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CAMBRIDGE SCHOOL AND COLLEGE TEXT BOOKS.

ELEMENTARY TRIGONOMETRY

WITH

A COLLECTION OF EXAMPLES.

BY

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PREFACE.

In this contribution to the series of Cambridge School and College Text Books now in course of publication by Messrs Deighton, Bell and Co. I have tried as far as possible to keep in view the objects stated in their Prospectus. I have thought it right therefore to explain in considerable detail such points as appeared to me likely to present difficulties to younger students, or to those who are reading without the assistance of a private Tutor; and I have moreover added a very copious collection of examples of various degrees of difficulty, of several of which I have given the working. The great majority of these examples have been selected from the Examination Papers of the several Colleges and of the University of Cambridge, but some few have been taken from Dr August Wiegand's *Sammlung trigonometrischer Aufgaben*. The results have been given in all cases where it was necessary, and considerable care has been taken to secure their accuracy. I shall however be thankful for corrections which any one may be kind enough to send me.

I must not omit to express my thanks to the friends who have given me assistance and advice from time to time during the progress of the work, and in particular to Mr Walton, of Trinity College, for his very kind interest and supervision.

T. P. H.

TRINITY COLLEGE, CAMBRIDGE,
May 16, 1862.

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CHAPTER I.

DEFINITIONS. UNITS AND MEASUREMENT OF ANGULAR MAGNITUDES.

1. THE object of that branch of mathematical science, which is called Trigonometry, is the investigation of all geometrical properties and relations in which angular magnitude is concerned. In the earlier stages of its progress it was, as its name implies*, applied exclusively to the measurement of *triangles*, and to the establishment of propositions connected immediately with them. Its methods, however, have now received an extension and a generality which render it a most valuable analytical instrument in the higher departments of mathematics. Of all the elementary branches of mathematical science it is perhaps the one of which the practical utility is most distinctly apparent. The student will, for instance, without difficulty foresee how indispensable such methods of calculation are to the surveyor, the navigator, and the astronomer.

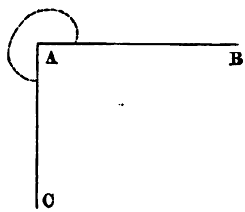
2. *Extension of the definition of an angle.*

Euclid defines a plane rectilineal angle to be the inclination of two straight lines to one another, which meet together but are not in the same straight line. He does not in his definition take into account the direction in which this inclination is supposed to be estimated, and, moreover, necessarily limits the signification of the word to angular magnitudes which are less than two right angles.

In Trigonometry, however, we regard an angle as capable of being of any magnitude whatever, and consequently

* τρίγωνον, a triangle, and μετρέω, I measure.

must have proper regard to the direction in which we estimate the inclination or opening between the two straight lines which contain the angle; *i.e.* to the direction in which one of the straight lines must be supposed to revolve from coincidence with the other in order to pass over the angular space in question.



For instance, the straight lines AB, AC, according to Euclid, would only bound one right angle, but in accordance with the more extended definition of an angle, they may also be considered as containing an angle whose magnitude is three right angles, the line AB in this case being supposed to revolve from right to left in order to move into coincidence with AC.

A trigonometrical angle then must be regarded not merely as the opening between two straight lines, but as the angular space swept over by a revolving line, which starts from coincidence with one of the bounding lines of the angle and moves into coincidence with the other. Moreover, in order to effect this, the revolving line may be supposed to have made any number of complete revolutions, so that under this supposition we can have angular space of any magnitude whatever.

For instance, the minute hand of a watch at a quarter past four o'clock will since twelve o'clock have revolved through an angle the magnitude of which is 17 right angles.

3. *Angular units.*

In order to measure angles some particular angle must be chosen as a standard or unit. This selection is of course quite arbitrary, and is influenced only by considerations of convenience.

4. *Degrees, minutes, seconds. Sexagesimal division of the right angle.*

The ninetieth part of a right angle is called a *degree*, the sixtieth part of a degree a *minute*, and the sixtieth part of a minute a *second*. The magnitude of any angle is evidently determined when we know how many degrees, minutes, and seconds it contains. The symbols employed to indicate degrees, minutes, and seconds respectively are $^\circ, ', ''$. Thus, 57 degrees, 15 minutes and 10 seconds are written, $57^\circ\ 15'\ 10''$.

5. *Grades, minutes, seconds. Centesimal division of the right angle.*

The hundredth part of a right angle is called a *grade*, the hundredth part of a grade a *minute*, and the hundredth part of a minute a *second*. The symbol expressing grades is ^{g}, and the minutes and seconds in this system of measurement (generally called the centesimal or French system) are generally indicated by `, ``, respectively, the strokes being turned in a direction opposite to that which they have when standing for the minutes and seconds of the sexagesimal division. Thus 38 grades, 17 minutes and 11 seconds would be written $38^g\ 17`\ 11``$. The advantage of this method of dividing the right angle is that any angle so expressed can at once be written down in grades and decimal parts of a grade, and *vice versâ*. Thus $38^g\ 17`\ 11``$ is 38·1711 grades and $15^g$·0235 is $15^g\ 2`\ 35``$. The sexagesimal method, however, is that which is now almost universally employed.

6. *Having given an angle expressed in degrees to express it in grades and vice versâ.*

Let D be the number (not necessarily an integer) of degrees in the given angle and G the number of grades in the same angle.

Then, since a right angle contains 90 degrees and 100 grades, we have

$$D : G :: 90 : 100;$$

$$\text{or} \qquad 9G = 10D;$$

$$\therefore \quad G = \frac{10}{9}D = D + \frac{1}{9}D.$$

Hence to reduce degrees to grades we must add one-ninth of the number of degrees.

Also $$D = \frac{9}{10}G = G - \frac{1}{10}G;$$

and therefore to reduce grades to degrees we must subtract one-tenth of the number of grades.

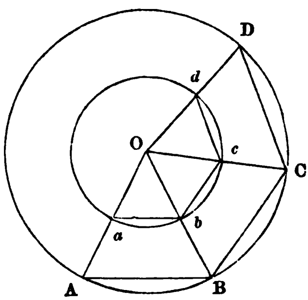
It will be found most convenient in performing these operations to use decimals.

Vide Examples.

7. *Circular measure.*

A third system of measurement is called *the circular measure.* This is very important, and the student is recommended to pay particular attention to the following articles.

8. *The circumference of a circle varies directly as its radius.*



Let ABC, abc be any two concentric circles. Describe in each of them regular polygons $ABCD...$ $abcd...$ having the same number of sides. Then clearly, by similar triangles,

$$\frac{AB}{ab} = \frac{OA}{Oa},$$

and therefore since the polygons are equilateral,

$$\frac{\text{perimeter of polygon } ABCD...}{\text{perimeter of polygon } abcd...} = \frac{OA}{Oa};$$

and this proportion is true whatever be the number of the sides of the polygons. But the more the number of the sides of the polygons is increased, the more nearly will their perimeters approach to those of the circles in which they are inscribed. Hence, supposing the number of the ·des to be indefiuitely increased, the difference between the perimeters of the polygons and those of the circles will be indefinitely diminished and the polygonal perimeters will merge into the circular ones, and therefore, the above proportion still holding, we have

$$\frac{\text{circumference of } circle\ ABCD...}{\text{circumference of } circle\ abcd...} = \frac{\text{radius } OA}{\text{radius } Oa}.$$

N.B. The fixed ratio which the circumference of a circle bears to its diameter is usually denoted by the Greek letter π. It is in reality a non-terminating decimal of which the first six figures are 3·14159.

9. The proportion which we have just proved will be true, not only of the whole circumferences, but also of any two corresponding portions of the circumferences respectively; that is, of any two arcs in the two circles which subtend the same angle at the centre.

From this we conclude at once that if the arc of a circle bears a certain fixed ratio to the radius, the angle which it subtends at the centre will be a fixed angle whatever be the magnitude of the circle.

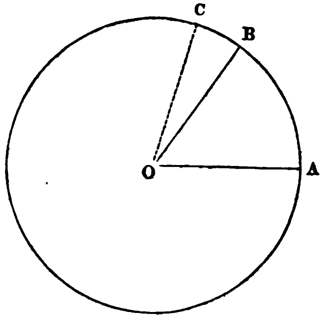
We are now able to define the angle which is taken as the unit in estimating angles by the circular measure.

The angle employed as a unit is that angle at the centre of a circle which is subtended by an arc *equal* to the radius, and which, as we have just explained, must be an invariable angle for all circles.

The circular measure then of an angle is the *ratio* which its magnitude bears to the magnitude of this invariable angle.

Thus, for instance, when we say that the circular measure of an angle is 3, we mean to express that the angle is three times as large as the unit of circular measure; or if we express it by symbols and call, for example, the circular measure π or θ, we must recollect that π and θ stand for *numerical quantities*, which are the *ratios* which the magnitudes of the angles represented respectively bear to the magnitude of that angle which is taken as the unit of circular measure.

10. ***To find the number of degrees in the unit of circular measure.***



Let AOB be the unit of circular measure; consequently the arc AB is equal to the radius OA.

Let x = number of degrees in AOB,

r = radius OA.

Then, since in a circle angles at the centre are as their subtending arcs, we shall have

$$\frac{x}{180} = \frac{\text{arc } AB}{\text{arc subtending 2 right angles}}.$$

But the arc subtending two right angles is the semicircumference; *i.e.* πr (since π is the ratio of the whole circumference to the diameter) and arc $AB = r$;

$$\therefore \frac{x}{180} = \frac{r}{\pi r};$$

$$\therefore x = \frac{180}{\pi} = \frac{180}{3{\cdot}14159} = 57{\cdot}29577 \text{ approximately.}$$

Hence, in the unit of circular measure there are 57·29577 degrees nearly.

11. ***The circular measure of any angle is the ratio*** $\dfrac{\textit{its subtending arc}}{\textit{radius}}$.

Let AOC be any angle.

Then we have $\dfrac{\text{angle } AOC}{\text{angle } AOB} = \dfrac{\text{arc } AC}{\text{arc } AB} = \dfrac{\text{arc } AC}{\text{radius}}$.

Now the ratio $\dfrac{\text{angle } AOC}{\text{angle } AOB}$ is the circular measure of the angle AOC, since the angle AOB is the unit of circular measure. Hence the proposition.

12. *Angle expressed in arc.*

If the radius of the circle be taken for the unit of length, the number expressing the length of the subtending arc and that representing the circular measure of the angle are the same, so that the arc may itself be taken as the measure of the angle, which is then said to be expressed *in arc.*

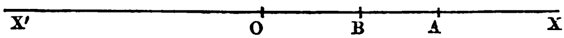
The circular measure of two right angles is manifestly π, since it is the ratio of the arc which subtends two right angles to the radius.

CHAPTER II.

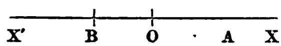
USE OF THE SIGNS + AND –. DEFINITIONS OF THE TRIGONOMETRICAL FUNCTIONS AND RELATIONS CONNECTING THEM. TRIGONOMETRICAL FUNCTIONS OF CERTAIN FIXED ANGLES.

1. *Use of the signs + and – to indicate contrariety of direction. Algebraical representation of straight lines and angles.*

Let O be a fixed point in a straight line $X'OX$, and let OA contain a units of length, and let AB contains b units,



AB being first supposed less than OA. Then OB will contain $a-b$ units of length. If we suppose any assigned point A to be reached by a point starting from O and travelling along the line OX, B will be determined by the point travelling back from A over b units of length. Suppose now that AB is greater than OA, B will in this case fall on the left of O and the *magnitude* of OB will be $b-a$.

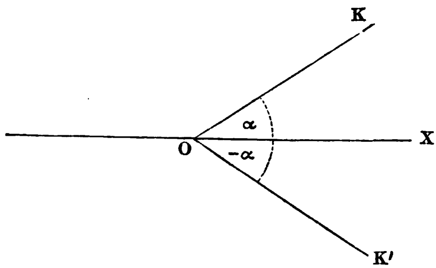


The necessity however of making any distinction between the cases in which b is less or greater than a, can be obviated if we agree to represent a distance measured to the left of O by a symbol with a negative sign prefixed. For, in accordance with this convention, $b-a$ being the *magnitude* of the distance OB, the *position* of the point B will be indicated by $-(b-a)$; *i.e.* $a-b$, as in the case in which b is less than a.

It is this generality of algebraical representation, this power of including all the possible cases of a theorem under one algebraical formula, which constitutes the principal utility of algebraical analysis as applied to geometry.

It will be readily seen that this same method of symbolical representation may be applied to angles, with reference to the direction in which a revolving line must move from its initial position in order that it may come into coincidence with any assigned position.

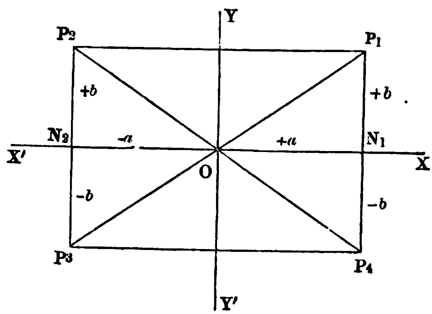
Thus, if OX be taken for the initial line, if the angle XOK be represented algebraically by $+a$, the angle XOK', which is drawn equal in magnitude to XOK, will be represented by $-a$.



For the revolving line, in order to pass over the angle XOK', must revolve in a direction opposite to that in which it moves in passing over the angle XOK.

2. *Position of a point in space and of the revolving line.*

Let $X'OX$, YOY' be two straight lines at right angles to each other.



The position of any point in space can be determined with reference to O by means of the above methods of representing the position and magnitude of straight lines. Suppose we agree to affix the positive sign to the symbols which represent the lengths of lines measured from O in the directions OX, OY; then if we know the proper symbols representing the distances of any point from $X'OX$ and $Y'OY$ respectively, the position of the point is determined.

Thus $\left.\begin{matrix}+a\\+b\end{matrix}\right\}$ indicate a point P_1 in the quarter XOY,

$\left.\begin{matrix}-a\\+b\end{matrix}\right\}$ .... P_2 .... $X'OY$,

$\left.\begin{matrix}-a\\-b\end{matrix}\right\}$ .... P_3 .... $X'OY'$,

$\left.\begin{matrix}+a\\-b\end{matrix}\right\}$ .... P_4 .... XOY'.

N.B. It is customary to represent lines measured on XX' by symbols, to which the positive sign is prefixed, when they are measured from O towards the right, and consequently lines measured to the left of O by symbols affected with the negative sign. Also the direction OY *above* XX' is that which is generally taken to correspond to symbols which have the positive sign in representing lines measured on YY'.

Similarly, symbols representing angles when affected with the sign + are generally supposed to represent angles traced by a revolving line, which moves from coincidence with the initial line in a direction contrary to that of the hands of a watch.

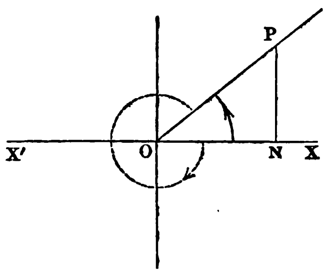
Such arrangements are of course perfectly arbitrary. It is manifestly a slovenly and inaccurate form of expression to talk of *positive or negative lines, positive or negative directions of revolution.* A line or a direction cannot be positive or negative, inasmuch as they are ideas which are strictly geometrical, and plainly quite distinct from the conceptions of algebraical representation with which they are associated.

To obviate this objection, therefore, I propose to call the directions OX, OY the *primary*, and OX', OY' the *secondary* directions of measurement. Similarly, I shall call the direction of revolution opposite to that of the hands of a watch the primary, and the opposite one the secondary direction of revolution.

The angle XOK I shall call a *primary* angle, and the angle XOK' a *secondary* angle.

3. *Definitions of the Trigonometrical Ratios.*

Let OX be the initial line from coincidence with which a line OP revolves in sweeping over any angle. Let OP be



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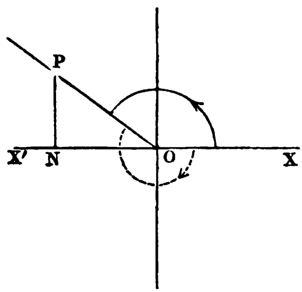

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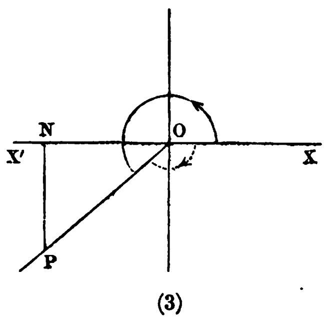

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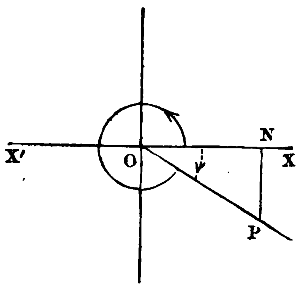


(4)

any position of the revolving line which will correspond to an angle whose symbol will be positive or negative according to the direction in which OP is supposed to move from coincidence with OX.

Let any point P be taken in OP and from it let a perpendicular be dropped upon the initial line (produced backwards, as OX', if necessary). Take proper algebraical symbols to represent the sides of the right-angled triangle PON so formed, and call the angle through which OP has revolved A.

The ratio of the algebraical representative of

PN to that of OP* is called the *sine* of A,

..................ON .... OP .... *cosine* of A,

..................PN .... ON .... *tangent* of A.

The reciprocals of the sine, cosine and tangent are called the *cosecant*, *secant* and *cotangent* of A respectively.

It is evident that the sine and cosine can neither of them exceed unity *in magnitude*, since the perpendicular and base are neither of them ever greater than the hypotenuse of a right-angled triangle.

Hence the cosecant and secant can never be less than unity.

The defect of the cosine from unity is called the versed sine. The versed sine of A is generally written versin A. Twice the sine of the half of A is called the chord of A, for a reason which will be hereafter explained, and is written chd A.

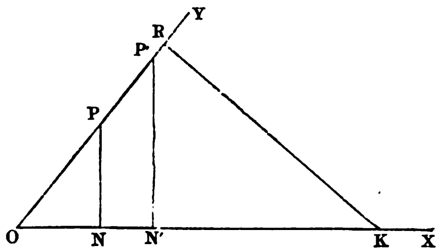
The definitions of the trigonometrical ratios (or functions as they are also called) which are ordinarily given are something as follows:

"If from a point in one of the straight lines containing an angle a perpendicular be let fall upon the other side, or the other side produced, the ratio of the perpendicular to the hypotenuse is called the sine of the angle."

* The symbol representing a line measured from O along OP is taken with a positive sign. If the sign were negative, then, in accordance with the rules of algebraical representation, the line indicated would be one measured from O along OP produced backwards, i.e. in the direction PO.

This, however, is not sufficiently general properly to include the trigonometrical ratios of angles greater than a right angle, and therefore some such definition as that given above, which introduces the idea of algebraical representation, is necessary.

4. *The trigonometrical functions are the same so long as the angle is the same.*



Let YOX be any angle. Take any points P, P', &c., in OY, and draw perpendiculars PN, $P'N'$ to OX.

Then, since by similar triangles,

$$PN : ON : OP = P'N' : ON' : OP',$$

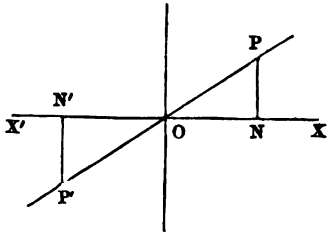
therefore the trigonometrical functions are the same so long as the angle is the same. If KR be a perpendicular to OP, then we have also by similar triangles

$$PN : ON : OP = KR : OR : OK,$$

which shews that it is indifferent upon which of the sides containing the angle the perpendicular is drawn (except so far as uniformity in the consideration of the signs of the algebraical symbols representing the sides is concerned).

5. It is very important to observe, that there is in every case an infinite number of angles corresponding to any one position of the revolving line which differ by multiples of four right angles, inasmuch as any number of complete revolutions in either direction will evidently not affect the position of this line.

Now as for any one position of the revolving line, eac' of the trigonometrical functions has only one value, i' evident that there is a certain group of angles, infinite in number, which have the same trigonometrical functions. Furthermore, for *more than one* position of the revolving line any one of the trigonometrical functions *may be* the same, as, for instance, the tangent is the same for the two positions OP, OP' of the revolving line, since the



ratios $\frac{PN}{ON}$ and $\frac{PN'}{ON'}$, are of the same magnitude, and their algebraical representatives are of the same sign.

We shall afterwards investigate what are the groups of angles which correspond to an assigned value of each of the trigonometrical functions.

We now proceed to prove some fundamental relations among the trigonometrical functions.

6. The defect of any angle from a right angle is called its *complement.*

The defect of any angle from two right angles is called its *supplement.*

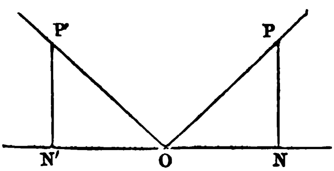
Thus the complement of an angle A expressed in degrees is $90° - A$, or, expressed in circular measure, $\frac{\pi}{2} - \frac{\pi A}{180}$.

The supplement similarly of A is $180° - A$.

7. *Having given the trigonometrical ratios of an angle to find those of its complement and supplement.*

Let PON be any angle A, of which the trigonometrical ratios are known.

Then OPN is evidently its complement.



Now,

$$\sin OPN = \frac{ON}{OP} = \cos PON, \text{ i.e. } \sin(90^\circ - A) = \cos A,$$

$$\tan OPN = \frac{ON}{PN} = \cot PON \ \ldots \ \tan(90^\circ - A) = \cot A,$$

$$\sec OPN = \frac{OP}{ON} = \operatorname{cosec} PON \ldots \ \sec(90^\circ - A) = \operatorname{cosec} A;$$

so that the sine, tangent and secant of an angle are respectively the cosine, cotangent and cosecant of its complement.

Again, draw OP' making the same angle with ON' which OP does with ON. Then the angle $P'ON$ is the supplement of PON or is $180^\circ - A$.

Take OP' equal to OP and draw $P'N'$ perpendicular to ON', so that ON' and $P'N'$ will be respectively equal to ON and PN. Let p be the number of linear units in PN, b the number in ON and r the number in OP.

Then,

$$\sin P'ON = \frac{p}{r} = \sin PON,$$

$$\cos P'ON = \frac{-b}{r} = -\cos PON,$$

$$\tan P'ON = \frac{+p}{-b} = -\tan PON,$$

$$\operatorname{cosec} P'ON = \frac{r}{p} = \operatorname{cosec} PON,$$

$$\sec P'ON = \frac{r}{-b} = -\sec PON,$$

$$\cot P'ON = \frac{-b}{+p} = -\cot PON;$$

or we may write these results as follows:

$$\sin(180^\circ - A) = \sin A,$$
$$\cos(180^\circ - A) = -\cos A,$$
$$\tan(180^\circ - A) = -\tan A,$$
$$\operatorname{cosec}(180^\circ - A) = \operatorname{cosec} A,$$
$$\sec(180^\circ - A) = -\sec A,$$
$$\cot(180^\circ - A) = -\cot A.$$

These relations have been proved only in the case in which A is less than a right angle. It will be a useful exercise for the student to establish their truth for any other position of OP.

8. The following simple formulæ are of constant occurrence:

$$\tan A = \frac{PN}{ON} = \frac{\frac{PN}{OP}}{\frac{ON}{OP}} = \frac{\sin A}{\cos A};$$

$$\therefore \cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}.$$

Also $$PN^2 + ON^2 = OP^2,$$

and, dividing both sides of the equation by OP^2,

$$\frac{PN^2}{OP^2} + \frac{ON^2}{OP^2} = 1;$$

$$\therefore \sin^2 A + \cos^2 A = 1.$$

This formula is the trigonometrical mode of expressing the 47th proposition of Euclid I.

Dividing by $\cos^2 A$ we have

$$\frac{\sin^2 A}{\cos^2 A} + 1 = \frac{1}{\cos^2 A},$$

$$\text{or } \tan^2 A + 1 = \sec^2 A.$$

Similarly, by dividing by $\sin^2 A$ we get

$$\cot^2 A + 1 = \operatorname{cosec}^2 A.$$

9. *Having given one trigonometrical function of an angle, to express all the others in terms of it.*

This may in all cases be effected by means of the formulæ we have just established.

I. Suppose $\sin A$ to be given and to be equal to s, it is required to find all the other trigonometrical ratios of A.

Since $$\cos^2 A = 1 - \sin^2 A = 1 - s^2;$$

$$\therefore \cos A = \pm\sqrt{(1-s^2)},$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\pm\sqrt{(1-\sin^2 A)}} = \pm\frac{s}{\sqrt{(1-s^2)}},$$

$$\cot A = \frac{1}{\tan A} = \pm\frac{\sqrt{(1-s^2)}}{s},$$

$$\sec A = \frac{1}{\cos A} = \pm\frac{1}{\sqrt{(1-s^2)}},$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{1}{s}.$$

Wherever the double sign occurs before the radicals in these results, it arises from the fact that there is more than one angle which corresponds to a given sine, and therefore since the sine of the angle, and not the angle itself is given, we do not know of which of the angles corresponding to the same sine we are to take the cosine, tangent, cotangent, and secant respectively; and since those angles for which the sine is the same have not all their other trigonometrical functions the same, we have ambiguities in the results, as ought to be the case. This point however will be more fully explained hereafter.

II. *Given* $\cos A = c$, *to find the other trigonometrical functions.*

$$\sin^2 A = 1 - \cos^2 A = 1 - c^2;$$

$$\therefore \sin A = \pm\sqrt{(1-c^2)},$$

$$\tan A = \frac{\sin A}{\cos A} = \pm\frac{\sqrt{(1-\cos^2 A)}}{\cos A} = \pm\frac{\sqrt{(1-c^2)}}{c},$$

$$\cot A = \frac{1}{\tan A} = \pm\frac{c}{\sqrt{(1-c^2)}},$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{c},$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \pm\frac{1}{\sqrt{(1-c^2)}}.$$

III. *Given* $\tan A = t$.

$$\sin^2 A = \frac{1}{\operatorname{cosec}^2 A} = \frac{1}{1 + \cot^2 A} = \frac{1}{1 + \dfrac{1}{\tan^2 A}} = \frac{\tan^2 A}{1 + \tan^2 A};$$

$$\therefore \sin A = \pm \frac{t}{\sqrt{(1 + t^2)}}, \text{ and } \operatorname{cosec} A = \pm \frac{\sqrt{(1 + t^2)}}{t},$$

$$\cos^2 A = \frac{1}{\sec^2 A} = \frac{1}{1 + \tan^2 A} = \frac{1}{1 + t^2};$$

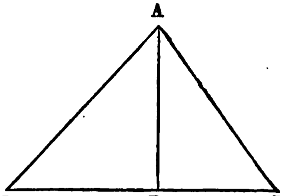
$$\therefore \cos A = \pm \frac{1}{\sqrt{(1 + t^2)}}, \quad \sec A = \pm \sqrt{(1 + t^2)},$$

$$\cot A = \frac{1}{\tan A} = \frac{1}{t}.$$

Similar remarks apply with reference to the double signs in these results as were made at the end of Case I.

The cosecant, secant, and cotangent being respectively the reciprocals of the sine, cosine and tangent, the cases in which any one of them is given to determine the other trigonometrical functions from, are too simple to require special proof.

10. *To find the trigonometrical functions of* 30° *and* 60°.



Let ABC be an equilateral triangle, each angle of which therefore is 60°. From A draw AD perpendicular to BC. Then the angle BAD is 30°;

$$\therefore \sin 30^\circ = \frac{BD}{AB} = \frac{\frac{1}{2}BC}{AB} = \frac{1}{2}; \quad \because BC = AB,$$

$$\text{s } 30^\circ = \frac{AD}{AB} = \frac{\sqrt{(AB^2 - BD^2)}}{AB} = \frac{\sqrt{(AB^2 - \frac{1}{4}AD^2)}}{AB} = \frac{\sqrt{3}}{2},$$

the positive signs being taken with the radical because we know that the cosine of 30° must be positive.

$$\therefore\ \tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}},$$

$$\sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}},$$

$$\text{cosec}\ 30^\circ = \frac{1}{\sin 30^\circ} = 2,$$

$$\cot 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3},$$

$$\sin 60^\circ = \sin (90^\circ - 30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2},$$

$$\cos 60^\circ = \cos (90^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2},$$

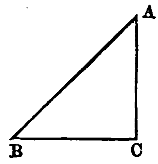
$$\tan 60^\circ = \tan (90^\circ - 30^\circ) = \cot 30^\circ = \sqrt{3},$$

and similarly, $\sec 60^\circ = 2,$

$$\text{cosec}\ 60^\circ = \frac{2}{\sqrt{3}},$$

$$\cot 60^\circ = \frac{1}{\sqrt{3}}.$$

11. *To find the trigonometrical functions of* 45°.



Let ACB be an isosceles right-angled triangle, the right angle being at C.

Then each of the angles A and B is 45°;

$$\therefore\ \sin 45^\circ = \frac{AC}{AB} = \frac{AC}{\sqrt{(BC^2 + AC^2)}} = \frac{AC}{\sqrt{(2AC^2)}} = \frac{1}{\sqrt{2}}.$$

The cosine of 45° is evidently equal to the sine because $BC = AC$;

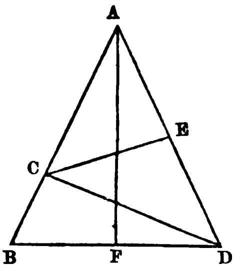
$$\therefore \cos 45° = \frac{1}{\sqrt{2}},$$

$$\tan 45° = \frac{AC}{BC} = 1.$$

The remaining trigonometrical functions are of course at once determined.

12. *To find the trigonometrical functions of* 36° *and* 18°.

In Euclid IV. 10, it is shewn that if a point C be taken in a straight line AB, so that the rectangle AB, BC is equal to the square on AC, and if an isosceles triangle be described with base BD equal to AC and with each of its equal sides AB, AD equal to AB, then the angle ABD or ADB is double the angle BAD, and also CD is equal to AC.



The angle BAD therefore being the fifth part of two right angles contains 36°.

From C draw CE at right angles to AD, and therefore bisecting AD since $CD = CA$.

$$\text{Then } \cos 36° = \frac{AE}{AC} = \frac{1}{2}\frac{AB}{AC},$$

$$\text{and } AB . BC = AC^2,$$

$$\text{or } AB(AB - AC) = AC^2;$$

dividing by AC^2, $\frac{AB^2}{AC^2} - \frac{AB}{AC} = 1$;

$$\text{and } \therefore \frac{AB}{AC} = \frac{\pm\sqrt{5}+1}{2} \text{ (by solving the quadratic)};$$

$$\therefore \cos 36° = \frac{\sqrt{5}+1}{4},$$

the positive sign being taken, since we know that cos 36° must be positive.

Hence,

$$\sin 36° = \sqrt{(1-\cos^2 36°)} = \sqrt{\left(1-\frac{6+2\sqrt{5}}{16}\right)} = \sqrt{\left(\frac{5-\sqrt{5}}{8}\right)}$$

and the other trigonometrical ratios of 36° can be determined from these.

Again, drawing a perpendicular AF to BD from A, we have the angle BAF equal to 18°;

$$\therefore \sin 18° = \frac{BF}{BA} = \frac{1}{2}\frac{BD}{AB} = \frac{1}{2}\frac{AC}{AB} = \frac{1}{\sqrt{5}+1}$$

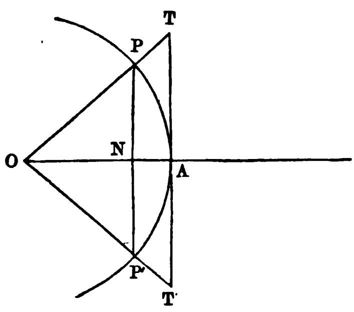
$$= \frac{\sqrt{5}-1}{\{\sqrt{5}+1\}\{\sqrt{5}-1\}}$$

$$= \frac{\sqrt{5}-1}{4},$$

which is the form in which this result is generally given.

From this value for the sine, let the student find the values of the other trigonometrical ratios of 18°.

13. *The circular measure of an angle is greater than the sine and less than the tangent of the angle.*



Let PAP' be an arc of a circle the centre of which is O.

Bisect the angle POP' by the line OA.

Draw the perpendicular PNP' and the tangent TAT'.

Then manifestly the curved line PAP' is greater than PP' and less than TAT';

$\therefore AP$ is greater than PN and less than AT;

$\therefore \dfrac{AP}{OP}$ is greater than $\dfrac{PN}{OP}$ and less than $\dfrac{AT}{OA}$.

But $\dfrac{AP}{OP}$ is the circular measure of the angle AOP, (Ch. I. 11).

$\dfrac{PN}{OP}$ is the sine ....

$\dfrac{AT}{OA}$ is the tangent ....

Hence calling the circular measure of the angle AOP θ, we have proved that θ is greater than $\sin\theta$ and less than $\tan\theta$.

14. *Limit of* $\dfrac{\sin\theta}{\theta}$ *and of* $\dfrac{\tan\theta}{\theta}$ *when* θ *is indefinitely diminished.*

We have proved in the last article that $\sin\theta$, θ, and $\tan\theta$ are in ascending order of magnitude; therefore, dividing each of them by $\sin\theta$, 1, $\dfrac{\theta}{\sin\theta}$ and $\dfrac{1}{\cos\theta}$ are in ascending order of magnitude. Now when θ is made equal to zero, $\dfrac{1}{\cos\theta}$ is equal to 1.

But $\dfrac{\theta}{\sin\theta}$, *whatever be the value of* θ, is always intermediate in value to 1 and $\dfrac{1}{\cos\theta}$. These limits however are both equal to unity when $\theta = 0$. Hence necessarily we conclude that also

$$\frac{\theta}{\sin\theta} = 1, \text{ when } \theta = 0;$$

$$\text{and} \therefore \text{of course } \frac{\sin\theta}{\theta} = 1, \text{ when } \theta = 0.$$

$$\text{Since } \frac{\tan\theta}{\theta} = \frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta},$$

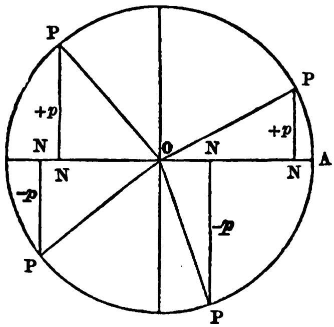
$$\text{also } \frac{\tan\theta}{\theta} = 1, \text{ when } \theta = 0.$$

CHAPTER III.

THE CHANGES IN SIGN AND MAGNITUDE OF THE TRIGONOMETRICAL FUNCTIONS. ANOTHER MODE OF DEFINING THE TRIGONOMETRICAL FUNCTIONS. THE GROUPS OF ANGLES CORRESPONDING TO AN ASSIGNED VALUE OF EACH OF THE TRIGONOMETRICAL FUNCTIONS.

1. *To trace the variations in the sign and magnitude of the sine of an angle, as the angle varies from* $0°$ *to* $360°$.

Let OP be the revolving line.



Let the perpendicular PN in any position be represented in magnitude by p, the base ON by b, and the radius OP by r. Then, representing the angle corresponding to any position of OP by A, we have in the

1st *Quadrant*,

$$\sin A = \frac{+p}{r}, \text{ and when } A = 0,\ p = 0, \text{ and therefore}$$

$$\sin \text{zero} = 0,$$

and as the angle increases p increases and b decreases. Hence the sine is positive and increases in magnitude

when $A = 90°$, $p = r$, and therefore $\sin 90° = 1$.

2nd *Quadrant*,

$$\sin A = \frac{+p}{r}, \text{ and is therefore positive ;}$$

as A increases p decreases, and therefore $\sin A$ decreases in magnitude ;

when $A = 180°$, $p = 0$, and therefore $\sin 180° = 0$.

3rd *Quadrant*,

$$\sin A = \frac{-p}{r}, \text{ and is therefore negative ;}$$

as A increases p increases in magnitude, and therefore $\sin A$ increases in magnitude;

when $A = 270°$ (3 right angles), $p = r$, and therefore

$$\sin 270° = \frac{-r}{r} = -1.$$

4th *Quadrant*,

$$\sin A = \frac{-p}{r}, \text{ and is therefore negative;}$$

as A increases p decreases in magnitude, and therefore $\sin A$ decreases in magnitude,

and when $A = 360°$, $p = 0$,

and therefore $\sin 360° = 0$,

as ought to be the case, since OP then occupies its initial position OA again.

2. *To trace the variations in the sign and magnitude of the cosine of an angle, as the angle varies from 0° to 360°.*

Using the same figure and notation as in the case of the sine.

1st *Quadrant*,

$$\cos A = \frac{+b}{r}, \text{ and is therefore positive;}$$

when $A = 0$, $b = r$, and therefore cos zero $= 1$;

as A increases b decreases in magnitude, and therefore $\cos A$ decreases;

when $A = 90°$, $b = 0$, and therefore $\cos 90° = 0$.

2nd *Quadrant*,

$$\cos A = \frac{-b}{r}, \text{ and is therefore negative;}$$

as A increases b increases, and therefore $\cos A$ increases in magnitude, and when $A = 180°$, $b = r$, and therefore

$$\cos 180° = \frac{-r}{r} = -1.$$

3rd *Quadrant*,

$$\cos A = \frac{-b}{r}, \text{ and is therefore negative;}$$

as A increases b decreases, and therefore $\cos A$ decreases in magnitude;

when $A = 270°$, $b = 0$, and therefore $\cos 270° = 0$.

4th *Quadrant*,

$$\cos A = \frac{+b}{r}, \text{ and is therefore positive;}$$

as A increases b increases, and therefore $\cos A$ increases, and when $A = 360°$, $b = r$, and therefore

$$\cos 360° = 1.$$

3. ***To trace the variations in the sign and magnitude of the tangent, as the angle varies from 0° to 360°.***

1st *Quadrant,*

$$\tan A = \frac{+p}{+b}, \text{ and is therefore positive;}$$

$$\text{when } A = 0, \; p = 0, \text{ and } b = r;$$

$$\therefore \tan \text{zero} = 0;$$

as A increases p increases and b decreases, and therefore $\tan A$ increases;

$$\text{when } A = 90°, \; p = r, \text{ and } b = 0;$$

$$\therefore \tan 90° = \frac{r}{0} = \infty \, *.$$

2nd *Quadrant,*

$$\tan A = \frac{+p}{-b}, \text{ and is therefore negative;}$$

as A increases p decreases in magnitude and b increases, and therefore $\tan A$ decreases in magnitude, and

$$\text{when } A = 180°, \; p = 0, \; b = r,$$

$$\text{and } \therefore \tan 180° = 0.$$

3rd *Quadrant,*

$$\tan A = \frac{-p}{-b} = \frac{p}{b}, \text{ and is therefore positive;}$$

as A increases p increases and b decreases, and therefore $\tan A$ increases in magnitude;

$$\text{when } A = 270°, \; p = r, \; b = 0,$$

$$\text{and } \therefore \tan 270° = \infty.$$

* Observe that when the angle is less than, but very nearly equal to 90°, the tangent is very large and positive, and that when the angle is very little greater than 90° the tangent is very large and negative. We express this by saying, that the tangent changes sign in passing through the value infinity.

4th *Quadrant,*

$$\tan A = \frac{-p}{b}, \text{ and is therefore negative;}$$

as A increases p decreases and b increases, and therefore $\tan A$ decreases in magnitude;

$$\text{when } A = 360°, \ p = 0, \text{ and } b = r;$$

$$\text{and } \therefore \tan 360° = 0.$$

The cosecant, secant, and cotangent being respectively the reciprocals of the sine, cosine, and tangent, require no special examination.

The variations of all the trigonometrical functions are subjoined in a tabular form:

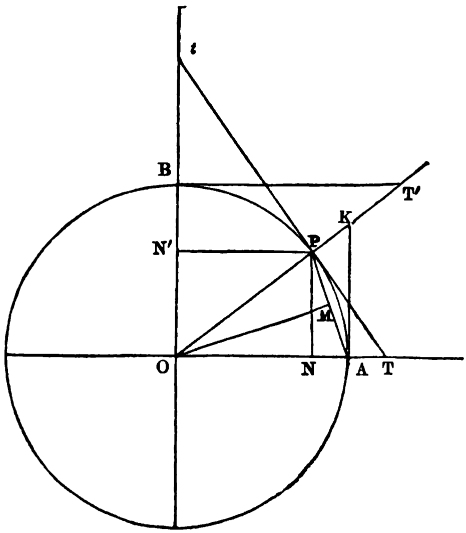
	0......90°	90°.....180°	180°...270°	270°...360°
sine	positive 0......1	positive 1.........0	negative 0...... −1	negative −1......0
cosine ...	positive 1......0	negative 0...... −1	negative −1......0	positive 0.........1
tangent..	positive 0.......∞	negative ∞.........0	positive 0.........∞	negative ∞.........0
cosecant.	positive ∞.......1	positive 1.........∞	negative ∞...... −1	negative −1......∞
secant ...	positive 1.......∞	negative ∞..... −1	negative −1......∞	positive ∞.........1
cotangent	positive ∞.......0	negative 0.........∞	positive ∞.........0	negative 0.........∞

4. *Line-definitions of the Trigonometrical Functions.*

The names sine, cosine, tangent, &c.... were originally applied to certain lines connected with the subtending arc rather than the angle.

Let AP be the arc of a circle subtending the angle AOP at the centre. Draw OB at right angles to OA. Draw TPt the tangent at P, and complete the figure.

