Florian Cajori,
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THE ELEMENTS

OF

ANALYTIC GEOMETRY

BY

ALBERT L. CANDY, PH. D.

ADJUNCT PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF NEBRASKA

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ALBERT L. CANDY

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PREFACE.

Analytic Geometry is a broader subject than Conic Sections. It is far more important to the student that he should acquire a familiarity with the analytic method, and thoroughly grasp the generality of its processes and the comprehensiveness of its results, than that he should obtain a detailed knowledge of any particular set of curves. Furthermore, all branches of mathematics are fundamentally and inseparably related. Any subject, therefore, should be presented in such a way as to keep it in touch with all that has preceded, and at the same time reach forward toward that which is immediately to follow, to the end that there may be no sudden transition in passing from one branch to another. Algebra and Geometry, Analytics and Calculus are mutually helpful, and should not be studied entirely apart. No one of these subjects can be *finished* before the others are begun.

The general plan and scope of this book is due to a firm conviction of the soundness of these statements. For this reason a fuller treatment than usual is given of the general analytic method before taking up the study of the conic sections, and subjects have been introduced not ordinarily treated in text books on Analytic Geometry. The method of the differential calculus is the only way of studying the slope of curves, and furnishes the best means of finding the equation of the tangent and the normal. The graphical method of illustration and the derivative are indispensable in the discussion of the Theory of Equations. The use of the "derivative curve" in the theory of equal roots, together with the fact that the ordinate of the derivative curve is the slope of the "integral curve," naturally suggests a possible converse relation, and leads easily and logically to the study of Quadrature, and Maxima and Minima.

It is believed that the elementary discussion of these subjects here given will tend to meet the needs of scientific and engineering students, who now require a knowledge of the graphic method and the simple elements of the calculus at the earliest possible moment; and that it will also be helpful to the general student who pursues the study of the subject no further. In the "*secant method*" of finding the equation of the tangent the reasoning is essentially the same as in the method here used, but the student never comprehends its significance; and furthermore, he never uses the method after he leaves the study of the conic sections.

Occasionally a subject should be presented from the most general point of view, when the proof required is not so difficult as to be beyond the student's comprehension. The disposition, strong in some students, to be satisfied with a numerical example or a special case should not be encouraged. It is not always desirable to use the simplest demonstration for a particular proposition, but rather that one which will teach the reader to use the best method. In order to put the student as far as possible on his own resources, the number of demonstrations given in the book has been reduced to a minimum, consistent with giving him a sufficient working basis, but these demonstrations have been made full and lucid. For this reason many theorems which are usually proved have been left as exercises. By means of suggestive questions the student is frequently urged to continue an investigation. The method of proof in analogous theorems is not always the same; and some important subjects are presented from more than one point of view. The prime object, which has been kept constantly in view, is to prevent mere routine work on the part of the student.

In finding the equations of loci special emphasis is given to the meaning of the parameters which enter the final equations, and the significance of a variation in their value; and a full discussion and thorough geometric interpretation of the result is rigidly insisted on from the beginning. *The teacher should never lose sight of this vital principle.*

Polar coordinates and their relations to rectangular coordinates have been introduced at any early stage. The transformation of coordinates, from the most general point of view, is treated merely as an application of the distance form of the equation of the straight line.

The conic section is first briefly studied geometrically. Its fundamental property is proved in this way, from which its general equation is shown to be of the second degree. A short discussion of the general equation of the second degree is then given, not only for the purpose of giving a general view of the conic section as a whole, but also of showing the correspondence between the geometric and the algebraic results, and at the same time pointing out the superiority of the analytic method. The student is thus made to see that all possible cases are wrapped up in a single equation, and that all are unfolded in a single investigation; and, furthermore, he gets a bird's eye view of the whole subject.

The two central conics are treated simultaneously by using the double sign in the standard equation. In this way much time is saved; and the similarities of the properties of the two conics are presented in a striking manner.

As the book is intended for beginners, numerous illustrative examples are given, and also a large number of exercises. The numerical examples have all been prepared especially for this book, and but few answers are submitted, as it is far better to check results in such cases by constructing a figure. The general theorems included among the exercises in the chapters on the conics have been taken chiefly from such English books as C. Smith's Conic Sections, Todhunter's Conic Sections, C. L. Loney's Coordinate Geometry, and Wolstenholme's Mathematical Problems. These have been selected with great care. The object has been to include a sufficient number of the easier ones to prevent the discouragement of the poorer students, and at the same time to give a large number that would test the power of the strongest. Altogether the book contains about 1,400 exercises.

If a short course is desired, the sections marked with a star can be omitted. In any case these may be omitted, and used merely for the purpose of reference at the discretion of the teacher.

I wish to thank most heartily all my colleagues in this university who have so kindly assisted me in this work. I am especially indebted to Prof. Ellery W. Davis, who has read the entire manuscript with great care, and given many valuable criticisms and helpful suggestions.

A. L. C.

THE UNIVERSITY OF NEBRASKA,
MARCH 12, 1900.

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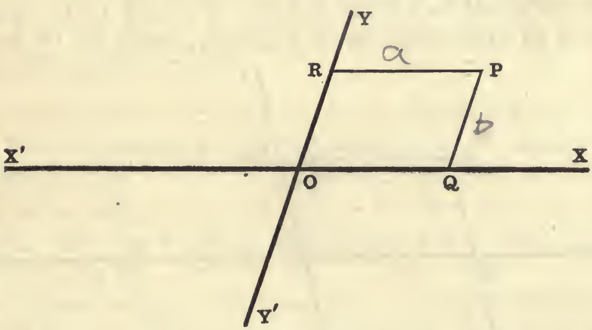
ANALYTIC GEOMETRY.

CHAPTER I.

COORDINATES, LENGTHS OF LINES AND AREAS OF POLYGONS.

Rectilinear Coordinates.

1. Let $X'X$ and $Y'Y$ be two fixed, non-parallel straight lines, intersecting in the point O. Let P be any point in the plane of these lines. Draw RP and QP parallel to $X'X$ and $Y'Y$ respectively.



These distances, RP and QP, *determine* the place of P within the angle XOY. That is, to every position of P there is one and only one pair of distances, to every pair of distances one and only one position of P. Moreover, the position of P can be found when the lengths of the lines RP and QP are given, and *vice versa*.

Suppose, for example, that we are given $RP = a$, $QP = b$, we need only measure $OQ = a$ and $OR = b$. and draw the parallels RP and QP, which will intersect in the required point.

2. The two lines RP and QP, or OQ and OR, which thus determine the position of the point P with reference to the lines

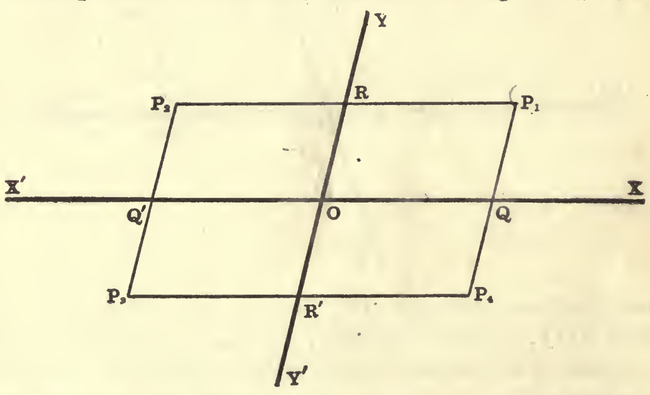
$X'X$ and $Y'Y$ are called the **Rectilinear** or **Cartesian*** **Coordinates** of the point P. QP is called the **Ordinate** of the point P, and is denoted by the letter y; RP, or its equal OQ, the intercept *cut off* by the ordinate, is called the **Abscissa**, and is denoted by the letter x.

The fixed lines $X'X$ and $Y'Y$ are called the **Axes of Coordinates**, and their point of intersection O is called the **Origin**. When the angle between the axes of coordinates is oblique, as in the preceding figure, the axes, and also the coordinates, are said to be **Oblique**; when the angle between the axes is right, the axes and the coordinates are said to be **Rectangular**.

If, in the preceding figure, OQ be a and OR be b, then at P, $x = a$ and $y = b$; at Q, $x = a$ and $y = 0$; at R, $x = 0$ and $y = b$; and at the origin O, $x = 0$ and $y = 0$.

The axis $X'X$ is called the **Axis of Abscissas**, or the **x-axis**; and $Y'Y$ is called the **Axis of Ordinates**, or the **y-axis**.

3. Let OQ and OQ' be equal in *magnitude* to a, and let OR and OR' be equal in *magnitude* to b. Through Q, Q', R and R' draw lines parallel to the axes, and intersecting in P_1, P_2, P_3, P_4.



Now at all of these four points $x = a$, in *magnitude*, and $y = b$, in *magnitude*. Hence in order that the equations $x = a$ and $y = b$

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* This method of determining the position of a point in a plane is due to the French Philosopher and Mathematician, Descartes. Hence the name *Cartesian*. The new method was first published in 1637.

"It is frequently stated that Descartes was the first to apply algebra to geometry. This statement is inaccurate, for Vieta and others had done this before him. Even the Arabs sometimes used algebra in connection with geometry. The new step that Descartes did take was the introduction into geometry of an analytical method based on the notion shall determine *only one* point, it is not sufficient to know the *lengths* of a and b, we must also know the *directions* in which they are measured.

In order to indicate the directions of lines we adopt the rule that *opposite directions shall be indicated by opposite signs.* It is *agreed*, as in Trigonometry, that distances measured in the directions OX (or to the right) and OY (or upwards) shall be considered *positive.* Hence distances measured in the directions OX' (or to the left) and OY' (or downwards) *must* be considered *negative.* Therefore

$$\text{at } P_1,\ x = a,\ y = b;\ \text{at } P_2,\ x = -a,\ y = b;$$
$$\text{at } P_3,\ x = -a,\ y = -b;\ \text{at } P_4,\ x = a,\ y = -b.$$

Thus the four points are easily and clearly distinguished, for no two *pairs* of values of x and y are the same.

If all possible values, positive and negative, be given to x and to y, or in other words, if both x and y be made to vary independently from $-\infty$ to $+\infty$, all points in the plane will be obtained. Moreover, to each *pair of values* of x and y there corresponds, in all the plane, *one* and *only one* point; to each point, one and only one pair of values.

4. For the sake of brevity, a point is usually represented by writing its coordinates within a parenthesis, the *abscissa* being always written *first.* Thus, in the preceding figure, P_1, P_2, P_3, P_4 are the points (a, b), $(-a, b)$, $(-a, -b)$, $(a, -b)$, respectively. In general, the point whose coordinates are x and y is called the point (x, y).

As in Trigonometry, it is convenient to distinguish the parts into which the axes divide the plane as first, second, third, and fourth quadrants.

Because of simplicity in formulæ and equations, it is *generally more convenient* to use rectangular axes.

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of variables and constants, which enabled him to represent curves by algebraic equations. In the Greek geometry, the idea of motion was wanting, but with Descartes it became a very fruitful conception. By him a point on a plane was determined in position by its distances from two fixed right lines or axes. These distances varied with every change of position in the point. This geometric idea of *coordinate representation*, together with the algebraic idea of *two variables in one equation* having an indefinite number of simultaneous values, furnished a method for the study of loci, which is admirable for the generality of its solutions. Thus the entire conic sections of Apollonius is wrapped up and contained in a single equation of the second degree."

[A History of Mathematics by Florian Cajori, p. 185.]

Accordingly, throughout this book, except when the contrary is expressly stated, the axes may be assumed rectangular.

EXAMPLES.

1. In what quadrants must a point lie if its coordinates have the same sign? different signs?

2. Locate the points $(1, -3)$, $(-2, 4)$, $(5, 0)$, $(-1, -3)$, $(4, 2)$, $(0, 3)$.

3. Construct the triangle whose vertices are the points $(0, 4)$, $(-5, -1)$ and $(4, -3)$.

4. Construct the triangle whose vertices are $(4, -1)$, $(1, 2)$, $(-1, -3)$.

5. Construct the quadrilateral whose vertices are the points $(3, 4)$, $(-1, 4)$, $(-1, -2)$, $(3, -2)$. What kind of a quadrilateral is it? Consider both oblique and rectangular axes.

6. Plot the points $(8, 0)$, $(5, 4)$, $(0, 4)$, $(-3, 0)$, $(0, -4)$, $(5, -4)$ and connect them by straight lines. What kind of a figure do these six lines enclose?

7. P is the point (x, y); P_1, P_2, P_3 are its symmetrical points with respect to the x-axis, y-axis, and origin, respectively. What are the coordinates of P_1, P_2, P_3?

8. The side of a square is $2a$. What are the coordinates of its vertices when the diagonals are the axes?

9. The side of an equilateral triangle is $2a$. What are the coordinates of its vertices, if one vertex is at the origin and one side coincides with the x-axis?

10. Where may a point be if its abscissa is 2? if its ordinate is -3?

11. Can a point move and yet always satisfy the condition $x = 0$? $y = 0$? both the conditions $x = 0$ and $y = 0$?

12. How must a point move so as to satisfy the condition $x = -c$? $y = d$? both these conditions?

13. If a point moves along either of the bisectors of the angles between the axes, what is the relation between its coordinates?

14. Where may a point be if its coordinates satisfy the condition $x^2 + y^2 = a^2$? What is the relation between the coordinates of a point which moves so that its distance from the origin is always 2?

15. If a line AB is two units to the left of the y-axis, what are the coordinates of a point whose distance from AB is three units?

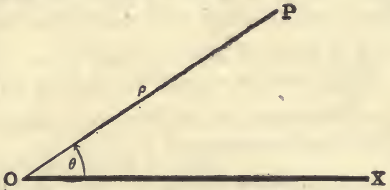
16. If P be any point on the bisector of the angle between the y-axis and a line three units above the x-axis, what is the general relation between the coordinates of P?

Polar Coordinates.

5. Let O be a fixed point called the **Pole**, and OX a fixed line called the **Initial Line.**

Take any other point P in the plane and draw OP. The position of the point P with reference to the line OX is known when the distance OP and the angle XOP are given.

The line OP is called the **Radius Vector** of the point P, and will be denoted by ρ; the angle XOP, which the radius vector makes with the initial line, is called the **Vectorial Angle** of the point P, and will be denoted by θ.



Then ρ and θ are the **Polar Coordinates*** of P; that is, P is the point (ρ, θ).

As in Trigonometry, it is agreed that the angle θ shall be positive when measured from OX counter clockwise; that ρ shall be positive when measured in the direction of the terminal line of the vectorial angle θ.

In determining the position of a point whose polar coordinates are given the following direction will be useful: Suppose I stand at O facing in the direction OX. To get to the point (ρ, θ), I turn through the angle θ to the *left* or *right* according as θ is *positive* or *negative*, then, keeping my new facing, I go a distance ρ *forward* or *backward* according as ρ is *positive* or *negative*.†

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* Whenever the position of a point in a plane is determined by any two magnitudes whatever, these two magnitudes are the coordinates of the point. Thus there may be an indefinite number of systems of coordinates. For an explanation of other systems which are in common use see Chap. I of Elements of Analytical Geometry by Briot and Bouquet, translated by J. H. Boyd. In this book we shall mainly use the two systems already explained, but a more general discussion of the subject is given in §§ 72–76.

† This method of locating points by means of coordinates is not altogether new to the student, neither is it confined to mathematics. For example, when we locate places on the surface of the earth by means of their latitude and longitude, we make use of a ***system of rectangular coordinates*** in which the axes are the equator and some chosen meridian. When we say the city B is forty miles north-east of the city A, we locate B with reference to A by means of a ***system of polar coordinates*** in which the initial line is the meridian through A, and A is the pole. Let the student suggest other familiar examples, if possible. How are places located in cities? in Washington, D. C.?

EXAMPLES.

Plot on one diagram the following points:

1. $(4, 30°)$, $(-3, 135°)$, $(3, 120°)$, $(-4, -30°)$.

2. $(5, 45°)$, $(-4, 120°)$, $(3, -150°)$, $(-6, -240°)$.

3. $(a, \frac{1}{4}\pi)$, $(-a, \frac{1}{6}\pi)$, $(a, -\frac{3}{4}\pi)$, $(2a, -\frac{2}{3}\pi)$, $(-\frac{1}{2}a, -\frac{4}{3}\pi)$, $(a, 0)$, $(2a, \pi)$.

4. $(5, \tan^{-1} 5)$, $(-2, \tan^{-1} 2)$, $(3, -\tan^{-1} 3)$, $(-4, \tan^{-1} -1)$.

5. $(a, \tan^{-1} 2)$, $(a, -\tan^{-1} 3)$, $(-a, \tan^{-1} \frac{1}{2})$, $(-a, -\tan^{-1} \frac{4}{3})$, $[a, \tan^{-1}(-4)]$.

6. Plot the points $(-6, 30°)$, $(2, 150°)$, $(2, -90°)$ and connect them by straight lines. What kind of a figure do these lines enclose?

7. Plot the points $(a, 60°)$, $(b, 150°)$, $(a, 240°)$, $(b, -30°)$ and join them by straight lines. What kind of a figure do these lines enclose?

8. Find the polar coordinates of the vertices of a square whose angular points in rectangular coordinates are $(3, 1)$, $(-1, -1)$, $(-1, 3)$, $(3, -3)$.

9. The side of an equilateral triangle is $2a$. If one vertex is at the pole and one side coincides with the initial line, what are the polar coordinates of its vertices? of the middle points of the sides?

10. Change "equilateral triangle" to "square" in Ex. 5.

11. Change "equilateral triangle" to "regular hexagon" in Ex. 5.

12. How must ρ and θ vary in order to obtain all points in the plane? (See § 3.)

13. Show that to each *pair* of values of ρ and θ there corresponds in all the plane one and only one point.

14. Show by plotting the four points, $(3, 60°)$, $(-3, 240°)$, $(3, -300°)$, $(-3, -120°)$, that the converse of Ex. 9 is not true.

15. Show that in general the same point is given by each of the four pairs of polar coordinates,

$$(\rho, \theta),\quad (-\rho, \pi+\theta),\quad [\rho, -(2\pi-\theta)],\quad [-\rho, -(\pi-\theta)].$$

16. Show that for all integral values of n the same point (ρ, θ) is also given by

$$(\rho, \theta \pm 2n\pi) \quad \text{and} \quad [-\rho, \theta \pm (2n+1)\pi].$$

17. Where does the point (ρ, θ) lie if $\theta = 0$? if $\theta = \pi$? if $\rho = 2$?

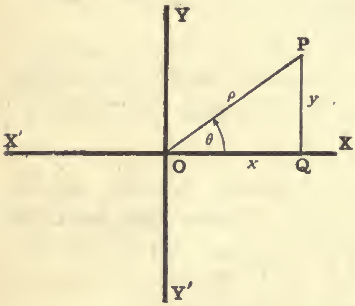
18. How can the point (ρ, θ) move if $\theta = \alpha$? if $\rho = a$? where α and a are constants.

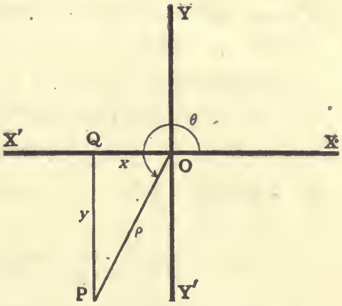
19. What condition must ρ and θ satisfy if the point (ρ, θ) moves along a line perpendicular to the initial line? parallel to the initial line?

20. What is the position of the point (ρ, θ) if $\rho = a \cos\theta$? $\rho = a \sin\theta$?

Relations Between Rectangular and Polar Coordinates.

6. Let P be any point whose rectangular coordinates are x and y, and whose polar coordinates, referred to O as pole and OX as initial line, are ρ and θ.





Draw PQ perpendicular to OX.

Then, according to the preceding definitions,

$$OQ = x, \quad QP = y, \quad OP = \rho, \quad \angle XOP = \theta.$$

From the right triangle PQO we have

$$OQ = OP \cos XOP \quad \text{and} \quad QP = OP \sin XOP.$$

$$\left.\begin{aligned} \therefore \quad x &= \rho \cos \theta. \\ y &= \rho \sin \theta. \\ x^2 + y^2 &= \rho^2. \end{aligned}\right\} \qquad (1)$$

These equations (1) express the rectangular coordinates in terms of the polar coordinates.

From equations (1) we find the corresponding equations expressing the polar coordinates in terms of the rectangular coordinates to be

$$\left.\begin{aligned} \rho &= \sqrt{x^2 + y^2}, & \theta &= \tan^{-1} \frac{y}{x}, \\ \sin \theta &= \frac{y}{\sqrt{x^2 + y^2}}, & \cos \theta &= \frac{x}{\sqrt{x^2 + y^2}}. \end{aligned}\right\} \qquad (2)$$

By means of equations (1) and (2) formulæ and equations in *either* system of coordinates can be changed into the *other* system of coordinates.

EXAMPLES.

1. Change the equation $\rho^2 = a^2 \cos 2\theta$ to rectangular coordinates.

Multiplying the equation by ρ^2, and putting $\cos 2\theta = \cos^2\theta - \sin^2\theta$ gives

$$\rho^4 = a^2(\rho^2 \cos^2\theta - \rho^2 \sin^2\theta).$$

Whence by substituting equations (1) we have

$$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

Change to polar coordinates the equations

2. $x^2 + y^2 = 2rx.$ Ans. $\rho = 2r \cos \theta.$

3. $x^2 - y^2 = a^2.$ Ans. $\rho^2 = a^2 \sec 2\theta.$

4. $(2x^2 + 2y^2 - ax)^2 = a^2(x^2 + y^2).$ Ans. $\rho^{\frac{1}{2}} = a^{\frac{1}{2}} \cos \frac{1}{2}\theta.$

Transform to rectangular coordinates

5. $\rho^2 \sin 2\theta = 2a^2.$ Ans. $xy = a^2.$

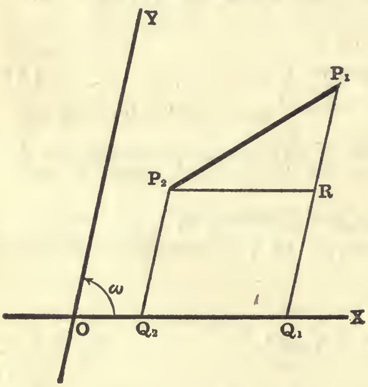
6. $\rho^{\frac{1}{2}} \cos \frac{1}{2}\theta = a^{\frac{1}{2}}.$ Ans. $y^2 + 4ax = 4a^2.$

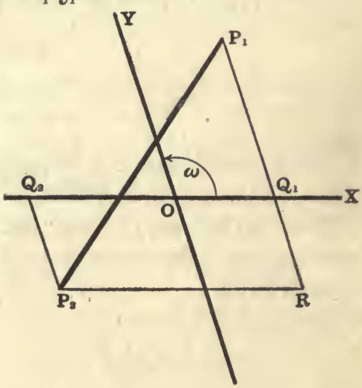
Distance Between Two Points.

7. *To find the distance between two points whose rectilinear coordinates are given.*

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be the given points, and let the axes be inclined at an angle ω.

Draw P_1Q_1 and P_2Q_2 parallel to OY, to meet OX in Q_1 and Q_2. Draw P_2R parallel to OX to meet P_1Q_1 in R.





Then $OQ_1 = x_1, \quad OQ_2 = x_2, \quad Q_1P_1 = y_1, \quad Q_2P_2 = y_2.$

$$\therefore \quad P_2R = Q_2Q_1 = OQ_1 - OQ_2 = x_1 - x_2,$$

and $$RP_1 = Q_1P_1 - Q_1R = Q_1P_1 - Q_2P_2 = y_1 - y_2.$$

Also $$\angle P_1RP_2 = \angle P_1Q_1O = \pi - \omega.$$

From the triangle P_1RP_2 we have, by the law of cosines in Trigonometry,

$$P_2P_1^2 = P_2R^2 + RP_1^2 - 2\, P_2R \cdot RP_1 \cos(\pi - \omega).$$

Whence by substitution, since $\cos(\pi - \omega) = -\cos \omega$,

$$P_2P_1 = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos \omega]^{\frac{1}{2}}. \quad (1)$$

If, as is usually the case, the axes are rectangular,

$$\omega = 90° \text{ and } \cos \omega = 0.$$

Hence for the distance between two points whose rectangular coordinates are given, we have the very useful formula

$$P_2P_1 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.^* \quad (2)$$

If the plus sign before the radicals in (1) and (2) gives P_2P_1, the minus sign will give P_1P_2.

It will aid the memory to observe that the meaning of (2) is expressed by writing

$$(\textit{Distance})^2 = (\textit{Easting})^2 + (\textit{Northing})^2.$$

Cor. If P_2 coincides with the origin $x_2 = y_2 = 0$, and then equations (1) and (2) give for the distance of a point $P_1(x_1, y_1)$ from the origin

$$OP_1 = \sqrt{x_1^2 + y_1^2 + 2\, x_1 y_1 \cos \omega}, \quad \text{for oblique axes,} \quad (3)$$

$$OP_1 = \sqrt{x_1^2 + y_1^2}, \quad \text{for rectangular axes.} \quad (4)$$

EXAMPLES.

1. Find the distance between (—5, 3) and (7, —2).

2. Show that if the axes are inclined at an angle of 60°, the distance between the points (—3, 3) and (4, —2) is $\sqrt{39}$.

3. Find the distance from the origin to the point (—2, 4) when the axes are inclined at angle of 120°.

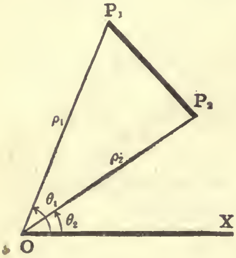
4. Find the lengths of the sides of the triangle whose vertices are (4, 1), (—2, 4), and (1, —2).

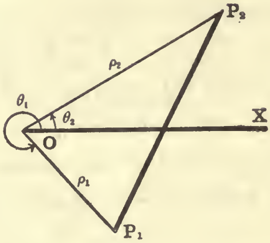
5. Show that the four points (2, 4), (1, 7), (—2, 4), and (—1, 1) are the angular points of a parallelogram.

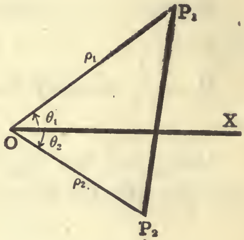
6. If the point (x, y) is 5 units distant from the point (3, 4), then will $x^2 + y^2 - 6x - 8y = 0$.

* The student should convince himself of the generality of equations (1) and (2) by constructing other special cases in which the given points lie in different quadrants. He will thus have one illustration of a general principle whose truth he will gradually see as he proceeds with the study of the subject; viz. that formulæ and equations deduced by considering points lying in the first quadrant, where both coordinates are positive, must, from the nature of the analytic method, hold true when the points are situated in any quadrant.

8. *To express the distance between two points in terms of their polar coordinates.*







Let $P_1(\rho_1, \theta_1)$ and $P_2(\rho_2, \theta_2)$ be the two given points.

Then $OP_1 = \rho_1$, $OP_2 = \rho_2$, $\angle XOP_1 = \theta_1$, $\angle XOP_2 = \theta_2$,

and $\angle P_2OP_1 = \theta_1 - \theta_2$.

From the triangle P_1OP_2, as in § 7, we have

$$P_1P_2^2 = OP_1^2 + OP_2^2 - 2\,OP_1 \cdot OP_2 \cos P_2OP_1.$$

$$\therefore \quad P_1P_2 = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_1 - \theta_2)}. \qquad (1)$$

Ex. 1. Derive equation (2), § 7, from equation (1), § 8.

Expanding the last term and squaring (1), § 8, gives

$$P_1P_2^2 = \rho_1^2 + \rho_2^2 - 2(\rho_1 \cos \theta_1)(\rho_2 \cos \theta_2) - 2(\rho_1 \sin \theta_1)(\rho_2 \sin \theta_2).$$

Substituting the values given in equations (1), § 6, we have

$$P_1P_2^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2x_1x_2 - 2y_1y_2.$$

$$\therefore \quad P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Ex. 2. Show that the distance between the points $(4, 90°)$ and $(-3, 30°)$ is $\sqrt{37}$.

Ex. 3. Find the distance between $(2a, 180°)$ and $(-a, 45°)$.

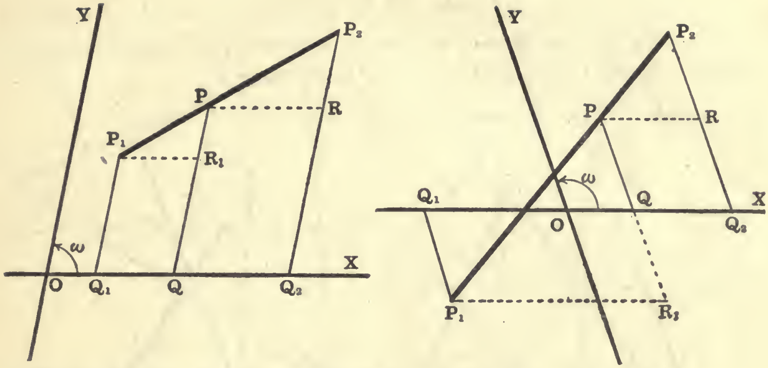
9. *To find the coordinates of the point which divides the line joining two given points in a given ratio* $(m_1 : m_2)$.

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be the two given points, and let $P(x, y)$ be the required point.

Draw P_1Q_1, PQ, P_2Q_2 parallel to the y-axis, and PR, P_1R_1 parallel to the x-axis.

Then $P_1R_1 = x - x_1$, $PR = x_2 - x$,

$R_1P = y - y_1$, $RP_2 = y_2 - y$.



From the similar triangles P_1PR_1 and PP_2R, we have

$$\frac{P_1P}{PP_2}=\frac{P_1R_1}{PR}=\frac{R_1P}{RP_2}=\frac{m_1}{m_2}.$$

That is, $$\frac{m_1}{m_2}=\frac{x-x_1}{x_2-x}=\frac{y-y_1}{y_2-y}.$$

$$\therefore\quad m_1(x_2-x)=m_2(x-x_1), \tag{1}$$

and $$m_1(y_2-y)=m_2(y-y_1). \tag{2}$$

Solving (1) and (2) for x and y, respectively, we obtain

$$x=\frac{m_1x_2+m_2x_1}{m_1+m_2},\qquad y=\frac{m_1y_2+m_2y_1}{m_1+m_2}. \tag{3}$$

When the division is internal, as we have so far assumed, the ratio of the segments is negative; that is,

$$\frac{P_1P}{P_2P}=-\frac{m_1}{m_2}.$$

But when the division is external, this ratio is positive. Hence for external division we have, by simply changing the sign of m_2 in equations (3),

$$x=\frac{m_1x_2-m_2x_1}{m_1-m_2},\qquad y=\frac{m_1y_2-m_2y_1}{m_1-m_2}. \tag{4}$$

If P be the middle point of P_1P_2, $m_1=m_2$, and therefore the coordinates of the middle of a line joining two given points are

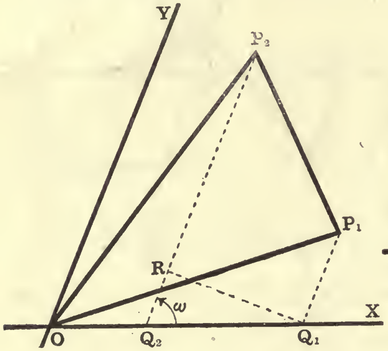
$$x=\tfrac{1}{2}(x_1+x_2),\qquad y=\tfrac{1}{2}(y_1+y_2). \tag{5}$$

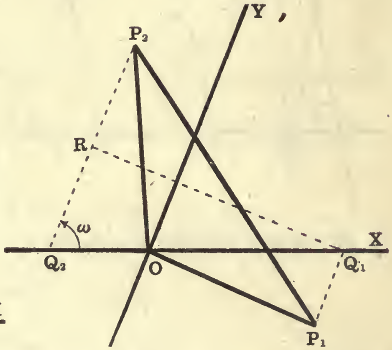
These formulæ, (3), (4), (5), are independent of the angle between the axes, and therefore they hold for rectangular as well as for oblique axes.

Areas of Polygons.

10. *To find the area of a triangle in terms of the coordinates of its vertices, the axes being inclined at an angle ω.*

Case I. *When one vertex is at the origin.*





Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be the other two vertices. Draw P_1Q_1, P_2Q_2 parallel to the y-axis, and Q_1R perpendicular to P_2Q_2.

Then $\quad OQ_1 = x_1,\ OQ_2 = x_2,\ Q_1P_1 = y_1,\ Q_2P_2 = y_2,$

$$RQ_1 = Q_2Q_1 \sin\omega = (x_1 - x_2)\sin\omega,$$

and
$$\begin{aligned}\triangle OP_1P_2 &= \triangle OQ_2P_2 + \text{trap. } Q_2Q_1P_1P_2{*} - \triangle OQ_1P_1,\\ &= \tfrac{1}{2}OQ_2\cdot Q_2P_2\sin\omega + \tfrac{1}{2}Q_2Q_1(Q_2P_2 + Q_1P_1)\sin\omega\\ &\qquad - \tfrac{1}{2}OQ_1\cdot Q_1P_1\sin\omega\\ &= \tfrac{1}{2}[x_2y_2 + (x_1 - x_2)(y_1 + y_2) - x_1y_1]\sin\omega\\ &= \tfrac{1}{2}(x_1y_2 - x_2y_1)\sin\omega. \qquad (1)\end{aligned}$$

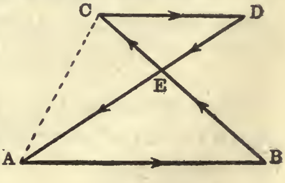
In the notation of determinants this may be written

$$\triangle OP_1P_2 = \tfrac{1}{2}\begin{vmatrix} x_1, & y_1 \\ x_2, & y_2 \end{vmatrix}\sin\omega. \qquad (2)$$

---

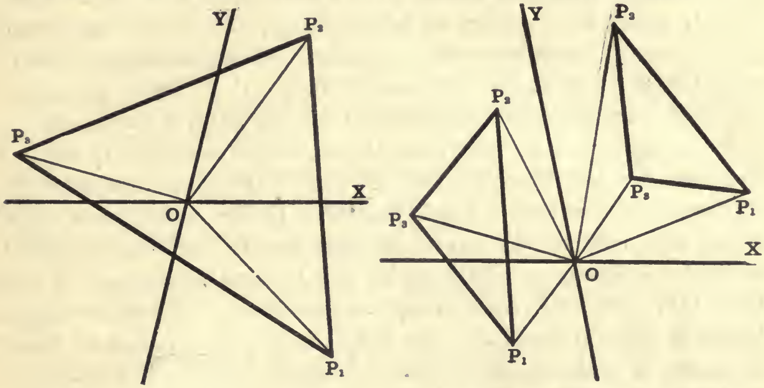
* The area of the crossed trapezoid *ABCD*, in which the non-parallel sides intersect, is the *difference* of the areas of the two triangles formed by drawing the diagonal *AC*. That is,
$$ABCD = ABC - ADC = ABE - CDE.$$

This is expressed analytically by saying that the area is the *algebraic sum* of the triangles. The base *CD* is then regarded as having changed its direction (and hence its sign) with reference to *AB*; for in going along the sides consecutively in the order *ABCD*, the base *CD* is traversed in the same direction as *AB*, which is not the case in the ordinary trapezoid.



This figure furnishes a good illustration of the principle that areas gone around so as to be on the *left* differ in *sign* from those gone around so as to be on the *right*. (See § 11.)

CASE II. *When the origin is not a vertex of the given triangle.*



Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ be the vertices of the given triangle.

Draw the lines OP_1, OP_2, OP_3.

Then by Case I we have

$$\triangle\ OP_1P_2 = \tfrac{1}{2}(x_1y_2 - x_2y_1)\sin\omega = \tfrac{1}{2}\begin{vmatrix} x_1, & y_1 \\ x_2, & y_2 \end{vmatrix}\sin\omega.$$

$$\triangle\ OP_2P_3 = \tfrac{1}{2}(x_2y_3 - x_3y_2)\sin\omega = \tfrac{1}{2}\begin{vmatrix} x_2, & y_2 \\ x_3, & y_3 \end{vmatrix}\sin\omega.$$

$$\triangle\ OP_3P_1 = \tfrac{1}{2}(x_3y_1 - x_1y_3)\sin\omega = \tfrac{1}{2}\begin{vmatrix} x_3, & y_3 \\ x_1, & y_1 \end{vmatrix}\sin\omega.$$

$$\therefore\quad \triangle\ P_1P_2P_2 = \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]\sin\omega \qquad (3)$$

$$= \tfrac{1}{2}\left\{\begin{vmatrix} x_1, & y_1 \\ x_2, & y_2 \end{vmatrix} + \begin{vmatrix} x_2, & y_2 \\ x_3, & y_3 \end{vmatrix} + \begin{vmatrix} x_3, & y_3 \\ x_1, & y_1 \end{vmatrix}\right\}\sin\omega$$

$$= \tfrac{1}{2}\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}\sin\omega. \qquad (4)$$

When the axes are rectangular $\sin\omega = 1$, and equations (1), (2), (3), (4), respectively, reduce to

$$\triangle\ OP_1P_2 = \tfrac{1}{2}(x_1y_2 - x_2y_1) = \tfrac{1}{2}\begin{vmatrix} x_1, & y_1 \\ x_2, & y_2 \end{vmatrix}. \qquad (5)$$

$$\triangle\ P_1P_2P_3 = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3) \qquad (6)$$

$$= \tfrac{1}{2}\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = \tfrac{1}{2}\begin{vmatrix} x_1 - x_2, & y_1 - y_2 \\ x_2 - x_3, & y_2 - y_3 \end{vmatrix}. \qquad (7)$$

11. When the origin is within the given triangle, the given triangle includes the three triangles OP_1P_2, OP_2P_3, OP_3P_1 (§ 10); hence the expressions $\frac{1}{2}(x_1y_2 - x_2y_1)$, $\frac{1}{2}(x_2y_3 - x_3y_2)$, and $\frac{1}{2}(x_3y_1 - x_1y_3)$ must have the *same sign.* When the origin is outside, the given triangle does not include all of these triangles, and therefore the above expressions can *not* have the *same sign.*

Suppose a person to start from O and walk consecutively around the triangles OP_1P_2, OP_2P_3, OP_3P_1 in the direction indicated by this order of vertices. This imaginary person would thus walk along each side of the given triangle *once* in the *same* direction around the figure, as indicated by $P_1P_2P_3$, and along each of the lines OP_1, OP_2, OP_3 *twice* in *opposite* directions. When the origin is inside the given triangle, he would walk around each of these triangles in such a manner that he would have *its area always on his left hand.* When the origin is outside, he would go around those triangles which include *no part of the given triangle,* in such a manner that he would have *their area always on his right hand.*

Thus direction around a triangle may be taken to indicate the *sign* of its area. (See note under § 10.)

The expressions for area in § 10 will be found to be *positive,* if the vertices are numbered so that in passing around in the direction thus indicated *the area is always on the left.*

Let the student show by trial that

$(x_1y_2 - x_2y_1)$ is $\pm$ according as angle P_1OP_2 is $\pm$;

angle P_1OP_2 is $\pm$ according as the cycle OP_1P_2 is $\pm$.

