power of h in $G_{m+1}(c+h)$ will be

$$\frac{\gamma_{m+1} f^{m+1}(c)}{f^m(c) \lfloor \rho - 1} Rh^{\rho-1}.$$

Hence finally $G_m(c+h)$ has the sign of Rh^ρ and $G_{m+1}(c+h)$ has the sign of $Rh^{\rho-1}$; so that $G_{m+1}(c+h)$ and $G_m(c+h)$ have contrary signs when h is negative, and have the same sign when h is positive.

Thus the simple elements which we have to consider have all the same sign; and the quadratic elements comprised between $G_{r+s}(c+h)$ and $G_r(c+h)$, both inclusive, have alternate signs when h is negative and have the same sign when h is positive.

We can now determine the number of double permanences; the following results will be easily obtained:

Suppose s even, and $G_{r+s}(c)$ and $G_{r-1}(c)$ of the same sign; when h is negative there is one double permanence, and when h is positive there are $s+1$: thus s double permanences are gained when t increases through the value c.

Suppose s even, and $G_{r+s}(c)$ and $G_{r-1}(c)$ of contrary signs; when h is negative there is no double permanence, and when h is positive there are s: thus s are gained.

Suppose s odd, and $G_{r+s}(c)$ and $G_{r-1}(c)$ of the same sign; when h is negative there is no double permanence, and when h is positive there are $s+1$: thus $s+1$ are gained.

Suppose s odd, and $G_{r+s}(c)$ and $G_{r-1}(c)$ of contrary signs; when h is negative there is one double permanence, and when h is positive there are s: thus $s-1$ are gained.

Hence an *even* number of double permanences is gained when t increases through the value c.

327. We now consider what takes place when an extreme term vanishes.

$f^n(t)$ and $G^n(t)$ are constants, and can never vanish; and $G(t)$ is essentially positive.

Suppose however that $f^{s-1}(c)=0$, $f^{s-2}(c)=0, \ldots f(c)=0$, so that c is a root repeated s times of the equation $f(x)=0$. Then, as in Art. 325, the last $s+1$ terms of the associated series will have the same signs when $t=c+h$, as

$$f^s(c),\quad hf^s(c),\qquad h^2f^s(c), \ldots\ldots\ldots\ldots h^{s-1}f^s(c),\quad h^sf^s(c),$$

$$+,\ 2-\gamma_{s-1},\ \ \frac{3}{2}-\gamma_{s-2} \ldots\ldots\ldots\ldots \frac{s}{s-1}-\gamma_1,\ +.$$

Here when h is negative there are no double permanences, and when h is positive there are s: thus s double permanences are gained when t increases through the value c.

328. This completes the demonstration of the theorem. The general result is that the number of double permanences belonging to the associated series is increased by at least as many units as there are real roots, equal or unequal, passed over as t increases from one specific value to another; and the excess, if any, of such number over the number of real roots will be an even number.

Thus, with the notation of Art. 319, we know that the number of real roots between λ and μ cannot exceed $\varpi(\mu)-\varpi(\lambda)$. If we *know* that some of the double permanences gained arise from the vanishing of any of the elements except $f(t)$ we can of course make a corresponding reduction in the extreme number of real roots. Thus, for example, suppose that s double permanences are gained in the manner considered in Art. 325, then the number of real roots between λ and μ is not greater than $\varpi(\mu)-\varpi(\lambda)-s$.

329. Some extension may be given to the theorem by ascribing another value to γ_r. The principal property of γ_r which is required in the preceding investigation is that used in Art. 322, namely,

$$\frac{1}{\gamma_{r+1}}=2-\gamma_r.$$

We may then examine what form of γ_r will satisfy this equation. We will solve the equation, though the process will require from the student a knowledge of the elements of the Calculus of Finite Differences.

Put $$\gamma_r = \frac{u_r}{u_{r+1}};$$

thus $$2 - \frac{u_r}{u_{r+1}} = \frac{u_{r+2}}{u_{r+1}};$$

therefore $$u_{r+2} - 2u_{r+1} + u_r = 0.$$

The solution of this equation is

$$u_r = A + B(r-1),$$

where A and B are constants.

Hence $$\gamma_r = \frac{A + B(r-1)}{A + Br}$$

$$= \frac{C + r - 1}{C + r},$$

where C stands for $\frac{A}{B}$.

The student who is not acquainted with the Calculus of Finite Differences may easily verify that this value of γ_r satisfies the relation

$$\frac{1}{\gamma_{r+1}} = 2 - \gamma_r.$$

We have also to satisfy the condition that γ_r shall be positive, and also the condition assumed in Art. 325, that $\frac{\mu+1}{\mu} - \gamma_m$ shall be positive; these conditions will be satisfied if C be any positive quantity, and also if C be negative provided it does not lie between 0 and $-n$.

330. Professor Sylvester observes that his theorem bears the same relation to Newton's rule which Fourier's theorem bears to Descartes's rule. Fourier's theorem may be stated thus:

Form the simple elements corresponding to $f(x+\lambda)$ and to $f(x+\mu)$. Let $p(\lambda)$ and $p(\mu)$ denote the corresponding numbers of permanences of sign; and suppose μ greater than λ. Then $p(\mu)-p(\lambda)$ is either equal to the number of roots of the equation $f(x)=0$ between λ and μ, or surpasses that number by some even integer.

331. We have given in Art. 106 a simple proposition which resembles a special case of Newton's rule; and it is easy to extend the proposition so as to convert the resemblance into a coincidence. For take the equation there obtained,

$$y^{r+1}+\frac{(n-r)p_r}{p_{r+1}}y^r+\frac{(n-r+1)(n-r)}{1\,.\,2}\frac{p_{r-1}}{p_{r+1}}y^{r-1}+\ldots=0\,;$$

this equation has at least as many imaginary roots as any of its derived equations. Take the $(r-1)^{\text{th}}$ derived equation, which is

$$\frac{(r+1)r}{1\,.\,2}y^2+\frac{r(n-r)p_r}{p_{r+1}}y+\frac{(n-r+1)(n-r)}{1\,.\,2}\frac{p_{r-1}}{p_{r+1}}=0.$$

This equation has imaginary roots if

$$(n-r)rp_r^2-(n-r+1)(r+1)p_{r-1}p_{r+1}$$

is negative; and hence in this case the original equation

$$p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0$$

has imaginary roots.

It will be found that the above condition is equivalent to having one of the *quadratic elements* negative; and as the first and last quadratic elements are positive, there must be at least two variations in the quadratic elements and therefore at least two imaginary roots. See Art. 316.

This special case of Newton's rule, and only this, had been established before Professor Sylvester's investigations.

332. If we consider the intrinsic beauty of the theorem which has now been expounded, the interest which belongs to the rule associated with the great name of Newton, and the long lapse of years during which the reason and extent of that rule remained

undiscovered by mathematicians, among whom Maclaurin, Waring, and Euler are explicitly included, we must regard Professor Sylvester's investigations as among the most important contributions made to the Theory of Equations in modern times, justly to be ranked with those of Fourier, Sturm, and Cauchy.

XXVII. REMOVAL OF TERMS FROM AN EQUATION.

333. We have already in Art. 56 shewn how to transform an equation into another which shall want an assigned term. We shall now consider this subject more generally, and shew how theoretically any number of terms may be removed. The method of transformation which we shall explain is called by the name of its inventor Tschirnhausen.

334. Suppose we have the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_{n-1}x + p_n = 0 \ldots\ldots\ldots\ldots(1).$$

Assume

$$y = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_mx^m \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where m is an integer less than n, and $a_0, a_1, \ldots a_m$ are constants at present undetermined. We propose to eliminate x, and thus form an equation in terms of y. Since there are as many values of y as of x the equation in y will be of the degree n.

The elimination may be effected thus: raise the equation (2) to the powers denoted by 2, 3, ... n; and by means of (1) depress the exponents of x, so that none of them shall exceed $n-1$, in the following way,

$$x^n = -p_1x^{n-1} - p_2x^{n-2} - \ldots\ldots - p_{n-1}x - p_n,$$

$$x^{n+1} = -p_1x^n - p_2x^{n-1} - \ldots\ldots - p_{n-1}x^2 - p_nx;$$

substitute for x^n its value from the preceding line, and we have x^{n+1} expressed in terms of x^{n-1} and lower powers of x; then multiply by x and substitute as before, and we have x^{n+2} so expressed; and so on. Thus we shall obtain results of the following form:

$$\left.\begin{aligned} y^2 &= b_0 + b_1x + b_2x^2 + \dots + b_{n-1}x^{n-1}, \\ y^3 &= c_0 + c_1x + c_2x^2 + \dots + c_{n-1}x^{n-1}, \\ &\dots\dots\dots\dots \\ y^n &= k_0 + k_1x + k_2x^2 + \dots + k_{n-1}x^{n-1}. \end{aligned}\right\} \dots\dots\dots\dots\dots (3).$$

Here $b_0, b_1, \dots b_{n-1}$ are integral homogeneous functions of the *second* degree of the undetermined quantities $a_0, a_1, \dots a_m$; also $c_0, c_1, \dots c_{n-1}$ are integral homogeneous functions of the *third* degree of $a_0, a_1, \dots a_m$; and so on.

Let $s_1, s_2, s_3, \dots$ denote the sums of the first, second, third, ... powers of the roots of (1); and let $S_1, S_2, S_3, \dots$ denote the sums of the first, second, third, ... powers of the roots of the required equation in y. Then from (2) and (3) we have

$$\left.\begin{aligned} S_1 &= na_0 + a_1s_1 + a_2s_2 + \dots + a_{m-1}s_{m-1} + a_ms_m, \\ S_2 &= nb_0 + b_1s_1 + b_2s_2 + \dots + b_{n-1}s_{n-1}, \\ S_3 &= nc_0 + c_1s_1 + c_2s_2 + \dots + c_{n-1}s_{n-1}, \\ &\dots\dots\dots\dots\dots \\ S_n &= nk_0 + k_1s_1 + k_2s_2 + \dots + k_{n-1}s_{n-1}. \end{aligned}\right\} \dots\dots (4).$$

Thus, as the sums of the powers of the roots of the equation in y are known we can construct the equation; see Art. 244.

Or we may proceed thus: from equations (3) we can obtain the values of $x, x^2, \dots x^{n-1}$ in terms of the powers of y; then by substituting in (2) we have the required equation in y. This method has the advantage of giving x as a rational function of y, and thus the value of each root of (1) will be known as soon as the equation in y is solved.

335. We may now take the hitherto undetermined quantities $a_0, a_1, \dots a_m$ so as to make some terms disappear from the equation in y. For example, suppose we wish to make the coefficients of the m terms which succeed the first disappear; it will be sufficient to put

$$S_1 = 0, \quad S_2 = 0, \dots\dots S_m = 0.$$

But from equations (4) we see that S_1 is of the first degree with respect to $a_0, a_1, \dots a_m$, that S_2 is of the second degree, S_3 of the third degree, and so on. Hence by Art. 259 the determination of these quantities $a_0, a_1, \dots a_m$, of which one may be assumed arbitrarily, will depend on the solution of an equation of the degree $\lfloor m$.

336. We shall make an application of the preceding method which is of especial interest in connexion with equations of the fifth degree: a preliminary proposition will be required, which we shall now give.

337. *An integral homogeneous function of the second degree of* n *variables can be expressed as the sum of the squares of* ν *linear functions, the number* ν *not being greater than* n.

Let V be an integral homogeneous function of the second degree of the n variables $x_1, x_2, \dots x_n$.

If $n = 1$, the function contains only one variable, so that it is of the form βx_1^2, that is, $(x_1\sqrt{\beta})^2$.

Suppose that n is greater than 1, and that V involves the square of one of the variables, say x_1^2; then by arranging in powers of x_1 we obtain

$$V = \beta x_1^2 + 2Qx_1 + R,$$

where β is a constant, Q is a linear function of the $n-1$ variables, $x_2, x_3, \dots x_n$, and R is an integral homogeneous function of the second degree of these $n-1$ variables.

Put $$X_1 = x_1 + \frac{Q}{\beta}, \quad V_1 = R - \frac{Q^2}{\beta};$$

thus $$V = (X_1\sqrt{\beta})^2 + V_1,$$

and V_1 is an integral homogeneous function of the second degree of $n-1$ variables at most.

Next suppose that V does not contain the square of any of the variables; then, arranging V with respect to two variables x_1 and x_2, we have

$$V = \beta x_1 x_2 + Qx_1 + Rx_2 + S$$

$$= \beta\left(x_1 + \frac{R}{\beta}\right)\left(x_2 + \frac{Q}{\beta}\right) + S - \frac{QR}{\beta}.$$

Put
$$X_1 = \frac{1}{2}\left(x_1 + x_2 + \frac{Q+R}{\beta}\right),$$

$$X_2 = \frac{1}{2}\left(x_1 - x_2 - \frac{Q-R}{\beta}\right),$$

$$V_2 = S - \frac{QR}{\beta};$$

thus
$$V = \beta\,(X_1^2 - X_2^2) + V_2 = (X_1\sqrt{\beta})^2 + (X_2\sqrt{-\beta})^2 + V_2.$$

Here X_1 and X_2 are linear functions which may involve the n variables $x_1, x_2, \ldots x_n$; and V_2 is an integral homogeneous function of the second degree which involves at most $n-2$ variables.

Thus in the first case the function V which involves n variables is made the sum of a certain square and of V_1, where V_1 involves only $n-1$ variables at most; and in the second case V is made the sum of two squares and of V_2, where V_2 involves only $n-2$ variables at most. Then by continuing the process on V_1 or V_2 we can finally express V as the sum of not more than n squares.

338. Let there be an equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0.$$

Assume
$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4.$$

Let the equation in y obtained by eliminating x be denoted by

$$y^n + q_1 y^{n-1} + q_2 y^{n-2} + \ldots + q_n = 0.$$

Now from Art. 334 it will follow that $q_1, q_2, q_3, \ldots$ are respectively of the first, second, third, ... degrees with respect to the quantities a_0, a_1, a_2, a_3, a_4. Suppose then that we wish to make the second, third, and fourth terms of the equation in y disappear. We put

$$q_1 = 0, \quad q_2 = 0, \quad q_3 = 0.$$

The first of these equations is of the first degree. Suppose we obtain a_0 from it in terms of a_1, a_2, a_3, a_4, and substitute in the second and third equations; and then denote these equations by

$$q'_2=0, \quad q'_3=0.$$

Here q'_2 is an integral homogeneous function of the second degree with respect to a_1, a_2, a_3, a_4; and q'_3 is an integral homogeneous function of the third degree.

By Art. 337 the equation $q'_2=0$ may be put in the form

$$f^2+g^2+h^2+k^2=0,$$

where f, g, h, and k are linear functions. This equation will be satisfied by putting

$$f=g\sqrt{-1}, \quad h=k\sqrt{-1};$$

these two equations are linear. Suppose we deduce from them the values of a_1 and a_2 in terms of a_3 and a_4 and substitute in the equation $q'_3=0$; and then denote this equation by

$$q''_3=0.$$

Here q''_3 is an integral homogeneous function of the third degree with respect to a_3 and a_4. One of these quantities may be taken arbitrarily, and the other can then be found by the solution of a cubic equation.

If we wish to make the second, third, and fifth terms disappear from the equation in y the process will be similar but the final equation will be of the fourth degree.

339. If with the transformation of Tschirnhausen we combine that of changing the unknown quantity into its reciprocal we can by the aid of a single equation of the third or fourth degree remove from an equation the three terms which precede the last, or the two terms which precede the last, together with the fifth from the end.

340. Thus we see that the general equation of the fifth degree can always be reduced to any one of the following forms:

$$x^5+px+q=0, \quad x^5+px^2+q=0, \quad x^5+px^3+q=0, \quad x^5+px^4+q=0.$$

341. The foregoing Articles of the present Chapter have been derived from Serret's *Cours d'Algèbre Supérieure.*

The reduction of the equation of the fifth degree to the form of the preceding Article was given by Mr Jerrard; it appears from a paper by Mr Harley in the *Quarterly Journal of Mathematics*, Vol. VI., that the result had been previously obtained by E. S. Bring, a Swedish mathematician.

Mr Jerrard considered that the algebraical solution of equations of the fifth degree could be effected; his proposed method formed the subject of an enquiry by Sir W. R. Hamilton in the *Reports of the British Association*, Vol. VI. Most mathematicians admit that Abel has demonstrated the impossibility of the algebraical solution of equations of a higher degree than the fourth. An abstract of Sir W. R. Hamilton's exposition of Abel's argument will be found in the *Quarterly Journal of Mathematics*, Vol. V.

A simpler demonstration due to Wantzel will be found in Serret's *Cours d'Algèbre Supérieure.*

An *Essay on the Resolution of Algebraical Equations* by the late Judge Hargreave has been recently published; the results arrived at are to some extent at variance with those of Abel and Sir W. R. Hamilton.

XXVIII. INTRODUCTION TO DETERMINANTS.

342. We now propose to give some account of the theory of determinants, a branch of Mathematics of comparatively recent origin, but already of great and rapidly increasing importance. In the present Chapter we shall consider some particular examples and illustrations which will enable the student to form a conception of the nature and properties of determinants; in the next Chapter we shall demonstrate the principal general theorems of the subject, and in the following Chapter we shall give some applications to the theory of equations.

Consider the simultaneous equations

$$a_1x+b_1y=c_1, \quad a_2x+b_2y=c_2;$$

$a_2x + b_2y = c_2$

from these equations we obtain

$$x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}, \quad y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}.$$

The common denominator $a_1b_2-a_2b_1$ is called the *determinant* of the four quantities a_1, b_1, a_2, b_2, and is denoted by the following symbol,

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}$$

The numerators of the values of x and y are also determinants; and we may exhibit the values of x and y thus,

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} x = \begin{vmatrix} c_1, & b_1 \\ c_2, & b_2 \end{vmatrix}, \qquad \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} y = \begin{vmatrix} a_1, & c_1 \\ a_2, & c_2 \end{vmatrix}$$

343. The determinants here considered are all said to be of the *second order*, because they consist of terms each of which is the product of *two* quantities. The quantities a_1, b_1, a_2, b_2 which occur in the determinant $a_1b_2-a_2b_1$ are called the *constituents* of the determinant; the products a_1b_2 and a_2b_1 are called the *elements* of that determinant. Thus a determinant of the second order consists of two elements involving four constituents. In the symbol used to denote this determinant the constituents are arranged in a square forming two *rows* or two *columns*.

344. We shall now indicate some properties of determinants of the second order.

Since we have

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} = a_1b_2-a_2b_1 = \begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix}$$

it follows that the determinant is not altered by changing the rows into columns.

345. The following identities may be easily verified.

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} = - \begin{vmatrix} b_1, & a_1 \\ b_2, & a_2 \end{vmatrix} = - \begin{vmatrix} a_2, & b_2 \\ a_1, & b_1 \end{vmatrix} = \begin{vmatrix} b_2, & a_2 \\ b_1, & a_1 \end{vmatrix}.$$

Thus in the determinant, if the two rows or the two columns are interchanged, the sign of the determinant is altered, but not its value; if both these interchanges are made, the determinant is unaltered.

346. We have

$$\begin{vmatrix} pa_1, & b_1 \\ pa_2, & b_2 \end{vmatrix} = p \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}, \qquad \begin{vmatrix} pa_1, & pb_1 \\ a_2, & b_2 \end{vmatrix} = p \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}$$

Thus if each constituent in one row or in one column is multiplied by a given quantity, the determinant is multiplied by that quantity.

347. We have

$$\begin{vmatrix} a_1, & b_1 \\ a_1, & b_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_1, & a_1 \\ a_2, & a_2 \end{vmatrix} = 0.$$

Thus if two rows or two columns are identical the determinant vanishes.

