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ELEMENTS

OF

ALGEBRA:

TRANSLATED FROM THE FRENCH OF

M. BOURDON.

REVISED AND ADAPTED TO THE COURSES OF MATHEMATICAL INSTRUCTION IN THE UNITED STATES;

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PREFACE.

The Treatise on Algebra, by Bourdon, is a work of singular excellence and merit. In France, it is one of the leading text books, and shortly after its publication, had passed through several editions. It has been translated, in part, by Professor De Morgan, of the London University, and it is now used in the University of Cambridge.

A translation was made by Lt. Ross, and published in 1831, since which time it has been adopted as a text book in the Military Academy, the University of the City of New-York, Union College, Princeton College, Geneva College, and in Kenyon College, in Ohio.

The original work is a full and complete treatise on the subject of Algebra, and contains six hundred and seventy pages octavo. The time which is given to the study of Algebra, even in those seminaries where the course of mathematics is the fullest, is too short to accomplish so voluminous a work, and hence it has been found necessary either to modify it, or abandon it altogether.

The work which is here presented to the public, is an abridgment of Bourdon; with such modifications, as experience in teaching it, and a very careful comparison with other standard works, have suggested.

It has been the intention to unite in this work, the scientific discussions of the French, with the practical methods of the English school; that theory and practice, science and art, may mutually aid and illustrate each other.

MILITARY ACADEMY, *March*, 1835.

CONTENTS.

CHAPTER I.

Preliminary Definition and Remarks.

ALGEBRAIC FRACTIONS.

CHAPTER II.

Equations of the First Degree.

CHAPTER III.

CHAPTER IV.

Formation of Powers, and Extraction of Roots of any degree whatever.

CHAPTER V.

Of Progressions, Continued Fractions, and Logarithms.

CHAPTER VI.

General Theory of Equations.

CHAPTER VII.

Resolution of Numerical Equations.

ALGEBRA.

CHAPTER I.

Preliminary Definitions and Remarks.

1. QUANTITY is a general term embracing every thing which admits of increase or diminution.

2. MATHEMATICS is the science of quantity.

3. ALGEBRA is that branch of mathematics in which the quantities considered are represented by letters, and the operations to be performed upon them are indicated by signs.

4. The sign $+$, is called *plus;* and indicates the addition of two or more quantities. Thus, $9+5$ is read, 9 plus 5, or 9 augmented by 5.

In like manner, $a+b$ is read, a plus b; and denotes that the quantity represented by a is to be added to the quantity represented by b.

5. The sign $-$, is called *minus;* and indicates that one quantity is to be subtracted from another. Thus, $9-5$ is read, 9 minus 5, or 9 diminished by 5.

In like manner, $a-b$, is read, a *minus* b, or a diminished by b.

6. The sign $\times$, is called the sign of multiplication; and when placed between two quantities, it denotes that they are to be multiplied together. The multiplication of two quantities is also frequently indicated by simply placing a point between them. Thus,

36×25, or 36.25, is read, 36 multiplied by 25, or the product of 36 by 25.

7. The multiplication of quantities, which are represented by letters, is indicated by simply writing them one after the other, without interposing any sign.

Thus, ab signifies the same thing as $a\times b$, or as $a.b$; and abc the same as $a\times b\times c$, or as $a.b.c$. It is plain that the notation ab, or abc, which is more simple than $a\times b$, or $a\times b\times c$, cannot be employed when the quantities are represented by figures. For example, if it were required to express the product of 5 by 6, and we were to write 5 6, the notation would confound the product with the number 56.

8. In the product of several letters, as abc, the single letters, a, b and c, are called *factors* of the product. Thus, in the product ab, there are two factors, a and b; in the product acd, there are three, a, c and d

9. There are three signs used to denote division. Thus,

$a\div b$ denotes that a is to be divided by b,

$\frac{a}{b}$ denotes that a is to be divided by b,

$a|\underline{b}$ denotes that a is to be divided by b.

10. The sign $=$, is called the sign of equality, and is read, *is equal to*. When placed between two quantities, it denotes that they are equal to each other. Thus, $9-5=4$: that is, 9 minus 5 is equal to 4: Also, $a+b=c$, denotes that the sum of the quantities a and b is equal to c.

11. The sign $>$, is called the sign of *inequality*, and is used to express that one quantity is greater or less than another.

Thus, $a>b$ is read, a greater than b; and $a<b$ is read, a less than b; that is, the opening of the sign is turned towards the greater quantity.

12. If a quantity is added to itself several times, as $a+a+a+a+a$, we generally write it but once, and then place a number before

it to express how many times it is taken. Thus,

$$a+a+a+a+a=5a.$$

The number 5 is called the *co-efficient* of a, and denotes that a is taken 5 times.

Hence, a *co-efficient* is a number prefixed to a quantity, denoting the number of times which the quantity is taken; and it also indicates the number of times *plus one*, that the quantity is added to itself. When no co-efficient is written, the co-efficient 1 is always understood.

13. If a quantity be multiplied continually by itself, as $a\times a\times a\times a\times a$, we generally express the product by writing the letter once, and placing a number to the right of, and a little above it: thus,

$$a\times a\times a\times a\times a=a^5$$

The number 5 is called the *exponent* of a, and denotes the number of times which a enters into the product as a factor.

Hence, the exponent of a quantity shows how many times the quantity is a factor; and it also indicates the number of times, *plus one*, that the quantity is to be multiplied by itself. When no exponent is written, the exponent 1 is always understood.

14. The product resulting from the multiplication of a quantity by itself any number of times, is called the *power* of that quantity; and the exponent, which always exceeds by one the number of multiplications to be made, denotes the *degree* of the power. Thus, a^5 is the fifth power of a. The exponent 5 denotes the *degree* of the power; and the power itself is formed by multiplying a four times by itself.

15. In order to show the importance of the exponent in algebra, suppose that we wish to express that a number a is to be multiplied three times by itself, that this product is to be multiplied three times by b, and that this new product is to be multiplied twice by c, we would write simply $a^4\ b^3\ c^2$.

If, then, we wish to express that this last result is to be added to itself six times, or is to be multiplied by 7, we would write, $7a^4b^3c^2$.

This gives an idea of the brevity of algebraic language.

16. The root of a quantity, is a quantity which being multiplied by itself a certain number of times will produce the given quantity.

The sign $\sqrt{}$, is called the radical sign, and when prefixed to a quantity, indicates that its root is to be extracted. Thus,

$\sqrt[2]{a}$ or simply $\sqrt{a}$ denotes the square root of a.

$\sqrt[3]{a}$ denotes the cube root of a.

$\sqrt[4]{a}$, denotes the fourth root of a.

The number placed over the radical sign is called the *index* of the root. Thus, 2 is the index of the square root, 3 of the cube root, 4 of the fourth root, &c.

17. Every quantity written in algebraic language; that is, with the aid of letters and signs, is called an *algebraic quantity*, or the *algebraic expression* of a quantity. Thus,

$3a$ } is the algebraic expression of three times the number a;

$5a^2$ } is the algebraic expression of five times the square of a;

$7a^3b^2$ } is the algebraic expression of seven times the product of the cube of a by the square of b;

$3a-5b$ } is the algebraic expression of the difference between three times a and five times b;

$2a^2-3ab+4b^2$ } is the algebraic expression of twice the square of a, diminished by three times the product of a by b, augmented by four times the square of b.

18. When an algebraic quantity is not connected with any other by the sign of addition or subtraction, it is called a *monomial*, or a quantity composed of a single term, or simply, a *term*.

Thus, $3a$, $5a^2$, $7a^3b^2$, are monomials, or single terms.

19. An algebraic expression composed of two or more parts, separated by the sign $+$ or $-$, is called a *polynomial*, or quantity involving two or more terms.

For example, $3a-5b$ and $2a^2-3cb+4b^2$ are polynomials.

20. A polynomial composed of two terms, is called a *binomial;* and a polynomial of three terms is called a *trinomial.*

21. The *numerical value* of an algebraic expression, is the number which would be obtained by giving particular values to the letters which enter it, and performing the arithmetical operations indicated. This numerical value evidently depends upon the particular values attributed to the letters, and will *generally* vary with them.

For example, the numerical value of $2a^3=54$ when we make $a=3$; for, the cube of $3=27$, and $2\times 27=54$.

The numerical value of the same expression is 250 when we make $a=5$; for, $5^3=125$, and $2\times 125=250$.

22. We have said, that the numerical value of an algebraic expression *generally* varies with the values of the letters which enter it: it does not, however, always do so. Thus, in the expression $a-b$, so long as a and b increase by the same number, the value of the expression will not be changed.

For example, make $a=7$ and $b=4$: there results $a-b=3$.

Now make $a=7+5=12$, and $b=4+5=9$, and there results $a-b=12-9=3$, as before.

23. The *numerical value* of a polynomial is not affected by changing the order of its terms, provided the signs of all the terms be preserved. For example, the polynomial $4a^3-3a^2b+5ac^2=5ac^2-3a^2b+4a^3=-3a^2b+5ac^2+4a^3$. This is evident, from the nature of arithmetical addition and subtraction.

24. Of the different terms which compose a polynomial, some are preceded by the sign $+$, and the others by the sign $-$. The first are called *additive terms*, the others, *subtractive terms.*

The first term of a polynomial is commonly not preceded by any sign, but then, it is understood to be affected with the sign $+$.

25. Each of the literal factors which compose a term is called a *dimension* of this term; and the *degree* of a term is the number of

these factors or dimensions. Thus,

$3a$ is a term of one dimension, or of the first degree.

$5ab$ is a term of two dimensions, or of the second degree.

$7a^3bc^2=7aaabcc$ is of six dimensions, or of the sixth degree.

In general, *the degree, or the number of dimensions of a term, is estimated by taking the sum of the exponents of the letters which enter this term.* For example, the term $8a^2bcd^3$ is of the seventh degree, since the sum of the exponents, $2+1+1+3=7$.

26. A polynomial is said to be *homogeneous,* when all its terms are of the same degree. The polynomial

$3a-2b+c$ is of the first degree and homogeneous.

$-4ab+b^2$ is of the second degree and homogeneous.

$5a^2c-4c^3+2c^2d$ is of the third degree and homogeneous.

$8a^3-4ab+c$ is not homogeneous.

27. A vinculum or bar ——, or a parenthesis ( ), is used to express that all the terms of a polynomial are to be considered together. Thus, $\overline{a+b+c}\times b$, or $(a+b\times c)\times b$ denotes that the trinomial $a+b+c$ is to be multiplied by b; also $\overline{a+b+c}\times\overline{c+d+f}$ or $(a+b+c)\times(c+d+f)$ denotes that the trinomial $a+b+c$ is to be multiplied by the trinomial $c+d+f$.

When the parenthesis is used, the sign of multiplication is usually omitted. Thus $(a+b+c)\times b$ is the same as $(a+b+c)\ b$.

The bar is also sometimes placed vertically. Thus,

$$\left.\begin{array}{r}+a\\+b\\+c\end{array}\right|x \quad \text{is the same as } (a+b+c)\ x \text{ or } \overline{a+b+c}\times x$$

28. The terms of a polynomial which are composed of the same letters, the same letters in each being affected with like exponents, are called *similar terms.*

Thus, in the polynomial $7ab+3ab-4a^3b^2+5a^3b^2$, the terms $7ab$ and $3ab$, are similar; and so also are the terms $-4a^3b^2$ and $5a^3b^2$, the letters and exponents in each being the same. But in the binomial $8a^2b+7ab^2$, the terms are not similar; for, although they are composed of the same letters, yet the same letters are not affected with like exponents.

29. When a polynomial contains several similar terms it may often be reduced to a simpler form.

Take the polynomial $4a^2b-3a^2c+7a^2c-2a^2b$.

It may be written (Art. 23), $4a^2b-2a^2b+7a^2c-3a^2c$.

But $4a^2b-2a^2b$ reduces to $2a^2b$, and $7a^2c-3a^2c$ to $4a^2c$.

Hence, $4a^2b-3a^2c+7a^2c-2a^2b=2a^2b+4a^2c$.

When we have a polynomial with similar terms, of the form

$$+2a^3bc^2-4a^3bc^2+6a^3bc^2-8a^3bc^2+11a^3bc^2,$$

Find the sum of the additive and subtractive terms separately, and take their difference: thus,

Additive terms.		*Subtractive terms.*
$+\ 2a^3bc^2$		$-\ 4a^3bc^2$
$+\ 6a^3bc^2$		$-\ 8a^3bc^2$
$+11a^3bc^2$	Sum	$-12a^3bc^2$
Sum $+19a^3bc^2$		

Hence, the given polynomial reduces to

$$19a^3bc^2-12a^3bc^2=7a^3bc^2.$$

It may happen that the sum of the subtractive terms exceeds the sum of the additive terms. In that case, subtract the positive coefficient from the negative, and prefix the minus sign to the remainder.

Thus, in the polynomial, $3a^2b+2a^2b-5a^2b-3a^2b$, in which the sum of the additive terms is $5a^2b$, and the sum of the subtractive terms $-8a^2b$, we say that the polynomial reduces to $-3a^2b$.

For, since $-8a^2b$ is equal to $-5a^2b-3a^2b$, we shall have,

$$5a^2b-8a^2b=5a^2b-5a^2b-3a^2b=-3a^2b.$$

Hence, for the reduction of the similar terms of a polynomial we have the following

RULE.

I. *Form a single additive term of all the terms preceded by the sign plus: this is done by adding together the co-efficients of those terms, and annexing to their sum the literal part.*

II. *Form, in the same manner, a single subtractive term.*

III. *Subtract the less sum from the greater, and prefix to the result the sign of the greater.*

REMARK.—It should be observed that the reduction affects only the co-efficients, and not the exponents.

EXAMPLES.

1. Reduce the polynomial $4a^2b-8a^2b-9a^2b+11a^2b$ to its simplest form. *Ans.* $-2a^2b$.

2. Reduce the polynomial $7abc^2-abc^2-7abc^2-8abc^2+6abc^2$ to its simplest form. *Ans.* $-3abc^2$.

3. Reduce the polynomial $9cb^3-8ac^2+15cb^3+8ca+9ac^2-24cb^3$ to its simplest form. *Ans.* ac^2+8ca.

The reduction of similar terms is an operation peculiar to algebra. Such reductions are constantly made in *Algebraic Addition*, *Subtraction*, *Multiplication*, and *Division*.

30. It has been remarked in Definition 3, that the quantities considered in algebra are represented by letters, and the operations to be performed upon them, are indicated by signs. The letters and signs are used to *abridge and generalize the reasoning required in the resolution of questions*.

31. There are two kinds of questions, viz. theorems and problems. If it is required to demonstrate the existence of certain properties relating to quantities, the question is called a *theorem;* but if it is proposed to determine certain quantities from the knowledge of others, which have with the first known relations, the question is called a *problem*.

The given or known quantities are generally represented by the first letters of the alphabet, a, b, c, d, &c. and the unknown or required quantities by the last letters, x, y, z, &c.

32. The following question will tend to show the utility of the algebraic analysis, and to explain the manner in which it abridges and generalizes the reasoning required in the resolution of questions.

Question.

The sum of two numbers is 67, *and their difference* 19; *what are the two numbers?*

Solution.

We will begin by establishing, with the aid of the conventional signs, a connexion between the given and unknown numbers of the question. If the least of the two required numbers was known, we would have the greater by adding 19 to it. This being the case, denote the least number by x: the greater may then be designated by $x+19$: hence their sum is $x+x+19$, or $2x+19$.

But from the enunciation, this sum is to be equal to 67. Therefore we have the equality or *equation*

$$2x+19=67.$$

Now, if $2x$ augmented by 19, gives 67, $2x$ alone is equal to 67 minus 19, or $2x=67-19$, or performing the subtraction, $2x=48$.

Hence x is equal to the half of 48, that is,

$$x=\frac{48}{2}=24.$$

The least number being 24, the greater is

$$x+19=24+19=43.$$

And indeed, we have $43+24=67$, and $43-24=19$.

Table of the Algebraic Operations.

Let x be the least number.

$x+19$ will be the greater.

Hence, $2x+19=67$, and $2x=67-19$; therefore $x=\frac{48}{2}=24$ and consequently $x+19=24+19=43$,

And indeed, $43+24=67$, $43-24=19$.

Another Solution.

Let x represent the greater number,

$x-19$ will represent the least.

Hence, $2x-19=67$, whence $2x=67+19$;

therefore, $x=\frac{86}{2}=43$

and consequently, $x-19=43-19=24$.

From this we see how we might, with the aid of algebraic signs, write down in a very small space, the whole course of reasoning which it would be necessary to follow in the resolution of a problem, and which, if written in common language, would often require several pages.

General Solution of this Problem.

The sum of two numbers is a, their difference is b. What are the two numbers?

Let x be the least number,

$x+b$ will represent the greater.

Hence, $2x+b=a$, whence $2x=a-b$,

therefore, $x=\frac{a-b}{2}=\frac{a}{2}-\frac{b}{2}$

and consequently, $x+b=\frac{a}{2}-\frac{b}{2}+b=\frac{a}{2}+\frac{b}{2}$

As the form of these two results is independent of any particular value attributed to the letters a and b, it follows that, *knowing the sum and difference of two numbers, we will obtain the greater by adding the half difference to the half sum, and the less, by subtracting the half difference from half the sum.*

Thus, when the given sum is 237, and the difference 99,

the greater is $\frac{237}{2}+\frac{99}{2}$, or $\frac{237+99}{2}=\frac{336}{2}=168$;

and the least $\frac{237}{2}-\frac{99}{2}$, or $\frac{138}{2}=69$,

And indeed, $168+69=237$, and $168-69=99$.

From the preceding question we perceive the utility of representing the given quantities of a problem by letters. As the arithmetical operations can only be indicated upon these letters, the result obtained, points out the operations which are to be performed upon the known quantities, in order to obtain the values of those required by the question.

The expressions $\frac{a}{2}+\frac{b}{2}$ and $\frac{a}{2}-\frac{b}{2}$ obtained in this problem, are called *formulas,* because they may be regarded as comprehending the solutions of all questions of the same nature, the enunciations of which differ only in the numerical values of the given quantities. Hence, *a formula is the algebraic enunciation of a general rule.*

From the preceding explanations, we see that Algebra may be regarded as a kind of language, composed of a series of signs, by the aid of which we can follow with more facility the train of ideas in the course of reasoning, which we are obliged to pursue, either to demonstrate the existence of a property, or to obtain the solution of a problem.

ADDITION.

33. Addition, in Algebra, consists in finding the simplest equivalent expression for several algebraic quantities, connected together by the sign plus or minus. Such equivalent expression is called their *sum.*

34. Let it be required to add together the expressions. $\left\{\begin{array}{l} 3a \\ 5b \\ 2c \end{array}\right.$

The result of the addition is . . . . $3a+5b+2c$

an expression which cannot be reduced to a more simple form.

Again, add together the monomials $\left\{\begin{array}{l} 4a^2b^3 \\ 2a^2b^3 \\ 7a^2b^3 \end{array}\right.$

The result, after reducing (Art. 29), is . . $13a^2b^3$

Let it be required to find the sum of the expressions.

$$\begin{cases} 3a^2-4ab \\ 2a^2-3ab+b^2 \\ \qquad 2ab-5b^2 \end{cases}$$

Their sum, after reducing (Art. 29), is . . $5a^2-5ab-4b^2$

35. As a course of reasoning similar to the above would apply to all polynomials, we deduce for the addition of algebraic quantities the following general

RULE.

I. *Write down the quantities to be added so that the similar terms shall fall under each other, and give to each term its proper sign.*

II. *Reduce the similar terms, and annex to the results, those terms which cannot be reduced, giving to each term its respective sign.*

EXAMPLES.

1. Add together the polynomials, $3a^2-2b^2-4ab$, $5a^2-b^2+2ab$, and $3ab-3c^2-2b^2$.

The term $3a^2$ being similar to $5a^2$, we write $8a^2$ for the result of the reduction of these two terms, at the same time slightly crossing them, as in the first term.

$$\begin{array}{r} 3\cancel{a}^2-4ab-2b^2 \\ 5a^2+2ab-\ b^2 \\ +3ab-2b^2-3c^2 \\ \hline 8a^2+\ ab-5b^2-3c^2 \end{array}$$

Passing then to the term $-4ab$, which is similar to $+2ab$ and $+3ab$, the three reduce to $+ab$, which is placed after $8a^2$, and the terms crossed like the first term. Passing then to the terms involving b^2, we find their sum to be $-5b^2$, after which we write $-3c^2$.

The marks are drawn across the terms, that none of them may be overlooked and omitted.

(2).

$$\begin{array}{lr} & 7x\ +3ab+2c \\ & -3x\ -3ab-5c \\ & 5x\ -9ab-9c \\ \hline \text{Sum.} & 9x\ -9ab-12c \end{array}$$

(3).

$$\begin{array}{r} 8\sqrt{x}+\ bc-2abc \\ -\ \sqrt{x}-9bc+6abc \\ -5\sqrt{x}+\ bc+\ abc \\ \hline 2\sqrt{x}-7bc+5abc. \end{array}$$

4. Add together the polynomials $5a^2b+6cx+9bc^2$, $7cx-8a^2b+\sqrt{a}$ and $-15cx-9bc^2+2a^2b$.

Ans. $\sqrt{a}-a^2b-2cx$.

5. Add together the polynomials $\sqrt{x}+ax-ab$, $ab-\sqrt{x}+xy$, $ax+xy-4ab$, $\sqrt{x}+\sqrt{x}-x$ and $xy+xy+ax$.

Ans. $2\sqrt{x}+3ax-4ab+4xy-x$.

6. Add together the polynomials $15axy+5bc^2+3af^2$, $3af^2+\sqrt{xy}-12xay$, $-5bc^2+\sqrt{ay}-3axy$, and $-2\sqrt{ay}-\sqrt{x}-6af^2$.

Ans. $\sqrt{xy}-\sqrt{ay}-\sqrt{x}$.

7. Add together the polynomials $7a^2b-3abc-8b^2c-9c^3+cd^3$, $8abc-5a^2b+3c^3-4b^2c+cd^2$ and $4a^2b-8c^3+9b^2c-3d^3$.

Ans. $6a^2b+5abc-3b^2c-14c^3+2cd^3-3d^3$.

SUBTRACTION.

36. Subtraction, in algebra, consists in finding the simplest expression for the difference between two algebraic quantities.

The result obtained by subtracting $4b$ from $5a$ is expressed by $5a-4b$.

In like manner, the difference between $7a^3b$ and $4a^3b$ is expressed by $7a^3b-4a^3b=3a^3b$.

Let it be required to subtract from . . . $4a$

the binomial . . . . $2b-3c$

In the first place, the result may be written thus, $4a-(2b-3c)$

by placing the quantity to be subtracted within the parenthesis, and writing it after the other quantity with the sign —. But the question frequently requires the difference to be expressed by a single polynomial; and it is in this that algebraic subtraction principally consists.

To accomplish this object, we will observe, that if a, b, c, were given numerically, the subtraction indicated by $2b-3c$, could be performed, and we might then substract this result from $4a$; but as this subtraction cannot be effected in the actual condition of the quantities, $2b$ is subtracted from $4a$, which gives $4a-2b$; but in subtracting the number of units contained in $2b$, the number taken away is too great by the number of units contained in $3c$, and the result is therefore too small by the same quantity; this result must therefore be corrected by adding $3c$ to it. Hence, there results from the proposed subtraction $4a-2b+3c$.

$$\begin{array}{l} 4a \\ \quad 2b-3c \\ \hline 4a-2b+3c \end{array}$$

Again, from . . . . . $8a^2-2ab$
subtract . . . . . . $5a^2-4ab+3bc-b^2$.

The difference is expressed by $8a^2-2ab-(5a^2-4ab+3bc-b^2)$
which is equal to . . . $8a^2-2ab-5a^2-3bc+4ab+b^2$.
or by reducing, equal to . . . . $3a^2+2ab-3bc+b^2$.

The reduction is made by observing, that to subtract $5a^2-4ab+3bc-b^2$, is to subtract the difference between the sum of the additive terms $5a^2+3bc$, and the sum of the substractive terms $4ab+b^2$. We can then first subtract $5a^2+3bc$, which gives $8a^2-2ab-5a^2-3bc$; and as this result is necessarily too small by $4ab+b^2$, this last quantity must be added to it, and it becomes $8a^2-2ab-5a^2-3bc+4ab+b^2$; and finally, after reducing, $3a^2+2ab-3bc+b^2$.

37. Hence, for the subtraction of algebraic quantities, we have the following general

RULE.

I. *Write the quantity to be subtracted under that from which it is to be taken, placing the similar terms, if there are any, under each other.*

II. *Change the signs of all the terms of the polynomial to be subtracted, or conceive them to be changed, and then reduce the polynomial result to its simplest form.*

	(1).	The same with the signs of the lower line changed.	(1).
From . .	$6ac-5ab+c^2$		$6ac-5ab+c^2$
Take . .	$3ac+3ab-7c$		$-3ac-3ab+7c$
Remainder	$3ac-8ab+c^2+7c.$		$3ac-8ab+c^2+7c.$

	(2).	(3).
From . .	$6\sqrt{2ay}-\sqrt{x}+3b^2$	$6yx-3x^2+5b$
Take. . .	$9\sqrt{2ay}-\ \ x+\ b^2$	$yx-3\ +\ a$
Remainder	$-3\sqrt{2ay}-\sqrt{x}+x+2b^2.$	$5yx-3x^2+3+5b-a.$

	(4).	(5).
From . .	$5a^3-4a^2b+3b^2c$	$4ab-\ cd+3a^2$
Take . .	$-2a^3+3a^2b-8b^2c$	$5ab-4cd+3a^2+5b^2$
Remainder	$7a^3-7a^2b+11b^2c.$	$-\ ab+3cd-5b^2$

7. From $8abc-12b^3a+5cx-7xy$, take $7cx-xy-13b^3a$.

Ans. $8abc+b^3a-2cx-6xy$.

38. By the rule for subtraction, polynomials may be subjected to certain transformations.

For example . . .	$6a^2-3ab+2b^2-2bc$,
becomes	$6a^2-(3ab-2b^2+2bc)$.
In like manner . .	$7a^3-8a^2b-4b^2c+6b^3$,
becomes	$7a^3-(8a^2b+4b^2c-6b^3)$;
or, again, ...	$7a^3-8a^2b-(4b^2c-6b^3)$.

These transformations consist in decomposing a polynomial into two parts, separated from each other by the sign —: they are very useful in algebra.

39. Remark.—From what has been shown in addition and subtraction, we deduce the following principles.

1st. In algebra, the words *add* and *sum* do not always, as in arithmetic, convey the idea of augmentation; for $a-b$, which results from the addition of $-b$ to a, is properly speaking, a difference between the number of units expressed by a, and the number of units expressed by b. Consequently, this result is less than a.

To distinguish this sum from an arithmetical sum, it is called the *algebraic sum.*

Thus, the polynomial $2a^2-3a^2b+3b^2c$ is an algebraic sum, so long as it is considered as the result of the union of the monomials $2a^3$, $-3a^2b$, $+3b^2c$, with their respective signs; and, in its *proper acceptation*, it is the arithmetical difference between the sum of the units contained in the additive terms, and the sum of the units contained in the subtractive terms.

It follows from this that an algebraic sum may, in the numerical applications, be reduced to a *negative* number, or a number affected with the sign —.

2d. The words *subtraction* and *difference* do not always convey the idea of diminution, for the difference between $+a$ and $-b$ being $a+b$, exceeds a. This result is an *algebraic difference*, and can be put under the form of $a-(-b)$.

MULTIPLICATION.

40. Algebraic multiplication has the same object as arithmetical, viz. to repeat the multiplicand as many times as there are units in the multiplier.

It is generally proved, in arithmetical treaties, that the product of two or more numbers is the same, in whatever order the multiplication is performed; we will, therefore, consider this principle demonstrated.

This being admitted, we will first consider the case in which it is required to multiply one monomial by another.

The expression for the product of . $7a^3b^2$ by $4a^2b$

may at once be written thus . . $7a^3b^2\times 4ba^2$

But this may be simplified by observing that, from the preceding principles and the signification of algebraic symbols, it can be written . . . $7\times 4aaaaabbb$.

Now, as the co-efficients are particular numbers, nothing prevents our forming a single number from them by multiplying them together, which gives 28 for the co-efficient of the product. As to the letters, the product *aaaaa*, is equivalent to a^5, and the product *bbb*, to b^3; therefore, the final result is . . . $28a^5b^3$.

Again, let us multiply . . . . . $12a^2b^4c^2$ by $8a^3b^2d^2$.
The product is $12\times 8aaaaabbbbbbccdd = 96a^5b^6c^2d^2$.

41. Hence, for the multiplication of monomials we have the following

RULE.

I. *Multiply the co-efficients together.*

II. *Write after this product all the letters which are common to the multiplicand and multiplier, affecting each letter with an exponent equal to the sum of the two exponents with which this letter is affected in the two factors.*

III. *If a letter enters into but one of the factors, write it in the product with the exponent with which it is affected in the factor.*

The reason for the rule relative to the co-efficients is evident. But in order to understand the rule for the exponents, it should be observed, that in general, a quantity a is found as many times a factor in the product, as it is in both the multiplicand and multiplier. Now the exponents of the letters denote the number of times they enter as factors (Art. 13.); hence the sum of the two exponents of the same letter denotes the number of times it is a factor in the required product.

From the above rule, it follows that,

$$8a^2bc^2\times 7abd^3 = 56a^3b^2c^2d^3$$
$$21a^3b^2dc\times 8abc^3 = 168a^4b^3c^4d$$
$$4abc\times 7df = 28abcdf.$$

Multiply .	$3a^2b$	$12\ a^2x$	$6xy\ z$	a^2xy
by . .	$2b\ a^2$	$12\ x^2y$	ay^2z	$2\ xy^2$
	$6a^4b^2$	$144a^2x^3y$	$6xay^3z^2$	$2a^2x^2y^3$

42. We will now proceed to the multiplication of polynomials. Take the two polynomials $a+b+c$, and $d+f$, composed entirely of additive terms; the product may be presented under the form $(a+b+c)\ (d+f)$. But it is often necessary to form a single

polynomial from this product, and it is in this that the multiplication of two polynomials consists.

Now it is evident, that to multiply . $a+b+c$
by . . . . . $d+f$

$$\begin{array}{r}
a+b+c \\
d+f \\
\hline
ad+bd+cd \qquad\qquad \\
+af+bf+cf \\
\hline
ad+bd+cd+af+bf+cf.
\end{array}$$

is the same thing as taking $a+b+c$ as many times as there are units in d, then as many times as there are units in f, and adding the two products together. But to multiply $a+b+c$ by d, is to take each of the parts of the multiplicand d times and add together the partial products, which gives $ad+bd+cd$. In like manner, to multiply $a+b+c$ by f, is to take each of the parts of the multiplicand, f times, and add together the partial products.

Hence, $(a+b+c)\,(d+f)=ad+bd+cd+af+bf+cf$.

Therefore, in order to multiply together two polynomials composed entirely of additive terms, *multiply successively each term of the multiplicand by each term of the multiplier, and add together all the products.*

If the terms are affected with co-efficients and exponents, observe the rule given for the multiplication of monomials (Art. 41).

For example, multiply . . $3a^2+4ab+b^2$
by . . . . . $2a+5b$

$$\begin{array}{r}
3a^2+4ab+b^2 \qquad\qquad \\
2a+5b \qquad\qquad \\
\hline
6a^3+8a^2b+2ab^2 \qquad\quad \\
+15a^2b+20ab^2+5b^3
\end{array}$$

The product, after reducing, becomes . . . $6a^3+23a^2b+22ab^2+5b^3$

$$\begin{array}{l}
x^2+y^2 \\
x+y \\
\hline
x^3+xy^2 \\
\quad +x^2y+y^3 \\
\hline
x^3+xy^2+x^2y+y^3
\end{array}
\qquad\qquad
\begin{array}{r}
x^5+xy^6+7ax \qquad\qquad \\
ax+5ax \qquad\qquad\qquad \\
\hline
ax^6+ax^2y^6+7a^2x^2 \qquad\quad \\
+5ax^6+5ax^2y^6+35a^2x^2 \\
\hline
6ax^6+6ax^2y^6+42a^2x^2.
\end{array}$$

43. In order to explain the most general case, we will first remark, that if the multiplicand contains additive and subtractive terms, it may be considered as expressing the difference between the number of units indicated by the sum of the additive terms, and the number of units indicated by the sum of the subtractive terms. The same reasoning applies to the multiplier; whence it follows, that the general case may be reduced to the multiplication of two binomials, such as $a-b$ and $c-d$; a denoting the sum of the additive terms, and b the sum of the subtractive terms of the multiplicand, c and d expressing similar values of the multiplier. We will then show how the multiplication expressed by $(a-b)\times(c-d)$ can be effected.

$$\begin{array}{l} a\ -b \\ c\ -d \\ \hline ac-bc \\ \quad\ -ad+bd \\ \hline ac-bc\ -ad+bd. \end{array}$$

Now, to multiply $a-b$ by $c-d$, is evidently the same thing as to take $a-b$ as many times as there are units in c, and then diminish this product by $a-b$, taken as many times as there are units in d; or to multiply $a-b$ by c, and subtract from this product that of $a-b$ by d. But to multiply $a-b$ by c, is to take $a-b$, c times. Now if we multiply a by c the product is ac, which is too large by b taken c times; therefore cb must be taken from it: hence, the product of $a-b$ by c, is $ac-bc$. In like manner, the product of $a-b$ by d, is $ad-bd$; and as we have just seen that this last product should be subtracted from the preceding $ac-bc$, it is necessary to change the signs of $ad-bd$, and write it under $ac-bc$, which (Art. 37), gives

$$(a-b)\ (c-d)=ac-bc-ad+bd.$$

If we suppose a and c each equal to 0, the product will reduce to $+bd$.

44. Hence, for the multiplication of one polynomial, by another we have the following

RULE.

I. *Multiply all the terms of the multiplicand, both additive and subtractive, by each additive term of the multiplier, and affect the partial products with the same signs as those with which the terms of the multiplicand are affected; also multiply all the terms of the multiplicand by each subtractive term of the multiplier, but affect the partial products with signs contrary to those with which the terms of the multiplicand are affected. Then reduce the polonomial result to its simplest form.*

Take, for an example, the two polynomials:

$$4a^3-5a^2b-8ab^2+2b^3$$

and $$2a^2-3ab-4b^2$$

$$\begin{array}{l} 8a^5-10a^4b-16a^3b^2+4a^2b^3 \\ \quad -12a^4b+15a^3b^2+24a^2b^3-6ab^4 \\ \quad\quad -16a^3b^2+20a^2b^3+32ab^4-8b^5 \\ \hline 8a^5-22a^4b-17a^3b^2+48a^2b^3+26ab^4-8b^5. \end{array}$$

After having arranged the polynomials one under the other, multiply each term of the first, by the term $2a^2$ of the second; this gives the polynomial $8a^5-10a^4b-16a^3b^2+4a^2b^3$, the signs of which are the same as those of the multiplicand. Passing then to the term $3ab$ of the multiplier, multiply each term of the multiplicand by it, and as it is affected with the sign $-$, affect each product with a sign contrary to that of the corresponding term in the multiplicand; this gives $-12a^4b+15a^3b^2+24a^2b^3-6ab^4$ for a product, which is written under the first.

The same operation is also performed with the term $4b^2$, which is also subtractive; this gives, $-16a^3b^2+20a^2b^3+32ab^4-8b^5$. The product is then reduced, and we finally obtain, for the most simple expression of the product,

$$8a^5-22a^4b-17a^3b^2+48a^2b^3+26ab^4-8b^5.$$

The rule for the signs, which is the most important to retain, in the multiplication of two polynomials, may be expressed thus: *When two terms of the multiplicand and multiplier are affected with the same sign, the corresponding product is affected with the sign +, and when* *they are affected with contrary signs, the product is affected with the sign* $-$.

Again, we say in algebraic language, that $+$ multiplied by $+$, or $-$ multiplied by $-$, gives $+$; $-$ multiplied by $+$, or $+$ multiplied by $-$, gives $-$. But this last enunciation, which does not in itself offer any reasonable direction, should only be considered as an abbreviation of the preceding.

This is not the only case in which algebraists, for the sake of brevity, employ incorrect expressions, but which have the advantage of fixing the rules in the memory.

EXAMPLES.

1. Multiply $12ax$ by $3a$. *Ans.* $36a^2x$.
2. Multiply $4x^2-2y$ by $2y$. *Ans.* $8x^2y-4y^2$.
3. Multiply $2x+4y$ by $2x-4y$. *Ans.* $4x^2-16y^2$.
4. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. *Ans.* x^4-y^4.
5. Multiply x^2+xy+y^2 by x^2-xy+y^2.
Ans. $x^4+x^2y^2+y^4$.
6. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$.
7. Multiply $3x^2-2xy+5$ by $x^2+2xy-3$.
8. Multiply $3x^3+2x^2y^2+3y^2$ by $2x^3-3x^2y^2+5y^3$.
9.

$$\begin{array}{l} 3a^2-5bd+cf \\ -5a^2+4bd-8cf. \\ \hline \end{array}$$

Prod. red. $-15a^4+37a^2bd-29a^2cf-20b^2d^2+44bcdf-8c^2f^2.$

10.

$$\begin{array}{l} 4a^3b^3-5a^2b^2c+8a^2bc^2-3a^2c^3-7abc^3 \\ 2ab^3-4abc-2bc^2+c^3. \\ \hline \end{array}$$

Prod. red. $\left\{\begin{array}{l} 8a^4b^4-10a^3b^4c+28a^3b^3c^2-34a^3b^2c^3 \\ -4a^2b^3c^3-16a^4b^3c+12a^3bc^4+7a^2b^2c^4 \\ +14a^2bc^5+14ab^2c^5-3a^2c^6-7abc^6. \end{array}\right.$

45. We will make some important remarks upon algebraic multiplication.

1st. If the polynomials proposed to be multiplied by each other are homogeneous, *the product of these two polynomials will also be ho-*

mogeneous. This is an evident consequence of the rules relative to the letters and exponents in the multiplication of monomials. Moreover, the degree of each term of the product should be equal to the sum of the degrees of any two terms of the multiplier and multiplicand. Thus, in example 9, all the terms of the multiplicand being of the second degree, as well as those of the multiplier, all the terms of the product are of the fourth degree. In example 10, the multiplicand being of the fifth degree, and the multiplier of the third, the product is of the eighth degree. This remark serves to discover any errors in the calculations with respect to the exponents. For example, if it is found that in one of the terms of a product that should be homogeneous, the sum of the exponents is equal to 7, while in all the others their sum is 8, there is a manifest error in the addition of the exponents, and the multiplication of the two terms which have formed this product must be revised.

2d. When, in the multiplication of two polynomials, the product does not present any similar terms for reduction, the total number of terms in the product is equal to the product of the number of terms in the multiplicand, multiplied by the number of terms in the multiplier. This is a consequence of the rule, (Art. 44). Thus, when there are five terms in the multiplicand, and four in the multiplier, there are 5×4, or 20, in the product. In general when the multiplicand is composed of m terms, and the multiplier of n terms, the product contains $m\times n$ terms.

3d. When some of the terms are similar, the total number of terms in the product, when reduced, may be much less. But we will remark, that among the different terms of the product, there are some that cannot be reduced with any others. These are, 1st. The term produced by the multiplication of the term of the multiplicand, affected with the highest exponent of a certain letter, by the term of the multiplier, affected with the highest exponent of the same letter. 2d. The term produced by the multiplication of the terms affected with the lowest exponents of the same letter. For these two partial products will contain this letter, affected with a higher or lower exponent than either of the other partial products, and consequently cannot be similar to any of them. This remark, the truth of which is deduced from the rule of the exponents, will be very useful in division.

46. To finish with what has reference to algebraic multiplication, we will make known a few results of frequent use in algebra.

1st. Let it be required to form the square or second power of the binomial, $(a+b)$. We have, from known principles,

$$(a+b)^2=(a+b)\ (a+b)=a^2+2ab+b^2.$$

That is, the square of the sum of two quantities is composed of the square of the first, plus twice the product of the first by the second, plus the square of the second.

Thus, to form the square of $5a^2+8a^2b$, we have, from what has just been said,

$$(5a^2+8a^2b)^2=25a^4+80a^4b+64a^4b^2.$$

2d. To form the square of a difference, $a-b$, we have

$$(a-b)^2=(a-b)\ (a-b)=a^2-2ab+b^2$$

That is, the square of the difference between two quantities is composed of the square of the first, minus twice the product of the first by the second, plus the square of the second.

Thus, $$(7a^2b^2-12ab^3)^2=49a^4b^4-168a^3b^5+144a^2b^6.$$

3d. Let it be required to multiply $a+b$ by $a-b$.

We have $$(a+b)\times(a-b)=a^2-b^2$$

Hence, the sum of two quantities, multiplied by their difference, gives the difference of their squares for a product.

Thus, $$(8a^3+7ab^2)\ (8a^3-7ab^2)=64a^6-49a^2b^4.$$

We can, by combining these different results, find the products of certain polynomials more promptly, than by the common process. For example, multiply $5a^2-4ab+3b^2$, by $5a^2-4ab-3b^2$. If we observe that the first of these two quantities is the sum of the two quantities $5a^2-4ab$, and $3b^2$, and that the second is the difference of the same quantities, we find immediately that the product is

$$(5a^2-4ab)^2-(3b^2)^2=25a^4-40a^3b+16a^2b^2-9b^4.$$

47. By reflecting upon the results of multiplication that we have

just obtained, it will be perceived that their composition, or the manner in which they are formed from the multiplicand and multiplier, is entirely independent of any particular values that may be attributed to the letters a and b which enter the two factors.

The manner in which an algebraic product is formed from its two factors, is called the *law* of this product; and this law remains always the same, whatever values may be attributed to the letters which enter into the two factors.

48. Lastly, a polynomial being given, it may sometimes be decomposed into factors merely by inspection.

Take for example the polynomial ab^2c+5ab^3+ac.

It is plain that a is a factor of all the terms. Hence, we may write $ab^2c+5ab^3+ac=a(b^2c+5b^3+c)$.

Take the polynomial $25a^4-30a^3b+15a^2b^2$, it is evident that 5 and a^2 are factors of each of the terms. We may, therefore, put the polynomial under the form $5a^2(5a^2-6ab+3b^2)$.

In the same way $64a^4b^6-25a^2b^8$ is transformed into

$$(8a^2b^3+5ab^4)\times(8a^2b^3-5ab^4).$$

DIVISION.

49. Algebraic division has the same object as arithmetical, viz. having given a product, and one of its factors, to find the other factor.

We will first consider the case of two monomials.

The division of $72a^5$ by $8a^3$ is indicated thus: $\dfrac{72a^5}{8a^3}$

It is required to find a third monomial, which, multiplied by the second, will produce the first. Now, by the rules for the multiplication of monomials, the required quantity must be such that its coefficient multiplied by 8 should give 72 for a product, and that the exponent of the letter a in this quantity, added to 3, the exponent of the letter a in the divisor, should give 5, the exponent of a in the dividend. This quantity may, therefore, be obtained by dividing 72 by 8 and subtracting the exponent 3 from the exponent 5, which gives . . . . $\dfrac{72a^5}{8a^3}=9a^2.$

Also, $\dfrac{35a^3b^2c}{7ab}=5a^{3-1}b^{2-1}c=5a^2bc;$

for, $7ab\times 5a^2bc=35a^3b^2c.$

50. Hence for the division of monomials we have the following

RULE.

I. *Divide the co-efficient of the dividend by the co-efficient of the divisor.*

II. *Write in the quotient, after the co-efficient, all the letters common to the dividend and divisor, and affect each with an exponent equal to the excess of its exponent in the dividend over that in the divisor.*

III. *Annex to these, those letters of the dividend, with their respective exponents, which are not found in the divisor.*

From these rules we find,

$$\frac{48a^3b^3c^2d}{12ab^2c}=4a^2bcd\,;\frac{150a^5b^8cd^3}{30a^3b^5d^2}=5a^2b^3cd.$$

1. Divide $16x^2$ by $8x$. *Ans.* $2x$.
2. Divide $15axy^3$ by $3ay$. *Ans.* $5xy^2$.
3. Divide $84ab^3x$ by $12b^2$. *Ans.* $7abx$.

51. It follows from the preceding rule that the division of monomials will be impossible,

1st. When the co-efficients are not divisible by each other.

2d. When the exponents of the same letter are greater in the divisor than in the dividend.

3d. When the divisor contains one or more letters which are not found in the dividend.

When either of these three cases occurs, the quotient remains under the form of a monomial fraction, that is, a monomial expression, necessarily containing the algebraic sign of division; but which may frequently be reduced.

Take for example, $12a^4b^2cd$ to be divided by $8a^2bc^2$

Here an entire monomial cannot be obtained for a quotient; that is to say, a monomial which does not contain the sign of division; for 12 is not divisible by 8, and moreover, the exponent of c is less in the dividend than in the divisor; therefore, the quotient is presented under the form $\frac{12a^4b^2cd}{8a^2b\,c^2}$; but this expression can be reduced, by observing that the factors 4, a^2, b and c being common to the two terms of the fraction, may be suppressed, and we have

$$\frac{3a^2bd}{2c} \text{ for the result.}$$

In general, to reduce a monomial fraction it is necessary

1st. *To suppress the greatest factor common to the two co-efficients.*

2d. *Subtract the less of the two exponents of the same letter, from the greater, and write the letter affected with this difference, in that term of the fraction corresponding with the greatest exponent.*

3d. *Write those letters which are not common, with their respective exponents, in the term of the fraction which contains them.*

From this new rule, we find,

$$\frac{48a^3b^2c\,d^3}{36a^2b^3c^2de}=\frac{4ad^2}{3bce} \text{ and } \frac{37a\,b^3c^5d}{6a^3b\,c^4d^2}=\frac{37b^2c}{6a^2d};$$

also,
$$\frac{7a^2b}{14a^3b^2}=\frac{1}{2ab}$$

In the last example, as all the factors of the dividend are found in the divisor, the numerator is reduced to *unity;* for it amounts to dividing both terms of the fraction by the numerator.

52. It often happens, that the exponents of certain letters, are the same in the dividend and divisor.

For example, divide $24a^3b^2$, by $8a^2b^2$; as the letter b is affected with the same exponent, it should not be contained in the quotient, and we have $\frac{24a^3b^2}{8a^2b^2}=3a$. But it is to be remarked, that this result, $3a$, can be put under a form which will preserve the trace of the letter b, this letter having disappeared in consequence of the reduction.

For if we apply, conventionally, the rule for the exponents, (Art. 50.), to the expression $\frac{b^2}{b^2}$, it becomes $\frac{b^2}{b^2}=b^{2-2}=b^0$: this new symbol b^0, indicates that the letter enters 0 times, as a factor in the quotient (Art. 13); or, which amounts to the same thing, that it does not enter it; but it indicates at the same time, that it was in the dividend and divisor, and that it has disappeared in consequence of the operation. This symbol has the advantage of preserving the trace of a quantity which constitutes a part of the question, that it has been our object to resolve, without changing the value of the result; for since b^0 is equivalent to $\frac{b^2}{b^2}$, which is, moreover, equivalent to 1, it follows that $3ab^0=3a\times1=3a$. In like manner,

$$\frac{15a^2b^3c^2}{3a^2b\,c^2}=5a^0b^2c^0=5b^2.$$

53. As it is important to have clear ideas of the origin and signification of the symbols employed in algebra, we will show that in general every quantity a affected with the exponent 0, is equivalent to 1; that is, we will have $a^0=1$.

For this expression arises, as has just been said, from the fact that a is affected with the same exponent in the divisor and dividend.

To make the case general, let m denote the entire number which is the exponent of a. We shall then have, $\frac{a^m}{a^m}=a^0$. But the quotient of any quantity divided by itself, is 1. Hence, $\frac{a^m}{a^m}=1$; therefore, we also have $a^0=1$

We observe again, that the symbol a^0 is only employed conventionally, to preserve in the calculation the trace of a letter which entered in the enunciation of a question, but which must disappear in consequence of a division; and it is often necessary to preserve this trace.

Division of Polynomials.

54. Let it be required to divide $51a^2b^2+10a^4-48a^3b-15b^4+4ab^3$ by $4ab-5a^2+3b^2$. In order that we may follow the steps of the operation more easily, we will arrange the quantities thus.

$$
\begin{array}{l|l}
\textit{Dividend.} & \textit{Divisor.} \\
10a^4-48a^3b+51a^2b^2+4ab^3-15b^4 & \underline{-5a^2+4ab+3b^2} \\
\underline{+10a^4-\ 8a^3b-\ 6a^2b^2} & -2a^2+8ab-5b^2 \\
\quad -40a^3b+57a^2b^2+4ab^3-15b^4 & \quad \textit{Quotient.} \\
\quad \underline{-40a^3b+32a^2b^2+24ab^3} & \\
\qquad 25a^2b^2-20ab^3-15b^4 & \\
\qquad \underline{25a^2b^2-20ab^3-15b^4} &
\end{array}
$$

The object of this operation is, as we have already said (Art. 49), to find a third polynomial, which, multiplied by the second, shall produce the first.

It follows from the definition and the rule for the multiplication of polynomials (Art. 43), that the dividend is the assemblage, after addition and reduction, of the partial products of each term of the divisor, multiplied by each term of the quotient sought. Hence, if we could discover a term in the dividend which was derived, without reduction, from the multiplication of one of the terms of the divisor, by a term of the quotient, then, by dividing the term of the dividend by that of the divisor, we would obtain a term of the required quotient.

Now, from the third remark of Art. 45, the term $10a^4$, affected with the highest exponent of the letter a, is derived, without reduction from the two terms of the divisor and quotient, affected with the highest exponent of the same letter. Hence, by dividing the term $10a^4$ by the term $-5a^2$, we will have a term of the required quotient. But here another difficulty presents itself, viz. to determine the sign with which the term of the quotient should be affected. In order that this subject may not impede our progress hereafter, we will establish a rule for the signs in division.

Since, in multiplication, the product of two terms having the same sign is affected with the sign $+$, and the product of two terms having contrary signs is affected with the sign $-$, we may conclude,

1st. That when the term of the dividend has the sign $+$, and that of the divisor the sign $+$, the term of the quotient must have the sign $+$.

2d. When the term of the dividend has the sign $+$, and that of the divisor the sign $-$, the term of the quotient must have the sign $-$, because it is only the sign $-$, which, combined with the sign $-$, can produce the sign $+$ of the dividend.

3d. When the term of the dividend has the sign $-$, and that of the divisor the sign $+$, the quotient must have the sign $-$.

That is, when the two terms of the dividend and divisor have the same sign, the quotient will be affected with the sign $+$, and when they are affected with contrary signs, the quotient will be affected with the sign $-$; again, for the sake of brevity, we say that

$+$ divided by $+$, and $-$ divided by $-$, give $+$;
$-$ divided by $+$, and $+$ divided by $-$, give $-$.

In the proposed example, $10a^4$ and $-5a^2$ being affected with contrary signs, their quotient will have the sign $-$; moreover, $10a^4$, divided by $5a^2$, gives $2a^2$; hence, $-2a^2$ is a term of the required quotient. After having written it under the divisor, multiply each term of the divisor by it, and subtract the product,

$$10a^4-8a^3b+6a^2b^2,$$

from the dividend, which is done by writing it below the dividend, conceiving the signs to be changed, and performing the reduction. Thus, the result of the first partial operation is

$$-40a^3b+57a^2b^2+4ab^3-15b^4.$$

This result is composed of the partial products of each term of the divisor, by all the terms of the quotient which remain to be determined. We may then consider it as a new dividend, and reason upon it as upon the proposed dividend. We will therefore take in this result, the term $-40a^3b$, affected with the highest exponent of a, and divide it by the term $-5a^2$ of the divisor. Now, from the preceding principles, $-40a^3b$, divided by $-5a^2$ gives $+8ab$ for a new term of the quotient, which is written on the right of the first. Multiplying each term of the divisor by this term, and writing the products underneath the second dividend, and making the subtraction, the result of the second operation is

$$25a^2b^2-20ab^3-15b^4;$$

then dividing $25a^2b^2$ by $-5a^2$, we have $-5b^2$ for the third term of the quotient. Multiplying the divisor by this term, and writing the terms of the product under the third dividend, and reducing, we obtain 0 for the result. Hence, $-2a^2+8ab-5b^2$, or $8ab-2a^2-5b^2$ is the required quotient, which may be verified by multiplying the divisor by it; the product should be equal to the dividend.

By reflecting upon the preceding reasoning, it will be perceived, that, in each partial operation, we divide that term of the dividend which is affected with the highest exponent of one of the letters, by that term of the divisor affected with the highest exponent of the same letter. Now, we avoid the trouble of looking out the term, by taking care, in the first place, *to write the terms of the dividend and divisor in such a manner that the exponents of the same letter shall go on diminishing from left to right.* This is what is called *arranging* the dividend and divisor with reference to a certain letter. By this preparation, the first term on the left of the dividend, and the first on the left of the divisor, are always the two which must be divided by each other in order to obtain a term of the quotient; and it is the same in all the following operations; because the partial quotients, and the products of the divisor by these quotients are always *arranged.*

55. Hence, for the division polynomials we have the following

RULE.

I. *Arrange the dividend and divisor with reference to a certain letter, and then divide the first term on the left of the dividend by the first term* *on the left of the divisor, the result is the first term of the quotient; multiply the divisor by this term, and subtract the product from the dividend.*

II. *Then divide the first term of the remainder by the first term of the divisor, which gives the second term of the quotient; multiply the divisor by this second term, and subtract the product from the result of the first operation. Continue the same process until you obtain* 0 *for a result ; in which case the division is said to be exact.*

When the first term of the arranged dividend is not exactly divisible by that of the arranged divisor, the complete division is impossible, that is to say, there is not a polynomial which, multiplied by the divisor, will produce the dividend. And in general, we will find that a division is impossible, when the first term of one of the partial dividends is not divisible by the first term of the divisor.

56. Though there is some analogy between arithmetical and algebraical division, with respect to the manner in which the operations are disposed and performed, yet there is this essential difference between them, that in arithmetical division the figures of the quotient are obtained by trial, while in algebraical division the quotient obtained by dividing the first term of the partial dividend by the first term of the divisor is always one of the terms of the quotient sought.

Besides, nothing prevents our commencing the operation at the right instead of the left, since it might be performed upon the terms affected with the lowest exponent of the letter, with reference to which the arrangement has been made. In arithmetical division the quotient can only be obtained by commencing on the left.

Lastly, so independent are the partial operations required by the process, that after having subtracted the product of the divisor by the first term found in the quotient, we could obtain another term of the quotient by dividing by each other the two terms of the new dividend and divisor, affected with the highest exponent of a different letter from the one first considered. If the same letter is preserved, it is because there is no reason for changing it, and because the two polynomials are already arranged with reference to it; the first terms on the left of the dividend and divisor being sufficient to obtain a term of the quotient; whereas, if the letter is changed, it would be necessary to seek again for the highest exponent of this letter.

SECOND EXAMPLE.

Divide . . . $21x^3y^2+25x^2y^3+68xy^4-40y^5-56x^5-18x^4y$ by $5y^2-8x^2-6xy$.

$$
\begin{array}{lrrrrrr|l}
& -40y^5 & +68xy^4 & +25x^2y^3 & +21x^3y^2 & -18x^4y & -56x^5 & \underline{5y^2-6xy-8x^2} \\
& \underline{-40y^5} & \underline{+48xy^4} & \underline{+64x^2y^3} & & & & -8y^3+4xy^2-3x^2y+7x^3 \\
\text{1st. rem.} & & 20\,xy^4 & -39x^2y^3 & +21x^3y^2 & & & \\
& & \underline{20\,xy^4} & \underline{-24x^2y^3} & \underline{-32\,x^3y^2} & & & \\
\text{2d. rem.} & & & -15x^2y^3 & +53x^3y^2 & -18x^4y & & \\
& & & \underline{-15x^2y^3} & \underline{+18x^3y^2} & \underline{+24x^4y} & & \\
& & & & 35x^3y^2 & -\ 42x^4y & -56x^5 & \\
& & & & \underline{35x^3y^2} & \underline{-\ 42x^4y} & \underline{-56x^5} & \\
\text{Final remainder} & & & & & 0. & &
\end{array}
$$

57. Remark.—In performing the division, it is not necessary to bring down all the terms of the dividend to form the first remainder, but they may be brought down in succession, as in the example.

As it is important that beginners should render themselves familiar with the algebraic operations, and acquire the habit of calculating promptly, we will treat of this last example in a different manner, at the same time indicating the simplifications which should be introduced.

As in arithmetic, they consist in subtracting each partial product from the dividend as soon as this product is formed.

$$
\begin{array}{lrrrrrr|l}
& \underline{-40y^5} & \underline{+68xy^4} & \underline{+25x^2y^3} & \underline{+21x^3y^2} & \underline{-18x^4y} & \underline{-56x^5} & \underline{5y^2-6xy-8x^2} \\
\text{1st. rem.} & & \underline{20xy^4} & \underline{-39x^2y^3} & \underline{+21\,x^3y^2} & & & -8y^3+4xy^2-3x^2y+7x^3 \\
\text{2d. rem.} & & & \underline{-15x^2y^3} & \underline{+\ 53\,x^3y^2} & \underline{-18x^4y} & & \\
\text{3d. rem.} & & & & \underline{+\ 35\,x^3y^2} & \underline{-42x^4y} & \underline{-56x^5} & \\
\text{Final rem.} & & & & & 0. & &
\end{array}
$$

First, by dividing $-40y^5$ by $5y^2$, we obtain $-8y^3$ for the quotient. Multiplying $5y^2$ by $-8y^3$, we have $-40y^5$, or by changing the sign, $+40y^5$, which destroys the first term of the dividend.

In like manner, $-6xy \times -8y^3$ gives $+48xy^4$ and for the subtraction $-48xy^4$, which reduced with $+68xy^4$, gives $20xy^4$ for a remainder. Again, $-8x^2 \times -8y^3$ gives $+$, and changing sign, $-64x^2y^3$, which reduced with $25x^2y^3$, gives $-39x^2y^3$. Hence the result of the first operation is $20xy^4-39x^2y^3$ followed by those terms of the dividend which have not been reduced with the partial products already obtained. For the second part of the operation, it is only necessary to bring down the next term of the dividend, separating this new dividend from the primitive by a line, and operate upon this new dividend in the same manner as we operated upon the primitive, and so on.

THIRD EXAMPLE.

Divide $95a-73a^2+56a^4-25-59a^3$ by $-3a^2+5-11a+7a^3$

	$56a^4-59a^3-73a^2+95a-25$	$7a^3-3a^2-11a+5$
1st. rem.	$-35a^3+15a^2+55a-25$	$8a-5$
2d. rem.	$0.$	

EXAMPLES.

1. Divide $18x^2$ by $9x$. *Ans.* $2x$.
2. Divide $10x^2y^2$ by $-5x^2y$. *Ans.* $-2y$.
3. Divide $-9ax^2y^2$ by $9x^2y$. *Ans.* $-ay$.
4. Divide $-8x^2$ by $-2x$. *Ans.* $+4x$.
5. Divide $10ab+15ac$ by $5a$. *Ans.* $2b+3c$.
6. Divide $30ax-54x$ by $6x$. *Ans.* $5a-9$.
7. Divide $10x^2y-15y^2-5y$ by $5y$. *Ans.* $2x^2-3y-1$.
8. Divide $13a+3ax-17x^2$ by $21a$.
9. Divide $3a^2-15+6a+3b$ by $3a$.
10. Divide $a^2+2ax+x^2$ by $a+x$. *Ans.* $a+x$.
11. Divide $a^3-3a^2y+3ay^2-y^3$ by $a-y$.
    Ans. $a^2-2ay+y^2$.

12. Divide 1 by $1-x$. *Ans.* $1+x+x^2+x^3$, &c.

13. Divide $6x^4-96$ by $3x-6$. *Ans.* $2x^3+4x^2+8x+16$.

14. Divide $a^6-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$.
Ans. $a^3-3a^2x+3ax^2-x^3$.

15. Divide $48x^3-76ax^2-64a^2x+105a^3$ by $2x-3a$.

16. Divide $y^6-3y^4x^2+3y^2x^4-x^6$ by $y^3-3y^2x+3yx^2-x^3$.

58. It may happen that one, or both, of the proposed polynomials contains in two or more terms the same power of the letter with reference to which the arrangement is to be made.

In this case, how should the arrangement be made, and the division be effected?

Divide $11a^2b-19abc+10a^3-15a^2c+3ab^2+15bc^2-5b^2c$
by $5a^2+3ab-5bc$.

In the first place, the two terms $11a^2b-15a^2c$, can be placed under the form $(11b-15c)\,a^2$, or $\left.\begin{array}{r} 11b \\ -15c \end{array}\right| a^2$, by writing the power a^2 once, and placing to the left of it, and in the same vertical column, the quantities by which this power is multiplied; this polynomial multiplier is then called the co-efficient of a^2.

The second manner of connecting the terms involving the same power, is preferable to the first, for two reasons. 1st. Because where there are many terms in the dividend and divisor, it would be difficult to write all on the same horizontal line. 2d. As the co-efficient of each power ought to be arranged with reference to a second letter, we are obliged, if the first term is subtractive, to subject the term to a modification, which might lead to error, in employing the first manner. Take, for example, $-15b^2a^2+7bca^2-8c^2a^2$ the modification consists in putting this expression under the form

$$-(15b^2-7bc+8c^2)a^2 \quad . \quad . \quad . \quad . \quad . \quad \text{(Art. 38).}$$

whereas, by the second, it is written thus: $\left.\begin{array}{r} -15b^2 \\ +\ 7bc \\ -\ 8c^2 \end{array}\right| a^2$, and by this manner we have the advantage of preserving to each term the sign with which it was at first affected.

In like manner, $-19abc+3ab^2$ is written: . . $\begin{array}{r|l} +\,3b^2 & a \\ -19bc & \end{array}$

This being understood the operation may be performed in the following manner.

$$\begin{array}{lr|r|l|l}
 & 10a^3+11b & a^2+\ \ 3b^2 & a\ \ -5b^2c+15bc^2 & \quad 5a^2+3ba-5bc \\
 & -15c & -19bc & & \quad \overline{2a+b-3c} \\
\hline
\text{1st. Rem.} & 5b & a^2+\ \ 3b^2 & a-5b^2c+15bc^2 & \\
 & -15c & -9bc & & \\
\hline
\text{2d. Rem.} & & 0. & &
\end{array}$$

First divide $10\,a^3$ by $5a^2$, the quotient is $2a$. Subtracting the product of the divisor by $2a$, we obtain the first remainder. Dividing the part involving a^2 in this remainder by $5a^2$, the quotient is $b-3c$. Multiplying successively each term of the divisor by $b-3c$, and subtracting the product, we have 0 for the result. Hence, $2a+b-3c$ is the required quotient.

59. Among the different examples of algebraic division, there is one remarkable for its applications. It is so often met with in the resolution of questions, that algebraists have made a kind of *theorem* of it.

We have seen (Art. 46), that

$$(a+b)(a-b)=a^2-b^2: \qquad \text{hence,}$$

$$\frac{a^2-b^2}{a-b}=a+b.$$

If we divide . . a^3-b^3 by $a-b$ we have

$$\frac{a^3-b^3}{a-b}=a^2+ab+b^2: \qquad \text{also}$$

$$\frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3$$

by performing the division.

These are results that may be obtained by the ordinary process of division. Analogy would lead to the conclusion that whatever may be the exponents of the letters a and b, the division could be performed exactly; but analogy does not always lead to certainty. To be certain on this point, denote the exponent by m; and proceed to divide a^m-b^m by $a-b$.

$$\begin{array}{ll} & a^m-b^m \quad \| \; a-b \\ \text{1st. Rem.} \quad . \quad . \quad . \quad . & \overline{a^{m-1}b-b^m} \mid \overline{a^{m-1}+} \\ \text{or} \quad . \quad . \quad . \quad . \quad . & b(a^{m-1}-b^{m-1}). \end{array}$$

Dividing a^m by a the quotient is a^{m-1}, by the rule for the exponents. The product of $a-b$ by a^{m-1} being subtracted from the dividend, the first remainder is $a^{m-1}b-b^m$, which can be put under the form $b\,(a^{m-1}-b^{m-1})$. Now, if $a^{m-1}-b^{m-1}$ is divisible by $a-b$, then will a^m-b^m also be divisible by $a-b$; that is, if the difference of the similar powers of two quantities of *a certain degree*, is exactly divisible by the difference of these quantities, *the difference of the powers of a degree greater by unity, is also divisible by it.*

But it has already been shown that a^4-b^4 is divisible by $a-b$: hence, a^5-b^5 is also divisible by $a-b$. Now, if a^5-b^5 is divisible by $a-b$, it must follow that a^6-b^6 is also divisible by $a-b$. In the same way it may be shown that the division is possible when the exponent is 7, 8, 9, &c.

Hence, generally, a^m-b^m is divisible by $a-b$.

This proposition may be verified by actually performing the division, and then multiplying the quotient by the divisor. Thus,

$$\frac{a^m-b^m}{a\;-b}=a^{m-1}+a^{m-2}b+a^{m-3}b^2 \quad . \quad . \quad . \quad +ab^{m-2}+b^{m-1}$$

$$\begin{array}{lllll} \text{But} \quad . \quad . \quad . & a^{m-1}+a^{m-2}b+a^{m-3}b^2 & . \quad . & +ab^{m-2}+b^{m-1} & \\ \text{multiplied by} \quad . & a\;-\;b & & & \\ & a^m+\;a^{m-1}b+a^{m-2}b^2 & . \quad . & +a^2b^{m-2}+ab^{m-1} & \\ & \quad -\;a^{m-1}b-a^{m-2}b^2 & . \quad . \quad . & -ab^{m-1}-b^m. & \\ \hline \text{is equal to} \quad . \quad . \quad . & \qquad a^m-\;b^m. & & & \end{array}$$

It will be perceived that the partial products a^m and $-b^m$ are the only ones that do not destroy each other in the reduction.

For example, multiplying $a^{m-2}b$ by a, the product is $a^{m-1}b$; but by multiplying a^{m-1} by $-b$, the product is $-a^{m-1}b$, and this term destroys the preceding. The other terms cancel in the same way.

The beginner should reflect upon the first method of demonstrating the proposition, as it is frequently employed in algebra.

60. We have given (Art. 51. and 55.), the principal circumstances by which it may be discovered that the division of monomial or polynomial quantities is not exact; that is, the case in which there does not exist a third entire algebraic quantity, which, multiplied by the second, will produce the first.

We will add, as to polynomials, that it may often be discovered by mere inspection that they cannot be divided by each other. When these polynomials contain two or more letters, before arranging them with reference to a particular letter, observe the two terms of the dividend and divisor, which are affected with the highest exponent of each of the letters. If for either of these letters, one of the terms with the highest exponent is not divisible by the other, we may conclude that the total division is impossible. This remark applies to each of the operations required by the process for finding the quotient.

Take, for example, $12a^3-5a^2b+7ab^2-11b^3$, to be divided by $4a^2-8ab+3b^2$.

By considering only the letter a, the division would appear possible; but regarding the letter b, the division is impossible, since $-11b^3$ is not divisible by $3b^2$.

One polynomial A, cannot be divided by another B containing a letter which is not found in the dividend; for it is impossible that a third quantity, multiplied by B which depends upon a certain letter, should give a product independent of this letter.

A monomial is never divisible by a polynomial, because every polynomial multiplied by another, gives a product containing at least two terms which are not susceptible of reduction.

61. Remark.—If the letter with reference to which the dividend is arranged, is not found in the divisor, *the divisor is said to be independent of that letter;* and in that case the exact division is impossible, *unless the divisor will divide separately the co-efficient of each term of the dividend.*

For example, if the dividend were $3ba^4+9ba^2+12b$, arranged with reference to the letter a, and the divisor $3b$, the divisor would be *independent* of the letter a; and it is evident that the exact division could not be performed unless the co-efficient of each term of the dividend were divisible by $3b$. The exponents of the leading letter in the quotient would be the same as in the dividend.

OF ALGEBRAIC FRACTIONS.

62. Algebraic fractions should be considered in the same point of view as arithmetical fractions, such as $\frac{3}{4}$, $\frac{1}{12}$, that is, we must conceive that the unit has been divided into as many equal parts as there are units in the denominator, and that one of these parts is taken as many times as there are units in the numerator. Hence, addition, subtraction, multiplication, and division, are performed according to the rules established for arithmetical fractions.

It will not, therefore, be necessary to demonstrate those rules, and in their application we must follow the procedures indicated for the calculus of entire algebraic quantities.

63. Every quantity which is not expressed under a fractional form is called an *entire* algebraic quantity.

64. An algebraic expression, composed partly of an entire quantity and partly of a fraction, is called a *mixed quantity.*

65. When a division of monomial or polynomial quantities cannot be performed exactly, it is indicated by means of the known sign, and in this case, the quotient is presented under the form of a fraction, which we have already learned how to simplify (Art. 51). With respect to polynomial fractions, the following are cases which are easily reduced.

Take, for example, the expression $\dfrac{a^2-b^2}{a^2-2ab+b^2}$

This fraction can take the form $\dfrac{(a+b)\ (a-b)}{(a-b)^2}$ (Art. 46).

Suppressing the factor $a-b$, which is common to the two terms,

we obtain . . . . $\frac{a+b}{a-b}$

Again, take the expression $\frac{5a^3-10a^2b+5ab^2}{8a^3-8a^2b}$

This expression can be decomposed thus: $\frac{5a(a^2-2ab+b^2)}{8a^2(a-b)}$

or . . . . $\frac{5a(a-b)^2}{8a^2(a-b)}$.

Suppressing the common factor, $a(a-b,)$ the result is . .

$$\frac{5(a-b)}{8a}.$$

The particular cases examined above, are those in which the two terms of the fraction can be decomposed into the product of the sum by the difference of two quantities, and into the square of the sum or difference of two quantities. Practice teaches the manner of performing these decompositions, when they are possible.

But the two terms of the fraction may be more complicated polynomials, and then, their decomposition into factors not being so easy, we have recourse to the process for finding *the greatest common divisor.*

CASE I.

Of the Greatest Common Divisor.