12. *To express the area of a triangle in terms of the polar coordinates of its vertices.*

Let $P_1(\rho_1, \theta_1)$, $P_2(\rho_2, \theta_2)$, $P_3(\rho_3, \theta_3)$ be the three vertices.

Then $x_1 = \rho_1 \cos \theta_1$, $x_2 = \rho_2 \cos \theta_2$, $x_3 = \rho_3 \cos \theta_3$,

$y_1 = \rho_1 \sin \theta_1$, $y_2 = \rho_2 \sin \theta_2$, $y_3 = \rho_3 \sin \theta_3$. [(1), § 6.]

Substituting these values in (5) and (6) of § 10 gives

$$\triangle OP_1P_2 = \tfrac{1}{2}\rho_1\rho_2 (\sin \theta_2 \cos \theta_1 - \cos \theta_2 \sin \theta_1) = \tfrac{1}{2}\rho_1\rho_2 \sin (\theta_2 - \theta_1). \quad (1)$$

$$\triangle P_1P_2P_3 = \tfrac{1}{2}[\rho_1\rho_2 \sin (\theta_2 - \theta_1) + \rho_2\rho_3 \sin (\theta_3 - \theta_2) + \rho_3\rho_1 \sin (\theta_1 - \theta_3)]. \quad (2)$$

From (1) it follows that the three terms of (2) represent, respectively, the areas of the triangles OP_1P_2, OP_2P_3, and OP_3P_1.

The signs of these terms are the signs of the *angle differences* (since ρ can always be made positive), and we therefore have an independent proof of the statements in § 11.

Let the student prove (1) and (2) directly from a figure.

13. *To find the area of any polygon when the rectangular coordinates of its vertices are known.*

Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, $P_4(x_4, y_4) \dots P_n(x_n, y_n)$ be the n vertices of the given polygon.

Then, we have, from (5) § 10,

$$\triangle\, OP_1P_2 = \tfrac{1}{2}\begin{vmatrix} x_1, & y_1 \\ x_2, & y_2 \end{vmatrix}, \quad \triangle\, OP_2P_3 = \tfrac{1}{2}\begin{vmatrix} x_2, & y_2 \\ x_3, & y_3 \end{vmatrix},$$

$$\triangle\, OP_3P_4 = \tfrac{1}{2}\begin{vmatrix} x_3, & y_3 \\ x_4, & y_4 \end{vmatrix}, \quad \triangle\, OP_4P_5 = \tfrac{1}{2}\begin{vmatrix} x_4, & y_4 \\ x_5, & y_5 \end{vmatrix},$$

. . . .

$$\triangle\, OP_nP_1 = \tfrac{1}{2}\begin{vmatrix} x_n, & y_n \\ x_1, & y_1 \end{vmatrix}.$$

$$\therefore \quad \text{Area } P_1P_2 \dots P_n = \tfrac{1}{2}\left\{ \begin{vmatrix} x_1, & y_1 \\ x_2, & y_2 \end{vmatrix} + \begin{vmatrix} x_2, & y_2 \\ x_3, & y_3 \end{vmatrix} + \begin{vmatrix} x_3, & y_3 \\ x_4, & y_4 \end{vmatrix} + \begin{vmatrix} x_4, & y_4 \\ x_5, & y_5 \end{vmatrix} + \dots \begin{vmatrix} x_n, & y_n \\ x_1, & y_1 \end{vmatrix} \right\}, \qquad (1)$$

since the area of the polygon is the *algebraic* sum of the areas of these triangles.

This formula is easy to remember, but by expanding the determinants and collecting the positive and negative terms it may be written,

$$\text{Area } P_1P_2 \dots P_n = \tfrac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_4 + \dots x_ny_1) - (y_1x_2 + y_2x_3 + y_3x_4 + \dots y_nx_1)], \qquad (2)$$

which gives the following simple rule for finding the area of a polygon when the rectangular coordinates of its vertices are known:

(1) *Number the vertices consecutively, keeping the area on the left.*

(2) *Multiply each abscissa by the next ordinate.*

(3) *Multiply each ordinate by the next abscissa.*

(4) *From the sum of the first set of products subtract the sum of the second set and take half of the result.*

If the axes are oblique, the second members of (1) and (2) must be multiplied by the sine of the angle between the axes.

The law of the *sign* of the area is the same as for the triangle.

EXAMPLES ON CHAPTER I.

Find the area of the polygons the coordinates of whose vertices taken in order are, respectively,

1. $(1, 3)$, $(-2, -4)$, and $(3, -1)$.

2. $(2, 5)$, $(-6, -2)$, and $(-1, 5)$, when $\omega = 60°$.

3. $(4, 15°)$, $(-5, 45°)$, and $(6, 75°)$.

4. $(3, -30°)$, $(-5, 150°)$, and $(4, 210°)$.

5. $(2, 15°)$, $(6, 75°)$, and $(5, 135°)$.

6. $(-a, \frac{1}{6}\pi)$, $(a, \frac{1}{2}\pi)$, and $(-2a, -\frac{2}{3}\pi)$.

7. $(a, b + c)$, $(a, b - c)$, and $(-a, c)$.

8. $(a, c + a)$, (a, c), and $(-a, c - a)$.

9. $(2, 3)$, $(-1, 4)$, $(-5, -2)$, and $(3, -2)$.

10. $(4, 1)$, $(1, 5)$, $(-2, 6)$, $(-5, 3)$, $(-1, -1)$, $(-3, -4)$, $(1, -2)$, and $(3, -4)$.

11. What are the rectangular coordinates of $(4, 30°)$, $(-2, 135°)$, $(-3, \frac{2}{3}\pi)$?

12. What are the polar coordinates of $(3, -4)$, $(-5, 12)$, $(1, 3)$?

13. Find the coordinates of the points which trisect the line joining the points $(-2, -1)$ and $(3, 2)$.

14. Find the coordinates of the point which divides the line joining $(3, -2)$ and $(-5, 4)$ internally in the ratio $3 : 4$.

15. Find the coordinates of the point which divides the line joining $(5, 3)$ and $(-1, 4)$ externally in the ratio $3 : 2$.

16. Find the length of the sides and medians of the triangle $(2, 6)$, $(7, -6)$, $(-5, -1)$. What kind of a triangle is it?

17. Find the length of the sides and the area of the triangle $(3, 4)$, $(-1, 0)$, $(2, -3)$. What kind of a triangle is it?

18. Find the sides and area of the quadrilateral whose vertices taken in order are $(5, -1)$, $(-1, 2)$, $(-5, 0)$, and $(1, -3)$. What kind of a quadrilateral is it?

Change to polar coordinates the equations

19. $x^2 + y^2 = r^2$.

20. $y = x \tan \alpha$.

21. $x^3 = y^2(2a - x)$.

Transform to Cartesian coordinates

22. $\theta = \tan^{-1} m$.

23. $\rho^2 = a^2 \sec 2\theta$.

24. $\rho = a \sin 2\theta$.

25. $\rho^{\frac{1}{2}} = a^{\frac{1}{2}} \sin \frac{1}{2}\theta$.

Prove analytically the following theorems:

26. The diagonals of a parallelogram bisect each other.

27. The lines joining the middle points of the adjacent sides of any quadrilateral form a parallelogram.

28. The three medians of a triangle meet in a point, which is one of their points of trisection.

29. The area of the triangle formed by joining the middle points of the sides of a given triangle is equal to one-fourth of the area of the given triangle.

30. If in any triangle a median be drawn from the vertex to the base, the sum of the squares of the other two sides is equal to twice the square of half the base plus twice the square of the median.

31. The sum of the squares of the four sides of any quadrilateral is equal to the sum of the squares of the diagonals plus four times the square of the line joining the middle points of the diagonals.

32. The lines joining the middle points of opposite sides of any quadrilateral and the line joining the middle points of its diagonals meet in a point and bisect one another.

33. $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, $P_4(x_4, y_4)$ . . . $P_n(x_n, y_n)$ are any n points in a plane. P_1P_2 is bisected at Q_1; Q_1P_3 is divided at Q_2 in the ratio $1:2$; Q_2P_4 is divided at Q_3 in the ratio $1:3$; Q_3P_5 at Q_4 in the ratio $1:4$, and so on till all the points are used. Show that the coordinates of the final point so obtained are

$$\frac{x_1 + x_2 + x_3 + x_4 + \cdots x_n}{n} \quad \text{and} \quad \frac{y_1 + y_2 + y_3 + y_4 + \cdots y_n}{n}.$$

Show that the result is independent of the order in which the points are taken.

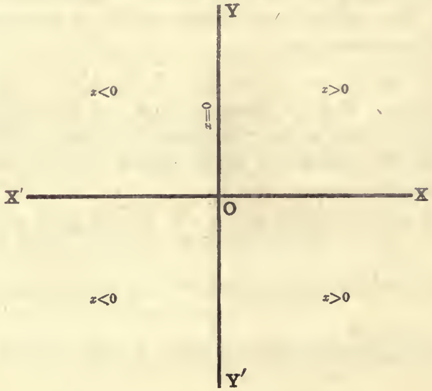
[This point is called the *Centre of Mean Position* of the n given points.]

CHAPTER II.

LOCI AND THEIR EQUATIONS.

14. It has been shown in § 3 that to each pair of values of x and y there corresponds in all the plane one and only one point, and that to each point corresponds one and only one pair of values. Also, if x and y vary independently and unconditionally from $-\infty$ to ∞ every point in the plane will be obtained.

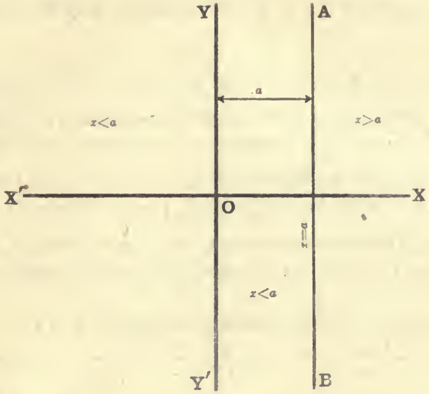
If, on the contrary, one or both of the coordinates cannot take all values, or if all values cannot be independently taken by both, the point cannot move to all positions in the plane.



If, for example, $x > 0$, the point (x, y) must lie to the *right* of the y-axis; if $x < 0$, the point must lie to the *left* of the y-axis; if x is neither *greater* nor *less* than zero, the point can lie neither to the *right* nor to the *left* of the y-axis; *i. e.*, if $x = 0$, the point must lie *on* the y-axis.

Ex. Where must the point (x, y) lie if $y > 0$? $y < 0$? $y = 0$?

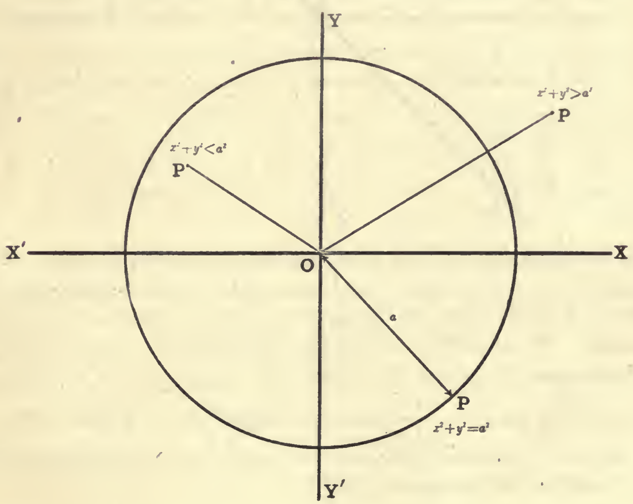
15. If $x > a$, the point (x, y) must lie to the *right* of the parallel AB, which is a units to the right of the y-axis; if $x < a$, the point must lie to the *left* of AB. Therefore, if $x = a$, the point will lie *on* the line AB.



Ex. 1. Where will the point (x, y) lie if $x > -3$? $x < -3$? $x = -3$?

Ex. 2. Where is the point (x, y) if $y > b$? $y < b$? $y = b$? $y > -b$? $y < -b$? $y = -b$?

16. Draw a circle with centre at the origin and radius equal to a.



Then the point $P(x, y)$ will be outside, inside, or on this circle according as

$$OP > a, \quad OP < a, \quad \text{or} \quad OP = a.$$

But $$OP^2 = x^2 + y^2. \qquad [(4), § 7.]$$

Therefore the point $P(x, y)$ is outside, inside, or on the circle according as

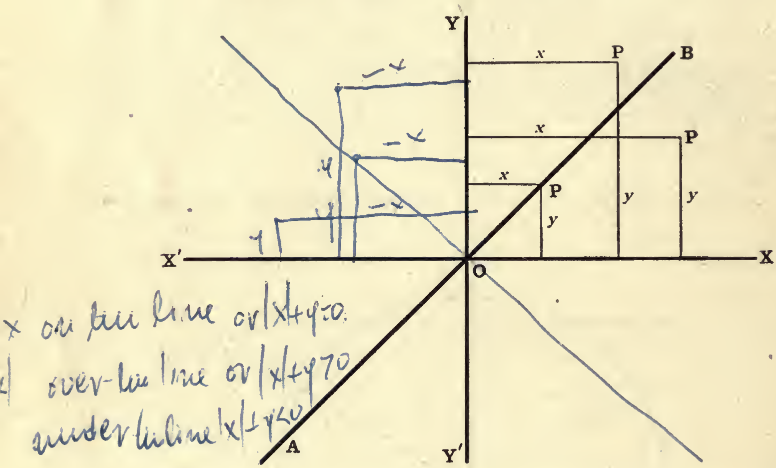
$$x^2 + y^2 > a^2, \quad x^2 + y^2 < a^2, \quad \text{or} \quad x^2 + y^2 = a^2.$$

Ex. 1. Write down the conditions that the point (x, y) shall be outside, inside, or on the circle whose centre is at the origin and radius 3.

Ex. 2. What are the conditions that the point (x, y) shall be outside, inside, or on a circle with centre at $(-3, 1)$ and radius 4?

Ex. 3. Draw a circle with centre at (a, b) and radius r, and write down the conditions that the point (x, y) shall be outside, inside, or on this circle.

17. Let the line AOB bisect the angle XOY.



Then every point on AB is equidistant from the axes. Hence the point $P(x, y)$ is *above* AB, *below* AB, or *on* AB according as

$$y > x, \quad y < x, \quad \text{or} \quad y = x,$$

or according as $\qquad y - x >, <, \text{ or } = 0;$

i. e. according as $y - x$ is positive, negative, or zero.

Ex. What are the conditions that the point (x, y) shall be above, below, or on the bisector of the angle $X'OY$?

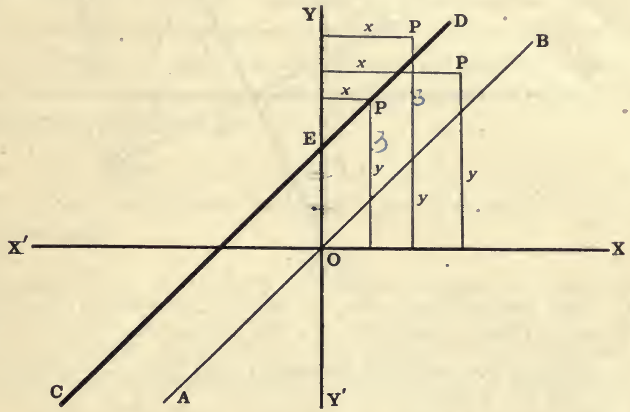
18. Draw CD parallel to AB, cutting the y-axis in E, three units above O.

Then every point on CD is three units farther from the x-axis than from the y-axis. Therefore the point $P(x, y)$ will be *above*

CD, below CD, or *on CD*, according as

$$y >, <, \text{ or } = x + 3;$$

i. e. according as $y - x - 3$ is positive, negative, or zero.



Ex. 1. Draw a line parallel to AB, cutting the y-axis two units below O; and write down the conditions that the point (x, y) shall be above, below, or on this line.

Ex. 2. What are the conditions that the point (x, y) shall be above, below, or on the line through E parallel to the bisector of the angle $X'OY$?

Ex. 3. Where is the point (x, y) if $y + x + 4 >, <, \text{ or } = 0$?

Ex. 4. Locate the point (x, y) if $y - 2x - 2 >, <, \text{ or } = 0$.

Ex. 5. Locate the point (x, y) if $2y + 3x - 1 >, <, \text{ or } = 0$.

19. Let CD be the perpendicular bisector of the line joining $A(-1, 1)$ and $B(3, -1)$.

Then all points on CD are equidistant from A and B, and all other points are not equally distant from A and B. Hence the point $P(x, y)$ will lie to the *right* of, to the *left* of, or *on CD* according as

$$AP >, <, \text{ or } = BP,$$

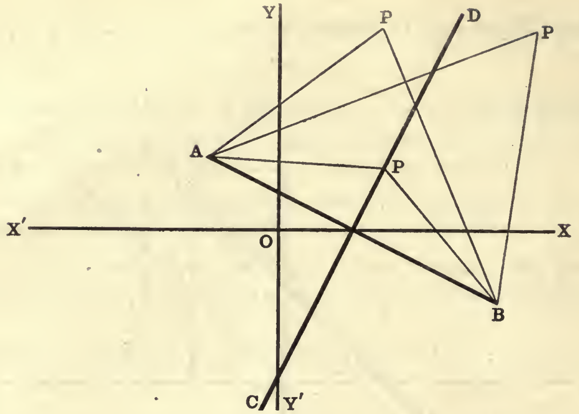
or according as

$$AP^2 >, <, \text{ or } = BP^2;$$

i. e. according as [(2), § 7]

$$(x+1)^2 + (y-1)^2 >, <, \text{ or } = (x-3)^2 + (y+1)^2;$$

whence $$2x - y - 2 >, <, \text{ or } = 0.$$



Ex. 1. Find the conditions that the point (x, y) shall be above, below, or on the perpendicular bisector of the line joining (2, 3) and (—1, —2).

Ex. 2. What is the condition that (x, y) shall be on the perpendicular bisector of the line joining (a, b) and $c, d)$?

20. The foregoing examples (§§ 14–19) illustrate certain general principles, of which we will here make only a preliminary statement.

I. All points whose coordinates satisfy an *equation of condition* (not an identity) lie on a certain line; and conversely, if a point lies on a fixed line, its coordinates must satisfy an *equation.*

II. Points whose coordinates satisfy a *condition of inequality* do not lie on any fixed line.

If $f(x, y)$ be used to represent any expression (which is not decomposable) containing the two variables x and y and certain constants, these principles may be stated more definitely, as follows:

I. All points whose coordinates make $f(x, y) = 0$, lie on a certain line; and conversely, the coordinates of all points on this line make $f(x, y) = 0$.

II. If $f(x_1, y_1) > 0$ and $f(x_2, y_2) < 0$, the two points (x_1, y_1) and (x_2, y_2) lie on *opposite* sides of the line the coordinates of whose points make $f(x, y) = 0$.

Hence every line, as well as the axes of coordinates, is said to have a *positive* and a *negative* side.

DEF. *The locus of a variable point subject to a given condition is the place, i. e. the totality of positions, where the point may lie and satisfy the given condition.*

DEF. *The line (or lines) containing all points, and no others, whose coordinates satisfy a given equation is called the* **Locus of the Equation**; *conversely, the equation satisfied by the coordinates of all points on a certain line (or lines) is called the* **Equation of the Line**, *or the* **Equation of the Locus.**

DEF. That part of the plane containing all points, and no others, whose coordinates satisfy a given *inequation* is the *Locus of the Inequation.*

Thus the *Locus* of a point in Plane Geometry is *not always a line.*

In the examples of §§ 14–19 only Cartesian coordinates have been used, but the fundamental principles there illustrated, and also the above definitions, hold for all systems of coordinates.

Let the student give some similar illustrations with polar coordinates.

EXAMPLES.

What is the locus of

1. $x^2+y^2=0$? $x^2+y^2>0$? $x^2+y^2<0$?

2. $x=\sqrt{x^2+y^2}$? $x>\sqrt{x^2+y^2}$? $x<\sqrt{x^2+y^2}$?

3. $\rho=a\sec\theta$? $\rho>a\sec\theta$? $\rho<a\sec\theta$?

4. $\rho=b\csc\theta$? $\rho>b\csc\theta$? $\rho<b\csc\theta$?

5. $4<x^2+y^2<9$?

6. $9<(x-2)^2+(y-3)^2<16$?

7. $a\sec\theta<\rho<b\sec\theta$?

8. $\rho=a\cos\theta$? $\rho>a\cos\theta$? $\rho<a\cos\theta$?

9. $a\cos\theta<\rho<b\cos\theta$?

10. $\rho=a\sin\theta$? $\rho>a\sin\theta$? $\rho<a\sin\theta$?

11. $\rho=a$? $\rho>a$? $\rho<a$?

12. What is the locus of a point moving so that the sum of its distances from the lines $x=0$ and $x=3$ is 1, 2, 3, 4?

To Find the Locus of a Given Equation.

21. If the locus of an equation is a straight line, the locus is easily drawn; it is only necessary to locate two points on it (preferably the intersections* with the axes) and draw a straight line through these points.

Likewise, if the locus is a circle, the complete locus can be drawn when the centre and radius are known.

It will be shown farther on that straight lines and circles can easily be recognized by the forms of the equations.

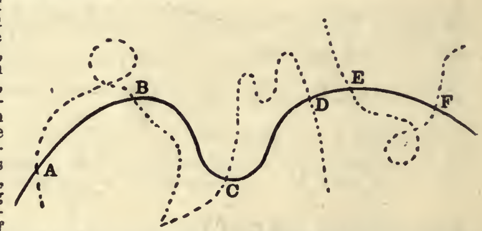
In general, having given an equation of condition between the coordinates (in any system) of a variable point, we may assign any value we please to one coordinate and find a corresponding† value, or values, of the other. To every such pair of corresponding values will correspond a definite point of the locus. Since these pairs of values may be as numerous as we please, we can in this way locate as many points of the locus as we please. A smooth curve drawn through these points will be an *approximation* to the locus of the given equation. The degree of approximation will depend upon the proximity of the points thus located. This method of constructing a locus is applicable to any equation that can be solved for one of the variables, and is called **Plotting‡ an Equation**, or **Plotting the Locus of an Equation.** The steps of this process are as follows:

---

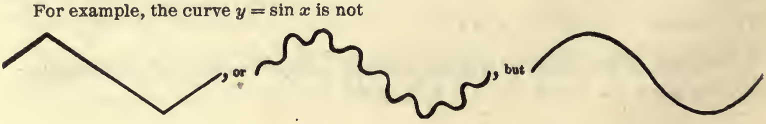
*Unless *both* intersections are near the origin, when the line will be inaccurately determined, or both at the origin, when its direction will be quite undetermined.

†"Corresponding values" of the variables, x and y say, involved in a given equation are a pair of values of x and y which satisfy the equation.

‡The logic of the process of plotting is that of induction, and should be so recognized by the student. Given the points A, B, C, D, E, F on a curve; then, in the absence of further knowledge, we take as a probable approximation a *smooth* curve drawn through them like the full curve in the figure. We are not warranted in drawing such a curve as the dotted one through the points, because it is unlikely that, taking points at random on such an irregular curve, the position of these points should fail to disclose any of the irregularity. The student should also be warned that sudden changes of slope or curvature are as unlikely as sudden changes in the value of an ordinate.



For example, the curve $y = \sin x$ is not



(1) Solve the equation with respect to *one* of the coordinates.

(2) Assign to the *other* coordinate a series of values differing but little from each other.

(3) Find each corresponding value, or values, of the *first* coordinate.

(4) Locate the point corresponding to each pair of corresponding values thus found.

(5) Join these points *in order* by a smooth curve, and this curve will be an approximation to the required locus. If there be doubt how to fill up any of the intervening spaces, more points must be interpolated.

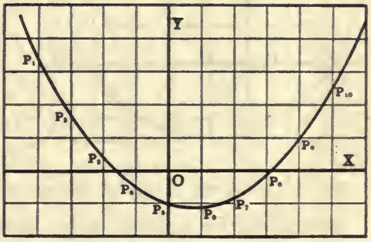
22. Illustrative Examples.

Ex. 1. *Plot the locus of the equation* $10y = x^2 - 3x - 20$.

Assigning to x values from -8 to $+10$, differing by two units, we find the following pairs of values of x and y to satisfy the equation:

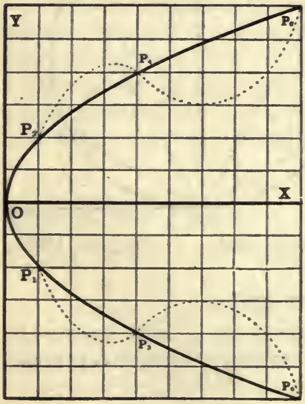
$x =$	-8	-6	-4	-2	0
$y =$	6.8	3.4	$.8$	-1	-2

$x =$	2	4	6	8	10
$y =$	-2.2	-1.6	$-.2$	2	5



Plotting the corresponding points P_1, P_2, P_3, etc., and drawing a smooth curve through them in the order of the increasing values of x, we find the locus to be approximately the curve drawn in the figure.

Ex. 2. *Plot the locus of the equation* $y^2 = 4x$.



Solving for y gives $y = \pm 2\sqrt{x}$.

When $x = 0, 1, 4, 9, \ldots$ to ∞,

$y = 0, \pm 2, \pm 4, \pm 6 \ldots$ to $\pm \infty$,

respectively.

The corresponding points of the locus are $O(0, 0)$, $P_1(1, -2)$, $P_2(1, 2)$, $P_3(4, -4)$, $P_4(4, 4)$, $P_5(9, -6)$ and $P_6(9, 6)$. . . .

When x is negative, y is imaginary. Therefore no points of the locus lie to the left of the y-axis. For every positive value of x there are two values of y numerically equal but opposite in sign. Hence the two corresponding points of the locus are equidistant from the x-axis. As x increases, both values of y increase numerically.

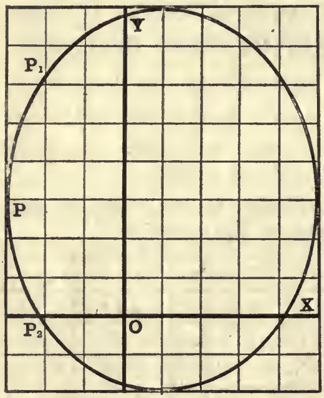
Therefore the locus can not be such a curve as that represented by the dotted line, but must be approximately that indicated by the full line.

Ex. 3. *Plot the locus of the equation* $25(x-1)^2+16(y-3)^2=400$.

Solving this equation for y gives

$$y=3\pm\tfrac{5}{4}\sqrt{16-(x-1)^2}.$$

This form of the equation shows that y is imaginary when $x<-3$, or $x>5$, since $16-(x-1)^2$ is then negative; and when x is neither less than -3 nor greater than 5 there are two real unequal values of y, one found by using the $+$ sign before the radical, the other found by using the $-$ sign. Hence the locus lies between the two parallel lines $x=-3$ and $x=5$.



The equation is satisfied by the following pairs of values of x and y:

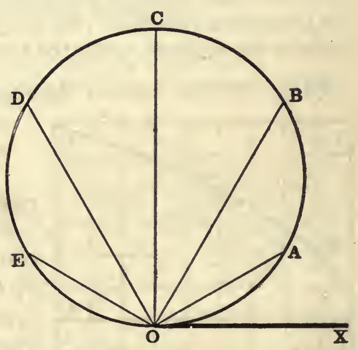
$x=$	-3	-2	-1	0
$y=$	3	6.3	7.3	7.8
$y=$	3	$-.3$	-1.3	-1.8

$x=$	1	2	3	4	5
$y=$	8	7.8	7.3	6.3	3
$y=$	-2	-1.8	-1.3	$-.3$	3

The corresponding points are $P(-3, 3)$, $P_1(-2, 6.3)$, $P_2(-2, -.3)$, etc., and the locus is approximately as shown in the figure.

Ex. 4. *Plot the locus of the equation* $\rho=2a\sin\theta$.

Here ρ has its greatest value when $\sin\theta$ has its greatest value, *i. e.* when $\theta=\frac{1}{2}\pi$. As ρ increases from 0 to $\frac{1}{2}\pi$, $\sin\theta$ increases from 0 to 1, and ρ increases from 0 to $2a$; as θ increases from $\frac{1}{2}\pi$ to π, $\sin\theta$ decreases from 1 to 0, and ρ decreases from $2a$ to 0. Hence the locus starts from the origin and returns to the origin as θ is made to vary from 0 to π.



Assigning to θ values from 0 to 180°, differing by 30° we find the following points are on the locus:

$O(0, 0)$, $A(a, 30°)$, $B(a\sqrt{3}, 60°)$, $C(2a, 90°)$, $D(a\sqrt{3}, 120°)$, $E(a, 150°)$, and $O(0, 180°)$.

The complete locus is the curve shown in the figure.

Ex. *a*. Show that the points $A, B, \ldots$ all lie on a circle tangent to OX at O and whose radius is a. Show also that every point on this circle satisfies the given equation.

Ex. *b*. Show that the same circle will be described as θ varies from 180° to 360°; also as θ varies from any value a to $a + \pi$.

We have in this example an illustration of a characteristic property of equations in polar coordinates containing a periodic function of θ. In such equations ρ takes all possible values as θ varies through a limited range of values called the period of the function. The complete locus is described *at least once* as θ varies through this period, and is repeated as θ varies through any other equal period.

The period of sin θ is 2π; hence in the above equation ρ takes all possible values from $-2a$ to $+2a$ as θ varies from 0 to 2π. The whole circle is described *twice* as θ varies through this period, *once* as θ varies from 0 to π with ρ positive, and *once* as θ varies from π to 2π with ρ negative. Also the whole circle is described twice if θ starts from any value and varies through 2π in either direction.

Ex. 5. *Plot the locus of the equation* $\rho = \sin 2\theta$.

This equation is satisfied by the following pairs of values of ρ and θ:

$\theta = 45°, 225°, \quad \rho = 1.$

$\theta = 135°, 315°, \quad \rho = -1.$

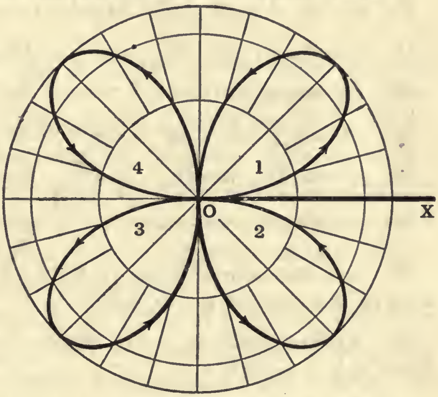
$\theta = 30°, 60°, 210°, 240°,$
$\rho = \frac{1}{2}\sqrt{3}.$

$\theta = 120°, 150°, 300°, 330°,$
$\rho = -\frac{1}{2}\sqrt{3}.$

$\theta = 15°, 75°, 195°, 255°,$
$\rho = \frac{1}{2}.$

$\theta = 105°, 165°, 285°, 345°,$
$\rho = -\frac{1}{2}.$

$\theta = 0°, 90°, 180°, 270°, 360°,$
$\rho = 0.$



The corresponding points are found by drawing three circles with centres at O and radii $\frac{1}{2}$, $\frac{1}{2}\sqrt{3}$, and 1, and then drawing radii dividing these circles into arcs of 15°.

The locus is the four-leaf curve shown in the figure.

As θ varies from 0 to 2π, the four leaves are described in the order 1, 2, 3, 4, and in the direction indicated by the arrow heads.

EXAMPLES.

Plot the loci of the following equations:*

1. $\left\{\begin{array}{l} 2x - 3y - 6 = 0. \\ 4x - 6y - 6 = 0. \\ 6x - 9y + 27 = 0. \end{array}\right\}$ †

2. $\left\{\begin{array}{l} 2x + 3y + 5 = 0. \\ 3x - 2y - 12 = 0. \\ 5x + 2y - 4 = 0. \end{array}\right\}$

* For convenience in plotting loci in rectangular coordinates the student should be supplied with "coordinate paper."

† Loci grouped under the same number should be plotted on the same diagram.

3. $\begin{cases} 2x+9y+13=0. \\ y=7x-3. \\ 2y-x=2. \end{cases}$

4. $(x-4)(y+3)=0.$

5. $(x^2-4)(y-2)=0.$

6. $4x^2-y^2=0.$

7. $6x^2+5xy-6y^2=0.$

8. $\begin{cases} x^2+y^2=4. \\ x^2-y^2=4. \end{cases}$

9. $\begin{cases} x^2+y^2=25. \\ (x-8)^2+(y-4)^2=25. \\ (x-4)^2+(y-2)^2=5. \end{cases}$

10. $\begin{cases} 4(x+1)=(y-2)^2. \\ 10y=(x+1)^2. \end{cases}$

11. $y=x^3-4x^2-4x+16.$

12. $\begin{cases} y=x^4-20x^2+64. \\ y^2=x^4-20x^2+64. \end{cases}$

13. $\begin{cases} y=(x^2-4)^2. \\ y^2=(x^2-4)^2. \end{cases}$

14. $(x^2+y^2)^2=a^2(x^2-y^2).$

15. $y=x,\ x^2,\ x^3,\ x^4,\ x^5 \ldots x^n$. What points are common to these curves? Consider the case $n=\infty$.

16. $y=(x-1),\ (x-1)^2,\ (x-1)^3 \ldots$ Compare with No. 15.

17. $y^2=x,\ x^2,\ x^3,\ x^4.$

18. $y=\sin x,\ \cos x.$

19. $y=\tan x,\ \cot x.$

20. $y=\sec x,\ \csc x.$

21. $\rho=\sin\theta,\ \cos\theta,\ \sec\theta,\ \csc\theta.$

22. $\rho=\sin 3\theta,\ \sin 4\theta.$

23. $\rho=\cos 2\theta,\ \cos 3\theta,\ \cos 4\theta.$

24. $\rho=\tan\theta,\ \cot\theta.$

25. $\rho=\sin\frac{1}{2}\theta,\ \cos\frac{1}{2}\theta.$

26. $\rho=\dfrac{2}{1-\cos\theta},\ \dfrac{6}{3-2\cos\theta}.$

27. Are the points (2, 9), (1, 5), (—1, —4) on the same or opposite sides of the locus of $y-3x=2$?

28. Are the points (9, — 10), (5, 12), (—8, 10) on, inside, or outside the circle $x^2+y^2=169$?

29. Are the points (3, 60°), ($\frac{3}{2}$, —90°) on the same or opposite sides of the loci of Ex. 26?

30. Which of the loci represented by the following equations pass through the origin?

(1) $2x+3y=0.$

(2) $x^2+y^2=1.$

(3) $y=3x^2-x.$

(4) $y^2-a^2x^2=0.$

(5) $ax+by+c=0.$

(6) $ax^2+by^2=1.$

(7) $y^2=4ax.$

(8) $y^2=4a(x+a).$

(9) $(x-a)^2+(y-b)^2=a^2-b^2.$

What is the necessary and sufficient condition that the locus of an equation in Cartesian coordinates shall pass through the origin?

Plot the following loci:

31. $\rho^2=\sin 2\theta,\ \cos 2\theta.$

32. $\rho^2=\sec 2\theta,\ \csc 2\theta.$

33. $y = \frac{x-2}{x-3}, \quad \frac{(x-1)(x-2)}{x-3.}$.

34. $y = \frac{(x-1)(x-2)}{(x-3)(x-4)}, \quad \frac{(x-1)(x-3)}{(x-2)(x-4)}$.

35. $y = \frac{x+2}{x+3}, \quad \frac{(x-1)(x-3)}{(x-2)}$.

36. $y = \frac{(x+1)(x-2)}{(x+3)(x-4)}, \quad \frac{(x+2)(x-4)}{(x-1)(x-3)}$.

37. $y = \frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}, \quad \frac{(x+1)(x-4)(x-6)}{(x-1)(x+2)(x-3)}$.

38. $y = \frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)}, \quad \frac{(x-1)(x+3)(x-5)}{(x-2)(x-4)}$.

The Use of Graphic Methods.

23. It has been shown in §§ 14–20 that whenever the relation between two quantities, whose values depend upon one another, can be definitely expressed by an equation, then the geometric or *graphic* representation of this relation is given by means of a curve. Such a curve often gives at a glance information which otherwise could be obtained only by considerable computation; and in many cases reveals facts of peculiar interest and importance which might otherwise escape notice.

The use of graphic methods in the study of physics, analytical mechanics, and engineering, as well as in many other branches of scientific investigation, is already extensive and is rapidly increasing. Graphic methods can be used, however, not only in examples where the equation connecting the two variable quantities is known, such as those already given, but also in examples where no such relation can be found; in these latter cases the graphic method furnishes almost the only practical means of studying the relations involved.

Comparative statistics, and results of experiments and direct observations, can frequently be more concisely and forcibly represented graphically than by tabulating numerical values. The following are simple examples of this kind:

1. The following table shows the net gold (to the nearest million of dollars) in the U. S. Treasury at intervals of one month, from Jan. 10, 1893 to Oct. 31, 1894 (Report of the Sec. of the Treas., 1894, p. 8):

1893.	Millions of Dollars.	1893.	Millions of Dollars.	1894.	Millions of Dollars.	1894.	Millions of Dollars.
Jan. 10..	120	July 10..	97	Jan. 10..	74	July 10..	65
Feb. 10..	112	Aug. 10..	103	Feb. 10..	104	Aug. 10..	52½
Mar. 10..	102	Sept. 9..	98	Mar. 10..	107	Sept. 10..	56
Apr. 10..	106	Oct. 10..	87	Apr. 10..	106	Oct. 10..	60
May 10..	99	Nov. 10..	85	May 10..	92	Oct. 31..	61
June 10..	91	Dec. 9..	84	June 9..	69	....	....

Using time (in months) as abscissas, and dollars (1,000,000 per unit) as ordinates, the separate points represented by the table have been plotted (Fig. 1) and then joined by a smooth curve.

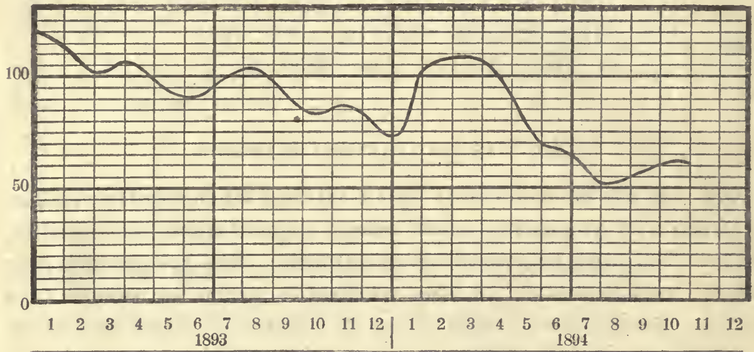


FIG. 1.

In this example the curve gives no *new* information, but it presents in a much more concise form the information given by the tabulated numbers. Observe also that *if the points are inaccurately located, the diagram becomes not only worthless, but misleading.*

2. An excellent example of the use and advantages of the graphic method of representing comparative statistics is found in the large engraved plate placed under the front cover of the Annual Report of the Secretary of the Treasury for 1894. This plate presents on a single sheet information that covers several pages when expressed in tabulated numbers. All of the curves given on this plate, except one, are shown (on a smaller scale) in Fig. 2. This figure should be carefully studied, and if possible the original plate should be consulted.