348. It may be proved by developing the determinants that

$$\begin{vmatrix} a_1 + a_1', & b_1 + b_1' \\ a_2 + a_2', & b_2 + b_2' \end{vmatrix} = \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} + \begin{vmatrix} a_1', & b_1 \\ a_2', & b_2 \end{vmatrix} + \begin{vmatrix} a_1, & b_1' \\ a_2, & b_2' \end{vmatrix} + \begin{vmatrix} a_1', & b_1' \\ a_2', & b_2' \end{vmatrix}$$

Thus the determinant, each of whose constituents is the sum of two terms, is equivalent to the four determinants which can be formed by taking instead of each column one of its partial columns. As a special case, suppose $a_1' = b_1$ and $a_2' = b_2$; then the second of the above four determinants vanishes by Art. 347, and we have

$$\begin{vmatrix} a_1 + b_1, & b_1 + b_1' \\ a_2 + b_2, & b_2 + b_2' \end{vmatrix} = \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} + \begin{vmatrix} a_1, & b_1' \\ a_2, & b_2' \end{vmatrix} + \begin{vmatrix} b_1, & b_1' \\ b_2, & b_2' \end{vmatrix}$$

349. By Art. 348 we have

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1, & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1, & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1, & a_1\alpha_2 \\ a_2\alpha_1, & a_2\alpha_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1, & b_1\beta_2 \\ b_2\beta_1, & b_2\beta_2 \end{vmatrix} + \begin{vmatrix} a_1\alpha_1, & b_1\beta_2 \\ a_2\alpha_1, & b_2\beta_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1, & a_1\alpha_2 \\ b_2\beta_1, & a_2\alpha_2 \end{vmatrix}$$

$$= \alpha_1\alpha_2 \begin{vmatrix} a_1, & a_1 \\ a_2, & a_2 \end{vmatrix} + \beta_1\beta_2 \begin{vmatrix} b_1, & b_1 \\ b_2, & b_2 \end{vmatrix} + \alpha_1\beta_2 \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} + \beta_1\alpha_2 \begin{vmatrix} b_1, & a_1 \\ b_2, & a_2 \end{vmatrix}$$

by Art. 346. By Art. 347 the first two of the four determinants just written vanish. And by Art. 345

$$\begin{vmatrix} b_1, & a_1 \\ b_2, & a_2 \end{vmatrix} = -\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}$$

Thus we have left

$$(\alpha_1\beta_2 - \beta_1\alpha_2)\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} \text{ that is } \begin{vmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \end{vmatrix} \times \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}$$

Therefore

$$\begin{vmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \end{vmatrix} \times \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1, & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1, & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$$

Thus the product of two determinants of the second order is a determinant of the second order.

As a particular case, suppose the constituents α_1, β_1, α_2, β_2 to be respectively equal to the constituents a_1, b_1, a_2, b_2; then we find that the square of the determinant

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}$$

is equal to the determinant

$$\begin{vmatrix} a_1^2 + b_1^2, & a_1a_2 + b_1b_2 \\ a_1a_2 + b_1b_2, & a_2^2 + b_2^2 \end{vmatrix}$$

350. We will now proceed to determinants of the *third* order. Consider the simultaneous equations

$$a_1x + b_1y + c_1z = d_1, \quad a_2x + b_2y + c_2z = d_2, \quad a_3x + b_3y + c_3z = d_3;$$

from these equations we obtain

$$x = \frac{d_1(b_2c_3 - b_3c_2) + d_2(b_3c_1 - b_1c_3) + d_3(b_1c_2 - b_2c_1)}{a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)},$$

and similar expressions for the values of y and z.

The denominator of the value of x is called a determinant of the third order, involving the nine constituents a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3; the determinant consists of six elements, each element being the product of three constituents. This determinant is denoted by the following symbol,

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$$

Since the value of this determinant is

$$a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1),$$

we may express it in terms of determinants of the second order thus

$$a_1 \begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3, & c_3 \\ b_1, & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix}$$

The numerator of the value of x is also a determinant of the third order; we have only to change a_1, a_2, a_3 into d_1, d_2, d_3 respectively in the symbolical expressions already given for the denominator, and we obtain symbolical expressions for the numerator.

We shall now see that determinants of the third order have the same properties as determinants of the second order.

351. Suppose $a_1=1$, $a_2=0$, and $a_3=0$; then we have

$$\begin{vmatrix} 1, & b_1, & c_1, \\ 0, & b_2, & c_2, \\ 0, & b_3, & c_3, \end{vmatrix} = \begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix}$$

Thus the determinant of the third order reduces in this case to a determinant of the second order. The values of b_1 and c_1 have no influence on the value of this determinant, and we may if we please suppose them zero.

Hence we see that when we have any relation holding among determinants of the third order we can deduce the corresponding relation for determinants of the second order by supposing certain constituents to vanish.

352. It may be shewn by developing the determinants that

$$a_1 \begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3, & c_3 \\ b_1, & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2, & b_3 \\ c_2, & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2, & c_3 \\ a_2, & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2, & a_3 \\ b_2, & b_3 \end{vmatrix}$$

that is,

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix}$$

Thus the determinant is not altered by changing the rows into columns.

353. The following identities may be easily verified, by expressing the determinants of the third order in terms of determinants of the second order and developing:

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = - \begin{vmatrix} b_1, & a_1, & c_1 \\ b_2, & a_2, & c_2 \\ b_3, & a_3, & c_3 \end{vmatrix} = \begin{vmatrix} b_1, & c_1, & a_1 \\ b_2, & c_2, & a_2 \\ b_3, & c_3, & a_3 \end{vmatrix}$$

Thus if two columns are interchanged the sign of the determinant is altered but not its value, and therefore if this operation is performed twice the determinant is unaltered. Hence, by Art. 352, if two rows are interchanged the sign of the determinant is altered but not its value, and therefore if this operation is performed twice the determinant is unaltered.

Hence too it follows that if two columns are interchanged and also two rows the determinant is unaltered; so that

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = \begin{vmatrix} b_2, & a_2, & c_2 \\ b_1, & a_1, & c_1 \\ b_3, & a_3, & c_3 \end{vmatrix}$$

354. As in Article 346 we may prove that if every constituent in one row or in one column is multiplied by a given quantity the determinant is multiplied by that quantity.

355. It is easy to shew that

$$\begin{vmatrix} a_1, & b_1, & b_1 \\ a_2, & b_2, & b_2 \\ a_3, & b_3, & b_3 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_2, & b_2, & c_2 \end{vmatrix} = 0.$$

Thus if two rows or two columns are identical the determinant vanishes.

356. It is easy to see that the determinant

$$\begin{vmatrix} a_1 + a_1' + a_1'', & b_1, & c_1 \\ a_2 + a_2' + a_2'', & b_2, & c_2 \\ a_3 + a_3' + a_3'', & b_3, & c_3 \end{vmatrix}$$

is equivalent to the sum of the three determinants

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_1', & b_1, & c_1 \\ a_2', & b_2, & c_2 \\ a_3', & b_3, & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_1'', & b_1, & c_1 \\ a_2'', & b_2, & c_2 \\ a_3'', & b_3, & c_3 \end{vmatrix};$$

and a similar result would be obtained if each constituent in the first column consisted of the sum of four terms, or of the sum of five terms, and so on. Again, if each of the constituents b_1, b_2, b_3 is replaced by three terms, each of the above three determinants becomes equivalent to the sum of three determinants; and so on.

In this way the following determinant may be seen to be equivalent to the sum of 27 determinants:

$$\begin{vmatrix} a_1+a_1'+a_1'', & b_1+b_1'+b_1'', & c_1+c_1'+c_1'' \\ a_2+a_2'+a_2'', & b_2+b_2'+b_2'', & c_2+c_2'+c_2'' \\ a_3+a_3'+a_3'', & b_3+b_3'+b_3'', & c_3+c_3'+c_3'' \end{vmatrix}$$

The 27 determinants are to be formed by taking instead of each column one of the partial columns; thus for example three of these determinants will be the three which are given above.

357. As a particular case of Art. 356 we will take the following determinant:

$$\begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1, & a_1\alpha_2+b_1\beta_2+c_1\gamma_2, & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1, & a_2\alpha_2+b_2\beta_2+c_2\gamma_2, & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1, & a_3\alpha_2+b_3\beta_2+c_3\gamma_2, & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \end{vmatrix}$$

It will be found that of the 27 determinants of which this may be considered the sum, all except 6 vanish by Arts. 354 and 355. For example, we have for one of the 27 determinants,

$$\begin{vmatrix} a_1\alpha_1, & a_1\alpha_2, & b_1\beta_3 \\ a_2\alpha_1, & a_2\alpha_2, & b_2\beta_3 \\ a_3\alpha_1, & a_3\alpha_2, & b_3\beta_3 \end{vmatrix} \text{ that is, } \alpha_1\alpha_2\beta_3 \begin{vmatrix} a_1, & a_1, & b_1 \\ a_2, & a_2, & b_2 \\ a_3, & a_3, & b_3 \end{vmatrix}$$

by Art. 354; and this determinant vanishes by Art. 355. One of the six determinants which remain is

$$\begin{vmatrix} a_1\alpha_1, & b_1\beta_2, & c_1\gamma_3 \\ a_2\alpha_1, & b_2\beta_2, & c_2\gamma_3 \\ a_3\alpha_1, & b_3\beta_2, & c_3\gamma_3 \end{vmatrix} \text{ that is, } \alpha_1\beta_2\gamma_3 \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$$

Another of the six determinants which remain is

$$\begin{vmatrix} a_1\alpha_1, & c_1\gamma_2, & b_1\beta_3 \\ a_2\alpha_1, & c_2\gamma_2, & b_2\beta_3 \\ a_3\alpha_1, & c_3\gamma_2, & b_3\beta_3 \end{vmatrix} \text{ that is, } \alpha_1\gamma_2\beta_3 \begin{vmatrix} a_1, & c_1, & b_1 \\ a_2, & c_2, & b_2 \\ a_3, & c_3, & b_3 \end{vmatrix}$$

$$\text{that is, } -\alpha_1\gamma_2\beta_3 \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} \text{ by Art. 353.}$$

The result is that the six determinants which do remain constitute

$$\left\{\alpha_1(\beta_2\gamma_3-\beta_3\gamma_2)+\alpha_2(\beta_3\gamma_1-\beta_1\gamma_3)+\alpha_3(\beta_1\gamma_2-\beta_2\gamma_1)\right\}\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2 & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$$

$$\text{that is,}\quad \begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 \end{vmatrix} \times \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$$

Hence we see that the product of two determinants of the third order can be exhibited as a determinant of the third order. If we suppose a_1, b_1, ... respectively equal to α_1, β_1,... we obtain a determinant of the third order which is equivalent to the square of a determinant of the third order.

358. We have now given sufficient examples of the nature and properties of determinants to enable the student to form a conception of the subject. We might have confined ourselves to determinants of the third order, because by Art. 351 the properties of determinants of the second order can be immediately derived from the corresponding properties of determinants of the third order, but the method we have adopted will be of service to the beginner. In the next Chapter we shall give general demonstrations applicable to determinants of any order.

It will be observed that we introduce the subject of determinants by considering the forms obtained in solving certain simultaneous equations. The student thus may see at once that the expressions called determinants do naturally present themselves in mathematics. It is however more convenient in treating the general theory to give an independent definition of a determinant, and this we shall do in the next Chapter. It will prepare the student for that definition if we here consider the determinant of the third order in this new light.

359. The value of the determinant

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$$

is $a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$.

The first element here is $a_1b_2c_3$, which is the product of constituents situated diagonally in the square symbol denoting the determinant. The other elements may all be deduced from the first element in a way which we shall now explain. The suffixes 1, 2, 3 are to be attached to the letters a, b, c in all the different ways in which permutations can be made of these suffixes; and the sign + or − is to be prefixed to any element according as it can be deduced from the first element by an even number or an odd number of mutual interchanges of two suffixes. Thus the second element given above is $a_1b_3c_2$; this can be derived from the first element by interchanging the suffixes 2 and 3, and so according to the rule it is to have the sign − prefixed. The third element is $a_2b_3c_1$; this can be derived from the second element by interchanging the suffixes 2 and 1, and therefore it can be derived from the first element by two interchanges of two suffixes, and so according to the rule it is to have the sign + prefixed. Similarly the remaining elements with their proper signs may be determined.

360. The following examples are particular cases of determinants of the third order, which the student may verify:

(1) $$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = abc - af^2 - bg^2 - ch^2 + 2fgh.$$

(2) $$\begin{vmatrix} 1, & x_1, & y_1 \\ 1, & x_2, & y_2 \\ 1, & x_3, & y_3 \end{vmatrix} = x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3.$$

(3) $$\begin{vmatrix} 1, & a_1 + a_2, & a_1a_2 \\ 1, & b_1 + b_2, & b_1b_2 \\ 1, & c_1 + c_2, & c_1c_2 \end{vmatrix} = (a_1 - b_2)(b_1 - c_2)(c_1 - a_2) + (a_2 - b_1)(b_2 - c_1)(c_2 - a_1).$$

(4) $$\begin{vmatrix} 1, & \gamma, & -\beta \\ -\gamma, & 1, & \alpha \\ \beta, & -\alpha, & 1 \end{vmatrix} = 1 + \alpha^2 + \beta^2 + \gamma^2.$$

XXIX. PROPERTIES OF DETERMINANTS.

361. Let there be n symbols $a_1, a_2, \ldots a_n$; then one of these symbols will be called *higher* than another when it has a greater suffix, so that for example a_3 is higher than a_2 or a_1, a_4 is higher than a_3 or a_2 or a_1, and so on.

Now suppose that permutations are formed of these symbols; then whenever in a permutation the higher of two symbols precedes the other there is said to be a *disarrangement*. Thus, for example, in the permutation $a_2a_4a_3a_1$ there are four disarrangements, namely a_2a_1, a_4a_3, a_4a_1, and a_3a_1.

362. The permutations of the symbols $a_1, a_2, \ldots a_n$ may be divided into two classes, those in which there is an *even* number of disarrangements and those in which there is an *odd* number.

363. *When in any permutation two symbols interchange their places while the others remain unchanged the number of disarrangements is increased or diminished by an odd number.*

Let g and k denote two symbols of which k is the higher. Let A denote the group of symbols before g and k, let B denote the group between g and k, and let C denote the group after g and k; so that the permutations which we have to compare may be denoted by $AgBkC$ and $AkBgC$. Then the difference of the numbers of the disarrangements depends upon the symbols which constitute the groups gBk and kBg. Let B consist of β symbols and suppose that β_1 of them are higher than g and β_2 of them higher than k. Then in the group gBk, besides the disarrangements in B itself, there are $\beta - \beta_1 + \beta_2$ disarrangements; for g is higher than $\beta - \beta_1$ of the symbols in B, and there are β_2 symbols in B higher than k. In the group kBg, besides the disarrangements in B itself, there are $\beta - \beta_2 + \beta_1 + 1$ disarrangements; for k is higher than $\beta - \beta_2$ symbols in B, and there are β_1 symbols in B higher than g, and k is higher than g. Therefore the difference of the numbers of the disarrangements is

$$\beta - \beta_2 + \beta_1 + 1 - (\beta - \beta_1 + \beta_2),$$

that is, $2(\beta_1 - \beta_2) + 1$; thus this difference is an *odd* number.

364. By repeated interchanges of two symbols all the permutations of a set of n symbols taken all together can be deduced from a given permutation. In this mode of deriving the permutations we shall, by Art. 363, obtain alternately permutations with an even number of disarrangements and permutations with an odd number of disarrangements. The whole number of the permutations of a set of symbols taken all together is an *even* number; hence it follows that there are as many permutations with an *even* number of disarrangements as there are with an *odd* number of disarrangements.

365. Let there be n^2 quantities arranged in the form of a square, thus

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & \dots\dots\dots & a_{1,n} \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & \dots\dots\dots & a_{2,n} \\ \dots & \dots & \dots & \dots\dots\dots & \dots \\ a_{n,1}, & a_{n,2}, & a_{n,3}, & \dots\dots\dots & a_{n,n} \end{vmatrix}$$

Here for any quantity $a_{r,k}$ the first suffix, r, indicates the row, and the second suffix, k, indicates the column in which the quantity $a_{r,k}$ appears.

The above symbol is used to denote the *determinant* of the n^2 quantities occurring in it; these quantities are called *constituents* of the determinant. The value of the determinant is found by taking the aggregate of a certain number of *elements*, each element being the product of n constituents. The first element is the product of the constituents $a_{1,1}$, $a_{2,2}$, $a_{3,3}$, $\dots a_{n,n}$, which lie in the diagonal drawn from the upper left-hand corner of the square to the opposite corner; we shall call this diagonal *the* diagonal of the square, for we shall only have occasion to refer to this diagonal. All the other elements are to be derived from the first element $a_{1,1}a_{2,2}a_{3,3}\ldots a_{n,n}$ by permutations of the second suffixes, the first suffixes being left unchanged. The sign + or − is to be prefixed to each element of the determinant according as it is or is not of the same class as the first element, the class being determined by the number of disarrangements in the permutations of the second suffixes; see Art. 362.

366. The above determinant is said to be of the n^{th} order because each element is the product of n constituents. The number of elements is the same as the number of the permutations of n things taken all together, that is $\lfloor n$; half of these elements will have the sign + prefixed, and half of them the sign − prefixed. It will be seen from the mode of formation of the elements, that each element involves one and only one constituent out of each row or each column in the symbol which denotes the determinant.

367. Instead of the above symbol for the determinant, it is sometimes denoted by $\Sigma \pm a_{1,1}a_{2,2}a_{3,3}\ldots a_{n,n}$; that is, the first element is written and the symbol $\Sigma\pm$ put before it to indicate the aggregate of elements which can be derived from the first element by suitable permutations and adjustment of the signs + and −. The constituents of the determinant may be denoted in various ways; thus sometimes (i, k) is used instead of $a_{i,k}$, and in this case we must remember that (i, k) and (k, i) in general denote different quantities. In examples of determinants of low orders, we may find it convenient to avoid double suffixes, and use the same letter for all the constituents in one column, distinguishing the constituents by single suffixes; this notation was adopted in the preceding Chapter.

368. The other elements of a determinant are derived from the first element by permutations of the second suffixes while the

first suffixes remain unchanged; these elements may however be derived in a different way, namely, by permutations of the first suffixes while the second suffixes remain unchanged. For suppose that $\alpha, \beta, \gamma, \ldots \nu$ represents a certain permutation of the n numbers $1, 2, 3, \ldots n$; then $a_{1,\alpha} a_{2,\beta} a_{3,\gamma} \ldots a_{n,\nu}$ is an element of the determinant which arises from the first element by changing the second suffixes $1, 2, \ldots n$, into $\alpha, \beta, \gamma, \ldots \nu$, respectively. This element may however also be derived from the first element $a_{1,1} a_{2,2} \ldots a_{n,n}$ if the second suffixes are left unchanged and the first suffixes are suitably changed, namely, α to 1, β to 2, γ to 3,...ν to n. In these two modes of derivation there is the same number of interchanges of two suffixes, and therefore the same sign is obtained to prefix to the element by the rule in Art. 365.

369. *The value of a determinant is not altered if the successive rows are changed into successive columns;* that is

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, \ldots\ldots\ldots & a_{1,n} \\ a_{2,1}, & a_{2,2}, \ldots\ldots\ldots & a_{2,n} \\ \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots \\ a_{n,1}, & a_{n,2}, \ldots\ldots\ldots & a_{n,n} \end{vmatrix} = \begin{vmatrix} a_{1,1}, & a_{2,1}, \ldots\ldots\ldots & a_{n,1} \\ a_{1,2}, & a_{2,2}, \ldots\ldots\ldots & a_{n,2} \\ \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots \\ a_{1,n}, & a_{2,n}, \ldots\ldots\ldots & a_{n,n} \end{vmatrix}$$

For it is obvious from Art. 365, that the elements in these determinants are of equal *value;* and they have the same *signs,* as appears from Art. 368.

370. *If two rows or two columns are interchanged, the sign of the determinant is changed.*

For let R denote the given determinant, R' that which arises from the interchange. Then the elements in R and R' are the same as to value, and we have only to examine their signs. The first element in R' can be derived from the first element in R by interchanging two of the second suffixes, and thus these elements have contrary signs in the two determinants. Then an element in R' which arises from the first element in R' by m interchanges of the second suffixes will be deducible from the first element in R by $m+1$ interchanges, and therefore it will appear in R and R' with contrary signs prefixed.

371. *If two rows or two columns are identical, the determinant vanishes.*

For by interchanging two rows or two columns, a determinant is changed from R to $-R$ by Art. 370. But if two rows or two columns are identical, the interchange of these rows or columns can have no influence on the determinant, so that $R=-R$; and therefore $R=0$.