66. The greatest common divisor of two polynomials, is the greatest polynomial, with reference to the exponents and co-efficients, that will exactly divide the proposed polynomials.

If two polynomials be divided by their greatest common divisor, the quotients will be *prime with respect to each other;* that is, they will no longer contain a common factor.

For, let A and B be the given polynomials, D their greatest common divisor, A′ and B′ the quotients after division*. Then

* NOTE.—When the same letter is used to designate different quantities, as above, the quantities having a certain connexion with each other, we read A′, B′, A prime, B prime, and if we have A″, B″, we say, A second, B second, &c.

$$\frac{A}{D}=A' \quad \text{and} \quad \frac{B}{D}=B'$$

Or . . . $A=A'\times D$ and $B=B'\times D$

now if A' and B' had a common factor d, it would follow that $d\times D$ would be a divisor, common to the two polynomials, and greater than D, either with respect to the exponents or the co-efficients, which would be contrary to the definition.

Again, since D exactly divides A and B, every factor of D will have a corresponding factor in both A and B. Hence,

1st. *The greater common divisor of two polynomials contains as factors, all the particular divisors common to the two polynomials, and does not contain any other factors.*

67. We will now show that the greatest common divisor of two polynomials will divide their remainder after division.

Let A and B be two polynomials, D their greatest common divisor, and suppose A to contain the highest exponent of the letter with reference to which they are arranged. Then,

$$\frac{A}{D}=A' \text{ and } \frac{B}{D}=B' \qquad \text{or,}$$

$$A=A'\times D \quad \text{and} \quad B=B'\times D.$$

Let us now represent the entire part of the quotient by Q and the remainder by R, and we shall have

$$\frac{A}{B}=\frac{A'\times D}{B'\times D}=Q+\frac{R}{B'\times D} \qquad \text{or}$$

$$A'\times D=B'\times D\times Q+R$$

hence, $$A'=B'\times Q+\frac{R}{D}.$$

But A' is an entire quantity, hence the quantity to which it is equal is also entire: and since $B'Q$ is entire, it follows, that $\frac{R}{D}$ is entire; that is, D will exactly divide R.

We will now show that if D will exactly divide B and R that it will also divide A. For, having divided A by B we have

$A=B\times Q+R$, and by dividing by D, we obtain

$$\frac{A}{D}=\frac{B}{D}\times Q+\frac{R}{D}.$$

But since we suppose B and R to be divisible by D, and know Q to be an entire quantity, the second part of the equality is entire; hence the first part, to which it is equal, is also entire; that is, A is exactly divisible by D. Hence,

2dly. *The greatest common divisor of two polynomials is the same as that which exists between the least polynomial and their remainder after division.*

These principles being established, let us suppose that it is required to find the greatest common divisor between the two polynomials

$$a^3-a^2b+3ab^2-3b^3, \text{ and } a^2-5ab+4b^2.$$

First Operation.

$$\begin{array}{r||l} a^3-a^2b+3ab^2-3b^3 & a^2-5ab+4b^2 \\ \hline 4a^2b-ab^2-3b^3 & a+4b \\ \hline \end{array}$$

1st. Rem. . . $19ab^2-19b^3$

or . . . $19b^2(a-b)$

Second Operation.

$$\begin{array}{r||l} a^2-5ab+4b^2 & a-b \\ \hline -4ab+4b^2 & a-4b \\ \hline 0. & \end{array}$$

Hence, $a-b$ is the greatest common divisor.

We begin by dividing the polynomial of the highest degree by that of the lowest degree; the quotient is, as we see in the above table, $a+4b$ and the remainder is $19ab^2-19b^3$.

By the second principle, the required common divisor is the same as that which exists between this remainder and the polynomial divisor.

But $19ab^2-19b^3$ can be put under the form $19b^2(a-b)$. Now the factor, $19b^2$, will divide this remainder without dividing

$$a^2-5ab+4b^2,$$

hence, by the first principle, this factor cannot enter into the greatest common divisor; we may therefore suppress it, and the question is reduced to finding the greatest common divisor between

$$a^2-5ab+b^2 \text{ and } a-b.$$

Dividing the first of these two polynomials by the second, there is an exact quotient, $a-4b$; hence $a-b$ is their greatest common divisor, and is consequently the greatest common divisor of the two proposed polynomials.

Again, take the same example, and arrange the polynomials with reference to b.

$$-3b^3+3ab^2-a^2b+a^3, \text{ and } 4b^2-5ab+a^2.$$

First Operation.

$$\begin{array}{l r||l}
 & -12b^3+12ab^2-4a^2b+4a^3 & 4b^2-5ab+a^2 \\
\hline
\text{1st. Rem.} & -\ 3ab^2-a^2b\ +4a^3 & -3b,\ -3a \\
 & -12ab^2-4a^2b+16a^3 & \\
\hline
\text{2d. Rem.}\ \ .\ \ . & -19a^2b+19a^3 & \\
\text{or}\ \ .\ \ . & 19a^2(-b+a). &
\end{array}$$

Second Operation.

$$\begin{array}{r||l}
4b^2-5ab+a^2 & -b+a \\
\hline
-ab+a^2 & -4b+a \\
\hline
0. &
\end{array}$$

Hence, $-b+a$, or $a-b$, is the greatest common divisor.

Here we meet with a difficulty in dividing the two polymonials, because the first term of the dividend is not exactly divisible by the first term of the divisor. But if we observe that the co-efficient 4 of this last, is not a factor of all the terms of the polynomial

$$4b^2-5ab+a^2,$$

and that therefore, by the first principle, 4 cannot form a part of the greatest common divisor, we can, without affecting this common divisor, introduce this factor into the dividend. This gives

$$-12b^3+12ab^2-4a^2b+4a^3,$$

and then the division of the first two terms is possible.

Effecting this division, the quotient is $-3b$, and the remainder is

$$-3ab^2-a^2b+4a^3.$$

As the exponent of b in this remainder is still equal to that of the divisor, the division may be continued, by multiplying this remainder by 4, in order to render the division of the first term possible.

This done, the remainder becomes $-12ab^2-4a^2b+16a^3$, which divided by $4b^2-5ab+a^2$, gives the quotient $-3a$, which should be separated from the first by a comma, having no connexion with it; and the remainder is . $-19a^2b+19a^3$.

Placing this last remainder under the form $19a^2(-b+a)$, and suppressing the factor $19a^2$, as forming no part of the common divisor, the question is reduced to finding the greatest common divisor between $4b^2-5ab+a^2$, and $-b+a$.

Dividing the first of these polynomials by the second, we obtain an exact quotient, $-4b+a$; hence $-b+a$, or $a-b$, is the greatest common divisor required.

68. In the above example, as in all those in which the exponent of the principal letter is greater by unity in the dividend than in the divisor, we can abridge the operation by multiplying every term of the dividend by the square of the co-efficient of the first term of the divisor. We may easily conceive that, by this means, the first partial quotient obtained will contain the first power of this co-efficient. Multiplying the divisor by the quotient, and making the reductions with the dividend thus prepared, the result will still contain the coefficient as a factor, and the division can be continued until a remainder is obtained of a lower degree than the divisor, with reference to the principal letter.

Take the same example as before, viz. $-3b^3+3ab^2-a^2b+a^3$ and $4b^2-5ab+a^2$; and multiply the dividend by the square of $4=16$: and we have

First Operation.

$$\begin{array}{rl|l} & -48b^3+48ab^2-16a^2b+16a^3 & 4b^2-5ab+a^2 \\ \hline & -12ab^2-4a^2b+16a^3 & -12b-3a \\ \hline \text{1st. Rem.} \ \ldots & -19a^2b+19a^3 & \\ \text{or} \ \ldots & 19a^2\,(-b+a) & \end{array}$$

Second Operation.

$$\begin{array}{rl|l} & 4b^2-5ab+a^2 & -b+a \\ \hline & -ab+a^2 & -4b+a \\ \hline \text{2d. Rem.} \ \ldots & -\ 0. & \end{array}$$

REMARK 1. When the exponent of the principal letter in the dividend exceeds that of the same letter in the divisor by two, three, &c. units, multiply the dividend by the third, fourth, &c. power of the co-efficient of the first term of the divisor. It is easy to see the reason of this.

2. It might be asked if the suppression of the factors, common to all the terms of one of the remainders, is *absolutely necessary*, or whether the object is merely to render the operations more simple. Now, it will easily be perceived that the suppression of these factors is necessary; for, if the factor $19a^2$ was not suppressed in the preceding example, it would be necessary to multiply the whole dividend by this factor, in order to render the first term of the dividend divisible by the first term of the divisor; but then, a factor would be introduced into the dividend which was also contained in the divisor; and consequently the required greatest common divisor would be combined with the factor $19a^2$, which should not form a part of it.

69. For another example, it is proposed to find the greatest common divisor between the two polynomials,

$$a^4+3a^3b+4a^2b^2-6ab^3+2b^4 \text{ and } 4a^2b+2ab^2-2b^3,$$

or simply, $2a^2+ab-b^2$, since the factor $2b$ can be suppressed, being a factor of the second polynomial and not of the first.

First Operation.

$$\begin{array}{l|l} 8a^4+24a^3b+32a^2b^2-48ab^3+16b^4 & 2a^2+ab-b^2 \\ \hline \quad +20a^3b+36a^2b^2-48ab^3+16b^4 & 4a^2+10ab+13b^2 \\ \qquad\quad +26a^2b^2-38ab^3+16b^4 & \end{array}$$

1st. Rem. . . . $-51ab^3+29b^4$

or, . . . $-\ b^3\,(51a-29b)$.

Second Operation.

Multiply by 2601, the square of 51.

$$\begin{array}{l|l} 5202a^2+2601ab-2601b^2 & 51a-\ 29b \\ \hline 5202a^2-2958ab & 102a+109b \end{array}$$

1st. Rem. . $+5559ab-2601b^2$

$5559ab-3161b^2$

2d. Rem. . . . $+\ 560b^2$.

The exponent of the letter a in the dividend, exceeding that of the same letter in the divisor by *two* units, we multiply the whole dividend by the cube of 2, or 8. This done, we perform three consecutive divisions, and obtain for the first principal remainder,

$$-51ab^3+29b^4.$$

Suppressing b^3 in this remainder, it becomes $-51a+29b$ for a new divisor, or, changing the signs, which is permitted, $51a-29b$: the new dividend is $2a^2+ab-b^2$.

Multiplying this dividend by the square of 51, or 2601, then effecting the division, we obtain for the second principal remainder, $+560b^2$, which proves that the two proposed polynomials are *prime with respect to each other*, that is, they have not a common factor. In fact it results from the second principle (Art. 67), that the greatest common divisor must be a factor of the remainder of each operation; therefore it should divide the remainder $560b^2$; but this remainder is *independent* of the principal letter a; hence, if the two polynomials have a common divisor, it must be *independent* of a, and will consequently be found as a factor in the co-efficients of the different powers of this letter, in each of the proposed polynomials; but it is evident that the co-efficients of these polynomials have not a common factor.

70. These examples are sufficient to point out the course the beginner is to pursue, in finding the greatest common divisor of two polynomials, which may be expressed by the following general

RULE.

I. *Take the first polynomial and suppress all the monomial factors common to each of its terms. Do the same with the second polynomial, and if the factors so suppressed have a common divisor, set it aside as forming a part of the common divisor sought.*

II. *Having done this, prepare the dividend in a such a manner that its first term shall be divisible by the first term of the divisor; then perform the division, which gives a remainder of a degree less than that of the divisor, in which suppress all the factors that are common to the co-efficients of the different powers of the principal letter. Then take this remainder as a divisor, and the second polynomial as a dividend, and continue the operation with these polynomials, in the same manner as with the preceding.*

III. *Continue this series of operations until a remainder is obtained which will exactly divide the preceding remainder; this last remainder will be the greatest common divisor; but if a remainder is obtained which is independent of the principal letter, and which will not divide the co-efficients of each of the proposed polynomials, it shows that the proposed polynomials are prime with respect to each other, or that they have not a common factor.*

EXAMPLES.

1. Find the greatest common divisor between the two polynomials.

$$ab+2a^2-3b^2-4bc-ac-c^2.$$

and . . $$9ac+2a^2-5ab+4c^2+8bc-12b^2$$

First Operation.

$$
\begin{array}{r|l||r|l}
2a^2+b & a-3b^2 & 2a^2-5b & a-12b^2 \\
-c & -4bc & +9c & +\ 8bc \\
 & -\ c^2 & & +\ 4c^2 \\
\hline
\end{array}
$$

$$
\begin{array}{lr|l|c}
\text{1st. Remainder} & 6b & a+9b^2 & 1 \\
 & -10c & -12bc & \\
 & & -\ 5\ c^2 &
\end{array}
$$

or . . . $(3b-5c)\ (2a+3b+c)$.

Second Operation.

$$
\begin{array}{r|l||l}
2a^2-5b & a-12b^2 & 2a+3b+c \\
+9c & +8bc & \\
\cline{3-3}
 & +4c^2 & a-4b \\
\hline
-8b & a-12b^2 & +4c \\
+8c & +\ 8bc & \\
 & +\ 4c^2 & \\
\hline
 & 0. &
\end{array}
$$

Hence, $2a+3b+c$ is the greatest common divisor.

After arranging the two polynomials, the division may be performed without any preparation, and the first remainder will be,

$$
\begin{array}{r|l}
6b & a+\ 9b^2 \\
-10c & -12bc \\
 & -\ 5c^2
\end{array}
$$

To continue the operation, it is necessary to take the second polynomial for a dividend, and this remainder for a divisor, and multiply the new dividend by $6b-10c$, or simply $3b-5c$, since 2 is a factor of the first term of the dividend. But we are not at liberty to multiply by $3b-5c$, if it is a factor of the remainder. Therefore, before effecting the multiplication, we must see if $3b-5c$ will exactly divide the first remainder ; we find that it does, and gives for a quotient $2a+3b+c$: whence it follows that the remainder can be put under the form

$$(3b-5c)\ (2a+3b+c).$$

Now, $3b-5c$ is a factor of this remainder, and is not a factor of the new dividend. For, being independent of the letter a, if it was a factor of the dividend it would necessarily divide the co-efficient of this letter in each of the terms, which it does not; we may therefore suppress it without affecting the greatest common divisor.

This suppression is indispensable, for otherwise a new factor would be introduced into the dividend, and then the two polynomials containing a factor they had not before, the greatest common divisor would be changed; it would be combined with the factor $3b-5c$, which should not form a part of it.

Suppressing this factor, and effecting the new division, we obtain an exact quotient; hence

$2a+3b+c$ is the greatest common divisor.

Remark. The rule for the greatest common divisor of two polynomials, may readily be extended to three or more polynomials. For, having the polynomials A, B, C, D, &c. if we find the greatest common divisor of A and B, and then the greatest common divisor of this result and C, the divisor so obtained will evidently be the greatest common divisor of A, B and C; and the same process may be applied to the remaining polynomials.

2. Find the greatest common divisor of x^4-1 and x^5+x^3.

Ans. $1+x^2$.

3. Find the greatest common divisor of $4a^3-2a^2-3a+1$ and $3a^2-2a-1$.

Ans. $a-1$.

4. Find the greatest common divisor of a^4-x^4 and $a^3-a^2x-ax^2+x^3$.

Ans. a^2-x^2.

5. Find the greatest common divisor of $36a^6-18a^5-27a^4+9a^3$ and $27a^5b^2-18a^4b^2-9a^3b^2$.

Ans. $9a^3(a-1)$.

6. Find the greatest common divisor of

$$qnp^3+3np^2q^2-2npq^3-2nq^4 \text{ and } 2mp^2q^2-4mp^4-mp^3q+3mpq^3.$$

Ans. $p-q$.

7. Find the greatest common divisor of the two polynomials

$$15a^5+10a^4b+4a^3b^2+6a^2b^3-3ab^4$$
$$12a^3b^2+38a^2b^3+16ab^4-10b^5.$$

Ans. $3a^2+2ab-b^2$.

CASE II.

71. To reduce a mixed quantity to the form of a fraction.

RULE.

Multiply the entire part by the denominator of the fraction: then connect this product with the terms of the numerator by the rules for addition, and under the result place the given denominator.

EXAMPLES.

1. Reduce $x-\frac{(a^2-x^2)}{x}$ to the form of a fraction.

$$x-\frac{a^2-x^2}{x}=\frac{x^2-(a^2-x^2)}{x}=\frac{2x^2-a^2}{x}. \quad \textit{Ans.}$$

2. Reduce $x-\frac{ax+x^2}{2a}$ to the form of a fraction.

Ans. $\frac{ax-x^2}{2a}$.

3. Reduce $5+\frac{2x-7}{3x}$ to the form of a fraction

Ans. $\frac{17x-7}{3x}$.

4. Reduce $1-\frac{x-a-1}{a}$ to the form of a fraction.

Ans. $\frac{2a-x+1}{a}$.

5. Reduce $1+2x-\frac{x-3}{5x}$ to the form of a fraction.

Ans. $\frac{10x^2+4x+3}{5x}$.

CASE III.

72. To reduce a fraction to an entire or mixed quantity.

RULE.

Divide the numerator by the denominator for the entire part, and place the remainder, if any, over the denominator for the fractional part.

EXAMPLES.

1. Reduce $\frac{ax+a^2}{x}$ to a mixed quantity.

$$\frac{ax+a^2}{x}=a+\frac{a^2}{x} \quad \textit{Ans.}$$

2. Reduce $\frac{ax-x^2}{x}$ to an entire or mixed quantity.

Ans. $a-x$.

3. Reduce $\frac{ab-2a^2}{b}$ to a mixed quantity.

Ans. $a-\frac{2a^2}{b}$.

4. Reduce $\frac{a^2-x^2}{a-x}$ to an entire quantity.

Ans. $a+x$

5. Reduce $\frac{x^3-y^3}{x-y}$ to an entire quantity.

Ans. x^2+xy+y^2.

6. Reduce $\frac{10x^2-5x+3}{5x}$ to a mixed quantity.

Ans. $2x-1+\frac{3}{5x}$.

CASE IV.

73. To reduce fractions having different denominators to equivalent fractions having a common denominator.

RULE.

Multiply each numerator into all the denominators except its own, for the new numerators, and all the denominators together for a common denominator.

EXAMPLES.

1. Reduce $\frac{a}{b}$ and $\frac{b}{c}$ to equivalent fractions having a common denominator.

$$\left.\begin{array}{l} a\times c=ac \\ b\times b=b^2 \end{array}\right\} \text{the new numerators.}$$

and . . $b\times c=bc$ the common denominator.

2. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to fractions, having a common denominator. *Ans.* $\frac{ac}{bc}$ and $\frac{ab+b^2}{bc}$

3. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and d, to fractions having a common denominator. *Ans.* $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$ and $\frac{6acd}{6ac}$.

4. Reduce $\frac{3}{4}$, $\frac{2x}{3}$, and $a+\frac{2x}{a}$, to fractions having a common denominator. *Ans.* $\frac{9a}{12a}$, $\frac{8ax}{12a}$, and $\frac{12a^2+24x}{12a}$.

5. Reduce $\frac{1}{2}$, $\frac{a^2}{3}$ and $\frac{a^2+x^2}{a+x}$, to fractions having a common denominator.

Ans. $\frac{3a+3x}{6a+6x}$, $\frac{2a^3+2a^2x}{6a+6x}$, and $\frac{6a^2+6x^2}{6a+6x}$.

CASE V.

74. To add fractional quantities together.

RULE.

Reduce the fractions, if necessary, to a common denominator: then add the numerators together and place their sum over the common denominator.

EXAMPLES.

1. Find the sum of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$.

Here, $\left.\begin{array}{l} a\times d\times f=adf \\ c\times b\times f=cbf \\ e\times b\times d=ebd \end{array}\right\}$ the new numerators.

And . . $b\times d\times f=bdf$ the common denominator.

Hence, $\frac{adf}{bdf}+\frac{cbf}{bdf}+\frac{ebd}{bdf}=\frac{adf+cbf+ebd}{bdf}$ the sum.

2. To $a-\frac{3x^2}{b}$ add $b+\frac{2ax}{c}$.

Ans. $a+b+\frac{2abx-3cx^2}{bc}$.

3. Add $\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$ together. *Ans.* $x+\frac{x}{12}$.

4. Add $\frac{x-2}{3}$ and $\frac{4x}{7}$ together. *Ans.* $\frac{19x-14}{21}$.

5. Add $x+\frac{x-2}{3}$ to $3x+\frac{2x-3}{4}$.

Ans. $4x+\frac{10x-17}{12}$.

6. It is required to add $4x$, $\frac{5x^2}{2a}$, and $\frac{x+a}{2x}$ together.

Ans. $4x+\frac{5x^3+ax+a^2}{2ax}$.

7. It is required to add $\frac{2x}{3}$, $\frac{7x}{4}$, and $\frac{2x+1}{5}$ together.

Ans. $2x+\frac{49x+12}{60}$.

8. It is required to add $4x$, $\frac{7x}{9}$, and $2+\frac{x}{5}$ together.

Ans $4x+\frac{44x+90}{45}$.

9. It is required to add $3x+\frac{2x}{5}$ and $x-\frac{8x}{9}$ together.

Ans. $3x+\frac{23x}{45}$.

CASE VI.

75. To subtract one fractional quantity from another.

RULE.

I. *Reduce the fractions to a common denominator.*

II. *Subtract the numerator of the fraction to be subtracted from the numerator of the other fraction, and place the difference over the common denominator.*

EXAMPLES.

1. Find the difference of the fractions $\frac{x-a}{2b}$ and $\frac{2a-4x}{3c}$

Here, $\left.\begin{array}{r}(x-a)\times 3c=3cx-3ac\\(2a-4x)\times 2b=4ab-8bx\end{array}\right\}$ the numerators

And, $2b\times 3c=6bc$ the common denominator.

Hence, $\frac{3cx-3ac}{6bc}-\frac{4ab-8bx}{6bc}=\frac{3cx-3ac-4ab+8bx}{6bc}$. *Ans.*

2. Required the difference of $\frac{12x}{7}$ and $\frac{3x}{5}$.

Ans. $\frac{39x}{35}$.

3. Required the difference of $5y$ and $\frac{3y}{8}$.

Ans. $\frac{37y}{8}$.

4. Required the difference of $\frac{3x}{7}$ and $\frac{2x}{9}$.

Ans. $\frac{13x}{63}$.

5. Required the difference between $\frac{x+a}{b}$ and $\frac{c}{d}$.

Ans. $\frac{dx+ad-bc}{bd}$.

6. Required the difference of $\frac{3x+a}{5b}$ and $\frac{2x+7}{8}$.

Ans. $\frac{24x+8a-10bx-35b}{40b}$.

7. Required the difference of $3x+\frac{x}{b}$ and $x-\frac{x-a}{c}$.

Ans. $2x+\frac{cx+bx-ab}{bc}$.

CASE VII.

76. To multiply fractional quantities together.

RULE.

If the quantities to be multiplied are mixed, reduce them to a fractional form; then multiply the numerators together for a numerator and the denominators together for a denominator.

EXAMPLES.

1. Multiply $a+\frac{bx}{a}$ by $\frac{c}{d}$.

$$a+\frac{bx}{a}=\frac{a^2+bx}{a}$$

Hence, ... $\frac{a^2+bx}{a}\times\frac{c}{d}=\frac{a^2c+bcx}{ad}$. Ans.

2. Required the product of $\frac{3x}{2}$ and $\frac{3a}{b}$.

Ans. $\frac{9ax}{2b}$.

3. Required the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$.

Ans. $\frac{3x^3}{5a}$.

4. Find the continued product of $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{3ac}{2b}$.

Ans. $9ax$.

5. It is required to find the product of $b+\frac{bx}{a}$ and $\frac{a}{x}$.

Ans. $\frac{ab+bx}{x}$.

6. Required the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$.

Ans. $\frac{x^4-b^4}{b^2c+bc^2}$.

7. Required the product of $x+\frac{x+1}{a}$, and $\frac{x-1}{a+b}$.

Ans. $\frac{ax^2-ax+x^2-1}{a^2+ab}$.

8. Required the product of $a+\frac{ax}{a-x}$ by $\frac{a^2-x^2}{x+x^2}$.

Ans. $\frac{a^4-a^2x^2}{ax+ax^2-x^2-x^3}$.

CASE VIII.

77. To divide one fractional quantity by another.

RULE.

Reduce the mixed quantities, if there are any, to a fractional form: then invert the terms of the divisor and multiply the fractions together as in the last case.

EXAMPLES.

1. Divide . . . . . $a-\frac{b}{2c}$ by $\frac{f}{g}$.

$$a-\frac{b}{2c}=\frac{2ac-b}{2c}.$$

Hence, $a-\frac{b}{2c}\div\frac{f}{g}=\frac{2ac-b}{2c}\times\frac{g}{f}=\frac{2acg-bg}{2cf}$. *Ans.*

2. Let $\frac{7x}{5}$ be divided by $\frac{12}{13}$. *Ans.* $\frac{91x}{60}$.

3. Let $\frac{4x^2}{7}$ be divided by $5x$. *Ans.* $\frac{4x}{35}$.

4. Let $\frac{x+1}{6}$ be divided by $\frac{2x}{3}$. *Ans.* $\frac{x+1}{4x}$.

5. Let $\frac{x}{x-1}$ be divided by $\frac{x}{2}$. *Ans.* $\frac{2}{x-1}$.

6. Let $\frac{5x}{3}$ be divided by $\frac{2a}{3b}$. *Ans.* $\frac{5bx}{2a}$.

7. Let $\frac{x-b}{8cd}$ be divided by $\frac{3cx}{4d}$. *Ans.* $\frac{x-b}{6c^2x}$.

8. Let $\frac{x^4-b^4}{x^2-2bx+b^2}$ be divided by $\frac{x^2+bx}{x-b}$.

Ans. $x+\frac{b^2}{x}$.

78. We will add but a single proposition more on the subject of fractions. It is this.

If the same number be added to each of the terms of a proper fraction, the new fraction resulting from this addition will be greater than the first; but if it be added to the terms of an improper fraction, the resulting fraction will be less than the first.

Let the fraction be expressed by $\frac{a}{b}$, and suppose $a<b$.

Let m represent the number to be added to the terms: the fraction then becomes $\frac{a+m}{b+m}$.

In order to compare the two fractions, they must be reduced to the same denominator, which gives $\frac{ab+am}{b^2+bm}$ for the first, and $\frac{ab+bm}{b^2+bm}$ for the second.

Now, the denominators being the same, that fraction will be the greatest which has the greater numerator. But the two numerators, $ab+am$, and $ab+bm$, have a common part ab; and the part bm of the second is greater than the part am of the first, since $b>a$. Hence the second fraction is greater than the first.

If the given fraction is improper, or $a>b$, it is plain that the numerator of the second fraction will be less than that of the first, since bm would be less than am.

CHAPTER II.

Of Equations of the First Degree.

79. An *Equation* is the expression of two equal quantities with the sign of equality placed between them. Thus, $x=a+b$ is an equation, in which x is equal to the sum of a and b.

80. By the definition, every equation is composed of two parts, separated from each other by the sign $=$. The part on the left of the sign, is called the *first member*, and the part on the right, is called the *second member;* and each member may be composed of one or more terms.

81. Every equation may be regarded as the enunciation, in algebraic language, of a particular problem. Thus, the equation $x+x=30$, is the algebraic enunciation of the following problem;

To find a number which, being added to itself, shall give a sum equal to 30.

Were it required to solve this problem we should first express it in algebraic language, which would give the equation

$$x+x=30.$$

By adding x to itself, we have . . . . . . $2x=30$.

and by dividing by 2, we obtain . . . . $x=15$.

Hence we see that the solution of a problem by algebra, consists of two distinct parts.

1*st*. *To express algebraically the relation between the known and unknown quantities.*

2*d*. *To find a value for the unknown quantity, in terms of those which are known, which substituted in its place in the given equation will satisfy the equation; that is, render the first member equal to the second.*

This latter part is called the *solution* of the equation.

82. An equation is said to be *verified*, when such a value is substituted for the unknown quantity as will prove the two members of the equation to be equal to each other.

83. Equations are divided into different classes. Those which contain only the first power of the unknown quantity, are called equations of the first degree. Thus,

$ax + b = cx + d$ is an equation of the 1st. degree.

$2x^2 - 3x = 5 - 2x^2$ is an equation of the 2d. degree.

$4x^3 - 5x^2 = 3x + 11$ is an equation of the 3d. degree.

In general, the *degree* of an equation is denoted by the greatest of the exponents with which the unknown quantity is affected.

84. Equations are also distinguished as *numerical equations* and *literal equations*. The first are those which contain numbers only, with the exception of the unknown quantity, which is always denoted by a letter. Thus, $4x-3=2x+5$, $3x^2-x=8$, are numerical equations. They are the algebraical translation of problems, in which the known quantities are particular numbers.

The equations $ax-b=cx+d$, $ax^2+bx=c$, are literal equations, in which the given quantities of the problem are represented by letters.

85. It frequently occurs in algebra, that the algebraic sign + or −, which is written, is not the true sign of the term before which it is placed. Thus, if it were required to subtract $-b$ from a, we should write

$$a-(-b)=a+b.$$

Here the true sign of the second term of the binomial is plus, although its algebraic sign, which is written in the first member of the equation, is $-$. This minus sign, operating upon the sign of b, which is also negative, produces a plus sign for b in the result. The sign which results, after combining the algebraic sign with the sign of the quantity, *is called the essential sign of the term*, and is often different from the algebraic sign.

By considering the nature of an equation, we perceive that it must possess the three following properties.

1st. The two members are composed of quantities of the same kind.

2d. The two members are equal to each other.

3d. The essential sign of the two members must be the same.

Equations of the First Degree involving but one unknown quantity.

86. An axiom is a self-evident proposition. We may here state the following.

1. If equal quantities be added to both members of an equation, the equality of the members will not be destroyed.

2. If equal quantities be subtracted from both members of an equation, the equality will not be destroyed.

3. If both members of an equation be multiplied by the same number, the equality will not be destroyed.

4. If both members of an equation be divided by the same number, the equality will not be destroyed.

87. The *transformation* of an equation consists in changing its form without affecting the equality of its members.

The following transformations are of continued use in the resolution of equations.

First Transformation.

88. When some of the terms of an equation are fractional, to reduce the equation to one in which the terms shall be entire.

Take the equation,

$$\frac{2x}{3}-\frac{3}{4}x+\frac{x}{6}=11.$$

First, reduce all the fractions to the same denominator, by the known rule; the equation becomes

$$\frac{48x}{72}-\frac{54x}{72}+\frac{12x}{72}=11$$

and since we can multiply both members by the same number without destroying the equality, we will multiply them by 72, which is the same as suppressing the denominator 72, in the fractional terms, and multiplying the entire term by 72; the equation then becomes

$$48x-54x+12x=792.$$

or dividing by 6 $\quad 8x-\ 9x+\ 2x=132.$

89. The last equation could have been found in another manner by employing the least common multiple of the denominators.

The *common multiple* of two or more numbers is any number which they will both divide without a remainder; and the *least* common multiple, is the least number which they will so divide. The least common multiple will be the product of all the numbers, when, in comparing either with the others, we find no common factors. But when there are common factors, the least common multiple will be the product of all the numbers divided by the product of the common factors.

The least common multiple, when the numbers are small, can generally be found by inspection. Thus, 24 is the least common multiple of 4, 6, and 8, and 12 is the least common multiple of 3, 4 and 6.

Take the last equation $\quad \frac{2x}{3}-\frac{3}{4}x+\frac{x}{6}=11.$

We see that 12 is the least common multiple of the denominators, and if we multiply all the terms of the equation by 12, and divide by the denominators, we obtain

$$8x-9x+2x=132.$$

the same equation as before found.

90. Hence, to make the denominators disappear from an equation, we have the following

RULE.

I. *Form the least common multiple of all the denominators.*

II. *Multiply each of the entire terms by this multiple, and each of the fractional terms by the quotient of this multiple divided by the denominator of the term thus multiplied, and omit the denominators of the fractional terms.*

EXAMPLES.

1. Clear the equation $\frac{x}{5}+\frac{x}{7}-4=3$ of its denominators.

Ans. $7x+5x-140=105$.

2. Clear the equation $\frac{a}{b}-\frac{c}{d}+f=g$.

Ans. $ad-bc+bdf=bdg$.

3. In the equation

$$\frac{ax}{b}-\frac{2c^2x}{ab}+4a=\frac{4bc^2x}{a^3}-\frac{5a^3}{b^2}+\frac{2c^2}{a}-3b.$$

the least common multiple of the denominators is a^3b^2; hence clearing the fractions, we obtain

$$a^4bx-2a^2bc^2x+4a^4b^2=4b^3c^2x-5a^6+2a^2b^2c^2-3a^3b^3.$$

Second Transformation.

91. When the two members of an equation are entire polynomials, to transpose certain terms from one member to the other.

Take for example the equation . . . . $5x-6=8+2x$.

If, in the first place we subtract $2x$ from both members, the equality will not be destroyed, aud we have . . . . $5x-6-2x=8$.

Whence we see that the term $2x$, which was additive in the second member becomes subtractive in the first.

In the second place if we add 6 to both members, the equality will still exist and we have . . . . $5x-6-2x+6=8+6$.

Or, since -6 and $+6$ destroy each other $5x-2x=8+6$.

Hence the term which was subtractive in the first member, passes into the second member with the sign of addition.

Again, take the equation . . . . $ax+b=d-cx$.

If we add cx to both members and subtract b from them, the equation becomes . . . . $ax+b+cx-b=d-cx+cx-b$.

or reducing . . . . $ax+cx=d-b$.

Therefore, for the transposition of the terms, we have the following

RULE.

Any term of an equation may be transposed from one member to the other by changing its sign.

92. We will now apply the preceding principles to the resolution of the equation,

$$4x-3=2x+5.$$

by transposing the terms -3 and $2x$ it becomes

$$4x-2x=5+3$$

Or reducing . $2x=8$

Dividing by 2 . $x=\frac{8}{2}=4$.

Now, if 4 be substituted in the place of x in the first equation, it becomes

$$4\times4-3=2\times4+5$$

or . . . . $13=13$.

Hence, the value of x is verified by substituting it for the unknown quantity in the given equation.

For a second example, take the equation

$$\frac{5x}{12}-\frac{4x}{3}-13=\frac{7}{8}-\frac{13x}{6}.$$

By making the denominators disappear, we have

$$10x-32x-312=21-52x$$

or, by transposing . $10x-32x+52x=21+312$

by reducing . . . $30x=333$

dividing by 30 . . $x=\frac{333}{30}=\frac{111}{10}=11{,}1.$

a result which may be verified by substituting it for x in the given equation.

For a third example let us take the equation

$$(3a-x)\ (a-b)+2ax=4b(x+a).$$

It is first necessary to perform the multiplications indicated, in order to reduce the two members to two polynomials, and thus be able to disengage the unknown quantity x, from the known quantities. Having done that, the equation becomes,

$$3a^2-ax-3ab+bx+2ax=4bx+4ab.$$

or by transposing . $-ax+bx+2ax-4bx=4ab+3ab-3a^2$

by reducing . . $ax-3bx=7ab-3a^2$

Or, (Art. 48). . . $(a-3b)x=7ab-3a^2$

Dividing both members by $a-3b$ we find

$$x=\frac{7ab-3a^2}{a-3b}.$$

93. Hence, in order to resolve any equation of the first degree, we have the following general

RULE.

I. *If there are any denominators, cause them to disappear, and perform, in both members, all the algebraic operations indicated: we thus obtain an equation the two members of which are entire polynomials.*

II. *Then transpose all the terms affected with the unknown quantity into the first member, and all the known terms into the second member.*

III. *Reduce to a single term all the terms involving x: this term will be composed of two factors, one of which will be x, and the other all the multipliers of x, connected with their respective signs.*

IV. *Divide both members by the number or polynomial by which the unknown quantity is multiplied.*

EXAMPLES.

1. Given $3x-2+24=31$ to find x. *Ans.* $x=3$.

2. Given $x+18=3x-5$ to find x. *Ans.* $x=11\frac{1}{2}$.

3. Given $6-2x+10=20-3x-2$ to find x. *Ans.* $x=2$.

4. Given $x+\frac{1}{2}x+\frac{1}{3}x=11$ to find x. *Ans.* $x=6$.

5. Given $2x-\frac{1}{2}x+1=5x-2$ to find x. *Ans.* $x=\frac{6}{7}$.

6. Given $3ax+\frac{a}{2}-3=bx-a$ to find x.

Ans. $x=\frac{6-3a}{6a-2b}$.

7. Given $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x-19}{2}$ to find x.

Ans. $x=23\frac{1}{4}$.

8. Given $\frac{x+3}{2}+\frac{x}{3}=4-\frac{x-5}{4}$ to find x.

Ans. $x=3\frac{6}{13}$.

9. Given $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}$ to find x.

Ans. $x=\frac{3b}{3a-2b}$.

10. Find the value of x in the equation

$$\frac{(a+b)\,(x-b)}{a-b}-3a=\frac{4ab-b^2}{a+b}-2x+\frac{a^2-bx}{b}.$$

Ans. $x=\frac{a^4+3a^3b+4a^2b^2-6ab^3+2b^4}{2b(2a^2+ab-b^2)}$.

Of Questions producing Equations of the First Degree involving but a single unknown quantity.

94. It has already been observed (Art. 81), that the solution of a problem by algebra, consists of two distinct parts.

1st. To express the conditions of the problem algebraically; and

2d. To disengage the unknown from the known quantities.

We have already explained the manner of finding the value of the unknown quantity, after the question has been stated; and it only remains to point out the best methods of enunciating a problem in the language of algebra.

This part of the algebraic resolution of a problem, cannot, like the second, be subjected to any well defined rule. Sometimes the enunciation of the problem furnishes the equation immediately; and sometimes it is necessary to discover, from the enunciation, new conditions from which an equation may be formed. The conditions enunciated are called *explicit conditions*, and those which are deduced from them, *implicit conditions*.

In almost all cases, however, we are enabled to discover the equation by applying the following

RULE.

Consider the problem solved; and then indicate, by means of algebraic signs, upon the known and unknown quantities, the same course of reasoning and operations which it would be necessary to perform, in order to verify the unknown quantity, had it been given.

QUESTIONS.

1. Find a number such, that the sum of one half, one third, and one fourth of it, augmented by 45, shall be equal to 448.

Let the required number be denoted by . . . x.

Then, one half of it will be denoted by . . . $\frac{x}{2}$.

one third of it . . by . . . $\frac{x}{3}$.

one fourth of it . . by . . . $\frac{x}{4}$.

And by the conditions, $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+45=448.$

Or by subtracting 45 from both members,

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=403.$$

By clearing the terms of their denominators, we obtain

$$6x+4x+3x=4836.$$

or . . $$13x=4836.$$

Hence . $$x=\frac{4836}{13}=372.$$

Let this result be verified.

$$\frac{372}{2}+\frac{372}{3}+\frac{372}{4}+45=186+124+93+45=448.$$

2. What number is that whose third part exceeds its fourth, by 16.

Let the required number be represented by x. Then,

$\frac{1}{3}x=$ the third part.

$\frac{1}{4}x=$ the fourth part.

And by the question $\frac{1}{3}x-\frac{1}{4}x=16.$

or, . . . $4x-3x=192.$

$$x=192.$$

Verification.

$$\frac{192}{3}-\frac{192}{4}=64-48=16.$$

3. Divide \$1000 between A, B, and C, so that A shall have \$72 more than B, and C \$100 more than A.

Let . . $x=$ B's share of the \$1000.

Then . . $x+72=$ A's share.

And . . $x+172=$ C's share.

Their sum $3x+244=1000$.

Whence, $3x=1000-244=756$

or $x=\frac{756}{3}=\$252=$ B's share.

$x+72=252+72=\$324=$ A's share.

And $x+172=252+172=\$424=$ C's share.

Verification.

$$252+324+424=1000.$$

4. Out of a cask of wine which had leaked away a third part, 21 gallons were afterwards drawn, and the cask being then gauged, appeared to be half full: how much did it hold?

Suppose the cask to have held x gallons.

Then, $\frac{x}{3}=$ what leaked away.

And $\frac{x}{3}+21=$ all that was taken out of it.

Hence, $\frac{x}{3}+21=\frac{1}{2}x$ by the question.

or $2x+126=3x$.

or $-x=-126$.

or $x=126$, by changing the signs of both members, which does not destroy their equality.

Verification.

$$\frac{126}{3}+21=42+21=63=\frac{126}{2}.$$

5. A fish was caught whose tail weighed 9*lb.*; his head weighed

as much as his tail and half his body, and his body weighed as much as his head and tail together; what was the weight of the fish?

Let . $2x=$ the weight of the body.

Then . $9+x=$ weight of the head.

And since the body weighed as much as both head and tail

$$2x=9+\ 9+x$$

or . . $2x-x=18$ and $x=18$.

Verification.

$2x=36\,lb=$ weight of the body.

$9+x=27\,lb=$ weight of the head.

$9\,lb=$ weight of the tail.

Hence, . $72\,lb=$ weight of the fish.

6. A person engaged a workman for 48 days. For each day that he laboured he received 24 cents, and for each day that he was idle, he paid 12 cents for his board. At the end of the 48 days, the account was settled, when the labourer received 504 cents. *Required the number of working days, and the number of days he was idle.*

If these two numbers were known, by multiplying them respectively by 24 and 12, then subtracting the last product from the first, the result would be 504. Let us indicate these operations by means of algebraic signs.

Let . . $x=$ the number of working days.

$48-x=$ the number of idle days.

Then $24\times x=$ the amount earned, and

$12(48-x)=$ the amount paid for his board.

Then $24x-12(48-x)\ =504$ what he received.

or $24x-576+12x=504$.

or $36x=504+576=1080$

and $x=\frac{1080}{36}=30$ the working days.

whence, $48-30=18$ the idle days.

Verification.

Thirty day's labor, at 24 cents a day, amounts to . . . . $30\times24=720$ cts.

And 18 day's board, at 12 cents a day, amounts to . . . . $18\times12=216$ cts.

And $720-216=504$, the amount received.

This question may be made general, by denoting the whole number of working and idle days, by n.

The amount received, for each day he worked, by a.

The amount paid for his board, for each idle day, ... by b.

And the balance due the laborer, or the result of the account, ... by c.

As before, let the number of working days be represented . . . . by x.

The number of idle days will be expressed . by $n-x$.

Hence, what he earns will be expressed . by ax.

and the sum to be deducted, on account of his board, by $b(n-x)$.

The equation of the problem therefore is,

$$ax-b(n-x)=c$$

whence

$$ax-bn+bx=c$$

$$(a+b)x=c+bn$$

$$x=\frac{c+bn}{a+b}$$

and consequently,

$$n-x=n-\frac{c+bn}{a+b}=\frac{an+bn-c-bn}{a+b}$$

or

$$n-x=\frac{an-c}{a+b}.$$

7. A fox, pursued by a greyhound, has a start of 60 leaps. He makes 9 leaps while the greyhound makes but 6 ; but three leaps of the greyhound are equivalent to 7 of the fox. How many leaps must the greyhound make to overtake the fox ?

From the enunciation, it is evident that the distance to be passed over by the greyhound is composed of the 60 leaps which the fox is in advance, plus the distance that the fox passes over from the moment when the greyhound starts in pursuit of him. Hence, if we can find the expression for these two distances, it will be easy to form the equation of the problem.

Let $x=$ the number of leaps made by the greyhound before he overtakes the fox.

Now, since the fox makes 9 leaps while the greyhound makes but 6, the fox will make $\frac{9}{6}$ or $\frac{3}{2}$ leaps while the greyhound makes 1; and, therefore, while the greyhound makes x leaps, the fox will make $\frac{3}{2}x$ leaps.

Hence, the distance which the greyhound must pass over, will be expressed by $60+\frac{3}{2}x$ leaps of the fox.

It might be supposed, that in order to obtain the equation, it would be sufficient to place x equal to $60+\frac{3}{2}x$; but in doing so, a manifest error would be committed; for the leaps of the greyhound are greater than those of the fox, and we would then equate heterogeneous numbers, that is, numbers referred to different units. Hence it is necessary to express the leaps of the fox by means of those of the greyhound, or reciprocally. Now, according to the enunciation, 3 leaps of the greyhound are equivalent to 7 leaps of the fox, then 1 leap of the greyhound is equivalent to $\frac{7}{3}$ leaps of the fox, and consequently x leaps of the greyhound are equivalent to $\frac{7x}{3}$ of the fox.

Hence, we have the equation $\frac{7x}{3}= 60+\frac{3}{2}x$;

making the denominators disappear $14x=360+9x$,

Whence . . . . . $5x=360$ and $x=72$.

Therefore, the greyhound will make 72 leaps to overtake the fox, and during this time the fox will make $72\times\frac{3}{2}$ or 108.

Verification.

The 72 leaps of the greyhound are equivalent to $\frac{72\times 7}{3}=168$ leaps of the fox.

And $60+108=168$, the leaps which the fox made from the beginning.

8. A father who had three children, ordered in his will, that his property should be divided amongst them in the following manner: the first to have a sum a, plus the nth part of what remained after subtracting a from the whole estate; the second, a sum $2a$ plus the nth part of what remained after subtracting from it the first part and $2a$; the third, to have a sum $3a$ plus the nth part of what remained after subtracting from it the first two parts and $3a$. In this manner his property was entirely divided; required the amount of it.

Let x denote the property of the father. If by means of this quantity, algebraic expressions can be formed for the three parts, we may subtract their sum from the whole property x, and the remainder placed equal to zero, will give the equation of the problem. We will then endeavour to determine successively these three parts.

Since x denotes the property of the father, $x-a$ is what remains after having subtracted a from it; therefore $x-a$ is the first remainder, and the part which the first child is to have, is $a+\frac{x-a}{n}$, or reducing to a common denominator,

$$\frac{an+x-a}{n}=\text{1st part.}$$

In order to form the 2d part, this first part and $2a$ must be subtracted from x: this gives $x-2a-\frac{(an+x-a)}{n}$, or reducing to a common denominator and subtracting,

$$\frac{nx-3an-x+a}{n} \quad \text{2d. remainder.}$$

Now, the second part is composed of $2a$, plus the nth part of this remainder ; therefore, it is $2a+\frac{nx-3an-x+a}{n^2}$, or reducing to a common denominator,

$$\frac{2an^2+nx-3an-x+a}{n^2}=\text{2d. part.}$$

Subtracting the two first parts plus $3a$, from x, we have

$$x-3a-\frac{(an+x-a)}{n}-\frac{(2an^2+nx-3an-x+a)}{n^2},$$

Or, reducing to a common denominator, and performing the operations indicated,

$$\frac{n^2x-6an^2-2nx+4an+x-a}{n^2} \quad \text{3d. remainder.}$$

Hence the 3d part is $3a+\frac{n^2x-6an^2-2nx+4an+x-a}{n^3}$.

Or, reducing to a common denominator,

$$\frac{3an^3+n^2x-6an^2-2nx+4an+x-a}{n^3}=\text{3d part.}$$

But from the enunciation, the estate of the father is found to be entirely divided. Hence, the difference between x, and the sum of the three parts should be equal to zero. This gives the equation

$$\left.\begin{array}{l} x-\dfrac{an+x-a}{n}-\dfrac{2an^2+nx-3an-x+a}{n^2} \\ -\dfrac{3an^3+n^2x-6an^2-2nx+4an+x-a}{n^3} \end{array}\right\}=0.$$

by making the denominators disappear, and performing the operations indicated, we have

$$n^3x-6an^3-3n^2x+10an^2+3nx-5an-x+a=0.$$

Whence,

$$x=\frac{6an^3-10an^2+5an-a}{n^3-3n^2+3n-1}=\frac{a(6n^3-10n^2+5n-1)}{n^3-3n^2+3n-1}.$$

A more simple equation and result may be obtained, by observing, that the part which goes to the third child is composed of $3a$, plus the nth part of what remains, and that the estate is then entirely exhausted; that is, the third child has only the sum $3a$, and the remainder just mentioned is nothing.

Now the expression for this remainder has been found to be

$$\frac{n^2x-6an^2-2nx+4an+x-a}{n^2}.$$

Placing this equal to zero, and making the denominator disappear, we have

$$n^2x-6an^2-2nx+4an+x-a=0.$$

Whence $$x=\frac{6an^2-4an+a}{n^2-2n+1}=\frac{a(6n^2-4n+1)}{n^2-2n+1}$$

Verification.

To prove the *numerical identity* of this expression with the preceding, it is only necessary to show that the second can be deduced from the first, by suppressing a factor common to its numerator and denominator. Now if we apply the rule for finding the greatest common divisor (Art. 70.), to the two polynomials

$$a(6n^3-10n^2+5n-1) \quad \text{and} \quad n^3-3n^2+3n-1,$$

it will be seen that $n-1$ is a common factor, and by dividing the numerator and denominator of the first expression by this factor, the result will be the second.

This problem shows the beginner how important it is to seize upon every circumstance in the enunciation of a question, which may facilitate the formation of the equation, otherwise he runs the risk of arriving at results more complicated than the nature of the the question requires.

The conditions which have served to form successively the expressions for the three parts, are the *explicit conditions* of the problem; and the condition which has served to determine the most simple equation of the problem, is an *implicit condition*, which a little attention has sufficed to show, was comprehended in the enunciation.

To obtain the values of the three parts, it is only necessary to substitute for x its value in the three expressions obtained for these parts.

Apply the formula $x=\dfrac{a(6n^2-4n+1)}{n^2-2n+1}$ to a particular example.

Let $a=10000, \quad n=5.$

We have

$$x=\frac{10000(6\times25-4\times5+1)}{25-10+1}=\frac{10000\times131}{16}=\frac{1310000}{16}=81875.$$

To verify the enunciation in this case:

The first child should have, $10000+\dfrac{81875-10000}{5}$, or 24375.

There remains then $81875-24375$, or 57500, to divide between the other two children.

The second should have, $20000+\dfrac{57500-20000}{5}$, or 27500.

Then there remains $57500-27500$, or 30000, for the third child. Now 30000 is triple of 10000; hence the problem is verified.

We can give a more simple and elegant solution to this problem, but it is less direct. It also depends upon the remark, that after having subtracted $3a$ and the two first parts from the whole estate, nothing remains.

Denote the three remainders mentioned in the enunciation by r, r', r''. The algebraic expressions for the three parts will be

$$a+\frac{r}{n}, \quad 2a+\frac{r'}{n}, \quad 3a+\frac{r''}{n}.$$

Now, 1st. From the enunciation, it is evident that $r''=0$.

Therefore the third part is $3a$.

2d. What remains after giving to the second child $2a+\frac{r'}{n}$ can be represented by $r'-\frac{r'}{n}$, or $\frac{(n-1)r'}{n}$.

Moreover, this remainder also forms the third part. Therefore we have

$$\frac{(n-1)r'}{n}=3a\text{; whence } r'=\frac{3an}{n-1}.$$

Then the second part is $2a+\frac{3an}{n-1}\div n=2a+\frac{3a}{n-1}$, or converting the whole number into a fraction, and reducing, $\frac{2an+a}{n-1}$.

3d. The remainder, after giving to the first $a+\frac{r}{n}$, can be expressed by $r-\frac{r}{n}$ or $\frac{(n-1)r}{n}$. Now this remainder should form the two other parts, or $3a+\frac{2an+a}{n-1}$.

Therefore, $\quad \frac{(n-1)r}{n}=3a+\frac{2an+a}{n-1}=\frac{5an-2a}{n-1}$.

Hence, $\quad r=\frac{5an-2a}{n-1}\times\frac{n}{(n-1)}=\frac{5an^2-2an}{(n-1)^2}$.

And consequently the first part is

$$a+\frac{5an^2-2an}{(n-1)^2}\div n=a+\frac{5an-2a}{(n-1)^2}.$$

$$=a+\frac{5an-2a}{n^2-2n+1}=\frac{an^2+3an-a}{n^2-2n+1}.$$

Ther the whole estate is

$$3a+\frac{2an+a}{n-1}+\frac{an^2+3an-a}{n^2-2n+1}.$$

Or, by reducing the whole number and fractions to a common denominator,

$$\frac{3a(n^2-2n+1)+(2an+a)\ (n-1)+an^2+3an-a}{n^2-2n+1}.$$

Or performing the operations indicated and reducing

$$\frac{6an^2-4an+a}{n^2-2n+1}=\frac{a(6n^2-4n+1)}{(n-1)^2},$$

which agrees with the preceding result.

This solution is more complete than the preceding, since we obtain from it the estate of the father, and the expressions for the three parts.

9. A father ordered in his will, that the eldest of his children should have a sum a, out of his estate, plus the nth part of the remainder; that the second should have a sum $2a$, plus the nth part of what remained after having subtracted from it the first part and $2a$; that the third should have a sum $3a$, plus the nth part of the new remainder—and so on. It is moreover supposed that the children share equally. Required, the value of the father's estate, the share of each child, and the number of children.

This problem is remarkable, because the number of conditions contained in the enunciation is greater than the number of unknown values required to be found.

Let the estate of the father be represented by x: then will $x-a$ express what remains after having taken from it the sum a. Therefore the share of the eldest is

$$a+\frac{x-a}{n} \text{ or } \frac{an+x-a}{n}=\text{1st. part.}$$

Subtracting the first part, and $2a$, from x, we have

$$x-2a-\frac{(an+x-a)}{n} \text{ or, } \frac{nx-3an-x+a}{n},$$

the nth part of which is, $\dfrac{nx-3an-x+a}{n^2}$.

Hence, the share of the second child is

$$2a+\frac{nx-3an-x+a}{n^2}, \text{ or } \frac{2an^2+nx-3an-x+a}{n^2}=\text{2d part.}$$

In like manner, the other parts might be formed, but as all the parts should be equal, it suffices to form the equation of the problem, to equate the two first parts, which gives

$$\frac{an+x-a}{n}=\frac{2an^2+nx-3an-x+a}{n^2},$$

whence,

$$x=an^2-2an+a=a(n-1)^2.$$

Substituting this value of x in the expression for the first part, we find

$$\frac{an+an^2-2an+a-a}{n};$$

or reducing,

$$\frac{an^2-an}{n}=an-a=a(n-1);$$

and as the parts are equal, by dividing the whole estate by the first part, we will obtain a quotient that will show the number of children; therefore, $\dfrac{an^2-2an+a}{an-a}$, or $n-1$, denotes the number of children.

The father's estate, . .	$an^2-2an+a=a(n-1)^2$.
The share of each child, .	$a(n-1)$.
Whole number of children, .	$(n-1)$.

It yet remains to be shown, that the other conditions of the problem are satisfied; that is, that by giving to the second child, $2a$ plus the nth part of what remains; to the third, $3a$ plus the nth part of what remains, &c., the share of each child is in fact $(n-1)a$.

The difference between the estate of the father and the first part being $a(n-1)^2-a(n-1)$, the share of the second child will be

$$2a+\frac{a(n-1)^2-a(n-1)-2a}{n}, \text{ or } \frac{2a(n-1)+a(n-1)^2-a(n-1)}{n},$$

and reducing

$$\frac{a(n-1)+a(n-1)^2}{n}, \text{ or } \frac{a(n-1)\,(1+n-1)}{n},$$

or $$a(n-1).$$

In like manner, the difference between $a(n-1)^2$ and the two first parts being, $a(n-1)^2-2a(n-1)$, the third part will be

$$3a+\frac{a(n-1)^2-2a(n-1)-3a}{n},$$

which being reduced, becomes

$$\frac{a(n-1)+a(n-1)^2}{n}, \text{ or } a(n-1).$$

In the same way we would obtain for the fourth part

$$4a+\frac{a(n-1)^2-3a(n-1)-4a}{n}, \text{ or } \frac{a(n-1)+a(n-1)^2}{n}, \text{ and so on.}$$

Hence all the conditions of the enunciation are satisfied.

10. What number is that from which, if 5 be subtracted, $\frac{2}{3}$ of the remainder will be 40? *Ans.* 65.

11. A post is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and ten feet above the water: what is the whole length of the post?

Ans. 24 feet.

12. After paying $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left in my purse: how many guineas were in it at first?

Ans. 120.

13. A person was desirous of giving 3 pence a piece to some beggars, but found he had not money enough in his pocket by 8 pence: he therefore gave them each 2 pence and had 3 pence remaining: required the number of beggars. *Ans.* 11.

14. A person in play lost $\frac{1}{4}$ of his money, and then won 3 shillings; after which he lost $\frac{1}{3}$ of what he then had; and this done, found that he had but 12 shillings remaining: what had he at first?

Ans. 20*s.*

15. Two persons, A and B, lay out equal sums of money in trade; A gains \$126, and B loses \$87, and A's money is now double of B's: what did each lay out? *Ans.* \$300.

16. A person goes to a tavern with a certain sum of money in his pocket, where he spends 2 shillings; he then borrows as much mo-

ney as he had left, and going to another tavern, he there spends 2 shillings also; then borrowing again as much money as was left, he went to a third tavern, where likewise he spent two shillings and borrowed as much as he had left; and again spending 2 shillings at a fourth tavern, he then had nothing remaining. What had he at first? *Ans.* 3*s.* 9*d.*

Of Equations of the First Degree involving two or more unknown quantities.

95. Although several of the questions hitherto resolved, contained in their enunciation more than one unknown quantity, we have resolved them by employing but one symbol. The reason of this is, that we have been able, from the conditions of the enunciation, to express easily the other unknown quantities by means of this symbol; but this is not the case in all problems containing more than one unknown quantity.

To ascertain how problems of this kind are resolved: first, take some of those which have been resolved by means of one unknown quantity.

1. Given the sum a, of two numbers, and their difference b, it is required to find these numbers.

Let $x=$ the greater, and y the less number.

Then by the conditions . . . . . $x+y=a$.

and . . . . . $x-y=b$.

By adding (Art. 86. Ax. 1.) . . . $2x=a+b$.

By subtracting (Art. 86. Ax. 2.) . . $2y=a-b$.

Each of these equations contains but one unknown quantity.

From the first we obtain . . . $x=\frac{a+b}{2}$.

And from the second . . . . $y=\frac{a-b}{2}$.

Verification.

$$\frac{a+b}{2}+\frac{a-b}{2}=\frac{2a}{2}=a\text{; and }\frac{a+b}{2}-\frac{a-b}{2}=\frac{2b}{2}=b.$$

For a second example, let us also take a problem that has been already solved.

2. A person engaged a workman for 48 days. For each day that he labored he was to receive 24 cents, and for each day that he was idle he was to pay 12 cents for his board. At the end of the 48 days, the account was settled, when the laborer received 504 cents. Required the number of working days and the number of days he was idle.

Let $x =$ the number of working days.
$y =$ the number of idle days.
$n =$ the whole number of days $= 48$.
$a =$ what he received per day for work $= 24$ cts.
$b =$ what he paid per day for board $= 12$ cts.
$c =$ what he received at the end of the time $= 504$.
Then, $ax =$ what he earned,
And $by =$ what he paid for his board.

We have by the question . . . $\begin{cases} x+y=n. \\ ax-by=c. \end{cases}$

It has already been shown that the two members of an equation can be multiplied by the same number, without destroying the equality; therefore the two members of the first equation may be multiplied by b, the co-efficient of y in the second, and we have

The equation . . . . . $bx+by=bn.$
Which, added to the second . . $ax-by=c.$
Gives . . . . . . $ax+bx=bn+c.$
Whence . . . . . . $x=\dfrac{bn+c}{a+b}.$

In like manner, multiplying the two members of the first equation by a, the co-efficient of x in the second, it becomes

$$ax+ay=an.$$

From which, subtract the second equation, $ax-by=c.$

And we obtain . . . . . . $ay+by=an-c.$

Whence . . . . $y=\frac{an-c}{a+b}.$

By introducing a symbol to represent each of the unknown quantities in the preceding problem, the solution which has just been given has the advantage of making known the two required numbers, independently of each other.

Elimination.

96. The method which has just been explained of combining two equations, involving two unknown quantities, and deducing therefrom a single equation involving but one, may be extended to three, four, or any number of equations, and is called *elimination.*

There are three principal methods of elimination:

1st. By addition and subtraction.

2d. By substitution.

3d. By comparison.

We will consider these methods separately.

Elimination by Addition and Subtraction.

97. Take the two equations . . . $\begin{cases} 5x+7y=43. \\ 11x+9y=69. \end{cases}$

which may be regarded as the algebraic enunciation of a problem containing two unknown quantities. If, in these equations, one of the unknown quantities was affected with the same co-efficient, we might, by a simple subtraction, form a new equation which would contain but one unknown quantity, and from which the value of this unknown quantity could be deduced.

Now, if both members of the first equation be multiplied by 9, the co-efficient of y in the second, and the two members of the second by 7, the co-efficient of y in the first, we will obtain

$$45x+63y=387,$$
$$77x+63y=483,$$

equations which may be substituted for the two first, and in which y is affected with the same co-efficient.

Subtracting, then, the first of these equations from the second, there results $32x=96$, whence $x=3$.

Again, if we multiply both members of the first equation by 11, the co-efficient of x in the second, and both members of the second by 5, the co-efficient of x in the first, we will form the two equations

$\left.\begin{array}{l}55x+77y=473,\\ 55x+45y=345,\end{array}\right\}$ which may be substituted for the two proposed equations, and in which the co-efficients of x are the same.

Subtracting, then, the second of these two equations from the first, there results $32y=128$, whence $y=4$.

Therefore $x=3$ and $y=4$, are the values of x and y, which should verify the enunciation of the question. Indeed we have,

1st. $5\times3+7\times4=15+28=43$;

2d. $11\times3+9\times4=33+36=69$.

The method of elimination, just explained is called the *method by addition and subtraction,* because the unknown quantities disappear by additions and subtractions, after having prepared the equations in such a manner that one unknown quantity shall have the same co-efficient in two of them.

Elimination by Substitution.

98. Take the same equations . . $\left\{\begin{array}{l}5x+7y=43.\\ 11x+9y=69.\end{array}\right.$

Find the value of x in the first equation, which gives

$$x=\frac{43-7y}{5}.$$

Substitute this value of x in the second equation, and we have

$$11\times\frac{43-7y}{5}+9y=69.$$

or . $473-77y+45y=345.$

or . $-32y=-128.$

Hence . $y=4.$

And . $x=\frac{43-28}{5}=3.$

This method, called the method by substitution, consists in finding the value of one of the unknown quantities in one of the equations, as if the other unknown quantities were already determined, and in substituting this value in the other equations; in this way new equations are formed, which contain one unknown quantity less than the given equations, and upon which we operate as upon the proposed equations.

Elimination by Comparison.

99. Take the same equations $\begin{cases} 5x+7y=43 \\ 11x+9y=69. \end{cases}$

Finding the value of x in the first equation, we have

$$x=\frac{43-7y}{5}.$$

And finding the value of x in the second, we obtain

$$x=\frac{69-9y}{11}.$$

Let these two values of x be placed equal to each other, and we have, . . $\frac{43-7y}{5}=\frac{69-9y}{11}$

Or, . . $473-77y=345-45y$

Or, . . $-32y=-128.$

Hence, . . $y=4$

And, . . $x=\frac{69-36}{11}=3.$

This method of elimination is called the method by comparison, and consists in finding the value of the same unknown quantity in all the equations, placing them equal to each other, two and two, which necessarily gives a new set of equations, containing one unknown quantity less than the other, upon which we operate as upon the proposed equations.

But there is an inconvenience in the two last methods, which the *method by addition and subtraction* is not subject to, viz.: they produce new equations, containing denominators, which it is afterwards necessary to make disappear. The *method by substitution* is, however, advantageously employed whenever the co-efficient of one of the unknown quantities is equal to unity in one of the equations, because then the inconvenience of which we have just spoken does not occur. We shall sometimes have occasion to employ it, but generally, the method by *addition and subtraction* is preferable. It moreover presents this advantage, viz.: when the co-efficients are not too great, we can perform the addition or subtraction at the same time with the multiplication which is necessary to render the co-efficients equal to each other.

100. Let us now consider the case of three equations involving three unknown quantities.

Take the equations, $$\begin{cases} 5x-6y+4z=15. \\ 7x+4y-3z=19. \\ 2x+\ y+6z=46. \end{cases}$$

To eliminate z by means of the first two equations, multiply the first by 3 and the second by 4, then since the co-efficients of z have contrary signs, add the two results together: this gives a new equation . . . . $43x-2y=121$

Multiplying the second equation by 2, a factor of the co-efficient of z in the third equation, and adding them together, we have . . $16x+9y=\ 84$

The question is then reduced to finding the values of x and y, which will satisfy these new equations.

Now, if the first be multiplied by 9, the second by 2, and the results be added together, we find

$$419x=1257, \text{ whence } x=3.$$

We might, by means of the two equations involving x and y, determine y in the same way we have determined x ; but the value of y may be determined more simply, by observing that the last of these two equations becomes, by substituting for x its value found above,

$$48+9y=84 \quad \text{whence} \quad y=\frac{84-48}{9}=4.$$

In the same manner the first of the three proposed equations, becomes, by substituting the values of x and y,

$$15-24+4z=15, \quad \text{whence} \quad z=\frac{24}{4}=6.$$

101. Hence, if there are m equations involving a like number of unknown quantities, the unknown quantities may be eliminated by the following

RULE.

I. *To eliminate one of the unknown quantities, combine any one of the equations with each of the* m$-$1 *others ; there will thus be obtained* m$-$1 *new equations containing* m$-$1 *unknown quantities.*

II. *Eliminate another unknown quantity by combining one of these new equations with the* m$-$2 *others ; this will give* m$-$2 *equations containing* m$-$2 *unknown quantities.*

III. *Continue this series of operations until a single equation containing but one unknown quantity is obtained, from which the value of this unknown quantity is easily found. Then by going back through the series of equations which have been obtained, the values of the other unknown quantities may be successively determined.*

102. It often happens that each of the proposed equations does not contain all the unknown quantities. In this case, with a little address, the elimination is very quickly performed.

Take the four equations involving four unknown quantities :

$$\left.\begin{array}{r} 2x-3y+2z=13 \\ 4u-2x=30 \end{array}\right\} \quad \begin{array}{l} \ldots (1) \\ \ldots (2) \end{array} \qquad \begin{array}{r} 4y+2z=14 \\ 5y+3u=32 \end{array} \quad \begin{array}{l} \ldots (3). \\ \ldots (4). \end{array}$$

By inspecting these equations, we see that the elimination of z in the two equations, (1) and (3), will give an equation involving x and y; and if we eliminate u in the equations (2) and (4), we will obtain a second equation, involving x and y. These two last unknown quantities may therefore be easily determined. In the first place, the elimination of z in (1) and (3) gives . . . $7y-2x=1$

That of u in (2) and (4), gives . . . $20y+6x=38$

Multiplying the first of these equations by 3, and adding . . . . $41y=41$

Whence . . . . $y=1$

Substituting this value in $7y-2x=1$, we find . $x=3$

Substituting for x its value in equation (2), it becomes $4u-6=30$, whence . . . $u=9$

And substituting for y its value in equation (3), there results . . . . . . $z=5$

EXAMPLES.

1. Given $2x+3y=16$, and $3x-2y=11$ to find the values of x and y. *Ans.* $x=5$, $y=2$.

2. Given $\frac{2x}{5}+\frac{3y}{4}=\frac{9}{20}$ and $\frac{3x}{4}+\frac{2y}{5}=\frac{61}{120}$ to find the values of x and y. *Ans.* $x=\frac{1}{2}$, $y=\frac{1}{3}$.

3. Given $\frac{x}{7}+7y=99$, and $\frac{y}{7}+7x=51$, to find the values of x and y. *Ans.* $x=7$, $y=14$.

4. Given $\frac{x}{2}-12=\frac{y}{4}+8$, and $\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27$, to find the values of x and y. *Ans.* $x=60$, $y=40$.

5. Given $\left\{\begin{array}{l} x+\ y+\ z=29 \\ x+2y+3z=62 \\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=10 \end{array}\right\}$ to find x, y and z.

Ans. $x=8$, $y=9$, $z=12$.

6. Given $\left\{\begin{array}{l} 2x+4y-3z=22 \\ 4x-2y+5z=18 \\ 6x+7y-z=63 \end{array}\right\}$ to find x, y and z.

Ans. $x=3,\ y=7,\ z=4.$

7. Given $\left\{\begin{array}{l} x+\frac{1}{2}y+\frac{1}{3}z=32 \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15 \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12 \end{array}\right\}$ to find x, y and z.

Ans. $x=12,\ y=20,\ z=30.$

8. Given $\left\{\begin{array}{l} 7x-2z+3u=17 \\ 4y-2z+t=11 \\ 5y-3x-2u=8 \\ 4y-3u+2t=9 \\ 3z+8u=33 \end{array}\right\}$ to find x, y, z, u, and t.

Ans. $x=2,\ y=4,\ z=3,\ u=3,\ t=1.$

103. In all the preceding reasoning, we have supposed the number of equations equal to the number of symbols employed to denote the unknown quantities. This must be the case in every problem involving two or more unknown quantities, in order that it may be *determinate*; that is, in order that it may not admit of an infinite number of solutions.

Suppose, for example, that a problem involving two unknown quantities, x and y, leads to the single equation, $5x-3y=12$; we deduce from it $x=\frac{12+3y}{5}$. Now, by making successively

$$y=1,\quad 2,\quad 3,\quad 4,\quad 5,\quad 6,\ \&c.,$$

there results,

$$x=3,\quad \frac{18}{5},\quad \frac{21}{5},\quad \frac{24}{5},\quad \frac{27}{5},\quad 6,\ \&c$$

and every system of values,

$$(y=1,\ x=3);\quad \left(y=2,\ x=\frac{18}{5}\right);\quad \left(y=3,\ x=\frac{21}{5}\right);\ \&c.$$

substituted for x and y in the equation, will satisfy it equally well.

If we had two equations involving three unknown quantities, we could in the first place eliminate one of the unknown quantities by means of the proposed equations, and thus obtain an equation, which, containing two unknown quantities, would be satisfied by an infinite number of systems of values taken for these unknown quantities. Therefore, *in order that a problem may be determined, its enunciation must* contain at least as many *different conditions as there are unknown quantities, and these conditions must be such, that each of them may be expressed by an independent equation; that is, an equation not produced by any combination of the others of the system.*

If, on the contrary, the number of *independent* equations exceeds the number of unknown quantities involved in them, the conditions which they express cannot be fulfilled.