The *line* PN was formerly called the sine, AK the tangent, and OK the secant of the arc AP. Similarly,



according to the same mode of definition, PN', BT', OT' are the sine, tangent, and secant of PB, which is the complement of arc AP. Hence they are the cosine, co-tangent, and cosecant of arc AP. Instead of PN', OK, OT', it is usually more convenient to use the equal lines ON, OT, Ot.

AN was called the versed-sine of arc AP.

The *co*-sine, *co*-tangent, and *co*-secant, are respectively the sine, tangent, and secant of the complement of the arc AP.

5. ***Case in which the line-definitions and the ratio-definitions of the Trigonometrical functions are identical.***

Denote the angle AOP by A.

Then using the ratio-definitions,

$$\sin A = \frac{PN}{OP},\ \cos A = \frac{ON}{OP},\ \tan A = \frac{AK}{OA},\ \sec A = \frac{OK}{OA},$$

$$\operatorname{cosec} A = \sec BOP = \frac{OT'}{OB},\ \cot A = \tan BOP = \frac{BT'}{OB},$$

$$\text{versed-sine}\, A = 1 - \cos A = 1 - \frac{ON}{OP} = \frac{AN}{OP}.$$

Observe that all the trigonometrical functions of A, as here expressed, have the radius of the circle as the denominator of the ratios which represent them.

If now we take the length of the radius as the unit of length by which we measure each of the lines PN, ON, AK, &c. ...... then the number of units of length in each of the lines PN, ON, AK, OK, OT', BT' will respectively be the same as the numerical quantities which are the values of $\sin A$, $\cos A$, $\tan A$, $\sec A$, $\operatorname{cosec} A$, $\cot A$, $\operatorname{versin} A$. Hence when the radius of the circle is taken equal to unity, the trigonometrical functions according to the two definitions are numerically identical. AP is called the chord of the arc AP. Draw a perpendicular OM to it. This manifestly bisects the angle AOP and the chord AP.

Hence, chord of $AP = 2AM = 2 \sin \frac{1}{2} \text{arc} AP$*.

So similarly, $\text{ch}^{\text{d}} A = 2 \sin \frac{A}{2}$.

If we denote what we may term the line-sine of A by $\text{Sin}\, A$, and the other or ratio-sine by $\sin A$, we have manifestly $\sin A = \frac{\text{Sin}\, A}{r}$ (r being the radius), and similarly for the other functions. Thus a formula expressed in accordance with one method of definition can be at once reduced to the corresponding formula when the other definition is employed.

* If OM be produced to meet the circle in M' and a perpendicular $M'N'$ be drawn to OA, $M'N'$ is the sine of half the arc AP, and it is plainly equal to AM.

For example, in the formula $\sec^2 A = 1 + \tan^2 A$, put $\frac{\text{Sec}\,A}{r}$ for $\sec A$ and $\frac{\text{Tan}\,A}{r}$ for $\tan A$, and we get

$$\frac{\text{Sec}^2 A}{r^2} = 1 + \frac{\text{Tan}^2 A}{r^2},$$

$$\text{or } \text{Sec}^2 A = r^2 + \text{Tan}^2 A,$$

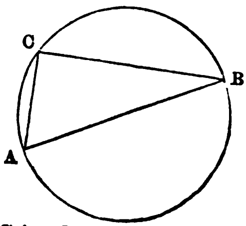
the corresponding formula when the line-definitions are employed.

The advantage of the ratio-definitions as compared with the line-definitions is manifestly that they are independent of the particular value of the radius, and are abstract numerical quantities depending only on the magnitude of the angle.

6. *To construct an angle which shall have its sine or cosine equal to a given quantity.*

The given quantity cannot be greater than unity. Let it then be $\frac{b}{a}$, where b is not greater than a.

Describe a circle with diameter AB equal to a units of length; with the point A as centre, describe another circle of radius equal to b units of length cutting the first circle in C.



$$\text{Then } \sin CBA = \frac{CA}{AB} = \frac{b}{a},$$

$$\text{and } \cos CAB = \frac{CA}{AB} = \frac{b}{a}.$$

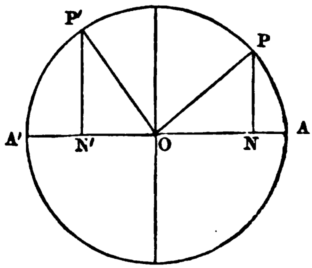
Therefore the angle CBA has its sine and the angle CAB its cosine equal to the given quantity.

The same construction manifestly serves for an angle which shall have its secant or cosecant equal to a given quantity.

The construction of an angle of which the tangent or cotangent is equal to a given quantity is so simple as not to require insertion.

7. In pursuance of the remarks made in Chap. II. Art. 5, we now proceed to determine what groups of angles correspond to an assigned value of each of the trigonometrical functions.

To find a general expression for all the angles, the sines of which have the same given value.



Let AOP be the least primary angle which has its sine equal to the given value, and let a be its circular measure. The angle AOP' will also have the same sine, $A'OP'$ being equal to AOP; for the sine of AOP' is evidently equal to that of AOP in magnitude, and the sign of the sine is the same in the first two quadrants. For every angle then corresponding to the positions OP, OP' of the revolving line, the sine will be the same and have the given value. These angles are evidently (using circular measure),

Primary angles, a, $\pi - a$, and these increased by any multiple of 2π,

Secondary angles, $-(\pi + a)$, $-(2\pi - a)$, and these increased by any multiple of 2π.

(The line in revolving from OA to OP' in the secondary direction sweeps over an angle whose magnitude is $\pi + a$, and the negative sign must be prefixed to shew the direction of revolution.)

Expressing "any multiple of" by m, these are

$$2m\pi + a,\ 2m\pi + \pi - a,\ -2m\pi - (\pi + a),\ -2m\pi - (2\pi - a),$$

or,

$$2m\pi + a,\ (2m+1)\pi - a,\ -(2m+1)\pi - a,\ -2(m+1)\pi + a.$$

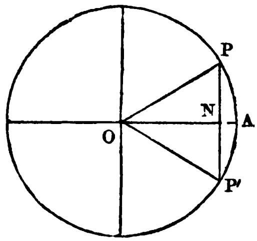
Now observing that when the + sign precedes α (i.e. in the first and last of these expressions) the multiple of π which is prefixed is *even*, and either positive or negative, and when the − sign precedes α, the multiple of π is *odd*, and either positive or negative; it is evident that all these angles are represented by the general expression

$$n\pi + (-1)^n \alpha,$$

where n stands for any integer, either even or odd, positive or negative.

Therefore $\sin\{n\pi + (-1)^n \alpha\} = \sin \alpha$.

8. *To find a general expression for all angles which have the same given value for their cosine.*



Let AOP be the least primary angle which has its cosine equal to the given value, and let α be its circular measure. Take AOP' equal to AOP in magnitude. Then since the sign of the cosine is the same for positions of the revolving line in the first and fourth quadrants, the cosine of all angles corresponding to the positions OP, OP' of the revolving line will be the same and equal to the given value. These angles are evidently

primary angles, α, $2\pi - \alpha$, and these increased by any multiple of 2π,

secondary angles, $-\alpha$, $-(2\pi - \alpha)$ and these increased by any multiple of 2π.

These are therefore

$$2m\pi + \alpha,\ 2m\pi + 2\pi - \alpha,\ -2m\pi - \alpha,\ -2m\pi - (2\pi - \alpha),$$

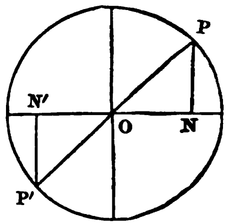
or $2m\pi + \alpha,\ 2(m+1)\pi - \alpha,\ -2m\pi - \alpha,\ -2(m+1)\pi + \alpha,$

all of which are manifestly included in the formula

$$2n\pi \pm \alpha,$$

so that $\cos(2n\pi \pm \alpha) = \cos \alpha$.

9. ***To find a general expression for all angles which have the same given value for their tangent.***



Denoting the circular measure of PON by a, as before, the two positions of the revolving line for which the tangent is the same as that of a are OP and OP'. The primary and secondary angles corresponding to these two positions are represented by

$$a,\ \pi+a,\ -(\pi-a),\ -(2\pi-a),$$

and these increased by any multiples of 2π, which give

$$2m\pi+a,\ 2m\pi+\pi+a,\ -2m\pi-(\pi-a),\ -2m\pi-(2\pi-a),$$

or, $2m\pi+a, (2m+1)\pi+a, -(2m+1)\pi+a, -2(m+1)\pi+a.$

Observing now that the sign prefixed to a is always positive while the mutiple of π is either even or odd, positive or negative, we see that all these angles are included in the one general formula

$$n\pi+a,$$

where n is any integer, odd or even, positive or negative, so that $\tan a=\tan(n\pi+a).$

10. It follows evidently from these results that

$$\operatorname{cosec}\{n\pi+(-1)^n a\}=\operatorname{cosec} a,$$

$$\sec(2n\pi\pm a)=\sec a,$$

$$\cot(n\pi+a)=\cot a.$$

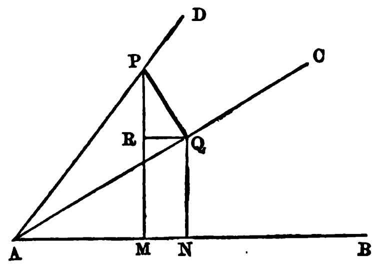
The results of the three preceding articles must be committed to memory, as they are of continual recurrence.

It may be observed that it is not necessary in these expressions that a should be absolutely the *least* primary angle which has the given value for the trigonometrical functions. They are equally true if a be *any* angle which has the given value for its sine, cosine, or tangent, as the case may be.

CHAPTER IV.

TRIGONOMETRICAL FUNCTIONS OF THE SUM AND DIFFERENCE OF TWO ANGLES AND OF THE MULTIPLES AND SUBMULTIPLES OF ANGLES.

1. *To find* $\sin(A+B)$ *in terms of the sines and cosines of* A *and* B.



Let the angle BAC be A, and CAD, B. Then the angle BAD is $A+B$.

Take any point P in AD. Draw PQ, PM at right angles to AC, AB respectively, and from Q draw QR, QN at right angles to PM, AB respectively. Then the angle RPQ = the angle RQA, for each is the complement of PQR, and therefore the angle RPQ is equal to A.

Now

$$\sin(A+B) = \frac{PM}{AP}$$

$$= \frac{RM+PR}{AP}$$

$$= \frac{QN}{AP} + \frac{PR}{AP}$$

$$= \frac{QN}{AQ} \cdot \frac{AQ}{AP} + \frac{PR}{PQ} \cdot \frac{PQ}{AP} = \sin A \cos B + \cos A \sin B.$$

2. *Cos $(A+B)$ in terms of sines and cosines of A and B.*

Employing the same fig. as in Art. 1, we have

$$\cos(A+B)=\frac{AM}{AP}=\frac{AN-MN}{AP}=\frac{AN}{AP}-\frac{QR}{AP}$$

$$=\frac{AN}{AQ}\cdot\frac{AQ}{AP}-\frac{QR}{QP}\cdot\frac{QP}{AP}$$

$$=\cos A\cos B-\sin A\sin B.$$

3. *Tan $(A+B)$ in terms of tangents of A and B.*

$$\tan(A+B)=\frac{PM}{AM}=\frac{RM+PR}{AN-MN}=\frac{QN+PR}{AN-QR}.$$

Dividing the numerator and denominator of this fraction by AN,

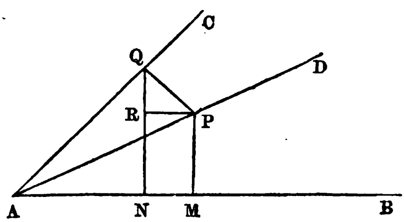
$$\tan(A+B)=\frac{\frac{QN}{AN}+\frac{PR}{AN}}{1-\frac{QR}{QN}\cdot\frac{QN}{AN}}.$$

Now $\frac{QN}{AN}=\tan A$, and by similar triangles PQR, QAN,

$$\frac{PR}{AN}=\frac{QR}{QN}=\frac{PQ}{AQ}=\tan B.$$

Therefore $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\,.\tan B}.$

4. *Sin $(A-B)$ and cos $(A-B)$ in terms of sines and cosines of A and B.*



Let BAC be represented by A and CAD by B.

Then

$BAD=A-B$.

Take any point P in AD and from P draw PQ at right angles to AC and PM at right angles to AB, and draw QN, PR respectively parallel to PM, AB.

Then the angle PQR, being the complement of AQN, is equal to BAC or A. Hence

$$\sin(A-B)=\frac{PM}{AP}=\frac{QN-QR}{AP}=\frac{QN}{AQ}\cdot\frac{AQ}{AP}-\frac{QR}{QP}\cdot\frac{QP}{AP}$$
$$=\sin A\cos B-\cos A\sin B,$$

and

$$\cos(A-B)=\frac{AM}{AP}=\frac{AN+PR}{AP}=\frac{AN}{AQ}\cdot\frac{AQ}{AP}+\frac{PR}{PQ}\cdot\frac{PQ}{AP}$$
$$=\cos A\cos B+\sin A\sin B.$$

5. ***Tan (A − B) in terms of tan A and tan B.***

$$\tan(A-B)=\frac{PM}{AM}=\frac{QN-QR}{AN+PR}.$$

Dividing the numerator and denominator of this fraction by AN, we get

$$\tan(A-B)=\frac{\frac{QN}{AN}-\frac{QR}{AN}}{1+\frac{PR}{AN}}=\frac{\frac{QN}{AN}-\frac{QR}{AN}}{1+\frac{PR}{QN}\cdot\frac{QN}{AN}},$$

and by similar triangles PQR, QAN,

$$\frac{QR}{AN}=\frac{PR}{QN}=\frac{PQ}{AQ}=\tan B,\quad\text{and}\quad\frac{QN}{AN}=\tan A;$$

$$\therefore\ \tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\,.\tan B}.$$

6. The expressions for $\tan(A+B)$ and $\tan(A-B)$ are also easily deduced from those for the sines and cosines of $A+B$ and $A-B$.

For $$\tan(A\pm B)=\frac{\sin(A\pm B)}{\cos(A\pm B)}=\frac{\sin A\cos B\pm\cos A\sin B}{\cos A\cos B\mp\sin A\sin B},$$
(Arts. 1 and 4)

or, dividing the numerator and denominator of this fraction by $\cos A\cos B$,

$$\tan(A\pm B)=\frac{\frac{\sin A}{\cos A}\pm\frac{\sin B}{\cos B}}{1\mp\frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}}=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B},$$

where the upper signs are to be taken together, and similarly for the lower signs.

7. Again, $\sin(A+B)$ and $\cos(A+B)$ might have been deduced from $\tan(A+B)$.

$$\text{For } \frac{\sin(A+B)}{\cos(A+B)}=\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A\sin B}{\cos A\cos B}};$$

$$\therefore \frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}.$$

Now the numerators of these fractions must bear to each other the same ratio which the denominators have to each other. Let this ratio be λ.

$$\text{Then } \lambda\sin(A+B)=\sin A\cos B+\cos A\sin B,$$

$$\lambda\cos(A+B)=\cos A\cos B-\sin A\sin B.$$

Squaring each of these and adding them,

$$\lambda^2=\sin^2A\cos^2B+\cos^2A\cos^2B+\cos^2A\sin^2B+\sin^2A\sin^2B$$

$$=(\cos^2B+\sin^2B)(\sin^2A+\cos^2A)=1;$$

$$\therefore \lambda=\pm 1;$$

$$\text{and } \therefore \sin(A+B)=\pm(\sin A\cos B+\cos A\sin B).$$

But the negative sign is not admissible because $(A+B)$, A and B being each less than 90°, their sines and cosines are all positive.

$$\text{Hence } \sin(A+B)=\sin A\cos B+\cos A\sin B;$$

$$\text{and } \cos(A+B)=\cos A\cos B-\sin A\sin B.$$

Similarly, $\sin(A-B)$ and $\cos(A-B)$ may be deduced from $\tan(A-B)$.

These formulæ have been established only in the case in which $A+B$ is less than $\frac{\pi}{2}$. They are however true for angles of all magnitudes, whether primary or secondary. This we proceed to shew*.

8. *Assuming that*

$$\sin(A+B)=\sin A\cos B+\cos A\sin B,$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B,$$

when $A+B$ is less than $\frac{\pi}{2}$, to shew that these formulæ are true for all values of A and B.

I. They are true for all values of A and B between o and $\frac{\pi}{2}$.

Let $A+B$ be $>\frac{\pi}{2}$ and let $A=\frac{\pi}{2}-A'$,

$$B=\frac{\pi}{2}-B'.$$

Then $A+B=\pi-(A'+B')$, and therefore $A'+B'$ must be $<\frac{\pi}{2}$.

Hence the formulæ are true for $A'+B'$.

But $\sin(A+B)=\sin\{\pi-(A'+B')\}$

$$=\sin(A'+B')$$
$$=\sin A'\cos B'+\cos A'\sin B'$$
$$=\sin\left(\frac{\pi}{2}-A\right)\cos\left(\frac{\pi}{2}-B\right)+\cos\left(\frac{\pi}{2}-A\right)\sin\left(\frac{\pi}{2}-B\right)$$
$$=\cos A\sin B+\sin A\cos B;$$

and similarly,

$$\cos(A+B)=\cos\{\pi-(A'+B')\}=-\cos(A'+B')$$
$$=-\cos A'\cos B'+\sin A'\sin B'$$
$$=-\sin A\sin B+\cos A\cos B.$$

Hence the formulæ are true for all values of A and B between o and $\frac{\pi}{2}$.

* The student may, if he chooses, omit Art. 8 on his first reading of the subject.

II. If the formulæ are true for two angles A and B, they are true if we add to these angles, one or both, either $\frac{\pi}{2}$ or any multiple of $\frac{\pi}{2}$.

$$\text{Assume } A' = \frac{\pi}{2} + A, \text{ and } \therefore A = A' - \frac{\pi}{2}.$$

Then

$$\sin(A'+B) = \sin\left(\frac{\pi}{2} + A + B\right) = \cos(A+B)$$

$$= \cos A \cos B - \sin A \sin B.$$

Now replacing A by $A' - \frac{\pi}{2}$ or $-\left(\frac{\pi}{2} - A'\right)$,

$$\sin(A'+B) = \cos\left\{-\left(\frac{\pi}{2} - A'\right)\right\}\cos B - \sin\left\{-\left(\frac{\pi}{2} - A'\right)\right\}\sin B$$

$$= \sin A' \cos B + \cos A' \sin B,$$

and

$$\cos(A'+B) = \cos\left\{\frac{\pi}{2} + (A+B)\right\} = -\sin(A+B)$$

$$= -\sin A \cos B - \cos A \sin B$$

$$= -\sin\left(A' - \frac{\pi}{2}\right)\cos B - \cos\left(A' - \frac{\pi}{2}\right)\sin B$$

$$= \cos A' \cos B - \sin A' \sin B.$$

Hence the formulæ are true for A', *i.e.* $A + \frac{\pi}{2}$ and B.

Similarly we may now shew that they are true also if $\frac{\pi}{2}$ be added to B. Hence they are true if $\frac{\pi}{2}$, or any multiple of $\frac{\pi}{2}$, be added to either or both.

Now this is the same thing as saying that they are true for primary angles of any magnitude whatever; for having shewn that they are true for all angles between o and $\frac{\pi}{2}$, we can, by adding multiples of $\frac{\pi}{2}$, increase them to any required magnitude.

III. They are also true for all secondary angles.

Suppose that B is a secondary angle and equal to $-B'$. Take n such an integer that $2n\pi$ is $> B'$.

Then

$$\sin(A+B) = \sin(A-B') = \sin(2n\pi + A - B')$$
$$= \sin\{A + (2n\pi - B')\}$$
$$= \sin A \cos(2n\pi - B') + \cos A \sin(2n\pi - B')$$

because $2n\pi - B'$ is a primary angle,

$$= \sin A \cos B' - \cos A \sin B'$$
$$= \sin A \cos(-B') + \cos A \sin(-B')$$
$$= \sin A \cos B + \cos A \sin B,$$

$$\cos(A+B) = \cos(A-B') = \cos\{A + (2n\pi - B')\}$$
$$= \cos A \cos(2n\pi - B') - \sin A \sin(2n\pi - B')$$
$$= \cos A \cos B' + \sin A \sin B'$$
$$= \cos A \cos(-B') - \sin A \sin(-B')$$
$$= \cos A \cos B - \sin A \sin B.$$

Next let A and B both be secondary angles, and let $A = -A'$, $B = -B'$.

Then

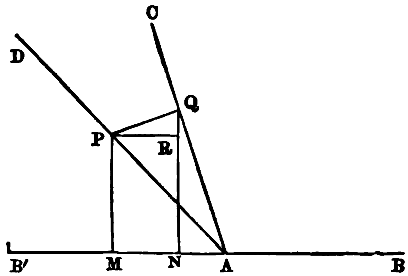
$$\sin(A+B) = \sin(-A' - B') = -\sin(A' + B')$$
$$= -\sin A' \cos B' - \cos A' \sin B'$$
$$= \sin(-A') \cos(-B') + \cos(-A') \sin(-B')$$
$$= \sin A \cos B + \cos A \sin B,$$

$$\cos(A+B) = \cos(-A' - B') = \cos(A' + B')$$
$$= \cos A' \cos B' - \sin A' \sin B'$$
$$= \cos(-A') \cos(-B') - \sin(-A') \sin(-B')$$
$$= \cos A \cos B - \sin A \sin B.$$

Hence the formulæ are true for angles of any magnitude whatever, whether primary or secondary.

9. Their truth can also be established in any assigned case from a geometrical figure, by a process similar to that given when $A + B$ is $< \frac{\pi}{2}$.

Suppose for instance the lines AC, AD to fall as in the figure.



Take any point P in AD, draw PQ perpendicular to AC, PM to AB produced, and draw PR, QN respectively parallel to AB, PM.

Then

$$\text{the angle } PQR = \frac{\pi}{2} - AQN = NAQ = \pi - BAC = \pi - A,$$

$$\text{and the angle } B'AD = \pi - (A+B).$$

Then $\sin(A+B) = \sin\{\pi - (A+B)\}$

$$= \sin B'AD = \frac{PM}{AP} = \frac{QN - QR}{AP}$$

$$= \frac{QN}{AQ}\cdot\frac{AQ}{AP} - \frac{QR}{QP}\cdot\frac{QP}{AP}$$

$$= \sin NAQ.\cos PAQ - \cos PQR.\sin PAQ$$

$$= \sin(\pi - A)\cos B - \cos(\pi - A)\sin B$$

$$= \sin A \cos B + \cos A \sin B.$$

Similarly for $\cos(A+B)$.

10. Collecting these formulæ, we have

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \ldots\ldots\ldots(1),$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B \ldots\ldots\ldots(2),$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \ldots\ldots\ldots(3),$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \ldots\ldots\ldots(4).$$

Then from these equations

$$\left.\begin{array}{ll} (1)+(2)\text{ gives} & \sin(A+B)+\sin(A-B)=2\sin A\cos B \\ (1)-(2)\ \ldots\ldots & \sin(A+B)-\sin(A-B)=2\cos A\sin B \\ (3)+(4)\ \ldots\ldots & \cos(A+B)+\cos(A-B)=2\cos A\cos B \\ (3)-(4)\ \ldots\ldots & \cos(A+B)-\cos(A-B)=-2\sin A\sin B \end{array}\right\}\ldots(A).$$

Now $A+B$ and $A-B$ are two angles, the magnitudes of which are independent of each other, and they may therefore be taken to represent *any two angles whatever*, and A is half their sum and B half their difference.

Hence we may put these formulæ in the following general form.

The sum of the sines of two angles
= 2 sin (half the sum of the angles)
× cos (half their difference)

The difference of the sines of two angles
= 2 sin (half the difference of the angles)
× cos (half their sum)

The sum of the cosines of two angles
= 2 cos (half the sum of the angles)
× cos (half their difference)

The difference of the cosines of two angles
= − 2 sin (half the sum of the angles)
× sin (half their sum) ...(B).

These formulæ are extremely important, and the student is recommended to familiarize himself with them under the form here given. It must be noticed that, in taking the difference of the angles in the second and fourth of these formulæ, the angles must be taken in the same order as the sines or the cosines on the left-hand side of the equations.

11. It is also necessary to be familiar with the converse of these formulæ, so as to express the product of two sines or two cosines, or of a sine and a cosine in terms of the sum or difference of two sines or two cosines.

This may be done by the same process as that by which formulæ (B) were obtained, only that instead of reading the results (A) from left to right, the right-hand side must be taken first. Hence we get

$$\left.\begin{array}{l}\text{The sine of an angle} \times \cos\text{ (any other angle)}\\ = \tfrac{1}{2}\{\sin\text{ (sum of the angles)} + \sin\text{ (their difference)}\}\\ \text{The cosine of an angle} \times \sin\text{ (any other angle)}\\ = \tfrac{1}{2}\{\sin\text{ (sum of the angles)} - \sin\text{ (their difference)}\}\\ \text{The cosine of an angle} \times \text{cosine (any other angle)}\\ = \tfrac{1}{2}\{\cos\text{ (sum of the angles)} + \cos\text{ (their difference)}\}\\ \text{The sine of an angle} \times \text{sine (any other angle)}\\ = \tfrac{1}{2}\{\cos\text{ (difference of the angles)} - \cos\text{ (their sum)}\}\end{array}\right\}\ldots\text{(C)}.$$

The reader will see that the first and second of the formulæ (C) are in reality the same, the apparent distinction arising from the order in which the difference of the angles is taken.

He is recommended before proceeding further to turn to 1, 2, 9, of the examples upon this chapter, where he will see applications of both sets of formulæ.

12. *Trigonometrical functions of* $2A$ and $\frac{A}{2}$.

Putting A for B in formulæ (1) and (3), and in the expression for $\tan(A+B)$, we get

$$\sin 2A = 2\sin A \,.\, \cos A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 \ldots\ldots(6),$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

In (5), (6), (7), putting A for $2A$, and therefore $\frac{A}{2}$ for A,

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

$$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 1 - 2\sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 \ldots\ldots(9),$$

$$\tan A = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

From (9), $2\sin^2\frac{A}{2} = 1 - \cos A$,

$$2\cos^2\frac{A}{2} = 1 + \cos A;$$

$$\therefore \sin\frac{A}{2} = \pm\sqrt{\left(\frac{1-\cos A}{2}\right)} \quad\text{..........(11)},$$

$$\cos\frac{A}{2} = \pm\sqrt{\left(\frac{1+\cos A}{2}\right)} \quad\text{..........(12)}.$$

From these results it appears that, when $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ are determined from $\cos A$, they have each two values. The explanation of this is analogous to that given of the double sign on page 17. It is a point however which will be more fully discussed hereafter.

So from (10) we see that, if $\tan A$ be given to determine $\tan\frac{A}{2}$, we shall have a quadratic equation which will give two values of $\tan\frac{A}{2}$ in terms of $\tan A$.

13. *Sine and cosine of* $3A$ *and* $\frac{A}{3}$.

$$\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A.$$

Substituting the values of $\sin 2A$ and $\cos 2A$ from (5) and (6),

$$\begin{aligned}\sin 3A &= 2\sin A\cos^2 A + (1 - 2\sin^2 A)\sin A\\ &= 2\sin A(1-\sin^2 A) + (1 - 2\sin^2 A)\sin A\\ &= 3\sin A - 4\sin^3 A \quad\text{..............................(13)},\end{aligned}$$

$$\begin{aligned}\cos 3A = \cos(2A + A) &= \cos 2A\cos A - \sin 2A\sin A\\ &= (2\cos^2 A - 1)\cos A - 2\sin^2 A\cos A\\ &\qquad\text{from (5) and (6)},\\ &= 2\cos^3 A - \cos A - 2\cos A(1-\cos^2 A)\\ &= 4\cos^3 A - 3\cos A \quad\text{..................(14)}.\end{aligned}$$

From these results we get, evidently,

$$\sin A = 3 \sin \frac{A}{3} - 4 \sin^3 \frac{A}{3} \quad \text{..................(15)},$$

$$\cos A = 4 \cos^3 \frac{A}{3} - 3 \cos \frac{A}{3} \quad \text{..................(16)}.$$

Hence if $\sin A$ be given to find $\sin \frac{A}{3}$, or $\cos A$ be given to determine $\cos \frac{A}{3}$, we have in each case a cubic equation, and therefore three values.

14. *To find the sine of* $18°$.

$$\text{Since } 54° = 90° - 36°;$$

$$\therefore \cos 54° = \sin 36°, \text{ or } \cos 3 \times 18° = \sin 2 \times 18°.$$

Hence from (15) and (6),

$$4 \cos^3 18° - 3 \cos 18° = 2 \sin 18° \cos 18°.$$

Dividing by $\cos 18°$, which cannot be zero, we get

$4 \cos^2 18 - 3 = 2 \sin 18°$, and putting x for $\sin 18°$,

$$1 - 4x^2 = 2x,$$

$$x^2 + \frac{1}{2}x = \frac{1}{4},$$

$$\therefore x = \frac{\pm\sqrt{5} - 1}{4}.$$

Now $\sin 18°$ cannot be negative, and therefore

$$\sin 18° = \frac{\sqrt{5} - 1}{4}.$$

15. *Given sin A to find sin* $\frac{A}{2}$ *and cos* $\frac{A}{2}$.

$$2 \sin \frac{A}{2} \cos \frac{A}{2} = \sin A,$$

$$\sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} = 1.$$

Adding and subtracting we get

$$\left(\sin \frac{A}{2} + \cos \frac{A}{2}\right)^2 = 1 + \sin A,$$

$$\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)^2 = 1-\sin A\,;$$

$$\left.\begin{array}{l}\therefore\ \sin\dfrac{A}{2}+\cos\dfrac{A}{2} = \pm\sqrt{(1+\sin A)} \\ \sin\dfrac{A}{2}-\cos\dfrac{A}{2} = \pm\sqrt{(1-\sin A)}\end{array}\right\} \ldots\ldots\ldots\ldots (17);$$

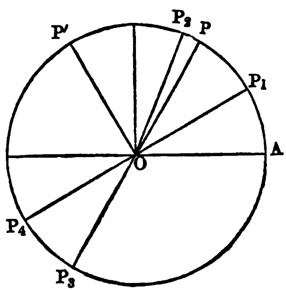
$$\therefore\ \sin\frac{A}{2} = \frac{1}{2}\{\pm\sqrt{(1+\sin A)}\pm\sqrt{(1-\sin A)}\} \ldots\ldots (18),$$

$$\cos\frac{A}{2} = \frac{1}{2}\{\pm\sqrt{(1+\sin A)}\mp\sqrt{(1-\sin A)}\} \ldots\ldots (19).$$

16. From these results we see that both $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ when determined in terms of $\sin A$ have four values according as we take the four combinations of the signs $\pm$ before each of the radicals.

This ambiguity arises from a reason similar to that mentioned on page 17.

It may be readily explained by reference to a figure.



Let OP, OP' be the two positions of the revolving line which correspond to the given value of the sine of A.

Bisecting the primary angle AOP by OP_1, and the primary angle AOP' by OP_2, we get two positions of the revolving line, for which the two sines are different, and also the two cosines different.

Again, bisecting the secondary angle AOP' and the secondary angle AOP, we get OP_3, OP_4 positions of the revolving line for which the sines and cosines are equal in magnitude, but opposite in sign to the sines and cosines of the angles corresponding to the positions OP_2, OP_1, respectively.

Hence we see that sin $\frac{A}{2}$ and cos $\frac{A}{2}$, when determined generally from sin A, must have four values, since each of them *may* correspond to any one of the four positions OP_1, OP_2, OP_3, OP_4 of the revolving line, which have their sines and cosines all different.

17. If we know however the limits between which A lies sufficiently to determine in which of certain quadrants the revolving line corresponding to $\frac{A}{2}$ falls, there is no longer any ambiguity in determining the value of sin $\frac{A}{2}$ and cos $\frac{A}{2}$. This we shall proceed to shew.

18. *To trace the change in the sign of sin $\frac{A}{2}$ + cos $\frac{A}{2}$ and of sin $\frac{A}{2}$ − cos $\frac{A}{2}$ as $\frac{A}{2}$ varies from 0° to 360°, that is, as A varies from zero to 720°.*

Observe that it is only necessary to take notice of the sign of the *greater* of the two functions sin $\frac{A}{2}$ and cos $\frac{A}{2}$, for the sign of the greater one will clearly determine the sign of the whole expression.

Now from −45° to +45° the cosine is greater than the sine and is positive.

Hence from −45° to +45°

$$\sin\frac{A}{2} + \cos\frac{A}{2} \text{ is positive,}$$

$$\text{and } \sin\frac{A}{2} - \cos\frac{A}{2} \text{ is negative;}$$

$$\therefore \sin\frac{A}{2} + \cos\frac{A}{2} = \sqrt{(1 + \sin A)},$$

$$\sin\frac{A}{2} - \cos\frac{A}{2} = -\sqrt{(1 - \sin A)}.$$

From 45° to 135° the sine is greater than the cosine, and is positive;

$$\therefore \sin\frac{A}{2}+\cos\frac{A}{2} \text{ is positive,}$$

$$\text{and } \sin\frac{A}{2}-\cos\frac{A}{2} \text{ is positive;}$$

$$\therefore \sin\frac{A}{2}+\cos\frac{A}{2}=\sqrt{(1+\sin A)},$$

$$\sin\frac{A}{2}-\cos\frac{A}{2}=\sqrt{(1-\sin A)}.$$

From 135° to 225° the cosine is greater numerically than the sine, and is negative;

$$\therefore \sin\frac{A}{2}+\cos\frac{A}{2} \text{ is negative,}$$

$$\text{and } \sin\frac{A}{2}-\cos\frac{A}{2} \text{ is positive;}$$

$$\therefore \sin\frac{A}{2}+\cos\frac{A}{2}=-\sqrt{(1+\sin A)},$$

$$\sin\frac{A}{2}-\cos\frac{A}{2}=\sqrt{(1-\sin A)}.$$

From 225° to 315° the sine is numerically greater than the cosine, and is negative;

$$\therefore \sin\frac{A}{2}+\cos\frac{A}{2} \text{ is negative,}$$

$$\text{and } \sin\frac{A}{2}-\cos\frac{A}{2} \text{ is negative;}$$

$$\therefore \sin\frac{A}{2}+\cos\frac{A}{2}=-\sqrt{(1+\sin A)},$$

$$\sin\frac{A}{2}-\cos\frac{A}{2}=-\sqrt{(1-\sin A)}.$$

Therefore for values of $\frac{A}{2}$ from

$-45°$ to $45°$, $\quad \sin\frac{A}{2}=\frac{1}{2}\{\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}\},$

$$\cos\frac{A}{2}=\frac{1}{2}\{\sqrt{(1+\sin A)}+\sqrt{(1-\sin A)}\},$$

45° to 135°, $\sin\frac{A}{2}=\frac{1}{2}\{\sqrt{(1+\sin A)}+\sqrt{(1-\sin A)}\},$

$$\cos\frac{A}{2}=\frac{1}{2}\{\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}\},$$

135° to 225°, $\sin\frac{A}{2}=\frac{1}{2}\{-\sqrt{(1+\sin A)}+\sqrt{(1-\sin A)}\},$

$$\cos\frac{A}{2}=\frac{1}{2}\{-\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}\},$$

225° to 315°, $\sin\frac{A}{2}=\frac{1}{2}\{-\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}\},$

$$\cos\frac{A}{2}=\frac{1}{2}\{-\sqrt{(1+\sin A)}+\sqrt{(1-\sin A)}\}.$$

Example. Given $\sin 30°=\frac{1}{2}$, find $\sin 15°$.

$$\sin\frac{30°}{2}+\cos\frac{30°}{2}=\sqrt{(1+\sin 30°)}=\sqrt{\left(1+\frac{1}{2}\right)},$$

$$\sin\frac{30°}{2}-\cos\frac{30°}{2}=-\sqrt{(1-\sin 30°)}=-\sqrt{\left(1-\frac{1}{2}\right)};$$

$$\therefore\ 2\sin 15°=\frac{\sqrt{3}-1}{\sqrt{2}},$$

$$2\cos 15°=\frac{\sqrt{3}+1}{\sqrt{2}},$$

$$\sin 15°=\frac{\sqrt{3}-1}{2\sqrt{2}},$$

$$\cos 15°=\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

These might of course be found as follows:

$\sin 15°=\sin(45°-30°)=\sin 45°\cos 30°-\cos 45°\sin 30°,$

$\cos 15°=\cos(45°-30°)=\cos 45°\cos 30°+\sin 45°\sin 30°,$

and the sines and cosines of 45° and 30° are known.

CHAPTER V.

DETERMINATION, A PRIORI, OF THE NUMBER OF VALUES WHICH ANY ASSIGNED TRIGONOMETRICAL FUNCTION MAY HAVE WHEN DETERMINED FROM ANY OTHER FUNCTION OF THE ANGLE, OR OF A MULTIPLE OR SUBMULTIPLE OF THE ANGLE.

THESE articles are given to illustrate the employment of the general formulæ given in Chap. III. Arts. 7, 8, 9, for equisinal, equicosinal angles, &c.

1. *To determine,* à priori, *how many values* $\sin A$ *will have when determined from* $\cos A$, *which is supposed to be known.*

If a be the circular measure of the least primary angle which has its cosine equal to $\cos A$, any of the angles included in the formula $2n\pi \pm a$ will have its cosine equal to $\cos A$. Hence, in finding the value of $\sin A$ from $\cos A$, we do not know which individual of this group we must take; for although the cosines of them all are the same, it does not follow that the sines are so. Hence all the values of $\sin A$ which can be got by taking all the angles whose cosine is $\cos A$ are included in the formula

$$\sin(2n\pi \pm a),$$

which is equal to $\sin(\pm a)$ or $\pm \sin a$;

so that $\sin A$, when determined from $\cos A$, has two values equal in magnitude but of opposite signs. This corresponds to and explains Chap. II. Art. 7 (II).

2. *Let* $\tan A$ *be given, to determine* $\sin A$.

Since all the angles which have a given tangent are included in the formula $n\pi + a$, where a is the least positive angle whose tangent is $\tan A$, all the values which $\sin A$ can have when determined from $\tan A$ will be included in $\sin(n\pi + \alpha)$, which is equal to $\pm\sin\alpha$, according as n is even or odd.

Hence $\sin A$, when determined from $\tan A$, has two values. Compare Chap. II. 7 (III).

3. *Given* $\cos A$, *to find how many values* $\sin\frac{A}{2}$ *and* $\cos\frac{A}{2}$ *will have when expressed in terms of it.*

Let α be the circular measure of the least primary angle whose cosine is $\cos A$. Then all the angles, the cosines of which are equal to $\cos A$, are included in the expression $2n\pi \pm \alpha$.

Hence all the values which $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ can have when expressed in terms of $\cos A$, will be included respectively in $\sin\frac{2n\pi \pm \alpha}{2}$ and $\cos\frac{2n\pi \pm \alpha}{2}$, or in $\sin\left(n\pi \pm \frac{\alpha}{2}\right)$ and $\cos\left(n\pi \pm \frac{\alpha}{2}\right)$, which are equal to $\pm\sin\frac{\alpha}{2}$ and $\pm\cos\frac{\alpha}{2}$,

for if n is even, they are equal to $\pm\sin\frac{\alpha}{2}$ and to $\cos\frac{\alpha}{2}$ respectively;

and if n is odd, they are equal to $\mp\sin\frac{\alpha}{2}$ and to $-\cos\frac{\alpha}{2}$ respectively.

This corresponds to and explains the results (11) and (12) in Chap. IV.

4. *Given* $\sin A$, *to determine,* à priori, *how many values* $\sin\frac{A}{2}$ *and* $\cos\frac{A}{2}$ *will have when expressed in terms of it.*

Let α be the circular measure of the least primary angle, the sine of which is equal to $\sin A$. Then all the angles whose sines are equal to $\sin A$ are included in the general formula

$$n\pi + (-1)^n\alpha.$$

Hence all the values which $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ can have when expressed in terms of $\sin A$ will be included in

$$\sin\left(\frac{n\pi+(-1)^n\alpha}{2}\right) \text{ and } \cos\frac{n\pi+(-1)^n\alpha}{2} \text{ respectively.}$$

Now n is either even or odd, *i.e.* it is of one of the forms 2λ or $2\lambda+1$.

Taking $n=2\lambda$, we have

$$\sin\frac{n\pi+(-1)^n\alpha}{2} = \sin\left(\lambda\pi+\frac{\alpha}{2}\right) = \pm\sin\frac{\alpha}{2},$$

according as λ is even or odd,

$$\text{and } \cos\frac{n\pi+(-1)^n\alpha}{2} = \cos\left(\lambda\pi+\frac{\alpha}{2}\right) = \pm\cos\frac{\alpha}{2},$$

according as λ is even or odd.

Taking $n=2\lambda+1$, we have

$$\sin\frac{n\pi+(-1)^n\alpha}{2} = \sin\left(\lambda\pi+\frac{\pi-\alpha}{2}\right) = \pm\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) = \pm\cos\frac{\alpha}{2},$$

and

$$\cos\frac{n\pi+(-1)^n\alpha}{2} = \cos\left(\lambda\pi+\frac{\pi-\alpha}{2}\right) = \pm\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) = \pm\sin\frac{\alpha}{2},$$

according as λ is even or odd.