372. *When all the constituents except one of a row or of a column vanish, the determinant reduces to the product of that constituent and of a determinant of the next inferior order.*

Consider, for example, the determinant

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ 0, & 0, & c_4, & 0 \end{vmatrix}.$$

By three successive interchanges of single rows we can bring the row which contains c_4 to be the highest row; and by two successive interchanges of single columns we can bring the column which contains c_4 to be the first column. Thus, by Art. 370,

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ 0, & 0, & c_4, & 0 \end{vmatrix} = (-1)^5 \times \begin{vmatrix} c_4, & 0, & 0, & 0 \\ c_1, & a_1, & b_1, & d_1 \\ c_2, & a_2, & b_2, & d_2 \\ c_3, & a_3, & b_3, & d_3 \end{vmatrix}$$

The first element of the determinant on the right-hand side is $c_4a_1b_2d_3$, and the other elements are to be derived from this by permutations of the suffixes. But c_4 is the only constituent with the suffix 4 which is not zero, and thus c_4 will be a factor of every element which does not vanish, and the other factor will be deducible from $a_1 b_2 d_3$ by permutations of the suffixes 1, 2, 3. Thus the original determinant reduces to

$$(-1)^5 c_4 \times \begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix}$$

This mode of demonstration applies, whatever may be the order of the proposed determinant.

The negative sign which arises in this example from $(-1)^5$ may if we please be removed by interchanging two rows or two columns in the determinant of the third order.

373. The top row of a determinant of the n^{th} order can be brought to the bottom by $n-1$ successive interchanges of two rows; and similarly, the first column can be brought to the end by $n-1$ successive interchanges of successive columns. Each of these is called a *cyclical* interchange, and it is sometimes convenient to effect any proposed interchange of rows or columns by a series of cyclical interchanges, for the sake of greater symmetry in the arrangement of rows and columns. In the preceding example we may bring c_4 to the place which we want it to occupy by performing three successive cyclical interchanges of rows and two successive cyclical interchanges of columns. Thus we obtain for the original determinant the following forms successively:

$$(-1)^3 \begin{vmatrix} a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ 0, & 0, & c_4, & 0 \\ a_1, & b_1, & c_1, & d_1 \end{vmatrix}, \quad (-1)^6 \begin{vmatrix} a_3, & b_3, & c_3, & d_3 \\ 0, & 0, & c_4, & 0 \\ a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \end{vmatrix}, \quad (-1)^9 \begin{vmatrix} 0, & 0, & c_4, & 0 \\ a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{vmatrix}$$

$$(-1)^{12} \begin{vmatrix} 0, & c_4, & 0, & 0 \\ b_1, & c_1, & d_1, & a_1 \\ b_2, & c_2, & d_2, & a_2 \\ b_3, & c_3, & d_3, & a_3 \end{vmatrix}, \quad (-1)^{15} \begin{vmatrix} c_4, & 0, & 0, & 0 \\ c_1, & d_1, & a_1, & b_1 \\ c_2, & d_2, & a_2, & b_2 \\ c_3, & d_3, & a_3, & b_3 \end{vmatrix}, \quad (-1)^{15} c_4 \begin{vmatrix} d_1, & a_1, & b_1 \\ d_2, & a_2, & b_2 \\ d_3, & a_3, & b_3 \end{vmatrix}.$$

374. *A determinant can always be expressed in the form of a determinant of any higher order.*

For example, by Art. 373,

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = \begin{vmatrix} 1, & 0, & 0, & 0 \\ \beta, & a_1, & b_1, & c_1 \\ \gamma, & a_2, & b_2, & c_2 \\ \delta, & a_3, & b_3, & c_3 \end{vmatrix} = \begin{vmatrix} 1, & 0, & 0, & 0, & 0 \\ \mu, & 1, & 0, & 0, & 0 \\ \nu, & \beta, & a_1, & b_1, & c_1 \\ \rho, & \gamma, & a_2, & b_2, & c_2 \\ \sigma, & \delta, & a_3, & b_3, & c_3 \end{vmatrix}$$

where β, γ, δ, μ, ν, ρ, σ, are any quantities. Similarly, we may carry on this process to any extent.

375. Let i and k denote any two suffixes out of the set 1, 2, ... n; let R denote the determinant $\Sigma \pm a_{1,1} a_{2,2} \ldots a_{n,n}$; and let $A_{i,k}$ denote the coefficient of $a_{i,k}$ in R. Then each of the expressions

$$a_{i,1}A_{k,1} + a_{i,2}A_{k,2} + \ldots + a_{i,n}A_{k,n},$$

and $$a_{1,i}A_{1,k} + a_{2,i}A_{2,k} + \ldots + a_{n,i}A_{n,k},$$

is equal to R or to 0, according as i and k are equal or unequal.

For every element of R contains as a factor one out of the constituents $a_{i,1}$, $a_{i,2}$, $a_{i,3}$, ... $a_{i,n}$, which form the i^{th} row. And since $A_{i,k}$ denotes the coefficient of $a_{i,k}$ in R, we have

$$R = a_{i,1}A_{i,1} + a_{i,2}A_{i,2} + \ldots + a_{i,n}A_{i,n}.$$

Similarly we have

$$R = a_{1,i}A_{1,i} + a_{2,i}A_{2,i} + \ldots + a_{n,i}A_{n,i}.$$

In the first of these expressions for R, put $a_{i,1} = a_{k,1}$, $a_{i,2} = a_{k,2}$, ... and so on; thus we obtain the value of a determinant with two rows identical, which is zero by Art. 371.

Similarly, in the second expression for R put $a_{1,i} = a_{1,k}$, $a_{2,i} = a_{2,k}$, ... and so on; thus we obtain the value of a determinant with two columns identical, which is zero by Art. 371.

376. *If every constituent in one row or one column is multiplied by a given quantity, the determinant is multiplied by that quantity.*

For $R = a_{i,1}A_{i,1} + a_{i,2}A_{i,2} + \dots + a_{i,n}A_{i,n}$; and if every term in the i^{th} row is multiplied by p we must put $pa_{i,1}$ for $a_{i,1}$, $pa_{i,2}$ for $a_{i,2}$, and so on; thus we obtain p times the former result for the new determinant.

Similarly, we may prove the theorem in the case in which all the constituents of a column are multiplied by a given quantity.

377. *If each of the constituents in one row or one column is the sum of* m *terms, the determinant can be considered as the sum of* m *determinants.*

Suppose, for example, that each constituent of the i^{th} row is the sum of m terms; and suppose that

$$a_{i,1} = p_1 + q_1 + r_1 + \dots$$
$$a_{i,2} = p_2 + q_2 + r_2 + \dots$$
$$a_{i,3} = p_3 + q_3 + r_3 + \dots$$
$$\dots\dots\dots\dots\dots$$

Then
$$\begin{aligned} R &= a_{i,1}A_{i,1} + a_{i,2}A_{i,2} + \dots\dots\dots + a_{i,n}A_{i,n} \\ &= p_1A_{i,1} + p_2A_{i,2} + \dots\dots\dots + p_nA_{i,n} \\ &+ q_1A_{i,1} + q_2A_{i,2} + \dots\dots\dots + q_nA_{i,n} \\ &+ r_1A_{i,1} + r_2A_{i,2} + \dots\dots\dots + r_nA_{i,n} \\ &+ \quad \dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

Thus R may be considered as the sum of m determinants which have for their i^{th} rows respectively

$$p_1,\ p_2,\ \dots\dots\dots\ p_n,$$
$$q_1,\ q_2,\ \dots\dots\dots\ q_n,$$
$$r_1,\ r_2,\ \dots\dots\dots\ r_n,$$
$$\dots\dots\dots\dots\dots$$

378. We shall now shew how the coefficient of $a_{i,k}$ in a determinant can be itself exhibited as a determinant. In order to obtain those elements of a determinant which involve a certain constituent $a_{i,k}$, and those alone, we may suppose all the constituents in the i^{th} row to be zero, except $a_{i,k}$; then putting 1 for $a_{i,k}$ we shall obtain the required coefficient. In this way we get

$$A_{i,k} = \begin{vmatrix} a_{1,1}, & \dots & a_{1,k-1}, & a_{1,k}, & a_{1,k+1}, & \dots & a_{1,n} \\ \multicolumn{7}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ a_{i-1,1}, & \dots & a_{i-1,k-1}, & a_{i-1,k}, & a_{i-1,k+1}, & \dots & a_{i-1,n} \\ 0, & \dots & 0, & 1, & 0, & \dots & 0 \\ a_{i+1,1}, & \dots & a_{i+1,k-1}, & a_{i+1,k}, & a_{i+1,k+1}, & \dots & a_{i+1,n} \\ \multicolumn{7}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ a_{n,1}, & \dots & a_{n,k-1}, & a_{n,k}, & a_{n,k+1}, & \dots & a_{n,n} \end{vmatrix}$$

Thus $A_{i,k}$ is here exhibited as a determinant of the n^{th} order. We may, without influencing the value of $A_{i,k}$, put 0 for each constituent in the k^{th} column except that which is 1.

By Art. 372, or by Art. 373, we may exhibit $A_{i,k}$ as a determinant of the $(n-1)^{\text{th}}$ order. Thus, adopting the method of Art. 373, we make $i-1$ cyclical changes in the rows and $k-1$ cyclical changes in the columns. Therefore

$$A_{i,k} = \epsilon \times \begin{vmatrix} a_{i+1,k+1}, & \dots & a_{i+1,n}, & a_{i+1,1}, & \dots & a_{i+1,k-1} \\ \multicolumn{6}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ a_{n,k+1}, & \dots & a_{n,n}, & a_{n,1}, & \dots & a_{n,k-1} \\ a_{1,k+1}, & \dots & a_{1,n}, & a_{1,1}, & \dots & a_{1,k-1} \\ \multicolumn{6}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ a_{i-1,k+1}, & \dots & a_{i-1,n}, & a_{i-1,1}, & \dots & a_{i-1,k-1} \end{vmatrix}$$

where $\epsilon = (-1)^{(i-1+k-1)(n-1)} = (-1)^{(i+k)(n-1)}$.

379. By the aid of Arts. 375 and 378 we can express any determinant of the n^{th} order as the aggregate of n terms, each of which is the product of one constituent and of a determinant of the $(n-1)^{\text{th}}$ order; the determinants of the $(n-1)^{\text{th}}$ order may themselves be similarly treated; and the process continued to any extent. For example,

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \\ b_4, & c_4, & d_4 \end{vmatrix} - b_1 \begin{vmatrix} c_2, & d_2, & a_2 \\ c_3, & d_3, & a_3 \\ c_4, & d_4, & a_4 \end{vmatrix} + c_1 \begin{vmatrix} d_2, & a_2, & b_2 \\ d_3, & a_3, & b_3 \\ d_4, & a_4, & b_4 \end{vmatrix} - d_1 \begin{vmatrix} a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \\ a_4, & b_4, & c_4 \end{vmatrix}$$

$$= a_1 \left\{ b_2 \begin{vmatrix} c_3, & d_3 \\ c_4, & d_4 \end{vmatrix} + c_2 \begin{vmatrix} d_3, & b_3 \\ d_4, & b_4 \end{vmatrix} + d_2 \begin{vmatrix} b_3, & c_3 \\ b_4, & c_4 \end{vmatrix} \right\}$$

$$- b_1 \left\{ c_2 \begin{vmatrix} d_3, & a_3 \\ d_4, & a_4 \end{vmatrix} + d_2 \begin{vmatrix} a_3, & c_3 \\ a_4, & c_4 \end{vmatrix} + a_2 \begin{vmatrix} c_3, & d_3 \\ c_4, & d_4 \end{vmatrix} \right\}$$

$$+ c_1 \left\{ d_2 \begin{vmatrix} a_3, & b_3 \\ a_4, & b_4 \end{vmatrix} + a_2 \begin{vmatrix} b_3, & d_3 \\ b_4, & d_4 \end{vmatrix} + b_2 \begin{vmatrix} d_3, & a_3 \\ d_4, & a_4 \end{vmatrix} \right\}$$

$$- d_1 \left\{ a_2 \begin{vmatrix} b_3, & c_3 \\ b_4, & c_4 \end{vmatrix} + b_2 \begin{vmatrix} c_3, & a_3 \\ c_4, & a_4 \end{vmatrix} + c_2 \begin{vmatrix} a_3, & b_3 \\ a_4, & b_4 \end{vmatrix} \right\}$$

380. We now proceed to an important part of the subject, that which relates to the multiplication of determinants.

Let there be two given sets of symbols, namely,

$$\begin{array}{l} a_{1,1}, \;\ldots\ldots\ldots\; a_{1,p}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n,1}, \;\ldots\ldots\ldots\; a_{n,p}, \end{array}$$

and

$$\begin{array}{l} b_{1,1}, \;\ldots\ldots\ldots\; b_{1,p}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_{n,1}, \;\ldots\ldots\ldots\; b_{n,p}. \end{array}$$

From these let a third set of symbols be formed,

$$\begin{array}{l} c_{1,1}, \;\ldots\ldots\ldots\; c_{1,n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_{n,1}, \;\ldots\ldots\ldots\; c_{n,n}, \end{array}$$

these symbols being determined by the general relation

$$c_{i,k} = a_{i,1}b_{k,1} + a_{i,2}b_{k,2} + \ldots\ldots + a_{i,p}b_{k,p}.$$

Let R denote the determinant $\Sigma \pm c_{1,1}c_{2,2} \ldots c_{n,n}$. We shall now prove the following results:

(1) Suppose p less than n; then $R=0$.

(2) Suppose $p=n$; then R is equal to the product of the two determinants which consist of the two given sets of symbols in the order they occupy.

(3) Suppose p greater than n; then R is equal to the sum of a set of products of pairs of determinants, each pair of determinants being formed by taking any n columns out of the first given set of symbols for one determinant, and the corresponding n columns out of the other given set of symbols for the other determinant.

The first element of R is $c_{1,1}c_{2,2}\dots c_{n,n}$, and the value of this is

$$(\Sigma a_{1,r}b_{1,r})(\Sigma a_{2,s}b_{2,s})(\Sigma a_{3,t}b_{3,t})\dots$$

where in the first factor Σ denotes a summation with respect to r, in the second factor Σ denotes a summation with respect to s, in the third factor Σ denotes a summation with respect to t, and so on; and all these summations extend from 1 to p, both inclusive. Thus the product may be obtained by taking the sum of the values of the expression

$$a_{1,r}a_{2,s}a_{3,t}\dots b_{1,r}b_{2,s}b_{3,t}\dots$$

where r, s, t, ... take all integral values from 1 to p.

We may denote this sum by

$$\Sigma_{r,s,t,\dots}(a_{1,r}a_{2,s}a_{3,t}\dots b_{1,r}b_{2,s}b_{3,t}\dots).$$

The other elements of R are derived from the first element by permutations of the second suffixes and prefixing the proper sign. Now from the general value of $c_{i,k}$ it follows that by changing the second suffixes of the symbol c no change is made in the suffixes of the symbol a, but the first suffixes of the symbol b are changed, and these alone.

Hence we obtain a result which we may denote thus,

$$R=\Sigma_{r,s,t,\dots}(a_{1,r}a_{2,s}a_{3,t}\dots\Sigma\pm b_{1,r}b_{2,s}b_{3,t}\dots).$$

Here $\Sigma \pm b_{1,r} b_{2,s} b_{3,t} \ldots$ constitutes a determinant of the n^{th} order, which is formed from the second given set of symbols by taking certain columns, and the Σ refers to changes of the first suffixes; see Art. 368. We shall denote this determinant by Q.

Now, in the first place, suppose p less than n. The suffixes $r, s, t, \ldots$ are n in number, and none of them can exceed p; hence it follows that there must be always two or more of them which have the same value. Thus Q always vanishes, by Art. 371; and therefore R vanishes.

Secondly, suppose $p = n$. Then the system of suffixes $r, s, t, \ldots$ can be a permutation of the n symbols $1, 2, \ldots n$; and they can be nothing else without making Q vanish. And by taking in succession different permutations the sign of Q will change, but not its value, by Art. 370. Thus the value of R reduces to the product of the determinant formed out of the second given set of symbols, into the sum of all the elements denoted by $\Sigma \pm a_{1,1} a_{2,2} \ldots a_{n,n}$, where Σ refers to changes of the second suffixes. Therefore when $p = n$,

$$R = \begin{vmatrix} a_{1,1}, \ldots a_{1,n} \\ \ldots\ldots\ldots\ldots \\ a_{n,1}, \ldots a_{n,n} \end{vmatrix} \times \begin{vmatrix} b_{1,1}, \ldots b_{1,n} \\ \ldots\ldots\ldots\ldots \\ b_{n,1}, \ldots b_{n,n} \end{vmatrix}$$

Lastly, suppose p greater than n. Then the system of suffixes $r, s, t, \ldots$ can be any combination of n numbers that can be formed out of the p numbers $1, 2, \ldots p$; and the number of such combinations is $\dfrac{\lfloor p}{\lfloor n \lfloor p-n}$. Let P denote what Q becomes by changing b into a. Hence, as in the second case, we shall obtain PQ for one term in R, which arises from the selection of a definite combination out of the $\dfrac{\lfloor p}{\lfloor n \lfloor p-n}$ possible combinations. Therefore when p is greater than n we have $R = \Sigma PQ$, where Σ refers to the summation of $\dfrac{\lfloor p}{\lfloor n \lfloor p-n}$ terms arising from all the possible combinations.

381. By the second case of the preceding Article we see that the product of two determinants of the order n can be exhibited as a determinant of the same order. Similarly, the product of three determinants of the order n can be exhibited as a determinant of the order n; for we can first exhibit the product of two of them as a new determinant of the order n, and then the product of this new determinant and the third of the original determinants can be exhibited as a determinant of the order n. Thus we see that the product of any number of determinants which are all of the same order can be exhibited as a determinant of that order.

Hence generally the product of any number of determinants of any orders can be exhibited as a determinant of the same order as that of the determinant of the highest order among the factors. For by Art. 374, all the other determinants may be made to be of the same order as that which is of the highest order; and then the product of these determinants of the same order can be exhibited as a determinant of that order.

382. Suppose we wish to form the product of the two determinants

$$\begin{vmatrix} a_{1,1}, \dots a_{1,n} \\ \dots\dots\dots\dots \\ a_{n,1}, \dots a_{n,n} \end{vmatrix}$$

$$\text{and} \quad \begin{vmatrix} b_{1,1}, \dots b_{1,n} \\ \dots\dots\dots\dots \\ b_{n,1}, \dots b_{n\,n} \end{vmatrix}$$

By Art. 369 we may change the successive rows into successive columns in either or both of these determinants. Thus, if we denote the product by

$$\begin{vmatrix} c_{1,1}, \dots c_{1,n} \\ \dots\dots\dots\dots \\ c_{n,1}, \dots c_{n,n} \end{vmatrix}$$

we may form the new constituents in four ways, for we may adopt either of the following laws throughout,

$$c_{i,k} = a_{i,1}b_{k,1} + a_{i,2}b_{k,2} + \ldots + a_{i,n}b_{k,n},$$

$$\text{or} \quad c_{i,k} = a_{i,1}b_{1,k} + a_{i,2}b_{2,k} + \ldots + a_{i,n}b_{n,k},$$

$$\text{or} \quad c_{i,k} = a_{1,i}b_{k,1} + a_{2,i}b_{k,2} + \ldots + a_{n,i}b_{k,n},$$

$$\text{or} \quad c_{i,k} = a_{1,i}b_{1,k} + a_{2,i}b_{2,k} + \ldots + a_{n,i}b_{n,k}.$$

383. Let $A_{i,k}$ denote the coefficient of $a_{i,k}$ in a determinant R. The system of symbols

$$\begin{array}{l} A_{1,1},\ A_{1,2}, \ldots A_{1,n} \\ A_{2,1},\ A_{2,2}, \ldots A_{2,n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_{n,1},\ A_{n,2}, \ldots A_{n,n} \end{array}$$

is called the *reciprocal* of the system of symbols

$$\begin{array}{l} a_{1,1},\ a_{1,2}, \ldots a_{1,n} \\ a_{2,1},\ a_{2,2}, \ldots a_{2,n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n,1},\ a_{n,2}, \ldots a_{n,n}. \end{array}$$

384. *The determinant of a system which is the reciprocal of a proposed system of* n^2 *symbols is the* $(n-1)^{\text{th}}$ *power of the determinant of the proposed system.*

If we multiply the determinants

$$\left| \begin{array}{l} A_{1,1}, \ldots A_{1,n} \\ \ldots\ldots\ldots\ldots \\ A_{n,1}, \ldots A_{n,n} \end{array} \right|$$

and

$$\left| \begin{array}{l} a_{1,1}, \ldots a_{1,n} \\ \ldots\ldots\ldots\ldots \\ a_{n,1}, \ldots a_{n,n} \end{array} \right|$$

we obtain for the product

$$\left| \begin{array}{l} c_{1,1}, \ldots c_{1,n} \\ \ldots\ldots\ldots\ldots \\ c_{n,1}, \ldots c_{n,n} \end{array} \right|$$

where $c_{i,k} = A_{i,1}a_{k,1} + A_{i,2}a_{k,2} + \ldots + A_{i,n}a_{k,n}$. Hence, by Art. 375, the constituents of the last determinant have the value R or 0 according as i and k are equal or unequal. Thus this determinant reduces to its first element $c_{1,1}c_{2,2}\ldots c_{n,n}$, that is, to R^n. Therefore

$$\begin{vmatrix} A_{1,1}, \ldots A_{1,n} \\ \ldots\ldots\ldots\ldots \\ A_{n,1}, \ldots A_{n,n} \end{vmatrix} R = R^n$$

therefore

$$\begin{vmatrix} A_{1,1}, \ldots A_{1,n} \\ \ldots\ldots\ldots\ldots \\ A_{n,1}, \ldots A_{n,n} \end{vmatrix} = R^{n-1}$$

385. Suppose we have a determinant of the n^{th} order, and in the square symbol denoting it suppose m columns and m rows destroyed; the remaining symbols may then be supposed moved close up so as to form a new square symbol which is a determinant of the order $n-m$. This determinant is called a *partial* determinant or a *minor* determinant, with respect to the original determinant. The symbols common to the m rows and columns will form a square symbol which is a determinant of the order m. This is also a partial determinant or minor determinant. The two partial or minor determinants are said to be *complementary* to each other.