3. The curves in figures 1 and 2 were constructed by locating separate points and then drawing a smooth curve through these isolated points. Such curves give no *new* information, but only represent graphically information already ascertained.

In some cases, however, curves can be drawn mechanically. When this is possible the curve is constructed, not for the purpose of exhibiting facts previously known, but for the purpose of *obtaining new information.* For instance, in the stations of the U. S. Weather Bureau an instrument called

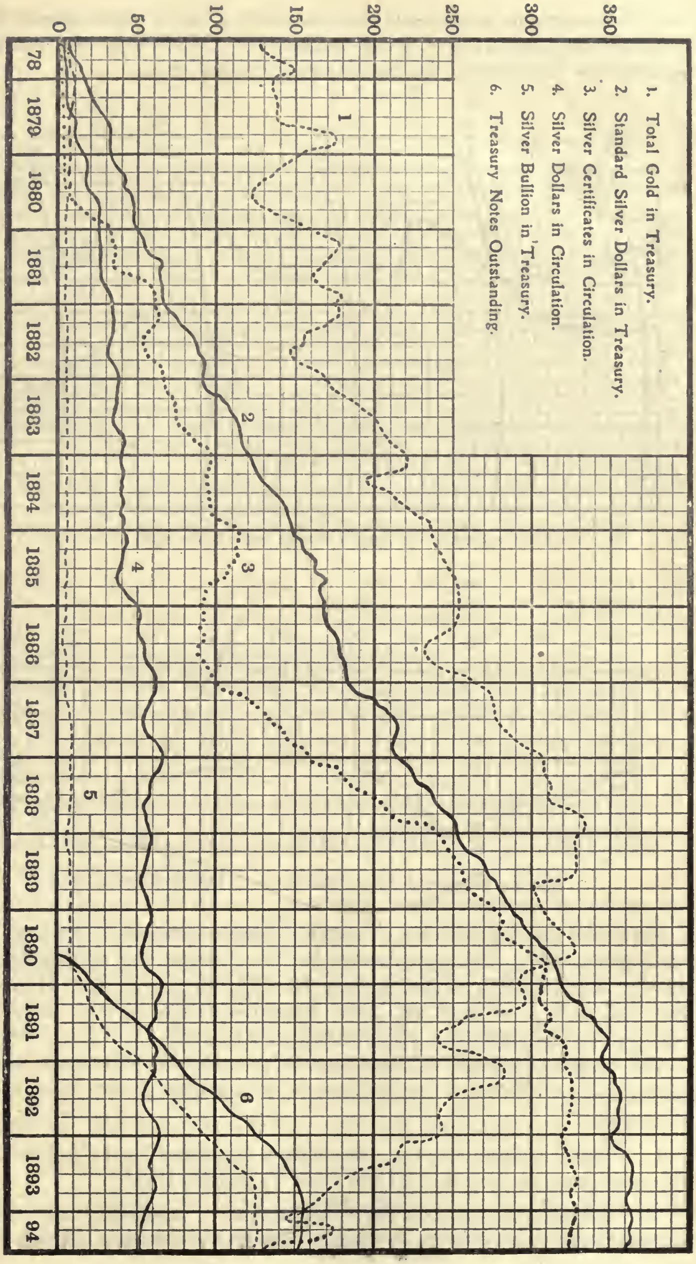


FIG. 2.—Gold and Silver in the Treasury of the United States, and Circulation of Silver and Silver Certificates, at the end of each quarter, in millions of dollars. (Report of the Secretary of the Treasury, 1894. Plate under front cover.)

the Thermograph* constructs automatically a curve which shows the continuous variation of the local temperature. Similarly the Barograph* records the variation of the barometric pressure, etc.

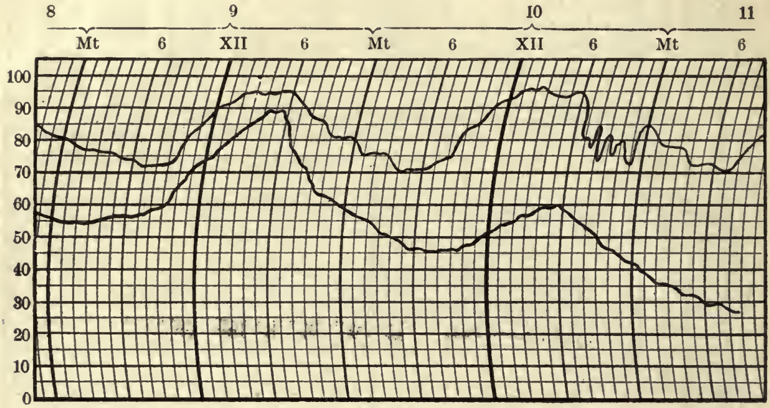


FIG. 3.—Thermographs for Aug. 9–10 and Sept. 27–28, 1899, at Lincoln, Neb.

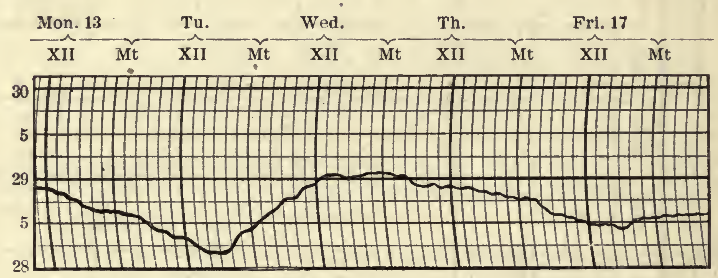


FIG. 4.—Barograph Sheet, March 13–17, 1899, at Lincoln, Neb.

Figures 3 and 4 are copies of curves thus constructed in the local station at Lincoln, Neb. The upper curve in Fig. 3 shows the temperature from 10 P. M. Aug. 8, 1899, to 9 A. M. Aug. 11, 1899; the lower from 11 P. M. Sept. 26, 1899, to 8 A. M. Sept. 29, 1899. Interpret these curves. Notice especially the record from 6 P. M. to midnight Aug. 10.

The varying pressure on the piston in the cylinder of a steam engine is determined in the same way by means of a similar instrument, called an Indicator.*

4. Exhibit graphically the information contained in the following table of wind velocities for Jan. 20 and June 15 and 25, 1894:

* For a description and cut of the "Thermograph," "Barograph," and "Indicator," see these words in the Century, Standard, or Webster's International Dictionary.

Day.	12–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9	9–10	10–11	11–12
Jan. 20, A. M. . .	3	6	7	7	8	9	12	15	15	19	12	21
June 15, A. M. . .	15	11	8	10	8	9	3	3	11	15	17	21
June 25, A. M. . .	17	14	13	13	11	23	23	13	9	4	2	10
Jan. 20, P. M. . .	22	22	18	19	14	9	6	7	5	6	5	4
June 15, P. M. . .	15	21	22	20	17	17	12	5	5	6	6	3
June 25, P. M. . .	12	15	11	12	12	5	1	3	6	7	7	3

Intersection of Loci.

24. *To find the points of intersection of two loci when their equations are known.*

Since the points of intersection of two loci lie on both curves, their coordinates must satisfy both equations. Therefore, to find the coordinates of the points of intersection of two loci we treat their equations simultaneously, regarding the coordinates as the unknown quantities, and thus find the values of the coordinates which satisfy both equations. A pair of values which satisfy both equations are the coordinates of a point of intersection of the two loci.

If the equations are both of the first degree, there will be but one pair of values of coordinates satisfying them, and therefore but one point of intersection of the loci.

If one or both of the equations be of a higher degree than the first, there will be several pairs of roots, and *one* point of intersection for *each pair*. The loci will then have several points of intersection.

If of a pair of roots *even one* is imaginary, there is no corresponding real point common to the two loci. We then say the intersection is imaginary.

Since imaginary roots of equations always occur in pairs, imaginary intersections of loci always occur in pairs; and hence the passage from a real pair of intersections to an imaginary one is through a coincident pair. Suppose, for example, a straight line intersects a circle in two real points. If the line be moved so that it becomes tangent to the circle, the two points of intersection coincide in the point of contact. If the line be moved still farther, the intersections are said to become imaginary.

25. *Intercepts on the axes of coordinates.*

This is a special and very important case of the preceding section in which one of the given equations is $x = 0$, or $y = 0$.

To find the points of intersection of a curve with the x-axis, put $y = 0$ in the equation of the curve and solve the resulting equation for x. The roots of this equation in x represent the distances from the origin to the points of intersection; and these distances are called the **x-intercepts** of the given curve.

Similarly, to find the **y-intercepts**, put $x = 0$ in the given equation and solve the resulting equation for y.

Ex. 1. How many x-intercepts may a curve of the nth degree have?

Ex. 2. What does it mean when in an equation in polar coordinates we put $\theta = 0$? $\rho = 0$?

26. A line may be defined as the path of a moving point. Then, if (x, y) be the moving point, both x and y are *variable* quantities, and are called the **variable** or **current coordinates** of the moving point. The path of the moving point is then determined by the condition that its coordinates must vary only in such a manner as always to satisfy a given equation; *i. e.* although *both coordinates vary the relation between them remains fixed.*

EXAMPLES.

Find the intercepts and the points of intersection of the following loci:

1. $2x + 3y = 12, \quad 4x - y = 5.$
2. $3x + 5y = 1, \quad x - 3y + 7 = 0.$
3. $5x - 2y + 4 = 0, \quad 10x - 4y + 3 = 0.$
4. $x + 3y = 15, \quad x^2 + y^2 = 25.$
5. $3x - 4y = 20, \quad x^2 + y^2 - 10x - 10y + 25 = 0.$
6. $5x + 4y = 20, \quad x^2 + y^2 = 4.$
7. $x - 3y = 0, \quad x^2 + y^2 + 20y = 0.$
8. $y^2 = 4ax, \quad 2xy = a^2.$
9. $y^2 = 4ax, \quad y^2 - x^2 = a^2.$

10. Find the points of intersection of the loci of Nos. 1, 2, 3, 9, 15, 17, 18, 19, 20, 21, 26 in the last preceding set of examples.

11. Find the intercepts of the loci of Nos. 7, 9, 10, 11, 12, 13, 14, 18, 19, 20 of the same set and check the results by the plots already made.

12. Find the area of the triangle whose sides are $x - 3y + 5 = 0$, $3x + 4y = 11$, $2x + 7y = 3$.

13. What is the area of the quadrilateral whose sides are $x = a$, $y = b$, $bx + ay = 0$, and $bx + ay = ab$?

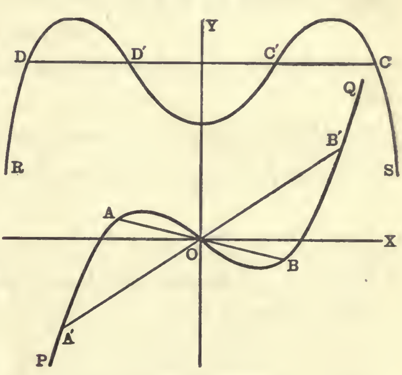
Symmetry of Loci.

27. The process of constructing a locus explained in § 21 is long and tedious. It may often be shortened by an examination of the peculiarities of the given equation, such as the limiting values of the variables for which both are real (see Ex. 3, § 22), symmetry, etc. Such considerations will often reveal the general form and limits of the curve and give all the information desired with little labor. The intercepts (§ 25) are almost always useful for this purpose.

Definitions. Two points A and B are said to be *symmetrical with respect to a centre* O when the line AB is bisected by O.

Two points A and B are said to be *symmetrical with respect to an axis* when the line AB is bisected at right angles by the axis.

The two points (x, y) and $(-x, -y)$ are symmetrical with respect to the origin; (x, y) and $(x, -y)$ with respect to the x-axis.



A curve is said to be *symmetrical with respect to a centre* O when all lines passing through O meet the curve in a pair, or pairs, of points symmetrical with respect to O.

A curve is said to be *symmetrical with respect to an axis* when all lines perpendicular to the axis meet the curve in a pair, or pairs, of points symmetrical with respect to the axis.

Or, in other words, a curve is symmetrical with respect to an axis, if the figure appears the same when a plane mirror is placed on the axis perpendicular to the plane of the curve.

The curve PQ is symmetrical with respect to the origin, and RS is symmetrical with respect to the y-axis.

28. *Equations in Cartesian Coordinates.*

(1) *If* $f(x, y) \equiv f(x, -y)$,* *the locus of the equation*

$$f(x, y) = 0$$

is symmetrical with respect to the x-axis; i. e.

If an equation is not altered when the sign of y is changed, its locus is symmetrical with respect to the x-axis.

Let (x', y') be any point on the locus $f(x, y) = 0$.

Then, since $f(x, y) \equiv f(x, -y)$, by hypothesis,

$$f(x', y') = f(x', -y') = 0.$$

That is, the point $(x', -y')$ is also on the locus. Therefore, if the line $x = x'$ meets the locus in any point (x', y'), it will also meet the locus in the symmetrical point $(x', -y')$, and the curve is symmetrical with respect to the x-axis.

Ex. Let $f(x, y) = y^2 - 4x$, then $f(x, -y) = (-y)^2 - 4x = y^2 - 4x$.

Therefore $f(x, y) \equiv f(x, -y)$ and the curve $y^2 - 4x = 0$ is symmetrical with respect to the x-axis. (See Ex. 2, § 22.)

(2) *Similarly, if* $f(x, y) \equiv f(-x, y)$ *the locus of*

$$f(x, y) = 0$$

is symmetrical with respect to the y-axis.

Ex. $$y - \cos x \equiv y - \cos(-x).$$

Therefore the locus of $y = \cos x$ is symmetrical with respect to the y-axis.

(3) *If* $f(x, y) \equiv \pm f(-x, -y)$ *the locus of*

$$f(x, y) = 0$$

is symmetrical with respect to the origin.

Let (x', y') be any point on the locus $f(x, y) = 0$.

Then, since $f(x, y) \equiv \pm f(-x, -y)$ by hypothesis,

$$f(x', y') = f(-x', -y') = 0.$$

Hence the straight line through the origin and the point (x', y') meets the locus again in the symmetrical point $(-x', -y')$. Therefore the curve is symmetrical with respect to the origin.

Ex. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \equiv \frac{x^2}{a^2} + \frac{(-y)^2}{b^2} - 1 \equiv \frac{(-x)^2}{a^2} + \frac{y^2}{b^2} - 1$$

$$\equiv \frac{(-x)^2}{a^2} + \frac{(-y)^2}{b^2} - 1.$$

---

*The sign "≡" means "identical with," i. e. the same for all values of x and y, and therefore that the two expressions vanish for the same values of x and y.

E. g. $(x + y)^2 \equiv x^2 + 2xy + y^2$, $\cos x \equiv \cos(-x)$.

Therefore the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is symmetrical with respect to both axes and the origin. (See Fig. of § 34.)

(4) *If* $f(x, y) \equiv f(y, x)$ *the locus of* $f(x, y) = 0$ *is symmetrical with respect to the bisector of the angle* XOY.

(5) *If* $f(-x, y) \equiv f(-y, x)$ *the locus of* $f(x, y) = 0$ *is symmetrical with respect to the bisector of the angle* $X'OY$.

Let the student prove propositions (4) and (5).

The foregoing conditions of symmetry are both *necessary* and *sufficient; i. e.* if either one of the conditions (3), for example, is satisfied, the locus is symmetrical with respect to the origin, otherwise not. Let the student examine the opposite propositions.

The following conditions, (6), (7), (8), are *sufficient*, but not *necessary:*

(6) *If an equation contains only even powers of* y, *its locus is symmetrical with respect to the* x*-axis.*

(7) *If an equation contains only even powers of* x, *its locus is symmetrical with respect to the* y*-axis.*

(8) *If an equation contains only even powers of both* x *and* y, *its locus is symmetrical with respect to both axes and also with respect to the origin.*

In an algebraic* equation either one of the following conditions is *sufficient*, and one or the other is *necessary.*

(9) *If all the terms of an algebraic equation are of even degree, or if all the terms are of odd degree, its locus is symmetrical with respect to the origin.*

Show that (6), (7), (8), and (9) follow from (1), (2), and (3).

Show that (6), (7), (8) are *necessary* conditions of symmetry if the equation is algebraic.

---

* A function in which the variables are involved in no other ways than by addition, subtraction, multiplication, division, and root extraction is called an *Algebraic Function.* All others are called *Transcendental Functions.*

E. g. $3x^2 - 2x + 4$, $x^2 - axy + by^2$, $\frac{ax^3 + by^2}{x + c} + n\sqrt{xy}$,

are algebraic functions; while a^x, $\sin x$, $\sec^{-1} y$, $\log (x^2 + y)$ are transcendental functions.

29. *Equations in Polar Coordinates.*

It has been shown in the first chapter that for all values of θ

$$(\rho, -\theta) \equiv (-\rho, \pi - \theta); \ (\rho, \pi - \theta) \equiv (-\rho, -\theta);$$
$$(\rho, \pi + \theta) \equiv (-\rho, \theta).$$

From the definitions of symmetry it follows that, for all values of θ,

(ρ, θ) and $(\rho, -\theta)$, or $(\rho, \pi - \theta)$ are symmetrical with respect to OX;

(ρ, θ) and $(\rho, \pi - \theta)$, or $(-\rho, -\theta)$ are symmetrical with respect to OY;*

(ρ, θ) and $(\rho, \pi + \theta)$, or $(-\rho, \theta)$ are symmetrical with respect to O.

Also from the definition of symmetrical curves we may say that *a curve is symmetrical with respect to a centre, or an axis, when every point on the curve has its symmetrical point with respect to the center, or axis, on the curve.*

Hence the following are *sufficient* conditions of symmetry for loci in polar coordinates:

(1) *If* $f(\theta) \equiv f(-\theta)$, *or, if* $f(\theta) \equiv -f(\pi - \theta)$, *the locus of* $\rho = f(\theta)$ *is symmetrical with respect to* OX.

For, in the first case, the value of ρ is the same when $\theta = \theta'$ as when $\theta = -\theta'$; and in the second case, the values of ρ corresponding to $\theta = \theta'$ and $\theta = \pi - \theta'$ differ only in *sign*.

Hence in either case, if any point (ρ', θ') is on the locus, its symmetrical point $(\rho', -\theta')$ is also on the locus; therefore the locus is symmetrical with respect to OX.

(2) *Similarly, if* $f(\theta) \equiv f(\pi - \theta)$, *or, if* $f(\theta) \equiv -f(-\theta)$, *the locus of* $\rho = f(\theta)$ *is symmetrical with respect to* OY.

(3) *If* $f(\theta) \equiv f(\pi + \theta)$, *the locus of* $\rho = f(\theta)$ *is symmetrical with respect to* O.

(4) *The locus of* $\rho^2 = f(\theta)$ *is symmetrical with respect to* O.

E. g. The locus $\rho = \cos\theta$ is symmetrical with respect to OX.

The locus $\rho = 2a \sin\theta$ is symmetrical with respect to OY. (See Ex. 4, § 22.)

---

* OY is assumed perpendicular to the initial line OX.

The locus $\rho = \sin 2\theta$ is symmetrical with respect to OX, OY, and O. (See Ex. 5, § 22.)

Show that

(5) *If* $f(45° + \theta) \equiv f(45° - \theta)$, *the locus of* $\rho = f(\theta)$ *is symmetrical with respect to the line* $\theta = 45°$.

(6) *If* $f(135° + \theta) \equiv f(135° - \theta)$, *the locus of* $\rho = f(\theta)$ *is symmetrical with respect to the line* $\theta = 135°$.

Are the conditions (5) and (6) satisfied by the equation of the locus shown in Ex. 5, § 22?

EXAMPLES.

In what respects are the loci of the following equations symmetrical?

1. $y = x^2$.
2. $y^2 = x$.
3. $y = x^3$.
4. $y^2 = x^3$.
5. $y^2 = x^2$.
6. $y^2 = x^4$.
7. $y^2 = x^6$.
8. $y^4 = x^6$.
9. $y = x^3 - x$.
10. $y = x^4 - x^2$.
11. $xy = a$.
12. $ax^2 + by^2 = 1$.
13. $ax^2 + 2bxy + cy^2 = 1$.
14. $ax^2 + 2bxy + ay^2 = 1$.
15. $axy + b(x + y) = c$.
16. $x^3 + y^3 = 1$.
17. $x^4 + y^4 = 1$.
18. $x^4 = y^2(4a^2 - x^2)$.
19. $x(y + x)^2 + a^2y = 0$.
20. $x^2y^2 = a^2(x^2 + y^2)$.
21. $xy^2 + 4(x + y) = 0$.
22. $x^{2/3} + y^{2/3} = a^{2/3}$.
23. $(a - x)y^2 = (a + x)x^2$.
24. $(a - x)y^2 + x^3 = 0$.
25. $\rho^2 = \sin 2\theta$.
26. $\rho^2 = \cos 2\theta$.

27. Point out the symmetric properties of the loci in the last two preceding sets of examples, especially those given in polar coordinates. Check the results by referring to the plots already made.

28. Show that if an equation is not altered when $-x$ is written in the place of y, and y in the place of x, its locus will show no change in position when the curve is turned about the origin through a right angle in its plane.

For example, the locus of the equation

$$x^4 + a^2xy - y^4 = 0$$

is such a curve.

To Find the Equation of a Locus, having given its Geometric Definition.

30. It should be borne in mind that to find the equation of a locus we have merely to find an equation that is satisfied by the coordinates of every point on the locus, and not satisfied by the coordinates of any other point. It is not easy to give specific directions which can be applied in all cases, but the following plan will be useful to the beginner, at least in the simpler cases:

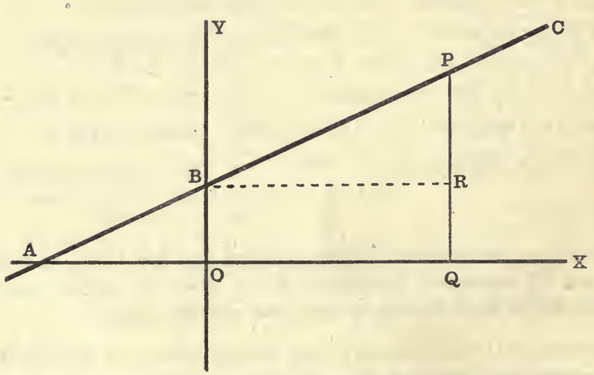
(1) Choose the system of coordinates best adapted to the locus under consideration, and select a convenient set of axes.

(2) Write down the *geometric equation* which expresses the given geometric definition, or any known geometric property of the locus.

(3) Express this geometric equation in terms of the chosen system of coordinates, and simplify the result.

The following examples will give a better idea of the method of procedure than any formal rules; they should be carefully studied:

31. *To find the equation of any straight line.*



Let ABC be any straight line meeting the axes in A and B.
Let $OB = b$, let $\tan XAC = m$.
Let $P(x, y)$ be any point on the line.
Draw PQ parallel to OY, and BR parallel to OX.

Then for the geometric equation we have

$$QP = QR + RP = OB + BR \tan PBR.$$

But $\quad QP = y, \quad OB = b, \quad BR = x, \quad \tan PBR = m.$

$$\therefore \quad y = mx + b, \tag{1}$$

which is the required equation.

For any particular straight line the quantities m and b remain the same, and are therefore called constants. Of these, m, the tangent of the angle between the line and the x-axis, is called the **Slope** of the line, while b is the y-intercept.

By giving suitable values to the constants m and b, (1) may be made to represent any straight line whatever, *e. g.*

If $b = 0$, we have

$$y = mx, \tag{2}$$

for the equation of *any* line through the origin.

Quantities entering into an equation, such as m and b, which remain constant so long as we consider any particular curve, but whose variation causes a change in the position, size, or shape of the curve, are called **Parameters** of the curve.*

Moreover, any equation that can be put in the form (1), *i. e. y equals some multiple of x plus a constant*, represents a straight line.

The general equation of the first degree

$$Ax + By + C = 0 \tag{3}$$

may be written $\quad y = -\frac{A}{B}x - \frac{C}{B},$

and therefore (3) represents a straight line whose slope is $-\frac{A}{B}$ and whose y-intercept is $-\frac{C}{B}$. (See § 43.)

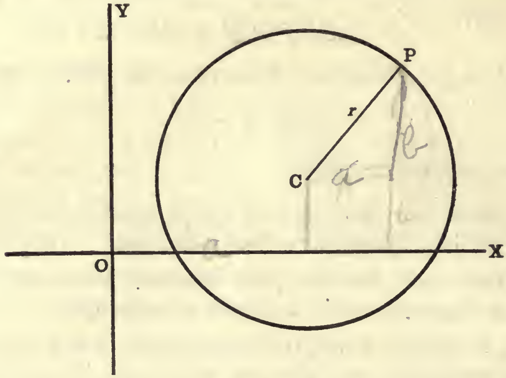
Ex. 1. If b varies in (1) while m remains constant, how will the line change position? If m varies while b remains constant? If m varies in (2)?

Ex. 2. What will be true of the signs of m and b when the line crosses the various quadrants?

---

* The difference between *parameters* and *coordinates* should be carefully noted; also the difference in the effect of a variation of the parameters of an equation and the variation of the current coordinates. (See § 26.)

32. *To find the equation of a circle referred to any rectangular axes.*



Let $r =$ radius, and let $C(a, b)$ be the centre.

Let $P(x, y)$ be any point on the circle.

Then $$CP = r. \quad \text{[Geometric equation.]}$$

But $$CP^2 = (x-a)^2 + (y-b)^2. \quad \text{[(2), § 7.]}$$

$$\therefore \quad (x-a)^2 + (y-b)^2 = r^2 \tag{1}$$

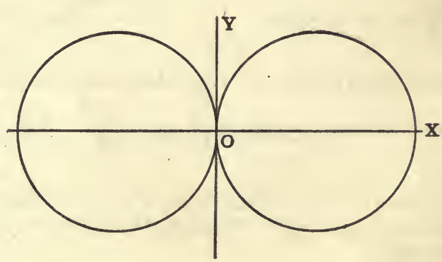
is the required equation.

If $a = r$ and $b = 0$, (1) reduces to

$$x^2 + y^2 - 2rx = 0. \tag{2}$$

If $a = -r$ and $b = 0$, (1) becomes

$$x^2 + y^2 + 2rx = 0. \tag{3}$$



The circle at the right in the figure is the locus of equation (2); the circle at the left is the locus of equation (3).

When the centre is at the origin, $a = b = 0$, and we have for the simplest equation of the circle in Cartesian coordinates the standard form (§ 16),

$$x^2 + y^2 = r^2. \tag{4}$$

Ex. 1. What is the form of the equation and the position of the circle, if $b = \pm r$ and $a = 0$?

Ex. 2. What are the parameters in these equations? Discuss the effect produced by their variation.

Ex. 3. Find the general equation of a circle which touches both axes.

33. *Polar equations of the circle.*

It follows from (1), § 8, that the polar equation of the circle whose centre is at the point (a, α) and whose radius is r, is

$$\rho^2 - 2a\rho \cos(\theta - \alpha) + a^2 - r^2 = 0. \tag{1}$$

If the pole is on the circle, the equation is

$$\rho = 2r \cos(\theta - \alpha); \tag{2}$$

if the centre is also on the initial line, the equation is

$$\rho = 2r \cos \theta; \tag{3}$$

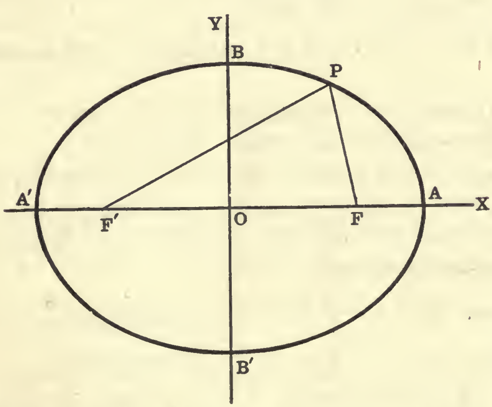
if the circle is above the initial and tangent to it at the pole, its equation is

$$\rho = 2r \sin \theta. \tag{4}$$

Ex. 1. Why is (1) of the second degree in ρ while (2), (3), and (4) are of the first degree? When is the pole outside, and when inside the circle? Discuss the effect of the variation of the parameters in these polar equations.

Ex. 2. Transform equations (1), (2), (3), (4) to rectangular coordinates.

34. THE ELLIPSE. *The ellipse is the locus of a point which moves so that the sum of its distances from two fixed points, called foci, is constant.*



Take the line through the foci as the x-axis, and the point midway between the foci as origin.

Let $2a =$ the sum of the distances from any point on the ellipse to the foci.

Let $F(c, 0)$ and $F'(-c, 0)$ be the two foci.

Let $P(x, y)$ be any point on the locus.

Then $FP + F'P = 2a$. [Geometric equation.]

But
$$FP = \sqrt{(x-c)^2 + y^2},$$
and
$$F'P = \sqrt{(x+c)^2 - y^2}. \qquad [(2), \S\ 7.]$$
$$\therefore \quad \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a. \qquad (1)$$

Transposing the first radical and squaring
$$(x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 - 4a\sqrt{(x-c)^2 + y^2},$$
or
$$a\sqrt{(x-c)^2 + y^2} = a^2 - cx.$$

Squaring and transposing again
$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

If we put $a^2 - c^2 = b^2$, we get the equation of the ellipse in the standard form,
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad (2)$$

35. An examination of this equation (2) as to symmetry, limiting values of the variables and intercepts, will give the general form and limits of the curve.

(1) Only the squares of the variables x and y appear in this equation.

Therefore the ellipse is symmetrical with respect to both axes, and also with respect to the origin. [(8), § 28.]

Hence every chord passing through O is bisected by O. For this reason, the point O is called the **Centre** of the ellipse. Likewise the lines AA' and BB' are called the **Major Axis** and **Minor Axis**, respectively.

(2) When $y = 0$, $x = \pm a$, x-intercepts.

When $x = 0$, $y = \pm b$, y-intercepts.

Therefore the curve cuts the x-axis a units to the right and a units to the left, the y-axis b units above and b units below the origin.

(3). Solving the equation (2) for y and x respectively we find

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}, \qquad x = \pm \frac{a}{b}\sqrt{b^2 - y^2}.$$

Hence y is imaginary when $x > a$, or $x < -a$; and x is imaginary when $y > b$, or $y < -b$.

Therefore the curve lies wholly within the rectangle formed by the lines $x = \pm a$ and $y = \pm b$.

Also, as either variable increases, the other diminishes. The form of the curve is shown in the figure.

Such an examination of an equation is called **A Discussion of the Equation.**

Ex. 1. Transform equation (2), § 34, to polar coordinates and show that ρ is finite for all values of θ.

Ex. 2. Where is the point (h, k) if $\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1 > 0$? < 0?

Ex. 3. Show the relation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$.

36. THE HYPERBOLA. *The hyperbola is the locus of a point which moves so that the difference of its distances from two fixed points (foci) is constant.*

Choose axes as in the case of the ellipse, let $2a$ be the constant difference, and show that when $b^2 = c^2 - a^2$ the equation of the hyperbola reduces to the standard form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad (1)$$

Ex. 1. Discuss equation (1).

Ex. 2. Show that the hyperbola (1) lies wholly between the two straight lines $y = \pm \frac{b}{a}x$, and that as x becomes infinite the ordinates of the lines become equal to the ordinates of the hyperbola. These lines are called the **Asymptotes*** of the hyperbola.

Ex. 3. Transform equation (1) to polar coordinates, and find the value of ρ when $\theta = \pm \tan^{-1}\frac{b}{a}$.

* See note under § 116.

37. THE PARABOLA. *The parabola is the locus of a point whose distance from a fixed straight line is equal to its distance from a fixed point.*

The fixed point is called the **Focus**; the fixed line the **Directrix.**

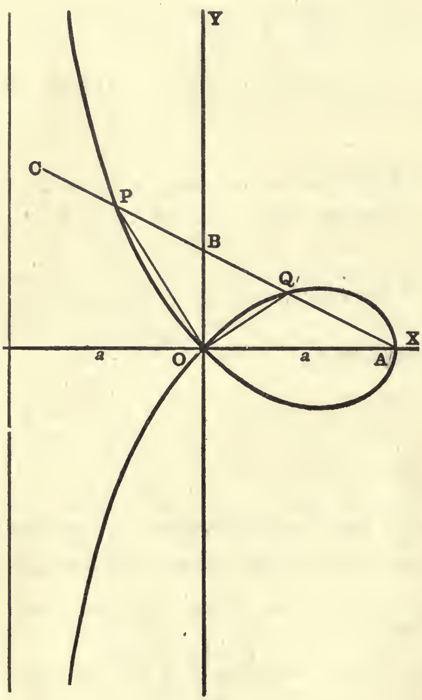
Take the line through the focus perpendicular to the directrix as the x-axis, and the origin midway between the focus and the directrix; let $2a$ denote the distance from the focus to the directrix.

Then show that the equation of the parabola is

$$y^2 = 4ax. \quad (1)$$

Discuss this equation (1) (see Ex. 2, § 22); also $y^2 = -4ax$ and $x^2 = \pm 4ay$.

STROPHOID.



38.* YOX is a right angle and A is a fixed point in OX; AC is any line through A cutting OY in B; P and Q are two points on AB such that

$$QB = BP = OB.$$

The locus of the points P and Q as AC turns about A is called the **Strophoid.**

To find the polar equation of the Strophoid, take the point O as the pole and OX as the initial line.

Let $OA = a$.

Let $Q(\rho, \theta)$ be any point on the locus.

In the isosceles triangle BOQ

$$\angle BOQ = \angle BQO = 90° - \theta,$$

and $\quad \angle OBQ = 2\theta.$

Hence in the triangle AOQ

$$\angle OAQ = 90° - 2\theta, \text{ and } \angle OQA = 90° + \theta;$$

whence $$\frac{\rho}{a}=\frac{\sin\,(90^\circ-2\theta)}{\sin\,(90^\circ+\theta)}.$$

$$\therefore\quad \rho=\frac{a\cos 2\theta}{\cos\theta}, \tag{1}$$

which is the required equation.

If we multiply (1) by ρ^2 it may be written

$$\rho^2(\rho\cos\theta)-a\rho^2(\cos^2\theta-\sin^2\theta)=0. \tag{2}$$

If OX and OY be taken as axes,

$$\rho^2=x^2+y^2,\quad \rho\cos\theta=x,\quad \rho\sin\theta=y.\quad [(1), \S\ 6.]$$

Substituting in (2) gives the rectangular equation

$$x(x^2+y^2)-a(x^2-y^2)=0. \tag{3}$$

Ex. 1. Show that the coordinates of P also satisfy equations (1) and (3).

Ex. 2. Trace the change in the values of ρ as θ varies from 0 to π in equation (1).

Ex. 3. From a discussion of equation (3) show that:

(1) The curve is symmetrical with respect to the x-axis.

(2) The curve cuts both axes twice at the origin and the x-axis once at the point $(a, 0)$.

(3) The curve lies wholly between the two lines $x=\pm a$.

(4) For all values of x between $+a$ and $-a$, y is finite, but for $x=-a$, y is infinite. Therefore the curve has infinite branches in the second and third quadrants.

The Rose of Four Branches.

39.* Given a line AB of constant length $(2a)$ whose extremities are free to move along two perpendicular lines OX and OY. Find the locus of P, the foot of the perpendicular drawn from O to AB.

Take O as the pole, and OX as the initial line.

Let $P(\rho, \theta)$ be any point on the locus.

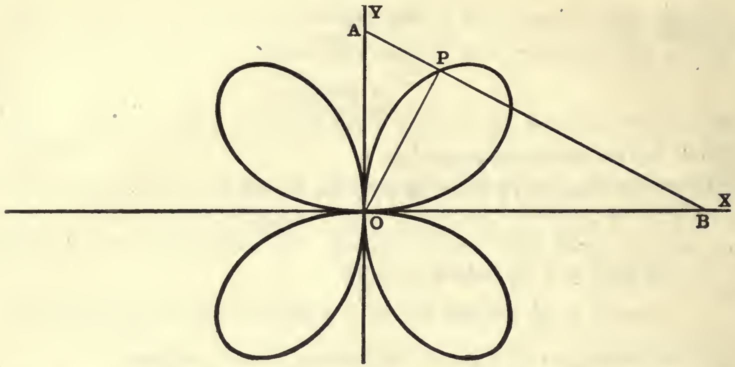
Then from the right triangles OPB and OAB,

$$\rho=OB\cos\theta,$$

and $$OB=AB\sin OAB=2a\sin\theta.$$

$$\therefore\quad \rho=a\sin 2\theta \tag{1}$$

is the polar equation of the locus. (See Ex. 5, § 22.)



Ex. 1. Show that the equation in rectangular coordinates is

$$(x^2 + y^2)^3 = 4a^2x^2y^2. \qquad (2)$$

Ex. 2. Find from equations (1) and (2):

(1) The smallest circle enclosing the curve.

(2) The number of times the curve passes through O, as θ varies from 0 to 2π.

(3) The different sorts of symmetry.

(4) Where the point (x_1, y_1) may be if $(x_1^2 + y_1^2)^3 - 4ax_1^2y_1^2 < 0$.

Also trace the curve directly as it is generated by the moving line AB.

EXAMPLES.

1. A moving point is always four times as far from the x-axis as from the y-axis. What is the equation of its locus?

2. Find the locus of a point which is equidistant from the two points (3, 2) and (−2, 1). Ans. $5x + y = 4$.

3. Find the locus of a point which is equidistant from the points (a, b) and (c, d).

4. A point moves so that its distance from the point (3, −4) is always 5. Find the equation of its locus. Does the locus pass through the origin? Why? Ans. $x^2 + y^2 - 6x + 8y = 0$.

5. Find the equation of a circle touching both axes and having its centre at the point (−3, 3).

6. Find the equation of a circle touching both axes and having a radius equal to 4.

7. A point P is two units from a circle with radius 4 and centre at (2, −6). What is the locus of P?

8. A point moves so that its distance from the origin is twice its distance from the x-axis. What is the equation of its locus? Ans. $x^2 - 3y^2 = 0$.

9. A point moves so that its distance from the x-axis is equal to its distance from the point $(2, -3)$. Show that the equation of its locus is $x^2 - 4x + 6y + 13 = 0$.

10. A point P moves so that its distances from the points $A(2, 2)$ and $B(-2, -2)$ satisfy the condition $AP + BP = 8$. Show that the equation of its locus is $3x^2 - 2xy + 3y^2 = 32$.

11. What is the locus of a point which moves so that (1) the sum, (2) the difference, (3) the product, (4) the quotient of its distances from the axes is constant (a)?

12. What is the locus of a point which moves so that (1) the sum, (2) the difference, (3) the product, (4) the quotient of the squares of its distances from the axes is constant (a^2)?

13. Find the locus of a point which moves so that the sum of the squares of its distances from the points $(a, 0)$ and $(-a, 0)$ is constant $(2c^2)$.

14. Find the locus of a point which moves so that the sum of the squares of its distances from the three points $(5, -1)$, $(3, 4)$, $(-2, -3)$ is always 64.

15. Show that the locus of a point, the sum of the squares of whose distances from n fixed points is constant, is a circle.

16. Find the locus of a point which moves so that the difference of the squares of its distances from $(a, 0)$ and $(-a, 0)$ is the constant $2c^2$.