For example, let it be required to find two numbers such that their sum shall be 100, their difference 80, and their product 700.

The equations expressing these conditions are,

$$x+y=100$$

$$x-y=80$$

and

$$x\times y=700.$$

Now, the first two equations determine the values of x and y, viz. $x=90$ and $y=10$. The product of the two numbers is therefore known, and equal to 900. Hence the third condition cannot be fulfilled.

Had the product been placed equal to 900, all the conditions would have been satisfied, in which case, however, the third would not have been an *independent* equation, since the condition expressed by it, is implied in the other two.

QUESTIONS.

1. What fraction is that, to the numerator of which, if 1 be add-

ed, its value will be $\frac{1}{3}$, but if one be added to its denominator, its value will be $\frac{1}{4}$.

Let the fraction be represented by $\frac{x}{y}$.

Then, by the question $\frac{x+1}{y}=\frac{1}{3}$ and $\frac{x}{y+1}=\frac{1}{4}$.

Whence $3x+3=y$, and $4x=y+1$.

Therefore, by subtracting, $x-3=1$ or $x=4$.

Hence, $12+3=y$: therefore $y=15$.

2. A market woman bought a certain number of eggs at 2 for a penny, and as many others, at 3 for a penny, and having sold them again altogether, at the rate of 5 for $2d$, found that she had lost $4d$: how many eggs had she?

Let $2x=$ the whole number of eggs.

Then $x=$ the number of eggs of each sort.

Then will $\frac{1}{2}x=$ the cost of the first sort.

And $\frac{1}{3}x=$ the cost of the second sort.

But $5 : 2 :: 2x : \frac{4x}{5}$ the amount for which the eggs were sold.

Hence, by the question $\frac{1}{2}x+\frac{1}{3}x-\frac{4x}{5}=4$.

Therefore . . $15x+10x-24x=120$.

Or, . . $x=120$ the number of eggs of each sort.

3. A person possessed a capital of 30,000 dollars for which he drew a certain interest; but he owed the sum of 20,000 dollars, for which he paid a certain interest. The interest that he received exceeded that which he paid by 800 dollars. Another person pos-

sessed 35,000 dollars, for which he received interest at the second of the above rates, but he owed 24,000 dollars, for which he paid interest at the first of the above rates. The interest that he received exceeded that which he paid by 310 dollars. Required, the two rates of interest.

Let x and y denote the two rates of interest: that is, the interest of \$100 for the given time.

To obtain the interest of \$30,000 at the first rate denoted by x, we form the proportion

$$100 : x :: 30{,}000 : \frac{30{,}000x}{100} \text{ or } 300x.$$

And for the interest \$20,000, the rate being y.

$$100 : y :: 20{,}000 : \frac{20{,}000y}{100} \text{ or } 200y.$$

But from the enunciation, the difference between these two interests is equal to 800 dollars.

We have, then, for the first equation of the problem,

$$300x - 200y = 800.$$

By writing algebraically the second condition of the problem, we obtain the other equation,

$$350y - 240x = 310.$$

Both members of the first equation being divisible by 100, and those of the second by 10, we may put the following, in place of them:

$$3x - 2y = 8, \qquad 35y - 24x = 31.$$

To eliminate x, multiply the first equation by 8, and then add it to the second; there results

$$19y = 95, \text{ whence } y = 5.$$

Substituting for y, in the first equation, its value, this equation becomes

$$3x - 10 = 8, \text{ whence } x = 6.$$

Therefore, the first rate is 6 per cent., and the second 5.

Verification.

\$30,000, placed at 6 per cent., gives $300\times 6, = \$1800.$
\$20,000, do. 5 do. $200\times 5, = \$1000.$
And we have $1800-1000=800$.

The second condition can be verified in the same manner.

4. There are three ingots composed of different metals mixed together. A pound of the first contains 7 ounces of silver, 3 ounces of copper, and 6 of pewter. A pound of the second contains 12 ounces of silver, 3 ounces of copper, and 1 of pewter. A pound of the third contains 4 ounces of silver, 7 ounces of copper, and 5 of pewter. It is required to find how much it will take of each of the three ingots to form a fourth, which shall contain in a pound, 8 ounces of silver, $3\frac{3}{4}$ of copper, and $4\frac{1}{4}$ of pewter.

Let x, y and z represent the number of ounces which it is necessary to take from the three ingots respectively, in order to form a pound of the required ingot. Since there are 7 ounces of silver in a pound, or 16 ounces, of the first ingot, it follows that one ounce of it contains $\frac{7}{16}$ of an ounce of silver, and consequently in a number of ounces denoted by x, there is $\dfrac{7x}{16}$ ounces of silver. In the same manner we would find that $\dfrac{12y}{16}$ and $\dfrac{4z}{16}$, express the number of ounces of silver taken from the second and third, to form the fourth; but from the enunciation, one pound of this fourth ingot contains 8 ounces of silver. We have, then, for the first equation

$$\frac{7x}{16}+\frac{12y}{16}+\frac{4x}{16}=8$$

$$\left.\begin{array}{lr}\text{or, making the denominators disappear.} & 7x+12y+4z=128\\ \text{As respects the copper, we should find} & 3x+3y+7z=60\\ \text{and with reference to the pewter} & 6x+y+5z=68\end{array}\right\}$$

As the co-efficients of y in these three equations, are the most simple, it is most convenient to eliminate this unknown quantity first.

Multiplying the second equation by 4, and subtracting the first equation from the product, we have . . . $5x+24z=112$

Multiplying the third equation by 3, and subtracting the second from the product . . $15x+\ 8z=144$

Multiplying this last equation by 3, and subtracting the preceding one from the product, we obtain $40x=320$, whence $x=8$.

Substitute this value for x in the equation $15x+8z=144$; it becomes

$$120+8z=144, \quad \text{whence} \quad z=3.$$

Lastly, the two values $x=8$, $z=3$, being substituted in the equation $6x+y+5z=68$, give $48+y+15=68$, whence $y=5$.

Therefore in order to form a pound of the fourth ingot, we must take 8 ounces of the first, 5 ounces of the second, and 3 of the third.

Verification.

If there be 7 ounces of silver in 16 ounces of the first ingot, in 8 ounces of it, there should be a number of ounces of silver expressed by $\frac{7\times 8}{16}$.

In like manner $\frac{12\times 5}{16}$ and $\frac{4\times 3}{16}$ will express the quantity of silver contained in 5 ounces of the second ingot, and 3 ounces of the third.

Now, we have $\frac{7\times 8}{16}+\frac{12\times 5}{16}+\frac{4\times 3}{16}=\frac{128}{16}=8$; therefore, a pound of the fourth ingot contains 8 ounces of silver, as required by the enunciation. The same conditions may be verified relative to the copper and pewter.

5. What two numbers are those, whose difference is 7, and sum 33? *Ans.* 13 and 20.

6. To divide the number 75 into two such parts, that three times the greater may exceed seven times the less by 15.

Ans. 54 and 21.

7. In a mixture of wine and cider, $\frac{1}{2}$ of the whole plus 25 gallons was wine, and $\frac{1}{3}$ part minus 5 gallons was cider; how many gallons were there of each?

Ans. 85 of wine, and 35 of cider.

8. A bill of £120 was paid in guineas and moidores, and the number of pieces of both sorts that were used was just 100; if the guinea be estimated at 21*s.* and the moidore at 27*s.* how many were there of each? *Ans.* 50 of each.

9. Two travellers set out at the same time from London and York, whose distance apart is 150 miles; one of them goes 8 miles a day, and the other 7; in what time will they meet?

Ans. In 10 days.

10. At a certain election, 375 persons voted for two candidates, and the candidate chosen had a majority of 91; how many voted for each? *Ans.* 233 for one, and 142 for the other.

11. A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 140; what is the age of each?

Ans. A's=84, B's=42, and C's=14.

12. A person bought a chaise, horse, and harness, for £60; the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness; what did he give for each?

Ans. { £13. 6*s.* 8*d.* for the horse.
£ 6. 13*s.* 4*d.* for the harness.
£40. for the chaise.

13. Two persons, A and B, have both the same income: A saves $\frac{1}{5}$ of his yearly, but B, by spending £50 per annum more than A, at the end of 4 years finds himself £100 in debt; what is their income? *Ans.* £125.

14. A person has two horses, and a saddle worth £50; now if the saddle be put on the back of the first horse, it will make his value double that of the second; but if it be put on the back of the second, it will make his value triple that of the first; what is the value of each horse?

Ans. One £30, and the other £40.

15. To divide the number 36 into three such parts that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, may be all equal to each other. *Ans.* 8, 12, and 16.

16. A footman agreed to serve his master for £8 a year and a livery, but was turned away at the end of 7 months, and received only £2. 13*s.* 4*d.* and his livery; what was its value? *Ans.* £4. 16*s.*

17. To divide the number 90 into four such parts, that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient so obtained, will be all equal to each other.

Ans. The parts are 18, 22, 10, and 40.

18. The hour and minute hands of a clock are exactly together at 12 o'clock; when are they next together?

Ans. 1 *h.* $5\frac{5}{11}$ *min.*

19. A man and his wife usually drank out a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days; how many days would the man alone be in drinking it?

Ans. 20 days.

20. If A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C in 10 days: how many days would it take each person to perform the same work alone?

Ans. A $14\frac{34}{49}$ days, B $17\frac{23}{41}$, and C $23\frac{7}{31}$.

21. A laborer can do a certain work expressed by a, in a time expressed by b; a second laborer, the work c in a time d; a third, the work e, in a time f. It is required to find the time it would take the three laborers, working together, to perform the work g.

Ans. $x=\frac{bdfg}{adf+bcf+bde}$.

Application.

$a=27$; $b=4$ | $c=35$; $d=6$ | $e=40$; $f=12$ | $g=191$;

x will be found equal to 12.

22. If 32 pounds of sea water contain 1 pound of salt, how

much fresh water must be added to these 32 pounds, in order that the quantity of salt contained in 32 pounds of the new mixture shall be reduced to 2 ounces, or $\frac{1}{8}$ of a pound?

Ans. 224 *lb.*

23. A number is expressed by three figures; the sum of these figures is 11; the figure in the place of units is double that in the place of hundreds; and when 297 is added to this number, the sum obtained is expressed by the figures of this number reversed. What is the number? *Ans.* 326

24. A person who possessed 100,000 dollars, placed the greater part of it out at 5 per cent. interest, and the other part at 4 per cent. The interest which he received for the whole amounted to 4640 dollars. Required, the two parts.

Ans. 64,000 and 36,000.

25. A person possessed a certain capital, which he placed out at a certain interest. Another person who possessed 10,000 dollars more than the first, and who put out his capital 1 per cent. more advantageously than he did, had an income greater by 800 dollars. A third person who possessed 15,000 dollars more than the first, and who put out his capital 2 per cent. more advantageously than he did, had an income greater by 1500 dollars. Required, the capitals of the three persons, and the three rates of interest.

Sums at interest,	\$30,000,	\$40,000,	\$45,000.	
Rates of interest,	4	5	6	per cent.

26. A banker has two kinds of money; it takes a pieces of the first to make a crown, and b of the second to make the same sum. Some one offers him a crown for c pieces. How many of each kind must the banker give him?

Ans. 1st kind, $\dfrac{a(c-b)}{a-b}$; 2d kind, $\dfrac{b(a-c)}{a-b}$.

27. Find what each of three persons A, B, C, are worth, knowing; 1st, that what A is worth added to l times what B and C are worth is equal to p; 2d, that what B is worth added to m times what

A and C are worth is equal to q; 3d, that what C is worth added to n times what A and B are worth is equal to r.

This question can be resolved in a very simple manner, by introducing an auxiliary unknown quantity into the calculus. This unknown quantity is equal to what A, B and C are worth.

28. Find the values of the estates of six persons, A, B, C, D, E, F, from the following conditions: 1st. The sum of the estates of A and B is equal to a; that of C and D is equal to b; and that of E and F is equal to c. 2d. The estate of A is worth m times that of C; the estate of D is worth n times that of E, and the estate of F is worth p times that of B.

This problem may be resolved by means of a single equation, involving but one unknown quantity.

Theory of Negative Quantities. Explanation of the terms, Nothing and Infinity.

104. The algebraic signs are an abbreviated language. They point out in the shortest and clearest manner the operations to be performed on the quantities with which they are connected.

Having once fixed the particular operation indicated by a particular sign, it is obvious that that operation should always be performed on every quantity before which the sign is placed. Indeed, the principles of algebra are all established upon the supposition, that each particular sign which is employed always means the same thing; and that whatever it requires is strictly performed. Thus, if the sign of a quantity is $+$, we understand that the quantity is to be added; if it is $-$, we understand that it is to be subtracted.

For example, if we have -4, we understand that this 4 is to be subtracted from some other number, or that it is the result of a subtraction in which the number to be subtracted was the greatest.

If it were required to subtract 20 from 16, the subtraction could not be made by the rules of arithmetic, since 16 does not contain

20; nor indeed can it be entirely performed by Algebra. We write the numbers for subtraction thus,

$$16-20=16-16-4=-4.$$

By decomposing -20 into -16 and -4, the -16 will cancel the $+16$, and leave -4 for a remainder.

We thus indicate that the quantity to be subtracted exceeds the quantity from which it is to be taken, by 4.

To show the necessity of giving to this remainder its proper sign, let us suppose that the difference of $16-20$ is to be added to 10. The numbers would then be written

$$\begin{array}{rr} 16-20= & -\ 4 \\ +10\qquad = & +10 \\ \hline 26-20= & +\ 6 \end{array}$$

105. If the sum of the negative quantities in the first member of the equation, exceeds the sum of the positive quantities, the second member of the equation will be negative, and the verification of the equation will show it to be so.

For example, if $$a-b=c,$$
and we make $a=15$ and $b=18$, c will be $=-3$. Now the essential sign of c is different from its algebraic sign in the equation. This arises from the circumstance, that the equation $a-b=c$ expresses *generally*, the difference between a and b, without indicating which of them is the greater. When, therefore, we attribute particular values to a and b, the *sign* of c, as well as its value, becomes known.

We will illustrate these remarks by a few examples.

1. To find a number which, added to the number b, gives for a sum the number a.

Let $x=$ the required number.

Then, by the condition $x+b=a$, whence $x=a-b$.

This expression, or *formula*, will give the algebraic value of x in all the particular cases of this problem.

For example, let $a=47$, $b=29$, then $x=47-29=18$.

Again, let $a=24$, $b=31$; then will $x=24-31=-7$.

This value obtained for x, is called a *negative solution.* How is it to be interpreted?

Considered arithmetically, the problem with these values of a and b, is impossible, since the number b is already greater than 24. Considered algebraically, however, it is not so; for we have found the value of x to be -7, and this number added, in the algebraic sense, to 31, gives 24 for the algebraic sum, and therefore satisfies both the equation and enunciation.

2. A father has lived a number a of years, his son a number of years expressed by b. Find in how many years the age of the son will be one fourth the age of the father.

Let $x=$ the required number of years.

Then $a+x=$ the age of the father
and $b+x=$ the age of the son
$\Big\}$ at the end of the required time.

Hence, by the question $\dfrac{a+x}{4}=b+x$; whence $x=\dfrac{a-4b}{3}$.

Suppose $a=54$, and $b=9$: then $x=\dfrac{54-36}{3}=\dfrac{18}{3}=6$.

The father having lived 54 years, and the son 9, in 6 years the father will have lived 60 years, and his son 15; now 15 is the fourth of 60; hence, $x=6$ satisfies the enunciation.

Let us now suppose $a=45$, and $b=15$: then $x=\dfrac{45-60}{3}=-5$.

If we substitute this value of x in the equation of condition, we obtain

$$\frac{45-5}{4}=15-5$$

or $$10=10.$$

Hence, -5 substituted for x verifies the equation, and therefore is the true answer.

Now, the positive result which has been obtained, shows that the age of the father will be four times that of the son at the expiration of 6 years from the time when their ages were considered ; while the negative result indicates that the age of the father was four times that of his son, 5 years *previous* to the time when their ages were compared.

The question, taken in its most general or algebraic sense, demands *the time*, at which the age of the father was four times that of the son. In stating it, we supposed that the age of the father was to be augmented ; and so it was, by the first supposition. But the conditions imposed by the second supposition, required the age of the father to be diminished, and the algebraic result conformed to this condition by appearing with a negative sign. If we wished the result, under the second supposition, to have a positive sign, we might alter the enunciation by demanding *how many years since the age of the father was four times that of his son.*

If $x=$ the number of years, we shall have

$$\frac{a-x}{4}=b-x: \quad \text{hence} \quad x=\frac{4b-a}{3}.$$

If $a=45$ and $b=15$, x will be equal to 5.

Reasoning from analogy, we establish the following general principles.

1st. *Every negative value found for the unknown quantity in a problem of the first degree, will, when taken with its proper sign, verify the equation from which it was derived.*

2d. *That this negative value, taken with its proper sign, will also satisfy the enunciation of the problem, understood in its algebraic sense.*

3d. *If the enunciation is to be understood in its arithmetical sense, in which the quantities referred to are always supposed to be positive, then this value, considered without reference to its sign, may be considered as the answer to a problem, of which the enunciation only differs from that of the proposed problem in this, that certain quantities which were additive, have become subtractive, and reciprocally.*

106. Take for example the problem of the labourer (Page. 88).

Supposing that the labourer receives a sum c, we have the equations.

$$\left.\begin{array}{r}x+\ y=n\\ ax-by=c\end{array}\right\},\quad\text{whence}\quad x=\frac{bn+c}{a+b},\ y=\frac{an-c}{a+b}.$$

But if we suppose that the labourer, instead of receiving, owes a sum c, the equations will then be

$$\left.\begin{array}{r}x+\ y=n\\ by-ax=c\end{array}\right\}\quad\text{or,}\quad\left\{\begin{array}{l}x+\ y=n,\\ ax-by=-c.\end{array}\right.$$

By changing the signs of the second equation.

Now it is visible that we can obtain immediately the values of x and y, which correspond to the preceding values, by merely changing the sign of c in each of those values; this gives

$$x=\frac{bn-c}{a+b},\ y=\frac{an+c}{a+b}.$$

To prove this rigorously, let us denote $-c$ by d;

The equations then become $\left\{\begin{array}{l}x+\ y=n\\ ax-\ by=d\end{array}\right.$ and they only differ from those of the first enunciation by having d in the place of c. We would, therefore, necessarily find

$$x=\frac{bn+d}{a+b},\ y=\frac{an-d}{a+b}.$$

And by substituting $-c$ for d, we have

$$x=\frac{bn+(-c)}{a+b};\ y=\frac{an-(-c)}{a+b};$$

or by applying the rules of Art. 85,

$$x=\frac{bn-c}{a+b};\ y=\frac{an+c}{a+b}.$$

The results, which agree to both enunciations, may be comprehended in the same formula, by writing

$$x=\frac{bn\pm c}{a+b};\ y=\frac{an\mp c}{a+b}.$$

The double sign $\pm$ is read *plus* or *minus*, the superior signs correspond to the case in which the labourer received, and the inferior signs to the case in which he owed a sum c. These formulas comprehend the case in which, in a settlement between the labourer and his employer, their accounts balance. This supposes $c=0$, which gives

$$x=\frac{bn}{a+b}; \quad y=\frac{an}{a+b}.$$

107. When a problem has been resolved generally, that is, by representing the given quantities by letters, it may be required to determine what the values of the unknown quantities become, when particular suppositions are made upon the given quantities. The determination of these values, and the interpretation of the peculiar results obtained, form what is called the *discussion of the problem.*

The discussion of the following question presents nearly all the circumstances which are met with in problems of the first degree.

108. Two couriers are travelling along the same right line and in the same direction from R′ towards R. The number of miles travelled by one of them per hour is expressed by m, and the number of miles travelled by the other per hour, is expressed by n. Now, at a given time, say 12 o'clock, the distance between them is equal to a number of miles expressed by a: required the time when they will be together.

R′ A B R.

At 12 o'clock suppose the forward courier to be at B, the other at A, and R to be the point at which they will be together.

Then, $AB=a$, their distance apart at 12 o'clock.

Let . . $t=$ the number of hours which must elapse, before they come together.

And . $x=$ the distance BR, which is to be passed over by the forward courier.

Then, since the rate per hour, multiplied by the number of hours, will give the distance passed over by each, we have,

$$t \times m = a + x = \text{AR}.$$
$$t \times n = x \quad = \text{BR}.$$

Hence by subtracting, $t(m-n) = a$

Therefore, . . $t = \dfrac{a}{m-n}$.

Now so long as $m > n$, t will be positive, and the problem will be solved in the arithmetical sense of the enunciation. For, if $m > n$ the courier from A will travel faster than the courier from B, and will therefore be continually gaining on him: the interval which separates them will diminish more and more, until it becomes 0, and then the couriers will be found upon the same point of the line.

In this case, the time t, which elapses, must be added to 12 o'clock, to obtain the time when they are together.

But, if we suppose $m < n$, then $m-n$ will be negative, and the value of t will be negative. How is this result to be interpreted?

It is easily explained from the nature of the question, which, considered in its most general sense, demands the time when the couriers are together.

Now, under the second supposition, the courier which is in advance, travels the fastest, and therefore will continue to separate himself from the other courier. At 12 o'clock the distance between them was equal to a: after 12 o'clock it is greater than a, and as the rate of travel has not been changed, it follows that previous to 12 o'clock the distance must have been less than a. At a certain hour, therefore, before 12 the distance between them must have been equal to nothing, or the couriers were together at some point R′. The precise hour is found by subtracting the value of t from 12 o'clock.

This example, therefore, conforms to the general principle, that, *if the conditions of a problem are such as to render the unknown quantity essentially negative, it will appear in the result with the minus sign, whenever it has been regarded as positive in the enunciation.*

If we wish to find the distances AR, and BR passed over by the two couriers before coming together, we may take the equation

$$t=\frac{a}{m-n}$$

and multiply both members by the rates of travel respectively: this will give

$$\mathrm{AR}=mt=\frac{ma}{m-n} \qquad \text{and}$$

$$\mathrm{BR}=nt=\frac{na}{m-n}.$$

Also, . . $\mathrm{AR}'=-mt=\frac{ma}{m-n}$

and . . $\mathrm{BR}'=-nt=\frac{na}{m-n}$.

from which we see that the two distances AR, BR, will both be positive when estimated towards the right, and that AR′, BR′ will both be negative when estimated in the contrary direction.

109. To explain the terms nothing and infinity, let us consider the equation

$$t=\frac{a}{m-n}.$$

If in this equation we make $m=n$, then $m-n=0$, and the value of t will reduce to

$$t=\frac{a}{0}.$$

In order to interpret this new result, let us go back to the enunciation, and it will be perceived that it is absolutely impossible to satisfy it for any finite value for t; for whatever time we allow to the two couriers they can never come together, since being once separated by an interval a, and travelling equally fast, this interval will always be preserved.

Hence, the result, $\frac{a}{0}$ may be regarded as a sign of impossibility for any *finite* value of t.

Nevertheless, algebraists consider the result

$$t=\frac{a}{0},$$

as forming a species of value, to which they have given the name of *infinite value*, for this reason :

When the difference $m-n$, without being absolutely nothing, is supposed to be very small, the result

$$t=\frac{a}{m-n},$$

is very great.

Take, for example, $m-n=0,01$.

Then $$t=\frac{a}{m-n}=\frac{a}{0,01}=100a\,;$$

Again, take $m-n=0,001$, and we have

$$\frac{a}{m-n}=\frac{a}{0,001}=1000a.$$

In short, if the difference between the rates is not zero, the couriers will come together at some point of the line, and the time will become greater and greater as this difference is diminished.

Hence, *if the difference between the rates is less than any assignable number, the time expressed by*

$$t=\frac{a}{m-n}=\frac{a}{0},$$

will be greater than any assignable or finite number. Therefore, for brevity, we say when $m-n=0$, the result

$$t=\frac{a}{m-n}$$

becomes equal to *infinity*, which we designate by the character ∞.

Again, as the value of a fraction increases as its numerator becomes greater with reference to its denominator, the expression $\frac{A}{0}$, A being any finite number, is a proper symbol to represent an *infinite* quantity ; that is, a quantity greater than any assignable quantity.

A quantity less than any given quantity may be expressed by $\frac{A}{\infty}$; for a fraction diminishes as its denominator becomes greater with reference to its numerator. Hence, 0 and $\frac{A}{\infty}$ are synonymous symbols, and so are $\frac{A}{0}$ and ∞.

We have been thus particular in explaining these ideas of infinity, because there are some questions of such a nature, that infinity may be considered as the true answer to the enunciation.

In the case, just considered, where $m=n$ it will be perceived that there is not, properly speaking, any solution in *finite and determinate numbers;* but the value of the unknown quantity is found to be infinite.

110. If, in addition to the hypothesis $m=n$, we suppose that $a=0$, we have

$$t=\frac{0}{0}.$$

To interpret this result, let us reconsider the enunciation, where it will be perceived, that if the two couriers travel equally fast, and are once at the same point, they ought always to be together, and consequently the required point is any point whatever of the line travelled over. Therefore, the expression $\frac{0}{0}$ is in this case, the symbol of an *indeterminate quantity.*

If the couriers do not travel equally fast, that is, if $m>$, or $m<n$, and $a=0$, then will $t=0$.

Indeed, it is evident, that if the couriers travel at different rates, and are together at 12 o'clock, they can never be together afterwards.

The preceding suppositions are the only ones that lead to remarkable results; and they are sufficient to show to beginners the manner in which the results of algebra answer to all the circumstances of the enunciation of a problem.

111. We will add another example to show, that the expression $\frac{0}{0}$ expresses, generally, an indeterminate quantity.

Take the expression, . . $\frac{1-x}{1-x}$.

Now, if we perform the division the quotient will be 1; and if we make $x=1$, there will result

$$\frac{1-x}{1-x}=\frac{0}{0}=1.$$

Let us next take the expression $\frac{1-x^2}{1-x}$.

If we perform the division, the quotient will be $1+x$; then making $x=1$, the expression becomes

$$\frac{1-x^2}{1-x}=\frac{0}{0}=2.$$

In like manner . $\frac{1-x^3}{1-x}=\frac{0}{0}=3$ when $x=1$.

and . . . . $\frac{1-x^n}{1-x}=\frac{0}{0}=n$ when $x=1$. (See Art. 59).

all of which goes to show that $\frac{0}{0}$ is the symbol of an indeterminate quantity.

112. We will add another example showing the value of the expressions $\frac{A}{0}$ and $\frac{0}{0}$.

Take the equation $ax=b$, involving one unknown quantity, whence $x=\frac{b}{a}$.

1st. If, for a particular supposition made with reference to the given quantities of the question, we have $a=0$, there results $x=\frac{b}{0}$.

Now in this case the equation becomes $0\times x=b$, and evidently cannot be satisfied by any finite value of x. We will however remark that, as the equation can be put under the form $\frac{b}{x}=0$, if we substitute for x, numbers greater and greater, $\frac{b}{x}$ will differ less and less from 0, and the equation will become more and more exact; so that, we may take a value for x so great that $\frac{b}{x}$ will be less than any assignable quantity, or $\frac{b}{\infty}=0$.

It is in consequence of this that algebraists say, that infinity satisfies the equation in this case; and there are some questions for which this kind of result forms a true solution; at least, it is certain that the equation does not admit of a solution in *finite* numbers, and this is all that we wish to prove.

2d. If we have $a=0$, $b=0$, at the same time, the value of x takes the form $x=\frac{0}{0}$.

In this case, the equation becomes $0\times x=0$, and *every finite number*, positive or negative, substituted for x, will satisfy the equation. Therefore *the equation, or the problem of which it is the algebraic translation, is indeterminate.*

113. It should be observed, that the expression $\frac{0}{0}$, does not always indicate an *indetermination,* it frequently indicates only *the existence of a common factor* to the two terms of the fraction, which factor becomes nothing, in consequence of a particular hypothesis.

For example, suppose that we find for the solution of a problem, $x=\frac{a^3-b^3}{a^2-b^2}$. If, in this formula, a is made equal to b, there results $x=\frac{0}{0}$.

But it will be observed, that a^3-b^3 can be put under the form $(a-b)\ (a^2+ab+b^2)$, (Art. 59), and that a^2-b^2 is equal to $(a-b)$

$(a+b)$, therefore the value of x becomes

$$x=\frac{(a-b)\,(a^2+ab+b^2)}{(a-b)\,(a+b)}.$$

Now, if we suppress the common factor $(a-b)$, before making the supposition $a=b$, the value of x becomes $x=\frac{a^2+ab+b^2}{a+b}$, which reduces to $x=\frac{3a^2}{2a}$, or $x=\frac{3a}{2}$, when $a=b$.

For another example, take the expression

$$\frac{a^2-b^2}{(a-b)^2}=\frac{(a+b)\,(a-b)}{(a-b)\,(a-b)}.$$

Making $a=b$, we find $x=\frac{0}{0}$, because the factor $(a-b)$ is common to the two terms; but if we first suppress this factor, there results $x=\frac{a+b}{a-b}$, which reduces to $x=\frac{2a}{0}$, when $a=b$.

From this we conclude, that *the symbol* $\frac{0}{0}$ *sometimes indicates the existence of a common factor to the two terms of the fraction which reduces to this form.* Therefore, before pronouncing upon the true value of the fraction, it is necessary to ascertain whether the two terms do not contain a common factor. If they do not, we conclude that the equation is really *indeterminate.* If they do contain one, suppress it, and then make the particular hypothesis; this will give the true value of the fraction, which will assume one of the three forms $\frac{A}{B}$, $\frac{A}{0}$, $\frac{0}{0}$, in which case, the equation is *determinate*, *impossible* in finite numbers, or *indeterminate.*

This observation is very useful in the discussion of problems.

Of Inequalities.

114. In the discussion of problems, we have often occasion to suppose several *inequalities*, and to perform transformations upon them, analogous to those executed upon *equalities.* We are often

obliged to do this, when, in discussing a problem, we wish to establish the necessary relations between the given quantities, in order that the problem may be susceptible of a direct, or at least a real solution, and to fix, with the aid of these relations, the limits between which the particular values of certain given quantities must be found, in order that the enunciation may fulfil a particular condition. Now, although the principles established for equations are in general applicable to inequalities, there are nevertheless some exceptions, of which it is necessary to speak, in order to put the beginner upon his guard against some errors that he might commit, in making use of the sign of inequality. These exceptions arise from the introduction of *negative expressions* into the calculus, as *quantities*.

In order that we may be clearly understood, we will take exar ples of the different transformations that inequalities may be subjected to, taking care to point out the exceptions to which these transformations are liable.

115. Two inequalities are said to subsist in the same sense, when the greater quantity stands at the left in both, or at the right in both ; and in a contrary sense, when the greater quantity stands at the right in one, and at the left in the other.

Thus, $25>20$ and $18>10$, or $6<8$ and $7<9$, are inequalities which subsist in the same sense ; and the inequalities $15>13$ and $12<14$, subsist in a contrary sense.

1. *If we add the same quantity to both members of an inequality, or subtract the same quantity from both members, the resulting inequality will subsist in the same sense.*

Thus, take $8>6$; by adding 5, we still have $8+5>6+5$ and $8-5>6-5$.

When the two members of an equality are both negative, that one is the least, algebraically considered, which contains the greatest number of units. Thus, $-25<-20$; and if 30 be added to both members, we have $5<10$. This must be understood entirely in an algebraic sense, and arises from the convention before established, to consider all quantities preceded by the minus sign, as subtractive.

The principle first enunciated, serves to transpose certain terms from one member of the inequality to the other. Take, for example, the inequality $a^2+b^2>3b^2-2a^2$; there will result from it

$$a^2+2a^2>3b^2-b^2, \text{ or } 3a^2>2b^2.$$

2. *If two inequalities subsist in the same sense, and we add them member to member, the resulting inequality will also subsist in the same sense.*

Thus, from $a>b$, $c>d$, $e>f$, there results $a+c+e>b+d+f$.

But this is not always the case, when we subtract, member from member, two inequalities established in the same sense.

Let there be the two inequalities $4<7$ and $2<3$, we have $4-2$ or $2<7-3$ or 4.

But if we have the inequalities $9<10$ and $6<8$, by subtracting we have $9-6$ or $3>10-8$ or 2.

We should then avoid this transformation as much as possible, or if we employ it, determine in what sense the resulting inequality exists.

3. *If the two members of an inequality be multiplied by a positive number, the resulting inequality will exist in the same sense.*

Thus, from $a<b$, we deduce $3a<3b$; and from $-a<-b$, $-3a<-3b$.

This principle serves to make the denominators disappear.

From the inequality $\dfrac{a^2-b^2}{2d}>\dfrac{c^2-d^2}{3a}$, we deduce, by multiplying by $6ad$,

$$3a(a^2-b^2)>2d(c^2-d^2).$$

The same principle is true for division.

But *when the two members of an inequality are multiplied or divided by a negative number, the inequality subsists in a contrary sense.*

Take, for example, $8>7$; multiplying by -3, we have $-24<-21$.

In like manner, $8>7$ gives $\frac{8}{-3}$, or $-\frac{8}{3}<-\frac{7}{3}$.

Therefore, when the two members of an inequality are multiplied or divided by a number expressed algebraically, it is necessary to ascertain whether the *multiplier* or *divisor* is negative; for, in that case, the inequality would exist in a contrary sense.

4. *It is not permitted to change the signs of the two members of an inequality unless we establish the resulting inequality in a contrary sense;* for this transformation is evidently the same as multiplying the two members by -1.

5. *Both members of an inequality between positive numbers can be squared, and the inequality will exist in the same sense.*

Thus from $5>3$, we deduce $25>9$; from $a+b>c$, we find $(a+b)^2>c^2$.

6. *When both members of the inequality are not positive, we cannot tell before the operation is performed, in which sense the resulting inequality will exist.*

For example, $-2<3$ gives $(-2)^2$ or $4<9$; but $-3>-5$ gives, on the contrary, $(-3)^2$ or $9<(-5)^2$ or 25.

We must then, before squaring, ascertain whether the two members can be considered as *positive numbers.*

EXAMPLES.

1. Find the limit of the value of x in the expression

$$5x-6>19. \qquad \textit{Ans.} \quad x>5.$$

2. Find the limit of the value of x in the expression

$$3x+\frac{14}{2}x-30>10 \qquad \textit{Ans.} \quad x>4.$$

3. Find the limit of the value of x in the expression

$$\frac{1}{6}x-\frac{1}{3}x+\frac{x}{2}+\frac{13}{2}>\frac{17}{2}. \qquad \textit{Ans.} \quad x>6.$$

4. Find the limit of the value of x in the inequalities

$$\frac{ax}{5}+bx-ab>\frac{a^2}{5}.$$

$$\frac{bx}{7}-ax+ab<\frac{b^2}{7}.$$

5. The double of a number diminished by 5 is greater than 25, and triple the number diminished by 7, is less than double the number increased by 13. Required a number which shall satisfy the conditions.

By the question, we have

$$2x-5>25.$$

$$3x-7<2x+13.$$

Resolving these inequalities, we have $x>15$ and $x<20$. Any number, therefore, either entire or fractional, comprised between 15 and 20, will satisfy the conditions.

6. A boy being asked how many apples he had in his basket, replied, that the sum of 3 times the number plus half the number, diminished by 5 is greater than 16; and twice the number diminished by one third of the number, plus 2 is less than 22. Required the number which he had.

Ans. 7, 8, 9, 10, or 11.

CHAPTER III.

Extraction of the Square Root of Numbers. Formation of the Square and Extraction of the Square Root of Algebraic Quantities. Calculus of Radicals of the Second Degree. Equations of the Second Degree.

116. The *square* or second power of a number, is the product which arises from multiplying that number by itself once: for example, 49 is the square of 7, and 144 is the square of 12.

The *square root* of a number is a second number of such a value, that, when multiplied by itself once the product is equal to the given number. Thus, 7 is the square root of 49, and 12 the square root of 144: for $7\times7=49$, and $12\times12=144$.

The square of a number, either entire or fractional, is easily found, being always obtained by multiplying this number by itself once. The extraction of the square root of a number, is however, attended with some difficulty, and requires particular explanation.

The first ten numbers are,

1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

and their squares,

1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

and reciprocally, the numbers of the first line are the square roots of the corresponding numbers of the second. We may also remark that, *the square of a number expressed by a single figure, will contain no figure of a higher denomination than tens.*

The numbers of the last line 1, 4, 9, 16, &c., and all other numbers which can be produced by the multiplication of a number by itself, are called *perfect squares.*

It is obvious, that there are but nine perfect squares among all the numbers which can be expressed by one or two figures: the square roots of all other numbers expressed by one or two figures will be found between two whole numbers differing from each other by unity. Thus, 55 which is comprised between 49 and 64, has for its square root a number between 7 and 8. Also, 91 which is comprised between 81 and 100, has for its square root a number between 9 and 10.

Every number may be regarded as made up of a certain number of tens and a certain number of units. Thus 64 is made up of 6 tens and 4 units, and may be expressed under the form $60+4=64$.

Now, if we represent the tens by a and the units by b, we shall have $a+b=64$

and . . . . . $(a+b)^2=(64)^2$

or . . . $a^2+2ab+b^2=4096.$

Which proves that the square of a number composed of tens and units contains, *the square of the tens plus twice the product of the tens by the units, plus the square of the units.*

117. If now, we make the units 1, 2, 3, 4, &c., tens, by annexing to each figure a cipher, we shall have,

10, 20, 30, 40, 50, 60, 70, 80, 90, 100

and for their squares,

100, 400, 900, 1600, 2500, 3600, 4900, 6400, 8100, 10000.

from which we see that the square of one ten is 100, the square of two tens 400; and generally, *that the square of tens will contain no figure of a less denomination than hundreds, nor of a higher name than thousands.*

Example I.—To extract the square root of 6084.

Since this number is composed of more than two places of figures its roots will contain more than one. 60.84
But since it is less than 10000, which is the square of 100, the root will contain but two figures: that is, units and tens.

Now, the square of the tens must be found in the two left hand figures which we will separate from the other two by a point. These parts, of two figures each, are called *periods.* The part 60 is comprised between the two squares 49 and 64, of which the roots are 7 and 8: hence, 7 *is the figure of the tens sought;* and the required root is composed of 7 tens and a certain number of units.

The figure 7 being found, we write it on the right of the given number, from which we separate it by a vertical line: then we subtract its square 49 from 60, which leaves a remainder of 11, to which we bring down the two next figures 84.

	60.84	78
	49	
$7 \times 2 = 14.8$	118.4	
	118 4	
	0	

The result of this operation 1184, contains *twice the product of the tens by the units plus the square of the units.* But since tens multiplied by units cannot give a product of a less name than tens, it follows that the last figure 4 can form no part of the double product of the tens by the units : this double product is therefore found in the part 118, which we separate from the units' place 4 by a point.

Now if we double the tens, which gives 14, and then divide 118 by 14, the quotient 8 *is the figure of the units*, or a figure greater than the units. This quotient figure can never be too small, since the part 118 will be at least equal to twice the product of the tens by the units : but it may be too large ; for the 118 besides the double product of the tens by the units, may likewise contain tens arising from the square of the units. To ascertain if the quotient 8 expresses the units, we write the 8 to the right of the 14, which gives 148, and then we multiply 148 by 8. Thus, we evidently form, 1st, the square of the units : and 2d, the double product of the tens by the units. This multiplication being effected, gives for a product 1184, a number equal to the result of the first operation. Having subtracted the product, we find the remainder equal to 0 : hence 78 is the root required.

Indeed, in the operations, we have merely subtracted from the given number 6084, 1*st*, the square of 7 tens or 70 ; 2*d*, twice the product of 70 by 8 ; and 3*d*, the square of 8 : that is, the three parts which enter into the composition of the square of $70+8$ or 78 ; and since the result of the subtraction is 0, it follows that 78 is the square root of 6084.

Ex. 2. To extract the square root of 841.

We first separate the number into periods, as in the last example. In the second period, which contains the square of the tens, there is but one figure. The greatest square contained in 8 is 4, the root of which is 2 : hence 2 is the figure of the tens in the required root.

	8.41	29
	4	
$2\times2=4.9$	44.1	
	44 1	
	0	

Subtracting its square 4 from 8, and bringing down 41, we obtain for a result 441.

If now, as in the last example, we separate the last figure 1 from the others by a point, and divide 44 by 4, which is double the tens, the quotient figure will be the units, or a figure greater than the units. Here the quotient is 11 ; but it is plain that it ought not to exceed 9, for if it could, the figure of the tens already found would be too small. Let us then try 9. Placing 9 in the root, and also on the right of the 4, and multiplying 49 by 9, we obtain for a product 441 : hence, 29 is the square root of 841.

Remark. The quotient figure 11, first found, was too large because the dividend 44 contained, besides the double product of the tens by the units, 8 tens arising from the square of the units. When the dividend is considerably augmented, by tens arising from the square of the units, the quotient figure will be too large.

Ex. 3. To extract the square root of 431649.

Since the given number exceeds 10,000 its root will be greater than 100 ; that is, it will contain more than two places of figures. But we may still regard the root as composed of tens and units, for every number may be expressed in tens and units. For example, the number 6758 is equal to 675 tens and 8 units, equal to 6750+8.

Now, we know that the square of the tens of the required root can make no part of the two right hand figures 49, which therefore, we separate from the others by a point, and the remaining figures 4316 contain the square of the tens of the required root. But since 4316 exceeds 100 the tens of the required root will contain more than one figure : hence 4316 must be separated into two parts, of which the right hand period 16 will contain no part of the square of that figure of the root, which is of the highest name, and for a similar reason we should separate again if the part to the left contained more than two figures.

	43.16.49	657
	36	
12.5	71.6	
5	62 5	
130.7	9 14.9	
	9 14 9	
	0	

Since 36 is the greatest square contained in 43, the first figure of the root is 6. We then subtract its square 36 from 43, and to the remainder 7 bring down the next period 16. Now, since the last figure 6 of the result 716, contains no part of the double product of the first figure of the tens by the second, it follows, that the second figure of the root will be obtained by dividing 71 by 12, double the first figure of the tens. This gives 5 for a quotient, which we place in the root, and at the right of the divisor 12. Then subtract the product of 125 by 5 from 716, and to the remainder bring down the next period, and the result 9149 will contain *twice the product of the tens of the root multiplied by the units, plus the square of the units.* If this result be then divided by twice 65, that is, by double the tens of the root, (which may always be found by adding the last figure of the divisor to itself), the quotient will be the units of the root.

Hence, for the extraction of the square root of numbers, we have the following

RULE.

I. *Separate the given number into periods of two figures each beginning at the right hand,—the period on the left will often contain but one figure.*

II. *Find the greatest square in the first period on the left, and place its root on the right after the manner of a quotient in division. Subtract the square of the root from the first period, and to the remainder bring down the second period for a dividend.*

III. *Double the root already found and place it on the left for a divisor. Seek how many times the divisor is contained in the dividend, exclusive of the right hand figure, and place the figure in the root and also at the right of the divisor.*

IV. *Multiply the divisor, thus augmented, by the last figure of the root, and subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.*

V. *Double the whole root already found, for a new divisor, and continue the operation as before, until all the periods are brought down.*

1st. Remark. If, after all the periods are brought down, there is no remainder, the proposed number is a perfect square. But if there is a remainder, you have only found the root of the greatest perfect square contained in the given number, or *the entire part of the root sought.*

For example, if it were required to extract the square root of 665, we should find 25 for the entire part of the root and a remainder of 40, which shows that 665 is not a perfect square. But is the square of 25 the greatest perfect square contained in 665? that is, is 25 the entire part of the root? To prove this, we will first show that, *the difference between the squares of two consecutive numbers, is equal to twice the less number augmented by unity.*

Let . . . a = the less number,

and . . . $a+1$ = the greater.

Then . . $(a+1)^2=a^2+2a+1$

and . . . $(a)^2=a^2$

Their difference is . . $= \overline{2a+1}$ as enunciated.

Hence, the entire part of the root cannot be augmented, unless the remainder exceed twice the root found, plus unity.

But $25\times2+1=51>40$ the remainder: therefore, 25 is the entire part of the root.

2d. Remark. The number of figures in the root will always be equal to the number of periods into which the given number is separated.

EXAMPLES.

1. To find the square root of 7225.
2. To find the square root of 17689.
3. To find the square root of 994009.
4. To find the square root of 85678973.
5. To find the square root of 67812675.

118. The square root of a number which is not a perfect square, is called *incommensurable* or *irrational*, because its exact root can-

not be found in terms of the numerical unit. Thus, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{7}$, are incommensurable numbers. They are also sometimes called *surds*.

In order to prove that the root of an imperfect power cannot be expressed by exact parts of unity, we must first show that,

Every number P, *which will exactly divide the product* $A \times B$ *of two numbers, and which is prime with one of them, will divide the other.*

Let us suppose that P will not divide A, and that A is greater than P. Apply to A and P the process for finding the greatest common divisor, and designate the quotients which arise by $Q, Q', Q'' \ldots$ and the remainders $R, R', R'' \ldots$ respectively. If the division be continued sufficiently far, we shall obtain a remainder equal to unity, for the remainder cannot be 0, since by hypothesis A and P are prime with each other. Hence we shall have the following equations.

$$\begin{aligned} A &= P\,Q + R \\ P &= R\,Q' + R' \\ R &= R'Q'' + R'' \\ R' &= R''Q''' + R''' \\ &\ldots\ldots \\ &\ldots\ldots \end{aligned}$$

Multiplying the first of these equations by B, and dividing by P, we have

$$\frac{AB}{P} = BQ + \frac{BR}{P}.$$

But, by hypothesis, $\frac{AB}{P}$ is an entire number, and since B and Q are entire numbers, the product BQ is an entire number. Hence it follows that $\frac{BR}{P}$ is an entire number.

If we multiply the second of the above equations by B, and divide by P, we have

$$B = \frac{BRQ'}{P} + \frac{BR'}{P}.$$

But we have already shown that $\frac{BR}{P}$ is an entire number; hence $\frac{BRQ'}{P}$ is an entire number. This being the case, $\frac{BR'}{P}$ must also be an entire number. If the operation be continued until the number which multiplies B becomes 1, we shall have $\frac{B\times 1}{P}$ equal to an entire number, which proves that P will divide B.

In the operations above we have supposed $A>P$, but if $P>A$ we should first divide P by A.

Hence, *if a number* P *will exactly divide the product of two numbers, and is prime with one of them, it will divide the other.*

We will now show that the root of an imperfect power cannot be expressed by a fractional number.

Let c be an imperfect square. Then if its exact root can be expressed by a fractional number, we shall have

$$\sqrt{c}=\frac{a}{b}$$

or . . . . $c=\frac{a^2}{b^2}$ by squaring both members.

But if c is not a perfect power, its root will not be a whole number, hence $\frac{a}{b}$ will at least be an irreducible fraction, or a and b will be prime to each other. But if a is not divisible by b, $a\times a$ or a^2 will not be divisible by b, from what has been shown above; neither then can a^2 be divisible by b^2. Since to divide by b^2 is but to divide a^2 twice by b. Hence, $\frac{a^2}{b^2}$ is an irreducible fraction, and therefore cannot be equal to the entire number c: therefore, we cannot assume $\sqrt{c}=\frac{a}{b}$, or the root of an imperfect power cannot be expressed by a fractional number that is rational.

Extraction of the square root of Fractions.

119. Since the square or second power of a fraction is obtained by squaring the numerator and denominator separately, it follows that the square root of a fraction will be equal to the square root of the numerator divided by the square root of the denominator.

For example, the square root of $\frac{a^2}{b^2}$ is equal to $\frac{a}{b}$: for

$$\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}.$$

But if neither the numerator nor the denominator is a perfect square, the root of the fraction cannot be exactly found. We can however, easily find the exact root to within less than one of the equal parts of the fraction.

To effect this, *multiply both terms of the fraction by the denominator, which makes the denominator a perfect square without altering the value of the fraction. Then extract the square root of the perfect square nearest the value of the numerator, and place the root of the denominator under it; this fraction will be the approximate root.*

Thus, if it be required to extract the square root of $\frac{3}{5}$, we multiply both terms by 5, which gives $\frac{15}{25}$: the square nearest 15 is 16: hence $\frac{4}{5}$ is the required root, and is exact to within less than $\frac{1}{5}$.

120. We may, by a similar method, determine approximatively the roots of whole numbers which are not perfect squares. Let it be required, for example, to determine the square root of an entire number a, nearer than the fraction $\frac{1}{n}$: that is to say, to find a

number which shall differ from the exact root of a, by a quantity less than $\frac{1}{n}$.

It may be observed that $a=\frac{an^2}{n^2}$. If we designate by r the entire part of the root of an^2, the number an^2 will then be comprised between r^2 and $(r+1)^2$; and $\frac{an^2}{n^2}$ will be comprised between $\frac{r^2}{n^2}$ and $\frac{(r+1)^2}{n^2}$; and consequently the true root of a is comprised between the root of $\frac{r^2}{n^2}$ and $\frac{(r+1)^2}{n^2}$; that is, between $\frac{r}{n}$ and $\frac{r+1}{n}$. Hence $\frac{r}{n}$ will represent the square root of a within less than the fraction $\frac{1}{n}$. Hence to obtain the root:

Multiply the given number by the square of the denominator of the fraction which determines the degree of approximation: then extract the square root of the product to the nearest unit, and divide this root by the denominator of the fraction.

Suppose, for example, it were required to extract the square root of 59, to within less than $\frac{1}{12}$.

Let us repeat on this example, the demonstration which has just been made.

The number 59 can be put under the form $\frac{59\times(12)^2}{(12)^2}$, or by multipliying by $(12)^2$, $\frac{8496}{(12)^2}$. But the root of 8496 to the nearest unit, is 92: hence it follows that $\frac{8496}{(12)^2}$ or 59, is comprised between $\frac{(92)^2}{(12)^2}$ and $\frac{(93)^2}{(12)^2}$. Then, the square root of 59 is itself comprised between $\frac{92}{12}$ and $\frac{93}{12}$: that is to say, the true root differs from $\frac{92}{12}$ by a fraction less than $\frac{1}{12}$.

Indeed the squares of $\frac{92}{12}$ and $\frac{93}{12}$ are $\frac{8464}{(12)^2}$ and $\frac{8649}{(12)^2}$, numbers which comprise $\frac{8496}{(12)^2}$ or 59.

2. To find the $\sqrt{11}$ to within less than $\frac{1}{15}$.

Ans. $3\frac{4}{15}$.

3. To find the $\sqrt{223}$ to within less than $\frac{1}{40}$.

Ans. $14\frac{37}{40}$.

121. The manner of determining the approximate root in decimals, is a consequence of the preceding rule.

To obtain the square root of an entire number within $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c.—it is necessary according to the preceding rule to multiply the proposed number by $(10)^2$, $(100)^2$, $(1000)^2$ . . . or, which is the same thing, *add to the right of the number, two, four, six, &c. ciphers: then extract the root of the product to the nearest unit, and divide this root by* 10, 100, 1000, &c., *which is effected by pointing off one, two, three, &c., decimal places from the right hand.*

Example 1. To extract the square root of 7 to within $\frac{1}{100}$.

Having added four ciphers to the right hand of 7, it becomes 70000, whose root extracted to the nearest unit is 264, which being divided by 100 gives 2,64 for the answer, which is true to within less than $\frac{1}{100}$.

$$
\begin{array}{r|l|l}
 & 7.00.00 & 2,64 \\
 & 4 & \\
46 & 300 & \\
 & 276 & \\
524 & 2400 & \\
 & 2096 & \\
 & 304 & \text{Rem}
\end{array}
$$

2. Find the $\sqrt{29}$ to within $\frac{1}{100}$.

Ans. 5,38.

3. Find the $\sqrt{227}$ to within $\frac{1}{10000}$.

Ans. 15,0665.

REMARK. The number of ciphers to be annexed to the whole number, is always double the number of decimal places required to be found in the root.

122. The manner of extracting the square root of decimal fractions is deduced immediately from the preceding article.

Let us take for example the number 3,425. This fraction is equivalent to $\frac{3425}{1000}$. Now 1000 is not a perfect square, but the denominator may be made such without altering the value of the fraction, by multiplying both the terms by 10; this gives $\frac{34250}{10000}$ or $\frac{34250}{(100)^2}$. Then extracting the square root of 34250 to the nearest unit, we find 185; hence $\frac{185}{100}$ or 1,85 is the required root to within $\frac{1}{100}$.

If greater exactness be required, it will be necessary to add to the number 3,4250 so many ciphers as shall make the periods of decimals equal to the number of decimal places to be found in the root. Hence, to extract the square root of a decimal fraction:

Annex ciphers to the proposed number until the decimal places shall be even, and equal to double the number of places required in the root. Then extract the root to the nearest unit, and point off from the right hand the required number of decimal places

Ex. 1. Find the $\sqrt{3271,4707}$ to within ,01.

Ans. 57,19.

2. Find the $\sqrt{31,027}$ to within ,01.

Ans. 5,57.

3. Find the $\sqrt{0,01001}$ to within ,00001.

Ans. 0,10004.

123. Finally, if it be required to find the square root of a vulgar fraction in terms of decimals: *Change the vulgar fraction into a decimal and continue the division until the number of decimal places is double the number of places required in the root. Then extract the root of the decimal by the last rule.*

Ex. 1. Extract the square root of $\frac{11}{14}$ to within ,001. This number, reduced to decimals, is 0,785714 to within 0,000001; but the root of 0,785714 to the nearest unit, is ,886 : hence 0,886 is the root of $\frac{11}{14}$ to within ,001.

2. Find the $\sqrt{2\frac{13}{15}}$ to within 0,0001.

Ans. 1,6931.

Extraction of the Square Root of Algebraic Quantities.

124. We will first consider the case of a monomial; and in order to discover the process, see how the square of the monomial is formed.

By the rule for the multiplication of monomials (Art. 41.), we have

$$(5a^2b^3c)^2 = 5a^2b^3c \times 5a^2b^3c = 25a^4b^6c^2;$$

that is, in order to square a monomial, it is necessary to *square its co-efficient, and double each of the exponents of the different letters.* Hence, to find the root of the square of a monomial, it is necessary, 1st. *To extract the square root of the co-efficient.* 2d. *To take the half of each of the exponents.*

Thus, $\sqrt{64a^6b^4} = 8a^3b^2$; for $8a^3b^2 \times 8a^3b^2 = 64a^6b^4$.

In like manner,

$$\sqrt{625a^2b^8c^6}=25ab^4c^3, \text{ for } (25ab^4c^3)^2=625a^2b^8c^6.$$

From the preceding rule, it follows, that, when a monomial is a perfect square, *its numerical co-efficient is a perfect square, and all its exponents even numbers.* Thus, $25a^4b^2$ is a perfect square, but $98ab^4$ is not a perfect square, because 98 is not a perfect square, and a is affected with an uneven exponent.

In the latter case, the quantity is introduced into the calculus by affecting it with the sign $\sqrt{}$, and it is written thus, $\sqrt{98ab^4}$. Quantities of this kind are called *radical quantities*, or *irrational quantities*, or simply *radicals of the second degree.*

125. These expressions may sometimes be simplified, upon the principle that, *the square root of the product of two or more factors is equal to the product of the square roots of these factors;* or, in algebraic language, $\sqrt{abcd}\ldots=\sqrt{a}.\sqrt{b}.\sqrt{c}.\sqrt{d}.\ldots$

To demonstrate this principle, we will observe, that from the definition of the square root, we have

$$(\sqrt{abcd}\ldots)^2=abcd\ldots$$

Again,

$$(\sqrt{a}\times\sqrt{b}\times\sqrt{c}\times\sqrt{d}\ldots)^2=(\sqrt{a})^2\times(\sqrt{b})^2\times(\sqrt{c})^2\times(\sqrt{d})^2\ldots$$
$$=abcd\ldots$$

Hence, since the squares of $\sqrt{abcd}\ldots$, and $\sqrt{a}.\sqrt{b}.\sqrt{c}.\sqrt{d}.\ldots$, are equal, the quantities themselves are equal.

This being the case, the above expression, $\sqrt{98ab^4}$, can be put under the form $\sqrt{49b^4\times 2a}=\sqrt{49b^4}\times\sqrt{2a}$. Now $\sqrt{49b^4}$ may be reduced to $7b^2$; hence $\sqrt{98ab^4}=7b^2\sqrt{2a}$.

In like manner,

$$\sqrt{45a^2b^3c^2d}=\sqrt{9a^2b^2c^2\times 5bd}=3abc\sqrt{5bd},$$
$$\sqrt{864a^2b^5c^{11}}=\sqrt{144a^2b^4c^{10}\times 6bc}=12ab^2c^5\sqrt{6bc}.$$

The quantity which stands without the radical sign is called the *co-efficient* of the radical. Thus, in the expressions $7b^2\sqrt{2a}$, $3abc\sqrt{5bd}$, $12ab^2c^5\sqrt{6bc}$, the quantities $7b^2$, $3abc$, $12ab^2c^5$, are called *co-efficients of the radicals.*

In general, to simplify an irrational monomial, *separate it into two parts, of which one shall contain all the factors that are perfect squares, and the other the remaining ones: then take the roots of the perfect squares and place them before the radical sign, under which, leave those factors which are not perfect squares.*

EXAMPLES.

1. To reduce $\sqrt{75a^3bc}$ to its simplest form.
2. To reduce $\sqrt{128b^5a^6d^2}$ to its simplest form.
3. To reduce $\sqrt{32a^9b^8c}$ to its simplest form.
4. To reduce $\sqrt{256a^2b^4c^8}$ to its simplest form.
5. To reduce $\sqrt{1024a^9b^7c^5}$ to its simplest form.
6. To reduce $\sqrt{728a^7b^5c^6d}$ to its simplest form.

126. Since like signs in both the factors give a plus sign in the product, the square of $-a$, as well as that of $+a$, will be a^2: hence the root of a^2 is either $+a$ or $-a$. Also, the square root of $25a^2b^4$ is either $+5ab^2$ or $-5ab^2$. Whence we may conclude, that if a monomial is positive, its square root may be affected either with the sign $+$ or $-$; thus, $\sqrt{9a^4}=\pm 3a^2$, for $+3a^2$ or $-3a^2$, squared, gives $9a^4$. The double sign $\pm$ with which the root is affected is read *plus or minus.*

If the proposed monomial were *negative*, it would be impossible to extract its root, since it has just been shown that the square of every quantity, whether positive or negative, is essentially positive. Therefore, $\sqrt{-9}$, $\sqrt{-4a^2}$, $\sqrt{-8a^2b}$, are algebraic symbols which indicate operations that cannot be performed. They are called *imaginary quantities*, or rather *imaginary expressions*, and are frequently met with in the resolution of equations of the second degree. These symbols can, however, by extending the rules, be simplified in the same manner as those irrational expressions which indicate operations that cannot be performed. Thus, $\sqrt{-9}$ may be reduced to (Art. 125.)

$$\sqrt{9}\times\sqrt{-1}, \text{ or, } 3\sqrt{-1}; \quad \sqrt{-4a^2}=\sqrt{4a^2}\times\sqrt{-1}=2a\sqrt{-1};$$

$$\sqrt{-8a^2b}=\sqrt{4a^2\times-2b}=2a\sqrt{-2b}=2a\sqrt{2b}\times\sqrt{-1}.$$

127. Let us now examine the *law of formation* for the square of any polynomial whatever; for, from this law, a rule is to be deduced for extracting the square root.

It has already been shown that the square of a binomial $(a+b)$ is equal to $a^2+2ab+b^2$ (Art. 46.).

Now to form the square of a trinomial $a+b+c$, denote $a+b$ by the single letter s, and we have

$$(a+b+c)^2=(s+c)^2=s^2+2sc+c^2.$$

But $s^2=(a+b)^2=a^2+2ab+b^2$; and $2sc=2(a+b)c=2ac+2bc$.

Hence $(a+b+c)^2=a^2+2ab+b^2+2ac+2bc+c^2$;

that is, *the square of a trinomial is composed of the sum of the squares of its three terms, and twice the products of these terms multiplied together two and two.*

If we take a polynomial of four or more terms, and square it, we shall find the same *law of formation*. We may, therefore, suppose the law to be proved for the square of a polynomial of m terms; and it then only remains to be shown that it will be true for a polynomial of $m+1$ terms.

Take the polynomial $(a+b+c\ldots+i)$, having m terms, and denote their sum by s: then the polynomial $(a+b+c\ldots+i+k)$ having $m+1$ terms, will be denoted by $(s+k)$.

Now, $(s+k)^2=s^2+2sk+k^2$, or by substituting for s.

$$(s+k)^2=(a+b+c\ldots+i)^2+2(a+b+c\ldots+i)k+k^2.$$

But by hypothesis, the first part of this expression is composed of *the squares of all the terms of the first polynomial and the double* *products of these terms taken two and two ;* the second part contains *the double products of all the terms of the* first polynomial by the additional term k ; and the third part *is the square of this term.* Therefore, the *law of composition*, announced above, is true for the new polynomial. But it has been proved to be true for a trinomial; hence it is true for a polynomial containing four terms ; being true for *four*, it is necessarily true for *five*, and so on. Therefore it is general. This *law* can be enunciated in another manner : viz.

The square of any polynomial contains the square of the first term, plus twice the product of the first by the second, plus the square of the second ; plus twice the product of the first two terms by the third, plus the square of the third ; plus twice the product of the first three terms by the fourth, plus the square of the fourth ; and so on.

This enunciation which is evidently comprehended in the first, shows more clearly the process for extracting the square root of a polynomial.

From this law,

$$(a+b+c)^2=a^2+2ab+b^2+2(a+b)c+c^2$$

$$(a+b+c+d)^2=a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2.$$

128. We will now proceed to extract the square root of a polynomial.

Let the proposed polynomial be designated by N, and its root, which we will suppose is determined, by R ; conceive, also, that these two polynomials are arranged with reference to one of the letters which they contain, a, for example.

Now it is plain that the first term of the root R may be found by extracting the root of the first term of the polynomial N ; and that the second term of the root may be found by dividing the second term of the polynomial N, by twice the first term of the root R.

If now we form the square of the binomial thus found, and subtract it from N, the first term of the remainder will be twice the product of the first term of R by the third term : hence, if this first term be divided by double the first term of R, the quotient will be the third term of R.

In order to obtain the fourth term of R, form the double products of the first and second terms, by the third, plus the square of the third; then subtract all these products from the remainder before found, and the first term of the result will be twice the product of the first term of the root by the 4th: hence, if it be divided by double the first term, the quotient will be the fourth term. In the same manner the next and subsequent terms may be found. Hence, for the extraction of the square root of a polynomial we have the following

RULE.

I. *Arrange the polynomial with reference to one of its letters and extract the square root of the first term: this will give the first term of the root.*

II. *Divide the second term of the polynomial by double the first term of the root, and the quotient will be the second term of the root.*

III. *Then form the square of the two terms of the root found, and subtract it from the first polynomial, and then divide the first term of the remainder by double the first term of the root, and the quotient will be the third term.*

IV. *Form the double products of the first and second terms, by the third, plus the square of the third; then subtract all these products from the last remainder, and divide the first term of the result by double the first term of the root, and the quotient will be the fourth term. Then proceed in the same manner to find the other terms.*

EXAMPLES.

1. Extract the square root of the polynomial

$$49a^2b^2-24ab^3+25a^4-30a^3b+16b^4.$$

First arrange it with reference to the letter a.

$$
\begin{array}{rl|l}
25a^4-30a^3b+49a^2b^2-24ab^3+16b^4 & & 5a^2-3ab+4b^3 \\ \cline{3-3}
\underline{25a^4-30a^3b+\ \ 9a^2b^2}\qquad\qquad\qquad & & 10a^2 \\
40a^2b^2-24ab^3+16b^4 & \text{1st. Rem.} & \\
\underline{40a^2b^2-24ab^3+16b^4} & & \\
0\quad .\quad .\quad . & \text{2d. Rem.} &
\end{array}
$$

After having arranged the polynomial with reference to a, extract the square root of $25a^4$, this gives $5a^2$, which is placed to the right of the polynomial; then divide the second term, $-30a^3b$, by the double of $5a^2$, or $10a^2$; the quotient is $-3ab$, and is placed to the right of $5a^2$. Hence, the first two terms of the root are $5a^2-3ab$. Squaring this binomial, it becomes $25a^4-30a^3b+9a^2b^2$, which, subtracted from the proposed polynomial, gives a remainder, of which the first term is $40a^2b^2$. Dividing this first term by $10a^2$, (the double of $5a^2$), the quotient is $+4b^2$; this is the third term of the root, and is written on the right of the first two terms. Forming the double product of $5a^2-3ab$ by $4b^2$, and the square of $4b^2$, we find the polynomial $40a^2b^2-24ab^3+16b^4$, which, subtracted from the first remainder, gives 0. Therefore $5a^2-3ab+4b^2$ is the required root.

2. Find the square root of

$$a^4+4a^3x+6a^2x^2+4ax^3+x^4.$$

3. Find the square root of

$$a^4-2a^3x+3a^2x^2-2ax^3+x^4.$$

4. Find the square root of

$$4x^6+12x^5+5x^4-2x^3+7x^2-2x+1.$$

5. Find the square root of

$$9a^4-12a^3b+28a^2b^2-16ab^3+16b^4.$$

6. Find the square root of

$$25a^4b^2-40a^3b^2c+76a^2b^2c^2-48ab^2c^3+36b^2c^4-30a^4bc+24a^3bc^2$$
$$-36a^2bc^3+9a^4c^2.$$

129. We will conclude this subject with the following remarks.

1st. A binomial can never be a perfect square, since we know that the square of the most simple polynomial, viz. a binomial, contains three distinct parts, which cannot experience any reduction amongst themselves. Thus, the expression a^2+b^2 is not a perfect square; it wants the term $\pm 2ab$ in order that it should be the square of $a\pm b$.

2d. In order that a trinomial, when arranged, may be a perfect square, its two extreme terms must be squares, and the middle term must be the double product of the square roots of the two others. Therefore, to obtain the square root of a trinomial when it is a perfect square; *Extract the roots of the two extreme terms, and give these roots the same or contrary signs, according as the middle term is positive or negative. To verify it, see if the double product of the two roots gives the middle term of the trinomial.* Thus,

$$9a^6-48a^4b^2+64a^2b^4 \quad \text{is a perfect square,}$$

since $\sqrt{9a^6}=3a^3$, and $\sqrt{64a^2b^4}=-8ab^2$,

and also $2\times 3a^3\times -8ab^2=-48a^4b^2$, the middle term.

But $4a^2+14ab+9b^2$ is not a perfect square: for although $4a^2$ and $+9b^2$ are the squares of $2a$ and $3b$, yet $2\times 2a\times 3b$ is not equal to $14ab$.

3d. In the series of operations required in a general problem, when the first term of one of the remainders is not exactly divisible by twice the first term of the root, we may conclude that the proposed polynomial is not a perfect square. This is an evident consequence of the course of reasoning, by which we have arrived at the general rule for extracting the square root.

4th. When the polynomial is *not a perfect square*, it may be simplified (See Art. 125.).

Take, for example, the expression $\sqrt{a^3b+4a^2b^2+4ab^3}$.

The quantity under the radical is not a perfect square; but it can be put under the form $ab(a^2+4ab+4b^2)$. Now, the factor between the parenthesis is evidently the square of $a+2b$, whence we may conclude that,

$$\sqrt{a^3b+4a^2b^2+4ab^3}=(a+2b)\sqrt{ab}.$$

Of the Calculus of Radicals of the Second Degree.

130. A *radical quantity* is the indicated root of an imperfect power.

The extraction of the square root gives rise to such expressions as $\sqrt{a}$, $3\sqrt{b}$, $7\sqrt{2}$, which are called *irrational quantities*, or *radicals of the second degree*. We will now establish rules for performing the four fundamental operations on these expressions.

131. Two radicals of the second degree are *similar*, when the quantities under the radical sign are the same in both. Thus, $3\sqrt{b}$ and $5c\sqrt{b}$ are similar radicals; and so also are $9\sqrt{2}$ and $7\sqrt{2}$.

Addition and Subtraction.

132. In order to add or subtract similar radicals, *add or subtract their co-efficients, then prefix the sum or difference to the common radical.*

Thus, . . . $3a\sqrt{b}+5c\sqrt{b}=(3a+5c)\sqrt{b}$.

And . . . $3a\sqrt{b}-5c\sqrt{b}=(3a-5c)\sqrt{b}$.

In like manner, $7\sqrt{2a}+3\sqrt{2a}=(7+3)\sqrt{2a}=10\sqrt{2a}$.

And . . . $7\sqrt{2a}-3\sqrt{2a}=(7-3)\sqrt{2a}=4\sqrt{2a}$.

Two radicals, which do not appear to be similar at first sight, may become so by simplification (Art. 125).

For example,

$$\sqrt{48ab^2}+b\sqrt{75a}=4b\sqrt{3a}+5b\sqrt{3a}=9b\sqrt{3a},$$

and $2\sqrt{45}-3\sqrt{5}=6\sqrt{5}-3\sqrt{5}=3\sqrt{5}$.

When the radicals are not similar, the addition or subtraction can only be indicated. Thus, in order to add $3\sqrt{b}$ to $5\sqrt{a}$, we write

$$5\sqrt{a}+3\sqrt{b}.$$

Multiplication.

133. To multiply one radical by another, *multiply the two quantities under the radical sign together, and place the common radical over the product.*

Thus, $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$; this is the principle of Art. 125, taken in an inverse order.

When there are co-efficients, *we first multiply them together, and write the product before the radical.* Thus,

$$3\sqrt{5ab}\times4\sqrt{20a}=12\sqrt{100a^2b}=120a\sqrt{b}.$$

$$2a\sqrt{bc}\times3a\sqrt{bc}=6a^2\sqrt{b^2c^2}=6a^2bc.$$

$$2a\sqrt{a^2+b^2}\times-3a\sqrt{a^2+b^2}=-6a^2(a^2+b^2).$$

Division.