386. Let R denote a determinant of the order n. A partial determinant of the reciprocal system of the order m is numerically equal to the product of R^{m-1} into the complementary of the corresponding partial determinant of the original system.

Let $f, g, \ldots r, s, \ldots$ denote one permutation of the n numbers $1, 2, \ldots n$; and let $i, k, \ldots u, v, \ldots$ denote another permutation. And suppose $f, g, \ldots$ and $i, k, \ldots$ to be groups of m numbers each, while $r, s, \ldots$ and $u, v, \ldots$ are groups of $n-m$ numbers each. Thus

$$\begin{vmatrix} A_{f,i}, & A_{f,k}, \ldots \\ A_{g,i}, & A_{g,k}, \ldots \\ \ldots\ldots & \ldots\ldots \end{vmatrix}$$

is a partial determinant of the reciprocal system of the order m; we shall denote it by S.

Now

$$\begin{vmatrix} a_{f,i}, & a_{f,k}, & \dots a_{f,u}, & a_{f,v}, & \dots \\ a_{g,i}, & a_{g,k}, & \dots a_{g,u}, & a_{g,v}, & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{r,i}, & a_{r,k}, & \dots a_{r,u}, & a_{r,v}, & \dots \\ a_{s,i}, & a_{s,k}, & \dots a_{s,u}, & a_{s,v}, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = \epsilon R$$

where ϵ is $+1$ or -1 according as the permutations $f, g, \dots r, s, \dots$ and $i, k, \dots u, v, \dots$ belong to the same class or to different classes.

We now propose to obtain the product of these two determinants. The determinant S may be raised to the order n by inserting additional constituents; see Art. 374. Thus we may put for S the following determinant,

$$\begin{vmatrix} A_{f,i}, & A_{f,k}, & \dots A_{f,u}, & A_{f,v}, & \dots \\ A_{g,i}, & A_{g,k}, & \dots A_{g,u}, & A_{g,v}, & \dots \\ \dots & \dots & \dots & \dots & \dots \\ B_{r,i}, & B_{r,k}, & \dots B_{r,u}, & B_{r,v}, & \dots \\ B_{s,i}, & B_{s,k}, & \dots B_{s,u}, & B_{s,v}, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

where the constituents denoted by the letter B with suffixes are all supposed zero, except those standing in *the* diagonal which are all supposed equal to unity.

Now form the product of S and ϵR, which will be a new determinant of the order n. Let the constituents of this new determinant be denoted by the letter c with two suffixes, the first of which indicates as usual the row and the second the column. By Art. 382 there are four ways by which we may determine the constituents in the product of S and ϵR; we shall select the first of these, according to which $c_{p,q}$ is obtained by multiplying respectively the constituents in the p^{th} row of S by those in the q^{th} row of ϵR. Thus

$$c_{1,1} = A_{f,i}a_{f,i} + A_{f,k}a_{f,k} + \dots + A_{f,u}a_{f,u} + A_{f,v}a_{f,v} + \dots$$
$$c_{1,2} = A_{f,i}a_{g,i} + A_{f,k}a_{g,k} + \dots + A_{f,u}a_{g,u} + A_{f,v}a_{g,v} + \dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Therefore by Art. 375, we have $c_{1,1}$, $c_{2,2}, \ldots c_{m,m}$ all equal to R, while all the other constituents in the first m rows of the determinant which is the product of S and ϵR are zero.

For the first term in the $(m+1)^{\text{th}}$ row, we have

$$c_{m+1,1} = B_{r,i}a_{f,i} + B_{r,k}a_{f,k} + \ldots + B_{r,u}a_{f,u} + B_{r,v}a_{f,v} + \ldots = a_{f,u},$$

because all the symbols denoted by B with suffixes which occur here are zero except $B_{r,u}$, and that is unity. For the second term in the $(m+1)^{\text{th}}$ row we have similarly

$$c_{m+1,2} = a_{g,u}.$$

Proceeding in this way, we find that the $(m+1)^{\text{th}}$ row in the product of S and ϵR is the same as the $(m+1)^{\text{th}}$ column in ϵR.

Similarly, the $(m+2)^{\text{th}}$ row in the product is the same as the $(m+2)^{\text{th}}$ column in ϵR.

The determinant then which is equivalent to $S\epsilon R$ reduces by Art. 372 to the product of R^m and the following determinant of the $(n-m)^{\text{th}}$ order,

$$\begin{vmatrix} a_{r,u}, & a_{r,v}, & \ldots \\ a_{s,u}, & a_{s,v}, & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$$

Thus $S = \epsilon R^{m-1} \begin{vmatrix} a_{r,u}, & a_{r,v}, & \ldots \\ a_{s,u}, & a_{s,v}, & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$

387. The following examples may be verified by the student. In examples (4), (5), and (6), we have determinants of which the constituents are themselves determinants.

(1) $\begin{vmatrix} 0, & \alpha, & \beta, & \gamma \\ \alpha, & 0, & \gamma_1, & \beta_1 \\ \beta, & \gamma_1, & 0, & \alpha_1 \\ \gamma, & \beta_1, & \alpha_1, & 0 \end{vmatrix} = \alpha^2\alpha_1^2 + \beta^2\beta_1^2 + \gamma^2\gamma_1^2 - 2\alpha\alpha_1\beta\beta_1 - 2\alpha\alpha_1\gamma\gamma_1 - 2\beta\beta_1\gamma\gamma_1$

(2) $$\begin{vmatrix} 0, & \alpha, & \beta, & \gamma \\ -\alpha, & 0, & \gamma_1, & \beta_1 \\ -\beta, & -\gamma_1, & 0, & \alpha_1 \\ -\gamma, & -\beta_1, & -\alpha_1, & 0 \end{vmatrix} = (\alpha\alpha_1 - \beta\beta_1 + \gamma\gamma_1)^2$$

(3) $$\begin{vmatrix} \theta, & \alpha, & \beta, & \gamma \\ -\alpha, & \theta, & \gamma_1, & \beta_1 \\ -\beta, & -\gamma_1, & \theta, & \alpha_1 \\ -\gamma, & -\beta_1, & -\alpha_1, & \theta \end{vmatrix} = \theta^4 + \theta^2(\alpha^2 + \beta^2 + \gamma^2 + \alpha_1^2 + \beta_1^2 + \gamma_1^2) + (\alpha\alpha_1 - \beta\beta_1 + \gamma\gamma_1)^2$$

(4) $$\begin{vmatrix} \begin{vmatrix} c, & g \\ g, & a \end{vmatrix} & \begin{vmatrix} g, & a \\ f, & h \end{vmatrix} \\ \begin{vmatrix} g, & a \\ f, & h \end{vmatrix} & \begin{vmatrix} a, & h \\ h, & b \end{vmatrix} \end{vmatrix} = a \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

(5) $$\begin{vmatrix} \begin{vmatrix} g, & a \\ f, & h \end{vmatrix} & \begin{vmatrix} f, & c \\ h, & g \end{vmatrix} \\ \begin{vmatrix} a, & h \\ h, & b \end{vmatrix} & \begin{vmatrix} h, & b \\ g, & f \end{vmatrix} \end{vmatrix} = h \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

(6) $$\begin{vmatrix} \begin{vmatrix} b, & f \\ f, & c \end{vmatrix} & \begin{vmatrix} f, & c \\ h, & g \end{vmatrix} & \begin{vmatrix} h, & b \\ g, & f \end{vmatrix} \\ \begin{vmatrix} f, & c \\ h, & g \end{vmatrix} & \begin{vmatrix} c, & g \\ g, & a \end{vmatrix} & \begin{vmatrix} g, & a \\ f, & h \end{vmatrix} \\ \begin{vmatrix} h, & b \\ g, & f \end{vmatrix} & \begin{vmatrix} g, & a \\ f, & h \end{vmatrix} & \begin{vmatrix} a, & h \\ h, & b \end{vmatrix} \end{vmatrix} = \text{the square of } \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

(7) $$\begin{vmatrix} a, & b \\ a_1, & b_1 \end{vmatrix} \begin{vmatrix} c, & d \\ c_1, & d_1 \end{vmatrix} + \begin{vmatrix} b, & c \\ b_1, & c_1 \end{vmatrix} \begin{vmatrix} a, & d \\ a_1, & d_1 \end{vmatrix} + \begin{vmatrix} c, & a \\ c_1, & a_1 \end{vmatrix} \begin{vmatrix} b, & d \\ b_1, & d_1 \end{vmatrix} = 0.$$

XXX. APPLICATIONS OF DETERMINANTS.

388. Suppose we have to find the values of n unknown quantities $x_1, x_2, \ldots x_n$ from the following n simple equations

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \ldots + a_{1,n}x_n = u_1,$$

$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \ldots + a_{2,n}x_n = u_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n,1}x_1 + a_{n,2}x_2 + a_{n,3}x_3 + \ldots + a_{n,n}x_n = u_n.$$

Let R denote the determinant $\Sigma \pm a_{1,1}a_{2,2}\ldots a_{n,n}$; and let $A_{i,k}$ denote the coefficient of $a_{i,k}$ in R. Then the values of the unknown quantities will be given by the formula

$$Rx_k = u_1A_{1,k} + u_2A_{2,k} + \ldots + u_nA_{n,k},$$

where k may have any value between 1 and n both inclusive.

For let the given equations be multiplied respectively by $A_{1,k}, A_{2,k}, \ldots A_{n,k}$; and add the results. The coefficient of x_k is then

$$a_{1,k}A_{1,k} + a_{2,k}A_{2,k} + \ldots + a_{n,k}A_{n,k},$$

which is equal to R by Art. 375. The coefficient of x_i is

$$a_{1,i}A_{1,k} + a_{2,i}A_{2,k} + \ldots + a_{n,i}A_{n,k},$$

which is zero by Art. 375.

We may write the formula which gives x_k thus

$$Rx_k = S,$$

where S is also a determinant, namely the determinant which is obtained from R by removing the k^{th} column of R and substituting for it the column formed of $u_1, u_2, \ldots u_n$.

389. Suppose that the determinant R vanishes; then the values of the unknown quantities become infinite. This indicates that the given equations are inconsistent; see *Algebra*, Chapter xv.

390. Suppose that $u_1, u_2, \ldots u_n$ vanish, and that R also vanishes. The method of Art. 388 gives for the unknown quantities the indeterminate form $\frac{0}{0}$. In this case we may take $n-1$ of the given equations, and these will be sufficient to determine the ratios of $n-1$ of the unknown quantities to the remaining unknown quantity.

These ratios can however be at once assigned: we shall have

$$x_1 : x_2 : x_3 : \ldots = A_{i,1} : A_{i,2} : A_{i,3} : \ldots$$

where i is any integer not greater than n.

For since $R=0$, we have by Art. 375, for all integral values of i and k between 1 and n,

$$a_{k,1}A_{i,1} + a_{k,2}A_{i,2} + a_{k,3}A_{i,3} + \ldots\ldots = 0;$$

and thus when $x_1, x_2, x_3, \ldots$ are taken in the ratios assigned above, we have

$$a_{k,1}x_1 + a_{k,2}x_2 + a_{k,3}x_3 + \ldots\ldots = 0.$$

By taking $n-1$ of the given equations, and supposing $u_1, u_2, \ldots u_n$ all zero, we shall obtain in general a *single definite* value for the ratio of each of $n-1$ of the unknown quantities to the remaining unknown quantity. Hence it follows that when $R=0$ the ratios

$$A_{i,1} : A_{i,2} : A_{i,3} : \ldots$$

are independent of i.

391. If $u_1, u_2, \ldots u_n$ all vanish, and R does not vanish, the system of equations in Art. 388 has no solutions, except we suppose $x_1, x_2, \ldots x_n$ all zero. The condition $R=0$ is thus necessary in order that the unknown quantities may have values which are not zero.

392. For example, in order that the equations

$$a_1x + b_1y + c_1z = 0,$$
$$a_2x + b_2y + c_2z = 0,$$
$$a_3x + b_3y + c_3z = 0,$$

may admit of solutions which are not zero we must have

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = 0.$$

If this condition is satisfied the equations may be satisfied by

$$x : y : z :: \begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix} : \begin{vmatrix} c_2, & a_2 \\ c_3, & a_3 \end{vmatrix} : \begin{vmatrix} a_2, & b_2 \\ a_3, & b_3 \end{vmatrix},$$

$$\text{or } x : y : z :: \begin{vmatrix} b_3, & c_3 \\ b_1, & c_1 \end{vmatrix} : \begin{vmatrix} c_3, & a_3 \\ c_1, & a_1 \end{vmatrix} : \begin{vmatrix} a_3, & b_3 \\ a_1, & b_1 \end{vmatrix},$$

$$\text{or } x : y : z :: \begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix} : \begin{vmatrix} c_1, & a_1 \\ c_2, & a_2 \end{vmatrix} : \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}.$$

These three forms of solution coincide by Art. 390..

393. From the given equations in Art. 388 we have deduced

$$u_1 A_{1,1} + u_2 A_{2,1} + u_3 A_{3,1} + \dots + u_n A_{n,1} = Rx_1,$$
$$u_1 A_{1,2} + u_2 A_{2,2} + u_3 A_{3,2} + \dots + u_n A_{n,2} = Rx_2,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$u_1 A_{1,n} + u_2 A_{2,n} + u_3 A_{3,n} + \dots + u_n A_{n,n} = Rx_n.$$

Let ρ denote the determinant $\Sigma \pm A_{1,1} A_{2,2} \dots A_{n,n}$; and let $\alpha_{i,k}$ denote the coefficient of $A_{i,k}$ in ρ. We may from the above equations find the values of $u_1, u_2, \dots u_n$; and by proceeding as in Art. 388 we shall obtain the general result

$$\rho u_k = R \left\{ x_1 \alpha_{k,1} + x_2 \alpha_{k,2} + \dots + x_n \alpha_{k,n} \right\}.$$

By comparing this result with the given equation in Art. 388,

$$a_{k,1} x_1 + a_{k,2} x_2 + \dots + a_{k,n} x_n = u_k,$$

we have, since the values of u_k must be identical,

$$\frac{R\alpha_{k,i}}{\rho} = a_{k,i}.$$

But $\rho = R^{n-1}$ by Art. 384; thus

$$a_{k,i} = R^{n-2} a_{k,i}.$$

394. We now proceed to apply determinants to another problem, that of forming the product of all the differences of given quantities.

Let n quantities be denoted by $a_1, a_2, \ldots a_n$. Let P denote the product of the differences obtained by subtracting each of these n quantities from all those which follow it, so that

$$P = (a_2 - a_1)(a_3 - a_1)\ldots(a_n - a_1)(a_3 - a_2)(a_4 - a_2) \ldots (a_n - a_{n-1}).$$

Then P may be exhibited as a determinant of the order n. For consider the determinant

$$\begin{vmatrix} 1, & a_1, & a_1^2, \ldots a_1^{n-1} \\ 1, & a_2, & a_2^2, \ldots a_2^{n-1} \\ \ldots & \ldots & \ldots \\ 1, & a_n, & a_n^2, \ldots a_n^{n-1} \end{vmatrix}$$

This determinant is a rational integral function of the quantities $a_1, a_2, \ldots a_n$; and it vanishes when any two of these quantities are equal, by Art. 371. It is therefore divisible by the product which we have denoted by P. Also both the determinant and the product P are of the degree $\frac{n(n-1)}{1 \cdot 2}$ in powers and products of $a_1, a_2, \ldots a_n$; therefore the quotient when the determinant is divided by P is some *number*. And this number must be unity, as we see by comparing the first element of the determinant with the product of the first terms of the binomial factors of which P is composed.

395. The above determinant of the n^{th} order consists of $\lfloor n$ terms. The product P prior to simplification and cancelling would involve a much larger number of terms, namely, $2^{\frac{n(n-1)}{2}}$. Thus the determinant is an advantageous form for the product on account of the saving in terms.

396. We have

$$P^2 = \begin{vmatrix} 1, & a_1, & a_1^2, \ldots & a_1^{n-1} \\ 1, & a_2, & a_2^2, \ldots & a_2^{n-1} \\ \multicolumn{4}{c}{\dotfill} \\ 1, & a_n, & a_n^2, \ldots & a_n^{n-1} \end{vmatrix} \times \begin{vmatrix} 1, & a_1, & a_1^2, \ldots & a_1^{n-1} \\ 1, & a_2, & a_2^2, \ldots & a_2^{n-1} \\ \multicolumn{4}{c}{\dotfill} \\ 1, & a_n, & a_n^2, \ldots & a_n^{n-1} \end{vmatrix}$$

Now the product of these determinants can be exhibited as a single determinant; adopting the last of the four methods given in Art. 382, we have

$$P^2 = \begin{vmatrix} s_0, & s_1, & s_2, & \ldots s_{n-1} \\ s_1, & s_2, & s_3, & \ldots s_n \\ \multicolumn{4}{c}{\dotfill} \\ s_{n-1}, & s_n, & s_{n+1}, & \ldots s_{2n-2} \end{vmatrix}$$

where $s_r = a_1^r + a_2^r + \ldots + a_n^r$.

397. Suppose, for example, that $a_1, a_2, \ldots a_n$ are the roots of an equation of the n^{th} degree; then P^2 is the product of the squares of the differences of the roots. Thus the product of the squares of the differences of all the roots of an equation can be exhibited as a determinant, the constituents of which are known in terms of the coefficients of the given equation, for s_r can be expressed in terms of the coefficients.

