GEOMETRY. EUCLID. I-III

E. ATKINS.

COLLINS' SCHOOL SERIES

GEOMETRY.

Collins' School Series.

ELEMENTS OF GEOMETRY

BASED ON EUCLID.

BOOKS I.—III.

FOR

Elementary and Middle Class Schools.

BY

EDWARD ATKINS, B.Sc.
AUTHOR OF "PURE MATHEMATICS."

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GEOMETRY.

EUCLID'S ELEMENTS, BOOK I.

Definitions.

1. A **point** is that which has position, but not magnitude.

2. A **line** is length without breadth.

3. The **extremities of a line** are points.

4. A **straight line** is that which lies evenly between its extreme points.

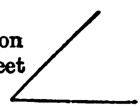
5. A **superficies** (or **surface**) is that which has only length and breadth.

6. The **extremities of a superficies** are lines.

7. A **plane superficies** is that in which any two points being taken, the straight line between them lies wholly in that superficies.

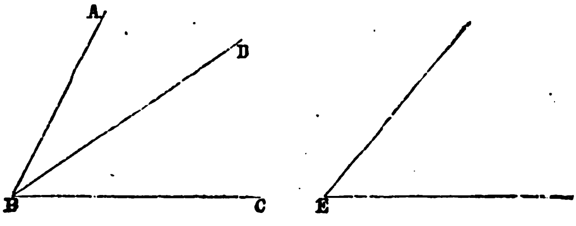
8. A **plane angle** is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

9. A **plane rectilineal angle** is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

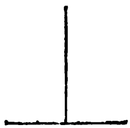


Note.—When several angles are at one point B, any one of them is expressed by three letters, of which the middle letter is B, and the first letter is on one of the straight lines which contain the angle, and the last letter on the other line.

Thus, the angle contained by the straight lines AB and BC is expressed either by ABC or CBA, and the angle contained by AB and

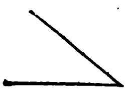


BD is expressed either by ABD or DBA. When there is only one angle at any given point, it may be expressed by the letter at that point, as the angle E.



10. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a **right angle;** and the straight line which stands on the other is called a **perpendicular** to it.

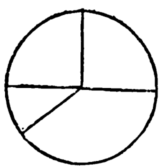
11. An **obtuse angle** is that which is greater than a right angle.



12. An **acute angle** is that which is less than a right angle.

13. A **term** or **boundary** is the extremity of anything.

14. A **figure** is that which is enclosed by one or more boundaries.



15. A **circle** is a plane figure contained by one line, which is called the **circumference,** and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

16. And this point is called the **centre** of the circle, [and any straight line drawn from the centre to the circumference is called a **radius** of the circle].

17. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

18. A **semicircle** is the figure contained by a diameter and the part of the circumference cut off by the diameter.

19. A **segment** of a circle is the figure contained by a straight line and the part of the circumference which it cuts off.

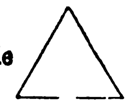
20. **Rectilineal figures** are those which are contained by straight lines.

21. **Trilateral figures, or triangles,** by three straight lines.

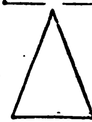
22. **Quadrilateral figures,** by four straight lines.

23. **Multilateral figures,** or **polygons,** by more than four straight lines.

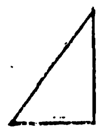
24. Of three-sided figures an **equilateral triangle** is that which has three equal sides.



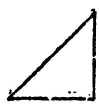
25. An **isosceles triangle** is that which has only two sides equal.



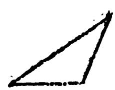
26. A **scalene triangle** is that which has three unequal sides.



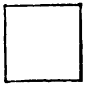
27. A **right-angled triangle** is that which has a right angle.



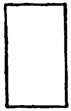
28. An **obtuse-angled triangle** is that which has an obtuse angle.



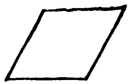
29. An **acute-angled triangle** is that which has three acute angles.



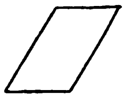
30. Of four-sided figures, a **square** is that which has all its sides equal, and all its angles right angles.



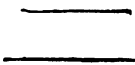
31. An **oblong** is that which has all its angles right angles, but not all its sides equal.



32. A **rhombus** is that which has all its sides equal, but its angles are not right angles.



33. A **rhomboid** is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.



34. **Parallel straight lines** are such as are in the same plane, and which being produced ever so far both ways do not meet.

35. A **parallelogram** is a four-sided figure of which the opposite sides are parallel; and the **diagonal** is the straight line joining two of its opposite angles. All other four-sided figures are called **trapeziums.**

Postulates.

1. Let it be granted that a straight line may be drawn from any one point to any other point.

2. That a terminated straight line may be produced to any length in a straight line.

3. And that a circle may be described from any centre, at any distance from that centre.

Axioms.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same are equal to one another.

7. Things which are halves of the same are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot inclose a space.

11. All right angles are equal to one another.

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced shall at length meet on that side on which are the angles which are less than two right angles.

Explanation of Terms and Abbreviations.

An **Axiom** is a truth admitted without demonstration.

A **Theorem** is a truth which is capable of being demonstrated from previously demonstrated or admitted truths.

A **Postulate** states a geometrical process, the power of effecting which is required to be admitted.

A **Problem** proposes to effect something by means of admitted processes, or by means of processes or constructions, the power of effecting which has been previously demonstrated.

A **Corollary** to a proposition is an inference which may be easily deduced from that proposition.

The sign = is used to express *equality*.

∠ means *angle*, and △ signifies *triangle*.

The sign $>$ signifies "is greater than," and $<$ "is less than."

+ expresses addition; thus AB + BC is the line whose length is the *sum* of the lengths of AB and BC.

− expresses subtraction; thus AB − BC is the excess of the length of the line AB above that of BC.

AB^2 means the square described upon the straight line AB.

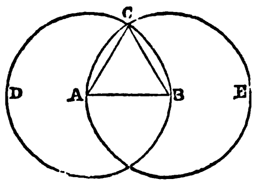
Proposition 1.—Problem.

To describe an equilateral triangle on a given finite straight line.

Let AB be the given straight line.

It is required to describe an equilateral triangle on AB.

From centres A and B, and radius = AB, describe circles.



CONSTRUCTION.—From the centre A, at the distance AB, describe the circle BCD (Post. 3).

From the centre B, at the distance BA, describe the circle ACE (Post. 3).

From the point C, in which the circles cut one another, draw the straight lines CA, CB to the points A and B (Post. 1).

Then ABC *shall be an equilateral triangle.*

AC=AB. PROOF.—Because the point A is the centre of the circle BCD, AC is equal to AB (Def. 15).

BC=AB. Because the point B is the centre of the circle ACE, BC is equal to BA (Def. 15).

AC and BC each=AB. Therefore AC and BC are each of them equal to AB.

AC=BC. But things which are equal to the same thing are equal to one another. Therefore AC is equal to BC (Ax. 1).

∴ AB=BC =CA. Therefore AB, BC, and CA are equal to one another.

Therefore the triangle ABC is equilateral, and it is described on the given straight line AB. *Which was to be done.*

Proposition 2.—Problem.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line.

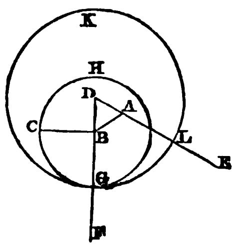
It is required to draw from the point A a straight line equal to BC.

CONSTRUCTION.—From the point A to B draw the straight line AB (Post. 1). Draw AB.

Upon AB describe the equilateral triangle DAB (Book I., Prop. 1). Δ DAB equilateral.

Produce the straight lines DA, DB, to E and F (Post. 2).

From the centre B, at the distance BC, describe the circle CGH, meeting DF in G (Post. 3). B as centre.



From the centre D, at the distance DG, describe the circle GKL, meeting DE in L (Post. 3). D as centre.

Then AL *shall be equal to* BC.

PROOF.—Because the point B is the centre of the circle CGH, BC is equal to BG (Def. 15). BC=BG.

Because the point D is the centre of the circle GKL, DL is equal to DG (Def. 15). DL=DG.

But DA, DB, parts of them, are equal (Construction). DA=DB.

Therefore the remainder AL is equal to the remainder BG (Ax. 3). ∴ AL = BG.

But it has been shown that BC is equal to BG.

Therefore AL and BC are each of them equal to BG. ∴ AL and BC each = BG.

But things which are equal to the same thing are equal to one another, therefore AL is equal to BC (Ax. 1).

Therefore from the given point A a straight line AL has been drawn equal to the given straight line BC. *Which was to be done.* ∴ AL = BC.

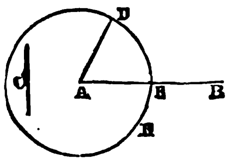
Proposition 3.—Problem.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater.

It is required to cut off from AB, the greater, a part equal to C, the less.

Make AD = C.



CONSTRUCTION.—From the point A draw the straight line AD equal to C (I. 2).

A as centre and radius AD.

From the centre A, at the distance AD, describe the circle DEF, cutting AB in E (Post. 3).

Then AE *shall be equal to* C.

PROOF.—Because the point A is the centre of the circle

AE=AD. DEF, AE is equal to AD (Def. 15).

AD=C. But C is also equal to AD (Construction).

AE and C each=AD. Therefore AE and C are each of them equal to AD.

∴ AE=C. Therefore AE is equal to C (Ax. 1).

Therefore, from AB, the greater of two given straight lines, a part AE has been cut off, equal to C, the less. *Q. E. F.**

Proposition 4.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another: they shall have their bases, or third sides, equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, viz., those to which the equal sides are opposite. Or,

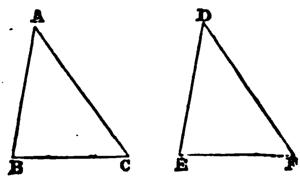
If two sides and the contained angle of one triangle be respectively equal to those of another, the triangles are equal in every respect.

Let ABC, DEF be two triangles which have

AB=DE. The two sides AB, AC, equal to the two sides DE, DF,

AC=DF. each to each, viz., AB equal to DE, and AC equal to DF.

∠ BAC = ∠ EDF. And the angle BAC equal to the angle EDF:—then—



The base BC shall be equal to the base EF;

The triangle ABC shall be equal to the triangle DEF;

* Q. E. F. is an abbreviation for *quod erat faciendum*, that is "*which was to be done.*"

And the other angles to which the equal sides are opposite, shall be equal, each to each, viz., the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

PROOF.—For if the triangle ABC be applied to (*or placed upon*) the triangle DEF, Suppose Δ ABC put upon Δ DEF.

So that the point A may be on the point D, and the straight line AB on the straight line DE,

The point B shall coincide with the point E, because AB is equal to DE (Hypothesis).

And AB coinciding with DE, AC shall coincide with DF, because the angle BAC is equal to the angle EDF (Hyp.).

Therefore also the point C shall coincide with the point F, because the straight line AC is equal to DF (Hyp.).

But the point B was proved to coincide with the point E.

Therefore the base BC shall coincide with the base EF.

Because the point B coinciding with E, and C with F, if the base BC do not coincide with the base EF, two straight lines would enclose a space, which is impossible (Ax. 10).

Therefore the base BC coincides with the base EF, and is therefore equal to it (Ax. 8). BC=EF.

Therefore the whole triangle ABC coincides with the whole triangle DEF, and is equal to it (Ax. 8). ∴ Δ ABC = Δ DEF.

And the other angles of the one coincide with the remaining angles of the other, and are equal to them, viz., the angle ABC to DEF, and the angle ACB to DFE. ∠ ABC = ∠ DEF. ∠ ACB = ∠ DFE.

Therefore, if two triangles have, &c. (see Enunciation). *Which was to be shown.*

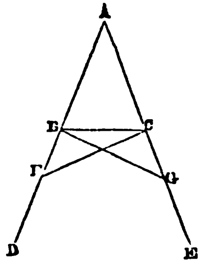
Proposition 5.—Theorem.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall also be equal.

Let ABC be an isosceles triangle, of which the side AB is equal to the side AC. AB = AC.

Let the straight lines AB, AC (*the equal sides of the triangle*), be produced to D and E.

The angle ABC shall be equal to the angle ACB (*angles at the base*), And the angle CBD shall be equal to the angle BCE (*angles upon the other side of the base*).



Construction.—In BD take any point F.

AG = AF.

From AE, the greater, cut off AG, equal to AF, the less (I. 3).

Join FC, GB.

Proof.—Because AF is equal to AG (Construction), and AB is equal to AC (Hyp.),

FA, AC respectively = GA, AB.

Therefore the two sides FA, AC are equal to the two sides GA, AB, each to each;

And they contain the angle FAG, common to the two triangles AFC, AGB.

∴ FC=GB and △ AFC = △ AGB.

Therefore the base FC is equal to the base GB (I. 4);

And the triangle AFC to the triangle AGB (I. 4);

∠ ACF = ∠ ABG. ∠ AFC = ∠ AGB.

And the remaining angles of the one are equal to the remaining angles of the other, each to each, to which the equal sides are opposite, viz., the angle ACF to the angle ABG, and the angle AFC to the angle AGB (I. 4).

And because the whole AF is equal to the whole AG, of which the parts AB, AC, are equal (Hyp.),

BF = CG.

The remainder BF is equal to the remainder CG (Ax. 3).

And FC was proved to be equal to GB;

Therefore the two sides BF, FC are equal to the two sides CG, GB, each to each.

And the angle BFC was proved equal to the angle CGB;

Therefore the triangles BFC, CGB are equal; and their other angles are equal, each to each, to which the equal sides are opposite (I. 4).

∴ ∠ FBC = ∠ GCB. ∠ BCF = ∠ CBG.

Therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG.

And since it has been demonstrated that the whole angle ABG is equal to the whole angle ACF, and that the parts of these, the angles CBG, BCF, are also equal,

∴ ∠ ABC = ∠ ACB.

Therefore the remaining angle ABC is equal to the remaining angle ACB (Ax. 3),

Which are the angles at the base of the triangle ABC.

And it has been proved that the angle FBC is equal to the angle GCB (Dem. 11),

Which are the angles upon the other side of the base,

Therefore the angles at the base, &c. (see Enunciation). *Which was to be shown.*

Corollary.—Hence every equilateral triangle is also equiangular.

Proposition 6.—Theorem.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB.

The side AB shall be equal to the side AC.

For if AB be not equal to AC, one of them is greater than the other. Let AB be the greater. Suppose AB > AC.

Construction.—From AB, the greater, cut off a part DB, equal to AC, the less (I. 3). Make DB = AC.

Join DC.

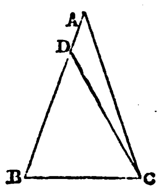
Proof.—Because in the triangles DBC, ACB, DB is equal to AC, and BC is common to both,

Therefore the two sides DB, BC are equal to the two sides AC, CB, each to each;

And the angle DBC is equal to the angle ACB (Hyp.)

Therefore the base DC is equal to the base AB (I. 4).

And the triangle DBC is equal to the triangle ACB (I. 4), the less to the greater, which is absurd. Δ DBC = Δ ACB.



Therefore AB is not unequal to AC, that is, it is equal to it.

Wherefore, if two angles, &c. *Q. E. D.* *

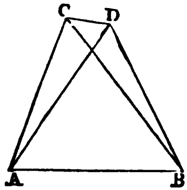
Corollary.—Hence every equiangular triangle is also equilateral.

* Q. E. D. is an abbreviation for *quod erat demonstrandum*, that is, "*which was to be shown or proved.*"

Proposition 7.—Theorem.

Upon the same base, and on the same side of it, there cannot be two triangles that have their sides, which are terminated in one extremity of the base, equal to one another, and likewise those which are terminated in the other extremity.

Let the triangles ACB, ADB, upon the same base AB, and on the same side of it, have, if possible,



Suppose CA = DA. Their sides CA, DA, terminated in the extremity A of the base, equal to one another;

CB = DB. And their sides CB, DB, terminated in the extremity B of the base, likewise equal to one another.

Case I.—Let the vertex of each triangle be without the other triangle.

Construction.—Join CD.

Proof.—Because AC is equal to AD (Hyp.),

∠ ACD = ∠ ADC. The triangle ADC is an isosceles triangle, and the angle ACD is therefore equal to the angle ADC (I. 5).

But the angle ACD is greater than the angle BCD (Ax. 9).

Therefore the angle ADC is also greater than BCD.

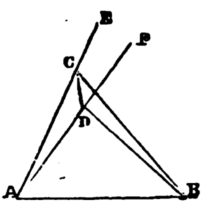
BDC > BCD. Much more then is the angle BDC greater than BCD.

Again, because BC is equal to BD (Hyp.),

∠ BDC = ∠ BCD. The triangle BCD is an isosceles triangle, and the angle BDC is equal to the angle BCD (I. 5).

But the angle BDC has been shown to be greater than the angle BCD (Dem. 5).

∠ BDC = and > ∠ BCD. Therefore the angle BDC is both equal to, and greater than the same angle BCD, which is impossible.



Case II.—Let the vertex of one of the triangles fall within the other.

Construction.—Produce AC, AD to E and F, and join CD.

Proof.—Because AC is equal to AD (Hyp.),

Again ∠ ECD = ∠ FDC. The triangle ADC is an isosceles triangle, and the angles ECD, FDC, upon the other side of its base CD, are equal to one another (I. 5),

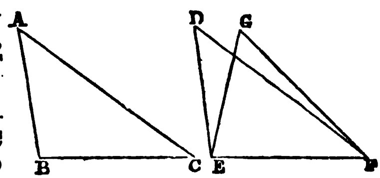
But the angle ECD is greater than the angle BCD (Ax. 9).
Therefore the angle FDC is likewise greater than BCD.
Much more then is the angle BDC greater than BCD. ∠ BDC > ∠ BCD.
Again, because BC is equal to BD (Hyp.),
The triangle BDC is an isosceles triangle, and the angle BDC is equal to the angle BCD (I. 5). ∠ BDC = ∠ BCD.
But the angle BDC has been shown to be greater than the angle BCD.
Therefore the angle BDC is both equal to, and greater than the same angle BCD, which is impossible. ∴ ∠ BDC = and > ∠ BCD.
Therefore, upon the same base, &c. *Q. E. D.*

Proposition 8.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides, equal to them, of the other. Or,

If two triangles have three sides of the one respectively equal to the three sides of the other, they are equal in every respect, those angles being equal which are opposite to the equal sides.

Let ABC, DEF be two triangles which have
The two sides AB, AC equal to the two sides DE, DF, each to each, viz., AB to DE, and AC to DF, Given AB = DE, AC = DF, and BC = EF.
And the base BC equal to the base EF.
The angle BAC shall be equal to the angle EDF.
Proof.—For if the triangle ABC be applied to the triangle DEF,
So that the point B may be on E, and the straight line BC on EF,
The point C shall coincide with the point F, because BC is equal to EF (Hyp.). Make BC coincide with EF.



Therefore, BC coinciding with EF, BA and AC shall coincide with ED and DF.
For if the base BC coincides with the base EF,

But the sides BA, AC, do not coincide with the sides ED, DF, but have a different situation, as EG, GF,

Then upon the same base, and on the same side of it, there will be two triangles, which have their sides terminated in one extremity of the base equal to one another, and likewise their sides, which are terminated in the other extremity. But this is impossible (I. 7).

∴ BA, AC respectively coincide with ED, DF.

Therefore, if the base BC coincides with the base EF, the sides BA, AC must coincide with the sides ED, DF.

Therefore the angle BAC coincides with the angle EDF, and is equal to it (Ax. 8).

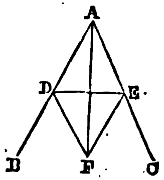
Also the triangle ABC coincides with the triangle DEF and is therefore equal to it in every respect (Ax. 8).

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 9.—Problem.

To bisect a given rectilineal angle, that is, to divide it into two equal parts.

Let BAC be the given rectilineal angle.



It is required to bisect it.

Construction.—Take any point D in AB.

Make AE = AD,

From AC cut off AE equal to AD (I. 3).

Join DE.

Δ DEF equilateral.

Upon DE, on the side remote from A, describe an equilateral triangle DEF (I. 1).

Join AF.

Then the straight line AF *shall bisect the angle* BAC.

Proof.—Because AD is equal to AE (Const.), and AF is common to the two triangles DAF, EAF;

The two sides DA, AF are equal to the two sides EA, AF, each to each;

And the base DF is equal to the base EF (Const.);

∴ ∠ DAF = ∠ EAF.

Therefore the angle DAF is equal to the angle EAF (I. 8).

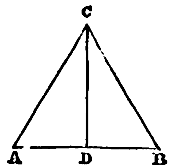
Therefore the given rectilineal angle BAC is bisected by the straight line AF. *Q. E. F.*

Proposition 10.—Problem.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given straight line.

It is required to divide it into two equal parts.



Make Δ ABC equilateral and ∠ ACD = ∠ BCD.

CONSTRUCTION.—Upon AB describe the equilateral triangle ABC (I. 1).

Bisect the angle ACB by the straight line CD (I. 9).

Then AB *shall be cut into two equal parts in the point* D.

PROOF.—Because AC is equal to CB (Const.), and CD common to the two triangles ACD, BCD;

The two sides AC, CD are equal to the two sides BC, CD, each to each;

And the angle ACD is equal to the angle BCD (Const.);

Therefore the base AD is equal to the base DB (I. 4). ∴ AD = DB.

Therefore the straight line AB is divided into two equal parts in the point D. *Q. E. F.*

Proposition 11.—Problem.

To draw a straight line at right angles to a given straight line from a given point in the same.

Let AB be the given straight line, and C a given point in it.

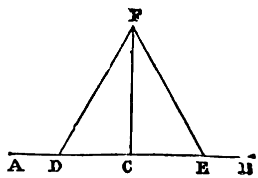
It is required to draw a straight line from the point C at right angles to AB.

CONSTRUCTION.—Take any point D in AC.

Make CE equal to CD (I. 3).

Upon DE describe the equilateral triangle DFE (I. 1). Make CE = CD and Δ DEF equilateral.

Join FC.



Then FC *shall be at right angles to* AB.

PROOF.—Because DC is equal to CE (Const.), and FC common to the two triangles DCF, ECF;

The two sides DC, CF, are equal to the two sides EC, CF, each to each;

And the base DF is equal to the base EF (Const.);

Therefore the angle DCF is equal to the angle ECF (I. 8); ∠ DCF = ∠ ECF.

And they are adjacent angles.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle (Def. 10);

∴ ∠ DCF, ∠ ECF are right angles.

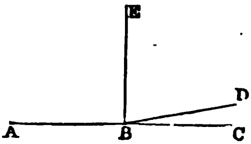
Therefore each of the angles DCF, ECF is a right angle.

Therefore from the given point C in the given straight line AB, a straight line FC has been drawn at right angles to AB. *Q. E. F.*

Corollary.—By help of this problem, it may be demonstrated that

Two straight lines cannot have a common segment.

If it be possible, let the two straight lines ABC, ABD, have the segment AB common to both of them.



Make ∠ ABE a right ∠

Construction.—From the point B, draw BE at right angles to AB (I. 11).

∠ CBE = ∠ EBA.

Proof.—Because ABC is a straight line, the angle CBE is equal to the angle EBA (Def. 10).

and ∠ DBE = ∠ EBA.

Also, because ABD is a straight line, the angle DBE is equal to the angle EBA (Def. 10).

∴ ∠ DBE = ∠ CBE.

Therefore the angle DBE is equal to the angle CBE. The less to the greater; which is impossible.

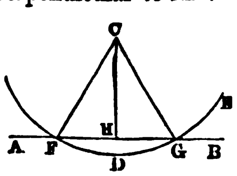
Therefore two straight lines cannot have a common segment.