134. To divide one radical by another, *divide one of the quantities under the radical sign by the other and place the common radical over the quotient.*

Thus, $\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}$; for the squares of these two expressions are equal to the same quantity $\dfrac{a}{b}$; hence the expressions themselves must be equal. When there are co-efficients, *write their quotient as a co-efficient of the radical.*

For example,

$$5a\sqrt{b}\div2b\sqrt{c}=\frac{5a}{2b}\sqrt{\frac{b}{c}},$$

$$12ac\sqrt{6bc}\div4c\sqrt{2b}=3a\sqrt{\frac{6bc}{2b}}=3a\sqrt{3c}.$$

135. There are *two transformations* of frequent use in finding the numerical values of radicals.

The first consists in passing the co-efficient of a radical under the sign. Take, for example, the expression $3a\sqrt{5b}$; it is equivalent to $\sqrt{9a^2} \times \sqrt{5b}$, or $\sqrt{9a^2.5b} = \sqrt{45a^2b}$, by applying the rule for the multiplication of two radicals; therefore, *to pass the co-efficient of a radical under the sign, it is only necessary to square it.*

The principal use of this transformation, is to find a number which shall differ from the proposed radical, *by a quantity less than unity.* Take, for example, the expression $6\sqrt{13}$; as 13 is not a perfect square, we can only obtain an approximate value for its root. This root is equal to 3, plus a certain fraction; this being multiplied by 6, gives 18, plus the product of the fraction by 6; and the entire part of this result, obtained in this way, cannot be greater than 18. The only method of obtaining the entire part exactly, is to put $6\sqrt{13}$ under the form .... $\sqrt{6^2 \times 13} = \sqrt{36 \times 13} = \sqrt{468}$.
Now 468 has 21 for the entire part of its square root; hence, $6\sqrt{13}$ is equal to 21, plus a fraction.

In the same way, we find that $12\sqrt{7} = 31$, plus a fraction.

136. The object of the second transformation is to convert the denominators of such expressions as $\dfrac{a}{p+\sqrt{q}}$, $\dfrac{a}{p-\sqrt{q}}$, into rational quantities, a and p being any numbers whatever, and q *not a perfect square.* Expressions of this kind are often met with in the resolution of equations of the second degree.

Now this object is accomplished by multiplying the two terms of the fraction by $p-\sqrt{q}$, when the denominator is $p+\sqrt{q}$, and by $p+\sqrt{q}$, when the denominator is $p-\sqrt{q}$. For multiplying in this manner, and recollecting that the sum of two quantities, multiplied by their difference, is equal to the difference of their squares, we have

$$\frac{a}{p+\sqrt{q}} = \frac{a(p-\sqrt{q})}{(p+\sqrt{q})(p-\sqrt{q})} = \frac{a(p-\sqrt{q})}{p^2-q} = \frac{ap-a\sqrt{q}}{p^2-q},$$

$$\frac{a}{p-\sqrt{q}}=\frac{a(p+\sqrt{q})}{(p-\sqrt{q})(p+\sqrt{q})}=\frac{a(p+\sqrt{q})}{p^2-q}=\frac{ap+a\sqrt{q}}{p^2-q},$$

in which the denominators are rational.

To form an idea of the utility of this method, suppose it is required to find the approximate value of the expression $\dfrac{7}{3-\sqrt{5}}$. It becomes $\dfrac{7(3+\sqrt{5})}{9-5}$, or $\dfrac{21+7\sqrt{5}}{4}$. Now $7\sqrt{5}$ is equivalent to $\sqrt{49\times5}$, or $\sqrt{245}$, which is equal to 15, within one of the true value.

Therefore, $\dfrac{7}{3-\sqrt{5}}=\dfrac{21+15+\text{a fraction}}{4}=\dfrac{36}{4}=9$, within a fraction of one *fourth*; that is, it differs from the true value by a quantity less than one *fourth*.

When we wish to have a more exact value for this expression, *extract the square root of 245 to a certain number of decimal places, add 21 to this root, and divide the result by 4.*

For another example, take $\dfrac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}}$, and find the value of it to within 0.01.

We have,

$$\frac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}}=\frac{7\sqrt{5}(\sqrt{11}-\sqrt{3})}{11-3}=\frac{7\sqrt{55}-7\sqrt{15}}{8}.$$

Now, $\quad 7\sqrt{55}=\sqrt{55\times49}=\sqrt{2695}=51,91$, within 0.01.

$\quad 7\sqrt{15}=\sqrt{15\times49}=\sqrt{735}=27,11\ \ldots\ldots$

therefore, $\dfrac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}}=\dfrac{51,91-27,11}{8}=\dfrac{24,80}{8}=3,10.$

Hence we have 3,10 for the required result. This is exact to within $\dfrac{1}{100}$.

By a similar process, it will be found that

$$\frac{3+2\sqrt{7}}{5\sqrt{12}-6\sqrt{5}}=2{,}123, \text{ exact to within } 0{,}001.$$

REMARK. Expressions of this kind might be calculated by approximating to the value of each of the radicals which enter the numerator and denominator. But as the value of the denominator would not be exact, we could not form a precise idea of the degree of approximation which would be obtained, whereas by the method just indicated, the denominator becomes *rational*, and we always know to what degree the approximation is made.

The principles for the extraction of the square root of particular numbers and of algebraic quantities, being established, we will proceed to the resolution of problems of the second degree.

Examples in the Calculus of Radicals.

1. Reduce $\sqrt{125}$ to its most simple terms.

Ans. $5\sqrt{5}$.

2. Reduce $\sqrt{\frac{50}{147}}$ to its most simple terms.

Ans. $\frac{5}{21}\sqrt{6}$.

3. Reduce $\sqrt{98a^2x}$ to its most simple terms.

Ans. $7a\sqrt{2x}$.

4. Reduce $\sqrt{(x^3-a^2x^2)}$ to its most simple terms.

5. Required the sum of $\sqrt{72}$ and $\sqrt{128}$.

Ans. $14\sqrt{2}$.

6. Required the sum of $\sqrt{27}$ and $\sqrt{147}$.

Ans. $10\sqrt{3}$

7. Required the sum of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{27}{50}}$.

Ans. $\frac{19}{30}\sqrt{6}$.

8. Required the sum of $2\sqrt{a^2b}$ and $3\sqrt{64bx^4}$.

9. Required the sum of $9\sqrt{243}$ and $10\sqrt{363}$.

10. Required the difference of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$.

Ans. $\frac{4}{45}\sqrt{15}$.

11. Required the product of $5\sqrt{8}$ and $3\sqrt{5}$.

Ans. $30\sqrt{10}$.

12. Required the product of $\frac{2}{3}\sqrt{\frac{1}{8}}$ and $\frac{3}{4}\sqrt{\frac{7}{10}}$.

Ans. $\frac{1}{40}\sqrt{35}$.

13. Divide $6\sqrt{10}$ by $3\sqrt{5}$. *Ans.* $2\sqrt{2}$.

Of Equations of the Second Degree.

137. When the enunciation of a problem leads to an equation of the form $ax^2=b$, in which the unknown quantity is multiplied by itself, the equation is said to be of the *second degree*, and the principles established in the two preceding chapters are not sufficient for the resolution of it; but since by dividing the two members by a, it becomes $x^2=\frac{b}{a}$, we see that the question is reduced to finding the square root of $\frac{b}{a}$.

138. Equations of the second degree are of two kinds, viz. equations involving *two terms*, or *incomplete* equations, and equations involving *three terms*, or *complete* equations.

The first are those which contain only terms involving the square of the unknown quantity, and known terms; such are the equations,

$$3x^2=5\,;\ \frac{1}{3}x^2-3+\frac{5}{12}x^2=\frac{7}{24}-x^2+\frac{299}{24}.$$

These are called equations involving *two terms* because they may be reduced to the form $ax^2=b$, by means of the two general transformations (Art. 90 & 91). For, let us consider the second equation, which is the most complicated; by clearing the fractions it becomes

$$8x^2-72+10x^2=7-24x^2+299,$$

or transposing and reducing

$$42x^2=378.$$

Equations involving three terms, or complete equations, are those which contain the square, and also the first power of the unknown quantity, together with a known term; such are the equations

$$5x^2-7x=34\,;\ \frac{5}{6}x^2-\frac{1}{2}x+\frac{3}{4}=8-\frac{2}{3}x-x^2+\frac{273}{12}.$$

They can always be reduced to the form $ax^2+bx=c$, by the two transformations already cited.

Of Equations involving two terms.

139. There is no difficulty in the resolution of the equation $ax^2=b$. We deduce from it $x^2=\frac{b}{a}$, whence $x=\sqrt{\frac{b}{a}}$.

When $\frac{b}{a}$ is a particular number, either entire or fractional, we can obtain the square root of it exactly, or by approximation. If $\frac{b}{a}$ is algebraic, we apply the rules established for algebraic quantities.

But as the square of $+m$ or $-m$, is $+m^2$, it follows that $\left(\pm\sqrt{\frac{b}{a}}\right)^2$ is equal to $\frac{b}{a}$. Therefore, x is susceptible of two values, viz. $x=+\sqrt{\frac{b}{a}}$, and $x=-\sqrt{\frac{b}{a}}$. For, substituting either of these values in the equation $ax^2=b$, it becomes

$$a\times\left(+\sqrt{\frac{b}{a}}\right)^2=b, \text{ or } a\times\frac{b}{a}=b,$$

and . . . $$a\times\left(-\sqrt{\frac{b}{a}}\right)^2=b, \text{ or } a\times\frac{b}{a}=b.$$

For another example take the equation $4x^2-7=3x^2+9$; by transposing, it becomes, $x^2=16$, whence $x=\pm\sqrt{16}=\pm4$.

Again, take the equation

$$\frac{1}{3}x^2-3+\frac{5}{12}x^2=\frac{7}{24}-x^2+\frac{299}{24}$$

We have already seen (Art. 138.), that this equation reduces to $42x^2=378$, and dividing by 42, $x^2=\frac{378}{42}=9$; hence $x=\pm3$.

Lastly, from the equation $3x^2=5$; we find

$$x=\pm\sqrt{\frac{5}{3}}=\pm\frac{1}{3}\sqrt{15}.$$

As 15 is not a perfect square, the values of x can only be determined by approximation

Of complete Equations of the Second Degree.

140. In order to resolve the general equation

$$ax^2+bx=c.$$

we begin by dividing both numbers by the co-efficient of x^2, which gives,

$$x^2+\frac{b}{a}x=\frac{c}{a}, \quad \text{or} \quad x^2+px=q$$

by making $$\frac{b}{a}=p \text{ and } \frac{c}{a}=q.$$

Now, if we could make the first member x^2+px the square of a binomial, the equation might be reduced to one of the first degree, by simply extracting the square root. By comparing this member with the square of the binomial $(x+a)$, that is, with $x^2+2ax+a^2$, it is plain that x^2+px is composed of the square of a first term x, plus the double product of this first term x by a second, which must be $\frac{p}{2}$, since $px=2\frac{p}{2}x$; therefore, if the square of $\frac{p}{2}$ or $\frac{p^2}{4}$, be added to x^2+px, the first member of the equation will become the square of $x+\frac{p}{2}$; but in order that the equality may not be destroyed $\frac{p^2}{4}$ must be added to the second member.

By this transformation, the equation $x^2+px=q$ becomes

$$x^2+px+\frac{p^2}{4}=q+\frac{p^2}{4}.$$

Whence by extracting the square root

$$x+\frac{p}{2}=\pm\sqrt{q+\frac{p^2}{4}}.$$

The double sign $\pm$ is placed here, because either

$$+\sqrt{q+\frac{p^2}{4}}, \text{ or } -\sqrt{q+\frac{p^2}{4}}, \text{ squared gives } q+\frac{p^2}{4}.$$

Transposing $\frac{p}{2}$, we obtain

$$x=-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}.$$

From this we derive, for the resolution of complete equations of the second degree, the following general

RULE.

After reducing the equation to the form $x^2+px=q$, *add the square of half of the co-efficient of* x, *or of the second term, to both members; then extract the square root of both members, giving the double sign* $\pm$ *to the second member; then find the value of* x *from the resulting equation.*

This formula for the value of x may be thus enunciated.

The value of the unknown quantity is equal to half the co-efficient *of* x, *taken with a contrary sign, plus or minus the square root of the known term increased by the square of half the co-efficient of* x.

Take, for an example, the equation

$$\frac{5}{6}x^2-\frac{1}{2}x+\frac{3}{4}=8-\frac{2}{3}x-x^2+\frac{273}{12}.$$

Clearing the fractions, we have

$$10x^2-6x+9=96-8x-12x^2+273,$$

or, transposing and reducing,

$$22x^2+2x=360,$$

and dividing both members by 22,

$$x^2+\frac{2}{22}x=\frac{360}{22}.$$

Add $\left(\frac{1}{22}\right)^2$ to both members, and the equation becomes

$$x^2+\frac{2}{22}x+\left(\frac{1}{22}\right)^2=\frac{360}{22}+\left(\frac{1}{22}\right)^2;$$

whence, by extracting the square root,

$$x+\frac{1}{22}=\pm\sqrt{\frac{360}{22}+\left(\frac{1}{22}\right)^2},$$

Therefore,

$$x=-\frac{1}{22}\pm\sqrt{\frac{360}{22}+\left(\frac{1}{22}\right)^2},$$

which agrees with the enunciation given above for the double value of x.

It remains to perform the numerical operations. In the first place, $\frac{360}{22}+\left(\frac{1}{22}\right)^2$ must be reduced to a single number, having $(22)^2$ for its denominator.

Now, $$\frac{360}{22}+\left(\frac{1}{22}\right)^2=\frac{360\times22+1}{(22)^2}=\frac{7921}{(22)^2};$$

extracting the square root of 7921, we find it to be 89; therefore,

$$\sqrt{\frac{360}{22}+\left(\frac{1}{22}\right)^2}=\frac{89}{22}.$$

Consequently, $$x=-\frac{1}{22}\pm\frac{89}{22}.$$

Separating the two values, we have

$$x=-\frac{1}{22}+\frac{89}{22}=\frac{88}{22}=4,$$

$$x=-\frac{1}{22}-\frac{89}{22}=-\frac{45}{11}.$$

Therefore, one of the two values which will satisfy the proposed equation, is a positive whole number, and the other a negative fraction.

For another example, take the equation

$$6x^2-37x=-57,$$

which reduces to

$$x^2-\frac{37}{6}x=-\frac{57}{6}.$$

If we add the square of $\frac{37}{12}$, or $\left(\frac{37}{12}\right)^2$ to both members, it becomes

$$x^2-\frac{37}{6}x+\left(\frac{37}{12}\right)^2=-\frac{57}{6}+\left(\frac{37}{12}\right)^2;$$

whence, by extracting the square root

$$x-\frac{37}{12}=\pm\sqrt{-\frac{57}{6}+\left(\frac{37}{12}\right)^2}.$$

Consequently,

$$x=\frac{37}{12}\pm\sqrt{-\frac{57}{6}+\left(\frac{37}{12}\right)^2}.$$

In order to reduce $\left(\frac{37}{12}\right)^2-\frac{57}{6}$ to a single number, we will observe, that

$$(12)^2=12\times12=6\times24;$$

therefore, it is only necessary to multiply 57 by 24, then 37 by itself, and divide the difference of the two products by $(12)^2$. Now,

$$37\times37=1369;\ 57\times24=1368;$$

therefore,

$$\left(\frac{37}{12}\right)^2-\frac{57}{6}=\frac{1}{(12)^2}.$$

the square root of which is $\frac{1}{12}$.

$$\text{Hence,}\quad x=\frac{37}{12}\pm\frac{1}{12},\quad\text{or}\quad \begin{cases} x=\frac{37}{12}+\frac{1}{12}=\frac{38}{12}=\frac{19}{6}.\\ x=\frac{37}{12}-\frac{1}{12}=\frac{36}{12}=3.\end{cases}$$

This example is remarkable, as both of the values are positive, and answer directly to the enunciation of the question, of which the proposed equation is the algebraic translation.

Let us now take the literal equation

$$4a^2-2x^2+2ax=18ab-18b^2.$$

By transposing, changing the signs, and dividing by 2, it becomes

$$x^2-ax=2a^2-9ab+9b^2;$$

whence, completing the square,

$$x^2-ax+\frac{a^2}{4}=\frac{9a^2}{4}-9ab+9b^2.$$

extracting the square root,

$$x=\frac{a}{2}\pm\sqrt{\frac{9a^2}{4}-9ab+9b^2}.$$

Now, the square root of $\frac{9a^2}{4}-9ab+9b^2$, is evidently, $\frac{3a}{2}-3b$.

Therefore,

$$x=\frac{a}{2}\pm\left(\frac{3a}{2}-3b\right),\quad\text{or}\quad\begin{cases} x=2a-3b,\\ x=-\,a+3b.\end{cases}$$

These two values will be positive at the same time, if $2a>3b$, and $3b>a$, that is if the numerical value of b is greater than $\frac{a}{3}$ and less than $\frac{2a}{3}$.

EXAMPLES.

$$x^2-7x+10=0 \; . \; . \; . \; . \text{ values } \left\{\begin{matrix} x=2 \\ x=5 \end{matrix}\right\},$$

$$\frac{1}{3}x-4-x^2+2x-\frac{4}{5}x^2=45-3x^2+4x \left\{\begin{matrix} x=7{,}12 \\ x=-5{,}73 \end{matrix}\right\} \text{ to within } 0{,}01.$$

3. Given $x^2-8x+10=19$, to find x. *Ans.* $x=9$.
4. Given $x^2-x-40=170$, to find x. *Ans.* $x=15$.
5. Given $3x^2+2x-9=76$, to find x. *Ans.* $x=5$.
6. Given $\frac{1}{2}x^2-\frac{1}{3}x+7\frac{3}{8}=8$, to find x. *Ans.* $x=1\frac{1}{2}$.
7. Given $a^2+b^2-2bx+x^2=\dfrac{m^2x^2}{n^2}$ to find x.

$$\textit{Ans.}\quad x=\frac{n}{n^2-m^2}\left(bn\pm\sqrt{a^2m^2+b^2m^2-a^2n^2}\right).$$

QUESTIONS.

1. Find a number such, that twice its square, increased by three times this number, shall give 65.

Let x be the unknown number, the equation of the problem will be

$$2x^2+3x=65,$$

whence,

$$x=-\frac{3}{4}\pm\sqrt{\frac{65}{2}+\frac{9}{16}}=-\frac{3}{4}\pm\frac{23}{4}.$$

Therefore,

$$x=-\frac{3}{4}+\frac{23}{4}=5, \text{ and } x=-\frac{3}{4}-\frac{23}{4}=-\frac{13}{2}.$$

Both these values satisfy the question in its algebraic sense.

For, $\quad 2\times(5)^2+3\times5=2\times25+15=65.$

and $\quad 2\left(-\frac{13}{2}\right)^2+3\times-\frac{13}{2}=\frac{169}{2}-\frac{39}{2}=\frac{130}{2}=65.$

But, if we wish to restrict the enunciation to its arithmetical sense, we will first observe, that when x is replaced by $-x$, in the equation $2x^2+3x=65$, the sign of the second term $3x$ only, is changed, because $(-x)^2=x^2$.

Therefore, instead of obtaining $x=-\frac{3}{4}\pm\frac{23}{4}$, we would find $x=\frac{3}{4}\pm\frac{23}{4}$, or $x=\frac{13}{2}$ and $x=-5$, values which only differ from the preceding by their signs. Hence, we may say that the negative solution $-\frac{13}{2}$, considered independently of its sign, satisfies this new enunciation, viz. : *To find a number such, that twice its square, diminished by three times this number, shall give* 65. In fact, we have

$$2\times\left(\frac{13}{2}\right)^2-3\times\frac{13}{2}=\frac{169}{2}-\frac{39}{2}=65.$$

2. A certain person purchased a number of yards of cloth for 240 cents. If he had received 3 yards less of the same cloth, for the same sum, it would have cost him 4 cents more per yard. How many yards did he purchase?

Let $x=$ the number of yards purchased.

Then $\frac{240}{x}$ will express the price per yard.

If, for 240 cents, he had received 3 yards less, that is $x-3$ yards, the price per yard, in this hypothesis, would have been represented by $\frac{240}{x-3}$. But, by the enunciation this last cost would exceed the first, by 4 cents. Therefore, we have the equation

$$\frac{240}{x-3}-\frac{240}{x}=4;$$

whence, by reducing $\qquad x^2-3x=180,$

$$x=\frac{3}{2}\pm\sqrt{\frac{9}{4}+180}=\frac{3\pm27}{2};$$

therefore

$$x=15, \text{ and } x=-12.$$

The value $x=15$ satisfies the enunciation; for, 15 yards for 240 cents, gives $\frac{240}{15}$, or 16 cents for the price of one yard, and 12 yards for 240 cents, gives 20 cents for the price of one yard, which exceeds 16 by 4.

As to the second solution, we can form a new enunciation, with which it will agree. For, go back to the equation, and change x into $-x$, it becomes,

$$\frac{240}{-x-3}-\frac{240}{-x}=4, \text{ or } \frac{240}{x}-\frac{240}{x+3}=4,$$

an equation which may be considered the algebraic translation of this problem, viz.: *A certain person purchased a number of yards of cloth for* 240 *cents: if he had paid the same sum for* 3 *yards more, it would have cost him* 4 *cents less per yard. How many yards did he purchase?* *Ans.* $x=12$, and $x=-15$.

REMARK. Hence the principles of (Arts. 104 and 105.) are confirmed for two problems of the second degree, as they were for all problems of the first degree.

3. A merchant discounted two notes, one of \$8776, payable in nine months, the other of \$7488, payable in eight months. He paid \$1200 more for the first than the second. At what rate of interest did he discount them?

To simplify the operation, denote the interest of \$100 for one month by x, or the annual interest by $12x$; $9x$ and $8x$ are the interests for 9 and 8 months. Hence $100+9x$, and $100+8x$, represent what the capital of \$100 will be at the end of 9 and 8 months. Therefore, to determine the *present values* of the notes for \$8776, and \$7488, make the two proportions,

$$100+9x : 100 :: 8776 : \frac{877600}{100+9x},$$

$$100+8x : 100 :: 7488 : \frac{748800}{100+8x};$$

and the fourth terms of these proportions will express what the merchant paid for each note. Hence, we have the equation

$$\frac{877600}{100+9x}-\frac{748800}{100+8x}=1200;$$

or, observing that the two members are divisible by 400,

$$\frac{2194}{100+9x}-\frac{1872}{100+8x}=3.$$

Clearing the fraction, and reducing, it becomes,

$$216x^2+4396x=2200;$$

whence

$$x=-\frac{2198}{216}\pm\sqrt{\frac{2200}{216}+\frac{(2198)^2}{(216)^2}}.$$

Reducing the two terms under the radical to the same denominator,

$$x=\frac{-2198\pm\sqrt{5306404}}{216},$$

or multiplying by 12,

$$12x=\frac{-2198\pm\sqrt{5306404}}{18}.$$

To obtain the value of $12x$ to within 0,01, we have only to extract the square root of 5306404 to within 0,1, since it is afterwards to be divided by 18.

This root is 2303,5 ; hence

$$12x=\frac{-2198\pm 2303{,}5}{18};$$

and consequently,

$$12x=\frac{105{,}5}{18}=5{,}86,$$

and

$$12x=\frac{-4501{,}5}{18}=-250{,}08.$$

The positive value, $12x=5{,}86$, therefore represents the rate of interest sought.

As to the negative solution, it can only be regarded as connected with the first by an equation of the second degree. By going back to the equation, and changing x into $-x$, we could with some trouble, translate the new equation into an enunciation analogous to that of the proposed problem.

4. A man bought a horse, which he sold after some time for 24 dollars. At this sale, he loses as much per cent. upon the price of his purchase, as the horse cost him. What did he pay for the horse?

Let x denote the number of dollars that he paid for the horse, $x-24$ will express the loss he sustained. But as he lost x per cent. by the sale, he must have lost $\frac{x}{100}$ upon each dollar, and upon x dollars he loses a sum denoted by $\frac{x^2}{100}$; we have then the equation

$$\frac{x^2}{100}=x-24, \text{ whence } x^2-100x=-2400.$$

and $$x=50\pm\sqrt{2500-2400}=50\pm10.$$

Therefore,

$$=60 \text{ and } x=40.$$

Both of these values satisfy the question.

For, in the first place, suppose the man gave \$60 for the horse and sold him for 24, he loses 36. Again, from the enunciation, he should lose 60 *per cent.* of 60, that is, $\frac{60}{100}$ of 60, or $\frac{60\times60}{100}$, which reduces to 36; therefore 60 satisfies the enunciation.

If he paid \$40 for the horse, he loses 16 by the sale; for, he should lose 40 *per cent.* of 40, or $40\times\frac{40}{100}$, which reduces to 16; therefore 40 verifies the enunciation.

5. A grazier bought as many sheep as cost him £60, and after

reserving fifteen out of the number, he sold the remainder for £54, and gained 2*s* a head on those he sold: how many did he buy?

Ans. 75.

6. A merchant bought cloth for which he paid £33 15*s*, which he sold again at £2 8*s* per piece, and gained by the bargain as much as one piece cost him: how many pieces did he buy?

Ans. 15.

7. What number is that, which, being divided by the product of its digits, the quotient is 3; and if 18 be added to it, the digits will be inverted? *Ans.* 24.

8. To find a number such that if you subtract it from 10, and multiply the remainder by the number itself, the product shall be 21.

Ans. 7 *or* 3.

9. Two persons, A and B, departed from different places at the same time, and travelled towards each other. On meeting, it appeared that A had travelled 18 miles more than B; and that A could have gone B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. How far did each travel?

$x = y + 18$

$x \div \frac{64y}{63} = y \div \frac{x}{28}$

Ans. $\left\{ \begin{array}{l} \text{A 72 } \textit{miles.} \\ \text{B 54 } \textit{miles.} \end{array} \right.$

Discussion of the General Equation of the Second Degree.

141. As yet we have only resolved problems of the second degree, in which the known quantities were expressed by particular numbers. To be able to resolve general problems, and interpret all of the results obtained, by attributing particular values to the given quantities, it is necessary to resume the general equation of the second degree, and to examine the circumstances which result from every possible hypothesis made upon its co-efficients. This is the object of *the discussion of the equation of the second degree.*

142. *A root* of an equation of the second degree, is such a number as being substituted for the unknown quantity, will satisfy the equation.

It has been shown (Art. 138), that every equation of the second degree can be reduced to the form

$$x^2+px=q \quad . \; . \; . \; . \; (1),$$

p and q being numerical or algebraic quantities, whole numbers or fractions, and their signs plus or minus.

If, in order to render the first member a perfect square, we add $\frac{p^2}{4}$ to both members, the equation becomes

$$x^2+px+\frac{p^2}{4}=q+\frac{p^2}{4}$$

$$\text{or} \quad \left(x+\frac{p}{2}\right)^2=q+\frac{p^2}{4}.$$

Whatever may be the value of the number expressed by $q+\frac{p^2}{4}$, its root can be denoted by m, and the equation becomes

$$\left(x+\frac{p}{2}\right)^2=m^2, \quad \text{or} \quad \left(x+\frac{p}{2}\right)^2-m^2=0.$$

But as the first member of this equation is the difference between two squares, it can be put under the form

$$\left(x+\frac{p}{2}-m\right)\cdot\left(x+\frac{p}{2}+m\right)=0\,; \quad . \; . \; . \; (2).$$

in which the first member is the product of two factors, and the second is 0. Now we can render the product equal to 0, and consequently satisfy the equation (2), in two different ways: viz.

By supposing $\quad x+\frac{p}{2}-m=0, \quad$ whence $\quad x=-\frac{p}{2}+m.$

or supposing $\quad x+\frac{p}{2}+m=0, \quad$ whence $\quad x=-\frac{p}{2}-m.$

Or substituting for m its value, $\left\{\begin{array}{l} x=-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}, \\ x=-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}, \end{array}\right.$

Now, either of these values, being substituted for x in its corresponding factor of equation (2) will satisfy that equation; and as equation (1) will always be satisfied when the derived equation (2) is satisfied, it follows, that either value will satisfy equation (1). Hence we conclude,

1st. *That every equation of the second degree has two roots, and only two.*

2d. *That every equation of the second degree may be decomposed into two binomial factors of the first degree with respect to* x, *having* x *for a common term, and the two roots, taken with their signs changed, for the second terms.*

For example, the equation $x^2+3x-28=0$ being resolved gives $x=4$ and $x=-7$; either of which values will satisfy the equation. We also have

$$(x-4)\ (x+7)\ =x^2+3x-28.$$

143. If we designate the two roots by x' and x'', we have

$$x'=-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}} \quad \text{and} \quad x''=-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}},$$

by adding the roots we obtain

$$x'+x''=-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}=-p;$$

and by multiplying them together, we have

$$x'x''=\left(-\frac{p}{2}+\sqrt{q^2+\frac{p^2}{4}}\right)\ \left(-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}\right)$$

$$=\frac{p^2}{4}-\left(q+\frac{p^2}{4}\right)=-q.$$

Hence, 1st. *The algebraic sum of the two roots is equal to the coefficient of the second term of the equation, taken with a contrary sign.* 2d. *The product of the two roots is equal to the second member of the equation, taken also with a contrary sign.*

REMARK. The preceding properties suppose that the equation has been reduced to the form $x^2+px=q$; that is, 1st. That every term of the equation has been divided by the co-efficient of x^2. 2d. That all the terms involving x have been transposed and arranged in the first member, and x^2 made positive.

144. There are four forms, under which the equation of the second degree may be written.

$$x^2+px=\quad q \qquad (1)$$
$$x^2-px=\quad q \qquad (2)$$
$$x^2+px=-q \qquad (3)$$
$$x^2-px=-q \qquad (4).$$

In which we suppose p and q to be positive.

These equations being resolved, give,

$$x=-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}} \qquad (1)$$

$$x=+\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}} \qquad (2)$$

$$x=-\frac{p}{2}\pm\sqrt{-q+\frac{p^2}{4}} \qquad (3)$$

$$x=+\frac{p}{2}\pm\sqrt{-q+\frac{p^2}{4}} \qquad (4).$$

In order that the value of x, in these equations, may be found, either exactly or approximatively, it is necessary that the quantity under the radical sign be positive (Art. 126).

Now, $\frac{p^2}{4}$ being necessarily positive, whatever may be the sign of p, it follows, that in the *first* and *second* forms all the values of x will be real. They will be determined exactly, when the quantity $q+\frac{p^2}{4}$ is a perfect square, and approximatively when it is not so.

In the first form, the *first* value of x, that is, the one arising from taking the plus value of the radical, is always positive; for the radical $\sqrt{q+\frac{p^2}{4}}$, being numerically greater than $\frac{p}{2}$, the expression $-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}$ is necessarily of the same sign as that of the radical. For the same reason, the second value is essentially negative, since it must have the same sign as that with which the radical is affected: but each root, taken with its proper sign, will satisfy the equation. The positive value will, in general, alone satisfy the problem understood in its arithmetical sense; the negative value, answering to a similar problem, differing from the first only in this; that a certain quantity which is regarded as additive in the one, is subtractive in the other, and the reverse.

In the second form, the first value of x is also positive, and the second negative, the positive value being the greater.

In the third and fourth forms, the values of x will be imaginary when

$$q>\frac{p^2}{4}, \quad \text{and } \textit{real} \text{ when} \quad q<\frac{p^2}{4}.$$

And since $\sqrt{-q+\frac{p^2}{4}}$ is less than $\frac{p}{2}$, it follows that the real values of x will both be negative in the third form, and both positive in the fourth.

145. The same general consequences which have just been remarked, would follow from the two properties of an equation of the second degree demonstrated in (Art. 143). The properties are:

The algebraic sum of the roots is equal to the co-efficient of the second term, taken with a contrary sign, and their product is equal to the second member, taken also with a contrary sign.

For, in the first two forms, q being positive in the second member, it follows that the product of the two roots is negative: hence, *they have contrary signs.* But in the third and fourth forms q being negative in the second member, it follows that the product of the two roots will be positive: hence, *they will have like signs*, viz. both *negative* in the third form, where p is positive, and both positive in the fourth form where p is negative.

Moreover, since the sum of the roots is affected with a sign contrary to that of the co-efficient p; it follows, that, *the negative root will be the greatest in the first form, and the least in the second.*

146. We will now show that, when in the third and fourth forms, we have $q>\frac{p^2}{4}$, the conditions of the question will be incompatible with each other, and therefore, the values of x ought to be imaginary.

Before showing this it will be necessary to establish a proposition on which it depends: viz.

If a given number be decomposed into two parts and those parts multiplied together, the product will be the greatest possible when the parts are equal.

Let p be the number to be decomposed, and d the difference of the parts. Then

$$\frac{p}{2}+\frac{d}{2}= \quad \text{the greater part (Art. 32).}$$

and $$\frac{p}{2}-\frac{d}{2}= \quad \text{the less part.}$$

and $$\frac{p^2}{4}-\frac{d^2}{4}=\text{P}, \quad \text{their product (Art. 46).}$$

Now it is plain that P will increase as d diminishes, and that it will be the greatest possible when $d=0$: that is,

$$\frac{p}{2}\times\frac{p}{2}=\frac{p^2}{4} \quad \text{is the greatest product.}$$

147. Now, since in the equation

$$x^2-px=-q$$

p is the sum of the roots, and q their product, it follows that q can never be greater than $\frac{p^2}{4}$. The conditions of the equation therefore fix a limit to the value of q, and if we make $q>\frac{p^2}{4}$, we express by the equation a condition which cannot be fulfilled, and, this contradiction is made apparent by the values of x becoming imaginary. Hence we may conclude that,

The value of the unknown quantity will always be imaginary when the conditions of the question are incompatible with each other.

REMARK. Since the roots of the equation, in the first and second forms, have contrary signs, the condition that their sum shall be equal to a given number p, does not fix a limit to their product: hence, in those two forms the roots are never imaginary.

148. We will conclude this discussion by the following remarks.

1st. If in the third and fourth forms, we suppose $q=\frac{p^2}{4}$, the radical part of the two values of x becomes 0, and both of these values reduce to $x=-\frac{p}{2}$: *the two roots are then said to be equal.*

In fact, by substituting $\frac{p^2}{4}$ for q in the equation, it becomes

$$x^2+px=-\frac{p^2}{4}, \text{ whence}$$

$$x^2+px+\frac{p^2}{4}=0, \text{ or } \left(x+\frac{p}{2}\right)^2=0.$$

In this case, the first member is the *product of two equal factors.* Hence we may also say, that the roots of the equation are equal, since in this case the two factors being placed equal to zero, give the same value for x.

2d. If, in the general equation, $x^2+px=q$, we suppose $q=0$, the two values of x reduce to $x=-\frac{p}{2}+\frac{p}{2}$, or $x=0$, and to

$$x=-\frac{p}{2}-\frac{p}{2}, \text{ or } x=-p.$$

In fact, the equation is then of the form $x^2+px=0$, or $x(x+p)=0$, which can be satisfied either by supposing $x=0$, or $x+p=0$, whence $x=-p$: that is, one of the roots is 0, and the other the co-efficient of x taken with a contrary sign.

3d. If in the general equation $x^2+px=q$, we suppose $p=0$, there will result $x^2=q$, whence $x=\pm\sqrt{q}$; that is, in this case *the two values of* x *are equal, and have contrary signs, real in the first and second forms, and imaginary in the third and fourth.*

The equation then belongs to the class of equations involving two terms, treated of in (Art. 139).

4th. Suppose we have at the same time $p=0$, $q=0$; the equation reduces to $x^2=0$, and gives two values of x, equal to 0.

149. There remains a singular case to be examined, which is often met with in the resolution of problems of the second degree.

To discuss it, take the equation $ax^2+bx=c$. This equation gives

$$x=\frac{-b\pm\sqrt{b^2+4ac}}{2a}.$$

Suppose now, that from a particular hypothesis made upon the given quantities of the question, we have $a=0$; the expression for x becomes

$$x=\frac{-b\pm b}{0}, \text{ whence } \begin{cases} x=\dfrac{0}{0}, \\ x=-\dfrac{2b}{0}. \end{cases}$$

The second value is presented under the form of infinity, and may be considered as an answer when the proposed questions will admit of answers in infinite numbers.

As to the first $\frac{0}{0}$, we must endeavour to interpret it.

By multiplying the numerator and denominator of the 2d member of the equation

$$x=\frac{-b+\sqrt{b^2+4ac}}{2a} \quad \text{by} \quad \frac{-b-\sqrt{b^2+4ac}}{2a}$$

we obtain

$$x=\frac{b^2-(b^2+4ac)}{2a(-b-\sqrt{b^2+4ac}}=\frac{-4ac}{2a(-b-\sqrt{b^2+4ac}},$$

or $$x=\frac{-2c}{-b-\sqrt{b^2+4ac}} \quad \text{by dividing by } 2a,$$

or $$x=\frac{c}{b} \quad \text{by making } a=0.$$

Hence we see that the indetermination arises from a common factor in the numerator and denominator.

If we had at the same time $a=0$, $b=0$, $c=0$, the proposed equation would be altogether indeterminate.

This is the only case of indetermination that the equation of the second degree presents.

We are now going to apply the principles of this general discussion to a problem which will give rise to most of the circumstances which are commonly met with in problems of the second degree.

Problem of the Lights.

150. Find upon the line which joins two lights, A and B, of different intensities, the point which is equally illuminated; admitting the following principle of physics, viz.: The intensity of the same light at two different distances, is in the inverse ratio of the squares of these distances.

Let the distance AB between the two lights be expressed by a; the intensity of the light A, at the units distance, by b; that of the light B, at the same distance, by c. Let C be the required point, and make $AC=x$, whence $BC=a-x$.

From the principle of physics, the intensity of A, at the *unity* of distance, being b, its intensity at the distances 2, 3, 4, &c., is $\frac{b}{4}$, $\frac{b}{9}$, $\frac{b}{16}$, &c., hence at the distance x it will be expressed by

$\frac{b}{x^2}$. In like manner, the intensity of B at the distance $a-x$, is $\frac{c}{(a-x)^2}$; but, by the enunciation, these two intensities are equal to each other, therefore we have the equation

$$\frac{b}{x^2}=\frac{c}{(a-x)^2}.$$

Whence, by developing and reducing,

$$(b-c)x^2-2abx=-a^2b.$$

This equation gives

$$x=\frac{ab}{b-c}\pm\sqrt{\frac{a^2b^2}{(b-c)^2}-\frac{a^2b}{b-c}},$$

or reducing,

$$x=\frac{a(b\pm\sqrt{bc})}{b-c}.$$

This expression may be simplified by observing, 1st. that $b\pm\sqrt{bc}$ can be put under the form $\sqrt{b}.\sqrt{b}\pm\sqrt{b}.\sqrt{c}$, or $\sqrt{b}(\sqrt{b}\pm\sqrt{c})$; 2d. that $b-c=(\sqrt{b})^2-(\sqrt{c})^2=(\sqrt{b}+\sqrt{c}).(\sqrt{b}-\sqrt{c}.)$ Therefore, by first considering the superior sign of the above expression, we have

$$x=\frac{a\sqrt{b}(\sqrt{b}+\sqrt{c})}{(\sqrt{b}+\sqrt{c}).(\sqrt{b}-\sqrt{c})}=\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}.$$

In like manner we obtain for the second value,

$$x=\frac{a\sqrt{b}(\sqrt{b}-\sqrt{c})}{(\sqrt{b}+\sqrt{c}).(\sqrt{b}-\sqrt{c})}=\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}.$$

Hence, we have

$$\left.\begin{aligned}&\text{1st} \ldots x=\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}},\\ &\text{2d} \ldots x=\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}},\end{aligned}\right\}\ \text{from which we obtain}\ \left\{\begin{aligned}&a-x=\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}},\\ &a-x=\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}},\end{aligned}\right.$$

1st. Suppose that $b>c$.

The first value of x, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is then positive and less than a, because $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is a proper fraction; thus this value gives for the required point, a point C, situated between the points A and B. We see moreover, that the point is nearer to B than A; for since $b>c$, we have $\sqrt{b}+\sqrt{b}$ or $2\sqrt{b}>(\sqrt{b}+\sqrt{c})$; whence $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{1}{2}$ and consequently, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{a}{2}$. In fact this ought to be the case, since the intensity of A was supposed to be greater than that of B.

The corresponding value of $a-x$, $\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$ is also positive, and less than $\frac{a}{2}$, as may easily be shown.

The second value of x, $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, is also positive, but greater than a; because $\frac{\sqrt{b}}{\sqrt{b}-\sqrt{c}}>1$. Hence this second value gives a second point C', situated upon the prolongation of AB, and to the right of the two lights. We may in fact conceive that the two lights, exerting their influence in every direction, should have upon the prolongation of AB, another point equally illuminated; but this point must be nearest that light whose intensity is the least.

We can easily explain, why these two values are connected by the same equation. If, instead of taking AC for the unknown quantity x, we had taken AC', there would have resulted $BC'=x-a$; and the equation $\frac{b}{x^2}=\frac{c}{(x-a)^2}$. Now, as $(x-a)^2$ is identical with $(a-x)^2$, the new equation is the same as that already established, which consequently should have given AC' as well as AC.

And since every equation is but the algebraic enunciation of a problem, it follows that, *when the same equation enunciates several problems, it ought by its different roots to solve them all.*

When the unknown quantity x represents the line AC', the second value of $a-x$, $\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, is negative, as it should be, since we have $x>a$; but by changing the signs in the equation $a-x=\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$ it becomes $x-a=\frac{a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$; and this value of $x-a$ represents the positive value of BC'.

2d. Let $b<c$.

The first value of x, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is always positive, but less than $\frac{a}{2}$, since we have

$$(\sqrt{b}+\sqrt{c})>(\sqrt{b}+\sqrt{b}) \text{ or than } 2\sqrt{b}.$$

The corresponding value of $a-x$, or $\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$ is positive, and greater than $\frac{a}{2}$.

Therefore in this hypothesis, the point C, situated between A and B, must be nearer A than B.

The second value of x, $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$ or $\frac{-a\sqrt{b}}{\sqrt{c}-\sqrt{b}}$, is essentially negative. To interpret it, let us take for the unknown quantity the distance AC'', and let us represent this distance by x, and at the same time consider, as we have a right to do, x as essentially negative. Then the general expression for BC'' being $a-x$, if we regard x as essentially negative, the true numerical value of $a-x$ is expressed by $a+x$. Hence as before, the equation or algebraic expression will be

$$\frac{b}{x^2}=\frac{c}{(a-x)^2} \text{ or } \frac{b}{x^2}=\frac{c}{(a+x)^2}$$

in the first of which equations x is essentially negative.

This equation ought to give a negative value for x, and a positive value for $BC''=a+x$. Indeeed, since the intensity of the light B is greater than that of A, the second required point ought to be nearer A than B. The algebraic value for BC'', which is

$a-x$, or $\dfrac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$ or $\dfrac{a\sqrt{c}}{\sqrt{c}-\sqrt{b}}$ is positive.

3d. Let $b=c$.

The first two values of x and $a-x$ reduce to $\frac{a}{2}$, which gives the middle of AB for the first required point. This result agrees with the hypothesis.

The two other values reduce to $\frac{\pm a\sqrt{b}}{0}$, or *infinity;* that is, the second required point is situated at a distance from the two points A and B, greater than any assignable quantity. This result agrees perfectly with the present hypothesis, because, by supposing the difference $b-c$ to be extremely small, without being absolutely nothing, the second point must be at a very great distance from the lights; this is indicated by the expression $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, the denominator of which is extremely small with respect to the numerator. And if we finally suppose $b=c$, or $\sqrt{b}-\sqrt{c}=0$, the required point cannot exist for a finite distance, or is situated at an *infinite* distance.

We will observe, that in the case of $b=c$, if we should consider the values before they were simplified, viz.

$$x=\frac{a(b+\sqrt{bc})}{b-c}, \text{ and } x=\frac{a(b-\sqrt{bc})}{b-c},$$

the first, which corresponds to $x=\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, would become $\frac{2ab}{0}$, and the second, which corresponds to $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$, would become $\frac{0}{0}$. But $\frac{0}{0}$ would be obtained in consequence of the existence of a common factor, $\sqrt{b}-\sqrt{c}$, between the two terms of the value of x (see Art. 113).

Let $b=c$, *and* $a=0$.

The first system of values for x and $a-x$, reduces to 0, and the second to $\frac{0}{0}$. This last symbol is that of *indetermination*; for, resuming the equation of the problem, $(b-c)x^2-2abx=-a^2b$, it reduces, in the present hypothesis to $0.x^2-0.x=0$, which may be satisfied by giving x any value whatever. In fact, since the two lights have the same intensity, and are placed at the same point, *they ought to illuminate equally each point of the line A B.*

The solution 0, given by the first system, is one of those solutions *in infinite numbers*, of which we have spoken.

Finally, suppose $a=0$, *and* b *and* c, *unequal*

Each of the two systems reduces to 0, which proves that there is but one point in this case equally illuminated, and *that is the point in which the two lights are placed.*

In this case, the equation reduces to $(b-c)x^2=0$, and gives the two equal values, $x=0$, $x=0$.

The preceding discussion presents another example of the precision with which algebra responds to all the circumstances of the enunciation of a problem.

Of Equations of the Second Degree, involving two or more unknown quantities.

151. A complete theory of this subject cannot be given here, because the resolution of two equations of the second degree involving two unknown quantities, in general depends upon the solution of an equation of the fourth degree involving one unknown quantity; but we will propose some questions, which depend only upon the solution of an equation of the second degree involving one unknown quantity.

1. Find two numbers such that the sum of their products by the respective numbers a and b, may be equal to $2s$, and that their product may be equal to p.

Let x and y be the required numbers, we have the equations,

$$ax+by=2s.$$

$$xy=p.$$

From the first $y=\frac{2s-ax}{b}$; whence, by substituting in the second, and reducing,

$$ax^2-2sx=-bp.$$

Therefore,

$$x=\frac{s}{a}\pm\frac{1}{a}\sqrt{s^2-abp},$$

and consequently,

$$y=\frac{s}{b}\mp\frac{1}{b}\sqrt{s^2-abp}.$$

This problem is susceptible of two direct solutions, because s is evidently $>\sqrt{s^2-abp}$, but in order that they may be real, it is necessary that $s^2>$ or $=abp$.

Let $a=b=1$; the values of x, and y, reduce to

$$x=s\pm\sqrt{s^2-p} \quad \text{and} \quad y=s\mp\sqrt{s^2-p}$$

Whence we see, that the two values of x are equal to those of y, taken in an inverse order; which shows, that if $s+\sqrt{s^2-p}$ represents the value of x, $s-\sqrt{s^2-p}$ will represent the corresponding value of y, and reciprocally.

This circumstance is accounted for, by observing, that in this particular case the equations reduce to $\left\{\begin{array}{c} x+y=2s, \\ xy=p; \end{array}\right.$ and then the question is reduced to, *finding two numbers of which the sum is* $2s$, *and their product* p, or in other words, *to divide a number* $2s$, *into two such parts, that their product may be equal to a given number* p.

2. Find four numbers in proportion, knowing the sum $2s$ of their extremes, the sum $2s'$ of the means, and the sum $4c^2$ of their squares.

Let u, x, y, z, denote the four terms of the proportion; the equations of the problem will be

$$u+z=2s$$
$$x+y=2s'$$
$$uz=xy$$
$$u^2+x^2+y^2+z^2=4c^2.$$

At first sight, it may appear difficult to find the values of the unknown quantities, but with the aid of an *unknown auxiliary* they are easily determined.

Let p be the unknown product of the extremes or means, we have

1st. The equations

$$\begin{cases} u+z=2s, \\ uz=p, \end{cases} \quad \text{which give} \quad \begin{cases} u=s+\sqrt{s^2-p}, \\ z=s-\sqrt{s^2-p}. \end{cases}$$

2d. The equations

$$\begin{cases} x+y=2s', \\ xy=p, \end{cases} \quad \text{which give} \quad \begin{cases} x=s'+\sqrt{s'^2-p}, \\ y=s'-\sqrt{s'^2-p}. \end{cases}$$

Hence, we see that the determination of the four unknown quantities depends only upon that of the product p.

Now, by substituting these values of u, x, y, z in the last of the equations of the problem, it becomes

$$\left(s+\sqrt{s^2-p}\right)^2+\left(s-\sqrt{s^2-p}\right)^2+\left(s'+\sqrt{s'^2-p}\right)^2$$
$$+\left(s'-\sqrt{s'^2-p}\right)^2=4c^2\,;$$

or, developing and reducing,

$$4s^2+4s'^2-4p=4c\,; \quad \text{hence} \quad p=s^2+s'^2-c^2.$$

Substituting this value for p, in the expressions for u, x, y, z, we find

$$\begin{cases} u=s+\sqrt{c^2-s'^2}, \\ z=s-\sqrt{c^2-s'^2}, \end{cases} \qquad \begin{cases} x=s'+\sqrt{c^2-s^2}, \\ y=s'-\sqrt{c^2-s^2}. \end{cases}$$

These four numbers evidently form a proportion; for we have

$$uz=\left(s+\sqrt{c^2-s'^2}\right)\left(s-\sqrt{c^2-s'^2}\right)=s^2-c^2+s'^2,$$
$$xy=\left(s'+\sqrt{c^2-s^2}\right)\left(s'-\sqrt{c^2-s^2}\right)=s'^2-c^2+s^2.$$

This problem shows how much the introduction of an *unknown auxiliary* facilitates the determination of the principal unknown quantities. There are other problems of the same kind, which lead to equations of a degree superior to the second, and yet they may be resolved by the aid of equations of the first and second degrees, by introducing *unknown auxiliaries.*

152. We will now consider the case in which a problem leads to two equations of the second degree, involving two unknown quantities.

An equation involving two unknown quantities is said to be of the *second degree*, when it contains a term in which *the sum of the exponents of the two unknown quantities is equal to* 2. Thus,

$$3x^2-4x+y^2-xy-5y+6=0, \quad 7xy-4x+y=0,$$

are equations of the second degree.

Hence, every general equation of the second degree, involving two unknown quantities, is of the form

$$ay^2+bxy+cx^2+dy+fx+g=0,$$

a, b, c, . . . representing known quantities, either numerical or algebraic.

Take the two equations

$$a y^2+b\,xy+c\,x^2+d\,y+f\,x+g=0,$$
$$a'y^2+b'xy+c'x^2+d'y+f'x+g'=0.$$

Arranging them with reference to x, they become

$$c\,x^2+(b\,y+f\,)x+a\,y^2+d\,y+g\,=0,$$
$$c'x^2+(b'y+f'\,)x+a'y^2+d'y+g'=0.$$

Now, if the co-efficients of x^2 in the two equations were the same, we could, by subtracting one equation from the other, obtain an equation of the first degree in x, which could be substituted for one of the proposed equations; from this equation, the value of x could be found in terms of y, and by substituting this value in one of the proposed equations, we would obtain an equation involving only the unknown quantity y.

By multiplying the first equation by c', and the second by c, they become

$$cc'x^2+(by+f)c'x+(ay^2+dy+g)c'=0,$$
$$cc'x^2+(b'y+f')c\,x+(a'y^2+d'y+g')c=0,$$

and these equations, in which the co-efficients of x^2 are the same, may take the place of the preceding.

Subtracting one from the other, we have

$$[(bc'-cb')y+fc'-cf']x+(ac'-ca')y^2+(dc'-cd')y+gc'-cg'=0,$$

which gives

$$x=\frac{(ca'-ac')y^2+(cd'-dc')y+cg'-gc'}{(bc'-cb')y+fc'-cf'}.$$

This expression for x, substituted in one of the proposed equations, will give *a final equation*, involving y.

But without effecting this substitution, which would lead to a very complicated result, it is easy to perceive that the equation involving y will be of the fourth degree; for the numerator of the expression for x being of the form my^2+ny+p, its square, or the expression for x^2, is of the fourth degree. Now this square forms one of the parts of the result of the substitution.

Therefore, in general, *the resolution of two equations of the second degree, involving two unknown quantities, depends upon that of an equation of the fourth degree, involving one unknown quantity.*

153. There is a class of equations of the fourth degree, that can be resolved in the same way as equations of the second degree; these are equations of the form $x^4+px^2+q=0$. They are called *trinomial equations*, because they contain but three kinds of terms; viz. terms involving x^4, those involving x^2, and terms entirely known.

In order to resolve the equation $x^4+px^2+q=0$, suppose $x^2=y$, we have

$$y^2+py+q=0, \text{ whence } y=-\frac{p}{2}\pm\sqrt{-q+\frac{p^2}{4}}.$$

But the equation $x^2=y$, gives $x=\pm\sqrt{y}$.

Hence, $$x=\pm\sqrt{-\frac{p}{2}\pm\sqrt{-q+\frac{p^2}{4}}}.$$

We perceive that the unknown quantity has four values, since each of the signs + and −, which affect the first radical, can be combined successively with each of the signs which affect the second; *but these values taken two and two are equal, and have contrary signs.*

Take for example the equation $x^4-25x^2=-144$;
by supposing $x^2=y$, it becomes $y^2-25y=-144$;
whence $y=16$, $y=9$.

Substituting these values in the equation $x^2=y$ there will result 1st. $x^2=16$, whence $x=\pm4$; 2d. $x^2=9$, whence $x=\pm3$.

Therefore the four values are $+4$, -4, $+3$ and -3.

Again, take the equation $x^4-7x^2=8$. Supposing $x^2=y$, the equation becomes $y^2-7y=8$; whence $y=8$, $y=-1$.

Therefore, 1st. $x^2=8$, whence $x=\pm2\sqrt{2}$; 2d. $x^2=-1$; whence $x=\pm\sqrt{-1}$; the two last values of x are imaginary.

Let there be the algebraic equation $x^4-(2bc+4a^2)x^2=-b^2c^2$;
taking $x^2=y$, the equation becomes $y^2-(2bc+4a^2)y=-b^2c^2$;
from which we deduce $y=bc+2a^2\pm2a\sqrt{bc+a^2}$.

And consequently $$x=\pm\sqrt{bc+2a^2\pm2a\sqrt{bc+a^2}}.$$

154. Every equation of the form $y^{2n}+py^n+q=0$, in which the exponent of the unknown quantity in one term is double that of the other, may be solved by the rules for equations of the second degree.

For, put $y^n=x$, then $y^{2n}=x^2$, and $y^{2n}+py^n+q=x^2+px+q=0$.

Hence $$x=-\frac{p}{2}\pm\sqrt{-q+\frac{p^2}{4}},$$

Or $$y^n=-\frac{p}{2}\pm\sqrt{-q+\frac{p^2}{4}}.$$

And $$y=\sqrt[n]{-\frac{p}{2}\pm\sqrt{-q+\frac{p^2}{4}}}.$$

Extraction of the Square Root of Binomials of the form $a \pm \sqrt{b}$.

155. The resolution of *trinomial equations of the fourth degree*, gives rise to a new species of algebraic operation: viz. the extraction of the square root of a quantity of the form $a \pm \sqrt{b}$, a and b being numerical or algebraic quantities.

By squaring the expression $3 \pm \sqrt{5}$, we have

$$(3 \pm \sqrt{5})^2 = 9 \pm 6\sqrt{5} + 5 = 14 \pm 6\sqrt{5}:$$

hence, *reciprocally* $\sqrt{14 \pm 6\sqrt{5}} = 3 \pm \sqrt{5}$.

In like manner, $(\sqrt{7} \pm \sqrt{11})^2 = 7 \pm 2\sqrt{7} \times \sqrt{11} + 11$
$= 18 \pm 2\sqrt{77}$.

Hence reciprocally $\sqrt{18 \pm 2\sqrt{77}} = \sqrt{7} \pm \sqrt{11}$.

Whence we see that an expression of the form $\sqrt{a \pm \sqrt{b}}$, may sometimes be reduced to the form $a' \pm \sqrt{b'}$ or $\sqrt{a'} \pm \sqrt{b'}$; and when this transformation is possible, it is advantageous to effect it, since in this case we have only to extract two simple square roots, whereas the expression $\sqrt{a \pm \sqrt{b}}$ requires the extraction of the square root of the square root.

156. If we let p and q denote two indeterminate quantities, we can always attribute to them such values as to satisfy the equations

$$\sqrt{a + \sqrt{b}} = p + q \quad \ldots\ldots \quad (1).$$
$$\sqrt{a - \sqrt{b}} = p - q \quad \ldots\ldots \quad (2).$$

These equations, being multiplied together, give

$$\sqrt{a^2 - b} = p^2 - q^2 \quad \ldots\ldots \quad (3).$$

Now, if p and q are irrational monomials involving only single radicals of the second degree, or if one is rational and the other irrational, it follows that p^2 and q^2 will be rational; in which case, $p^2 - q^2$, or its value, $\sqrt{a^2 - b}$, is necessarily a rational quantity, or $a^2 - b$ is a perfect square.

When this is the case, the transformation can always be effected. For, take a^2-b, a perfect square, and suppose $\sqrt{a^2-b}=c$; the equation (3) becomes

$$p^2-q^2=c.$$

Moreover, the equations (1) and (2) being squared, give

$$p^2+q^2+2pq=a+\sqrt{b},$$

$$p^2+q^2-2pq=a-\sqrt{b};$$

whence, by adding member to member,

$$p^2+q^2=a \;\ldots\ldots\; (4);$$

but

$$p^2-q^2=c \;\ldots\ldots\; (5).$$

Hence, by adding these last equations, and subtracting the second from the first, we obtain

$$2p^2=a+c,$$

$$2q^2=a-c;$$

and consequently,

$$\left\{\begin{array}{l} p=\pm\sqrt{\dfrac{a+c}{2}}, \\ q=\pm\sqrt{\dfrac{a-c}{2}}, \end{array}\right.$$

Therefore,

$$\sqrt{a+\sqrt{b}}, \quad \text{or} \quad p+q=\pm\sqrt{\frac{a+c}{2}}\pm\sqrt{\frac{a-c}{2}},$$

$$\sqrt{a-\sqrt{b}}, \quad \text{or} \quad p-q=\pm\sqrt{\frac{a+c}{2}}\mp\sqrt{\frac{a-c}{2}};$$

or

$$\sqrt{a+\sqrt{b}}=\pm\left(\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}\right) \;\ldots\ldots\; (6),$$

$$\sqrt{a-\sqrt{b}}=\pm\left(\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}}\right) \;\ldots\ldots\; (7).$$

These two formulas can be verified; for by squaring both members of the first, it becomes

$$a+\sqrt{b}=\frac{a+c}{2}+\frac{a-c}{2}+2\sqrt{\frac{a^2-c^2}{4}}=a+\sqrt{a^2-c^2};$$

but the relation $\sqrt{a^2-b}=c$, gives $c^2=a^2-b$.

Hence, $a+\sqrt{b}=a+\sqrt{a^2-a^2+b}=a+\sqrt{b}$.

The second formula can be verified in the same manner.

157. REMARK. As the accuracy of the formulas (6) and (7) is proved, whatever may be the quantity c, or $\sqrt{a^2-b}$, it follows, that when this quantity is not a perfect square, we may still replace the expressions $\sqrt{a+\sqrt{b}}$ and $\sqrt{a-\sqrt{b}}$, by the second members of the equalities (6) and (7); but then we would not simplify the expression, since the quantities p and q would be of the same form as the proposed expression.

We would not, therefore, in general, use this transformation, unless a^2-b is a perfect square.

EXAMPLES

158. Take the numerical expression $94+42\sqrt{5}$, which reduces to $94+\sqrt{8820}$. We have

$$a=94,\ b=8820,$$

whence $$c=\sqrt{a^2-b}=\sqrt{8836-8820}=4,$$

a rational quantity; therefore the formula (6) is applicable to this case.

It becomes

$$\sqrt{94+42\sqrt{5}}=\pm\left(\sqrt{\frac{94+4}{2}}+\sqrt{\frac{94-4}{2}}\right),$$

or, reducing, $$=\pm(\sqrt{49}+\sqrt{45});$$

therefore, $$\sqrt{94+42\sqrt{5}}=\pm(7+3\sqrt{5}).$$

In fact, $$(7+3\sqrt{5})^2=49+45+42\sqrt{5}=94+42\sqrt{5}.$$

Again, take the expression

$$\sqrt{np+2m^2-2m\sqrt{np+m^2}};$$

we have $$a=np+2m^2,\quad b=4m^2(np+m^2),$$

whence $$a^2-b=n^2p^2,$$

and $$c \text{ or } \sqrt{a^2-b}=np;$$

therefore the formula (7) is applicable. It gives for the required root

$$\pm\left(\sqrt{\frac{np+2m^2+np}{2}}-\sqrt{\frac{np+2m^2-np}{2}}\right),$$

or, reducing, $\pm(\sqrt{np+m^2}-m)$.

In fact, $(\sqrt{np+m^2}-m)^2=np+2m^2-2m\sqrt{np+m^2}$.

For another example, take the expression

$$\sqrt{16+30\sqrt{-1}}+\sqrt{16-30\sqrt{-1}},$$

and reduce it to its simplest terms. By applying the preceding formulas, we find

$$\sqrt{16+30\sqrt{-1}}=5+3\sqrt{-1},\quad \sqrt{16-30\sqrt{-1}}=5-3\sqrt{-1}.$$

Hence, $\sqrt{16+30\sqrt{-1}}+\sqrt{16-30\sqrt{-1}}=10.$

This last example shows, better than any of the others, the utility of the general problem; because it proves that *imaginary expressions* combined together, may produce *real*, and even *rational results.*

$$\sqrt{28+10\sqrt{3}}=5+\sqrt{3}\,;\quad \sqrt{1+4\sqrt{-3}}=2+\sqrt{-3},$$

$$\sqrt{bc+2b\sqrt{bc-b^2}}+\sqrt{bc-2b\sqrt{bc-b^2}}=\pm 2b\,;$$

$$\sqrt{ab+4c^2-d^2+2\sqrt{4abc^2-abd^2}}=\sqrt{ab}+\sqrt{4c^2-d^2}.$$

Examples of Equations of the Second Degree, which either involve Radicals, or two unknown quantities.

1. Given $x+\sqrt{a^2+x^2}=\dfrac{2a^2}{\sqrt{a^2+x^2}}$ to find x.

$$x\sqrt{a^2+x^2}+a^2+x^2=2a^2$$

$$x\sqrt{a^2+x^2}=a^2-x^2 \qquad \text{by transposing.}$$

$$a^2x^2+x^4=a^4-2a^2x^2+x^4, \quad \text{by squaring,}$$

hence $$3a^2x^2=x^4.$$

or $$x=\pm\sqrt{\frac{a}{3}}.$$

2. Given $\sqrt{\frac{a^2}{x^2}+b^2}-\sqrt{\frac{a^2}{x^2}-b^2}=b$ to find x.

$$\sqrt{\frac{a^2}{x^2}+b^2}=\sqrt{\frac{a^2}{x^2}-b^2}+b, \quad \text{by transposing.}$$

$$\frac{a^2}{x^2}+b^2=\frac{a^2}{x^2}-b^2+2b\sqrt{\frac{a^2}{x^2}-b^2}+b^2,$$

hence . . . . . $$b^2=2b\sqrt{\frac{a^2}{x^2}-b^2},$$

or . . . . $$b=2\sqrt{\frac{a^2}{x^2}-b^2},$$

hence . . . . . $$b^2=\frac{4a^2}{x^2}-4b^2,$$

$$x^2=\frac{4a^2}{5b^2},$$

hence . . . . . . $$x=\pm\frac{2a}{b\sqrt{5}}.$$

3. Given $\frac{a}{x}+\frac{\sqrt{a^2-x^2}}{x}=\frac{x}{b}$ to find x.

Ans. $x=\pm\sqrt{2ab-b^2}$.

4. Given
$$\left.\begin{array}{l}\dfrac{\dfrac{xy}{\sqrt{x}}}{\sqrt{y}}=48 \\ \text{and} \quad \dfrac{xy}{\sqrt{x}}=24\end{array}\right\} \text{ to find } x \text{ and } y.$$

Dividing the first equation by the second, we have

$$\frac{\sqrt{x}}{\sqrt{\frac{x}{y}}}=\sqrt{y}=2, \text{ or } y=4.$$

Whence from the second equation $\dfrac{4x}{\sqrt{x}}=4\sqrt{x}=24,$

hence $\sqrt{x}=6$ and $x=36.$

5. Given $\sqrt{\dfrac{x+a}{x}}+2\sqrt{\dfrac{a}{x+a}}=b^2\sqrt{\dfrac{x}{x+a}}$ to fina x.

Ans. $x=\dfrac{a}{(b\mp 1)^2}$.

6. Given $\left.\begin{array}{l} x+\sqrt{xy}+y=19 \\ x^2+xy+y^2=133 \end{array}\right\}$ to find x and y.

Dividing the second equation by the first, we have

$$x-\sqrt{xy}+y=7$$

but . . . $x+\sqrt{xy}+y=19$

hence . . . . $2x+2y=26$ by addition,

or . . . . $x+y=13$

and . . . . $\sqrt{xy}+13=19$ by substituting in the 1st eq.

or . . . . $\sqrt{xy}=6$

and . . . . $xy=36$

From 2d equation, $x^2+xy+y^2=133$

and from the last $3xy \quad =108$

Subtracting . . $x^2-2xy+y^2=25$

Hence . . . . . $x-y=\pm 5$

But . . . . . $x+y=13$

Hence . . $x=9$ or 4; and $y=4$ or 9.

7. Given $\dfrac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b,$ to find x.

Ans. $x=\pm\dfrac{2a\sqrt{b}}{1+b}$.

8. Given $\dfrac{\sqrt{x}+\sqrt{x-a}}{\sqrt{x}-\sqrt{x-a}}=\dfrac{n^2a}{x-a}$ to find x.

Ans. $x=\dfrac{a(1\pm n)^2}{1\pm 2n}$.

9. Given $\dfrac{\sqrt{a+x}}{\sqrt{x}}+\dfrac{\sqrt{a-x}}{\sqrt{x}}=\sqrt{\dfrac{x}{b}}$ to find x.

Ans. $x=\pm 2\sqrt{ab-b^2}$.

10. Given $\left\{\begin{array}{l} x^2y+xy^2=6. \\ x^3y^2+x^2y^3=12. \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{lll} x=2 & \text{or} & 1 \\ y=1 & \text{or} & 2. \end{array}\right.$

11. Given $\left\{\begin{array}{l} (x^2+y^2)(x+y)=2336 \\ (x^2-y^2)(x-y)=576. \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{lll} x=11 & \text{or} & 5 \\ y=5 & \text{or} & 11. \end{array}\right.$

12. Given $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x}=b$, to find x.

Ans. $x=\dfrac{\pm a(1\mp\sqrt{2b-b^2})}{\sqrt{2b-b^2}}$.

13. Given $\left\{\begin{array}{r} x^2+x+y=18-y^2 \\ xy=\ 6 \end{array}\right\}$ to find x and y

Ans. $\left\{\begin{array}{lllll} x=3 & \text{or} & 2 & \text{or} & -3\pm\sqrt{3} \\ y=2 & \text{or} & 3 & \text{or} & -3\mp\sqrt{3}. \end{array}\right.$

14. Given the sum of two numbers equal to a, and the sum of their cubes equal to c, to find the numbers

By the conditions $\left\{\begin{array}{l} x+y=a \\ x^3+y^3=c. \end{array}\right.$

Putting $x=s+z$, and $y=s-z$, we have $a=2s$,

and . . . . $\left\{\begin{array}{l} x^3=s^3+3s^2z+3sz^2+z^3 \\ y^3=s^3-3s^2z+3sz^2-z^3 \end{array}\right.$

Hence, by addition, $x^3+y^3=2s^3+6sz^2=c$

Whence $z^2=\dfrac{c-2s^3}{6s}$ and $z=\pm\sqrt{\dfrac{c-2s^3}{6s}}$,

or $x=s\pm\sqrt{\dfrac{c-2s^3}{6s}}$; and $y=s\mp\sqrt{\dfrac{c-2s^3}{6s}}$,

Or by putting for s its value,

$$x=\frac{a}{2}\pm\sqrt{\left(\frac{c-\frac{a^3}{4}}{3a}\right)}=\frac{a}{2}\pm\sqrt{\frac{4c-a^3}{12a}},$$

and $$y=\frac{a}{2}\mp\sqrt{\left(\frac{c-\frac{a^3}{4}}{3a}\right)}=\frac{a}{2}\mp\sqrt{\frac{4c-a^3}{12a}}.$$

QUESTIONS.

1. There are two numbers whose difference is 15, and half their product is equal to the cube of the lesser number. What are those numbers? *Ans.* 3 and 18.

2. What two numbers are those whose sum, multiplied by the greater, is equal to 77; and whose difference, multiplied by the lesser, is equal to 12?

Ans. 4 and 7, or $\frac{3}{2}\sqrt{2}$ and $\frac{11}{2}\sqrt{2}$.

3. To divide 100 into two such parts, that the sum of their square roots may be 14. *Ans.* 64 and 36.

4. It is required to divide the number 24 into two such parts, that their product may be equal to 35 times their difference.

Ans. 10 and 14.

5. The sum of two numbers is 8, and the sum of their cubes is 152. What are the numbers? *Ans.* 3 and 5.

6. The sum of two numbers is 7, and the sum of their 4th powers is 641. What are the numbers? *Ans.* 2 and 5.

7. The sum of two numbers is 6, and the sum of their 5th powers is 1056. What are the numbers? *Ans.* 2 and 4.

8. Two merchants each sold the same kind of stuff; the second sold 3 yards more of it than the first, and together, they receive 35

Jacob N Lesler

by R. M.......

crowns. The first said to the second, I would have received 24 crowns for your stuff; the other replied, and I would have received $12\frac{1}{2}$ crowns for yours. How many yards did each of them sell?

$$Ans. \quad \left\{\begin{array}{lll} \text{1st merchant } x=15 & \text{or} & x=5 \\ \text{2d} \;\;\ldots\;\; y=18 & & y=8 \end{array}\right\}.$$

9. A widow possessed 13,000 dollars, which she divided into two parts, and placed them at interest, in such a manner, that the incomes from them were equal. If she had put out the first portion at the same rate as the second, she would have drawn for this part 360 dollars interest, and if she had placed the second out at the same rate as the first, she would have drawn for it 490 dollars interest. What were the two rates of interest?

Ans. 7 and 6 per cent.

CHAPTER IV.

Formation of Powers, and Extraction of Roots of any degree whatever.

159. The resolution of equations of the second degree supposes the process for extracting the square root to be known; in like manner the resolution of equations of the third, fourth, &c. degree, requires that we should know how to extract the third, fourth, &c. root of any numerical or algebraic quantity.

It will be the principal object of this chapter to explain the raising of powers, the extraction of roots, and the calculus of radicals.