Hence $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$, when expressed in terms of $\sin A$, will each have four values, which corresponds to Arts. 15 and 16 of Chap. IV.

5. *Given* $\cos A$, *to determine,* à priori, *how many values* $\cos\frac{A}{3}$ *will have when expressed in terms of it.*

Let α be, as before, the circular measure of the least primary angle whose cosine is equal to $\cos A$. Then all the angles whose cosines are equal to $\cos A$ will be included in the expression $2n\pi\pm\alpha$, and consequently all the different values which $\cos\frac{A}{3}$ can have when expressed in terms of $\cos A$ will be included in

$$\cos\frac{2n\pi \pm a}{3}.$$

Now n must be of one of the forms 3λ, $3\lambda+1$, $3\lambda+2$, since every number is either exactly divisible by 3, or divisible by 3 with a remainder, which will be either 1 or 2.

Taking $n=3\lambda$, we have

$$\cos\frac{2n\pi \pm a}{3} = \cos\left(2\lambda\pi \pm \frac{a}{3}\right) = \cos\left(\pm\frac{a}{3}\right) = \cos\frac{a}{3}.$$

Taking $n=3\lambda+1$,

$$\cos\frac{2n\pi \pm a}{3} = \cos\left(2\lambda\pi + \frac{2\pi \pm a}{3}\right) = \cos\frac{2\pi \pm a}{3}.$$

Taking $n=3\lambda+2$,

$$\cos\frac{2n\pi \pm a}{3} = \cos\left(2\lambda\pi + \frac{4\pi \pm a}{3}\right)$$

$$= \cos\frac{4\pi \pm a}{3}$$

$$= \cos\left(2\pi - \frac{2\pi \mp a}{3}\right)$$

$$= \cos\frac{2\pi \mp a}{3}.$$

Hence $\cos\frac{A}{3}$, when determined from $\cos A$, has three values, viz. $\cos\frac{a}{3}$, $\cos\frac{2\pi + a}{3}$ and $\cos\frac{2\pi - a}{3}$.

6. In Chap. IV. article 13, we proved that

$$\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}.$$

Hence if $\cos A$ be given and be equal to c suppose, we have a cubic equation

$$4\cos^3\frac{A}{3} - 3\cos\frac{A}{3} = c \text{......}(1)$$

to solve, in order to find the value of $\cos\frac{A}{3}$ in terms of c.

We see, from the last article, without endeavouring to solve the equation, that these values are $\cos\frac{a}{3}$, $\cos\frac{2\pi+a}{3}$ and $\cos\frac{2\pi-a}{3}$, where a is the least primary angle, the cosine of which is equal to c.

The general form of a cubic equation whose roots are a, β, γ, is

$$(x-a)(x-\beta)(x-\gamma)=0,$$

and the coefficient of x^2 in this is $-(a+\beta+\gamma)$. Hence, if in a cubic equation there is no term involving x^2, we must have the sum of the roots equal to zero.

Now, since there is no term involving $\cos^2\frac{A}{3}$ in (1), we see that

$$\cos\frac{a}{3}+\cos\frac{2\pi+a}{3}+\cos\frac{2\pi-a}{3}=0.$$

That this is true is evident otherwise; for (Art. 10, Ch. IV.)

$$\cos\frac{2\pi+a}{3}+\cos\frac{2\pi-a}{3}=2\cos\frac{2\pi}{3}\cos\frac{a}{3},$$

$$\text{and }\cos\frac{2\pi}{3}=\cos 120^\circ=-\frac{1}{2};$$

$$\therefore\ \cos\frac{a}{3}+\cos\frac{2\pi+a}{3}+\cos\frac{2\pi-a}{3}=\cos\frac{a}{3}-\cos\frac{a}{3}=0.$$

7. *Given sin A, to determine how many values sin mA and cos mA will have when expressed in terms of it, m being an integer.*

Let a represent the circular measure of the least primary angle, the sine of which is equal to $\sin A$.

Then all the angles whose sines are equal to $\sin A$ are included in the formula $n\pi+(-1)^n a$; and therefore all the values which $\sin mA$ and $\cos mA$ can have when determined from $\sin A$ will be included in

$\sin m\{n\pi+(-1)^n a\}$ and $\cos m\{n\pi+(-1)^n a\}$ respectively.

1st. Let n be even.

These are equal to $\sin ma$ and $\cos ma$ respectively, whether m be even or odd.

2nd. Let n be odd.

Then, if m be even,

$$\sin m\{n\pi+(-1)^n a\}=\sin(mn\pi-ma)=\sin(-ma)=-\sin ma,$$

$$\cos m\{n\pi+(-1)^n a\}=\cos(mn\pi-ma)=\cos(-ma)=\cos ma;$$

and if m be odd, since then mn is odd,

$$\sin m\{n\pi+(-1)^n a\}=\sin(mn\pi-ma)=\sin(\pi-ma)=\sin ma,$$

$$\cos m\{n\pi+(-1)^n a\}=\cos(mn\pi-ma)=\cos(\pi-ma)$$
$$=-\cos ma;$$

$\sin mA$ and $\cos mA$ therefore can each only have two values, viz. $\pm\sin ma$ and $\pm\cos ma$.

If m were not an integer we should have to proceed somewhat differently.

We subjoin an example in a particular case.

8. *Given* $\sin A$, *to determine*, à priori, *how many values* $\sin \frac{3}{4}A$ *will have when expressed in terms of it.*

Let a represent the circular measure of the least primary angle, the sine of which is equal to $\sin A$. Then all the angles whose sines are equal to $\sin A$ are included in the expression

$$n\pi+(-1)^n a,$$

and therefore all the values which $\sin\frac{3}{4}A$ can have when expressed in terms of $\sin A$ are included in

$$\sin\tfrac{3}{4}\{n\pi+(-1)^n a\}.$$

Now n is of one of the forms 4λ, $4\lambda+1$, $4\lambda+2$, $4\lambda+3$.

If $n=4\lambda$,

$$\sin\tfrac{3}{4}\{n\pi+(-1)^n a\}=\sin(3\lambda\pi+\tfrac{3}{4}a)=\pm\sin\tfrac{3}{4}a,$$

according as λ is even or odd.

If $n=4\lambda+1$,

$$\sin\tfrac{3}{4}\{n\pi+(-1)^n a\}=\sin\{3\lambda\pi+\tfrac{3}{4}(\pi-a)\}=\pm\sin\tfrac{3}{4}(\pi-a),$$

according as λ is even or odd.

If $n=4\lambda+2$,

$$\sin\tfrac{3}{4}\{n\pi+(-1)^n a\}=\sin\left(3\lambda\pi+\frac{3\pi}{2}+\frac{3a}{4}\right)$$

$$=\pm\sin\left(\frac{3\pi}{2}+\frac{3a}{4}\right), \text{ according as } \lambda \text{ is even or odd,}$$

$$=\mp\cos\frac{3a}{4}.$$

If $n=4\lambda+3$,

$$\sin\tfrac{3}{4}\{n\pi+(-1)^n a\}=\sin\left(3\lambda\pi+\frac{9\pi-3a}{4}\right)$$

$$=\pm\sin\frac{9\pi-3a}{4}, \text{ according as } \lambda \text{ is even or odd.}$$

$$=\pm\sin\left(2\pi+\frac{\pi-3a}{4}\right)$$

$$=\pm\sin\frac{\pi-3a}{4}.$$

Therefore $\sin\frac{3}{4}A$, when determined in terms of $\sin A$, has *eight* values, which are represented by

$$\pm\sin\tfrac{3}{4}a,\quad \pm\sin\tfrac{3}{4}(\pi-a),\quad \pm\cos\tfrac{3}{4}a,\quad \pm\sin\frac{\pi-3a}{4},$$

all of which are different.

The examples already given will be found sufficient to indicate the method to be pursued in any analogous questions which can be proposed.

CHAPTER VI.

INVERSE TRIGONOMETRICAL FUNCTIONS.

1. THE symbols $\sin^{-1} a$, $\cos^{-1} b$, $\tan^{-1} c$, &c., are used to denote respectively an angle the sine of which is a, an angle of which the cosine is b, and an angle the tangent of which is c.

We can write therefore, if $\sin\theta = a$, $\cos\phi = b$, $\tan\psi = c$, $\theta = \sin^{-1} a$, $\phi = \cos^{-1} b$, $\psi = \tan^{-1} c$.

These latter equations are true, but we must recollect that the expressions $\sin^{-1} a$, $\cos^{-1} b$, $\tan^{-1} c$, do not stand only for certain individual angles θ, ϕ, ψ respectively, but for any one of the angles of which the sines, cosines, and tangents respectively are equal to $\sin\theta$, $\cos\phi$, and $\tan\psi$, so that we should more correctly write

$$\left.\begin{aligned} \sin^{-1} a &= n\pi + (-1)^n \theta, \\ \cos^{-1} b &= 2n\pi \pm \phi, \\ \tan^{-1} c &= n\pi + \psi, \end{aligned}\right\} \text{ where } n \text{ is any integer, and may be affected either with a positive or negative sign.}$$

The functions $\sin^{-1} a$, $\cos^{-1} b$, &c. are called *inverse* trigonometrical functions from the nature of the notation, since in algebra x^{-1} is the inverse of x, and therefore by analogy $\sin^{-1}$, $\tan^{-1}$, &c. are called the inverse of sine and tangent respectively.

2. *Given $\sin^{-1} a$ and $\sin^{-1} b$, to find $\sin^{-1} a + \sin^{-1} b$ and $\sin^{-1} a - \sin^{-1} b$.*

$$\text{Let } \sin^{-1} a = \theta, \quad \sin^{-1} b = \phi.$$

$$\text{Then } a = \sin\theta, \quad b = \sin\phi,$$

and $\cos^2\theta = 1 - \sin^2\theta = 1 - a^2$;

$$\therefore\ \cos\theta = \pm\sqrt{(1-a^2)}.$$

Similarly, $\cos\phi = \pm\sqrt{(1-b^2)}$,

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$= \pm a\sqrt{(1-b^2)} \pm b\sqrt{(1-a^2)};$$

and $\therefore\ \theta+\phi = \sin^{-1}\{\pm a\sqrt{(1-b^2)} \pm b\sqrt{(1-a^2)}\}$,

or $\sin^{-1}a + \sin^{-1}b = \sin^{-1}\{\pm a\sqrt{(1-b^2)} \pm b\sqrt{(1-a^2)}\}$.

Similarly,

$$\sin^{-1}a - \sin^{-1}b = \sin^{-1}\{\pm a\sqrt{(1-b^2)} \mp b\sqrt{(1-a^2)}\}.$$

3. *To prove that* $\tan^{-1}a + \tan^{-1}b = \tan^{-1}\dfrac{a+b}{1-2b}$.

Let $\tan^{-1}a = \theta$, $\tan^{-1}b = \phi$.

Then $a = \tan\theta$, $b = \tan\phi$,

and $\tan(\theta+\phi) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta \,.\, \tan\phi} = \dfrac{a+b}{1-ab}$;

$$\therefore\ \tan^{-1}a + \tan^{-1}b = \theta + \phi = \tan^{-1}\frac{a+b}{1-ab}.$$

The exact meaning of this formula is worthy of consideration. Since there are several angles of which the tangents are a, and again several of which the tangents are b, and also several of which the tangent is $\dfrac{a+b}{1-ab}$, we should notice that the above formula expresses the following fact;

The sum of *any* two of the angles of which the tangents are a and b respectively is equal to *some one* of the angles of which the tangent is $\dfrac{a+b}{1-ab}$.

Similar remarks of course apply to the expressions for $\sin^{-1}a + \sin^{-1}b$, and in fact to all formulæ in which these inverse functions are combined.

Evidently by the same process we get

$$\tan^{-1} a - \tan^{-1} b = \tan^{-1} \frac{a-b}{1+ab}.$$

4. Again, if $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c$ be required, we have

$$\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \tan^{-1} \frac{a+b}{1-ab} + \tan^{-1} c$$

$$= \tan^{-1} \frac{\frac{a+b}{1-ab} + c}{1 - \frac{a+b}{1-ab} c}$$

$$= \tan^{-1} \frac{a+b+c-abc}{1-(ab+bc+ca)},$$

and similarly for the sum of any number of angles.

The reader is referred to the examples upon this chapter for a number of questions which will illustrate and further explain the methods of combining these inverse functions.

CHAPTER VII.

TRIGONOMETRICAL FUNCTIONS OF THREE ANGLES AND RELATIONS AMONG THE TRIGONOMETRICAL FUNCTIONS OF THE ANGLES OF A TRIANGLE.

DENOTE the sum of any three angles θ_1, θ_2, θ_3 by S, and their sines and cosines respectively by s_1, s_2, s_3, c_1, c_2, c_3.

Then,

$$\begin{aligned}\sin S = \sin(\theta_1+\theta_2+\theta_3) &= \sin\{(\theta_1+\theta_2)+\theta_3\}\\ &= \sin(\theta_1+\theta_2)\cos\theta_3+\cos(\theta_1+\theta_2)\sin\theta_3\\ &= s_1c_2c_3+s_2c_3c_1+s_3c_1c_2-s_1s_2s_3.\end{aligned}$$

Now, recollecting that

$$\sin(-A) = -\sin A \text{ and } \cos(-A) = \cos A,$$

we can obtain $\sin(-\theta_1+\theta_2+\theta_3)$, i.e. $\sin(S-2\theta_1)$, from $\sin S$ by simply changing the sign of s_1. Hence we have

$$\sin S = s_1c_2c_3+s_2c_3c_1+s_3c_1c_2-s_1s_2s_3 \quad\dots\dots\dots(1),$$

$$\sin(S-2\theta_1) = -s_1c_2c_3+s_2c_3c_1+s_3c_1c_2+s_1s_2s_3 \quad\dots\dots\dots(2),$$

$$\sin(S-2\theta_2) = s_1c_2c_3-s_2c_3c_1+s_3c_1c_2+s_1s_2s_3 \quad\dots\dots\dots(3),$$

$$\sin(S-2\theta_3) = s_1c_2c_3+s_2c_3c_1-s_3c_1c_2+s_1s_2s_3 \quad\dots\dots\dots(4).$$

From these expressions we can deduce a variety of results.

Adding (2), (3), (4), and subtracting (1), we get

$$\sin(S-2\theta_1)+\sin(S-2\theta_2)+\sin(S-2\theta_3)-\sin S = 4s_1s_2s_3 \quad\dots\dots\dots(\text{A}).$$

If $\theta_1+\theta_2+\theta_3=180°$, *i.e.* if θ_1, θ_2, θ_3 are the angles of a triangle, recollecting that $\sin(180°-\theta)=\sin\theta$, (A) becomes

$$\sin 2\theta_1+\sin 2\theta_2+\sin 2\theta_3=4\sin\theta_1 \sin\theta_2 \sin\theta_3 \text{.........(B)}.$$

If instead of θ_1, θ_2, θ_3 in the general formula (A), we write $\frac{\theta_1}{2}$, $\frac{\theta_2}{2}$, $\frac{\theta_3}{2}$, we shall have $S=\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}=90°$; and therefore

$$\cos\theta_1+\cos\theta_2+\cos\theta_3=1+4\sin\frac{\theta_1}{2}.\sin\frac{\theta_2}{2}.\sin\frac{\theta_3}{2} \text{..........(C)}.$$

Dividing (1) by $c_1c_2c_3$, we have

$$\tan\theta_1+\tan\theta_2+\tan\theta_3=\tan\theta_1.\tan\theta_2.\tan\theta_3 \text{......(D)},$$

and dividing (1) by $s_1s_2s_3$,

$$\cot\theta_1\cot\theta_2+\cot\theta_2\cot\theta_3+\cot\theta_3\cot\theta_1=1 \text{......(E)}.$$

Again,

$$\cos(\theta_1+\theta_2+\theta_3)=\cos(\theta_1+\theta_2)\cos\theta_3-\sin(\theta_1+\theta_2)\sin\theta_3,$$

$$\text{or } \cos S=c_1c_2c_3-c_1s_2s_3-c_2s_3s_1-c_3s_1s_2 \text{............(7)}.$$

Hence changing the sign of s_1, we get $\cos(-\theta_1+\theta_2+\theta_3)$, or

$$\cos(S-2\theta_1)=c_1c_2c_3-c_1s_2s_3+c_2s_3s_1+c_3s_1s_2 \text{.....(8)}.$$

Similarly, $\cos(S-2\theta_2)=c_1c_2c_3+c_1s_2s_3-c_2s_3s_1+c_3s_1s_2$.....(9),

$$\cos(S-2\theta_3)=c_1c_2c_3+c_1s_2s_3+c_2s_3s_1-c_3s_1s_2 \text{...(10)}.$$

Adding (7), (8), (9), (10), we get

$$\cos(S-2\theta_1)+\cos(S-2\theta_2)+\cos(S-2\theta_3)+\cos S$$
$$=4\cos\theta_1\cos\theta_2\cos\theta_3 \text{......(F)},$$

and if $\theta_1+\theta_2+\theta_3=180°$, (F) becomes

$$\cos 2\theta_1+\cos 2\theta_2+\cos 2\theta_3+4\cos\theta_1\cos\theta_2\cos\theta_3+1=0 \text{..(G)}.$$

Again, since $\cos 2\theta_1=2\cos^2\theta_1-1$,

we get from (G)

$$\cos^2\theta_1+\cos^2\theta_2+\cos^2\theta_3+2\cos\theta_1\cos\theta_2\cos\theta_3=1 \text{......(H)}.$$

Putting $\frac{\theta_1}{2}, \frac{\theta_2}{2}, \frac{\theta_3}{2}$, respectively for $\theta_1, \theta_2, \theta_3$ in (F),

and remembering that $\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} = 90^\circ$,

we get

$$\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 4\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2} \quad \text{.......(I).}$$

Many more symmetrical relations might be proved to subsist among the trigonometrical functions of the angles of a triangle. (See examples upon Chap. IV.)

CHAPTER VIII.

LOGARITHMS, LOGARITHMIC AND TRIGONOMETRICAL TABLES.

1. LET x be that power to which a number a must be raised in order to be equal to a number N. Then

$$a^x = N.$$

The number x is called the logarithm of N with reference to a, or, as it is usually expressed, to the *base a*.

2. *The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.*

Let a be the base, M, N the numbers, and x and y their logarithms respectively to the base a.

Then by the definitions $M = a^x$,

$$N = a^y;$$

$$\text{and } \therefore\ MN = a^x \times a^y = a^{x+y};$$

therefore by definition $x+y$ is the logarithm of MN to base a, or denoting the logarithm of M by $\log M$, we have

$$\log MN = x + y = \log M + \log N.$$

3. *The logarithm of the quotient of two numbers is the difference of the logarithms of the two numbers.*

Using the same notation

$$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y};$$

$$\therefore\ \log \frac{M}{N} = x - y = \log M - \log N.$$

4. *The logarithm of the m^{th} power of any number is m times the logarithm of the number.*

Let $a^x = M$ as before; *i.e.* $x = \log M$.

$$\text{Then } M^m = (a^x)^m = a^{mx};$$

$$\therefore \log M^m = mx = m \log M,$$

and this is true whether m be an integer or a fraction.

5. We thus see, from the preceding articles, that, by means of logarithms, the processes of multiplication and division are reduced to those of addition and subtraction, and those of evolution and involution to multiplication and division.

For, suppose we know the separate logarithms of M and N, we have only to add them together to obtain the log of MN, and if we have the means of at once determining the number corresponding to a given logarithm, we can find the number corresponding to $\log MN$, *i.e.* MN itself. By this means laborious arithmetical processes are very materially facilitated. It is of course necessary for this purpose that the calculator should be in possession of a "Table of Logarithms," that is, of a table giving the logarithm of any number he may have to employ. The construction and use of such a table we now proceed to explain. Although theoretically any base might be employed, that which is most convenient in practice is 10, the radix of the ordinary scale of arithmetical notation. Tables of logarithms are also calculated to a base e, where e stands for a certain decimal 2·718218, but we shall not use them in this treatise. They are called Napierian logarithms, from Lord Napier their inventor.

When therefore we speak of logarithms, we must be understood to mean logarithms to the base 10, unless the contrary is specified.

6. *Characteristic and mantissa of a logarithm.*

$$\text{Since } 10^0 = 1,$$
$$10^1 = 10,$$
$$10^2 = 100,$$
$$10^3 = 1000,$$
and so on.

Therefore for any number between 1 and 10 the logarithm is between 0 and 1.

For any number between 10 and 100 the logarithm is between 1 and 2.

For any number between 100 and 1000 the logarithm is between 2 and 3.

And similarly for any number between 10^n and 10^{n+1} the logarithm is between n and $n+1$.

Now 10^n has $n+1$ digits, being expressed by writing down unity followed by n zeros, and 10^{n+1} similarly has $n+2$ digits. Hence any number from 10^n to 10^{n+1} (including 10^n and excluding 10^{n+1}) will have $n+1$ digits if it be a whole number, or $n+1$ integral places followed by a decimal if it is not integral. From this we conclude that the logarithm of any number containing $n+1$ digits in its integral part will be n followed by some decimal.

Thus log 23453487 will be the whole number 7 followed by a decimal part, being in fact 7·3702074.

This decimal part of the logarithm is called the *mantissa,* the integral part n being termed the characteristic.

$$\text{Again,}\quad 10^0 = 1 = 1,$$

$$10^{-1} = \frac{1}{10} = {\cdot}1,$$

$$10^{-2} = \frac{1}{100} = {\cdot}01,$$

$$10^{-3} = \frac{1}{1000} = {\cdot}001,$$

and so on.

Hence for any number between 1 and ·1 the logarithm is between 0 and -1.

For any number between ·1 and ·01 the logarithm is between -1 and -2.

For any number between ·01 and ·001 the logarithm is between -2 and -3.

And similarly for any number between 10^{-n} and $10^{-(n+1)}$ the logarithm is between $-n$ and $-(n+1)$; that is, it will be a negative number consisting of an integral part $-n$ as characteristic, and a negative decimal as mantissa. Or if we wish to have the characteristic only negative, we must add $+1$ to the mantissa, and -1 to the characteristic, so that we shall have then $-(n+1)$ as the characteristic, which will be followed by a positive mantissa.

Now 10^{-n} is expressed by writing down unity, prefixing $n-1$ ciphers, and placing a decimal point before these n figures; and similarly $10^{-(n+1)}$ will be a decimal expressed by n zeros followed by unity.

Hence any number lying between 10^{-n} and $10^{-(n+1)}$ will be a decimal, having n ciphers preceding its first significant digit. The rule then for obtaining the characteristic of any number less than unity is, to add unity to the number of ciphers preceding the first significant digit of the number, it being always recollected that on this supposition the mantissa is positive.

Thus the logarithm of ·0023453487 is $\bar{3}·3702074$ the minus sign being written above the characteristic, to shew that it alone, and not the mantissa, is negative.

7. If we know the logarithm of any number we can at once write down the logarithm of any other number which has the same digits, but only differs from the given number in the position of the decimal point.

The position of the decimal point in any decimal is affected by multiplying or dividing by some power of 10.

Let N be the number, the logarithm of which is known.

Then $10^n \times N$ is a number with the same digits as N, but with the decimal point moved n places to the right, and $\frac{N}{10^n}$ is a number with the same digits as N, but with the decimal point moved n places to the left.

Now $\log 10^n \times N = \log 10^n + \log N = n + \log N,$

and $\log \frac{N}{10^n} = \log N - \log 10^n = -n + \log N.$

Hence, n being an integer, it is only the characteristic of the logarithm which differs in these two results from the logarithm of N. Therefore by giving the logarithm its proper characteristic in accordance with the rules investigated in Art. 5, we can deduce the logarithms of $10^n \times N$ and $\frac{N}{10^n}$ from that of N.

Thus, for instance, knowing the logarithm of 3·4567 to be ·5386617, we can at once write down the logarithms of 34567, 3456·7, 345·67, 34·567, ·34567, ·034567, and any number formed by adding ciphers to 34567, or any decimal formed by prefixing ciphers to the same digits.

$$\text{Thus } \log 345{\cdot}67 = 2{\cdot}5386617,$$
$$\log 3456700 = 6{\cdot}5386617,$$
$$\log {\cdot}00034567 = \bar{4}{\cdot}5386617,$$

so that the one register ·5386617 in the tables is sufficient for the logarithms of all three numbers.

It is this which gives logarithms calculated to the base 10 the advantage over all others.

Vide the examples on this chapter.

8. The trigonometrical functions, being themselves numerical quantities, have corresponding logarithms, which are of continual occurrence in trigonometrical calculations. The sines and cosines of all angles, and the tangents of all angles, less than 45°, being less than unity, the log sines, log cosines of all angles, and log taus of angles less than 45°, will be negative quantities. This being the case, in the tables which are drawn up of the logarithms of the trigonometrical functions, and which are generally calculated at intervals of 10″, the logarithms are increased by 10 for convenience. The logarithms so increased are called *tabular logarithms* of the trigonometrical functions, and the tabular logarithm of sin θ or cos θ, for instance, is generally written $L \sin\theta$ and $L \cos\theta$, to distinguish it from the actual logarithms which would be indicated by $\log\sin\theta$ and $\log\cos\theta$, the difference being that

$$L\sin\theta = \log\sin\theta + 10,$$
$$L\cos\theta = \log\cos\theta + 10.$$

Tables are also calculated not only of the logarithms of the trigonometrical functions, but also of the values of the functions themselves for all angles from 0° to 90°, at intervals of 10″ or at intervals of 60″. For the method however of calculating such tables the reader is referred to more advanced works upon the subject.

9. *Logarithms of numbers, of functions, &c. not given in the tables.*

We often require to find the logarithms of a number which does not occur in the tables; for instance, supposing that the tables do not give the logarithm of any numbers containing more than five digits, it may be necessary to find that of a number of six digits. The calculation is made by "*the method of proportional parts,*" the statement of which is, *The increase of the logarithm is proportional to the increase of the number, the logarithm of which is taken.* The method of proceeding is as follows:

Let δ be the difference between the logarithms of the two consecutive numbers given in the tables, between which the number lies, the logarithm of which is required; and let d be the difference between the least of these two numbers and the number of which the logarithm is required; let x be what must be added to the logarithm of the least of the two numbers given in the tables, to produce the required logarithm. Then, since 1 is the difference of the two consecutive numbers in the tables, we have the proportion

$$\frac{x}{\delta} = \frac{d}{1},$$

which gives x.

It is necessary to observe that we can, in finding the logarithm, always consider a number containing more digits than those which have their logarithms given in the tables, as intermediate in value to two numbers there given, even though it may not be so in reality; for we can suppose a decimal point to be placed before the additional digits, since the logarithm of the number so altered will only differ from the logarithm of the actual number given in its characteristic, which can be at once assigned.

Thus, if in tables calculated only for 5 digits, the logarithm of 3573489 were required, we should proceed to find in reality the logarithm of 35734·89, and then by assigning the proper characteristic we should obtain log 3573489.

10. The method will however be best illustrated by an example.

Given from the tables

$$\log 35734 = 4\cdot5530816,$$

$$\log 35735 = 4\cdot5530938,$$

required log 3573489.

log 35734·89 is intermediate in value to log 35735 and log 35734, the difference between which is ·0000122.

This corresponds to δ, and d is ·89.

Hence we have

$$\frac{x}{\cdot0000122} = \cdot89;$$

$$\therefore\ x = \cdot000010858,$$

$$\text{or } x = \cdot0000109,$$

omitting the figures beyond the 7th place of decimals and replacing the 8 in the 7th place by 9, since 58 is more than 50;

$$\therefore \log 3573489 = 6 + \cdot5530816$$

$$+ \cdot0000109$$

$$= 6\cdot5530925$$

11. The converse process, viz. that of finding the number corresponding to a given logarithm which is not registered in the tables, is exactly analogous.

Suppose we have given 6·5530925, and are required to find the number of which it is the logarithm. Referring to the table we find that the numbers given there to which ·5530925 is intermediate are ·5530816 and ·5530938. These are the logarithms of 35734 and 35735, and the difference between them is ·0000122. The difference between the lowest of these and ·5530925 is ·0000109. Hence if d is what must be added to 35734 to get the number of which 6·5530925 is the logarithm, we have the proportion

$$\frac{d}{1} = \frac{\cdot 0000109}{\cdot 0000122} = \frac{109}{122}$$

$$= \cdot 89\,;$$

therefore the number, the logarithm of which is 6·5530925, is 3573489.

12. The tables of the trigonometrical functions and of their logarithms being calculated for angles at intervals of 10″ or of 60″, it is frequently necessary to calculate the functions or logarithms of the functions of angles which do not occur in the tables.

This is also effected by the method of proportional parts, which when applied to angles is stated thus:

The increase or decrease of the trigonometrical function is proportional to the increase of the angle.

The increase or decrease of the logarithm of a trigonometrical function is proportional to the increase of the angle.

It should be noticed that as the angle increases from zero to 90°, the sine, tangent, and secant increase in magnitude; but that the cosine, cotangent, and cosecant decrease in magnitude.

For instances of the employment of this rule the reader is referred to the examples.

The truth of it of course requires proof, which, however, I think it better to omit in this elementary work.

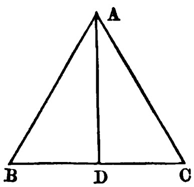
The results given by it are moreover only approximate, but the error is *in general* so small as to be quite safely neglected. We say, *in general*, because there are certain cases in the calculation of angles where the error is too large to allow of its being left out of consideration. In these cases the method of proportional parts is said to fail, and other processes of calculation have to be employed. The discussion of these however is beyond the scope of this treatise.

CHAPTER IX.

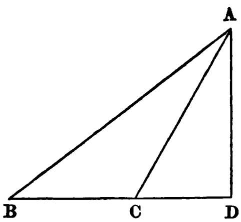
SOLUTION OF TRIANGLES.

IF ABC be a triangle, the lengths of the sides opposite to the angles A, B, C respectively are usually denoted by a, b, c, and accordingly this is the notation which we shall employ.

1. *Proportionality of the sides and the sines of the opposite angles.*



(1)



(2)

From the angle A of a triangle ABC draw a perpendicular AD upon the opposite side, which will fall within or without the triangle, according as the angle C is acute or obtuse, as is represented in figs. (1) and (2) respectively.

Then fig. (1), $\dfrac{AD}{AB} = \sin B$ and $\dfrac{AD}{AC} = \sin C$,

$$\text{or } AD = c \sin B = b \sin C,$$

and fig. (2), $\frac{AD}{AB} = \sin B,$

$$\text{and } \frac{AD}{AC} = \sin ACD = \sin (180^\circ - C) = \sin C;$$

$$\therefore AD = c \sin B = b \sin C.$$

Hence in each case

$$c \sin B = b \sin C,$$

$$\text{or } \frac{\sin B}{b} = \frac{\sin C}{c}.$$

An exactly similar proof would shew that

$$\frac{\sin A}{a} = \frac{\sin B}{b}.$$

$$\text{Hence } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

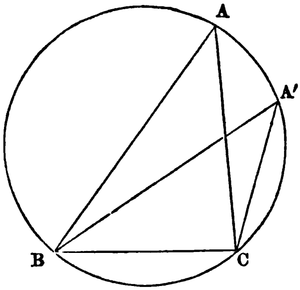
2. This relation between the sides and the sines of the opposite angles furnishes us with two equations involving the six parts of which a triangle is composed, viz. the three sides and the three angles. We know also that

$$A + B + C = 180^\circ.$$

This is another equation, so that we have three equations between the three sides and the three angles. If then three of these six parts be known, we have, by substituting the three given values in these equations, three equations containing only three unknown quantities, and therefore we can, by solving the equations, find the values of the three remaining parts. The finding the magnitudes of the sides and angles of a triangle from given data, is called solving the triangle, and theoretically the three equations we have obtained are sufficient for our purpose. There are, however, several deductions from these equations which in practice are frequently more convenient to employ than the fundamental equations themselves, and which we proceed to investigate.

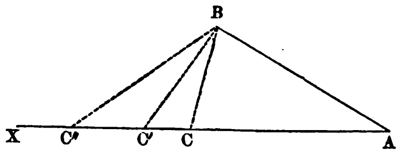
Three parts of the triangle must be given in order to determine the remaining parts. For suppose two parts only were given, say two sides. We might represent this case by a pair of compasses, whose legs are the same lengths respectively as the given sides of the triangle, and whose extremities are joined by an elastic string. By altering the inclination of the legs we can form an infinite number of triangles, all of which have two of their sides of the given length.

Or if one side and its opposite angle were given.



Suppose BC is the given side and BAC the given angle. Describe a circle about the triangle ABC. Then any triangle, which has BC for its base and a point on the circle for its vertex, will fulfil the required conditions.

Or again, if one side and an adjacent angle be given as the angle XAB and the side AB. Then any straight line



drawn from B to intersect the indefinite line AX will form a triangle fulfilling the required conditions.

If two angles (or which is the same thing, the three angles) be given, we may have a triangle of any magnitude with the required angles.

Hence we conclude, that in order to solve a triangle it is necessary and sufficient that three parts should be known, and further, that one of the parts at least should be a side.

It sometimes happens that the magnitudes given, from which the triangle is to be determined, are not actually three of the six simple parts of the triangle. It will always be found, however, that three of these parts are virtually given, *i. e.* can be determined from the data. Such a case, for instance, would be: "Given the difference of two sides of a triangle and the angles opposite to their sides, to determine the triangle."

3. *To prove that* $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$.

In figure (1) on p. 72, $BC = BD + DC = c\cos B + b\cos C$,

...... (2), $BC = BD - DC = c\cos B - b\cos(180° - C)$

$$= c\cos B + b\cos C;$$

therefore in every triangle,

$$a = c\cos B + b\cos C \;..........(1).$$

Similarly, $b = a\cos C + c\cos A \;..........(2)$,

and $c = b\cos A + a\cos B \;..........(3)$.

Multiplying (1) by a, (2) by b, (3) by c, and then subtracting the last result from the sum of the two others, we have

$$a^2+b^2-c^2 = 2ab\cos C,$$

$$\text{or}\ \cos C = \frac{a^2+b^2-c^2}{2ab}.$$

$$\text{Similarly,}\ \cos A = \frac{b^2+c^2-a^2}{2bc},$$

$$\text{and}\ \cos B = \frac{c^2+a^2-b^2}{2ca}.$$

From these values the other trigonometrical functions of A, B and C can of course be determined in terms of the sides a, b, c.

4. This expression might also have been obtained by assuming the results of the 12th and 13th propositions of the 2nd book of Euclid.

For fig. (1) p. 72, by Euc. II. 13,

$$AB^2 = BC^2 + AC^2 - 2BC.CD,$$

and $CD = AC\cos C$;

$$\therefore\ c^2 = a^2 + b^2 - 2ab\cos C.$$

And fig. (2) by Euc. II. 12,

$$AB^2 = BC^2 + AC^2 + 2BC.CD,$$

and $CD = AC\cos(180^\circ - C) = -AC\cos C$;

$$\therefore\ c^2 = a^2 + b^2 - 2ab\cos C,$$

in this case also.

This result is in fact nothing more than the results of the 12th and 13th propositions of Euc. II., translated into analytical language. It is a very good example of the generality of symbolical representation as compared with that of geometry, the one result including both the cases given in Euclid.

5. *Expressions for the trigonometrical functions of* $\frac{A}{2}, \frac{B}{2}, \frac{C}{2}$ *in terms of* a, b, c.

$$\sin^2\frac{A}{2} = \frac{1-\cos A}{2} = \frac{1 - \dfrac{b^2+c^2-a^2}{2bc}}{2}, \text{ by the last Art.}$$

$$= \frac{2bc - b^2 - c^2 + a^2}{4bc};$$

$$\therefore\ \sin^2\frac{A}{2} = \frac{a^2 - (b-c)^2}{4bc} = \frac{\dfrac{(a-b+c)}{2}\cdot\dfrac{(a+b-c)}{2}}{bc}.$$

Let $\frac{a+b+c}{2} = s$. Then $\frac{a-b+c}{2} = \frac{a+b+c}{2} - b = s - b$.

Similarly, $\frac{a+b-c}{2} = s - c$;

$$\therefore\ \sin\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}.$$

Similarly

$$\cos^2\frac{A}{2} = \frac{1+\cos A}{2} = \frac{1+\dfrac{b^2+c^2-a^2}{2bc}}{2} = \frac{2bc+b^2+c^2-a^2}{4bc};$$

$$\therefore \cos^2\frac{A}{2} = \frac{(b+c)^2-a^2}{4bc} = \frac{\dfrac{(a+b+c)}{2}\cdot\dfrac{(b+c-a)}{2}}{bc} = \frac{s\,.\,s-a}{bc};$$

$$\therefore \cos\frac{A}{2} = \sqrt{\left\{\frac{s\,.\,(s-a)}{bc}\right\}};$$

$$\therefore \tan\frac{A}{2} = \frac{\sin\dfrac{A}{2}}{\cos\dfrac{A}{2}} = \frac{\sqrt{\left\{\dfrac{(s-b)(s-c)}{bc}\right\}}}{\sqrt{\left\{\dfrac{s\,.\,(s-2)}{bc}\right\}}} = \sqrt{\left\{\frac{(s-b)(s-c)}{s\,.\,(s-a)}\right\}}.$$

The analogous expressions for the sine, cosine, and tangents of $\frac{B}{2}$ and $\frac{C}{2}$ can be written down from the symmetry. For, noticing that the value of the sine, for instance, has in the numerator the sides adjacent to the angle subtracted from s, and the same sides in the denominator, we can at once write down

$$\sin\frac{B}{2} = \sqrt{\left\{\frac{(s-c)(s-a)}{ca}\right\}} \text{ and } \sin\frac{C}{2} = \sqrt{\left\{\frac{(s-a)(s-b)}{ab}\right\}},$$

and, from similar considcrations,

$$\cos\frac{B}{2} = \sqrt{\left\{\frac{s\,(s-b)}{ac}\right\}}, \qquad \cos\frac{C}{2} = \sqrt{\left\{\frac{s\,(s-c)}{ab}\right\}};$$

$$\therefore \tan\frac{B}{2} = \sqrt{\left\{\frac{(s-c)(s-a)}{s\,(s-b)}\right\}},\ \tan\frac{C}{2} = \sqrt{\left\{\frac{(s-a)(s-b)}{s\,(s-c)}\right\}}.$$

6. The positive sign is taken with the functions, because, A being $< 180^\circ$, $\frac{A}{2}$ is less than 90°, and consequently its trigonometrical functions are all positive.

We may also remark that the expressions

$$\frac{a+b-c}{2},\ \frac{a-b+c}{2},\ \frac{a+b-c}{2},\text{ that is } s-a,\ s-b,\ s-c,$$

are all necessarily positive, since any two sides of a triangle must be greater than the third.

We may also notice that these values of the sines and cosines are less than unity, as ought to be the case.

For, taking $\sin\frac{A}{2}$ for example, it will be less than 1 if

$$\frac{a^2-(b-c)^2}{4bc}<1,$$

$$\text{i. e. if } a^2-b^2+2bc-c^2<4bc,$$
$$\text{if } a^2<b^2+2bc+c^2,$$
$$\text{if } a<b+c,$$

which we know to be the case.

7. *Area of a triangle in terms of the sides.*

$$\sin B=2\sin\frac{B}{2}\cos\frac{B}{2}=2\sqrt{\left\{\frac{(s-a)(s-c)}{ac}\right\}}\sqrt{\left\{\frac{s.(s-b)}{ac}\right\}}$$
$$=2\sqrt{\left\{\frac{s.(s-a)(s-b)(s-c)}{ac}\right\}}.$$

Now both in figs (1) and (2) on p. 72,

the area of the triangle $ABC=\frac{1}{2}BC.AD$,

and $AD=c\sin B$;

$\therefore$ the area of triangle $ABC=\frac{1}{2}ac\sin B$,

and, substituting the value just found for $\sin B$ in terms of the sides,

the area of the triangle $=\sqrt{\{s.(s-a)(s-b)(s-c)\}}$.

8. *To prove that* $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$.

$$\frac{\sin A}{\sin B}=\frac{a}{b};$$

$$\therefore \frac{\sin A+\sin B}{\sin A-\sin B}=\frac{a+b}{a-b},$$

$$\text{or } \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}=\frac{a+b}{a-b};$$

$$\therefore \tan\frac{A-B}{2}=\frac{a-b}{a+b}\tan\frac{A+B}{2},$$

$$\text{and } \frac{A+B}{2}=\frac{180^\circ-C}{2};$$

$$\therefore \tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

9. *To prove that* $c=\dfrac{(a-b)\cos\frac{C}{2}}{\sin\frac{A-B}{2}}$.

$$c=\frac{a\sin C}{\sin A}=\frac{2a\sin\frac{C}{2}\cos\frac{C}{2}}{\sin A},$$

and $\sin\frac{C}{2}=\cos\frac{A+B}{2}$, because $\frac{A+B}{2}=90^\circ-\frac{C}{2}$;

$$\therefore c=\frac{2a\cos\frac{C}{2}\cos\frac{A+B}{2}}{\sin A}\ \text{............(1)}.$$

Again, since $\dfrac{\sin B}{\sin A}=\dfrac{b}{a}$,

$$\frac{\sin A-\sin B}{\sin A}=\frac{a-b}{a},$$

$$\text{or } \frac{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{\sin A}=\frac{a-b}{a};$$

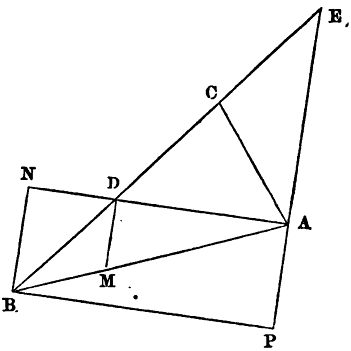
$$\therefore 2a\frac{\cos\frac{A+B}{2}}{\sin A}=\frac{a-b}{\sin\frac{A-B}{2}};$$

$$\therefore \text{ from (1), } c=\frac{(a-b)\cos\frac{C}{2}}{\sin\frac{A-B}{2}}.$$

10. We subjoin a geometrical proof of the two relations

$$\tan\frac{A-B}{2}=\frac{a-b}{a\times b}\cot\frac{C}{2},$$

$$c=\frac{(a-b)\cos\frac{C}{2}}{\sin\frac{A-B}{2}}.$$



Let ABC be the triangle and let a be greater than b.