Proposition 12.—Problem.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced to any length both ways, and let C be a point without it.

It is required to draw from the point C, a straight line perpendicular to AB.



Construction.—Take any point D upon the other side of AB.

CD as radius.

From the centre C, at the distance CD, describe the circle EGF, meeting AB in F and G (Post. 3).

Bisect FG in H.

Bisect FG in H (I. 10).

Join CF, CH, CG.

Then CH *shall be perpendicular to* AB.

Proof.—Because FH is equal to HG Const.), and HC common to the two triangles FHC, GHC.

The two sides FH, HC are equal to the two sides GH, HC, each to each;

And the base CF is equal to the base CG (Def. 15);

Therefore the angle CHF is equal to the angle CHG (I. 8), and they are adjacent angles.

∴ adjacent angles CHF, CHG are equal.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it (Def. 10).

Therefore, from the given point C, a perpendicular has been drawn to the given straight line AB. *Q. E. F.*

Proposition 13.—Theorem.

The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD.

These angles shall either be two right angles, or shall together be equal to two right angles.

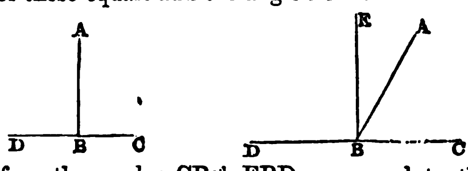
PROOF.—If the angle CBA be equal to the angle ABD, each of them is a right angle (Def. 10).

But if the angle CBA be not equal to the angle ABD, from the point B, draw BE at right angles to CD (I. 11).

Therefore the angles CBE, EBD, are two right angles.

Now the angle CBE is equal to the two angles CBA, ABE; to each of these equals add the angle EBD.

Make ∠ CBE = ∠ EBD = a right ∠.



Therefore the angles CBE, EBD, are equal to the three angles CBA, ABE, EBD (Ax. 2).

Again, the angle DBA is equal to the two angles DBE, EBA; to each of these equals add the angle ABC.

Therefore the angles DBA, ABC, are equal to the three angles DBE, EBA, ABC (Ax. 2).

∴ ∠ CBE + ∠ EBD = ∠ CBA + ∠ ABE + ∠ EBD, also ∠ DBA + ∠ ABC = ∠ DBE + ∠ EBA + ∠ ABC.

But the angles CBE, EBD have been shown to be equal to the same three angles;

And things which are equal to the same thing are equal to one another;

∴ ∠ CBE + ∠ EBD = ∠ DBA + ∠ ABC.

Therefore the angles CBE, EBD, are equal to the angles DBA, ABC (Ax. 1).

But the angles CBE, EBD are two right angles.

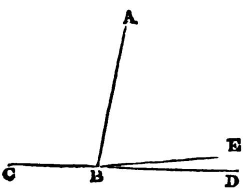
Therefore the angles DBA, ABC, are together equal to two right angles (Ax. 1).

Therefore, the angles which one straight line, &c. *Q. E. D.*

Proposition 14.—Theorem.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

Given ∠ ABC + ∠ ABD = two right angles.



At the point B in the straight line AB, let the two straight lines BC, BD, upon the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles.

BD shall be in the same straight line with BC.

If possible, let CBE be a straight line.

For if BD be not in the same straight line with BC, let BE be in the same straight line with it.

PROOF.—Because CBE is a straight line, and AB meets it in B.

Therefore the adjacent angles ABC, ABE are together equal to two right angles (I. 13).

But the angles ABC, ABD, are also together equal to two right angles (Hyp.);

Therefore the angles ABC, ABE, are equal to the angles ABC, ABD (Ax. 1).

Take away the common angle ABC.

∴ ∠ ABE = ∠ ABD.

The remaining angle ABE is equal to the remaining angle ABD (Ax. 3), the less to the greater, which is impossible;

Therefore BE is not in the same straight line with BC.

And, in like manner, it may be demonstrated that no other can be in the same straight line with it but BD.

Therefore BD is in the same straight line with BC.

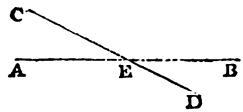
Therefore, if at a point, &c. *Q. E. D.*

Proposition 15.—Theorem.

If two straight lines cut one another, the vertical, or opposite angles shall be equal.

Let the two straight lines AB, CD cut one another in the point E.

The angle AEC shall be equal to angle DEB, and the angle CEB to the angle AED.



PROOF.—Because the straight line AE makes with CD, the angles CEA, AED, these angles are together equal to two right angles (I. 13). ∠ CEA + ∠ AED = 2 right angles.

Again, because the straight line DE makes with AB the angles AED, DEB, these also are together equal to two right angles (I. 13). ∠ AED + ∠ DEB = 2 right angles.

But the angles CEA, AED have been shown to be together equal to two right angles,

Therefore the angles CEA, AED are equal to the angles AED, DEB (Ax. 1).

Take away the common angle AED.

The remaining angle CEA is equal to the remaining angle DEB (Ax. 3). ∴ ∠ CEA = ∠ DEB.

In the same manner it can be shown that the angles CEB, AED are equal.

Therefore, if two straight lines, &c. *Q. E. D.*

COROLLARY 1.—From this it is manifest that if two straight lines cut one another, the angles which they make at the point where they cut, are together equal to four right angles.

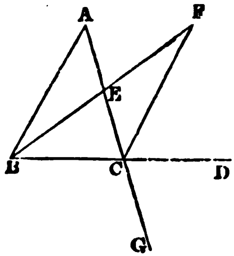
COROLLARY 2.—And, consequently, that all the angles made by any number of lines meeting in one point are together equal to four right angles, provided that no one of the angles be included in any other angle.

Proposition 16.—Theorem.

If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

Let ABC be a triangle, and let its side BC be produced to D.

The exterior angle ACD shall be greater than either of the interior opposite angles CBA, BAC.



Make AE = EC. and EF = BE.

CONSTRUCTION.—Bisect AC in E (I. 10).

Join BE, and produce it to F, making EF equal to BE (I. 3), and join FC

PROOF.—Because AE is equal to EC, and BE equal to EF (Const.),

AE, EB are equal to CE, EF, each to each ;

And the angle AEB is equal to the angle CEF, because they are opposite vertical angles (I. 15).

Therefore the base AB is equal to the base CF (I. 4);

And the triangle AEB to the triangle CEF (I. 4);

And the remaining angles to the remaining angles, each to each, to which the equal sides are opposite.

∴ ∠BAE = ∠ECF.

Therefore the angle BAE is equal to the angle ECF (I. 4).

But the angle ECD is greater than the angle ECF (Ax. 9);

Therefore the angle ACD is greater than the angle BAE.

∴ ∠ACD > ∠BAE.

In the same manner, if BC be bisected, and the side AC be produced to G, it may be proved that the angle BCG (or its equal ACD), is greater than the angle ABC.

Therefore, if one side, &c. *Q. E. D.*

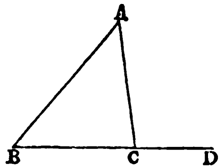
Proposition 17.—Theorem.

Any two angles of a triangle are together less than two right angles.

Let ABC be any triangle.

Any two of its angles together shall be less than two right angles.

CONSTRUCTION.—Produce BC to D.



PROOF.—Because ACD is the exterior angle of the triangle ABC, it is greater than the interior and opposite angle ABC (I. 16). ∠ ACD > ∠ ABC.

To each of these add the angle ACB. Add to each ∠ ACB.

Therefore the angles ACD, ACB are greater than the angles ABC, ACB (Ax. 4). ∠ ABC + ∠ ACB < 2 right angles.

But the angles ACD, ACB are together equal to two right angles (I. 13);

Therefore the angles ABC, ACB are together less than two right angles.

In like manner, it may be proved that the angles BAC, ACB, as also the angles CAB, ABC are together less than two right angles.

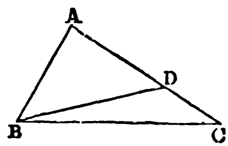
Therefore, any two angles, &c. *Q. E. D.*

Proposition 18.—Theorem.

The greater side of every triangle is opposite the greater angle.

Let ABC be a triangle, of which the side AC is greater than the side AB. AC > AB.

The angle ABC shall be greater than the angle BCA.



CONSTRUCTION.—Because AC is greater than AB, make AD equal to AB (I. 3), and join BD. Make AD = AB.

PROOF.—Because ADB is the exterior angle of the triangle BDC, it is greater than the interior and opposite angle BCD (I. 16). ∠ ADB > ∠ BCD, and

But the angle ADB is equal to the angle ABD; the triangle BAD being isosceles (I. 5), ∠ ADB = ∠ ABD and

Therefore the angle ABD is greater than the angle BCD (*or* ACB). ∴ ∠ ABD > ∠ BCD.

Much more then is the angle ABC greater than the angle ACB.

Therefore, the greater side, &c. *Q. E. D.*

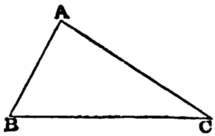
Proposition 19.—Theorem.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Given ∠ ABC > ∠ BCA.

Let ABC be a triangle, of which the angle ABC is greater than the angle BCA;

The side AC shall be greater than the side AB.



Proof.—If AC be not greater than AB, it must either be equal to or less than AB.

It is not equal, for then the angle ABC would be equal to the angle BCA (I. 5); but it is not (Hyp.);

AC not = AB.

Therefore AC is not equal to AB.

Neither is AC less than AB, for then the angle ABC would be less than the angle BCA (I. 18); but it is not (Hyp.);

AC not < AB.

Therefore AC is not less than AB.

And it has been proved that AC is not equal to AB;

Therefore AC is greater than AB.

Therefore, the greater angle, &c. *Q. E. D.*

Proposition 20.—Theorem.

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle;

Any two sides of it are together greater than the third side.

Make AD = AC.

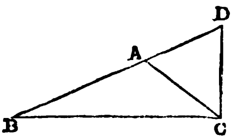
Construction.—Produce BA to the point D, making AD equal to AC (I. 3), and join DC.

Proof.—Because DA is equal to AC, the angle ADC is equal to the angle ACD (I. 5).

But the angle BCD is greater than the angle ACD (Ax. 9);

∠ BCD > ∠ BDC.

Therefore the angle BCD is greater than the angle ADC (*or* BDC).



And because the angle BCD of the triangle DCB is greater than its angle BDC, and that the greater angle is subtended by the greater side;

∴ DB > BC.

Therefore the side DB is greater than the side BC (I. 19).

But BD is equal to BA and AC;
Therefore BA, AC are greater than BC. ∴ BA + AC > BC.
In the same manner it may be proved that AB, BC are greater than AC; and BC, CA greater than AB.
Therefore any two sides, &c. *Q. E. D.*

Proposition 21.—Theorem.

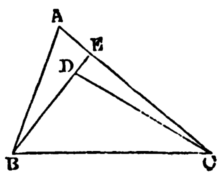
If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be a triangle, and from the points B, C, the ends of the side BC, let the two straight lines BD, CD be drawn to the point D within the triangle;

BD, DC shall be less than the sides BA, AC;

But BD, DC shall contain an angle BDC greater than the angle BAC.

Construction.—Produce BD to E.



Proof.—1. Because two sides of a triangle are greater than the third side (I. 20), the two sides BA, AE, of the triangle BAE are greater than BE.

To each of these add EC.

Therefore the sides BA, AC, are greater than BE, EC (Ax. 4). BA + AC > BE + EC.

Again, because the two sides CE, ED, of the triangle CED are greater than CD (I. 20),

To each of these add DB.

Therefore CE, EB are greater than CD, DB (Ax. 4).

But it has been shown that BA, AC are greater than BE, EC; ∴ EC + EB > CD + DB.

Much more then are BA, AC greater than BD, DC.

Proof. — 2. Again, because the exterior angle of a triangle is greater than the interior and opposite angle (I. 16), therefore BDC, the exterior angle of the triangle CDE, is greater than CED or CEB. Again ∠ BDC > ∠ CEB,

For the same reason, CEB, the exterior angle of the triangle ABE, is greater than the angle BAE or BAC. and ∠ CEB > ∠ BAE.

And it has been shown that the angle BDC is greater than CEB;

∴ ∠ BDC > ∠ BAC. Much more then is the angle BDC greater than the angle BAC.

Therefore, if from the ends, &c. *Q. E. D.*

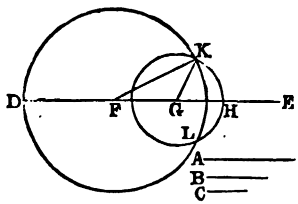
Proposition 22.—Problem.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these lines must be greater than the third (I. 20).

Let A, B, C be the three given straight lines, of which any two whatever are greater than the third—namely, A and B greater than C, A and C greater than B, and B and C greater than A;

It is required to make a triangle of which the sides shall be equal to A, B, and C, each to each.

CONSTRUCTION.—Take a straight line DE terminated at the point D, but unlimited towards E.



DF, FG, GH respectively = A, B, C. Make DF equal to A, FG equal to B, and GH equal to C (I. 3).

FD as radius. From the centre F, at the distance FD, describe the circle DKL (Post. 3).

and GH as radius. From the centre G, at the distance GH, describe the circle HLK (Post. 3).

Join KF, KG.

Then the triangle KFG *shall have its sides equal to the three straight lines* A, B, C.

PROOF.—Because the point F is the centre of the circle DKL, FD is equal to FK (Def. 15).

But FD is equal to A (Const.);

FK = A. Therefore FK is equal to A (Ax. 1).

Again, because the point G is the centre of the circle HLK, GH is equal to GK (Def. 15).

But GH is equal to C (Const.);

GK = C. Therefore GK is equal to C (Ax. 1), And FG is equal to B (Const.); FG = B

Therefore the three straight lines KF, FG, GK are equal to the three A, B, C, each to each.

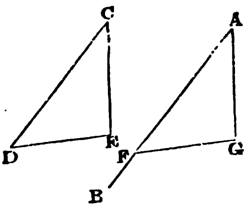
Therefore the triangle KFG has its three sides KF, FG, GK equal to the three given straight lines A, B, C. *Q. E. F.*

Proposition 23.—Problem.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle.

It is required to take an angle at the point A, in the straight line AB, equal to the rectilineal angle DCE.



Construction. — In CD, CE, take any points D, E, and join DE.

On AB construct a triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, EC — namely, AF equal to CD, FG to DE, and AG to EC (I. 22);

Make △ AFG so that AF = CD FG = DE AG = CE.

Then the angle FAG *shall be equal to the angle* DCE.

Proof.—Because DC, CE are equal to FA, AG, each to each, and the base DE equal to the base FG (Const.),

The angle DCE is equal to the angle FAG (I. 8). Then ∠ DCE = ∠ FAG

Therefore, at the given point A, in the given straight line AB, the angle FAG has been made equal to the given rectilineal angle DCE. *Q. E. F.*

Proposition 24.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle shall be greater than the base of the other.

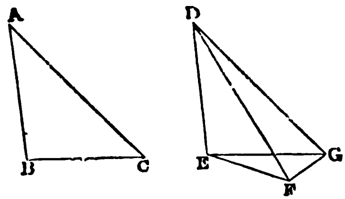
Let ABC, DEF, be two triangles which have

The two sides AB, AC equal to the two DE, DF, each to each—namely, AB to DE,and AC to DF,

But the angle BAC greater than the angle EDF;

The base BC shall be greater than the base EF.

Suppose DF > DE. CONSTRUCTION.—Let the side DF of the triangle DEF be greater than its side DE.



Make ∠ EDG = ∠ BAC. Then at the point D, in the straight line ED, make the angle EDG equal to the angle BAC (I. 23).

Make DG = AC, and = DF. Make DG equal to AC or DF (I. 3).

Join EG, GF.

PROOF.—Because AB is equal to DE (Hyp.), and AC to DG (Const.), the two sides BA, AC are equal to the two ED, DG, each to each;

And the angle BAC is equal to the angle EDG (Const.);

∴BC=EG. Therefore the base BC is equal to the base EG (I. 4).

And because DG is equal to DF (Const.), the angle DFG is equal to the angle DGF (I. 5).

But the angle DGF is greater than the angle EGF (Ax. 9);

and ∠ EFG > ∠ EGF. Therefore the angle DFG is greater than the angle EGF;

Much more then is the angle EFG greater than the angle EGF.

And because the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater angle is subtended by the greater side,

∴EG>EF. Therefore the side EG is greater than the side EF (I. 19).

But EG was proved equal to BC;

Therefore BC is greater than EF.

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 25.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base shall be greater than the angle contained by the sides equal to them of the other.

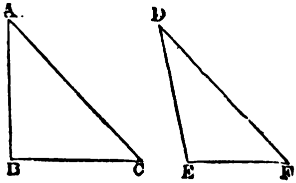
Let ABC, DEF, be two triangles, which have

The two sides AB, AC equal to the two sides DE, DF, each to each—namely, AB to DE, and AC to DF,

But the base BC greater than the base EF;

The angle BAC shall be greater than the angle EDF.

PROOF.—For if the angle BAC be not greater than the angle EDF, it must either be equal to it or less.



But the angle BAC is not equal to the angle EDF, for then the base BC would be equal to the base EF (I. 4), but it is not (Hyp.);

Therefore the angle BAC is not equal to the angle EDF; ∠ BAC not = ∠ EDF.

Neither is the angle BAC less than the angle EDF, for then the base BC would be less than the base EF (I. 24), but it is not (Hyp.),

Therefore the angle BAC is not less than the angle EDF. ∠ BAC not < ∠ EDF.

And it has been proved that the angle BAC is not equal to the angle EDF;

Therefore the angle BAC is greater than the angle EDF.

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 26.—Theorem.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side—namely, either the side adjacent to the equal angles in each, or the side opposite to them; then shall the other sides be equal, each to each; and also the third angle of the one equal to the third angle of the other. Or,

If two angles and a side in one triangle be respectively equal to two angles and a corresponding side in another triangle, the triangles shall be equal in every respect.

Let ABC, DEF be two triangles, which have

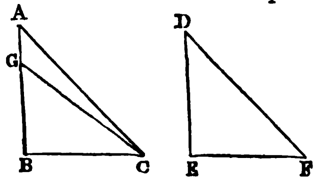
The angles ABC, BCA equal to the angles DEF, EFD, each to each—namely, ABC to DEF, and BCA to EFD;

Also one side equal to one side.

CASE 1.—First, let the sides adjacent to the equal angles in each be equal—namely, BC to EF; Given BC = EF.

Then shall the side AB be equal to DE, the side AC to DF, and the angle BAC to the angle EDF.

Suppose AB > DE. For if AB be not equal to DE, one of them must be greater than the other. Let AB be the greater of the two.



Make BG = DE. CONSTRUCTION.—Make BG equal to DE (I. 3), and join GC.

PROOF.—Because BG is equal to DE (Const.), and BC is equal to EF (Hyp.), the two sides GB, BC are equal to the two sides DE, EF, each to each.

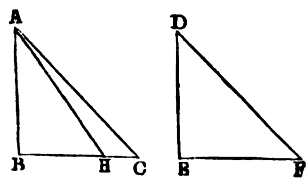
And the angle GBC is equal to the angle DEF (Hyp.);
Therefore the base GC is equal to the base DF (I. 4),
And the triangle GBC to the triangle DEF (I. 4),
And the other angles to the other angles, each to each, to which the equal sides are opposite;

∴ ∠ GCB = ∠ DFE. Therefore the angle GCB is equal to the angle DFE (I. 4).
But the angle DFE is equal to the angle BCA (Hyp.);
Therefore the angle GCB is equal to the angle BCA (Ax. 1), the less to the greater, which is impossible;

AB not unequal to DE. Therefore AB is not unequal to DE, that is, it is equal to it; and BC is equal to EF (Hyp.);
Therefore the two sides AB, BC are equal to the two sides DE, EF, each to each,
And the angle ABC is equal to the angle DEF (Hyp.);
Therefore the base AC is equal to the base DF (I. 4),
And the third angle BAC to the third angle EDF (I. 4).

CASE 2.—Next, let the sides which are opposite to the equal angles in each triangle be equal to one another—namely, AB equal to DE.

Likewise in this case the other sides shall be equal, AC to DF, and BC to EF; and also the angle BAC to the angle EDF.



Suppose BC > EF. For if BC be not equal to EF, one of them must be greater than the other. Let BC be the greater of the two.

Make BH = EF. CONSTRUCTION.—Make BH equal to EF (I. 3), and join AH.

PROOF.—Because BH is equal to EF (Const.), and AB is equal to DE (Hyp.), the two sides AB, BH are equal to the two sides DE, EF, each to each, BH = EF, AB = DE.

And the angle ABH is equal to the angle DEF (Hyp.); ∠ ABH = ∠ DEF.

Therefore the base AH is equal to the base DF (I. 4),

And the triangle ABH to the triangle DEF (I. 4),

And the other angles to the other angles, each to each, to which the sides are opposite;

Therefore the angle BHA is equal to the angle EFD (I. 4). ∴ ∠ BHA = ∠ EFD = ∠ BCA.

But the angle EFD is equal to the angle BCA (Hyp.);

Therefore the angle BHA is also equal to the angle BCA (Ax. 1);

That is, the exterior angle BHA of the triangle AHC, is equal to its interior and opposite angle BCA, which is impossible (I. 16);

Therefore BC is not unequal to EF—that is, it is equal to it; and AB is equal to DE (Hyp.); BC not unequal to EF.

Therefore the two sides AB, BC are equal to the two sides DE, EF, each to each,

And the angle ABC is equal to the angle DEF (Hyp.);

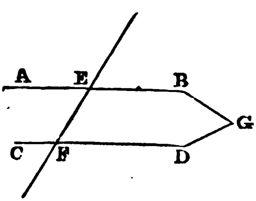
Therefore the base AC is equal to the base DF (I. 4),

And the third angle BAC is equal to the third angle EDF (I. 4).

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 27.—Theorem.

If a straight line falling upon two other straight lines make the alternate angles equal to one another, these two straight lines shall be parallel.



Let the straight line EF, which falls upon the two straight lines AB, CD, make the alternate angles AEF, EFD, equal to one another. Given ∠ AEF = ∠ EFD.

AB shall be parallel to CD.

For if AB and CD be not parallel, they will meet if produced, either towards B, D, or towards A, C. Let them be produced, and meet towards B, D, in the point G.

∠ AEF > ∠ EFG, and also = ∠ EFG.

PROOF.—Then GEF is a triangle, and its exterior angle AEF is greater than the interior and opposite angle EFG (I. 16).

But the angle AEF is also equal to EFG (Hyp.), which is impossible;

Therefore AB and CD, being produced, do not meet towards B, D.

In like manner it may be shown that they do not meet towards A, C.

But those straight lines in the same plane which being produced ever so far both ways do not meet are parallel (Def. 34);

Therefore AB is parallel to CB.

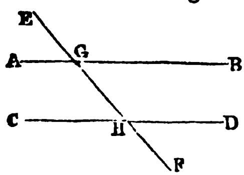
Therefore, if a straight line, &c. *Q. E. D.*

Proposition 28.—Theorem.

If a straight line falling upon two other straight lines make the exterior angle equal to the interior and opposite upon the same side of the line, or make the interior angles upon the same side together equal to two right angles, the two straight lines shall be parallel to one another.

Let the straight line EF, which falls upon the two straight lines AB, CD, make—

The exterior angle EGB equal to the interior and opposite angle GHD, upon the same side;



Or make the interior angles on the same side, the angles BGH, GHD, together equal to two right angles;

AB shall be parallel to CD.

PROOF 1.—Because the angle EGB is equal to the angle GHD (Hyp.),

And the angle EGB is equal to the angle AGH (I. 15);

∠ AGH = ∠ GHD.