17. Find the locus of a point such that the sum of the squares of its distances from the sides of a square is constant.

18. Find the locus of the centre of a variable circle which touches a fixed circle and a fixed straight line.

19. Find the locus of the centre of a circle which touches two fixed circles. Four cases should be considered. What does the locus become when the fixed circles are equal?

20. Find the locus of the middle points of all chords of a given circle which pass through a fixed point. [Take the fixed point as pole, and use the polar equation of the given circle.]

21. A straight rod moves so that its ends constantly touch two fixed perpendicular rods. Find the locus of any point P on the moving rod.

22. On a level plain the crack of a rifle and the thud of the ball striking the target are heard at the same instant. Find the locus of the hearer. [S. L. Loney's Coordinate Geometry, p. 283.]

23. In a given circle let AOB be a fixed diameter, OC any radius, CD the perpendicular from C on AB; let P and Q be two points on the line through O and C such that $QO = OP = CD$. Find the locus of P and Q as OC turns about O.

24. A and B are two fixed points, and P moves so that $PA = n \cdot PB$. Find the locus of P.

25. AOB and COD are two straight lines which bisect one another at right angles. Find the locus of a point P such that $PA \cdot PB = PC \cdot PD$.

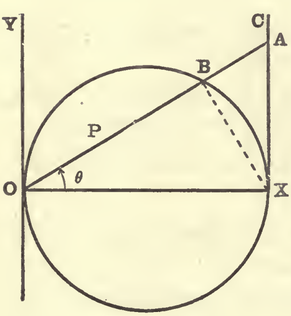
26. If ABC is an equilateral triangle, find the locus of a point P such that $PA = PB + PC$.

27. AB is a fixed diameter of a given circle and AC is any chord; P and Q are two points on the line AC such that $QC = CP = CB$. Find the locus of P and Q as AC turns about A.

28. Any straight line is drawn from a fixed point O meeting a fixed straight line in P, and a point Q is taken in this line such that $OP \cdot OQ$ is constant. Find the locus of Q.

29. Any straight line is drawn from a fixed point O meeting a fixed circle in P, and on this line a point Q is taken such that $OP \cdot OQ$ is constant. Show that the locus of Q is a circle. [See suggestion under Ex. 20.]

30. *The Cissoid of Diocles.**



Let OX be a fixed diameter of a given circle and CX a tangent line; let OA be any line through O meeting CX in A and the circle in B; on this line take $OP = BA$. Then the locus of P as OA revolves about O is the Cissoid.

Using OX and OY as axes show that the equations of the Cissoid are

$$\rho = 2r\frac{\sin^2\theta}{\cos\theta},$$

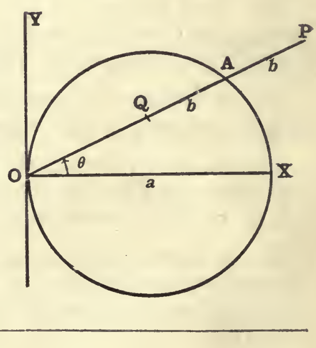
and

$$y^2 = \frac{x^3}{2r - x},$$

where r is the radius of the given circle.

31. *The Limacon of Pascal.**

Through a fixed point O on a given circle draw any secant cutting the circle again in A; and on this line take $QA = AP = b$, a constant. The locus of the points P and Q as OA turns about O is called the Limacon.



Show that the equations of the Limacon referred to OX and OY as axes are

$$\rho = a\cos\theta \pm b,$$

and

$$(x^2 + y^2 - ax)^2 = b^2(x^2 + y^2).$$

Notice the three different forms of the curve according as $b >, =,$ or $< a$, the diameter of the circle.

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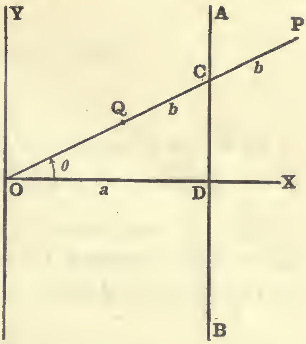
* See note on page 51.

32. *The Conchoid of Nicomedes.** Through a fixed point O draw any secant meeting a fixed straight line AB in C, and on this secant lay off $QC = CP = b$, a constant. The locus of the points P and Q as OC revolves about O is called the **Conchoid**.



Take OX, the perpendicular to AB, as initial line and x-axis, and show that the equations of the Conchoid are

$$\rho = a \sec \theta \pm b,$$

and $$(x^2 + y^2)(x - a)^2 = b^2 x^2.$$

Consider the forms of the curve when $b >, =$, and $< a$, where $a = OD$.

*For a historic account of the invention of the Cissoid and Conchoid, and the cause which led to their invention, see Cajori, "A History of Mathematics," 1894, p. 50, and Ball, "A Short History of Mathematics," 1888, p. 78.

For the reason why the ancient geometers desired to *Duplicate the Cube*, see Cajori, p. 25, and Ball, pp. 38 and 75. For biography of Pascal, see Cajori, pp. 175–77, and Ball, pp. 249–56.

CHAPTER III.

THE STRAIGHT LINE.

40. It was shown in § 31 that the equation of any straight line when expressed in terms of its slope m and y-intercept b is an equation of the first degree,

$$y = mx + b; \qquad (1)$$

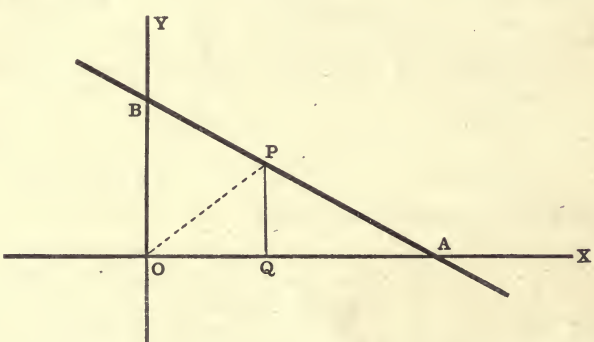
and also that the general equation of the first degree,

$$Ax + By + C = 0,$$

represents a straight line.

It is sometimes more convenient, however, to write the equation of the straight line in other forms; *i. e.*, to express it in terms of some other *pair* of parameters.

41. *To find the equation of the straight line in terms of its intercepts on the axes.*



Let A and B be the points in which the straight line meets the axes and let $OA = a$, and $OB = b$.

Let $P(x, y)$ be any point on the line.

Draw PQ parallel to the y-axis, and join O and P.

Then $\qquad \triangle OAP + \triangle OBP = \triangle OAB.$

Hence $\qquad bx + ay = ab,$

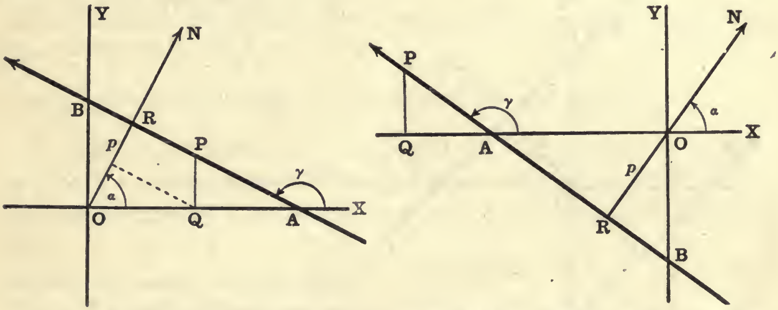
or

$$\frac{x}{a} + \frac{y}{b} = 1. \qquad (1)$$

If $l = \frac{1}{a}$ and $m = \frac{1}{b}$, the equation may be written

$$lx + my = 1. \tag{2}$$

42. *To find the equation of a straight line in terms of the length of the perpendicular from the origin upon the line and the angle which that perpendicular makes with the x-axis.*



Let ON be perpendicular to the straight line AB, and intersect it in R.

Let $OR = p$, and angle $XON = \alpha$.

Let $P(x, y)$ be any point on the line.

Then since $OQPR$ is a closed polygon, OR is equal to the sum of the projections of OQ, QP, and PR upon OR. That is,

$$OR = \text{proj. of } OQ + \text{proj. of } QP + \text{proj. of } PR.$$

$$= OQ \cos \alpha + QP \sin \alpha + 0.$$

$$\therefore \quad x \cos \alpha + y \sin \alpha = p, \tag{1}$$

which is the required equation.

Let angle $XAP = \gamma = 90° + \alpha$.

Then $\qquad \cos \alpha = \sin \gamma, \quad \sin \alpha = -\cos \gamma,$

and, by substituting in (1), the equation of the line becomes

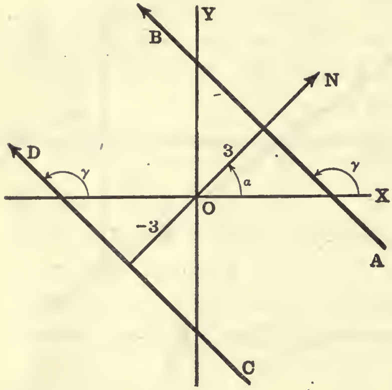
$$x \sin \gamma - y \cos \gamma = p. \tag{2}$$

Since equations (1) and (2) involve the trigonometric functions, *sin* and *cos*, ON and AB must be regarded as *directed* lines. As in Trigonometry, we will consider the directions of the terminal lines of α and γ as the *positive* directions of these lines.

If $\gamma = 90° + \alpha$, as assumed above, then standing at R facing the positive direction of ON, the positive direction of AB is to the *left;* and standing at R facing the positive direction of AB, the positive direction of ON is *from AB toward the right.*

This will be called the *positive side** of the line AB.

Then in equations (1) and (2) p is positive when taken in the positive direction of ON. Hence when p is *positive* the *origin* is on the *negative* side of the line.



E. g. In the equations

$$\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}=\pm 3,$$

$$\cos a=\sin a=\frac{1}{\sqrt{2}}.$$

$\therefore\ a=45°$ and $\gamma=135°$ for both lines; but

for AB $p=3$,

for CD $p=-3$.

Hence the two lines are parallel but on opposite sides of O. Also O is on the positive side of CD and on the negative side of AB.

Since

$$\sin(\theta\pm\pi)=-\sin\theta \quad\text{and}\quad \cos(\theta\pm\pi)=-\cos\theta,$$

if the signs of all the terms in (1), or (2), be changed, the direction of AB, and also of ON, will be changed by $\pm\pi$; and therefore the positive and negative sides of the line will be reversed. That is, the equation of a line may be written so as to make either side of the line positive or negative, just as we choose.

E. g. The equation of the line AB,

$$\frac{x}{2}-\frac{\sqrt{3}y}{2}=-2, \qquad (1)$$

may also be written

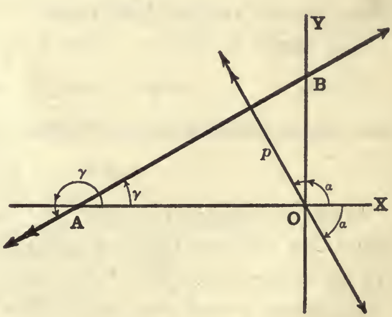
$$-\frac{x}{2}+\frac{\sqrt{3}y}{2}=2. \qquad (2)$$

In (1) $p=-2$,

$$\cos a=\sin\gamma=\frac{1}{2},$$

$$\sin a=-\cos\gamma=-\frac{\sqrt{3}}{2}.$$

$\therefore\ a=-60°$ and $\gamma=30°$.



* This holds for all lines except the x-axis. (See § 50.)

In (2) $p=2, \quad \cos\alpha=\sin\gamma=-\frac{1}{2}, \quad \sin\alpha=-\cos\gamma=\frac{\sqrt{3}}{2}.$

$$\therefore \quad \alpha=120° \quad \text{and} \quad \gamma=210°.$$

Angles and directions corresponding to (1) are denoted by single arrow-heads, those corresponding to (2) by double arrow-heads.

The origin is on the positive or negative side of the line according as the equation is written in the form (1) or (2).

Ex. Point out the combinations of signs of $\cos\alpha$, $\sin\alpha$, and p when the line crosses the different quadrants.

43. *Transformation of the equations of the straight line.*

In §§ 31, 41, and 42 we have found, by independent methods, the three *standard forms* of the equation of a straight line involving different pairs of parameters, m and b, a and b, α or γ, and p; viz.:

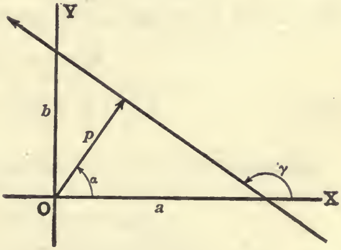
$$y=mx+b, \quad \textit{Slope form,} \qquad (1)$$

$$\frac{x}{a}+\frac{y}{b}=1, \quad \textit{Intercept form,} \qquad (2)$$

$$\left.\begin{cases} x\cos\alpha+y\sin\alpha=p, \\ x\sin\gamma-y\cos\gamma=p, \end{cases}\right\} \quad \textit{Distance or Normal form.} \qquad (3)$$

Any one of these forms of the equation may, however, be deduced from any other.

I. From the figure we obtain directly the relations



$$m=\tan\gamma=\frac{\sin\gamma}{\cos\gamma}=-\frac{\cos\alpha}{\sin\alpha}=-\frac{b}{a},$$

and $\quad p=a\cos\alpha=b\sin\alpha$

$$=-b\cos\gamma=a\sin\gamma.$$

Then substituting these values of m in (1), for example, gives

$$y=-\frac{b}{a}x+b,$$

$$y=-\frac{\cos\alpha}{\sin\alpha}x+b,$$

and

$$y=\frac{\sin\gamma}{\cos\gamma}x+b.$$

Whence, since $b \sin \alpha = -b \cos \gamma = p$, we get

$$\frac{x}{a} + \frac{y}{b} = 1,$$

$$x \cos \alpha + y \sin \alpha = p,$$

and $$x \sin \gamma - y \cos \gamma = p.$$

Moreover, the general equation of the first degree,

$$Ax + By + C = 0, \qquad (4)$$

can be transformed into any one of the three standard forms.

II. Solving (4) for y gives (see § 31)

$$y = -\frac{A}{B}x - \frac{C}{B}. \quad \textit{Slope form.} \qquad (5)$$

III. If we transpose and divide by C, (4) may be written

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1. \quad \textit{Intercept form.} \qquad (6)$$

IV. *To reduce the general equation* (4) *to the distance form.*

In this case we are to transform (4) so that the sum of the squares of the resulting coefficients of x and y shall be unity. Hence, if we assume the transformed equation to be

$$KAx + KBy + KC = 0, \qquad (7)$$

then $$K^2A^2 + K^2B^2 = \cos^2 \alpha + \sin^2 \alpha = 1.$$

Whence $$K = \frac{1}{\sqrt{A^2 + B^2}}.$$

$$\therefore \quad \frac{A}{\sqrt{A^2 + B^2}}x + \frac{B}{\sqrt{A^2 + B^2}}y = -\frac{C}{\sqrt{A^2 + B^2}} \qquad (8)$$

is the required equation.

Hence, *to reduce the general equation* (4) *to the distance form, transpose C and divide by* $\sqrt{A^2 + B^2}$.

The general equation of the first degree must therefore represent a straight line, since, by transposing and multiplying by a suitable constant, it can be reduced to any one of the standard forms of the equation of the straight line. (*Cf.* § 31.)

V. *Values of parameters in terms of A, B, and C.*

Comparing coefficients in (1) and (5), (2) and (6), (3) and (8), we get

$$a = -\frac{C}{A},\quad b = -\frac{C}{B},\quad m = -\frac{A}{B},\quad p = \frac{-C}{\sqrt{A^2+B^2}},$$

$$\cos\alpha = \sin\gamma = \frac{A}{\sqrt{A^2+B^2}},\quad \sin\alpha = -\cos\gamma = \frac{B}{\sqrt{A^2+B^2}}.$$

Observe that the values of a and b thus obtained are the same as those found by putting $y=0$, then $x=0$ in (4); also that $m = -\frac{A}{B} = -\frac{b}{a}$, as found above directly from the figure. Then $\sin\alpha$, $\cos\alpha$, and p can be found by Trigonometry and the relations obtained from the figure.

EXAMPLES.

1. When is it impossible to write the equation of a straight line in the *intercept form?* in the *slope form?*

Change the following equations to the standard forms and thus determine their parameters. Also draw the lines:

2. $x + \sqrt{3}y + 10 = 0.$

3. $4y = 3x + 24.$

4. $y = x - 6.$

5. $5x + 4y = 20.$

6. $5x - 12y = 13.$

7. $2x - 4y + 9 = 0.$

8. $2x - 3y = 4.$

9. $2x + 3y = 0.$

10. $x - a = 0.$

11. $y = 4.$

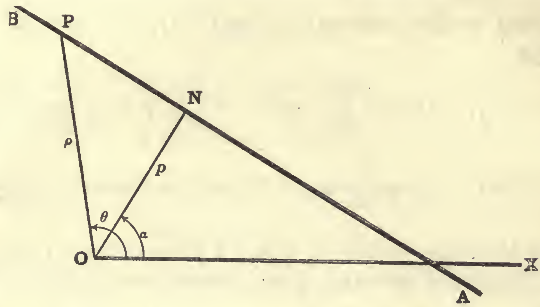
12. Transform $Ax + By + C = 0$ so that the sum of the three coefficients shall be K; so that the square of the first shall be three times the second; so that the product of the three shall be twice their sum.

13. Transform $5x + 4y - 20 = 0$ so that the sum of the three coefficients is 22; so that the product of the first and third shall be equal to the second.

14. Transform $3x - 4y + 12 = 0$ so that the square of the second coefficient shall be equal to twice the third minus four times the first; so that the product of the three shall be minus three times the last.

15. Transform $5x - 2y - 3 = 0$ so that the product of the first and second coefficients minus ten times the third shall be equal to -40; so that the square of the second plus twice the sum of the first and third shall be equal to 24.

44. *To find the polar equation of a straight line.*



Let $N(p, a)$ be the foot of the perpendicular from O upon the given line AB.

Let $P(\rho, \theta)$ be any other point on AB.

Then $$\angle NOP = (\theta - a),$$

and $$OP \cos NOP = ON.$$

$$\therefore \quad \rho \cos (\theta - a) = p, \tag{1}$$

which is the required equation.

EXAMPLES.

1. Transform $x \cos a + y \sin a = p$ to polar coordinates.

2. What is the polar equation of any straight line through the pole? of the initial line?

3. What locus is represented by $\sin \theta = 0$? $\sin 2\theta = 0$? $\sin 3\theta = 0$? . . . $\sin n\theta = 0$?

4. What is the locus of $\cos n\theta = 0$ when $n = 1, 2, 3 \ldots$?

Find the parameters and draw the lines whose equations are

5. $\rho \cos (\theta - 30°) = 2$.

6. $\rho \cos (\theta - 60°) = 1$.

7. $\rho \cos (\theta + 45°) = 3$.

8. $\rho \cos (\theta + 120°) + 4 = 0$.

9. $\rho \cos (\theta - 120°) + 1 = 0$.

10. $\rho \cos (\theta + 60°) + 5 = 0$.

11. Find the coordinates of the point of intersection of $\rho \cos (\theta \pm 45°) = 1$.

12. Find the polar equations of the bisectors of the angles between the lines
$$\rho \cos (\theta - 60°) = 2, \quad \text{and} \quad \rho \cos (\theta - 30°) = 2.$$

13. What is the polar equation of a line perpendicular to the initial line? parallel to the initial line?

14. Show that the equations

$$A \cos\theta + B \sin\theta + \frac{C}{\rho} = 0, \quad A \cot\theta = K,$$

$$\rho = k \sec(\theta - a), \qquad \rho = l \csc(\theta - \beta),$$

represent straight lines.

15. What will the equation $\rho \cos(\theta - a) = p$ become if the lines $\theta = a$, $\theta = -a$, $\theta = \beta$, $\theta = 90°$ be taken as the initial line?

45. If we wish to find the equation of a straight line which satisfies any two conditions, we may take for its equation any one of the forms

$$y = mx + b, \quad \frac{x}{a} + \frac{y}{b} = 1, \quad lx + my = 1,$$

$$x \cos a + y \sin a = p, \quad Ax + By + C = 0.$$

The given conditions must be expressed by *two* equations involving the parameters of the line. From these *two* equations of condition we have then to determine the values of one of the *pairs* of parameters,

$$m \text{ and } b, \quad a \text{ and } b, \quad l \text{ and } m, \quad p \text{ and } a, \quad \text{or} \quad \frac{A}{C} \text{ and } \frac{B}{C}.$$

If the line be made to satisfy only *one* condition, there will be only *one* conditional equation involving the parameters, and consequently only *one* parameter can be eliminated; then by assigning a suitable value to the remaining parameter, the line may be made to satisfy any other given condition.

Ex. Find the equation of a straight line passing through the point $(-4, 1)$ and making equal intercepts on the axes.

Let $\frac{x}{a} + \frac{y}{b} = 1$ be the equation of the line.

Then, since the intercepts are equal, $a = b$.

Also, since $(-4, 1)$ is on the line, $\frac{-4}{a} + \frac{1}{b} = 1$.

$\therefore \ a = b = -3$, and $x + y + 3 = 0$ is the equation required.

46. *To find the equation of a straight line passing through a fixed point (x_1, y_1) in a given direction.*

Let the line make with the x-axis an angle $\tan^{-1} m$.

Its equation will then be (where b is unknown)

$$y = mx + b; \tag{1}$$

and since the line passes through (x_1, y_1),

$$y_1 = mx_1 + b. \tag{2}$$

Whence, by subtracting (2) from (1),

$$y - y_1 = m(x - x_1). \qquad (3)$$

The line given by (3) will pass through the point (x_1, y_1) for all values of m; and may be made to represent any line through (x_1, y_1) by giving to m a suitable value.

If then we know a line passes through a certain point we may write its equation in the form (3), and determine the value of m from the other condition the line is made to satisfy.

Since $m = \tan \gamma = \dfrac{\sin \gamma}{\cos \gamma}$ (§ 42), equation (3) may be written in the form

$$\frac{x - x_1}{\cos \gamma} = \frac{y - y_1}{\sin \gamma} = r, \qquad (4)$$

where r is the variable distance from the fixed point (x_1, y_1) to any point (x, y) on the line.

Let the student prove (4) directly from a figure.

47. *To find the equation of a straight line which passes through two given points (x_1, y_1) and (x_2, y_2).*

Since the line passes through (x_1, y_1) its equation will be of the form [(3), § 46]

$$y - y_1 = m(x - x_1); \qquad (1)$$

then, since (x_2, y_2) is also on the line, we have

$$y_2 - y_1 = m(x_2 - x_1). \qquad (2)$$

Dividing (1) by (2) gives the required equation

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}. \qquad (3)$$

Equation (3) may also be written

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} = 0, \qquad (4)$$

which is obvious, since the area of the triangle formed by (x_1, y_1) (x_2, y_2) and any other point (x, y) on the line is zero.

EXAMPLES.

Find the equation of the straight line

1. if $b=\frac{3}{5}$ and $\gamma=\tan^{-1}\frac{1}{5}$.

2. if $a=b$ and $p=5$.

3. if $\gamma=30^\circ$ and $p=4$.

4. if $b=-3$ and $\gamma=150^\circ$.

5. if $\gamma=\tan^{-1}2$ and the line passes through $(3, -4)$.

6. if $\gamma=\tan^{-1}\dfrac{a}{b}$ and the line passes through $(-a, b)$.

7. passing through the pairs of points $(2, 3)$ and $(-6, 1)$; $(-1, 3)$ and $(6, -7)$; (a, b) and $(a+b, a-b)$.

8. Find the equations of the sides of the triangle whose vertices are the points $(1, 3)$, $(3, -5)$ and $(-1, -3)$.

9. Find the equations of the three medians of this triangle, and show that they meet in a point.

10. Find the equation of a line passing through $(-1, 4)$ and having intercepts (1) equal in length, (2) equal in length but opposite in sign.

11. Show that the equations of the lines passing through the point $(4, 4)$ and whose distance from the origin is 2 are $x(1\pm\sqrt{7})+y(1\mp\sqrt{7})=8$.

12. Find the equation of the line through (a, b) parallel to the line joining $(0, -a)$ and $(b, 0)$.

13. What is the equation of the line through $(4, -5)$ parallel to $2x+3y=6$?

14. Find the equations of the lines which pass through $(-2, 1)$ and cut off equal lengths from the axes.

15. Show that the three lines $2x-y=4$, $x+2y=7$, and $3x+y=11$ meet in a point.

16. Show that the three points $(1, 3)$, $(-1, 4)$, and $(9, -1)$ are on a straight line; also $(3a, 0)$, $(0, 3b)$, and $(a, 2b)$.

17. Show that the equation of the line passing through the points $(a\cos\alpha, b\sin\alpha)$ and $(a\cos\beta, b\sin\beta)$ is

$$bx\cos\tfrac{1}{2}(\alpha+\beta)+ay\sin\tfrac{1}{2}(\alpha+\beta)=ab\cos\tfrac{1}{2}(\alpha-\beta).$$

18. Show that the equation of the line which passes through the points $(a\sec\alpha, b\tan\alpha)$ and $(a\sec\beta, b\tan\beta)$ is

$$bx\cos\tfrac{1}{2}(\alpha-\beta)-ay\sin\tfrac{1}{2}(\alpha+\beta)=ab\cos\tfrac{1}{2}(\alpha+\beta).$$

19. Find the equations of the lines which bisect the opposite sides of the quadrilateral $(3, 4)$, $(5, 1)$, $(-3, 4)$, and $(5, -1)$.

20. Find the equations of the lines which go through the origin and trisect that portion of the line $3x-2y=18$ which is intercepted between the axes.

21. For what value of m will the line $y = mx - 4$ pass through (4, 2)? be 2 units distant from the origin?

22. A line is 3 units distant from O and makes an angle of 60° with OX. What is its polar equation? its rectangular equation?

23. Find the locus of all points which are equidistant from the two lines $3x - 2y = 8$ and $3x - 2y + 2 = 0$.

24. What is the distance between the parallel lines

$$3x + 4y = 5 \quad \text{and} \quad 6x + 8y + 15 = 0?$$

25. Show, by the use of (1), § 44, or by transforming (3), § 46, that the polar equation of a line passing through the fixed point (ρ_1, θ_1) may be written

$$\rho \cos(\theta - \alpha) = \rho_1 \cos(\theta_1 - \alpha).$$

26. Show, directly from a figure, or by transforming (3), § 47, that the polar equation of the straight line which passes through the two fixed points (ρ_1, θ_1) and (ρ_2, θ_2) is

$$\rho_1\rho_2 \sin(\theta_2 - \theta_1) + \rho_2\rho \sin(\theta - \theta_2) + (\rho\rho_1 \sin(\theta_1 - \theta) = 0.$$

27. Show that the three straight lines

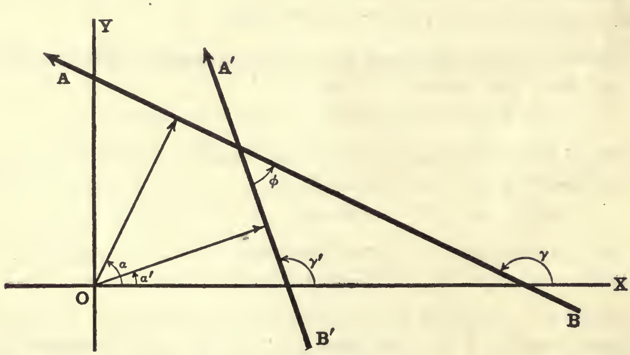
$$a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0, \quad a_3x + b_3y + c_3 = 0$$

will meet in a point if

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = 0.$$

28. Find the determinant expressions for the coordinates of the vertices, and for the area of the triangle formed by the three lines in Ex. 27, and show that the determinant there given is a *square factor* of the determinant expression for the area of the triangle.

48. *To find the angle between two straight lines whose equations are given.*



Let AB and $A'B'$ be the given lines.

Let φ be the required angle.

Then, using the same notation and the same convention as to direction of the lines as in § 42,

$$\varphi = \alpha - \alpha' = \gamma - \gamma'. \tag{1}$$

I. If the equations of the given lines be

$$x \cos \alpha + y \sin \alpha = p \quad \text{and} \quad x \cos \alpha' + y \sin \alpha' = p',$$

$\cos \varphi$ can be found by direct substitution in

$$\cos \varphi = \cos \alpha \cos \alpha' + \sin \alpha \sin \alpha'. \tag{2}$$

II. If the equations of the given lines be

$$y = mx + b \quad \text{and} \quad y = m'x + b',$$

we have from (1), since $\tan \gamma = m$, and $\tan \gamma' = m'$,

$$\tan \varphi = \tan (\gamma - \gamma') = \frac{\tan \gamma - \tan \gamma'}{1 + \tan \gamma \tan \gamma'} = \frac{m - m'}{1 + mm'}. \tag{3}$$

$$\therefore \quad \varphi = \tan^{-1}\left(\frac{m - m'}{1 + mm'}\right). \tag{4}$$

When $m = m'$, $\tan \varphi = 0$, and the lines are parallel.

When $1 + mm' = 0$, $\tan \varphi$ is infinite.

Therefore, when $m' = -\frac{1}{m}$ the lines are perpendicular to one another.

III. If the equations of the lines be

$$Ax + By + C = 0 \quad \text{and} \quad A'x + B'y + C' = 0,$$

then $m = -\frac{A}{B}, \quad m' = -\frac{A'}{B'}$; and therefore, from (3),

$$\tan \varphi = \frac{A'B - AB'}{AA' + BB'}. \tag{5}$$

If $A'B - AB' = 0$, *i. e.* if $\frac{A}{A'} = \frac{B}{B'}$, the lines will be parallel.

If $AA' + BB' = 0$, the lines will be at right angles to one another.

It should be noticed that (2) gives the angle between two directed lines. For if all the signs in *one* of the equations in I. be changed, the direction of the line will be changed by $\pm \pi$, (§ 42).

The *sign* of $\cos\varphi$ given by (2) will also be changed and φ becomes the supplement of its former value. But if all the signs in both equations be changed, φ is unaltered.

If the equations be so written that the origin is on the *same* side (either positive or negative) of both lines, it will be in the obtuse angle between the lines when $\cos\varphi$ is positive, and in the acute angle when $\cos\varphi$ is negative.

If m and m' be so taken that $m' > m$, then $\gamma' > \gamma$ and (3) will give $\tan(-\varphi) = -\tan\varphi$, instead of $\tan\varphi$.

49. *To find the equations of two lines passing through a given point* (x_1, y_1), *the one parallel, the other perpendicular to a given line.*

Let the given line be

$$Ax + By + C = 0.$$

Then the parallel line is

$$Ax + By + K = 0, \qquad [\S\ 48,\ \text{III.}] \quad (1)$$

and the perpendicular line is

$$Bx - Ay + K' = 0, \qquad [\S\ 48,\ \text{III.}] \quad (2)$$

where K and K' are constants to be determined.

Since both (1) and (2) are to go through (x_1, y_1) these constants are such that

$$\left.\begin{aligned} &Ax_1 + By_1 + K = 0 \\ \text{and}\quad &Bx_1 - Ay_1 + K' = 0, \end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned} \text{i. e.}\quad &K = -(Ax_1 + By_1) \\ \text{and}\quad &K' = -(Bx_1 - Ay_1). \end{aligned}\right\} \quad (4)$$

Therefore, the required equations are, respectively,

$$A(x - x_1) + B(y - y_1) = 0 \quad (5)$$

and

$$B(x - x_1) - A(y - y_1) = 0. \quad (6)$$

If the equation of the given line is in the form

$$y = mx + b,$$

the required equations may be written [(3), § 46, and II., § 48]

$$y - y_1 = m(x - x_1) \quad (7)$$

and

$$y - y_1 = -\frac{1}{m}(x - x_1). \quad (8)$$

EXAMPLES.

Find the angles between the following pairs of lines:

1. $3x+4y=8$ and $7y-x+14=0$.

2. $2x+3y=6$ and $2y=3x-12$.

3. $x+4=2y$ and $x+3y=9$.

4. $3y+12x+16=0$ and $2y=4x+5$.

5. $\frac{x}{a}-\frac{y}{b}=1$ and $\frac{y}{a}-\frac{x}{b}=1$.

6. Prove that the points (1, 3), (5, 0), (0, —4), and (—4, —1) are the vertices of a parallelogram, and find the angle between its diagonals.

Find the equations of the two straight lines

7. passing through the point (2, 3), the one parallel, the other perpendicular to the line $4x-3y=6$.

8. passing through (4, —2), the one parallel, the other perpendicular to the line $y=2x+4$.

9. passing through the intersection of $4x+y+5=0$ and $2x-3y+13=0$, one parallel, the other perpendicular to the line through the two points (3, 1) and (—1, —2).

10. Find the equation of the perpendicular bisector of the line joining the points (3, —1) and (—2, 1).

11. Find the equations of the lines perpendicular to the line joining (2, 1) and (—3, —2) at the points which divide it internally and externally in the ratio 2 : 3.

12. What is the equation of a line parallel to $3x+4y=12$ and at a distance 4 from the origin?

13. Show that two parallel lines intersect at infinity.

The vertices of a triangle are (3, 1), (—2, 3), and (2, —4):

14. Find the equations of its altitudes and show that they meet in a point.

15. Find the equations of the perpendicular bisectors of its sides, and show that they meet in a point which is equidistant from the three vertices.

16. Find its interior angles.

17. Find the equations of two lines through the origin, each making an angle of 30° with the line $4x+y+4=0$.

18. Show that the equations of the two straight lines through a given point (x_1, y_1) making a given angle ϕ with the line $y=mx+b$ are

$$y-y_1=\frac{m\pm\tan\phi}{1\mp m\tan\phi}(x-x_1).$$

19. Show that the equations of the lines passing through $(-3, 2)$ and inclined at an angle of 60° to the line $\sqrt{3}y - x = 3$ are

$$x + 3 = 0 \quad \text{and} \quad \sqrt{3}y + x + 3 = 0.$$

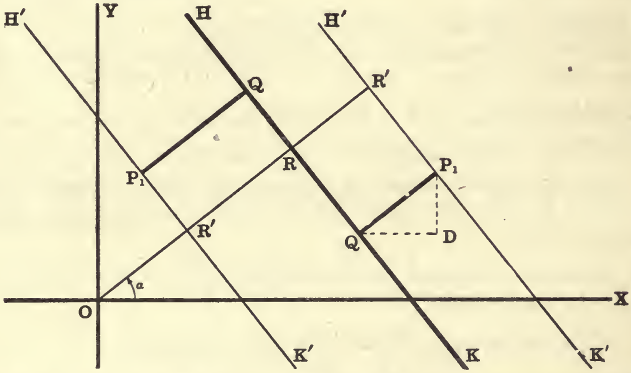
20. Find the equations of the sides of a square of which the points $(2, 2)$ and $(-2, 1)$ are opposite vertices.

21. What are the equations of the sides of a rhombus if two opposite vertices are at the points $(-1, 3)$ and $(5, -3)$, and the interior angles at these vertices are each 60°?

22. Prove that the equation of the straight line which passes through the point $(a\cos^3\theta, a\sin^3\theta)$ and is perpendicular to the straight line $x\sec\theta + y\csc\theta = a$ is

$$x\cos\theta - y\sin\theta = a\cos 2\theta.$$

50. *To find the perpendicular distance from a given straight line to a given point $P_1(x_1, y_1)$.*



Let HK be the given line, and let $H'K'$ be parallel to HK and pass through P_1.

Let P_1Q be the perpendicular from P_1 on HK, and OR, OR' the perpendiculars from O on HK and $H'K'$.

Let the equation of HK be

$$x\cos\alpha + y\sin\alpha = p. \tag{1}$$

Then the equation of $H'K'$ is

$$x\cos\alpha + y\sin\alpha = p + RR' = p + QP_1; \tag{2}$$

and since this line (2) goes through $P_1(x_1, y_1)$,

$$x_1\cos\alpha + y_1\sin\alpha = p + QP_1. \tag{3}$$

$$\therefore \quad QP_1 = x_1 \cos \alpha + y_1 \sin \alpha - p,^* \tag{4}$$

which is the distance *from* the line α, p *to* the point (x_1, y_1).

If the equation of the given line be

$$Ax + By + C = 0,$$

$$\cos \alpha = \frac{A}{\sqrt{A^2 + B^2}}, \quad \sin \alpha = \frac{B}{\sqrt{A^2 + B^2}},$$

$$p = \frac{-C}{\sqrt{A^2 + B^2}}; \qquad [\S\ 43, \text{V.}]$$

and substituting these values in (4) gives

$$QP_1 = \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}, \tag{5}$$

which is the distance from line A, B, C to the point (x_1, y_1).

Hence the length of the perpendicular from a given line to a given point is found by substituting the coordinates of the point in the equation of the line reduced to the distance form with all the terms transposed to the first member.

The expression (5) will be positive or negative according as $Ax_1 + By_1 + C$ is positive or negative (if $\sqrt{A^2 + B^2}$ be positive). If $Ax_1 + By_1 + C$ is positive, the point (x_1, y_1) is said to be on the *positive side* of the line $Ax + By + C = 0$; if $Ax_1 + By_1 + C$ is negative, (x_1, y_1) is said to be on the *negative side* of the line. If the equation of the line be written so that p is *positive*, the expression (5) will be found to be positive when P_1 and O are on opposite sides of the line. (*Cf.* § 42.)

Hence the points (x_1, y_1) and (x_2, y_2) are on the *same* side or *opposite* sides of the line $Ax + By + C = 0$ according as $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ have the *same* sign or *opposite* signs.

This proves for the straight line the principles illustrated in §§ 14–20.

**Another proof.* Let the coordinates of Q be x_2, y_2; of D, x_1, y_2; then $x_2 \cos \alpha + y_2 \sin \alpha = p$, since Q is on (1). Projecting on QP_1 gives

$$QP_1 = \text{proj. } QD + \text{proj. } DP_1,$$

$$\therefore \quad QP_1 = (x_1 - x_2) \cos \alpha + (y_1 - y_2) \sin \alpha$$

$$= (x_1 \cos \alpha + y_1 \sin \alpha) - (x_2 \cos \alpha + y_2 \sin \alpha)$$

$$= x_1 \cos \alpha + y_1 \sin \alpha - p.$$

51. *To find the equations of the bisectors of the angles between the lines*

$$Ax + By + C = 0, \quad or \quad x \cos \alpha + y \sin \alpha - p = 0, \qquad (1)$$

and $$A'x + B'y + C' = 0, \quad or \quad x \cos \alpha' + y \sin \alpha' - p' = 0. \qquad (2)$$

Suppose the equations of the lines written so that the origin is on the *same side* of both lines.

Then for any point (x, y) on the bisector of the angle which includes the origin,

$$\textit{Dist. from } (1) \textit{ to } (x, y) = \textit{Dist. from } (2) \textit{ to } (x, y);$$

and for any point (x, y) on the other bisector,

$$\textit{Dist. from } (1) \textit{ to } (x, y) = -\textit{Dist. from } (2) \textit{ to } (x, y).$$

Therefore the required equations are [§ 50]

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = \pm \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}}, \qquad (3)$$

or $$x \cos \alpha + y \sin \alpha - p = \pm\, x \cos \alpha' + y \sin \alpha' - p'. \qquad (4)$$

Ex. Show that these two lines are perpendicular to each other. [Use (4).]