Although any power of a number can be obtained from the rules of multiplication, yet this power is subjected to a certain *law of composition* which it is absolutely necessary to know, in order *to deduce the root from the power*. Now, the law of composition of the square of a numerical or algebraic quantity, is deduced from the expression for the square of a binomial (Art. 117); so likewise, the law of a power of any degree, is deduced from the same power of a binomial. We will therefore determine *the development of any power of a binomial.*

160. By multiplying the binomial $x+a$ into itself several times the following results are obtained;

$$(x+a)=x+a,$$
$$(x+a)^2=x^2+2ax+a^2,$$
$$(x+a)^3=x^3+3ax^2+3a^2x+a^3,$$
$$(x+a)^4=x^4+4ax^3+6a^2x^2+4a^3x+a^4,$$
$$(x+a)^5=x^5+5ax^4+10a^2x^3+10a^3x^2+5a^4x+a^5$$

By inspecting these developments it is easy to discover *a law* according to which the exponents of x and a decrease and increase in the successive terms; it is not, however, so easy to discover a law for the co-efficients. Newton discovered one, by means of which, any power of a binomial can be formed, without first obtaining all of the inferior powers. He did not however explain the course of reasoning which led him to the discovery of it; but the existence of this law has since been demonstrated in a rigorous manner. Of all the known demonstrations of it, the most elementary is that which is founded upon the *theory of combinations.* However, as it is rather complicated, we will, in order to simplify the exposition of it, begin by resolving some problems relative to combinations, from which it will be easy to deduce the *formula for the binomial,* or the development of any power of a binomial.

Theory of Permutations and Combinations.

161. Let it be proposed to determine the *whole number* of ways in which several letters, a, b, c, d, &c. can be written one after the other. The results corresponding to each change in the position of any one of these letters, are called *permutations.*

Thus, the two letters a and b furnish the two *permutations* ab and ba.

In like manner, the three letters a, b, c, furnish six permutations.

$$\left\{\begin{array}{l} abc \\ acb \\ cab \\ bac \\ bca \\ cba \end{array}\right.$$

Permutations, *are the results obtained by writing a certain number of letters one after the other, in every possible order, in such a manner that all the letters shall enter into each result, and each letter enter but once.*

PROBLEM 1. *To determine the number of permutations of which* n *letters are susceptible.*

In the first place, two letters a and b evidently give two permutations.

$$\left\{\begin{array}{l} ab \\ ba \end{array}\right.$$

Therefore, the number of permutations of two letters is 1×2.

Take the three letters a, b, and c. Reserve either of the letters, as c, and permute the other two, giving . . . .

$$\left\{\begin{array}{l} c \\ \;ab \\ \;ba \end{array}\right.$$

Now, the third letter c may be placed before ab, between a and b, and at the right of ab; and the same for ba: that is, in *one of the first permutations the reserved letter* c *may have three different places, giving three permutations.* Now, as the same may be shown for *each* of the first permutations, it follows that the whole number of permutations of three letters will be expressed by $1\times 2\times 3$.

$$\left\{\begin{array}{l} cab \\ acb \\ abc \\ cba \\ bca \\ bac \end{array}\right.$$

If now, a fourth letter d be introduced, it can have four places in *each* of the six permutations of three letters: hence all the permutations of four letters will be expressed by $1\times 2\times 3\times 4$.

In general, let there be n letters a, b, c, &c. and suppose the total number of permutations of $n-1$ letters to be known; and let Q denote that number. Now, in *each* of the $n-1$ permutations the reserved letter may have n places, giving n permutations: hence, when it is so placed in all of them, the number of permutations will be expressed by $Q \times n$.

Let $n=2$. Q will then denote the number of permutations that can be made with a single letter; hence $Q=1$, and in this particular case we have $Q \times n = 1 \times 2$.

Let $n=3$. Q will then express the number of permutations of $3-1$ or 2 letters, and is equal to 1×2. Therefore $Q \times n$ is equal to $1 \times 2 \times 3$.

Let $n=4$. Q in this case denotes the number of permutations of 3 letters, and is equal to $1 \times 2 \times 3$. Hence, $Q \times n$ becomes $1 \times 2 \times 3 \times 4$, and similarly when there are more letters.

162. Suppose we have a number m, of letters a, b, c, d, &c., if they are written one after the other, 2 and 2, 3 and 3, 4 and 4 . . . in every possible order, in such a manner, however, that the number of letters in each result may be less than the number of given letters, we may demand the *whole number* of results thus obtained. These results are called *arrangements*.

Thus *ab*, *ac*, *ad*, . . . *ba*, *bc*, *bd*, . . . *ca*, *cb*, *cd*, . . . are *arrangements* of m letters taken 2 and 2, or in sets of 2 each.

In like manner, *abc*, *abd*, . . . *bac*, *bad*, . . . *acb*, *acd*, . . . are *arrangements* taken in sets of 3.

Arrangements, *are the results obtained by writing a number* m *of letters one after the other in every possible order, in sets of* 2 *and* 2, 3 *and* 3, 4 *and* 4 . . . n *and* n; m *being* $>$n: that is, the number of letters in each set being less than the whole number of letters considered. However, if we suppose $n=m$, the *arrangements* taken n and n, will become simple *permutations*.

Problem 2. *Having given a number* m *of letters* a, b, c, d . . ., *to determine the total number of arrangements that may be formed of them by taking them* n *at a time;* m *being supposed greater than* n.

Let it be proposed, in the first place, to arrange the three letters a, b, and c in sets of two each.

First, arrange the letters in sets of one each, in which case we say there are two letters reserved: the reserved letters for either arrangement, being those which do not enter it.

$$\left\{\begin{array}{l} a \\ b \\ c \end{array}\right.$$

Now, to any one of the letters, as a, annex, in succession, the reserved letters b and c: to the second arrangement b, annex the reserved letters a and c; and to the third arrangement c, annex the reserved letters a and b: this gives . . . .

$$\left\{\begin{array}{l} ab \\ ac \\ ba \\ bc \\ ca \\ cb \end{array}\right.$$

Hence, we see, *that the arrangements of three letters taken two in a set, will be equal to the arrangements of the same number of letters taken one in a set, multiplied by the number of reserved letters.*

Let it be required to form the arrangement of four letters, a, b, c, and d, taken 3 in a set.

First, arrange the four letters two in a set: there will then be two reserved letters. Take one of the sets and write after it, in succession, each of the reserved letters: we shall thus form as many sets of three letters each as there are reserved letters; these sets differing from each other by at least the last letter. Take another of the first arrangements, and annex in succession the reserved letters; we shall again form as many different arrangements, as there are reserved letters. Do the same for all of the first arrangements, and it is plain, that the whole number of arrangements which will be formed, of four letters, taken 3 and 3, *will be equal to the arrangements of the same letters, taken two in a set, multiplied by the number of reserved letters.*

$$\left\{\begin{array}{l} ab \\ ba \\ ac \\ ca \\ ad \\ da \\ bc \\ cb \\ bd \\ db \\ cd \\ dc \end{array}\right.$$

In order to resolve this question in a general manner, suppose the total number of *arrangements* of the m letters taken $n-1$ in a set to be known, and denote this number by P.

Take any one of these arrangements, and annex to it each of the reserved letters, of which the number is $m-(n-1)$, or

$m-n+1$; it is evident, that we shall thus form a number $m-n+1$ of arrangements of n letters, differing from each other by the last letter. Now take another of the arrangements of $n-1$ letters, and annex to it each of the $m-n+1$ letters which do not make a part of it; we again obtain a number $m-n+1$ of arrangements of n letters, differing from each other, and from those obtained as above, at least in the disposition of one of the $n-1$ first letters. Now, as we may in the same manner take all the P arrangements of the m letters, taken $n-1$ in a set, and annex to them successively the $m-n+1$ other letters, it follows that the total number of arrangements of m letters taken n in a set, is expressed by

$$P(m-n+1).$$

To apply this to the particular cases of the number of arrangements of m letters taken 2 and 2, 3 and 3, 4 and 4, make $n=2$, whence $m-n+1=m-1$; P will in this case express the total number of arrangements, taken $2-1$ and $2-1$, or 1 and 1, and is consequently equal to m; therefore the formula becomes $m(m-1)$.

Let $n=3$, whence $m-n+1=m-2$; P will then express the number of arrangements taken 2 and 2, and is equal to $m(m-1)$; therefore the formula becomes $m(m-1)(m-2)$.

Again, take $n=4$, whence $m-n+1=m-3$; P will express the number of arrangements taken, 3 and 3, or is equal to

$$m(m-1)(m-2);$$

therefore the formula becomes

$$m(m-1)(m-2)(m-3).$$

REMARK. From the manner in which the particular cases have been deduced from the general formula, we may conclude that it reduces to

$$m(m-1)(m-2)(m-3)\ldots.(m-n+1);$$

that is, *it is composed of the product of the* n *consecutive numbers comprised between* m *and* m$-$n$+$1, *inclusively.*

From this formula, that of the preceding Art. can easily be deduced, viz. the development of the value of $Q \times n$.

For, we see that the *arrangements* become permutations when the number of letters composing each arrangement is supposed equal to the total number of letters considered.

Therefore, to pass from the total number of arrangements of m letters, taken n and n, to the number of permutations of n letters, it is only necessary to make $m=n$ in the above development, which gives

$$n(n-1)\,(n-2)\,(n-3)\,\ldots\ldots\,1.$$

By reversing the order of the factors, observing that the last is 1, the next to the last 2, which is preceded by 3 . . ., it becomes

$$1,\,2,\,3,\,4\,\ldots\ldots\,(n-2)\,(n-1)n,$$

for the development of $Q \times n$.

This is nothing more than the series of natural numbers comprised between 1 and n, inclusively.

163. When the letters are disposed, as in the arrangements, 2 and 2, 3 and 3, 4 and 4, &c., it may be required that no two of the results, thus formed, shall be composed of the same letters, in which case the products of the letters will be different; and we may then demand the whole number of results thus obtained. In this case, the results are called *combinations*.

Thus, *ab*, *ac*, *bc*, . . . *ad*, *bd*, . . . are *combinations* of the letters taken 2 and 2.

In like manner, *abc*, *abd*, . . . *acd*, *bcd* . . . are *combinations* of the letters taken 3 and 3.

Combinations, *are arrangements in which any two will differ from each other by at least one of the letters which enter them.*

Hence, there is an essential difference in the signification of the words, *permutations*, *arrangements*, and *combinations*.

PROBLEM 3. *To determine the total number of different combinations that can be formed of* m *letters, taken* n *in a set.*

Let X denote the total number of arrangements that can be formed of m letters, taken n and n : Y the number of permutations of n letters; and Z the total number of *different combinations* taken n and n.

It is evident, that all the possible arrangements of m letters, taken n at a time, can be obtained, by subjecting the n letters of each of the Z *combinations*, to all the *permutations* of which these letters are susceptible. Now a single combination of n letters gives, by hypothesis Y permutations; therefore Z combinations will give $Y \times Z$ . . . arrangements, taken n and n; and as X denotes the total number of arrangements, it follows that the three quantities X, Y, and Z, give the relations $X = Y \times Z$; whence $Z = \frac{X}{Y}$.

But we have (Art. 162), $\quad X = P(m-n+1)$

and (Art. 161), $\quad Y = Q \times n$.

Therefore, $\quad Z = \frac{P(m-n+1)}{Q \times n} = \frac{P}{Q} \times \frac{m-n+1}{n}$.

Since P expresses the total number of arrangements, taken $n-1$ and $n-1$, and Q the number of permutations of $n-1$ letters, it follows that $\frac{P}{Q}$ expresses the number of different combinations of m letters taken $n-1$ and $n-1$.

To apply this to the particular case of combinations of m letters taken 2 and 2, 3 and 3, 4 and 4 . . .

Make $n = 2$, in which case $\frac{P}{Q}$ expresses the number of combinations of the letters taken $2-1$ and $2-1$ or 1 and 1, and is equal to m; the above formula becomes

$$m \times \frac{m-1}{2} \quad \text{or} \quad \frac{m(m-1)}{1.2}$$

Let $n = 3$, $\frac{P}{Q}$ will express the number of combinations taken 2 and 2, and is equal to $\frac{m(m-1)}{1.2}$; and the formula becomes

$$\frac{m(m-1)\ (m-2)}{1.2.3}.$$

In like manner, we would find the number of combinations of m letters taken 4 and 4, to be $\dfrac{m(m-1)\,(m-2)\,(m-3)}{1.2.3.4}$; and in general, the number of combinations of m letters taken n and n, is expressed by

$$\frac{m(m-1)\,(m-2)\,(m-3)\,\ldots\,(m-n+1)}{1.2.3.4\,\ldots\,(n-1).n};$$

which is the development of the expression

$$\frac{P(m-n+1)}{Q\times n}.$$

Demonstration of the Binomial Theorem.

164. In order to discover more easily the law for the development of the mth power of the binomial $x+a$, we will observe the law of the product of several binomial factors $x+a$, $x+b$, $x+c$, $x+d\,\ldots$ of which the first term is the same in each, and the second terms different.

$$\begin{array}{ll}
 & x+a \\
 & x+b \\
\hline
\text{1st. product} \;.\;. & \begin{array}{l|l} x^2+a & x+ab \\ +b & \end{array} \\
 & x+c \\
\hline
\text{2d.} \;.\;.\;.\;.\;. & \begin{array}{l|l|l} x^3+a & x^2+ab & x+abc \\ +b & +ac & \\ +c & +bc & \end{array} \\
 & x+d \\
\hline
\text{3d.} \;.\;.\;.\;.\;. & \begin{array}{l|l|l|l} x^4+a & x^3+ab & x^2+abc & x+abcd \\ +b & +ac & +abd & \\ +c & +ad & +acd & \\ +d & +bc & +bcd & \\ & +bd & & \\ & +cd & & \end{array}
\end{array}$$

From these products, obtained by the common rule for algebraic multiplication, we discover the following laws:

1st. With respect to the exponents; the exponent of x, in the first term, is equal to the number of binomial factors employed. In the following terms, this exponent diminishes by unity to the last term, where it is 0.

2d. With respect to the co-efficients of the different powers of x: that of the first term is unity; the co-efficient of the second term is equal to the sum of the second terms of the binomials; the co-efficient of the third term is equal to the sum of the products of the different second terms taken two and two; the co-efficient of the fourth term is equal to the sum of their different products taken three and three. Reasoning from *analogy*, we may conclude that the co-efficient of the term which has n terms before it, is equal to the sum of the different products of the m second terms of the binomials taken n and n. The last term is equal to the continued product of the second terms of the binomials.

In order to be certain that this law of composition is general, suppose that it has been proved to be true for a number m of binomials; let us see if it be true when a new factor is introduced into the product.

For this purpose, suppose

$$x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}\ldots+Mx^{m-n+1}+Nx^{m-n}+\ldots+U,$$

to be the product of m binomial factors, Nx^{m-n} representing the term which has n terms before it, and Mx^{m-n+1} that which immediately precedes.

Let $x+K$ be the new factor, the product when arranged according to the powers of x, will be

$$\begin{array}{r|r|r|r|r} x^{m+1}+A & x^{m}+B & x^{m-1}+C & x^{m-2}+\ldots+N & x^{m-n+1}+\ldots \\ +K & +AK & +BK & +MK & +UK. \end{array}$$

From which we perceive that *the law of the exponents* is evidently the same.

With respect to the co-efficients, 1st. That of the first term *is* *unity.* 2d. $A+K$, or the co-efficient of x^m, is also the *sum of the second terms of the* m+1 *binomials.*

3d. B is by hypothesis the sum of the different products of the second terms of the m binomials, and $A.K$ expresses the sum of the products of each of the second terms of the m first binomials, by the new second term K; therefore $B+AK$ *is the sum of the different products of the second terms of the* m+1 *binomials, taken two and two.*

In general, since N expresses the sum of the products of the second terms of the m first binomials, taken n and n; and as MK represents the sum of the products of these second terms, taken $n-1$ and $n-1$, multiplied by the new *second term* K, it follows that $N+MK$, or the co-efficient of the term which has n terms before it, is equal to the sum of the different products of the second terms of the $m+1$ binomials, taken n and n. The last term is equal to the continued product of the $m+1$ second terms.

Therefore, the law of composition, supposed true for a number m of binomial factors, is also true for a number denoted by $m+1$. It is therefore general.

Let us suppose, that in the product resulting from the multiplication of the m binomial factors, $x+a$, $x+b$, $x+c$, $x+d$ . . . we make $a=b=c=d$ . . ., the indicated expression of this product, $(x+a)(x+b)(x+c)$, will be changed into $(x+a)^m$. With respect to its development, the co-efficients being $a+b+c+d$ . . ., $ab+ac+ad+$. . ., $abc+abd+acd$ . . ., the co-efficient of x^{m-1}, or $a+b+c+d$ . . ., becomes $a+a+a+a+$ . . ., that is, a taken as many times as there are letters a, b, c . . ., and is therefore equal to ma. The co-efficient of x^{m-2}, or $ab+ac+ad+$ . . ., reduces to $a^2+a^2+a^2$ . . ., or to a^2 taken as many times as we can form different combinations with m letters, taken two and two, or to $m \cdot \frac{m-1}{2} a^2$. (Art. 163).

The co-efficient of x^{m-3} reduces to the product of a^3, multiplied

by the number of different combinations of m letters, taken 3 and 3, or to
$$m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}a^3, \text{ \&c.}$$

In general, if the term, which has n terms before it, is denoted by Nx^{m-n}, the co-efficient, which in the hypothesis of the second terms being different, is equal to the sum of their products, taken n and n, reduces, when all of the terms are supposed equal, to a^n multiplied by the number of different combinations that can be made with m letters, taken n and n. Therefore
$$N=\frac{P(m-n+1)}{Q\times n}a^n. \quad \text{(Art. 163).}$$

From which we have the formula
$$(x+a)^m=x^m+max^{m-1}+m\cdot\frac{m-1}{2}a^2x^{m-2}$$
$$+m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}a^3x^{m-3}\ldots+\frac{P(m-n+1)}{Q\cdot n}a^nx^{m-n}\ldots+a^m.$$

165. By inspecting the different terms of this development, a *simple law* will be perceived, by means of which the co-efficient of any term is formed from the co-efficient of the preceding term.

The co-efficient of any term is formed by multiplying the co-efficient of the preceding term by the exponent of x *in that term, and dividing the product by the number of terms which precede the required term.*

For, take the *general term* $\frac{P(m-n+1)}{Q\times n}a^nx^{m-n}$. This is called the *general term*, because by making $n=2, 3, 4\ldots$, all of the others can be deduced from it. The term which immediately precedes it, is evidently $\frac{P}{Q}a^{n-1}x^{m-n+1}$, since $\frac{P}{Q}$ expresses the number of combinations of m letters taken $n-1$ and $n-1$. Here we see that the co-efficient $\frac{P(m-n+1)}{Q\times n}$ is equal to the co-efficient

$\frac{P}{Q}$ which precedes it, multiplied by $m-n+1$, the exponent of x in that term, and divided by n, the number of terms preceding the required term. This law serves to develop a particular power, without our being obliged to have recourse to the general formula.

For example, let it be required to develop $(x+a)^6$. From this law we have,

$$(x+a)^6=x^6+6ax^5+15a^2x^4+20a^3x^3+15a^4x^2+6a^5x+a^6.$$

After having formed the first two terms from the terms of the general formula $x^m+max^{m-1}+\ldots$, multiply 6, the co-efficient of the second term, by 5, the exponent of x in this term, then divide the product by 2, which gives 15 for the co-efficient of the third term. To obtain that of the fourth, multiply 15 by 4, the exponent of x in the third term, and divide the product by 3, the number of terms which precede the fourth, this gives 20 ; and the co-efficients of the other terms are found in the same way.

In like manner we find

$$(x+a)^{10}=x^{10}+10ax^9+45a^2x^8+120a^3x^7+210a^4x^6,$$
$$+252a^5x^5+210a^6x^4+120a^7x^3+45a^8x^2+10a^9x+a^{10}.$$

166. It frequently occurs that the terms of the binomial are affected with co-efficients and exponents, as in the following example.

Let it be required to raise the binomial $3a^2c-2bd$ to the 4th power.

Placing $3a^2c=x$ and $-2bd=y$, we have

$$(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4.$$

Substituting for x and y their values, we have

$$(3a^3c-2bd)^4=(3a^2c)^4+4(3a^2c)^3(-2bd)+6(3a^2c)^2(-2bd)^2+$$
$$4(3a^2c)\ (-2bd)^3+(-2bd)^4,$$

or, by performing the operations indicated

$$(3a^3c-2bd)^4=81a^8c^4-216a^6c^3bd+216a^4c^2b^2d^2-96a^2cb^3d^3$$
$$+16b^4d^4.$$

The terms of the development are alternately plus and minus, as they should be, since the second term is $-$.

167. The powers of any polynomial may easily be found by the binomial theorem.

For example, raise $a+b+c$ to the third power.

First, put . . . . $b+c=d$.

Then $(a+b+c)^3=(a+d)^3=a^3+3a^2d+3ad^2+d^3$.

Or, by substituting for the value of d,

$$
\begin{aligned}
(a+b+c)^3=a^3+3a^2b&+3ab^2+b^3\\
3a^2c&+3b^2c+6abc\\
&+3ac^2+3bc^2\\
&+\quad c^3.
\end{aligned}
$$

This expression is composed of *the cubes of the three terms, plus three times the square of each term by the first powers of the two others, plus six times the product of all three terms.* It is easily proved that this *law* is true for any polynomial.

To apply the preceding formula to the development of the cube of a trinomial, in which the terms are affected with co-efficients and exponents, *designate each term by a single letter, then replace the letters introduced, by their values, and perform the operations indicated.*

From this rule, we will find that

$$
\begin{aligned}
(2a^2-4ab+3b^2)^3=8a^6-48a^5b+132a^4b^2-208a^3b^3\\
+198a^2b^4-108ab^5+27b^6.
\end{aligned}
$$

The fourth, fifth, &c. powers of any polynomial can be developed in a similar manner.

Consequences of the Binomial Formula.

168. *First.* The expression $(x+a)^m$ being such, that x may be substituted for a, and a for x, without altering its value, it follows that the same thing can be done in the development of it; therefore, if this development contains a term of the form $\mathrm{K}a^nx^{m-n}$, it must have another equal to $\mathrm{K}x^na^{m-n}$ or $\mathrm{K}a^{m-n}x^n$. These two terms of the development are evidently at equal distances from the two extremes; for the number of terms which precede any term, being indicated by the exponent of a in that term, it follows that

the term $Ka^n x^{m-n}$ has n terms before it; and that the term $Ka^{m-n}x^n$ has $m-n$ terms before it, and consequently n terms after it, since the whole number of terms is denoted by $m+1$.

Therefore, *in the development of any power of a binomial, the co-efficients at equal distances from the two extremes are equal to each other.*

REMARK. In the terms $Ka^n x^{m-n}$, $Ka^{m-n}x^n$, the first co-efficient expresses the number of different combinations that can be formed with m letters taken n and n; and the second, the number which can be formed when taken $m-n$ and $m-n$; we may therefore conclude that, *the number of different combinations of* m *letters taken* n *and* n, *is equal to the number of combinations of* m *letters taken* m—n *and* m—n.

For example, *twelve* letters combined 5 and 5, give the same number of combinations as these twelve letters taken 12—5 and 12—5, or 7 and 7. Five letters combined 2 and 2, give the same number of combinations as five letters combined 5—2 and 5—2, or 3 and 3.

169. *Second.* If in the general formula,

$$(x+a)^m = x^m + max^{m-1} + m\frac{m-1}{2}a^2x^{m-2} +, \text{ \&c.}$$

we suppose $x=1$, $a=1$, it becomes

$$(1+1)^m \text{ or } 2^m = 1 + m + m\frac{m-1}{2} + m\frac{m-1}{2}\cdot\frac{m-2}{3} +, \text{ \&c.}$$

That is, *the sum of the co-efficients of the different terms of the formula for the binomial, is equal to the* m*th power of* 2.

Thus, in the particular case

$$(x+a)^5 = x^5 + 5ax^4 + 10a^2x^3 + 10a^3x^2 + 5a^4x + a^5,$$

the sum of the co-efficients $1+5+10+10+5+1$ is equal to 2^5 or 32. In the 10th power developed, the sum of the co-efficients is equal to 2^{10} or 1024.

170. *Third.* In a series of numbers decreasing by unity, of which

the first term is m and the last $m-p$, m and p being entire numbers, the continued product of all these numbers is divisible by the continued product of all the natural numbers from 1 to $p+1$ inclusively.

That is, $\frac{m(m-1)\,(m-2)\,(m-3)\ldots(m-p)}{1\,.\,2\,.\,3\,.\,4\ldots.(p+1)}$ is a whole number. For, from what has been said in (Art. 163), this expression represents the number of different combinations that can be formed of m letters taken $p+1$ and $p+1$. Now this number of combinations is, from its nature, an entire number; therefore the above expression is necessarily a whole number.

Of the Extraction of the Roots of particular numbers.

171. The *third power* or *cube* of a number, is the product arising from multiplying this number by itself twice; and the *third* or *cube root*, is a number which, being raised to the third power, will produce the proposed number.

The ten first numbers being

	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
their cubes are	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000.

Reciprocally, the numbers of the first line are the cube roots of the numbers of the second.

By inspecting these lines, we perceive that there are but nine *perfect cubes* among numbers expressed by one, two, or three figures; each of the other numbers has for its cube root a whole number, plus a fraction *which cannot be expressed exactly by means of unity*, as may be shown, by a course of reasoning entirely similar to that pursued in the latter part of (Art. 118).

172. The difference between the cubes of two consecutive numbers increases, when the numbers are increased.

Let a and $a+1$, be two consecutive whole numbers; we have

$$(a+1)^3=a^3+3a^2+3a+1;$$

whence

$$(a+1)^3-a^3=3a^2+3a+1.$$

That is, *the difference between the cubes of two consecutive whole numbers, is equal to three times the square of the least number, plus three times this number, plus* 1.

Thus, the difference between the cube of 90 and the cube of 89, is equal to $3(89)^2+3\times 89+1=24031$.

173. In order to extract the cube root of an entire number, we will observe, that when the figures expressing the number do not exceed three, *its root is obtained by merely inspecting the cubes of the first nine numbers.* Thus, the cube root of 125 is 5; the cube root of 72 is 4 plus a fraction, or is within one of 4; the cube root of 841 is within one of 9, since 841 falls between 729, or the cube of 9, and 1000, or the cube of 10.

When the number is expressed by more than three figures, the process will be as follows. Let the proposed number be 103823.

$$
\begin{array}{rr|l}
 & 103.823 & 47 \\
 & 64 & \overline{8} \\ \cline{2-2}
4^2\times 3=48 & 398.23 &
\end{array}
$$

$$
\begin{array}{rcr}
48 & & 47 \\
48 & & 47 \\ \hline
384 & & 329 \\
192 & & 188 \\ \hline
2304 & & 2209 \\
48 & & 47 \\ \hline
18432 & & 15463 \\
9216 & & 8836 \\ \hline
110592 & & 103823
\end{array}
$$

This number being comprised between 1,000, which is the cube of 10, and 1,000,000, which is the cube of 100, its root will be expressed by two figures, or by tens and units. Denoting the tens by a, and the units by b, we have (Art. 160),

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

Whence it follows, that the cube of a number composed of tens and units, *is equal to the cube of the tens, plus three times the product* *of the square of the tens by the units, plus three times the product of the tens by the square of the units, plus the cube of the units.*

This being the case, the cube of the tens, giving at least, *thousands*, the last three figures to the right cannot form a part of it: the cube of the tens must therefore be found in the part 103 which is separated from the last three figures by a point. Now the root of the greatest cube contained in 103 being 4, this is the number of tens in the required root; for 103823 is evidently comprised between $(40)^3$ or 64,000, and $(50)^3$ or 125,000; hence the required root is composed of 4 tens, plus a certain number of units less than *ten*.

Having found the number of tens, subtract its cube 64 from 103; there remains 39, and bringing down the part 823, we have 39823, which contains *three times the square of the tens by the units*, plus the two parts before mentioned. Now, as the square of a number of tens gives at least hundreds, it follows that three times the square of the tens by the units, must be found in the part 398, to the left of 23, which is separated from it by a point. Therefore, dividing 398 by three times the square of the tens, which is 48, the quotient 8 will be the unit of the root, or something greater, since 398 hundreds is composed of three times the square of the tens by the units, together with the two other parts. We may ascertain whether the figure 8 is too great, by forming the three parts which enter into 39823, by means of the figure 8 and the number of tens 4; but it is much easier to cube 48, as has been done in the above table. Now the cube of 48 is 110592, which is greater than 103823; therefore 8 is too great. By cubing 47 we obtain 103823; hence the proposed number is a perfect cube, and 47 is the cube root of it.

Remark. The units figures could not be first obtained; because the cube of the units might give tens, and even hundreds, and the tens and hundreds would be confounded with those which arise from other parts of the cube.

Again, extract the cube root of 47954

	47.954	36		
	27			
$3^2 \times 3 = 27$	209		36	37
			36	37
	47954		216	259
	46656		108	111
	1298		1296	1369
			36	37
			7776	9583
			3888	4107
			46656	50653

The number 47954 being below 1,000,000, its root contains only two figures, viz. tens and units. The cube of the tens is found in 47 *thousands*, and we can prove, as in the preceding example, that 3, the root of the greatest cube contained in 47, expresses the tens. Subtracting the cube of 3 or 27, from 47, there remains 20; bringing down to the right of this remainder the figure 9 from the part 954, the number 209 hundreds, is composed of three times the square of the tens by the units, plus the number arising from the other two parts. Therefore, by forming three times the square of the tens, 3, which is 27, and dividing 209 by it, the quotient 7 will be the units of the root, or something greater. Cubing 37, we have 50653, which is greater than 47954; then cubing 36, we obtain 46656, which subtracted from 47954, gives 1298 for a remainder. Hence the proposed number is not a perfect cube; but 36 is its root to *within unity.* In fact, the difference between the proposed number and the cube of 36, is, as we have just seen, 1298, which is less than $3(36)^2+3\times36+1$, for in verifying the result we have obtained 3888 for three times the square of 36.

174. Again, take for another example, the number, 43725658 containing more than 6 figures.

	43.725.658	352		
	27			
$3^2\times3=27$	167		35	352
			35	352
$35^2\times3=3675$	43 725		175	704
	42 875		105	1760
	8506		1225	1056
			35	123904
	43725658		6125	352
	43614208		3675	247808
Rem. . . .	111450		42875	619520
				371712
				43614208

Now the required root contains more than one figure, and may be considered as composed of units and tens only, the tens being expressed by one or more figures.

Since the cube of the tens gives at least thousands, it must be found in the part which is to the left of the last three figures 658. I say now that if we extract the root of the greatest cube contained in the part 43725, considered with reference to its absolute value, we shall obtain the whole number of tens of the root; for let a be the root of 43725, to within unity, that is, such that 43725 shall be comprised between a^3 and $(a+1)^3$; then will 43725000 be comprehended between $a^3\times1000$ and $(a+1)^3\times1000$; and as these two last numbers differ from each other by more than 1000, it follows that the proposed number itself, 43725658, is comprised between $a^3\times1000$ and $(a+1)^3\times1000$; therefore the required root is comprised between that of $a^3\times1000$, and $(a+1)^3\times1000$, that is, between $a\times10$ and $(a+1)\times10$. It is therefore composed of a tens, plus a certain number of units less than *ten*.

The question is then reduced to extracting the cube root of 43725; but this number having more than three figures, its root will contain more than one, that is, it will contain tens and units. To obtain the tens, point off the last three figures, 725, and extract the root of the greatest cube contained in 43.

The greatest cube contained in 43 is 27, the root of which is 3; this figure will then express the tens of the root of 43725, or the figure in the place of hundreds in the total root. Subtracting the cube of 3, or 27, from 43, we obtain 16 for a remainder, to the right of which bring down the first figure 7, of the second period 725, which gives 167.

Taking three times the square of the tens, 3, which is 27, and dividing 167 by it, the quotient 6 is the unit figure of the root of 43725, or something greater. It is easily seen that this number is in fact too great; we must therefore try 5. The cube of 35 is 42875, which, subtracted from 43725, gives 850 for a remainder, which is evidently less than $3\times(35)^2+3\times35+1$. Therefore, 35 is the root of the greatest cube contained in 43725; hence it is the number of tens in the required root.

To obtain the units, bring down to the right of the remainder 850, the first figure, 6, of the last period, 658, which gives 8506; then take 3 times the square of the tens, 35, which is 3675, and divide 8506 by it; the quotient is 2, which we try by cubing 352: this gives 43614208, which is less than the proposed number, and subtracting it from this number, we obtain 111450 for a remainder. Therefore 352 is the cube root of 43725658, to within unity. Hence, for the extraction of the cube root we have the following

RULE.

I. *Separate the given number into periods of three figures each, beginning at the right hand: the left hand period will often contain less than three places of figures.*

II. *Seek the greatest cube in the first period, at the left, and set its root on the right, after the manner of a quotient in division. Subtract the cube of this figure of the root from the first period, and to the re-*

mainder bring down the first figure of the next period, and call this number the dividend.

III. *Take three times the square of the root just found for a divisor, and see how often it is contained in the dividend, and place the quotient for a second figure of the root. Then cube the figures of the root thus found, and if their cube be greater than the first two periods of the given number, diminish the last figure; but if it be less, subtract it from the first two periods, and to the remainder bring down the first figure of the next period, for a new dividend.*

IV. *Take three times the square of the whole root for a new divisor, and seek how often it is contained in the new dividend: the quotient will be the third figure of the root. Cube the whole root and subtract the result from the three first periods of the given number, and proceed in a similar way for all the periods.*

Remark. If any of the remainders are equal to, or exceed, *three times the square of the root obtained plus three times this root, plus one*, the last figure of the root is too small and must be augmented by at least unity (Art. 172).

EXAMPLES.

1. $\sqrt[3]{48228544}=364$.
2. $\sqrt[3]{27054036008}=3002$.
3. $\sqrt[3]{483249}=78$, with a remainder 8697;
4. $\sqrt[3]{91632508641}=4508$, with a remainder 20644129·
5. $\sqrt[3]{32977340218432}=32068$.

To extract the n^{th} root of a whole number.

175. In order to generalize the process for *the extraction of roots*, we will denote the proposed number by N, and the degree of the root to be extracted by n. If the number of figures in N, does not exceed n, the root will be expressed by a single figure, and is obtained immediately by forming the n^{th} power of each of the whole

numbers comprised between 1 and 10; for the n^{th} power of 9 is the largest perfect power which can be expressed by n figures.

When N contains more than n figures, there will be more than one figure in the root, which may then be considered as composed of tens and units. Designating the tens by a, and the units by b, we have (Art. 166),

$$N=(a+b)^n=a^n+na^{n-1}b+n\frac{n-1}{2}a^{n-2}b^2+, \text{\&c.};$$

that is, the proposed number contains *the* nth *power of the tens, plus* n *times the product of the* n$-$1 *power of the tens by the units,* plus a series of other parts which it is not necessary to consider.

Now, as the n^{th} power of the tens cannot give units of an order inferior to unity followed by n ciphers, the last n figures on the right, cannot make a part of it. They must then be pointed off, and the root of the greatest n^{th} power contained in the figures on the left should be extracted; this root will be *the tens of the required root.*

If this part on the left should contain more than n figures, the n figures on the right of it, must be separated from the rest, and the root of the greatest n^{th} power contained in the part on the left extracted, and so on. Hence the following

RULE.

I. *Divide the number* N *into periods of* n *figures each, beginning at the right hand; extract the root of the greatest* nth *power contained in the left hand period, and subtract the* nth *power of this figure from the left hand period.*

II. *Bring down to the right of the remainder corresponding to the first period, the first figure of the second period, and call this number the dividend.*

III. *Form the* n$-$1 *power of the first figure of the root, multiply it by* n, *and see how often the product is contained in the dividend: the quotient will be the second figure of the root, or something greater.*

IV. *Raise the number thus formed to the* nth *power, then subtract this result from the two first periods, and to the new remainder bring down the first figure of the third period: then divide the number thus* *formed by* n *times the* n−1 *power of the two figures of the root already found, and continue this operation until all the periods are brought down.*

EXAMPLES.

Extract the 4th root of 531441.

$$\begin{array}{rl|l} & 53.1441 & 27 \\ 2^4= & 16 & \\ \hline 4\times 2^3=32 \mid & 371 & \\ (27)^4= & 531441. & \end{array}$$

We first divide off, from the right hand, the period of four figures, and then find the greatest fourth root contained in 53, the first period to the left, which is 2. We next subtract the 4th power of 2, which is 16, from 53, and to the remainder 37 we bring down the first figure of the next period. We then divide 371 by 4 times the cube of 2, which would give 8 for a quotient; but by raising 28 to the 4th power, we discover that 8 is too large, then trying 7 we find the exact root to be 27.

176. REMARK. When the degree of the root to be extracted is a multiple of two or more numbers, as 4, 6, ..., *the root can be obtained by extracting the roots of more simple degrees, successively.* To explain this, we will remark that,

$$(a^3)^4=a^3\times a^3\times a^3\times a^3=a^{3+3+3+3}=a^{3\times 4}=a^{12}.$$

and that in general $(a^m)^n=a^m\times a^m\times a^m\times a^m\ .\ .\ .=a^{m\times n}$ (Art. 13). Hence, *the* n^{th} *power of the* m^{th} *power of a number, is equal to the* mn^{th} *power of this number.*

Reciprocally, *the* mn^{th} *root of a number is equal to the* n^{th} *root of the* m^{th} *root of this number,* or algebraically

$$\sqrt[mn]{a}=\sqrt[n]{\sqrt[m]{a}}=\sqrt[m]{\sqrt[n]{a}}.$$

For, let . . . $\sqrt[n]{\sqrt[m]{a}}=a'$, raising both members to the n^{th} power there will result . . . $\sqrt[m]{a}=a'^n$; for from the definition of a root, we have $(\sqrt[n]{K})^n=K$.

Again, by raising both members to the m^{th} power, we obtain $a=(a'^n)^m=a'^{mn}$. Extracting the mn^{th} root of both members, $\sqrt[mn]{a}=a'$; but we already have $\sqrt[n]{\sqrt[m]{a}}=a'$; hence $\sqrt[mn]{a}=\sqrt[n]{\sqrt[m]{a}}$.

In a similar manner we might find $\sqrt[mn]{a}=\sqrt[m]{\sqrt[n]{a}}$.

By this method we find that

$$\sqrt[4]{256}=\sqrt{\sqrt{256}}=\sqrt{16}=4;$$

$$\sqrt[6]{2985984}=\sqrt[3]{\sqrt{2985984}}=\sqrt[3]{1728}=12;$$

$$\sqrt[6]{1771561}=\sqrt[3]{\sqrt{1771561}}=11;$$

$$\sqrt[8]{1679616}=\sqrt[4]{1296}=\sqrt{\sqrt{1296}}=6.$$

Remark. Although the successive roots may be extracted in any order whatever, it is better to extract the roots of the lowest degree first, for then the extraction of the roots of the higher degrees, which is a more complicated operation, is effected upon numbers containing fewer figures than the proposed number.

Extraction of Roots by approximation.

177. When it is required to extract the n^{th} root of a number which is not a *perfect power*, the method of (Art. 175), will give only the entire part of the root, or the root to within unity. As to the fraction which is to be added, in order to complete the root, it cannot be obtained exactly, but we can approximate as near as we please to the required root.

Let it be required to extract the n^{th} root of the whole number a, to within a fraction $\frac{1}{p}$; that is, so near it, that the error shall be less than $\frac{1}{p}$.

We will observe that a can be put under the form $\frac{a\times p^n}{p^n}$. If we denote the root of ap^n to within unity, by r, the number $\frac{a\times p^n}{p^n}$ or a, will be comprehended between $\frac{r^n}{p^n}$ and $\frac{(r+1)^n}{p^n}$; therefore the $\sqrt[n]{a}$ will be comprised between the two numbers, $\frac{r}{p}$ and $\frac{r+1}{p}$. Hence $\frac{r}{p}$ is the required root, to within the fraction $\frac{1}{p}$.

Hence, *to extract the root of a whole number to within a fraction* $\frac{1}{\mathrm{p}}$, *multiply the number by* p^n; *extract the* n^{th} *root of the product to within unity, and divide the result by* p.

178. Again, suppose it is required to extract the n^{th} root of the fraction $\frac{a}{b}$.

Multiply each term of the fraction by

b^{n-1}; it becomes $\frac{a}{b}=\frac{ab^{n-1}}{b^n}$.

Let r denote the n^{th} root of ab^{n-1}, to within unity;

$\frac{ab^{n-1}}{b^n}$ or $\frac{a}{b}$, will be comprised between $\frac{r^n}{b^n}$ and $\frac{(r+1)^n}{b^n}$

Therefore, *after having made the denominator of the fraction a perfect power of the* n^{th} *degree, extract the* n^{th} *root of the numerator, to within unity, and divide the result by the root of the new denominator.*

When a greater degree of exactness is required than that indicated by $\frac{1}{b}$, extract the root of ab^{n-1} to within any fraction $\frac{1}{p}$; and designate this root by $\frac{r'}{p}$. Now, since $\frac{r'}{p}$ is the root of the numerator to within $\frac{1}{p}$, it follows, that $\frac{r'}{bp}$ is the true root of the fraction to within $\frac{1}{bp}$.

179. Suppose it is required to extract the cube root of 15, to within $\frac{1}{12}$. We have $15 \times 12^3 = 15 \times 1728 = 25920$. Now the cube root of 25920, to within unity, is 29; hence the required root is $\frac{29}{12}$ or $2\frac{5}{12}$.

Again, extract the cube root of 47, to within $\frac{1}{20}$.

We have $47 \times 20^3 = 47 \times 8000 = 376000$. Now the cube root of 376000, to within unity, is 72; hence $\sqrt[3]{47} = \frac{72}{20} = 3\frac{12}{20}$, to within $\frac{1}{20}$.

Find the value of $\sqrt[3]{25}$ to within 0,001.

To do this, multiply 25 by the cube of 1000, or by 1000000000, which gives 25000000000. Now, the cube root of this number, is 2920; hence $\sqrt[3]{25} = 2{,}920$ to within 0,001.

In general, *in order to extract the cube root of a whole number to within a given decimal fraction, annex three times as many ciphers to the number, as there are decimal places in the required root; extract the cube root of the number thus formed to within unity, and point off from the right of this root the required number of decimals.*

180. We will now explain the method of *extracting the cube root of a decimal fraction.* Suppose it is required to extract the cube root of 3,1415.

As the denominator 10000, of this fraction, is not a perfect cube, it is necessary to make it one, by multiplying it by 100, *which amounts to annexing two ciphers to the proposed decimal,* and we have 3,141500. Extract the cube root of 3141500, that is, of the number considered independent of the comma, to within unity; this gives 146. Then divide by 100, or $\sqrt[3]{1000000}$, and we find $\sqrt[3]{3{,}1415} = 1{,}46$ to within 0,01.

Hence, to extract the cube root of a decimal number, we have the following

RULE

Annex ciphers to the decimal part, if necessary, until it can be divided into exact periods of three figures each, observing that the number of periods must be made equal to the number of decimal places required in the root. Then, extract the root as in entire numbers, and point off as many places for decimals as there are periods in the decimal part of the number.

To extract the cube root of a vulgar fraction to within a given decimal fraction, the most simple method is to *reduce the proposed fraction to a decimal fraction, continuing the operation until the number of decimal places is equal to three times the number required in the root.* The question is then reduced to extracting the cube root of a decimal fraction.

181. Suppose it is required to find the sixth root of 23, to within 0,01.

Applying the rule of Art. 177 to this example, we multiply 23 by 100^6, or annex *twelve* ciphers to 23, extract the sixth root of the number thus formed to within unity, and divide this root by 100, or point off two decimals on the right.

In this way we will find that $\sqrt[6]{23}=1,68$, to within 0,01.

EXAMPLES.

1. Find the $\sqrt[3]{473}$ to within $\frac{1}{20}$. *Ans.* $7\frac{3}{4}$.
2. Find the $\sqrt[3]{79}$ to within ,0001. *Ans.* 4,2908.
3. Find the $\sqrt[6]{13}$ to within ,01. *Ans.* 1,53.
4. Find the $\sqrt[3]{3,00415}$ to within ,0001. *Ans.* 1,4429.
5. Find the $\sqrt[3]{0,00101}$ to within ,01. *Ans.* 0,10.
6. Find the $\sqrt[3]{\frac{14}{25}}$ to within ,001. *Ans.* 0,824.

Formation of Powers and Extraction of Roots of Algebraic Quantities. Calculus of Radicals.

We will first consider monomials.

182. Let it be required to form the fifth power of $2a^3b^2$. We have

$$(2a^3b^2)^5=2a^3b^2\times 2a^3b^2\times 2a^3b^2\times 2a^3b^2\times 2a^3b^2,$$

from which it follows, 1st. That the co-efficient 2 must be multiplied by itself four times, or raised to the fifth power. 2d. That each of the exponents of the letters must be added to itself four times, or multiplied by 5.

Hence, $\qquad (2a^3b^2)^5=2^5.a^{3\times 5}b^{2\times 5}=32a^{15}b^{10}.$

In like manner, $\qquad (8a^2b^3c)^3=8^3.a^{2\times 3}b^{3\times 3}c^3=512a^6b^9c^3.$

Therefore, in order to raise a monomial to a given power, *raise the co-efficient to this power, and multiply the exponent of each of the letters by the exponent of the power.*

Hence, reciprocally, to extract any root of a monomial, 1st. *Extract the root of the co-efficient.* 2d. *Divide the exponent of each letter by the index of the root.*

$$\sqrt[3]{64a^9b^3c^6}=4a^3bc^2; \quad \sqrt[4]{16a^8b^{12}c^4}=2a^2b^3c.$$

From this rule, we perceive, that in order that a monomial may be a perfect power of the degree of the root to be extracted, 1st. its co-efficient must be a perfect power; and 2d. the exponent of each letter must be divisible by the *index* of the root to be extracted. It will be shown hereafter, how the expression for the root of a quantity which is not a perfect power is reduced to its simplest terms.

183. Hitherto, we have paid no attention to the sign with which the monomial may be affected; but if we observe, that whatever may be the sign of a monomial, *its square is always positive,* and that every power of an even degree, $2n$, can be considered as the n^{th} power of the square, that is, $a^{2n}=(a^2)^n$, it will follow that,

every power of a quantity, of an even degree, whether positive or negative, is essentially positive.

Thus, $(\pm 2a^2b^3c)^4 = +16a^8b^{12}c^4.$

Again, as a power of an uneven degree, $2n+1$, is the product of a power of an even degree, $2n$, by the first power, it follows that, *every power of an uneven degree of a monomial, is affected with the same sign as the monomial.*

Hence, $(+4a^2b)^3 = +64a^6b^3$; $(-4a^2b)^3 = -64a^6b^3$.

From this it is evident, 1st. That when the degree of the root of a monomial is uneven, the root will be affected with the same sign as the quantity.

Therefore,

$$\sqrt[3]{+8a^3} = +2a; \quad \sqrt[3]{-8a^3} = -2a; \quad \sqrt[5]{-32a^{10}b^5} = -2a^2b.$$

2d. When the degree of the root is even, and the monomial a positive quantity, the root is affected either with $+$ or $-$.

Thus, $\sqrt[4]{81a^4b^{12}} = \pm 3ab^3$; $\sqrt[6]{64a^{18}} = \pm 2a^3$.

3d. *When the degree of the root is even, and the monomial negative, the root is impossible*; for, there is no quantity which, raised to a power of an even degree, can give a negative result. Therefore, $\sqrt[4]{-a}$, $\sqrt[6]{-b}$, $\sqrt[8]{-c}$, are symbols of operations which it is impossible to execute. They are, like $\sqrt{-a}$, $\sqrt{-b}$, *imaginary expressions* (Art. 126).

184. In order to develop $(a+y+z)^3$, we will place $y+z=u$, and we have

$$(a+u)^3 = a^3 + 3a^2u + 3au^2 + u^3,$$

or by replacing u by its value, $y+z$

$$(a+y+z)^3 = a^3 + 3a^2(y+z) + 3a(y+z)^2 + (y+z)^3,$$

or performing the operations indicated

$$(a+y+z)^3 = a^3 + 3a^2y + 3a^2z + 3ay^2 + 6ayz + 3az^2 + y^3 + 3y^2z + 3yz^2 + z^3.$$

When the polynomial is composed of more than three terms, as

$a+y+z+x \ldots . p$, let, as before, $u=$ the sum of all the terms after the first. Then, $a+u$ will be equal to the given polynomial, and

$$(a+u)^3=a^3+3a^2u+3au^2+u^3.$$

From which we see, that by cubing a polynomial, we obtain *the cube of the first term, plus three times the square of the first term multiplied by each of the remaining terms,* plus other terms.

It often happens that u contains a, as in the polynomial a^2+ax+b, where $u=ax+b$. But since we suppose the polynomial arranged with reference to a, it follows that a will have a less exponent in u than in the first term.

In this case also, the co-efficient of u, multiplied by the first term of u, will be irreducible with the remaining terms of the development, because that product will involve a to a higher power than the other terms: and when a does not enter u, the product of that co-efficient by all the terms of u, will be irreducible with all the other terms of the development.

185. As to the extraction of roots of polynomials, we will first explain the method for the cube root; it will afterwards be easy to generalize.

Let N be the polynomial, and R its cube root. Conceive the two polynomials to be arranged with reference to some letter, a, for example. It results from the law of composition of the cube of a polynomial (Art. 184), that the cube of R contains two parts, which cannot be reduced with the others; these are, the cube of the first term, and three times the square of the first term by the second.

Hence, the cube root of that term of N which contains a, affected with the highest exponent, will be the first term of R: and the second term of R will be found by dividing the second term of N by three times the square of the first term of R.

If then, we form the cube of the two terms of the root already found, and subtract it from N, and divide the first term of the remainder by 3 times the square of the first term of R, the quotient will be the third term of the root. Therefore, having arranged the terms of N, we have the following

RULE.

I. *Extract the cube root of the first term.*

II. *Divide the second term of* N *by three times the square of the first term of* R *: the quotient will be the second term of* R.

III. *Having found the two first terms of* R, *form the cube of the binomial and subtract it from* N *; after which, divide the first term of the remainder by three times the square of the first term of* R *: the quotient will be the third term of* R.

IV. *Cube the three terms of the root found, and subtract the cube from* N *: then divide the first term of the remainder by the divisor already used : the quotient will be the fourth term of the root, and the remaining terms, if there are any, may be found in a similar manner.*

EXAMPLES.

1. Extract the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

$$\begin{array}{rl|l}
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1 & x^2-2x+1 \\
(x^2-2x)^3= & \underline{x^6-6x^5+12x^4-\ \ 8x^3} & 3x^4 \\
\text{1st Rem.} \quad \ldots\ldots & 3x^4-12x^3+, \text{ \&c.} & \\
\end{array}$$

$$(x^2-2x+1)^3=x^6-6x^5+15a^4-20x^3+15x^2-6x+1.$$

In this example, we first extract the cube root of x^6, which gives x^2, for the first term of the root. Squaring x^2, and multiplying by 3, we obtain the divisor $3x^4$: this is contained in the second term $-6x^5$, $-2x$ times. Then cubing the root, and subtracting, we find that the first term of the remainder $3x^4$, contains the divisor once. Cubing the whole root, we find the cube equal to the given polynomial.

Remark. The rule for the extraction of the cube root is easily extended to a root with a higher index. For,

Let $a+b+c+\ldots f$, be any polynomial.

Let $s=$ the sum of all the terms after the first.

Then $a+s=$ the given polynomial : and

$$(a+s)^n=a^n+na^{n-1}s+ \text{ other terms.}$$

That is, *the* n^{th} *power of a polynomial, is equal to the* n^{th} *power of the first term, plus* n *times the first term raised to the power* $n-1$, *multiplied by each of the remaining terms* ; + *other terms of the development.*

Hence, we see, that the rule for the cube root will become the rule for the n^{th} root, by first extracting the n^{th} root of the first term, taking for a divisor n times this root raised to the $n-1$ power, and raising the partial roots to the n^{th} power, instead of to the cube.

2. Extract the 4th root of

$$16a^4-96a^3x+216a^2x^2-216ax^3+81a^4.$$

$$\begin{array}{r|l} 16a^4-96a^3x+216a^2x^2-216ax^3+81a^4 & 2a-3x \\ \hline (2a-3x)^4=16a^4-96a^3x+216a^2x^2-216ax^3+81a^4 & 32a^3=4\times(2a)^3 \end{array}$$

We first extract the 4th root of $16a^4$, which is $2a$. We then raise $2a$ to the third power, and multiply by 4, the index of the root: this gives the divisor $32a^3$. This divisor is contained in the second term $-96a^3x$, $-3x$ times, which is the second term of the root. Raising the whole root to the 4th power, we find the power equal to the given polynomial.

3. Find the cube root of

$$x^6+6x^5-40x^3+96x-64.$$

4. Find the cube root of

$$15x^4-6x+x^6-6x^5-20x^3+15x^2+1.$$

5. Find the 5th root of

$$32x^5-80x^4+80x^3-40x^2+10x-1.$$

Calculus of Radicals.

186. When it is required to extract a certain root of a monomial or polynomial which is not a perfect power, it can only be indicated by writing the proposed quantity after the sign $\sqrt{}$, and placing over this sign the number which denotes the degree of the root to be extracted. This number is called the *index of the root, or of the radical.*

A radical expression may be reduced to its simplest terms, by observing that, *the* n[th] *root of a product is equal to the product of the* n[th] *roots of its different factors.*

Or, in algebraic terms:

$$\sqrt[n]{abcd}=\sqrt[n]{a}\times\sqrt[n]{b}\times\sqrt[n]{c}\times\sqrt[n]{d}.$$

For, raising both members to the n^{th} power, we have for the first,

$$\left(\sqrt[n]{abcd}\right)^n=abcd\ldots,\quad\text{and for the second,}$$

$$(\sqrt[n]{a}\times\sqrt[n]{b}\times\sqrt[n]{c}\times\sqrt[n]{d}\ldots)^n=(\sqrt[n]{a})^n.(\sqrt[n]{b})^n.(\sqrt[n]{c})^n.(\sqrt[n]{d})^n\ldots=abcd.$$

Therefore, since the n^{th} powers of these quantities are equal, the quantities themselves must be equal.

Let us take the expression $\sqrt[3]{54a^4b^3c^2}$, which cannot be replaced by a rational monomial, since 54 is not a perfect cube, and the exponents of a and c are not divisible by 3: but we can put it under the form

$$\sqrt[3]{54a^4b^3c^2}=\sqrt[3]{27a^3b^3}.\sqrt[3]{2ac^2}=3ab\sqrt[3]{2ac^2}.$$

In like manner,

$$\sqrt[3]{8a^2}=2\sqrt[3]{a^2};\quad\sqrt[4]{48a^5b^8c^6}=2ab^2c\sqrt[4]{3ac^2};$$

$$\sqrt[6]{192a^7bc^{12}}=\sqrt[6]{64a^6c^{12}}\times\sqrt[6]{3ab}=2ac^2\sqrt[6]{3ab}.$$

In the expressions, $3ab\sqrt[3]{2ac^2}$, $2\sqrt[3]{a^2}$, $2ab^2c\sqrt[4]{3ac^2}$, the quantities placed before the radical, are called *co-efficients* of the radical.

187. The rule of (Art. 125) gives rise to another kind of simplification.

Take, for example, the radical expression, $\sqrt[6]{4a^2}$; from this rule we have, $\sqrt[6]{4a^2}=\sqrt[3]{\sqrt{4a^2}}$, and as the quantity affected with the radical of the second degree $\sqrt{\ }$, is a perfect square, its root can be extracted, hence

$$\sqrt[6]{4a^2}=\sqrt[3]{2a}.$$

In like manner,

$$\sqrt[4]{36a^2b^2}=\sqrt{\sqrt{36a^2b^2}}=\sqrt{6ab}.$$

In general, $\sqrt[mn]{a^n}=\sqrt[m]{\sqrt[n]{a^n}}=\sqrt[m]{a}$; that is, when the index of a radical is multiplied by any number n, and the quantity under the the radical sign is an exact n^{th} power, *we can, without changing the value of the radical, divide its index by* n, *and extract the* n$^{\text{th}}$ *root of the quantity under the sign.*

This proposition is the inverse of another, not less important, viz. *we can multiply the index of a radical by any number, provided we raise the quantity under the sign to a power of which this number denotes the degree.*

Thus, $\sqrt[m]{a}=\sqrt[mn]{a^n}$. For, a is the same thing as $\sqrt[n]{a^n}$; hence,

$$\sqrt[m]{a}=\sqrt[m]{\sqrt[n]{a^n}}=\sqrt[mn]{a^n}.$$

This last principle serves to reduce two or more radicals to the same index.

For example, let it be required to reduce the two radicals $\sqrt[3]{2a}$ and $\sqrt[4]{(a+b)}$ to the same index.

By multiplying the index of the first by 4, the index of the second, and raising the quantity $2a$ to the fourth power; then multiplying the index of the second by 3, the index of the first, and cubing $a+b$, the values of the radicals will not be changed, and the expressions will become

$$\sqrt[3]{2a}=\sqrt[12]{2^4a^4}=\sqrt[12]{16a^4};\quad \sqrt[4]{(a+b)}=\sqrt[12]{(a+b)^3}.$$

188. Hence to reduce radicals to a common index we have the following

RULE.

Multiply the index of each radical by the product of the indices of all the other radicals, and raise the quantity under each radical sign to a power denoted by this product.

This rule, which is analogous to that given for the reduction of fractions to a common denominator, is susceptible of some modifications.

For example, reduce the radicals $\sqrt[4]{a}$, $\sqrt[6]{5b}$, $\sqrt[8]{a^2+b^2}$, to the same index.

As the numbers 4, 6, 8, have common factors, and 24 is the most simple multiple of the three numbers, it is only necessary to multiply the first by 6, the second by 4, and the third by 3, and to raise the quantities under each radical sign to the 6th, 4th, and 3d powers respectively, which gives

$$\sqrt[4]{a}=\sqrt[24]{a^6};\quad \sqrt[6]{5b}=\sqrt[24]{5^4b^4},\quad \sqrt[8]{a^2+b^2}=\sqrt[24]{(a^2+b^2)^3}.$$

In applying the above rules to numerical examples, beginners very often make mistakes similar to the following, viz.: In reducing the radicals $\sqrt[3]{2}$ and $\sqrt{3}$ to a common index, after having multiplied the index of the first (3), by that of the second (2), and the index of the second by that of the first, then, instead of multiplying the *exponent* of the quantity under the first sign by 2, and the *exponent* of that under the second by 3, they often multiply the *quantity* under the first sign by 2, and the *quantity* under the second by 3. Thus, they would have

$$\sqrt[3]{2}=\sqrt[6]{2\times2}=\sqrt[6]{4},\quad\text{and}\quad \sqrt{3}=\sqrt[6]{3\times3}=\sqrt[6]{9}.$$

Whereas, they should have, by the foregoing rule,

$$\sqrt[3]{2}=\sqrt[6]{(2)^2}=\sqrt[6]{4},\quad\text{and}\quad \sqrt{3}=\sqrt[6]{(3)^3}=\sqrt[6]{27}.$$

Reduce $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt[5]{\frac{1}{2}}$, to the same index.

Addition and Subtraction of Radicals.

189. Two radicals are *similar*, when they have the same index, and the same quantity, under the sign. Thus, $3\sqrt{ab}$ and $7\sqrt{ab}$, are similar radicals, as also $3a^2\sqrt[3]{b^2}$, and $9c^3\sqrt[3]{b^2}$.

Therefore, to add or subtract similar radicals, *add or subtract their co-efficients, and prefix the sum or difference to the common radical.*

Thus, $3\sqrt[3]{b}+2\sqrt[3]{b}=5\sqrt[3]{b}$, $3\sqrt[3]{b}-2\sqrt[3]{b}=\sqrt[3]{b}$,

$$3a\sqrt[4]{b}\pm2c\sqrt[4]{b}=(3a\pm2c)\sqrt[4]{b}.$$

Sometimes when two radicals are dissimilar, they can be reduced to similar radicals by Arts. 186 and 187. For example,

$$\sqrt{48ab^2}+b\sqrt{75a}=4b\sqrt{3a}+5b\sqrt{3a}=9b\sqrt{3a}.$$

$$\sqrt[3]{8a^3b+16a^4}-\sqrt[3]{b^4+2ab^3}=2a\sqrt[3]{b+2a}-b\sqrt[3]{b+2a}$$
$$=(2a-b)\sqrt[3]{b+2a};$$

$$3\sqrt[6]{4a^2}+2\sqrt[3]{2a}=3\sqrt[3]{2a}+2\sqrt[3]{2a}=5\sqrt[3]{2a}.$$

When the radicals are dissimilar, and irreducible, they can only be added or subtracted by means of the signs $+$ or $-$.

Multiplication and Division.

190. We will first suppose that the radicals have a common index.

Let it be required to multiply or divide $\sqrt[n]{a}$ by $\sqrt[n]{b}$. We have

$$\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}, \text{ and } \sqrt[n]{a}\div\sqrt[n]{b}=\sqrt[n]{\frac{a}{b}}.$$

For by raising $\sqrt[n]{a}\,.\sqrt[n]{b}$ and $\sqrt[n]{ab}$ to the n^{th} power, we obtain the same result ab; hence the two expressions are equal.

In like manner, $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\sqrt[n]{\dfrac{a}{b}}$ raised to the n^{th} power give $\dfrac{a}{b}$: hence these two expressions are equal. Therefore we have the following

RULE.

Multiply or divide the quantities under the sign by each other, and give to the product, or quotient, the common radical sign. If they have co-efficients, first multiply or divide them separately.

Thus,

$$2a\sqrt[3]{\frac{a^2+b^2}{c}}\times-3a\sqrt[3]{\frac{(a^2+b^2)^2}{d}}=-6a^2\sqrt[3]{\frac{(a^2+b^2)^3}{cd}}.$$

or, reducing to its simplest terms,

$$-\frac{6a^2(a^2+b^2)}{\sqrt[3]{cd}};$$

$$3a\sqrt[4]{8a^2}\times 2b\sqrt[4]{4a^2c}=6ab\sqrt[4]{32a^4c}=12a^2b\sqrt[4]{2c}.$$

$$\frac{\sqrt[3]{a^2b^2+b^4}}{\sqrt[3]{\frac{a^2-b^2}{8b}}}=\sqrt[3]{\frac{8b(a^2b^2+b^4)}{a^2-b^2}}=2b\sqrt[3]{\frac{a^2+b^2}{a^2-b^2}}.$$

When the radicals have not a common index, they should be reduced to one.

For example, $3a\sqrt[6]{b}\times 5b\sqrt[8]{2c}=15ab\times\sqrt[24]{8b^4c^3}.$

EXAMPLES.

1. Multiply $\sqrt{2}\times\sqrt[3]{3}$ by $\sqrt[4]{\frac{1}{2}}\times\sqrt[3]{\frac{1}{3}}$

Ans. $\sqrt[12]{8}.$

2. Multiply $2\sqrt{15}$ by $3\sqrt[3]{10}$

Ans. $6\sqrt[6]{337500}.$

3. Multiply $4\sqrt[5]{\frac{2}{3}}$ by $2\sqrt{\frac{3}{4}}.$

Ans. $8\sqrt[10]{\frac{27}{256}}.$

4. Reduce $\dfrac{2\sqrt{3}\times\sqrt[3]{4}}{\frac{1}{2}\sqrt[4]{2}\times\sqrt[3]{3}}$ to its lowest terms.

Ans. $4\sqrt[12]{288}.$

5. Reduce $\sqrt{\dfrac{\sqrt{\frac{1}{2}}\times 2\sqrt[3]{3}}{4\sqrt[3]{2}\times\sqrt{3}}}$ to its lowest terms.

Ans. $\frac{1}{2}\sqrt[12]{\frac{2}{3}}$

6. Multiply $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$ to ether.

Ans. $\sqrt[12]{648000}.$

7. Multiply $\sqrt[7]{\frac{4}{3}}$, $\sqrt[3]{\frac{1}{2}}$ and $\sqrt[14]{6}$ together.

Ans. $\sqrt[42]{\frac{2}{27}}$.

8. Multiply $\left(4\sqrt{\frac{7}{3}}+5\sqrt{\frac{1}{2}}\right)$ by $\left(\sqrt{\frac{7}{3}}+2\sqrt{\frac{1}{2}}\right)$.

Ans. $\frac{43}{3}+\frac{13}{6}\sqrt{42}$.

9. Divide $\frac{1}{2}\sqrt{\frac{1}{2}}$ by $\left(\sqrt{2}+3\sqrt{\frac{1}{2}}\right)$.

Ans. $\frac{1}{10}$.

10. Divide 1 by $\sqrt[4]{a}+\sqrt[4]{b}$.

Ans. $\frac{\sqrt[4]{a^3}-\sqrt[4]{a^2b}+\sqrt[4]{ab^2}-\sqrt[4]{b^3}}{a-b}$

11. Divide $\sqrt[4]{a}+\sqrt[4]{b}$ by $\sqrt[4]{a}-\sqrt[4]{b}$.

Ans. $\frac{a+b+2\sqrt{ab}+2\sqrt[4]{a^3b}+2\sqrt[4]{ab^3}}{a-b}$.

Formation of Powers, and Extraction of Roots.

191. By raising $\sqrt[m]{a}$ to the n^{th} power, we have

$$(\sqrt[m]{a})^n=\sqrt[m]{a}\times\sqrt[m]{a}\times\sqrt[m]{a}\ .\ .\ .\ =\sqrt[m]{a^n},$$

by the rule just given for the multiplication of radicals. Hence, for raising a radical to any power, we have the following

RULE.

Raise the quantity under the sign to the given power, and affect the result with the radical sign, having the primitive index. If it has a co-efficient, first raise it to the given power.

Thus, $(\sqrt[4]{4a^3})^2=\sqrt[4]{(4a^3)^2}=\sqrt[4]{16a^6}=2a\sqrt[4]{a^2}$;

$$(3\sqrt[3]{2a})^5=3^5.\sqrt[3]{(2a)^5}=243\sqrt[3]{32a^5}=486a\sqrt[3]{4a^2}$$

When the index of the radical is a multiple of the power, the result can be reduced.

For, $\sqrt[4]{2a}=\sqrt[4]{\sqrt{2a}}$ (Art. 176): hence, to square $\sqrt[4]{2a}$, we have only to omit the first radical, which gives $\left(\sqrt[4]{2a}\right)^2=\sqrt{2a}$.

Again, to square $\sqrt[6]{3b}$, we have $\sqrt[6]{3b}=\sqrt{\sqrt[3]{3b}}\cdot$ hence

$$\left(\sqrt[6]{3b}\right)^2=\sqrt[3]{3b}.$$

Consequently, *when the index of the radical is divisible by the exponent of the power, perform this division, leaving the quantity under the radical unchanged.*

To extract the root of a radical, *multiply the index of the radical by the index of the root to be extracted, leaving the quantity under the sign unchanged.*

Thus, $\sqrt[3]{\sqrt[4]{3c}}=\sqrt[12]{3c}$; $\sqrt{\sqrt[3]{5c}}=\sqrt[6]{5c}$.

This rule is nothing more than the principle of Art. 176, enunciated in an inverse order.

When the quantity under the radical is a perfect power, of the degree of either of the roots to be extracted, the result can be reduced.

Thus, $\sqrt[3]{\sqrt[4]{8a^3}}$ being equal to $\sqrt[4]{\sqrt[3]{8a^3}}$ it reduces to $\sqrt[4]{2a}$.

In like manner, $\sqrt{\sqrt[5]{9a^2}}=\sqrt[5]{\sqrt{9a^2}}=\sqrt[5]{3a}$.

It is evident that $\sqrt[n]{\sqrt[m]{a}}=\sqrt[m]{\sqrt[n]{a}}$; because both expressions are equal to $\sqrt[mn]{a}$ (Art. 176).

192. The rules just demonstrated for the calculus of radicals, principally depend upon the fact that the n^{th} root of the product of several factors is equal to the product of the n^{th} roots of these factors; and the demonstration of this principle depends upon this: *When the powers, of the same degree, of two expressions are equal,*

the expressions are also equal. Now this last proposition, which is true for absolute numbers, is not always true for algebraic expressions.

To prove this, we will show that the same number can have *more than one square root, cube root, fourth root, &c.*

For, denote the general expression of the square root of a by x, and the *arithmetical* value of it by p; we have the equation $x^2=a$, or $x^2=p^2$, whence $x=\pm p$. Hence we see that the square of p, which is the root of a, will give a, whether its sign be $+$ or $-$.

In the second place, let x be the general expression of the cube root of a, and p the numerical value of this root; we have the equation

$$x^3=a, \text{ or } x^3=p^3.$$

This equation is satisfied by making $x=p$.

Observing that the equation $x^3=p^3$ can be put under the form $x^3-p^3=0$, and that the expression x^3-p^3 is divisible by $x-p$, (Art. 59), which gives the exact quotient, x^2+px+p^2, the above equation can be transformed into

$$(x-p)\,(x^2+px+p^2)=0.$$

Now, every value of x which will satisfy this equation will satisfy the first equation. But this equation can be verified by supposing $x-p=0$, whence $x=p$; or by supposing

$$x^2+px+p^2=0,$$

from which last we have

$$x=-\frac{p}{2}\pm\frac{p}{2}\sqrt{-3}, \text{ or } x=p\left(\frac{-1\pm\sqrt{-3}}{2}\right).$$

Hence, *the cube root of* a, *admits of three different algebraic values, viz.*

$$p,\ \ p\left(\frac{-1+\sqrt{-3}}{2}\right), \text{ and } p\left(\frac{-1-\sqrt{-3}}{2}\right).$$

Again, resolve the equation $x^4=p^4$, in which p denotes the arithmetical value of $\sqrt[4]{a}$. This equation can be put under the form $x^4-p^4=0$. Now this expression reduces to $(x^2-p^2)\,(x^2+p^2)$.

Hence the equation reduces to $(x^2-p^2)(x^2+p^2)=0$, and can be satisfied by supposing $x^2-p^2=0$, whence $x=\pm p$; or by supposing $x^2+p^2=0$, whence $x=\pm\sqrt{-p^2}=\pm p\sqrt{-1}$.