Therefore the angle AGH is equal to the angle GHD (Ax. 1), and these angles are alternate;

Therefore AB is parallel to CD (I. 27).

PROOF 2.—Again, because the angles BGH, GHD are equal to two right angles (Hyp.),

And the angles BGH, AGH are also equal to two right angles (I. 13).

Therefore the angles BGH, AGH are equal to the angles BGH, GHD (Ax. 1). ∴ ∠ BGH + ∠ AGH = ∠ BGH + ∠ GHD.

Take away the common angle BGH.

Therefore the remaining angle AGH is equal to the remaining angle GHD (Ax. 3), and they are alternate angles. ∴ ∠ AGH = ∠ GHD.

Therefore AB is parallel to CD (I. 27).

Therefore, if a straight line, &c. *Q. E. D.*

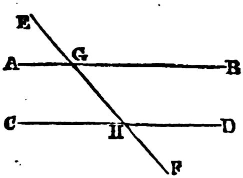
Proposition 29.—Theorem.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite upon the same side; and also the two interior angles upon the same side together equal to two right angles.

Let the straight line EF fall upon the parallel straight lines AB, CD;

The alternate angles AGH, GHD shall be equal to one another.

The exterior angle EGB shall be equal to GHD, the interior and opposite angle upon the same side;



And the two interior angles on the same side BGH, GHD shall be together equal to two right angles.

For if AGH be not equal to GHD, one of them must be greater than the other. Let AGH be the greater. ∠ AGH > GHD. (suppose.)

PROOF.—Then the angle AGH is greater than the angle GHD; to each of them add the angle BGH.

Therefore the angles BGH, AGH are greater than the angles BGH, GHD (Ax. 4).

But the angles BGH, AGH are together equal to two right angles (I. 3).

Therefore the angles BGH, GHD are less than two right angles. ∴ ∠ BGH + ∠ GHD <two right angles.

But if a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced, shall at length meet on that side on which are the angles which are less than two right angles (Ax. 12);

Hence AB and CD meet, and are parallel.

Therefore the straight lines AB, CD will meet if produced far enough.

But they cannot meet, because they are parallel straight lines (Hyp.);

∴ ∠ AGH, not unequal to ∠ GHD.

Therefore the angle AGH is not unequal to the angle GHD—that is, it is equal to it.

But the angle AGH is equal to the angle EGB (I. 15);

and ∠ EGB = ∠ GHD,

Therefore the angle EGB is equal to the angle GHD (Ax. 1).

Add to each of these the angle BGH.

Therefore the angles EGB, BGH, are equal to the angles BGH, GHD (Ax. 2).

But the angles EGB, BGH, are equal to two right angles (I. 13).

also ∠ BGH + ∠ GHD = two right angles.

Therefore also BGH, GHD, are equal to two right angles (Ax. 1).

Therefore, if a straight line, &c. *Q. E. D.*

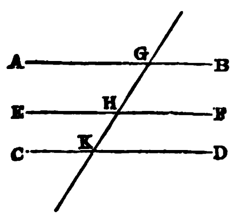
Proposition 30.—Theorem.

Straight lines which are parallel to the same straight lines are parallel to one another.

Let AB, CD be each of them parallel to EF;

AB shall be parallel to CD.

Construction.—Let the straight line GHK cut AB, EF, CD.



∠ AGH or ∠ AGK = ∠ GHF,

Proof.—Because GHK cuts the parallel straight lines AB, EF, the angle AGH is equal to the angle GHF (I. 29).

and ∠ GHF = ∠ GKD.

Again, because GK cuts the parallel straight lines EF, CD, the angle GHF is equal to the angle GKD (I. 29).

And it was shown that the angle AGK is equal to the angle GHF;

∴ ∠ AGK = ∠ GKD.

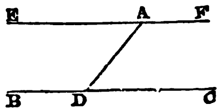
Therefore the angle AGK is equal to the angle GKD (Ax. 1), and they are alternate angles;

Therefore AB is parallel to CD (I. 27).
Therefore, straight lines, &c. *Q. E. D.*

Proposition 31.—Problem.

To draw a straight line through a given point, parallel to a given straight line.

Let A be the given point, and BC the given straight line.
It is required to draw a straight line through the point A, parallel to BC.

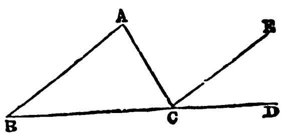


CONSTRUCTION.—In BC take any point D, and join AD.
At the point A, in the straight line AD, make the angle DAE equal to the angle ADC (I. 23). Make ∠ DAE = ∠ ADC.
Produce the straight line EA to F.
Then EF *shall be parallel to* BC.
PROOF.—Because the straight line AD, which meets the two straight lines BC, EF, makes the alternate angles EAD, ADC equal to one another; They are alternate angles.
Therefore EF is parallel to BC (I. 27).
Therefore, the straight line EAF is drawn through the given point A, parallel to the given straight line BC. *Q. E. F.*

Proposition 32.—Theorem.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, and let one of its sides BC be produced to D;
The exterior angle ACD shall be equal to the two interior and opposite angles CAB, ABC;
And the three interior angles of the triangle—namely, ABC, BCA, CAB, shall be equal to two right angles.



CONSTRUCTION.—Through the point C, draw CE parallel to AB (I. 31). Make CE parallel to AB.

Then ∠ BAC = ∠ ACE, and ∠ ECD = ∠ ABC.

PROOF.—Because AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are equal (I. 29).

Again, because AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and opposite angle ABC (I. 29).

But the angle ACE was shown to be equal to the angle BAC;

∴ ∠ ACD = ∠ BAC + ∠ ABC. Add ∠ ACB.

Therefore the whole exterior angle ACD is equal to the two interior and opposite angles BAC, ABC (Ax. 2).

To each of these equals add the angle ACB.

Therefore the angles ACD, ACB are equal to the three angles CBA, BAC, ACB (Ax. 2).

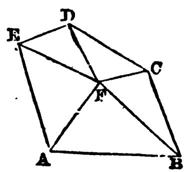
But the angles ACD, ACB are equal to two right angles (I. 13);

Therefore also the angles CBA, BAC, ACB are equal to two right angles (Ax. 1).

Therefore, if a side of any triangle, &c. *Q. E. D.*

COROLLARY 1.—*All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*

For any rectilineal figure ABCDE can, by drawing straight lines from a point F within the figure to each angle, be divided into as many triangles as the figure has sides.



And, by the preceding proposition, the angles of each triangle are equal to two right angles.

Therefore all the angles of the triangles are equal to twice as many right angles as there are triangles; that is, as there are sides of the figure.

But the same angles are equal to the angles of the figure, together with the angles at the point F;

And the angles at the point F, which is the common vertex of all the triangles, are equal to four right angles (I. 15, Cor. 2);

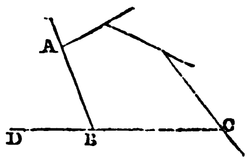
Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Corollary 2.—*All the exterior angles of any rectilineal figure are together equal to four right angles.*

The interior angle ABC, with its adjacent exterior angle ABD, is equal to two right angles (I. 13);

Therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as the figure has sides.

But all the interior angles, together with four right angles, are equal to twice as many right angles as the figure has sides (I. 32, Cor. 1);



Therefore all the interior angles, together with all the exterior angles, are equal to all the interior angles and four right angles (Ax. 1).

Take away the interior angles which are common;

Therefore all the exterior angles are equal to four right angles (Ax. 3).

Proposition 33.—Theorem.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are also themselves equal and parallel.

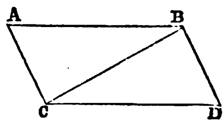
Let AB and CD be equal and parallel straight lines joined towards the same parts by the straight lines AC and BD;

AC and BD shall be equal and parallel.

Construction.—Join BC.

Proof.—Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal (I. 29). ∠ ABC = ∠ BCD.

Because AB is equal to CD, and BC common to the two triangles ABC, DCB, the two sides AB, BC are equal to the two sides DC, CB, each to each;



And the angle ABC was proved to be equal to the angle BCD;

Therefore the base AC is equal to the base BD (I. 4), ∴ AC = BD,

And the triangle ABC is equal to the triangle BCD (I. 4),

And the other angles are equal to the other angles, each to each, to which the equal sides are opposite;

and ∠ ACB = ∠ CBD. Therefore the angle ACB is equal to the angle CBD.

And because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another;

Therefore AC is parallel to BD (I. 27); and it was shown to be equal to it.

Therefore, the straight lines, &c. *Q. E. D.*

Proposition 34.—Theorem.

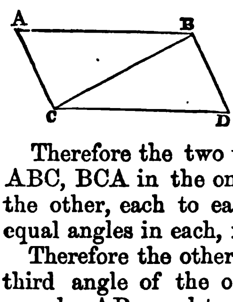
The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects the parallelogram—that is, divides it into two equal parts.

Let ACDB be a parallelogram, of which BC is a diagonal;

The opposite sides and angles of the figure shall be equal to one another,

And the diagonal BC shall bisect it.

∠ ABC = ∠ BCD, PROOF.—Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal to one another (I. 29);



and ∠ ACB = ∠ CBD. Because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal to one another (I. 29);

Therefore the two triangles ABC, BCD have two angles, ABC, BCA in the one, equal to two angles, BCD, CBD in the other, each to each; and the side BC, adjacent to the equal angles in each, is common to both triangles.

Therefore the other sides are equal, each to each, and the third angle of the one to the third angle of the other—

∴ AB = CD, AC = BD, ∠ BAC = ∠ CDB, namely, AB equal to CD, AC to BD, and the angle BAC to the angle CDB (I. 26).

And because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB,

and ∴ ∠ ABD = ∠ ACD, Therefore the whole angle ABD is equal to the whole angle ACD (Ax. 2).

And the angle BAC has been shown to be equal to the angle BDC; therefore the opposite sides and angles of a parallelogram are equal to one another.

Also the diagonal bisects it.

For AB being equal to CD, and BC common,

The two sides AB, BC are equal to the two sides CD and CB, each to each.

And the angle ABC has been shown to be equal to the angle BCD;

Therefore the triangle ABC is equal to the triangle BCD (I. 4), also △ ABC = △ BCD.

And the diagonal BC divides the parallelogram ABCD into two equal parts.

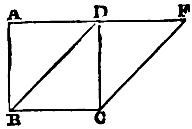
Therefore, the opposite sides, &c. *Q. E. D.*

Proposition 35.—Theorem.

Parallelograms upon the same base, and between the same parallels, are equal to one another.

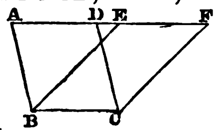
Let the parallelograms ABCD, EBCF be on the same base BC, and between the same parallels AF, BC;

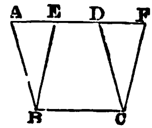
The parallelogram ABCD shall be equal to the parallelogram EBCF.



CASE 1.—If the sides AD, DF of the parallelograms ABCD, DBCF, opposite to the base BC, be terminated in the same point D, it is plain that each of the parallelograms is double of the triangle DBC (I. 34), and that they are therefore equal to one another (Ax. 6).

CASE 2.—But if the sides AD, EF, opposite to the base BC, of the parallelograms ABCD, EBCF, be not terminated in the same point, then—





PROOF.—Because ABCD is a parallelogram, AD is equal to BC (I. 34). AD = BC.

For the same reason EF is equal to BC; EF = BC.

Therefore AD is equal to EF (Ax. 1), and DE is common;

Therefore the whole, or the remainder, AE, is equal to the whole, or the remainder, DF (Ax. 2, or 3), ∴AE=DF.

And AB is equal to DC (I. 34).

Therefore the two EA, AB are equal to the two FD, DC, each to each;

And the exterior angle FDC is equal to the interior EAB (I. 29);

Hence △ EAB = △ FDC.

Therefore the base EB is equal to the base FC (I. 4),

And the triangle EAB equal to the triangle FDC (I. 4).

Take the triangle FDC from the trapezium ABCF, and from the same trapezium ABCF, take the triangle EAB, and the remainders are equal (Ax. 3)

That is, the parallelogram ABCD is equal to the parallelogram EBCF.

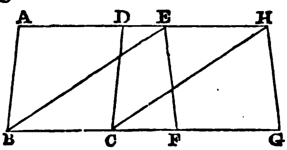
Therefore, parallelograms, &c. *Q. E. D.*

Proposition 36.—Theorem.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Let ABCD, EFGH be parallelograms on equal bases BC, FG, and between the same parallels AH, BG;

The parallelogram ABCD shall be equal to the parallelogram EFGH.



CONSTRUCTION.—Join BE, CH.

PROOF.—Because BC is equal to FG (Hyp.), and FG to EH (I. 34),

BC = EH,

Therefore BC is equal to EH (Ax. 1); and they are parallels, and joined towards the same parts by the straight lines BE, CH.

But straight lines which join the extremities of equal and parallel straight lines towards the same parts, are themselves equal and parallel (I. 33);

and BE = CH.

Therefore BE, CH are both equal and parallel;

EBCH a parallelogram, equal each of the given ones.

Therefore EBCH is a parallelogram (Def. 35),

And it is equal to the parallelogram ABCD, because they are on the same base BC, and between the same parallels BC, AH (I. 35).

For the like reason, the parallelogram EFGH is equal to the same parallelogram EBCH;

Therefore the parallelogram ABCD is equal to the parallelogram EFGH (Ax. 1).

Therefore, parallelograms, &c. *Q. E. D.*

Proposition 37.—Theorem.

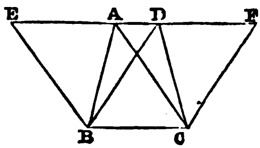
Triangles upon the same base, and between the same parallels, are equal to one another.

Let the triangles ABC, DBC be on the same base BC, and between the same parallels AD, BC;

The triangle ABC shall be equal to the triangle DBC.

Construction.—Produce AD both ways, to the points E, F.

Through B draw BE parallel to CA, and through C draw CF parallel to BD (I. 31).



Proof. — Then each of the figures EBCA, DBCF, is a parallelogram (Def. 35), and they are equal to one another, because they are on the same base BC, and between the same parallels BC, EF (I. 35.); Figures EBCA and DBCF are equal;

And the triangle ABC is half of the parallelogram EBCA, because the diagonal AB bisects it (I. 34); and the triangles are respectively half of these.

And the triangle DBC is half of the parallelogram DBCF, because the diagonal DC bisects it (I. 34).

But the halves of equal things are equal (Ax. 7);

Therefore the triangle ABC is equal to the triangle DBC.

Therefore, triangles, &c. *Q. E. D.*

Proposition 38.—Theorem.

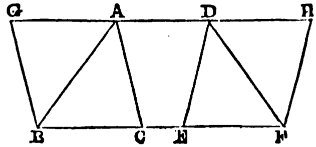
Triangles upon equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF, be on equal bases BC, EF, and between the same parallels BF, AD.

The triangle ABC shall be equal to the triangle DEF.

Construction.—Produce AD both ways to the points G, H.

Through B draw BG parallel to CA, and through F draw FH parallel to ED (I. 31).



Proof.—Then each of the figures GBCA, DEFH, is a Figures GBCA and DEFH are equal;

parallelogram (Def. 35), and they are equal to one another, because they are on equal bases BC, EF, and between the same parallels BF, GH (I. 36);

and the triangles are half of these respectively.

And the triangle ABC is half of the parallelogram GBCA, because the diagonal AB bisects it (I. 34);

And the triangle DEF is half of the parallelogram DEFH, because the diagonal DF bisects it (I. 34).

But the halves of equal things are equal (Ax. 7);

Therefore the triangle ABC is equal to the triangle DEF.

Therefore, triangles, &c. *Q. E. D.*

Proposition 39.—Theorem.

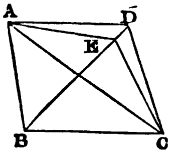
Equal triangles upon the same base, and on the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC be upon the same base BC, and on the same side of it;

They shall be between the same parallels.

CONSTRUCTION.—Join AD; AD shall be parallel to BC.

AE parallel to BC suppose.



For if it is not, through A draw AE parallel to BC (I. 31), and join EC.

PROOF.—The triangle ABC is equal to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE (I. 37).

Then △ DBC = △ EBC, an absurdity.

But the triangle ABC is equal to the triangle DBC (Hyp.);

Therefore the triangle DBC is equal to the triangle EBC (Ax. 1), the greater equal to the less, which is impossible;

Therefore AE is not parallel to BC.

In the same manner, it can be demonstrated that no line passing through A can be parallel to BC, except AD;

Therefore AD is parallel to BC.

Therefore, equal triangles, &c. *Q. E. D.*

Proposition 40.—Theorem.

Equal triangles upon the same side of equal bases, that are in the same straight line, are between the same parallels.

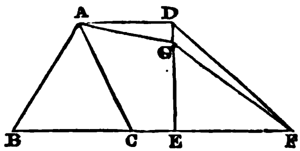
Let the equal triangles ABC, DEF, be upon the same side of equal bases BC, EF, in the same straight line BF.

The triangles ABC, DEF shall be between the same parallels.

Construction.—Join AD; AD shall be parallel to BF.

For if it is not, through A draw AG parallel to BF (I. 31), and join GF. AG parallel to BF suppose.

Proof.—The triangle ABC is equal to the triangle GEF, because they are upon equal bases BC, EF, and are between the same parallels BF, AG (I. 38).



But the triangle ABC is equal to the triangle DEF;

Therefore the triangle DEF is equal to the triangle GEF (Ax. 1), the greater equal to the less, which is impossible; △ DEF = △ GEF, an absurdity.

Therefore AG is not parallel to BF.

In the same manner, it can be demonstrated that no line, passing through A, can be parallel to BF, except AD;

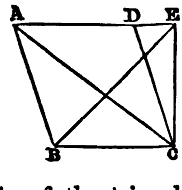
Therefore AD is parallel to BF.

Therefore, equal triangles, &c.

Proposition 41.—Theorem.

If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram shall be double of the triangle.

Let the parallelogram ABCD, and the triangle EBC be upon the same base BC, and between the same parallels BC, AE;



The parallelogram ABCD shall be double of the triangle EBC.

Construction.—Join AC.

Proof.—The triangle ABC is equal to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE (I. 37). △ ABC = △ EBC.

But the parallelogram ABCD is double of the triangle ABC, because the diagonal AC bisects the parallelogram (I. 34). And parallelogram = 2 △ ABC.

Therefore the parallelogram ABCD is also double of the triangle EBC (Ax. 1).

Therefore, if a parallelogram, &c. *Q. E. D.*

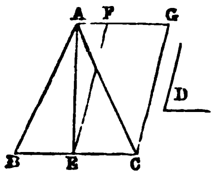
Proposition 42.—Problem.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle;

It is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.

Make BE = EC and ∠ CEF = D



Construction.—Bisect BC in E (I. 10), and join AE.

At the point E, in the straight line CE, make the angle CEF equal to D (I. 23).

Through A draw AFG parallel to EC (I. 31).

Through C draw CG parallel to EF (I. 31).

Then FECG *is the parallelogram required.*

Proof.—Because BE is equal to EC (Const.), the triangle ABE is equal to the triangle AEC, since they are upon equal bases and between the same parallels (I. 38);

△ ABC = 2 △ AEC, and also figure FECG = 2 △ AEC.

Therefore the triangle ABC is double of the triangle AEC.

But the parallelogram FECG is also double of the triangle AEC, because they are upon the same base, and between the same parallels (I. 41);

Therefore the parallelogram FECG is equal to the triangle ABC (Ax. 6),

And it has one of its angles CEF equal to the given angle D (Const.).

Therefore a parallelogram FECG has been described equal to the given triangle ABC, and having one of its angles CEF equal to the given angle D. *Q. E. F.*

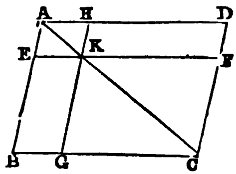
Proposition 43.—Theorem.

The complements of the parallelograms which are about the diagonal of any parallelogram are equal to one another.

Let ABCD be a parallelogram, of which the diagonal is AC; and EH, GF parallelograms about AC, that is, through which AC passes; and BK, KD the other parallelograms, which make up the whole figure ABCD, and are therefore called the complements.

The complement BK shall be equal to the complement KD.

PROOF.—Because ABCD is a parallelogram, and AC its diagonal, the triangle ABC is equal to the triangle ADC (I. 34). △ ABC = △ ADC.



Again, because AEKH is a parallelogram, and AK its diagonal, the triangle AEK is equal to the triangle AHK (I. 34). Also △ AEK = △ AHK.

For the like reason the triangle KGC is equal to the triangle KFC. And △ KGC = △ KFC.

Therefore, because the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC,

The triangles AEK, KGC are equal to the triangles AHK, KFC (Ax. 2).

But the whole triangle ABC was proved equal to the whole triangle ADC;

Therefore the remaining complement BK is equal to the remaining complement KD (Ax. 3). ∴ BK = KD.

Therefore, the complements, &c. *Q. E. D.*

Proposition 44.—Problem.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, C the given triangle, and D the given angle.

It is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to D.

Make parallelogram BEFG = △ C, and ∠ at B = ∠ D, and EBA a straight line.

CONSTRUCTION 1.—Make the parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D (I. 42);

And let the parallelogram BEFG be made so that BE may be in the same straight line with AB.

Produce FG to H.

Through A draw AH parallel to BG or EF (I. 31).

Join HB.

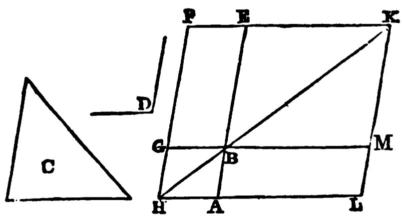
PROOF 1.—Because the straight line HF falls on the parallels AH, EF, the angles AHF, HFE are together equal to two right angles (I. 29).

Therefore the angles BHF, HFE are together less than two right angles (Ax. 9). But straight lines which with another straight line make the interior angles on the same side together less than two right angles, will meet on that side, if produced far enough (Ax. 12);

HB and FE meet.

Therefore HB and FE shall meet if produced.

CONSTRUCTION 2.—Produce HB and FE towards BE, and let them meet in K.



Through K draw KL parallel to EA or FH (I. 31).

Produce HA, GB to the points L, M.

Then LB *shall be the parallelogram required.*

PROOF 2.—Because HLKF is a parallelogram, of which the diagonal is HK; and AG, ME are the parallelograms about HK; and LB, BF are the complements;

Figures LB = BF.

Therefore the complement LB is equal to the complement BF (I. 43).

But BF is equal to the triangle C (Const.);
Therefore LB is equal to the triangle C (Ax. 1).
And because the angle GBE is equal to the angle ABM (I. 15), and likewise to the angle D (Const.);
Therefore the angle ABM is equal to the angle D (Ax. 1).
Therefore, the parallelogram LB is applied to the straight line AB, and is equal to the triangle C, and has the angle ABM equal to the angle D. *Q. E. F.*

But BF = △ C, ∴ LB = △ C;

Also ∠ ABM = ∠ GBE = ∠ D.

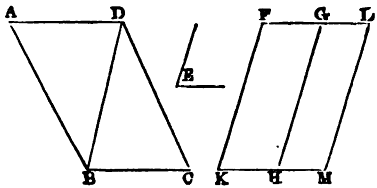
Proposition 45.—Problem.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle.

It is required to describe a parallelogram equal to ABCD, and having an angle equal to E.

Construction.—Join DB.



Describe the parallelogram FH equal to the triangle ADB, and having the angle FKH equal to the angle E (I. 42).

To the straight line GH apply the parallelogram GM equal to the triangle DBC, and having the angle GHM equal to the angle E (I. 44).

Make FH = △ ADB. Apply to GH, GM = △ DBC, with ∠ GHM = ∠ E.