Cut off from BC CD equal to CA, and produce BC to E, making CE equal to CA or CD.

Join AE. Then, clearly, since

$$CD=CA=CE,$$

EAD is the angle in a semicircle, and is therefore a right angle. Draw DM parallel to AE. Then MDA is a right angle.

We have $BD=a-b$, $BE=a+b$,

$$BAD=A-CAD=A-CDA=A-(B+BAD)$$

$$\therefore 2BAD=A-B,$$

$$\text{or } BAD=\frac{A-B}{2}, \text{ and } E=\frac{C}{2}.$$

$$\text{Now } \tan\frac{A-B}{2}=\frac{DM}{AD}=\frac{DM}{AE}\cdot\frac{AE}{AD}=\frac{BD}{BE}\cot E,$$

by similar triangles BMD, BAE;

$$\therefore \tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}.$$

Again, completing the rectangular parallelogram $APBN$,

$$AB \sin BAN = BN = BD \cos DBN,$$

$$\text{therefore } c\sin\frac{A-B}{2} = (a-b)\cos\frac{C}{2};$$

$$\text{or } c = \frac{(a-b)\cos\frac{C}{2}}{\sin\frac{A-B}{2}}.$$

11. *Different cases of solution.*

The formulæ which we have proved are sufficient to enable us to determine a triangle, where three of the six parts are given as already explained.

We proceed now to distinguish the cases of solution, and to explain the method of proceeding in each case.

The only cases which can occur are the following four:

(1) When two angles and a side are given.

(2) When two sides and an angle opposite to one of these sides are given.

(3) When two sides and the included angle are given.

(4) When the three sides are given.

Case (1). Let a, B, and C be given.

Then A is of course known, because $A = 180° - (B + C)$.

$$\text{And, since } b = a\frac{\sin B}{\sin A}$$

$$c = a\frac{\sin C}{\sin A},$$

the sides are given in a form adapted to logarithmic computation.

If a, A, B be given, $C = 180° - (A + B)$, and b and c are found as before.

Case (2). *The ambiguous case.* Suppose A, a, and b to be given.

$$\text{Then } \sin B = \frac{b}{a}\sin A.$$

Hence $\sin B$ is given in a form adapted to logarithmic computation. B therefore is found, and $C = 180° - (A + B)$,

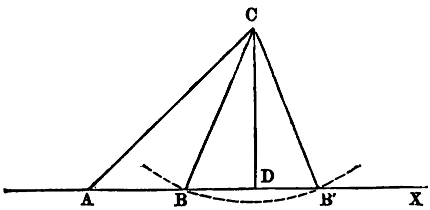
$$\text{and } c = a\frac{\sin C}{\sin A}.$$

In determining B however from the equation

$$\sin B = \frac{b}{a}\sin A,$$

we do not know in general which angle to take, the least positive angle which has its sine equal to $\frac{b}{a}\sin A$, or its supplement. Hence this is called the ambiguous case, since there may be two triangles answering to the required conditions. The ambiguity in some instances however does not exist, as we proceed to shew.

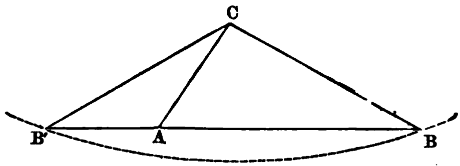
Let CAX be the given angle A. Take AC equal to b, and with centre C and radius equal to a describe a circle.



If a is less than the perpendicular CD, *i.e.* if a is less than $b \sin A$, the circle does not cut the line AX at all, and there is no triangle answering to the conditions, as is evident, because the formula $\sin B = \frac{b}{a}\sin A$ is impossible if a is less than $b \sin A$, or $\sin B$ greater than unity. If $a = CD$, then the circle touches AX, and the required triangle is right-angled, B being the right angle, because then $\sin B = 1$. If, however, $a > b \sin A$, the circle will cut AX in two points, B and B'.

Now if $a < CA$, *i.e.* b, the points B and B' will fall on the same side of A, and the two triangles CAB, CAB' will each fulfil the required conditions, because they each have the angle A and two sides equal to a and b.

But if CB, *i.e.* a, is $> b$, B and B' fall upon opposite sides of A, and then only the triangle CAB fulfils the



given conditions, because the triangle CAB' has for its angle opposite to CB' or b, not A but $180° - A$.

Hence the solution is not ambiguous in the case in which the side opposite the given angle is greater than the other side.

Case (3). *a, b, and C given.*

From Art. 8 we have

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}.$$

Now, since a and b are known, $a-b$ and $a+b$ are known. And

$$L \tan \frac{A-B}{2} = \log (a-b) - \log (a+b) + L \cot \frac{C}{2}.$$

Hence $L \tan \frac{A-B}{2}$, and therefore $\frac{A-B}{2}$, *i.e.* $A-B$, is known; and $A+B$ is known, because $A+B = 180° - C$. Hence $A-B$ and $A+B$ being known, A and B are known,

$$\text{and } c = \frac{a \sin C}{\sin A}.$$

Hence the triangle is completely determined.

c might evidently be determined by means of the formula proved in Art. 9, which is in a form adapted to logarithmic computation.

c might also be found without determining the angles A and B from the formula

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

This however in its present state is practically inconvenient, since it is not in a form adapted to logarithmic computation.

It may however be reduced to a logarithmic form as follows:

$$c^2 = (a^2+b^2)\left(\cos^2\frac{C}{2} + \sin^2\frac{C}{2}\right) - 2ab\left(\cos^2\frac{C}{2} - \sin^2\frac{C}{2}\right)$$

$$= (a^2+b^2-2ab)\cos^2\frac{C}{2} + (a^2+b^2+2ab)\sin^2\frac{C}{2}$$

$$= (a-b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2}$$

$$= (a-b)^2\cos^2\frac{C}{2}\left\{1 + \frac{(a+b)^2\sin^2\frac{C}{2}}{(a-b)^2\cos^2\frac{C}{2}}\right\}$$

$$= (a-b)^2\cos^2\frac{C}{2}\left\{1 + \left(\frac{a+b}{a-b}\tan\frac{C}{2}\right)^2\right\}.$$

Now since the tangent of an angle may be of any magnitude, there will be some angle whose tangent is equal to $\frac{a+b}{a-b}\tan\frac{C}{2}$. This can be found from the tables, for if we call it ϕ, we have

$$L\tan\phi = \log(a+b) - \log(a-b) + L\tan\frac{C}{2};$$

and therefore ϕ is known since $a+b$, $a-b$ and C are known.

Hence $$c^2 = (a-b)^2\cos^2\frac{C}{2}(1+\tan^2\phi)$$

$$= (a-b)^2\cos^2\frac{C}{2}\sec^2\phi;$$

$$\therefore\ c=(a-b)\cos\frac{C}{2}\sec\phi,$$

which is in a form adapted to logarithmic computation, and we have

$$\log c=\log(a-b)+\log\cos\frac{C}{2}+\log\sec\phi,$$

and $\log\sec\phi$ is known because ϕ is known. Hence c is known.

There is however no particular advantage in this method of finding c, for the determination of the subsidiary angle ϕ involves exactly the same amount of labour as is necessary to find $A-B$.

Case (4). *Given a, b, c.*

We have $$\sin\frac{A}{2}=\sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}},$$

$$\sin\frac{B}{2}=\sqrt{\left\{\frac{(s-c)(s-a)}{ca}\right\}}.$$

Hence A and B are known in a form adapted to logarithmic computation.

It is better in practice to use the formulæ

$$\tan\frac{A}{2}=\sqrt{\left\{\frac{(s-b)(s-c)}{s.(s-a)}\right\}},$$

$$\tan\frac{B}{2}=\sqrt{\left\{\frac{(s-c)(s-a)}{s.(s-b)}\right\}},$$

because this only necessitates finding

$$\log s,\ \log(s-a),\ \log(s-b),\ \log(s-c),$$

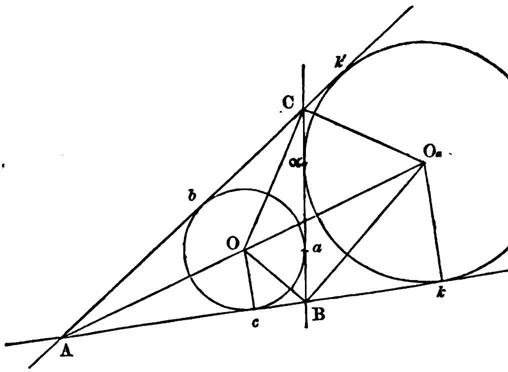
whereas the formulæ for the sines require, in addition to these, $\log a$, $\log b$ and $\log c$.

A and B having been found, C is of course known, because $C=180^\circ-(A+B)$.

CHAPTER X.

CIRCLES INSCRIBED IN AND CIRCUMSCRIBED ABOUT A TRIANGLE, POLYGONS, AREA OF A CIRCLE, &c.

1. *Inscribed and escribed circles.*



If the three sides of a triangle ABC be produced, four circles can be described, each of which touches the three sides; viz. one which touches all the sides internally as the circle *abc*, and three which touch two sides internally and the other externally, as the circle *kak'*.

The first-mentioned circle is called the inscribed circle, and the three others are termed the *escribed* circles of the triangle.

The centre O of the inscribed circle is determined, as we know from Euclid IV. 4, by drawing the lines AO, BO,

CO to bisect the angles A, B, C respectively. By an exactly similar process of reasoning, it may be shewn that the centre of any one of the escribed circles is determined by bisecting the exterior angles CBk, BCk', and the angle A by the lines BO_a, CO_a, AO_a respectively, the centre O_a being their common point of concourse.

The circle $k'ak$ being escribed opposite to the angle A, we call its centre O_a, and shall denote its radius by r_a. Let r be the radius of the inscribed circle, and let Δ denote the area of the triangle ABC.

2. *To find r, r_a, r_b, r_c in terms of the sides of the triangle.*

$$\Delta = \text{area } AOB + \text{area } BOC + \text{area } COA,$$

$$\text{and area } AOB = \tfrac{1}{2}Oc\,.\,AB = \tfrac{1}{2}rc.$$

$$\text{Similarly area } BOC = \tfrac{1}{2}ra,$$

$$\text{area } COA = \tfrac{1}{2}rb;$$

$$\therefore\ \Delta = \tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc = r \times \frac{a+b+c}{2} = r\,.\,s;$$

$$\therefore\ r = \frac{\Delta}{s}, \text{ and } \Delta = \sqrt{\{s\,.(s-a)(s-b)(s-c)\}}.$$

Again, the quadrilateral $O_aBAC = \text{area } O_aAB + \text{area } O_aAC$,

also the quadrilateral $O_aBAC = \text{area } O_aBC + \Delta$;

$$\therefore\ O_aAB + O_aAC = O_aBC + \Delta.$$

$$\text{Now area } O_aAB = \tfrac{1}{2}O_ak\,.\,AB = \tfrac{1}{2}r_a\,.\,c,$$

$$\text{area } O_aAC = \tfrac{1}{2}O_ak'\,.\,AC = \tfrac{1}{2}r_a\,.\,b,$$

$$\text{area } O_aBC = \tfrac{1}{2}O_aa\,.\,BC = \tfrac{1}{2}r_a\,.\,a;$$

$$\therefore\ \tfrac{1}{2}r_a\,.\,c + \tfrac{1}{2}r_a\,.\,b = \tfrac{1}{2}r_a\,.\,a + \Delta;$$

$$\therefore\ r_a\,.\,\frac{-a+b+c}{2} = \Delta,$$

$$\text{or } r_a = \frac{\Delta}{s-a}.$$

$$\text{Similarly } r_b = \frac{\Delta}{s-b},$$

$$\text{and } r_c = \frac{\Delta}{s-c};$$

$$\therefore \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}$$

$$= \frac{3s-(a+b+c)}{\Delta} = \frac{s}{\Delta} = \frac{1}{r},$$

which is a formula connecting the radii of the inscribed and escribed circles.

3. *Lengths of tangents to the inscribed and escribed circles from the angles of a triangle.*

If the inscribed circle touches the triangle in the points a, b, c, since tangents drawn to a circle from the same point are equal,

$$Ac = Ab, \quad Ba = Bc, \quad Cb = Ca;$$

$$\text{and } \therefore \ Cb + CB + Bc = 2a;$$

and $\therefore$ $Ac + Ab$; *i.e.* $2Ac$ or $2Ab = \text{perimeter} - 2a = 2s - 2a$;

$$\therefore \ Ac = Ab = s - a.$$

Similarly tangents from $B = s - b$,

tangents from $C = s - c$.

$$\text{Again,} \quad Ak = AB + Bk = AB + Ba,$$

$$Ak' = AC + Ck' = AC + Ca.$$

But $Ak = Ak'$;

$$\therefore \ 2Ak = 2Ak' = AB + AC + Ba + Ca$$

$$= AB + AC + BC$$

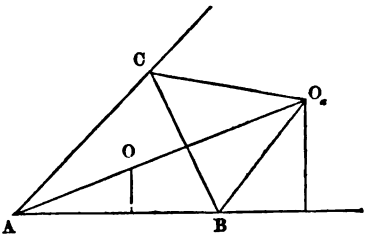
$$= a + b + c;$$

$$\therefore \ Ak = Ak' = \frac{a+b+c}{2} = s,$$

$$\text{and } Ba = Bk = Ak - AB = s - c,$$

$$Ca = Ck' = Ak' - AC = s - b.$$

4. **A simple expression may be obtained for the distance between the centre of the inscribed and that of one of the escribed circles.**



For $OO_a = O_aA - OA$

$$= \frac{r_a}{\sin\frac{A}{2}} - \frac{r}{\sin\frac{A}{2}}$$

$$= \frac{1}{\sin\frac{A}{2}}\left\{\frac{\text{area}}{s-a} - \frac{\text{area}}{s}\right\}$$

$$= \frac{\text{area}}{\sin\frac{A}{2}} \cdot \frac{a}{s.(s-a)} = \frac{a}{\sin\frac{A}{2}}\sqrt{\left\{\frac{(s-b)(s-c)}{s.(s-a)}\right\}}$$

$$= a\sqrt{\left\{\frac{bc}{(s-b)(s-c)}\right\}} \cdot \sqrt{\left\{\frac{(s-b)(s-c)}{s.(s-a)}\right\}}$$

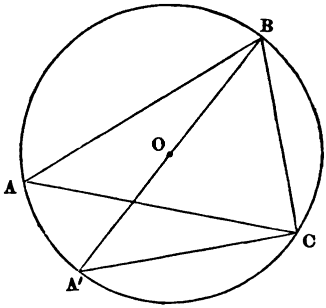
$$= \frac{a}{\sqrt{\left\{\frac{s.(s-a)}{bc}\right\}}}$$

$$= \frac{a}{\cos\frac{A}{2}}.$$

Similarly $OO_b = \dfrac{b}{\cos\frac{B}{2}}$,

$$OO_c = \frac{c}{\cos\frac{C}{2}}.$$

5. ***Radius of a circle circumscribing a triangle ABC in terms of the sides.***



Let ABC be the triangle, O the centre of the circumscribing circle, which, as we know from Euclid IV. 5, is determined by bisecting the sides of the triangle, and drawing perpendiculars from the points of bisection to the sides respectively. The common point of concourse of the three perpendiculars is the center O.

Draw the diameter BOA' and join $A'C$. Then,

$$\text{angle } BA'C = \text{angle } BAC,$$

because they are in the same segment of the circle, and BCA', being in a semi-circle, is a right angle;

$$\therefore \text{ diameter } BA' = \frac{BC}{\sin BA'C} = \frac{a}{\sin A},$$

or, calling the radius R,

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}.$$

$$\text{Now } \sin A = \frac{2\Delta}{bc}. \quad \text{(Chap. IX. 7.)}$$

$$\text{Hence } R = \frac{abc}{4\Delta} = \frac{abc}{4\sqrt{\{s.(s-a)(s-b)(s-c)\}}}.$$

6. ***Formulæ connecting r and R and also r_a and R.***

Referring to the figure of Art. 1, we have

$$c = AB = Ac + Bc = r\cot\frac{A}{2} + r\cot\frac{B}{2} = r\left\{\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}} + \frac{\cos\frac{B}{2}}{\sin\frac{B}{2}}\right\}.$$

$$\text{Now } R = \frac{c}{2 \sin C}; \quad i.e.\ c = 2R \sin C;$$

$$\therefore 2R \sin C = r . \frac{\cos\frac{A}{2}\sin\frac{B}{2} + \sin\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}} = r . \frac{\sin\frac{A+B}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}.$$

$$\text{Now } \sin\frac{A+B}{2} = \sin\frac{\pi - C}{2} = \cos\frac{C}{2},$$

$$\text{and } \sin C = 2\sin\frac{C}{2}\cos\frac{C}{2};$$

$$\therefore 4R\sin\frac{C}{2}\cos\frac{C}{2} = r\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}};$$

$$\therefore r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Again, referring to the same figure,

$$a = BC = Ba + Ca = O_a a \cot O_a Ba + O_a a \cot O_a Ca$$

$$= O_a a\left\{\cot\left(\frac{\pi}{2} - \frac{B}{2}\right) + \cot\left(\frac{\pi}{2} - \frac{C}{2}\right)\right\}$$

$$= r_a\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right);$$

$$\therefore 2R\sin A = r_a\left\{\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right\} = r_a\frac{\sin\frac{B+C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}};$$

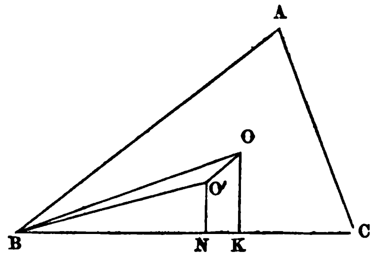
$$\therefore 4R\sin\frac{A}{2}\cos\frac{A}{2} = r_a\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}};$$

$$\therefore r_a = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Analogous expressions of course obtain for r_b and r_c in terms of R.

7. *Expression for the distance between the centres of the inscribed and circumscribed circles of a triangle in terms of the radii of these two circles.*

Let O be the centre of the inscribed, and O' of the circumscribed circle.



Then $O'B=R$, $OK=r$, $O'BC=90^\circ-A$, because $BO'N=A$, it being half the angle at the centre, which is $2A$.

$$OB=r\operatorname{cosec}\frac{B}{2}.$$

Then $OO'^2=R^2+r^2\operatorname{cosec}^2\frac{B}{2}-2Rr\operatorname{cosec}\frac{B}{2}\cos OBO'$,

and

$$OBO'=\tfrac{1}{2}B-(90^\circ-A)=90^\circ-\frac{A}{2}-\frac{C}{2}-(90^\circ-A)=\frac{A-C}{2};$$

$$\therefore\ OO'^2=R^2+r^2\operatorname{cosec}^2\frac{B}{2}-2Rr\operatorname{cosec}\frac{B}{2}\cos\frac{A-C}{2};$$

$$=R^2-2Rr\left\{\frac{\cos\frac{A}{2}\cos\frac{C}{2}+\sin\frac{A}{2}\sin\frac{C}{2}}{\sin\frac{B}{2}}-\frac{r\operatorname{cosec}^2\frac{B}{2}}{2R}\right\}.$$

Substituting for r, in the second term of the expression within the bracket, the value

$$4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}, \quad \text{(Art. 6,)}$$

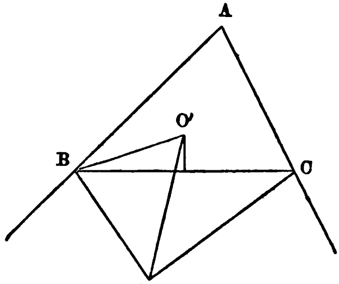
we get

$$OO'^2 = R^2 - 2Rr \left\{ \frac{\cos \frac{A}{2} \cos \frac{C}{2} + \sin \frac{A}{2} \sin \frac{C}{2} - 2 \sin \frac{A}{2} \sin \frac{C}{2}}{\sin \frac{B}{2}} \right\}$$

$$= R^2 - 2Rr \frac{\cos \frac{A+C}{2}}{\cos \frac{A+C}{2}}$$

$$= R^2 - 2Rr.$$

8. Again, O_a being the centre of the escribed circle opposite to A,



The angle $O'BO_a = O_aBC + CBO'$

$$= \tfrac{1}{2}(180° - B) + 90° - A$$

$$= 180° - \frac{B}{2} - A = 90° - \frac{A-C}{2};$$

$$\therefore\ O'O_a^2 = R^2 + r_a^2 \sec^2 \frac{B}{2} - 2Rr_a \sec \frac{B}{2} \sin \frac{A-C}{2},$$

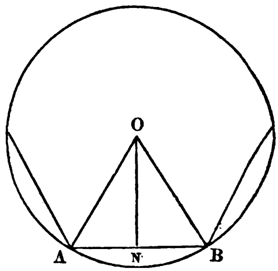
$$\text{and } r_a = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}. \quad \text{(Art. 6.)}$$

Hence, by a substitution similar to that in Art. 8,

$$O_aO'^2 = R^2 + 2Rr_a.$$

Similar expressions of course hold for OO_b and OO_c.

9. *Polygon inscribed in a circle.*



Let AB be a side of a regular polygon of n sides inscribed in a circle, the centre of which is O, and radius r. Join AO, BO.

Then the triangle AOB is $\frac{1}{n}^{\text{th}}$ of the area of the polygon.

Draw ON perpendicular to AB.

Then the angle $AOB = \frac{2\pi}{n}$, and $\therefore$ $AON = \frac{\pi}{n}$;

$$\therefore\ AB = 2AN = 2AO \sin AON = 2r \sin\frac{\pi}{n};$$

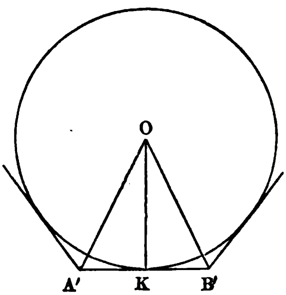
and the area of the triangle AOB

$$= \tfrac{1}{2}AO.OB \sin AOB = \tfrac{1}{2}r^2 \sin\frac{2\pi}{n};$$

$\therefore$ the perimeter of the polygon $= 2nr \sin\frac{\pi}{n}$;

and the area of the polygon $= \frac{1}{2}nr^2 \sin\frac{2\pi}{n}$.

10. *Circumscribed polygon.*



Suppose a regular polygon of n sides to be circumscribed about the same circle, and let $A'B'$ be one of its sides.

Then since, as before,

$$AOB' = \frac{2\pi}{n},$$

$$A'B' = 2A'K = 2OK \tan A'OK = 2r \tan\frac{\pi}{n},$$

$$\text{triangle } A'OB' = \tfrac{1}{2}A'B'.OK = r^2 \tan\frac{\pi}{n};$$

$$\therefore \text{ the perimeter of the polygon} = 2nr \tan\frac{\pi}{n},$$

$$\text{and the area of the polygon} = nr^2 \tan\frac{\pi}{n}.$$

11. *Circumference and area of a circle.*

It may be assumed that the circumference of the circle is intermediate in length to the perimeters of the inscribed and circumscribed polygons, and that the smaller the sides of the polygons are made, the nearer do their perimeters approach each other and the circumference of the circle in magnitude.

The circumference then of the circle is intermediate in magnitude to

$$2nr \sin\frac{\pi}{n}, \text{ and } 2r \tan\frac{\pi}{n};$$

that is, to

$$2\pi r \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}, \text{ and } 2\pi r \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}.$$

Now, when n is made very great, $\frac{\pi}{n}$ is very small; and, when n is made infinite, $\frac{\pi}{n}$ becomes zero.

But (Chap. II. Art. 14) when $\frac{\pi}{n}$ is zero,

$$\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} = 1 = \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}.$$

Hence, when n is made infinite, each of the two limits, between which the circumference of the circle lies, becomes $2\pi r$.

Hence the circumference of the circle is $2\pi r$.

Again, similarly, the area of the circle always lies between

$$\tfrac{1}{2}nr^2 \sin\frac{2\pi}{n}, \text{ and } nr^2 \tan\frac{\pi}{n},$$

or between

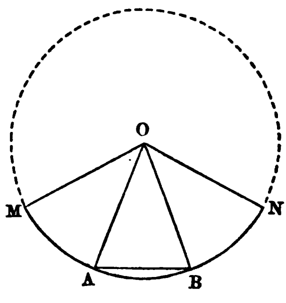
$$\pi r^2 \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}, \text{ and } \pi r^2 \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}},$$

each of which when $n = \infty$ becomes πr^2.

Hence the area of the circle, which is always intermediate in magnitude to those of the two polygons, is πr^2.

12. *To find the area of a sector of a circle.*

Let θ be the *circular measure* of the angle which the arc of the sector MON subtends at the centre O. Divide the angle MON into n equal parts of which AOB is one. Then the circular measure of AOB is $\frac{\theta}{n}$.



The area of the triangle

$$AOB = \frac{1}{2} r^2 \sin\frac{\theta}{n}.$$

Hence, as in the case of the whole circle, the area of the sector MON will be equal to the value of $\tfrac{1}{2}nr^2 \sin\frac{\theta}{n}$, when n is increased without limit.

Now $$\tfrac{1}{2}nr^2 \sin\frac{\theta}{n} = \frac{1}{2}\theta r^2 \left\{\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right\};$$

and, when n is increased without limit, $\frac{\theta}{n}$ is zero;

$$\therefore \text{ when this is the case } \frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}} = 1;$$

$\therefore$ the area of the sector is $\frac{1}{2}\theta r^2$.

CHAPTER XI.

APPLICATION OF TRIGONOMETRY TO MEASURING HEIGHTS AND DISTANCES.

1. IT is by means of the relations between the sides and angles of triangles that the operations of surveying are conducted. To go into the details of this branch of the subject with any degree of completeness would alone require as large a volume as has been devoted to this treatise. It is however the expedients to which it is necessary to have recourse in actual practice, which constitute a large portion of a work specially devoted to surveying. The trigonometrical principles of the calculations involve but little difficulty, and we shall be able by the solution of a few general problems to illustrate them sufficiently for the purposes of this work.

2. In making any measurement the first requisite is a fixed standard length very accurately measured, to which we can refer other lengths as a unit. On paper, and for small distances, an accurately marked ruler or some equivalent instrument might be employed. When however the measurements are on a large scale, a straight line is very accurately measured, and by means of the principles of Trigonometry, we are able to express the other distances we require in terms of this length. This line or *base*, as it is technically termed, answers the same purpose as a ruler or measuring rod, except that it is stationary, and the other

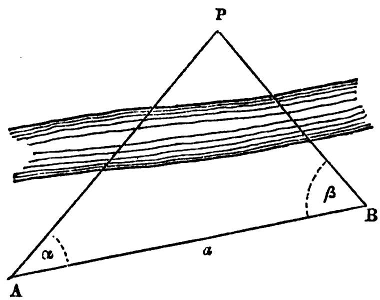
distances to be measured are calculated by referring them to the base, by means of trigonometrical calculations.

There are various instruments which enable the surveyor to measure the angles which any given point subtends at two other points. He must also be furnished with a set of logarithmic and trigonometrical tables.

3. The *angle of elevation* of a point anywhere situated above the eye of the observer, is the angle which the straight line joining his eye and the point makes with the horizon.

The *angle of depression* of a point situated below the eye of the observer, is the angle which the line joining his eye and the point makes with the horizon.

4. *To find the distance of an inaccessible object upon a horizontal plane.*



Let P be the distant object which owing to some obstacle cannot be reached. Let A be the place of the observer. Measure a base AB, the length of which call a. Measure the angles PAB and PBA, which suppose to be found to be α and β respectively. Then we have

$$\frac{AP}{AB} = \frac{\sin PBA}{\sin APB},$$

$$\text{and } \frac{BP}{AB} = \frac{\sin PAB}{\sin APB},$$

$$\therefore AP = a\frac{\sin\beta}{\sin(\alpha+\beta)},$$

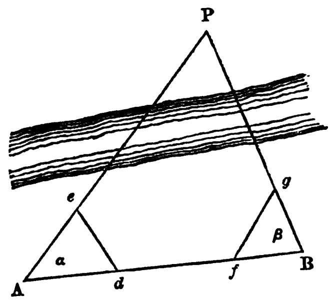
$$BP = a\frac{\sin\alpha}{\sin(\alpha+\beta)};$$

$$\therefore \log AP = \log a + \log\sin\beta - \log\sin(\alpha+\beta),$$

$$\text{and } \log BP = \log a + \log\sin\alpha - \log\sin(\alpha+\beta).$$

Thus AP and BP are known from the tables.

5. This problem may also be solved without any angular measurement, which in the absence of instruments may be very convenient. The method is due to Colonel Everest, who surveyed India. It is also given in Mr Galton's *Art of Travel*, page 287 (3rd edition).



Having measured the base AB as before, measure any length Ad in AB and an equal one Ae along AP. Then measure de. Similarly measure Bf and Bg each equal to Ad or Ae, and then measure fg.

Since eAd is an isosceles triangle, a perpendicular from A upon de will bisect de.

$$\therefore de = 2Ad\sin\frac{PAB}{2},$$

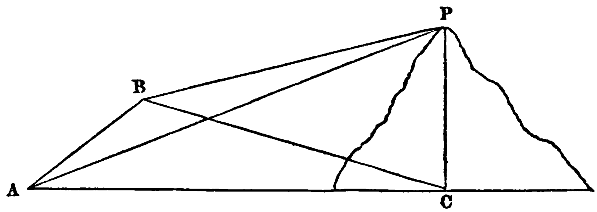
$$\text{or } \sin\frac{\alpha}{2} = \frac{1}{2}\frac{de}{Ad}.$$

$$\text{Similarly } \sin\frac{\beta}{2} = \frac{1}{2}\frac{fg}{Bf}.$$

Thus α and β are known by the aid of the tables, and the problèm is now completed as before.

Mr Galton has drawn up a very ingenious table in which, by referring to a value of de given in one column and the value of fg given in another, the magnitudes of the angles A and B and the distances AP, BP are given. In the table the length of the lines Ad, Ae, Bf, Bg is supposed to be one-tenth of the base AB.

6. To measure the distance of the summit P of a hill from a point A, and the height of the hill above the horizontal plane in which A lies.



Measure a base AB of length a say.

Observe the angles BAP (α), ABP (β), and the angle of elevation PAC (γ).

Then, from the triangle PAB, we have

$$PA = AB\frac{\sin ABP}{\sin BPA},$$

$$\text{or } PA = a\frac{\sin\beta}{\sin(\alpha+\beta)}.$$

$$\text{Now } PC = PA\sin\gamma,$$

$$AC = PA\cos\gamma;$$

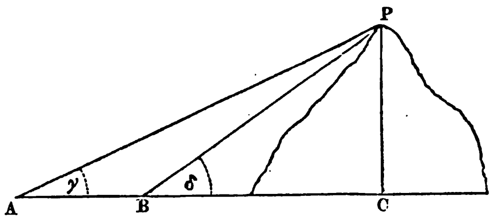
$$\therefore\ PC = a\frac{\sin\beta\sin\gamma}{\sin(\beta+\gamma)},$$

$$AC = a\frac{\sin\beta\cos\gamma}{\sin(\beta+\gamma)},$$

which are both in a form adapted to logarithmic computation.

N.B. The base AB need not necessarily in this problem be horizontal.

7. The distances AC and PC might have been found by measuring only two angles if AB had been measured along AC.



For by measuring PAB (γ) and PBC (δ) we have

$$PB = AB\frac{\sin\gamma}{\sin(\delta-\gamma)},$$

$$\text{and } \therefore\ PC = PB\sin\delta$$

$$= a\frac{\sin\gamma\sin\delta}{\sin(\delta-\gamma)},$$

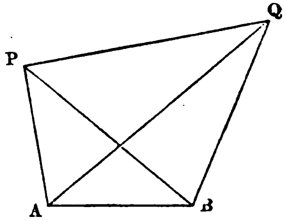
$$\text{and } AC = AB + BC$$

$$= a + a\frac{\sin\gamma\cos\delta}{\sin(\delta-\gamma)}$$

$$= a\frac{\sin\delta\cos\gamma}{\sin(\delta-\gamma)}.$$

8. *To find the distance between two inaccessible objects in a horizontal plane.*

Let P and Q be the two objects.



Measure a line AB equal to a.

Observe the angles PAB, QAB, ABP, QBA, which call α, β, γ, δ respectively.

Then from the triangle ABP

$$PA = AB\frac{\sin PBA}{\sin APB}.$$

Similarly $$AQ = AB\frac{\sin ABQ}{\sin AQB},$$

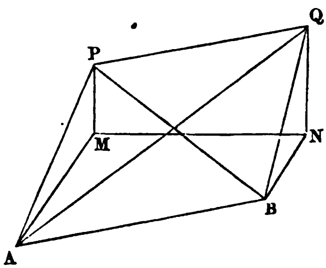
or $$PA = a\frac{\sin\gamma}{\sin(\alpha+\gamma)},$$

and $$AQ = a\frac{\sin\delta}{\sin(\beta+\delta)}.$$

Hence in the triangle APQ we have AP, AQ and the included angle α given, and therefore PQ can be found by the method of Chap. IX. Art. 11, Case (3).

9. Suppose that the two points P and Q are above the horizontal plane in which AB lies, and that the four points A, B, P, Q are not in the same plane. To find the distance PQ and the vertical height of P and Q.

Let PM, QN be the vertical heights of P and Q above the horizontal plane in which AB lies. The lines PB, AQ will not meet each other if Q is in a different plane to A, B and P.



Observe the angles PAB, PBA and the

angle of elevation PAM. Then exactly, as in Article 6, the height of PM can be found.

Again, observe the angles QBA, QAB and the angle of elevation QBN, and thence similarly QN is known.

PQ can de determined as in Art. 8.

10. *Three objects being visible, the places of which upon a map are known, to find the position upon the map of the place of observation.*

Let A, B, C be the three objects visible from P the place of observation, the situations of which upon the map are known.

Observe the angles APB, BPC subtended at P by the two sides of the triangle AB, BC respectively.

Upon AB, [figs. (1) and (2),] on the map describe a segment of a circle containing an angle equal to the observed angle APB.

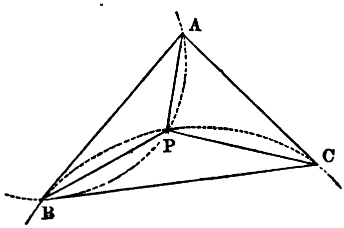


Fig. (1).

Similarly upon BC describe a segment containing an angle equal to the observed angle BPC.

The point of intersection of these two segments of circles will clearly be the position upon the map of the place of observation.

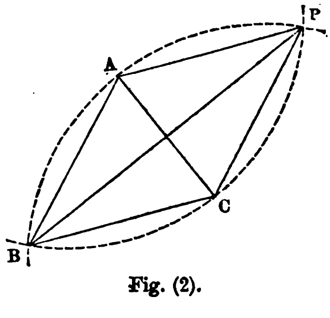
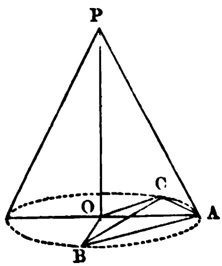


Fig. (2).

The construction obviously fails if the quadrilateral $ABCP$ be such as can be inscribed in a circle, i.e. if P lie anywhere on the circumference of the circle circumscribing the triangle ABC, for then the arcs BAP, BCP are arcs of the same circle, and we get no point of intersection to determine P.

11. The distances between three points A, B, C in a horizontal plane being given from which the elevations of the summit of an inaccessible object are the same, to find the height of the object.



Let P be the object. Then since the elevations of P as seen from A, B and C are the same, A, B, C clearly lie upon the surface of a cone of which P is the vertex. Hence if PO be a vertical through P intersecting the horizontal plane in O, O will be the centre of the circle circumscribing the triangle ABC. Hence if α be the angle of elevation of P from A, B and C, and a, b, c be the distances BC, CA, AB respectively,

$$OA = \frac{abc}{4\sqrt{\{s.(s-a)(s-b)(s-c)\}}},$$

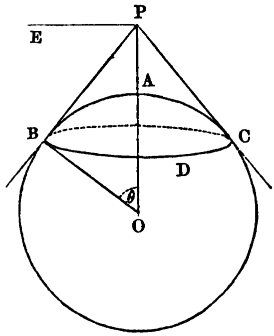
and $OP = OA \tan \alpha$,

$$= \frac{abc \tan \alpha}{4\sqrt{\{s.(s-a)(s-b)(s-c)\}}}.$$

12. *Horizon. Dip of the Horizon, &c.*

If straight lines be drawn from an observer's eye to touch the surface of the earth, the circle in which the points of contact lie evidently forms the boundary of that portion of the earth's surface which is visible to him. This circle is the terrestrial horizon.

Let P be the observer's eye at a height h above the earth's surface. Draw the tangents PB, PC to the earth's surface. Then the circle BCD is the horizon. Let R be the earth's radius, which will be very large in comparison with h being nearly 4000 miles in length. Let θ be the circular measure of the angle AOB, which will be a very small angle, since the arc AB will be very small compared with the radius of the earth.



If d be the length of the arc AB,

$$\theta = \frac{d}{R}.$$

Then $\cos\theta = \frac{OB}{OP} = \frac{R}{R+h} = \frac{1}{1+\frac{h}{R}} = \left(1+\frac{h}{R}\right)^{-1}.$

$$\therefore (1-\sin^2\theta)^{\frac{1}{2}} = \left(1+\frac{h}{R}\right)^{-1}.$$

Expanding by the Binomial Theorem,

$$1 - \frac{1}{2}\sin^2\theta - \frac{1}{8}\sin^4\theta + \text{\&c.} = 1 - \frac{h}{R} + \frac{h^2}{R^2} - \text{\&c.}$$

Now θ being small we may write θ instead of $\sin\theta$ (Chap. II. Art. 14,) or $\frac{d}{R}$ instead of $\sin\theta$, without serious error.

We shall then have

$$1 - \frac{1}{2}\frac{d^2}{R^2} - \frac{1}{8}\frac{d^4}{R^4}\text{\&c.} = 1 - \frac{h}{R} + \frac{h^2}{R^2} - \text{\&c.}$$

and $\frac{d}{R}$ being a small fraction, we may neglect $\frac{d^4}{R^4}$ and all the other terms of the series without introducing any serious error. Similarly (as will better be seen by reference to the subjoined numerical example) $\frac{h^2}{R^2}$ may be neglected in comparison with $\frac{d^2}{R^2}$ and $\frac{h}{R}$. Hence we get, approximately,

$$d^2 = 2Rh,$$

$$\text{and} \therefore h \text{ varies as } d^2.$$

If PE be a horizontal line, the angle BPE, which is equal to θ, is called the "*dip of the horizon*."

13. Suppose h to be 50 feet, to find d approximately, and the amount of error in the value obtained of d if $\frac{h^2}{R^2}$ be neglected.

Taking R to be 4000 miles, by substitution in the formula

$$d^2 = 2Rh,$$

we shall get

$$d = 551478{\cdot}09 \text{ inches},$$

which is between 8 and 9 miles.

Now, if $\frac{h^2}{R^2}$ were retained, we should have

$$d^2 = 2Rh - 2h^2,$$

which would give a value of d less than that we have obtained by a distance which is less than an inch.

Hence the error in d would be practically quite inconsiderable.

14. We might also proceed thus,

$$(R + h)\cos\theta = R,$$

$$\therefore\ R=\frac{h\cos\theta}{1-\cos\theta}=\frac{h\cos\theta}{2\sin^2\frac{\theta}{2}},$$

$$\therefore\ \log 2R=\log h+\log\cos\theta-2\log\sin\frac{\theta}{2}.$$

From this formula R can be found if θ and h are known, or h can be found if θ and R are known.

A

COLLECTION OF EXAMPLES

IN ILLUSTRATION OF THE PRECEDING CHAPTERS.

EXAMPLES ILLUSTRATING CHAPTER I.

1. Reduce $35^\circ\ 17'\ 35''$ to grades.

$$\begin{array}{r|l} 60 & 35{\cdot}000 \\ \hline 60 & 17{\cdot}58333 \\ \hline & 35{\cdot}293055 \end{array}$$

$\therefore\ 35^\circ\ 17'\ 35'' = 35{\cdot}2930\dot{5}$;

$$\therefore \text{ the number of grades in the angle} = \frac{10}{9}(35{\cdot}2930\dot{5})$$
$$= \frac{1}{9}(352{\cdot}930\dot{5})$$
$$= 39{\cdot}214506\ldots$$
$$= 39^g\ 21'\ 45''\ {\cdot}06\ldots,$$

if we omit the decimals after the first two places.