EXAMPLES.

Find the following distances:

1. From $3x + 4y + 10 = 0$ to $(1, 12)$, $(-3, -9)$, $(3, 4)$.
2. From $x - 3y = 7$ to $(3, 2)$, $(6, 3)$, $(2, -5)$.
3. From $5x + 12y = 13$ to $(3, -2)$, $(-3, 2)$, $(4, -7)$.
4. From $b(x - a) + a(y - b) = 0$ to $(-a, -b)$, $(-b, -a)$, (b, a).
5. From $4(x - 3) = 3(y + 1)$ to $(6, 1)$, $(4, -5)$, $(-7, 2)$.

Are the above points on the same or opposite sides of the lines?

Find the equations of the bisectors of the angles between the lines

6. $3x + 4y + 12 = 0$ and $4x - 3y = 12$.
7. $3x - 4y + 5 = 0$ and $12x + 5y + 14 = 0$.
8. $y = 2x + 5$ and $x - 2y = 8$.
9. $y = \sqrt{3}x + 3$ and $x + \sqrt{3}y = 9$.
10. Find the lengths of the altitudes of the triangle whose vertices are $(3, 4)$, $(-4, 1)$, and $(-1, -5)$.
11. What is the locus of a point which is 3 units distant from the line $2x - 4y = 9$?

12. Find the points on the axes which are 4 units from the line $x - 7y + 21 = 0$.

13. Show that the perpendiculars let fall from any point of $22x - 4y = 15$ upon the lines $24x + 7y = 20$ and $4x - 3y = 2$ are equal. Find another line of which this statement is true.

14. Find the perpendicular distance of the point (l, m) from the line through (a, b) perpendicular to $lx + my = 1$.

15. Show that the bisectors of the interior angles of a triangle meet in a point.

16. Find the locus of a point which is equally distant from the lines $5x - 3y = 15$ and $3y = 5x + 6$.

17. Show that the locus of a point which moves so that the *sum* of its distances from the two lines

$$x \cos \alpha + y \sin \alpha = p \quad \text{and} \quad x \cos \alpha' + y \sin \alpha' = p'$$

is constant and equal to K is the straight line

$$x \cos \tfrac{1}{2}(\alpha + \alpha') + y \sin \tfrac{1}{2}(\alpha + \alpha') = 2(p + p' + K) \sec \tfrac{1}{2}(\alpha + \alpha').$$

Show that the locus is parallel to *one* of the bisectors of the angles formed by the two given lines.

Show also that if the *difference* of the distances from the two given lines is constant, the locus is a straight line parallel to the other bisector.

18. If p and p' be the perpendiculars from the origin upon the straight lines whose equations are

$$x \sec \theta + y \csc \theta = a \quad \text{and} \quad x \cos \theta - y \sin \theta = a \cos 2\theta,$$

prove that

$$4p^2 + p'^2 = a^2.$$

52. *To find the equation of a straight line passing through the intersection of two given straight lines.*

The most obvious method of finding the required equation is to find the coordinates x', y' of the point of intersection of the two given lines, and then substitute these values in equation (3), § 46.

The following method of dealing with this class of problems is, however, sometimes preferable, both on account of its generality and because it saves the labor of solving the two given equations:

Let the equations of the two given straight lines be

$$Ax + By + C = 0, \qquad (1)$$

and

$$A'x + B'y + C' = 0. \qquad (2)$$

The required equation is then written

$$Ax + By + C + \lambda(A'x + B'y + C') = 0, \qquad (3)$$

where λ is any constant.

Equation (3) is of the first degree, and therefore represents a straight line; if (x', y') is the point common to (1) and (2), we have

$$Ax' + By' + C = 0$$

and

$$A'x' + B'y' + C' = 0.$$

$$\therefore \quad Ax' + Bx' + C + \lambda(A'x' + B'y' + C') = 0,$$

which shows that the point (x', y') is also on (3).

Hence (3) is the equation of a straight line passing through the point of intersection of the two given lines. Moreover, equation (3) contains one arbitrary parameter, λ, and therefore, by giving a suitable value to λ, the line may be made to satisfy any other given condition; it may, for example, be made to pass through any other given point, may be made parallel, or perpendicular to a given line. Hence equation (3) represents, for different values of λ, all straight lines through the point of intersection of (1) and (2).

The other condition which any particular line is made to satisfy will give an equation for the determination of the value of λ.

Ex. Find the equation of a straight line passing through the point of intersection of $2x + 5y - 4 = 0$ and $4x - 2y + 2 = 0$, and perpendicular to the line

$$2x - 4y = 7. \qquad (1)$$

Any line through the intersection is given by

$$2x + 5y - 4 + \lambda(4x - 2y + 2) = 0,$$

or

$$(2 + 4\lambda)x + (5 - 2\lambda)y + (2\lambda - 4) = 0. \qquad (2)$$

Now (2) is perpendicular to (1) if (§ 48, III.)

$$2(2 + 4\lambda) - 4(5 - 2\lambda) = 0; \quad i.\ e., \text{ if } \lambda = 1.$$

$$\therefore \quad 6x + 3y = 2 \text{ is the required equation.}$$

EXAMPLES.

1. Find the equations of the lines joining the points (0, 0), (4, 2), (—1, 3), (—3, —4) to the point of intersection of the lines $2x + y = 2$ and $2x - 3y = 6$.

2. What is the equation of the straight line passing through the intersection of $4x - 2y = 4$ and $7x - 3y + 21 = 0$, and parallel to $9x - 4y = 0$?

3. Find the equations of the two lines passing through the intersection of $x-2y=1$ and $2x+5y+4=0$, the one parallel, the other perpendicular to $x+2y=0$.

4. Find the equations of the two lines passing through the intersection of $7x-5y=35$ and $8x-3y+24=0$, the one parallel to $y=2x$, the other perpendicular to $3x+4y=0$.

5. What is the equation of a line passing through the intersection of $3x-2y+12=0$ and $x+4y=20$, and (a) equally inclined to the axes? (b) whose slope is -2?

6. The distance of a line from the origin is 5, and it passes through the intersection of $2x+3y+11=0$ and $3x-5y=16$. Find its equation.

7. Find the equations of the two lines which pass through the intersection of $x+2y=0$ and $2x-y+8=0$, and touch the circle

$$x^2+y^2=9.$$

8. Find the equations of the two lines which pass through the intersection of $x+3y+9=0$ and $3x=y+13$, and touch the circle

$$(x+2)^2+(y-3)^2=25.$$

9. Find the equations of the diagonals of the rectangle whose sides are $x+2y=10$, $x+2y+2=0$, $2x-y=12$, and $2x-y=16$, without finding the coordinates of its vertices.

10. Show that if $S=0$ and $S'=0$ represent the equations of any two loci with terms all transposed to the first member, and λ denotes an arbitrary constant, then the locus represented by the equation

$$S+\lambda S'=0$$

will pass through all the common points of the two given loci.

Consider the two cases $\lambda=0$, and $\lambda=\infty$.

11. Find the equation of the circle which passes through the origin and the common points of the circles

$$x^2+y^2=25 \quad \text{and} \quad x^2+y^2-18x+20=0.$$

12. Find the equation of the circle which passes through the common points of

$$x^2+y^2=16 \quad \text{and} \quad x-y=4,$$

and (1) passes through the origin, (2) touches the x-axis.

13. A circle passes through the common points of

$$x^2+y^2=25 \quad \text{and} \quad x-4y+13=0,$$

and cuts the x-axis in two coincident points. Find its equation.

EQUATIONS REPRESENTING TWO OR MORE STRAIGHT LINES.

53.* *The straight lines represented by* n *equations of the first degree may be represented by a single equation of the* nth *degree.*

Let
$$S_1 \equiv A_1X + B_1y + C_1 = 0,$$
$$S_2 \equiv A_2X + B_2y + C_2 = 0,$$
$$S_3 \equiv A_3X + B_3y + C_3 = 0,$$
$$\cdots\cdots\cdots\cdots$$
$$S_n \equiv A_nX + B_ny + C_n = 0,$$
be the equations of n straight lines.

Taking the product of these n expressions gives
$$S_1S_2S_3 \ldots S_n = 0. \qquad (1)$$

Equation (1) is satisfied by the coordinates of all the points, and no others, which satisfy the separate equations $S_1 = 0, \ldots S_n = 0$; because the product $S_1S_2S_3 \ldots S_n = 0$ when, and only when, at least one of its factors is zero. Therefore all points which are on the n given lines, and no other points, are on the locus of (1). But (1) is an equation of the nth degree, hence the proposition.

Conversely, if an expression of the nth *degree can be separated into* n *factors of the first degree, it will represent* n *straight lines** *when equated to zero.*

Observe that in the given equations all terms must be transposed to the first member before we multiply or factor.

E. g. Let the given equations be $y - x - a = 0$, and $y + x - a = 0$.

Since $$(y - x - a)(y + x - a) \equiv y^2 - x^2 - 2ay + a^2, \qquad (1)$$
the same locus will be represented by
$$(y - x - a)(y + x - a) = 0 \quad \text{and} \quad y^2 - x^2 - 2ay + a^2 = 0. \qquad (2)$$

The given equations may, however, be written
$$(3)\ \ y = a + x, \quad \text{or} \quad (4)\ \ y - x = a,$$
and
$$(5)\ \ y = a - x, \quad \text{or} \quad (6)\ \ y + x = a.$$

Multiplying (3) by (5) and (4) by (6) gives, respectively,
$$(7)\ \ y^2 = a^2 - x^2, \quad \text{or} \quad x^2 + y^2 = a^2, \quad \text{a circle;}$$
and
$$(8)\ \ y^2 - x^2 = a^2, \quad \text{a hyperbola (§ 36), instead of two lines.}$$

The first members of (2) are identities, *i. e.*, the same for all values of x and y; but equations (3), (5), (7), and (8) are merely consistent; *i. e.*, they

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* For this reason factors of the first degree are sometimes called *linear factors.*

are satisfied by the pair of values $x=0$, $y=a$, and by no other *real* pair. Hence we observe that the loci of identities equated to zero *coincide*, whereas the loci of consistent equations merely *concur*.

Ex. Show that the equation $S_1 S_2 S_3 \ldots S_n = 0$ represents all the loci of $S_1=0$, $S_2=0$, $S_3=0 \ldots S_n=0$, whatever the form of the expressions $S_1, S_2, S_3 \ldots S_n$ may be.

Is the locus of $\dfrac{S_1}{S_2}=0$, or $S_1+S_2=0$, the same as the loci of $S_1=0$ and $S_2=0$?

The equation of any two straight lines may therefore be written in the form

$$(lx+my+n)(l'x+m'y+n')=0,$$

or

$$ll'x^2+(lm'+l'm)xy+mm'y^2+(ln'+l'n)x + (mn'+m'n)y+nn'=0.$$

This equation contains terms involving x^2, y^2, xy, x, y, and a constant, or all possible terms involving x and y of a degree not higher than the second. A notation which is in general use for such an expression is

$$ax^2+2hxy+by^2+2gx+2fy+c,$$

and when equated to zero is called **The General Equation of the Second Degree.**

Hence, the most general equation which represents two straight lines is a form assumed by the general equation of the second degree. An equation of the second degree, however, can not be separated into linear factors unless a certain relation holds between the coefficients of its terms, and therefore does not always represent a line pair. *E. g.*, $x^2 \pm y^2 = 1$, $y^2 = x$, $xy \doteq a^2$ are not line pairs.

In general, as will be shown in Chap. VII, an equation of the second degree represents a *Conic Section.*

54.* *To find the condition that the general equation of the second degree may represent two straight lines.*

The necessary and sufficient condition that (§ 53)

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

may represent a pair of straight lines is

$$ax^2+2hxy+by^2+2gx+2fy+c \equiv (lx+my+n)(l'x+m'y+n').$$

These two expressions are identically equal if their coefficients are respectively equal; *i. e.*, if (§ 53)

$$a = ll', \quad b = mm', \quad c = nn',$$

$$2h = lm' + l'm, \quad 2g = ln' + l'n, \quad 2f = mn' + m'n.$$

The continued multiplication of the last three of these equations gives

$$8fgh = 2ll'mm'nn' + ll'(m^2n'^2 + m'^2n^2) + mm'(n^2l'^2 + n'^2l^2) + nn'(l^2m'^2 + l'^2m^2)$$

$$= 2ll'mm'nn' + ll'[(mn' + m'n)^2 - 2mm'nn'] + mm'[(nl' + n'l)^2 - 2nn'll'] + nn'[(lm' + l'm)^2 - 2ll'mm']$$

$$= 2abc + a(4f^2 - 2bc) + b(4g^2 - 2ac) + c(4h^2 - 2ab).$$

$$\therefore \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0, \qquad (1)$$

or

$$\triangle \equiv \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0, \qquad (2)$$

is the required condition.

This determinant is called the **Discriminant** of the General Equation. The general equation of the second degree therefore represents two straight lines if its discriminant vanishes.

Def. If the sum of the exponents of x and y is the same in each term of an algebraic function (§ 28), it is called a *homogeneous* function of x and y.

E. g. $2x^3 - 4x^2y + 3xy^2 - y^3$ is a homogeneous function of x and y of the third degree.

55.* *A homogeneous function of the* n*th degree can be separated into* n *homogeneous linear factors, and therefore, when equated to zero, represents* n *straight lines, real or imaginary, through the origin.*

Let the function be

$$y^n + K_1y^{n-1}x + K_2y^{n-2}x^2 + K_3y^{n-3}x^3 + \ldots + K_nx^n. \quad (1)$$

This is identically equal to

$$x^n\left[\left(\frac{y}{x}\right)^n + K_1\left(\frac{y}{x}\right)^{n-1} + K_2\left(\frac{y}{x}\right)^{n-2} + K_3\left(\frac{y}{x}\right)^{n-3} + \ldots + K_n\right]. \quad (2)$$

The polynomial factor in (2) is a function of the nth degree in $\left(\frac{y}{x}\right)$, and therefore has n roots, real or imaginary. (§ 90.)

Let $m_1, m_2, m_3 \ . \ . \ . \ m_n$ be these roots. Then (2) is identically equal to (§ 89)

$$x^n\left[\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m^2\right)\left(\frac{y}{x} - m_3\right) \ . \ . \ . \ \left(\frac{y}{x} - m_n\right)\right], \qquad (3)$$

or $$(y - m_1x)(y - m_2x)(y - m_3x) \ . \ . \ . \ (y - m_nx). \qquad (4)$$

When (4), which is identically equal to (1), is equated to zero, it represents the n straight lines (§ 53),

$$y - m_1x = 0, \quad y - m_2x = 0, \quad y - m \quad = 0, \ . \ . \ . \ y - m_nx = 0,$$

all of which go through the origin.

56.* *To find the equation of the lines joining the origin to the common points of*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \qquad (1)$$

and $$lx + my = n. \qquad (2)$$

Equation (2) may be written

$$\frac{lx + my}{n} = 1. \qquad (3)$$

Making equation (1) homogeneous and of the second degree by means of (3), we get

$$ax^2 + 2hxy + by^2 + 2(gx + fy)\left(\frac{lx + my}{n}\right) + c\left(\frac{lx + my}{n}\right)^2 = 0, \qquad (4)$$

which is the required equation.

Equation (4), being homogeneous and of the second degree, represents two straight lines through the origin (§ 55). Moreover, the coordinates of the common points of the two given loci satisfy both equation (1) and equation (3), and therefore satisfy (4). For values of x and y which satisfy (3) make $\frac{lx + my}{n}$ equal to unity, and therefore give the same result when substituted in (4) as when substituted in (1); *i. e.*, if they satisfy (1) they will also satisfy (4).

Therefore the two lines (4) pass through the common points of (1) and (2).

In the same manner we may make the equation of any curve homogeneous by means of (3) and obtain the equation of the straight lines joining the origin to the points common to the line (2) and the given curve.

Ex. 1. Find the equation of the lines through the origin and the points common to

$$x^2 + xy - 6x - 3y + 9 = 0, \quad \text{and} \quad y + 3x = 7.$$

The equation required is

$$x^2 + xy - 3(2x + y)\left(\frac{y+3x}{7}\right) + 9\left(\frac{y+3x}{7}\right)^2 = 0,$$

which on reduction gives

$$2x^2 - xy - 6y^2 \equiv (x - 2y)(2x + 3y) = 0.$$

Ex. 2. If S and S' be two homogeneous functions of the same degree, and K, K' be two constants, show that

$$(S + K) + \lambda(S' + K') = 0$$

will be the equation of the straight lines through the origin and the common points of $S + K = 0$ and $S' + K' = 0$, if λ be so chosen that $K + \lambda K' = 0$.

57.* *To find the angle between the two straight lines represented by the homogeneous equation*

$$ax^2 + 2hxy + by^2 = 0. \qquad (1)$$

Let the separate equations of the two lines be

$$y - m_1x = 0, \quad \text{and} \quad y - m_2x = 0. \qquad (2)$$

Then $y^2 + 2\frac{h}{b}xy + \frac{a}{b}x^2 \equiv (y - m_1x)(y - m_2x)$ [§ 55] (3)

$$\equiv y_2 - (m_1 + m_2)xy + m_1m_2x^2. \qquad (4)$$

Equating the coefficients of xy and x^2 in (4) gives

$$m_1 + m_2 = -2\frac{h}{b}, \quad \text{and} \quad m_1m_2 = \frac{a}{b}. \qquad (5)$$

Whence $m_1 - m_2 \equiv \sqrt{(m_1 + m_2)^2 - 4m_1m_2} = \frac{2}{b}\sqrt{h^2 - ab}.$ (6)

If φ be the angle between the lines (2),

$$\tan\varphi = \frac{m_1 - m_2}{1 + m_1m_2}. \qquad [(4), \S\ 48.]$$

Therefore from (5) and (6) we get

$$\tan\varphi = \frac{2\sqrt{h^2 - ab}}{a + b}. \qquad (7)$$

If $h^2 - ab > 0$, the lines (1) are real.

If $h^2 - ab = 0$, the lines (1) are coincident.

If $h^2 - ab < 0$, the lines (1) are imaginary.

If $a + b = 0$, *i. e.*, if the *sum of the coefficients of x^2 and y^2 is zero*, $\tan \varphi = \infty$, and the two lines given by (1) are at right angles to one another.

Ex. Show that equation (7) also gives the angle between the two lines represented by the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

when the discriminant is zero.

58.* *To find the equation of the straight lines bisecting the angles between the two lines given by*

$$ax^2 + 2hxy + by^2 = 0. \qquad (1)$$

Let γ_1 and γ_2 be the angles which the lines given by (1) make with the x-axis; then

$$y^2 + 2\frac{h}{b}xy + \frac{a}{b}x^2 \equiv (y - x \tan \gamma_1)(y - x \tan \gamma_2).$$

$$\therefore \quad \tan \gamma_1 + \tan \gamma_2 = -\frac{2h}{b}, \quad \tan \gamma_1 \tan \gamma_2 = \frac{a}{b}. \quad [\S\, 57, (5).] \qquad (2)$$

Let γ be the angle that *one* of the bisectors makes with the x-axis; then will

$$\gamma = \tfrac{1}{2}(\gamma_1 + \gamma_2), \quad \text{or} \quad \gamma = \tfrac{1}{2}(\gamma_1 + \gamma_2 + \pi).$$

Hence for either value of γ,

$$\tan 2\gamma = \tan (\gamma_1 + \gamma_2);$$

i. e.

$$\frac{2 \tan \gamma}{1 - \tan^2 \gamma} = \frac{\tan \gamma_1 + \tan \gamma_2}{1 - \tan \gamma_1 \tan \gamma_2}. \qquad (3)$$

If (x, y) be any point on either bisector, then

$$\tan \gamma = \frac{y}{x}. \qquad (4)$$

Substituting (2) and (4) in (3) gives

$$\frac{2xy}{y^2 - x^2} = \frac{-2h}{a - b}. \qquad (5)$$

$$\therefore \quad h(x^2 - y^2) = (a - b)xy \qquad (6)$$

is the required equation, since it is the relation between the coordinates of any point on either of the bisectors.

EXAMPLES.

Find the separate equations of the lines represented by the following equations, and determine the angle between each pair:

1. $x^2-9=0$.
2. $xy+3x-2y=6$.
3. $8x^2+24xy+10y^2=0$.
4. $x^3-6x^2+11x-6=0$.
5. $4x^2-24xy+11y^2=0$.
6. $4x^2+20xy+9y^2=0$.
7. $2x^2+3xy-2y^2=0$.
8. $y^3+xy^2-14x^2y-24x^3=0$.
9. $x^2+2xy\sec\theta+y^2=0$.
10. $x^2+2xy\cot 2\theta-y^2=0$.
11. $x^2+2xy\csc 2\theta+y^2=0$.
12. $y^2\cot^2\theta+2xy+x^2\sin^2\theta=0$.

What loci are represented by

13. $y^4=16a^2x^2$.
14. $(x^2+y^2)^2-4r^2x^2=0$.
15. $y^2+(x^3-x^4)y-x^7=0$.
16. $y^4+(x-x^3)y^2-x^4=0$.

17. Find the equations of the bisectors of the angles between the pairs of lines given in examples 3, 5, 6, 7, 9, and 10.

18. Show that the two straight lines

$$(x^2-y^2)\sin 2\phi+2(x\sin\phi-y\cos\phi)^2\cot\theta=2xy\cos 2\phi$$

include an angle θ.

Show that the following equations represent straight lines; find their point of intersection and the angle between them: [Solve for x or y.]

19. $x^2+3xy+2y^2-3x-3y=0$.
20. $10x^2-13xy-3y^2+16x+10y-8=0$.
21. $x^2-xy-2y^2-x-4y-2=0$.
22. $2x^2+5xy-3y^2+6x-10y-8=0$.
23. $2x^2-3xy-2y^2-10x-10y-12=0$.
24. $4x^2+12xy+9y^2-18x-27y+18=0$.

Find the respective values of λ for which the following equations represent line pairs:

25. $x^2-4y^2-2x+8y+\lambda=0$.
26. $12x^2-xy+\lambda y^2+2x+7y-2=0$.
27. $\lambda x^2-7xy-5y^2-14x+32y-12=0$.
28. $x^2-4xy+4y^2+3x-6y+\lambda=0$.
29. $\lambda xy+3y^2-2x-12y+12=0$.

30. $x^2+5xy+6y^2+\lambda x-12y-48=0.$

31. $12x^2+2\lambda xy-3y^2+10x+25y-28=0.$

32. $6x^2+xy-15y^2+\lambda x+50y-40=0.$

33. $x^2-6xy+\lambda y^2-8x+8\lambda y+16=0.$

34. $x^2+\lambda xy+9y^2-8x-4\lambda y+16=0.$

35. What are the conditions that the equations

$$ax^2+by^2+gx+gy=0 \quad \text{and} \quad ay^2+hxy+gx+fy=0,$$

may each represent a pair of straight lines?

36. Find the equations of the straight lines passing through the origin and the points of intersection of

(1) $y^2=4x$ and $2x+y=12.$

(2) $(x-6)^2+(y-2)^2=20$ and $y+3x=10.$

What is the angle between the last pair of lines?

37. Find the angle between the lines which join the origin to the common points of

$$3x^2-xy-2y^2+10x+8=0 \quad \text{and} \quad 4y-3x=10.$$

38. Show that the lines through the origin and the points of intersection of

$$x^2+y^2=2 \quad \text{and} \quad y=mx+2$$

are at right angles if $m=\pm\sqrt{3}$.

39. Show that the straight lines joining the origin to the points of intersection of the straight line $x-y=2$ and the curve

$$2y^2-2xy-3x^2-4x+4y+4=0$$

make equal angles with the axes.

40. Prove that the angle between the lines joining the origin to the points common to the straight line $x+2y=1$ and the curve

$$10y^2-7xy+4x^2-2x-4y+1=0 \text{ is } \tan^{-1}\frac{7}{9}.$$

41. Find the equation of the straight lines passing through the origin and the common points of

$$\frac{x^2}{25}+\frac{y^2}{9}=1 \quad \text{and} \quad x^2+y^2=16.$$

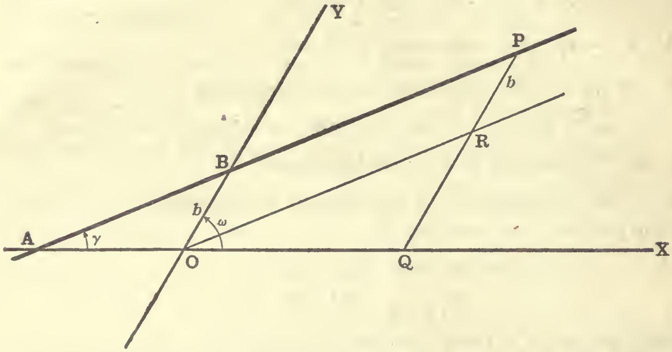
42. Show that the lines passing through the origin and the points common to

$$\frac{x^2}{4a^2}+\frac{y^2}{a^2}=1 \quad \text{and} \quad 5(x^2+y^2)=8a^2$$

are perpendicular to each other.

Oblique Axes.

59. *To find the equation of a straight line referred to axes inclined at an angle ω.*



Let ABP be any line meeting the y-axis at a distance b from the origin, and making an angle γ with the x-axis.

Draw PQ parallel to the y-axis and OR parallel to the given line ABP.

Let $P(x, y)$ be any point on the line ABP; then

$$OQ = x, \quad \text{and} \quad QR = QP - RP = y - b.$$

Since $\angle ORQ = \angle ROY = \omega - \gamma$, we also have

$$\frac{y-b}{x} = \frac{QR}{OQ} = \frac{\sin QOR}{\sin ORQ} = \frac{\sin \gamma}{\sin(\omega - \gamma)}.$$

$$\therefore \quad y = \frac{\sin \gamma}{\sin(\omega - \gamma)} x + b, \tag{1}$$

which is the required equation.

Let $$m = \frac{\sin \gamma}{\sin(\omega - \gamma)} = \frac{\tan \gamma}{\sin \omega - \cos \omega \tan \gamma}. \tag{2}$$

Then $$\tan \gamma = \frac{m \sin \omega}{1 + m \cos \omega}, \tag{3}$$

and equation (1) becomes

$$y = mx + b, \tag{4}$$

which in oblique coordinates represents a straight line inclined to the x-axis at an angle $\tan^{-1}\left(\frac{m \sin \omega}{1 + m \cos \omega}\right)$.

60. Some of the investigations in the preceding sections of this chapter apply to oblique as well as to rectangular axes. Let the student show that this is true of the following equations:

$$\frac{x}{a}+\frac{y}{b}=1, \qquad [(1), \S\ 41]$$

$$y-y_1=m(x-x_1), \qquad [(3), \S\ 46]$$

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}. \qquad [(3), \S\ 47]$$

61.* *To find the angle between two straight lines whose equations, referred to axes inclined at an angle* ω, *are*

$$y=mx+b \quad \textit{and} \quad y=m'x+b'.$$

If γ and γ' are the angles which these lines make respectively with the x-axis, then [§ 59, (3)]

$$\tan\gamma=\frac{m\sin\omega}{1+m\cos\omega}, \quad \tan\gamma'=\frac{m'\sin\omega}{1+m'\cos\omega}. \qquad (1)$$

Whence
$$\tan(\gamma-\gamma')=\frac{\dfrac{m\sin\omega}{1+m\cos\omega}-\dfrac{m'\sin\omega}{1+m'\cos\omega}}{1+\dfrac{m\sin\omega}{1+m\cos\omega}\cdot\dfrac{m'\sin\omega}{1+m'\cos\omega}}. \qquad (2)$$

$$\therefore \quad \tan\varphi=\frac{(m-m')\sin\omega}{1+(m+m')\cos\omega+mm'}, \qquad (3)$$

where $\varphi=\gamma-\gamma'$, the angle between the lines.

The two given lines are parallel if $m=m'$.

They are perpendicular to one another if

$$1+(m+m')\cos\omega+mm'=0. \qquad (4)$$

If the equations of the given lines are

$$Ax+By+C=0 \quad \text{and} \quad A'x+B'y+C'=0,$$

then
$$m=-\frac{A}{B} \quad \text{and} \quad m'=-\frac{A'}{B'}.$$

Substituting these values in (3) we have

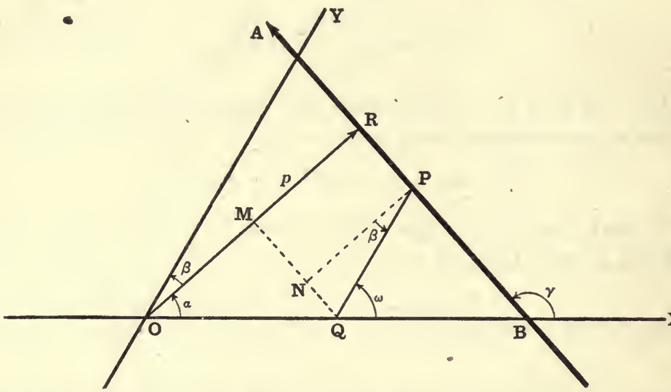
$$\tan\varphi=\frac{(A'B-AB')\sin\omega}{AA'+BB'-(AB'+A'B)\cos\omega}. \qquad (5)$$

The lines will be parallel if $A'B - AB' = 0$.

They will be perpendicular to one another if

$$AA' + BB' - (AB' + A'B) \cos \omega = 0. \tag{6}$$

62. *To find the equation of a straight line in terms of p, the perpendicular upon it from the origin, and the angles α, β which p makes with the axes.*



Let AB be the given line and OR the perpendicular on it from the origin.

Let $P(x, y)$ be any point on AB. Draw QP parallel to the y-axis, and NP parallel to OR.

Then $\angle YOR = \angle NPQ = \beta$, $OQ = x$, $QP = y$.

Projecting OQ and QP on OR gives

$$OQ \cos \alpha + QP \cos \beta = OR.$$

$$\therefore \quad x \cos \alpha + y \cos \beta = p, \tag{1}$$

is the required equation.

Let $\gamma = \angle XBA = \angle XQM = \alpha + 90°$.

Then $\angle PQM = 90° - \beta = \gamma - \omega$.

$\therefore \quad \cos \alpha = \sin \gamma \quad \cos \beta = \sin (\gamma - \omega) = -\sin (\omega - \gamma)$,

and the equation (1) of the line may be written

$$x \sin \gamma - y \sin (\omega - \gamma) = p. \tag{2}$$

Equation (1), or (2), is called the *normal*, or *distance form* of the equation of the straight line when the axes are oblique.

The conventions as to the direction of p and AB, and the positive and negative sides of AB, are the same as in § 42.

63. *To change the general equation*

$$Ax + By + C = 0, \tag{1}$$

referred to oblique axes, to the distance form

$$x \cos \alpha + y \cos \beta - p = 0. \tag{2}$$

Since (1) and (2) represent the same line their first members are identically equal, and therefore their coefficients are proportional, *i. e.*

$$\frac{\cos \alpha}{A} = \frac{\cos \beta}{B} = \frac{-p}{C}. \tag{3}$$

$$\therefore \quad \frac{p^2}{C^2} = \frac{\cos^2 \alpha}{A^2} = \frac{\cos^2 \beta}{B^2} = \frac{2 \cos \alpha \cos \beta \cos (\alpha + \beta)}{2AB \cos (\alpha + \beta)}$$

$$= \frac{\cos^2 \alpha + \cos^2 \beta - 2 \cos \alpha \cos \beta \cos (\alpha + \beta)}{A^2 + B^2 - 2AB \cos (\alpha + \beta)}. \tag{4}$$

But

$$\cos^2 \alpha + \cos^2 \beta - 2 \cos \alpha \cos \beta \cos (\alpha + \beta) = \sin^2 (\alpha + \beta) = \sin^2 \omega,$$

since $\alpha + \beta = \omega$, the angle between the axes.

$$\therefore \quad \frac{\cos \alpha}{A} = \frac{\cos \beta}{B} = \frac{-p}{C} = \frac{\sin \omega}{\sqrt{A^2 + B^2 - 2AB \cos \omega}}. \tag{5}$$

Whence
$$\cos \alpha = \frac{A \sin \omega}{\sqrt{A^2 + B^2 - 2AB \cos \omega}}, \tag{6}$$

$$\cos \beta = \frac{B \sin \omega}{\sqrt{A^2 + B^2 - 2AB \cos \omega}}, \tag{7}$$

and
$$p = \frac{-C \sin \omega}{\sqrt{A^2 + B^2 - 2AB \cos \omega}}. \tag{8}$$

Substituting (6), (7), and (8) in (2) gives

$$\frac{(Ax + By + C) \sin \omega}{\sqrt{A^2 + B^2 - 2AB \cos \omega}} = 0, \tag{9}$$

which is the distance form of the general equation (1).

64. *To find the perpendicular distance of a given point* (x_1, y_1) *from the line whose equation is*

$$x \cos \alpha + y \cos \beta - p = 0, \tag{1}$$

$$x \sin \gamma - y \sin (\omega - \gamma) - p = 0, \tag{2}$$

or

$$Ax + By + C = 0, \tag{3}$$

where $\omega = \alpha + \beta$ *is the angle between the axes.*

The demonstration given in § 50 applies also when the axes are oblique. The required results are, respectively,

$$x_1 \cos \alpha + y_1 \cos \beta - p, \tag{4}$$

$$x_1 \sin \gamma - y_1 \sin (\omega - \gamma) - p, \tag{5}$$

and

$$\frac{(Ax_1 + By_1 + C) \sin \omega}{\sqrt{A^2 + B^2 - 2AB \cos \omega}}. \tag{6}$$

Observe that formulæ previously found independently for rectangular axes can now be derived from those here obtained for oblique axes by simply putting $\omega = 90°$. (*Cf.* §§ 7 and 10.)

EXAMPLES.

1. The axes being inclined at an angle of 60°, find the inclination to the x-axis of the straight lines

$$y = x - 3, \quad (\sqrt{3} - 1)y = 2x + (\sqrt{3} + 1), \quad 2y + x = 4.$$

2. If $\omega = 120°$, find the angle between the two lines

(1) $y + 3x = 3$ and $4y = x + 8$,

(2) $y = 3x - 2$ and $2y + x = 4$.

3. Show that when the angle between the axes is ω, the angle between the two lines

$$y - mx = 0 \text{ and } my + x = 0 \text{ is } \tan^{-1}\left(\frac{m^2+1}{m^2-1}\tan \omega\right).$$

4. Prove that the straight lines

$$y = x + c \quad \text{and} \quad y + x = b$$

are at right angles, whatever be the angle between the axes.

5. If the lines

$$3y - 2x = 3 \quad \text{and} \quad 7y + 8x + 14 = 0$$

are at right angles, what is the value of ω?

6. Show that the two points (x_1, y_1) and (x_2, y_2) are on the same side or on opposite sides of the line $Ax + By + C = 0$ according as $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ have the same sign or opposite signs, the axes being oblique.

7. Find the equations of the bisectors of the angles between the following lines when $\omega = 60°$:

(1) $y = x + 2$ and $y + 2x + 5 = 0$.

(2) $x + y + 3 = 0$ and $2y + (1 + \sqrt{6})x + 4 = 0$.

(3) $3x + 6y + 8 = 0$ and $6x + 3y = 10$.

8. If $\omega = 30°$, find the equation of the line through the point (2, 3), and (1) parallel to, (2) perpendicular to the line $3y = 2x$.

9. Find the length of the perpendicular drawn from the point (2, — 4) upon the line $3x + 6y + 11 = 0$, when $\omega = 60°$.

10. Find the equation to, and the length of the perpendicular drawn from the point (— 3, — 2) to the line $4x + 3y = 6$, when $\omega = 60°$.

11. Prove that the equation of the line which passes through the point (x_1, y_1) and is perpendicular to

(1) $y = 0$,

(2) $x = 0$,

(3) $x \sin\gamma + y \sin(\gamma - \omega) = p$

is, respectively,

(1) $(x - x_1) + (y - y_1)\cos\omega = 0$,

(2) $(x - x_1)\cos\omega + y - y_1 = 0$,

(3) $(x - x_1)\cos\gamma + (y - y_1)\cos(\gamma - \omega) = 0$.

12. Show that the equation of the line through the point (x_1, y_1) perpendicular to the line

$$y = mx + b$$

may be written $$y - y_1 = -\frac{1 + m\cos\omega}{m + \cos\omega}(x - x_1).$$

13. Show that the lines

$$x + y\cos\omega = b\cos\omega \quad \text{and} \quad x\cos\omega + y = b$$

are perpendicular to the axes of x and y respectively.

14. If $y = mx + b$ and $y = m'x + b'$ make equal angles with the x-axis and are not parallel, prove that

$$m + m' + 2mm'\cos\omega = 0.$$

15. Find the equations of the sides of a regular hexagon when two of the sides which meet in a vertex are the axes of coordinates.

16. PA and PB are the perpendiculars upon the axes from the point $P(a, b)$; if ω be the angle between the axes, prove that

$$AB = \sin\omega\sqrt{a^2 + b^2 + 2ab\cos\omega}.$$

17. Show also that the length of the perpendicular from P on AB in Ex. 16 is

$$\frac{ab \sin^2 \omega}{\sqrt{a^2 + b^2 + 2ab \cos \omega}},$$

and that its equation is $ax - by = a^2 - b^2$.

18. From each vertex of a parallelogram a perpendicular is drawn upon the diagonal which does not pass through that vertex, and these are produced to form another parallelogram; show that its diagonals are perpendicular to the sides of the first parallelogram and that they both have the same centre.

19. If the axes be inclined at an angle ω, show that the equation of a straight line through (x_1, y_1) making a given angle γ with the x-axis may be written

$$(x - x_1) \sin \gamma - (y - y_1) \sin (\omega - \gamma) = 0,$$

and also that this equation is in the distance form.

20. Find the angle between the lines

$$ax^2 + 2hxy + by^2 = 0,$$

when the axes are inclined at an angle ω. Show also that these lines are at right angles to one another if

$$a + b - 2h \cos \omega = 0.$$

Examples on Chapter III.

1. What are the loci of the following equations?

(1) $x^2 + axy = 0$. (2) $x^3 - xy^2 = 0$.

(3) $x^3 + y^3 = 0$. (4) $x^3 - y^3 = 0$.

(5) $a^2x^2 - b^2y^2 = 0$. (6) $a^2x^2 + b^2y^2 = 0$.

(7) $(x^2 - 1)(y^2 - 4) = 0$. (8) $(ax + by)^2 = c^2$.

(9) $y^2 - (x - a)^2 = 0$. (10) $(x - a)^2 + (y - b)^2 = 0$.

(11) $(x - a)^2 - (y - b)^2 = 0$. (12) $x^3 - x^2y + xy^2 - y^2 = 0$.

(13) $\rho = a \sec (\theta - \alpha)$.

2. Find the angle between the two lines $3x = 4y + 7$ and $5y = 12x + 6$; also the equations of the two lines which pass through the point $(4, 5)$ and make equal angles with the two given lines.

3. Find the length of the perpendicular from the origin upon the line passing through the points

$$(a \cos \alpha, a \sin \alpha) \quad \text{and} \quad (a \cos \beta, a \sin \beta).$$

4. What is the equation of the line through the intersection of the two straight lines

$$bx + ay = ab \quad \text{and} \quad y = mx,$$

and perpendicular to the former?