Then the figure FKML *shall be the parallelogram required.*

Proof.—Because the angle E is equal to each of the angles FKH, GHM (Const.),

Therefore the angle FKH is equal to the angle GHM (Ax. 1).

Add to each of these equals the angle KHG;

Therefore the angles FKH, KHG are equal to the angles KHG, GHM (Ax. 2).

But FKH, KHG are equal to two right angles (I. 29);

Therefore also KHG, GHM are equal to two right angles (Ax. 1).

And because at the point H, in the straight line GH, the two straight lines KH, HM, on the opposite sides of it, make the adjacent angles together equal to two right angles,

Then KHM is a straight line;

Therefore KH is in the same straight line with HM (I. 14).

And because the straight line HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal (I. 29).

Add to each of these equals the angle HGL;

Therefore the angles MHG, HGL are equal to the angles HGF, HGL (Ax. 2).

But the angles MHG, HGL are equal to two right angles (I. 29);

Therefore also the angles HGF, HGL are equal to two right angles,

and FGL is a straight line.

And therefore FG is in the same straight line with GL (I. 14).

And because KF is parallel to HG, and HG parallel to ML (Const.);

Therefore KF is parallel to ML (I. 30).

And KM, FL are parallels (Const);

∴ KFLM a parallelogram.

Therefore KFLM is a parallelogram (Def. 35).

And because the triangle ABD is equal to the parallelogram HF, and the triangle DBC equal to the parallelogram GM (Const.),

And the figure ABCD is equal to it.

Therefore the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM (Ax. 2).

Therefore, the parallelogram KFLM has been described equal to the given rectilineal figure ABCD, and having the angle FKM equal to the given angle E. *Q. E. F.*

Corollary.—From this it is manifest how to apply to a given straight line a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure—namely, by applying to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle; and so on (I. 44).

Proposition 46.—Problem.

To describe a square upon a given straight line.

Let AB be the given straight line;

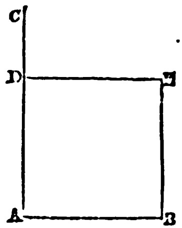
It is required to describe a square upon AB.

CONSTRUCTION.—From the point A draw AC at right angles to AB (I. 11),

And make AD equal to AB (I. 3).

Through the point D draw DE parallel to AB (I. 31).

Through the point B draw BE parallel to AD (I. 31).



Then ADEB *shall be the square required.*

PROOF.—Because DE is parallel to AB, and BE parallel to AD (Const.), therefore ADEB is a parallelogram; ADEB a parallelogram.

Therefore AB is equal to DE, and AD to BE (I. 34).

But AB is equal to AD (Const.);

Therefore the four straight lines BA, AD, DE, EB are equal to one another (Ax. 1),

And the parallelogram ADEB is therefore equilateral. It is equilateral.

Likewise all its angles are right angles.

For since the straight line AD meets the parallels AB, DE, the angles BAD, ADE are together equal to two right angles (I. 29).

But BAD is a right angle (Const.);

Therefore also ADE is a right angle (Ax. 3).

But the opposite angles of parallelograms are equal (I. 34);

Therefore each of the opposite angles ABE, BED is a right angle (Ax. 1);

Therefore the figure ADEB is rectangular; and it has been proved to be equilateral; therefore it is a square (Def. 30). It is rectangular.

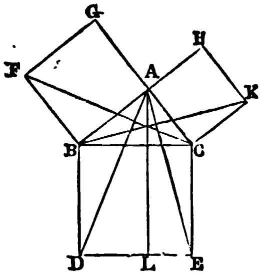
Therefore, the figure ADEB is a square, and it is described upon the given straight line AB. *Q. E. F.* ∴ a square.

COROLLARY.—Hence every parallelogram that has one right angle has all its angles right angles.

Proposition 47.—Theorem.

In any right-angled triangle, the square which is described upon the side opposite to the right angle is equal to the squares described upon the sides which contain the right angle.

Let ABC be a right-angled triangle, having the right angle BAC;



The square described upon the side BC shall be equal to the squares described upon BA, AC.

CONSTRUCTION. — On BC describe the square BDEC (I. 46).

On BA, AC describe the squares GB, HC (I. 46).

Through A draw AL parallel to BD or CE (I. 31).

Join AD, FC.

PROOF.—Because the angle BAC is a right angle (Hyp.), and that the angle BAG is also a right angle (Def. 30),

The two straight lines AC, AG, upon opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles;

C G is a straight line.

Therefore CA is in the same straight line with AG (I. 14).

BAH is a straight line.

For the same reason, AB and AH are in the same straight line.

Now the angle DBC is equal to the angle FBA, for each of them is a right angle (Ax. 11); add to each the angle ABC.

Therefore the whole angle DBA is equal to the whole angle FBC (Ax. 2).

And because the two sides AB, BD are equal to the two sides FB, BC, each to each (Def. 30), and the angle DBA equal to the angle FBC;

△ ABD = △ FBC.

Therefore the base AD is equal to the base FC, and the triangle ABD to the triangle FBC (I. 4).

Now the parallelogram BL is double of the triangle ABD, because they are on the same base BD, and between the same parallels BD, AL (I. 41).

And the square GB is double of the triangle FBC, because they are on the same base FB, and between the same parallels FB, GC (I. 41).

Hence parallelogram BL = square GB, and parallelogram CL = square HC.

But the doubles of equals are equal (Ax. 6), therefore the parallelogram BL is equal to the square GB.

In the same manner, by joining AE, BK, it can be shown that the parallelogram CL is equal to the square HC.

Therefore the whole square BDEC is equal to the two squares GB, HC (Ax. 2);

And the square BDEC is described on the straight line BC, and the squares GB, HC upon BA, AC. ∴ $BC^2 = BA^2 + AC^2$

Therefore the square described upon the side BC is equal to the squares described upon the sides BA, AC.

Therefore, in any right-angled triangle, &c. *Q. E. D.*

Proposition 48.—Theorem.

If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.

Let the square described upon BC, one of the sides of the triangle ABC, be equal to the squares described upon the other sides BA, AC;

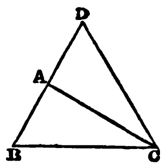
The angle BAC shall be a right angle.

Construction.—From the point A draw AD at right angles to AC (I. 11). Draw AD at right angles to AC. *(Do not produce BA.)*

Make AD equal to BA (I. 3), and join DC.

Proof.—Because DA is equal to AB, the square on DA is equal to the square on BA.

To each of these add the square on AC.



Therefore the squares on DA, AC are equal to the squares on BA, AC (Ax. 2).

But because the angle DAC is a right angle (Const.), the square on DC is equal to the squares on DA, AC (I. 47),

And the square on BC is equal to the squares on BA, AC (Hyp.);

Therefore the square on DC is equal to the square on BC (Ax. 1);

And therefore the side DC is equal to the side BC. Then $DC^2 = BC^2$, and $DC = BC$.

And because the side DA is equal to AB (Const.), and AC common to the two triangles DAC, BAC, the two sides DA, AC are equal to the two sides BA, AC, each to each.

And the base DC has been proved equal to the base BC;

Hence ∠DAC = ∠BAC.

Therefore the angle DAC is equal to BAC (I. 8).
But DAC is a right angle (Const.);
Therefore also BAC is a right angle (Ax. 1).
Therefore, if the square, &c. *Q. E. D.*

EXERCISES ON BOOK I.

PROP. 1—15.

1. From the greater of two given straight lines to cut off a portion which is three times as long as the less.
2. The line bisecting the vertical angle of an isosceles triangle also bisects the base.
3. Prove Euc. I. 5, by the method of *super-position.*
4. In the figure to Euc. I. 5, show that the line joining A with the point of intersection of BG and FC, makes equal angles with AB and AC.
5. ABC is an isosceles triangle, whose base is BC, and AD is perpendicular to BC; every point in AD is equally distant from B and C.
6. Show that the sum of the sum and difference of two given straight lines is twice the greater, and that the difference of the sum and difference is twice the less.
7. Prove the same property with regard to angles.
8. Make an angle which shall be three-fourths of a right angle.
9. If, with the extremities of a given line as centres, circles be drawn intersecting in two points, the line joining the points of intersection will be perpendicular to the given line, and will also bisect it.
10. Find a point which is at a given distance from a given point and from a given line.
11. Show that the sum of the angles round a given point are together equal to four right angles.
12. If the exterior angle of a triangle and its adjacent interior angle be bisected, the bisecting lines will be at right angles.
13. If three points, A, B, C, be taken not in the same straight line, and AB and AC be joined and bisected by perpendiculars which meet in D, show that DA, DB, DC are equal to each other.

PROP. 16—32.

14. The perpendiculars from the angular points upon the opposite sides of a triangle meet in a point.
15. To construct an isosceles triangle on a given base, the sides being each of them double the given base.

16. Describe an isosceles triangle having a given base, and whose vertical angle is half a right angle.

17. AB is a straight line, C and D are points on the same side of it; find a point E in AB such that the sum of CE and ED shall be a minimum.

18. Having given two sides of a triangle and an angle, construct the triangle. Examine the cases when there will be (1.) one solution; (2.) two solutions; (3.) none.

19. Given an angle of a triangle and the sum and difference of the two sides including the angle, to construct the triangle.

20. Show that each of the angles of an equilateral triangle is two-thirds of a right angle, and hence show how to trisect a right angle.

21. If two angles of a triangle be bisected by lines drawn from the angular points to a given point within, then the line bisecting the third angle will pass through the same point.

22. The difference of any two sides of a triangle is less than the third side.

23. If the angles at the base of a right-angled isosceles triangle be bisected, the bisecting line includes an angle which is three halves of a right angle.

24. The sum of the lines drawn from any point within a polygon to the angular points is greater than half the sum of the sides of the polygon.

Prop. 33—48.

25. Show that the diagonals of a square bisect each other at right angles, and that the square described upon a semi-diagonal is half the given square.

26. Divide a given line into any number of equal parts, and hence show how to divide a line similarly to a given line.

27. If D and E be respectively the middle points of the sides BC and AC of the triangle ABC, and AD and BE be joined, and intersect in G, show that GD and GE are respectively one-third of AD and BE.

28. The lines drawn to the bisections of the sides of a triangle from the opposite angles meet in a point.

29. Describe a square which is five times a given square.

30. Show that a square, hexagon, and dodecagon will fill up the space round a point.

31. Divide a square into three equal areas, by lines drawn parallel to one of the diagonals.

32. Upon a given straight line construct a regular octagon.

33. Divide a given triangle into equal triangles by lines drawn from one of the angles.

34. If any two angles of a quadrilateral are together equal to two right angles, show that the sum of the other two is two right angles.

35. The area of a trapezium having two parallel sides is equal to half the rectangle contained by the perpendicular distance between the parallel sides of the trapezium, and the sum of the parallel sides.

36. The area of any trapezium is half the rectangle contained by one of the diagonals of the trapezium, and the sum of the perpendiculars let fall upon it from the opposite angles.

37. If the middle points of the sides of a triangle be joined, the lines form a triangle whose area is one-fourth that of the given triangle.

38. If the sides of a triangle be such that they are respectively the sum of two given lines, the difference of the same two lines, and twice the side of a square equal to the rectangle contained by these lines, the triangle shall be right-angled, having the right angle opposite to the first-named side.

39. If a point be taken within a triangle such that the lengths of the perpendiculars upon the sides are equal, show that the area of the rectangle contained by one of the perpendiculars and the perimeter of the triangle is double the area of the triangle.

40. In the last problem, if O be the given point, and OD, OE, OF the respective perpendiculars upon the sides BC, AC, and AB, show that the sum of the squares upon AD, OB, and DC exceeds the sum of the squares upon AF, BD, and CD by three times the square upon either of the perpendiculars.

41. Having given the lengths of the segments AF, BD, CE, in Problem 40, construct the triangle.

42. Draw a line, the square upon which shall be seven times the square upon a given line.

43. Draw a line, the square upon which shall be equal to the sum or difference of two given squares.

44. Reduce a given polygon to an equivalent triangle.

45. Divide a triangle into equal areas by drawing a line from a given point in a side.

46. Do the same with a given parallelogram.

47. If in the fig., Euc. I. 47, the square on the hypothenuse be on the other side, show how the other two squares may be made to cover exactly the square on the hypothenuse.

48. The area of a quadrilateral whose diagonals are at right angles is half the rectangle contained by the diagonals.

49. Bisect a given triangle by a straight line drawn from one of its angles.

50. Do the same with a given rectilineal figure ABCDEF.

51. If from the angle A of a triangle ABC a perpendicular be drawn meeting the base or base produced in D, show that the difference of the squares of AB and AC is equal to the difference of the squares of BD and DC.

52. If a straight line join the points of bisection of two sides of a triangle, the base is double the length of this line.

53. ABCD is a parallelogram, and E a point within it, and lines are drawn through E parallel to the sides of the parallelogram, show that E must lie on the diagonal AC when the figures BC and DE are equal.

54. If AD, BE, CF, are the perpendiculars from the angular points

of the triangle ABC upon the opposite sides, show that the sum of the squares upon AE, CD, BF, is equal to the sum of the squares upon CE, BD, AF.

55. The diagonals of a parallelogram bisect one another.

56. Write out at full length a definition of parallelism, and then prove that the alternate angles are equal when a straight line meets two parallel straight lines.

57. ABCDE are the angular points of a regular pentagon, taken in order. Join AC and BD meeting in H, and show that AEDH is an equilateral parallelogram.

58. Having given the middle points of the sides of a triangle, show how to construct the triangle.

59. Show that the diagonal of a parallelogram diminishes while the angle from which it is drawn increases. What is the limit to which the diagonal approaches as the angle approaches respectively zero and two right angles?

60. A, B, C, are three angles taken in order of a regular hexagon, show that the square on AC is three times the square upon a side of the hexagon.

EUCLID'S ELEMENTS, BOOK II.

The student, as he works through this book of Euclid's Elements, will not fail to perceive how easily many geometrical truths may be proved, and how concisely they may be expressed by means of Algebra. We have not space here to at all discuss the relation between Geometry and Algebra, but we will express a few of the propositions of this book in Algebraical symbols.

I............$a(b + c + d) = ab + ac + cd$.
II............$(a + b)a + (a + b)b = (a + b)^2$.
III............$(a + b)a = a^2 + ab$.
IV............$(a + b)^2 = a^2 + b^2 + 2ab$.
V............$(a + x)(a - x) + x^2 = a^2$.
VI............$(2a + x)x + a^2 = (a + x)^2$.
VII............$(a + b)^2 + b^2 = 2(a + b)b + a^2$.
VIII............$4(a + b)a + b^2 = (2a + b)^2$.

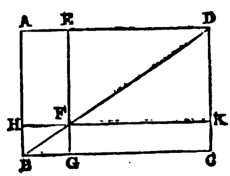
EUCLID'S ELEMENTS, BOOK II.

Definitions.

1. **A rectangle, or right-angled parallelogram,** is said to be contained by any two of the straight lines which contain one of the right angles.

2. **In any parallelogram,** the **figure** which is composed of either of the parallelograms about a diameter, together with the two complements, is called a *gnomon.*

Thus the parallelogram HG, together with the complements AF, FC, is a gnomon, which is briefly expressed by the letters AGK, or EHC, which are at the opposite angles of the parallelograms which make the gnomon.



The rectangle under, or contained by two lines, as AB and BC, is concisely expressed thus;—AB, BC.

Proposition 1.—Theorem.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A and BC be two straight lines; and let BC be divided into any parts in the points D, E;

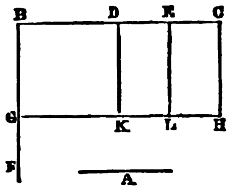
The rectangle contained by the straight lines A and BC shall be equal to the rectangle contained by A and BD, together with that contained by A and DE, and that contained by A and EC. A·BC = A·BD + A·DE + A·EC.

CONSTRUCTION.—From the point B draw BF at right angles to BC (I. 11),

And make BG equal to A (I. 3).

Through G draw GH parallel to BC (I. 31),

And through the points D, E, C, draw DK, EL, CH parallel to BG (I. 31).



PROOF.—Then the rectangle BH is equal to the rectangles BK, DL, EH. Since BH = BK + DL + EH.

But BH is contained by A and BC, for it is contained by GB and BC, and GB is equal to A (Const.);

And BK is contained by A and BD, for it is contained by GB and BD, and GB is equal to A;

And DL is contained by A and DE, because DK is equal to BG, which is equal to A (I. 34);

And in like manner EH is contained by A and EC;

Therefore the rectangle contained by A and BC is equal to the several rectangles contained by A and BD, by A and DE, and by A and EC.

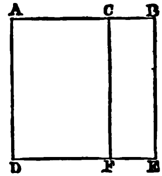
Therefore, if there be two straight lines, &c. *Q. E. D.*

Proposition 2.—Theorem.

If a straight line be divided into any two parts, the rectangles contained by the whole line and each of its parts are together equal to the square on the whole line.

Let the straight line AB be divided into any two parts in the point C;

AB · BC + AB · AC = AB². The rectangle contained by AB and BC, together with the rectangle contained by AB and AC, shall be equal to the square on AB.



CONSTRUCTION.—Upon AB describe the square ADEB (I. 46).

Through C draw CF parallel to AD or BE (I. 31).

PROOF.—Then AE is equal to the rectangles AF and CE.

For AB² is the sum of its parts AF + CE. But AE is the square on AB;

Therefore the square on AB is equal to the rectangles AF and CE.

And AF is the rectangle contained by BA and AC, for it is contained by DA and AC, of which DA is equal to BA;

And CE is contained by AB and BC, for BE is equal to AB.

And ∴ = AB · AC + AB · BC. Therefore the rectangle AB, AC, together with the rectangle AB, BC, is equal to the square on AB.

Therefore, if a straight line, &c. *Q. E. D.*

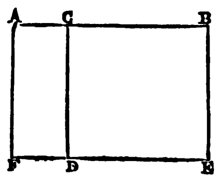
Proposition 3.—Theorem.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the square on that part, together with the rectangle contained by the two parts.

Let the straight line AB be divided into any two parts in the point C;

AB · BC = BC² + AC · BC. The rectangle AB · BC shall be equal to the square on BC, together with the rectangle AC · CB.

CONSTRUCTION.—Upon BC describe the square CDEB (I. 46).



Produce ED to F; and through A draw AF parallel to CD or BE (I. 31).

For AE = AD + CE. PROOF.—Then the rectangle AE is equal to the rectangles AD and CE.

But AE is the rectangle contained by AB and BC, for it is contained by AB and BE, of which BE is equal to BC;

And AD is contained by AC and CB, for CD is equal to CB, And CE is the square on BC.

Therefore the rectangle AB, BC is equal to the square on BC, together with the rectangle AC, CB.

Therefore, if a straight line, &c. *Q. E. D.*

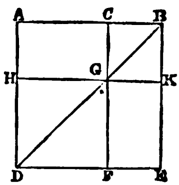
Proposition 4.—Theorem.

If a straight line be divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts, together with twice the rectangle contained by the parts.

Let the straight line AB be divided into any two parts in C;

The square on AB shall be equal to the squares on AC and CB, together with twice the rectangle contained by AC and CB. $AB^2 = AC^2 + CB^2 + 2AC \cdot CB$.

Construction.—Upon AB describe the square ADEB (I. 46), and join BD.



Through C draw CGF parallel to AD or BE (I. 31).

Through G draw HGK parallel to AB or DE (I. 31).

Proof.—Because CF is parallel to AD, and BD falls upon them,

Therefore the exterior angle BGC is equal to the interior and opposite angle ADB (I. 29).

Because AB is equal to AD, being sides of a square, the angle ADB is equal to the angle ABD (I. 5);

Therefore the angle CGB is equal to the angle CBG (Ax. 1); Show first that CK is a square $= CB^2$

Therefore the side BC is equal to the side CG (I. 6).

But CB is also equal to GK, and CG to BK (I. 34);

Therefore the figure CGKB is equilateral.

It is likewise rectangular.

For since CG is parallel to BK, and CB meets them, the angles KBC and GCB are together equal to two right angles (I. 29).

But KBC is a right angle (Const.), therefore GCB is a right angle (Ax. 3).

Therefore also the angles CGK, GKB, opposite to these, are right angles (I. 34).

Therefore CGKB is rectangular; and it has been proved equilateral; therefore it is a square; and it is upon the side CB.

So also $HF = AC^2$ For the same reason HF is also a square, and it is on the side HG, which is equal to AC (I. 34).

Therefore HF and CK are the squares on AC and CB.

And because the complement AG is equal to the complement GE (I. 43),

And $AG + GE = 2AC \cdot CB$ And that AG is the rectangle contained by AC and CG, that is, by AC and CB,

Therefore GE is also equal to the rectangle AC, CB;

Therefore AG, GE are together equal to twice the rectangle AC, CB;

And HF, CK are the squares on AC and CB.

Therefore the four figures HF, CK, AG, GE are equal to the squares on AC and CB, together with twice the rectangle AC, CB.

But HF, CK, AG, GE, make up the whole figure ADEB, which is the square on AB;

∴ whole figure or $AB^2 = AC^2 + BC^2, + 2AC \cdot CB$. Therefore the square on AB is equal to the squares on AC and CB and twice the rectangle AC · CB.

Therefore, if a straight line, &c. *Q. E. D.*

Corollary.—From this demonstration it follows that the parallelograms about the diameter of a square are likewise squares.

Proposition 5.—Theorem.

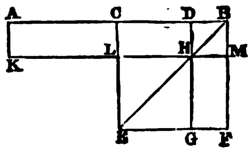
If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Let the straight line AB be bisected in C, and divided unequally in D;

$AD \cdot DB + CD^2 = CB^2$. The rectangle AD, BD, together with the square on CD, shall be equal to the square on CB.

Construction.—Upon CB describe the square CEFB (I. 46), and join BE.

Through D draw DHG parallel to CE or BF (I. 31).
Through H draw KLM parallel to CB or EF.
And through A draw AK parallel to CL or BM.



PROOF.—Then the complement CH is equal to the complement HF (I. 43).

To each of these add DM; therefore the whole CM is equal to the whole DF (Ax. 2).

But CM is equal to AL (I. 36), because AC is equal to CB (Hyp.); For AL=CM =DF.

Therefore also AL is equal to DF (Ax. 1).

To each of these add CH; therefore the whole AH is equal to DF and CH (Ax. 2). ∴ AH= DF+CH.

But AH is contained by AD and BD, since DH is equal to DB (II. 4, cor.),

And DF, together with CH, is the gnomon CMG;

Therefore the gnomon CMG is equal to the rectangle AD, DB. ∴ CMG. =AD·DB.

To each of these equals add LG, which is equal to the square on CD (II. 4, cor., and I · 34); Add to each LG or CD^2.

Therefore the gnomon CMG, together with LG, is equal to the rectangle AD, DB, together with the square on CD.

But the gnomon CMG and LG make up the whole figure CEFB, which is the square on CB;

Therefore the rectangle AD, DB, together with the square on CD, is equal to the square on CB. ∴ CB^2 =AD·DB +CD^2.

Therefore, if a straight line, &c. *Q. E. D.*

COROLLARY.—From this proposition it is manifest that the difference of the squares on two unequal lines AC, CD is equal to the rectangle contained by their sum and difference.

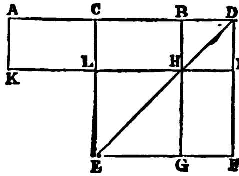
Proposition 6.—Theorem.

If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to D;

AD · DB + CB² = CD². The rectangle AD, DB, together with the square on CB, shall be equal to the square on CD.

Construction.—Upon CD describe the square CEFD (I. 46), and join DE.