We therefore obtain four *different algebraic* expressions for the fourth root of a.

For another example, resolve the equation . . . . $x^6=p^6$,
which can be put under the form . . . . $x^6-p^6=0$.
Now x^6-p^6 reduces to . . . . $(x^3-p^3)(x^3+p^3)$,
therefore the equation becomes . . . . $(x^3-p^3)(x^3+p^3)=0$.
But $x^3-p^3=0$, gives

$$x=p,\ \text{and}\ x=p\left(\frac{-1\pm\sqrt{-3}}{2}\right).$$

And if in the equation $x^3+p^3=0$, we make $p=-p'$, it becomes $x^3-p'^3=0$ from which we deduce $x=p'$, and

$$x=p'\left(\frac{-1\pm\sqrt{-3}}{2}\right);$$

or, substituting for p' its value, $-p$,

$$x=-p\ \text{and}\ x=-p\left(\frac{-1\pm\sqrt{-3}}{2}\right).$$

Therefore the value of x, in the equation $x^6-p^6=0$, and consequently the 6th root of a, admits of *six* values, p, ap, $a'p$, $-p$, $-ap$, $-a'p$, by making

$$a=\frac{-1+\sqrt{-3}}{2},\quad a'=\frac{-1-\sqrt{-3}}{2}.$$

We may then conclude from analogy, that x in every equation of the form $x^m-a=0$, or $x^m-p^m=0$, is susceptible of m different values, that is, the m^{th} root of a number admits of m *different algebraic values.*

193. If in the preceding equations and the results corresponding to them, we suppose as a particular case $a=1$, whence $p=1$, we shall obtain the second, third, fourth, &c. roots of unity. Thus $+1$ and -1 are the two *square roots of unity*, because the equation $x^2-1=0$, gives $x=\pm1$.

In like manner $+1$, $\dfrac{-1+\sqrt{-3}}{2}$, $\dfrac{-1-\sqrt{3}}{2}$, are the three cube roots of unity, or the roots of $x^3-1=0$. And

$+1$, -1, $+\sqrt{-1}$, $-\sqrt{-1}$, are the four fourth roots of unity, or the roots of $x^4-1=0$.

194. It results from the preceding analysis, that the rules for the calculus of radicals, which are exact when applied to absolute numbers, are susceptible of some *modifications*, when applied to *expressions or symbols which are purely algebraic*; these modifications are more particularly necessary when applied to *imaginary expressions*, and are a consequence of what has been said in (Art. 192).

For example, the product of $\sqrt{-a}$ by $\sqrt{-a}$, by the rule of (Art. 190), would be

$$\sqrt{-a}\times\sqrt{-a}=\sqrt{+a^2}.$$

Now, $\sqrt{a^2}$ is equal to $\pm a$ (Art. 192); there is, then, apparently, an uncertainty as to the sign with which a should be affected. Nevertheless, the true answer is $-a$; for, in order to square $\sqrt{m}$, it is only necessary to suppress the radical; but the $\sqrt{-a}\times\sqrt{-a}$ reduces to $(\sqrt{-a})^2$, and is therefore equal to $-a$.

Again, let it be required to form the product $\sqrt{-a}\times\sqrt{-b}$, by the rule of (Art. 190), we shall have

$$\sqrt{-a}\times\sqrt{-b}=\sqrt{+ab}.$$

Now, $\sqrt{ab}=\pm p$ (Art. 192), p being the arithmetical value of the square root of ab; but I say that the true result should be $-p$ or $-\sqrt{ab}$, so long as both the radicals $\sqrt{-a}$ and $\sqrt{-b}$ are considered to be affected with the sign $+$.

For, $\quad \sqrt{-a}=\sqrt{a}.\sqrt{-1}$ and $\sqrt{-b}=\sqrt{b}.\sqrt{-1}$;

hence

$$\sqrt{-a}\times\sqrt{-b}=\sqrt{a}.\sqrt{-1}\times\sqrt{-b}\times\sqrt{-1}=\sqrt{ab}(\sqrt{-1})^2$$
$$=\sqrt{ab}\times-1=-\sqrt{ab}.$$

Upon this principle we find the different powers of $\sqrt{-1}$ to be, as follows:

$$\sqrt{-1}=\sqrt{-1},\quad (\sqrt{-1})^2=-1,$$
$$(\sqrt{-1})^3=(\sqrt{-1})^2.\sqrt{-1}=-\sqrt{-1},$$

and $$(\sqrt{-1})^4=(\sqrt{-1})^2.(\sqrt{-1})^2=-1\times-1=+1.$$

Again, let it be proposed to determine the product of $\sqrt[4]{-a}$ by the $\sqrt[4]{-b}$ which, from the rule, will be $\sqrt[4]{+ab}$, and consequently will give the four values (Art. 192).

$$+\sqrt[4]{ab},\ -\sqrt[4]{ab},\ +\sqrt[4]{ab}.\sqrt{-1},\ -\sqrt[4]{ab}.\sqrt{-1}.$$

To determine the true product, observe that

$$\sqrt[4]{-a}=\sqrt[4]{a}.\sqrt[4]{-1},\quad \sqrt[4]{-b}=\sqrt[4]{b}.\sqrt[4]{-1}.$$

But $$\sqrt[4]{-1}\times\sqrt[4]{-1}=(\sqrt[4]{-1})^2=\left(\sqrt{\sqrt{-1}}\right)^2=\sqrt{-1};$$

hence $$\sqrt[4]{-a}.\sqrt[4]{-b}=\sqrt[4]{ab}.\sqrt{-1}.$$

We will apply the preceding calculus to the verification of the expression $\dfrac{-1+\sqrt{-3}}{2}$, considered as a root of the equation $x^3-1=0$, that is, as the cube root of 1 (Art. 192).

From the formula $\qquad (a+b)^3=a^3+3a^2b+3ab^2+b^3,$

we have $$\left(\frac{-1+\sqrt{-3}}{2}\right)^3$$

$$=\frac{(-1)^3+3(-1)^2.\sqrt{-3}+3(-1).(\sqrt{-3})^2+(\sqrt{-3})^3}{8}$$

$$=\frac{-1+3\sqrt{-3}-3\times-3-3\sqrt{-3}}{8}=\frac{8}{8}=1.$$

The second value, $\dfrac{-1-\sqrt{-3}}{2}$, may be verified in the same manner.

Theory of Exponents.

195. In extracting the n^{th} root of a quantity a^m, we have seen that when m is a multiple of n, we should divide the exponent m by

n the index of the root; but when m is not divisible by n, in which case the root cannot be extracted algebraically, it has been agreed to indicate this operation by indicating the division of the two exponents.

Hence, $\sqrt[n]{a^m}=a^{\frac{m}{n}}$, from *a convention founded upon the rule for the exponents, in the extraction of the roots of monomials. In such expressions, the numerator indicates the power to which the quantity is to be raised, and the denominator, the root to be extracted.*

Therefore, $\sqrt[3]{a^2}=a^{\frac{2}{3}}$; $\sqrt[4]{a^7}=a^{\frac{7}{4}}$.

In like manner, suppose it is required to divide a^m by a^n. We know that the exponent of the divisor should be subtracted from the exponent of the dividend, when $m>n$, which gives $\frac{a^m}{a^n}=a^{m-n}$. But when $m<n$, in which case the division cannot be effected algebraically, it has been agreed to subtract the exponent of the divisor from that of the dividend. Let p be the absolute difference between n and m; then will $n=m+p$, whence $\frac{a^m}{a^{m+p}}=a^{-p}$; but $\frac{a^m}{a^{m+p}}$ reduces to $\frac{1}{a^p}$; hence $a^{-p}=\frac{1}{a^p}$.

Therefore, the *expression* a^{-p} is the symbol of a division which it has been impossible to perform; and its true value is the quotient represented by unity divided by the letter a, affected with the exponent p, taken positively. Thus,

$$a^{-3}=\frac{1}{a^3};\ a^{-5}=\frac{1}{a^5}.$$

The notation of fractional exponents has the advantage of giving an entire form to fractional expressions.

From the combination of the extraction of a root, and an impossible division, there results another notation, viz. *negative fractional exponents.*

In extracting the n^{th} root of $\frac{1}{a^m}$, we have first $\frac{1}{a^m}=a^{-m}$, hence $\sqrt[n]{\frac{1}{a^m}}=\sqrt[n]{a^{-m}}=a^{-\frac{m}{n}}$, by substituting a fractional exponent for the radical sign.

Hence, $a^{\frac{m}{n}}$, a^{-p}, $a^{-\frac{m}{n}}$, *are conventional expressions, founded upon preceding rules*, and equivalent to $\sqrt[n]{a^m}$, $\frac{1}{a^p}$, $\sqrt[n]{\frac{1}{a^m}}$.

We may therefore substitute the second for the first, or reciprocally.

As a^p is called a to the p power, when p is a positive whole number, so by analogy, $a^{\frac{m}{n}}$, a^{-p}, $a^{-\frac{m}{n}}$, are called a to the $\frac{m}{n}$ power, a to the $-p$ power, a to the $-\frac{m}{n}$ power, which has induced algebraists to generalize the word *power*; but it would, perhaps, be more accurate to say, a, exponent $-\frac{m}{n}$, exponent $-p$, exponent $-\frac{m}{n}$ using the word *power* only when we wish to designate the product of a number multiplied by itself two or more times.

Since a^{-p} and $\frac{1}{a^p}$ are equivalent expressions, also a^p and $\frac{1}{a^{-p}}$, we conclude that *any factor may be transferred from the numerator to the denominator, or from the denominator to the numerator, by changing the sign of its exponent.*

Multiplication of Quantities affected with any Exponents.

196. In order to multiply $a^{\frac{3}{5}}$ by $a^{\frac{2}{3}}$, it is only necessary to *add the two exponents*, and we have

$$a^{\frac{3}{5}}\times a^{\frac{2}{3}}=a^{\frac{3}{5}+\frac{2}{3}}=a^{\frac{19}{15}}.$$

For, by (Art. 195), $a^{\frac{3}{5}}=\sqrt[5]{a^3}$; $a^{\frac{2}{3}}=\sqrt[3]{a^2}$

hence, $$a^{\frac{3}{5}}\times a^{\frac{2}{3}}=\sqrt[5]{a^3}\times\sqrt[3]{a^2};$$

or, performing the multiplication by the rule of (Art. 190),

$$a^{\frac{3}{5}} \times a^{\frac{2}{3}} = \sqrt[15]{a^{19}} = a^{\frac{19}{15}}.$$

Again, multiplying $a^{-\frac{3}{4}}$ by $a^{\frac{5}{6}}$, we have

$$a^{-\frac{3}{4}} \times a^{\frac{5}{6}} = a^{-\frac{3}{4}+\frac{5}{6}} = a^{-\frac{9}{12}+\frac{10}{12}} = a^{\frac{1}{12}};$$

for, $$a^{-\frac{3}{4}} = \sqrt[4]{\frac{1}{a^3}},\ a^{\frac{5}{6}} = \sqrt[6]{a^5};$$

hence

$$a^{-\frac{3}{4}} \times a^{\frac{5}{6}} = \sqrt[4]{\frac{1}{a^3}} \times \sqrt[6]{a^5} = \sqrt[12]{\frac{1}{a^9}} \times \sqrt[12]{a^{10}} = \sqrt[12]{\frac{a^{10}}{a^9}} = \sqrt[12]{a} = a^{\frac{1}{12}}$$

In general, multiplying $a^{-\frac{m}{n}}$ by $a^{\frac{p}{q}}$; we have

$$a^{-\frac{m}{n}} \times a^{\frac{p}{q}} = a^{-\frac{m}{n}+\frac{p}{q}} = a^{\frac{np-mq}{nq}}.$$

Therefore, in order to multiply two monomials affected with any exponents whatever, *add together the exponents of the same letter ;* this rule is the same as that given in (Art. 41), for quantities affected with entire exponents.

From this rule we will find that

$$a^{\frac{3}{4}} b^{-\frac{1}{2}} c^{-1} \times a^2 b^{\frac{2}{3}} c^{\frac{3}{5}} = a^{\frac{11}{4}} b^{\frac{1}{6}} c^{-\frac{2}{5}};$$

and $$3a^{-2} b^{\frac{2}{3}} \times 2a^{-\frac{4}{5}} b^{\frac{1}{2}} c^2 = 6a^{-\frac{14}{5}} b^{\frac{7}{6}} c^2.$$

Division.

197. To divide one monomial by another when both are affected with any exponent whatever, follow the rule given in Art. 50 for quantities affected with entire and positive exponents; that is, *subtract the exponents of the letters in the divisor from the exponents of the same letters in the dividend.*

For, the exponent of each letter in the quotient must be such, that added to that of the same letter in the divisor, the sum shall be equal to the exponent of the letter the dividend; hence the exponent in the quotient is equal to the difference between the exponent in the dividend and that in the divisor

EXAMPLES.

$$a^{\frac{2}{3}} \div a^{-\frac{3}{4}} = a^{\frac{2}{3}-(-\frac{3}{4})} = a^{\frac{17}{12}};$$

$$a^{\frac{3}{4}} \div a^{\frac{4}{5}} = a^{\frac{3}{4}-\frac{4}{5}} = a^{-\frac{1}{20}};$$

$$a^{\frac{2}{5}} \times b^{\frac{3}{4}} \div a^{-\frac{1}{2}} b^{\frac{7}{8}} = a^{\frac{9}{10}} b^{-\frac{1}{8}}.$$

Formation of Powers.

198. To form the n^{th} power of a monomial, affected with any exponent whatever, observe the rule given in Art. 182, viz. *multiply the exponent of each letter by the exponent* m *of the power;* for, to raise a quantity to the m^{th} power, is the same thing as to multiply it by itself $m-1$ times; therefore, by the rule for multiplication, the exponent of each letter must be added to itself $m-1$ times, or multiplied by m.

$$\text{Thus,}\quad \left(a^{\frac{3}{4}}\right)^5 = a^{\frac{15}{4}};\ \left(a^{\frac{2}{3}}\right)^3 = a^{\frac{6}{3}} = a^2;$$

$$\left(2a^{-\frac{1}{2}} b^{\frac{3}{4}}\right)^6 = 64a^{-3} b^{\frac{9}{2}};\ \left(a^{-\frac{5}{6}}\right)^{12} = a^{-10}.$$

Extraction of Roots.

199. To extract the n^{th} root of a monomial, follow the rule given in Art. 182, viz. *divide the exponent of each letter by the index of the root.*

For, the exponent of each letter in the result should be such, that multiplied by n, the index of the root to be extracted, there will be produced the exponent with which the letter is affected in the proposed monomial; therefore, the exponents in the result must be respectively equal to the quotients arising from the division of the exponents in the proposed monomial, by n, the index of the root.

$$\text{Thus,}\quad \sqrt[3]{a^{\frac{2}{3}}} = a^{\frac{2}{9}};\ \sqrt[4]{a^{\frac{8}{11}}} = a^{\frac{2}{11}};\ \sqrt{a^{-\frac{3}{4}}} = a^{-\frac{3}{8}}$$

$$\sqrt[3]{a^{\frac{3}{5}} b^{-2}} = a^{\frac{1}{5}} b^{-\frac{2}{3}}.$$

The last three rules have been easily deduced from the rule for multiplication; but we might give a direct demonstration for them, by going back to the origin of quantities affected with fractional and negative exponents.

We will terminate this subject by an operation which contains implicitly the demonstration of the two preceding rules.

Let it be required to raise $a^{\frac{m}{n}}$ to the $-\frac{r}{s}$ power;

We say that,

$$\left(a^{\frac{m}{n}}\right)^{-\frac{r}{s}} = a^{\frac{m}{n} \times -\frac{r}{s}} = a^{-\frac{mr}{ns}}.$$

For, by going back to the origin of these notations, we find that

$$\left(a^{\frac{m}{n}}\right)^{-\frac{r}{s}} = \sqrt[s]{\frac{1}{\left(a^{\frac{m}{n}}\right)^r}} = \sqrt[s]{\frac{1}{(\sqrt[n]{a^m})^r}} = \sqrt[s]{\frac{1}{\sqrt[n]{a^{mr}}}}$$

$$= \sqrt[s]{\sqrt[n]{\frac{1}{a^{mr}}}} = \sqrt[ns]{a^{-mr}} = a^{-\frac{mr}{ns}}.$$

The advantage derived from the use of exponents consists principally in this: The operations performed upon expressions of this kind require no other rules than those established for the calculus of quantities affected with entire exponents. Besides, this calculus is reduced to simple operations upon fractions, with which we are already familiar.

200. REMARK. In the resolution of certain questions, we shall be led to consider quantities affected with *incommensurable exponents.* Now, it would seem that the rules just established for commensurable exponents, ought to be demonstrated for the case in which the exponents are incommensurable; but we will observe, that an incommensurable, such as $\sqrt{3}$, $\sqrt[3]{11}$, is by its nature composed of an entire part, and a fraction which cannot be expressed exactly, but *to which it is possible to approximate as near as we please,* so that we may always conceive the incommensurable to be replaced by an exact fraction, which only differs from it by a quan-

tity less than any given quantity; and in applying the rules to the symbol which designates the incommensurable, it is necessary to understand that we apply it to the exact fraction which represents it approximatively.

EXAMPLES.

Reduce $\dfrac{2\sqrt{2}\times(3)^{\frac{1}{3}}}{\frac{1}{2}\sqrt{2}}$ to its simplest terms.

Ans. $4\sqrt[3]{3}$.

Reduce $\left\{\dfrac{\frac{1}{2}(2)^{\frac{1}{2}}\sqrt[3]{3}}{2\sqrt[4]{2}(3)^{\frac{1}{2}}}\right\}^4$ to its simplest terms.

Ans. $\dfrac{1}{384}\sqrt[3]{3}$.

Reduce $\sqrt{\left\{\dfrac{(\frac{1}{2})^3+\sqrt{3\frac{1}{2}}}{2\sqrt{2}.(\frac{3}{4})^{\frac{1}{2}}}\right\}^{\frac{1}{2}}}$ to its simplest terms.

Ans. $\sqrt[4]{\dfrac{1}{6}\left(\dfrac{1}{8}\sqrt{6}+\sqrt{21}\right)}$.

Demonstration of the Binomial Theorem in the case of any Exponent whatever.

201. Since the rules for the calculus of entire and positive exponents may be extended to the case of any exponent whatever, it is natural to suppose that the binomial formula, which serves to develop the m^{th} power of a binomial when m is entire and positive, will also effect this when m is any exponent whatever. In fact, analysts have discovered that this is the case, and they have deduced important consequences from it, *both for the extraction of roots by approximation, and the development of algebraic expressions into series.*

The following is a modification of Euler's demonstration.

We will remark, in the first place, that the binomial $x+a$ can be put under the form $x\left(1+\dfrac{a}{x}\right)$; whence there results

$$(x+a)^m = x^m\left(1+\frac{a}{x}\right)^m = x^m(1+z)^m, \text{ by making } \frac{a}{x}=z.$$

Therefore, if the formula

$$(1+z)^m = 1+mz+m\frac{m-1}{2}z^2+m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}z^3+, \&c. \quad (A)$$

is proved to be correct for any value of m, we may consider the formula

$$(x+a)^m = x^m+max^{m-1}+m\,.\,\frac{m-1}{2}a^2x^{m-2}$$

$$+m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}a^3x^{m-3}+, \&c. \quad (B)$$

exact for any value of m. For, by substituting $\frac{a}{x}$ for z in the formula (A), and multiplying by x^m, we obtain

$$(x+a)^m = x^m\left(1+m\frac{a}{x}+m\,.\,\frac{m-1}{3}\,.\,\frac{a^2}{x^2}+, \&c.\right),$$

from which, by performing the operations indicated, we obtain the formula (B).

Now, when m is a whole number, we have

$$(1+z)^m = 1+mz+m.\frac{m-1}{2}z^2+m.\frac{m-1}{2}\,.\,\frac{m-2}{3}z^3+, \&c.$$

But, if m is a fraction $\frac{p}{q}$, we do not know from what algebraic expression the development

$$1+mz+m\frac{m-1}{2}z^2+m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}z^3+, \&c. \ldots \text{ is derived.}$$

Denoting this unknown expression by y, we have the equation

$$y = 1+mz+m\,.\,\frac{m-1}{2}z^2+m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}z^3+, \&c. \ldots\ldots (1).$$

and it is now required to prove that $y=(1+z)^m$.

If m' is another fractional exponent, we shall have in like manner,

$$y' = 1+m'z+m'\,.\,\frac{m'-1}{2}z^2+m'\,.\,\frac{m'-1}{2}\,.\,\frac{m'-2}{3}z^3+, \&c. \ldots (2).$$

Multiplying the equations (1) and (2), member by member, we shall have for the first member of the result yy'. As to the second, it would be very difficult to obtain its true form, by the common rule for the multiplication of polynomials; but by observing that *the form of a product does not depend upon the particular values of the letters which enter into its two factors* (Art. 47), we see that the above product will be of the same form as in the case where m and m' are positive whole numbers. Now in this case we have

$$1+mz+m\cdot\frac{m-1}{2}z^2+\ldots=(1+z)^m,$$

$$1+m'z+m'\cdot\frac{m'-1}{2}z^2+\ldots=(1+z)^{m'},$$

whence

$$\left(1+mz+m\cdot\frac{m-1}{2}z^2+\ldots\right)\left(1+m'z+m'\cdot\frac{m'-1}{2}z^2+\ldots\ldots\right)$$

$$=(1+z)^{m+m'}=1+(m+m')z+(m+m')\frac{m+m'-1}{2}z^2+\ldots\ldots;$$

Therefore this form is true in the case in which m and m' are any quantities whatever, and we have

$$yy'=1+(m+m')z+(m+m')\frac{m+m'-1}{2}\cdot z^2+\ldots\ldots(3);$$

Let m'' be a third positive fractional exponent, we shall have

$$y''=1+m''z+m''\frac{m''-1}{2}z^2+\ldots$$

Multiplying the two last equations member by member, we have

$$yy'y''=1+(m+m'+m'')z+(m+m'+m'')\frac{m+m'+m''-1}{2}z^2+\ldots\ldots$$

Suppose the fractional exponent $m=\frac{p}{q}$. Take as many exponents m, m', m'', m''', &c. as there are units in q; we shall have, by making r equal to the sum of the exponents $m+m'+m''+m'''+\ldots$

$$yy'y''y'''=1+rz+r\cdot\frac{r-1}{2}z^2+r\cdot\frac{r-1}{2}\cdot\frac{r-2}{3}z^3+\ldots(4).$$

And by supposing $m=m'=m''=m'''$ . . . in which case

$$r=m+m+m+m+\ldots=mq,$$

the equation (4) becomes

$$y^q=1+mq\,.\,z+mq\,.\,\frac{mq-1}{2}z^2+mq\,.\,\frac{mq-1}{2}\,.\,\frac{mq-2}{3}z^3+\ldots$$

Now we have by hypothesis, $m=\frac{p}{q}$, or $mq=p$;

hence $$y^q=1+p\,.\,z+p\,.\,\frac{p-1}{2}z^2+p\,.\,\frac{p-1}{2}\,.\,\frac{p-2}{3}z^3+\ldots$$

but p is a whole number, therefore the second member of this equation is the development of $(1+z)^p$, which gives $y^q=(1+z)^p$, whence $y=(1+z)^{\frac{p}{q}}=(1+z)^m$; consequently

$$(1+z)^m=1+mz+m\frac{m-1}{2}z^2+m\frac{m-1}{2}\,.\,\frac{m-2}{3}z^3+\ldots$$

m being any positive fraction.

To demonstrate this formula, for the case in which m is a negative fraction or whole number, it is only necessary to suppose, $m'=-m$, in the equation (3) obtained from the equations (1) and (2), for when $m+m'=0$, the equation (3) reduces to $yy'=1$; whence $y=\frac{1}{y'}$.

But since m is negative by hypothesis, m' or $-m$, must be positive, and we have

$$y'=(1+z)^{m'},\ \text{hence}\ y=\frac{1}{(1+z)^{m'}}=(1+z)^{-m'}=(1+z)^m,$$

and consequently

$$(1+z)^m=1+mz+m\frac{m-1}{2}\,.\,z^2+\ldots\quad\text{or}$$

$$(1+z)^{-m'}=1-m'z-m'\frac{-m'-1}{2}z^2-m'\frac{(-m'-1)(-m'-2)}{1\,.\,2\,.\,3}z^3+,\ \&c.$$

Applications of the Binomial Theorem.

202. If in the formula $(x+a)^m=$

$$x^m\left(1+m\,.\,\frac{a}{x}+m\,.\,\frac{m-1}{2}\,.\,\frac{a^2}{x^2}+m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,.\,\frac{a^3}{x^3}+\,.\,.\,.\right)$$

we make $m=\frac{1}{n}$, it becomes $(x+a)^{\frac{1}{n}}$ or $\sqrt[n]{x+a}=$

$$x^{\frac{1}{n}}\left(1+\frac{1}{n}\,.\,\frac{a}{x}+\frac{1}{n}\,.\,\frac{\frac{1}{n}-1}{2}\,.\,\frac{a^2}{x^2}+\frac{1}{n}\,.\,\frac{\frac{1}{n}-1}{2}\,.\,\frac{\frac{1}{n}-2}{3}\,.\,a^3+\,.\,.\,.\right)$$

or, reducing, $\sqrt[n]{x+a}=$

$$x^{\frac{1}{n}}\left(1+\frac{1}{n}\,.\,\frac{a}{x}-\frac{1}{n}\,.\,\frac{n-1}{2n}\,.\,\frac{a^2}{x^2}+\frac{1}{n}\,.\,\frac{n-1}{2n}\,.\,\frac{2n-1}{3n}\,.\,\frac{a^3}{x^3}-\,.\,.\,.\,.\,.\right)$$

The fifth term can be found by multiplying the fourth by $\frac{3n-1}{4n}$ and by $\frac{a}{x}$, then changing the sign of the result, and so on.

203. Remark. When the terms of a series go on decreasing in value, the series is called a *decreasing* or *converging* series; and when they go on increasing in value, it is called a *diverging* series.

In a converging series the greater number of terms we take in the series, the nearer will we approximate to the true value of the proposed series. When the terms of the series are *alternately positive and negative*, we can, by taking a given number of terms, determine the *degree of approximation.*

For, let $a-b+c-d+e-f+\ .\ .\ .$, &c. be a decreasing series; b, c, d . . . being positive quantities, and let x denote the number represented by this series.

The numerical value of x is contained between any two consecutive sums of the terms of the series. For take any two consecutive sums,

$$a-b+c-d+e-f, \text{ and } a-b+c-d+e-f+g.$$

In the first, the terms which follow f, are $\overline{g-h}$, $+\overline{k-l}+$. but since the series is decreasing, the partial differences $g-h$, $k-l$, ... are positive numbers; therefore, in order to obtain the complete value of x, a certain positive number must be added to the sum $a-b+c-d+e-f$. Hence we have

$$a-b+c-d+e-f<x.$$

In the second series, the terms which follow $+g$ are $-\overline{h+k}$, $-\overline{l+m}$ . . . . Now, the partial differences $-h+k$, $-l+m$ . . ., are negative; therefore, in order to obtain the sum of the series, a negative quantity must be added to

$$a-b+c-d+e-f+g,$$

or, in other words, it is necessary to diminish it. Consequently

$$a-b+c-d+e-f+g>x.$$

Therefore, *x is comprehended between these two sums.*

The difference between these two sums is equal to g. But since x is comprised between them, their difference g must be greater than the difference between x and either of them; hence, the *error committed by taking a certain number of terms*, a$-$b$+$c$-$d$+$e$-$f, *for the value of* x, *is numerically less than the following term.*

206. The binomial formula also serves to develop algebraic expressions into series.

Take for example, the expression $\frac{1}{1-z}$, we have

$$\frac{1}{1-z}=(1-z)^{-1}.$$

In the binomial formula, make $m=-1$, $x=1$, and $a=-z$, it becomes

$$(1-z)^{-1}=1-1\,.\,(-z)-1\,.\,\frac{-1-1}{2}\,.\,(-z)$$

$$-1\,.\,\frac{-1-1}{2}\,.\,\frac{-1-2}{3}\,.\,(-z)^3-\,.\,.\,.$$

or. performing the operations, and observing that each term is composed of an even number of factors affected with the sign $-$,

$$(1-z)^{-1}=\frac{1}{1-z}=1+z+z^2+z^3+z^4+z^5+\ \ldots\ .$$

The same result will be obtained by applying the rule for division (Art. 55).

$$\begin{array}{lr|l}
 & 1 & 1-z \\ \cline{3-3}
\text{1st. remainder} \ . & +z & 1+z+z^2+z^3+z^4+\ \ldots \\
\text{2d.} \ \ . \ \ . \ \ . \ \ . & +z^2 & \\
\text{3d.} \ \ . \ \ . \ \ . \ \ . & +z^3 & \\
\text{4th.} \ \ . \ \ . \ \ . \ \ . & +z^4 & \\
 & +\ \ldots &
\end{array}$$

Again, take the expression $\frac{2}{(1-z)^3}$, or $2(1-z)^{-3}$.

We have $2(1-z)^{-3}=$

$$2[1-3.(-z)-3.\frac{-3-1}{2}.(-z)^2-3.\frac{-3-1}{2}.\frac{-3-2}{3}.(-z)^3-.]$$

or $\qquad 2(1-z)^{-3}=2(1+3z+6z^2+10z^3+15z^4+\ \ldots\ .)$

To develop the expression $\sqrt[3]{2z-z^2}$ which reduces to

$$\sqrt[3]{2z}\left(1-\frac{z}{2}\right)^{\frac{1}{3}}, \quad \text{we first find}$$

$$\left(1-\frac{z}{2}\right)^{\frac{1}{3}}=1+\frac{1}{3}\left(-\frac{z}{2}\right)+\frac{1}{3}.\frac{\frac{1}{3}-1}{2}.\left(-\frac{z}{2}\right)^3+\ \ldots$$

$$=1-\frac{1}{6}z-\frac{1}{36}z^2-\frac{5}{648}z^3-\ \ldots\ ;$$

hence $\quad \sqrt[3]{2z-z^2}=\sqrt[3]{2z}\left(1-\frac{1}{6}z-\frac{1}{36}z^2-\frac{5}{648}z^3-, \text{\&c.}\right)$

EXAMPLES.

1. To find the value of $\frac{1}{(a+b)^2}$, or its equal $(a+b)^{-2}$ in an infinite series.

2. To find the value of $\frac{r^2}{r+x}$, in an infinite series.

$$Ans.\quad r-x+\frac{x^2}{r}-\frac{x^3}{r^2}+\frac{x^4}{r^3}, \text{\&c.}$$

3. Required the square root of $\frac{a^2+x^2}{a^2-x^2}$ in an infinite series.

$$Ans.\quad 1+\frac{x^2}{a^2}+\frac{x^4}{2a^4}+\frac{x^6}{2a^6}, \text{\&c.}$$

4. Required the cube root of $\frac{a^2}{(a^2+x^2)^2}$ in an infinite series.

$$Ans.\quad \frac{1}{a^{\frac{2}{3}}}\times\left(1-\frac{2x^2}{3a^2}+\frac{5x^4}{9a^4}-\frac{40x^6}{81a^6}, \text{\&c.}\right)$$

Method of Indeterminate Co-efficients. Recurring Series

207. Algebraists have invented another method of developing algebraic expressions into series, which is in general, more simple than those we have just considered, and more extensive in its applications, as it can be applied to algebraic expressions of any nature whatever.

Before considering this method, it will be necessary to explain what is meant by the term *function.*

Let $a=b+c$. In this equation, a, b and c, mutually depend on each other for their values. For,

$$a=b+c,\ b=a-c, \text{ and } c=a-b.$$

The quantity a is said to be a *function* of b and c, b a *function* of a and c, and c a *function* of a and b. And generally, *when one quantity depends on others for its value, it is said to be a function of those quantities on which it depends*

In order to give some idea of this method of development, we will suppose it is required to develop the expression $\frac{a}{a'+b'x}$ into a series arranged according to the ascending powers of x. It is plain that the expression can be developed; for $\frac{a}{a'+b'x}$ reduces to $a(a'+b'x)^{-1}$; and by applying the binomial formula to it, we should evidently obtain a series of terms arranged according to the ascending powers of x. We may therefore assume

$$\frac{a}{a'+b'x}=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\ \dots\dots\ (1)$$

the co-efficients A, B, C, D, . . . being functions of a, a', b', but independent of x, it is required to determine these co-efficients, which are called *indeterminate co-efficients*.

For this purpose, multiply both members of the equation (1) by by $a'+b'x$; arranging the result with reference to the powers of x, and transposing a, it becomes

$$0=\left\{\begin{array}{c|c|c|c|l} Aa'+Ba' & x+Ca' & x^2+Da' & x^3+Ea' & x^4+\ \dots\ (2). \\ -a+Ab' & +Bb' & +Cb' & +Db' & \end{array}\right.$$

Now if the values of A, B, C, D, . . . were determined, the equation (1) would be satisfied by any value given to x; this must therefore be the case also in the equation (2).

But by supposing $x=0$, this equation becomes,

$$0=Aa'-a;$$

Whence $$A=\frac{a}{a'};$$

A being equal to $\frac{a}{a'}$, when $x=0$, this must be the value of it when x is any quantity whatever, since A is independent of x by hypothesis; therefore whatever may be the value of x, the equation (2) reduces to

$$0=\left\{\begin{array}{c|c|c|l} Ba' & x+Ca' & x^2+Da' & x^3+\ \dots\dots; \\ +Ab' & +Bb' & +Cb' & \end{array}\right. \text{ or, dividing by } x,$$

$$0=\left\{\begin{array}{c|c|c|l} Ba' & +Ca' & x+Da' & x^2+\ \dots\dots\ (3). \\ +Ab' & +Bb' & +Cb' & \end{array}\right.$$

This equation being also satisfied by any value for x, by making $x=0$, it becomes $$Ba'+Ab'=0.$$

Whence $\quad B=-\dfrac{Ab'}{a'}$, or $B=-\dfrac{a}{a'}\times\dfrac{b'}{a'}=-\dfrac{ab'}{a'^2}$.

As this must be the value of B whatever may be that of x, we will suppress the first term $Ba'+Ab'$ of the equation (3), which this value of B makes equal to zero, and divide by x; it thus becomes

$$0=\left\{\begin{array}{c|c|l} Ca'+Da' & x+Ea' & x^2+\ \ldots \\ +Bb'+Cb' & +Db' & \end{array}\right.$$

Making $x=0$, there results

$$Ca'+Bb'=0.$$

Whence $C=-\dfrac{Bb'}{a'}$, or $C=-\left(-\dfrac{ab'}{a'^2}\right)\times\dfrac{b'}{a'}=\dfrac{ab'^2}{a'^3}$.

In the same way we should find

$$Da'+Cb'=0,$$

Whence $D=-\dfrac{Cb'}{a'}$, or $D=\dfrac{ab'^2}{a'^3}\times-\dfrac{b'}{a'}=-\dfrac{ab'^3}{a'^4}$; and so on.

It is easily perceived that any co-efficient is formed from that which precedes it, by multiplying by $-\dfrac{b'}{a'}$; therefore we have,

$$\frac{a}{a'+b'x}=\frac{a}{a'}-\frac{ab'}{a'^2}x+\frac{ab'^2}{a'^3}x^2-\frac{ab'^3}{a'^4}x^3+\frac{ab'^4}{a'^5}x^4-\ \ldots$$

208. By reflecting upon the preceding reasoning, we perceive, that the fundamental principle of the method of indeterminate co-efficients, depends upon this, viz., *when an equation of the form* $0=M+Nx+Px^2+Qx^3+\ \ldots$ ($M, N, P, Q, \ldots$ being independent of x), *is verified by any value of* x *whatever, each of the co-efficients must necessarily be equal to* 0.

For since these co-efficients are independent of x, when they are determined by any particular hypothesis made with respect to x, the values must answer for any value of x whatever. Now, making $x=0$, we find $M=0$, and dividing the equation by x, it reduces to

$$0=N+Px+Qx^2+\ \ldots$$

making $x=0$ in this equation, it becomes $N=0$, and dividing the equation by x, it reduces to $0=P+Qx+\ .\ .\ .$ and so on. Hence we have

$$M=0,\ N=0,\ P=0,\ Q=0\ .\ .\ .\ ;$$

in this manner we obtain as many equations as there are co-efficients to be determined.

This principle may be enunciated in another manner, viz.

When an equation of the form

$$a+bx+cx^2+dx^3+\ .\ .\ .\ =a'+b'x+c'x^2+d'x^3+\ .\ .\ .$$

is satisfied by any value given to x, the terms involving the same powers in the two members are respectively equal; for, by transposing all the terms into the second member, the equation will take the form $0=M+Px+Qx^2+\ .\ .\ .$, whence

$$a'-a=0,\ b'-b=0,\ c'-c=0\ .\ .\ .\ .\ ,$$

and consequently,

$$a'=a,\ b'=b,\ c'=c,\ d'=d\ .\ .\ .\ .,$$

Every equation in which the terms are arranged with reference to a certain letter, and which is satisfied by any value which can be given to this letter, is called an *identical equation*, in order to distinguish it from a *common equation*, that is, an equation which can only be satisfied by giving particular values to this letter.

209. The method of *indeterminate co-efficients* requires that we should know the form of the development, with reference to the exponents of x. The development is generally supposed to be arranged according to the ascending powers of x, commencing with the power x^0; sometimes, however, this form is not exact; in this case, the calculus detects the error in the supposition.

For example, develop the expression $\dfrac{1}{3x-x^2}$.

Suppose that $\dfrac{1}{3x-x^2}=A+Bx+Cx^2+Dx^3+\ .\ .\ .\ .,$

whence, by clearing the fraction, and arranging the terms,

$$0=-1+3Ax+3B\left|\begin{matrix}x^2+3C\\-B\end{matrix}\right|\begin{matrix}x^3+3D\\-C\end{matrix}\left|x^4+\ldots,\right.$$
$$\qquad\quad -A$$

whence (Art. 208),

$$-1=0,\ 3A=0,\ 3B-A=0\ldots\ldots$$

Now the first equation, $-1=0$, is absurd, and indicates that the above form is not a suitable one for the expression $\frac{1}{3x-x^2}$; but if we put this expression under the form $\frac{1}{x}\times\frac{1}{3-x}$, and suppose that

$$\frac{1}{x}\times\frac{1}{3-x}=\frac{1}{x}(A+Bx+Cx^2+Dx^3+\ldots.),$$

it will become, after the reductions are made,

$$0=\left\{\begin{matrix}3A+3B\\-1-A\end{matrix}\right.\left|\begin{matrix}x+3C\\-B\end{matrix}\right|\begin{matrix}x^2+3D\\-C\end{matrix}\left|x^3+\ldots,\right.$$

which gives the equations

$$3A-1=0,\ 3B-A=0,\ 3C-B=0\ldots,$$

whence $\qquad A=\frac{1}{3},\ B=\frac{1}{9},\ C=\frac{1}{27},\ D=\frac{1}{81}\ldots$

Therefore,

$$\frac{1}{3x-x^2}=\frac{1}{x}\left(\frac{1}{3}+\frac{1}{9}x+\frac{1}{27}x^2+\frac{1}{81}x^3+\ldots\right),$$

or
$$=\frac{1}{3}x^{-1}+\frac{1}{9}x^0+\frac{1}{27}x+\frac{1}{81}x^2+\ldots;$$

that is, the development contains a term affected with a negative exponent.

Recurring Series.

210. The development of algebraic fractions by the method of indeterminate co-efficients, gives rise to certain series, called *recurring series.*

A recurring series is the development of a rational fraction involving x, *made according to a fixed law, and containing the ascending powers of* x *in its different terms.*

It has been shown in Art. 207, that the development of the expression $\frac{a}{a'+b'x}$ is the series $\frac{a}{a'}-\frac{ab'}{a'^2}x+\frac{ab'^2}{a'^3}x^2-\frac{ab'^3}{a'^4}x^3+\ldots$, in which each term is formed by multiplying that which precedes it by $-\frac{b'}{a'}x$.

This property is not peculiar to the proposed fraction; it belongs to all rational algebraic fractions, and it consists in this, viz.: *Every rational fraction involving* x, *may be developed into a series of terms, each of which is equal to the algebraic sum of the products which arise from multiplying certain terms of a particular expression, by certain of the preceding terms of the series.*

The particular expression, from which any term of the series may be found, when the preceding terms are known, is called the *scale* of the series; and that from which the co-efficient may be formed, the *scale* of the co-efficients.

In the preceding series, the *scale* is $-\frac{b'}{a'}x$, and the series is called *a recurring series of the first order*, and $-\frac{b'}{a'}$ is the scale of the co-efficients.

Let it be required to develop $\frac{a+bx}{a'+b'x+c'x^2}$ into a series.

Assume $\frac{a+bx}{a'+b'x+c'x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\ldots$

Clearing the fraction and transposing, we have,

$$0=\left\{\begin{array}{l|l|l|l|l} Aa'+Ba' & x+Ca' & x^2+Da' & x^3+Ea' & x^4+\ldots \\ -a+Ab' & +Bb' & +Cb' & +Db' & \\ -b & +Ac' & +Bc' & +Cc' & \end{array}\right.$$

which gives the equations

$$Aa'-a=0, \text{ or } A=\frac{a}{a'}$$

$$Ba'+Ab'-b=0, \text{ or } B=-\frac{b'}{a'}A+\frac{b}{a'}$$

$$Ca'+Bb'+Ac'=0, \text{ or } C=-\frac{b'}{a'}B-\frac{c'}{a'}A$$

$$Da'+Cb'+Bc'=0, \text{ or } D=-\frac{b'}{a'}C-\frac{c'}{a'}B$$

$$Ea'+Db'+Cc'=0, \text{ or } E=-\frac{b'}{a'}D-\frac{c'}{a'}C.$$

. . . .

. . . .

Whence we perceive that the two first co-efficients are not obtained by any law; but commencing at the third, each co-efficient is formed by multiplying the two which precede it respectively by $-\frac{b'}{a'}$ and $-\frac{c'}{a'}$, viz. that which immediately precedes the required co-efficient by $-\frac{b'}{a'}$, that which precedes it two terms by $-\frac{c'}{a'}$, and taking the algebraic sum of the products. Hence,

$$\left(-\frac{b'}{a'}-\frac{c'}{a'}\right)$$

is the scale of the co-efficients.

From this law of the formation of the co-efficients, it follows that the third term of the series, Cx^2 is equal to

$$-\frac{b'}{a'}Bx^2-\frac{c'}{a'}Ax^2$$

or
$$-\frac{b'}{a'}xB\,.\,x-\frac{c'}{a'}x^2\,.\,A.$$

In like manner, we have for Dx^3

$$-\frac{b'}{a'}Cx^3-\frac{c'}{a'}Bx^3$$

or
$$-\frac{b'}{a'}x\,.\,Cx^2-\frac{c'}{a'}x^2\,.\,Bx.$$

Hence, each term of the required series, commencing at the third, is obtained by multiplying the two terms which precede, respectively by

$$-\frac{b'}{a'}x-\frac{c'}{a'}x^2,$$

and taking the sum of the products: hence, this last expression is the *scale* of the series.

211. Recurring series are divided into orders, and the order is estimated by the number of terms contained in the *scale*.

Thus, the expression $\frac{a}{a'+b'x}$ gives a recurring series of *the first order*, the scale of which is $-\frac{b'}{a'}x$.

The expression $\frac{a+bx}{a'+b'x+c'x^2}$ will give a recurring series *of the second order*, of which the scale will be

$$\left(-\frac{b'}{a'}x, \ -\frac{c'}{a'}x^2\right).$$

The series obtained in the preceding Art. is of the second order. In general, an expression of the form

$$\frac{a+bx+cx^2+\ .\ .\ .\ kx^{n-1}}{a'+b'x+c'x^2+\ .\ .\ .\ k'x^n},$$

gives a recurring series *of the* n[th] *order*, the scale of which is

$$\left(-\frac{b'}{a'}x, \ -\frac{c'}{a'}x^2\ .\ .\ .\ -\frac{k'}{a'}x^n\right).$$

Remark. It is here supposed that the degree of x in the numerator is less than it is in the denominator. If it was not, it would first be necessary to perform the division, arranging the quantities with reference to x, which would give an entire quotient, plus a fraction similar to the above.

Thus, in the expression $\frac{1-x-3x^2+4x^3+x^4}{2-5x+3x^2-x^3}$.

$$\begin{array}{l|l} x^4+4x^3-3x^2-x+1 & -x^3+3x^2-5x+2 \\ \hline +7x^3-8x^2+x & -x-7 \\ \hline +13x^2-34x+15. & \end{array}$$

Performing the division, we find the quotient to be $-x-7$, plus the fraction.

$$\frac{13x^2-34x+15}{-x^3+3x^2-5x+2}, \text{ or } \frac{15-34x+13x^2}{2-5x+3x^2-x^3}.$$

CHAPTER V.

Of Progressions, Continued Fractions, and Logarithms.

212. This chapter is naturally connected with the last, as it explains the properties of two kinds of series, and also presents an application of the theory of exponents. It moreover completes that part of algebra which is absolutely necessary for the study of *Trigonometry*, and the *Application of Algebra to Geometry*.

Progressions by Differences.

213. A *progression by differences*, or an *Arithmetical progression*, is a series in which the successive terms continually increase or decrease by a constant quantity, which is called the *common difference* of the progression.

Thus, in the two series

$$1,\ 4,\ 7,\ 10,\ 13,\ 16,\ 19,\ 22,\ 25\ldots$$
$$60,\ 56,\ 52,\ 48,\ 44,\ 40,\ 36,\ 32,\ 28\ldots$$

The first is called an *increasing progression*, of which the common difference is 3, and the second a *decreasing progression*, of which the common difference is 4.

214. If there are four quantities a, b, c, d, in arithmetical progression, a is said to be to b, as c to d : and a and c are called *antecedents*, and b and d *consequents*.

In general, let $a, b, c, d, e, f, \ldots$ designate the terms of a progression by differences; it has been agreed to write them thus:

$$a.b.c.d.e.f.g.h.i.k.\ldots$$

This series is read, a is to b, as b is to c, as c is to d, as d is to e, &c. or a is to b, is to c, is to d, is to e, &c. This is a series of *continued equi-differences*, in which each term is at the same time a consequent and antecedent, with the exception of the first term, which is only an *antecedent*, and the last, which is only a *consequent*.

215. Let r represent the common difference of the progression, which we will consider as increasing. In the case of a decreasing progression, it will only be necessary to change r into $-r$, in the results.

From the definition of the progression, it evidently follows that

$$b=a+r,\ c=b+r=a+2r,\ d=c+r=a+3r\,;$$

and in general, any term of the series is equal to, *the first plus as many times the common difference as there are preceding terms.*

Thus, let l be any term, and n the number which marks the place of it, the expression for this *general term*, is

$$l=a+(n-1)r.$$

That is, *the last term is equal to the first term, plus the product of the common difference by the number of terms less one.*

If we suppose n successively equal to 1, 2, 3, 4, &c. we shall obtain the first, second, third, fourth, &c. term of the progression.

The formula $l=a+(n-1)r$, serves to find any term whatever, without our being obliged to determine all those which precede it.

Thus, by making $n=50$, we find the 50^{th} term of the progression,

$$1.4.7.10.13.16.19\ldots \qquad l=1+49\times 3=148.$$

216. If the progression were a decreasing one, we should have

$$l=a-(n-1)r.$$

That is, *in a decreasing arithmetical progression, the last term is* *equal to the first term minus the product of the common difference by the number of terms less one.*

217. A progression by differences being given, it is proposed to prove that, *the sum of any two terms, taken at equal distances from the two extremes, is equal to the sum of the two extremes.*

Let $a \,.\, b \,.\, c \,.\, d \,.\, e \,.\, f \,.\,.\,.\,.\, i \,.\, k \,.\, l$, be the proposed progression, and n the number of terms.

We will first observe that, if x denotes a term which has p terms before it, and y a term which has p terms after it, we have, from what has been said, ... $x=a+p\times r$,

and . . . . $y=l-p\times r$;

whence, by addition, ... $x+y=a+l$.

which demonstrates the proposition.

This being the case, write the progression below itself, but in an inverse order, viz.

$$a \,.\, b \,.\, c \,.\, d \,.\, e \,.\, f \,.\,.\,.\,.\, i \,.\, k \,.\, l.$$
$$l \,.\, k \,.\, i \,.\,.\,.\,.\,.\,.\,.\,.\,.\, c \,.\, b \,.\, a.$$

Calling S the sum of the terms of the first progression, $2S$ will be the sum of the terms in both progressions, and we shall have

$$2S=(a+l)+(b+k)+(c+i)\ldots+(i+c)+(k+b)+(l+a);$$

or, since the number of the parts $a+l$, $b+k$, $c+i$ . . . . . is equal to n,

$$2S=(a+l)n, \text{ or } S=\frac{(a+l)n}{2}.$$

That is, *the sum of a progression by differences, is equal to half the sum of the two extremes, multiplied by the number of terms.*

If in this formula we substitute for l its value, $a+(n-1)r$, we obtain

$$S=\frac{[2a+(n-1)r]n}{2};$$

but the first expression is the most useful.

218. The formulas $l=a+(n-1)r$, $S=\dfrac{(a+l)n}{2}$, contain five

quantities, a, r, n, l and S, and consequently give rise to the following general problem, viz.: *Any three of these five quantities being given, to determine the other two.*

There will, therefore, be as many different cases as there can be formed combinations of five letters taken three in a set: that is,

$$5 \cdot \frac{5-1}{2} \cdot \frac{5-2}{3} = 10. \text{ (Art. 163).}$$

Of these cases we shall consider only the most important.

We already know the value of S in terms of a, n and r.

From the formula $l=a+(n-1)r$, we find

$$a=l-(n-1)r.$$

That is, *the first term of an increasing arithmetical progression is equal to the last term, minus the product of the common difference by the number of terms less one.*

From the same formula, we also find,

$$r=\frac{l-a}{n-1}.$$

That is, *in any arithmetical progression, the common difference is equal to the difference between the two extremes divided by the number of terms less one.*

219. The last principle affords a solution to the following question.

To find a number m *of arithmetical means between two given numbers* a *and* b.

To resolve this question, it is first necessary to find the common difference. Now we may regard a as the first term of an arithmetical progression, b as the last term, and the required means as intermediate terms. The number of terms of this progression will be expressed by $m+2$.

Now, by substituting in the above formula, b for l, and $m+2$ for n, it becomes $r=\frac{b-a}{m+2-1}$, or $r=\frac{b-a}{m+1}$; that is, *the common difference* of the required progression is obtained by dividing the difference between the given numbers a and b, by one more than the required number of means.

Having obtained the common difference, form the second term of the progression, or the *first arithmetical mean*, by adding r, or $\frac{b-a}{m+1}$, to the first term a. The *second mean* is obtained by augmenting the first by r, &c.

For example, let it be required to find 12 arithmetical means between 12 and 77. We have $r=\frac{77-12}{13}=\frac{65}{13}=5$, which gives the progression $12 . 17 . 22 . 27 \ldots 72 . 77$.

220. Remark. If the same number of arithmetical means are inserted between all of the terms, taken two and two, these terms, and the arithmetical means united, will form but one and the same progression.

For, let $a . b . c . d . e . f \ldots$ be the proposed progression, and m the number of means to be inserted between a and b, b and c, c and $d \ldots$

From what has just been said, the common difference of each partial progression will be expressed by $\frac{b-a}{m+1}, \frac{c-b}{m+1}, \frac{d-c}{m+1} \ldots$, which are equal to each other, since $a, b, c \ldots$ are in progression: therefore, the common difference is the same in each of the partial progressions; and since the *last term* of the first, forms the *first term* of the second, &c. we may conclude that all of these partial progressions form a single progression.

EXAMPLES.

1. Find the sum of the first fifty terms of the progression $2 . 9 . 16 . 23 \ldots$

For the 50th term we have $l=2+49\times7=345$.

Hence $$S=(2+345)\times\frac{50}{2}=347\times25=8675.$$

2. Find the 100th term of the series $2 . 9 . 16 . 23 \ldots$

Ans. 695.

3. Find the sum of 100 terms of the series $1 . 3 . 5 . 7 . 9 \ldots$

Ans. 10000.

4. The greatest term is 70, the common difference 3, and the number of terms 21, what is the least term and the sum of the series? *Ans.* Least term 10 : sum of series 840.

5. The first term of a decreasing arithmetical progression is 10, the common difference $\frac{1}{3}$, and the number of terms 21 : required the sum of the series. *Ans.* 140.

6. In a progression by differences, having given the common difference 6, the last term 185, and the sum of the terms 2945, find the first term, and the number of terms.

Ans. First term $=5$, number of terms 31.

7. Find 9 arithmetical means between each antecedent and consequent of the progression $2 . 5 . 8 . 11 . 14 \ldots$

Ans. Ratio, or $r=0,3$.

8. Find the number of men contained in a triangular battalion, the first rank containing 1 man, the second 2, the third 3, and so on to the n^{th}, which contains n. In other words, find the expression for the sum of the natural numbers 1, 2, 3 . . ., from 1 to n, inclusively. *Ans.* $S=\frac{n(n+1)}{2}$.

9. Find the sum of the n first terms of the progression of uneven numbers 1, 3, 5, 7, 9 . . . *Ans.* $S=n^2$.

10. One hundred stones being placed on the ground, in a straight line, at the distance of 2 yards from each other: how far will a person travel, who shall bring them one by one to a basket, placed at 2 yards from the first stone?

Ans. 11 miles, 840 yards.

Geometrical Progression, or Progressions by Quotients.

221. A *Geometrical progression*, or *progression by quotients*, is a series of terms, each of which is equal to the product of that which precedes it, by a *constant number*, which number is called the *ratio* of the progression; thus in the two series:

$$3,\quad 6,\ 12,\ 24,\ 48,\ 96\ \ldots$$

$$64,\ 16,\quad 4,\quad 1,\ \frac{1}{4},\ \frac{1}{16}\ \ldots$$

each term of the first contains that which precedes it *twice*, or is equal to double that which precedes it; and each term of the second is contained in that which precedes it four times, or is a *fourth* of that which precedes it; they are then progressions by quotients, of which the ratio is 2 for the first, and $\frac{1}{4}$ for the second.

Let $a, b, c, d, e, f \ldots$ be numbers in a progression by quotients, they are written thus; $a : b : c : d : e : f : g \ldots$, and it is enunciated in the same manner as a progression by differences; however it is necessary to make the distinction that one is a series of equal differences, and the other a series of equal quotients or ratios, in which each term is at the same time an antecedent and a consequent, except the first, which is only an antecedent, and the last, which is only a consequent.

222. Let q denote the ratio of the progression $a : b : c : d \ldots$, q being >1 when the progression is *increasing*, and $q<1$ when it is *decreasing:* we deduce from the definition, the following equalities.

$$b=aq,\quad c=bq=aq^2,\quad d=cq=aq^3,\quad e=dq=aq^4\ \ldots$$

and in general, any term n, that is, one which has $n-1$ terms before it, is expressed by aq^{n-1}.

Let l be this term; we have the formula $l=aq^{n-1}$, by means of which we can obtain the value of any term without being obliged to find the values of all those which precede it. That is, *the last term of a geometrical progression is equal to the first term multiplied* *by the ratio raised to a power whose exponent is one less than the number of terms.*

For example, the 8^{th} term of the progression $2 : 6 : 18 : 54 \ldots,$ is equal to $2\times 3^7=2\times 2187=4374.$

In like manner, the 12^{th} term of the progression

$$64 : 16 : 4 : 1 : \frac{1}{4} \ldots \text{ is equal to}$$

$$64\left(\frac{1}{4}\right)^{11}=\frac{4^3}{4^{11}}=\frac{1}{4^8}=\frac{1}{65536}.$$

223. We will now proceed to determine the sum of n terms of the progression $a : b : c : d : e : f : \ldots : i : k : l$, l denoting the n^{th} term.

We have the equations (Art. 222),

$$b=aq, \quad c=bq, \quad d=cq, \quad e=dq, \ldots k=iq, \quad l=kq;$$

and by adding them all together, member to member, we deduce

$$b+c+d+e+ \ldots +k+l=(a+b+c+d+ \ldots +i+k)q;$$

or, representing the required sum by S,

$$S-a=(S-l)q=Sq-lq, \quad \text{or} \quad Sq-S=lq-a;$$

whence $$S=\frac{lq-a}{q-1};$$

That is, to obtain the sum of a certain number of terms of a progression by quotients, *multiply the last term by the ratio, subtract the first term from this product, and divide the remainder by the ratio diminished by unity.*

When the progression is decreasing, we have $q<1$ and $l<a$; the above formula is then written under the form $S=\frac{a-lq}{1-q}$, in order that the two terms of the fraction may be positive.

By substituting aq^{n-1} for l in the two expressions for S, they become,

$$S=\frac{aq^n-a}{q-1}, \text{ and } S=\frac{a-aq^n}{1-q}.$$

EXAMPLES.

1. Find the first eight terms of the progression

$$2 : 6 : 18 : 54 : 162 \ldots : 2\times3^7 \text{ or } 4374$$

$$S=\frac{lq-a}{q-1}=\frac{13122-2}{2}=6560.$$

2. Find the sum of the first twelve terms of the progression

$$64 : 16 : 4 : 1 : \frac{1}{4} : \ldots : 64\left(\frac{1}{4}\right)^{11}, \text{ or } \frac{1}{65536},$$

$$S=\frac{a-lq}{1-q}=\frac{64-\frac{1}{65536}\times\frac{1}{4}}{\frac{3}{4}}=\frac{256-\frac{1}{65536}}{3}=85+\frac{65535}{196608}$$

We perceive that the principal difficulty consists in obtaining the numerical value of the last term, a tedious operation, even when the number of terms is not very great.

3. What debt may be discharged in a year, or twelve months, by paying \$1 the first month, \$2 the second month, \$4 the third month, and so on, each succeeding payment being double the last; and what will be the last payment?

Ans. Debt, \$4095 : last payment, \$2048.

224. REMARK. If, in the formula $S=\frac{a(q^n-1)}{q-1}$, we suppose

$$q=1, \text{ it becomes } S=\frac{0}{0}.$$

This result, which is sometimes a symbol of indetermination, is also often a consequence of the existence of a common factor (Art. 113), which becomes nothing by making a particular hypothesis respecting the given question. This, in fact, is the case in the present question; for the expression q^n-1 is divisible by $q-1$, (Art. 59), and gives the quotient

$$q^{n-1}+q^{n-2}+q^{n-3}+\ \ldots\ +q+1;$$

hence the value of S takes the form

$$S=aq^{n-1}+aq^{n-2}+aq^{n-3}+\ \ldots\ +aq+a.$$

Now, making $q=1$, we have $S=a+a+a+\ \ldots\ +a=na$.

We can obtain the same result by going back to the proposed progression, $a:b:c:\ \ldots\ :l$, which, in the particular case of $q=1$, reduces to $a:a:a:\ \ldots\ :a$, the sum of which series is equal to na.

The result $\frac{0}{0}$, given by the formula, may be regarded as indicating that the series is characterized by some particular property. In fact, the progression, being entirely composed of equal terms, is no more a progression by quotients than it is a progression by differences. Therefore, in seeking for the sum of a certain number of the terms, there is no reason for using the formula $S=\frac{a(q^n-1)}{q-1}$, in preference to the formula $S=\frac{(a+l)n}{2}$, which gives the sum in the progression by differences.

Of Progressions having an infinite number of terms.

225. Let there be the decreasing progression $a:b:c:d\ \ e:f:\ldots$, containing an indefinite number of terms. The formula $S=\frac{a-aq^n}{1-q}$, which represents the sum of n of its terms, can be put under the form

$$S=\frac{a}{1-q}-\frac{aq^n}{1-q}.$$

Now, since the progression is decreasing, q is a fraction; and q^n is also a fraction, which diminishes as n increases. Therefore the greater the number of terms we take, the more will $\frac{a}{1-q}\times q^n$ diminish, and consequently, the more will the partial sum of these terms approximate to an equality with the first part of S, that is, to

$\frac{a}{1-q}$. Finally, when n is taken greater than any given number, or $n=\infty$, then $\frac{a}{1-q}\times q^n$ will be less than any given number, or will become equal to 0; and the expression $\frac{a}{1-q}$ will represent the true value of the sum of all the terms of the series.

Whence we may conclude, that the expression for *the sum of the terms of a decreasing progression, in which the number of terms is infinite, is*

$$S=\frac{a}{1-q}.$$

This is, properly speaking, the *limit* to which the *partial sums* approach, by taking a greater number of terms in the progression. The difference between these sums and $\frac{a}{1-q}$ can become as small as we please, and will only become *nothing* when the number of terms taken is infinite.

EXAMPLES.

1. Find the sum of

$$1:\frac{1}{3}:\frac{1}{9}:\frac{1}{27}:\frac{1}{81} \quad \text{to infinity.}$$

We have for the expression of the sum of the terms

$$S=\frac{a}{1-q}=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}.$$

The error committed by taking this expression for the value of the sum of the n first terms, is expressed by

$$\frac{a}{1-q}\cdot q^n=\frac{3}{2}\left(\frac{1}{3}\right)^n.$$

First take $n=5$; it becomes

$$\frac{3}{2}\left(\frac{1}{3}\right)^5=\frac{1}{2\,.\,3^4}=\frac{1}{162}.$$

When $n=6$, we find

$$\frac{3}{2}\left(\frac{1}{3}\right)^6=\frac{1}{162}\cdot\frac{1}{3}=\frac{1}{486}.$$

Whence we see that the *error committed*, when $\frac{3}{2}$ is taken for the sum of a certain number of terms, is less in proportion as this number is greater.

Again take the progression

$$1 : \frac{1}{2} : \frac{1}{4} : \frac{1}{8} : \frac{1}{16} : \frac{1}{32} : \&c. \ldots$$

We have $$S=\frac{a}{1-q}=\frac{1}{1-\frac{1}{2}}=2.$$

226. The expression $S=\frac{a}{1-q}$, can be obtained directly from the progression $a : b : c : d : e : f : g : \ldots$

For, take the equations $b=aq$, $c=bq$, $d=cq$, $e=dq \ldots\ldots$ of which the number is indefinite, and add them together, member to member; we have

$$b+c+d+e+\ldots=(a+b+c+d+\ldots)q.$$

Now, the first member is evidently the proposed series, diminished by the first term a; it is therefore expressed by $S-a$; the second member is q multiplied by the entire series, since there is no last term, or rather this last term is nothing; hence the expression for this member is qS, and the above equality becomes $S-a=qS$,

whence $$S=\frac{a}{1-q}.$$

In fact, by developing $\frac{a}{1-q}$ into a series by the rule for division, we shall find the result to be $a+aq+aq^2+aq^3+\ldots$, which is nothing more than the proposed series, having $b, c, d \ldots$ replaced by their values in functions of a.

227. When the series is increasing, the expression $\frac{a}{1-q}$ cannot be considered as a *limit of the partial sums;* because, the sum of a given number of terms being $S=\frac{a}{1-q}-\frac{aq^n}{1-q}$, (Art. 225), the second part $\frac{aq^n}{1-q}$ augments numerically, in proportion to the increase of n; hence the greater the number of terms taken, the more the expression of their sum will differ numerically from $\frac{a}{1-q}$.

The formula $S=\frac{a}{1-q}$ is, in this case, merely the algebraic expression which, by its development, gives the series

$$a+aq+aq^2+aq^3 \ldots$$

There is another circumstance presents itself here, which appears very singular at first sight. Since $\frac{a}{1-q}$ is the fraction which generates the above series, we should have

$$\frac{a}{1-q}=a+aq+aq^2+aq^3+aq^4+\ldots$$

Now, by making $a=1$, $q=2$, this equality becomes

$$\frac{1}{1-2} \text{ or } -1=1+2+4+8+16+32+\ldots$$

an equation of which the first member is negative whilst the second is positive, and greater in proportion to the value of q.

To interpret this result, we will observe that, when in the equation $\frac{a}{1-q}=a+aq+aq^2+aq^3+\ldots$, we stop at a certain term of the series, it is necessary to complete the quotient in order that the equality may subsist. Thus, in stopping, for example, at the fourth term, aq^3.

$$\begin{array}{lr|l} & a & 1-q \\ \hline \text{1st remainder} & +\ aq & a+aq+aq^2+aq^3+\dfrac{aq^4}{1-q}. \\ \text{2d.} & +\ aq^2 & \\ \text{3d.} & +\ aq^3 & \\ \text{4th.} & +\ aq^4 & \end{array}$$

It is necessary to add the fractional expression $\frac{aq^4}{1-q}$ to the quotient, which gives rigorously,

$$\frac{a}{1-q}=a+aq+aq^2+aq^3+\frac{aq^4}{1-q}.$$

If in this exact equation we make $a=1$, $q=2$, it becomes

$$-1=1+2+4+8+\frac{16}{-1}=1+2+4+8-16,$$

which verifies itself.

In general, when an expression involving x, designated by $f(x)$, which is called *a function* of x, is developed into a series of the form $a+bx+cx^2+dx^3+\ .\ .\ .\ .\ .$, we have not rigorously $f(x)=a+bx+cx^2+dx^3+\ .\ .\ .$, unless we conceive that, in stopping at a certain term in the second member, the series is completed by a certain expression involving x.

When, in particular applications, the series is *decreasing* (Art. 203), the expression which serves to complete it may be obtained as near as we please, by prolonging the series; but the contrary is the case when the series is *increasing*, for then it must not be neglected. This is thereason why increasing series cannot be used for approximating to the value of numbers. It is for this reason, also, that algebraists have called those series which go on diminishing from term to term, *converging series*, and those in which the terms go on increasing, *diverging series*. In the first, the greater the number of terms taken, the nearer the sum approximates numerically to the expression of which this series is the development; whilst in the others, the more terms we take, the more their sum differs from the numerical value of this expression.

228. The consideration of the five quantities a, q, n, l and S, which enter in the two formulas $l=aq^{n-1}$, $S=\frac{lq-a}{q-1}$ (Arts. 222 & 223), gives rise to ten problems, as in the progression by differences (Art. 218). Of these cases, we shall consider here, as we did there, only the most important. We will first find the values of S and q in terms of a, l and n.

The first formula gives $q^{n-1}=\frac{l}{a}$, whence $q=\sqrt[n-1]{\frac{l}{a}}$. Substituting this value in the second formula, the value of S will be obtained.

The expression $q=\sqrt[n-1]{\frac{l}{a}}$ furnishes the means for resolving the following question, viz.

To find m *mean proportionals between two given numbers* a *and* b; that is, *to find a number* m *of means, which will form with* a *and* b, *considered as extremes, a progression by quotients.*

For this purpose, it is only necessary to know the *ratio;* now the required number of means being m, the total number of terms is equal to $m+2$. Moreover, we have $l=b$, therefore the value of q becomes $q=\sqrt[m+1]{\frac{b}{a}}$; that is, we must *divide one of the given numbers* (b) *by the other* (a), *then extract that root of the quotient whose index is one more than the required number of means.*

Hence, the progression is

$$a : a\sqrt[m+1]{\frac{b}{a}} : a\sqrt[m+1]{\frac{b^2}{a^2}} : a\sqrt[m+1]{\frac{b^3}{a^3}} : \ldots b.$$

Thus, to insert six mean proportionals between the numbers 3 and 384, we make $m=6$, whence $q=\sqrt[7]{\frac{384}{3}}=\sqrt[7]{128}=2$; whence we deduce the progression

$$3 : 6 : 12 : 24 : 48 : 96 : 192 : 384.$$

REMARK. *When the same number of mean proportionals are in-*

serted between all the terms of a progression by quotients, taken two and two, all the progressions thus formed will constitute a single progression.

229. Of the ten principal problems that may be proposed in progressions, *four* are susceptible of being easily resolved. The following are the enunciations, with the formulas relating to them.

1st. $a, q, n,$ being given, to find l and S.

$$l=aq^{n-1};\ S=\frac{lq-a}{q-1}=\frac{a(q^n-1)}{q-1}.$$

2d. $a, n, l,$ being given, to find q and S.

$$q=\sqrt[n-1]{\frac{l}{a}};\ S=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}.$$

3d. $q, n, l,$ being given, to find a and S.

$$a=\frac{l}{q^{n-1}};\ S=\frac{l(q^n-1)}{q^{n-1}(q-1)}.$$

4th. $q, n, S,$ being given to find a and l.

$$a=\frac{S(q-1)}{q^n-1},\ l=\frac{Sq^{n-1}(q-1)}{q^n-1}.$$

Two other problems depend upon the resolution of equations of a degree superior to the second; they are those in which the unknown quantities are supposed to be a and q, or l and q.

For, from the second formula we deduce

$$a=lq-Sq+S;$$

Whence, by substituting this value of a in the first $l=aq^{n-1}$,

$$l=(lq-Sq+S)q^{n-1},$$

or,

$$(S-l)q^n-Sq^{n-1}+l=0.$$

an equation of the n^{th} degree.

In like manner, in determining l and q, we should obtain the equation $aq^n-Sq+S-a=0$.

230. Finally, the other *four* problems lead to the resolution of

equations of a peculiar nature; they are those in which n and one of the other four quantities are unknown.

From the second formula it is easy to obtain the value of one of the quantities a, q, l, and S, *in functions* of the other three; hence the problem is reduced to finding n by means of the formula

$$l=aq^{n-1}.$$

Now this equality becomes $q^n=\frac{lq}{a}$, an equation of the form $a^x=b$, a and b being known quantities. Equations of this kind are called *exponential equations*, to distinguish them from those previously considered, in which the unknown quantity is raised to a power denoted by a known number.

Before, however, we can solve the exponential equation $a^x=b$, we must understand the elementary properties of *Continued Fractions*, which are now to be explained.

Of Continued Fractions.