398. Suppose we have to find the values of the n unknown quantities $x_1, x_2, \ldots x_n$ from the equations

$$\begin{aligned} x_1 + x_2 + x_3 + \ldots + x_n &= 1, \\ x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots + x_n a_n &= t, \\ x_1 a_1^2 + x_2 a_2^2 + x_3 a_3^2 + \ldots x_n a_n^2 &= t^2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_1 a_1^{n-1} + x_2 a_2^{n-1} + x_3 a_3^{n-1} + \ldots + x_n a_n^{n-1} &= t^{n-1}. \end{aligned}$$

The values of the unknown quantities will be determined by the formula

$$x_i = \frac{(a_1 - t)(a_2 - t) \ldots (a_{i-1} - t)(a_{i+1} - t) \ldots (a_n - t)}{(a_1 - a_i)(a_2 - a_i) \ldots (a_{i-1} - a_i)(a_{i+1} - a_i) \ldots (a_n - a_i)}.$$

For by Art. 388 we have $Rx_i = S$,

$$\text{where } R = \begin{vmatrix} 1, & 1, & 1, & \ldots 1 \\ a_1, & a_2, & a_3, & \ldots a_n \\ a_1^2, & a_2^2, & a_3^2, & \ldots a_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ a_1^{n-1}, & a_2^{n-1}, & a_3^{n-1}, & \ldots a_n^{n-1} \end{vmatrix}$$

$$\text{and } S = \begin{vmatrix} 1, & \ldots 1, & 1, & 1, & \ldots 1 \\ a_1, & \ldots a_{i-1}, & t, & a_{i+1}, & \ldots a_n \\ a_1^2, & \ldots a_{i-1}^2, & t^2, & a_{i+1}^2, & \ldots a_n^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1^{n-1}, & \ldots a_{i-1}^{n-1}, & t^{n-1}, & a_{i+1}^{n-1}, & \ldots a_n^{n-1} \end{vmatrix}$$

Now let the i^{th} column in R be placed first, and the i^{th} column in S be placed first; see Art. 373. Then let the two determinants be changed into products of differences by Art. 394; and by cancelling common factors in the numerator and denominator we obtain the value of x_i in the form assigned above.

As a verification we observe that if $a_i = t$ the equations are obviously satisfied by supposing $x_i = 1$, and all the other unknown quantities zero.

399. The method of determinants may also be used to obtain the resulting equation when certain quantities are eliminated from given equations. Suppose we have to eliminate x from the equations $f(x) = 0$ and $\phi(x) = 0$, where

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3, \qquad \phi(x) = b_0 + b_1x + b_2x^2.$$

We may proceed thus

$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + 0, \\ xf(x) &= 0 + a_0x + a_1x^2 + a_2x^3 + a_3x^4, \\ \phi(x) &= b_0 + b_1x + b_2x^2 + 0 + 0, \\ x\phi(x) &= 0 + b_0x + b_1x^2 + b_2x^3 + 0, \\ x^2\phi(x) &= 0 + 0 + b_0x^2 + b_1x^3 + b_2x^4. \end{aligned}$$

Let $R = \begin{vmatrix} a_0, & a_1, & a_2, & a_3, & 0 \\ 0, & a_0, & a_1, & a_2, & a_3 \\ b_0, & b_1, & b_2, & 0, & 0 \\ 0, & b_0, & b_1, & b_2, & 0 \\ 0, & 0, & b_0, & b_1, & b_2 \end{vmatrix}$;

then since by supposition $f(x) = 0$ and $\phi(x) = 0$, and therefore also $xf(x)$, $x\phi(x)$, and $x^2\phi(x)$ are all zero, it follows by Art. 391 that $R = 0$ is the necessary relation which must hold among the coefficients of $f(x)$ and $\phi(x)$.

400. We have given a particular example in the preceding Article, as the general investigation to which we now proceed will thus be more intelligible. Let

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_mx^m = 0,$$
$$\phi(x) = b_0 + b_1x + b_2x^2 + \dots + b_nx^n = 0;$$

and suppose we have to eliminate x between these equations. We have

$$\begin{aligned}
f(x) &= a_0 + a_1x + a_2x^2 + \dots + a_mx^m, \\
xf(x) &= \qquad a_0x + a_1x^2 + \dots + a_{m-1}x^m + a_mx^{m+1}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
x^{n-1}f(x) &= \qquad\qquad a_0x^{n-1} + a_1x^n + \dots \\
\phi(x) &= b_0 + b_1x + b_2x^2 + \dots + b_nx^n, \\
x\phi(x) &= \qquad b_0x + b_1x^2 + \dots + b_{n-1}x^n + b_nx^{n+1}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
x^{m-1}\phi(x) &= \qquad\qquad b_0x^{m-1} + b_1x^m + \dots
\end{aligned}$$

Let R denote the determinant of the order $m+n$ which has for its first n rows

$$\begin{array}{l}
a_0,\ a_1,\ a_2, \dots a_m,\quad 0,\quad 0,\quad 0, \dots \\
0,\ a_0,\ a_1, \dots a_{m-1},\ a_m,\quad 0,\quad 0, \dots \\
0,\ 0,\ a_0, \dots a_{m-2},\ a_{m-1},\ a_m,\ 0, \dots \\
\dots\dots\dots\dots\dots\dots\dots\dots,
\end{array}$$

and for its next m rows

$$\begin{array}{llllllll} b_0, & b_1, & b_2, \dots b_n, & 0, & 0, & 0, \dots \\ 0, & b_0, & b_1, \dots b_{n-1}, & b_n, & 0, & 0, \dots \\ 0, & 0, & b_0, \dots b_{n-2}, & b_{n-1}, & b_n, & 0, \dots \\ \dots & \dots & \dots & \dots & \dots & \dots ; \end{array}$$

then $R=0$ is the necessary relation among the coefficients in order that $f(x)$ and $\phi(x)$ may simultaneously vanish.

The relation $R=0$ has been called the *resultant* or the *eliminant* of the proposed equations $f(x)=0$ and $\phi(x)=0$.

401. The terms in the quotient obtained by dividing one algebraical expression by another may be exhibited as determinants.

Let $\phi(x)=a_0x^m+a_1x^{m-1}+a_2x^{m-2}+\dots+a_rx^{m-r}+\dots,$

$$\psi(x)=b_0x^n+b_1x^{n-1}+b_2x^{n-2}+\dots+b_rx^{n-r}+\dots,$$

and let the quotient of $\phi(x)$ divided by $\psi(x)$ be denoted by

$$q_0x^{m-n}+q_1x^{m-n-1}+\dots+q_rx^{m-n-r}+\dots$$

Multiply by the denominator, and equate the coefficients of x^{m-r} on both sides. Thus

$$a_r=q_rb_0+q_{r-1}b_1+q_{r-2}b_2+\dots+q_0b_r.$$

Similarly,

$$\begin{aligned} a_{r-1} &= q_{r-1}b_0+q_{r-2}b_1+\dots+q_0b_{r-1}, \\ a_{r-2} &= q_{r-2}b_0+\dots+q_0b_{r-2}, \\ &\dots\dots\dots\dots\dots\dots \\ a_1 &= q_1b_0+q_0b_1, \\ a_0 &= q_0b_0. \end{aligned}$$

We may regard these as $r+1$ equations for finding $q_r, q_{r-1}, \dots q_0$.

We have

$$q_r \begin{vmatrix} b_0, & b_1, & b_2, & \dots b_r \\ 0, & b_0, & b_1, & \dots b_{r-1} \\ \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & \dots b_0 \end{vmatrix} = \begin{vmatrix} a_r, & b_1, & b_2, & \dots & b_r \\ a_{r-1}, & b_0, & b_1, & \dots & b_{r-1} \\ a_{r-2}, & 0, & b_0, & \dots & b_{r-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_0, & 0, & 0, & \dots & b_0 \end{vmatrix}$$

Therefore by evaluating the determinant on the left-hand side, and rearranging that on the right-hand side, we obtain

$$q_r = \frac{1}{b_0^{r+1}} \begin{vmatrix} b_0, & 0, & 0, & 0, & \dots\dots a_0 \\ b_1, & b_0, & 0, & 0, & \dots\dots a_1 \\ b_2, & b_1, & b_0, & 0, & \dots\dots a_2 \\ b_3, & b_2, & b_1, & b_0, & \dots\dots a_3 \\ \dots & \dots & \dots & \dots & \dots \\ b_r, & b_{r-1}, & b_{r-2}, & b_{r-3}, & \dots\dots a_r \end{vmatrix}$$

402. We will now give some applications of the theory of determinants which occur in a case of the transformation of functions by linear substitutions.

Let there be any function of the n independent variables $x_1, x_2, \dots x_n$; and let these variables be expressed in terms of n new independent variables $y_1, y_2, \dots y_n$ by means of the following n linear equations:

$$\left.\begin{aligned} x_1 &= a_{1,1}y_1 + a_{1,2}y_2 + \dots + a_{1,n}y_n \\ x_2 &= a_{2,1}y_1 + a_{2,2}y_2 + \dots + a_{2,n}y_n \\ &\dots\dots\dots\dots\dots\dots \\ x_n &= a_{n,1}y_1 + a_{n,2}y_2 + \dots + a_{n,n}y_n \end{aligned}\right\} \dots\dots\dots\dots\dots(1);$$

then by substituting the values of $x_1, x_2, \dots x_n$, the assigned function becomes a function of $y_1, y_2, \dots y_n$.

Suppose now that we impose the condition that

$$x_1^2 + x_2^2 + \dots + x_n^2 = y_1^2 + y_2^2 + \dots + y_n^2 \dots\dots\dots\dots(2);$$

then certain relations will hold among the coefficients of the linear equations (1); these relations we shall now demonstrate.

I. For every value of i and k between 1 and n inclusive

and
$$\left.\begin{array}{r} a_{1,i}^2 + a_{2,i}^2 + \dots\dots + a_{n,i}^2 = 1 \\ a_{1,i}a_{1,k} + a_{2,i}a_{2,k} + \dots\dots + a_{n,i}a_{n,k} = 0 \end{array}\right\} \dots\dots\dots\dots (3).$$

Substitute the values of $x_1, x_2, \dots x_n$ from (1) in the identity (2); and then by comparing the coefficients of like terms we obtain (3).

II. From (1) we can express $y_1, y_2, \dots y_n$ in terms of $x_1, x_2, \dots x_n$; we shall shew that for every value of i between 1 and n inclusive

$$y_i = a_{1,i}x_1 + a_{2,i}x_2 + \dots\dots + a_{n,i}x_n \dots\dots\dots\dots\dots\dots (4).$$

To establish this it will be sufficient to verify the statement: substitute for $x_1, x_2, \dots x_n$ from (1) in (4); then by means of (3) it will be found that the right-hand member of (4) reduces to y_i.

III. In the same way as we obtained (3) by substituting from (1) in (2) we may obtain, by substituting from (4) in (2), the following results for every value of i and k between 1 and n both inclusive:

$$\left.\begin{array}{r} a_{i,1}^2 + a_{i,2}^2 + \dots + a_{i,n}^2 = 1, \\ a_{i,1}a_{k,1} + a_{i,2}a_{k,2} + \dots + a_{i,n}a_{k,n} = 0. \end{array}\right\} \dots\dots\dots\dots\dots (5).$$

IV. The square of the following determinant is equal to unity:

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & \dots & a_{1,n} \\ a_{2,1}, & a_{2,2}, & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{n,1}, & a_{n,2}, & \dots & a_{n,n} \end{vmatrix}.$$

Denote the proposed determinant by R: then R^2, by Art. 380, is equal to the determinant

$$\begin{vmatrix} c_{1,1}, & c_{1,2}, & \dots & c_{1,n} \\ c_{2,1}, & c_{2,2}, & \dots & c_{2,n} \\ \dots & \dots & \dots & \dots \\ c_{n,1}, & c_{n,2}, & \dots & c_{n,n} \end{vmatrix}.$$

where
$$c_{i,k} = a_{i,1}a_{k,1} + a_{i,2}a_{k,2} + \dots + a_{i,n}a_{k,n}.$$

Thus, by (5), we have $c_{i,k}=0$ when i is not equal to k, and $c_{i,i}=1$. Thus the latter determinant reduces to its first element, that is, to unity: therefore $R^2=1$.

V. Let $A_{i,k}$ have the same meaning as in Art. 388: then we obtain from (1)

$$y_i=\frac{1}{R}\left\{A_{1,i}x_1+A_{2,i}x_2+\ldots+A_{n,i}x_n\right\}.$$

Hence, comparing this result with (4), we have

$$a_{k,i}=\frac{A_{k,i}}{R}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

VI. The following partial determinants which can be formed of the constituents of R are numerically equal:

$$\begin{vmatrix} a_{m+1,m+1}, & a_{m+1,m+2}, & \ldots & a_{m+1,n} \\ a_{m+2,m+1}, & a_{m+2,m+2}, & \ldots & a_{m+2,n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n,m+1}, & a_{n,m+2}, & \ldots\ldots & a_{n,n} \end{vmatrix}$$

and

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & \ldots & a_{1,m} \\ a_{2,1}, & a_{2,2}, & \ldots & a_{2,m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m,1}, & a_{m,2}, & \ldots & a_{m,m} \end{vmatrix}$$

Denote the first determinant by P, and the second by Q; then by Art. 386,

$$\begin{vmatrix} A_{1,1}, & A_{1,2}, & \ldots & A_{1,m} \\ A_{2,1}, & A_{2,2}, & \ldots & A_{2,m} \\ \ldots & \ldots & \ldots & \ldots \\ A_{m,1}, & A_{m,2}, & \ldots & A_{m,m} \end{vmatrix}$$

is numerically equal to $R^{m-1}P$.

And by Art. 376 and equation (6),

$$\begin{vmatrix} A_{1,1}, & A_{1,2}, & \ldots & A_{1,m} \\ A_{2,1}, & A_{2,2}, & \ldots & A_{2,m} \\ \ldots & \ldots & \ldots & \ldots \\ A_{m,1}, & A_{m,2}, & \ldots & A_{m,m} \end{vmatrix}=R^mQ.$$

Hence, since $R^2=1$, we see that P and Q are numerically equal.

403. We will finish with some examples.

(1) Shew that $\begin{vmatrix} a, & b, & c \\ c, & a, & b \\ b, & c, & a \end{vmatrix} = a^3+b^3+c^3-3abc.$

Shew that $a+b+c$ must be a factor of this determinant.

(2) Shew that

$$\begin{vmatrix} a, & b, & c, & d \\ d, & a, & b, & c \\ c, & d, & a, & b \\ b, & c, & d, & a \end{vmatrix} = a^4-b^4+c^4-d^4-2a^2c^2+2b^2d^2$$
$$-4a^2bd+4b^2ac-4c^2bd+4d^2ac.$$

Shew that $a+b+c+d$ must be a factor of this determinant.

(3) Let there be a determinant of the order $n+1$ in which all the constituents are equal to unity except those which form the diagonal series, and these are 1, $1+a_1$, $1+a_2$, ... $1+a_n$: the value of this determinant is $a_1a_2 \ldots a_n$.

For if any one of the quantities a_1, a_2, ... a_n vanishes the determinant vanishes, because it then has two rows identical; thus the determinant is divisible by $a_1a_2 \ldots a_n$. And the quotient of this division must be unity, as we see by considering the first element of the determinant.

(4) Let there be a determinant of the order n in which all the constituents are unity except those which form the diagonal series, and these are $1+a_1$, $1+a_2$, ... $1+a_n$: the value of this determinant is

$$a_1a_2 \ldots a_n \left\{1+\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right\}.$$

For if any one of the quantities a_1, a_2, ... a_n vanishes the determinant reduces to a case of the preceding example; and the term $a_1a_2 \ldots a_n$ is found by considering the first element of the determinant.

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XXXI. TRIGONOMETRICAL FORMULÆ.

404. We will give in the present Chapter a few propositions which bring the Theory of Equations into connexion with Trigonometry.

405. In Art. 272 of the *Plane Trigonometry* we have an expression for $\tan n\theta$ in powers of $\tan\theta$, supposing n to be a positive integer. Suppose now that $\tan n\theta$ is given, and we require $\tan\theta$. Clear of fractions, and thus we obtain the following equation of the n^{th} degree for determining $\tan\theta$;

$$\tan n\theta\left\{1-\frac{n(n-1)}{\lfloor 2}\tan^2\theta+\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\tan^4\theta-\ldots\right\}$$

$$=n\tan\theta-\frac{n(n-1)(n-2)}{\lfloor 3}\tan^3\theta$$

$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5}\tan^5\theta-\ldots\ldots(1).$$

Now the value of $\tan n\theta$ is not changed if we put instead of θ any one of the following angles :

$$\theta+\frac{\pi}{n},\quad \theta+\frac{2\pi}{n},\quad \theta+\frac{3\pi}{n},\ \ldots\ldots\ \theta+\frac{n-1}{n}\pi.$$

Hence we infer that the roots of (1) are

$$\tan\theta,\quad \tan\left(\theta+\frac{\pi}{n}\right),\quad \tan\left(\theta+\frac{2\pi}{n}\right)\ldots\ldots\tan\left(\theta+\frac{n-1}{n}\pi\right).$$

Let S denote the sum, and P the product, of the n quantities just expressed; then, by the aid of Art. 45, we may deduce from (1) values for S and P: but for this purpose we shall have to consider separately two cases.

I. Suppose n *even*. Then (1) becomes

$$\tan n\theta\left\{1-\frac{n(n-1)}{\lfloor 2}\tan^2\theta+\ldots\ldots+(-1)^{\frac{n}{2}}\tan^n\theta\right\}$$

$$=n\tan\theta-\frac{n(n-1)(n-2)}{\lfloor 3}\tan^3\theta+\ldots+n(-1)^{\frac{n-2}{2}}\tan^{n-1}\theta.$$

In this case in order to put our equation in the standard form, that is, with unity for the coefficient of the highest power of the unknown quantity, we must divide by $(-1)^{\frac{n}{2}}\tan n\theta$. Thus we obtain

$$S=\frac{n(-1)^{\frac{n-2}{2}}}{(-1)^{\frac{n}{2}}\tan n\theta}=-n\cot n\theta,\quad P=\frac{(-1)^n}{(-1)^{\frac{n}{2}}}=(-1)^{\frac{n}{2}}\ldots(2).$$

II. Suppose n *odd*. Then (1) becomes

$$\tan n\theta\left\{1-\frac{n(n-1)}{\lfloor 2}\tan^2\theta+\ldots+n(-1)^{\frac{n-1}{2}}\tan^{n-1}\theta\right\}$$

$$=n\tan\theta-\frac{n(n-1)(n-2)}{\lfloor 3}\tan^3\theta+\ldots+(-1)^{\frac{n-1}{2}}\tan^n\theta.$$

In this case in order to put our equation in the standard form, we must divide by $(-1)^{\frac{n-1}{2}}$. Thus we obtain

$$S=n\tan n\theta,\quad P=(-1)^{\frac{n-1}{2}}\tan n\theta\ldots\ldots\ldots\ldots(3).$$

406. Again, take equation (1) of Art. 405, and multiply by $\cot^n\theta$; thus we have

$$\tan n\theta\left\{\cot^n\theta-\frac{n(n-1)}{\lfloor 2}\cot^{n-2}\theta+\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\cot^{n-4}\theta-\ldots\right\}$$

$$=n\cot^{n-1}\theta-\frac{n(n-1)(n-2)}{\lfloor 3}\cot^{n-3}\theta$$

$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5}\cot^{n-5}\theta-\ldots$$

Divide by $\tan n\theta$, and we obtain an equation in the standard form for determining $\cot\theta$ when $\tan n\theta$ is given. Hence, proceeding as in Art. 404, we have

$$\cot\theta + \cot\left(\theta + \frac{\pi}{n}\right) + \cot\left(\theta + \frac{2\pi}{n}\right) + \ldots\ldots + \cot\left(\theta \quad \frac{n-1}{n}\pi\right)$$
$$= n\cot n\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

407. From equations (3) and (4) we see that, if n be any *odd* integer, the product

of $$\left\{\tan\theta + \tan\left(\theta + \frac{\pi}{n}\right) + \ldots\ldots + \tan\left(\theta + \frac{n-1}{n}\pi\right)\right\}$$

into $$\left\{\cot\theta + \cot\left(\theta + \frac{\pi}{n}\right) + \ldots\ldots + \cot\left(\theta + \frac{n-1}{n}\pi\right)\right\}$$
$$= n^2.$$

408. Propositions like those of Arts. 405 and 406 may be easily deduced from other formulæ of Trigonometry. We will give one more.