Through B draw BHG parallel to CE or DF (I. 31).

Through H draw KLM parallel to AD or EF.

And through A draw AK parallel to CL or DM.

For AL = CH = HF. Proof.—Because AC is equal to CB (Hyp.), the rectangle AL is equal to CH (I. 36).

But CH is equal to HF (I. 43), therefore AL is equal to HF (Ax. 14).

∴ AM or AD · DB = CMG. To each of these add CM; therefore the whole AM is equal to the gnomon CMG (Ax. 2).

But AM is the rectangle contained by AD and DB, since DM is equal to DB (II. 4, cor.);

Therefore the gnomon CMG is equal to the rectangle AD, DB (Ax. 1).

Add to each LG or CB². Add to each of these LG, which is equal to the square on CB (II. 4, cor., and I. 34);

Therefore the rectangle AD, DB, together with the square on CB, is equal to the gnomon CMG and the figure LG.

But the gnomon CMG and LG make up the whole figure CEFD, which is the square on CD;

∴ AD · DB + CB² = CD². Therefore the rectangle AD, DB, together with the square on CB, is equal to the square on CD.

Therefore, if a straight line, &c. *Q. E. D.*

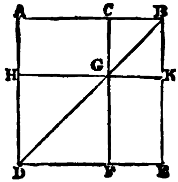
Proposition 7.—Theorem.

If a straight line be divided into any two parts, the squares on the whole line and on one of the parts are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

Let the straight line AB be divided into any two parts in the point C;

AB² + BC² = 2 AB · BC + AC². The squares on AB and BC shall be equal to twice the rectangle AB, BC, together with the square on AC.

CONSTRUCTION.—Upon AB describe the square ADEB (I. 46), and join BD.



Through C draw CGF parallel to AD or BE (I. 31).

Through G draw HGK parallel to AB or DE (I. 31).

PROOF.—Then AG is equal to GE (I. 43).

To each of these add CK; therefore the whole AK is equal to the whole CE; For AK=CE.

Therefore AK and CE are double of AK.

But AK and CE are the gnomon AKF, together with the square CK;

Therefore the gnomon AKF, together with the square CK, is double of AK. ∴ AKF + CK = 2AB·BC.

But twice the rectangle AB, BC is also double of AK, for BK is equal to BC (II. 4, cor.);

Therefore the gnomon AKF, together with the square CK, is equal to twice the rectangle AB, BC.

To each of these equals add HF, which is equal to the square on AC (II. 4, cor., and I. 34); Add HF or AC^2 to each equal.

Therefore the gnomon AKF, together with the squares CK and HF, is equal to twice the rectangle AB, BC, together with the square on AC.

But the gnomon AKF, together with the squares CK and HF, make up the whole figure ADEB and CK, which are the squares on AB and BC; ∴ $AB^2 + BC^2 = 2\,AB\cdot BC + AC^2$.

Therefore the squares on AB and BC are equal to twice the rectangle AB, BC, together with the square on AC.

Therefore, if a straight line, &c. *Q. E. D.*

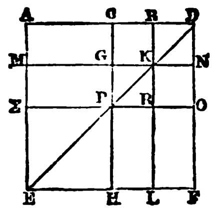
Proposition 8.—Theorem.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole line and the first mentioned part.

Let the straight line AB be divided into any two parts in the point C;

4 AB·BC + AC² = (AB+BC)². Four times the rectangle AB, BC, together with the square on AC, shall be equal to the square on the straight line made up of AB and BC together.

Construction.—Produce AB to D, so that BD may be equal to CB (Post. 2, and I. 3).



Upon AD describe the square AEFD (I. 46),

And construct two figures such as in the preceding propositions.

Proof.—Because CB is equal to BD (Const.), CB to GK, and BD to KN (Ax. 1),

For the same reason PR is equal to RO.

For BN=CK =GR=RN. And because CB is equal to BD, and GK to KN,

Therefore the rectangle CK is equal to BN, and GR to RN (I. 36).

But CK is equal to RN, because they are the complements of the parallelogram CO (I. 43);

Therefore also BN is equal to GR (Ax. 1).

∴ the four together = 4 CK. Therefore the four rectangles BN, CK, GR, RN are equal to one another, and so the four are quadruple of one of them, CK.

And the rectangles AG, MP, PL, RF, are equal to each other, and are together = 4 AG. Again, because CB is equal to BD (Const.);

And that BD is equal to BK, that is CG (II. 4, Cor., and I. 34);

And that CB is equal to GK, that is GP (I. 34, and II. 4, cor.);

Therefore CG is equal to GP (Ax. 1).

And because CG is equal to GP, and PR to RO,

The rectangle AG is equal to MP, and PL to RF (I. 36).

But MP is equal to PL, because they are complements of the parallelogram ML (I. 43), and AG is equal to RF (Ax. 1);

Therefore the four rectangles AG, MP, PL, RF are equal to one another, and so the four are quadruple of one of them, AG.

And it was demonstrated that the four CK, BN, GR, and RN are quadruple of CK;

Therefore the eight rectangles which make up the gnomon AOH are quadruple of AK.

And because AK is the rectangle contained by AB and BC, for BK is equal to BC; ∴ Gnomon AOH = 4(CK+AG)

Therefore four times the rectangle AB, BC is quadruple of AK. = 4 AK = 4 AB·BC

But the gnomon AOH was demonstrated to be quadruple of AK;

Therefore four times the rectangle AB, BC is equal to the gnomon AOH (Ax. 1).

To each of these add XH, which is equal to the square on AC (II. 4, cor., and I. 34); Hence adding XH or AC^2,

Therefore four times the rectangle AB, BC, together with the square on AC, is equal to the gnomon AOH and the square XH.

But the gnomon AOH and the square XH make up the figure AEFD, which is the square on AD;

Therefore four times the rectangle AB, BC, together with the square on AC, is equal to the square on AD, that is, on the line made up of AB and BC together. $4 AB \cdot BC + AC^2 = AF = (AB+BC)^2$.

Therefore, if a straight line, &c. *Q. E. D.*

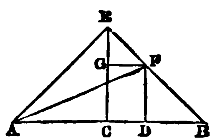
Proposition 9.—Theorem.

If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line, and of the square on the line between the points of section.

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts in the point D;

The squares on AD and DB shall be together double of the squares on AC and CD. $AD^2+DB^2 = 2(AC^2+CD^2)$.

Construction.—From the point C draw CE at right angles to AB (I. 11), and make it equal to AC or CB (I. 3), and join EA, EB.



Through D draw DF parallel to CE (I. 31).

Through F draw FG parallel to BA (I. 31), and join AF.

Proof.—Because AC is equal to CE (Const.), the angle EAC is equal to the angle AEC (I. 5).

And because the angle ACE is a right angle (Const.), the angles AEC and EAC together make one right angle (I. 32), and they are equal to one another;

For AEC = ½ a right ∠ = CEB.

Therefore each of the angles AEC and EAC is half a right angle.

For the same reason, each of the angles CEB and EBC is half a right angle;

∴ AEB is a right ∠.

Therefore the whole angle AEB is a right angle.

And because the angle GEF is half a right angle, and the angle EGF a right angle, for it is equal to the interior and opposite angle ECB (I. 29),

Therefore the remaining angle EFG is half a right angle;

Therefore the angle GEF is equal to the angle EFG, and the side EG is equal to the side GF (I. 6).

EG = GF.

Again, because the angle at B is half a right angle, and the angle FDB a right angle, for it is equal to the interior and opposite angle ECB (I. 29),

Therefore the remaining angle BFD is half a right angle;

Therefore the angle at B is equal to the angle BFD, and the side DF is equal to the side DB (I. 6).

And DF = DB.

And because AC is equal to CE (Const.), the square on AC is equal to the square on CE;

Therefore the squares on AC and CE are double of the square on AC.

But $AE^2 = 2AC^2$.

But the square on AE is equal to the squares on AC and CE, because the angle ACE is a right angle (I. 47);

Therefore the square on AE is double of the square on AC.

Again, because EG is equal to GF (Const.), the square on EG is equal to the square on GF;

Therefore the squares on EG and GF are double of the square on GF.

So also $EF^2 = 2GF^2 = 2CD^2$.

But the square on EF is equal to the squares on EG and GF, because the angle EGF is a right angle (I. 47);

Therefore the square on EF is double of the square on GF.

And GF is equal to CD (I. 34);

Therefore the square on EF is double of the square on CD.

But it has been demonstrated that the square on AE is also double of the square on AC;

∴ $AE^2 + EF^2 = 2(AC^2 + CD^2)$.

Therefore the squares on AE and EF are double of the squares on AC and CD.

But the square on AF is equal to the squares on AE and EF, because the angle AEF is a right angle (I. 47);

But $AE^2+EF^2 = AF^2 = AD^2+DF^2 = AD^2+DB^2$.

Therefore the square on AF is double of the squares on AC and CD.

But the squares on AD and DF are equal to the square on AF, because the angle ADF is a right angle (I. 47);

Therefore the squares on AD and DF are double of the squares on AC and CD.

And DF is equal to DB; therefore the squares on AD and DB are double of the squares on AC and CD.

$\therefore AD^2+DB^2 = 2(AC^2+CD^2)$

Therefore, if a straight line, &c. *Q.E.D.*

Proposition 10.—Theorem.

If a straight line be bisected and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected, and of the square on the line made up of the half and the part produced.

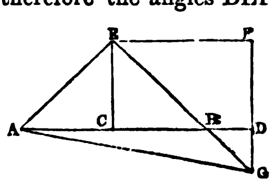
Let the straight line AB be bisected in C, and produced to D;

$AD^2+DB^2 = 2(AC^2+CD^2)$.

The squares on AD and DB shall be together double of the squares on AC and CD.

CONSTRUCTION.—From the point C draw CE at right angles to AB, and make it equal to AC or CB (I. 11, I. 3), and join AE, EB.

Through E draw EF parallel to AB, and through D draw DF parallel to CE (I. 31). Then because the straight line EF meets the parallels EC, FD, the angles CEF, EFD are equal to two right angles (I. 29); therefore the angles BEF, EFD are less than two right angles; therefore EB, FD will meet, if produced towards B and D (Ax. 12).



Let them meet in G, and join AG.

PROOF.—Because AC is equal to CE (Const.), the angle CEA is equal to the angle EAC (I. 5).

And the angle ACE is a right angle; therefore each of the angles CEA and EAC is half a right angle (I. 32).

As in Prop. 9. For the same reason each of the angles CEB and EBC is half a right angle;

AEB is a right ∠. Therefore the whole angle AEB is a right angle.

And because the angle EBC is half a right angle, the angle DBG, which is vertically opposite, is also half a right angle (I. 15);

But the angle BDG is a right angle, because it is equal to the alternate angle DCE (I. 29);

Therefore the remaining angle DGB is half a right angle, and is therefore equal to the angle DBG;

And BD=DG. Therefore also the side BD is equal to the side DG (I. 6).

Again, because the angle EGF is half a right angle, and the angle at F a right angle, for it is equal to the opposite angle ECD (I. 34);

Therefore the remaining angle FEG is half a right angle (I. 32), and therefore equal to the angle EGF;

Also GF=FE. Therefore also the side GF is equal to the side FE (I. 6).

And because EC is equal to CA, the square on EC is equal to the square on CA;

Therefore the squares on EC and CA are double of the square on CA.

Again as in Prop. 9. But the square on AE is equal to the squares on EC and CA (I. 47);

$AE^2 = 2AC^2$, Therefore the square on AE is double of the square on AC.

Again, because GF is equal to FE, the square on GF is equal to the square on FE;

Therefore the squares on GF and FE are double of the square on FE.

But the square on EG is equal to the squares on GF and FE (I. 47);

Therefore the square on EG is double of the square on FE.

And FE is equal to CD (I. 34),

And $EG^2 = 2CD^2$. $\therefore AE^2 + EG^2 = 2(AC^2 + CD^2)$. Therefore the square on EG is double of the square on CD.

But it has been demonstrated that the square on AE is double of the square on AC;

Therefore the squares on AE and EG are double of the squares on AC and CD.

But $AE^2 + EG^2 = AG^2$. But the square on AG is equal to the squares on AE and EG (I. 47);

Therefore the square on AG is double of the squares on AC and CD. $\therefore AG^2 = AD^2 + DB^2.$

But the squares on AD and DG are equal to the square on AG (I. 47); $\therefore AD^2 + DB^2 = 2(AC^2 + CD^2).$

Therefore the squares on AD and DG are double of the squares on AC and CD.

And DG is equal to DB; therefore the squares on AD and DB are double of the squares on AC and CD.

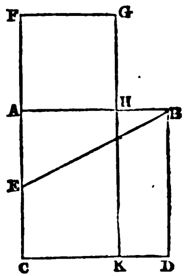
Therefore, if a straight line, &c. *Q.E.D.*

Proposition 11.—Problem.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square on the other part.

Let AB be the given straight line.

It is required to divide AB into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square on the other part.



Construction.—Upon AB describe the square ABDC (I. 46).

Bisect AC in E (I. 10), and join BE.

Produce CA to F, and make EF equal to EB (I. 3).

Upon AF describe the square AFGH (I. 46).

Produce GH to K.

Then AB shall be divided in H, so that the rectangle AB, BH is equal to the square on AH.

Proof.—Because the straight line AC is bisected in E, and produced to F,

The rectangle CF, FA, together with the square on AE, is equal to the square on EF (II. 6). $CF \cdot FA + AE^2 = EF^2$

But EF is equal to EB (Const.); $= EB^2$

Therefore the rectangle CF, FA, together with the square on AE, is equal to the square on EB.

But the square on EB is equal to the squares on AE and AB, because the angle EAB is a right angle (I. 47); $= AB^2 + AE^2.$

Therefore the rectangle CF, FA, together with the square on AE, is equal to the squares on AE and AB.

Take away the square on AE, which is common to both;

CF·FA =AB². Therefore the remaining rectangle CF, FA is equal to the square on AB (Ax. 3).

But the figure FK is the rectangle contained by CF and FA, for FA is equal to FG;

And AD is the square on AB;

∴FK=AD Therefore the figure FK is equal to AD.

Take away AK, then FH=HD or AB·BH =AH². Take away the common part AK, and the remainder FH is equal to the remainder HD (Ax. 3).

But HD is the rectangle contained by AB and BH, for AB is equal to BD;

And FH is the square on AH;

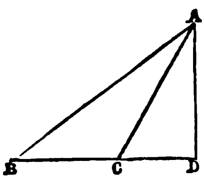
Therefore the rectangle AB,BH is equal to the square on AH.

Therefore the straight line AB is divided in H, so that the rectangle AB, BH is equal to the square on AH. *Q.E.F.*

Proposition 12.—Theorem.

In obtuse-angled triangles if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB; and from the point A let AD be drawn perpendicular to BC produced.



For $AB^2=AC^2+CB^2+2\,BC\cdot CD$.

The square on AB shall be greater than the squares on AC and CB by twice the rectangle BC, CD.

PROOF.—Because the straight line BD is divided into two parts in the point C,

$BD^2=BC^2+CD^2+2BC\cdot CD$. The square on BD is equal to the squares on BC and CD, and twice the rectangle BC, CD (II. 4).

To each of these equals add the square on DA;

Therefore the squares on BD and DA are equal to the squares on BC, CD, DA, and twice the rectangle BC, CD. $\therefore BD^2+DA^2$ or $AB^2 = BC^2+(CD^2+DA^2)+2BC\cdot CD = BC^2+AC^2+2BC\cdot CD.$

But the square on BA is equal to the squares on BD and DA, because the angle at D is a right angle (I. 47);

And the square on CA is equal to the squares on CD and DA (I. 47);

Therefore the square on BA is equal to the squares on BC and CA, and twice the rectangle BC, CD; that is, the square on BA is greater than the squares on BC and CA by twice the rectangle BC, CD.

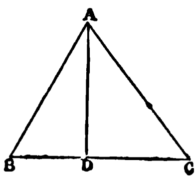
Therefore, in obtuse-angled triangles, &c. *Q.E.D.*

Proposition 13.—Theorem.

In every triangle, the square on the side subtending an acute angle is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle and the acute angle.

Let ABC be any triangle, and the angle at B an acute angle; and on BC, one oi the sides containing it, let fall the perpendicular AD from the opposite angle (I. 12).

The square on AC opposite to the angle B, shall be less than the squares on CB and BA, by twice the rectangle CB, BD. $AC^2 = CB^2 + AB^2 - 2CB\cdot BD.$



CASE I.—First, let AD fall within the triangle ABC.

PROOF.—Because the straight line CB is divided into two parts in the point D,

The squares on CB and BD are equal to twice the rectangle contained by CB, BD, and the square on DC (II. 7). For $CB^2+BD^2 = 2CB\cdot BD + DC^2.$

To each of these equals add the square on DA.

Therefore the squares on CB, BD, DA are equal to twice the rectangle CB, BD, and the squares on AD and DC. $\therefore CB^2+(BD^2+DA^2) = 2CB\cdot BD+(AD^2+DC^2),$ or $CB^2+BA^2 = 2CB\cdot DB + AC^2.$

But the square on AB is equal to the squares on BD and DA, because the angle BDA is a right angle (I. 47);

And the square on AC is equal to the squares on AD and DC (I. 47);

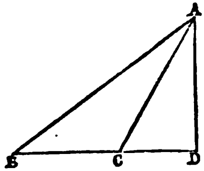
Therefore the squares on CB and BA are equal to the

$\therefore AC^2 = CB^2 + BA^2 - 2CB \cdot BD.$

square on AC, and twice the rectangle CB, BD; that is, the square on AC alone is less than the squares on CB and BA by twice the rectangle CB, BD.

CASE II.—Secondly, let AD fall without the triangle ABC.

PROOF.—Because the angle at D is a right angle (Const.), the angle ACB is greater than a right angle (I. 16);



$AB^2 = AC^2 + CB^2 + BC \cdot CD.$

Therefore the square on AB is equal to the squares on AC and CB, and twice the rectangle BC, CD (II. 12).

To each of these equals add the square on BC.

$\therefore AB^2 + BC^2 = AC^2 + 2(BC^2 + BC \cdot CD).$

Therefore the squares on AB and BC are equal to the square on AC, and twice the square on BC, and twice the rectangle BC, CD (Ax. 2).

But because BD is divided into two parts at C,

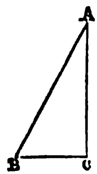
The rectangle DB, BC is equal to the rectangle BC, CD and the square on BC (II. 3);

Now $DB \cdot BC = BC \cdot CD + BC^2.$

And the doubles of these are equal, that is, twice the rectangle DB, BC is equal to twice the rectangle BC, CD and twice the square on BC;

$\therefore AC^2 = AB^2 + BC^2 - 2BC \cdot DB.$

Therefore the squares on AB and BC are equal to the square on AC, and twice the rectangle DB, BC; that is, the square on AC alone is less than the squares on AB and BC by twice the rectangle DB, BC.



CASE III.—Lastly, let the side AC be perpendicular to BC.

PROOF.—Then BC is the straight line between the perpendicular and the acute angle at B; and it is manifest that the squares on AB and BC are equal to the square on AC, and twice the square on BC (I. 47, and Ax. 2).

Therefore, in every triangle, &c. *Q.E.D.*

Proposition 14.—Problem.

To describe a square that shall be equal to a given rectilineal figure.

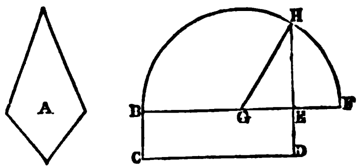
Let A be the given rectilineal figure.

It is required to describe a square that shall be equal to A.

CONSTRUCTION.—Describe the rectangular parallelogram BCDE equal to the rectilineal figure A (I. 45).

If then the sides of it, BE, ED, are equal to one another, it is a square, and what was required is now done.

But if they are not equal, produce one of them, BE, to F, and make EF equal to ED (I. 3).



Bisect BF in G (I. 10), and from the centre G, at the distance GB, or GF, describe the semicircle BHF;

Produce DE to H, and join GH;

Then the square described upon EH shall be equal to the rectilineal figure A.

PROOF.—Because the straight line BF is divided into two equal parts in the point G, and into two unequal parts in the point E,

The rectangle BE,EF, together with the square on GE, is equal to the square on GF (II. 5). $BE \cdot EF + GE^2 = GF^2 = GH^2 = GE^2 + EH^2.$

But GF is equal to GH;

Therefore the rectangle BE,EF, together with the square on GE, is equal to the square on GH.

But the square on GH is equal to the squares on GE and EH (I. 47);

Therefore the rectangle BE,EF, together with the square on GE, is equal to the squares on GE and EH. $\therefore BE \cdot EF$ or $BD = EH^2.$

Take away the square on GE, which is common to both;

Therefore the rectangle BE,EF is equal to the square on EH (Ax. 3).

But the rectangle contained by BE and EF is the parallelogram BD, because EF is equal to ED (Const.);

Therefore BD is equal to the square on EH.

But BD is equal to the rectilineal figure A (Const.);

Therefore the square on EH is equal to the rectilineal figure A. Hence $EH^2 = A.$

Therefore, a square has been made equal to the given rectilineal figure A, viz., the square described on EH. *Q.E.F.*

EXERCISES ON BOOK II.

PROP. 1—11.

1. Divide a given straight line into two such parts that the rectangle contained by them may be the greatest possible.

2. The sum of the squares of two straight lines is never less than twice the rectangle contained by the straight lines.

3. Divide a given straight line into two parts such that the squares of the whole line and one of the parts shall be equal to twice the square of the other part.

4. Given the sum of two straight lines and the difference of their squares, to find the lines.

5. In any triangle the difference of the squares of the sides is equal to the rectangle contained by the sum and difference of the parts into which the base is divided by a perpendicular from the vertical angle.

6. Divide a given straight line into such parts that the sum of their squares may be equal to a given square.

7. If ABCD be any rectangle, A and C being opposite angles, and O any point either within or without the rectangle—$OA^2 + OC^2 = OB^2 + OD^2$.

8. Let the straight line AB be divided into any two parts in the point C. Bisect CB in D, and take a point E in AC such that $EC = CD$. Then shall $AD^2 = AE^2 + AC \cdot CB$.

9. If a point C be taken in AB, and AB be produced to D so that BD and AC are equal, show that the squares described upon AD and AC together exceed the square upon AB by twice the rectangle contained by AE and AC.

10. From the hypothenuse of a right-angled triangle portions are cut off equal to the adjacent sides. Show that the square on the middle segment is equal to twice the rectangle under the extreme segments.

11. If a straight line be divided into any number of parts, the square of the whole line is equal to the sum of the squares of the parts, together with twice the rectangles of the parts taken two and two together.

12. If ABC be an isosceles triangle, and DE be drawn parallel to the base BC, cutting in D and E either the side or sides produced, and EB be joined; prove that $BE^2 = BC \cdot DE + CE^2$.

PROP. 12—14.

13. In any triangle show that the sum of the squares on the sides is equal to twice the square on half the base, and twice the square on the line drawn from the vertex to the middle of the base.

14. If squares are described on the sides of any triangle, find the difference between the sum of two of the squares and the third square, and show from your result what this becomes when the angle opposite the third square is a right angle.

15. Show also what the difference becomes when the vertex of the triangle is depressed until it coincide with the base.

16. The square on any straight line drawn from the vertex of an isosceles triangle, together with the rectangle contained by the segments of the base, is equal to the square upon a side of the triangle.

17. If a side of a triangle be bisected, and a perpendicular drawn from the middle point of the base to meet the side, then the square of the altitude of the triangle exceeds the square upon half the base by twice the rectangle contained by the side and the straight line between the points of section of the side.