2. If n yards be taken as the unit of length, what will be the numerical measure of m feet?

The numerical measure of any magnitude is the ratio which it bears to the magnitude which is taken as the unit of measurement. Now the ratio of m feet or $\frac{m}{3}$ yards to n yards is $\frac{m}{3n}$, which is therefore the numerical measure of m feet required.

3. If one French minute be taken as the unit of angular magnitude, what is the measure of an angle of one English minute?

$$\text{One English minute} = \frac{1}{60} \text{ of a degree,}$$

$$= \frac{10}{9} \times \frac{1}{60} \text{ of a grade,}$$

$$= \frac{1}{54} \text{ of a grade,}$$

$$= \frac{100}{54} \text{ of a French minute,}$$

$$= \frac{50}{27} \text{ of a French minute,}$$

Hence $\frac{50}{27}$ (being the ratio of the given angle which is to be measured to the angle which is taken as the unit of angular measurement) is the measure required.

4. What must be the unit angle if the sum of the measures of a degree and a grade is 1?

Let x be the number of degrees in the unit angle,

then $\frac{1}{x}$ is the measure of 1^0.

And in 1 grade there are $\frac{9}{10}$ of a degree,

$\therefore \frac{9}{10x}$ is the measure of 1 grade.

And, by the conditions,

$$\frac{1}{x} + \frac{9}{10x} = 1,$$

$$\therefore x = 1 + \frac{9}{10},$$

$$= 1.9;$$

$\therefore$ the unit is an angle which contains 1.9 degrees.

5. If the measure of an angle be equal to the sum of the number of degrees, and half the number of grades in it, what is the unit of angular measure?

Let x be the number of degrees in the angle.

Then $\frac{10}{9}x$ is the number of grades in it.

The measure of the angle is therefore represented by the number $x+\frac{5}{9}x$ or by $\frac{14}{9}x$.

Let y be the number of degrees in the unit.

Then $\frac{14}{9}x\times y=$ the number of degrees in the angle,

$$=x,$$

$$\therefore\ y=\frac{9}{14}.$$

Or the unit of measurement is $\frac{9}{14}$ ths of a degree.

6. The measures of the three angles of a triangle expressed respectively A in degrees, B in grades, and C in circular measure, are numerically equal to one another; find A.

Let x, y, and z be the number of degrees in the three angles.

Then since they are the angles of a triangle,

$$x+y+z=180.\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now y degrees contain $\frac{10}{9}y$ grades,

and the circular measure of z degrees is

$$\frac{z}{\frac{180}{\pi}}\text{ or }\frac{\pi z}{180}.$$

Hence, by the conditions of the problem,

$$x=\frac{10}{9}y=\frac{\pi z}{180}.\ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Substituting in (1) the values of y and z in terms of x obtained from (2), we have

$$x+\frac{9}{10}x+\frac{180}{\pi}x=180,$$

$$\therefore\ x\left(\frac{10\pi+9\pi+1800}{10\pi}\right)=180,$$

$$\therefore\ x=\frac{1800\pi}{19\pi+1800},$$

which is the number of degrees in the angle A.

7. Reduce $135^{0}\,47'\,52''$ to grades. Ans. $150^{g}\,88^{`}\,64^{``}\cdot1\dot{9}\ldots$

8. Reduce $25^{g}\,32^{`}\,50^{``}$ to degrees &c. Ans. $22^{0}\,47'\,33''$.

9. Find the number of grades in $53^0 4' 21''$ and in $27^0 19' 1''$.
Ans. $58^g 96^{\backprime} 94^{\backprime\backprime}\cdot 4$ and $30^g 35^{\backprime} 21^{\backprime\backprime}\cdot 6\ldots$

10. Find the number of degrees in $27^g 19^{\backprime} 1^{\backprime\backprime}$ and in $53^g 4^{\backprime} 21^{\backprime\backprime}$.
Ans. $24^0 28' 15''\cdot 924$ and $47^0 44' 16''\cdot 404$.

11. Find the number of degrees and minutes in $\cdot 255$ of a right angle. Ans. $22^0 57'$.

12. The angles of a quadrilateral inscribed in a circle, taken in order, when multiplied by 1, 2, 2, 3 respectively, are in arithmetical progression; find their values.

Ans. $\frac{12}{17}, \frac{14}{17}, \frac{22}{17}, \frac{20}{17}$ of a right angle.

13. How many sides has a polygon which contains as many grades in all its angles together as there are degrees in all the angles of a dodecagon? Ans. 11.

14. The numbers of the sides of two regular polygons are as 2 : 3, and the number of grades in an angle of one equals the number of degrees in an angle of the other. Find how many sides they each have.

Ans. Eight and twelve.

15. The interior angles of an irregular polygon are in arithmetical progression; the least angle is 120^0 and the common difference 5^0. Find the number of sides. Ans. 16 or 9 sides.

16. The number of grades in an angle of a regular polygon is to the number of degrees in an angle of another as 5 : 3. Find the number of sides in each, shewing that there are only three different solutions.

Ans. 3 and 4 sides,
4 and 8 sides,
5 and 20 sides.

17. Find the circular measure of $10''$ of $36^0 15' 22''$ and of π grades. Ans. $\cdot 000015\pi$,
$\cdot 6328$ nearly,
$\frac{\pi^2}{200}$.

18. Find the number of degrees, minutes and seconds in the angles of which the circular measures are $\pi + 1$ and $\frac{11}{21}$.

Ans. $237^0 17' 44''$ nearly, and $30^0 0' 43''\cdot 45$.

19. If one-third of a right angle be assumed as the unit of angular measurement, what number will represent an angle of 75^0? Ans. $\frac{5}{2}$.

20. Determine the number of degrees in the unit of angular measurement when an angle of $66 \cdot \dot{6}$ grades is represented by 20.
Ans. 3 degrees.

21. Find the circular measure of 42^0. Ans. ·73303.

22. The length of an arc of 45^0 in one circle is equal to that of 60^0 in another, find the circular measure of an angle which would be subtended at the centre of the first by an arc equal to the radius of the second. Ans. $\frac{3}{4}$.

23. If the unit of angle be an angle the subtending arc of which is 3 times the radius, what number would represent an angle of 45^0? Ans. $\frac{\pi}{12}$.

24. One angle of a triangle is π grades, and another is π degrees, find the circular measure of the third.

Ans. $\pi - \frac{19\pi^2}{1800}$.

25. Express in degrees, minutes, and seconds, the angle subtended at the centre of a circle of radius 5 feet by an arc whose length is 7 inches. Ans. $6^0\, 40'\, 54''\, 5\dot{2}$.

26. Shew that the length of an arc subtending an angle of $14^0\, 29'\, 24''$ in a circle, the radius of which is 1000 yards, is 252 yds. 2 ft. 8 in. nearly.

27. Assuming $\pi = 3.14159$, find the number of linear inches in a circle the radius of which is 2 ft. 6 inches.

Express also the length of each inch of circumference (1) in degrees, minutes, and seconds, and (2) in grades, French minutes, and seconds.

Ans. 188.4954 inches; $1^0\, 54'\, 35\frac{1}{2}''$ nearly; $2^g\, 12^`\, 20^{``}\frac{1}{2}$ nearly.

28. Find the circular measure of an angle in which the number of degrees is equal to the number of grades in its complement. Ans. $\frac{5\pi}{19}$.

29. The angles of a triangle are in arithmetical progression, and the number of grades in the least is to the circular measure of the greatest as 30 to π. Find the angles.

Ans. $\frac{2\pi}{23}$, $\frac{\pi}{3}$ and $\frac{40\pi}{69}$.

30. If the unit of angle be $\frac{1}{32}$ of the angular magnitude determined by the complete revolution in one plane of a straight line, find in degrees the angle expressed by 5.09296.

Ans. $57^0.29580$.

31. If with two units of angular measurement differing by 10^0, the measures of an angle are as 3 : 2, what are these units?

Ans. Angles of 20^0 and 30^0.

32. If half a right angle be taken for the unit of measurement, express the following angles: $\frac{3}{2}$, 4, π, $4n+\frac{1}{3}$,

(1) in degrees, (2) in circular measure.

33. If the circumference of a circle be divided into n equal parts, how many of them would an arc equal to the radius contain?

Ans. $\frac{n}{2\pi}$.

34. What is the length of an arc of 1^0 of the meridian, the earth being supposed spherical and its radius 4000 miles?

Ans. $\frac{200\pi}{9}$ miles.

35. Find the relation between the numbers a, b, c, that an angle containing a English degrees may be expressed by b in a scale of which the unit of measurement is one c^{th} of the circular unit of angle.

Ans. $b=\frac{\pi ac}{180}$.

36. Find the velocity of the earth in its orbit (supposed circular), the distance of the sun from it being 94,000,000 miles.

Ans. $\frac{11750}{1971}\pi$ miles a second.

37. The numerical measures of the angles of a quadrilateral, when referred to units containing 1^0, 2^0, 3^0, 4^0 respectively, are in Arithmetical Progression, and the difference of the second and fourth is equal to a right angle. Find the angles.

Ans. 30^0, 66^0, 108^0, 156^0.

38. What must be the unit of measurement, that the numerical measure of an angle may be equal to the difference between its numerical measures as expressed in degrees and in the common circular measure?

Ans. An angle of $\frac{180}{180-\pi}$ degrees.

39. Two wheels with fixed centres roll upon each other, and the circular measure of the angle through which one turns gives the number of degrees through which the other turns in the same time. Compare the radii of the wheels.

Ans. The ratio is $180 : \pi$.

EXAMPLES ILLUSTRATING CHAPTER II.

1. Given $\tan\theta = \frac{a}{b}$, find $\sin\theta$ and $\cos\theta$.

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{a}{b};$$

$\therefore$ $\sin\theta$ bears the same ratio to a that $\cos\theta$ bears to b.

Let this ratio be λ.

Then $$\lambda\sin\theta = a,$$
$$\lambda\cos\theta = b.$$

Squaring and adding,

$$\lambda^2(\sin^2\theta + \cos^2\theta) = a^2 + b^2.$$

But $$\sin^2\theta + \cos^2\theta = 1;$$

$$\therefore\ \lambda = \pm\sqrt{(a^2+b^2)};$$

$$\therefore\ \sin\theta = \frac{a}{\lambda} = \pm\frac{a}{\sqrt{(a^2+b^2)}},$$

$$\cos\theta = \frac{b}{\lambda} = \pm\frac{b}{\sqrt{(a^2+b^2)}}.$$

These results should be recollected.

This same use of an arbitrary multiplier λ will be found in Chap. II. Art. 6.

2. If $\frac{\tan\alpha}{\sin\theta} - \frac{\tan\beta}{\tan\theta} = (\tan^2\alpha - \tan^2\beta)^{\frac{1}{2}}$,

then $$\cos\theta = \frac{\tan\beta}{\tan\alpha}.$$

Dividing both sides of the equation by $\tan\alpha$, and writing n for the ratio $\dfrac{\tan\beta}{\tan\alpha}$ for shortness, we have

$$\frac{1}{\sin\theta}-\frac{n}{\tan\theta}=(1-n^2)^{\frac{1}{2}};$$

$$\therefore\ \frac{1}{\sin\theta}-\frac{n\cos\theta}{\sin\theta}=(1-n^2)^{\frac{1}{2}}.$$

Squaring both sides,

$$\frac{1-2n\cos\theta+n^2\cos^2\theta}{\sin^2\theta}=1-n^2;$$

$$\therefore\ 1-2n\cos\theta+n^2\cos^2\theta=(1-n^2)(1-\cos^2\theta)$$
$$=1-n^2-\cos^2\theta+n^2\cos^2\theta;$$

$$\therefore\ \cos^2\theta-2n\cos\theta+n^2=0,$$

$$\text{or } (\cos\theta-n)^2=0;$$

$$\therefore\ \cos\theta=n=\frac{\tan\beta}{\tan\alpha}.$$

3. Find the limit of $\dfrac{\tan n'+\tan n''}{n}$ when n is indefinitely diminished.

The circular measure of n' is $\dfrac{n}{\dfrac{180\times 60}{\pi}}=\dfrac{n\pi}{180\times 60}$,

and that of n'' is $\dfrac{n}{\dfrac{180\times 60\times 60}{\pi}}=\dfrac{n\pi}{180\times 60\times 60}$;

$$\therefore\ \frac{\tan n'}{n}\text{ is }\frac{\tan\dfrac{n\pi}{180\times 60}}{n}=\frac{\pi}{180\times 60}\cdot\frac{\tan\dfrac{n\pi}{180\times 60}}{\dfrac{n\pi}{180\times 60}},$$

$$\text{and }\frac{\tan n''}{n}\text{ is }\frac{\tan\dfrac{n\pi}{180\times 60\times 60}}{n}=\frac{\pi}{180\times 60\times 60}\cdot\frac{\tan\dfrac{n\pi}{180\times 60\times 60}}{\dfrac{n\pi}{180\times 60\times 60}}.$$

Now since, when θ is indefinitely diminished, $\dfrac{\tan\theta}{\theta}=1$,

$\therefore$ when n is indefinitely diminished,

$$\frac{\tan\dfrac{n\pi}{180\times 60}}{\dfrac{n\pi}{180\times 60}}=1,\quad\text{and}\quad\frac{\tan\dfrac{n\pi}{180\times 60\times 60}}{\dfrac{n\pi}{180\times 60\times 60}}=1,$$

∴ when n is indefinitely diminished, the limit of

$$\frac{\tan n'}{n}+\frac{\tan n''}{n} \text{ is } \frac{\pi}{180\times 60}+\frac{\pi}{180\times 60\times 60},$$

or

$$\frac{61\,\pi}{180\times 60\times 60}.$$

4. Is the equation $\sec^2\theta=\frac{4ab}{(a+b)^2}$ a possible equation?

5. Prove that $\sin^2\frac{\pi}{2}\cdot\frac{a}{a+b}+\sin^2\frac{\pi}{2}\cdot\frac{b}{a+b}=1$.

6. Prove that $\cos^4\theta-\sin^4\theta=\cos^2\theta-\sin^2\theta$.

7. Find the supplement and complement of $149^0\ 25'\ 50''.4$.

Ans. $30^0\ 34'\ 9''6,\ -59^0\ 25'\ 50''.4$.

8. If $\text{versin}\,A=\frac{1}{13}$, find the other trigonometrical functions.

Ans. $\cos A=\frac{12}{13}$, $\sin A=\pm\frac{5}{13}$,

$\tan A=\pm\frac{5}{12}$, $\sec A=\frac{13}{12}$.

9. Prove that $\sin A\,(1+\tan A)+\cos A\,(1+\cot A)$
$=\sec A+\text{cosec}\,A$.

10. Given $\sin\theta=\frac{3}{10}$, find the other trigonometrical functions of θ.

Ans. $\cos\theta=\pm\frac{\sqrt{91}}{10}$, $\tan\theta=\pm\frac{3}{\sqrt{91}}$.

11. Prove that
$(\sec\theta\cot\theta)^2+(\text{cosec}\,\theta\tan\theta)^2=(\sec\theta\,\text{cosec}\,\theta)^2$.

12. Shew that if $\tan\theta+\cot\theta=2$, then $\sin\theta+\cos\theta=\sqrt{2}$.

13. If $\tan\theta=-\frac{3}{4}$, find the other trigonometrical functions.

Ans. $\sin\theta=\pm\frac{3}{\sqrt{5}}$, $\cos\theta=\mp\frac{4}{\sqrt{5}}$.

14. Prove that $\sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta=\sin^2\alpha-\sin^2\beta$.

15. If $\text{versin}\,\theta=\frac{1}{3}$, find the other trigonometrical functions.

Ans. $\sin\theta=\pm\frac{\sqrt{5}}{3}$, $\tan\theta=\pm\frac{\sqrt{5}}{2}$.

16. If $\tan\theta = \frac{a}{b}$, find the values of

$$a\cos\theta - b\sin\theta, \qquad \frac{a\sin\theta - b\cos\theta}{a\sin\theta + b\cos\theta},$$

$$\sin^2\theta\,(1 + \text{versin}^2\,\theta).$$

Ans. $0, \quad \frac{a^2-b^2}{a^2+b^2}, \quad \frac{a^2}{(a^2+b^2)^2}\{2a^2 + 3b^2 - 2b\sqrt{(a^2+b^2)}\}.$

17. If $\tan\theta + 3\cot\theta = 4$, find $\sin\theta$. Ans. $\pm\frac{1}{\sqrt{2}}$ or $\pm\frac{3}{\sqrt{10}}$.

18. Eliminate θ and ϕ between the equations

$$\sin\theta = m\cos\phi + n\sin\phi,$$
$$\cos\theta = m\sin\phi - n\cos\phi.$$

Ans. $m^2 + n^2 = 1$.

19. Given $\text{cosec}\,\theta = \frac{4}{3}$, find the other functions.

Ans. $\sin\theta = \frac{3}{4}, \quad \cos\theta = \pm\frac{\sqrt{7}}{4},$

$\tan\theta = \pm\frac{3}{\sqrt{7}}, \quad \cot\theta = \pm\frac{\sqrt{7}}{3},$

$\sec\theta = \pm\frac{4}{\sqrt{7}}, \quad \text{vers}\,\theta = \frac{4\mp\sqrt{7}}{4}.$

20. Prove that

$$\tan\theta + \cot\theta = \frac{1}{\sin\theta\cos\theta} = \sqrt{(\sec^2\theta + \text{cosec}^2\,\theta)}.$$

21. Prove that $\cot^2\theta - \cos^2\theta = \cot^2\theta\cos^2\theta.$

22. Prove that $\frac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta} = \tan\alpha\tan\beta.$

23. Prove that

$$2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1 = 0.$$

24. Prove that

$$(\sin^2\alpha - \sin^2\alpha\sin^2\beta)(\cos^2\alpha - \cos^2\alpha\cos^2\beta) = (\sin^2\beta - \sin^2\alpha\sin^2\beta) \times (\cos^2\beta - \cos^2\alpha\cos^2\beta).$$

25. Prove that $\frac{\sec\theta\cot\theta - \text{cosec}\,\theta\tan\theta}{\cos\theta - \sin\theta} = \text{cosec}\,\theta\sec\theta.$

26. If $\frac{\cot x - \sec x}{\cot x} = \frac{1}{16},$

then $\tan x = \pm\frac{3}{4}.$

27. If $\cos\theta = \tan\theta$, then $\sin\theta = \frac{\sqrt{5}-1}{2}$.

28. If $\tan\theta = \frac{m}{n}\cos\alpha$, and $\sin\theta = \frac{\cos\alpha}{\sin\beta}$,

$$\sin\beta = \pm\frac{\sqrt{(n^2+m^2\cos^2\alpha)}}{m}.$$

29. Given
$$\sin\theta = m\cos\phi = n\tan\psi,$$
$$\sec\psi = \sin\phi + \cos\phi,$$
shew that either $\sin\theta = \frac{2mn^2}{\sqrt{(m^4+4n^4)}}$, or $\sin\theta = 0$.

30. Prove that $\frac{(\operatorname{cosec}\theta + \sec\theta)^2}{\operatorname{cosec}^2\theta + \sec^2\theta} = 1 + 2\sin\theta\cos\theta$.

31. Eliminate θ from the equations
$$\operatorname{cosec}^2\theta = m\tan\theta,$$
$$\sec^2\theta = n\cot\theta.$$
Ans. $(mn)^{\frac{2}{3}} = (\sqrt{m}+\sqrt{n})^2$.

32. Prove that
$$\operatorname{cosec}\theta(\sec\theta - 1) + \sin\theta = \cot\theta(1-\cos\theta) + \tan\theta.$$

33. Prove that
$$\sec^4\theta + \tan^4\theta = 1 + 2\sec^2\theta.\tan^2\theta.$$

34. If $\sin\alpha = m\sin\beta$, and $\tan\alpha = n\tan\beta$,
$$\cos\alpha = \sqrt{\left(\frac{m^2-1}{n^2-1}\right)}.$$

35. If $\tan\theta + \sin\theta = m$, and $\tan\theta - \sin\theta = n$,
$$(m^2-n^2)^2 = 16mn.$$

36. If
$$a = \sec\theta + \operatorname{cosec}\theta\tan^3\theta(\operatorname{cosec}^2\theta + 1),$$
$$b = \tan\theta - \tan^3\theta(\operatorname{cosec}^2\theta + 1),$$
then
$$a^{\frac{2}{3}} - b^{\frac{2}{3}} = 2^{\frac{2}{3}}.$$

37. Prove that
$$\sin^2\theta\tan\theta + \cos^2\theta\cot\theta + 2\sin\theta\cos\theta = \tan\theta + \cot\theta.$$

38. Given $\frac{1+\tan\theta}{1-\tan\theta} = \frac{a}{b}$, find $\sin\theta$ and $\cos\theta$.

Ans. $$\left.\begin{aligned}\sin\theta &= \pm\frac{a-b}{\sqrt{\{2(a^2+b^2)\}}}\\ \cos\theta &= \pm\frac{a+b}{\sqrt{\{2(a^2+b^2)\}}}\end{aligned}\right\}$$

39. If $\tan\theta = \frac{4}{3}$, find the value of

$$(\sin\theta + \tan\theta + \sec\theta)\ (\cos\theta + \cot\theta + \operatorname{cosec}\theta).$$

Ans. $9\frac{22}{25}$.

40. Eliminate a, b and c from the equations

$$\begin{cases} a - b\cos C - c\cos B = 0, \\ b - c\cos A - a\cos C = 0, \\ c - a\cos B - b\cos A = 0. \end{cases}$$

Ans. $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$.

41. If $\frac{\tan^2\theta}{\tan^2\alpha} + \frac{\tan^2\phi}{\tan^2\beta} = 1$ and $\frac{\sin\theta}{\sin\alpha} = \frac{\sin\phi}{\sin\beta}$, shew that

$$\sin\theta = \pm\frac{\sin\alpha}{\sqrt{(1 \pm \cos\alpha\cos\beta)}}.$$

42. Eliminate θ between the equations

$$\frac{x}{a\sin\theta} + \frac{y}{b\cos\theta} = 1,$$

$$\frac{x}{a\sin^2\theta} + \frac{y}{b\cos^2\theta} = 0,$$

Ans. $\sqrt{(bx)}\sqrt{(bx - ay)} + \sqrt{(ay - bx)}\sqrt{(ay)} = ab$.

43. Eliminate p, q, s and ϕ from the following equations:

$$\alpha = a\cos\phi + (qc - sb)\sin\phi,$$
$$\beta = b\cos\phi + (sa - pc)\sin\phi,$$
$$\gamma = c\cos\phi + (pb - qa)\sin\phi,$$
$$pa + qb + sc = 0,$$
$$p^2 + q^2 + s^2 = 1.$$

Ans. $a^2 + b^2 + c^2 = \alpha^2 + \beta^2 + \gamma^2$.

44. Eliminate ϕ and i from the equations

$$A = \left(\frac{\sin^2\phi}{a^2} + \frac{\cos^2\phi}{b^2}\right)\cos^2 i + \frac{\sin^2 i}{c^2},$$

$$B = \frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2},$$

$$C = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\sin\phi\cos\phi\cos i.$$

Ans. $\left(A - \frac{1}{c^2}\right)\left(B - \frac{1}{a^2}\right)\left(B - \frac{1}{b^2}\right) + C^2\left(\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2} - B\right) = 0.$

45. Eliminate θ between the equations

$$\frac{x}{a\sin\theta}+\frac{y}{b\cos\theta}=1,$$

$$\frac{x}{a\sin^3\theta}+\frac{y}{b\cos^4\theta}=0.$$

Ans. $\pm\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}}\mp\frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}}=\frac{a^{\frac{1}{3}}b^{\frac{1}{3}}}{\sqrt{\{(bx)^{\frac{2}{3}}+(ay)^{\frac{2}{3}}\}}}$.

46. When $\theta=30^0$, find the value of $3\sin\theta-2\cos^2\theta$.
Ans. zero.

47. Shew that a value of θ which satisfies the equation $2\tan\theta=\cos\theta$ lies between 18^0 and 30^0.

48. Solve the equation
$4\sin^2 30^0+\tan^2 45^0+\sec^2 60^0=x\operatorname{cosec} 30^0$. Ans. $x=3$.

49. The altitude of a tower at the extremity of a horizontal line measured from its foot, 100 yards long, is 30^0. Find the height of the tower to two places of decimals. Ans. 57.73 yards.

50. Find the limit to which $\frac{\sin m\theta}{\sin n\theta}$ approximates when θ is diminished without limit. Ans. $\frac{m}{n}$.

51. Find the value to which the ratio $\frac{\sin n''}{n}$ approximates as n is diminished without limit. Ans. $\frac{\pi}{648000}$.

52. Find at what distance from the eye a shilling, five-eighths of an inch in diameter, will completely cover the moon's disc, considered to subtend a degree. Ans. $\frac{225}{2\pi}$ inches nearly.

EXAMPLES ILLUSTRATING CHAPTER III.

1. If $\sin\theta=a$, find the sine and secant of $(6n+3)\frac{\pi}{2}+\theta$.

$$\sin\left\{(6n+3)\frac{\pi}{2}+\theta\right\}=\sin\left(3n\pi+\frac{3\pi}{2}+\theta\right),$$

denoting $\frac{3\pi}{2}+\theta$ by a, we have

$$\sin(3n\pi+a) = \pm\sin a, \text{ according as } n \text{ is even or odd;}$$

$$\text{or, } \sin(3n\pi+a) = (-1)^n \sin a,$$

$$\text{and } \sin a = \sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$$

$$= -\sqrt{(1-a^2)};$$

$$\therefore \sin\left\{(6n+3)\frac{\pi}{2}+\theta\right\} = (-1)^{n+1}\sqrt{(1-a^2)},$$

$$\text{and } \sec(3n\pi+a) = \frac{1}{\cos(3n\pi+a)} = \frac{(-1)^n}{\cos a} = \frac{(-1)^n}{\sin\theta};$$

$$\therefore \sec\left\{(6n+3)\frac{\pi}{2}+\theta\right\} = \frac{(-1)^n}{a}.$$

2. If $\tan(\pi\cot\theta) = \cot(\pi\tan\theta)$, shew that

$$\tan\theta = \frac{2n+1}{4} \pm \frac{\sqrt{(4n^2+4n-15)}}{4},$$

n being any integer.

Since $\tan(\pi\cot\theta) = \cot(\pi\tan\theta)$,

$$\tan(\pi\cot\theta) = \tan\left(\frac{\pi}{2}-\pi\tan\theta\right).$$

Now, since $n\pi+a$ is an expression including all angles which have the same tangent as a, $\pi\cot\theta$ *may be* any one of the angles included in the expression $n\pi+\frac{\pi}{2}-\pi\tan\theta$. To get then *all* the values of $\tan\theta$ which satisfy the given equation, we must write

$$\pi\cot\theta = n\pi+\frac{\pi}{2}-\pi\tan\theta,$$

or, dividing by π and writing $\frac{1}{\tan\theta}$ for $\cot\theta$,

$$\tan^2\theta - \frac{2n+1}{2}\tan\theta = -1,$$

a quadratic equation, which gives

$$\tan\theta = \frac{2n+1}{4} \pm \frac{\sqrt{(4n^2+4n-15)}}{4}.$$

3. Trace the variations in the sign and magnitude of $\sin(\pi\cos\theta)$ as θ varies from 0° to 360°.

Since $\cos\theta$ can never be numerically greater than unity, $\sin(\pi\cos\theta)$ will be positive or negative according as $\cos\theta$ is positive or negative.

Hence it is positive throughout the first and fourth quadrants, and negative throughout the second and third.

As regards magnitude.

When $\theta = 0$, $\sin(\pi\cos\theta) = \sin\pi = 0$.

As θ increases, $\pi\cos\theta$ decreases, and therefore $\sin(\pi\cos\theta)$ increases until $\theta = 60^0$, when $(\pi\cos\theta) = 1$.

From $\theta = 60^0$ to $\theta = 90^0$ $\sin(\pi\cos\theta)$ decreases in magnitude, and when $\theta = 90^0$ it is $= 0$.

Similarly from $\theta = 90^0$ to $\theta = 120^0$ it increases in magnitude, and, when $\theta = 120^0$, $\sin(\pi\cos\theta) = -1$.

From $\theta = 120^0$ to $\theta = 180^0$ it decreases in magnitude, and, when $\theta = 180^0$ it is $= 0$.

From $\theta = 180^0$ to $\theta = 240^0$ it increases in magnitude, and, when $\theta = 240^0$, it $= -1$.

From $\theta = 240^0$ to $\theta = 270^0$ it decreases in magnitude, and, when $\theta = 270^0$, it $= 0$.

From $\theta = 270^0$ to $\theta = 330^0$ it increases in magnitude, and, when $\theta = 330^0$, it $= 1$.

From $\theta = 330^0$ to $\theta = 360^0$ it decreases in magnitude, and, when $\theta = 360^0$, it $= 0$.

[N.B. Observe that, π and $\cos\theta$ being each numerical quantities, $\pi\cos\theta$ is a numerical quantity, and that therefore $\sin(\pi\cos\theta)$ denotes the sine of the angle of which the circular measure is $\pi\cos\theta$.]

4. Find the sines and cosines of 240^0, 300^0 and 162^0.

$$\text{Ans. } \sin 240^0 = -\frac{\sqrt{3}}{2},\ \cos 240^0 = -\frac{1}{2},$$

$$\sin 300^0 = -\frac{\sqrt{3}}{2},\ \cos 300^0 = \frac{1}{2},$$

$$\sin 162^0 = \frac{\sqrt{5}-1}{4},\ \cos 162^0 = -\frac{\sqrt{(10+2\sqrt{5})}}{4}.$$

5. Find all the trigonometrical functions of 3480^0.

$$\text{Ans. } \sin 3480^0 = -\frac{\sqrt{3}}{2},\ \cos 3480^0 = -\frac{1}{2},\ \tan 3480 = \sqrt{3}.$$

6. **Find the trigonometrical functions of 1845^0 and 2178^0.**
Ans. They are the same as those of 45^0 and 18^0 respectively.

7. **Find the trigonometrical functions of 4356^0 and 3690^0.**

$$\text{Ans. } \sin 4356^0 = \frac{\sqrt{(10-2\sqrt{5})}}{4}, \ \cos 4356^0 = \frac{\sqrt{5}+1}{4}.$$

Ans. The functions of 3690^0 are the same as those of 90^0.

8. **Write down all the values of A lying between zero and $\pm 360^0$ which give**

(1) $\sin A = \sin 100^0$,
(2) $\tan A = \cot 75^0$.

Ans. (1) 80^0, -260^0 and -280^0,
(2) 15^0 and -345^0, 195 and -165^0.

9. **Find the sine and tangent of $\left(\frac{3\pi}{2}+\theta\right)$ in its simplest form.** **Ans.** $-\cos\theta$ and $-\cot\theta$.

10. **Find the sine and tangent of $(2n \pm 3)\frac{\pi}{2}+\theta$ in their simplest forms, n being an integer.** **Ans.** $\pm\cos\theta$ and $-\cot\theta$.

11. **Find the general values of the limits between which A lies when $\sin^2 A$ is greater than $\cos^2 A$.**

Ans. Between $2n\pi + \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$,
$2n\pi + \frac{5\pi}{4}$ and $2n\pi + \frac{7\pi}{4}$.

12. **The sine and tangent of an angle are both negative, and the ratio of one to the other is $\sqrt{2}$. Find an algebraical expression for all the angles which have this property.** **Ans.** $2n\pi - \frac{\pi}{4}$.

13. **Prove that vers $(2n\pi+\theta)$ + versin $\{(2n+1)\pi-\theta\} = 2$.**

14. **If $\sin\theta = a$, find the sine, cosine, and tangent of $\left(m+\frac{1}{2}\right)\pi - \theta$.** **Ans.** $(-1)^m\sqrt{(1-a^2)}$, $(-1)^m a$, $\frac{\sqrt{(1-a^2)}}{a}$.

15. **Express in terms of θ all the angles of which the sine is $-\sin\theta$.** **Ans.** $n\pi + (-1)^{n+1}\theta$.

16. Prove that, n being any *odd* integer,

$$\cos n\left(\frac{\pi}{2}-A\right)=(-1)^{\frac{n-1}{2}}\sin nA.$$

17. Shew that all the angles which have the same sine as α are included in the formula $n\pi+\alpha\cos n\pi$.

Find all the values of θ which satisfy each of the following equations:

18. $\sin\theta=0$. Ans. $n\pi$.

19. $\cos\theta=-1$. Ans. $(2n+1)\pi$.

20. $\sin\theta=\frac{1}{\sqrt{2}}$. Ans. $n\pi+(-1)^n\frac{\pi}{4}$.

21. $\cos\theta=\frac{\sqrt{3}}{2}$. Ans. $2n\pi\pm\frac{\pi}{6}$.

22. $\sin^2\theta=\frac{1}{4}$. Ans. $n\pi\pm\frac{\pi}{6}$.

23. $\cos^2\theta-\sin^2\theta=\frac{1}{2}$. Ans. $n\pi\pm\frac{\pi}{6}$.

24. $\tan\theta=-1$. Ans. $n\pi-\frac{\pi}{4}$.

25. $\cot\theta=-\sqrt{3}$. Ans. $n\pi-\frac{\pi}{6}$.

26. $\sec\theta=2\tan\theta$. Ans. $n\pi+(-1)^n\frac{\pi}{6}$.

27. $\sin\theta=\cos m\theta$. Ans. $\frac{4n\pm1}{2(m\pm1)}\pi$.

28. $4-5\cos\theta-2\sin^2\theta=0$. Ans. $2n\pi\pm\frac{\pi}{3}$.

29. $\tan\theta+\cot\theta=2\sec\theta$. Ans. $n\pi+(-1)^n\frac{\pi}{6}$.

30. $\frac{2}{\sin\theta}-\cot\theta=\sqrt{3}$. Ans. $2n\pi\pm\frac{\pi}{3}$.

31. $$\frac{\sec\theta\cot\theta-\operatorname{cosec}\theta\tan\theta}{\cos\theta-\sin\theta}=2.$$

Ans. $n\pi+\frac{\pi}{4}$.

32. $\sin^2\theta = \frac{3\cos\theta}{2}$. Ans. $2n\pi \pm \frac{\pi}{3}$.

33. $\sec^2\theta - \frac{5}{2}\sec\theta + 1 = 0$. Ans. $2n\pi \pm \frac{\pi}{3}$.

34. Trace the variation in the sign and magnitude of the tangent as the angle varies from 0^0 to 360^0, by means of the "*line-definition.*"

35. Trace the variation in the *versed sine,* using the same method of definition.

36. Construct the angle of which the tangent is $3 - \sqrt{2}$.

37. Construct the angle of which the secant is $\sqrt{5} - 1$.

Trace the variations in sign and magnitude of the following expressions, as θ varies from 0 to 2π.

38. $\sin\theta + \cos\theta$.

39. $\sin\theta - \cos\theta$.

40. $\sin\theta\cos\theta$.

41. $\tan 2\theta$.

42. $\frac{\cos 2\theta}{\sin\theta}$.

43. $\frac{\cos 2\theta}{\cos\theta}$.

44. $\cos\theta + \sqrt{3}.\sin\theta$.

45. $\tan\pi(1 + \cos\theta)$.

46. $\tan\pi(1 - \cos\theta)$.

47. $a\cos\theta - b\sin\theta$, a and b being positive and $b < a$.

48. $\frac{a\sin\theta - b\cos\theta}{a\cos\theta + b\sin\theta}$, the same conditions holding.

49. $\frac{\sin(\pi\cos\theta)}{\cos(\pi\sin\theta)}$.

50. $3\tan^2\theta - 10 + 3\cot^2\theta$.

EXAMPLES ILLUSTRATING CHAPTER IV.

1. To prove that

$$\cos(\alpha+\beta+\gamma)+\cos(-\alpha+\beta+\gamma)+\cos(-\beta+\gamma+\alpha)$$
$$+\cos(-\gamma+\alpha+\beta)=4\cos\alpha\cos\beta\cos\gamma.$$

Referring to Ch. IV. Art. 10, (B), we see that

$$\cos(\alpha+\beta+\gamma)+\cos(-\alpha+\beta+\gamma)=2\cos(\beta+\gamma)\cos\alpha,$$

and $\cos(-\beta+\gamma+\alpha)+\cos(-\gamma+\alpha+\beta)=2\cos\alpha\cos(\beta-\gamma)$.

Hence, adding,

$$\cos(\alpha+\beta+\gamma)+\cos(-\alpha+\beta+\gamma)+\cos(-\beta+\gamma+\alpha)+\cos(-\gamma+\alpha+\beta)$$
$$=2\cos\alpha\times\{\cos(\beta+\gamma)+\cos(\beta-\gamma)\}$$
$$=4\cos\alpha\cos\beta\cos\gamma.$$

To prove the converse, we have, by Ch. IV. Art. 11, (C),

$$2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta),$$

$$\therefore\ 4\cos\alpha\cos\beta\cos\gamma=2\cos(\alpha+\beta)\cos\gamma+2\cos(\alpha-\beta)\cos\gamma$$
$$=\cos(\alpha+\beta+\gamma)+\cos(\alpha+\beta-\gamma)+\cos(\alpha-\beta+\gamma)$$
$$+\cos(-\alpha+\beta+\gamma).$$

2. Prove that

$$\cos 9A+3\cos 7A+3\cos 5A+\cos 3A=8\cos^3 A\cos 6A,$$
$$\cos 9A+\cos 3A=2\cos 6A\cos 3A,$$
$$3(\cos 7A+\cos 5A)=6\cos 6A\cos A,$$

$$\therefore\ \cos 9A+3\cos 7A+3\cos 5A+\cos 3A$$
$$=2\cos 6A(\cos 3A+3\cos A)=8\cos^3 A\cos 6A,$$

since $\cos 3A=4\cos^3 A-3\cos A$.

3. To shew that all the values of θ which satisfy the equations

$$\sin\theta+\sin\phi=m,$$
$$\cos\theta+\cos\phi=n,$$

are contained in the formula $n\pi-\alpha+(-1)^n\beta$, where α and β are the least angles which satisfy the equations

$$\tan\alpha=\frac{n}{m},\quad \sin\beta=\frac{1}{2}(m^2+n^2)^{\frac{1}{2}}.$$

We have
$$\sin\phi=m-\sin\theta,$$
$$\cos\phi=n-\cos\theta.$$

Squaring and adding these equations, we have

$$1 = m^2 + n^2 - 2(m\sin\theta + n\cos\theta) + 1,$$

$$\therefore\ m\sin\theta + n\cos\theta = \frac{1}{2}(m^2+n^2),$$

$$\therefore\ \frac{m}{(m^2+n^2)^{\frac{1}{2}}}\sin\theta + \frac{n}{(m^2+n^2)^{\frac{1}{2}}}\cos\theta = \frac{1}{2}(m^2+n^2)^{\frac{1}{2}},$$

$$\text{or}\quad \cos\alpha\sin\theta + \sin\alpha\cos\theta = \sin\beta,$$

$$\text{or}\quad \sin(\theta+\alpha) = \sin\beta,$$

$\therefore$ the general value of $\theta+\alpha$ is $n\pi + (-1)^n\beta$,

or of θ is $n\pi - \alpha + (-1)^n\beta$.

4. If $\tan\phi = \mu\tan\phi'$ and $\phi > \phi' < \frac{\pi}{2}$, shew that $\phi - \phi'$ is a maximum when $\tan\phi = \sqrt{\mu}$.

$$\tan(\phi-\phi') = \frac{\tan\phi - \tan\phi'}{1+\tan\phi\tan\phi'} = \frac{\tan\phi\left(1 - \frac{\tan\phi}{\mu}\right)}{1 + \frac{\tan^2\phi}{\mu}} = \frac{(\mu-1)\tan\phi}{\mu+\tan^2\phi}.$$

This is a maximum when $\dfrac{\tan\phi}{\mu+\tan^2\phi}$ is a maximum.

Let $\dfrac{\tan\phi}{\mu+\tan^2\phi} = u$.

Then we have the quadratic,

$$\tan^2\phi - \frac{1}{u}\tan\phi = -\mu,$$

which gives

$$\tan\phi = \frac{1}{2u} \pm \frac{\sqrt{(1-4\mu u^2)}}{2u}.$$

Hence the greatest value of u^2, consistent with $\tan\phi$ having a possible value, is $\frac{1}{4\mu}$.

And, in this case,

$$\tan\phi = \frac{1}{2u} = \sqrt{\mu},$$

because, ϕ being less than 90^0, the value

$$-\frac{1}{2\sqrt{\mu}} \text{ of } u \text{ is inadmissible.}$$

5. Solve the equation

$$4\cot 2\theta = \cot^2\theta - \tan^2\theta.$$

$$4\cot 2\theta = 4\frac{1-\tan^2\theta}{2\tan\theta} = 2(\cot\theta - \tan\theta);$$

$\therefore$ we have $\quad 2(\cot\theta - \tan\theta) = \cot^2\theta - \tan^2\theta,$

and $\quad \cot\theta - \tan\theta$ is a factor of the equation.

$$\cot\theta - \tan\theta = 0 \text{ gives}$$

$$\tan\theta = \pm 1, \text{ or } \theta = n\pi \pm \frac{\pi}{4}.$$

Again, $2 = \cot\theta + \tan\theta$ gives

$$2 = \frac{\sin^2\theta + \cos^2\theta}{\frac{1}{2}\sin 2\theta}, \text{ or } \sin 2\theta = 1;$$

$$\therefore 2\theta = n\pi + (-1)^n\frac{\pi}{2},$$

$$\text{and } \theta = \frac{n\pi}{2} + (-1)^n\frac{\pi}{4},$$

which is included in the formula $\theta = n\pi \pm \dfrac{\pi}{4}$.