231. Having given a fraction of the form $\frac{65}{149}$, in which the terms are large, and prime with respect to each other, we are unable to discover its precise value, either by inspection or by any mode of reduction yet explained. The manner of approximating to the value of such a fraction gives rise to a series of numbers, which taken together, form what is called a *continued fraction.*

232. If we take, for example, the fraction $\frac{65}{149}$, and divide both its terms by the numerator 65, the value of the fraction will not be changed, and we shall have

$$\frac{65}{149}=\frac{1}{\frac{149}{65}},$$

or effecting the division, $$\frac{65}{149}=\frac{1}{2+\frac{19}{65}}.$$

Now, if we neglect the fractional part $\frac{19}{65}$ of the denominator, we shall obtain $\frac{1}{2}$ for the approximate value of the given fraction. But this value would be too large, since the denominator used was too *small.*

If, on the contrary, instead of neglecting the part $\frac{19}{65}$, we were to replace it by 1, the approximate value would be $\frac{1}{3}$, which must be too small, since the denominator 3 is too *large.* Hence

$$\frac{65}{149}<\frac{1}{2} \text{ and } \frac{65}{149}>\frac{1}{3},$$

therefore the value of the fraction is comprised between $\frac{1}{2}$ and $\frac{1}{3}$.

If we wish a nearer approximation, it is only necessary to operate on the fraction $\frac{19}{65}$ as we did on the given fraction $\frac{65}{149}$, and we obtain

$$\frac{19}{65}=\frac{1}{3+\frac{8}{19}},$$

hence

$$\frac{65}{149}=\frac{1}{2+\cfrac{1}{3+\frac{8}{19}}}.$$

If now, we neglect the part $\frac{8}{19}$, the denominator 3 will be less than the true denominator, and $\frac{1}{3}$ will be *larger* than the number which ought to be added to 2; hence, 1 divided by $2+\frac{1}{3}$ will be *less* than the value of the fraction: that is, if we reject the fractional part after the second reduction, we shall have

$$\frac{65}{149} > \frac{3}{7}.$$

If we wish to approximate still nearer to the value of the given fraction, we find

$$\frac{8}{19} = \cfrac{1}{2 + \cfrac{3}{8}},$$

and by substituting this value, we have

$$\frac{65}{149} = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{3}{8}}}}$$

Now, if we neglect the fractional part $\frac{3}{8}$, after the third reduction, we see that 2 will be less than the real denominator; hence $\frac{1}{2}$ will be larger than the number to be added to 3: that is,

$$3 + \frac{1}{2} = \frac{7}{2} \quad \text{is too large; hence}$$

$$\cfrac{1}{\cfrac{7}{2}} = \frac{2}{7} \quad \text{is too small, and}$$

$$2 + \frac{2}{7} = \frac{16}{7} \quad \text{is too small; therefore}$$

$$\cfrac{1}{\cfrac{16}{7}} = \frac{7}{16} \quad \text{is too large, and hence}$$

$$\frac{65}{149} < \frac{7}{16}.$$

Now, as the same train of reasoning may be pursued for the reductions which follow, and as all the results are independent of particular numbers, it follows that, *if we stop at an odd reduction, and neglect the fractional part, the result will be too great; but if we stop at an even reduction, and neglect the fractional part, the result will be too small.*

Making two more reductions, in the last example, we have,

$$\frac{65}{149}=\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2.}}}}}}$$

233. Let us take, as a general case, the continued fraction

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{f+, \text{ \&c.}}}}}}$$

Hence we see, *that a continued fraction has for its numerator the unit* 1, *and for its denominator a whole number, plus a fraction which has* 1 *for its numerator and for its denominator a whole number plus a fraction, and so on.*

234. The fractions

$$\frac{1}{a},\quad \cfrac{1}{a+\cfrac{1}{b}},\quad \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}}, \text{ \&c.}$$

are called *approximating fractions,* because each affords, in succession, a nearer value of the given fraction.

The fractions $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, &c. are called *integral fractions.*

When the continued fraction can be exactly expressed by a vulgar fraction, as in the numerical examples already given, the integral fractions $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, &c. will terminate, and we shall obtain an expression for the exact value of the given fraction by taking them all.

235. We will now explain the manner in which any approximating fraction may be found from those which precede it.

1. $\frac{1}{a}$ . . . . $= \frac{1}{a}$ 1st. app. fraction.

2. $\cfrac{1}{a+\cfrac{1}{b}}$ . . . . . . $= \frac{b}{ab+1}$ 2d. app. fraction

3. $\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}}$ . . . . . . $=\frac{bc+1}{(ab+1)c+a}$ 3d. app. fraction.

By examining the third approximating fraction, we see, that its numerator is formed by multiplying the numerator of the preceding fraction by the denominator of the third integral fraction, and adding to the product the numerator of the first approximating fraction: and that the denominator is formed by multiplying the denominator of the last fraction by the denominator of the third integral fraction, and adding to the product the denominator of the first approximating fraction.

We should infer, from analogy, that this law of formation is general. But to prove it rigorously, let $\frac{P}{P'}$, $\frac{Q}{Q'}$, $\frac{R}{R'}$, be the three approximating fractions for which the law is already established. Since c is the denominator of the last integral fraction, we have from what has already been proved

$$\frac{R}{R'}=\frac{Qc+P}{Q'c+P'}.$$

Let us now add a new integral fraction $\frac{1}{d}$ to those already reduced, and suppose $\frac{S}{S'}$ to express the 4th approximating fraction.

It is plain that $\frac{R}{R'}$ will become $\frac{S}{S'}$ by simply substituting for c, $c+\frac{1}{d}$: hence,

$$\frac{S}{S'}=\frac{Q\left(c+\frac{1}{d}\right)+P}{Q'\left(c+\frac{1}{d}\right)+P'}=\frac{(Qc+P)d+Q}{(Q'c+P')d+Q'}=\frac{Rd+Q}{R'd+Q'}$$

Hence we see that the fourth approximating fraction is deduced from the two immediately preceding it, in the same way that the third was deduced from the first and second; and as any fraction may be deduced from the two immediately preceded in a similar manner, we conclude, that, *the numerator of the* n^{th} *approximating fraction is formed by multiplying the numerator of the preceding fraction by the denominator of the* n^{th} *integral fraction, and adding to the product the numerator of the* $n-2$ *fraction; and the denominator is formed according to the same law, from the two preceding denominators.*

236. If we take the difference between any two of the consecutive approximating fractions, we shall find, after reducing them to a common denominator, that the difference of their numerators will be equal to ± 1; and the denominator of this difference will be the product of the denominators of the fractions.

Taking, for example, the consecutive fractions $\frac{1}{a}$, and $\frac{b}{ab+1}$, we have,

$$\frac{1}{a}-\frac{b}{ab+1}=\frac{ab+1-ab}{a(ab+1)}=\frac{+1}{a(ab+1)},$$

and $$\frac{b}{ab+1}-\frac{bc+1}{(ab+1)c+a}=\frac{-1}{(ab+1)\big((ab+1)c+a\big)}.$$

To prove this property in a general manner, let $\frac{P}{P'}$, $\frac{Q}{Q'}$, $\frac{R}{R'}$, be three consecutive approximating fractions. Then

$$\frac{P}{P'}-\frac{Q}{Q'}=\frac{PQ'-P'Q}{P'Q'}.$$

But
$$\frac{Q}{Q'}-\frac{R}{R'}=\frac{R'Q-RQ'}{Q'R'}.$$

But $R=Qc+P$ and $R'=Q'c+P'$ (Art. 235).

Substituting these values in the last equation, we have

$$\frac{Q}{Q'}-\frac{R}{R'}=\frac{(Q'c+P')Q-(Qc+P)Q'}{R'Q'}$$

or reducing

$$\frac{Q}{Q'}-\frac{R}{R'}=\frac{P'Q-PQ'}{R'Q'}.$$

From which we see that the numerator of the difference $\frac{P}{P'}-\frac{Q}{Q'}$ is equal, with a contrary sign, to that of the difference $\frac{Q}{Q'}-\frac{R'}{R'}$. That is, *the difference between the numerators of any two consecutive approximating fractions, when reduced to a common denominator, is the same with a contrary sign, as that which exists between the last numerator and the numerator of the fraction immediately following.*

But we have already seen that the difference of the numerators of the 1st and 2d fractions is equal to $+1$; also that the difference between the numerators of the 2d and 3d fractions is equal to -1; hence the difference between the numerators of the 3d and 4th is equal to $+1$; and so on for the following fractions.

Since the odd approximating fractions are all greater than the true value of the continued fraction, and the even ones all less (Art. 232), it follows, that when a fraction of an even order is subtracted from one of an odd order, the difference should have a plus sign; and on the contrary, it ought to have a minus sign, when one of an odd order is subtracted from one of an even.

237. It has already been shown (Art. 232), that each of the approximating fractions corresponding to the odd numbers, exceeds the true value of the continued fraction; while each of those corresponding to the even numbers is less than it. Hence, the difference between any two consecutive fractions is greater than the difference between either of them and the true value of the continued fraction. Therefore, stopping at the n^{th} fraction, the result will be true to within 1 divided by the denominator of the n^{th} fraction, multiplied by the denominator of the fraction which follows. Thus, if Q' and R' are the denominators of consecutive fractions, and we stop at the fraction whose denominator is Q', the result will be true to within $\frac{1}{Q'R'}$. But since a, b, c, d, &c. are entire numbers, the denominator R' will be greater than Q', and we shall have

$$\frac{1}{Q'R'} < \frac{1}{Q'^2},$$

hence, if the result be true to within $\frac{1}{Q'R'}$ it will certainly be true to within less than the larger quantity

$$\frac{1}{Q'^2},$$

that is, *the approximate result which is obtained, is true to within unity divided by the square of the denominator of the last approximating fraction that is employed.*

If we take the fraction $\frac{829}{347}$ we have

$$\frac{829}{347} = 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{19}}}}}.$$

Here we have in the quotient the whole number 2, which may either be set aside and added to the fractional part after its value shall have been found, or we may place 1 under it for a denominator and treat it as an approximating fraction.

Of Exponential Quantities.

Resolution of the Equation $a^x=b$

238. The object of this question is, to find the exponent of the power to which it is necessary to raise a given number a, in order to produce another given number b.

Suppose it is required to resolve the equation $2^x=64$. By raising 2 to its different powers, we find that $2^6=64$; hence $x=6$ will satisfy the conditions of the equation.

Again, let there be the equation $3^x=243$. The solution is $x=5$. In fact, so long as the second member b is a *perfect power* of the given number a, x will be an entire number which may be obtained by raising a to its successive powers, commencing at the first.

Suppose it were required to resolve the equation $2^x=6$. By making $x=2$, and $x=3$, we find $2^2=4$ and $2^3=8$: from which we perceive that x has a value comprised between 2 and 3.

Suppose then, that $x=2+\frac{1}{x'}$, in which case $x'>1$.

Substituting this value in the proposed equation, it becomes,

$$2^{2+\frac{1}{x'}}=6 \quad \text{or} \quad 2^2\times 2^{\frac{1}{x'}}=6\,; \quad \text{hence} \quad 2^{\frac{1}{x'}}=\frac{3}{2},$$

or $\left(\frac{3}{2}\right)^{x'}=2$, by changing the members, and raising both to the x' power.

To determine x', make successively $x'=1$ and 2; we find $\left(\frac{3}{2}\right)^1=\frac{3}{2}$ less than 2, and $\left(\frac{3}{2}\right)^2=\frac{9}{4}$, which is greater than 2; therefore x' is comprised between 1 and 2.

Suppose $x'=1+\frac{1}{x''}$, in which $x''>1$.

By substituting this value in the equation $\left(\frac{3}{2}\right)^{x'}=2$

$$\left(\frac{3}{2}\right)^{1+\frac{1}{x''}}=2 \text{ or } \frac{3}{2}\times\left(\frac{3}{2}\right)^{\frac{1}{x''}}=2$$

$$\left(\frac{4}{3}\right)^{x''}=\frac{3}{2} \quad \text{by reducing.}$$

The two hypotheses $x''=1$ and $x''=2$, give $\frac{4}{3}$ which is less than $\frac{3}{2}$, and $\left(\frac{4}{3}\right)^2=\frac{16}{9}=1+\frac{7}{9}$ which is greater than $\frac{3}{2}$; therefore x'' is comprised between 1 and 2.

Let $x''=1+\frac{1}{x'''}$, there will result

$$\left(\frac{4}{3}\right)^{1+\frac{1}{x'''}}=\frac{3}{2} \text{ or } \frac{4}{3}\times\left(\frac{4}{3}\right)^{\frac{1}{x'''}}=\frac{3}{2},$$

whence $$\left(\frac{9}{8}\right)^{x'''}=\frac{4}{3} \quad \text{by reducing.}$$

Making successively $x'''=1, 2, 3$, we find for the two last hypotheses $\left(\frac{9}{8}\right)^2=\frac{81}{64}=1+\frac{17}{64}$, which is $<1+\frac{1}{3}$, and . . . . . $\left(\frac{9}{8}\right)^3=\frac{729}{512}=1+\frac{217}{512}$, which is $>1+\frac{1}{3}$: therefore x''' is comprised between 2 and 3.

Let $x'''=2+\frac{1}{x^{\text{IV}}}$, the equation involving x''' becomes

$$\left(\frac{9}{8}\right)^{2+\frac{1}{x^{\text{IV}}}}=\frac{4}{3}, \text{ or } \frac{81}{64}\left(\frac{9}{8}\right)^{\frac{1}{x^{\text{IV}}}}=\frac{4}{3};$$

and consequently $$\left(\frac{256}{243}\right)^{x^{\text{IV}}}=\frac{9}{8}.$$

Operating upon this exponential equation in the same manner as upon the preceding equations, we shall find two entire numbers k and $k+1$, between which x^{IV} will be comprised. Making $x^{\text{IV}}=k+\frac{1}{x^{\text{V}}}$, x^{V} can be determined in the same manner as x^{IV}, and so on.

Making the necessary substitutions in the equations

$$x=2+\frac{1}{x'},\ x'=1+\frac{1}{x''},\ x''=1+\frac{1}{x'''},\ x'''=2+\frac{1}{x^{\text{IV}}}\ \ldots,$$

we obtain the value of x under the form of a continued fraction

$$x=2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{x^{\text{IV}}}}}}.$$

Hence we find the first three approximating fractions to be.

$$\frac{1}{1},\quad \frac{1}{2},\quad \frac{3}{5},$$

and the fourth is equal to

$$\frac{3\times2+1}{5\times2+2}=\frac{7}{12}\ \text{(Art. 235)},$$

which is the value of the fractional part to within

$$\frac{1}{(12)^2}\ \text{or}\ \frac{1}{144}\ \text{(Art. 217)}.$$

Therefore $x=2+\frac{7}{12}=\frac{31}{12}$ to within $\frac{1}{144}$, and if a greater degree of exactness is required, we must take a greater number of integral fractions.

EXAMPLES.

$3^x = 15 \ldots\ldots x =$ 2,46 to within 0,01.

$10^x = 3 \ldots\ldots x =$ 0,477 ..... 0,001.

$5^x = \frac{2}{3} \ldots\ldots x = -$ 0,25 ...... 0,01.

Theory of Logarithms.

239. If we suppose a to preserve the same value in the equation

$$a^x = y,$$

and y to be replaced by all possible positive numbers, it is plain that x will undergo changes corresponding to those made in y. Now, by the method explained in the last Article, we can determine for each value of y, the corresponding value of x, either exactly or approximatively.

First suppose $a>1$.

Making in succession $\quad x=0, \;\; 1, \;\; 2, \;\; 3, \;\; 4, \;\; 5, \ldots$ &c.

there will result $\quad y=a^0=1, \;\; a, \;\; a^2, \;\; a^3, \;\; a^4, \;\; a^5, \ldots$ &c.

hence, *every value of* y *greater than unity, is produced by the powers of* a, *the exponents of which are positive numbers, entire or fractional; and the values of* y *increase with* x.

Make now $\quad x=0, \;\; -1, \;\; -2, \;\; -3, \;\; -4, \;\; -5, \ldots$ &c.

there will result $\quad y=a^0=1, \;\; \frac{1}{a}, \;\; \frac{1}{a^2}, \;\; \frac{1}{a^3}, \;\; \frac{1}{a^4}, \;\; \frac{1}{a^5}, \ldots$ &c.

hence, *every value of* y *less than unity, is produced by the powers of* a, *of which the exponents are negative; and the value of* y *diminishes as the value of* x *increases negatively.*

Suppose $a<1$ *or equal to the proper fraction* $\frac{1}{a'}$.

Making . . . . . $x=0, \;\; 1, \;\; 2, \;\; 3, \;\; 4, \ldots$ &c.

we fina . . . $y=\left(\frac{1}{a'}\right)^0=1, \;\; \frac{1}{a'}, \;\; \frac{1}{a'^2}, \;\; \frac{1}{a'^3}, \;\; \frac{1}{a'^4}, \ldots$ &c.

Making . . . . . $x=0, \;\; -1, \;\; -2, \;\; -3, \;\; -4,$

we obtain . . $y=\left(\frac{1}{a'}\right)^0=1, \;\; a', \;\; a'^2, \;\; a'^3, \;\; a'^4, \ldots$ &c.

That is, in the hypothesis $a<1$, all numbers are formed with the different powers of a, in the inverse order of that in which they are formed when we suppose $a>1$.

Hence, *every possible positive number can be formed with any constant positive number whatever, by raising it to suitable powers.*

REMARK. The number a must always be *different from unity*, because all the powers of 1 are equal to 1.

240. By conceiving that a table has been formed, containing in one column, every entire number, and in another, the *exponents of the powers* to which it is necessary to raise *an invariable number*, to form all these numbers, an idea will be had of a table of *logarithms*. Hence,

The logarithm of a number, is the exponent of the power to which it is necessary to raise a certain invariable number, in order to produce the first number.

Any number, except 1, may be taken for the invariable number; but when once chosen, it must remain the same for the formation of all numbers, and it is called the *base* of the system of logarithms.

Whatever the base of the system may be, *its logarithm is unity, and the logarithm of* 1 *is* 0.

For, let a be the base: then

1st, we have $a^1=a$, whence $\log a=1$.

2d, $a^0=1$, whence $\log 1=0$.

The word logarithm is commonly denoted by the first three letters *log*, or simply by the first letter *l*.

We will now show some of the advantages of tables of logarithms in making numerical calculations.

Multiplication and Division.

241. Let a be the base of a system of logarithms, and suppose the table to be calculated. Let it be required to multiply together a series of numbers by means of their logarithms. Denote the numbers by y, y', y'', y''' . . . &c., and their corresponding logarithms

by x, x', x'', x''', &c. Then by definition (Art. 240), we have

$$a^x=y, \quad a^{x'}=y', \quad a^{x''}=y'', \quad a^{x'''}=y''' \ldots \&c.$$

Multiplying these equations together, member by member, and applying the rule for the exponents, we have

$$a^{x+x'+x''+x'''} \ldots = y\,y'y''y''' \ldots\ldots\ldots \text{ or}$$

$$x+x'+x''+x''' \ldots = \log y + \log y' + \log y'' + \log y''' \ldots$$
$$= \log.\ y\,y'y''y''',$$

that is, *the sum of the logarithms of any number of factors is equal to the logarithm of the product of those factors.*

242. Suppose it were required to divide one number by another. Let y and y' denote the numbers, and x and x' their logarithms. We have the equations

$$a^x=y \quad \text{and} \quad a^{x'}=y';$$

hence by division

$$a^{x-x'}=\frac{y}{y'},$$

or

$$x-x' = \log y - \log y' = \log \frac{y}{y'},$$

that is, *the difference between the logarithm of the dividend and the logarithm of the divisor, is equal to the logarithm of the quotient.*

Consequences of these properties. A multiplication can be performed by taking the logarithms of the two factors from the tables, and *adding* them together; this will give the logarithm of the product. Then finding this new logarithm in the tables, and taking the number which corresponds to it, we shall obtain the required product. Therefore, *by a simple addition, we find the result of a multiplication.*

In like manner, when one number is to be divided by another, subtract the logarithm of the divisor from that of the dividend, then find the number corresponding to this difference; this will be the required quotient. Therefore, *by a simple subtraction, we obtain the quotient of a division.*

Formation of Powers and Extraction of Roots.

243. Let it be required to raise a number y to any power denoted by $\frac{m}{n}$. If a denotes the base of the system, and x the logarithm of y, we shall have

$$a^x = y \quad \text{or} \quad y = a^x,$$

whence, raising both members to the power $\frac{m}{n}$,

$$y^{\frac{m}{n}} = a^{\frac{m}{n}x}.$$

Therefore,
$$\log y^{\frac{m}{n}} = \frac{m}{n} \, . \, x = \frac{m}{n} \, . \log y,$$

that is, *if the logarithm of any number be multiplied by the exponent of the power to which the number is to be raised, the product will be equal to the logarithm of that power.*

As a particular case, take $n=1$; there will result $m . \log y = \log y^m$; an equation which is susceptible of the above enunciation.

244. Suppose, in the first equation, $m=1$; there will result

$$\frac{1}{n} \log y = \log y^{\frac{1}{n}} = \log \sqrt[n]{y},$$

that is, *the logarithm of any root of a number is obtained by dividing the logarithm of the number by the index of the root.*

Consequence. To form any power of a number, take the logarithm of this number from the tables, multiply it by the exponent of the power; then the number corresponding to this product will be the required power.

In like manner, to extract the root of a number, divide the logarithm of the proposed number by the index of the root, then the number corresponding to the quotient will be the required root. Therefore, *by a simple multiplication, we can raise a quantity to a power, and extract its root by a simple division.*

245. The properties just demonstrated are independent of any system of logarithms; but the consequences which have been deduced from them, that is, the use that may be made of them in numerical calculations, supposes the construction of a table, containing all the numbers in one column, and the *logarithms* of these numbers in another, calculated from a given *base*. Now, in calculating this table, it is necessary, in considering the equation $a^x=y$, to make y pass through all possible states of magnitude, and determine the value of x corresponding to each of the values of y, by the method of Art. 238.

The tables in common use, are those of which the base is 10, and their construction is reduced to the resolution of the equation $10^x=y$. Making in this equation, y successively equal to the series of natural numbers, 1, 2, 3, 4, 5, 6, 7 . . ., we have to resolve the equations

$$10^x=1, \quad 10^x=2, \quad 10^x=3, \quad 10^x=4 \ldots$$

We will moreover observe, that it is only necessary to calculate directly, by the method of Art. 238, the logarithms of the prime numbers 1, 2, 3, 5, 7, 11, 13, 17 . . .; for as all the other entire numbers result from the multiplication of these factors, their logarithms may be obtained by the addition of the logarithms of the prime numbers (Art. 241).

Thus, since 6 can be decomposed into 2×3, we have

$$\log 6=\log 2+\log 3;$$

in like manner, $24=2^3\times3$; hence $\log 24=3\log 2+\log 3$.

Again, $360=2^3\times3^2\times5$; hence

$$\log 360=3\log 2+2\log 3+\log 5.$$

It is only necessary to place the logarithms of the entire numbers in the tables; for, by the property of division (Art. 242), we obtain the logarithm of a fraction by subtracting the logarithm of the divisor from that of the dividend.

246. Resuming the equation $10^x=y$, if we make

$$x=0,\quad 1,\quad 2,\quad 3,\quad 4,\quad 5,\quad \ldots\quad n-1,\quad n.$$

we have

$$y=1,\quad 10,\quad 100,\ 1000,\quad 10000,\ 100000,\ldots\quad 10^{n-1},\quad 10^n.$$

And making

$$x=0,\ -1,\quad -2,\ -3,\quad -4,\quad -5,\quad \ldots -(n-1),\quad -n.$$

we have

$$y=1,\quad \frac{1}{10},\ \frac{1}{100},\ \frac{1}{1000},\ \frac{1}{10000},\ \frac{1}{100000},\ \ldots\quad \frac{1}{10^{n-1}},\ \frac{1}{10^n}.$$

From which we see that, *the logarithm of a whole number will become the logarithm of a corresponding decimal by changing its sign from plus to minus.*

247. Resume the equation $a^x=y$, in which we will first suppose

$$a>1.$$

Then, if we make $y=1$ we shall have

$$a^0=1.$$

If we make $y<1$ we shall have

$$a^{-x}=y \quad \text{or} \quad \frac{1}{a^x}=y<1.$$

If now, y diminishes x will increase, and when y becomes 0, we have $a^{-x}=\frac{1}{a^x}=0$ or $a^x=\infty$ (Art. 112); but no finite power of a is infinite, hence $x=\infty$: and therefore, *the logarithm of* 0 *in a system of which the base is greater than unity, is an infinite number and negative.*

248. Again take the equation $a^x=y$, and suppose the base $a<1$. Then making, as before, $y=1$, we have $a^0=1$.

If we make y less than 1 we shall have

$$a^x=y<1.$$

Now, if we diminish y, x will increase; for since $a<1$ its powers will diminish as the exponent x increases, and when $y=0$, x must

24*

be infinite, for no finite power of a fraction is 0. Hence, *the logarithm of* 0 *in a system of which the base is less than unity, is an infinite number, and positive.*

Logarithmic and Exponential Series.

349. The method of resolving the equation $a^x=b$, explained in Art. 238, is sufficient to give an idea of the construction of logarithmic tables; but this method is very laborious when we wish to approximate very near the value of x. Analysts have discovered much more expeditious methods for constructing new tables, or for verifying those already calculated. These methods consist in the development of logarithms into series.

Taking again the equation $a^x=y$, it is proposed to develop the logarithm of y into a series involving the powers of y, and co-efficients independent of y.

It is evident, that the same number y will have a different logarithm in different systems; hence the log y, will depend for its value, 1st. on the value of y; and 2dly, on a, the base of the system of logarithms. Hence the development must contain y, or some quantity dependent on it, and some quantity dependent on the base a.

To find the form of this development, we will assume

$$\log y = A + By + Cy^2 + Dy^3 +, \text{ \&c.},$$

in which A, B, C, &c. are independent of y, and dependent on the base a.

Now, if we make $y=0$, the log y becomes infinite, and is either negative or positive according as the base a is greater or less than unity (Arts. 247 & 248). But the second member under this supposition, reduces to A, a finite number: hence the development cannot be made under that form.

Again, assume

$$\log y = Ay + By^2 + Cy^3 + Dy^4 +, \text{ \&c.}$$

If we make $y=0$, we have

$$\log y = \pm \infty = 0,$$

which is absurd, and hence the development cannot be made under the last form. Hence we conclude that, *the logarithm of a number cannot be developed in the powers of that number.*

Let us now place for y, $1+y$, and we shall have

$$\log(1+y)=Ay+By^2+Cy^3+Dy^4+ \text{\&c.} \ldots (1),$$

making $y=0$, the equation is reduced to $\log 1=0$, which does not present any absurdity.

In order to determine the co-efficients $A, B, C \ldots$, we will follow the process of Art. 207. Substituting z for y, the equation becomes

$$\log(1+z)=Az+Bz^2+Cz^3+Dz^4+ \ldots (2).$$

Subtracting the equation (2) from (1), we obtain

$$\log(1+y)-\log(1+z)=A(y-z)+B(y^2-z^2)+C(y^3-z^3)+\ldots(3).$$

The second member of this equation is divisible by $y-z$; we will see, if we can by any artifice, put the first under such a form that it shall also be divisible by $y-z$.

We have, $\log(1+y)-\log(1+z)=\log\dfrac{1+y}{1+z}=\log\left(1+\dfrac{y-z}{1+z}\right)$;

but since $\dfrac{y-z}{1+z}$ can be regarded as a single number u, we can develop $\log(1+u)$, or $\log\left(1+\dfrac{y-z}{1+z}\right)$, in the same manner as . . . . . $\log(1+y)$, which gives

$$\log\left(1+\frac{y-z}{1+z}\right)=A\frac{y-z}{1+z}+B\left(\frac{y-z}{1+z}\right)^2+C\left(\frac{y-z}{1+z}\right)^3+\ldots$$

Substituting this development for $\log(1+y)-\log(1+z)$ in the equation (3), and dividing both members by $y-z$, it becomes

$$A\cdot\frac{1}{1+z}+B\frac{y-z}{(1+z)^2}+C\frac{(y-z)^2}{(1+z)^3}+\ldots$$
$$=A+B(y+z)+C(y^2+yz+z^2)+\ldots$$

Since this equation, like the preceding, must be verified by all values of y and z, make $y=z$, and there will result

$$\frac{A}{1+y}=A+2By+3Cy^2+4Dy^3+5Ey^4+\ \ldots$$

Whence, clearing the fraction, and transposing

$$\begin{array}{l|l|l|l|l} 0=A+2B & y+3C & y^2+4D & y^3+5E & y^4+\ \ldots \\ \ \ -A+\ A & \ +2B & \ +3C & \ +4D & \end{array}$$

Putting the co-efficients of the different powers of y equal to zero, we obtain the series of equations

$$A-A=0,\quad 2B+A=0,\quad 3C+2B=0,\quad 4D+3C=0\ \ldots;$$

whence

$$A=A,\quad B=-\frac{A}{2},\quad C=-\frac{2B}{3}=+\frac{A}{3},\quad D=-\frac{3C}{4}=-\frac{A}{4}\ldots$$

The law of the series is evident; the co-efficient of the n^{th} term is equal to $\mp\frac{A}{n}$, according as n is even or odd; hence we shall obtain for the development of log $(1+y)$,

$$\log(1+y)=Ay-\frac{A}{2}y^2+\frac{A}{3}y^3-\frac{A}{4}y^4\ \ldots$$

$$=A\left(y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}\ \ldots\right)\quad (4).$$

If we substitute $-y$ for y, we shall have

$$\log(1-y)=A\left(-y-\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^4}{4}+\ \&c.\right)\quad (5).$$

Hence, although the logarithm of a number cannot be developed in the powers of that number, yet *it may be developed in the powers of a number greater or less by unity.*

By the above method of development, the co-efficients B, C, D, E, &c. have all been determined in functions of A; but the relation between A and the base of the system is yet undetermined.

The number A is called the *modulus* of the system of logarithms in which the log $(1+y)$, or log $(1-y)$, is taken. Hence, *The modulus of a system of logarithms depends for its value on the base, and if a certain function of any number be multiplied by it, the product will be the logarithm, of that number augmented by unity.*

250. If we take the logarithm of $1+y$ in a new system, and denote it by $l'(1+y)$, we shall have

$$\mathrm{l}'(1+y)=A'\left(y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}-\text{ \&c.}\right) \quad (6)$$

in which A' is the modulus of the new system.

If we suppose y to have the same value as in equation (4), we shall have

$$\mathrm{l}'(1+y) : \mathrm{l}(1+y) : : A' : A,$$

for, since the series in the second members are the same they may be omitted. Therefore,

The logarithms of the same number, taken in two different systems, are to each other as the moduli of those systems.

251. If we make the modulus $A'=1$, the system of logarithms which results is called the *Naperian System*. This was the first system known, and was invented by Baron Napier, a Scotch Mathematician.

With this modification the proportion above becomes

$$\mathrm{l}'(1+y) : \mathrm{l}(1+y) : : 1 : A$$

or

$$A \,.\, \mathrm{l}'(1+y)=\mathrm{l}(1+y).$$

Hence we see that, *the Naperian logarithm of any number, multiplied by the modulus of another system, will give for a product the logarithm of the same number in that system.*

252. Again, $A \,.\, \mathrm{l}'(1+y)=\mathrm{l}(1+y)$ gives

$$\mathrm{l}'(1+y)=\frac{\mathrm{l}(1+y)}{A}$$

That is, *the logarithm of any number divided by the modulus of the system, is equal to the Naperian logarithm of the same number.*

253. If we take the Naperian logarithm and make $y=1$, equation (6) becomes

$$l'z=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\ldots$$

a series which does not converge rapidly, and in which it would be necessary to take a great number of terms for a near approximation. In general, this series will not serve for determining the logarithms of entire numbers, since for every number greater than 2 we should obtain a series in which the terms would go on increasing continually.

The following are the principal transformations for converting the above series into converging series, for the purpose of obtaining the logarithms of entire numbers, which are the only logarithms placed in the tables.

First Transformation.

Taking the Naperian logarithm in equation (6), making $y=\frac{1}{z}$, and observing that

$$l'\left(1+\frac{1}{z}\right)=l'(1+z)-l'z, \quad \text{it becomes}$$

$$l'(1+z)-l'z=\frac{1}{z}-\frac{1}{2z^2}+\frac{1}{3z^3}-\frac{1}{4z^4}+\text{ \&c.} \quad (7).$$

This series becomes more converging as z increases; besides the first member of this equation expresses the difference between two consecutive logarithms.

Making $z=1, 2, 3, 4, 5$, &c. we have

$$l'2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\text{ \&c.}$$

$$l'3-l'2=\frac{1}{2}-\frac{1}{8}+\frac{1}{24}-\frac{1}{64}+\ldots$$

$$\mathrm{l}'4-\mathrm{l}'3=\frac{1}{3}-\frac{1}{18}+\frac{1}{81}-\frac{1}{324}+\ldots$$

$$\mathrm{l}'5-\mathrm{l}'4=\frac{1}{4}-\frac{1}{32}+\frac{1}{192}-\frac{1}{1024}.$$

The first series will give the logarithm of 2; the second series will give the logarithm of 3 by means of the logarithm of 2; the third, the logarithm of 4, in functions of the logarithm of 3 . . . &c. The degree of approximation can be estimated, since the series are composed of terms alternately positive and negative (Art. 203).

Second Transformation.

A much more converging series is obtained in the following manner.

In the series

$$\mathrm{l}'(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$$

substitute $-x$ for x; and it becomes

$$\mathrm{l}'(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots$$

Subtracting the second series from the first, observing that

$$\mathrm{l}'(1+x)-\mathrm{l}'(1-x)=\mathrm{l}'\frac{1+x}{1-x},\text{ we obtain}$$

$$\mathrm{l}'\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\frac{x^9}{9}+\ldots\right)$$

This series will not converge very rapidly unless x is a very small fraction, in which case, $\frac{1+x}{1-x}$ will be greater than unity, but will differ very little from it.

Take $\frac{1+x}{1-x}=1+\frac{1}{z}$, z being an entire number;

we have $(1+x)z=(1-x)(z+1)$: whence $x=\frac{1}{2z+1}$.

Hence the preceding series becomes $l'\left(1+\frac{1}{z}\right)$ or

$$l'(z+1)-l'z=2\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\ldots\right)$$

This series also gives the difference between two consecutive logarithms, but it converges much more rapidly than the series (7).

Making successively $z=1, 2, 3, 4, 5 \ldots$, we find

$$l'2=2\left(\frac{1}{3}+\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}+\frac{1}{7\cdot 3^7}+\ldots\right),$$

$$l'3-l'2=2\left(\frac{1}{5}+\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}+\frac{1}{7\cdot 5^7}+\ldots\right),$$

$$l'4-l'3=2\left(\frac{1}{7}+\frac{1}{3\cdot 7^3}+\frac{1}{5\cdot 7^5}+\frac{1}{7\cdot 7^7}+\ldots\right),$$

$$l'5-l'4=2\left(\frac{1}{9}+\frac{1}{3\cdot 9^3}+\frac{1}{5\cdot 9^5}+\frac{1}{7\cdot 9^7}\ldots\right).$$

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Let $z=100$; there will result

$$l'101=l'100+2\left(\frac{1}{201}+\frac{1}{3(201)^3}+\frac{1}{5(201)^5}+\ldots\right);$$

whence we see, that knowing the logarithm of 100, the first term of the series is sufficient for obtaining that of 101 to seven places of decimals.

The Naperian logarithm of 10 may be deduced from the third and fourth of the above equations, by simply adding the logarithm of 2 to that of 5 (Art. 241). This number has been calculated with great exactness, and is 2,302585093.

There are formulas more converging than the above, which serve to obtain logarithms in functions of others already known, but the preceding are sufficient to give an idea of the facility with which tables may be constructed. We may now suppose the Naperian logarithms of all numbers to be known.

254. We have already observed that the base of the common system of logarithms is 10 (Art. 245). We will now find its modulus.

$$l'(1+y) : l(1+y) : : 1 : A \quad \text{(Art. 250).}$$

If we make $y=9$, we shall have

$$l'10 : l10 : : 1 : A.$$

But the $l'10=2{,}302585093\ldots$ and $l10=1$ (Art. 245); hence $A=\frac{1}{2{,}302585093}=0{,}434294482$ the modulus of the common system.

If now, we multiply the Naperian logarithms before found, by this modulus, we shall obtain a table of common logarithms (Art. 251).

255. All that now remains to be done is to find the base of the Naperian system. If we designate that base by e, we shall have (Art. 250),

$$l'e : le : : 1 : 0{,}434294482.$$

But $l'e=1$ (Art. 240): hence

$$1 : le : : 1 : 0{,}434294482,$$

or

$$le=0{,}434294482.$$

But as we have already explained the method of calculating the common tables, we may use them to find the number whose logarithm is 0,434294482, which we shall find to be 2,718281828: hence

$$e=2{,}718281828.$$

We see from the last equation but one that, *the modulas of the common system is equal to the common logarithm of the Naperian base.*

CHAPTER VI.

General Theory of Equations.

256. THE most celebrated analysts have tried to resolve equations of any degree whatever, but hitherto their efforts have been unsuccessful with respect to equations of a higher degree than the fourth. However, their investigations on this subject have conducted them to some properties common to equations of every degree, which they have since used, either to resolve certain classes of equations, or to reduce the resolution of a given equation to that of one more simple. In this chapter it is proposed to make known these properties, and their use in facilitating the resolution of equations.

257. The development of the properties relating to equations of every degree, leads to the consideration of polynomials of a particular nature, and entirely different from those considered in the first chapter. These are, expressions of the form

$$Ax^m+Bx^{m-1}+Cx^{m-2}+\ldots+Tx+U,$$

in which m is a positive whole number; but the co-efficients A, B, C, . . . T, U, denote any quantities whatever, that is, entire or fractional quantities, commensurable or incommensurable. Now, in algebraic division, as explained in Chapter I, the object was this, viz.: *given two polynomials* entire, with reference to all the letters and particular numbers involved in them, *to find a third polynomial of the same kind, which, multiplied by the second, would produce the first.*

But when we have two polynomials,

$$A\,x^m+B\,x^{m-1}+C\,x^{m-2}+\ldots+T\,x+U,$$
$$A'x^n+B'x^{n-1}+C'x^{n-2}+\ldots+T'x+U\,,$$

which are *necessarily* entire only with respect to x, and in which the co-efficients A, B, C . . ., A′, B′, C′ . . ., may be any quantities whatever, it may be proposed *to find a third polynomial*, of the same form and nature as the two preceding, *which multiplied by the second, will re-produce the first.*

The process for effecting this division is analogous to that for common division; but there is this difference, viz.: In this last, *the first term of each partial dividend must be exactly divisible by the first term of the divisor;* whereas, in the new kind of division, we divide the first term of each partial dividend, that is, the part affected with the highest power of the principal letter, by the first term of the divisor, whether the co-efficient of the corresponding partial quotient is entire or fractional; and the operation is continued *until a quotient is obtained, which, multiplied by the divisor, will cancel the last partial dividend*, in which case the division is said to be exact; or, *until a remainder is obtained, of a degree less than that of the divisor*, with reference to the principal letter, in which case the division is considered impossible, since by continuing the operation, quotients would be obtained containing the principal letter affected with *negative exponents*, or this same letter in the denominator of them, which would be contrary to the nature of the question, which requires that the quotient should be of the same form as the proposed polynomials.

258. To distinguish polynomials which are entire with reference to a letter, x for example, but the co-efficients of which are any quantities whatever, from ordinary polynomials, that is, from polynomials which are entire with reference to all the letters and particular numbers involved in them, it has been agreed to call the first *entire functions of x*, and the second, *rational and entire polynomials.*

General Properties of Equations.

259. Every complete equation of the m^{th} degree, m being a positive whole number, may, by the transposition of terms, and by the division of both members by the co-efficient of x^m, be put under the form

$$x^m+Px^{m-1}+Qx^{m-2}+\ldots+Tx+U=0;$$

P, Q, R . . . T, U, being co-efficients taken in the most general algebraic sense.

Any expression, whatever the nature of it may be, that is, numerical or algebraic, real or imaginary, which, substituted in place of x *in the equation, renders its first member equal to* 0, *is called a root of this equation.*

260. As every equation may be considered as the algebraic translation of the relations which exist between the given and unknown quantities of a problem, we are naturally led to this principle, viz. EVERY EQUATION *has at least one root.* Indeed, the conditions of the enunciation may be incompatible, but then we must suppose that we shall be warned of it by some *symbol of absurdity*, such as a formula, containing as a necessary operation, the extraction of an even root of a negative quantity; yet there will still exist an expression which, substituted for x in the equation, will satisfy it. We will admit this principle, which we shall have occasion to verify hereafter for most equations.

The following proposition may be regarded as the fundamental property of the theory of equations.

First Property.

261. *If* a *is a root of the equation*

$$x^m+Px^{m-1}+Qx^{m-2}+\ldots Tx+U=0,$$

the first member of it is divisible by $x-a$; and reciprocally, *if a factor of the form* x—a, *will divide the first member of the proposed equation,* a *is a root of it.*

For, perform the division, and see what takes place when the operation is continued until the exponent of x, in the first term of the dividend, becomes 0.

This operation is of the nature of that spoken of in Art. 257, since a, P, Q, . . . are any quantities whatever.

$$
\begin{array}{l|l}
x^m+\mathrm{P}x^{m-1}+\mathrm{Q}x^{m-2}+\ldots+\mathrm{T}x+\mathrm{U} & x-a \\
\hline
\left.\begin{array}{l}+a\\+\mathrm{P}\end{array}\right| x^{m-1} &
x^{m-1}+\left.\begin{array}{l}a\\+\mathrm{P}\end{array}\right| x^{m-2}+\left.\begin{array}{l}a^2\\+\mathrm{P}a\\+\mathrm{Q}\end{array}\right| x^{m-3}+\ldots+\begin{array}{l}a^{m-1}\\+\mathrm{P}a^{m-2}\\+\mathrm{Q}a^{m-3}\\+\ldots\\+\mathrm{T}\end{array} \\
\qquad\left.\begin{array}{l}+a^2\\+\mathrm{P}a\\+\mathrm{Q}\end{array}\right| x^{m-2}+\ldots &
\end{array}
$$

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By reflecting a little upon the manner in which the partial quotients are obtained, we shall first discover from analogy, and afterwards by a method employed several times (Arts. 59 & 127), *a law of formation* for the co-efficients of these quotients; and we may conclude, 1st. that there will be m partial quotients, 2d. that the co-efficient of the m^{th} quotient, that is of x^0, must be

$$a^{m-1}+\mathrm{P}a^{m-2}+\mathrm{Q}a^{m-3}+\ \ldots\ +\mathrm{T},$$

T being the co-efficient of the last term but one of the proposed equation.

Hence, by multiplying the divisor by this quotient, and reducing it with the dividend, we obtain for a remainder

$$a^m+\mathrm{P}a^{m-1}+\mathrm{Q}a^{m-2}+\ \ldots\ +\mathrm{T}a+\mathrm{U}.$$

Now, by hypothesis a is a root of the equation; hence, *this remainder is nothing*, since it is nothing more than the result of the substitution of a for x in the equation; *therefore the division is exact.*

Reciprocally, if $x-a$ is an exact divisor of $x^m+\mathrm{P}x^{m-1}+\ldots$, the remainder $a^m+\mathrm{P}a^{m-1}+\ldots$ will be *nothing;* therefore (Art. 259), a is a root of the equation.

262. From this it results that, in order to discover whether a binomial of the form $x-a$ is an exact divisor of a polynomial involv-

ing x, it will be sufficient to see if the result of the substitution of a for x, is equal to 0.

To ascertain whether a is a root of a polynomial involving x, which is placed equal to 0, it will be sufficient to try the division of it by $x-a$. If it is exact, we may be certain that a is a root of the equation.

263. Remark. By inspecting the quotient of the division in Art. 261, we perceive the following law for the co-efficients: *Each co-efficient is obtained by multiplying that which precedes it by the root* a, *and adding to the product that co-efficient of the proposed equation which occupies the same rank as that which we wish to obtain in the quotient.*

Thus, the co-efficient of the 3d term, a^2+Pa+Q, is equal to $(a+P)a+Q$, or to the product of the preceding co-efficient $a+P$, by the root a, augmented by the co-efficient Q of the 3d term of the proposed equation.

The co-efficient of the 4th term is

$$(a^2+Pa+Q)a+R, \quad \text{or} \quad a^3+Pa^2+Qa+R.$$

This law should be remembered.

Second Property.

264. *Every equation involving but one unknown quantity, has as many roots as there are units in the exponent of its degree, and no more.*

Let the proposed equation be

$$x^m+Px^{m-1}+Qx^{m-2}+\ .\ .\ .\ +Tx+U=0.$$

Since every equation has at least one root (Art. 260), if we de note that root by a, the first member will be divisible by $x-a$, and we shall have the identical equation

$$x^m+Px^{m-1}+\ .\ .\ .\ =(x-a)\ (x^{m-1}+P'x^{m-2}+\ .\ .\ .)\ .\ .\ .\ (1).$$

But by supposing

$$x^{m-1}+P'x^{m-2}+\ .\ .\ .\ =0,$$

we obtain an equation which has at least one root.

Denote this root by b, we have (Art. 261),

$$x^{m-1}+P'x^{m-2}+\ldots=(x-b)\,(x^{m-2}+P''x^{m-3}+\ldots).$$

Substituting the 2d member for its value, in equation (1), and we have,

$$x^m+Px^{m-1}+\ldots=(x-a)\,(x-b)\,(x^{m-2}+P''x^{m-3}+\ldots)\ldots(2).$$

Reasoning upon the polynomial $x^{m-2}+P''x^{m-3}+\ldots$ as upon the preceding polynomial, we have

$$x^{m-2}+P''x^{m-3}+\ldots=(x-c)\,(x^{m-3}+P'''x^{m-4}+\ldots),$$

and by substitution

$$x^m+Px^{m-1}+\ldots=(x-a)\,(x-b)\,(x-c)\,(x^{m-3}+\ldots)\ldots(3).$$

Observe that for each indicated factor of the first degree with reference to x, the degree of x in the polynomial is diminished by unity; therefore, after $m-2$ factors of the first degree have been divided out, the exponent of x will be reduced to $m-(m-2)$, or 2; that is, we shall obtain a polynomial of the second degree with reference to x, which can be decomposed into the product of two factors of the first degree, $(x-k)\,(x-l)$ (Art. 142). Now, as the $m-2$ factors of the first degree have already been indicated, it follows that we have the identical equation,

$$x^m+Px^{m-1}+\ldots=(x-a)\,(x-b)\,(x-c)\ldots(x-k)\,(x-l).$$

From which we see, that the *first member of the proposed equation is decomposed into* m *factors of the first degree.*

As there is *a root* corresponding to each divisor of the first degree (Art. 261), it follows that the m factors of the first degree $x-a$, $x-b$, $x-c\ldots$, give the m roots $a, b, c\ldots$ for the proposed equation.

Hence, the equation can have no other roots than $a, b, c\ldots k, l$, since if it had a root α, different from $a, b, c\ldots l$, it would follow that it would have a divisor $x-\alpha$, different from $x-a$, $x-b$, $x-c\ldots x-l$, which is impossible.

Finally, *every equation of the* m[th] *degree has* m *roots, and can have no more.*

265. There are some equations in which the number of roots is apparently less than the number of units in the exponent of their degree. They are those in which the first member is the product of equal factors, such as the equation

$$(x-a)^4(x-b)^3(x-c)^2(x-d)=0,$$

which has but *four* different roots, although it is of the 10th degree.

It is evident that no quantity α, different from a, b, c, d, can verify it; for if it had this root α, the first member would be divisible by $x-\alpha$, which is impossible.

But this is no reason why the proposed equation should not have ten roots, *four* of which are equal to a, *three* equal to b, *two* equal to c, and *one* equal to d.

266. Consequence of the second property.

The first member of every equation of the m^{th} degree, having m divisors of the first degree, of the form

$$x-a,\ x-b,\ x-c, \ldots x-k,\ x-l,$$

if we multiply these divisors together, *two and two, three and three* . . ., we shall obtain as many divisors of the second, third, &c., degree with reference to x, as we can form different combinations of m quantities, taken two and two, three and three, &c. Now the number of these combinations is expressed by

$$m \cdot \frac{m-1}{2}, m \cdot \frac{m-2}{3} \ldots \text{ (Art. 163).}$$

Thus, the proposed equation has $m \cdot \dfrac{m-1}{2}$ divisors of the second degree, $m \cdot \dfrac{m-1}{2} \cdot \dfrac{m-2}{3}$ divisors of the third degree, and so on.

Composition of Equations.

267. If in the identical equation

$$x^m+Px^{m-1}+ \ldots =(x-a)(x-b)(x-c) \ldots (x-l),$$

we perform the multiplication of four factors, we have

$$\left.\begin{array}{l|l|l|l} x^4-a & x^3+ab & x^2-abc & x+abcd \\ \quad -b & \quad +ac & \quad -abd & \\ \quad -c & \quad +ad & \quad -acd & \\ \quad -d & \quad +bc & \quad -bcd & \\ & \quad +bd & & \\ & \quad +cd & & \end{array}\right\}=0.$$

If we perform the multiplication of the m factors of the second member, and compare the terms of the two members, we shall find the following relations between the co-efficients P, Q, R, . . . T, U, and the roots a, b, c, . . . k, l, of the proposed equation, viz.

$$-a-b-c\ \ldots\ -k-l=\mathrm{P},\ \text{or}\ a+b+c+\ \ldots\ +k+l=-\mathrm{P};$$
$$ab+ac+\ \ldots\ +kl=\mathrm{Q}\ \ldots\ldots\ldots\ldots$$
$$-abc-abd\ \ldots\ -ikl=\mathrm{R},\ \text{or}\ abc+abd\ \ldots\ldots\ +ikl=-\mathrm{R};$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$\pm abcd\ \ldots\ kl=\mathrm{U},\ \text{or}\ abcd\ \ldots\ kl=\pm\mathrm{U}.$$

The double sign has been placed in the last relation, because the product $-a\times-b\times-c\ \ldots\ \times-l$ will be *plus* or *minus* according as the degree of the equation is *even* or *odd*.

Hence, 1st. The algebraic sum of the roots, taken with contrary signs, is equal to the co-efficient of the second term ; or, the algebraic sum of the roots themselves, is equal to the co-efficient of the second term taken with a contrary sign.

2d. The sum of the products of the roots taken two and two, with their respective signs, is equal to the co-efficient of the third term.

The sum of the products of the roots taken three and three with their signs changed, is equal to the co-efficient of the fourth term ; or the co-efficient of the fourth term, taken with a contrary sign, is equal to the sum of the products of the roots taken three and three ; and so on.

Finally, the product of all the roots, is equal to the last term ; that is, the product of all the roots, taken with their respective signs, is equal to the last term of the equation, taken with its sign, *when the equation is of an even degree,* and with a contrary sign, *when the equation is of an odd degree. If one of the roots is equal to* 0, *the absolute term will be* 0.

The properties demonstrated (Art. 142), with respect to equations of the second degree, are only particular cases of the above. The last term, taken with its sign, is equal to the product of the roots themselves, because the equation is of an even degree.

Remarks on the Greatest Common Divisor.

268. The properties of the greatest common divisor of two polynomials, were explained in Arts. 66 & 67. We shall here offer a few remarks to serve as a guide in determining it.

Let A be a rational and entire polynomial, supposed to be arranged with reference to one of the letters involved in it, a, for example.

If this polynomial is not *absolutely prime,* that is, if it can be decomposed into rational and entire factors, it may be regarded as the product of three principal factors, viz.

1st. Of a monomial factor A_1, common to all the terms of A. This factor is composed of the greatest common divisor of all the numerical co-efficients, multiplied by the product of the literal factors which are common to all the terms.

2d. Of a polynomial factor A_2, independent of a, which is common to all the co-efficients of the different powers of a, in the arranged polynomial.

3d. Of a polynomial factor A_3, depending upon a, and in which the co-efficients of the different powers of a are prime with each other; so that we shall have

$$A = A_1 \times A_2 \times A_3.$$

Sometimes one or both of the factors A_1, A_2 reduce to unity, but this is the general form of *rational and entire* polynomials. It follows from this, that when there is a greatest common divisor of two polynomials A and B, we shall have

$$D=D_1.D_2.D_3;$$

D_1 denoting the greatest monomial common factor, D_2 the greatest polynomial factor independent of a, and D_3 the greatest polynomial factor depending upon this letter.

In order to obtain D_1, *find the monomial factor* A_1 *common to all the terms of* A. This factor is in general composed of literal factors, which are found by inspecting the terms, and of a numerical co-efficient, obtained by finding the greatest common divisor of the numerical co-efficients in A.

In the same way, find the monomial B_1 *common to all the terms of* B; then *determine the greatest factor* D_1 *common to* A_1 *and* B_1.

This factor D_1, is set aside, as forming the first part of the required common divisor. *The factors* A_1 *and* B_1 *are also suppressed in the proposed polynomials*, and the question is reduced to finding the greatest common divisor of two new polynomials A' and B' which do not contain a common monomial factor. It is then to be understood that the process developed below, is to be applied to these two polynomials.

269. Several circumstances may occur as regards the number of letters that may be contained in A' and B'.

1*st. When A' and B' contain but one letter* a.

When A' and B' are arranged with reference to a, the coefficients will necessarily be *prime with each other;* therefore in this case, we shall only have to seek for the greatest common factor depending upon a, viz. D_3.

In order to obtain it, we must first prepare the polynomial of the highest degree, so that its first term may be exactly divisible by the first term of the divisor. This preparation consists in *multiplying the whole dividend by the co-efficient of the first term of the divisor, or by a factor of this co-efficient, or by a certain power of it,* in order that we may be able to execute several operations, without any new preparations (Art. 68).

The division is then performed, continuing the operation until a remainder is obtained of a lower degree than the divisor.

If there is a factor common to all the co-efficients of the remainder, it must be suppressed, as it cannot form a part of the required divisor; after which, *we operate with the second polynomial, and this remainder, in the same way we did with the polynomials* A' *and* B'.

Continue this series of operations until a remainder is obtained which will exactly divide the preceding remainder, this remainder will be the greatest common divisor D_3 of A' and B'; and $D_1 \times D_3$ will express the greatest common divisor of A and B; *or, continue the operation until a remainder is obtained independent of a,* that is, a *numerical* remainder, in which case, the two polynomials, A' and B' will be *prime with each other.*

2d. *When* A' *and* B' *contain two letters* a *and* b.

After having arranged the polynomials with reference to a, we first find the polynomial factor which is *independent of a,* if there is one.

To do this, we determine the greatest common divisor A_2 of all the co-efficients of the different powers of a in the polynomial A'. This common divisor is obtained by applying the rule for finding the greatest common divisor of several polynomials, as well as the rule for the last case, since these co-efficients contain only one letter b. *In the same way we determine the greatest common divisor* B_2 *of all the co-efficients of* B'. Then comparing A_2 and B_2, *we set aside their greatest common divisor* D_2, as forming a part of the required greatest common divisor; and we also suppress the factors A_2 and B_2, in A' and B'; which produces two new polynomials A'' and B'', the co-efficients of which are *prime with each other,* and to which we may consequently apply the rule for the first case.

Care must always be taken to ascertain, in each remainder, *whether* *the co-efficients of the different powers of the letter* a, *do not contain a common factor, which must be suppressed,* as not forming a part of the common divisor. We have already seen that the suppression of these factors is absolutely necessary (Art. 68).

We shall in this way obtain the common divisor D_2, of A'' and B'', and $D_1 \times D_2 \times D_3$, for the greatest common divisor of the polynomials A and B.

REMARK. In applying the rule for the first case to A'' and B'', we could ascertain when these two polynomials *were prime with each other,* from this circumstance, viz: *a remainder would be obtained which would be either numerical, or a function of* b, *but independent of* a. The greatest common divisor of A and B would then be $D_1 \times D_2$.

3d. *When* A′ *and* B′ *contain three letters,* a, b, c.

After arranging the two polynomials with reference to a, we determine the greatest common divisor independent of a, which is done by applying to the co-efficients of the different powers of a, in both polynomials, the process for the second case, since these polynomial co-efficients contain but two letters, b and c.

The independent polynomial D_2 being thus obtained, and the factor A_2 and B_2, which have given it, being suppressed in A' and B', there will result two polynomials A'' and B'', having their co-efficients *prime with each other,* and to which the rules for the preceding cases may be applied, and so on.

EXAMPLES.

1. Let there be the two polynomials

$$a^2d^2-c^2d^2-a^2c^2+c^4, \quad \text{and} \quad 4a^2d-2ac^2+2c^3-4acd.$$

The second contains a monomial factor 2. Suppressing it, and arranging the polynomials with reference to d, we have

$$(a^2-c^2)d^2-a^2c^2+c^4, \quad \text{and} \quad (2a^2-2ac)d-ac^2+c^3.$$

It is first necessary to ascertain whether there is a common divisor independent of d.

By considering the co-efficients a^2-c^2, and $-a^2c^2+c^4$, of the first polynomial, it will be seen that $-a^2c^2+c^4$ can be put under the form $-c^2(a^2-c^2)$; hence a^2-c^2 is a common factor of the co-efficients of the first polynomial. In like manner, the co-efficients of the second, $2a^2-2ac$, and $-ac^2+c^3$, can be reduced to $2a(a-c)$, and $-c^2(a-c)$; therefore $a-c$ is a common factor of these co-efficients.

Comparing the two factors a^2-c^2 and $a-c$, as this last will divide the first, it follows that $a-c$ is a common factor of the proposed polynomials, and it is that part of their greatest common divisor *which is independent of* d.

Suppressing a^2-c^2 in the first polynomial, and $a-c$ in the second, we obtain the two polynomials d^2-c^2 and $2ad-c^2$, to which the ordinary process must be applied.

$$\begin{array}{r|l}
d^2-c^2 & 2ad-c^2 \\ \cline{2-2}
4a^2d^2-4a^2c^2 & 2ad+c^2 \\ \cline{1-1}
+2ac^2d-4a^2c^2 & \\ \cline{1-1}
-4a^2c^2+c^4. &
\end{array}$$

Explanation. After having multiplied the dividend by $4a^2$, and performed two consecutive divisions, we obtain a remainder $-4a^2c^2+c^4$, independent of the letter d; hence the two polynomials d^2-c^2, and $2ad-c^2$, are prime with each other. Therefore the greatest common divisor of the proposed polynomials is $a-c$.

Again, taking the same example, and arranging with reference to a, it becomes, after suppressing the factor 2 in the second polynomial,

$$(d^2-c^2)a^2-c^2d^2+c^4, \text{ and } 2da^2-(2cd+c^2)a+c^3.$$

It is easily perceived, that the co-efficient of the different powers of a in the second polynomial are prime with each other. In the first polynomial, the co-efficient $-c^2d^2+c^4$, of the second term, or of a^0, becomes $-c^2(d^2-c^2)$; whence d^2-c^2 is a common factor of the two co-efficients, and since it is not a factor of the second polynomial, it may be suppressed in the first, as not forming a part of the common divisor.

By suppressing this factor, and taking the second polynomial for a dividend and the first for a divisor, (in order to avoid preparation), we have

$$\begin{array}{l}
\text{1st.}\quad 2da^2 - \left.\begin{array}{r}2cd\\ -\ c^2\end{array}\right| a + c^3 \ \left|\,\begin{array}{l}a^2-c^2\\ \hline 2d\end{array}\right. .\\
\hline
\text{Rem. . .}\quad \left.\begin{array}{r}-2cd\\ -\ c^2\end{array}\right| \begin{array}{l}a+2dc^2\\ \ \ +\ \ c^3\end{array}\\
\text{or, ...}\ a-c,
\end{array}$$

by suppressing the common factor $(-2cd-c^2)$;

$$\text{2d.}\qquad \begin{array}{r|l}
a^2 - c^2 & a-c\\ \cline{2-2}
\hline
+ac-c^2 & a+c\\
\hline
0 &
\end{array}.$$

Explanation. After having performed the first division, a remainder is obtained which contains $-2cd-c^2$, as a factor of its two co-efficients; for $2dc^2+c^3=-c(-2cd-c^2)$. This factor being suppressed, the remainder is reduced to $a-c$, which will exactly divide a^2-c^2.

Hence $a-c$ is the required greatest common divisor.

270. There is a remarkable case, in which, the greatest common divisor may be obtained more easily than by the general method; it is when *one of the two polynomials contains a letter which is not contained in the other.*

In this case, as it is evident that the greatest common divisor is independent of this letter, it follows that, by arranging the polynomial which contains it, with reference to this letter, *the required common divisor will be the same as that which exists between the co-efficients of the different powers of the principal letter and the second polynomial, which, by hypothesis, is independent of it.*

By this method, we are led to determine the greatest common divisor between three or more polynomials; but they will be more simple than the proposed polynomials. It often happens, that some of the co-efficients of the arranged polynomial are monomials, or, that we may discover by simple inspection that they are prime with each other; and, in this case, we are certain that the proposed polynomials are prime with each other.

Thus, in the example of Art. 269, treated by the first method, after having suppressed the common factor $a-c$, which gives the results,

$$d^2-c^2 \quad \text{and} \quad 2ad-c^2,$$

we know immediately that these two polynomials are prime with each other; for, since the letter a is contained in the second and not in the first, it follows from what has just been said, that the common divisor must divide the co-efficients $2d$ and $-c^2$, which is evidently impossible; hence, &c.

2. We will apply this last principle to the two polynomials

$$3bcq+30mp+18bc+5mpq,$$

and

$$4adq-42fg+24ad-7fgq.$$

Since q is the only letter common to the two polynomials, which, moreover, do not contain any common monomial factors, we can arrange them with reference to this letter, and follow the ordinary rule. But as b is found in the first polynomial and not in the second, if we arrange the first with reference to b, which gives

$$(3cq+18c)b+30mp+5mpq,$$

the required greatest common divisor will be the same as that which exists between the second polynomial and the two co-efficients

$$3cq+18c \quad \text{and} \quad 30mp+5mpq.$$

Now the first of these co-efficients can be put under the form $3c(q+6)$, and the other becomes $5mp(q+6)$; hence $q+6$ is a common factor of these co-efficients. It will therefore be sufficient to ascertain whether $q+6$, which is a *prime* divisor, is a factor of the second polynomial.

Arranging this polynomial with reference to q, it becomes

$$(4ad-7fg)q-42fg+24ad\,;$$

as the second part $24ad-42fg$ is equal to $6(4ad-7fg)$, it follows that this polynomial is divisible by $q+6$, and gives the quotient $4ad-7fg$. Therefore $q+6$ is the greatest common divisor of the proposed polynomials.

271. Remark. It may be ascertained that $q+6$ is an exact divisor of the polynomial $(4ad-7fg)q+24ad-42fg$, by a method derived from the property proved in Art. 261.

Make $q+6=0$ or $q=-6$ in this polynomial; it becomes

$$(4ad-7fg)\times-6+24ad-42fg,$$

which reduces to 0; hence $q+6$ is a divisor of this polynomial.

This method may be advantageously employed in nearly all the applications of the process. It consists in this, viz. after obtaining a remainder of the first degree with reference to a, when a is the principal letter, *make this remainder equal to* 0, *and deduce the value of* a *from this equation.*

If this value, substituted in the remainder of the 2d degree, *destroys it,* then the remainder of the 1st degree, simplified Art. 68, is a common divisor. If the remainder of the 2d degree does not reduce to 0 by this substitution, we may conclude that there is no common divisor depending upon the principal letter.

Farther, having obtained a remainder of the 2d degree with reference to a, it is not necessary to continue the operation any farther. For,

Decompose this polynomial into two factors of the 1*st degree,* which is done by placing it equal to 0, and resolving the resulting equation of the second degree.

When each of the values of a thus obtained, substituted in the remainder of the 3d degree, *destroys it,* it is a proof that the remainder of the 2d degree, *simplified*, is a common divisor; when only one of the values destroys the remainder of the 3d degree, the common divisor is the factor of the 1st degree with respect to a, which corresponds to this value.

Finally, when neither of these values destroys the remainder of the 3d degree, we may conclude that there is not a common divisor depending upon the letter a.

It is here supposed that the two factors of the 1st degree with reference to a, are rational, otherwise it would be more simple to perform the division of the remainder of the 3d degree by that of the second, and when this last division cannot be performed exactly, we may be certain that there is no rational common divisor, for if there was one, it could only be of the first degree with respect to a, and should be found in the remainder of the second degree, which is contrary to hypothesis.

3. Find the greatest common divisor of the two polynomials

$$6x^5-4x^4-11x^3-3x^2-3x-1$$

and

$$4x^4+2x^3-18x^2+3x-5$$

Ans. $2x^3-4x^2+x-1$.

4. Find the greatest common divisor of the polynomials

$$20x^6-12x^5+16x^4-15x^3+14x^2-15x+4.$$

and

$$15x^4-\ 9x^3+47x^2-21x+28.$$

Ans. $5x^2-3x+4$.

5. Find the greatest common divisor of the two polynomials

$$5a^4b^2+2a^3b^3+ca^2-3a^2b^4+bca$$

and

$$a^5+5a^3d-a^3b^2+5a^2bd.$$

Ans. a^2+ab.

Transformation of Equations.

The transformation of an equation consists in changing its form without affecting the equality of its members. The object of a transformation, is to change an equation from one form to another that is more easily resolved.

First Transformation.

To make the second term disappear from an equation.

272. The difficulty of resolving an equation generally diminishes with the number of terms involving the unknown quantity; thus, the equation $x^2=p$, gives immediately $x=\pm\sqrt{q}$, whilst the complete equation $x^2+px+q=0$, requires preparation before it can be resolved.

Now, any equation being given, it can always be *transformed* into another, in which the second term is wanting.

For, let there be the general equation

$$x^m+Px^{m-1}+Qx^{m-2}+\ .\ .\ .\ +Tx+U=0.$$

Suppose $x=u+x'$, u being unknown, and x' an *indeterminate quantity;* by substituting $u+x'$ for x, we obtain

$$(u+x')^m+P(u+x')^{m-1}+Q(u+x')x^{m-2}+\ .\ .\ .\ .\ +T(u+x')+U=0\ ;$$

developing by the binomial formula, and arranging according to the decreasing powers of u, we have

$$\left.\begin{array}{l|l|ll}
u^m+mx' & u^{m-1}+m.\dfrac{m-1}{2}x'^2 & u^{m-2}+\ .\ .\ .\ & +x'^m \\
+P & \quad +(m-1)Px' & & +Px'^{m-1} \\
 & \quad +Q & & +Qx'^{m-2} \\
 & & & +\ .\ .\ . \\
 & & & .\ .\ .\ .\ . \\
 & & & +Tx' \\
 & & & +U
\end{array}\right\}=0.$$

Since x' is entirely arbitrary, we may dispose of it in such a way that we shall have $mx'+P=0$; whence $x'=-\dfrac{P}{m}$. Substituting this value of x' in the last equation, we shall obtain an equation of the form,

$$u^m+Q'u^{m-2}+R'u^{m-3}+\ .\ .\ .\ +T'u+U'=0.$$

in which the second term is wanting.

If this equation was resolved, we could obtain the values of x

corresponding to those of u, by substituting each of the values of u in the equation $x=u+x'$, or $x=u-\frac{\mathrm{P}}{m}$.

Whence we may deduce the following general rule:

In order to make the second term of an equation disappear, *substitute for the unknown quantity a new unknown quantity, united with the co-efficient of the second term, taken with a contrary sign, and divided by the exponent of the degree of the equation.*

Let us apply the preceding rule to the equation $x^2+px=q$. If we take $x=u-\frac{p}{2}$, it becomes $\left(u-\frac{p}{2}\right)^2+p\left(u-\frac{p}{2}\right)=q$, or, by performing the operations, and reducing, $u^2-\frac{p^2}{4}=q$, this equation gives $u=\pm\sqrt{\frac{p^2}{4}+q}$, consequently we obtain for the two corresponding values of x,

$$x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}.$$

273. Instead of making the second term disappear, an equation may be required, which shall be deprived of its third, fourth, &c. term; this can be obtained by placing the co-efficient of u^{m-2}, u^{m-3} . . . equal to 0. For example, to make the third term disappear, we make in the above transformed equation

$$m\frac{m-1}{2}x'^2+(m-1)\mathrm{P}x'+\mathrm{Q}=0;$$

from which we obtain two values for x', which substituted in the transformed equation reduces it to the form

$$u^m+\mathrm{P}'u^{m-1}+\mathrm{R}'u^{m-3}+\ .\ .\ .\ \mathrm{T}'u+\mathrm{U}'=0.$$

Beyond the third term it will be necessary to resolve equations of a degree superior to the second, to obtain the value of x': thus to cause the last term to disappear, it will be necessary to resolve the equation

$$x'^m+Px'^{m-1}+\ldots Tx'+U=0,$$

which is nothing more than what the proposed equation becomes when x' is substituted for x.

It may happen that the value $x'=-\frac{P}{m}$ which makes the second term disappear, causes also the disappearance of the third or some other term. For example, in order that the second and third terms may disappear at the same time, it is necessary that the equation $x'=-\frac{P}{m}$ should agree with

$$m\frac{m-1}{2}x'^2+(m-1)Px'+Q=0.$$

Now if in this last equation, we replace x' by $-\frac{P}{m}$ it becomes

$$m\frac{m-1}{2}\cdot\frac{P^2}{m^2}-(m-1)\frac{P^2}{m}+Q=0,\quad\text{or}\quad(m-1)P^2-2mQ=0;$$

therefore, whenever this relation exists between the co-efficients P and Q, the disappearance of the second term involves that of the third.

Remarks upon the preceding Transformation. Formation of derived Polynomials.

274. The relation $x=u+x'$, of which we have made use in the two preceding articles, indicates that the roots of the transformed equations are equal to those of the proposed, diminished or increased by a certain quantity. Sometimes this quantity is introduced in the calculus, as an indeterminate quantity, the value of which is afterwards fixed in such a manner as to satisfy a given condition; sometimes it is a particular number of a given value, which expresses a *constant difference* between the roots of a primitive equation and those of another equation which we wish to form.

In short, the transformation which consists in substituting $u+x'$ for x, in an equation, is of very frequent use in the theory of equations. Now there is a very simple method of obtaining, in practice, the transformation which results from this substitution.