By Art. 287 of the *Plane Trigonometry* we have, when n is even,

$$\cos n\theta = 1 - \frac{n^2}{\lfloor\underline{2}}\sin^2\theta + \frac{n^2(n^2-2^2)}{\lfloor\underline{4}}\sin^4\theta - \ldots\ldots\ldots(5).$$

Let $\cos n\theta = 0$; then we may put for θ any one of the following n values:

$$\pm\frac{\pi}{2n}, \quad \pm\frac{3\pi}{2n}, \quad \pm\frac{5\pi}{2n}, \ldots\ldots\ldots \pm\frac{n-1}{2n}\pi.$$

Let $m = \frac{n}{2}$, and $x = \operatorname{cosec}^2\theta$; then dividing (5) by $\sin^n\theta$, we get

$$0 = x^m - \frac{n^2}{\lfloor\underline{2}}x^{m-1} + \frac{n^2(n^2-2^2)}{\lfloor\underline{4}}x^{m-2} - \ldots\ldots$$

The m values of x are

$$\operatorname{cosec}^2\frac{\pi}{2n}, \quad \operatorname{cosec}^2\frac{3\pi}{2n}, \ldots\ldots \operatorname{cosec}^2\frac{n-1}{2n}\pi;$$

hence, by Art. 45, the sum of these m quantities $= \frac{n^2}{\lfloor\underline{2}}$.

Thus, if n be an *even* integer,

$$\operatorname{cosec}^2 \frac{\pi}{2n} + \operatorname{cosec}^2 \frac{3\pi}{2n} + \dots\dots + \operatorname{cosec}^2 \frac{n-1}{2n}\pi = \frac{n^2}{2}.$$

409. We see by Art. 142 that any algebraical quantity has n different n^{th} roots. If then we have found an expression for the n^{th} root of an algebraical quantity that expression must be susceptible of n different values, unless some restriction has been introduced in our reasoning by virtue of which this multiplicity of values has been excluded. In other words, if two expressions are asserted to be equal, one of them must in general admit of as many values as the other.

Various Trigonometrical formulæ involving expansions were given by some of the older mathematicians, as for instance by Euler and Lagrange, which were not in accordance with the principle here stated, and which have been shewn to be inaccurate by Poinsot in a memoir, published in 1825, entitled *Recherches sur l'analyse des sections angulaires.* A memoir by Abel also treats on the same subject: see his *Oeuvres Complètes*, Vol. I. page 91. We will illustrate the point by considering one case, and will follow Poinsot, though his method is not very rigorous: for a more elaborate investigation we refer to Abel.

410. Let it be required to investigate a series for $(2\cos\theta)^n$ in terms of cosines or sines of multiples of θ.

The case in which n is a positive integer is treated in the *Plane Trigonometry*, Art. 280; we proceed to the more general proposition in which n is not restricted to be an integer, though it is assumed to be positive.

First suppose $\cos\theta$ *positive;* and let ρ denote the arithmetical value of $(2\cos\theta)^n$. Then we may put

$$(2\cos\theta)^n = 1^n\rho \quad\dots\dots\dots\dots(1).$$

Now $2\cos\theta = e^{\iota\theta} + e^{-\iota\theta}$, where ι is used for $\sqrt{-1}$; thus

$$(2\cos\theta)^n = (e^{\iota\theta} + e^{-\iota\theta})^n$$
$$= e^{n\iota\theta} + ne^{(n-2)\iota\theta} + \frac{n(n-1)}{\lfloor 2} e^{(n-4)\iota\theta} + \dots$$

But
$$e^{n\iota\theta} = \cos n\theta + \iota \sin n\theta,$$
$$e^{(n-2)\iota\theta} = \cos (n-2)\,\theta + \iota \sin (n-2)\,\theta,$$
and so on.

Hence
$$(2 \cos \theta)^n = c + \iota s,$$
where c stands for a certain series involving cosines, and s for a corresponding series involving sines.

Again,
$$1^n \rho = (\cos 2n\mu\pi + \iota \sin 2n\mu\pi)\,\rho,$$
where μ denotes any integer.

If then we were to equate this to $c + \iota s$ we should fall into the error against which we are warned in Art. 409. We observe that (1) remains unchanged when θ is increased by any even multiple of 2π. Let then c_m and s_m denote what c and s respectively become when in them θ is changed to $\theta + 2m\pi$. Then we may put
$$c_m + \iota s_m = (\cos 2n\mu\pi + \iota \sin 2n\mu\pi)\,\rho \qquad\text{..............(2).}$$

411. If we suppose n an integer, we have c_m and s_m coinciding with c and s respectively. Then equating the real and imaginary parts of (2) we obtain
$$c = \rho \text{ and } s = 0.$$
The former agrees with the result which is obtained and more closely discussed in the *Plane Trigonometry*, Art. 280.

412. But we now suppose that n is not an integer. The first point to be established is that in equation (2) we must take $m = \mu$. This point has sometimes been assumed; but Poinsot gives a reason for it in the following manner. Let us suppose θ to diminish without limit. Then it will be found that
$$c_m = \cos 2nm\pi \left\{1 + n + \frac{n(n-1)}{\lfloor 2} + \ldots\right\},$$
$$s_m = \sin 2nm\pi \left\{1 + n + \frac{n(n-1)}{\lfloor 2} + \ldots\right\}.$$

The series within the brackets may be regarded as equal to 2^n, by the Binomial Theorem; so that

$$c_m = 2^n \cos 2nm\pi = \rho \cos 2nm\pi,$$

$$s_m = 2^n \sin 2nm\pi = \rho \sin 2nm\pi.$$

Hence by (2) we get

$$\cos 2nm\pi + \iota \sin 2nm\pi = \cos 2n\mu\pi + \iota \sin 2n\mu\pi;$$

from which we conclude that $m = \mu$.

413. Thus, when n is not an integer, we have from (2)

$$\rho = \frac{c_m}{\cos 2nm\pi} \text{ and } \rho = \frac{s_m}{\sin 2nm\pi} \quad \ldots\ldots\ldots\ldots\ldots(3);$$

so that ρ may be expressed either in a series of cosines or in a series of sines.

414. If we put $m = 0$ in the first of equations (3) we obtain

$$\rho = c;$$

this coincides with Art. 280 of the *Plane Trigonometry* in form, and we see that it is true so long as $\cos\theta$ is positive.

Again, put $m = 0$ in the second of equations (3); then, since $\sin 2nm\pi$ vanishes with m, it follows that $s = 0$ so long as $\cos\theta$ is positive.

415. Let us now suppose that $\cos\theta$ is *negative*; and let ρ denote as before the arithmetical value of $(2\cos\theta)^n$. Then we may put

$$(2\cos\theta)^n = (-1)^n \rho.$$

Also $$(-1)^n = \cos(2\mu + 1)n\pi + \iota \sin(2\mu + 1)n\pi.$$

Hence instead of (2) we now obtain

$$c_m + \iota s_m = \{\cos(2\mu + 1)n\pi + \iota \sin(2\mu + 1)n\pi\}\rho \ldots\ldots\ldots(4).$$

416. If we suppose n an integer, we have c_m and s_m coinciding with c and s respectively. Then equating the real and imaginary parts of (4) we obtain

$$c = -\rho \text{ and } s = 0.$$

The former agrees substantially with the result obtained in the *Plane Trigonometry*, Art. 280.

417. But we now suppose that n is not an integer. We first shew, as in Art. 412, that $m = \mu$; then, as in Art. 413, we have

$$\rho = \frac{c_m}{\cos(2m+1)n\pi} \text{ and } \rho = \frac{s_m}{\sin(2m+1)n\pi} \quad \text{.......(5).}$$

If we put $m = 0$ in the first of equations (5) we obtain

$$\rho = \frac{c}{\cos n\pi}.$$

This shews that when $\cos\theta$ is negative the numerical value of $(2\cos\theta)^n$ is not equal to c, but to c divided by $\cos n\pi$; and this divisor is in general not equal to unity.

Also, if we put $m = 0$ in the second of equations (5) we obtain

$$\rho = \frac{s}{\sin n\pi};$$

thus s is in general not zero.

418. Return to equation (3) of Art. 413; and let us determine when ρ can be expressed by cosines only, and when by sines only.

We may suppose that n is equal to some integer together with a proper fraction; let this proper fraction in its lowest terms be denoted by $\frac{r}{s}$; then we shall not require to consider a value of m greater than $s - 1$.

If ρ can be expressed by cosines only, it is obvious we must have $\sin 2nm\pi = 0$; thus $m = 0$ is one value of m, and if s be even, $m = \frac{s}{2}$ is another.

If ρ can be expressed by sines only, it is obvious we must have $\cos 2nm\pi = 0$; therefore $\frac{2rm\pi}{s}$ must be an odd multiple of $\frac{\pi}{2}$; thus if r is odd and s a multiple of 4 we may take $m = \frac{s}{4}$ or $= \frac{3s}{4}$.

419. Again, take the equations (5) of Art. 417; and let us determine when ρ can be expressed by cosines only, and when by sines only.

Use the same notation as before.

If ρ can be expressed by cosines only, it is obvious we must have $\sin(2m+1)\,n\pi = 0$; therefore $\sin(2m+1)\frac{r}{s}\pi = 0$; thus, if s is odd, we can take $2m+1 = s$.

If ρ can be expressed by sines only, it is obvious we must have $\cos(2m+1)\,n\pi = 0$; therefore $\cos(2m+1)\frac{r}{s}\pi = 0$; thus, if r is odd, and $\frac{s}{2}$ an odd integer, we may take $2m+1 = \frac{s}{2}$ or $2m+1 = \frac{3s}{2}$.

420. Abel shews that the formulæ here obtained for $(2\cos\theta)^n$ hold when n has any positive value; and also when n has any negative value numerically less than unity, except for those values of θ which make $\cos\theta$ vanish.

421. We might investigate series for $(2\sin\theta)^n$ in the same way as for $(2\cos\theta)^n$; or we may deduce the results by putting $\frac{\pi}{2} - \theta$ for θ in the formulæ already obtained.

422. In the *Plane Trigonometry*, Chapter XXIII., the expression $x^{2n} - 2x^n \cos\alpha + 1$ is resolved into quadratic factors by a process which depends on De Moivre's Theorem, and which therefore involves the use of the imaginary symbol $\sqrt{-1}$. It has been

lately shewn by mathematicians that the result can be obtained without the use of the imaginary symbol; we will here reproduce the process employed for this purpose by Professor Adams in Vol. XI. of the Transactions of the *Cambridge Philosophical Society*.

423. The relation between successive values of $x^m + \frac{1}{x^m}$ corresponding to successive integral values of m is given by the formula

$$x^{m+1} + \frac{1}{x^{m+1}} = \left(x + \frac{1}{x}\right)\left(x^m + \frac{1}{x^m}\right) - \left(x^{m-1} + \frac{1}{x^{m-1}}\right);$$

when $m = 1$ this becomes

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)\left(x + \frac{1}{x}\right) - 2.$$

An exactly similar relation holds between the successive values of $2 \cos m\theta$; thus

$$2 \cos (m+1)\theta = (2 \cos \theta)(2 \cos m\theta) - 2 \cos (m-1)\theta;$$

when $m = 1$ this becomes

$$2 \cos 2\theta = (2 \cos \theta)(2 \cos \theta) - 2.$$

Now let $v_0, v_1, v_2, \ldots v_n$ be a series of quantities the successive terms of which are connected by the same relation as that which we have just seen to hold for the successive values of $x^m + \frac{1}{x^m}$ and of $2 \cos m\theta$; that is to say, let

$$v_{m+1} = v_1 v_m - v_{m-1}.$$

Also, as in those cases, let $v_0 = 2$, but let v_1 be any quantity whatever; thus we have

$$v_2 = v_1 v_1 - 2,$$

$$v_3 = v_1 v_2 - v_1 = v_1^3 - 3v_1,$$

and so on.

Hence it is obvious (1) that v_n is a definite integral function of v_1 of n dimensions, and that the coefficient of v_1^n is unity; (2) that if $v_1 = x + \frac{1}{x}$, then $v_n = x^n + \frac{1}{x^n}$; (3) that if $v_1 = 2\cos\theta$, then $v_n = 2\cos n\theta$.

Hence $v_n - 2\cos n\alpha$ will vanish when v_1 is equal to any one of the following n quantities:

$$2\cos\alpha,\quad 2\cos(\alpha+\beta),\quad 2\cos(\alpha+2\beta),\ \dots\ 2\cos(\alpha+\overline{n-1}\beta),$$

where β is put for $\frac{2\pi}{n}$. Therefore $v_n - 2\cos n\alpha =$

$$\left\{v_1 - 2\cos\alpha\right\}\left\{v_1 - 2\cos(\alpha+\beta)\right\}\left\{v_1 - 2\cos(\alpha+2\beta)\right\}\dots$$
$$\dots\left\{v_1 - 2\cos(\alpha+\overline{n-1}\beta)\right\}.$$

Now put $x + \frac{1}{x}$ for v_1; thus we obtain

$$x^n + \frac{1}{x^n} - 2\cos n\alpha =$$

$$\left\{x + \frac{1}{x} - 2\cos\alpha\right\}\left\{x + \frac{1}{x} - 2\cos(\alpha+\beta)\right\}\left\{x + \frac{1}{x} - 2\cos(\alpha+2\beta)\right\}\dots$$
$$\dots\left\{x + \frac{1}{x} - 2\cos(\alpha+\overline{n-1}\beta)\right\}.$$

This gives the required resolution.

Similarly if we put $2\cos\theta$ for v_1 we obtain $2\cos n\theta - 2\cos n\alpha =$

$$\left\{2\cos\theta - 2\cos\alpha\right\}\left\{2\cos\theta - 2\cos(\alpha+\beta)\right\}\left\{2\cos\theta - 2\cos(\alpha+2\beta)\right\}\dots$$
$$\dots\left\{2\cos\theta - 2\cos(\alpha+\overline{n-1}\beta)\right\}.$$

Hence we see that the two equations just found are particular cases of the general equation from which they have been derived; v_1 being in the former case numerically not less than 2, and in the latter case numerically not greater than 2. Two special examples may be formed by taking first $x = 1$ or $\theta = 0$, and then $x = -1$ or $\theta = \pi$.

424. Various theorems may be obtained by the aid of the imaginary symbol, which can be verified if required by other methods.

For example, we have by *Plane Trigonometry*, Art. 287, if n be an *even* integer,

$$\cos n\theta = 1 - \frac{n^2}{\lfloor 2} \sin^2\theta + \frac{n^2(n^2-2^2)}{\lfloor 4} \sin^4\theta - \frac{n^2(n^2-2^2)(n^2-4^2)}{\lfloor 6} \sin^6\theta - \ldots\ldots \text{ (1)};$$

and if n be an *odd* integer,

$$\sin n\theta = n \sin\theta - \frac{n(n^2-1)}{\lfloor 3} \sin^3\theta + \frac{n(n^2-1)(n^2-3^2)}{\lfloor 5} \sin^5\theta - \ldots (2).$$

Now substitute $\frac{1}{2\iota}\left(z - \frac{1}{z}\right)$ for $\sin\theta$, where ι denotes $\sqrt{-1}$; and for brevity put p for $z - \frac{1}{z}$.

Then from (1) we deduce, when n is *even*,

$$\frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = 1 + \frac{n^2}{2^2 \lfloor 2} p^2 + \frac{n^2(n^2-2^2)}{2^4 \lfloor 4} p^4 + \frac{n^2(n^2-2^2)(n^2-4^2)}{2^6 \lfloor 6} p^6 + \ldots\ldots \text{(3)};$$

and from (2) we deduce, when n is *odd*,

$$\frac{1}{2}\left(z^n - \frac{1}{z^n}\right) = \frac{n}{2} p + \frac{n(n^2-1)}{2^3 \lfloor 3} p^3 + \frac{n(n^2-1)(n^2-3^2)}{2^5 \lfloor 5} p^5 + \ldots\ldots (4).$$

Thus we obtain the algebraical identities (3) and (4).

These may be verified by the aid of Art. 244. For suppose $f(x) = x^2 - px - 1$, so that the roots of $f(x) = 0$ are of the form z and $-\frac{1}{z}$, and $z - \frac{1}{z} = p$. Then

$$\frac{xf'(x)}{f(x)} = \frac{x(2x-p)}{x^2 - px - 1} = 1 + \frac{x^2+1}{x^2 - px - 1} = 1 + \left(1 + \frac{1}{x^2}\right)\left(1 - \frac{1}{x^2} - \frac{p}{x}\right)^{-1}$$

$$= 1 + \left(1 + \frac{1}{x^2}\right)\left\{\left(1 - \frac{1}{x^2}\right)^{-1} + \left(1 - \frac{1}{x^2}\right)^{-2}\frac{p}{x} + \left(1 - \frac{1}{x^2}\right)^{-3}\frac{p^2}{x^2} + \ldots\right\}.$$

First suppose n *even;* then by Art. 244 we have $z^n + \left(-\frac{1}{z}\right)^n$, that is $z^n + \frac{1}{z^n}$, equal to the coefficient of $\frac{1}{x^n}$ in the expression just given. The terms involving *odd* powers of p will not furnish any part of the coefficient, since n is *even.*

Now the coefficient of $\frac{1}{x^n}$ in $\left(1+\frac{1}{x^2}\right)\left(1-\frac{1}{x^2}\right)^{-1}$ is $1+1$, that is 2.

The coefficient of $\frac{1}{x^n}$ in $\left(1+\frac{1}{x^2}\right)\left(1-\frac{1}{x^2}\right)^{-3}\frac{p^2}{x^2}$ is

$$\frac{p^2}{\underline{|2}}\left\{\left(\frac{n-2}{2}+1\right)\left(\frac{n-2}{2}+2\right)+\left(\frac{n-4}{2}+1\right)\left(\frac{n-4}{2}+2\right)\right\},$$

that is $\frac{p^2}{\underline{|2}}\left\{\frac{n}{2}\left(\frac{n}{2}+1\right)+\left(\frac{n}{2}-1\right)\frac{n}{2}\right\}$, that is $\frac{2p^2}{\underline{|2}}\frac{n^2}{2^2}$.

The coefficient of $\frac{1}{x^n}$ in $\left(1+\frac{1}{x^2}\right)\left(1-\frac{1}{x^2}\right)^{-5}\frac{p^4}{x^4}$ is

$$\frac{p^4}{\underline{|4}}\left\{\left(\frac{n-4}{2}+1\right)\left(\frac{n-4}{2}+2\right)\left(\frac{n-4}{2}+3\right)\left(\frac{n-4}{2}+4\right)\right.$$
$$\left.+\left(\frac{n-6}{2}+1\right)\left(\frac{n-6}{2}+2\right)\left(\frac{n-6}{2}+3\right)\left(\frac{n-6}{2}+4\right)\right\},$$

that is $\frac{p^4}{\underline{|4}}\left\{\left(\frac{n}{2}-1\right)\frac{n}{2}\left(\frac{n}{2}+1\right)\left(\frac{n}{2}+2\right)+\left(\frac{n}{2}-2\right)\left(\frac{n}{2}-1\right)\frac{n}{2}\left(\frac{n}{2}+1\right)\right\}$;

that is $\frac{p^4}{\underline{|4}}\left(\frac{n}{2}-1\right)\frac{n}{2}\left(\frac{n}{2}+1\right)\frac{2n}{2}$, that is $\frac{2p^4}{\underline{|4}}\frac{n^2(n^2-2^2)}{2^4}$.

And so on. Thus we obtain (3). Similarly by supposing n odd we obtain (4).

EXAMPLES.

I.

1. FIND the quotient and the remainder when

$$x^5+7x^4+3x^3+17x^2+10x-14$$

is divided by $x-4$.

2. Expand $(a+bx)^n$ in powers of x, and then obtain the *first derived function* of $(a+bx)^n$.

3. Shew that the equation $x^3+3x^2+x-6=0$ has one root and only one between 1 and 2.

II.

1. Find a root of the equation $x^4=+\sqrt{-1}$.

2. Find a root of the equation $x^6=-\sqrt{-1}$.

III.

1. Form the equation whose roots are 1, 1, 1, -1, -2.

2. Form the equation whose roots are $1\pm\sqrt{-2}$ and $2\pm\sqrt{-3}$.

3. Form the equation of the eighth degree one of whose roots is $\sqrt{2}+\sqrt{3}+\sqrt{-1}$.

4. Solve the following six equations in each of which *one* root is given:

(1) $x^3-x^2+3x+5=0$; $1-2\sqrt{-1}$.

(2) $x^4+4x^3+6x^2+4x+5=0$; $\sqrt{-1}$.

(3) $x^4+x^3-25x^2+41x+66=0$; $3+\sqrt{-2}$.

(4) $x^4+2x^3-4x^2-4x+4=0$; $\sqrt{2}$.

(5) $x^4-2x^3-5x^2-6x+2=0$; $2+\sqrt{3}$.