6. Having given $\sin 210^0 = -\dfrac{1}{2}$, find $\cos 105^0$.

Referring to Chap. IV. Art. 15, we have

$$\sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{(1+\sin A)},$$

$$\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{(1-\sin A)}.$$

Now $\sin 105^0$ is greater than $\cos 105^0$ and is positive, and therefore $\sin 105^0 + \cos 105^0 = +\sqrt{(1+\sin 210^0)} = \sqrt{\frac{1}{2}}$,

$$\sin 105^0 - \cos 105^0 = +\sqrt{(1-\sin 210^0)} = \sqrt{\tfrac{3}{2}}.$$

Therefore, by subtracting,

$$\cos 105^0 = \frac{1-\sqrt{3}}{2\sqrt{2}}.$$

7. If $\cos\theta = \cos\alpha\cos\beta$, $\cos\theta' = \cos\alpha'\cos\beta$,

$$\text{and } \tan\frac{\theta}{2}\tan\frac{\theta'}{2} = \tan\frac{\beta}{2},$$

then $\quad \sin^2\beta = (\sec\alpha - 1)(\sec\alpha' - 1),$

$\tan^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta} = \dfrac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta}$, and similarly for $\tan\dfrac{\theta'}{2}$.

$$\therefore \frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta}\times\frac{1-\cos\alpha'\cos\beta}{1+\cos\alpha'\cos\beta}=\tan^2\frac{\beta}{2}=\frac{1-\cos\beta}{1+\cos\beta};$$

$$\therefore \frac{\sec\alpha-\cos\beta}{\sec\alpha+\cos\beta}\times\frac{\sec\alpha'-\cos\beta}{\sec\alpha'+\cos\beta}=\frac{1-\cos\beta}{1+\cos\beta};$$

$$\therefore \frac{\sec\alpha\sec\alpha'-(\sec\alpha+\sec\alpha')\cos\beta+\cos^2\beta}{\sec\alpha\sec\alpha'+(\sec\alpha+\sec\alpha')\cos\beta+\cos^2\beta}=\frac{1-\cos\beta}{1+\cos\beta}.$$

$$\text{or } 2\sec\alpha\sec\alpha'\cos\beta+2\cos^3\beta-2(\sec\alpha+\sec\alpha')\cos\beta=0;$$

$$\therefore \cos^2\beta=\sec\alpha+\sec\alpha'-\sec\alpha\sec\alpha';$$

$$\therefore \sin^2\beta=1-\cos^2\beta=1-\sec\alpha-\sec\alpha'+\sec\alpha\sec\alpha'$$
$$=(\sec\alpha-1)(\sec\alpha'-1).$$

8. To find all the values of x which satisfy the equation

$$\cos 3x+\sin 3x=\frac{1}{\sqrt{2}}.$$

Multiplying both sides of the equation by $\frac{1}{\sqrt{2}}$, we have

$$\frac{1}{\sqrt{2}}\cos 3x+\frac{1}{\sqrt{2}}\sin 3x=\frac{1}{2},$$

$$\text{or}\quad \cos\frac{\pi}{4}\cos 3x+\sin\frac{\pi}{4}\sin 3x=\frac{1}{2},$$

$$\text{or}\quad \cos\left(3x-\frac{\pi}{4}\right)=\frac{1}{2};$$

$\therefore$ the general value of $3x-\frac{\pi}{4}$ is $2n\pi\pm\frac{\pi}{3}$.

And therefore the general value of x is

$$\frac{1}{3}\left(2n\pi+\frac{\pi}{4}\pm\frac{\pi}{3}\right),$$

$$\text{or}\quad \frac{8n+1}{12}\pi\pm\frac{\pi}{9}.$$

N.B. Several general expressions for the root of a trigonometrical equation can sometimes exist. Thus, in the present instance, $\sin\left(3x+\frac{\pi}{4}\right)=\frac{1}{2}=\sin\left\{n\pi+(-1)^n\frac{\pi}{6}\right\}$;

$$\therefore 3x+\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{6},$$

$$\text{or } x=\frac{(4n-1)\pi}{12}+(-1)^n\frac{\pi}{18}.$$

These expressions however are in reality in all cases equivalent, and the apparent difference is only one of form.

9. If $\frac{6\sin\beta}{\cos(\alpha+\beta)} = \frac{3\sin 2\beta}{\cos(\alpha+2\beta)} = \frac{2\sin 3\beta}{\cos(\alpha+3\beta)}$ shew that $\beta = n\pi$.

Putting each ratio equal to λ,

$$6\sin\beta = \lambda\cos(\alpha+\beta) \ldots\ldots (1),$$
$$3\sin 2\beta = \lambda\cos(\alpha+2\beta) \ldots\ldots (2),$$
$$2\sin 3\beta = \lambda\cos(\alpha+3\beta) \ldots\ldots (3).$$

(1) + (3) gives

$$6\sin\beta + 2\sin 3\beta = 2\lambda\cos(\alpha+2\beta)\cos\beta \quad \text{(Ch. IV. Art. 10, (B))}$$
$$= 6\sin 2\beta\cos 2\beta \text{ by } (2)$$
$$= 3\sin 3\beta + 3\sin\beta,$$
$$\therefore \sin 3\beta = 3\sin\beta.$$

But we know that $\sin 3\beta = 3\sin\beta - 4\sin^3\beta$.

Hence $\sin^3\beta = 0$,

and therefore $\beta = n\pi$.

10. Eliminate θ and ϕ from the equations

$$\tan\theta + \tan\phi = a \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$\tan\theta\tan\phi(\operatorname{cosec} 2\theta + \operatorname{cosec} 2\phi) = b \ldots (2),$$
$$\cos(\theta+\phi) = c\cos(\theta-\phi) \ldots\ldots\ldots\ldots (3).$$

expanding (3),

$$\cos\theta\cos\phi - \sin\theta\sin\phi = c\cos\theta\cos\phi + c\sin\theta\sin\phi;$$
$$\therefore \tan\theta\tan\phi = \frac{1-c}{1+c} \ldots\ldots\ldots (4).$$

Again,

since $\operatorname{cosec} 2\theta = \frac{1}{2\sin\theta\cos\theta} = \frac{\sec^2\theta}{2\tan\theta} = \frac{1+\tan^2\theta}{2\tan\theta}$,

we have from (2), $\tan\phi(1+\tan^2\theta) + \tan\theta(1+\tan^2\phi) = 2b$,

or $(\tan\theta + \tan\phi)(1 + \tan\theta\tan\phi) = 2b$;

$$\therefore a\left(1 + \frac{1-c}{1+c}\right) = 2b,$$

or $a = b(1+c)$.

11. Prove geometrically that $\cos 2A = \cos^2 A - \sin^2 A$.

12. Prove geometrically that $\tan(45^\circ + A) = \frac{1}{\tan(45^\circ - A)}$.

13. Solve the equation $\tan(45^\circ + A) = 3\tan(45^\circ - A)$.

Ans. $A = \frac{(4n-1)\pi}{4} \pm \frac{\pi}{3}$.

14. Prove that $3\sin A - \sin 3A = 2\sin A(1 - \cos 2A)$.

15. Find the least positive value of A which satisfies the equation $\tan A + \tan(45^0 + A) = 2$. Ans. $A = 15^0$.

16. Solve the equation $4 \sin\theta \sin(\theta - \alpha) = 2\cos\alpha - 1$.

Ans. $\theta = \frac{\alpha}{2} + n\pi \pm \frac{\pi}{6}$.

17. Solve the equation $m \operatorname{versin}\theta = n \operatorname{versin}(\alpha - \theta)$.

Ans. $\cos\theta = \dfrac{\pm 2n\sqrt{mn}\sin\alpha\sin\frac{\alpha}{2} + (m-n)(m - n\cos\alpha)}{m^2 - 2mn\cos\alpha + n^2}$.

18. Simplify the expression $\dfrac{\sin(a+b) + \sin(a-b)}{\cos(a+b) - \cos(a-b)}$.

Ans. $-\cot b$.

19. Prove that $\dfrac{\sin 3A - \cos 3A}{\sin A + \cos A} = 2\sin 2A - 1$.

20. Prove that $\sin\alpha = 4\sin\frac{\alpha}{3}\sin\frac{\pi - \alpha}{3}\sin\frac{\pi + \alpha}{3}$.

21. Prove that $\operatorname{cosec} A = \frac{1}{2}\left(\tan\frac{A}{2} + \cot\frac{A}{2}\right)$.

22. Prove that $\mathrm{ch^d}(\theta + \phi) + \mathrm{ch^d}(\theta - \phi) = \mathrm{ch^d}\,\theta \,.\, \mathrm{ch^d}(\pi - \phi)$.

23. Simplify the expression

$\cos^2(\theta + \phi) + \cos^2(\theta - \phi) - \cos 2\theta \cos 2\phi$. Ans. 1.

24. Prove that $\dfrac{2\cos 2A - 3}{2\cos 2A + 3} = \dfrac{\cos 3A - 2\cos A}{\sin 3A + 2\sin A}\tan A$.

25. Prove that

$\sin 3(A - 15^0) = 4\cos(A - 45^0)\cos(A + 15^0)\sin(A - 15^0)$.

26. $\cot(A + B) + \tan A = \dfrac{\cos B}{\cos A}\ \dfrac{1}{\sin(A+B)}$.

27. Prove that

$\cos(\alpha + \beta)\sin\beta - \cos(\alpha + \gamma)\sin\gamma$
$= \sin(\alpha + \beta)\cos\beta - \sin(\alpha + \gamma)\cos\gamma$.

28. Prove that

$$\frac{\tan\left(\frac{\pi}{6} + \theta\right) + \tan\left(\frac{\pi}{6} - \theta\right)}{\tan\left(\frac{\pi}{3} + \theta\right) + \tan\left(\frac{\pi}{3} - \theta\right)} = \frac{\tan\left(\frac{\pi}{6} + \theta\right) - \tan\left(\frac{\pi}{6} - \theta\right)}{\tan\left(\frac{\pi}{3} + \theta\right) - \tan\left(\frac{\pi}{3} - \theta\right)}.$$

29. Prove that

$$\tan^2 A + \tan^2 B + \cot^2(A + B) = 1 + \tfrac{1}{2}(\tan A - \tan B)^2 + \tfrac{1}{2}\{\tan A - \cot(A + B)\}^2 + \tfrac{1}{2}\{\tan B - \cot(A + B)\}^2.$$

30. Prove that $\dfrac{\cos nA - \cos (n+2)A}{\sin (n+2)A - \sin nA} = \tan (n+1)A.$

31. Solve the equation $\cos 4\theta + \cos 2\theta + \cos \theta = 0.$

Ans. $\left\{ \begin{array}{l} 2n\pi \pm \dfrac{\pi}{2} \\ \text{and } \dfrac{(6n \pm 2)\pi}{9}. \end{array} \right.$

32. If $\sin (\alpha - \beta) = m \sin (\alpha + \beta)$, shew that

$$\cot (\alpha + \beta) = \frac{1}{2} \left\{ \frac{m-1}{m} \cot \beta - \frac{m+1}{m} \tan \beta \right\}.$$

33. Prove that

$$\sin A + \sin (72^0 + A) + \sin (36^0 - A) = \sin (72^0 - A) + \sin (36^0 + A).$$

34. Prove that $\tan^2 \theta - \tan^2 \phi = \dfrac{\sin (\theta + \phi) \sin (\theta - \phi)}{\cos^2 \theta \cos^2 \phi}.$

35. Prove the formulæ

$$\cos^2 \theta - \cos^2 3\theta = \sin 4\theta \,.\, \sin 2\theta,$$

$$\frac{1 + \tan^2 \theta}{(1 + \tan \theta)^2} = 1 - \tan 2\theta \cot (45^0 + \theta).$$

36. If $\tan^2 x = \tan (\alpha - x) \tan (\alpha + x)$,

then will $\sin 2x = \sqrt{2} \sin \alpha.$

37. Prove that $\sin 2A \sin A = \cos A - \cos A \cos 2A.$

38. Prove that

$$\{(1 - e^2 \sin^2 \theta)(1 - e^2 \sin^2 (\theta - \alpha)\} - \{\cos \alpha - e^2 \sin \theta \sin (\theta - \alpha)\}^2 = (1 - e^2) \sin^2 \alpha.$$

39. Prove that $\dfrac{\cot A}{\cot A - \cot 3A} + \dfrac{\tan A}{\tan A - \tan 3A} = 1.$

40. Prove that $\tan 3\phi = \tan \phi \cot \left(\dfrac{\pi}{6} - \phi\right) \cot \left(\dfrac{\pi}{6} + \phi\right).$

41. Find all the values of A less than 360^0 which satisfy the equation $\sin 5A \cos 3A = \sin 4A \cos 4A.$

Ans. $0, 90^0, 180^0, 270^0.$

42. $(\sec A - \tan A)^2 \left(\cos \dfrac{A}{2} + \sin \dfrac{A}{2}\right)^2 = 1 - \sin A.$

43. Prove that $\sin^4 \theta + 2 \cos \alpha \sin^2 \theta \cos^2 \theta + \cos^4 \theta$

$$= 1 - \left(\sin \frac{\alpha}{2} \sin 2\theta\right)^2.$$

44. Prove that $\sin\theta+\sin 2\theta+\sin 3\theta=4\cos\frac{\theta}{2}\cos\theta\sin\frac{3\theta}{2}$.

45. If $\phi-\alpha$, ϕ, $\phi+\alpha$, be three angles whose cosines are in harmonical progression, shew that

$$\cos\phi=\sqrt{2}\,.\cos\frac{\alpha}{2}.$$

46. If $\sin(\alpha+\beta)\cos\gamma=\sin(\alpha+\gamma)\cos\beta$, prove that either $\beta-\gamma$ is a multiple of π, or α an odd multiple of $\frac{\pi}{2}$.

47. Find θ and ϕ from the equations

$$\sin\theta+\sin\phi+\sin\alpha=\cos\theta+\cos\phi+\cos\alpha,$$
$$\theta+\phi=2\alpha.$$

Ans. $\alpha-\theta=2n\pi\pm\frac{2\pi}{3}$.

48. If $\tan(\phi+\alpha)$, $\tan\phi$, $\tan(\phi+\beta)$ be in arithmetical progression, then are also $\cot\alpha$, $\tan\phi$, $\cot\beta$ in arithmetical progression.

49. Solve the equation

$$\cos x\,.\cos 3x=\cos 5x\cos 7x.$$

Ans. $x=\frac{n\pi}{8}$, or $\frac{n\pi}{4}$.

50. Solve the equation

$$\sec 2\theta=\operatorname{cosec} 3\theta.$$

Ans. $\theta=\frac{2}{5}\left(n\pi+\frac{\pi}{4}\right)$ or $2n\pi+\frac{\pi}{2}$.

51. Prove that

$$8\sin\theta\sin 2\theta\sin\phi\sin 2\phi=\cos(\theta+\phi)+\cos(\theta-\phi)+\cos 3(\theta+\phi)$$
$$+\cos 3(\theta-\phi)-\cos(3\theta+\phi)-\cos(3\theta-\phi)-\cos(\theta+3\phi)$$
$$-\cos(\theta-3\phi).$$

52. Prove that

$$\cos^2(\alpha-\theta)+\cos^2\theta-2\cos(\alpha-\theta)\cos\theta\cos\alpha=\sin^2(\alpha-\theta)+\sin^2\theta$$
$$+2\sin(\alpha-\theta)\sin\theta\cos\alpha=\sin^2\alpha.$$

53. If $\tan\theta=n\tan\phi$, $\tan(\theta\sim\phi)$ cannot exceed $\frac{n\sim 1}{2\sqrt{n}}$.

54. If $a\cos\phi=b\cos\theta$, then $\cot\frac{\theta+\phi}{2}\cot\frac{\theta-\phi}{2}=\frac{a+b}{a-b}$.

55. If $(1+\sin\theta)(1+\sin\phi)(1+\sin\psi)=\cos\theta\cos\phi\cos\psi$, shew that $\sec^2\theta+\sec^2\phi+\sec^2\psi-2\sec\theta\sec\phi\sec\psi=1$.

56. Prove that

$$\sin\alpha\sin\beta\sin(\alpha-\beta)+\sin\beta\sin\gamma\sin(\beta-\gamma)+\sin\gamma\sin\alpha\sin(\gamma-\alpha)$$
$$=\sin(\beta-\alpha)\cos(\gamma-\beta)\cos(\alpha-\gamma)+\sin(\gamma-\beta)\cos(\alpha-\gamma)\cos(\beta-\alpha)$$
$$+\sin(\alpha-\gamma).\cos(\beta-\alpha).\cos(\gamma-\beta)$$
$$=\frac{1}{4}\{\sin 2(\alpha-\beta)+\sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)\}.$$

57. Find the value of $\tan 165^0$ from the known value of $\tan 330^0$. Ans. $\sqrt{3}-2$.

58. If $\tan A=\sqrt{2}+1$, find $\cos 2A$, and thence the general value of A.

Ans. $\cos 2A=-\frac{1}{\sqrt{2}}$.

$A=n\pi\pm\frac{3\pi}{8}$.

59. Prove that $\sin A=\dfrac{1}{\cot\frac{A}{2}-\cot A}$.

60. Change $8\sin^2\frac{\phi}{2}\cos^2\frac{\theta}{2}$ into additions and subtractions.

Ans. $2+2(\cos\theta-\cos\phi)-\cos(\phi+\theta)-\cos(\phi-\theta)$.

61. Prove that $(\text{chord } A)^2=4\cos\frac{A}{2}\text{ versin }\frac{A}{2}\left(1+\sec\frac{A}{2}\right)$.

62. Prove that

$$\sin^2\left(\frac{\pi}{8}+\frac{\theta}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{\theta}{2}\right)=\frac{1}{\sqrt{2}}\sin\theta.$$

63. Prove that $1+8\cos\alpha\cos^2\frac{\alpha}{2}=\dfrac{\sin^2\frac{3\alpha}{2}}{\sin\frac{\alpha}{2}}$.

64. Prove that

$$(2^{n+1}\operatorname{cosec}2^{n+1}\alpha)^2-(2^n\operatorname{cosec}2^n\alpha)^2-(2^n\sec 2^n\alpha)^2=0.$$

65. Prove that

$$2^{n-1}\sin\frac{\alpha}{2^n}-2^{n-2}\sin\frac{\alpha}{2^{n-1}}=2^n\sin\frac{\alpha}{2^n}\left(\sin\frac{\alpha}{2^{n+1}}\right)^2.$$

66. If $\sin\left(\theta+\frac{\alpha}{2}\right)\sin\frac{\alpha}{2}=\cos^2\frac{\theta}{2}$, all the values of θ which satisfy this equation are included in $m\pi-\alpha$, where m is any odd integer.

67. Prove that

$$\sin A = \tan \frac{1}{2} A + 2 \sin^2 \frac{1}{2} A \cot A.$$

68. Eliminate θ between the equations

$$\frac{x}{a} = \cos \theta + \cos 2\theta,$$

$$\frac{y}{b} = \sin \theta + \sin 2\theta.$$

Ans. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 - 3\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - \frac{2x}{a} = 0.$

69. Find θ from the equation

$$4 \cot 2\theta = \cot^2 \theta - \tan^2 \theta.$$

Ans. $\theta = n\pi \pm \frac{\pi}{4}.$

70. Express $\cot 2A$ in terms of $\cot A$.

Ans. $\cot 2A = \frac{\cot^2 A - 1}{2 \cot A}.$

71. Prove that

$$\sin 4A = \frac{4 \tan A (1 - \tan^2 A)}{(1 + \tan^2 A)^2}.$$

72. Prove that

$$\tan \theta - \tan \frac{\theta}{2} = \tan \frac{\theta}{2} \sec \theta.$$

73. If $\tan \alpha = m$, $\tan \beta = n$, $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} = c$, prove that

$$(1 - c^2)^2 = 4c\left(\frac{1}{m} + \frac{c}{n}\right)\left(\frac{1}{n} + \frac{c}{m}\right).$$

74. Find $\sin A$ and $\sin B$ from the equations

$$\left.\begin{aligned} a \sin 2A - b \sin 2B = 0, \\ a \sin B - b \sin A = 0. \end{aligned}\right\}$$

Ans. $\begin{cases} \sin A = \pm a \left(\dfrac{a^2 + b^2}{a^4 + a^2b^2 + b^4}\right)^{\frac{1}{2}}. \\ \sin B = \pm b \left(\dfrac{a^2 + b^2}{a^4 + a^2b^2 + b^4}\right)^{\frac{1}{2}}. \end{cases}$

75. If $\cos \theta = \frac{a \cos \phi - b}{a - b \cos \phi},$

then $$\tan\frac{\theta}{2} = \sqrt{\left(\frac{a+b}{a-b}\right)}\tan\frac{\phi}{2}.$$

76. Prove that

$$\text{cosec}\,\theta\,\text{cosec}\,2\theta\,\text{cosec}\,4\theta = \frac{\sec^2\theta\sec 2\theta}{2\sec\frac{\pi}{4}}\left\{\frac{\sec 2\theta}{\sec 2\theta - 1}\right\}^{\frac{3}{2}}.$$

77. If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, prove that

$$\sin 2\theta = \pm\left(\frac{3}{4} + 2m - 4m^2\right),$$

m being any integer.

78. Solve the equation

$$\tan(\alpha+\theta)\tan(\alpha-\theta) = \tan^2 2\alpha - \tan^2 2\theta.$$

Ans. $\theta = n\pi \pm \alpha$ or $\sec 2\theta = \pm\sqrt{2} - \sec 2\alpha$.

79. If $\tan\frac{\theta}{2} = \frac{\tan\theta + m - 1}{\tan\theta + m + 1}$, shew that m cannot lie between 1 and -1.

80. If $\tan^2 A = 2\tan^2 B + 1$, then $\cos 2B = 2\cos 2A + 1$.

81. Prove that

$$\sin^4 A + 2\cos B\sin^2 A\cos^2 A + \cos^4 A = 1 - \left(\sin\frac{B}{2}\sin 2A\right)^2.$$

82. If $\cos\theta = \frac{\cos\alpha + \cos\beta}{1 + \cos\alpha\cos\beta}$, then $\tan\frac{\theta}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$.

83. Eliminate θ and θ' from the equations

$$\cot\theta = \frac{\lambda-a}{\mu},\quad \cot\theta' = \frac{\lambda+a}{\mu},\quad \tan\frac{\theta}{2} = m\tan\frac{\theta'}{2}.$$

Ans. $\pm m = \frac{\sqrt{\{\mu^2 + (\lambda-a)^2\}} - \lambda + a}{\sqrt{\{\mu^2 + (\lambda+a)^2\}} - \lambda - a}$.

84. If $x = r\sin\frac{1}{2}(\theta-\alpha)$ and $y = r\sin\frac{1}{2}(\theta+\alpha)$, shew that $$x^2 - 2xy\cos\alpha + y^2 = r^2\sin^2\alpha.$$

85. Find all the values of θ and ϕ which satisfy the equation

$$2(\sin 2\theta + \sin 2\phi) = 1 = 2\sin(\theta+\phi).$$

Ans. The values are given by $\theta - \phi = 2n\pi \pm \frac{\pi}{3}$ and

$$\theta + \phi = m\pi + (-1)^m\frac{\pi}{6}.$$

86. If $\cos\alpha=\cos\beta\cos\phi=\cos\beta'\cos\phi'$ and

$$\sin\alpha=2\sin\frac{\phi}{2}\sin\frac{\phi'}{2},$$

shew that

$$\tan\frac{\alpha}{2}=\tan\frac{\beta}{2}\tan\frac{\beta'}{2}.$$

87. If $\tan\frac{\theta}{2}=\operatorname{cosec}\theta-\sin\theta$, $\cos\theta=\frac{\sqrt{5}-1}{2}$.

88. If $$\cot\frac{\phi}{2}=\cot\frac{\alpha}{2}\tan\frac{3\pi-2\beta}{4},$$

then $$\tan(\pi-\phi)=\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}.$$

89. Solve the equation

$$\sec\theta+3\sec\frac{\theta}{3}-4\cos\frac{\theta}{3}=0.$$

Ans. $2n\pi$, $2n\pi\pm\pi$, $2n\pi\pm\frac{\pi}{4}$, $2n\pi\pm\frac{3\pi}{4}$.

90. Shew that all the values of θ which satisfy the equation

$$\sin^3\theta\frac{\cos 3\theta}{3}+\cos^3\theta\frac{\sin 3\theta}{3}=\frac{m}{4}$$

are contained in the formula

$$n\pi+(-1)^n\frac{\alpha}{4},$$

where α is the least angle which satisfies the equation

$$\sin\alpha=m.$$

91. Prove that the tangents of $\frac{\pi}{3}$, $\frac{\pi}{4}$ and $\frac{\pi}{12}$ are in arithmetical progression.

92. If $3\tan x\tan 3x+1=0$, then $\cos 2x=\frac{\pm\sqrt{3}+1}{2}$.

93. Prove that $\tan 52\frac{1}{2}^\circ=(\sqrt{3}+\sqrt{2})(\sqrt{2}-1)$.

94. Prove that $\sin 3\theta\sin^3\theta+\cos 3\theta\cos^3\theta=\cos^3 2\theta$.

95. If $$\frac{\cos x}{a_1}=\frac{\cos 2x}{a_2}=\frac{\cos 3x}{a_3},$$

then $$\sin^2\frac{x}{2}=\frac{2a_2-a_1-a_3}{4a_2}.$$

96. Find the value of sin 165° and cos 165° from the known value of sin 330°. Ans. $\frac{\sqrt{3}-1}{2\sqrt{2}}$, and $-\frac{\sqrt{3}+1}{2\sqrt{2}}$.

97. Given

$$a = b\cos C + c\cos B,$$
$$b = c\cos A + a\cos C,$$
$$c = a\cos B + b\cos A,$$

eliminate from them (1) A and B, (2) C and c.

Ans. $\cos C = \frac{a^2+b^2-c^2}{2ab}$, $\frac{\sin A}{a} = \frac{\sin B}{b}$.

98. If

$$\tan\phi = \frac{m^2+\sin^2\theta}{m^2+\cos^2\theta}\tan\theta,$$

then $\sin(3\theta+\phi) - (3+4m^2)\sin(\theta-\phi)$

is zero.

99. Express $\sin(A-B+C-D)$ and $\cos(A-B+C-D)$ in terms of sines and cosines of A, B, C, and D.

100. Express $(\sin\theta+\cos\theta)^3$ as the sum of simple sines and cosines. Ans. $\frac{3}{2}(\sin\theta+\cos\theta)+\frac{1}{2}(\sin 3\theta-\cos 3\theta)$.

101. Prove that $\cos 42° = \cdot 74314\ldots$

102. Find sin 63°. Ans. $\frac{\sqrt{(10+2\sqrt{5})}+\sqrt{5}-1}{4\sqrt{2}}$.

103. Prove that

$$\sin^4\theta + \sin^4 2\theta = \frac{3}{4}(1-\cos\theta\cos 3\theta) - \frac{1}{4}\sin 3\theta\sin 5\theta.$$

104. If n angles be in Arithmetic Progression having a common difference β, shew that the difference of the products of the sines of the r^{th} from the beginning and end, and of the sines of the s^{th} from the beginning and end,

$$= \sin(r\sim s)\beta\sin\{n+1-(r+s)\}\beta.$$

105. If $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ and $A+B+C = 180°$,

prove that $a\cos B + b\cos A = c$.

106. If $\frac{\cos\alpha}{a}=\frac{\cos\beta}{b}=\frac{\cos\gamma}{c}$ and $\alpha+\beta+\gamma=\frac{\pi}{2}$, express $\sin\alpha$ in terms of a, b, and c.

Ans. $\sin\alpha=\frac{b^2+c^2-a^2}{2bc}$.

107. If the sum of four angles be two right angles, the sum of their tangents is equal to the sum of their products taken three and three.

108. If $A+B+C=180^0$ and n be an integer, prove that

$$\tan nA+\tan nB+\tan nC=\tan nA\ \tan nB\ \tan nC.$$

109. Find the general value of α which satisfies the equation

$$\tan\alpha+\tan m\alpha+\tan n\alpha=\tan\alpha\ \tan n\alpha\ \tan n\alpha.$$

110. If $A+B+C+D=2\pi$ and $\tan A$, $\tan B$, $\tan C$, $\tan D$ be in geometrical progression, shew that

$$\tan A\ \tan D=\tan B\ \tan C=1.$$

111. If $A+B+C=180^0$, $\cos C=\frac{1}{2}\frac{\sin A}{\sin B}$,

and $\sin^2 A=\sin^2 B+\sin^2 C$; find A, B and C.

Ans. $B=C=45^0$,
$A=90^0$.

112. If

$$\cos 60^0=\sin 36^0\cos A,$$
$$\cos 36^0=\sin 60^0\cos B,$$
$$\cos C=\cos A\ \cos B,$$

there is one value of $A+B+C$ equal to 90^0.

113. If $\alpha+\beta+\gamma+\delta=2\pi$, prove that

$$\tan\alpha\ \tan\beta\ \tan\gamma\ \tan\delta=\frac{\tan\alpha+\tan\beta+\tan\gamma+\tan\delta}{\cot\alpha+\cot\beta+\cot\gamma+\cot\delta}.$$

114. If $A+B+C=180^0$,

$$\frac{\sin A+\sin B-\sin C}{\sin A+\sin B+\sin C}=\tan\frac{A}{2}\ \tan\frac{B}{2}.$$

115. If $A+B+C=180^0$, prove that

$$\frac{\tan\frac{A}{2}}{\tan\frac{C}{2}}=\frac{1-\cos A+\cos B+\cos C}{1-\cos C+\cos A+\cos B}.$$

116. If $\theta+\phi+\psi=180^0$, prove that

$$\cos\psi=\frac{\sin^2\theta+\sin^2\phi-\sin^2\psi}{2\sin\theta\sin\phi}.$$

On the three following examples, if $A+B+C=180^0$, prove that

117. $$\sin\left(C+\frac{A}{2}\right)+\sin\left(A+\frac{B}{2}\right)+\sin\left(B+\frac{C}{2}\right)$$
$$+1=4\cos\frac{A-B}{4}\cos\frac{B-C}{4}\cos\frac{C-A}{4}.$$

118. $$\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$
$$+2\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}+2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

119. $$\sin^3 A+\sin^3 B+\sin^3 C=3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$
$$+\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}.$$

120. If $\cos\theta=\cot\beta\cot\gamma$, $\cos\phi=\cot\gamma\cot\alpha$,
$\cos\psi=\cot\alpha\cot\beta$ and $\theta+\phi+\psi=\pi$;
prove that $$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$$

EXAMPLES ILLUSTRATING CHAPTER VI.

1. To prove that $\cos\sin^{-1}\cos\sin^{-1}x=\pm x$.

Let $\sin^{-1}x=\theta$, then $x=\sin\theta$,
and $\cos\sin^{-1}x=\cos\theta=\pm\sqrt{(1-x^2)}$.
Let $\sin^{-1}\{\pm\sqrt{(1-x^2)}\}=\phi$;
and therefore $\pm\sqrt{(1-x^2)}=\sin\phi$,
and $$\cos\sin^{-1}\cos\sin^{-1}x=\cos\phi$$
$$=\pm\sqrt{\{1-(1-x^2)\}}$$
$$=\pm x.$$

2. Prove that

$$\cot(\theta+\tan^{-1}\tan^3\theta)+2\cot 2\theta.$$

Let $\tan^{-1}\tan^3\theta=\phi$,
then $\tan\phi=\tan^3\theta$,

$$\text{and } \cot(\theta+\phi)=\frac{\cot\theta\cot\phi-1}{\cot\theta+\cot\phi}$$

$$=\frac{\cot^4\theta-1}{\cot\theta+\cot^3\theta}=\frac{\cot^2\theta-1}{\cot\theta}$$

$$=\frac{\cos 2\theta}{\sin\theta\cos\theta}$$

$$=2\cot 2\theta.$$

3. Solve the equation

$$\sin(\tan^{-1}x)+\tan(\sin^{-1}x)=mx.$$

Let $\sin^{-1}x=\theta$, or $x=\sin\theta$,

$$\text{then } \tan\theta=\frac{\sin\theta}{\sqrt{(1-\sin^2\theta)}}=\frac{x}{\sqrt{(1-x^2)}}.$$

And, if $\tan^{-1}x=\phi$, so that $x=\tan\phi$,

$$\sin\phi=\frac{\tan\phi}{\sqrt{(1+\tan^2\phi)}}=\frac{x}{\sqrt{(1+x^2)}};$$

$$\therefore \sin(\tan^{-1}x)=\frac{x}{\sqrt{(1+x^2)}}.$$

Hence the equation becomes

$$\frac{x}{\sqrt{(1+x^2)}}+\frac{x}{\sqrt{(1-x^2)}}=mx;$$

$\therefore x=0$ is one root,

and also $\sqrt{(1-x^2)}+\sqrt{(1+x^2)}=m\sqrt{(1-x^4)}$.

Squaring, $2+2\sqrt{(1-x^4)}=m^2(1-x^4)$;

$$\therefore 1-x^4-\frac{2}{m^2}\sqrt{(1-x^4)}=\frac{2}{m^2},$$

a quadratic in $\sqrt{(1-x^4)}$ of which the roots are $\dfrac{1\pm\sqrt{(2m^2+1)}}{m^2}$.

$$\text{Hence } 1-x^4=\frac{2+2m^2\pm2\sqrt{(2m^2+1)}}{m^4},$$

$$\therefore x^4=\frac{m^4-2m^2\mp2\sqrt{(2m^2+1)}-2}{m^4}.$$

Prove that

4. $\sin^{-1}x=\cos^{-1}\sqrt{(1-x^2)}$.

5. $\cos^{-1}x=\sin^{-1}\sqrt{(1-x^2)}$.

6. $\cos^{-1} x = \sec^{-1} \frac{1}{x}$.

7. $\sec^{-1} x = \sin^{-1} \frac{\sqrt{(x^2 - 1)}}{x}$.

8. $\tan^{-1} x = \sec^{-1} \sqrt{(1 + x^2)}$.

9. $\sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{(1 - x^2)}}$.

10. $\cos^{-1} x = \operatorname{versin}^{-1} (1 - x)$.

11. $\cos^{-1} x = 2 \sin^{-1} \sqrt{\left(\frac{1 - x}{2}\right)}$.

12. $\cos^{-1} x = 2 \cos^{-1} \sqrt{\left(\frac{1 + x}{2}\right)}$.

13. $\cot^{-1} x - \cot^{-1} y = \cot^{-1} \frac{1 + xy}{y - x}$.

14. $\cos \tan^{-1} \sin \cot^{-1} x = \sqrt{\left(\frac{x^2 + 1}{x^2 + 2}\right)}$.

15. $\cos \sec^{-1} \sin \tan^{-1} \cos \tan^{-1} \sin \cos^{-1} \tan \sin^{-1} x$

$$= \sqrt{\left(\frac{3 - 4x^2}{1 - x^2}\right)}.$$

16. Write down the general values of

$$\cos^{-1} (\sin \theta), \quad \tan^{-1} (\cot \theta).$$

Ans. $2n\pi \pm \left(\frac{\pi}{2} - \theta\right), \quad n\pi + \frac{\pi}{2} - \theta$.

Prove that

17. $\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{4}{5} = \frac{\pi}{2}$.

18. $\tan^{-1} \frac{3}{5} + \cot^{-1} \frac{7}{3} = \cot^{-1} \frac{13}{18}$.

19. $\sin^{-1} x + \cos^{-1} x = 90^\circ$.

20. $\sec^{-1} 3 + \tan^{-1} 2\sqrt{2} = -\tan^{-1} \frac{4\sqrt{2}}{7}$.

21. $\tan^{-1} \frac{2}{3} = \frac{1}{2} \tan^{-1} \frac{12}{5}$.

22. $\cos^{-1} \frac{1}{2} + 2 \sin^{-1} \frac{1}{2} = 120^\circ$.

23. If $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$, prove that

$$\theta=-\frac{1}{2}\sin^{-1}\frac{3}{4}.$$

24. Find x from the equation

$$\tan^{-1}2x+\tan^{-1}(3x)=\frac{\pi}{4}.$$

Ans. $x=\frac{1}{6}$ or -1.

25. Prove that

$$\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=45^{\circ}.$$

26. Solve the equation $\tan^{-1}\dfrac{1}{x-1}-\tan^{-1}\dfrac{1}{x}=\dfrac{\pi}{4}$.

Ans. $x=2$, or -1.

27. Solve the equation

$$\cos^{-1}(1-x)+\cos^{-1}x=\cos^{-1}(x-x^2)^{\frac{1}{2}}.$$

Ans. $x=0$, 1, or $\frac{1}{2}$.

28. Prove that

$$\tan^{-1}\{(\sqrt{2}+1)\tan\alpha\}-\tan^{-1}\{(\sqrt{2}-1)\tan\alpha\}=\tan^{-1}(\sin 2\alpha).$$

29. Solve the equation $\cos^{-1}x+\cos^{-1}(a-x)=\dfrac{\pi}{2}$.

Ans. $x=\frac{1}{2}\{a\pm\sqrt{(2-a^2)}\}$.

30. Prove that $\sin^{-1}\dfrac{4}{5}+\sin^{-1}\dfrac{5}{13}+\sin^{-1}\dfrac{16}{65}=\dfrac{\pi}{2}$.

31. Prove that

$$\tan^{-1}\frac{x\cos\phi}{1-x\sin\phi}-\tan^{-1}\frac{x-\sin\phi}{\cos\phi}=\phi.$$

32. If $\cot\theta=m\cot(\alpha-\theta)$, then $\theta=\dfrac{1}{2}\left\{\alpha-\sin^{-1}\dfrac{m-1}{m+1}\sin\alpha\right\}$.

33. The only possible solution of $\dfrac{\pi}{2}-\sin^{-1}x=\tan^{-1}x$ is

$$x=\frac{\sqrt{(\sqrt{5}-1)}}{\sqrt{2}},$$

where all the radicals are taken with positive signs.

34. Prove that

$$\tan^{-1}\frac{m\sin\theta}{1+m\cos\theta}+\tan^{-1}\frac{\sin\theta}{m+\cos\theta}=n\pi+\theta;$$

where n is any integer.

35. If $\tan(n \cot\theta) = \cot(n \tan\theta)$, prove that

$$\theta = \frac{m\pi}{2} + (-1)^m \frac{1}{2} \sin^{-1} \frac{4n}{(2r+1)\pi},$$

m and r being any integers.

36. Find the values of

$$\tan(\tan^{-1}x + \cot^{-1}x), \text{ and } \sin\left(\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2}\right).$$

Ans. ∞, and 1, or $-\frac{1}{2}$.

37. Prove that

$$2\tan^{-1}\left[\tan\frac{a}{2}\sqrt{\left\{\tan\left(\frac{\pi}{4} - b\right)\right\}}\right] = \cos^{-1}\left(\frac{\cos a + \tan b}{1 + \cos a \tan b}\right).$$

38. Find x from the equation

$$\tan^{-1}\frac{1}{x-1} - \tan^{-1}\frac{1}{x+1} = \frac{\pi}{12}.$$

Ans. $x = \pm(1 + \sqrt{3})$.

39. Find x from the equation

$$\sec^{-1}\frac{x}{a} - \sec^{-1}\frac{x}{b} + \sec^{-1}a - \sec^{-1}b = 0.$$

Ans. $x = \pm ab$.

40. Find x from the equation

$$\text{vers}^{-1}(1+x) - \text{vers}^{-1}(1-x) = \tan^{-1}2\sqrt{(1-x^2)}.$$

Ans. $x = \frac{1}{2}$, or -1.

41. The sum of any number of angles, $\sin^{-1}\frac{2ab}{a^2+b^2}$, $\sin^{-1}\frac{2a'b'}{a'^2+b'^2}$, &c. may be expressed in the form $\sin^{-1}\frac{2mn}{m^2+n^2}$, where m and n are rational functions of a, b, a', b', &c.

EXAMPLES ILLUSTRATING CHAPTER VIII.

The following logarithms occur frequently in the subjoined examples,

$$\log 2 = \cdot 30103, \ \log 3 = \cdot 4771213.$$

1. Find x from the equation $12^x = 180$, $\log 2$ and $\log 3$ being known.

Since $12^x = 180$, therefore the logarithms of both sides are equal,

$$\therefore x \log 12 = \log 180,$$

$$\begin{aligned} \log 12 = \log 4 + \log 3 &= 2\log 2 + \log 3 \\ &= \cdot 6020600 + \cdot 4771213 \\ &= 1\cdot 0791813, \end{aligned}$$

$$\begin{aligned} \log 180 = \log(3^2 \times 2 \times 10) &= 2\log 3 + \log 2 + \log 10 \\ &= \cdot 9542426 + \cdot 3010300 + 1 \\ &= 2\cdot 2552726. \end{aligned}$$

Hence $x = \dfrac{2\cdot 2552726}{1\cdot 0791813} = 2\cdot 089$ nearly.