5. Prove that the equation of the two straight lines which pass through the origin and make an angle α with the line $y + x = 0$ is

$$x^2 + 2xy \sec 2\alpha + y^2 = 0.$$

6. Find the perimeter, altitudes, and area of the triangle whose vertices are at the points (3, 5), (7, 9), (9, 11).

7. If p and p' be the perpendiculars from the points $(\pm\sqrt{a^2 - b^2}, 0)$ upon the line

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1,$$

prove

$$pp' = b^2.$$

8. Find the equations of the sides of the square of which the points (2, −3) and (6, 5) are two opposite vertices.

9. Show that the equation

$$y^3 - x^3 + 3xy(y - x) = 0$$

represents three straight lines equally inclined to one another.

10. Prove that the equation

$$y^2(\cos\alpha + \sqrt{3}\sin\alpha)\cos\alpha - xy(\sin 2\alpha - \sqrt{3}\cos 2\alpha) + x^2(\sin\alpha - \sqrt{3}\cos\alpha)\sin\alpha = 0$$

represents two lines inclined at an angle of 60° to each other.

11. For what value of m will the lines

$$bx + ay = ab, \quad ax + by = ab, \quad y = mx$$

meet in a point?

12. Show that the lines

$$y = m_1x + b_1, \quad y = m_2x + b_2, \quad y = m_3x + b_3$$

will meet in a point if

$$\frac{m_3 - m_1}{m_2 - m_1} = \frac{b_3 - b_1}{b_2 - b_1}.$$

13. From a point P perpendiculars PM and PN are drawn upon two fixed lines which are inclined at an angle ω and meet in O. Take the two fixed lines as axes of coordinates and find the locus of P if

(1) $OM + ON = 2c$. (2) $OM - ON = 2c$.

(3) $PM + PN = 2c$. (4) $PM - PN = 2c$.

(5) $MN = 2c$. Ans. to (5). $x^2 + 2xy\cos\omega + y^2 = 4c^2\csc^2\omega$.

14. Find the points of intersection of the loci

$$\rho \cos(\theta - \tfrac{1}{3}\pi) = a \quad \text{and} \quad \rho \cos(\theta - \tfrac{1}{6}\pi) = a;$$

15. also of $\rho \cos(\theta - \tfrac{1}{2}\pi) = \tfrac{3}{4}a$ and $\rho = a \sin\theta$.

16. OA and OB are two fixed straight lines, A and B being fixed points; P and Q are any two points on these lines such that the ratio $AP:BQ$ is constant. Show that the locus of the middle point of PQ is a straight line.

17. PM and PN are perpendiculars from a point P on two fixed straight lines which meet in O; MQ and NQ are drawn parallel to the fixed lines to meet in Q; prove that if the locus of P is a straight line, the locus of Q will also be a straight line.

18. $ABCD$ is a parallelogram. Taking A as pole and AB as initial line, find the polar equations of the four sides and two diagonals.

19. A straight line moves so that the sum of the reciprocals of its intercepts on two fixed intersecting lines is constant; show that it passes through a fixed point.

20. The distance of a point (x_1, y_1) from each of two straight lines, which pass through the origin, is d; prove that the two lines are given by

$$(x_1 y - x y_1)^2 = d^2(x^2 + y^2).$$

21. Show that the six bisectors of the angles formed by the lines

$$x \cos a_1 + y \sin a_1 = p_1,$$

$$x \cos a_2 + y \sin a_2 = p_2, \quad x \cos a_3 + y \sin a_3 = p_3,$$

meet in sets of three in four different points. What are these four points?

22. Prove that the three altitudes of a triangle meet in a point.

23. Prove that the three perpendicular bisectors of the sides of a triangle meet in a point.

24. Find the area of the triangle formed by the lines

$$y + 3x = 6, \quad y = 2x - 4, \quad y = 4x + 3.$$

25. Show that the area of the triangle formed by the lines

$$y = m_1 x + b_1, \quad y = m_2 x + b_2, \quad \text{and} \quad x = 0,$$

is

$$\tfrac{1}{2}\frac{(b_1 - b_2)^2}{m_1 - m_2}.$$

26. Show that the area of the triangle formed by the lines

$$y = m_1 x + b_1, \quad y = m_2 x + b_2, \quad \text{and} \quad y = m_3 x + b_3$$

is

$$\tfrac{1}{2}\left[\frac{(b_1 - b_2)^2}{m_1 - m_2} + \frac{(b_2 - b_3)^2}{m_2 - m_3} + \frac{(b_3 - b_1)^2}{m_3 - m_1}\right]. \quad \text{(Use Ex. 25.)}$$

27. What is the area of the triangle whose sides are the lines

$$3x + 4y + 12 = 0, \quad 2x + y = 4, \quad 5x - 3y = 15?$$

28. Find the equation of the pair of lines joining the origin to the intersections of the straight line $y = mx + b$ and the circle $x^2 + y^2 = r^2$. Show that these lines will be at right angles if

$$2b^2 = r^2(1 + m^2);$$

and coincident if $$b^2 = r^2(1 + m^2).$$

29. Prove that the straight lines joining the origin to the points of intersection of the line $bx + ay = 2ab$ with the circle

$$(x - a)^2 + (y - b)^2 = r^2$$

will be at right angles if $a^2 + b^2 = r^2$.

30. Show that the two straight lines joining the origin to the other points of intersection of the two curves

$$ax^2 + 2hxy + by^2 + 2gx = 0$$

and $$a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$$

will be perpendicular to one another if

$$g'(a + b) = g(a' + b').$$

31. Prove that the angle between the two lines joining the origin to the intersections of the line $2y = 3x + 2$ with the curve

$$10x^2 - 14xy + 3y^2 - 5x + 2y - 2 = 0 \text{ is } \tan^{-1}\tfrac{5}{8}.$$

32. Show that $bx^2 - 2hxy + ay^2 = 0$ represents two straight lines at right angles respectively to the two straight lines

$$ax^2 + 2hxy + by^2 = 0.$$

33. If the pairs of straight lines

$$x^2 - 2pxy - y^2 = 0 \quad \text{and} \quad x^2 - 2qxy - y^2 = 0$$

be such that each pair bisects the angles between the other pair, prove that $$pq = -1.$$

34. Find the locus of the vertex of a triangle which has a given base and a given difference of base angles.

35. The product of the perpendiculars drawn from a point $P(x', y')$ on the lines

$$x \cos\theta + y \sin\theta = a \quad \text{and} \quad x \cos\phi + y \sin\phi = a$$

is equal to the square of the perpendicular drawn from P on the line

$$x \cos\tfrac{1}{2}(\theta + \phi) + y \sin\tfrac{1}{2}(\theta + \phi) = a \cos\tfrac{1}{2}(\theta - \phi).$$

Show that the equation of the locus of P is

$$x'^2 + y'^2 = a^2.$$

36. Prove that the straight lines

$$ax^2 + 2hxy + by^2 = 0,$$

make equal angles with the x-axis if $h = a \cos \omega$, the axes being inclined at an angle ω.

37. If ω be the angle between the axes, show that the lines given by the equation

$$x^2 + 2xy \cos \omega + y^2 \cos 2\omega = 0$$

are at right angles to one another.

38. Prove that the general equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents two parallel straight lines if

$$h^2 = ab \quad \text{and} \quad bg^2 = af^2.$$

Prove also that the distance between them is

$$2\sqrt{\frac{g^2 - ac}{a(a+b)}}.$$

39. Show that the product of the perpendiculars from the point (x', y') upon the two lines

$$ax^2 + 2hxy + by^2 = 0$$

is

$$\frac{ax'^2 + 2hx'y' + by'^2}{\sqrt{(a-b)^2 + 4h^2}}.$$

40. Show that the pair of lines given by

$$ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$$

is equally inclined to the pair given by

$$ax^2 + 2hxy + by^2 = 0. \qquad \text{(Use § 58.)}$$

41. Show also that the pair

$$a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$$

is equally inclined to the same pair.

42. If the general equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a pair of straight lines, prove that the equation of the other pair of lines meeting the axes in the same points is

$$ax^2 + 2\left(\frac{2fg}{c} - h\right)xy + by^2 + 2gx + 2fy + c = 0.$$

43. Prove that the three lines represented by the equation

$$x(x^2 - 3y^2) = my(y^2 - 3x^2)$$

make equal angles with one another.

(HINT. Show that the polar equation is $m \tan 3\theta + 1 = 0$.)

44. Show that the condition that two of the lines represented by the equation

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 = 0$$

may be at right angles is

$$A^2 + 3AC + 3BD + D^2 = 0.$$

SUG. The given equation must be equivalent to

$$(lx + my)(x^2 + \lambda xy - y^2) = 0.)$$

45. Show that $\left(\frac{x}{y} - \frac{y}{x}\right)^2 + \left(\frac{x}{y} - \frac{y}{x}\right) - 6 = 0$ represents two pairs of perpendicular lines through the origin.

46. If $z \equiv \frac{x}{y} - \frac{y}{x}$, show that

$$z^n + az^{n-1} + bz^{n-2} + \ldots kz + l = 0$$

represents n pairs of perpendicular lines through the origin.

47. Show that $x^4 + 4(x^2 - y^2)xy - x^2y^2 + y^4 = 0$ represents two pairs of perpendicular lines through the origin.

48. Show that the equation

$$a(x^4 + y^4) - 4bxy(x^2 - y^2) + 6cx^2y^2 = 0$$

represents two pairs of straight lines at right angles to one another, and that the two pairs will coincide if

$$2b^2 = a^2 + 3ac.$$

CHAPTER IV.

TRANSFORMATION OF COORDINATES, OR CHANGE OF AXES.

65. The formulæ for changing an equation from rectangular to polar coordinates and *vice versa* have already been found in § 6, and their usefulness amply illustrated. Moreover, the equation of a curve in any system of coordinates is sometimes greatly simplified by referring it to a new set of axes of the same system. Hence, it is also desirable to be able to deduce from the equation of a curve referred to one set of axes its equation referred to another set of axes of the same system. Either of these operations is known as a **Transformation of Coordinates,** or **Change of Axes.**

The equations, which express the relations between the two sets of coordinates of *the same point*, and by means of which these operations are performed, are called **Formulæ of Transformation.**

Change of Axes in Cartesian Coordinates.

66. *To change from one set of rectilinear axes inclined at an angle ω to any other set inclined at an angle ω'.*

Let OX and OY be the positive directions of the original axes, $O'X'$ and $O'Y'$ the positive directions of the new axes; let $XOY = \omega$, and $X'O'Y' = \omega'$, be positive angles less than π.

Let $O'X'$ meet OX and OY in A and B respectively.

Let $\angle XAX' = \theta$, then $\angle X'BY = \omega - \theta$.

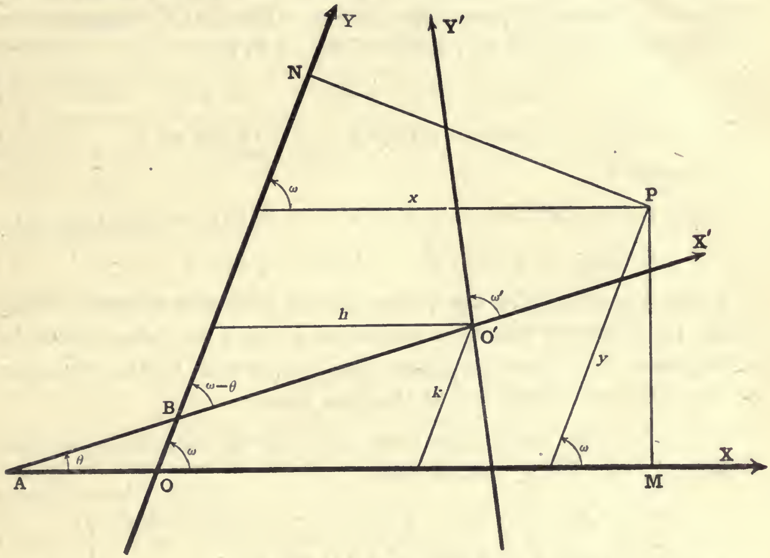
Let h, k be the coordinates of the new origin O' referred to the original axes.

The equation of any line referred to the new axes may be written in the distance form [(2), § 62]

$$x' \sin \gamma' - y' \sin (\omega' - \gamma') = p', \qquad (1)$$

where γ' is the angle the line makes with $O'X'$, p' is the distance from O' to the line, and the primes are used to denote that the equation is referred to the new axes.

For the line OX the positive direction of p' is *downward* (§ 42), while the positive direction of k is *upward;* hence p' and k have the *same* sign. For OY the positive direction of p' is toward the *right,* or the same as the positive direction of h; hence p' and h have *opposite* signs.



Therefore, for all relative positions of the two pairs of axes we have, since $\sin\omega$ is positive,

for OX, $\quad \gamma' = \angle X'AX = -\angle XAX' = -\theta,$

$p' = \textit{Dist. from } O' \textit{ to } OX = k \sin\omega;$

for OY, $\quad \gamma' = \angle X'BY = \omega - \theta,$

$p' = \textit{Dist. from } O' \textit{ to } OY = -h\sin\omega.$

Therefore the equations of OX and OY referred to the new axes are, respectively, from (1)

$$x'\sin(-\theta) - y'\sin[\omega' - (-\theta)] = k\sin\omega, \qquad (2)$$

and $$x'\sin(\omega-\theta) - y'\sin[\omega' - (\omega-\theta)] = -h\sin\omega. \qquad (3)$$

When (2) and (3) are written in the form

$$x'\sin\theta + y'\sin(\omega'+\theta) + k\sin\omega = 0, \qquad (4)$$

and $$x'\sin(\omega-\theta) - y'\sin(\omega' - \omega + \theta) + h\sin\omega = 0, \qquad (5)$$

the positive sides of OX and OY with reference to these equations (§ 50) are the same as their positive sides when they are considered as the original axes of coordinates.

Let P be any point whose coordinates are x, y referred to OX and OY, and x', y' referred to $O'X'$ and $O'Y'$.

Draw PM and PN perpendicular to OX and OY, respectively.

Then from (5) and (4) we get [(5), § 64].

$$NP = x \sin \omega = x' \sin(\omega - \theta) - y' \sin(\omega' - \omega + \theta) + h \sin \omega, \quad (6)$$

$$MP = y \sin \omega = x' \sin \theta + y' \sin(\omega' + \theta) + k \sin \omega. \quad (7)$$

Whence

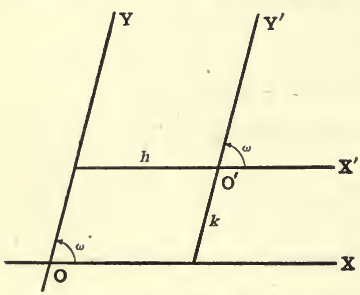
$$\left. \begin{aligned} x &= [x' \sin(\omega - \theta) - y' \sin(\omega' - \omega + \theta)] \csc \omega + h, \\ y &= [x' \sin \theta + y' \sin(\omega' + \theta)] \csc \omega + k. \end{aligned} \right\} \quad (8)$$

These formulæ give the values of the old coordinates of any point in terms of the new coordinates; and if these values be substituted in a given equation, the result will be the equation of the *same* curve referred to the new axes.

When the origin remains the same, and only the direction of the axes is changed, $h = k = 0$, *and we have*

$$\left. \begin{aligned} x &= [x' \sin(\omega - \theta) - y' \sin(\omega' - \omega + \theta)] \csc \omega, \\ y &= [x' \sin \theta + y' \sin(\omega' + \theta)] \csc \omega. \end{aligned} \right\} \quad (9)$$

These general formulæ (8) and (9) are rarely used in the manner suggested above, but some of the forms which they take in certain particular cases are of great importance.



To change the origin to the point (h, k) *without changing the direction of the axes.*

Since in this case the new axes are parallel respectively to the old,

$$\omega' = \omega$$

and

$$\theta = 0.$$

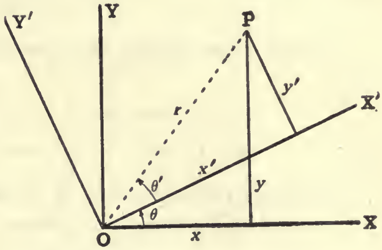
Substituting these values in (8) we get

$$\left. \begin{aligned} x &= x' + h, \\ y &= y' + k. \end{aligned} \right\} \quad (10)$$

As these equations are independent of ω, they hold for both rectangular and oblique coordinates.

Hence to find what a given equation becomes when the origin is moved to the point (h, k), the new axes being parallel to the old, substitute $x' + h$ for x and $y' + k$ for y. After the substitution is made we can write x and y instead of x' and y'; so that practically this transformation is effected by simply *writing* $x + h$ *in the place of* x, *and* $y + k$ *in the place of* y.

To turn a set of rectangular axes through an angle θ *without changing the origin.*



In this case

$$\omega = \omega' = 90°,$$

and the general formulæ (9) reduce to

$$\left.\begin{aligned} x &= x' \cos\theta - y' \sin\theta, \\ y &= x' \sin\theta + y' \cos\theta. \end{aligned}\right\} \qquad (11)$$

If at the same time the origin be changed to the point (h, k), *the required formulæ will be*

$$\left.\begin{aligned} x &= x' \cos\theta - y' \sin\theta + h. \\ y &= x' \sin\theta + y' \cos\theta + k. \end{aligned}\right\} \qquad (12)$$

This transformation is clearly obtained by combining the two formulæ (10) and (11).

Ex. 1. Prove by means of (11)

$$\cos(\theta + \theta') = \cos\theta \cos\theta' - \sin\theta \sin\theta',$$

$$\sin(\theta + \theta') = \sin\theta \cos\theta' + \cos\theta \sin\theta'.$$

Ex. 2. Show by the use of (12) that the area of a triangle is the same function* of the coordinates of its vertices referred to any set of rectangular axes.

To turn a set of oblique axes through an angle θ *without changing the origin, we have, since* $\omega' = \omega$,

$$\left.\begin{aligned} x &= [x' \sin(\omega - \theta) - y' \sin\theta] \csc\omega, \\ y &= [x' \sin\theta + y' \sin(\omega + \theta)] \csc\omega. \end{aligned}\right\} \qquad (13)$$

* It can now be shown that formulæ proved for points in the first quadrant will hold for points in any quadrant. (See note under § 7.)

To pass from rectangular axes to oblique without changing the origin, we have

$$\left.\begin{aligned} x &= x' \cos\theta + y' \cos(\omega' + \theta), \\ y &= x' \sin\theta + y' \sin(\omega' + \theta). \end{aligned}\right\} \qquad (14)$$

If, in making this transformation, the x-axis is not changed, $\theta = 0$ and the required formulæ are

$$\left.\begin{aligned} x &= x' + y' \cos\omega', \\ y &= y' \sin\omega'. \end{aligned}\right\} \qquad (15)$$

To pass from oblique to rectangular axes having the same origin, we have

$$\left.\begin{aligned} x &= [x' \sin(\omega - \theta) - y' \cos(\omega - \theta)] \csc\omega, \\ y &= [x' \sin\theta + y' \cos\theta] \csc\omega. \end{aligned}\right\} \qquad (16)$$

If this transformation is effected without changing the x-axis, $\theta = 0$ and these formulæ reduce to

$$\left.\begin{aligned} x &= x' - y' \cot\omega. \\ y &= y' \csc\omega. \end{aligned}\right\} \qquad (17)$$

What do the formulæ (13), (14), (15), (16) become when the origin is also changed to the point (h, k)?

Observe that in making all these transformations attention must be paid to the *signs* of h, k, and θ.

67. *The degree of an equation can not be altered by any change of the axes.*

The expressions giving x and y in terms of x' and y' are linear, that is, always

$$x = lx' + my' + n, \quad y = l'x' + m'y' + n',$$

where either l or m is not zero, and also either l' or m'.

Any term in $f(x, y)$, say $ax^p y^q$, becomes

$$a(lx' + my' + n)^p (l'x' + m'y' + n')^q,$$

and contains at least one term of degree $p + q$, either

$$all'x'^{p+q}, \quad alm'x'^p y'^q, \quad al'mx'^p y'^q, \quad \text{or} \quad amm'y'^{p+q};$$

perhaps other terms also, but none of *higher* degree.

Hence the degree of an equation can not be *raised* by any transformation of coordinates. Neither can it be *lowered;* for if it were, by changing back to the original axes, and therefore to the original equation, the degree would be raised.

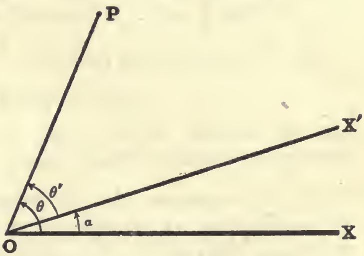
Otherwise. The degree of an equation (and also of its locus) is the number of points in which its locus is cut by a straight line; and this number can not be altered by any transformation of coordinates.

TRANSFORMATION IN POLAR COORDINATES.

68. *To turn the initial line through an angle α without changing the pole.*

Let angle $XOX' = \alpha$.

Let P be any point whose coordinates are ρ, θ, referred to OX, and ρ, θ', referred to OX'.



Then $\theta = \theta' + \alpha$, while ρ is not changed.

Hence the desired transformation is effected by simply writing $\theta + \alpha$ in the place of θ. [*Cf.* § 66, (10).]

69. *To change the pole, the direction of the initial line remaining the same, or changed by an angle α.*

This transformation can be performed by first changing the given equation to rectangular coordinates [§ 6, (2)]; then moving the origin to the new pole [§ 66, (10)]; then transforming back to polar coordinates [§ 6, (1)]; and finally, if desired, turning the initial line through the angle α (§ 68).

EXAMPLES.

Transform to parallel axes through the point $(-3, 2)$

1. $y^2 - 4x + 4y + 16 = 0$.
2. $2x^2 + 3y^2 - 12x + 2y + 29 = 0$.

What are the equations of the following loci when referred to parallel axes throughout the point (a, b)?

3. $(x-a)^2 + (y-b)^2 = r^2$.
4. $xy - ax - by + ab = a^2$.
5. $y^2 - 2by + 4ax = 4a^2 - b^2$.
6. $b^2(x^2 - 2ax) + a^2(y^2 - 2by) + a^2b^2 = 0$.

Transform by turning rectangular axes through an angle of 45°.

7. $x^2 - y^2 = a^2$.

8. $7x^2 - 2xy + 7y^2 = 2$.

9. $2(y + x) = (y - x)^3$.

10. $ax^2 + 2hxy + ay^2 = 1$.

11. $x^4 + 6x^2y^2 + y^4 = 2$.

12. $2xy(x^2 + y^2) + 1 = 0$.

13. Transform $\frac{x}{a} + \frac{y}{b} = 1$ by turning the axes through $\tan^{-1} \frac{a}{b}$.

14. What does $2x^2 - 3xy - 2y^2 = 5a^2$ become when the axes are turned through $\tan^{-1} 2$?

15. If the axes be turned through an angle of 30°, what does the equation $9x^2 - 2\sqrt{3}xy + 11y^2 = 4$ become?

16. Show that the equation

$$2x^2 + xy - y^2 + 3x - y + 2 = 0$$

can be reduced to $2x^2 + xy - y^2 = 0$, by transforming to parallel axes through a properly chosen point.

17. The equation of a line referred to axes inclined at 30° is $y = 3x - 2$. Show that its equation referred to axes inclined at 60°, the origin and x-axis not being changed, is $(3 + \sqrt{3})y = 3x - 2$.

18. The equation of a curve referred to axes inclined at 60° is $2x^2 + 5xy + y^2 = 2$. Find its equation referred to rectangular axes such that the two x-axes coincide.

19. Transform $y^2 + 4ay \cot \beta = 4ax$ from rectangular to oblique axes meeting at an angle β, leaving the x-axis and origin unchanged.

20. Show that the formulæ for transforming from rectangular axes OX, OY to oblique axes OX', OY', such that the angle $X'OY' = \omega$, and OX bisects the angle $X'OY'$, are

$$x = (y' + x') \cot \tfrac{1}{2}\omega, \quad y = (y' - x') \sin \tfrac{1}{2}\omega.$$

21. Apply the formulæ of Ex. 20 to the equations

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1, \quad \text{when } \omega = 2 \tan^{-1} \frac{b}{a}.$$

22. Prove that the formulæ for passing from axes inclined at an angle ω to axes bisecting the angles between the original axes are

$$x = \tfrac{1}{2}(x' \sec \tfrac{1}{2}\omega - y' \csc \tfrac{1}{2}\omega), \quad y = \tfrac{1}{2}(x' \sec \tfrac{1}{2}\omega + y' \csc \tfrac{1}{2}\omega).$$

Use the formulæ of Ex. 22 and thus transform

23. $x^2 + xy + y^2 = 8$, when $\omega = 60°$.

24. $\left\{ \begin{aligned} 2(x^2 + y^2) &= a^2 + b^2 \\ 4xy &= a^2 + b^2 \end{aligned} \right\}$, when $\omega = 2 \tan^{-1} \frac{b}{a}$.

Transformation of Functions of Two Linear Expressions.

70.* The equations giving the values of x and y in terms of x' and y' [(8), § 66] may be written

$$\left.\begin{aligned} x \sin \omega &= \lambda x' + \mu y' + \nu, \\ y \sin \omega &= \lambda' x' + \mu' y' + \nu'. \end{aligned}\right\} \quad (1)$$

In like manner we should find the equations giving x' and y' in terms of x and y to be

$$\left.\begin{aligned} x' \sin \omega' &= lx + my + n, \\ y' \sin \omega' &= l'x + m'y + n', \end{aligned}\right\} \quad (2)$$

where
$$lx + my + n = 0 \quad (3)$$
and
$$l'x + m'y + n' = 0 \quad (4)$$
are the equations, *in the distance form*, of the *new* axes $O'Y'$ and $O'X'$, respectively, referred to the *old;* and ω' is the angle between $O'X'$ and $O'Y'$.

If, then, the given equation is a function of the linear expressions $lx + my + n$ and $l'x + m'y + n'$, the new equation is obtained at once by writing $x' \sin \omega'$ in the place of $lx + my + n$, and $y' \sin \omega'$ in the place of $l'x + m'y + n'$.

That is, if the given equation be

$$f(lx + my + n,\ l'x + m'y + n') = 0, \quad (5)$$

the new equation referred to the lines (3) and (4) will be

$$f(x' \sin \omega',\ y' \sin \omega') = 0; \quad (6)$$

or, if the new axes be rectangular,

$$f(x', y') = 0. \quad (7)$$

71. Illustrative Examples.

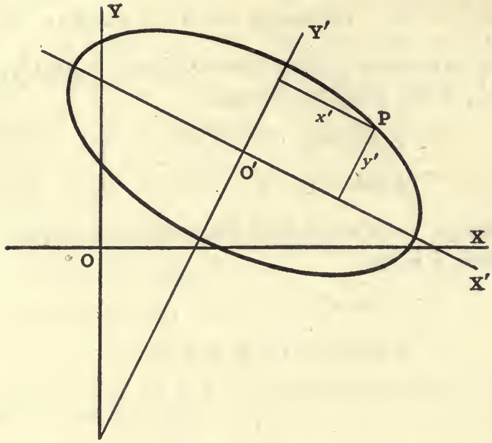
Ex. 1. *What is the locus of the equation*

$$(2x - y - 4)^2 + 4(x + 2y - 7)^2 = 80? \quad (1)$$

Dividing by 5 gives

$$\left(\frac{2x - y - 4}{\sqrt{5}}\right)^2 + 4\left(\frac{x + 2y - 7}{\sqrt{5}}\right)^2 = 16, \quad (2)$$

where the linear expressions within the parentheses are both in the distance form.



Take $x+2y-7=0$ for the new x-axis, $O'X'$, and $2x-y-4=0$ for the new y-axis, $O'Y'$.

Then, since the new axes are rectangular,

$$x'=\frac{2x-y-4}{\sqrt{5}},\qquad y'=\frac{x+2y-7}{\sqrt{5}}.\qquad [(2),\ \S\ 70]$$

Writing x' in the place of $\frac{2x-y-4}{\sqrt{5}}$, and y' in the place of $\frac{x+2y-7}{\sqrt{5}}$ in (2) gives for the *new* equation

$$x'^2+4y'^2=16,\quad\text{or}\quad \frac{x'^2}{16}+\frac{y'^2}{4}=1, \tag{3}$$

which represents an ellipse (§ 34) whose semi-axes are 4 and 2.

Ex. 2. *What is the locus of*

$$(3x-4y+12)^2=5(4x+3y+4)\,? \tag{1}$$

Dividing by 25 gives

$$\left(\frac{3x-4y+12}{5}\right)^2=\left(\frac{4x+3y+4}{5}\right). \tag{2}$$

Take $3x-4y+12=0$ for the new x-axis, $O'X'$, and $4x+3y+4=0$ for the new y-axis, $O'Y'$.

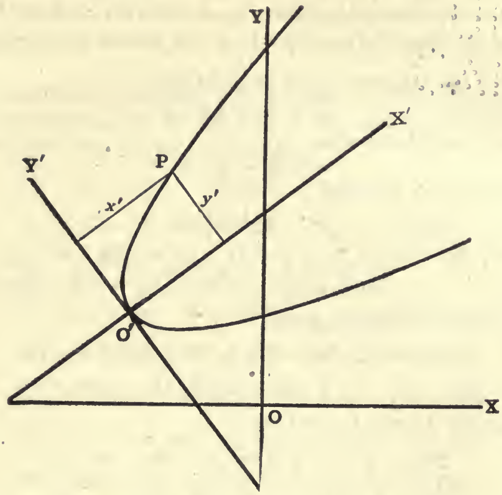
Then since angle $X'O'Y'=90°$,

$$x'=\frac{4x+3y+4}{5},\quad y'=\frac{3x-4y+12}{5}.\qquad [(2),\ \S\ 70]$$

Therefore the new equation referred to $O'X'$ and $O'Y'$ is

$$y'^2=x'. \tag{3}$$

Hence the locus is a parabola (§ 37), and lies on the *positive* side of the line $4x+3y+4=0$.



Ex. 3. *Find the equation of the locus represented by*

$$5(2x-y-3)^2+(3x-4y-8)^2=45, \tag{1}$$

when the new x-axis is the line

$$3x-4y-8=0, \tag{2}$$

and the new y-axis is the line

$$2x-y-3=0. \tag{3}$$

Equation (1) may be written

$$\left(\frac{2x-y-3}{\sqrt{5}}\right)^2+\left(\frac{3x-4y-8}{5}\right)^2=\frac{9}{5}. \tag{4}$$

If ω' be the angle between the lines (2) and (3), then from equation (5), § 48, $\tan\omega'=\frac{1}{2}$, and therefore $\sin\omega'=\frac{1}{\sqrt{5}}$.

Let p and q be the perpendiculars drawn from any point on (4) to the lines (3) and (2) respectively; then from (2) § 70, or (5) § 50,

$$p=x'\sin\omega'=\frac{x'}{\sqrt{5}}=\frac{2x-y-3}{\sqrt{5}}, \tag{5}$$

$$q=y'\sin\omega'=\frac{y'}{\sqrt{5}}=\frac{3x-4y-8}{5}. \tag{6}$$

$$\therefore\ \left(\frac{x'}{\sqrt{5}}\right)^2+\left(\frac{y'}{\sqrt{5}}\right)^2=\frac{9}{5},\quad\text{or}\quad x'^2+y'^2=9, \tag{7}$$

is the required equation.

The locus is enclosed by the lines $x'=\pm 3$, and $y'=\pm 3$; construct it.

Observe that if we substitute p and q in (4) we get

$$p^2+q^2=\frac{9}{5}, \tag{8}$$

which is also an equation of the locus referred to the same new axes, but expressed in terms of *perpendiculars* upon the axes instead of *parallels* to the axes; *i. e.* we have the equation in a new system of coordinates.

Ex. 4. *Find the equation of the straight line*

$$9x + 7y + 14 = 0 \tag{1}$$

when the new x-axis is the line

$$x + 3y + 6 = 0,$$

and the new y-axis is the line

$$3x - y + 3 = 0.$$

Assume $9x + 7y + 14 \equiv l(3x - y + 3) + m(x + 3y + 6) + k$

$$\equiv (3l + m)x + (3m - l)y + (3l + 6m + k). \tag{2}$$

Equating coefficients in (2) gives

$$3l + m = 9, \quad 3m - l = 7, \quad 3l + 6m + k = 14.$$

Whence $\qquad l = 2, \quad m = 3, \quad k = -10.$

Hence equation (1) may be written

$$2(3x - y - 3) + 3(x + 3y + 6) - 10 = 0, \tag{3}$$

or
$$2\left(\frac{3x - y - 3}{\sqrt{10}}\right) + 3\left(\frac{x + 3y + 6}{\sqrt{10}}\right) - \sqrt{10} = 0. \tag{4}$$

$$\therefore \quad 2x' + 3y' = \sqrt{10} \tag{5}$$

is the required equation, since the new axes are rectangular.

EXAMPLES.

If the lines $x - y + 1 = 0$ and $x + y = 2$ be taken as new axes, what are the equations of the lines

1. $x = 2, y = 3$?
2. $5x + y - 4 + 3\sqrt{2} = 0$?
3. $x - 11y + 16 = 0$?
4. $ax + by + c = 0$?

When referred to the lines $3x - 4y + 4 = 0$ and $4x + 3y = 6$ as axes, what are the equations of the lines

5. $18x + y = 4$?
6. $x - 18y + 14 = 0$?
7. $8x - 31y = 20$?
8. $22x - 21y + 6 = 0$?
9. $7x - y = 0$?
10. $4x + 5y + 20 = 0$?

Find the equations of the following lines when the lines $2x - y = 4$ and $x + 2y = 6$ are taken as axes:

11. $2x - 11y - 12 + \sqrt{5} = 0.$
12. $7x - 6y - 5 = 0.$
13. $12x - y = 22.$
14. $x + 12y = 7.$

Find the equations of the following straight lines in oblique coordinates, the new axes being

$$y = 3x + 6 \quad \text{and} \quad 3y = x + 3:$$

15. $4y - 4x - 9 + \sqrt{10} = 0.$

16. $11x + 3y = 8.$

17. $x + 5y + 5 = 0.$

18. $11y - 17x = 26.$

Find the equations of the loci represented by the following equations when the lines represented by the linear expressions which they contain are chosen for the new axes of coordinates:

19. $(4x + 3y + 15)^2 = 5(3x - 4y).$

20. $9(2x - 3y + 4)^2 + 4(3x + 2y - 5)^2 = 468.$

21. $(x + y - 4)^2 + 4(x - y + 2) = 0.$

22. $3(x + 3y - 4)^2 - 4(3x - y + 6)^2 = 120.$

23. $4(2x - 4y + 7)^2 + 5(2x - y + 7)^2 = 80.$

24. $4(5x + 12y + 24)^2 - (12x - 5y + 15)^2 = 676.$

25. $(y - 3x + 3)^2 = 20(3y - x - 6).$

26. $3(3x - 4y - 12)^2 + 10(2x - y + 4)^2 = 150.$

27. $5(x - 3y - 4)^2 + 4(x + 2y + 2)^2 = 200.$

28. $(y - 3x + 3)^2 - 2(y + 2x - 4)^2 = 8.$

29. $2(x + y) = (y - x)^3.$

30. $\sqrt{2}(y - x)^2 = (x + y - 2)^3.$

31. $(x + 2y + 4)(2x - y)^2 = 50\sqrt{5}.$

32. $(x - y)^2 + 2(x - y)(x + y + 1) - (x + y + 1)^2 = 2.$

33. $(lx + my + n)(l'x + m'y + n') = 0.$

34. $(y - 3x + 3)(y + 2x - 4) = 25\sqrt{2}.$

35. Show that when the lines $2x - y + 2 = 0$ and $x + 2y = 0$ are the axes of coordinates the equation of the locus given by

$$2x^2 + 3xy + 23y^2 + 2x - 26y + 13 = 0$$

is

$$x^2 - 3xy + 4y^2 + 2\sqrt{5}(x - 2y) + 1 = 0.$$

36. Transform the equation

$$9x^2 - 16xy - 4y^2 + 2x - 9y + 48 = 0$$

to the oblique axes whose equations are

$$y - 3x = 0 \quad \text{and} \quad y + 2x + 1 = 0.$$

37. Find the equations of the old axes referred to the new, and check the results obtained in numbers 35 and 36 by passing back to the original equations.

Parameters of Two Loci as Coordinates of Points.

72.* The point (a, b) has been defined as the point for which $x = a$ and $y = b$; *i. e.* the point of intersection of the two lines whose equations are

$$x - a = 0 \quad \text{and} \quad y - b = 0.$$

Now a and b are the parameters of these lines, and for every pair of values of a and b they both have a definite position. If a and b vary continuously and in such a manner that $b = f(a)$, these lines change their positions simultaneously and continuously, and their common point will describe a continuous* curve whose equation is $y = f(x)$.

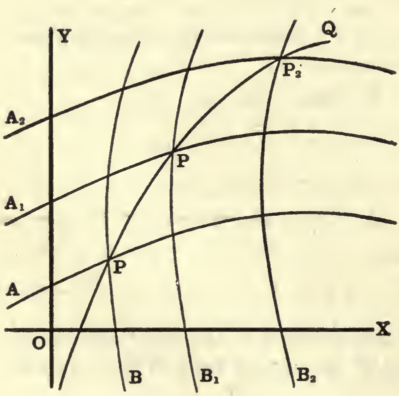
Likewise in polar coordinates the point (r, α) is the intersection of the circle and the straight line

$$\rho = r \quad \text{and} \quad \theta = \alpha;$$

and here also the coordinates of the point are the parameters of the curves whose intersection determines the position of the point. If these parameters vary so that $r = f(\alpha)$, the intersection of these two curves will move so that $\rho = f(\theta)$.

Hence, in both the Cartesian and polar systems, the *coordinates of a point* may be regarded as the *parameters of two loci* whose intersection determines the position of the point, and *vice versa.*

These are but special cases of the following general principle:



73.* Let

$$F(x, y, a) = 0 \qquad (1)$$

and $$F_1(x, y, b) = 0 \qquad (2)$$

be algebraic equations of two curves, a and b being arbitrary parameters.

If particular values be assigned to these parameters, two fixed curves A and B will be obtained which intersect in P. Now if a varies while b

* See second note under § 73.

remains constant, the curve A will change its position while the curve B remains fixed, and hence P will move along B. In like manner, if b varies while a is constant, P will move along the fixed curve A. If all possible values be assigned to a and b independently, the curves A and B move independently and all points in the plane will be obtained, provided the curves A and B both sweep over the whole plane. Moreover, to each pair of values of a and b there corresponds the same finite number of points, the number depending upon the degree of (1) and (2) in x and y; and to each point a finite number of values of a and b depending upon the degree of (1) and (2) in a and b, respectively.*

Hence a and b may be called the **Parameter Coordinates** of the point P.

If we solve (1) and (2) for a and b respectively, the results may be written in the form

$$a = \varphi_1(x, y), \qquad b = \varphi_2(x, y), \tag{3}$$

which express the relations between parameter and Cartesian coordinates.