(6) $x^6-x^5-8x^4+2x^3+21x^2-9x-54=0$; $\sqrt{2}+\sqrt{-1}$.

5. Solve the equation $x^5 - x^4 + 8x^2 - 9x - 15 = 0$, one root being $\sqrt{3}$, and another $1 - 2\sqrt{-1}$.

6. The equation $x^3 - 4x^2 + x + c = 0$ has one root $= 3$; find c and the other roots. = 6

7. Find the sum of the reciprocals of the roots, the sum of the squares of the roots, and the sum of the squares of the reciprocals of the roots of $x^6 - 6x^5 + 40x^3 + 60x^2 - x - 1 = 0$.

8. The equation $x^4 - 21x^3 + 166x^2 - 546x + 580 = 0$, has roots of the form α, β, $\alpha + \beta + (\alpha - \beta)\sqrt{-1}$; solve the equation.

9. Find the sum of the cubes of the roots of a given equation.

10. Form the equation the roots of which α, β, γ, δ, are

$$\frac{1}{2}\left(1 + \sqrt{3} \pm \sqrt{2\sqrt{3}}\right), \text{ and } \frac{1}{2}\left(1 - \sqrt{3} \pm \sqrt{-2\sqrt{3}}\right);$$

and thence shew that $\dfrac{\alpha^2 + \beta^2}{\alpha\beta} + \dfrac{\alpha^2 + \gamma^2}{\alpha\gamma} + \ldots = 0$.

11. If a, b, c, ... are the roots of an equation, find the value of

$$\frac{a^2}{b^2} + \frac{a^2}{c^2} + \ldots + \frac{b^2}{a^2} + \frac{b^2}{c^2} + \ldots$$

12. Assuming that the arithmetic mean of any number of positive quantities is greater than their geometric mean, shew that if $p_1^2 - 2p_2$ is less than $np_n^{\frac{2}{n}}$, the equation has impossible roots.

13. If a, b, c, ... are the roots of an equation in its simplest form, shew that

$$(1 - p_2 + p_4 - \ldots)^2 + (p_1 - p_3 + p_5 - \ldots)^2 = (1 + a^2)(1 + b^2)(1 + c^2)\ldots$$

14. If a, b, c, ... are the roots of an equation in its simplest form, shew that

$$p_2^2 - 2p_1p_3 + 2p_4 = a^2b^2 + a^2c^2 + b^2c^2 + \ldots$$

IV.

1. Transform each of the following three equations into another the roots of which are formed by adding to the roots of the original equation the number assigned :

(1) $x^5 - 3x^4 - x^2 + 4 = 0$; 1. (2) $x^6 + x + 1 = 0$; 3.

(3) $x^5 + 4x^3 - x^2 + 11 = 0$; -3.

2. Transform each of the following four equations into another wanting the second term :

(1) $x^3 - 3x^2 + 4x - 4 = 0$. (2) $x^3 - 6x^2 + 12x + 19 = 0$.

(3) $x^4 - 8x^3 + 5 = 0$. (4) $x^5 + 5x^4 + 3x^3 + x^2 + x - 1 = 0$.

3. Transform each of the following four equations into two others each wanting the third term :

(1) $x^3 + 5x^2 + 8x - 1 = 0$. (2) $x^3 - 6x^2 + 9x - 10 = 0$.

(3) $x^4 - 8x^3 + 18x^2 - 15x + 14 = 0$. (4) $x^4 - 18x^3 - 60x^2 + x - 2 = 0$.

4. Transform the equation $x^3 + 2x^2 + \frac{1}{4}x + \frac{1}{9} = 0$ into another with integral coefficients, and unity for the coefficient of the first term.

5. Remove the second term and solve the equation

$$x^3 - 18x^2 + 157x - 510 = 0.$$

6. Transform each of the following two equations into another whose roots are the squares of the differences of its roots; and discuss the nature of the roots :

(1) $x^3 + 7x - 1 = 0$. (2) $x^3 - 6x + 6 = 0$.

7. Transform $x^4 - 12x^2 + 12x - 3 = 0$ into an equation whose roots shall be the reciprocals of those of the given equation; and then diminish the roots of the transformed equation by unity.

8. Shew that the equation $x^4 + x^2 - 8x - 15 = 0$ has two real roots of contrary signs, and that it cannot have more real roots; and that they lie between -2 and 3.

9. The roots of the equation $x^3+px^2+qx+r=0$ are denoted by a, b, c; transform the equation into others which have the roots assigned in the following fourteen cases:

(1) a^2, b^2, c^2. (2) $b+c, c+a, a+b$.

(3) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$. (4) $\frac{a}{bc}, \frac{b}{ca}, \frac{c}{ab}$.

(5) b^2c^2, c^2a^2, a^2b^2. (6) $\sqrt{(ka)}, \sqrt{(kb)}, \sqrt{(kc)}$.

(7) $\frac{1}{2}(b+c-a), \frac{1}{2}(c+a-b), \frac{1}{2}(a+b-c)$.

(8) $b+c+ka, c+a+kb, a+b+kc$.

(9) $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$.

(10) $bc+\frac{1}{a}, ca+\frac{1}{b}, ab+\frac{1}{c}$. (11) $b^2+c^2, c^2+a^2, a^2+b^2$.

(12) $\frac{b}{c}+\frac{c}{b}, \frac{c}{a}+\frac{a}{c}, \frac{a}{b}+\frac{b}{a}$. (13) $\frac{b^2+c^2}{b^2c^2}, \frac{c^2+a^2}{c^2a^2}, \frac{a^2+b^2}{a^2b^2}$.

(14) $b-c, c-b, c-a, a-c, a-b, b-a$.

10. The roots of the equation $x^3+qx+r=0$ are denoted by a, b, c; transform the equation into others which have the roots assigned in the following two cases:

(1) $\left(\frac{a}{b-c}\right)^2, \left(\frac{b}{c-a}\right)^2, \left(\frac{c}{a-b}\right)^2$.

(2) $ba+ac, cb+ba, ac+cb$.

11. If a, b, c denote the roots of $x^3-6x^2+11x-6=0$, form the equation whose roots are

$$\frac{1}{b^2+c^2}, \frac{1}{c^2+a^2}, \frac{1}{a^2+b^2}.$$

12. If a, b, c denote the roots of $x^3-2x^2+2=0$, form the equation whose roots are

$$\frac{b^3+c^3}{a^3}, \frac{c^3+a^3}{b^3}, \frac{a^3+b^3}{c^3}.$$

13. Shew that the third term of the equation

$$x^3+px^2+qx+r=0$$

cannot be removed if p^2 be less than $3q$.

14. Shew that the second and fourth terms of the equation

$$x^4+p_1x^3+p_2x^2+p_3x+p_4=0$$

can be removed by the same transformation if $8p_3=p_1(4p_2-p_1^2)$.

15. Solve the following two equations:

(1) $x^4+4x^3+7x^2+6x-10=0$. (2) $x^4+4x^3+3x^2-2x-6=0$.

16. Shew that the equation $x^3+4x^2+6x+3=0$ does not admit of the second and third terms being removed by the same transformation, but that it does if multiplied by x.

17. Shew that it is possible to remove the second and third terms of an equation of the n^{th} degree if

$n \times$ (sum of squares of roots) = square of sum of roots.

V.

1. Shew that the equation $x^5-4x^2+3=0$ has at least two imaginary roots.

2. Shew that the equation $x^7-2x^4+x^3-1=0$ has at least four imaginary roots.

3. What may be inferred respecting the roots of the following two equations?

(1) $x^{10}-5x^6+x^2-x-1=0$. (2) $x^{3n}-x^{2n}+x^n+x+1=0$.

VI.

1. Solve the following twenty equations, each of which has equal roots:

(1) $x^3-7x^2+16x-12=0$. (2) $x^3-3x^2-9x+27=0$.

(3) $x^3-x^2-8x+12=0$. (4) $x^3-5x^2-8x+48=0$.

(5) $x^3-x-\dfrac{2}{3\sqrt{3}}=0$. (6) $x^3-x+\dfrac{2}{3\sqrt{3}}=0$.

(7) $x^3+8x^2+20x+16=0$. (8) $x^4-\dfrac{1}{2}x+\dfrac{3}{16}=0$.

(9) $x^4 - 11x^2 + 18x - 8 = 0$.

(10) $x^4 - 2x^3 - x^2 - 4x + 12 = 0$.

(11) $x^4 - 7x^3 + 13x^2 + 3x - 18 = 0$.

(12) $x^4 - 4x^3 - 6x^2 + 36x - 27 = 0$.

(13) $x^4 + 13x^3 + 33x^2 + 31x + 10 = 0$.

(14) $2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0$.

(15) $x^4 + 16x^3 + 79x^2 + 126x + 98 = 0$.

(16) $8x^4 + 4x^3 - 18x^2 + 11x - 2 = 0$.

(17) $x^5 - x^4 - 2x^3 + 2x^2 + x - 1 = 0$.

(18) $x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6 = 0$.

(19) $x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0$.

(20) $x^6 - 3x^5 + 6x^3 - 3x^2 - 3x + 2 = 0$.

2. Find the condition that $x^n - px^2 + r = 0$ may have equal roots.

3. Shew that $x^4 + qx^2 + s = 0$ cannot have three equal roots.

4. If $x^n + p_1x^{n-1} + \dots + p_n = 0$ have two roots equal to a, shew that $p_1x^{n-1} + 2p_2x^{n-2} + \dots + np_n = 0$ has a root equal to a.

5. If $x^5 + qx^3 + rx^2 + t = 0$ has two equal roots, prove that one of them will be a root of the quadratic

$$x^2 - \frac{2q^2}{5r}x + \frac{5t}{3r} - \frac{4q}{15} = 0.$$

VII.

1. Find limits to the positive and negative roots of

$$x^6 - 5x^5 + x^4 + 12x^3 - 12x^2 + 1 = 0.$$

2. Write $x^4 - 8x^3 + 12x^2 + 16x - 39 = 0$ so as to shew that 6 is a superior limit of the positive roots.

3. Shew that the real roots of the following six equations lie between the limits respectively assigned:

(1) $x^4 - x^3 + 4x^2 - 3x + 1 = 0$; $\frac{1}{4}$ and 1.

(2) $x^4 + x^3 - 10x^2 - x + 15 = 0$; -4 and 3.

(3) $x^5 + 5x^4 + x^3 - 16x^2 - 20x - 16 = 0$; -5 and 3, by Art. 92.

(4) $(x^3 - 26)(x^2 + 5x + 1) + 60x = 0$; -5 and 3.

(5) $(x^2 - 4x - 2)^2 - 43 = 0$; -2 and 6.

(6) $x^5 + x^4 + x^2 - 25x - 36 = 0$; -3 and 3, by Art. 92.

4. Find by Newton's method limits to the roots of the following five equations:

(1) $x^4 - x^3 - 5x^2 + 8x - 9 = 0$. (2) $x^4 - 5x^2 + 6x - 1 = 0$.

(3) $x^4 - x^3 + 4x^2 + x - 4 = 0$. (4) $x^4 - 5x^3 + 11x^2 - 20 = 0$.

(5) $x^4 - 2x^3 - 3x^2 - 15x - 3 = 0$.

5. Prove that $x^5 + 5x^4 - 20x^2 - 19x - 2 = 0$ has one root between 2 and 3, but none greater than 3, and one root between -5 and -4, but none less than -5.

6. Apply the method of Art. 102 to find the number and situation of the real roots of the following six equations:

(1) $x^3 - 12x + 17 = 0$. (2) $x^4 - 32x + 20 = 0$.

(3) $x^3 - 3x + 3 = 0$. (4) $4x^3 + 9x^2 - 12x + 2 = 0$.

(5) $x^7 - a^5x^2 + c^7 = 0$. (6) $x^{2n} - px^2 + r = 0$.

7. Shew that the equation $3x^4 + 8x^3 - 6x^2 - 24x + r = 0$ will have four real roots if r is less than -8 and greater than -13, and two real roots if r is greater than -8 and less than 19, and no real root if r is greater than 19.

VIII.

1. Obtain the commensurable roots of the following twelve equations:

(1) $x^3 - 106x - 420 = 0$. (2) $x^3 - 9x^2 + 22x - 24 = 0$.

(3) $x^3 - 2x^2 - 25x + 50 = 0$. (4) $2x^3 - 3x^2 + 2x - 3 = 0$.

(5) $3x^3-2x^2-6x+4=0.$ (6) $3x^3-26x^2+34x-12=0.$

(7) $x^4-2x^3+8x-16=0.$ (8) $x^4-x^3-13x^2+16x-48=0.$

(9) $x^4-x^3-x^2+19x-42=0.$ (10) $x^4+8x^3-7x^2-49x+56=0.$

(11) $x^5-3x^4-9x^3+21x^2-10x+24=0.$

(12) $x^6-7x^5+11x^4-7x^3+14x^2-28x+40=0.$

2. The coefficients of the equation $f(x)=0$ are all integers: shew that if $f(0)$ and $f(1)$ are both odd numbers the equation can have no integral roots.

IX.

1. Solve the following four equations each of which has two roots of the form $a, -a$:

(1) $x^4-2x^3-2x^2+8x-8=0.$ (2) $x^4+3x^3-7x^2-27x-18=0.$

(3) $x^4+3x^3+2x^2+9x-3=0.$ (4) $x^4+x^3-11x^2-9x+18=0.$

2. Solve the following four equations in each of which the roots are in Arithmetical Progression:

(1) $x^3-6x^2+11x-6=0.$ (2) $x^3-9x^2+23x-15=0.$

(3) $x^4-8x^3+14x^2+8x-15=0.$ (4) $x^4+4x^3-4x^2-16x=0.$

3. Solve the following six equations in which certain conditions relative to the roots are given:

(1) $3x^3-2x^2-27x+18=0$; product of two roots is 2.

(2) $x^4-3x^2-6x-2=0$; product of two roots is -1.

(3) $x^4-4x^3+5x^2-16x+4=0$; product of two roots is 1.

(4) $2x^4-5x^3+11x^2-11x+6=0$; product of two roots is 1.

(5) $x^4-45x^2-40x+84=0$; difference of two roots is 3.

(6) $x^5-7x^4+15x^3-15x^2+14x-8=0$; one root double another.

4. Solve the following six equations in which the roots are of the forms respectively assigned:

(1) $x^3-10x^2+27x-18=0$; a, $3a$, $6a$.

(2) $x^4-10x^3+35x^2-50x+24=0$; $a+1$, $a-1$, $b+1$, $b-1$.

(3) $6x^4-43x^3+107x^2-108x+36=0$; a, b, $\frac{a}{b}$, $\frac{b}{a}$.

(4) $x^5+8x^4+5x^3-50x^2-36x+72=0$; a, $2a$, b, $2b$, $a+b$.

(5) $x^6-4x^5+10x^4-16x^3+44x^2-16x+56=0$; $a\pm\sqrt{2}\pm\sqrt{b}$, $\pm\sqrt{c}$.

(6) $x^6-12x^4-2x^3+37x^2+10x-10=0$; $1\pm\sqrt{a}$, $b\pm\sqrt{2}$, $\pm\sqrt{c}$.

5. Solve the following two pairs of equations, each pair having a root in common:

(1) $x^3-3x^2-16x-12=0$; $x^3-7x^2+5x+13=0$.

(2) $x^3-3x^2+11x-9=0$; $x^3-5x^2+11x-7=0$.

6. Solve $x^3-7x^2+36=0$, and $x^3-3x^2-10x+24=0$, the former of which has a root equal to three times one of the roots of the latter.

7. Solve the following two equations which have two roots in common:

$$x^4-2x^3-7x^2+26x-20=0;\quad x^4+4x^3-2x^2-12x+8=0.$$

8. Find in terms of m and a the roots of the equation

$$x^4+pax^3+(m^2+m)\,a^2x^2+qa^3x+a^4=0,$$

which are in Geometrical Progression; and determine p and q in terms of m and a.

X.

1. Solve the following ten reciprocal equations:

(1) $x^4-2x^3+3x^2-2x+1=0$. (2) $x^4+4x^3-5x^2+4x+1=0$.

(3) $2x^4-5x^3+6x^2-5x+2=0$. (4) $x^4+4x^3-10x^2+4x+1=0$.

(5) $x^5-2x^4-19x^3-19x^2-2x+1=0$. (6) $x^5-4x^4+x^3+x^2-4x+1=0$.

(7) $6x^5-11x^4-33x^3+33x^2+11x-6=0$.

(8) $2x^6-5x^5+4x^4-4x^2+5x-2=0$.

(9) $8x^6-16x^4-25x^3-16x^2+8=0$. (10) $1+x^5=a\,(1+x)^5$.

2. Obtain roots of the following four equations, and depress the equations:

(1) $x^7 - 2x^5 + x^4 + x^3 - 2x^2 + 1 = 0.$

(2) $x^7 + 2x^6 - 8x^5 - 7x^4 - 7x^3 - 8x^2 + 2x + 1 = 0.$

(3) $x^8 + 2x^7 + 3x^6 + 2x^5 - 2x^3 - 3x^2 - 2x - 1 = 0.$ (4) $x^{10} - 1 = 0.$

3. Exhibit the roots of $x^4 + px^2 + 1 = 0$ in the form

$$a,\ b,\ \frac{1}{a},\ \frac{1}{b}.$$

4. If $a, b, c, \ldots$ denote the roots of the recurring equation

$$x^n + px^{n-1} + qx^{n-2} + \ldots + qx^2 + px + 1 = 0,$$

$$\frac{a^2}{b^2} + \frac{a^2}{c^2} + \ldots + \frac{b^2}{a^2} + \frac{b^2}{c^2} + \ldots + \frac{c^2}{a^2} + \ldots = (p^2 - 2q)^2 - n.$$

5. In the recurring equation $x^{2n} - px^{2n-1} + \ldots = 0$, if the terms are alternately positive and negative and p less than $2n$, the roots cannot be all real.

XI.

1. Solve the following three equations:

(1) $x^6 - 1 = 0.$ (2) $x^8 - 1 = 0.$ (3) $x^8 + 1 = 0.$

2. Shew that the factors of $a^3 + b^3 + c^3 - 3abc$ are of the form $a + bi + ci^2$, where $i^3 - 1 = 0$.

3. Shew that the factors of

$$a^2(a^2 - 4bd - c^2) - b^2(b^2 - 4ac - d^2) + c^2(c^2 - 4bd - a^2) - d^2(d^2 - 4ac - b^2)$$

are of the form $a + bk + ck^2 + dk^3$, where $k^4 - 1 = 0$.

XII.

1. Solve the following eight equations:

(1) $x^3 - 3x - 2 = 0.$ (2) $x^3 - 9x - 28 = 0.$

(3) $x^3 - x + 6 = 0.$ (4) $x^3 + 3x = \frac{3}{2}.$

(5) $3x^3 - 6x^2 - 2 = 0.$ (6) $x^3 - 15x^2 - 33x + 847 = 0.$

(7) $x^3 + 6ax^2 = 36a^3.$ (8) $x^3 - 3(a^2 + b^2)x = 2a(a^2 - 3b^2).$

2. Determine the relation between q and r necessary in order that the equation $x^3 + qx + r = 0$ may be put into the form

$$x^4 = (x^2 + ax + b)^2;$$

and hence solve the equation $8x^3 - 36x + 27 = 0$.

3. If the roots of the equation $x^3 + px^2 + qx + r = 0$ are in Geometrical Progression, $rp^3 = q^3$. Hence solve the equation

$$x^3 - x^2 + 2x - 8 = 0.$$

4. If the roots of the equation $x^3 + qx + r = 0$ are diminished by h, shew that the transformed equation will have its roots in Geometrical Progression if h be such that $27rh^3 - 9q^2h^2 - q^3 = 0$.

5. If the roots of the equation $x^3 + 3px^2 + 3qx + r = 0$ are in Harmonical Progression, $2q^3 = r(3pq - r)$.

6. If the roots of the equation $x^3 + 3px^2 + 3qx + r = 0$ are in Harmonical Progression, the equation $rx^2 + 2q^2x + qr = 0$ contains the greatest and least of them.

7. The impossible roots of $x^3 + qx + r = 0$ being put under the form $\alpha \pm \beta\sqrt{-1}$, shew that $\beta^2 = 3\alpha^2 + q$.