2. Given $\quad L \tan 30^0\ 21' = 9\cdot 7035329,$

$\qquad\qquad L \tan 30^0\ 22' = 9\cdot 7037486,$

find A from the equation

$$L \tan A = 9\cdot 7036421.$$

The difference for $60''$ is $\cdot 0002157$, or, as it is generally written, 2157, in which case it must be recollected that the figure in the units' place really represents a figure in the 7^{th} decimal place,

$$\text{and } L \tan A - L \tan 30^0\ 21' = 1092.$$

Hence, if x be the number of seconds by which A exceeds $30^0\ 21'$, we have

$$x = 60 \times \frac{1092}{2157} = \frac{65520}{2157} = 30\cdot 3.$$

Therefore $A = 30^0\ 21'\ 30''\cdot 3$ very nearly.

3. Having given

$$L \cos 34^0\ 18' = 9\cdot 9170317,$$

$$L \cos 34^0\ 19' = 9\cdot 9169455,$$

find $\qquad L \cos 34^0\ 18'\ 25''.$

The difference for 60″ is ·0000862,

and therefore for 25″ it is $\frac{25}{60} \times$ ·0000862,

or $\frac{5}{12}$ ·0000862,

or 5(·0000071) nearly,

or ·0000355.

Now, since the cosine *decreases* as the angle increases, the logarithm of the cosine decreases as the angle increases, and therefore we must *subtract* the difference, corresponding to an increase of 25″ in the angle, from the value of $L \cos 34^0\ 18'$.

Therefore $L \cos 34^0\ 18'\ 25'' = 9{\cdot}9169962$.

N.B. The same remark about *subtracting*, instead of adding, the difference from the logarithm of the angle next below the given one in the tables, is applicable to the logarithms of the cotangent and cosecant, since each of these functions *decreases* as the angle increases. *Vid.* Ch. VIII. Art. 12.

4. Having given

$$L \operatorname{cosec} 34^0\ 31' = 10{\cdot}2466882,$$
$$L \operatorname{cosec} 34^0\ 32' = 10{\cdot}2465046,$$

find A from the equation

$$L \operatorname{cosec} A = 10{\cdot}2466153.$$

The difference for 60″ is 1836,

and $L \operatorname{cosec} 34^0\ 31' - L \operatorname{cosec} A = 729$.

Let x be the number of seconds by which A exceeds $34^0\ 31'$,

then $$x = 60\frac{729}{1836} = \frac{7290}{306} = 23{\cdot}8.$$

Therefore $A = 34^0\ 31'\ 23''{\cdot}8$ nearly.

5. Shew that $7 \log \frac{15}{16} + 6 \log \frac{8}{3} + 5 \log \frac{2}{5} + \log \frac{32}{25} = \log 3$.

6. Find x from the equation $5^x = 20$, having given log 2.
Ans. $x = 1{\cdot}86$.

7. Find x from the equation $10^x = 2$. Given $\log 8 = {\cdot}90309$; also, if $4^x = 32$, shew that $x = \frac{5}{2}$. Ans. $x = {\cdot}30103$.

8. Find x from the equation $8^x = 100$, having given log 2.
Ans. $x = {\cdot}221$.

9. Prove that $\log_{\tan\theta} \cot\theta = -1$.

10. Find the logarithm of $\sqrt{\left(\frac{\sqrt[4]{32} \times \sqrt[3]{48}}{2\sqrt{27}}\right)}$, having log 2 and log 3 given. Ans. $-\cdot 0400053$.

11. Find the value of x from the equation

$$(4)^{2x} - 8\,(4)^x + 7 = 0,$$

having given log 2 and log 7 = ·845098.

Ans. $x = 0$ or $1\cdot 4$.

12. Find $\log_5 3125$. Ans. 3·494850.

13. Given $\log_{10} 2$ and $\log_{10} 7 = \cdot 845098$;

find $\log_{10} 98$ and $\log_{1000} \sqrt{\frac{4}{343}}$.

Ans. 1·991226 and $-\cdot 322205$.

14. Given log 24 = 1·38021, log 25 = 1·39794, log 26 = 1·41497; find log 117 and log 156.

Ans. 2·06818 and 2·19312.

15. Given log 2 and log 5·743491 = ·7591760; find $\sqrt[5]{\cdot 0625}$.

Ans. ·5743491.

16. Given log 3 and log 4·2366 = ·6270227;

find $\left(\frac{1}{3}\right)^{\frac{187}{14}}$.

Ans. ·00000042366.

17. Given $\left.\begin{array}{l}\log 76\cdot 563 = 1\cdot 8840189 \\ \log 76\cdot 564 = 1\cdot 8840246\end{array}\right\}$; find log 765·6372.

Find also the number of which the log is 3·8840213 to three places of decimals. Ans. 2·8840230 and 7656·342.

18. Given $\log_{10} 71968 = 4\cdot 8571394$, diff. for 1 = 60;

find the value of $\sqrt[8]{(\cdot 0719686)}$ to 7 places of decimals.

Ans. ·7196858.

19. Given log 247·63 = 2·3938033, log 247·64 = 2·3938208;

find the number of which the log is 2·3938134.

Ans. 247·6357.

20. Given $\log 3\cdot409 = \cdot5326 27$,

$\log 3\cdot410 = \cdot532754$;

find $\log 34\cdot0926$.

Ans. $1\cdot532660$.

21. Given $\log 8804 = 3\cdot9446800$,

$\log 8805 = 3\cdot9447294$;

find $\log 880467$.

Ans. $5\cdot9447130$.

22. Given $\log_{10} 1873\cdot8 = 3\cdot2727232$, diff. for $1 = 232$;

find the value of $\sqrt[12]{1873\cdot8}$.

Ans. $1\cdot873816$.

23. Given $L \sin 32^0\ 18' = 9\cdot7278277$, diff. for $1' = 1998$,

$L \cos 32^0\ 18' = 9\cdot9269913$, diff. for $1' = 799$;

find L sine, L cosine and L tan of $32^0\ 18'\ 24''\cdot6$.

Ans. $9\cdot7279096$, $9\cdot9269585$, $9\cdot8009511$ respectively.

24. Given $L \sin 59^0\ 37'\ 40'' = 9\cdot9358894$, diff. for $10'' = 124$;
find A from the equation $L \sin A = 9\cdot9358921$.

Ans. $A = 59^0\ 37'\ 42''\cdot18$.

25. Given $L \operatorname{cosec} 41^0\ 26' = 10\cdot1793073$,

$L \operatorname{cosec} 41^0\ 27' = 10\cdot1791642$;

find $L \operatorname{cosec} 41^0\ 26'\ 30''$.

Ans. $10\cdot1792358$.

26. Given $L \tan 34^0\ 5' = 9\cdot8303492$,

$L \tan 34^0\ 6' = 9\cdot8306213$;

find A from the equation $L \tan A = 9\cdot8305129$.

Ans. $A = 34^0\ 5'\ 36''$ nearly.

27. Given $L \cot 34^0\ 5' = 10\cdot1696508$,

$L \cot 34^0\ 6' = 10\cdot1693787$;

find A from the equation $L \cot A = 10\cdot1694531$.

Ans. $34^0\ 5'\ 43''\cdot5$ nearly.

28. Prove that $\log_a N = \log_a b \log_b N$.

29. If P be the number of the integers whose logarithms have the characteristic p, and Q the number of the integers the logarithms of whose reciprocals have the characteristic $-q$, prove that $\log P - \log Q = p - q + 1$.

EXAMPLES ILLUSTRATING CHAPTER IX.

1. If, in a triangle ABC, $b=a(\sqrt{3}-1)$ and $C=30^\circ$, find the angles A and B.

$$\text{Since } \frac{b}{a}=\sqrt{3}-1,$$

$$\frac{a-b}{a+b}=\frac{2-\sqrt{3}}{\sqrt{3}}.$$

$$\text{Now } \tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{2-\sqrt{3}}{\sqrt{3}}\sqrt{\left(\frac{1+\cos 30^\circ}{1-\cos 30^\circ}\right)}$$

$$=\frac{2-\sqrt{3}}{\sqrt{3}}\sqrt{\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)}$$

$$=\sqrt{\left\{\frac{(2-\sqrt{3})(2+\sqrt{3})}{\sqrt{3}}\right\}}$$

$$=\frac{1}{\sqrt{3}}=\tan 30^\circ;$$

$$\therefore \frac{A-B}{2}=30^\circ;$$

$$\therefore A-B=60^\circ,$$

$$\text{and } A+B=150.$$

$$\text{Therefore } A=105^\circ,$$

$$B=45^\circ.$$

2. Given $a=1900$, $b=100$, $C=60^\circ$, determine the other angles of the triangle.

Given $\log 3=\cdot4771213$, $L\tan 57^\circ\ 19'=10\cdot1927506$,
$L\tan 57^\circ\ 20'=10\cdot1930286$.

$$\text{Since } \tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$$

$$\tan\frac{A-B}{2}=\frac{1800}{2000}\cot 30^\circ=\frac{9}{10}\sqrt{3};$$

$$\therefore L\tan\frac{A-B}{2}=\frac{5}{2}\log 3-1+10,$$

$$=10\cdot1928030.$$

$$\text{Now} \quad L\tan\frac{A-B}{2} - L\tan 57^\circ\, 19' = 524,$$

$$\text{Difference for } 60'' = 2780.$$

Hence, if x'' be the excess of $\frac{A-B}{2}$ over $57^\circ\, 19'$,

$$x = \frac{524}{2780} \times 60 = 11\cdot 3.$$

$$\text{Hence} \quad \frac{A-B}{2} = 57^\circ\, 19'\, 11''\cdot 3,$$

$$\text{and} \quad \frac{A+B}{2} = 60^\circ;$$

$$\therefore\ A = 117^\circ\, 19'\, 11''\cdot 3,$$

$$B = 2^\circ\, 40'\, 48''\cdot 7.$$

3. The sides of a triangle are 1, 7, $\sqrt{56}$; find the angles in a form adapted to logarithmic computation, having given

$$\log 2 = \cdot 3010300, \quad \log 7 = \cdot 8450980,$$
$$L \sin 32^\circ\, 18'\, 40'' = 9\cdot 7279609,$$
$$L \sin 32^\circ\, 18'\, 50'' = 9\cdot 7279942.$$

Let A, B, C be the angles subtending the sides $\sqrt{56}$, 1, 7 respectively.

Then $\cos\frac{A}{2} = \sqrt{\frac{s.\ s-a}{bc}}$, and in this case $s = 4 + \sqrt{14}$,

$$s - a = 4 - \sqrt{14},$$
$$b = 1,$$
$$c = 7;$$

$$\therefore\ \cos\frac{A}{2} = \sqrt{\frac{2}{7}};$$

$$\therefore\ L\cos\frac{A}{2} = \tfrac{1}{2}\log 2 - \tfrac{1}{2}\log 7 + 10,$$

$$= 9\cdot 7279660.$$

Now $L\cos 57^\circ\, 41'\, 20'' = 9\cdot 7279609$, since $32^\circ\, 18'\, 40''$ is the complement of $57^\circ\, 41'\, 20''$,

$$L\cos 57^\circ\, 41'\, 10'' = 9\cdot 7279942.$$

$$\text{Diff. for } 10'' = 333,$$

$$\text{and} \quad L\cos\frac{A}{2} - L\cos 57^\circ\, 41'\, 20'' = 51.$$

Hence, if x'' be the excess of $57^0\ 41'\ 20''$ over $\frac{A}{2}$, we have

$$x = \frac{51}{333} \times 10 = \frac{510}{333} = 1\cdot5\ ;$$

$$\therefore\ \frac{A}{2} = 57^0\ 41'\ 18''\cdot5,$$

$$\text{or}\quad A = 115^0\ 22'\ 37''.$$

Again, $\quad \sin B = \frac{b}{a}\sin A = \frac{1}{\sqrt{14}}\sin A\ ;$

$$\therefore\ L\sin B = L\sin A - \tfrac{1}{2}(\log 2 + \log 7),$$

$$\text{and}\quad \log\sin A = \log 2\sin\frac{A}{2}\cos\frac{A}{2}\ ;$$

$$\therefore\ L\sin A = \log 2 + L\sin\frac{A}{2} + L\cos\frac{A}{2} - 10,$$

$$\text{and}\quad \sin\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}$$

$$= \sqrt{\left\{\frac{(\sqrt{14}+3)(\sqrt{14}-3)}{7}\right\}} = \sqrt{\frac{5}{7}}\ ;$$

$$\therefore\ L\sin\frac{A}{2} = 10 + \tfrac{1}{2}(\log 5 - \log 7),$$

$$= 10 + \tfrac{1}{2} - \tfrac{1}{2}\log 2 - \tfrac{1}{2}\log 7,$$

$$= 9\cdot9269360,$$

$$\therefore\ L\sin A = 9\cdot9559320\ ;$$

$$\therefore\ L\sin B = 9\cdot9559320 - \cdot5730640,$$

$$= 9\cdot3828680.$$

4. If $A = 41^0\ 10'$, $a = 145\cdot3$, $b = 178\cdot3$, determine the angles of the triangle, having given

$L\sin 41^0\ 10' = 9\cdot8183919$, $\quad L\sin 53^0\ 52' = 9\cdot9072216$,
$\log 1453 = 3\cdot1622656$, $\quad L\sin 53^0\ 53' = 9\cdot9073138$,
$\log 1783 = 3\cdot2511513$.

Since a is less than b, there are two solutions corresponding to the given values (*vide* Ch. IX. 11, Case (2)).

$$\text{Sin}\ B = \frac{b}{a}\sin A = \frac{178\cdot3}{145\cdot3}\sin 41^0\ 10'\ ;$$

$$\therefore\ L\sin B = \log 178\cdot3 - \log 145\cdot3 + L\sin 41^0\ 10',$$

$$= \cdot0888857 + 9\cdot8183919,$$

$$= 9\cdot9072776.$$

The difference for 60″ is 922,

and $L \sin B - L \sin 53^{0}\, 52' = 560$.

Hence, if x'' be the excess of B over $53^{0}\, 52'$, we have

$$x = \frac{560}{922}\, 60 = 36 \text{ nearly.}$$

$$\text{Hence } B = 53^{0}\, 52'\, 36'',$$

$$\text{or } B = 126^{0}\, 7'\, 24''.$$

C of course is known, because $C = 180^{0} - (A + B)$.

5. The cotangents of the half angles of a triangle are three successive integers. Prove that the triangle is right-angled, and find the ratios of the sides.

Let $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ be denoted by $x - 1$, x, $x + 1$.

$$\text{Now, since } \tan\frac{B}{2} = \cot\frac{B+C}{2} = \frac{\cot\frac{A}{2}\cot\frac{B}{2} - 1}{\cot\frac{A}{2} + \cot\frac{B}{2}};$$

$$\therefore \frac{1}{x} = \frac{(x-1)(x+1) - 1}{x - 1 + x + 1} = \frac{x^2 - 2}{2x};$$

$$\therefore x = \pm 2.$$

Now, since $\frac{B}{2}$ is less than 90^{0}, x cannot be negative;

$$\therefore \cot\frac{A}{2} = 1, \cot\frac{B}{2} = 2, \cot\frac{C}{2} = 3;$$

$$\text{and } \therefore \frac{A}{2} = 45^{0}, \text{ or } A = 90^{0}.$$

$$\text{And } \frac{b}{c} = \tan B = \frac{2\tan\frac{B}{2}}{1 - \tan^2\frac{B}{2}} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3},$$

$$\text{and } \frac{b}{a} = \sin B = \frac{4}{5};$$

$$\therefore a : b : c :: 5 : 4 : 3.$$

6. In a triangle ABC, if $\sin A$, $\sin B$, $\sin C$ are in Harmonical Progression, then $1 - \cos A$, $1 - \cos B$, $1 - \cos C$ are so also.

$$\text{Since } 1 - \cos A = 2\sin^2\frac{A}{2},$$

we have to prove that $2\sin^2\frac{A}{2}$, $2\sin^2\frac{B}{2}$, $2\sin^2\frac{C}{2}$ are in harmonical progression.

$$\text{Now}\quad 2\sin^2\frac{A}{2}=2\frac{(s-b)(s-c)}{bc},$$

$$2\sin^2\frac{B}{2}=2\frac{(s-c)(s-a)}{ca},$$

$$2\sin^2\frac{C}{2}=2\frac{(s-a)(s-b)}{ab}.$$

And therefore these are in harmonical progression, if

$$\frac{bc}{(s-b)(s-c)}+\frac{ab}{(s-a)(s-b)}=\frac{2ca}{(s-c)(s-a)};$$

$$\text{or if}\quad bc(s-a)+ab(s-c)=2ca(s-b);$$

$$\text{or if}\quad bc+ab=2ca,$$

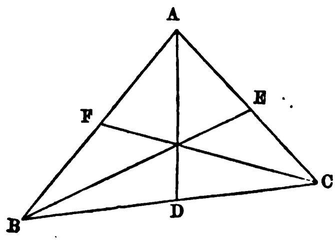
$$\text{i.e. if}\quad \frac{1}{a}+\frac{1}{c}=\frac{2}{b};$$

$$\text{or}\quad \frac{1}{\sin A}+\frac{1}{\sin C}=\frac{2}{\sin B};$$

i.e. if $\sin A$, $\sin B$, $\sin C$ are in harmonical progression.

7. If p, q, r be the length of the lines drawn from the angles of a triangle bisecting them and terminated by the opposite sides, then

$$\frac{\cos\frac{A}{2}}{p}+\frac{\cos\frac{B}{2}}{q}+\frac{\cos\frac{C}{2}}{r}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$



We have $\frac{BD}{AD}=\frac{\sin\frac{A}{2}}{\sin B}$,

or $BD=p\frac{\sin\frac{A}{2}}{\sin B}$.

Similarly $CD=p\frac{\sin\frac{A}{2}}{\sin C}$;

$$\therefore\ a=BD+CD=p\sin\frac{A}{2}\left(\frac{1}{\sin B}+\frac{1}{\sin C}\right)$$

$$=\frac{p}{2\cos\frac{A}{2}}\left(\frac{\sin A}{\sin B}+\frac{\sin A}{\sin C}\right)$$

$$=\frac{p}{2\cos\frac{A}{2}}\left(\frac{a}{b}+\frac{a}{c}\right);$$

$$\therefore\ \frac{\cos\frac{A}{2}}{p}=\frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}\right).$$

Similarly $\frac{\cos\frac{B}{2}}{q}=\frac{1}{2}\left(\frac{1}{c}+\frac{1}{a}\right)$,

and $\frac{\cos\frac{C}{2}}{r}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$,

$\therefore$ by addition

$$\frac{\cos\frac{A}{2}}{p}+\frac{\cos\frac{B}{2}}{q}+\frac{\cos\frac{C}{2}}{r}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

8. In any triangle, prove that

$$\cot B+\cot A=\frac{c\ \mathrm{cosec}\ A}{b},$$

and $a^2\sin A+ab\sin B+ac\sin C=(a^2+b^2+c^2)\sin A$.

9. In any triangle,

$$\frac{\sin^2 \frac{B}{2}}{b} + \frac{\sin^2 \frac{C}{2}}{c} = \frac{s-a}{bc}.$$

10. If in a triangle $(a^3+b^3+c^3)=c^2(a+b+c)$,

and $$\sin A \sin B = \sin^2 C,$$

the triangle is equilateral.

11. If

$A+B+C=180^0$, and $\tan A : \tan B : \tan C :: \sqrt{a} : \sqrt{b} : \sqrt{a}+\sqrt{b}$,

then $$\tan A \tan B = 2.$$

12. In any triangle,

$$\sin A = \frac{1}{4} \frac{\sin 2B + \sin 2C - \sin 2A}{\cos B \cos C}.$$

13. In every triangle,

$$(a+c) \sin \frac{B}{2} = b \cos \frac{A-C}{2}.$$

14. If in a triangle ABC, $\sin A$, $\sin B$, $\sin C$ be in arithmetical progression, then

$$\tan \frac{A}{2} \tan \frac{C}{2} = \frac{1}{3}.$$

15. If $\cos\theta = \frac{a}{c+b}$, $\cos\phi = \frac{b}{a+c}$, $\cos\psi = \frac{c}{a+b}$,

then $$\tan\frac{\theta}{2} \tan\frac{\phi}{2} \tan\frac{\psi}{2} = \tan\frac{A}{2} \tan\frac{B}{2} \tan\frac{C}{2},$$

and $$\tan^2\frac{\theta}{2} + \tan^2\frac{\phi}{2} + \tan^2\frac{\psi}{2} = 1.$$

16. If the sides of a triangle are x^2+x+1, $2x+1$, x^2-1, shew that the greatest angle is 120^0.

17. In any triangle ABC,

(1) $$\cot\frac{A}{2} \cot\frac{B}{2} = \frac{a+b+c}{a+b-c},$$

(2) $$\frac{a^2-b^2}{c^2} \sin C + \frac{b^2-c^2}{a^2} \sin A + \frac{c^2-a^2}{b^2} \sin B$$

$$+ 4 \sin\frac{A-B}{2} \sin\frac{B-C}{2} \sin\frac{C-A}{2} = 0.$$

18. **Having given one side and the opposite angle (120°) and the line joining the given angle with the point of bisection of the opposite side, solve the triangle and find its area.**

Ans. If $AB=c$, D the middle point of AB, $CD=d$,

$$a=\sqrt{\left(\frac{3c^2-4d^2}{8}\right)}\pm\sqrt{\left(\frac{12d^2-c^2}{8}\right)},$$

$$b=\sqrt{\left(\frac{3c^2-4d^2}{8}\right)}\mp\sqrt{\left(\frac{12d^2-c^2}{8}\right)}.$$

19. **If the tangents of the semi-angles of a plane triangle be in arithmetical progression, shew that the cosines of the whole angles will also be in arithmetical progression.**

20. **The straight line which bisects the vertical angle A of a triangle divides the base into two parts which are in the ratio $m : n$. Having given A, find $\tan B$,**

$$\tan B=\frac{m\sin A}{n-m\cos A}.$$

21. **The sides of a triangle being a, b, c; and θ, ϕ, ψ being angles determined by the equations**

$$\cos\theta=\frac{b-c}{a},\ \cos\phi=\frac{c-a}{b},\ \cos\psi=\frac{a-b}{c},$$

shew that $\tan\frac{\theta}{2}\tan\frac{\phi}{2}\tan\frac{\psi}{2}=\pm 1$.

22. **If ACB be a triangle having a right angle at C, and AE, BD, drawn perpendicular to AB respectively, meet BC, AC produced in E and D; prove that $\tan CED=\tan^3 BAC$, and that the triangle ECD is equal to the triangle ACB.**

23. **In the triangle ABC, $AC=2BC$. If CD, CE respectively bisect the angle C and the exterior angle formed by producing AC, prove that the triangles CBD, ACD, ABC, CDE have their areas as 1 : 2 : 3 : 4.**

24. **Three indefinite straight lines intersect in A, B, and C; any other straight line cuts AB in C', BC in A', and CA in B'; prove that the product of the areas of the triangles $A'BC'$, $B'CA'$, $C'AB'$**

$$=\frac{(A'B'.B'C'.C'A'\sin A'\sin B'\sin C')^2}{8\sin A\sin B\sin C}.$$

25*. ABC is a right-angled triangle in which C is the right angle. Prove the following relations:

$$\frac{\cos 2B - \cos 2A}{\sin 2A} = \tan A - \tan B,$$

$$2 \operatorname{cosec} 2A \cot B = \frac{c^2}{b^2},$$

$$\tan 2A - \sec 2B = \frac{b+a}{b-a}.$$

26. If, in a triangle ABC, $\cos A \cos B \sin C = \dfrac{\sin A + \sin B}{\sec A + \sec B}$, prove that it is a right-angled triangle.

27. If in a triangle $1 + \cot\left(\dfrac{\pi}{4} - B\right) = \dfrac{2}{1 - \tan A}$, and its area is equal to $\dfrac{c^2}{4}$, then the triangle is right-angled and isosceles.

ABC is a triangle of which C is a right angle. Solve the triangle in each of the following cases:

28. Given $a+b$ (equal to δ) and A.

Ans. $c = \dfrac{\delta}{\sin A + \cos A}$, $a = \dfrac{\delta \sin A}{\sqrt{2} \cos (A - 45^0)}$, $b = \dfrac{\delta \cos A}{\sqrt{2} \cos (A - 45^0)}$.

29. Given $a - b$ (equal to δ) and A.

Ans. $c = \dfrac{\delta}{\sqrt{2} \sin (A - 45^0)}$, $b = \dfrac{\delta \cos A}{\sqrt{2} \sin (A - 45^0)}$, $a = \dfrac{\delta \sin A}{\sqrt{2} \sin (A - 45^0)}$.

30. Given $b + c$ (equal to δ) and A.

Ans. $c = \dfrac{\delta}{1 + \cos A}$, $b = \delta \cot A \tan \dfrac{A}{2}$, $a = \dfrac{\delta \sin A}{2 \cos^2 \dfrac{A}{2}}$.

31. Given $b - c$ (equal to δ) and A.

Ans. $c = \dfrac{\delta}{2 \sin^2 \dfrac{A}{2}}$, $b = \delta \cot \dfrac{A}{2} \cot A$, $a = \delta \cot \dfrac{A}{2}$.

* The Examples from Nos. 25 to 40 inclusive are taken from Dr August Wiegand's *Sammlung trigonometrischer Aufgaben.*

32. Given $a+b+c$ (equal to δ) and A.

$$\text{Ans.}\quad c=\frac{\delta}{1+\sin A+\cos A},\quad b=\frac{\delta\cos A}{2\sqrt{2}\cos\frac{A}{2}\cos\left(45^0-\frac{A}{2}\right)},$$

$$a=\frac{\delta\sin\frac{A}{2}}{\sqrt{2}\cos\left(45^0-\frac{A}{2}\right)}.$$

33. Given $a+b-c$ (equal to δ) and the angle A.

$$\text{Ans.}\ c=\frac{\delta}{2\sqrt{2}\sin\frac{A}{2}\sin\left(45^0-\frac{A}{2}\right)},\quad b=\frac{\delta\cos A}{2\sqrt{2}\sin\frac{A}{2}\sin\left(45^0-\frac{A}{2}\right)},$$

$$a=\frac{\delta\cos\frac{A}{2}}{\sqrt{2}\sin\left(45^0-\frac{A}{2}\right)}.$$

34. Given $a+b$ (equal to δ) and the hypothenuse c.

$$\text{Ans.}\quad \sin(45^0+A)=\frac{\delta}{\sqrt{2c}},$$

$$a=\frac{\delta+\sqrt{(2c^2-\delta^2)}}{2},$$

$$b=\frac{\delta-\sqrt{(2c^2-\delta^2)}}{2},$$

$$\text{area}=\frac{1}{4}(\delta+c)(\delta-c).$$

35. Given $a-b$ (equal to δ) and the hypothenuse c.

$$\text{Ans.}\quad \sin(A-45^0)=\frac{\delta}{\sqrt{2c}},$$

$$a=\frac{1}{2}\{\delta+\sqrt{(2c^2-\delta^2)}\},$$

$$b=\frac{1}{2}\{-\delta+\sqrt{(2c^2-\delta^2)}\},$$

$$\text{area}=\frac{1}{4}(c+\delta)(c-\delta).$$

36. If the perpendicular from C upon the hypothenuse divides it into two segments of which the lengths are c_1, c_2, prove that

$$a=\sqrt{\{c_1(c_1+c_2)\}},$$

$$b=\sqrt{\{c_2(c_1+c_2)\}}.$$

37. c_1 and c_2 having the same meaning as in question 36, if $c_1 - c_2$ (equal to δ) be given, and the hypothenuse c.

Ans. $a=\sqrt{\left\{\frac{c(c+\delta)}{2}\right\}}$, $b=\sqrt{\left\{\frac{c(c-\delta)}{2}\right\}}$, $\tan A=\sqrt{\left(\frac{c+\delta}{c-\delta}\right)}$.

38. Given the radius (r) of the inscribed circle and one angle A.

$$\text{Ans. } b=\sqrt{2}\,.\,r\,.\,\frac{\sin\left(45^0+\frac{A}{2}\right)}{\sin\frac{A}{2}},\quad a=\sqrt{2}\,.\,r\,.\,\frac{\cos\frac{A}{2}}{\sin\left(45^0-\frac{A}{2}\right)}.$$

39. Given the radius (r) of the inscribed circle and one side a. Ans. $b=\frac{2r(a-r)}{a-2r}$, $\tan A=\frac{a(a-2r)}{2r(a-r)}$.

40. If $2a=b$, then the radii of the escribed circles are in arithmetic progression.

41. In any triangle ABC, given $A=60^0$, $a=\sqrt{6}$, $b=2$, solve the triangle. Ans. $B=45^0$.

42. Given $A=30^0$, $a=\sqrt{2}$, $b=2$, solve the triangle. Ans. $B=45^0$, or 135^0.

43. Given $A=135^0$, $a=2$, $b=\sqrt{6}$. There is no solution.

44. Given $A=135^0$, $a=2$, $b=\sqrt{2}$, solve the triangle. Ans. $B=30^0$.

45. Given $\sin B=\cdot 25$, $a=5$, $b=2\cdot 5$. Ans. $A=30^0$ or 150^0.

46. In the case where the solution of a triangle is ambiguous, if k and k' be the areas of the two triangles which satisfy the given conditions, prove that

$$\frac{k^2+k'^2-2kk'\cos 2A}{(k+k')^2}=\frac{a^2}{b^2},$$

A, a, and b, being given.

47. In the ambiguous case where a, b and A are given to determine the triangle, if c' c'' be the two values found for the third side of the triangle, prove that

$$c'^2-2c'c''\cos 2A+c''^2=4a^2\cos^2 A.$$

48. Given $\frac{b}{a}=\frac{1}{2}$, $C=60^0$, find the other angles.

Ans. $A=90^0$, $B=30^0$.

49. Given $a=18$, $b=2$, and $C=55^0$, find A and B.

$\log 2 = \cdot 30103$, $L \cot 27^0\ 30' = 10\cdot 2835233$,
$L \tan 56^0\ 56' = 10\cdot 1863769$, Diff. for $1' = 2763$.

Ans. $\begin{cases} A = 119^0 26' 51''\cdot 3; \\ B = 5^0\ 33'\ 8''\cdot 7. \end{cases}$

50. If $a=9$, $b=7$, $C=64^0\ 12'$, find the other angles.
Given $\log 2 = \cdot 3010300$, $L \tan 11^0\ 16' = 9\cdot 2993216$,
Diff. for $1' = 6588$.

Ans. $A = 69^0\ 10'\ 10''$,
$B = 46^0\ 37'\ 50''$.

51. Given $a = 85\cdot 63$, $b = 78\cdot 21$, $C = 48^0\ 24'$, solve the triangle.
Given $\log 16384 = 4\cdot 2144199$, $L \cot 24^0\ 12' = 10\cdot 3473497$,
$\log 742 = 2\cdot 8704039$, $L \tan 5^0\ 45' = 9\cdot 0030066$,
Diff. for $1' = 12655$.
$\log 67502 = 4\cdot 8293166$, $L \sin 24^0\ 12' = 9\cdot 6127023$,
$\log 67501 = 4\cdot 8293102$, $L \cos 5^0\ 45'\ 15'' = 9\cdot 9978062$.

Ans. $A = 71^0\ 33'\ 15''$, $B = 60^0\ 2'\ 45''$,
$c = 67\cdot 502$.

52. If $a = 1\cdot 5$, $b = 13\cdot 5$, $C = 65^0$, determine the other angles.
Given $\log 2 = \cdot 3010300$, $L \cot 32^0\ 30' = 10\cdot 1958127$,
$L \tan 51^0\ 28' = 10\cdot 0988763$,
$L \tan 51^0\ 29' = 10\cdot 0991355$.

Ans. $108^0\ 58'\ 6''\cdot 1$,
$6^0\ 1'\ 53''\cdot 9$.

53. In a triangle ABC,
if $a = 30$, $b = 20$, $C = 78^0$, find c.

Given
$\log 2 = \cdot 30103$, $L \sin 39^0 = 9\cdot 79887$, $L \cot 39^0 = 10\cdot 09163$,
$L \tan 13^0\ 52' = 9\cdot 39245$, $L \cos 13^0\ 53' = 9\cdot 98712$,
$L \tan 13^0\ 53' = 9\cdot 39299$, $L \cos 13^0\ 52' = 9\cdot 98715$,
$\log 3\cdot 2412 = \cdot 51070$.

Ans. $c = 32\cdot 412$.

54. Two sides of a triangle are 85 and 75 yards respectively, and the included angle is 75^0. Determine the triangle,

$\log 160 = 2\cdot 20412$, $L \tan 52^0\ 30' = 10\cdot 11502$,
$L \tan 4^0\ 40' = 8\cdot 9109$.

Ans. The other angles are $57^0\ 10'$, and $47^0\ 50'$.

55. Given that the ratio of two of the sides of a triangle is 7 : 3, and that the angle they include is $6^0\ 37'\ 24''$, find the other angles, having given

$$\log 2 = \cdot 3010300,\ L\tan 3^0\ 18'\ 42'' = 8\cdot 7624080,$$

$$L\tan 8^0\ 13'\ 50'' = 9\cdot 1603083,\ L\tan 8^0\ 14' = 9\cdot 1604569.$$

Ans. $\begin{cases} 4^0\ 55'\ 10''\cdot 6, \\ 168^0\ 27'\ 25''\cdot 4. \end{cases}$

56. The ratio of two sides of a triangle is 9 : 7, and the included angle is $47^0\ 25'$, find the other angles.

Given $\log 2 = \cdot 3010300,$

$$L\tan 66^0\ 17'\ 30'' = 10\cdot 3573942,$$

$$L\tan 15^0\ 53' = 9\cdot 4541479,$$

difference for $1' = 4797$.

Ans. $\begin{cases} A = 82^0\ 10'\ 49'', \\ B = 50^0\ 24'\ 11'', \end{cases}$ nearly.

57. Given $a = 18$, $b = 20$, $c = 22$, find $L\tan\frac{A}{2}$,

given $\log 2 = \cdot 30103$, $\log 3 = \cdot 4771213$.

Ans. $L\tan\frac{A}{2} = 9\cdot 6733937$.

58. The sides of a triangle are as 4 : 5 : 6, find the angle B.

Given $\log 2 = \cdot 3010299$, $L\cos 27^0\ 53' = 9\cdot 9464040$,
$\log 5 = \cdot 6989700$, $L\cos 27^0\ 54' = 9\cdot 9463371$.

Ans. $B = 55^0\ 46'\ 16''$.

59*. In a triangle ABC, given $a + b = \delta$, c, and the angle C, prove that

$$a = \delta\cos^2\frac{\phi}{2},\ b = \delta\sin^2\frac{\phi}{2},$$

where ϕ is given by the equation

$$\sin\phi = \pm\frac{\sqrt{(\delta^2 - c^2)}}{\delta}\sec\frac{C}{2}.$$

60. O is a point within a triangle ABC at which the sides subtend equal angles, and α, β, γ are the distances of O from

* The Examples from Nos. 59 to 70 are taken from Dr August Wiegand's *Sammlung trigonometrischer Aufgaben.*

the angles A, B, C respectively; find the sides and the area of the triangle.

Ans. The sides are $\sqrt{(\beta^2+\gamma^2+\beta\gamma)}$, $\sqrt{(\gamma^2+\alpha^2+\gamma\alpha)}$, $\sqrt{(\alpha^2+\beta^2+\alpha\beta)}$,

and the area is $\dfrac{\sqrt{3}}{4}(\alpha\beta+\beta\gamma+\gamma\alpha)$.

61. In any triangle, if $a-b=d$, prove that

$$a=\frac{d\sin A}{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}},\quad c=\frac{d\sin\frac{A+B}{2}}{\sin\frac{A-B}{2}}.$$

62. In any triangle, given a, A, and $a+b$ (equal to δ), to solve the triangle.

$$\text{Ans.}\quad \cos\frac{B-C}{2}=\frac{\delta\sin\frac{A}{2}}{a},$$

whence B and C are found easily, since $B+C$ is known.

63. Given a, A, and $a-b$ (equal to δ), to solve the triangle.

$$\text{Ans.}\quad \sin\frac{B-C}{2}=\frac{\delta\cos\frac{A}{2}}{a},\ \&c...$$

64. Given a, B, and $b+c$ (equal to δ), to solve the triangle.

$$\text{Ans.}\quad \tan\frac{C}{2}=\frac{\delta-a}{\delta+a}\cot\frac{B}{2},\ \&c...$$

65. Given a, B, and $b-c$ (equal to δ), to solve the triangle.

$$\text{Ans.}\quad \tan\frac{C}{2}=\frac{a-\delta}{a+\delta}\tan\frac{B}{2},\ \&c...$$

66. Given $a+b+c=2s$, and the angles A, B, C, to solve the triangle.

$$\text{Ans.}\quad a=\frac{s\sin\frac{A}{2}}{2\cos\frac{B}{2}\cos\frac{C}{2}},\ \&c...$$

67. If a_1, a_2 be the segments of BC made by the perpendicular upon it from A, prove that

$$\sin(B-C)=\frac{a_1-a_2}{a_1+a_2}\sin A.$$

68. Given a, p_1 the perpendicular upon BC from A, and also $b+c$ equal to δ, prove that

$$\tan\frac{A}{2}=\frac{2p_1a}{\delta^2-a^2}.$$

69. Given a, p_1, and $b-c$ (equal to δ), prove that

$$\tan\frac{A}{2}=\frac{a^2-\delta^2}{2p_1a}.$$

70. Given a, $b+c$ (equal to δ), and r the radius of the inscribed circle, prove that

$$\tan\frac{A}{2}=\frac{2r}{\delta-a}.$$

71. Given that the angle C is 120^0, and that the line joining C with the middle point of AB is t, prove that

$$a=\frac{(3c^2-4t^2)^{\frac{1}{2}}\pm(12t^2-c^2)^{\frac{1}{2}}}{2\sqrt{2}},$$

$$b=\frac{(3c^2-4t^2)^{\frac{1}{2}}\mp(12t^2-c^2)^{\frac{1}{2}}}{2\sqrt{2}}.$$

72. Upon the altitude of an equilateral triangle another equilateral triangle is described; upon the altitude of this again an equilateral triangle is constructed, and so on, *ad infinitum*. Prove that the sum of the areas of all these triangles is equal to three times the area of the original triangle.

73. If α, β, γ be the three perpendiculars from the angles of a triangle upon the opposite sides, then

$$\frac{\alpha^2}{\beta\gamma}=\frac{bc}{a^2}.$$

74. A circle is inscribed in a triangle ABC, and α, β, γ are the angles which the sides subtend at the centre of the circle; shew that

$$4\sin\alpha\sin\beta\sin\gamma=\sin A+\sin B+\sin C.$$

75. If $A+B+C=180^0$, the expression $\cot A+\dfrac{\sin A}{\sin B\sin C}$ will retain the same value if any two of the angles A, B, C be interchanged.

76. If the perpendiculars upon the opposite sides from the angles of a triangle ABC meet in O.

$$\frac{OB \cdot OC}{AB \cdot AC} + \frac{OC \cdot OA}{BC \cdot BA} + \frac{OA \cdot OB}{CA \cdot CB} = 1.$$

77. On the sides of an equilateral triangle three squares are described. Compare the area of the triangle formed by joining their centres, with the area of the original triangle.

Ans. $1 + \frac{\sqrt{3}}{2} : 1.$

78. The hypothenuse (c) of a right-angled triangle ABC is trisected in D and E. Prove that if CD, CE be joined, the sum of the squares of the sides of the triangle CDE is $\frac{2c^2}{3}$.

79. If the tangents of the angles of a triangle are as $1:2:3$, find them. Ans. 1, 2, 3 are the tangents.

80. If an equilateral triangle have its angular points in three parallel straight lines, of which the middle one is distant from the outside ones by a and b, each of its sides is equal to

$$2\sqrt{\left(\frac{a^2 + ab + b^2}{3}\right)}.$$

81. If the sides a, b, c of a triangle be increased in the ratios $l:1$, $m:1$, $n:1$, and the angle C be thereby halved and the other two doubled, prove that

$$(la^2 + mb^2)(l + m) = c^2(lm + n^2).$$

82. $ABCD$ is a parallelogram, P any point within it; shew that $\frac{\Delta APC}{\tan APC} - \frac{\Delta BPD}{\tan BPD}$ is a constant quantity.

83. In any triangle

$$\cot A + \cot B + \cot C > \frac{1}{2}(\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C).$$

84. If the sides a, b, c of a triangle are in harmonic progression, then

$$\cos\frac{B}{2} = \sqrt{\left(\frac{\sin A \sin C}{\cos A + \cos C}\right)}.$$

85. If A, B, C be the angles of a triangle,

$$8 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} \text{ is less than } 1,$$

except when the triangle is equilateral.

86. If θ and ϕ be the greatest and least angles of a triangle, the sides of which are in arithmetic progression, prove that

$$4(1-\cos\theta)(1-\cos\phi)=\cos\theta+\cos\phi.$$

87. In any triangle,

$$\text{if } \cot\frac{A}{2}+\cot\frac{C}{2}=2\cot\frac{B}{2},$$

$$\text{then } \sin A+\sin C=2\sin B.$$

88. In any triangle ABC, prove that

$$\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}}=\frac{1}{8}\,\frac{(a+b+c)^3}{abc}.$$

89. The perpendicular drawn from any point of a circle upon a chord of the circle is a mean proportional between the perpendiculars drawn from the same point upon the tangents at the extremity of the chord.