To show this we shall invert the order of the terms in $u+x'$, that is, for x substitute $x'+u$ in the equation

$$x^m+Px^{m-1}+Qx^{m-2}+Rx^{m-3}+\ldots Tx+U=0;$$

it becomes, by developing and arranging according to the ascending powers of u,

$$\begin{array}{l|l|l}
x'^m \quad +mx'^{m-1} & u+m\dfrac{m-1}{2}x'^{m-2} & u^2+\ldots u^m=0 \\
+Px'^{m-1}+(m-1)Px'^{m-2} & +(m-1)\dfrac{m-2}{2}Px'^{m-3} & \\
+Qx'^{m-2}+(m-2)Qx'^{m-3} & +(m-2)\dfrac{m-3}{2}Qx'^{m-4} & \\
+\ldots+\ldots & +\ldots & \\
+Tx' \quad +T & & \\
+U & &
\end{array}$$

If we observe how the co-efficients of the different powers of u are composed, we shall see that the co-efficient of u^0 is nothing more than what the first member of the proposed equation becomes when x' is substituted in place of x; we shall hereafter denote it by X'.

The co-efficient of u^1 is formed by means of the preceding, or X', by multiplying each of the terms of X' by the exponent of x' in this term, and then diminishing this exponent by unity; we shall call this co-efficient Y'.

The co-efficient of u^2 is formed from Y' by multiplying each of the terms of Y' by the exponent of x' in this term, dividing the product by 2, and then diminishing the exponent by unity. By calling this co-efficient $\dfrac{Z'}{2}$ it is evident that Z' is formed from Y' in the same manner that Y' is formed from X'.

In general, the co-efficient of any term in the above transformed equation, is formed from the preceding one, by multiplying each of its terms by the exponent of x' in this term, dividing the product by the number of co-efficients preceding the one required, and then diminishing the exponents of x' by unity.

This law, by which the co-efficients X', Y', $\frac{Z'}{2}$, $\frac{V'}{2.3}$ are derived from each other, is evidently entirely similar to that which regulates the different terms of the formula for the binomial (Art. 165).

The expressions Y', Z', V', W' . . . are called derived polynomials of X', because Z' is deduced or derived from Y', as Y' is derived from X': V' is derived from Z', as Z' is derived from Y', and so on. Y' is called *the first derived polynomial*, Z' *the second*, *&c.* Recollect that X' is what the first member of the proposed equation becomes, when x' is substituted for x.

The co-efficient of the first term of the proposed equation has been supposed equal to unity; when this is not the case, the law of formation for the co-efficients of the transformed equations is entirely the same, and the co-efficient of u^m is equal to that of x^m.

275. To show the use of this law in practice, let it be required to make the co-efficient of the second term of the following equation disappear.

$$x^4-12x^3+17x^2-9x+7=0.$$

According to the rule of Art. 272, take $x=u+\frac{12}{4}$, or $x=3+u$, which will give a transformed equation of the 4th degree, and of the form

$$X'+Y'u+\frac{Z'}{2}u^2+\frac{V'}{2\times 3}u^3+u^4=0,$$

and the operation is reduced to finding the values of

$$X',\quad Y',\quad \frac{Z'}{2},\quad \frac{V'}{2.3}.$$

Now it follows from the preceding law, that.

$$X' = (3)^4-12.(3)^3+17.(3)^2-9.(3)^1+7, \text{ or } \quad X'=-110;$$

$$Y' =4.(3)^3-36.(3)^2+34.(3)^1-9, \text{ or } \quad . \quad . \quad Y'=-123;$$

$$\frac{Z'}{2}=6.(3)^2-36.(3)^1+17, \text{ or } \quad . \quad . \quad . \quad . \quad . \quad \frac{Z'}{2}=-\ 37;$$

$$\frac{V'}{2.3}=4.(3)^1-12, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \frac{V}{2.3}=0.$$

Therefore the transformed equation becomes

$$u^4-37u^2-123u-110=0.$$

Again, transform the equation

$$4x^3-5x^2+7x-9=0$$

into another, the roots of which exceed the roots of the proposed equation by unity

Take $u=x+1$; there will result $x=-1+u$, which gives the transformed equation

$$X'+Y'u+\frac{Z'}{2}u^2+4u^3=0.$$

$$X' = 4.(-1)^3-5.(-1)^2+7.(-1)^1-9, \text{ or } \quad X'=-25;$$

$$Y' =12.(-1)^2-10.(-1)^1+7. \quad . \quad . \quad . \quad . \quad Y'=\ 29;$$

$$\frac{Z'}{2}=12.(-1)^1-5. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \frac{Z'}{2}=-17;$$

$$\frac{V'}{2.3}=4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \frac{V'}{2.3}=\ 4.$$

Therefore the transformed equation becomes

$$4u^3-17u^2+29u-25=0.$$

The following examples may serve the student for exercises:

Make the second term vanish from the following equations.

1st. $\quad x^5-10x^4+7x^3+4x-9=0.$

Ans. $\quad u^5-33u^3-118u^2-152u-73=0.$

2d. $\quad 3x^3+15x^2+25x-3=0.$

Ans. $\quad 3u^3-\frac{152}{9}=0.$

Transform the equation $3x^4-13x^3+7x^2-8x-9=0$ into another, the roots of which shall be less than the roots of the proposed by the fraction $\frac{1}{3}$.

Ans. $3u^4-9u^3-4u^2-\frac{65}{9}u-\frac{102}{9}=0.$

We shall frequently have occasion for the law of formation of derived polynomials.

276. These polynomials have the following remarkable properties.

Let X or $x^m+Px^{m-1}+Qx^{m-2}\ldots=0$, be the proposed equation, and a, b, c, l, its m roots, we shall then have (Art. 244),

$$x^m+Px^{m-1}+\ldots=(x-a)(x-b)(x-c)\ldots(x-l).$$

Substituting $x'+u$ (or to avoid the accents), $x+u$ in the place of x; it becomes,

$$(x+u)^m+P(x+u)^{m-1}+\ldots=(x+u-a)(x+u-b)\ldots;$$

or changing the order of the terms in the second member, and regarding $x-a$, $x-b$, ... each as a single quantity,

$$(x+u)^m+P(x+u)^{m-1}\ldots=(u+\overline{x-a})(u+\overline{x-b})\ldots(u+\overline{x-b}).$$

Now, by performing the operations indicated in the two members, we shall, by the preceding Article, obtain for the first member,

$$X+Yu+\frac{Z}{2}u^2+\ldots u^m;$$

X being the first member of the proposed equation, and Y, Z ... the derived polynomials of this member.

With respect to the second member, it follows from Art. 247,

1st. That the part involving u^0, or the last term, is equal to the product $(x-a)(x-b)\ldots(x-l)$ of the factors of the proposed equation;

2d. The co-efficient of u is equal to the sum of the products of these m factors taken $m-1$ and $m-1$.

3d. The co-efficient of u^2 is equal to the sum of the products of these m factors taken $m-2$ and $m-2$; and so on.

Moreover, the two members of the last equation are identical; therefore, the co-efficients of the same powers are equal. Hence

$$X=(x-a)(x-b)(x-c)\ldots(x-l),$$

which was already known. Hence also, Y, or the first derived polynomial, *is equal to the sum of the products of the* m *factors of the first degree in the proposed equation, taken* m-1 *and* m-1; *or equal to the sum of all the quotients that can be obtained by dividing* X *by each of the* m *factors of the first degree in the proposed equation; that is,*

$$Y=\frac{X}{x-a}+\frac{X}{x-b}+\frac{X}{x-c}+\ldots\frac{X}{x-l}.$$

$\frac{Z}{2}$ or the second derived polynomial, divided by 2, is equal to the sum of the products of the m factors of the proposed equation taken $m-2$ and $m-2$, or equal to the sum of the quotients that can be obtained by dividing X by each of the factors of the second degree; that is,

$$\frac{Z}{2}=\frac{X}{(x-a)(x-b)}+\frac{X}{(x-a)(x-c)}\ldots\frac{X}{(x-k)(x-l)},$$

and so on.

Second Transformation.

To make the denominators disappear from an equation.

277. Having given an equation, we can always transform it into another of which the roots will be equal to a given *multiple* or *submultiple* of those of the proposed equation.

Take the equation

$$x^m+Px^{m-1}+Qx^{m-2}+\ldots Tx+U=0,$$

and denote by y the unknown quantity of a new equation, of which the roots are K times greater than those of the proposed equation. If we take $y=Kx$, there will result $x=\frac{y}{K}$; whence, substituting and multiplying every term by K^m, we have

$$y^m+PKy^{m-1}+QK^2y^{m-2}+RK^3y^{m-3}+\ldots+TK^{m-1}y+UK^m=0.$$

an equation of which the co-efficients are equal to those of the proposed equation multiplied respectively by K^0, K^1, K^2, K^3, K^4, &c.

This transformation is principally used *to make the denominators disappear from an equation, when the co-efficient of the first term is unity.*

To fix the ideas, take the equation of the 4th degree

$$x^4+\frac{a}{b}x^3+\frac{c}{d}x^2+\frac{e}{f}x+\frac{g}{h}=0,$$

if in this equation we make $x=\frac{y}{K}$, y being a new unknown and K an indeterminate quantity, it becomes

$$y^4+\frac{aK}{b}y^3+\frac{cK^2}{d}y^2+\frac{eK^3}{f}y+\frac{gK^4}{h}=0.$$

Now, there may be two cases,

1st. Where the denominators b, d, f, h, are prime with each other; in this hypothesis, as K is altogether arbitrary, take $K=bdfh$, *the product of the denominators*, the equation will then become

$$y^4+adfh\,.\,y^3+cb^2df^2h^2\,.\,y^2+eb^3d^3f^2h^3\,.\,y+gb^4d^4f^4h^3=0,$$

an equation the co-efficients of which are entire, and that of its first term unity.

We have besides, the equation $x=\frac{y}{bdfh}$, to determine the values of x corresponding to those of y.

2d. When the denominators contain common factors, we shall evidently render the co-efficients entire by taking for K the smallest multiple of all the denominators. But we can simplify this still more, by observing, that it is reduced to determining K in such a manner that K^1, K^2, K^3 . . . shall contain the prime factors which compose b, d, f, h, raised to powers at least equal to those which are found in the denominators.

Thus, let the equation

$$x^4-\frac{5}{6}x^3+\frac{5}{12}x^2-\frac{7}{150}x-\frac{13}{9000}=0.$$

Take $x=\frac{y}{k}$, it becomes

$$y^4-\frac{5k}{6}y^3+\frac{5k^2}{12}y^2-\frac{7k^3}{150}y-\frac{13k^4}{9000}=0.$$

First make $k=9000$, which is a multiple of all the other denominators, it is clear that the co-efficients become whole numbers.

But if we decompose 6, 12, 150 and 9000 into their factors, we find

$$6=2\times3,\quad 12=2^2\times3,\quad 150=2\times3\times5^2,\quad 9000=2^3\times3^2\times5^3;$$

and by simply making $k=2\times3\times5$, the product of the different simple factors, we obtain

$$k^2=2^2\times3^2\times5^2,\quad k^3=2^3\times3^3\times5^3,\quad k^4=2^4\times3^4\times5^4,$$

whence we see that the values of k, k^2, k^3, k^4, contain the prime factors of 2, 3, 5, raised to powers at least equal to those which enter in 6, 12, 150 and 9000.

Hence the hypothesis $k=2\times3\times5$ is sufficient to make the denominators disappear. Substituting this value, the equation becomes

$$y^4-\frac{5.2.3.5}{2.3.}y^3+\frac{5.2^2.3^2.5^2}{2^2.3}y^2-\frac{7.2^3.3^3.5^3}{2.3.5^2}y-\frac{13.2^4.3^4.5^4}{2^3.3^2.5^3}=0,$$

which reduces to

$$y^4-5.5y^3+5.3.5^2y^2-7.2^2.3^2.5y-13.2.3^2.5=0;$$

or $$y^4-25y^3+375y^2-1260y-1170=0.$$

Hence, we perceive the necessity of taking k as small a number as possible: otherwise, we should obtain a transformed equation, having its co-efficients very great, as may be seen by reducing the transformed equation resulting from the supposition $k=9000$ in the preceding equation.

EXAMPLES.

$$1\text{st.}\quad x^3-\frac{7}{3}x^2+\frac{11}{36}x-\frac{25}{72}=0\ ;\ x=\frac{y}{6},$$

whence

$$y^3-14y^2+11y-75=0\ ;$$

$$2\text{d.}\ x^5-\frac{13}{12}x^4+\frac{21}{40}x^3-\frac{32}{225}x^2-\frac{43}{600}x-\frac{1}{800}=0\ ;\ x=\frac{y}{2^2.3.5},$$

$$\text{or } x=\frac{y}{60}$$

whence

$$y^5-65y^4+1890y^3-30720y^2-928800y+972000=0.$$

278. The preceding transformations are those most frequently used; there are others very useful, of which we shall speak as they present themselves; they are too simple to be treated of separately.

In general, the problem of the transformation of equations should be considered as an application of the problem of *elimination* between two equations of any degree whatever, involving two unknown quantities. In fact, an equation being given, suppose that we wish to transform it into another, of which the roots have, with those of the proposed equation, a determined relation.

Denote the proposed equation by $F(x)=0$, (enunciated function of x equal to zero), and the algebraic expression of the relation which should exist between x and the new unknown quantity y, by $F'(x, y)=0$; the question is reduced to finding, by means of these two equations, a new equation involving y, which will be the required equation. When the unknown quantity x is only of the first degree in $F'(x, y)=0$, the transformed equation is easily obtained, but if it is raised to the second, third . . . power, we must have recourse to the methods of elimination.

Elimination.

279. To eliminate between two equations of any degree whatever, involving two unknown quantities, is to obtain, by a series of operations, performed on these equations, *a single equation which contains but one of the unknown quantities*, and which gives all the values of this unknown quantity that will, taken in connection with the corresponding values of the other unknown quantity, satisfy at the same time both the given equations.

This new equation, *which is a function of one of the unknown quantities*, is called *the final equation*, and the values of the unknown quantity found from this equation, are called *compatible values.*

Of all the known methods of elimination, *the method of the common divisor*, is, in general, the most expeditious; it is the method which we are going to develop.

Let $F(x, y)=0$ and $F'(x, y)=0$ be any two equations whatever, or, more simply,

$$A=0, \quad B=0.$$

Suppose the final equation involving y obtained, and let us try to discover some property of the roots of this equation, which may serve to determine it.

Let $y=a$ be one of the compatible values of y; it is clear, that since this value satisfies the two equations, at the same time as a certain value of x, it is such, that by substituting it in both of the equations, which will then contain only x, *the equations will admit of at least one common value of x;* and to this common value there will necessarily be a corresponding common divisor involving x. Art. 262. This common divisor will be of the first, or a higher degree with respect to x, according as the particular value of $y=a$ corresponds to one or more values of x.

Reciprocally, every value of y, *which,* substituted in the two equations, gives a common divisor involving x, *is necessarily a compatible value,* because it then evidently satisfies the two equations at the same time with the value or values of x found from this common divisor when put equal to 0.

280. We will remark, that, before *the substitution, the first members of the equations cannot*, in general, *have a common divisor*, which is a function of one or both of the unknown quantities.

In fact, let us suppose for a moment that the equations $A=0$, $B=0$, are of the form

$$A' \times D=0, \quad B' \times D=0.$$

D being a function of x and y.

Making separately $D=0$, we obtain a single equation involving two unknown quantities, which can be satisfied with an *infinite number of systems of values*. Moreover, every system which renders D equal to 0, would at the same time cause $A'D$, $B'D$ to vanish, and would consequently satisfy the equations $A=0$, $B=0$.

Thus, the hypothesis of a common divisor of the two polynomials A and B, containing x and y, would bring with it as a consequence that the proposed equations were indeterminate. Therefore, if there exists a common divisor, involving x and y, of the two polynomials A and B, the proposed equations will be *indeterminate*, that is, they may be satisfied by an infinite number of systems of values of x and y. Then there are no data to determine a *final equation* in y, since the number of values of y is *infinite*.

If the two polynomials A and B were of the form $A' \times D$, $B' \times D$, D being a function of x only, we might conceive the equation $D=0$ resolved with reference to x, which would give one or more values for this unknown. Each of these values substituted in $A' \times D=0$ and $B' \times D=0$, at the same time with any *arbitrary* value of y, would verify these two equations, since D must be nothing, in consequence of the substitution of the value of x. Therefore, in this case, the proposed equations would admit of a *finite number of values* for x, but of an infinite number of values for y; then there could not exist a final equation in y.

Hence, when the equations $A=0$, $B=0$, are determinate, that is, when they only admit of a *limited number* of systems of values for x and y, their first members cannot have a *function of these unknown quantities for a common divisor*, unless a particular substitution has been made for one of them.

281. From this it is easy to deduce a process for obtaining the *final equation* involving y.

Since the characteristic property of every compatible value of y is, that being substituted in the first members of the two equations, it gives them a common divisor involving x, which they had not before, (unless the equations are indeterminate, which is contrary to the supposition), it follows, that if to the two proposed polynomials, arranged with reference to x, we apply the process for the greatest common divisor, we generally shall not find one; but, by continuing the operation properly, we shall arrive at a remainder independent of x, and which is a function of y, which, placed equal to 0, will give the required *final equation;* for every value of y found from this equation, reduces to nothing the last remainder of the operation for finding the common divisor; it is, then, such, that substituted in the preceding remainder, it will render this remainder a common divisor of the first members A and B. Therefore, each of the roots of the equation thus formed is a compatible value of y.

282. Admitting that the final equation may be completely resolved, which would give all the compatible values, it would afterwards be necessary to obtain the corresponding values of x. Now it is evident that it would be sufficient for this, to substitute the different values of y in the remainder preceding the last, put the polynomial involving x which results from it equal to 0, and find from it the values of x; for these polynomials are nothing more than the divisors involving x, which become common to A and B.

But as the final equation is generally of a degree superior to the second, we cannot here explain the methods of finding the values of y. Indeed, our design was principally to show that, *two equations of any degree being given, we can, without supposing the resolution* *of any equation, arrive at another equation, containing only one of the unknown quantities which enter into the proposed equations.*

Of Equal Roots.

283. An equation is said to contain equal roots, when its first member contains equal factors. When this is the case, the derived polynomial, which is the sum of the products of the m factors taken $m-1$ and $m-1$ (Art. 276), contains a factor in its different parts, which is two or more times a factor of the proposed equation.

Hence, *there must be a common divisor between the first member of the proposed equation and its first derived polynomial.*

It remains to ascertain the manner in which this common divisor is composed of the equal factors.

284. *Having given an equation, it is required to discover whether it has equal roots, and to determine these roots if possible.*

Let X denote the first member of the equation

$$x^m+Px^{m-1}+Qx^{m-2}+\ .\ .\ .\ +Tx+U=0,$$

and suppose that it contains n factors equal to $x-a$, n' factors equal to $x-b$, n'' factors equal to $x-c$ . . ., and contains also the simple factors $x-p$, $x-q$, $x-r$ . . .; so that we may have

$$X=(x-a)^n(x-b)^{n'}(x-c)^{n''}\ .\ .\ .\ (x-p)\ (x-q)\ (x-r)\ .\ .\ .$$

With respect to Y, or the derived polynomial of X, we have seen (Art. 276), that it is *the sum of the quotients obtained by dividing* X *by each of the* m *factors of the first degree in the proposed equation.* Now since X contains n factors equal to $x-a$, we shall have n partial quotients equal to $\dfrac{X}{x-a}$; the same reasoning applies to each of the general factors, $x-b$, $x-c$. . . . Moreover we can form but one quotient equal to

$$\frac{X}{x-p},\ \frac{X}{x-q},\ \frac{X}{x-r}\ .\ .\ .$$

Therefore, Y is necessarily of the form

$$Y=\frac{nX}{x-a}+\frac{n'X}{x-b}+\frac{n''X}{x-c}+\ldots+\frac{X}{x-p}+\frac{X}{x-q}+\frac{X}{x-r}+\ldots$$

From this composition of the polynomial Y, it is plain that

$$(x-a)^{n-1},\quad (x-b)^{n'-1},\quad (x-c)^{n''-1}\ldots$$

are factors common to all its terms ; hence the product

$$(x-a)^{n-1}\times(x-b)^{n'-1}\times(x-c)^{n''-1}\ldots$$

is a common divisor of Y ; moreover, it is evident that this product will also divide X, it is therefore a common divisor of X and Y ; and it is their greatest common divisor. For, the prime factors of X are $x-a$, $x-b$, $x-c$ . . . and $x-p$, $x-q$, $x-r$ . . . ; now $x-p$, $x-q$, $x-r$, cannot divide Y, since some one of them will be wanting in each of the parts of Y, while it will be a factor of all the other parts.

Hence, the greatest common divisor of X and Y is

$$D=(x-a)^{n-1}(x-b)^{n'-1}(x-c)^{n''-1}\ldots;$$

that is, *the greatest common divisor is composed of the product of those factors which enter two or more times in the proposed equation, each raised to a power less by unity than in the given equation.*

285. From the above we deduce the following method :

To discover whether an equation $X=0$ contains any equal roots, *form* Y or *the derived polynomial of* X ; *then seek for the greatest common divisor between* X and Y ; if one cannot be obtained, the equation has no equal roots, or equal factors.

If we find a common divisor D, and it is of the first degree, or of the form $x-h$, make $x-h=0$, whence $x=h$; *we may then conclude, that the equation has two roots equal to* h, *and has but one species of equal roots*, from which it may be freed by dividing X by $(x-h)^2$.

If D is of the second degree with reference to x, *resolve the equation* $D=0$; there may be two cases ; the two roots will be equal, or they will be unequal. 1st. When we find $D=(x-h)^2$, *the equation has three roots equal to* h, *and has but one species of equal roots*, from which it can be freed by dividing X by $(x-h)^3$; 2d, when D is of the form $(x-h)$ $(x-h')$, the proposed equation *has two roots equal to* h, and *two equal to* h', from which it may be freed by dividing X by $(x-h)^2(x-h')^2$, or by D^2.

Suppose now that D is of any degree whatever; *it is necessary*, in order to know the species of equal roots, and the number of roots of each species, *to resolve completely the equation* $D=0$; *and every simple root of* D *will be twice a root of the proposed equation; every double root of* D *will be three times a root of the proposed equation;* and so on.

EXAMPLES.

1. Determine whether the equation

$$2x^4-12x^3+19x^2-6x+9=0$$

contains equal roots.

We have (Art. 274), for the derived polynomial

$$8x^3-36x^2+38x-6.$$

Now, seeking for the greatest common divisor of these polynomials, we find $D=x-3=0$, whence $x=3$; hence the proposed equation has *two* roots equal to 3.

Dividing its first member by $(x-3)^2$, we obtain

$$2x^2+1=0;\ \text{whence}\ x=\pm\frac{1}{2}\sqrt{-2}.$$

Thus the equation is completely resolved, and its roots are

$$3,\ 3,\ +\frac{1}{2}\sqrt{-2}\ \text{and}\ -\frac{1}{2}\sqrt{-2}.$$

2. For a second example take $x^5-2x^4+3x^3-7x^2+8x-3=0$;
the first derived polynomial is $\qquad 5x^4-8x^3+9x^2-14x+8$,
and the common divisor $\qquad x^2-2x+1$, or $(x-1)^2$,
hence the proposed equation has *three* roots equal to 1.

Dividing its first member by $(x-1)^3$ or by x^3-3x^2+3x-1, the quotient is

$$x^2+x+3=0\ ;\ \text{whence}\ x=\frac{-1\pm\sqrt{-11}}{2};$$

thus the equation is completely resolved.

3. For a third example, take the equation

$$x^7+5x^6+6x^5-6x^4-15x^3-3x^2+8x+4=0\ ;$$

the derived polynomial is

$$7x^6+30x^5+30x^4-24x^3-45x^2-6x+8\ ;$$

and the common divisor is

$$x^4+3x^3+x^2-3x-2.$$

The equation $x^4+3x^3+x^2-3x-2=0$ cannot be resolved directly, but by applying the method of equal roots to it, that is, by seeking for a common divisor between its first member and its derived polynomial, $4x^3+9x^2+2x-3$, we find a common divisor, $x+1$; which proves that the *square* of $x+1$ is a factor of $x^4+3x^3+x^2-3x-2$, and the *cube* of $x+1$, a factor of the first member of the proposed equation.

Dividing $x^4+3x^3+x^2-3x-2$ by $(x+1)^2$ or x^2+2x+1, we have x^2+x-2, which placed equal to zero, gives the two roots $x=1$, $x=-2$, or the two factors $x-1$ and $x+2$. Hence we have

$$x^4+3x^3+x^2-3x-2=(x+1)^2(x-1)\ (x+2).$$

Therefore the first member of the proposed equation is equal to

$$(x+1)^3(x-1)^2(x+2)^2\ ;$$

or the proposed equation has *three* roots equal to -1, *two* equal to $+1$, and *two* equal to -2.

Take the examples,

1st. $$x^7-7x^6+10x^5+22x^4-43x^3-35x^2+48x+36=0,$$

$$(x-2)^2(x-3)^2(x+1)^3=0.$$

2d. $$x^7-3x^6+9x^5-19x^4+27x^3-33x^2+27x-9=0,$$

$$(x-1)^3(x^2+3)^2=0.$$

286. When, in the application of the above method, we obtain

an equation $D=0$, of a degree superior to the second, since this equation may itself be subjected to the method, we are often able to decompose D into its factors, and in this way to find the different species of equal roots contained in the equation $X=0$, and the number of roots of each species. As to the simple roots of $X=0$, we begin by freeing this equation from the equal factors contained in it, and the resulting equation, $X'=0$, will make known the simple roots.

CHAPTER VII.

Resolution of Numerical Equations, involving one or more Unknown Quantities.

287. The principles established in the preceding chapter, are applicable to all equations, whether their co-efficients are numerical or algebraic, and these principles should be regarded as the elements which have been employed in the resolution of equations of the higher degrees.

It has been said already, that analysts have hitherto been able to resolve only the general equations of the third and fourth degree. The formulas they have obtained for the values of the unknown quantities are so complicated and inconvenient, when they can be applied, (which is not always possible), that the problem of the resolution of algebraic equations, of any degree whatever, may be regarded as more curious than useful. Therefore, analysts have principally directed their researches to the resolution of *numerical equations*, that is, to those which arise from the algebraic translation of a problem in which the given quantities are particular numbers; and methods have been found, by means of which we can always determine the roots of a *numerical equation of any given degree.*

It is proposed to develop these methods in this chapter.

To render the reasoning general, we will represent the proposed equation by

$$x^m+Px^{m-1}+Qx^{m-2}+\ldots=0,$$

in which P, Q . . . denote particular numbers, real, positive, or negative.

First Principle.

288. *When two numbers* p *and* q, *substituted in the place of* x *in a numerical equation, give two results, affected with contrary signs, the proposed equation contains a real root, comprehended between these two numbers*

Let the proposed equation be

$$x^m+Px^{m-1}+Qx^{m-2}+\ldots Tx+U=0.$$

The first member will, in general, contain both positive and negative terms; denote the sum of the positive terms by A, and the sum of the negative terms by B, the equation will then take the form

$$A-B=0.$$

Suppose $p<q$, and that p substituted for x gives a *negative* result, and q a *positive* result.

Since the first member becomes negative by the substitution of p, and positive by the substitution of q, it follows that we have in the first case $A<B$, and in the second $A>B$. Now it results from the nature of the quantities A and B, that they both increase as x increases, since they contain only positive numbers, and positive and entire powers of x; therefore, by making x augment by insensible degrees, from p to q, the quantities A and B will also increase by insensible degrees. Now since A, by hypothesis, from being less than B, afterwards becomes greater than it, A must necessarily have a more rapid increment than B, *which insensibly destroys the excess that* B *had over* A, *and finally produces an excess of* A *over* B. From this, we conceive that in the passage from $A<B$ to $A>B$, there must be an intermediate value for which A becomes equal to B, and the value which produces this result is a root of the equation, since it verifies $A-B=0$, or the proposed equation. Hence, the proposition is proved.

In the preceding demonstration, p and q have been supposed to be positive numbers; but the proposition is not less true, whatever may be the signs with which p and q are affected. For we will remark, in the first place, that the above reasoning applies equally to the case in which one of the numbers p and q, p for example, is 0; that is, it could be proved, in this case, that there was at least one real root between 0 and q.

Let both p and q be *negative*, and represent them by $-p'$ and $-q'$.

If, in the equation

$$x^m+Px^{m-1}+Qx^{m-2}+\ldots Tx+U=0,$$

we change x into $-y$, which gives the transformation

$$(-y)^m+P(-y)^{m-1}+Q(-y)^{m-2}+\ldots T(-y)+U=0,$$

it is evident that substituting $-p'$ and $-q'$ in the proposed equation, amounts to the same thing as substituting p' and q' in the transformation, for the results of these substitutions are in both cases

$$(-p')^m+P(-p')^{m-1}+Q(-p')^{m-2}+\ldots T(-p')+U,$$

and $$(-q')^m+P(-q')^{m-1}+Q(-q')^{m-2}+\ldots T(-q')+U;$$

Now, since p and q, or $-p'$ and $-q'$, substituted in the proposed equation, give results with contrary signs, it follows that the numbers p' and q', substituted in the transformation, also give results with contrary signs; therefore, by the first part of the proposition, there is at least one real root of the transformation contained between p' and q'; and in consequence of the relation $x=-y$, there is at least one value of x comprehended between $-p'$ and $-q'$, or p and q. This demonstration applies to cases in which $p=0$ or $q=0$.

Lastly, suppose p *positive* and q *negative* or equal to $-q'$: by making $x=0$ in the equation, the first member will reduce to its last term, which is necessarily affected with a sign contrary to that of p, or that of $-q'$; whence we may conclude that there is a root comprehended between 0 and p, or between 0 and $-q'$, and consequently between p and $-q$.

Second Principle.

289. When two numbers, substituted in place of x, in an equation, give results affected with contrary signs, we may conclude that there is at least one real root comprehended between them, but we are not certain that there are no more, and there may be any odd number of roots comprised between them. We therefore enunciate the second principle thus.

When an uneven number $(2n+1)$ *of the real roots of an equation, are comprehended between two numbers, the results obtained by substituting these numbers for* x, *are affected with contrary signs, and if they comprehend an even number* 2n, *the results obtained by their substitution are necessarily affected with the same sign.*

To make this proposition as clear as possible, denote those roots of the proposed equation, $X=0$, which are supposed to be comprehended between p and q, by a, b, c, . . ., and by Y, the product of the factors of the first degree, with reference to x, corresponding both to those real roots which are not comprised between them and to the imaginary roots; the signs of p and q being arbitrary.

The first member, X, can be put under the form

$$(x-a)(x-b)(x-c)\ldots\times Y.$$

Now substitute in X, or the preceding product, p and q in place of x; we shall obtain the two results

$$(p-a)(p-b)(p-c)\ldots\times Y',$$
$$(q-a)(q-b)(q-c)\ldots\times Y'',$$

Y' and Y'' representing what Y becomes, when we replace x by p and q; these two quantities are necessarily affected with the same sign, for if they were not, by the first principle $Y=0$ would give at least one real root comprised between p and q, which is contrary to the hypothesis.

To determine the signs of the above results more easily, divide the first by the second, we obtain

$$\frac{(p-a)\,(p-b)\,(p-c)\,.\,.\,.\times Y'}{(q-a)\,(q-b)\,(q-c)\,.\,.\,.\times Y''}$$

which can be written thus;

$$\frac{p-a}{q-a}\times\frac{p-b}{q-b}\times\frac{p-c}{q-c}\times\,.\,.\,.\,\frac{Y'}{Y''}$$

Now, since the roots $a, b, c, \ldots$ are comprised between p and q, we have

$$p \gtrless a,\ b,\ c,\ d\ \ldots,$$

but

$$q \lessgtr a,\ b,\ c,\ d\ \ldots;$$

whence we deduce

$$p-a,\ p-b,\ p-c,\ \ldots \gtrless 0,$$

and

$$q-a,\ q-b,\ q-c,\ \ldots \lessgtr 0.$$

hence, since $p-a$ and $q-a$ are affected with contrary signs, as well as $p-b$ and $q-b$, $p-c$ and $q-c\ \ldots$, the partial quotients

$$\frac{p-a}{q-a},\ \frac{p-b}{q-b},\ \frac{p-c}{q-c},\ \&c.$$

are all *negative;* moreover $\frac{Y'}{Y''}$ is essentially positive, since Y' and Y'' are affected with the same sign; therefore the product

$$\frac{p-a}{q-a}\times\frac{p-b}{q-b}\times\frac{p-c}{q-c}\times\,.\,.\,.\,\frac{Y'}{Y''},$$

will be *negative*, when the number of roots, $a, b, c\ \ldots$, comprehended between p and q, is uneven, and *positive* when the number is even.

Consequently, the two results $(p-a)(p-b)(p-c)\ldots\times Y'$, and $(q-a)(q-b)(q-c)\ldots\times Y''$, will have contrary or the same signs, according as the number of roots comprised between p and q is *uneven* or *even*.

Limits of the real Roots of Equations.

290. The different methods for resolving numerical equations, consist generally in substituting particular numbers in the proposed equation, in order to discover if these numbers verify it, or whether there are roots comprised between these numbers. But by reflecting a little upon the composition of the first member, the first term being positive, and affected with the highest power of x, which is greater with respect to that of the inferior degree in proportion to the value of x, we are sensible that there are certain numbers, above which it would be useless to substitute, because all of these numbers would give positive results.

291. Every number which exceeds the greatest of the positive roots of an equation, is called a superior limit of the positive roots.

From this definition, it follows that the limit is susceptible of an infinite number of values; for when a number is found to exceed the greatest positive root, every number greater than this, is, for a still stronger reason, a superior limit. But it may be proposed to determine the simplest possible limit. Now we are sure of having one of the limits, when we obtain *a number, which, substituted in place of* x *renders the first member positive, and which, at the same time, is such, that every greater number will also give a positive result.*

We will determine such a number.

292. Before resolving this question, we will propose a more simple one. viz.

To determine a number, which, substituted in place of x *in an equation, will render the first term* x^m *greater than the arithmetical sum of all the others.*

Suppose that all the terms of the equation are negative, except the first, so that

$$x^m - Px^{m-1} - Qx^{m-2} - \ldots - Tx - U = 0.$$

It is required to find a number for x which will render

$$x^m > Px^{m-1} + Q^{m-2} + \ldots + Tx + U.$$

Let k denote the greatest co-efficient, and substitute it in place of the co-efficients ; the inequality will become

$$x^m > kx^{m-1} + kx^{m-2} + \ldots + kx + k.$$

It is evident that every number substituted for x which will satisfy this condition, will for a stronger reason, satisfy the preceding. Now, dividing this inequality by x^m, it becomes

$$1 > \frac{k}{x} + \frac{k}{x^2} + \frac{k}{x^3} + \ldots + \frac{k}{x^{m-1}} + \frac{k}{x^m}.$$

Making $x=k$, the second member becomes $\frac{k}{k}$, or 1 plus a series of positive fractions; then the number k will not satisfy the inequality ; but by supposing $x=k+1$, we obtain for the second member the series of fractions

$$\frac{k}{k+1} + \frac{k}{(k+1)^2} + \frac{k}{(k+1)^3} + \ldots + \frac{k}{(k+1)^{m-1}} + \frac{k}{(k+1)^m},$$

which, considered in an inverse order, is an increasing geometrical progression, the first term of which is $\frac{k}{(k+1)^m}$, the ratio $k+1$, and the last term $\frac{k}{k+1}$; hence the expression for the sum of all the terms is, (Art. 223),

$$\frac{\frac{k}{k+1}.(k+1) - \frac{k}{(k+1)^m}}{k+1-1}, \text{ or } 1 - \frac{1}{(k+1)^m},$$

which is evidently less than unity.

Any number $>k+1$, put in place of x, will render the sum of the fractions $\frac{k}{x}+\frac{k}{x^2}+\ldots$ still less. Therefore,

The greatest co-efficient of the equation plus unity, or any greater number, being substituted for x, will render the first term x^m greater than the arithmetical sum of all the others.

Ordinary limit of the Positive Roots.

293. The number obtained above may be considered a prime limit, since this number, or any greater number, rendering the first term superior to the sum of all the others, the results of the substitution of these numbers for x must be constantly positive; but this limit is commonly much too great, because, in general, the equation contains several positive terms. We will, therefore, seek for a limit suitable for all equations.

Let x^{m-n} denote the power of x, corresponding to the first negative term which follows x^m, and we will consider the most unfavourable case, viz. that in which all of the succeeding terms are negative, and affected with the greatest of the negative co-efficients in the equation.

Let S be this co-efficient, and try to satisfy the condition

$$x^m > Sx^{m-n}+Sx^{m-n-1}+\ldots Sx+S;$$

or, dividing both members of this inequality by x^m,

$$1>\frac{S}{x^n}+\frac{S}{x^{n+1}}+\frac{S}{x^{n+2}}+\ldots+\frac{S}{x^{m-1}}+\frac{S}{x^m}.$$

Now by supposing $x^n=S$ or $x=\sqrt[n]{S}$, the second member becomes $\frac{S}{S}$, or 1, plus a series of positive fractions; but by making $x=\sqrt[n]{S}+1$, or (supposing, for simplicity, $\sqrt[n]{S}=S'$, whence $S=S'^n$), $x=S'+1$, the second member becomes

$$\frac{S'^n}{(S'+1)^n}+\frac{S'^n}{(S'+1)^{n+1}}+\ldots+\frac{S'^n}{(S'+1)^{m-1}}+\frac{S'^n}{(S'+1)^m},$$

which is a progression by quotients, $\frac{S'^n}{(S'+1)^m}$ being the first term, $S'+1$ the ratio, and $\frac{S'^n}{(S'+1)^n}$ the last term. Hence the expression for the sum of all these fractions is

$$\frac{\frac{S'^n}{(S'+1)^n}.(S'+1)-\frac{S'^n}{(S'+1)^m}}{S'+1-1}=\frac{S'^{n-1}}{(S'+1)^{n-1}}-\frac{S'^{n-1}}{(S'+1)^m}.$$

which is evidently less than 1.

Moreover, every number $>S'+1$ or $\sqrt[n]{S}+1$, will, when substituted for x, render the sum of the fractions $\frac{S}{x^n}+\frac{S}{x^{n+1}}+\ldots$ still smaller, since the numerators remaining the same, the denominator will increase. Hence $\sqrt[n]{S}+1$, and any greater number, will render the first term x^n greater than the arithmetical sum of all the negative terms of the equation, and will consequently give a positive result for the first member.

Therefore $\sqrt[n]{S}+1$, or *unity increased by that root of the greatest negative co-efficient whose index is the number of terms which precede the first negative term, is a superior limit of the positive roots of the equation.*

Make $n=1$, in which case the first negative term is the second term of the equation; the limit becomes $\sqrt[1]{S}+1$, or $S+1$; that is, *the greatest negative co-efficient plus unity.*

Let $n=2$, then the two first terms are positive, or the term x^{m-1} is wanting in the equation; the limit is then $\sqrt[2]{S}+1$.

When $n=3$ the limit is $\sqrt[3]{S}+1$ . . .

Find the superior limits for the positive roots in the following examples :

$x^4-5x^3+37x^2-3x+39=0$; $\quad\sqrt[n]{S}+1=\sqrt[1]{5}+1=6$;

$x^5+7x^4-12x^3-49x^2+52x-13=0$; $\quad\sqrt[n]{S}+1=\sqrt{49}+1=8$;

$\sqrt{25}+1=6$

$x^4+11x^2-25x-67=0;$ $\qquad \sqrt[n]{S}+1=\cancel{\sqrt[3]{67}}+1$ or $6;$

$3x^3-2x^2-11x+4=0;$ $\qquad \sqrt[n]{S}+1=\frac{11}{3}\;+1$ or $5.$

Newton's method for determining the smallest limit in entire numbers.

294. Let $X=0$, be the proposed equation; if in this equation we make $x=x'+u$, x' being indeterminate, we shall obtain (Art. 274),

$$X'+Y'u+\frac{Z'}{2}u^2+\ \ldots\ +u^m=0. \quad (1)$$

Conceive, that after successive trials we have determined a number for x, which, substituted in X', Y', $\frac{Z'}{2}$ . . ., renders all these co-efficients positive at the same time; this number will be greater than the greatest positive root of the equation $X=0$.

For, the co-efficients of the equation (1) being all positive, no positive number can verify it; therefore *all* of the real values of u must be *negative;* but from the equation $x=x'+u$, we have $u=x-x'$; and in order that the values of u corresponding to each of the values of x and x' (already determined) may be negative, it is absolutely necessary that the greatest positive value of x should be less than the value of x'.

EXAMPLE.

$$x^4-5x^3-6x^2-19x+7=0.$$

As x' is indeterminate, the letter x may be retained in the formation of the derived polynomials, and we have

$$X\ =x^4-5x^3-6x^2-19x+7,$$
$$Y\ =4x^3-15x^2-12x-19,$$
$$\frac{Z}{2}=6x^2-15x-6,$$
$$\frac{V}{2\,.\,3}=4x-5.$$

The question is, as stated above, reduced to finding the smallest number which, substituted in place of x, will render all of these polynomials positive.

It is plain that 2 and every number >2, will render the polynomial of the first degree positive.

But 2, substituted in the polynomial of the second degree, gives a negative result; and 3, or any number >3, gives a positive result.

Now 3 and 4, substituted in the polynomial of the third degree, give a negative result; but 5 and any greater number, give a positive result.

Lastly, 5 substituted in X, gives a negative result, and so does 6; for the three first terms $x^4-5x^3-6x^2$ are equivalent to the expression $x^3(x-5)-6x^2$, which is reduced to 0 when $x=6$; but $x=7$ evidently gives a positive result. Hence 7 *is a superior limit of the positive roots of the proposed equation;* and since it has been shown that 6 gives a negative result, it follows that there is at least one real root between 6 and 7.

Applying this method to the equation

$$x^5-3x^4-8x^3-25x^2+4x-39=0,$$

the superior limit will be found to be 6.

We should find 7, for the superior limit of the positive roots of the equation

$$x^5-5x^4-13x^3+17x^2-69=0.$$

This method is scarcely ever used, except in finding incommensurable roots.

295. It remains to determine the *superior limit* of the negative roots, and the *inferior limits* of the positive and negative roots.

Hereafter we shall designate the superior limit of the positive roots of an equation by the letter L.

1st. If in the equation $X=0$, we make $x=-y$, which gives the transformed equation $Y=0$, it is clear that the positive roots of this new equation, taken with the sign $-$, will give the negative roots of the proposed equation; therefore, determining, by the known methods, the superior limit L' of the positive roots of the equation $Y=0$, we shall have $-L'$ for the *superior limit* (numerically) *of the negative roots of the proposed equation.*

2d. If in the equation $X=0$, we make $x=\frac{1}{y}$, which gives the equation $Y=0$, it follows from the relation $x=\frac{1}{y}$ that the greatest positive values of y correspond to the smallest of x; hence, designating the superior limit of the positive roots of the equation $Y=0$ by L'', we shall have $\frac{1}{L''}$ for *the inferior limit of the positive roots of the proposed equation.*

3d. Finally, if we replace x, in the proposed equation, by $-\frac{1}{y}$, and find the superior limit L''' of the transformed equation $Y=0$, $-\frac{1}{L'''}$ will be *the inferior limit* (numerically) *of the negative roots of the proposed equation.*

296. *Every equation in which there are no variations in the signs*, that is, *in which all the terms are positive, must have all of its real roots negative;* for every positive number substituted for x will render the first member essentially positive.

Every complete equation, having its terms alternately positive and negative, must have its real roots all positive; for every negative number substituted for x in the proposed equation, would render all the terms positive, if the equation was of an even degree, and all of them negative if it was of an odd degree. Hence the sum would not be equal to zero in either case.

This is also true for *every incomplete equation, in which there results, by substituting* $-y$ *for* x, *an equation having all of its terms affected with the same sign.*

Consequences deduced from the preceding Principles.

First.

297. Every equation of an odd degree, the co-efficients of which are real, *has at least one real root affected with a sign contrary to that of its last term.*

For, let $x^m + Px^{m-1} + \ldots Tx \pm U = 0$, be the proposed equation; and first consider the case in which the last term is *negative.*

By making $x=0$ the first member becomes $-U$. But by giving a value to x equal to the greatest negative co-efficient plus unity, or $(K+1)$, the first term x^m will become greater than the arithmetical sum of all the others (Art. 292), the result of this substitution will therefore be *positive;* hence, *there is at least one real root comprehended between* 0 *and* $K+1$, which root is positive, and consequently affected with a sign *contrary* to that of the last term.

Suppose now that the last term is *positive.*

Making $x=0$, we obtain $+U$ for the result; but by putting $-(K+1)$ in place of x, we shall obtain a *negative* result, since the first term becomes negative by this substitution; hence the equation has at least one real root comprehended between 0 and $-(K+1)$, which is negative, or *affected with a sign contrary* to that of the last term.

Second.

298. *Every equation of an even degree,* involving only real co-efficients *of which the last term is negative, has at least two real roots, one positive and the other negative.* For, let $-U$ be the last term; making $x=0$, there results $-U$. Now substitute either $K+1$, or $-(K+1)$, K being the greatest negative co-efficient of the equation: as m is an even number, the first term x^m will remain positive; besides, by these substitutions, it becomes greater than the sum of all the others; therefore the results obtained by these substitutions are both *positive,* or affected with a sign contrary to that given by the hypothesis $x=0$; hence the equation *has at least two real roots.*

one comprehended between 0 and $K+1$, or *positive,* and the other between 0 and $-(K+1)$, or *negative.*

Third.

299. *If an equation, involving only real co-efficients, contains imaginary roots, the number of these roots must be even.*

For, conceive that the first member has been divided by all the simple factors corresponding to the real roots; the co-efficients of the *quotient* will be real (261); *and the equation must also be of an even degree;* for if it was uneven, by placing it equal to zero, we should obtain an equation that would contain at least one real root, which, from the nature of the equation, it cannot have.

REMARK. 300. There is a property of the above polynomial quotient which belongs exclusively to equations containing only imaginary roots; viz. *every such equation always remains positive for any real value substituted for* x.

For, if it could become negative, since we could also obtain a positive result, by substituting $K+1$ or the greatest negative co-efficient plus unity for x, it would follow that this polynomial placed equal to zero, would have at least one real root comprehended between $K+1$ and the number which would give a negative result.

It also follows, that the last term of this polynomial must be *positive,* otherwise $x=0$ would give a negative result.

Fourth.

301. *When the last term of an equation is positive, the number of its real positive roots is even; and when it is negative this number is uneven.*

For, first suppose that the last term is $+U$, or *positive.* Since by making $x=0$, there will result $+U$, and by making $x=K+1$, the result will also be positive, it follows that 0 and $K+1$ give two results affected with the same sign, and consequently (289), the number of real roots, (if any), comprehended between them, is even.

When the last term is $-U$, then 0 and $K+1$ give two results affected with contrary signs, and consequently comprehend either a *single real root*, or *an odd number of them.*

The *reciprocal* of this proposition is evident.

Descartes' Rule.

302. *An equation of any degree whatever cannot have a greater number of positive roots than there are* variations *in the signs of its terms, nor a greater number of negative roots than there are* permanences *of these signs.*

In the equation $x-a=0$, there is one variation, that is a change of sign in passing along the terms, and one positive root, $x=a$. And in the equation $x+b=0$, there is one permanence, and one negative root, $x=-b$.

If these equations be multiplied together, there will result an equation of the second degree,

$$\left.\begin{array}{r|l} x^2-a & x-ab \\ +b & \end{array}\right\}=0.$$

If a is less than b, the equation will be of the first form (Art. 144); and if $a>b$ the equation will be of the second form: that is

$$\begin{array}{lll} a<b & \text{gives} & x^2+px-q=0 \quad \text{and} \\ a>b & & x^2-px-q=0 \end{array}$$

In either case, there is one variation, and one permanence, and since in either form, one root is positive and one negative, it follows that there are as many positive roots as there are variations, and as many negative roots as there are permanences.

The proposition would evidently be demonstrated in a general manner, if it were shown that the multiplication of the first member by a factor $x-a$, corresponding to a *positive* root, would introduce *at least one variation*, and that the multiplication by a factor $x+a$, would introduce *at least one permanence.*

Let there be the equation

$$x^m \pm Ax^{m-1} \pm Bx^{m-2} \pm Cx^{m-3} \pm \ldots \pm Tx \pm U = 0,$$

in which the signs succeed each other in any manner whatever; by multiplying it by $x-a$, we have

$$\begin{array}{r|r|r|r|l}
x^{m+1} \pm A & x^m \pm B & x^{m-1} \pm C & x^{m-2} \pm \ldots \pm U & x \\
-a & \mp Aa & \mp Ba & \mp Ta & \mp Ua
\end{array}$$

The co-efficients which form the first horizontal line of this product, are those of the proposed equation, taken with the same sign; and the co-efficients of the second line are formed from those of the first, multiplied by a, taken with contrary signs, and advanced one rank towards the right.

Now, so long as each co-efficient of the upper line is greater than the corresponding one in the lower, it will determine the sign of the total co-efficient; hence, in this case there will be, from the first term to that preceding the last, inclusively, the same variations and the same permanences as in the proposed equation; but the last term $\mp Ua$ having a sign contrary to that which immediately precedes it, there must be one or more variations than in the proposed equation.

When a co-efficient in the lower line is affected with a sign contrary to the one corresponding to it in the upper, and is also greater than this last, there is a change from a permanence of sign to a variation; for the sign of the term in which this happens, being the same as that of the inferior co-efficient, must be contrary to that of the preceding term, which has been supposed to be the same as that of its superior co-efficient. Hence, each time we descend from the upper to the lower line, in order to determine the sign, there is a variation which is not found in the proposed equation; and if, after passing into the lower line, we continue in it throughout, we shall find for the remaining terms the same variations and the same permanences as in the proposed equation, since the co-efficients of this line are all affected with signs contrary to those of the primitive co-efficients. This supposition would therefore give us one variation for each positive root. But if we ascend from the lower to the upper line, there may be either a variation or a permanence. But even by supposing that this passage produces permanences in all cases, since the last term $\mp Ua$ forms a part of the lower line, it will be necessary to go once more from the upper line to the lower, than from the lower to the upper. Hence the new equation *must have at least one more variation than the proposed;* and it will be the same for each positive root introduced into it.

It may be demonstrated, in an analogous manner, that *the multiplication by a factor $x+a$,* corresponding to a negative root, *would introduce one permanence more.* Hence, in any equation the number of positive roots cannot be greater than the number of VARIATIONS of sign, nor the number of negative roots greater than the number of PERMANENCES.

303. *Consequence.* When the roots of an equation are all real, *the number of positive roots is equal to the number of variations, and the number of negative roots is equal to the number of permanences.*

For, let m denote the degree of the equation, n the number of variations of the signs, p the number of permanences; we shall have $m=n+p$. Moreover, let n' denote the number of positive roots, and p' the number of negative roots, we shall have $m=n'+p'$; whence

$$n+p=n'+p'$$

or, $$n-n'=p'-p.$$

Now, we have just seen that n' cannot be $>n$, and p' cannot be $>p$; therefore we must have $n'=n$, and $p'=p$.

REMARK. 304. When an equation wants some of its terms, we can often discover the presence of imaginary roots, by means of the above rule.

For example, take the equation

$$x^3+px+q=0,$$

p and q being essentially positive; introducing the term which is wanting, by affecting it with the co-efficient ± 0, it becomes

$$x^3 \pm 0.x^2 + px + q = 0.$$

By considering only the superior signs we should only obtain permanences, whereas the inferior sign would give two variations. This proves that the equation has some imaginary roots; for if they were all three real, it would be necessary by virtue of the superior sign, that they should be all negative, and, by virtue of the inferior sign, that two of them should be positive and one negative, which are *contradictory results.*

We can conclude nothing from an equation of the form

$$x^3 - px + q = 0;$$

for introducing the term $\pm 0.x^2$, it becomes

$$x^3 \pm 0 . x^2 - px + q = 0,$$

which contains one permanence and two variations, whether we take the superior or inferior sign. Therefore this equation may have its three roots real, viz. two positive and one negative; or, two of its roots may be imaginary and one negative, since its last term is negative (Art. 301).

Of the Commensurable Roots of Numerical Equations.

305. Every equation in which the co-efficients are whole numbers, that of the first term being unity, can only have whole numbers for its commensurable roots.

For, let there be the equation

$$x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U = 0;$$

in which P, Q . . . T, U, are whole numbers, and suppose that it could have a commensurable fraction $\frac{a}{b}$ for a root. Substituting this fraction for x, the equation becomes

$$\frac{a^m}{b^m} + P\frac{a^{m-1}}{b^{m-1}} + Q\frac{a^{m-2}}{b^{m-2}} + \ldots + T\frac{a}{b} + U = 0;$$

whence, multiplying the whole equation by b^{m-1}, and transposing,

$$\frac{a^m}{b}=-\mathrm{P}a^{m-1}-\mathrm{Q}a^{m-2}b- \ldots -\mathrm{T}ab^{m-2}-\mathrm{U}b^{m-1};$$

but the second member of this equation is composed of a series of entire numbers, whilst the first is essentially fractional, for a and b being prime with each other, a^m and b will also be prime with each other (Art. 118), hence this equality cannot exist; for, an irreducible fraction cannot be equal to a whole number.

Therefore it is impossible for any commensurable fraction to satisfy the equation. Now it has been shown (Art. 277), that an equation containing rational, but fractional co-efficients, can be transformed into another in which the co-efficients are whole numbers, that of the first term being unity. Hence *the research of the commensurable roots*, entire or fractional, *can always be reduced to that of the entire roots.*

306. This being the case, take the general equation

$$x^m+\mathrm{P}x^{m-1}+\mathrm{Q}x^{m-2}+ \ldots +\mathrm{R}x^3+\mathrm{S}x^2+\mathrm{T}x+\mathrm{U}=0,$$

and let a denote any entire number, positive or negative, which will verify it.

Since a is a root, we shall have the equation

$$a^m+\mathrm{P}a^{m-1}+ \ldots +\mathrm{R}a^3+\mathrm{S}a^2+\mathrm{T}a+\mathrm{U}=0 \ldots (1);$$

replacing a by all the entire positive and negative numbers between 1 and the limit $+\mathrm{L}$, and between -1 and $-\mathrm{L}'$, those which verify the above equality will be the roots of the equation. But these trials being long and troublesome, we will deduce from equation (1), other conditions equivalent to this, and easier verified.

Transposing all the terms except the last, and dividing by a, the equation (1) becomes

$$\frac{\mathrm{U}}{a}=-a^{m-1}-\mathrm{P}a^{m-2}- \ldots -\mathrm{R}a^2-\mathrm{S}a-\mathrm{T} \ldots (2);$$

now, the second member of this equation is an entire number, hence

$\frac{\mathrm{U}}{a}$ must be an entire number; therefore *the entire roots of the equation are comprised among the divisors of the last term.*

Transposing $-\mathrm{T}$ in the equation (2) and dividing by a, and making $\frac{\mathrm{U}}{a}+\mathrm{T}=\mathrm{T}'$; it becomes

$$\frac{\mathrm{T}'}{a}=-a^{m-2}-\mathrm{P}a^{m-3}\ldots-\mathrm{R}a-\mathrm{S}\ldots(3);$$

the second member of this equation being an entire number, $\frac{\mathrm{T}'}{a}$ or, *the quotient of the division of* $\frac{\mathrm{U}'}{a}+\mathrm{T}$ *by* a, *is an entire number.*

Transposing the term $-\mathrm{S}$ and dividing by a, it becomes, by supposing . . . . $\frac{\mathrm{T}'}{a}+\mathrm{S}=\mathrm{S}'$,

$$\frac{\mathrm{S}'}{a}=-a^{m-3}-\mathrm{P}a^{m-4}-\ldots-\mathrm{R}\ldots(4),$$

the second member of this equation being an entire number, $\frac{\mathrm{S}'}{a}$ or, *the quotient of the division of* $\frac{\mathrm{T}'}{a}+\mathrm{S}$ *by* a, *is an entire number.*

By continuing to transpose the terms of the second member into the first, we shall, after $m-1$ transformations, obtain an equation of the form . . . . $\frac{\mathrm{Q}'}{a}=-a-\mathrm{P}$,

Then, transposing the term $-\mathrm{P}$, dividing by a, and making

$$\frac{\mathrm{Q}'}{a}+\mathrm{P}=\mathrm{P}', \text{ we shall find } \frac{\mathrm{P}'}{a}=-1, \text{ or } \frac{\mathrm{P}'}{a}+1=0.$$

This equation, which is only a transformation of the equation (1), is the *last condition which it is requisite and necessary* that the entire number a should satisfy, in order that it may be known to be a root.

From the preceding conditions we may conclude that, in order that an entire number a, positive or negative, may be a root of the proposed equation, it is necessary

That *the quotient of the last term, divided by* a, *should be an entire number ;*

Adding to this quotient the co-efficient of x^1, taken with its sign, *the quotient of this sum divided by* a, *must be entire ;*

Adding the co-efficient of x^2 to this quotient, *the quotient of this new sum by* a, *must be entire;* and so on.

Finally, adding the co-efficient of the second term, or of x^{m-1}, to the preceding quotient, *the quotient of this sum divided by* a, *must be entire and equal to* -1 ; or, *the result of the addition of unity, or the co-efficient of* x^m, *to the preceding quotient, must be equal to* 0.

Every number which will satisfy these conditions will be a root, and those which do not satisfy them should be rejected.

All the entire roots may be determined at the same time, as follows.

After having determined all the divisors of the last term, write those which are comprehended between the limits $+L$ *and* $-L'$ *upon the same horizontal line ; then underneath these divisors write the quotients of the last term by each of them.*

Add the co-efficient of x^1 *to each of these quotients, and write the sums underneath the quotients which correspond to them ; then divide these sums by each of the divisors, and write the quotients underneath the corresponding sums ;* taking care to reject the fractional quotients and the divisors which produce them ; and so on.

When there are terms wanting in the proposed equation, their co-efficients, which are to be regarded as equal to 0, must be taken into consideration.

EXAMPLE.

$$x^4-x^3-13x^2+16x-48=0.$$

The superior limit of the positive roots of this equation is $13+1$ or 14 (Art. 293). The co-efficient 48 is not considered, since the two last terms can be put under the form $16(x-3)$; hence when $x>3$ this part is essentially positive.

The superior limit of the negative roots is $-(1+\sqrt{48})$, or -8 (Art. 295).

Therefore, the divisors are 1, 2, 3, 4, 6, 8, 12; moreover, neither $+1$, nor -1, will satisfy the equation, because the co-efficient -48 is itself greater than the sum of all the others; we should therefore try only the *positive divisors* from 2 to 12, and the *negative divisors* from -2 to -6 inclusively.

By observing the rule given above, we have

12,	8,	6,	4,	3,	2,	$-$ 2,	$-$ 3,	$-$ 4,	$-$ 6
$-$ 4,	$-$ 6,	$-$8,	$-$12,	$-$16,	$-$24,	+24,	+16,	+12,	+ 8
+12,	+10,	+8,	+ 4,	0,	$-$ 8,	+40,	+32,	+28,	+24
+ 1,	..,	..,	+ 1,	0,	$-$ 4,	$-$20,	..,	$-$ 7,	$-$ 4
$-$12,	..,	..,	$-$12,	$-$13,	$-$17,	$-$33,	..,	$-$20,	$-$17
$-$ 1,	..,	..,	$-$ 3,	..,	..,	..,	..,	+ 5,	..
$-$ 2,	..,	..,	$-$ 4,	..,	..,	..,	..,	+ 4,	..
..,	..,	..,	$-$ 1,	..,	..,	..,	..,	$-$ 1,	..

The *first* line contains the divisors, the *second* contains the quotients of the division of the last term -48, by each of the divisors. The *third* line contains these quotients augmented by the co-efficient $+16$, and the *fourth* the quotients of these sums by each of the divisors; this second condition excludes the divisors $+8$, $+6$, and -3.

The *fifth* is the preceding line of quotients, augmented by the co-efficient -13, and the *sixth* is the quotients of these sums by each of the divisors; this third condition excludes the divisors 3, 2, -2 and -6.

Finally, the *seventh* is the third line of quotients, augmented by the co-efficient -1, and the *eighth* is the quotients of these sums by each of the divisors. The divisors $+4$ and -4 are the only ones which give -1; hence $+4$ and -4 are the only entire roots of the equation.

In fact, if we divide $x^4-x^3-13x^2+16x-48$, by the product

$(x-4)(x+4)$, or x^2-16, the quotient will be x^2-x+3, which placed equal to zero, gives

$$x=\frac{1}{2}\pm\frac{1}{2}\sqrt{-11},$$

therefore, the four roots are

$$4, \quad -4, \quad \frac{1}{2}+\frac{1}{2}\sqrt{-11} \quad \text{and} \quad \frac{1}{2}-\frac{1}{2}\sqrt{-11}.$$

EXAMPLES.

1st. $x^4-5x^3+25x-21=0.$

2d. $15x^5-19x^4+6x^3+15x^2-19x+6=0.$

3d. $9x^6+30x^5+22x^4+10x^3+17x^2-20x+4=0.$

Of Real and Incommensurable Roots.

307. When an equation has been freed from all the divisors of the first degree which correspond to its commensurable roots, the resulting equation contains the *incommensurable roots* of the proposed equation, *either real or imaginary.*

The true form of the real incommensurable roots of an equation will remain unknown, so long as there is not a general method for resolving equations of the higher degrees. Although this problem has not been resolved, yet there are methods for approximating as near as we please to the numerical values of these roots.

We shall here consider only the case in which the difference between any two roots of the proposed equation is greater than unity, omitting as too difficult for an elementary treatise, the cases in which this difference is less than unity.

We will also suppose, in what follows, that we have obtained the narrowest limits $+L$ and $-L'$, by Newton's method (Art. 294).

308. Each of the incommensurable roots being necessarily composed of *an entire part* and *a part less than unity*, we shall first determine the entire part of each root.

For this purpose, it is necessary to substitute, in the equation, for x, the series of natural numbers 0, 1, 2, 3 ... and -1, -2, -3..., comprised between $+L$ and $-L'$. Since there must be a real root between two numbers, which, by their substitution produce results affected with different signs (Art. 288), it follows *that each pair of consecutive numbers giving results affected with contrary signs, will comprehend a real root, and but one,* since by hypothesis the difference between any two of the roots is greater than unity. The entire part of the root will be the smallest of the two numbers substituted.

There are two cases which may occur; viz. by these different substitutions there may be *as many changes of* sign as there are units in the degree of the equation; in which case we may conclude that *all the roots are real. Or,* the number of changes of the sign will be less than the degree of the equation, and, in this case, it will have as many real roots as there are changes of sign; the other roots will be imaginary. In both cases, this method makes known the *entire part* of each of the real roots.

It now remains to determine *the part which is less than unity.*

Newton's Method of Approximation.

309. In order that this method may be more easily comprehended, we shall take the equation

$$x^3-5x-3=0 \quad . \; . \; . \quad (1).$$

The superior limits of the positive and negative roots being $+3$ and -2, we make

$$x=-2,\ -1,\ 0,\ 1,\ 2,\ 3\,;$$

whence

$x=-2$	the result is	-1,
$x=-1$	. . .	$+1$,
$x=\ \ 0$	. . .	-3,
$x=\ \ 1$	. . .	-7,
$x=\ \ 2$	. . .	-5,
$x=\ \ 3$	. . .	$+9$.

As there are three changes of sign, it follows that the three roots of the equation are real; viz. *one positive* contained between 2 and 3, *two negative*, one of which is contained between 0 and -1, the other between -1 and -2.

We shall first consider the positive value between 2 and 3.

The required root being between 2 and 3, we will try to contract these limits, by taking the mean $2\frac{1}{2}$, or 2,5, and substituting it in the equation $x^3-5x-3=0$; the result of which is $+0{,}125$. Now 2 has already given -5 for a result, therefore the root is between 2 and 2,5.

We will now consider another number, between 2 and 2,5; but as, from the results given from 2 and 2,5, it is to be presumed that the root is nearer 2,5 than 2, suppose $x=2{,}4$; we shall obtain $-1{,}176$; whereas 2,5 has given $+0{,}125$. Therefore the root is between 2,4 and 2,5.

By continuing to take the means, we should be able to contract the two limits of the roots more and more. But when we have once obtained, as in the above case, the value of x to at least 0,1, we may approximate nearer in another way, and it is in this that Newton's method principally consists.

In the equation $x^3-5x-3=0$, make $x=2{,}4+u$.

There will result (Art. 274), the transformation

$$X'+Y'u+\frac{Z'}{2}u^2+u^3=0;$$

in which
$$X'=(2{,}4)^3-5(2{,}4)-3=-1{,}176,$$
$$Y'=3(2{,}4)^2-5=12{,}28,$$
$$\frac{Z'}{2}=3(2{,}4)=7{,}2.$$

The equation involving u, being of the third degree, cannot be resolved directly, but by transposing all the terms except $Y'u$, and dividing both members by Y', it can be put under the form

$$u=-\frac{X'}{Y'}-\frac{Z'}{2\,.\,Y'}u^2-\frac{1}{Y'}u^3.$$

This being the case, since one of the three roots of this equation must be less than $\frac{1}{10}$, from the relation $x=2{,}4+u$, the corresponding values of u^2 and u^3 are less than $\frac{1}{100}$ and $\frac{1}{1000}$. Moreover, the inspection of the numerical values of Y' and Z', proves that $\frac{Z'}{2.Y'}$ is <1; therefore the value of u only differing numerically from $-\frac{X'}{Y'}$ by the quantity $\frac{Z'}{2.Y'}u^2+\frac{1}{Y'}u^3$, (which most frequently is less than $\frac{1}{100}$), is expressed by $-\frac{X'}{Y'}$ to within 1,01.

As, in this example,

$$-\frac{X'}{Y'}=\frac{+1{,}176}{12{,}28}=\frac{1176}{12280}=0{,}09\ldots,$$

there will result $u=0{,}09$, to within $\frac{1}{100}$, and consequently

$$x=2{,}4+0{,}09=2{,}49, \quad \text{to within} \quad \frac{1}{100}.$$

In fact, 2,49 substituted in the first member of the proposed equation, gives $\qquad -0{,}011751$;

whilst 2,250 gives $\qquad +0{,}125$.

To obtain a new approximation, make $x=2{,}49+u'$ in the proposed equation, and we have

$$X''+Y''u'+\frac{Z''}{2}u'^2+u'^3=0;$$

in which
$$X''=(2{,}49)^3-5(2{,}49)-3=-0{,}011751,$$
$$Y''=3(249)^2-5=13{,}6003,$$
$$\frac{Z''}{2}=3(2{,}49)=7{,}47.$$

But the equation involving u' may be written thus:

$$u'=-\frac{X''}{Y''}-\frac{Z''}{2.Y''}u'^2-\frac{1}{Y''}u'^3.$$

And since one of the values of u' must be less than $\frac{1}{100}$, the corresponding values of u'^2, u'^3, are less than $\frac{1}{10000}$, $\frac{1}{1000000}$; hence $-\frac{X''}{Y''}$ will represent the value of u' to within $\frac{1}{10000}$.

Since we have

$$-\frac{X''}{Y''}=\frac{0,011751}{13,6003}=\frac{11751}{13600300}=0,0008\ldots,$$

it follows that $u'=0,0008$, to within $\frac{1}{10000}$, and consequently

$$x=2,49+0,0008=2,4908, \text{ to within } \frac{1}{10000}.$$

Again, by supposing $x=2,4908+u''$, we could obtain a value of x to within $\frac{1}{100000000}$.

Each operation commonly gives the root to twice as many places of decimals as the previous operation.

310. Generally, let p and $p+1$ be two numbers between which one of the roots of the equation $X=0$ is comprised.

First determine the value of this root to within $\frac{1}{10}$, by substituting a series of numbers comprised between p and $p+1$, until two numbers are obtained which do not differ from each other by more than $\frac{1}{10}$.

Then, calling x' the value of x *obtained to within* $\frac{1}{10}$ *suppose* $x=x'+u$ *in the equation* $X=0$;
which gives

$$X'+Y'u+\frac{Z'}{2}u^2 \ldots +u^m=0;$$

which can be put under the form

$$u=-\frac{X'}{Y'}-\frac{Z'}{2.Y'}u^2 \ldots -\frac{1}{Y'}u^m,$$

X', Y' Z' . . . being easily calculated. (Art. 309).

Since the sum of the terms, which follow $-\frac{X'}{Y'}$ in the second member of this equation is, commonly, less than $\frac{1}{100}$, they can be neglected, and calculating $-\frac{X'}{Y'}$ to within $\frac{1}{100}$, we add the result to x', which gives a new value x'' approximating *to within* $\frac{1}{100}$ of the exact value.

To obtain a 3d approximation, *we suppose* $x=x''+u'$ *in the proposed equation, which gives*

$$X''+Y''u'+\frac{Z''}{2}u'^2+ \ldots u'^m=0;$$

whence $$u'=-\frac{X''}{Y''}-\frac{Z''}{2.Y''}u'^2- \ldots -\frac{1}{Y''}u'^m.$$

Neglecting the terms $-\frac{Z''}{2.Y''}u'^2- \ldots -\frac{1}{Y''}u'^m$ which are supposed to be less than 0,0001, we calculate the value of $-\frac{X''}{Y''}$, continuing the operation to the $\frac{1}{10000}$ place of decimals, and add the result to x''; this gives a third approximation x''', exact to within $\frac{1}{10000}$.

Repeat this series of operations for each of the positive roots. As for the negative roots, they are found in the same way as the positive roots, by changing x into $-x$ in the proposed equation, which then becomes,

$$-x^3+5x-3=0, \text{ or } x^3-5x+3=0$$

in which the positive roots taken with a negative sign, are the negative roots of the proposed equation. These roots are

$$x=-1{,}8342 \text{ and } x-=0{,}6566$$

to within 0,0001.

Pg 158. 9th Example

X = distance

$X-18$ = "

THE END.

$X-18 \; : \; 15\frac{3}{4} \; :: \; X$

$X \; : \; 28$

30*

$$\frac{28(X-18)}{X}$$

$$\frac{28(X-18)}{X} = \frac{63X}{4(X-18)}$$

$$\frac{4(X-18)}{X} = \frac{9X}{4(X-18)}$$

$$16(X-18)^2 = 9X^2$$

$$4(X-18) = 3X$$

$$4X-72 = 3X$$

$$4X-3X = 72$$

[illegible]