Suppose the parameters a and b are not both arbitrary, but must satisfy the equation

$$b = f(a). \tag{4}$$

Then a variation in a, however small, causes a variation in b; and for every displacement of the curve A, however small, there is a simultaneous displacement of the curve B. Hence P can take only a finite number of positions on any single curve; *i. e.* P can not move to all places in the plane. If we assign to a the particular value a_1, b will have the particular value b_1, and we obtain the two curves A_1 and B_1 which intersect in P_1. Likewise when $a = a_2$, $b = b_2$, and we have the curves A_2 and B_2 meeting in P_2, and so on.

If the parameter a varies continuously,† then will b also vary continuously, the two curves A and B will be displaced in a continuous manner, and therefore P will describe a continuous curve PQ.

* There is not a *one to one* correspondence between a pair of values of a and b, and the position of P, unless (1) and (2) are of the first degree in x and y, and also in a and b.

† A quantity is said to vary continuously from one value p to another value q when it passes through all values intermediate to p and q without at any stage making a sudden jump.

It is here assumed that b is a continuous function of a. See § 87; also Chrystal's Algebra, Vol. I, Chap. XV, § 2 and § 5.

The form of PQ will evidently depend upon equation (4), which may therefore be called the equation of the locus of the intersection of (1) and (2) expressed in *parameter coordinates.*

74.* *To find the locus of the common points of two curves whose equations involve one and the same independent variable parameter.*

Let
$$F(x, y, a) = 0 \tag{1}$$
and
$$F_1(x, y, b) = 0 \tag{2}$$
be the equations of two loci involving the variable parameters a and b which are connected by the equation
$$b = f(a). \tag{3}$$

Eliminating b from (2) by means of (3) gives
$$F_1[x, y, f(a)] = 0, \tag{4}$$
and we have the equations (1) and (4) of the two given curves expressed in terms of the same arbitrary parameter. If we treat (1) and (4) simultaneously and eliminate a we obtain an equation of the form
$$\varphi(x, y) = 0. \tag{5}$$

Now (1), (4), and (5) form a consistent system of equations; *i. e. all values of x and y which satisfy both* (1) *and* (4) *also satisfy* (5). But values of x and y which satisfy both (1) and (4) are the coordinates of the common points of their loci. Hence the coordinates of all points common to the two given curves satisfy equation (5). Moreover, since (5) does not involve a, it is satisfied by the coordinates of all points common to the loci of (1) and (4) whatever the value of a may be; *i. e.* by the coordinates of all points on the curve described by these common points as a varies.

Therefore (5) is the equation of the required locus.

Hence, to find the locus of the common points of two curves whose equations involve one arbitrary parameter, treat the two equations simultaneously and eliminate the arbitrary parameter.

When the given equations contain two dependent parameters, as (1) and (2), connected by an equation, such as (3), the required equation (5) is found by eliminating a and b from the three equations (1), (2), and (3), as has just been shown. This can be accomplished directly by substituting equations (3), § 73, in equation (3) above.

Hence equations (3), § 73, may be called the formulæ of transformation for changing an equation from parameter to Cartesian coordinates.

If one or both of the given equations contain both a and b, the equations can each be expressed in terms of the same parameter; or, values of a and b can be found in terms of x and y as before. Hence the locus is found in the same manner.

Likewise if the given equations contain n parameters connected by $n-1$ relations, only one parameter can be arbitrary; for by means of the $n-1$ equations between the parameters the values of all can be found in terms of one, and the given equations can then be expressed in terms of that one.

In such cases the locus is found by eliminating the n parameters between the $n+1$ given equations.*

A few examples will suffice to make this general theory clear.

Bi-Polar Coordinates.

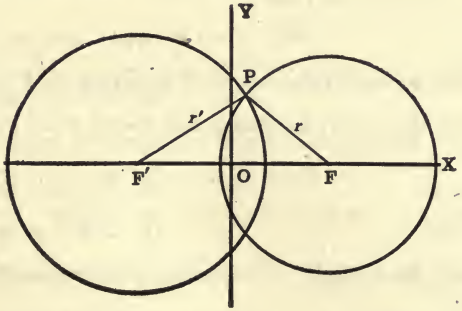
75.* Let the variable curves be the two circles whose equations are (§ 32)

$$(x-c)^2+y^2=r^2 \qquad (1)$$

and

$$(x+c)^2+y^2=r'^2, \qquad (2)$$

r and r' being the variable parameters.



For each pair of values of r and r', such that $r+r'>2c$ and $|r-r'|<2c$, these circles intersect in two real points P. If all

* A fuller discussion of this subject is given in Chap. III, Book II, of Briot and Bouquet's Elements of Analytical Geometry, translated by J. H. Boyd.

positive values, which satisfy these conditions, be assigned to r and r', every point in the plane will be obtained.

The variables r and r', which represent the distances of the point P from the fixed points F and F', are called the **Bi-Polar Coordinates** of the point P.

If r and r' vary continuously in such a manner that

$$r' = f(r), \tag{3}$$

then the circles are displaced simultaneously and continuously and their common points describe a continuous curve, of which (3) is the *bi-polar* equation.

Solving (1) and (2) for r and r', respectively, gives

$$r = \sqrt{(x-c)^2 + y^2}, \qquad r' = \sqrt{(x+c)^2 + y^2}, \tag{4}$$

which are the formulæ for passing from bi-polar to rectangular coordinates; where $(c, 0)$ and $(-c, 0)$ are the two fixed points of the bi-polar system.

EXAMPLES.

Find the locus of P in rectangular coordinates when its bi-polar coordinates satisfy the following equations:

1. $r = r'$.
2. $r \pm r' = 2c$.
3. $r + r' = 2a$. (*Cf.* § 34.)
4. $r - r' = 2a$. (*Cf.* § 36.)
5. $r' = nr$. (See Ex. 24, p. 50.)
6. $rr' = c^2$. Ans. $(x^2+y^2)^2 = 2c^2(x^2-y^2)$.
7. Trace $rr' = K$ for various values of K, taking $(\pm 1, 0)$ for poles.

Using the points $(\pm c, 0)$ for poles, find the bi-polar equations of the following loci:

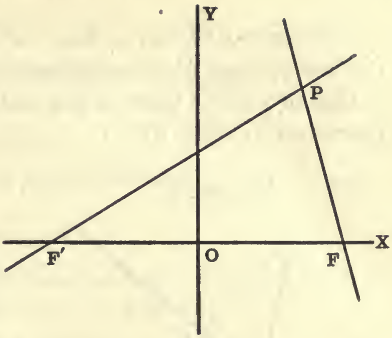
8. $x^2 + y^2 = c^2$.
9. $x^2 + y^2 = a^2$.
10. $x = a$, $x = c$, $y = b$.
11. $2(x^2 - y^2) = c^2$.

12. Show that the formulæ for changing from rectangular to bi-polar coordinates are

$$x = \frac{r'^2 - r^2}{4c}, \quad y = \frac{\sqrt{16c^2r^2 - (r'^2 - r^2 - 4c^2)^2}}{4c}.$$

13. Show that the bi-polar equation of parabola $y^2 = 4cx$ may be written

$$r + 2c = \sqrt{r'^2 + 8c^2}.$$



76.* Let the two given equations be

$$y = m(x - c) \qquad (1)$$

and $$y = m'(x + c), \qquad (2)$$

m and m' being arbitrary parameters.

These lines pass through the fixed points $F(c, 0)$ and $F'(-c, 0)$, respectively, for all values of m and m'. When the values of m and m' are given, the directions of these lines are determined and the position of P can be found.

Hence m and m' are coordinates of the point P.

If m and m' may take any values independently, P will move to any position in the plane, but if they are connected by the equation

$$m' = f(m), \qquad (3)$$

P will describe a definite curve (§ 73) whose form depends upon equation (3). The equation of this curve in rectangular coordinates will be found by substituting in (3) the values

$$m = \frac{y}{x - c}, \qquad m' = \frac{y}{x + c}, \qquad (4)$$

given by (1) and (2). (See § 74.)

EXAMPLES.

1. What are the coordinates of O, F, F', $(0, c)$, (c, c), $(-c, c)$, $(c, 0)$, $(-c, 0)$ in terms of m and m'?

Find the locus of P in rectangular coordinates when

2. $m' = km$. Consider the special case $k = -1$.

3. $mm' = k$. Discuss the result for positive and negative values of k, especially ± 1.

4. $m + m' = k$.

5. $m' - m = k$.

6. $m(a - c) = m'(a + c)$.

7. $2cmm' = b(m - m')$.

8. $2mm' = a(m + m')$.

9. $m^2m'^2 = m^2 - m'^2$.

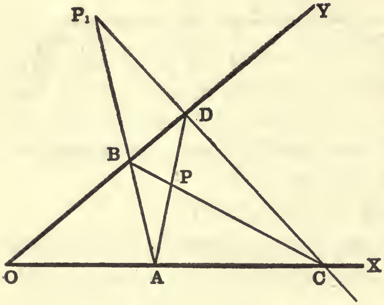
10. $\angle FPF' = \alpha$.

11. $\angle XFP + \angle XF'P = \alpha$.

Consider both acute and obtuse values of the constant angle α.

Let angle $XFP = \gamma = \tan^{-1} m$, and angle $XF'P = \delta = \tan^{-1} m'$. Then γ, δ are also coordinates of P, such that for every *pair* of values of γ and δ there is *one* and *only one* position of P, except for points on the line FF'.*

77.* EXAMPLE.



Let OX and OY be two fixed lines and $P_1(x_1, y_1)$ a fixed point. Through P_1 draw a fixed secant P_1BA (meeting OX in A, OY in B), and the variable secant P_1DC (meeting OX in C, OY in D); also draw the lines AD and BC meeting in P. Find the locus of P.

Take the lines OX and OY as axes of coordinates.

The equation of P_1BA may be written (§ 60)

$$y - y_1 = m_1(x - x_1), \tag{1}$$

where m_1 is constant.

Similarly the equation of the variable secant P_1DC is

$$y - y_1 = m(x - x_1), \tag{2}$$

where m is the variable parameter.

Putting $x = 0$, and $y = 0$ in (1) and (2) we find

$$OA = x_1 - \frac{y_1}{m_1}, \qquad OB = y_1 - m_1 x_1,$$

$$OC = x_1 - \frac{y_1}{m}, \qquad OD = y_1 - m x_1.$$

Hence the equations of AD and BC are (§ 41 and § 60)

$$\frac{x}{x_1 - \frac{y_1}{m_1}} + \frac{y}{y_1 - m x_1} = 1 \tag{3}$$

and

$$\frac{x}{x_1 - \frac{y_1}{m}} + \frac{y}{y_1 - m_1 x_1} = 1. \tag{4}$$

Equations (3) and (4) contain one and the same variable parameter, m; hence the locus of P is found by eliminating m between these two equations. (§ 74.)

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*As a result of this general theory we may say that, in the most general sense, the point (a, b) is the intersection of two curves whose equations involve a and b as arbitrary parameters. When the two curves are straight lines parallel to two fixed lines we have Cartesian coordinates; when one is a circle with a fixed centre and the other a straight line through its centre, we have polar coordinates; when both curves are circles having fixed centres, we have bi-polar coordinates, etc.

By subtracting (4) from (3) we obtain

$$x\left(\frac{m}{y_1 - mx_1} - \frac{m_1}{y_1 - m_1x_1}\right) + y\left(\frac{1}{y_1 - mx_1} - \frac{1}{y_1 - m_1x_1}\right) = 0, \tag{5}$$

which simplified gives

$$y_1x + x_1y = 0. \tag{6}$$

Therefore the locus is a straight line passing through O.

EXAMPLES.

1. A trapezoid is formed by drawing a line parallel to the base of a triangle. Find the locus of the intersection of its diagonals.

2. Through a fixed point O a variable secant is drawn meeting two fixed parallel lines in R and Q; through R and Q straight lines are drawn in fixed directions, meeting in P. Find the locus of P.

3. The hypotenuse of a given right triangle slides between the axes of coordinates, its ends always touching the axes. Find the locus of the vertex of the right angle.

4. Find the locus of the intersection of the lines

$$\frac{x}{a} + \frac{y}{m} = 1 \quad \text{and} \quad \frac{x}{l} + \frac{y}{b} = 1,$$

where l and m are variable parameters, such that

$$a + m = b + l.$$

5. Find the locus of the centres of all rectangles which may be inscribed in a given triangle.

6. A variable quadrilateral is inscribed in a given rectangle so that its diagonals are perpendicular to each other and parallel to the sides of the rectangle. Find the locus of the intersection of its opposite sides.

7. Find the equation of the locus of a point at which two given portions of the same straight line subtend equal angles.

8. Find the locus of the intersection of the two lines

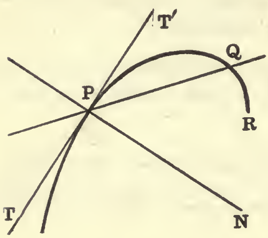
$$y - mx = a\sqrt{1 + m^2} \quad \text{and} \quad my + x = a\sqrt{1 + m^2}$$

for all values of m.

CHAPTER V.

SLOPE, TANGENTS AND NORMALS.

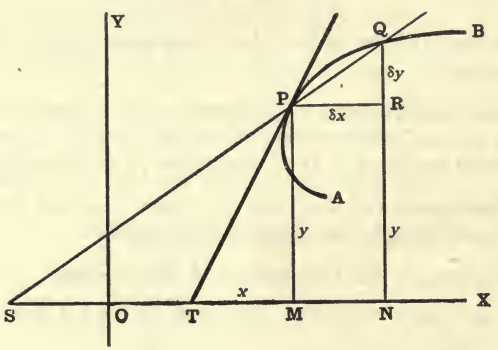
78. Definitions. Let two points P and Q be taken on any curve PQR, and let the point Q move along the curve nearer and nearer to P; the limiting position, TT', of the secant PQ when the point Q *moves up to and ultimately coincides with* P is called the **Tangent** * to the curve at the point P.



The straight line PN through the point P, perpendicular to the tangent TT', is called the **Normal** to the curve at the point P.

The **Slope,** or **Gradient,** of a curve at any point is the slope of the straight line tangent to the curve at that point.

79. *To find the slope of a curve at any point.*†



Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two points close together on any curve AB; then δx is the difference of the abscissas, δy the difference of the ordinates of P and Q.

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* This definition was first suggested by Roberval (1602–1675), but was stated more concisely by Fermat and Des Cartes. (History of Math.—Cajori, p. 173; Ball, p. 243.)

† Read Ex. 1, § 81, in connection with this general demonstration.

Let the secant PQ meet the x-axis in S, and let the tangent line at P meet the x-axis in T.

Draw the ordinates MP, NQ, and draw PR parallel to the x-axis.

Then $$PR = \delta x, \quad RQ = \delta y.$$

Let the equation of the curve be

$$y = f(x). \qquad (1)$$

Then at the points P and Q we have

$$OM = x, \qquad MP = y = f(x),$$
$$ON = x + \delta x, \quad NQ = y + \delta y = f(x + \delta x).$$
$$\therefore \quad \delta y = f(x + \delta x) - f(x).$$

Also $$\tan XSQ = \tan RPQ = \frac{RQ}{PR} = \frac{\delta y}{\delta x}.$$

$$\therefore \quad \tan XSQ = \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}. \qquad (2)$$

The slope of the tangent TP, which is the slope of the curve at the point P, is the ultimate slope of the secant SPQ when the point Q moves along the curve close up to P; *i. e.*

$$\tan XTP = \lim \tan XSQ = \lim \frac{\delta y}{\delta x} \text{ as } Q \text{ approaches } P.$$

When the point Q approaches the position of P as a limit, the differences δx and δy simultaneously approach zero as a limit, and the *limiting value of the ratio* $\frac{\delta y}{\delta x}$ is denoted by $\frac{dy}{dx}$; therefore in the limit we have

$$\tan XTP = \frac{dy}{dx} = \lim_{\delta x = 0}{}^{*} \frac{f(x + \delta x) - f(x)}{\delta x}. \qquad (3)$$

The ratio represented by the last member of equation (3) is also a function of x; and if, x being regarded as fixed, this ratio has a definite limiting value as δx becomes zero, this limiting value is called the **Derived Function,** or the **Derivative** of $f(x)$ with respect to x, and will be denoted by $f'(x)$; *i. e.* if

$$y = f(x), \text{ then } \frac{dy}{dx} = f'(x).$$

---

* The sign "=" in these conditions for a limit ($\delta x = 0$) is to be understood to mean *becomes equal.*

9

Hence to find the slope at any point of a curve whose equation is in the form $y = f(x)$ we find $f'(x)$, the derivative of $f(x)$ with respect to x, and in this substitute the *abscissa of the given point.*

To find the derivative of a function of x, denoted by $f(x)$, we assign a small increment δx to x, producing an increment, denoted by $f(x + \delta x) - f(x)$, in the function, and then find the limiting value of the ratio

$$\frac{f(x+\delta x) - f(x)}{\delta x},$$

as δx vanishes.

E. g., let $f(x) \equiv x^5$, then

$$f'(x) = \lim_{\delta x = 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x = 0} \frac{(x+\delta x)^5 - x^5}{\delta x}$$
$$= \lim_{\delta x = 0} (5x^4 + 10x^3\delta x + 10x^2\delta x^2 + 5x\delta x^3 + \delta x^4) = 5x^4.$$

The operation of finding the derivative of a function is called *differentiation.*

It is to be carefully noticed that in the definition of a derivative given above we speak of the *limiting value of the ratio* $\frac{\delta y}{\delta x}$, and not of the *ratio of the limiting values of* δy and δx. The latter ratio is indeterminate, on the face of it, being of the form $\frac{0}{0}$. To give the latter definiteness we now define it as equal to the former. That is, the ratio of the vanishing increments of function and variable is the limit that the ratio of their finite increments approaches when these finite increments at last vanish.

80. *Examples of limiting values of ratios of vanishing quantities.*

(1.) Let K be the area of a square whose side is x.

Then
$$\left[\frac{\lim \text{area}}{\lim \text{side}}\right]_{x=0} = \frac{0}{0}.$$

But
$$\lim_{x=0} \frac{K}{x} = \lim_{x=0} \frac{x^2}{x} = \lim_{x=0} (x) = 0.$$

(2.) Let K be the area of a rectangle with a constant base b and a variable altitude x.

Then
$$\left[\frac{\lim K}{\lim x}\right]_{x=0} = \frac{0}{0}.$$

But
$$\lim_{x=0} \frac{K}{x} = \lim_{x=0} \frac{bx}{x} = b.$$

(3.) Let V be the volume, T the total surface, C the circumference of the base of a right circular cylinder whose altitude is constant and radius variable.

Then $$\left[\frac{\lim T}{\lim C}\right]_{r=0}=\frac{0}{0}, \qquad \left[\frac{\lim T}{\lim V}\right]_{r=0}=\frac{0}{0}.$$

But $$\lim_{r=0}\frac{T}{C}=\lim_{r=0}\frac{2\pi r(r+h)}{2\pi r}=\lim_{r=0}(r+h)=h,$$

and $$\lim_{r=0}\frac{T}{V}=\lim_{r=0}\frac{2\pi r(r+h)}{\pi r^2 h}=\lim_{r=0}\frac{2(r+h)}{rh}=\frac{2h}{0}=\infty.$$

If S be the convex surface, find $\displaystyle\lim_{r=0}\frac{T}{S}$.

(4.) $$\left[\frac{\lim (x-a)^2}{\lim (x^2-a^2)}\right]_{x=a}=\frac{0}{0};$$

but $$\lim_{x=a}\frac{(x-a)^2}{x^2-a^2}=\lim_{x=a}\frac{x-a}{x+a}=0.$$

(5.) Find $$\lim_{x=0}\frac{1-\sqrt{1-x^2}}{x^2}.$$

Multiplying both numerator and denominator by $1+\sqrt{1-x^2}$ gives

$$\lim_{x=0}\frac{1-\sqrt{1-x^2}}{x^2}=\lim_{x=0}\frac{x^2}{x^2(1+\sqrt{1-x^2})}=\lim_{x=0}\frac{1}{1+\sqrt{1-x^2}}=\frac{1}{2}.$$

EXAMPLES.

Find the limits indicated in the following expressions:

1. $\displaystyle\lim_{x=a}\frac{x^3-a^3}{x^2-a^2}$.

2. $\displaystyle\lim_{x=a}\frac{x^4-a^4}{x^2-a^2}$.

3. $\displaystyle\lim_{x=a}\frac{(x-a)^3}{x^3-ax^2-a^2x+a^3}$.

4. $\displaystyle\lim_{x=1}\frac{3x^2-6x+3}{2x^2-4x+2}$.

5. $\displaystyle\lim_{x=0}\frac{x^2}{a-\sqrt{a^2-x^2}}$.

6. $\displaystyle\lim_{x=\infty}\frac{2x^2+x-1}{x^2-x+2}$.

7. $\displaystyle\lim_{x=0}\frac{\sqrt{4+x}-\sqrt{4-x}}{x}$.

8. $\displaystyle\lim_{x=\infty}\sqrt{a^2+x^2}-x$.

9. $\displaystyle\lim_{x=0}\frac{\sin x}{\tan x}=1$.

10. $\displaystyle\lim_{x=90^\circ}\frac{\sec x}{\tan x}=1$.

11. $\displaystyle\lim_{x=0}\frac{1-\cos x}{\sin^2 x}=\frac{1}{2}$.

12. $\displaystyle\lim_{x=0}\frac{\tan x-\sin x}{1-\cos x}=0$.

13. $\displaystyle\lim_{x=0}\frac{\sin x}{x}=\lim_{x=0}\frac{\tan x}{x}=1$.

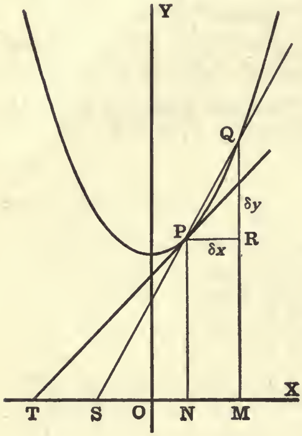
14. If V be the volume, T the total surface, S the convex surface, C the circumference of the base of a cone of revolution whose altitude h is constant, show that

$$\lim_{r=0}\frac{T}{C}=\frac{1}{2}, \quad \lim_{r=0}\frac{T}{V}=\infty, \quad \lim_{r=0}\frac{T}{S}=1, \quad \lim_{r=\infty}\frac{T}{S}=2.$$

81. *Examples of derivatives and slope of curves.*

Ex. 1. *Find the slope of the curve whose equation is*

$$y = x^2 + a. \quad (1)$$



Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be any two points close together on the curve; and let TP be the tangent at P.

Then at P, $\quad y = x^2 + a, \quad (2)$

and at Q, $\quad y + \delta y = (x + \delta x)^2 + a. \quad (3)$

Whence

$$\frac{(y + \delta y) - y}{\delta x} = \frac{(x + \delta x)^2 + a - (x^2 + a)}{\delta x}$$

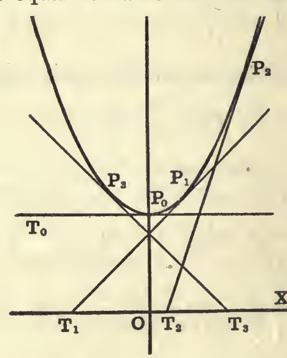
$$= \tan RPQ. \quad (4)$$

$$\therefore \quad \frac{\delta y}{\delta x} = 2x + \delta x = \tan XSQ. \quad (5)$$

When Q coincides with P, or as we say, proceeding to the limit $\delta x = 0$, we have (§ 79)

$$\frac{dy}{dx} = 2x = \tan XTP. \quad (6)$$

Hence the slope of the curve at any point is equal to twice the abscissa of the point.



At P_0, $\quad x = 0$.

$\quad \therefore \quad P_0T_0$ is parallel to the x-axis.

At P_1, $\quad x = \frac{1}{2}$,

$\quad \therefore \quad \tan XT_1P_1 = 1$.

At P_2, $\quad x = \frac{3}{2}$,

$\quad \therefore \quad \tan XT_2P_2 = 3$.

At P_3, $\quad x = -\frac{1}{2}$,

$\quad \therefore \quad \tan XT_3P_3 = -1$.

Ex. 2. *Find the slope of the curve* $y = \dfrac{1}{x}$.

We now have

$$\delta y \equiv f(x + \delta x) - f(x) = \frac{1}{x + \delta x} - \frac{1}{x} = -\frac{\delta x}{x(x + \delta x)}.$$

Whence

$$\frac{\delta y}{\delta x} = -\frac{1}{x(x + \delta x)}.$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x = 0}\left(-\frac{1}{x(x + \delta x)}\right) = -\frac{1}{x^2}.$$

That is, the slope is always negative and varies inversely as the square of the abscissa of the point.

Ex. 3. *Let $y = \sqrt{x}$ be the given curve.*

Then
$$\delta y = \sqrt{x + \delta x} - \sqrt{x} = \frac{\delta x}{\sqrt{x+\delta x} + \sqrt{x}},$$

and
$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x+\delta x} + \sqrt{x}}.$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x = 0} \frac{1}{\sqrt{x+\delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Verify the results found in Exs. 2 and 3 by constructing the loci.

EXAMPLES.

Find the slope at the points where $x = 0, \pm 1, \pm 2$, etc., of the curves whose equations are

1. $y = x^3$.

2. $y = x^4$.

3. $y = \frac{1}{x^2}$.

4. $y^2 = x^3$.

5. $y = x^3 - 4x$.

6. $y = x^4 - 20x^2 + 64$.

7. Find the slope of $y = \sqrt{a^2 + x^2}$, where $x = 0, \pm a, \infty$.

8. Find the slope of $y = \sqrt{a^2 - x^2}$, where $x = 0, \pm a, \pm \frac{1}{2}a$.

9. Find the slope of $10y = x^2 - 3x - 20$, where $x = 0, \pm 1, \pm 4$. [Ex. 1, § 22.]

10. Find the slope of $y = x$ and $y = mx + b$.

Find the derivatives of the functions

11. $y = \frac{1}{x+a}$.

12. $y = \frac{1}{x^2 - a^2}$.

13. $y = \frac{x+1}{2x+3}$.

14. $y = ax^2 + bx + c$.

15. $y = \sin x$. [See Ex. 1, § 104.]

16. $y = \cos x$.

17.* If $y = \frac{1}{f(x)}$, show that $\frac{dy}{dx} = -\frac{f'(x)}{[f(x)]^2}$.

18.* If $y = f(x) \cdot \phi(x)$, show that $\frac{dy}{dx} = f(x) \cdot \phi'(x) + \phi(x) \cdot f'(x)$.

19.* If $y = \frac{f(x)}{\phi(x)}$, show that $\frac{dy}{dx} = \frac{\phi(x) \cdot f'(x) - f(x) \cdot \phi'(x)}{[\phi(x)]^2}$.

20.* If $y = [f(x)]^n$, show that $\frac{dy}{dx} = n[f(x)]^{n-1} \cdot f'(x)$. [Use § 82.]

21.* Find the derivatives of the functions

$$y = \tan x, \cot x, \sec x, \csc x, \sin x, \cos x, \cos^2 x, \frac{x+a}{x-a}, \frac{ax-b}{ax+b}.$$

82. *To prove that for all rational values of* n

$$\lim_{x=a}\frac{x^n-a^n}{x-a}=na^{n-1}.$$

I. When n is a positive integer, we have

$$\lim_{x=a}\frac{x^n-a^n}{x-a}=\lim_{x=a}(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots+xa^{n-2}+a^{n-1})$$

$$=a^{n-1}+a^{n-1}+\ldots \text{ to } n \text{ terms}$$

$$=na^{n-1}.$$

II. When $n=\frac{p}{q}$, a positive fraction, we put

$$x=y^q \quad\text{and}\quad a=b^q;$$

then $x^{\frac{p}{q}}=y^p$, $a^{\frac{p}{q}}=b^p$, and when $y=b$, $x=a$.

$$\therefore \lim_{x=a}\frac{x^n-a^n}{x-a}=\lim_{x=a}\frac{x^{\frac{p}{q}}-a^{\frac{p}{q}}}{x-a}=\lim_{y=b}\frac{y^p-b^p}{y^q-b^q}.$$

$$=\lim_{y=b}\frac{\dfrac{y^p-b^p}{y-b}}{\dfrac{y^q-b^q}{y-b}}=\frac{pb^{p-1}}{qb^{q-1}} \qquad \text{(Case I.)}$$

$$=\frac{p}{q}b^{p-q}=\frac{p}{q}a^{\frac{p}{q}-1}=na^{n-1}.$$

III. When $n=-m$, we have

$$\lim_{x=a}\frac{x^n-a^n}{x-a}=\lim_{x=a}\frac{x^{-m}-a^{-m}}{x-a}=\lim_{x=a}\left(-\frac{1}{x^ma^m}\cdot\frac{x^m-a^m}{x-a}\right)$$

$$=-\frac{1}{a^{2m}}\cdot ma^{m-1}=-ma^{-m-1}=na^{n-1},$$

whether m is an integer or a fraction.. (Cases I and II.)

Ex. 1. Show that

$$\operatorname*{limit}_{f(x)=\phi(x)}\frac{[f(x)]^n-[\phi(x)]^n}{f(x)-\phi(x)}=n[\phi(x)]^{n-1}.$$

Ex. 2. Show that

$$\lim_{x=45^\circ}\frac{\sin^3 x-\cos^3 x}{\sin x-\cos x}=\frac{3}{2}.$$

General Rules for Differentiation.

83. *Differentiation of an integral algebraic function of one variable with rational exponents.*

The most general form of such a function is

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots + kx + l.$$

Let $$y = f(x) \equiv ax^n + bx^{n-1} + cx^{n-2} + \ldots + kx + l. \qquad (1)$$

If we let $\delta x = h = (x+h) - x$, for convenience, then will

$$\delta y = f(x+h) - f(x); \qquad [\S\ 79]$$

that is,

$$\delta y = a[(x+h)^n - x^n] + b[(x+h)^{n-1} - x^{n-1}]$$
$$+ c[(x+h)^{n-2} - x^{n-2}] + \ldots + k[(x+h) - x]. \qquad (2)$$

$$\therefore \quad \frac{\delta y}{\delta x} = a\left[\frac{(x+h)^n - x^n}{(x+h) - x}\right] + b\left[\frac{(x+h)^{n-1} - x^{n-1}}{(x+h) - x}\right]$$
$$+ c\left[\frac{(x+h)^{n-2} - x^{n-2}}{(x+h) - x}\right] + \ldots + k\left[\frac{(x+h) - x}{(x+h) - x}\right]. \qquad (3)$$

Proceeding to the limit $\delta x = h = 0$, we obtain, by applying § 82 to each term of the second member of (3),

$$\frac{dy}{dx} = nax^{n-1} + b(n-1)x^{n-2} + c(n-2)x^{n-3} + \ldots k = f'(x). \qquad (4)$$

Hence, if $f(x)$ is an integral algebraic function, we find $f'(x)$ by multiplying the coefficient of each term by the exponent of x in that term and diminishing each exponent by unity.

E. g., if $f(x) = x^4 - 2x^3 + 3x^2 + x - 4x^0 - 6x^{-1} + 2x^{-2} - 3x^{-3}$,

$$f'(x) = 4x^3 - 6x^2 + 6x + 1 + 6x^{-2} - 4x^{-3} + 9x^{-4}.$$

Observe from (4) and (1) that the *derivative of the sum* of a number of terms is the *sum of the derivatives* of the separate terms, and also that the derivative of a constant term is zero.

Ex. 1. If $f(x) = f_1(x) + f_2(x)$, show that $f'(x) = f_1'(x) + f_2'(x)$.

Ex. 2. Show by constructing a figure that the slope of $y = f_1(x) + f_2(x)$ is the sum of the separate slopes of

$$y = f_1(x) \quad \text{and} \quad y = f_2(x).$$

84. *To find the derivative of a function of the type $F(x, y)=0$.*

When we desire to differentiate a function of the type $F(x, y)=0$, we may try first to solve the equation with respect to y, so as to put it in the form $y=f(x)$; or to solve with respect to x, so as to bring it to the form $x=f_1(y)$. It is useful, however, to have a rule to meet cases when this process would be inconvenient or impracticable. It will be sufficient for the purpose of this book to illustrate the rule by considering the general equation of the second degree (§ 53).

Let $$F(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad (1)$$

Let $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ be two points close together on the locus of (1); then at P and Q, respectively,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad (2)$$

$$a(x+\delta x)^2 + 2h(x+\delta x)(y+\delta y) + b(y+\delta y)^2 + 2g(x+\delta x) + 2f(y+\delta y) + c = 0. \quad (3)$$

Subtracting (2) from (3) gives

$$a(2x\delta x + \delta x^2) + 2h(y\delta x + x\delta y + \delta x\delta y) + b(2y\delta y + \delta y^2) + 2g\delta x + 2f\delta y = 0. \quad (4)$$

Whence $$\frac{\delta y}{\delta x} = -\frac{2ax + 2hy + 2g + a\delta x + 2h\delta y}{2hx + 2by + 2f + b\delta y}. \quad (5)$$

Proceeding to the limit, when $\delta x = \delta y = 0$, we have

$$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}. \quad (6)$$

Now apply to (1) the rule deduced in § 83 and differentiate first with respect to x *regarding* y *as constant;* then differentiate with respect to y *regarding* x *as constant.* Denoting these partial derivatives respectively by $F_x'(x, y)$ and $F_y'(x, y)$, we thus obtain

$$F_x'(x, y) = 2(ax + hy + g) \quad (7)$$

and $$F_y'(x, y) = 2(hx + by + f). \quad (8)$$

$$\therefore \frac{dy}{dx} = -\frac{F_x'(x, y)}{F_y'(x, y)} = -\frac{ax + hy + g}{hx + by + f}, \quad (9)$$

which expresses the rule for differentiating any function of the type $F(x, y)=0$.

TANGENTS AND NORMALS.

85. *To find the equations of the tangent, and the normal at any point (x', y') of a curve.*

For the tangent, $$m = \frac{dy'}{dx'}. \qquad (§ 79.)$$

For the normal, $$m = -\frac{dx'}{dy'}. \qquad (§ 78 \text{ and } § 48.)$$

Since both lines pass through the point (x', y'), the equation of the tangent is (§ 49)

$$y - y' = \frac{dy'}{dx'}(x - x'); \qquad (1)$$

and the equation of the normal is

$$y - y' = -\frac{dx'}{dy'}(x - x'). \qquad (2)$$

The primes in $\frac{dy'}{dx'}$ denote that the coordinates x', y' of the point of contact are to be substituted in the derivative of the equation.

COR. If the axes are oblique,

$$\frac{dy}{dx} = \frac{\sin \gamma}{\sin(\omega - \gamma)} = m. \qquad (§ 59.)$$

Hence equation (1) holds also for oblique axes.*

EXAMPLES ON CHAPTER V.

Find the equations of the tangent and normal to each of the following curves at the point (x', y'):

1. $y = x^2$. Ans. $\frac{2x}{x'} - \frac{y}{y'} = 1$.
2. $y^2 = x$. Ans. $\frac{2y}{y'} - \frac{x}{x'} = 1$.
3. $y = x^3$. Ans. $\frac{3x}{2x'} - \frac{y}{2y'} = 1$.
4. $y^2 = x^3$. Ans. $\frac{3x}{x'} - \frac{2y}{y'} = 1$.
5. $xy = 1$. Ans. $\frac{x}{2x'} + \frac{y}{2y'} = 1$.
6. $x^2 + y^2 = 1$. Ans. $xx' + yy' = 1$.

* The theory of this chapter proves what has hitherto been assumed (see note on logic of plotting, § 21), viz., that loci of equations are usually smooth curves without sudden changes in slope or curvature. For, since the slope of a curve $f(x, y) = 0$ at any point (x, y) is a function of x and y, a small change in x and y will ordinarily produce only a small change in the slope.

7. $x^2 - y^2 = 1.$

8. $x^3 + y^3 = 1.$ Ans. $xx'^2 + yy'^2 = 1.$

9. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

10. $x^n + y^n = 1.$

11. What are the equations of the tangents to 6, 7, 8, 9, 10 at the point (1, 0); and to 6, 9, 10 at the point (0, 1)?

Find the equation of the tangent to

12. $y^4 = 4x - 3x^2$, at the point (1, 1).

13. $10y = (x + 1)^2$ at the point where $x = 9$. (Ex. 10, p. 28.)

14. $4(x + 1) = (y - 2)^2$ at the point where $x = 3$. (Ex. 10, p. 28.)

15. $(x - 8)^2 + (y - 2)^2 = 25$ at the point where $x = 4$.

16. $x(x^2 + y^2) = a(x^2 - y^2)$ at the point where $x = 0$, and $\pm a$. (§ 38.)

17. Find the equation of the tangent to $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$, and show that at the point (a, b) it is the same for all values of n.

18. Show that the curve $x^{2/3} + y^{2/3} = a^{2/3}$ becomes steeper as it approaches the y-axis, and is tangent to the axes at the points $(\pm a, 0)$ and $(0, \pm a)$.

19. Let $y = f(x)$ and $y = F(x)$ be two curves intersecting in the point (x_1, y_1), and let ϕ be the angle at which they intersect. Show that

$$\tan \phi = \frac{f'(x_1) - F'(x_1)}{1 + f'(x_1) \cdot F'(x_1)}.$$

What is the condition that the two curves shall meet at right angles? be tangent to each other?

[The angle at which two curves intersect is the angle between their tangents at the point of intersection of the curves.]

20. Find the angle of intersection between the parabolas

$$y^2 = 4ax \quad \text{and} \quad x^2 = 4ay.$$

21. Show that the confocal parabolas

$$y^2 = 2a(x + a) \quad \text{and} \quad y^2 = -2b(x - b)$$

intersect at right angles.

22. At what angle do the rectangular hyperbolas

$$x^2 - y^2 = a^2 \quad \text{and} \quad xy = b$$

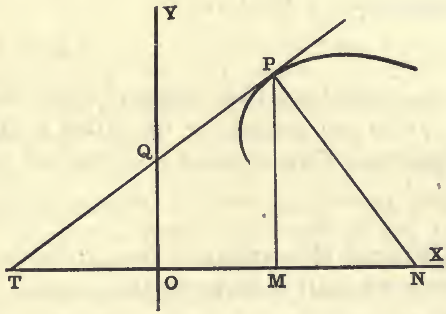
intersect? Draw several sets of these curves by assigning different values to a and b.

23. Find the angle at which the circle $x^2+y^2+2x=12$ intersects the parabola $y^2=9x$.

24. Find the angle of intersection between

$$x^2+y^2=25 \quad \text{and} \quad 4y^2=9x.$$

25. Let the tangent and the normal at any point $P(x, y)$ of a curve meet the x-axis in T and N respectively, and let M be the foot of the ordinate of P.



Prove, by the use of equations (1) and (2), § 85, the following formulæ:

$$\textit{Subtangent,} \quad TM = y\frac{dx}{dy}.$$

$$\textit{Subnormal,} \quad MN = y\frac{dy}{dx}.$$

$$\textit{Tangent,} \quad TP = y\sqrt{1+\left(\frac{dx}{dy}\right)^2}.$$

$$\textit{Normal,} \quad PN = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

$$\textit{Intercept,} \quad OT = x - y\frac{dx}{dy}.$$

$$\textit{Intercept,} \quad OQ = y - x\frac{dy}{dx}.$$

26. Find the intercepts of the tangent, the subtangent, and the subnormal of the parabola

$$y^2=4ax.$$

CHAPTER VI.