8. If r, $\alpha + \sqrt{\beta}$, $\alpha - \sqrt{\beta}$, are the three roots of the equation $x^3 + p_1x^2 + p_2x + p_3 = 0$, of which r is real, and if $x^3 + m_1x^2 + m_2x = 0$ is the equation resulting from the diminution of all the roots by r, shew that $\alpha = -\frac{m_1}{2} + r$ and $\beta = -\frac{1}{4}(m_2 + 3p_2 - p_1^2)$.

9. Reduce the equation $x^3 + px^2 + qx + r = 0$ to the form $y^3 - 3y + m = 0$, by assuming $x = ay + b$; and solve this equation by assuming $y = z + \frac{1}{z}$. Hence shew that if the original equation has equal roots,

$$4(p^2 - 3q)^3 = (2p^3 - 9pq + 27r)^2.$$

10. If the roots of the equation $x^3 + px^2 + qx + r = 0$ are in Harmonical Progression, so also are the roots of the equation

$$(pq - r)y^3 - (p^3 - 2pq + 3r)y^2 + (pq - 3r)y - r = 0.$$

XIII.

1. Solve the following four equations :

(1) $x^4+4x^3+3x^2-44x-84=0$. (2) $x^4-6x^2-8x-3=0$.

(3) $x^4-12x^3+49x^2-78x+40=0$.

(4) $x^4-2ax^3+(a^2-2b^2)x^2+2ab^2x-a^2b^2=0$. (Art. 192.)

2. If $r^2-p^2s=0$ the equation $x^4+px^3+qx^2+rx+s=0$ may be solved as a quadratic.

3. If s and p are positive, and $27p^4$ less than $256s$, the roots of the equation $x^4+px^3+s=0$ are all imaginary.

4. Assuming that the equation $x^4+qx^2+rx+s=0$ has roots of the form $\alpha\pm\beta\sqrt{-1}$, shew that the values of α and β may be found by the equations

$$64\alpha^6+32q\alpha^4+(4q^2-16s)\alpha^2-r^2=0,\quad \beta^2=\alpha^2+\frac{q}{2}+\frac{r}{4\alpha}.$$

XIV.

1. Apply Sturm's Theorem to determine the situation of the real roots of the following fivo equations in which the values of some of Sturm's functions are assigned :

(1) $x^4-4x^3-3x+23=0$; $f_3(x)=-491x+1371$, $f_4(x)=-$.

(2) $x^4-4x^3+x^2+6x+2=0$; $f_2(x)=5x^2-10x-7$, $f_3(x)=x-1$, $f_4(x)=+$.

(3) $x^4+x^3+x-1=0$; $f_2(x)=3x^2-12x+17$, Art. 199.

(4) $x^5-2x^4+x^3-8x+6=0$; $f_3(x)=16x^2-23x+9$.

(5) $x^5+5x^4-20x^2-19x-2=0$; $f_2(x)=20x^3+60x^2+36x-9$, $f_3(x)=96x^2+187x+67$, $f_4(x)=43651x+54571$, $f_5(x)=+$.

2. Apply Sturm's Theorem to shew that each of the following two equations has only one real root; and determine its situation :

(1) $x^3+6x^2+10x-1=0$. (2) $x^3-6x^2+8x+40=0$.

3. Determine the situation of the positive roots of the equation $x^6 - 2x^3 + 3x^2 - 5x - 1 = 0$, having given

$$f_2(x) = 6x(x-1)^2 + 19x + 6.$$

4. Apply Sturm's Theorem to the following four equations:

(1) $x^3 + x^2 - 2x - 1 = 0$. (2) $x^3 - 4x^2 - 4x + 20 = 0$.

(3) $x^4 + 2x^2 - 4x + 10 = 0$. (4) $x^n - x + 1 = 0$.

XV.

1. Shew that the equation

$$x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101 = 0$$

has all its real roots between -10 and 10, that it has one real root between -10 and -1, one between -1 and 0, no root between 0 and 1, and one at least between 1 and 10.

2. Apply Fourier's Theorem to the equation

$$x^4 + 3x^3 + 7x^2 + 10x + 1 = 0.$$

XVI.

1. Approximate by Lagrange's method to the positive root of the equation $3x^2 - 4x - 1 = 0$.

2. Approximate by Lagrange's method to the root of the equation $x^4 + x^3 - 2x^2 - 3x - 3 = 0$, which lies between 1 and 2.

XVII.

1. Apply Newton's method to calculate the root which is situated between the assigned limits in the following five equations:

(1) $x^3 - 4x - 12 = 0$; root between 2 and 3.

(2) $x^3 - 4x^2 - 7x + 24 = 0$; root between 2 and 3.

(3) $x^3 - 24x + 44 = 0$; root between 3·2 and 3·3.

(4) $x^3 - 15x - 5 = 0$; root between 4 and 4·1.

(5) $x^4 - 8x^3 + 12x^2 + 8x - 4 = 0$; root between 0 and 1.

2. Apply Newton's method to calculate a root of the following two equations:

(1) $x^3 + 3x - 5 = 0$. (2) $x^3 - 3x^2 - 3x + 20 = 0$.

XVIII.

1. Apply Horner's method to calculate the root which is situated between the assigned limits in the following three equations:

(1) $x^3 + 10x^2 + 6x - 120 = 0$; root between 2 and 3.

(2) $x^4 - 2x^3 + 21x - 23 = 0$; root between 1 and 2.

(3) $x^4 - 5x^3 + 3x^2 + 35x - 70 = 0$; root between 2 and 3.

2. Solve the equation $x^3 - 17 = 0$ by Horner's method.

3. Calculate the real roots of the following four equations by Horner's method:

(1) $x^3 + x - 3 = 0$. (2) $x^3 + 2x - 20 = 0$.

(3) $3x^3 + 5x - 40 = 0$. (4) $x^3 + 10x^2 + 8x - 120 = 0$.

XIX.

1. Find the value of the following seven symmetrical functions of the roots a, b, c of the equation $x^3 + px^2 + qx + r = 0$:

(1) $(a + b + ab)(b + c + bc)(c + a + ca)$.

(2) $(a + b - 2c)(b + c - 2a)(a + c - 2b)$.

(3) $\Sigma (a + b)^2 (a + c)$. (4) $\Sigma (a + b - 2c)(b + c - 2a)$.

(5) $\Sigma \dfrac{ab}{a + b}$. (6) $\Sigma \dfrac{a^2}{bc}\left(1 + \dfrac{b}{a}\right)^2 \left(1 + \dfrac{c}{a}\right)^2$.

(7) $(b - c)^2 (c - a)^2 (a - b)^2$.

2. If a, b, c, d are the roots of the equation

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

find the value of $\Sigma (a + b)(c + d)$.

3. In the equation $x^n+p_1x^{n-1}+\ldots+p_{n-1}x+p_n=0$, supposing the roots to be $a, b, c, \ldots l$ find

(1) Σa^2b. (2) $\Sigma(a+b)(a+c)\ldots(a+l)$.

(3) $\Sigma\dfrac{(a+b)^2}{ab}$. (4) $\Sigma\dfrac{a^2}{b}$.

4. Form the equation the roots of which are the squares of the sums of every three roots of the equation $x^4+px^3+rx+s=0$. Also form the equation the roots of which are the sums of the squares of every three roots of the same equation.

5. If the equation $x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\ldots+p_n=0$ is transformed into another of which the roots are the sum of every pair of roots of the original equation, find the first three coefficients of the transformed equation.

XX.

1. Transform the following three equations into others whose roots are the squares of the differences of their roots:

(1) $x^3-4x+2=0$. (2) $x^4+4x+3=0$. (3) $x^4+1=0$.

2. Eliminate x from the equations

$$ax^2+bx+c=0, \qquad a'x^2+b'x+c'=0.$$

XXI.

1. Find the sum of the assigned powers of the roots of the following five equations:

(1) $x^4-x^3-19x^2+49x-30=0$; the cubes.

(2) $x^5-3x^3-5x+1=0$; the fourth powers.

(3) $x^5-2x^4-22x^3-28x^2+72x+144=0$; the cubes.

(4) $x^4+2x+1=0$; the inverse squares.

(5) $x^3-x-1=0$; the sixth powers.

2. If $a, b, c,\ldots$ are the roots of $x^n - 1 = 0$, find $\Sigma a^m b^p$.

3. If the sum of the r^{th} powers of the roots of the equation $x^n + x + 1 = 0$ be expressed by S_r, and the sum of the r^{th} powers of their reciprocals by Σ_r, prove that

$$S_{n-1} - S_n = 1, \text{ and } \Sigma_{n-1} - \Sigma_n = n - 2(-1)^n.$$

4. In the equation $x^n - x^2 + 1 = 0$, find Σa^{n-3}, Σa^{n-2}, and Σa^n; supposing n greater than 3.

5. Find the sums of the r^{th} and $(2n)^{\text{th}}$ powers of the roots of the equation $x^{2r} - px^r + q = 0$, supposing n greater than r.

XXII.

1. Solve the equations

$$\left.\begin{array}{l}(y-1)x^2 + yx + y^2 - 2y = 0\\(y-1)x + y = 0\end{array}\right\}.$$

2. Solve the equations

$$\left.\begin{array}{l}(y-1)x^3 + y(y+1)x^2 + (3y^2 + y - 2)x + 2y = 0\\(y-1)x^2 + y(y+1)x + 3y^2 - 1 = 0\end{array}\right\}.$$

3. Shew that the following equations have no solution:

$$\left.\begin{array}{l}yx^3 - (y^3 - 3y - 1)x + y = 0\\x^2 - y^2 + 3 = 0\end{array}\right\}.$$

XXIII.

1. Find the first term of each value of y when expanded in descending powers of x from the equation

$$y^4x - y^3x^2 + 3yx^3 - y^2x + 4y - 2x = 0.$$

2. Find the first term of each value of y when expanded in ascending powers of x from the equation

$$x^{12} + x^{14} + x^{11}y - x^8y^2 + 2x^7y^3 - x^4y^4 + y^6 - 3xy^9 + x^{14}y^{13} = 0.$$

MISCELLANEOUS EXAMPLES.

1. If there be n quantities a, b, c..., and if n functions of them be taken of the form

$$\frac{(x-b)(x-c)\dots}{(a-b)(a-c)\dots},$$

shew that the sum of these functions is unity.

2. Remove the term which involves the cube of the unknown quantity from the equation

$$x^6+5x^4+200x^3-11x+6=0.$$

3. Shew how to transform an equation which has both changes and continuations of signs (1) into one which has only continuations of sign, (2) into one which has only changes of sign.

4. If p and q are positive, the equation $x^{2n}-px^{2r}+q=0$ has four different real roots or none according as $\left(\frac{rp}{n}\right)^n$ is greater or less than $\left(\frac{rq}{n-r}\right)^{n-r}$; and it has two pairs of equal roots if $\left(\frac{rp}{n}\right)^n=\left(\frac{rq}{n-r}\right)^{n-r}$.

5. If $-p_{n-q}x^{n-q}$, $-p_{n-r}x^{n-r}$, $-p_{n-s}x^{n-s}$, ... are the negative terms of an equation of the n^{th} degree, then the greatest root of the equation will be less than the sum of the two greatest of the quantities $(p_{n-q})^{\frac{1}{q}}$, $(p_{n-r})^{\frac{1}{r}}$, $(p_{n-s})^{\frac{1}{s}}$, ...

6. If k be the last term of an equation of the n^{th} degree whose roots are in geometrical progression, shew that $-k^{\frac{1}{n}}$ is a root, if n be odd. Shew that, in a similar manner, one root of an equation of an odd degree whose roots are either in arithmetical or harmonical progression may be found.

7. Find the greatest common measure of x^3-x^2-3x-1 and $x^5-6x^4+7x^3+7x^2-6x-3$.

Solve the equation $x^5-6x^4+7x^3+7x^2-6x-3=0$.

8. Diminish by h the roots of the equation

$$x^4 + qx^2 + rx + s = 0;$$

give such a value to h that the roots of the transformed equation may be of the form a, $\frac{m}{a}$, b, $\frac{m}{b}$, and shew how this equation may be solved. Example. $x^4 - 2x^2 + 16x + 1 = 0$.

9. Shew by the process for extracting the square root of an algebraical expression that the equation $x^4 + px^3 + qx^2 + rx + s = 0$ can be immediately reduced to quadratics if $p^2s - 4qs + r^2 = 0$, or if $p^3 - 4pq + 8r = 0$.

10. Prove that the equation $x^4 + \frac{3}{2}qx^2 + rx + s = 0$ cannot have all its roots real if $q^3 + r^2$ is positive.

11. If $f(x)$ be a rational integral function of x, either $f(x) = 0$ or $f'(x) = 0$ has certainly a real root.

12. Shew how to find the value of the semi-symmetrical function $a^2b + b^2c + c^2a$ of the roots of a cubic equation.

13. Let $a, b, c, \ldots k$ denote the roots of the equation $\phi(x) = 0$, which is of the n^{th} degree and in its simplest form, and suppose these roots all unequal: shew that the expression

$$\frac{a^r}{\phi'(a)} + \frac{b^r}{\phi'(b)} + \frac{c^r}{\phi'(c)} + \ldots + \frac{k^r}{\phi'(k)}$$

is equal to unity if $r = n - 1$, and is zero if r is zero or any positive integer less than $n - 1$.

Shew also, that if $r = -1$ the expression $= \frac{(-1)^{n-1}}{abc \ldots k}$.

14. If $\phi(x) = x^n - 1$, and $a, b, c, \ldots$ are the roots of $\phi(x) = 0$, shew that

$$\frac{nx^{n-1}}{x^n - 1} = \frac{1}{x - a} + \frac{1}{x - b} + \frac{1}{x - c} + \ldots$$

15. Shew that the integral part of $\frac{1}{\sqrt{3}}(\sqrt{3} + \sqrt{5})^{2n-1}$ is divisible by 2^n.

ANSWERS.

I. 1. $x^4+11x^3+47x^2+205x+830$; remainder 3306.

II. 1. $\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^{\frac{1}{2}}+\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{\frac{1}{2}}\sqrt{-1}$. 2. $\frac{1}{\sqrt{2}}+\frac{\sqrt{-1}}{\sqrt{2}}$.

III. 7. -1; 36; 121. 8. $\alpha=5$; $\beta=2$.

9. $-p_1^3+3p_1p_2-3p_3$. 10. $x^4-2x^3-2x+1=0$; then see Art. 48.

11. $(p_1^2-2p_2)\frac{p^2_{n-1}-2p_np_{n-2}}{p_n^2}-n$. 13. In the identity of Art. 45 substitute successively $\sqrt{-1}$ and $-\sqrt{-1}$ for x.

IV. 5. The roots are 6, $6\pm7\sqrt{-1}$. 7. $y^4-2y^2+\frac{2}{3}=0$.

8. See Arts. 22 and 50. 15. Apply Example 14.

VI. 1. (15) -7 is a root. (16) $\frac{1}{2}$ is a root. (17) The root 1 occurs three times. (18) The root -1 occurs three times. (19) 2 and 3 are roots. (20) The roots 1 and -1 are repeated.

3. Denote the root which is repeated by a, and the other by b; then the left-hand member of the proposed equation must be identical with $(x-a)^3(x-b)$; then we may equate coefficients.

VII. 7. The roots of $f'(x)=0$ are -2, -1, 1; use Art. 102.

VIII. 1. (4) $\frac{3}{2}$. (6) $\frac{2}{3}$.

IX. 2. (3) -1, 1, 3, 5. (4) -4, -2, 0, 2.

3. (1) 3, $\frac{2}{3}$. (2) $1\pm\sqrt{2}$. (3) $2\pm\sqrt{3}$. (4) $\frac{1}{4}(3\pm\sqrt{-7})$.
(5) -2, 1. (6) 1, 2.

4. (1) $a=1$. (2) $a=3$, $b=2$. (3) $a=2$, $b=3$. (4) $a=1$, $b=-3$.
(5) $a=1$, $b=-3$, $c=-2$. (6) $a=3$, $b=-1$, $c=5$.

5. (1) -1. (2) 1. 6. The roots are 6 and 2.

7. The common roots are given by $x^2+2x-4=0$.

8. Denote the roots by $\frac{\alpha}{\beta^3}$, $\frac{\alpha}{\beta}$, $\alpha\beta$, $\alpha\beta^3$; equate their product to a^4, and the sum of the products of every pair to $(m^2+m)\,a^2$. It may be shewn that p must be equal to q.

XII. 1. (1) 2. (2) 4. (3) -2. (4) $2^{\frac{1}{3}}-2^{-\frac{1}{3}}$. (5) $\frac{1}{3}\left(2^{\frac{1}{3}}+2+2^{\frac{5}{3}}\right)$.

(6) The root 11 occurs twice. (7) $\dfrac{6a}{2^{\frac{2}{3}}+2^{\frac{1}{3}}}$. (8) $2a$.

XIII. 1. (1) 3, -2. (2) The root -1 is repeated.
(3) Diminish the roots by 3, then the biquadratic can be solved.

XIV. 1. (1) A root between 2 and 3, another between 3 and 4, and two impossible roots. (2) Two roots between 0 and -1, and two between 2 and 3.

XVIII. 1. (1) 2·833066480704857...
(2) 1·157451508098991... (3) 2·6457513110645905...
2. 2·5712815906582353...
3. (1) 1·2134116627622296... (2) 2·4695456501065939...
(3) 2·13781194169747... (4) 2·76834546088879...

XIX. 1. (1) $(r-q)^2+p\,(r-q)+r$. (2) $2p^3-9pq+27r$.

(3) $-2p^3+pq-3r$. (4) $9q-3p^2$. (5) $\dfrac{q^2+rp}{r-pq}$.

(6) $\dfrac{q^3}{r^2}-\dfrac{(q-p^2)\,p}{r}+3$. (7) $\frac{4}{3}(3q-p^2)(3pr-q^2)-\frac{1}{3}(pq-9r)^2$.

2. $2q$. 3. (1) $3p_3-p_1p_2$. (2) If we denote the equation by $f(x)=0$, the proposed expression following the symbol Σ becomes $\dfrac{f(-a)}{2a\,(-1)^n}$. Hence the required sum is

$$\frac{1}{2}\left\{S_{n-1}-p_1S_{n-2}+p_2S_{n-3}-\ldots+(-1)^n p_nS_{-1}\right\}.$$

(3) $\dfrac{p_1p_{n-1}}{p_n}+n^2-2n$. (4) $p_1-\dfrac{(p_1^{\,2}-2p_2)\,p_{n-1}}{p_n}$.

5. Let the transformed equation be

$$x^m + q_1 x^{m-1} + q_2 x^{m-2} + q_3 x^{m-3} + \dots = 0;$$

then $m = \frac{n(n-1)}{2}$. Find the sums of the powers of the roots of the transformed equation, and then the coefficients by Art. 244. We shall get $q_1 = (n-1)p_1$; $q_2 = \frac{(n-1)(n-2)}{2} p_1^2 + (n-2)p_2$;

$$q_3 = \frac{(n-1)(n-2)(n-3)}{\underline{|3}} p_1^3 + (n-2)^2 p_1 p_2 + (n-4) p_3.$$

XXII. 1. The solutions are given by

$$y^2 - 2y = 0 \text{ and } (y-1)x + y = 0.$$

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2. The solutions are given by

$$y^2 - 1 = 0 \text{ and } (y-1)x + 2y = 0.$$

XXIII. 1. $y = x + \dots$; $y = \pm\sqrt{3x} + \dots$; $y = \frac{2}{3x^2} + \dots$

2. Six values of the form $y = x^2(u + U)$, where u is to be determined from $1 - u^2 - u^4 + u^6 = 0$; three values of the form $y = x^{-\frac{1}{3}}(u + U)$, where u is to be determined from $1 - 3u^3 = 0$; and four values of the form $y = x^{-\frac{13}{4}}(u + U)$, where u is to be determined from $3 - u^4 = 0$.

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MISCELLANEOUS EXAMPLES.

1. Call the sum $\phi(x)$; then shew that $\phi(x) - 1$ is identically zero by Art. 39.

2. $y^6 - 12y^5 + 65y^4 - 840y^2 + 2037y - 1428 = 0$.

15. Form a quadratic with roots $\sqrt{3} + \sqrt{5}$ and $\sqrt{3} - \sqrt{5}$; then use Art. 261; see also *Algebra*, Art. 526.

CAMBRIDGE: PRINTED AT THE UNIVERSITY PRESS.