18. In any triangle ABC, if perpendiculars be drawn from each of the angles upon the opposite sides, or opposite sides produced, meeting them respectively in D, E, F, show that—

$$BA^2 + AC^2 + CB^2 = 2\,AE \cdot AC + 2\,CD \cdot CB + 2\,BF \cdot BA;$$

all lines being measured in the *same direction* round the triangle.

19. Construct a square equal to the sum of the areas of two given rectilinear figures.

20. The base of a triangle is 63 ft., and the sides 25 ft. and 52 ft. respectively. Show that the segments of the base, made by a perpendicular from the vertex, are 15 ft. and 48 ft. respectively, and that the area of the triangle is 630 sq. ft.

21. In the same triangle, show that the length of the line joining the vertex with the middle of the base is 22·9 ft.

22. A ladder, 45 ft. long, reaches to a certain height against a wall, but, when turned over without moving the foot, must be shortened 6 ft. in order to reach the same height on the opposite side. Supposing the width of the street to be 42 ft., show that the height to which the ladder reaches is 36 ft.

23. The base and altitude of a triangle are 8 in. and 9 in. respectively; show that its area is equal to a square whose side is 6 in. Prove your result by construction.

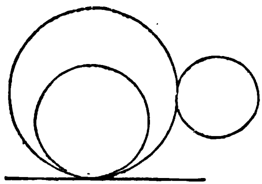
24. On the supposition that lines can be always expressed *exactly* in terms of some unit of length, what geometrical propositions may be deduced from the following algebraical identities?—

(1.) $(a + b)^2 = a^2 + 2\,ab + b^2$
(2.) $(a + b)\,(a - b) + b^2 = a^2$
(3.) $(2\,a + b)\,b + a^2 = (a + b)^2$
(4.) $(a + b)^2 + b^2 = 2\,(a + b)\,b + a^2$
(5.) $4\,(a + b)\,b + b^2 = (a + 2\,b)^2$
(6.) $(a + b)^2 + (a - b)^2 = 2\,a^2 + 2\,b^2$
(7.) $(2\,a + b)^2 + b^2 = 2\,a^2 + 2\,(a + b)^2$

EUCLID'S ELEMENTS, BOOK III.

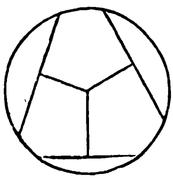
Definitions.

1. **Equal circles** are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.



2. **A straight line is said to touch a circle,** or to be a tangent to it, when it meets the circle, and being produced does not cut it.

3. **Circles** are said to **touch one another,** which meet, but do not cut one another.

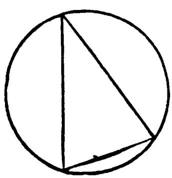


4. Straight lines are said to be **equally distant** from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.

5. And the straight line on which the greater perpendicular falls, is said to be **farther** from the centre.

6. A **segment of a circle** is the figure contained by a straight line and the circumference which it cuts off.

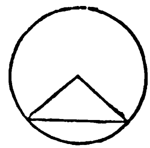
7. The **angle of a segment** is that which is contained by the straight line and the circumference.



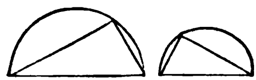
8. An **angle in a segment** is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line, which is the base of the segment.

9. An angle is said to **insist** or **stand upon** the circumference intercepted between the straight lines that contain the angle.

10. A **sector** of a circle is the figure contained by two straight lines drawn from the centre and the circumference between them.



11. **Similar segments** of circles are those which contain equal angles.



[Any portion of the circumference is called an *arc*, and the *chord* of an arc is the straight line joining its extremities.]

Proposition 1.—Problem.

To find the centre of a given circle.

Let ABC be the given circle.

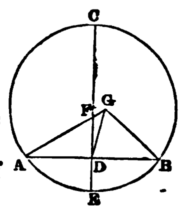
It is required to find its centre.

Construction.—Draw within the circle any chord AB, and bisect it in D (I. 10).

From the point D draw DC at right angles to AB (I. 11).

Produce CD to meet the circumference in E, and bisect CE in F (I. 10).

Then the point F shall be the centre of the circle ABC.



Proof.—For if F be not the centre, if possible let G be the centre; and join GA, GD, GB. Suppose G the centre.

Then, because DA is equal to DB (Const.), and DG common to the two triangles ADG, BDG;

The two sides AD, DG are equal to the two sides BD, DG, each to each;

And the base GA is equal to the base GB, being radii of the same circle;

Therefore the angle ADG is equal to the angle BDG (I. 8). ∴ ∠ADG = ∠BDG.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle (I. Def. 10).

Therefore the angle GDB is a right angle.

But the angle FDB is also a right angle (Const.);

∴ ∠GDB = ∠FDB. Therefore the angle GDB is equal to the angle FDB (Ax. 11), the less to the greater; which is impossible.

Therefore G is not the centre of the circle ABC.

In the same manner it may be shown that no point which is not in CE can be the centre.

And since the centre is in CE, it must be in F, its point of bisection.

Therefore F is the centre of the circle ABC: which was to be found.

Corollary.—From this it is manifest that, if in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

Proposition 2.—Theorem.

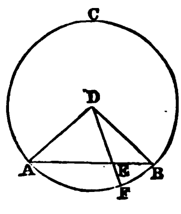
If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Let ABC be a circle, and A and B any two points in the circumference.

The straight line drawn from A to B shall fall within the circle.

Construction.—Find D the centre of the circle ABC (III. 1), and join DA, DB.

In AB take any point E; join DE, and produce it to the circumference in F.



Proof.—Because DA is equal to DB, the angle DAB is equal to the angle DBA (I. 5).

∠DBE = ∠DAB, and ∴ ∠DEB > DBE.

And because AE, a side of the triangle DAE, is produced to B, the exterior angle DEB is greater than the interior and opposite angle DAE (I. 16).

But the angle DAE was proved to be equal to the angle DBE;

Therefore the angle DEB is also greater than DBE.

But the greater angle is subtended by the greater side (I. 19);

∴ DB > DE. Therefore DB is greater than DE.

But DB is equal to DF; therefore DF is greater than DE, and the point E is therefore within the circle.

In the same manner it may be proved that every point in AB lies within the circle.

Therefore the straight line AB lies within the circle.

Therefore, if any two points, &c. *Q.E.D.*

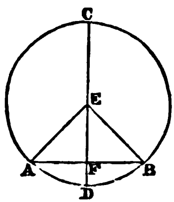
Proposition 3.—Theorem.

If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and conversely, if it cut it at right angles, it shall bisect it.

Let ABC be a circle, and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre.

CD shall cut AB at right angles.

CONSTRUCTION.—Take E, the centre of the circle (III. 1), and join EA, EB.



PROOF.—Because AF is equal to FB (Hyp.), and FE common to the two triangles AFE, BFE, and the base EA equal to the base EB (I. Def. 15),

Therefore the angle AFE is equal to the angle BFE (I. 8);

Therefore each of the angles AFE, BFE is a right angle (I. def. 10);

Triangles AFE and BEF are equal in every respect.

Therefore the straight line CD, drawn through the centre, bisecting another, AB, that does not pass through the centre, also cuts it at right angles.

Conversely, let CD cut AB at right angles.

CD shall bisect AB; that is, AF shall be equal to FB—the same construction being made.

PROOF.—Because the radii EA, EB are equal, the angle EAF is equal to the angle EBF (I. 5),

And the angle AFE is equal to the angle BFE (Hyp.),

Therefore, in the two triangles EAF, EBF, there are two angles in the one equal to two angles in the other, each to each, and the side EF, which is opposite to one of the equal angles in each, is common to both;

Therefore their other sides are equal (I. 26);

Therefore AF is equal to FB.

Therefore, if a straight line, &c. *Q.E.D.*

Proposition 4.—Theorem.

If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect each other.

Let ABCD be a circle, and AC, BD two straight lines in it, which cut one another at the point E, and do not both pass through the centre.

AC, BD shall not bisect one another.

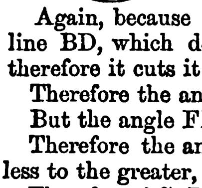
If one of the lines pass through the centre, it is plain that it cannot be bisected by the other, which does not pass through the centre.

If they bisect each other,

But if neither of them pass through the centre, if possible, let AE be equal to EC, and BE to ED.

CONSTRUCTION.—Take F, the centre of the circle (III. 1), and join EF.

EF bisects AC at right angles.



Because FE, a straight line drawn through the centre, bisects another line AC, which does not pass through the centre (Hyp.), therefore it cuts it at right angles (III. 3);

Therefore the angle FEA is a right angle.

And EF bisects BD at right angles.

Again, because the straight line FE bisects the straight line BD, which does not pass through the centre (Hyp.), therefore it cuts it at right angles (III. 3);

Therefore the angle FEB is a right angle.

But the angle FEA was shown to be a right angle;

Therefore the angle FEA is equal to the angle FEB, the less to the greater, which is impossible;

∴ ∠FEA = ∠FEB.

Therefore AC, BD do not bisect each other.

Therefore, if in a circle, &c. *Q.E.D.*

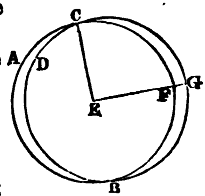
Proposition 5.—Theorem.

If two circles cut one another, they shall not have the same centre.

Let the two circles ABC, CDG cut one another in the points B, C.

They shall not have the same centre.

For, if it be possible, let E be their centre; join EC, and draw any straight line EFG, meeting the circles in F and G.



PROOF.—Because E is the centre of the circle ABC, EC is equal to EF (I. Def. 15). EC=EF

And because E is the centre of the circle CDG, EC is equal to EG. and =EG.

But EC was shown to be equal to EF;

Therefore EF is equal to EG (Ax. 1), the less to the greater, which is impossible; ∴EF=EG.

Therefore E is not the centre of the circles ABC, CDG.

Therefore, if two circles, &c. *Q.E.D.*

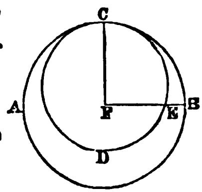
Proposition 6.—Theorem.

If one circle touch another internally, they shall not have the same centre.

Let the circle CDE touch the circle ABC internally in the point C.

They shall not have the same centre.

CONSTRUCTION.—For, if it be possible, let F be their centre; join FC, and draw any straight line FEB, meeting the circles in E and B. If they have the same centre F,



PROOF.—Because F is the centre of the circle ABC, FC is equal to FB (I Def. 15).

And because F is the centre of the circle CDE, FC is equal to FE. FC=FE and=FB.

But FC was shown to be equal to FB;

Therefore FE is equal to FB (Ax. 1), the less to the greater, which is impossible; ∴FE=FB

Therefore F is not the centre of the circles ABC, CDE.

Therefore, if one circle, &c. *Q.E.D.*

Proposition 7.—Theorem.

If any point be taken in the diameter of a circle, which is not the centre of all the straight lines which can be drawn from this point to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and, of any others, that which is nearer to the straight line which passes through the centre is always greater than one more remote; and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the shortest line.

Let ABCD be a circle, AD its diameter, and E its centre, in which let any point F be taken which is not the centre.

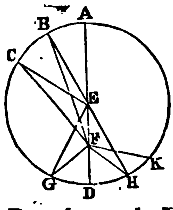
FA > FB > FC > FG &c., > FD the least. Of all the straight lines FB, FC, FG, &c., that can be drawn from F to the circumference, FA, which passes through the centre, shall be the greatest;

FD, the other part of the diameter AD, shall be the least;

And of the others, FB, the nearer to FA, shall be greater than FC, the more remote; and FC greater than FG.

Construction.—Join BE, CE, GE.

For BE + EF or AF > BF. Proof.—Because any two sides of a triangle are greater than the third side, BE, EF are greater than BF (I. 20).



But AE is equal to BE; therefore AE, EF, that is, AF is greater than BF.

Again, because BE is equal to CE, and EF common to the two triangles BEF, CEF, the two sides BE, EF are equal to the two CE, EF, each to each.

But the angle BEF is greater than the angle CEF;

And base BF > base FC. Therefore the base FB is greater than the base FC (I. 24).

In the same manner it may be shown that FC is greater than FG.

Again GF + FE > EG. Again, because GF, FE are greater than EG (I. 20), and that EG is equal to ED;

and ∴ > ED. Therefore GF, FE are greater than ED.

∴ GF > FD. Take away the common part FE, and the remainder GF is greater than the remainder FD (Ax. 5).

Therefore FA is the greatest, and FD the least, of all the straight lines from F to the circumference; and FB is greater than FC, and FC than FG.

Also, there cannot be drawn more than two equal straight lines from the point F to the circumference, one on each side of the shortest line.

CONSTRUCTION.—At the point E, in the straight line EF, make the angle FEH equal to the angle FEG (I. 23), and join FH.

PROOF.—Because EG is equal to EH, and EF common to the two triangles GEF, HEF, the two sides EG, EF are equal to the two sides EH, EF, each to each; **Triangles GEF and HEF are equal in every respect.**

And the angle GEF is equal to the angle HEF (Const.);

Therefore the base FG is equal to the base FH (I. 4).

But, besides FH, no other straight line can be drawn from F to the circumference equal to FG.

For, if it be possible, let FK be equal to FG;

Then, because FK is equal to FG, and FH is also equal to FG, therefore FH is equal to FK; **And if FK = FG, it also = FH, which is impossible.**

That is, a line nearer to that which passes through the centre is equal to a line which is more remote; which is impossible by what has been already shown.

Therefore, if any point, &c. *Q.E.D.*

Proposition 8.—Theorem.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, one of which passes through the centre; of those which fall on the concave circumference, the greatest is that which passes through the centre, and of the rest, that which is nearer to the one passing through the centre is always greater than one more remote; but of those which fall on the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote; and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the least line.

Let ABC be a circle, and D any point without it, and from D let the straight lines DA, DE, DF, DC be drawn to the circumference, of which DA passes through the centre.

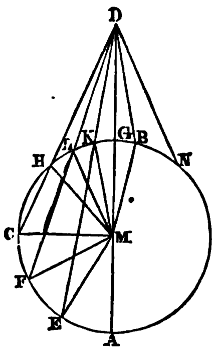
$DA > DE > DF > DC$,

Of those which fall on the concave part of the circumference AEFC, the greatest shall be DA, which passes through the centre, and the nearer to it shall be greater than the more remote, viz., DE greater than DF, and DF greater than DC.

and $DG < DK < DL < DH$

But of those which fall on the convex circumference GKLH, the least shall be DG between the point D and the diameter AG, and the nearer to it shall be less than the more remote, viz., DK less than DL, and DL less than DH.

Construction.—Take M, the centre of the circle ABC (III. 1), and join ME, MF, MC, MH, ML, MK.



Proof.—Because any two sides of a triangle are greater than the third side, EM, MD are greater than ED (I. 20).

But EM is equal to AM; therefore AM, MD are greater than ED—that is, AD is greater than ED.

For $EM + MD$ or $AD > ED$.

Again, because EM is equal to FM, and MD common to the two triangles EMD, FMD; the two sides EM, MD are equal to the two sides FM, MD, each to each;

But the angle EMD is greater than the angle FMD;

Therefore the base ED is greater than the base FD (I. 24).

Also base $ED >$ base FD, &c.

In like manner it may be shown that FD is greater than CD;

Therefore DA is the greatest, and DE greater than DF and DF greater than DC.

Again $MK + KD > MD$, or $> MK + DG$. $\therefore KD > DG$, and $\therefore DG < KD$, &c.

Again, because MK, KD are greater than MD (I. 20), and MK is equal to MG;

The remainder KD is greater than the remainder GD—that is, GD is less than KD.

And because MK, DK are drawn to the point K within the triangle MLD from M and D, the extremities of its side MD;

Therefore MK, DK are less than ML, LD (I. 21).

But MK is equal to ML; therefore the remainder KD is less than the remainder LD.

In like manner it may be shown that LD is less than HD.

Therefore DG is the least, and KD less than DL, and DL less than DH.

Also, there can be drawn only two equal straight lines from the point D to the circumference, one on each side of the least line.

CONSTRUCTION.—At the point M, in the straight line MD, make the angle DMB equal to the angle DMK (I. 23), and join DB.

PROOF.—Because MK is equal to MB, and MD common to the two triangles KMD, BMD; the two sides KM, MD are equal to the two sides BM, MD, each to each; Triangles KMD and BMD are equal in every respect.

And the angle DMK is equal to the angle DMB (Const);

Therefore the base DK is equal to the base DB (I. 4).

But, besides DB, no other straight line can be drawn from D to the circumference equal to DK.

For, if it be possible, let DN be equal to DK.

Then, because DN is equal to DK, and DB is also equal to DK, therefore DB is equal to DN (Ax. 1); DN=DK and ∴ =DB.

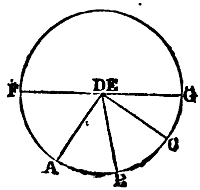
That is, a line nearer to the least is equal to one which is more remote; which is impossible by what has been already shown. which is impossible.

Therefore, if any point, &c. *Q.E.D.*

Proposition 9.—Theorem.

If a point be taken within a circle, from which there can be drawn more than two equal straight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC, from which to the circumference there can be drawn more than two equal straight lines, viz., DA, DB, DC.



The point D shall be the centre of the circle.

CONSTRUCTION.—For if not, let E be the centre; join DE, and produce it to the circumference in F and G.

PROOF.—Then FG is a diameter of the circle ABC (I. Def. 17).

If D be not the centre. And because in FG, the diameter of the circle ABC, there is taken the point D, not the centre;

DG > DC > DB > DA. Therefore DG is the greatest straight line from D to the circumference, and DC is greater than DB, and DB greater than DA (III. 7);

But they are also equal. But these lines are likewise equal, by hypothesis; which is impossible.

Therefore E is not the centre of the circle ABC.

In like manner it may be demonstrated that any other point than D is not the centre;

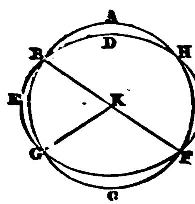
Therefore D is the centre of the circle ABC.

Therefore, if a point, &c. *Q.E.D.*

Proposition 10.—Theorem.

One circumference of a circle cannot cut another in more than two points.

If possible, CONSTRUCTION.—If it be possible, let the circumference ABC cut the circumference DEF in more than two points, viz., in the points B, G, F.



Take the centre K of the circle ABC (III. 1), and join KB, KG, KF.

PROOF.—Because K is the centre of the circle ABC, the radii KB, KG, KF are all equal.

the two circles have the same centre. And because within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal straight lines KB, KG, KF, therefore K is the centre of the circle DEF (III. 9).

But K is also the centre of the circle ABC (Const.);

Therefore the same point is the centre of two circles which cut one another; which is impossible (III. 5).

Therefore, one circumference, &c. *Q.E.D.*

Proposition 11.—Theorem.

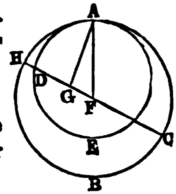
If one circle touch another internally in any point, the straight line which joins their centres, being produced, shall pass through that point.

Let the circle ADE touch the circle ABC internally in the point A; and let F be the centre of the circle ABC, and G the centre of the circle ADE.

The straight line which joins their centres, being produced, shall pass through the point of contact A.

CONSTRUCTION.—For, if not, let it pass otherwise, if possible, as FGDH. Join AF and AG. If not,

PROOF.—Because AG, GF are greater than AF (I. 20), and AF is equal to HF (I. def. 15);



Therefore AG, GF are greater than HF.

Take away the common part GF, and the remainder AG is greater than the remainder HG. AG > HG.

But AG is equal to DG (I. Def. 15);

Therefore DG is greater than HG, the less than the greater; which is impossible. But AG = DG. ∴ DG > HG.

Therefore the straight line which joins the centres, being produced, cannot fall otherwise than upon the point A, that is, it must pass through it.

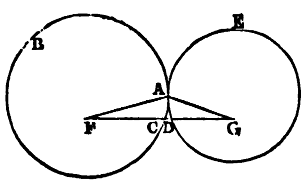
Therefore, if one circle, &c. *Q.E.D.*

Proposition 12.—Theorem.

If two circles touch each other externally in any point, the straight line which joins their centres shall pass through that point.

Let the two circles ABC, ADE touch each other externally in the point A; and let F be the centre of the circle ABC, and G the centre of the circle ADE;

The straight line which joins their centres shall pass through the point of contact A.



CONSTRUCTION.—For, if not, let it pass otherwise, if possible, as FCDG. Join FA and AG. If not,

PROOF.—Because F is the centre of the circle ABC, FA is equal to FC (I. Def. 15).

And because G is the centre of the circle ADE, GA is equal to GD;

Therefore FA, AG are equal to FC, DG (Ax. 2).

FG is > FA+AG, but it is also *less*.

Therefore the whole FG is greater than FA, AG.

But FG is also less than FA, AG (I. 20), which is impossible.

Therefore the straight line which joins the centres of the circles shall not pass otherwise than through the point A, that is, it must pass through it.

Therefore, if two circles, &c. *Q.E.D.*

Proposition 13.—Theorem.

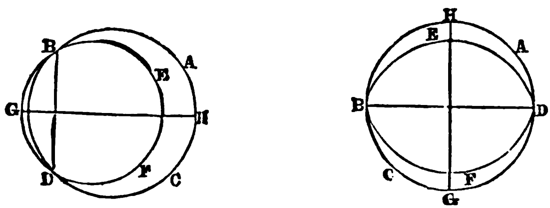
One circle cannot touch another in more points than one, whether it touch it internally or externally.

I. First, let the circle EBF touch the circle ABC internally in the point B.

Then EBF cannot touch ABC in any other point.

If possible, let it touch in D also;

CONSTRUCTION.—If it be possible, let EBF touch ABC in another point D; join BD, and draw GH bisecting BD at right angles (I. 10, 11).



PROOF.—Because the two points B, D are in the circumference of each of the circles, the straight line BD falls within each of them (III. 2).

Therefore the centre of each circle is in the straight line GH, which bisects BD at right angles (III. 1 cor.)

then GH passes through the point of contact, which it does not.

Therefore GH passes through the point of contact (III. 11).

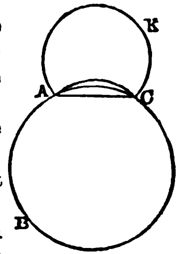
But GH does not pass through the point of contact, because the points B, D are out of the line of GH; which is absurd.

Therefore one circle cannot touch another internally in more points than one.

II. Next, let the circle ACK touch the circle ABC externally in the point A.

Then ACK cannot touch ABC in any other point.

CONSTRUCTION.—If it be possible, let ACK touch ABC in another point C. Join AC. *If possible let it touch in C also;*



PROOF.—Because the points A, C are in the circumference of the circle ACK, the straight line AC must fall within the circle ACK (III. 2).

But the circle ACK is without the circle ABC (Hyp.);

Therefore the straight line AC is without the circle ABC.

But because the two points A, C are in the circumference of the circle ABC, the straight line AC falls within the circle ABC (III. 2); which is absurd. *then AC falls without the circle ABC, which is absurd.*

Therefore one circle cannot touch another externally in more points than one.

And it has been shown that one circle cannot touch another internally in more points than one.

Therefore, one circle, &c. *Q.E.D.*

Proposition 14.—Theorem.

Equal straight lines in a circle are equally distant from the centre; and, conversely, those which are equally distant from the centre are equal to one another.

Let the straight lines AB, CD, in the circle ABDC, be equal to one another.

Then they shall be equally distant from the centre.

CONSTRUCTION.—Take E, the centre of the circle ABDC (III. 1).

From E draw EF, EG, perpendiculars to AB, CD (I. 12).

Join EA, EC.

PROOF.—Because the straight line EF, passing through the centre, cuts the straight line AB, which does not pass through the centre, at right angles, it also bisects it (III. 3).