EXAMPLES ILLUSTRATING CHAPTER X.

1. If R be the radius of the circumscribing and r_a, r_b, r_c, r the radii of the inscribed and escribed circles of a triangle, then

$$r_a+r_b+r_c-r=4R.$$

If Δ = the area of the triangle

$$r_a+r_b+r_c-r=\Delta\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}\right) \quad \text{(Ch. x. Art. 2)}$$

$$=\frac{\Delta}{s(s-a)(s-b)(s-c)}\{s(s-b)(s-c)+s(s-c)(s-a)$$
$$+s(s-a)(s-b)-(s-a)(s-b)(s-c)\}$$

$$=\frac{1}{\Delta}\{s[3s^2-2(a+b+c)s+bc+ca+ab]$$
$$-(s-a)(s-b)(s-c)\}$$

$$=\frac{1}{\Delta}\{s[bc+ca+ab-s^2]-s^3+(a+b+c)s^2$$
$$-(ab+bc+ca)s+abc\}$$

$$=\frac{1}{\Delta}\{-2s^3+(a+b+c)s^2+abc\}$$

$$=\frac{abc}{\Delta}=4R.$$

2. The radius of the circumscribing circle of a triangle ABC is equal to

$$\frac{1}{8}(a+b+c)\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}.$$

We have by Ch. x. Art. 5, if R be the radius,

$$2R=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C};$$

$$\text{therefore } 2R=\frac{a+b+c}{\sin A+\sin B+\sin C}.$$

$$\text{Now } \sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$=2\cos\frac{C}{2}\cos\frac{A-B}{2};\ \text{ since } \frac{A+B}{2}=90^0-\frac{C}{2};$$

$$\therefore\ \sin A+\sin B+\sin C=2\cos\frac{C}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right)$$

$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

$$\text{Hence } 2R=\frac{a+b+c}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}},$$

$$\text{or } R=\frac{1}{8}(a+b+c)\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}.$$

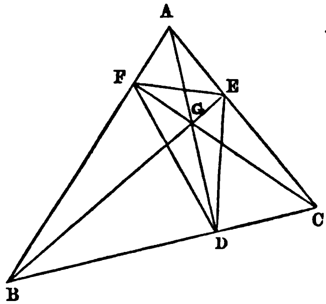
3. Perpendiculars AD, BE, CF are drawn from the angles of a triangle ABC to the opposite sides, meeting each other in G. If R be the radius of the circumscribing circle, then

$$AG=2R\cos A,$$
$$DG=2R\cos B\cos C,$$
$$EF=R\sin 2A,$$
$$\text{area } ABC=\tfrac{1}{2}R\,(DE+EF+FD).$$

The triangles AGE, ADC are similar.

$$\text{Hence } \frac{AG}{AE}=\frac{AC}{AD} \text{ or } AG=\frac{AE\,.\,AC}{AD}$$

$$=\frac{bc\cos A}{AD}.$$



Now $AD = b \sin C$;

$\therefore\ AG = \frac{c \cos A}{\sin C}$ and $2R = \frac{c}{\sin C}$;

$\therefore\ AG = 2R \cos A$.

Again, $DG = AD - AG = b \sin C - 2R \cos A$

$= 2R \{\sin B \sin C + \cos (B + C)\}$

$= 2R \cos B \cos C$.

Since a circle can be described about the quadrilateral $AFGE$, we have

$EF . AG = EG . AF + GF . AE$..................(1).

Now $AG = 2R \cos A$, $EG = 2R \cos A \cos C$,

$AF = b \cos A = 2R \sin B \cos A$,

$GF = 2R \cos A \cos B$,

$AE = c \cos A = 2R \sin C \cos A$.

Substituting these values in (1) and dividing by $2R$, we have

$EF \cos A = (\cos^2 A \cos C \sin B + \cos^2 A \sin C \cos B)\ 2R$

$= 2R \cos^2 A \sin (B + C)$;

$\therefore\ EF = 2R \sin A \cos A = R \sin 2A$.

Again,

$DE + EF + FD = R (\sin 2A + \sin 2B + \sin 2C)$

$= 4R \sin A \sin B \sin C$ [see Ch. VII. (B)],

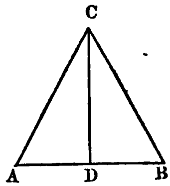
$\therefore\ \frac{1}{2} R (DE + EF + FD) = 2R^2 \sin A \sin B \sin C$

$= \frac{1}{2} ab \sin C$

= the area of the triangle ABC.

4. If the areas of a regular pentagon and decagon are equal to each other, the ratio of their sides will be $\sqrt[4]{20} : 1$.

Let CAB be one of the five equal triangles of which the pentagon is composed. $AB = x$. Draw the perpendicular CD to AB.



The angle ACB is $\frac{2\pi}{5}$,

and therefore angle ACD is $\frac{\pi}{5}$.

The area of the pentagon is $5\Delta ACB$ or $5AD.CD$;

$$\therefore \text{ the area of pentagon} = 5\frac{x}{2}.\frac{x}{2}\cot\frac{\pi}{5} = 5\frac{x^2}{4}\cot\frac{\pi}{5}.$$

Similarly, the area of the decagon, if y be the length of each of its sides, will be

$$10\frac{y^2}{4}\cot\frac{\pi}{10};$$

$$\text{therefore we have} \quad 10\,y^2\cot\frac{\pi}{10} = 5x^2\cot\frac{\pi}{5},$$

$$\text{or} \quad \frac{x^2}{y^2} = \frac{2\tan\frac{\pi}{5}}{\tan\frac{\pi}{10}},$$

$$= \frac{4}{1 - \tan^2 18^0}.$$

$$\text{Now} \quad \sin 18^0 = \frac{\sqrt{5} - 1}{4};$$

$$\therefore \sin^2 18 = \frac{6 - 2\sqrt{5}}{16} \text{ and } \cos^2 18 = \frac{2\sqrt{5} + 10}{4};$$

$$\therefore 1 - \tan^2 18^0 = 1 - \frac{3 - \sqrt{5}}{\sqrt{5} + 5} = \frac{2\sqrt{5} + 2}{\sqrt{5} + 5} = \frac{2(\sqrt{5} + 1)}{\sqrt{5}(\sqrt{5} + 1)}$$

$$= \frac{2}{\sqrt{5}}.$$

$$\text{Hence} \quad \frac{x^2}{y^2} = 2\sqrt{5} = 20,$$

$$\text{or} \quad \frac{x}{y} = \sqrt[4]{20}.$$

5. **Prove that**

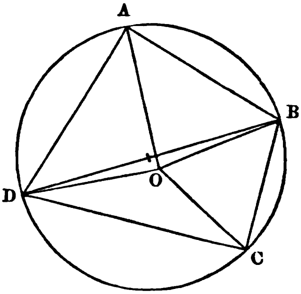
$$\sin(\alpha+\beta)\sin(\beta+\gamma)=\sin\alpha\sin\gamma+\sin\beta\sin(\alpha+\beta+\gamma),$$

and apply this formula to shew that the rectangle under the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the rectangles under the opposite sides.

$$\begin{aligned}\sin(\alpha+\beta)\sin(\beta+\gamma)&=\tfrac{1}{2}\{\cos(\alpha-\gamma)-\cos(\alpha+2\beta+\gamma)\}\\&=\tfrac{1}{2}\{\cos(\alpha-\gamma)-\cos(\alpha+\gamma)+\cos(\alpha+\gamma)\\&\qquad-\cos(\alpha+2\beta+\gamma)\}\\&=\sin\alpha\sin\gamma+\sin\beta\sin(\alpha+\beta+\gamma)\ldots\ldots\ldots\ldots(1)\end{aligned}$$

[Ch. IV. Art. 11, (B)].

Now let $ABCD$ be a quadrilateral inscribed in a circle, the centre of which is O.



Let R be the radius of the circle.

Denote the angles AOB, BOC, COD by 2α, 2β and 2γ respectively.

Then, since a perpendicular from O upon any one of the sides of the quadrilateral bisects it, we have

$$AB=2R\sin\alpha,\ BC=2R\sin\beta,\ CD=2R\sin\gamma,$$

$$DA=2R\sin\{\pi-(\alpha+\beta+\gamma)\}=2R\sin(\alpha+\beta+\gamma).$$

Similarly, $$AC=2R\sin\frac{AOC}{2}=2R\sin(\alpha+\beta),$$

$$BD=2R\sin\frac{BOD}{2}=2R\sin(\beta+\gamma).$$

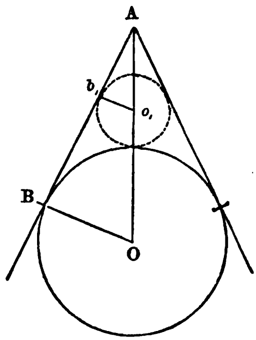
Hence, from formula (1), by substitution,

$$AC\times BD=AB\times CD+BC\times AD.$$

6. Two tangents intersecting in A are drawn to a circle the radius of which is r. Between these tangents and the circle, another circle is inscribed; then another is inscribed between the same tangents and this last circle, and so on *ad infinitum*. If a be the distance from A of the centre of the original circle, shew that the ratio of the sum of the areas of all the other circles to the area of this circle is $\dfrac{(a-r)^2}{4ar}$.

$AO=a,$

$OB=r.$



Let o_1 be the centre of the first circle inscribed, r_1 its radius, and denote Ao_1 by a_1.

Then, by similar triangles,

$$Ao_1b_1,\ AOB,$$

$$r_1=r.\frac{Ao_1}{AO}=r.\frac{a-(r_1+r)}{a};$$

$$\therefore\ r_1=r.\frac{a-r}{a+r}.$$

Similarly $r_2=r_1\dfrac{a_1-r_1}{a_1+r_1}=r.\left(\dfrac{a-r}{a+r}\right)^2$ by substitution.

Similarly $r_3=r\left(\dfrac{a-r}{a+r}\right)^3.$

Denote $\dfrac{a-r}{a+r}$ by λ,

then the sum of all the areas of the circles is

$$\pi r^2(\lambda^2+\lambda^4+\lambda^6+\&c.\ ad\ inf.)=\pi r^2\frac{\lambda^2}{1-\lambda^2}.$$

Hence, the required ratio

$$=\frac{\lambda^2}{1-\lambda^2}=\frac{\left(\dfrac{a-r}{a+r}\right)^2}{1-\left(\dfrac{a-r}{a+r}\right)^2}=\frac{(a-r)^2}{4ar}.$$

7. If A, A_1, A_2, A_3 be the areas of the four circles which touch the sides of a triangle, A being that of the inscribed circle,

$$\frac{1}{\sqrt{A}}=\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}.$$

8. If a polygon of an odd number of sides $(2n+1)$ be inscribed in a circle and a diameter be drawn from one of its angles (P) meeting the opposite side (AB) in Q, then the line PQ is

$$\tfrac{1}{2} AB \cot \frac{\pi}{2(2n+1)}.$$

9. $ABCD$ is a quadrilateral capable of being inscribed in a circle, shew that

$$AC \sin A = BD \sin B.$$

10. The sides of a quadrilateral inscribed in a circle taken in order are 3, 3, 4, 4; find the area and the radii of the inscribed and circumscribed circles.

Ans. area $=12$, rad. of circum. circle $=5$, inscrib. $=\dfrac{12}{7}$.

11. If r, r' be the radii of the escribed circles touching externally the sides of a right-angled triangle which are adjacent to the right angle, the area of the triangle is equal to rr'.

12. $ABCDEF$ being a regular hexagon, and AC, BD, CE, DF, EA, FB joined and meeting in a, b, c, d, e, f; shew that $abcdef$ is a regular hexagon and that its area is one-third of $ABCDEF$.

13. If α, β, γ be the distances of the centre of the inscribed circle of a triangle from the angles, and r_a, r_b, r_c be the radii of the escribed circles, prove that if

$$\frac{2}{\beta^2} = \frac{1}{\alpha^2} + \frac{1}{\gamma^2},$$

then r_a, r_b, r_c are in arithmetical progression.

14. If the radii of the escribed circles of a triangle be in arithmetic progression, the tangents of the half angles are in arithmetic progression.

15. If the radii of the escribed circles are in harmonic progression,

$$\cot \frac{A}{2} + \cot \frac{B}{2} = 2 \cot \frac{B}{2}.$$

16. A triangle ABC is described about a circle, and any three triangles are cut off by tangents to the circle. If r_1, r_2, r_3 be the radii of the circles inscribed in these three triangles, then the sum of the areas of the triangles is

$$\frac{a}{2}(-r_1+r_2+r_3) + \frac{b}{2}(r_1-r_2+r_3) + \frac{c}{2}(r_1+r_2-r_3).$$

17. Prove that, if the radius of the circumscribing circle of a triangle be equal to the perpendicular drawn from one of the angles on the opposite side, the product of the sines of the angles adjacent to that side is $\frac{1}{2}$.

18. If p_1, p_2, p_3 be the perpendiculars from the angles of a plane triangle upon the opposite sides, r the radius of the inscribed circle, and r_a, r_b, r_c those of the escribed circles, prove that

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}.$$

19. With the same notation as in the last question, prove that

$$p_1p_2p_3=\frac{(a+b+c)^3r^3}{abc}.$$

20. A circle is inscribed in a triangle and another triangle formed by joining the points of contact; within this latter triangle a circle is inscribed and another triangle formed as before, and so on continually. Shew that the triangles thus formed ultimately become equilateral.

21. The area of a triangle ABC is

$$Rr(\sin A+\sin B+\sin C),$$

R, r being the radii of the circumscribed and inscribed circles.

22. If lines be drawn from the angles of a triangle ABC to the centre of the inscribed circle, cutting the circumference in the points D, E, F; shew that the angles D, E, F, of the triangle formed by joining these points, are respectively equal to

$$\frac{\pi+A}{4},\ \frac{\pi+B}{4},\ \frac{\pi+C}{4}.$$

23. The area of the triangle, of which the centres of the escribed circles are the angular points, is $\frac{abc}{2r}$, where r is the radius of the inscribed circle.

24. Three circles are described, each of which touches one side of a triangle and the other two sides produced. If D be the point of contact of the side BC, E that of AC, and F that of AB, shew that

$$AE=BD,\ BF=CE,\ CD=AF.$$

25. Prove that $\sqrt{(rr_ar_br_c)}=$ the area of a triangle ABC.

26. Two similar triangles have a common escribed circle touching sides not homologous a_1, b_2; shew that

$$a_1:a_2=-\sin A+\sin B+\sin C:\sin A-\sin B+\sin C.$$

27. If α, β, γ be the distances from the angles of a triangle ABC to the points of contact of the inscribed circle, the radius of the circle is equal to

$$\left(\frac{\alpha\beta\gamma}{\alpha+\beta+\gamma}\right)^{\frac{1}{2}}.$$

28. If α, β, γ be the distances of the centre of the inscribed circle from the angles of a triangle ABC, and r the radius of the circle, then

$$a\alpha^2+b\beta^2+c\gamma^2=abc, \text{ and } r=\frac{1}{2}\frac{\alpha\beta\gamma}{abc}(a+b+c).$$

29. With the same notation as the last question, prove that, if R be the radius of the circumscribing circle,

$$\frac{1}{a\alpha^2}+\frac{1}{b\beta^2}+\frac{1}{c\gamma^2}=\frac{1}{abc}\left(1+\frac{4R}{r}\right).$$

30. With the same notation shew that

$$\frac{\alpha^6}{b^3c^3}+\frac{\beta^6}{a^3c^3}+\frac{\gamma^6}{a^3b^3}=1-\frac{24abc}{(a+b+c)^3}.$$

31. If in a circle any chord MN is drawn parallel to the diameter AB in which any point P is taken; then

$$AP^2+PB^2=PM^2+PN^2.$$

32. If p, q, r be the lines drawn from A, B, C, bisecting the angles of a triangle ABC and terminated by the circumference of the circumscribed circle, then

$$p\cos\frac{A}{2}+q\cos\frac{B}{2}+r\cos\frac{C}{2}=a+b+c.$$

33. O_a, O_b, O_c are the centres of the escribed circle of a triangle, of which the area is A. If A' is the area of the triangle $O_aO_bO_c$, shew that

$$4AA'=(a+b+c)\,abc.$$

34. If the middle points of the sides of a triangle be joined with the opposite angles, and R_1, R_2, R_3...&c. be the radii of the circles described about the six triangles so formed; r_1, r_2, r_3...&c. the radii of the circles inscribed in the same; prove that

$$R_1R_3R_5=R_2R_4R_6, \text{ and also that } \frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_5}=\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6}.$$

35. If R be the radius of the circumscribing circle of a triangle ABC,

$$a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C.$$

36. If A', B', C' be the feet of the perpendiculars from the angle of a triangle ABC upon the sides, prove that the radius of the circle circumscribing the triangle $A'B'C'$ is half that of the circle circumscribing ABC.

37. A circle is described about a triangle ABC, and a new triangle formed by joining the points of bisection of the arcs subtended by the sides of ABC. Shew that the sides of the triangle are

$$\frac{a}{2 \sin \frac{A}{2}}, \quad \frac{b}{2 \sin \frac{B}{2}}, \quad \frac{c}{2 \sin \frac{C}{2}},$$

and that its area : area of ABC :: $1 : 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.

38. A hexagon is inscribed in a circle of radius r and the alternate angles are joined, the joining lines forming another hexagon. Prove that the area of this last hexagon is $\frac{\sqrt{3}}{2} r^2$.

39. If α, β, γ be the distances between the centres of the escribed circles of a triangle ABC, prove that

$$\frac{r_a r_b r_c}{\alpha\beta\gamma} = \frac{1}{8} \sin A \sin B \sin C.$$

40. Compare the radii of the circles in Euclid's figure (Book IV. Prop. 10), and shew that $\frac{1}{AB} + \frac{1}{BC} = \frac{\sqrt{5}}{AC}$, C being the point which divides AB.

Shew also that the areas of the circles are as $5 + \sqrt{5} : 2$.

41. The sides of a triangle are in arithmetical progression, and the distance of the centre of the inscribed and circumscribed circle is a mean proportional between the greatest and least; shew that the sides are as the numbers

$$\sqrt{5} - 1 : \sqrt{5} : \sqrt{5} + 1.$$

42. From the arc AB of a circle of which the centre is O, AC is cut off equal to the sine of AB, shew that the sector COB = segment ACB.

43. AB, AC are the chords of arcs of 60^0 and 90^0 in a circle of which the centre is O; shew that if AB and OC be pro-

duced to meet in D, the arc of which the chord is equal to DC has its cosine and chord equal.

44. If, in a regular polygon of n sides, straight lines be drawn from the extremities of a side (a) to those of any other side, so as to cross each other; shew that the locus of their intersection is a circle of which the radius $=\frac{a}{2}\operatorname{cosec}\frac{2\pi}{n}$.

45. a and a' are homologous sides of two similar triangles described, one about and the other within a circle. Prove that

$$\frac{a'}{a}=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

46. From the extremity A of the diameter ACB of a semicircle of which C is the centre, a line AP is drawn dividing the area of the semicircle into two equal parts. If θ be the circular measure of the complement of PCB, prove that $\cos\theta=\theta$.

47. A triangle ABC is inscribed in a circle, and from A and C lines are drawn bisecting these angles and meeting the circumference in a and c. Shew that the line ca is divided by AB, CB, into segments which are in the ratio

$$\sin^2\frac{C}{2} : 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} : \sin^2\frac{A}{2}.$$

48. If R be the radius of the circumscribing circle of a triangle, and p_1, p_2, p_3 the perpendiculars from its centre upon the sides,

$$R^3-(p_1^2+p_2^2+p_3^2)R-2p_1p_2p_3=0.$$

49. From the extremity B of the radius AB of a circle, a straight line BC is placed in the circle $=\frac{1}{2}$ (radius); from C another straight line is drawn bisecting AB and is produced to the circumference: compare the areas of the triangles formed by joining its extremities with A and B.

If the angles thus formed at A be θ and ϕ, ($\theta<\phi$), shew that

$$\operatorname{cosec}\phi+\cot\theta+3\cot(\phi+\theta)=0.$$

Ans. $\sqrt{\frac{20}{27}}$.

50. A quadrilateral is described about a circle, such that the sides adjacent to one of its angles are equal to one another. A circle is inscribed in the triangle formed by the bisecting

diameter of the quadrilateral and the two unequal sides: shew that if r_2 be the radius of this circle, r_1 that of the first, and 2δ the length of the other diameter of the quadrilateral,

$$\frac{1}{r_2} - \frac{1}{r_1} = \frac{1}{\delta}.$$

51. ABC is a triangle inscribed in a circle, and the tangent at A meets BC produced in P, find PA in terms of the sides.

Shew also that if Q and R be the points of intersection of the tangents at B and C respectively with AC and AB, then

$$\frac{1}{PA} + \frac{1}{RC} = \frac{1}{QB}.$$

Ans. $PA = \dfrac{abc}{b^2 - c^2}.$

52. A polygon of $2n$ sides, n of which are equal to a and the remaining n to b, is inscribed in a circle; shew that the radius of the circle is

$$\frac{1}{2\sin\frac{\pi}{n}}\sqrt{\left(a^2 + 2ab\cos\frac{\pi}{n} + b^2\right)}.$$

53. If the angle A of a triangle ABC be joined to the centre of the inscribed circle and also to that of the circle touching BC and AB, AC, produced, find the ratio (r_1) of these two joining lines. If r_2, r_3 be similar ratios with respect to B and C, shew that

$$r_1 + r_2 + r_3 = 1.$$

54. If r_a, r_b, r_c are the radii of the escribed circles of a triangle, prove that

$$\cos^{-1}\left(\frac{r_b r_c}{bc}\right)^{\frac{1}{2}} + \cos^{-1}\left(\frac{r_c r_a}{ca}\right)^{\frac{1}{2}} + \cos^{-1}\left(\frac{r_a r_b}{ab}\right)^{\frac{1}{2}} = \frac{\pi}{2}.$$

55. Three circles (radii r_1, r_2, r_3) touch each other, O_1, O_2, O_3 being their centres, and G the point where their common tangents at the points of contact meet. If $GO_1 = p_1$, $GO_2 = p_2$, $GO_3 = p_3$, and R be the radius of the circle circumscribing the triangle $O_1O_2O_3$, prove that

$$p_1p_2p_3 = 4R\frac{r_1r_2r_3}{r_1 + r_2 + r_3}.$$

56. Three circles whose radii are r_1, r_2, r_3, touch one another externally. Then one of the sides of the triangle made by joining the points of contact

$$=2r_1\sqrt{\left(\frac{r_2 r_3}{(r_1+r_2)(r_1+r_3)}\right)},$$

$$\text{and its area}=2(r_1 r_2 r_3)^{\frac{3}{2}}\frac{\sqrt{(r_1+r_2+r_3)}}{(r_1+r_2)(r_2+r_3)(r_3+r_1)}.$$

57. Each of three circles within the area of a triangle touches the other two, touching also two sides of the triangle: if a be the distance between the points of contact of one of the sides, and b, c be like distances on the other two sides, prove that the area of the triangle, of which the centres of the circles are the angular points, is equal to

$$\tfrac{1}{4}(b^2c^2+c^2a^2+a^2b^2)^{\frac{1}{2}}.$$

EXAMPLES ILLUSTRATING CHAPTER XI.

ON HEIGHTS AND DISTANCES.

1. Two towers stand on a horizontal plane 144 feet apart. A person standing at the foot of each tower observes the angle of elevation of one tower to be double that of the other, but when he is halfway between them, the angle of elevation of one tower is the complement of the angle of elevation of the other. Shew that the heights of the towers are 108 and 48 feet respectively.

2. A fortress was observed by a ship at sea to bear E.N.E., and after sailing four miles to the East it was found to bear N.N.E. Shew that the distances of the fortress from the ship at the first and second observations respectively were

$$\sqrt{(16+8\sqrt{2})} \text{ and } \sqrt{(16-8\sqrt{2})} \text{ miles.}$$

3. The length of a road in which the ascent is 1 foot in 5, from the foot of a hill to the top, is $1\frac{3}{5}$ miles. What will be the length of a zigzag road in which the ascent is 1 foot in 12? Ans. 4 miles.

4. The angular elevation of a tower at a place A due S. of it is 30°; at a place B, due W. of A, and at a distance

a from it, the elevation is 18°. Shew that the height of the tower is

$$\frac{a}{\sqrt{(2\sqrt{5}+2)}}.$$

5. At each end of a horizontal base, measured in a known direction from the place of an observer, the angle which the distance of the other end and a certain object subtends is observed, and also the angle of elevation of the object at one end of the base. Find the height of the object.

If α be the angle of elevation, β, γ the other two angles, a the length of the base, then the height of the object is

$$\frac{a \sin \alpha \sin \beta}{\sin (\beta+\gamma)}.$$

6. The angular altitude and breadth of a cylindrical tower are observed to be α and β respectively, but at a point a feet nearer to the tower they are α' and β'. Find its height and circumference.

$$\text{Ans.} \begin{cases} \text{Height} = \dfrac{a}{\cot \alpha - \cot \alpha'}. \\ \text{Breadth} = 2a. \dfrac{\sin \frac{\beta}{2} \sin \frac{\beta'}{2}}{\sin \frac{\beta'}{2} - \sin \frac{\beta}{2}}. \end{cases}$$

7. A person, wishing to know the height of an inaccessible tower, measures equal distances AB, BC in a horizontal line, and observes the angles of elevation at A, B, C to be 30°, 45°, and 60° respectively. Find the height of the tower and its distance from the line ABC. Ans. $\sqrt{\frac{3}{2}}.AB$ and $\frac{1}{\sqrt{2}}AB$.

8. A circular ring is placed, in a vertical plane through the sun's centre, on the top of a vertical staff, the height of which is eight times its radius; and the extremity of the shadow of the ring is observed to be at a distance from the foot of the staff equal to the staff's height. Determine the sun's altitude.

Ans. $\tan^{-1}\frac{4}{3}$.

9. A tower 51 feet high has a mark at the height of 25 feet from the ground; find at what distance the two parts subtend equal angles to an eye at the height of 5 feet from the ground.

Ans. 160 feet.

10. A person on a level plain, on which stands a tower surmounted by a spire, observes that when he is a feet distant from the foot of the tower, its top is in a line with that of a mountain. From a point further from the tower by a distance b feet, he finds that the spire subtends at his eye the same angle as before, and has its top in a line with that of the mountain. Shew that, if the height of the tower above the horizontal plane through the observer's eye be c feet, the height of the mountain above that plain will be

$$\frac{abc}{c^2 - a^2} \text{ feet.}$$

11. A person observes the angle of elevation of a mountain to be 30^0, and, approaching 600 yards nearer, to be 60^0. Find its height.

Given $\log 3 = \cdot 47712$, $\log 51 = 1 \cdot 70757$,
$\log 52 = 1 \cdot 71600$.

Ans. $519 \cdot 6$ yards.

12. A, B, C are three inaccessible points upon a horizontal plane. D and E are points in the sides of AB, AC produced; DE is measured and found equal to a. The angles BDE, CDE, DEB, DEC, being measured, are found to be α, β, γ, δ respectively. Find the mutual distances of A, B and C.

$$AC = a \frac{\sin \delta \sin (\alpha - \beta)}{\sin (\beta + \delta) \sin (\alpha + \delta)}.$$

$$AB = a \frac{\sin \alpha \sin (\delta - \gamma)}{\sin (\alpha + \gamma) \sin (\alpha + \delta)}.$$

BC is also at once determined.

13. The shadows of two walls which run at right angles to each other, and which are respectively a and a' feet high, are observed when the sun is due S. and found to be b and b' feet broad respectively. Find the sun's altitude and the inclination of the first wall to the meridian.

Ans. $\begin{cases} \text{Sun's altitude} = \cot^{-1} \sqrt{\left\{\left(\frac{b}{a}\right)^2 + \left(\frac{b'}{a'}\right)^2\right\}}. \\ \text{The inclination of the wall to the meridian is } \cot^{-1} \frac{ab'}{a'b}. \end{cases}$

14. A person stationed on a promontory first observes a ship at a point due N. of him; in a quarter of an hour it bears due E.; and after another quarter of an hour disappears at the

S.E. point of his horizon. Required the course which the ship was steering; and to shew that it was nearest to the observer 12 minutes after he first saw her.

Ans. An angle $\tan^{-1}\frac{1}{2}$ to the East of South.

15. A cloud is observed due S. at an elevation of α^0, and after an interval it is observed in the S.W. at the same elevation. Assuming it to move in a straight line parallel to the horizon, find the direction of its course. Ans. W.N.W.

16. The angular elevation of a tower at a place A due south of it is α, and at a place β due west of A, and at a distance a from it, the elevation is β. Find the height of the tower.

Ans. $\dfrac{a \sin \alpha \sin \beta}{\sqrt{\{\sin(\alpha+\beta)\sin(\alpha-\beta)\}}}$.

17. A tower 150 feet high throws a shadow 75 feet long upon the horizontal plane on which it stands. Find the Sun's altitude,

$\log 2 = \cdot 30103$ $\quad L \tan 63^0\, 26' = 10\cdot 3009994,$
$L \tan 63^0\, 27' = 10\cdot 3013153.$

Ans. $63^0\, 26'\, 5''.81$ nearly.

18. A stick lying on a slope makes angles α and β with the vertical and with the horizontal line drawn on the slope; shew that the inclination of the slope to the horizon is

$$\sin^{-1}\frac{\cos\alpha}{\sin\beta}.$$

19. To find the length of an inaccessible wall an observer placed himself due south of one end of the wall, and then due west of the other end, in such positions that the angle θ which the wall subtended at the two stations was the same, the distance of the stations being a. Shew that the length of the wall was $a \tan\theta$.

20. The angles of elevation α, β, γ of a mountain above three points A, B, C in the same horizontal line are measured, the distances AB, BC being a and b respectively, shew that its height is

$$\left\{\frac{ab(a+b)}{a\cot^2\gamma + b\cot^2\alpha - (a+b)\cot^2\beta}\right\}^{\frac{1}{2}}.$$

21. A person standing on the sea shore can just see the top of a mountain, the height of which he knows to be 1284·80 yards.

After ascending vertically to the height of 3 miles in a balloon, he observes the angle of depression of the mountain's summit to be $2^0\ 15'$. Find the earth's radius and the distance of the mountain from the first place of observation.

Given $\log 3 = \cdot 4771213$ $\qquad \cot 2^0\ 15' = 11\cdot 4057168$,

$\log \cdot 73 = \bar{1}\cdot 863229$ $\qquad \log 7986\cdot 4 = 3\cdot 9023533$,

$\log 76\cdot 3551 = 1\cdot 8828381$.

Ans. { Distance $= 76\cdot 3551$ miles.
Earth's radius $= 3992\cdot 835$ miles.

22. The Peak of Teneriffe is $2\frac{1}{2}$ miles high, and the angular depression of the horizon from its summit is $2^0\ 1'\ 47''$; find the earth's diameter.

Given $\log 2 = \cdot 3010300$ $\qquad L \cos 2^0\ 2' = 9\cdot 9997265$,

$L \sin 1^0 = 8\cdot 2418553$ $\qquad L \cos 2^0\ 1' = 9\cdot 9997309$,

$L \sin 1^0\ 1' = 8\cdot 2490332$ $\qquad \log 79646 = 5\cdot 9011640$,

$\log 79647 = 5\cdot 9011694$.

Ans. 7964 miles nearly.

23. Solve the previous question also by the method given in Chap. XI. Art. 12. Ans. $7963\cdot 6$ miles.

24. A staff one foot long stands at the top of a tower 200 feet high. Shew that the angle which it subtends at a point in the horizontal plane 100 feet from the base of the tower is $6'\ 51''$ nearly.

25. Two chimneys are of equal height. A person standing between them in the line joining their bases observes the elevation of the nearer one to him to be 60^0. After walking 80 feet in a direction at right angles to the line joining their bases, he observes the elevations of the two to be 45^0 and 30^0 respectively. Find their heights and the distance between them.

Ans. Height $= 40\sqrt{6}$ feet.
Distance $= 40\sqrt{2}(1 + \sqrt{7})$.

26. The height of an inaccessible object being required, its angles of elevation (α, β, γ) are taken at three points in the same horizontal line. If a and a' be the distances between the extreme points and middle point of observation, find the height of the object.

Answer:

$$\sin\alpha \sin\beta \sin\gamma \frac{\{aa'(a+a')\}^{\frac{1}{2}}}{\{a \sin^2\alpha \sin(\beta-\gamma)\sin(\beta+\gamma) + a'\sin^2\gamma \sin(\beta-\alpha)\sin(\beta+\alpha)\}^{\frac{1}{2}}}.$$

27. From a point on a hill side the angle of elevation of the top of an obelisk on its summit is observed to be α, and, a feet nearer to the top of the hill, to be β; shew that, if h be the height of the obelisk, the inclination of the hill to the horizon at the lower place of observation will be

$$\cos^{-1}\left\{\frac{a}{h}\,\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}\right\}.$$

28. A person standing in the valley between mounts Ebal and Gerizim, observes the elevation of Ebal to be 50^0: he then ascends mount Gerizim to the height of a 1000 feet, and at that point finds that the angle between his previous station in the valley and the summit of mount Ebal is 50^0, while the angle of depression of his previous station is $28^0\,58'$. Find the height of mount Ebal, all the observations being taken in the same vertical plane.

$$L\sin 50^0 = 9\cdot8842540,$$
$$L\sin 28^0\,58' = 9\cdot6851151,$$
$$\log 2\cdot501 = \cdot39811137,$$
$$\log 2\cdot502 = \cdot39828 73.$$

Ans. 2501·94 feet.

29. An observer stands so that the top of one tower is just in a line with the top of another, his distances then from the bases of the two towers being b and a respectively. He then walks in the horizontal line joining the bases of the two towers until the elevation of the top of the nearer tower is double that of the other. If his distance from the nearer tower then is c, shew that the heights of the towers are

Ans. $$\frac{b}{a}\left\{(a-b+c)\left(a-b+c-\frac{2ac}{b}\right)\right\}^{\frac{1}{2}}$$

and $$\left\{(a-b+c)\left(a-b+c-\frac{2ac}{b}\right)\right\}^{\frac{1}{2}}.$$

30. A person walking on a straight road observes two objects P and Q in the same straight line with himself, and measures the angle (α) which their direction makes with the direction of the road. After having walked a distance (a) he finds they subtend the same angle at his eye, and at a farther distance (a) that they again subtend the same angle (α) at his eye. Find the distance PQ.

Ans. $$PQ = \frac{a}{6}\cdot\frac{1+8\sin^2\alpha}{\cos\alpha}.$$

31. Two posts AB, CD are placed at the edge of a river at a distance AC equal to AB, the height of CD being such that

AB and *CD* subtend equal angles at *E*, a point on the other bank exactly opposite to *A*: shew that the breadth of the river is

$$\frac{AB^2}{\sqrt{(CD^2 - AB^2)}}.$$

32. The three segments *AB*, *BC*, *CD* of a straight line *AD* subtend respectively angles α, β, γ at a point *E* without the line. If $AB=a$, $CD=b$, shew that if $BC=x$ it is determined by the equation

$$x^2 + (a+b)\,x = ab\,\frac{\sin\alpha \sin(\alpha-\beta)}{\sin\gamma \sin(\beta-\gamma)}.$$

33. On the bank of a river there is a column 200 feet high, supporting a statue 30 feet in height: the statue, to an observer on the opposite bank, subtends the same angle as a man 6 feet high standing at the base of the column; shew that the breadth of the river is 107 feet nearly.

34. At one end of the level embankment at Portmadoc, the elevation of the top of Snowdon is observed to be $5^0\,25'$, and the angle subtended by its summit and the other extremity is $93^0\,22'$. At the other end of the embankment ($1\frac{1}{2}$ miles off) the angle subtended by the top and the first station is 75^0. Find the distance of Portmadoc from Snowdon, and the vertical height of Snowdon in feet.

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Ans. The height of Snowdon is ·678272 } miles.
Distance of Portmadoc is 7·153164 }

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Deiicit Herminium, nudo cui vertice fulva
Caesaries, nudique humeri; nec volnera terrent;
Tantus in arma patet. Latos huic hasta per armos
Acta tremit, duplicatque virum transfixa dolore.
Funditur ater ubique cruor; dant funera ferro
Certantes, pulchramque petunt per volnera mortem.
At medias inter caedes exsultat Amazon,
Unum exserta latus pugnae, pharetrata Camilla;
Et nunc lenta manu spargens hastilia denset,
Nunc validam dextra rapit indefessa bipennem;
Aureus ex humero sonat arcus et arma Dianae.
Illa etiam, si quando in tergum pulsa recessit,
Spicula converso fugientia dirigit arcu.
At circum lectae comites, Larinaque virgo
Tullaque et aeratam quatiens Tarpeia securim,
Italides, quas ipsa decus sibi dia Camilla
Delegit, pacisque bonas bellique ministras:
Quales Threiciae cum flumina Thermodontis
Pulsant et pictis bellantur Amazones armis,
Seu circum Hippolyten, seu cum se Martia curru
Penthesilea refert, magnoque ululante tumultu
Feminea exsultant lunatis agmina peltis.
Quem telo primum, quem postremum, aspera virgo,
Deiicis? aut quot humi morientia corpora fundis?
Euneum Clytio primum patre; cuius apertum
Adversi longa transverberat abiete pectus.

ὑμῖν, ἀπ' ἐμαυτοῦ οὐ λαλῶ· ὁ δὲ πατὴρ ὁ[16] ἐν ἐμοὶ
μένων, **αὐτὸς ποιεῖ τὰ ἔργα**[17] [18]. πιστεύετέ μοι ὅτι 11
ἐγὼ ἐν τῷ πατρί, καὶ ὁ πατὴρ ἐν ἐμοί[19]· εἰ δὲ μή,
διὰ τὰ ἔργα αὐτὰ πιστεύετέ **μοι**[20]. Ἀμὴν ἀμὴν λέγω 12
ὑμῖν, ὁ πιστεύων εἰς ἐμέ, τὰ ἔργα ἃ ἐγὼ ποιῶ, κἀ-
κεῖνος ποιήσει, καὶ μείζονα τούτων ποιήσει· ὅτι
ἐγὼ πρὸς τὸν πατέρα **μου**[21] πορεύομαι. καὶ ὅ τι ἂν 13
αἰτήσητε ἐν τῷ ὀνόματί μου, τοῦτο ποιήσω· ἵνα
δοξασθῇ ὁ πατὴρ ἐν τῷ υἱῷ. ἐάν τι αἰτήσητε[22] ἐν 14
τῷ ὀνόματί μου, **ἐγὼ**[23] ποιήσω.
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σατε. **καὶ ἐγὼ**[24] ἐρωτήσω τὸν πατέρα, καὶ ἄλλον 16
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αἰῶνα**[25], τὸ πνεῦμα τῆς ἀληθείας, ὃ ὁ κόσμος οὐ 17
δύναται λαβεῖν, ὅτι οὐ θεωρεῖ αὐτό, οὐδὲ γινώσκει
αὐτό[26]· ὑμεῖς **δὲ**[27] γινώσκετε αὐτό, ὅτι παρ' ὑμῖν
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ἐμοί, κἀγὼ ἐν ὑμῖν· ὁ ἔχων τὰς ἐντολάς μου καὶ 21
τηρῶν αὐτάς, ἐκεῖνός ἐστιν ὁ ἀγαπῶν με· ὁ δὲ ἀγα-
πῶν με, ἀγαπηθήσεται ὑπὸ τοῦ πατρός μου· **καὶ
ἐγὼ**[32] ἀγαπήσω αὐτόν, καὶ ἐμφανίσω αὐτῷ ἐμαυτόν."
Λέγει αὐτῷ Ἰούδας, οὐχ ὁ Ἰσκαριώτης, "Κύριε, [33]τί 22
γέγονεν ὅτι ἡμῖν μέλλεις ἐμφανίζειν σεαυτόν, καὶ
οὐχὶ τῷ κόσμῳ;" Ἀπεκρίθη ὁ[34] Ἰησοῦς καὶ εἶπεν 23
αὐτῷ, "Ἐάν τις ἀγαπᾷ με, τὸν λόγον μου τηρήσει,
καὶ ὁ πατήρ μου ἀγαπήσει αὐτόν, καὶ πρὸς αὐτὸν
ἐλευσόμεθα, καὶ μονὴν παρ' αὐτῷ **ποιήσομεν**[35]. ὁ μὴ 24

[16] –ὁ T[L]. [17] ποιεῖ τὰ ἔργα αὐτός T. [18] +[αὐτοῦ] L. [19] +ἐστιν BE. [20] –μοι T. [21] –μου LT. [22] +[με] L. [23] τοῦτο L *marg.* [24] κἀγὼ LT. [25] μεθ' ὑμῶν εἰς τὸν αἰῶνα ᾖ T, ᾖ μεθ' ὑμῶν εἰς τὸν αἰῶνα T. [26] [αὐτό] L. [27] –δὲ T[L]. [28] ἐστίν L. [29] οὐκέτι LT. [30] ζήσετε T. [31] [ὑμεῖς] γνώσεσθε L. [32] Κἀγὼ LT. [33] +καὶ T. [34] –ὁ LT. [35] ποιησόμεθα LT.

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