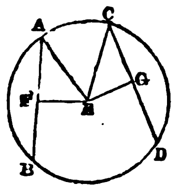
Therefore AF is equal to FB, and AB is double of AF.

For the like reason CD is double of CG.

AF = CG,

But AB is equal to CD (Hyp.); therefore AF is equal to CG (Ax. 7).

And because AE is equal to CE, the square on AE is equal to the square on CE.



But the squares on AF, FE are equal to the square on AE, because the angle AFE is a right angle (I. 47).

For the like reason the squares on CG, GE are equal to the square on CE;

and $AF^2+FE^2 = CG^2 + EG^2$.

Therefore the squares on AF, FE are equal to the squares on CG, GE (Ax. 1).

But the square on AF is equal to the square on CG, because AF is equal to CG;

Therefore the remaining square on FE is equal to the remaining square on GE (Ax. 3);

∴ EF=EG.

And therefore the straight line EF is equal to the straight line EG.

But straight lines in a circle are said to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal (III. Def. 4);

Therefore AB, CD are equally distant from the centre.

Conversely, let the straight lines AB, CD be equally distant from the centre, that is, let EF be equal to EG;

Here EF = EG, and $AF^2+EF^2 = CG^2 + EG^2$.

Then AB shall be equal to CD.

PROOF.—The same construction being made, it may be demonstrated, as before, that AB is double AF, and CD double of CG, and that the squares on EF, FA are equal to the squares on EG, GC.

But the square on EF is equal to the square on EG, because EF is equal to EG (Hyp.);

Therefore the remaining square on FA is equal to the remaining square on GC (Ax. 3),

∴ AF = CG, &c.

And therefore the straight line AF is equal to the straight line CG.

But AB was shown to be double of AF, and CD double of CG;

Therefore AB is equal to CD (Ax. 6);

Therefore, equal straight lines, &c. *Q.E.D.*

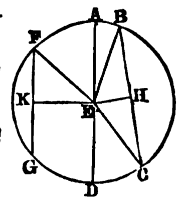
Proposition 15.—Theorem.

The diameter is the greatest straight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.

Let ABCD be a circle, of which AD is a diameter, and E the centre; and let BC be nearer to the centre than FG.

Then AD shall be greater than any straight line CB, which is not a diameter; and BC shall be greater than FG.

CONSTRUCTION.—From the centre E draw EH, EK perpendiculars to BC, FG (I. 12), and join EB, EC, EF.



PROOF.—Because AE is equal to BE, and ED to EC,

Therefore AD is equal to BE, EC.

But BE, EC are greater than BC (I. 20);

Therefore also AD is greater than BC. BE + EC or AD > BC,

And because BC is nearer to the centre than FG (Hyp.), EH is less than EK (III. Def. 5). and EH < EK.

But, as was demonstrated in the preceding proposition, BC is double of BH, and FG double of FK, and the squares on EH, HB are equal to the squares on EK, KF. ∴ since $EH^2 + HB^2 = EK^2 + KF^2$,

But the square on EH is less than the square on EK, because EH is less than EK;

Therefore the square on HB is greater than the square on KF, and the straight line BH greater than FK; HB > FK.

And therefore BC is greater than FG.

Conversely, let BC be greater than FG.

Then BC shall be nearer to the centre than FG, that is, the same construction being made, EH shall be less than EK.

PROOF.—Because BC is greater than FG, BH is greater than FK.

But the squares on BH, HE are equal to the squares on FK, KE;

And the square on BH is greater than the square on FK, because BH is greater than FK;

Therefore the square on HE is less than the square on KE, and the straight line EH less than EK;

And therefore BC is nearer to the centre than FG (III. def. 5).

Therefore, the diameter, &c. *Q.E.D.*

Proposition 16.—Theorem.

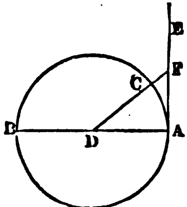
The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and a straight line, making an acute angle with the diameter at its extremity, cuts the circle.

Let ABC be a circle, of which D is the centre, and AB a diameter, and AE a line drawn from A perpendicular to AB.

The straight line AE shall fall without the circle.

Take any point F in AE,

CONSTRUCTION.—In AE take any point F; join DF, and let DF meet the circle in C.



then DF > DA and ∴ > DC.

PROOF.—Because DAF is a right angle, it is greater than the angle AFD (I. 17);

Therefore DF is greater than DA (I. 19).

But DA is equal to DC; therefore DF is greater than DC.

Therefore the point F is without the circle.

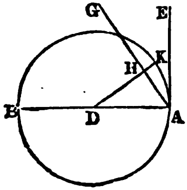
In the same manner it may be shown that any other point in AE, except the point A, is without the circle.

Therefore AE falls without the circle.

Again, let AG make with the diameter the angle DAG less than a right angle.

The line AG shall fall within the circle, and hence cut it.

Draw DH at right angles to HG, then DH < DA, and ∴ < DK.



CONSTRUCTION.—From D draw DH at right angles to AG, and meeting the circumference in K (I. 12).

PROOF.—Because DHA is a right angle, and DAH less than a right angle;

Therefore the side DH is less than the side DA (I. 19).

But DK is equal to DA; therefore DH is less than DK.

Therefore the point H is within the circle.

Therefore the straight line AG cuts the circle.
Therefore, the straight line, &c. *Q.E.D.*

Corollary.—From this it is manifest that the straight line which is drawn at right angles to the diameter of a circle, from the extremity of it, touches the circle (III. Def. 2); and that it touches it only in one point, because if it did meet the circle in two points it would fall within it (III. 2). Also it is evident that there can be but one straight line which touches the circle in the same point.

Proposition 17.—Problem.

To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

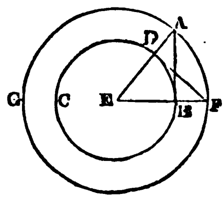
First, let the given point A be without the given circle BCD.

It is required to draw from A a straight line which shall touch the given circle.

Construction.—Find the centre E of the circle (III. 1), and join AE.

From the centre E, at the distance EA, describe the circle AFG.

From the point D draw DF at right angles to EA (I. 11), and join EBF and AB.



Then AB shall touch the circle BCD.

Proof.—Because E is the centre of the circles AFG, BCD, EA is equal to EF, and ED to EB;

Therefore the two sides AE, EB are equal to the two sides FE, ED, each to each; EA, ED respectively = EF, EB and ∠ E common;

And the angle at E is common to the two triangles AEB, FED;

Therefore the base AB is equal to the base FD, and the triangle AEB to the triangle FED, and the other angles to the other angles, each to each, to which the equal sides are opposite (I. 4);

Therefore the angle ABE is equal to the angle FDE. ∴ ∠ ABE = ∠ FDE a right angle.

But the angle FDE is a right angle (Const.);

Therefore the angle ABE is a right angle (Ax. 1).

And EB is drawn from the centre (Const.)

But the straight line drawn at right angles to a diameter of a circle, from the extremity of it, touches the circle (III. 16, cor.);

∴ AB touches the circle.

Therefore AB touches the circle, and it is drawn from the given point A.

Next, let the given point be in the circumference of the circle, at the point D.

Draw DE to the centre E, and DF at right angles to DE;

Then DF touches the circle (III. 16, cor.)

Therefore, from the given points A and D, straight lines, AB and DF, have been drawn, touching the given circle BCD. *Q.E.F.*

Proposition 18.—Theorem.

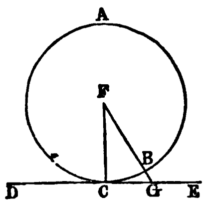
If a straight line touch a circle, the straight line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.

Let the straight line DE touch the circle ABC in the point C; take the centre F (III. 1), and draw the straight line FC.

FC shall be perpendicular to DE.

If not, suppose FG perpendicular.

CONSTRUCTION.—For, if not, let FG be drawn from the point F perpendicular to DE, meeting the circumference in B.



PROOF.—Because FGC is a right angle (Hyp.), FCG is an acute angle (I. 17), and to the greater angle the greater side is opposite (I. 19);

Therefore FC is greater than FG.

Then must FB > FG.

But FC is equal to FB; therefore FB is greater than FG; the part greater than the whole, which is impossible.

Therefore FG is not perpendicular to DE.

In the same manner it may be shown that no other straight line from F is perpendicular to DE, but FC; therefore FC is perpendicular to DE.

Therefore, if a straight line, &c. *Q.E.D.*

Proposition 19.—Theorem.

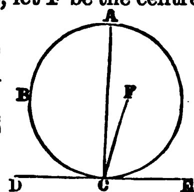
If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let the straight line DE touch the circle ABC in C, and from C let CA be drawn at right angles to DE.

The centre of the circle shall be in CA.

CONSTRUCTION.—For, if not, if possible, let F be the centre, and join CF.

If not, take F the centre, out of the line.



PROOF.—Because DE touches the circle ABC, and FC is drawn from the assumed centre to the point of contact,

Therefore FC is perpendicular to DE (III. 18);

Therefore FCE is a right angle.

But the angle ACE is also a right angle (Const.);

Therefore the angle FCE is equal to the angle ACE; the less to the greater, which is impossible.

Then ∠ACE = ∠FCE, being right angles.

Therefore F is not the centre of the circle ABC.

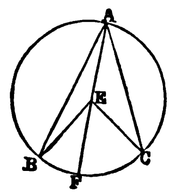
In the same manner it may be shown that no other point which is not in CA is the centre; therefore the centre is in CA.

Therefore, if a straight line, &c. *Q.E.D.*

Proposition 20.—Theorem.

The angle at the centre of a circle is double of the angle at the circumference, upon the same base, that is, upon the same arc.

Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the same arc BC for their base.



The angle BEC shall be double of the angle BAC.

CASE I.—First, let the centre E of the circle be within the angle BAC.

CONSTRUCTION.—Join AE, and produce it to the circumference in F.

PROOF.—Because EA is equal to EB, the angle EAB is equal to the angle EBA (I. 5);

Therefore the angles EAB, EBA are double of the angle EAB.

∠ BEF = ∠ EAB + ∠ EBA = 2 ∠ EAB, and so ∠ FEC = 2 ∠ EAC.

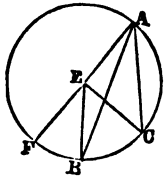
But the angle BEF is equal to the angles EAB, EBA (I. 32);

Therefore the angle BEF is double of the angle EAB.

For the same reason the angle FEC is double of the angle EAC.

∴ ∠ BEC = 2 ∠ BAC.

Therefore the whole angle BEC is double of the whole angle BAC.



CASE II.—Next, let the centre E of the circle be without the angle BAC.

CONSTRUCTION.—Join AE, and produce it to meet the circumference in F.

∠ FEC = 2 ∠ EAC, and ∠ FEB = 2 ∠ CAB.

PROOF.—It may be demonstrated, as in the first case, that the angle FEC is double of the angle FAC, and that FEB, a part of FEC, is double of FAB, a part of FAC;

∴ taking the difference ∠ BEC = 2 ∠ BAC.

Therefore the remaining angle BEC is double of the remaining angle BAC.

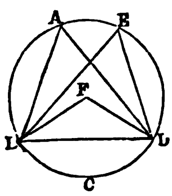
Therefore, the angle at the centre, &c. *Q.E.D.*

Proposition 21.—Theorem.

The angles in the same segment of a circle are equal to one another.

Let ABCD be a circle, and BAD, BED angles in the same segment BAED.

The angles BAD, BED shall be equal to one another.



CASE I.—First, let the segment BAED be greater than a semicircle.

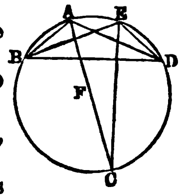
CONSTRUCTION.—Take F, the centre of the circle ABCD (III. 1), and join BF, DF.

PROOF.—Because the angle BFD is at the centre, and the angle BAD at the circumference, and that they have the same arc for their base, namely, BCD;

∠ BFD = 2 ∠ BAD,

Therefore the angle BFD is double of the angle BAD (III. 20).

For the same reason, the angle BFD is double of the angle BED; *and also = 2 ∠ BED.*



Therefore the angle BAD is equal to the angle BED (Ax. 7). *∴ ∠ BAD = ∠ BED.*

CASE II.—Next, let the segment BAED be not greater than a semicircle.

CONSTRUCTION.—Draw AF to the centre, and produce it to C, and join CE.

PROOF.—Then the segment BADC is greater than a semicircle, and therefore the angles BAC, BEC in it are equal by the first case. *∠ BAC = ∠ BEC,*

For the same reason, because the segment CBED is greater than a semicircle, the angles CAD, CED are equal. *and ∠ CAD = ∠ CED.*

Therefore the whole angle BAD is equal to the whole angle BED (Ax. 2). *∴ ∠ BAD = ∠ BED.*

Therefore, the angles in the same segment, &c. *Q.E.D.*

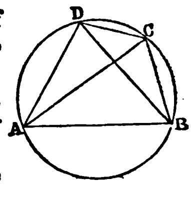
Proposition 22.—Theorem.

The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

Let ABCD be a quadrilateral figure inscribed in the circle ABCD.

Any two of its opposite angles shall be together equal to two right angles.

CONSTRUCTION.—Join AC, BD.



PROOF.—Because the three angles of every triangle are together equal to two right angles (I. 32),

The three angles of the triangle CAB, namely, CAB, ACB, ABC, are together equal to two right angles. *∠ CAB + ∠ ACB + ∠ ABC = 2 right angles.*

But the angle CAB is equal to the angle CDB, because they are in the same segment CDAB (III. 21); *The first two together = ∠ CDB + ∠ ADB = ∠ ADC.*

And the angle ACB is equal to the angle ADB, because they are in the same segment ADCB;

Therefore the two angles CAB, ACB are together equal to the whole angle ADC (Ax. 2).

To each of these equals add the angle ABC;

Therefore the three angles CAB, ACB, ABC are equal to the two angles ABC, ADC.

But the angles CAB, ACB, ABC are together equal to two right angles (I. 32);

∴ ∠ ADC + ∠ ABC = 2 right angles.

Therefore also the angles ABC, ADC are together equal to two right angles.

In like manner it may be shown that the angles BAD, BCD are together equal to two right angles.

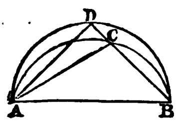
Therefore, the opposite angles, &c. *Q.E.D.*

Proposition 23.—Theorem.

Upon the same straight line, and on the same side of it, there cannot be two similar segments of circles not coinciding with one another.

If possible,

If it be possible, on the same straight line AB, and on the same side of it, let there be two similar segments of circles ACB, ADB not coinciding with one another.



Construction.—Then, because the circle ACB cuts the circle ADB in the two points A, B, they cannot cut one another in any other point (III. 10);

Therefore one of the segments must fall within the other.

Let ACB fall within ADB; draw the straight line BCD, and join AC, AD.

Proof.—Because the segment ACB is similar to the segment ADB (Hyp.), and that similar segments of circles contain equal angles (III. Def. 11);

exterior ∠ ACB = interior and opposite ∠ ADC.

Therefore the angle ACB is equal to the angle ADB; that is, the exterior angle of the triangle ACD, equal to the interior and opposite angle; which is impossible (I. 16).

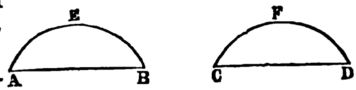
Therefore, there cannot be two similar segments of circles on the same straight line, and on the same side of it, which do not coincide. *Q.E.D.*

Proposition 24.—Theorem.

Similar segments of circles upon equal straight lines are equal to one another.

Let AEB, CFD be similar segments of circles upon the equal straight lines AB, CD.

The segment AEB shall be equal to the segment CFD.



They are equal, because they must coincide by Prop. 23.

PROOF.—For if the segment AEB be applied to the segment CFD, so that the point A may be on the point C, and the straight line AB on the straight line CD,

Then the point B shall coincide with the point D, because AB is equal to CD.

And the straight line AB coinciding with CD, the segment AEB must coincide with the segment CFD (III. 23); and is therefore equal to it.

Therefore, similar segments, &c. *Q.E.D.*

Proposition 25.—Problem.

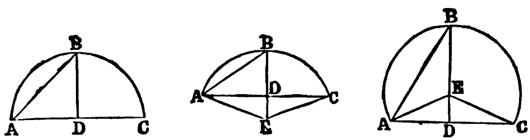
A segment of a circle being given, to describe the circle of which it is the segment.

Let ABC be the given segment of a circle.

It is required to describe the circle of which ABC is a segment.

CONSTRUCTION.—Bisect AC in D (I. 10).

From the point D draw DB at right angles to AC (I. 11), and join AB.



CASE I.—First, let the angles ABD, BAD be equal to one another.

Then D shall be the centre of the circle required.

When ∠ ABD = ∠ BAD, DA = DB = DC.

PROOF.—Because the angle ABD is equal to the angle BAD (Hyp.);

Therefore DB is equal to DA (I. 6).

But DA is equal to DC (Const.);

Therefore DB is equal to DC (Ax. 1).

Therefore the three straight lines DA, DB, DC are all equal;

and ∴ D the centre.

And therefore D is the centre of the circle (III. 9).

Hence, if from the centre D, at the distance of any of the three lines, DA, DB, DC a circle be described, it will pass through the other two points, and be the circle required.

CASE II.—Next, let the angles ABD, BAD be not equal to one another.

Make ∠ BAE = ∠ ABD.

CONSTRUCTION.—At the point A, in the straight line AB, make the angle BAE equal to the angle ABD (I. 23);

Produce BD, if necessary, to E, and join EC.

Then E shall be the centre of the circle required.

∴ EA = EB

PROOF.—Because the angle BAE is equal to the angle ABE (Const.), EA is equal to EB (I. 6).

And because AD is equal to CD (Const.), and DE is common to the two triangles ADE, CDE,

The two sides AD, DE are equal to the two sides CD, DE, each to each;

And the angle ADE is equal to the angle CDE, for each of them is a right angle (Const.);

and EA = EC.

Therefore the base EA is equal to the base EC (I. 4).

But EA was shown to be equal to EB;

Therefore EB is equal to EC (Ax. 1).

∴ EA = EB = EC,

Therefore the three straight lines EA, EB, EC are all equal;

and therefore E is the centre.

And therefore E is the centre of the circle (III. 9).

Hence, if from the centre E, at the distance of any of the three lines EA, EB, EC, a circle be described, it will pass through the other two points, and be the circle required.

In the *first* case, it is evident that, because the centre D is in AC, the segment ABC is a semicircle.

In the *second* case, if the angle ABD be greater than BAD, the centre E falls without the segment ABC, which is therefore less than a semicircle;

But if the angle ABD be less than the angle BAD, the centre E falls within the segment ABC, which is therefore greater than a semicircle.

Therefore, a segment of a circle being given, the circle has been described of which it is a segment. *Q.E.F.*

Proposition 26.—Theorem.

In equal circles, equal angles stand upon equal arcs, whether they be at the centres or at the circumferences.

Let ABC, DEF be equal circles, having the equal angles BGC, EHF at their centres, and BAC, EDF at their circumferences.

The arc BKC shall be equal to the arc ELF.

CONSTRUCTION.—Join BC, EF.

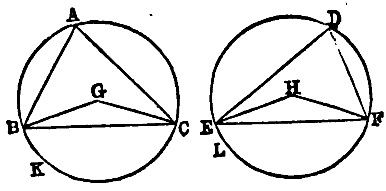
PROOF.—Because the circles ABC, DEF are equal (Hyp.), the straight lines from their centres are equal (III. def. 1);

Therefore the two sides BG, GC are equal to the two sides EH, HF, each to each; Triangles BGC and EHF are equal in every respect.

And the angle at G is equal to the angle at H (Hyp.);

Therefore the base BC is equal to the base EF (I. 4).

And because the angle at A is equal to the angle at D (Hyp.),



The segment BAC is similar to the segment EDF (III. def. 11), ∴ segments BAC and EDF are similar and on equal straight lines.

And they are on equal straight lines BC, EF.

But similar segments of circles on equal straight lines are equal to one another (III. 24);

Therefore the segment BAC is equal to the segment EDF. ∴ are equal.

But the whole circle ABC is equal to the whole circle DEF (Hyp.);

Therefore the remaining segment BKC is equal to the remaining segment ELF (Ax. 3).

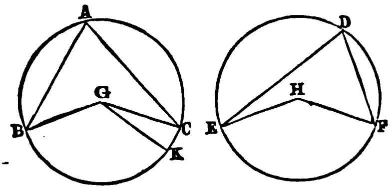
∴ arc BKC = arc ELF. Therefore the arc BKC is equal to the arc ELF.
Therefore, in equal circles, &c. *Q.E.D.*

Proposition 27.—Theorem.

In equal circles, the angles which stand upon equal arcs are equal to one another, whether they be at the centres or at the circumferences.

Let ABC, DEF be equal circles, and let the angles BGC, EHF, at their centres, and the angles BAC, EDF, at their circumferences, stand on equal arcs BC, EF.

The angle BGC shall be equal to the angle EHF, and the angle BAC equal to the angle EDF.



CONSTRUCTION.—If the angle BGC be equal to the angle EHF, it is manifest that the angle BAC is also equal to the angle EDF (III. 20, ax. 7).

But, if not, one of them must be the greater. Let BGC be the greater, and at the point G, in the straight line BG, make the angle BGK equal to the angle EHF (I. 23).

If one ∠ is greater than the other, the corresponding arc is greater.

PROOF.—Because the angle BGK is equal to the angle EHF, and that in equal circles equal angles stand on equal arcs, when they are at the centres (III. 26);

Therefore the arc BK is equal to the arc EF.

But the arc EF is equal to the arc BC (Hyp.);

Therefore the arc BK is equal to the arc BC (Ax. 1); the less to the greater, which is impossible.

Therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it.

And the angle at A is half of the angle BGC, and the angle at D is half of the angle EHF (III. 20);

Therefore the angle at A is equal to the angle at D (Ax. 7).

Therefore, in equal circles, &c. *Q.E.D.*

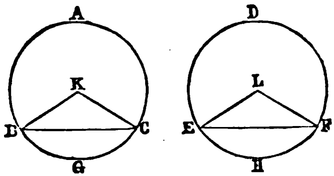
Proposition 28.—Theorem.

In equal circles, equal chords cut off equal arcs, the greater equal to the greater, and the less equal to the less.

Let ABC, DEF be equal circles, and BC, EF equal chords in them, which cut off the two greater arcs BAC, EDF, and the two less arcs BGC, EHF.

The greater arc BAC shall be equal to the greater arc EDF, and the less arc BGC equal to the less arc EHF.

CONSTRUCTION.—Take K, L, the centres of the circles (III. 1), and join BK, KC, EL, LF. Take K and L the centres.



PROOF.—Because the circles ABC, DEF are equal, their radii are equal (III. def. 1).

Therefore the two sides BK, KC are equal to the two sides EL, LF, each to each;

And the base BC is equal to the base EF (Hyp.);

Therefore the angle BKC is equal to the angle ELF (I. 8). Triangles BKC and ELF are equal in every respect.

But in equal circles equal angles stand on equal arcs, when they are at the centres (III. 26);

Therefore the arc BGC is equal to the arc EHF.

But the whole circle ABC is equal to the whole circle DEF (Hyp.);

Therefore the remaining arc BAC is equal to the remaining arc EDF (Ax. 3).

Therefore, in equal circles, &c. *Q.E.D.*

Proposition 29.—Theorem.

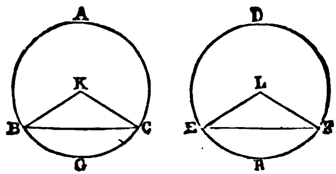
In equal circles equal arcs are subtended by equal chords.

Let ABC, DEF be equal circles, and let BGC, EHF be equal arcs in them, and join BC, EF.

The chord BC shall be equal to the chord EF.

Take K and L the centres.

CONSTRUCTION.—Take K, L, the centres of the circles (III. 1), and join BK, KC, EL, LF.



Then ∠ BKC = ∠ ELF,

PROOF.—Because the arc BGC is equal to the arc EHF (Hyp.), the angle BKC is equal to the angle ELF (III. 27).

And because the circles ABC, DEF are equal (Hyp.), their radii are equal (III. def. 1).

Therefore the two sides BK, KC are equal to the two sides EL, LF, each to each; and they contain equal angles;

and so base BC = base EF.

Therefore the base BC is equal to the base EF (I. 4).

Therefore, in equal circles, &c. *Q.E.D.*

Proposition 30.—Problem.

To bisect a given arc, that is, to divide it into two equal parts.

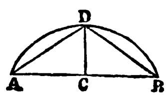
Let ADB be the given arc.

It is required to bisect it.

CONSTRUCTION.—Join AB, and bisect it in C (I. 10).

From the point C draw CD at right angles to AB (I. 11), and join AD and DB.

Then the arc ABD shall be bisected in the point D.



In the triangles ADC and CDB,

PROOF.—Because AC is equal to CB (Const.), and CD is common to the two triangles ACD, BCD;

The two sides AC, CD are equal to the two sides BC, CD, each to each;

And the angle ACD is equal to the angle BCD, because each of them is a right angle (Const.);

Therefore the base AD is equal to the base BD (I. 4). base AD = base BD.

But equal chords cut off equal arcs, the greater equal to the greater, and the less equal to the less (III. 28);

And each of the arcs AD, DB is less than a semicircle, because DC, if produced, is a diameter (III. 1, cor.);

Therefore the arc AD is equal to the arc DB.

Therefore, the given arc is bisected in D. *Q.E.F.*

Proposition 31.—Theorem.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let ABC be a circle, of which BC is a diameter, and E the centre; and draw CA, dividing the circle into the segments ABC, ADC, and join BA, AD, DC.

The angle in the semicircle BAC shall be a right angle;

The angle in the segment ABC, which is greater than a semicircle, shall be less than a right angle;

The angle in the segment ADC, which is less than a semicircle, shall be greater than a right angle.

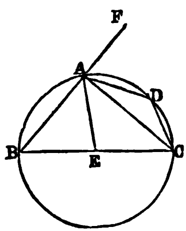
Construction.—Join AE, and produce BA to F.

Proof.—Because EA is equal to EB (I. Def. 15),

The angle EAB is equal to the angle EBA (I. 5);

And, because EA is equal to EC,

The angle EAC is equal to the angle ECA;



Therefore the whole angle BAC is equal to the two angles ABC, ACB (Ax. 2).

∠ BAE + ∠ EAC, or ∠ BAC = ∠ ABC + ∠ ACB = ∠ FAC and ∴ a right angle.

But FAC, the exterior angle of the triangle ABC, is equal to the two angles ABC, ACB (I. 32).

Therefore the angle BAC is equal to the angle FAC (Ax. 1), And therefore each of them is a right angle (I. Def. 10);

Therefore the angle in a semicircle BAC is a right angle.

And because the two angles ABC, BAC, of the triangle ABC, are together less than two right angles (I. 17), and that BAC has been shown to be a right angle;

∴ ∠ ABC < a right angle.

Therefore the angle ABC is less than a right angle.

Therefore the angle in a segment ABC, greater than a semicircle, is less than a right angle.

And, because ABCD is a quadrilateral figure in a circle, any two of its opposite angles are together equal to two right angles (III. 22);

Therefore the angles ABC, ADC are together equal to two right angles.

But the angle ABC has been shown to be less than a right angle;

Hence ∠ ADC > a right angle, by Prop. 32.

Therefore the angle ADC is greater than a right angle;

Therefore the angle in a segment ADC, less than a semicircle, is greater than a right angle.

Therefore, the angle, &c. *Q.E.D.*

Corollary.—From this demonstration it is manifest that, if one angle of a triangle be equal to the other two, it is a right angle.

For the angle adjacent to it is equal to the samo two angles (I. 32).

And, when the adjacent angles are equal, they are right angles (I. def. 10).

Proposition 32.—Theorem.

The angles contained by a tangent to a circle and a chord drawn from the point of contact are equal to the angles in the alternate segments of the circle.

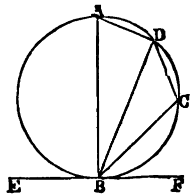
Let EF be a tangent to the circle ABCD, and BD a chord drawn from the point of contact B, cutting the circle.

The angles which BD makes with the tangent EF shall be equal to the angles in the alternate segments of the circle;

That is, the angle DBF shall be equal to the angle in the segment BAD, and the angle DBE shall bo equal to the angle in the segment BCD.

CONSTRUCTION.—From the point B draw BA at right angles to EF (I. 11).

Take any point C in the circumference BD, and join AD, DC, CB.



PROOF.—Because the straight line EF touches the circle ABCD at the point B (Hyp.), and BA is drawn at right angles to the tangent from the point of contact B (Const.),

The centre of the circle is in BA (III. 19). The centre is in BA.

Therefore the angle ADB, being in a semicircle, is a right angle (III. 31). ∴ ADB is a right angle,

Therefore the other two angles BAD, ABD are equal to a right angle (I. 32). and ∠ BAD + ∠ ABD = a right angle

But ABF is also a right angle (Const.);

Therefore the angle ABF is equal to the angles BAD, ABD. = ABF.

From each of these equals take away the common angle ABD;

Therefore the remaining angle DBF is equal to the remaining angle BAD, which is in the alternate segment of the circle (Ax. 3). ∴ ∠ BAD = ∠ DBF.

And because ABCD is a quadrilateral figure in a circle, the opposite angles BAD, BCD are together equal to two right angles (III. 22). Also, ∠ BCD + ∠ BAD = 2 right angles

But the angles DBF, DBE are together equal to two right angles (I. 13);

Therefore the angles DBF, DBE are together equal to the angles BAD, BCD.

And the angle DBF has been shown equal to the angle BAD; = ∠ DBF + ∠ DBE.

Therefore the remaining angle DBE is equal to the angle BCD, which is in the alternate segment of the circle (Ax. 3). ∴ ∠ DBE = ∠ BCD.

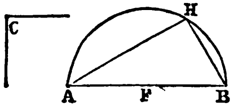
Therefore, the angles, &c. *Q.E.D.*

Proposition 33.—Problem.

Upon a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.

Let AB be the given straight line, and C the given rectilineal angle.

It is required to describe, on the given straight line AB, a segment of a circle, containing an angle equal to the angle C.



CASE I.—Let the angle C be a right angle.

CONSTRUCTION.—Bisect AB in F (I. 10).

From the centre F, at the distance FB, describe the semicircle AHB.

Then AHB shall be the segment required.

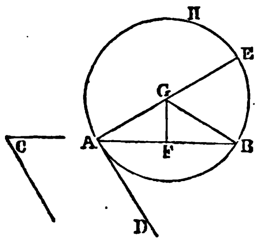
Angle in a semicircle is a right angle.

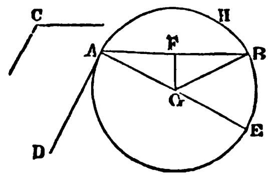
PROOF.—Because AHB is a semicircle, the angle AHB in it is a right angle, and therefore equal to the angle C (III. 31).

CASE II.—Let C be not a right angle.

At point A make ∠ BAD = C;

CONSTRUCTION.—At the point A, in the straight line AB, make the angle BAD equal to the angle C (I. 23).





and draw AE at right angles to AD. From F, middle of AB, draw perpendicular, meeting AE in G.

From the point A draw AE at right angles to AD (I. 11).

Bisect AB in F (I. 10).

From the point F draw FG at right angles to AB (I. 11), and join GB.

Because AF is equal to BF (Const.), and FG is common to the two triangles AFG, BFG;

The two sides AF, FG are equal to the two sides BF, FG, each to each;

And the angle AFG is equal to the angle BFG (Const.);

Therefore the base AG is equal to the base BG (I. 4).

Then G is the centre of a circle passing through A and B,

And the circle described from the centre G, at the distance GA, will therefore pass through the point B.

Let this circle be described; and let it be AHB.

The segment AHB shall contain an angle equal to the given rectilineal angle C.

PROOF.—Because from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE (Const.); And AD touches the circle,

Therefore AD touches the circle (III. 16, cor.)

Because AB is drawn from the point of contact A, the angle DAB is equal to the angle in the alternate segment AHB (III. 32).

But the angle DAB is equal to the angle C (Const.); and ∴ ∠ in AHB = ∠ DAB or C.

Therefore the angle in the segment AHB is equal to the angle C (Ax. 1).

Therefore, on the given straight line AB, the segment AHB of a circle has been described, containing an angle equal to the given angle C. *Q.E.F.*

Proposition 34.—Problem.

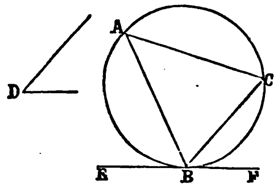
From a given circle to cut off a segment which shall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle.

It is required to cut off from the circle ABC a segment that shall contain an angle equal to the angle D.

CONSTRUCTION.—Draw the straight line EF touching the circle ABC in the point B (III. 17); Draw tangent EBF,

And at the point B, in the straight line BF, make the angle FBC equal to the angle D (I. 23). and make ∠ FBC = given ∠.



Then the segment BAC shall contain an angle equal to the given angle D.

PROOF.—Because the straight line EF touches the circle ABC, and BC is drawn from the point of contact B (Const.);

Therefore the angle FBC is equal to the angle in the alternate segment BAC of the circle (III. 32).

But the angle FBC is equal to the angle D (Const.);

Therefore the angle in the segment BAC is equal to the angle D (Ax. 1). ∴ ∠ BAC = ∠ FBC = ∠ D.

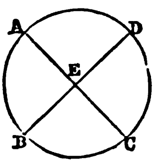
Therefore, from the given circle ABC, the segment BAC has been cut off, containing an angle equal to the given angle D. *Q.E.F.*

Proposition 35.—Theorem.

If two straight lines within a circle cut one another, the rectangle contained by the segments of one of them shall be equal to the rectangle contained by the segments of the other.

Let the two straight lines AC, BD cut one another in the point E, within the circle ABCD.

The rectangle contained by AE and EC shall be equal to the rectangle contained by BE and ED.

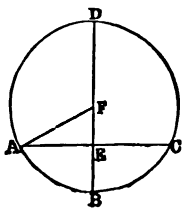


CASE I.—Let AC, BD pass each of them through the centre.

PROOF.—Because E is the centre, EA, EB, EC, ED are all equal (I. def. 15);

Therefore the rectangle AE, EC is equal to the rectangle BE, ED.

CASE II.—Let one of them, BD, pass through the centre, and cut the other, AC, which does not pass through the centre, at right angles, in the point E.



CONSTRUCTION.—Bisect BD in F, then F is the centre of the circle; join AF.

PROOF.—Because BD, which passes through the centre, cuts AC, which does not pass through the centre, at right angles in E (Hyp.);

AE = EC. Therefore AE is equal to EC (III. 3).

And because BD is cut into two equal parts in the point F, and into two unequal parts in the point E,

$BE \cdot ED + EF^2 = FB^2 = AF^2 = AE^2 + EF^2$. The rectangle BE, ED, together with the square on EF, is equal to the square on FB (II. 5); that is, the square on AF.

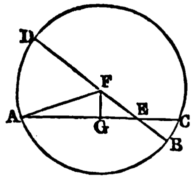
But the square on AF is equal to the squares on AE, EF (I. 47);

Therefore the rectangle BE, ED, together with the square on EF, is equal to the squares on AE, EF (Ax. 1).

Take away the common square on EF;

$\therefore BE \cdot ED = AE^2 = AE \cdot EC$. Then the remaining rectangle BE, ED is equal to the remaining square on AE; that is, to the rectangle AE, EC, since AE is equal to EC.

CASE III.—Let BD, which passes through the centre, cut the other AC, which does not pass through the centre, in the point E, but not at right angles.



CONSTRUCTION.—Bisect BD in F, then F is the centre of the circle.

Bisect BD in F the centre.

Join AF, and from F draw FG perpendicular to AC (I. 12).

Draw FG at right angles to AC.

PROOF.—Then AG is equal to GC (III. 3).

∴ AG = GC.

Therefore the rectangle AE, EC, together with the square on EG, is equal to the square on AG (II. 5).

Now, $AE \cdot EC + EG^2 = AG^2$.

To each of these equals add the square on GF;

Then the rectangle AE, EC, together with the squares on EG, GF, is equal to the squares on AG, GF (Ax. 2).

But the squares on EG, GF are equal to the square on EF;

And the squares on AG, GF are equal to the square on AF (I. 47).

Therefore the rectangle AE, EC, together with the square on EF, is equal to the square on AF; that is, the square on FB.

∴ $AE \cdot EC + EF^2 = AF^2 = FB^2$

But the square on FB is equal to the rectangle BE, ED, together with the square on EF (II. 5);

$= BE \cdot ED + EF^2$.

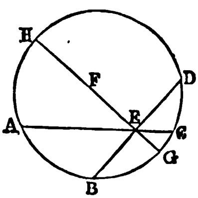
Therefore the rectangle AE, EC, together with the square on EF, is equal to the rectangle BE, ED, together with the square on EF.

Take away the common square on EF;

And the remaining rectangle AE, EC is equal to the remaining rectangle BE, ED (Ax. 3).

∴ $AE \cdot EC = BE \cdot ED$.

CASE IV.—Let neither of the straight lines AC, BD pass through the centre.



CONSTRUCTION.—Take the centre F (III. 1), and through E, the intersection of the lines AC, BD, draw the diameter GEFH.

PROOF.—Because the rectangle GE, EH is equal, as has been shown, to the rectangle AE, EC, and also to the rectangle BE, ED;

Again, $GE \cdot EH = AE \cdot EC$ and $= BE \cdot ED$, as just shown.

Therefore the rectangle AE, EC is equal to the rectangle BE, ED (Ax. 1).

∴ $AE \cdot EC = BE \cdot ED$.

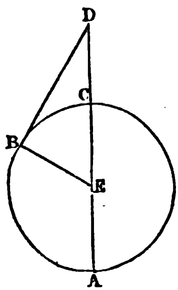
Therefore, if two straight lines, &c. *Q.E.D.*

Proposition 36.—Theorem.

If from a point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square on the line which touches it.

Let D be any point without the circle ABC, and let DCA, DB be two straight lines drawn from it, of which DCA cuts the circle, and DB touches it.

The rectangle AD, DC shall be equal to the square on DB.



CASE I.—Let DCA pass through the centre E, and join EB.

PROOF.—Then EBD is a right angle (III. 18).

And because the straight line AC is bisected in E, and produced to D, the rectangle AD, DC, together with the square on EC, is equal to the square on ED (II. 6).

AD·DC + EC² = ED².

But EC is equal to EB;

Therefore the rectangle AD, DC, together with the square on EB, is equal to the square on ED.

But the square on ED is equal to the squares on EB, BD, because EBD is a right angle (I. 47);

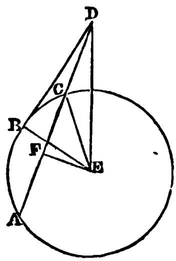
∴ AD·DC + EB² = BD² + EB².

Therefore the rectangle AD, DC, together with the square on EB, is equal to the squares on EB, BD.

Take away the common square on EB;

∴ AD·DC = BD².

Then the remaining rectangle AD, DC is equal to the square on DB (Ax. 3).



CASE II.—Let DCA not pass through the centre of the circle ABC.

CONSTRUCTION.—Take the centre E (III. 1), and draw EF perpendicular to AC (I. 12), and join EB, EC, ED.

Draw EF perpendicular to AC.

PROOF.—Because the straight line EF, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles, it also bisects it (III. 3);

Therefore AF is equal to FC.

$\therefore AF = FC.$

And because the straight line AC is bisected in F and produced to D, the rectangle AD, DC, together with the square on FC, is equal to the square on FD (II. 6).

$\therefore AD \cdot DC + FC^2 = FD^2.$

To each of these equals add the square on FE;

Therefore the rectangle AD, DC, together with the squares on CF, FE, is equal to the squares on DF, FE (Ax. 2).

But the squares on CF, FE are equal to the square on CE, because CFE is a right angle (I. 47);

And the squares on DF, FE are equal to the square on DE;

Therefore the rectangle AD, DC, together with the square on CE, is equal to the square on DE.

$\therefore AD \cdot DC + EC^2 = DE^2.$

But CE is equal to BE;

Therefore the rectangle AD, DC, together with the square on BE, is equal to the square on DE.

But the square on DE is equal to the squares on DB, BE, because EBD is a right angle (I. 47);

Therefore the rectangle AD, DC, together with the square on BE, is equal to the squares on DB, BE.

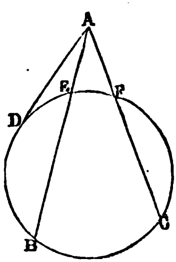
$\therefore AD \cdot DC + BE^2 = DB^2 + BE^2.$

Take away the common square on BE;

Then the remaining rectangle AD, DC is equal to the square on DB (Ax. 3).

$\therefore AD \cdot DC = DB^2.$

Therefore, if from any point, &c. *Q.E.D.*



Corollary.—If from any point without a circle there be drawn two straight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another; namely, the rectangle BA, AE is equal to the rectangle CA, AF; for each of them is equal to the square on the straight line AD, which touches the circle.

Proposition 37.—Theorem.

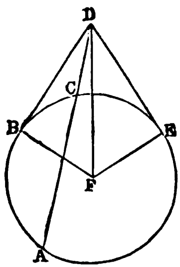
If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it, and if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square on *the line which meets the circle, the line which meets the circle shall touch it.*

Let any point D be taken without the circle ABC, and from it let two straight lines, DCA, DB, be drawn, of which DCA cuts the circle, and DB meets it; and let the rectangle AD, DC be equal to the square on DB.

Then DB shall touch the circle.

Draw DE touching the circle.

CONSTRUCTION.—Draw the straight line DE, touching the circle ABC (III. 17);



Find F the centre (III. 1) and join FB, FD, FE.

PROOF.—Then FED is a right angle (III. 18).

And because DE touches the circle ABC, and DCA cuts it, the rectangle AD, DC is equal to the square on DE (III. 36).

But the rectangle AD, DC is equal to the square on DB (Hyp.);

Therefore the square on DE is equal to the square on DB (Ax. 1);

Then DE = DB.

Therefore the straight line DE is equal to the straight line DB.

And EF is equal to BF (I. Def. 15);

Therefore the two sides DE, EF are equal to the two sides DB, BF, each to each;

And triangles DBF and DEF are equal in every respect.

And the base DF is common to the two triangles DEF, DBF;

Therefore the angle DEF is equal to the angle DBF (I. 8).

But DEF is a right angle (Const.);

Therefore also DBF is a right angle (Ax. 1).

∴ DBF is a right angle;

And BF, if produced, is a diameter; and the straight line which is drawn at right angles to a diameter, from the extremity of it, touches the circle (III. 16, Cor.);

and therefore DB touches the circle.

Therefore DB touches the circle ABC.

Therefore, if from a point, &c. *Q.E.D.*

EXERCISES ON BOOK III.

Prop. 1—15.

1. Two straight lines intersect. Describe a circle passing through the point of intersection and two other points, one in each straight line.

2. If two circles cut each other, any two parallel straight lines drawn through the points of section to cut the circumferences are equal.

3. Show that the centre of a circle may be found by drawing perpendiculars from the middle points of any two chords.

4. Through a given point, which is not the centre, draw the least line to meet the circumference of a given circle, whether the given point be within or without the circle.

5. The sum of the squares of any two chords in a circle, together with four times the sum of the squares of the perpendiculars on them from the centre, is equal to twice the square of the diameter.

6. With a given radius, describe a circle passing through the centre of a given circle and a point in its circumference.

7. If two chords of a circle are given in magnitude and position, describe the circle.

8. Describe a circle which shall touch a given circle in a given point, and shall also touch another given circle.

9. If, from any point in the diameter of a circle, straight lines be drawn to the extremities of a parallel chord, the squares of these lines are together equal to the squares of the segments into which the diameter is divided.

10. If two circles touch each other externally, and parallel diameters be drawn, the straight line joining extremities of these diameters will pass through the point of contact.

11. Draw three circles of given radii touching each other.

12. If a circle of constant radius touch a given circle, it will always touch the same concentric circle.

13. If a chord of constant length be inscribed in a circle, it will always touch the same concentric circle.

14. The locus of the middle points of chords parallel to a given straight line is a line drawn through the centre perpendicular to the parallel chords.

Prop. 16—30.

15. Show that the two tangents from an external point are equal in length.

16. Draw a tangent to a given circle, making a given angle with a given straight line.

17. If a polygon having an even number of sides be inscribed in a circle, the sums of the alternate angles are equal.

18. If such a polygon be described about a circle, the sums of the alternate sides are each equal to half the perimeter of the polygon.

19. If a polygon be inscribed in a circle, the sum of the angles in the segments exterior to the polygon, together with two right angles, is equal to twice as many right angles as the polygon has sides.

20. Draw the common tangents to two given circles.

21. From a given point draw a straight line cutting a given circle, so that the intercepted segment of the line may have a given length.

22. The straight line which joins the extremities of equal arcs towards the same parts are parallel.

23. Any parallelogram described about a circle is equilateral, and any parallelogram inscribed in a circle is rectangular.

24. Two opposite sides of a quadrilateral circumscribing a circle touch the circle at extremities of a diameter. Show that the area of the quadrilateral is equal to one-half the rectangle contained by the diameter, and the sum of the other sides.

Prop. 31—37.

25. A tangent is drawn to a circle of 21 inches diameter from a point 17·5 inches from the centre. Find the length of the tangent.

26. Show that a man 6 feet high, standing at the sea level, has a view of 3 miles (approximately) in every direction, along a horizontal plane passing through his eye.

27. The angle between a tangent to a circle and the chord through the point of contact is equal to half the angle which the chord subtends at the centre.

28. From a given point P, within or without a circle, draw a straight line cutting the circle in A and B such that PA shall be three-fourths of PB.

Ex. Let the circle be of 1·5 inches radius, and point P 3·5 inches from its centre. Prove your construction by scale.

29. The greatest rectangle which can be inscribed in a circle is a square whose area is equal to half that of the square described upon the diameter as side.

30. If the base and vertical angle of a triangle remain constant in magnitude while the sides vary, show that the locus of the middle point of the base is a circle.

31. Given the vertical angle, the difference of the two sides containing it, and the difference of the segments of the base made by a perpendicular from the vertex, to construct the triangle.

32. Show that the locus of the middle point of a straight line, which moves with its extremities upon two straight lines at right angles to each other, is a circle.

33. Show how to produce a straight line, that the rectangle contained by the given line, and the whole line thus produced, may be equal to the square of the part produced.

Ex. If the length of the given line be 2 inches, show geometrically that the length of the part produced is $(\sqrt{5} + 1)$ inches.

34. Given the height and chord of a segment of a circle to find the radius of the circle.

Ex. If the chord be 24 inches, and the height of the segment be 4 inches, show that the radius of the circle is 20 inches.

35. Show that the locus of the middle points of chords which pass through a fixed point is the circle described as diameter upon the line joining the fixed point and the centre of the given circle.

36. Let ACDB be a semicircle whose diameter is AB, and AD, BC any two chords intersecting in P; prove that

$$AB^2 = DA \cdot AP + CB \cdot BP.$$

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