THEORY OF EQUATIONS.

86. An expression of the form

$$ax^n + bx^{n-1} + cx^{n-2} + \dots + kx + l, \qquad (1)$$

where n is a finite positive integer and the coefficients $a, b, c, \dots k, l$ do not contain x, is called a **Rational and Integral Algebraic Function** of x of the nth degree; and

$$ax^n + bx^{n-1} + cx^{n-2} + \dots + kx + l = 0 \qquad (2)$$

is called the **General Equation** of the nth degree. This is the kind of equation we shall consider in this section.

If we divide the left side of equation (2) by a, the coefficient of x^n, we shall obtain the general equation of the nth degree in the *standard form*,

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0, \qquad (3)$$

where $p_1, p_2, \dots p_{n-1}, p_n$ do not contain x, but are otherwise unrestricted. As will be seen hereafter, some of the properties of equations can be stated more concisely when the equation is in the standard form.

In this section the symbol $f(x)$ will be used to denote a rational integral function of x, such as (1) or the left member of (3).

Any quantity which substituted for x in $f(x)$ makes $f(x)$ vanish is called a **Root of** $f(x)$; or a **Root of the Equation** $f(x) = 0$.

If we put $y = f(x)$ and plot the locus of this equation, we shall obtain a curve which is called the **Graph** of $f(x)$. *The real roots of $f(x)$ are, therefore, the x-intercepts of its graph.*

87. *A rational integral function of x is continuous, and finite for any finite value of x.*

Let $f(x) \equiv p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n. \qquad (1)$

Then each term will be finite, provided x is finite; and therefore, as the number of terms is finite, the sum of them all, that is $f(x)$, will be finite for any finite value of x.

Now suppose x receives a small increment h, producing in $f(x)$ the increment $f(x+h)-f(x)$; then

$$f(x+h)-f(x)=p_0[(x+h)^n-x^n]+p_1[(x+h)^{n-1}-x^{n-1}] + \ldots + p_{n-1}[(x+h)-x]. \quad (2)$$

Each of the terms in the right member of (2) will become indefinitely small when h is indefinitely small; hence their sum will become indefinitely small. Therefore $f(x+h)-f(x)$ *can be made as small as we please by making h sufficiently small.* This shows that as x changes from any value a to another value b, $f(x)$ will change gradually and without interruption, *i. e. without any sudden jump*, from $f(a)$ to $f(b)$; so that $f(x)$ must pass *at least once* through every value intermediate to $f(a)$ and $f(b)$. That is, $f(x)$ is a *continuous function.*

Hence *the graph of $f(x)$ is a continuous curve with finite ordinates for finite values of x.*

88. *To calculate the numerical value of $f(a)$.*

$$\text{Let } f(x) \equiv p_0x^3+p_1x^2+p_2x+p_3. \quad (1)$$

Then we wish to calculate the numerical value of

$$f(a)=p_0a^3+p_1a^2+p_2a+p_3. \quad (2)$$

This result is most easily obtained as follows:

Multiply p_0 by a and add to p_1, this gives p_0a+p_1;

Multiply this by a and add to p_2, this gives $p_0a^2+p_1a+p_2$;

Multiply this by a and add to p_3, this gives $p_0a^3+p_1a^2+p_2a+p_3$.

The process may be arranged in the following way:

p_0	p_1	p_2	p_3
	p_0a	$p_0a^2+p_1a$	$p_0a^3+p_1a^2+p_2a$
p_0	p_0a+p_1	$p_0a^2+p_1a+p_2$	$p_0a^3+p_1a^2+p_2a+p_3$.

We may proceed in the same way, whatever the degree of $f(x)$.

Ex. Find the numerical value of $f(3)$ if

$$f(x) \equiv 2x^4 - 7x^3 + 13x - 16.$$

2	-7	0	13	-16
	6	-3	-9	12
	-1	-3	4	-4

$$\therefore\ f(3) = -4.$$

This process is called **Synthetic Substitution.**

89. *To find the remainder and the quotient when $f(x)$ is divided by $x - a$, where a is any constant.*

Divide $f(x)$ by $x - a$ until the remainder no longer contains x. Let Q denote the quotient and R the remainder. We then have the identical equation

$$f(x) \equiv Q(x - a) + R, \qquad (1)$$

which must be satisfied when any value whatever is substituted for x. Let $x = a$, then

$$f(a) = Q(a - a) + R = R; \qquad (2)$$

for $Q(a - a) = 0$, since by § 87 Q remains finite. That is, the remainder is equal to the result obtained by substituting a for x in the given function.

Cor. *If a is a root of $f(x)$, then $f(x)$ is divisible by $x - a$.*

Conversely, if $f(x)$ is divisible by $x - a$, then a is a root of $f(x)$.

For, if either $f(a) = 0$, or $R = 0$, in (2) the other is also equal to zero, which proves the proposition.

Let $\qquad f(x) \equiv p_0x^3 + p_1x^2 + p_2x + p_3, \qquad$ for example.

By actual division we find

$$Q = p_0x^2 + (p_0a + p_1)x + (p_0a^2 + p_1a + p_2),$$

and $\qquad R = p_0a^3 + p_1a^2 + p_2a + p_3.$

By comparing these expressions with the results found in § 88 we see that R and the coefficients in Q are the same as the sums obtained by synthetic substitution.

Ex. Find Q and R when $3x^5-2x^4-16x^3-x+7$ is divided by $(x+2)$.

3	-2	-16	0	-1	$+7$
	-6	$+16$	0	0	$+2$
3	-8	0	0	-1	$+9$

Thus $Q=3x^4-8x^3-1$, and $R=9$.

$$\therefore\ 3x^5-2x^4-16x^2-x+7\equiv(x+2)(3x^4-8x^3-1)+9.$$

This process can be applied to any function of any degree, and is a particular case of **Synthetic Division**. (See Todhunter's Algebra, Chap. LVIII.)

90. *An equation of the* n*th degree has* n *roots, real or imaginary.*

Let the equation be

$$f(x)\equiv x^n+p_1x^{n-1}+p_2x^{n-2}+\ .\ .\ .\ +p_n=0. \qquad (1)$$

Let a_1 be one root* of the equation $f(x)=0$, then $f(x)$ is divisible by $(x-a_1)$. (§ 89.)

$$\therefore\ f(x)\equiv(x-a_1)f_1(x), \qquad (2)$$

where $f_1(x)$ is an integral function of x of degree $(n-1)$.

In like manner if a_2 is a root of $f_1(x)$, then

$$f_1(x)\equiv(x-a_2)f_2(x), \qquad (3)$$

where $f_2(x)$ is an integral function of x of degree $(n-2)$.

Proceeding in this way we shall find n factors of the form $(x-a_r)$, and we have finally,

$$f(x)\equiv(x-a_1)(x-a_2)(x-a_3)\ .\ .\ .\ (x-a_n)=0. \qquad (4)$$

It is now clear that $a_1, a_2, a_3\ .\ .\ .\ a_n$ are roots of the equation $f(x)=0$; and as no other value of x will make $f(x)$ vanish, the equation can have no other roots.

The factors of $f(x)$ need not all be different from one another; thus we may have

$$f(x)\equiv(x-a_1)^p(x-a_2)^q(x-a_3)^r\ .\ .\ .\ , \qquad (5)$$

where $p+q+r+\ .\ .\ .\ =n.$

*We here assume the fundamental theorem that *every equation has one root*, real or imaginary. Proofs of this theorem have been given by Argand, Cauchy, Clifford, and others, but they are too difficult to be included in this book. The student, however, is already familiar with the fact that every equation of the first degree has one root; that every equation of the second degree has two roots, real or imaginary; and it will be shown in § 94 that *every equation of an odd degree has one real root.*

In this case $f(x)$ has p roots each a_1, q roots each a_2, etc., the whole number of roots being

$$p+q+r+\ldots=n.$$

Therefore the graph of $f(x)$ will cut the x-axis in n points, which may be real, coincident, or imaginary; and the *real roots* are its x-intercepts.

Hence the real roots of a function may be found exactly or approximately by constructing its graph.

EXAMPLES.

1. Divide $2x^5-6x^4-5x^2+10x+18$ by $x-3$.

Find the other roots of the following equations:

2. Two roots of $x^4-12x^3+49x^2-78x+40=0$ are 1 and 5.
3. One root of $x^3-16x^2+20x+112=0$ is -2.
4. Two roots of $x^4+8x^3-22x^2-16x+40=0$ are 2 and -10.
5. Two roots of $x^4-12x^3+48x^2-68x+15=0$ are 5 and 3.
6. Three roots of $6x^5+11x^4-21x^3+7x^2+15x-18=0$ are ± 1 and -3.

Find graphically the exact or approximate roots of

7. $x^3-2x^2-11x+12=0.$
8. $x^4-8x^3+14x^2+8x-15=0.$
9. $x^4-2x^3-13x^2-14x+24=0.$
10. $x^3-8x^2-28x+80=0.$
11. $6x^3-13x^2-21x+18=0.$
12. $8x^3-18x^2-71x+60=0.$
13. $x^4-6x^3-5x^2+56x-30=0.$

91. *Relations between the roots and the coefficients of an equation.*

If there are two roots, a_1 and a_2, we have (§ 90)

$$\begin{aligned} x^2+p_1x+p_2 &\equiv (x-a_1)(x-a_2) \\ &\equiv x^2-(a_1+a_2)x+a_1a_2. \qquad (1)\end{aligned}$$

$$\therefore\quad a_1+a_2=-p_1, \qquad a_1a_2=p_2.$$

If there are three roots a_1, a_2, and a_3, we have

$$\begin{aligned} x^3+p_1x^2+p_2x+p_3 &\equiv (x-a_1)(x-a_2)(x-a_3) \\ &\equiv x^3-(a_1+a_2+a_3)x^2+(a_1a_2+a_2a_3+a_3a_1)x-a_1a_2a_3. \qquad (2)\end{aligned}$$

$$\therefore\quad a_1+a_2+a_3=-p_1, \quad a_1a_2+a_2a_3+a_3a_1=p_2, \quad a_1a_2a_3=-p_3.$$

In like manner if the equation is of the nth degree and therefore has n roots $a_1, a_2 \ldots a_r \ldots a_n$, then

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_rx^{n-r} + \ldots + p_n$$
$$\equiv (x - a_1)(x - a_2) \ldots (x - a_r) \ldots (x - a_n) \quad (3)$$
$$\equiv x^n - S_1x^{n-1} + S_2x^{n-2} - \ldots + (-1)^rS_rx^{n-r}$$
$$\pm \ldots + (-1)^nS_n, \quad (4)$$

where S_r is the sum of all the products of $a_1, a_2, \ldots a_r \ldots a_n$ taken r together.

Equating the coefficients of the same powers of x on the two sides of the identity (4) gives

$$S_1 = -p_1, \quad S_2 = p_2, \quad S_r = (-1)^rp_r,$$
$$S_n = (-1)^np_n = (-1)^na_1a_2 \ldots a_r \ldots a_n.$$

The absolute term, p_n, is divisible by each of the roots.

If $p_n = 0$, one root is zero; if $p_n = p_{n-1} = 0$, two roots are zero; if $p_n = p_{n-1} = \ldots p_{n-r} = 0$, $r + 1$ roots are zero.

EXAMPLES.

Find the other roots of the following equations:

1. Two roots of $x^3 + x^2 - 4x - 4 = 0$ are 2 and -1.
2. Two roots of $x^3 - 4x^2 - 3x + 12 = 0$ are 4 and $\sqrt{3}$.
3. Two roots of $x^3 - 13x + 12 = 0$ are 1 and 3.
4. Three roots of $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$ are 1, 2, and 3.
5. One root of $x^3 - 4x^2 - x = 0$ is $-(2 + \sqrt{5})$.
6. One root of $x^5 - 6x^4 + 12x^3 = 0$ is $3 - \sqrt{-3}$.
7. Two roots of $6x^4 - 7x^3 - 14x^2 + 15x = 0$ are 1 and $\frac{5}{3}$.
8. Two roots of $4x^5 - 5x^4 + 2x^3 + 6x^2 = 0$ are $1 \pm \sqrt{-1}$.

Form the equations whose roots are

9. $1, 3, -5$.
10. $-2, 3, -4, 6$.
11. $\frac{1}{3}, -\frac{7}{2}, \frac{3}{5}$.
12. $\pm 1, \pm 4$.
13. $0, 1, -4, 5$.
14. $\pm \sqrt{2}, \pm \sqrt{3}$.
15. $0, -2, \pm \sqrt{-2}$.
16. $3, 5 \pm \sqrt{5}$.
17. $4 \pm \sqrt{3}, -1 \pm \sqrt{6}$.
18. $1, -2, 3, -4, 5$.
19. $0, 2 \pm \sqrt{-1}, -3 \pm \sqrt{6}$.
20. $0, 0, \frac{1}{2}, -\frac{2}{3}, 1 \pm \sqrt{2}$.
21. $1 \pm \sqrt{-5}, -2 \pm \sqrt{-7}$.
22. $-3, 2 \pm \sqrt{-3}, -3 \pm \sqrt{-2}$.

92. *The first term of $f(x)$ can be made to exceed the sum of all the other terms by giving to x a value sufficiently great.*

Let $$f(x) \equiv p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n,$$
and let k be the greatest of the coefficients; then

$$\frac{p_0x^n}{p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n} > \frac{p_0x^n}{k(x^{n-1} + x^{n-2} + \ldots + 1)}$$
$$> \frac{p_0x^n(x-1)}{k(x^n-1)} > \frac{p_0x^n(x-1)}{kx^n} > \frac{p_0}{k}(x-1).$$

Now $\frac{p_0}{k}(x-1)$ can be made as great as we please by sufficiently increasing x, which gives the proposition.

93. *An* even *number, or an* odd *number, of real roots of $f(x) = 0$ lie between a and b according as $f(a)$ and $f(b)$ have the* same *sign, or* opposite *signs.*

The two points $A[a, f(a)]$ and $B[b, f(b)]$ are on the *same* side, or on *opposite* sides, of the x-axis according as $f(a)$ and $f(b)$ have the *same* sign, or *opposite* signs.

Therefore, since the graph of $f(x)$ is a continuous curve (§ 87), in passing from A to B along the graph the x-axis will be crossed an *even* number, or an *odd* number, of times according as $f(a)$ and $f(b)$ have the *same* sign, or *opposite* signs. This proves the proposition.

E. g., if $f(x) \equiv x^3 - 3x + 1$, then $f(1) = -1$ and $f(2) = 3$.

$\therefore$ At least one real root of $x^3 - 3x + 1 = 0$ lies between 1 and 2.

94. *An equation of an* odd degree *has at least one real root.*

Let the given equation be
$$f(x) \equiv x^{2n+1} + p_1x^{2n} + p_2x^{2n-1} + \ldots + p_{2n+1} = 0.$$

Let a be a value of x sufficiently large to make the first term of $f(a)$ greater than the sum of all the other terms (§ 92). Then the sign of $f(a)$ will be the same as the sign of a^{2n+1}, *i. e.* the same as the sign of a.

Hence if a be sufficiently great, $f(a)$ is positive, $f(0) = p_{2n+1}$, and $f(-a)$ is negative.

Therefore in all cases there is one real root, which is *positive* or *negative* according as p_{2n+1} is *negative* or *positive* (§ 93).

Hence the graph of a function of an odd degree in the standard form extends to infinity in the first and third quadrants.

95. *An equation of an* even degree *in the standard form with the* last term negative *has at least two real roots with opposite signs.*

Let the given equation be

$$f(x) \equiv x^{2n} + p_1 x^{2n-1} + p_2 x^{2n-2} + \ . \ . \ . \ + p_{2n} = 0.$$

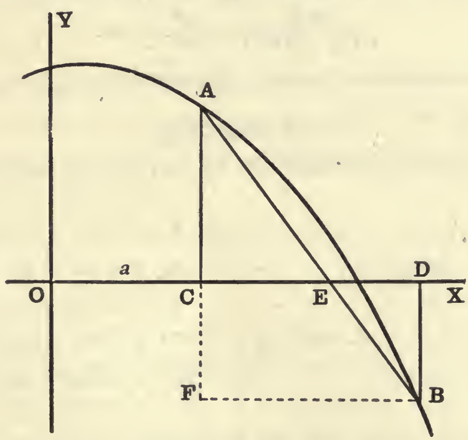
If a is taken sufficiently great, $f(a)$ will have the same sign as a^{2n} (§ 92), which is positive for both positive and negative values of a; that is, $f(a)$ and $f(-a)$ will both be positive, while $f(0) = p_{2n}$, which by hypothesis is negative.

Therefore there is at least one real root between 0 and a, and another between 0 and $-a$ (§ 93).

The graph of a function of an even degree in the standard form extends to infinity in the first and second quadrants.

96. *To find approximately the real roots of* $f(x) = 0$.

Construct the graph of $f(x)$ and thus determine the pairs of consecutive integers between which the roots lie.



Suppose $f(a) = CA$, a positive number; and $f(a+1) = DB$, a negative number.

Then there is at least one real root (§ 93) between a and $a+1$.

Draw the chord AB cutting the x-axis in E; draw BF parallel to the x-axis meeting AC produced in F.

Then, if there is *only one* root between a and $a+1$, it is approximately equal to OE; if the graph were a straight line, it would be *exactly* equal to OE.

Since the triangles ACE and AFB are similar,

$$CE = \frac{FB \cdot CA}{FA} = \frac{CA}{CA + BD} = \frac{f(a)}{f(a) - f(a+1)}. \qquad (1)$$

If we use *numerical* values of $f(a)$ and $f(a+1)$, we shall then have for all cases

$$OE = a + \frac{f(a)}{f(a) + f(a+1)}. \qquad (2)$$

Ex. Find the roots of $x^3 - 29x + 42 = 0$.

Here $f(4) = -10$ and $f(5) = 22$. Hence there is *one* root between 4 and 5. Substituting in (2) gives

$$OE = 4 + \frac{10}{10 + 22} = 4.4 -.$$

Then $f(4.4) = -.456$ and $f(4.5) = 3.081$.

Hence the root lies between 4.4 and 4.5.

When the root is greater than OE, as in the diagram and also in this example, it is better to try the figure next greater than that given by the quotient.

The next figure of the root may now be approximated in the same way.

Thus $$\frac{f(4.4)}{f(4.4) + f(4.5)} = \frac{.456}{3.537} = .01.$$

∴ The approximate root is 4.41. The exact root is $(3 + \sqrt{2})$.

EXAMPLES.

Calculate to two places of decimals the real roots of the following equations:

1. $x^3 - 3x - 1 = 0$.
2. $x^3 - 7x + 7 = 0$.
3. $x^3 + 2x^2 - 3x - 9 = 0$.
4. $x^3 + 2x^2 - 4x - 43 = 0$.
5. $x^3 - 15x + 21 = 0$.
6. $x^4 - 12x + 7 = 0$.
7. $x^4 - 5x^3 + 2x^2 - 13x + 55 = 0$.
8. $x^3 - 3x^2 - 2x + 5 = 0$.
9. $x^5 - 81x + 40 = 0$.
10. $x^4 - 55x^2 - 30x + 400 = 0$.

97. *In any equation with real coefficients imaginary roots occur in pairs.*

I. Let $f(x) = 0$ be an equation with real coefficients having r real roots and the other roots imaginary. Then

$$f(x) \equiv (x - a_1)(x - a_2) \ldots (x - a_r)\varphi(x) = 0, \qquad (\S\, 90) \qquad (1)$$

where $\varphi(x)$ is a function with real coefficients whose roots are all the imaginary roots of $f(x)$, and no others. Hence $\varphi(x)$ must be of even degree, and therefore has an even number of roots. Otherwise it would have at least one real root (§ 94).

Therefore (1) has an *even number* of imaginary roots.

II. *If $a + b\sqrt{-1}$ is a root of an equation with real coefficients, then $a - b\sqrt{-1}$ is also a root.*

Let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0. \quad (2)$$

Substituting $a + b\sqrt{-1}$ for x in (2), we have

$$(a + b\sqrt{-1})^n + p_1(a + b\sqrt{-1})^{n-1} + p_2(a + b\sqrt{-1})x^{n-2} + \dots + p_n = 0. \quad (3)$$

Expanding by the binomial theorem, and collecting together the real and imaginary terms, we shall have a result in the form

$$P + Q\sqrt{-1} = 0. \quad (4)$$

In order that this equation may hold we must have

$$P = Q = 0. \quad (5)$$

Since P and Q are *real*, they contain only even powers of $\sqrt{-1}$, and hence will not be changed by changing the sign of $\sqrt{-1}$. Therefore, when $a - b\sqrt{-1}$ is substituted for x in (2), the result will be $P - Q\sqrt{-1}$.

But from (5) $\qquad P - Q\sqrt{-1} = 0.$

$$\therefore \quad a - b\sqrt{-1} \text{ is also a root of (2).}$$

Corresponding to the roots $a \pm b\sqrt{-1}$ of $f(x) = 0$, $f(x)$ will have the real quadratic factor $[(x - a)^2 + b^2]$.

The two quantities $a \pm b\sqrt{-1}$ are called *conjugate imaginary expressions.*

Show that the locus of the equation $y = x^2 + k$ cuts the x-axis in two points which are real and distinct, real and coincident, or imaginary according as k is negative, zero, or positive. Hence illustrate graphically the preceding theorem by showing that, as the absolute term of $f(x)$ is changed, real intersections of its graph with the x-axis disappear or reappear in pairs; and that the passage from a pair of real distinct roots to a pair of imaginary roots is through a pair of real coincident roots.

EXAMPLES.

1. Show that if either $a \pm \sqrt{b}$ is a root of an equation with rational coefficients, the other is also a root.

2. Solve the equation $x^4 - 2x^3 - 22x^2 + 62x - 15 = 0$, having given that one root is $2 + \sqrt{3}$.

3. Solve the equation $2x^3 - 15x^2 + 46x - 42 = 0$, having given that one root is $3 + \sqrt{-5}$.

4. If $\sqrt{a} + \sqrt{b}$ is a root of an equation with rational coefficients, $\sqrt{a}$ and $\sqrt{b}$ not being similar surds, show that $\pm \sqrt{a} \pm \sqrt{b}$ will all four be roots.

5. Form the biquadratic equation with rational coefficients one root of which is $\sqrt{2} + \sqrt{3}$.

6. Show that Ex. 4 holds when either or both a and b are negative.

7. Find the biquadratic equation with rational coefficients one root of which is $\sqrt{2} + \sqrt{-3}$.

8. Solve the equation $2x^6 - 3x^5 + 5x^4 + 6x^3 - 27x + 81 = 0$, having given that one root is $\sqrt{2} + \sqrt{-1}$.

Transformation of Equations.

98. I. *To find an equation whose roots are those of a given equation with opposite signs.*

If the given equation is $f(x) = 0$, the required equation will be $f(-x) = 0$. For, when $x = a$, $f(x) = f(a)$, and when $x = -a$, $f(-x) = f(a)$; hence, if a is a root of $f(x) = 0$, then $-a$ will be a root of $f(-x) = 0$.

The graph of $f(-x)$ is the reflection of the graph of $f(x)$ in a mirror through the y-axis perpendicular to the plane; *i. e.* the two graphs are symmetrical with respect to the y-axis, which proves the transformation for real roots.

If $f(x) \equiv f(-x)$ [§ 28, (2)], the two graphs will coincide, and the roots of $f(x)$ will occur in symmetric pairs of the form $\pm a$.

If the equation is complete, this transformation is effected by simply changing the sign of every other term beginning with the second.

II. *To find an equation whose roots are those of a given equation, each diminished by the same given quantity.*

If we put $x = x' + h$, the origin will be moved to the right a distance equal to h [§ 66, (10)].

Hence the x-intercepts of the graph of $f(x)$, *i. e.* the *real* roots of $f(x)$, will each be diminished by h.

Therefore, if $f(x) = 0$ is the given equation, the required equation will be $f(x+h) = 0$. For, when $x = a$, $f(x) = f(a)$, and when $x = a - h$, $f(x+h) = f(a)$; hence, if a is a root of $f(x) = 0$, then $a - h$ is also a root of $f(x+h) = 0$, whether a is real or imaginary.

The coefficients of the new equation can be found by synthetic substitution as follows:

Ex. Find the equation whose roots are those of

$$x^4 - 3x^3 - 15x^2 + 49x - 12 = 0$$

each diminished by 2.

Operation

1	−3	−15	+49	−12
	2	−2	−34	+30
1	−1	−17	+15	+18
	2	2	−30	
1	+1	−15	−15	
	2	+6		
1	+3	−9		
	2			
1	5			

$\therefore$ $x^4 + 5x^3 - 9x^2 - 15x + 18 = 0$ is the required equation.

If we put $x = x' - \frac{p_1}{n}$, where p_1 is the coefficient of x^{n-1}, each root will be diminished by $\left(-\frac{p_1}{n}\right)$, and therefore the sum of the roots will be diminished by $n\left(-\frac{p_1}{n}\right) = -p_1$.

Hence the sum of the roots of the new equation will be zero (§ 91); *i. e. the coefficient of the second term will be zero.*

Ex. Transform the equation $x^3 + 6x^2 + 4x + 5 = 0$ into another in which the coefficient of x^2 is zero.

Let $x = x' - 2$, since $p_1 = 6$ and $n = 3$; then we obtain

$$\begin{array}{llll|l} 1 & +6 & +4 & +5 \\ & -2 & -8 & +8 \\ \hline 1 & +4 & -4 & +13 \\ & -2 & -4 & \\ \hline 1 & +2 & -8 & \\ & -2 & & \\ \hline 1 & 0 & & \end{array}$$

$\therefore\ x^3 - 8x + 13 = 0$ is the required equation.

III. *To find an equation whose roots are the reciprocals of the roots of a given equation.*

Let the given equation be

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0. \qquad (1)$$

Substituting $\frac{1}{z}$ for x in (1) gives

$$p_0\left(\frac{1}{z}\right)^n + p_1\left(\frac{1}{z}\right)^{n-1} + p_2\left(\frac{1}{z}\right)^{n-2} + \ldots + p_{n-1}\left(\frac{1}{z}\right) + p_n = 0, \qquad (2)$$

which is the required equation, for (2) is satisfied by the reciprocal of any quantity which satisfies (1).

Multiplying (2) by z^n gives

$$p_nz^n + p_{n-1}z^{n-1} + p_{n-2}z^{n-2} + \ldots + p_1z + p_0 = 0. \qquad (3)$$

Therefore the required equation is obtained by merely *reversing the order of the coefficients* of the given equation.

If $p_n = 0$, one root of (1) is zero, and hence the corresponding root of (2) is infinite. Therefore, *as the coefficient of the highest power of x in $f(x)$ approaches zero, one root of $f(x)$ approaches infinity.*

If the coefficients of (1) are the same (or differ only in sign) when read in order backwards as when read in order forwards, the roots of (1) and (3) are the same. That is, the roots of (1) will then occur in pairs of the form a and $\frac{1}{a}$.

An equation in which the reciprocal of any root is also a root is called a **Reciprocal Equation**.

E. g., $6x^3 - 19x^2 + 19x - 6 = 0$ is a reciprocal equation in which the coefficients differ in sign when read in order backwards and forwards; two roots are $\frac{2}{3}$ and $\frac{3}{2}$.

EXAMPLES.

Find the equations whose roots are those of the following equations with opposite signs:

1. $x^2 - 4x - 5 = 0.$
2. $x^3 + 6x^2 - 7x - 60 = 0.$
3. $x^3 - 8x^2 - 28x + 80 = 0.$
4. $x^4 - 12x^2 + 12x - 3 = 0.$

Find the equation whose roots are those of

5. $x^3 - 16x^2 + 20x + 112 = 0$, each diminished by 4.
6. $x^4 - 12x^3 + 49x^2 - 78x + 40 = 0$, each diminished by 2.
7. $x^4 - 3x^3 - 6x^2 + 14x + 12 = 0$, each diminished by -2.

Transform the following equations so as to make the second terms disappear:

8. $x^2 - 4x - 21 = 0.$
9. $x^3 - 6x^2 + 8x - 2 = 0.$
10. $x^4 + 4x^3 - 29x^2 - 156x + 180 = 0.$

11. Find the equation whose roots are those of $x^3 + 6x^2 - 15x + 12 = 0$ each diminished by c, and find what c must be in order that, in the transformed equation, (1) the sum of the roots, and (2) the sum of the products of the roots two together, may be zero.

12. Transform the equation $x^3 + 3x^2 - 9x - 27 = 0$ into another in which the coefficient of x shall be zero.

Find the equation whose roots shall be the reciprocals of the roots of

13. $x^2 - 8x - 9 = 0.$
14. $2x^3 + 3x^2 - 13x - 12 = 0.$
15. $6x^4 - 5x^3 - 30x^2 + 20x + 24 = 0.$

16. Show that a reciprocal equation of an odd degree whose corresponding coefficients have the same sign has one root equal to -1.

17. Show that a reciprocal equation of an odd degree in which corresponding coefficients have opposite signs has one root equal to $+1$.

18. Show that a reciprocal equation of an even degree in which corresponding coefficients have opposite signs has the two roots ± 1.

Solve the following equations:

19. $2x^3 - 7x^2 + 7x - 2 = 0.$
20. $6x^3 - 7x^2 - 7x + 6 = 0.$
21. $3x^3 + 5x^2 + 5x + 3 = 0.$
22. $5x^3 - 7x^2 + 7x - 5 = 0.$
23. $2x^4 + 5x^3 - 5x^2 - 2 = 0.$
24. $12x^4 - 25x^3 + 25x - 12 = 0.$
25. $6x^4 - 7x^3 + 7x - 6 = 0.$

26. Solve the equation $2x^4 - 3x^3 - 16x^2 - 3x + 2 = 0$, having given that one root is -2.

27. Solve the equation $14x^5 - 3x^4 - 34x^3 - 34x^2 - 3x + 14 = 0$, having given that one root is 2.

28. Solve the equation $10x^6 - 21x^5 + 21x - 10 = 0$, having given that one root is 2.

99. *Successive Derivatives.* If $f(x)$ denote any function of x, its derivative $f'(x)$, (§ 79), will in general be a function of x that can also be differentiated. The result of differentiating $f'(x)$ is called the **Second Derivative** of $f(x)$. If this, again, can be differentiated, the result is called the **Third Derivative,** and so on.

The successive derivatives of $f(x)$ will be denoted by

$$f'(x),\quad f''(x),\quad f'''(x)\ \ldots\ f^{n)}(x).$$

Let $\quad f(x) \equiv A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots + A_nx^n.$

Then $\quad f'(x) = A_1 + 2A_2x + 3A_3x^2 + \ldots + nA_nx^{n-1}, \quad$ (§ 83)

$$f''(x) = 2A_2 + 2\cdot 3A_3x + \ldots + n(n-1)A_nx^{n-2},$$

$$f'''(x) = 1\cdot 2\cdot 3A_3 + \ldots + n(n-1)(n-2)A_nx^{n-3},$$

. . . .

$$f^{n)}(x) = n(n-1)(n-2)\ \ldots + 3\cdot 2\cdot 1A_n = A_n\cdot n!.$$

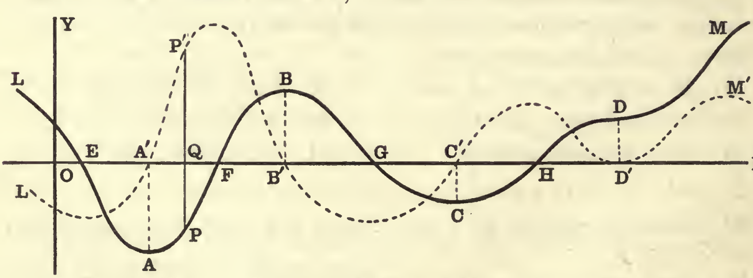
E. g., if $f(x) \equiv x^4 - 3x^3 - 5x^2 + 2x - 1$,

then $\quad f'(x) = 4x^3 - 9x^2 - 10x + 2, \qquad f'''(x) = 24x - 18,$

$\quad f''(x) = 12x^2 - 18x - 10, \qquad f''''(x) = 24 = 4!.$

Hence the rth derivative of a rational integral function of the nth degree is itself a rational integral function of degree $(n-r)$, (where r is not greater than n); and the nth derivative is a constant. Therefore the preceding theorems pertaining to a rational integral function $f(x)$ will also hold for its derivatives.

100. *The Derivative Curve, and Elbows.*



Let the curves LM and $L'M'$ be the loci, respectively, of the equations

$$y = f(x) \qquad (1)$$

and

$$y = f'(x). \qquad (2)$$

We will call $L'M'$, the locus of (2), the **Derivative Curve**, (or D. C.), and LM the **Integral Curve**. (See § 102.)

Draw any line parallel to the y-axis meeting the x-axis in Q, and the curves in P and P'.

We will call P and P' **corresponding points**.

Then, if $OQ = a$, we have by § 79

$$QP' = f'(a) = \textit{slope of } LM \textit{ at } P.$$

Hence the D. C. is a curve such that its *ordinate* at any point is the *slope* of the integral curve at the corresponding point.

Let A, B, C, D be the points on LM where the slope, *i. e.* $f'(x)$, is zero; then the ordinates of the corresponding points A', B', C', D' on $L'M'$ are zero. Hence A', B', C', D' are the intersections of $L'M'$ with the x-axis. Between A and B the slope of LM is positive, between B and C negative, etc. Therefore, between A' and B' the curve $L'M'$ is above the x-axis between B' and C' below, etc.

It will be convenient to call such points as A B, C, D, **Elbows** of the curve. Then the abscissas of the elbows of the graph of $f(x)$ are the roots of $f'(x)$, and may therefore be found by plotting the D. C. or by solving the equation $f'(x) = 0$.

Since $f'(x)$ is of degree $(n - 1)$, (§ 99,) the graph of $f(x)$ can not have more than $(n - 1)$ elbows.

If $f(x)$ is of an odd degree, its graph will have an even number of elbows, and therefore $f(x)$ will have at least one real root. (*Cf.* § 94.)

If the roots of $f'(x)$ are imaginary, the graph of $f(x)$ will have no elbows.

If two roots of $f'(x)$ are equal, its graph will touch the x-axis, as at D', and the two corresponding elbows of the integral curve will coincide as shown at D. Hence the slope of LM has the same sign on both sides of D. The integral curve therefore changes the direction of its curvature at D, and crosses its own tangent, which it cuts in *three coincident points*. Such a point is called a **Point of Inflection**.

Ex. Find the coordinates of the elbows of the following loci:

1. $y = x^3 - 12x$.
2. $y = 2x^3 - 15x^2 + 24x + 5$.
3. $y = x^3 - 6x^2 + 32$.
4. $y = 3x^4 - 20x^3 + 18x^2 + 108x$.
5. $y = 3x^5 - 20x^3 + 10$.
6. $y = 3x^4 - 8x^3 - 66x^2 + 144x$.

Equal Roots

101. Rolle's Theorem. *At least one real root of the equation*

$$f'(x) = 0 \tag{1}$$

lies between any two consecutive real roots of

$$f(x) = 0. \tag{2}$$

For there is at least one elbow of the integral curve, LM (§ 100), between any two consecutive intersections of it with the x-axis.

Conversely, LM can not meet the x-axis more than once between any two of its consecutive elbows.

Therefore, *at most one* real root of (2) lies between any two consecutive real roots of (1).

That is, the real roots of (1) separate those of (2).

If by a continuous modification of the form of $f(x)$—for example, by the addition or subtraction of a constant (§ 97)—two roots are made equal, the root of $f'(x)$ lying between them must approach the same value. Hence a double root of (2) is also a root of (1).

In general, if $f(x)$ has an r-fold root, such a root being regarded as due to the coalescence of r distinct roots, then will $f'(x)$ have an $(r-1)$-fold root due to the coalescence of the $(r-1)$ intervening roots. That is, if $f(x)$ has r roots each equal to a, $f'(x)$ will have $(r-1)$ roots each equal to a.

Then, by the application of Rolle's theorem to $f'(x)$ and $f''(x)$, $f''(x)$ and $f'''(x)$, and so on,

$$\left.\begin{array}{ll} \text{if} & f(x) \equiv (x-a)^r\varphi(x), \\ \text{we have} & f'(x) = (x-a)^{r-1}\varphi_1(x), \\ & f''(x) = (x-a)^{r-2}\varphi_2(x), \\ & \dots\dots\dots\dots \\ & f^{(r-1)}(x) = (x-a)\varphi_{r-1}(x). \end{array}\right\} \tag{3}$$

Conversely, if r roots of $f'(x)$ coalesce and become equal to a, the corresponding r elbows of the integral curve LM will coalesce; then, if a is a root of $f(x)$, this r-fold elbow will rest on the x-axis and give an $(r+1)$-fold root of $f(x)$.

Hence by induction, if

$$f^{(r-1)}(a) = f^{(r-2)}(a) = f^{(r-3)}(a) = \dots f'(a) = f(a) = 0,$$

and a is a single root of $f^{(r-1)}(x)$, then a is a double root of $f^{(r-2)}(x)$, a triple root of $f^{(r-3)}(x)$, . . . an $(r-1)$-fold root of $f'(x)$, and an r-fold root of $f(x)$.

This suggests an easy method of finding real multiple roots of an equation, when the roots are all equal except one or two.

E. g., if $$f(x) \equiv x^5 - 5x^4 + 40x^2 - 80x + 48 = 0,$$
we have
$$f'(x) = 5x^4 - 20x^3 + 80x - 80,$$
$$f''(x) = 20x^3 - 60x^2 + 80,$$
$$f'''(x) = 60x^2 - 120x.$$

The roots of $60x^2 - 120x = 0$ are 0 and 2.

Since $f'''(2) = f''(2) = f'(2) = f(2) = 0$, 2 is a fourfold root of $f(x) = 0$. Hence all its roots are 2, 2, 2, 2, -3.

Equations (3) are true whether a is real or imaginary. For suppose $f(x)$ has an r-fold root equal to a, then, whether a is real or imaginary, we have (§ 89 and § 90)

$$f(x) \equiv (x-a)^r \varphi(x). \tag{4}$$

Then

$$f'(x) = \lim_{h=0} \frac{(x-a+h)^r \varphi(x+h) - (x-a)^r \varphi(x)}{h}. \quad (\S 79) \tag{5}$$

Expanding $[(x-a)+h]^r$ by the binomial theorem gives

$$f'(x) = \lim_{h=0} \left\{ \frac{[(x-a)^r + r(x-a)^{r-1}h]\phi(x+h)}{h} \right.$$
$$\left. + \frac{[\frac{1}{2}r(r-1)(x-a)^{r-2}h^2 + \ldots + h^r]\phi(x+h) - (x-a)^r\phi(x)}{h} \right\} \tag{6}$$

$$= \lim_{h=0} \left\{ (x-a)^r \frac{\phi(x+h) - \phi(x)}{h} \right.$$
$$\left. + [r(x-a)^{r-1} + \tfrac{1}{2}r(r-1)(x-a)^{r-1}h + \ldots + h^{r-1}]\phi(x+h) \right\} \tag{7}$$

$$= (x-a)^r \phi'(x) + r(x-a)^{r-1}\phi(x) \tag{8}$$

$$= (x-a)^{r-1}[(x-a)\phi'(x) + r\phi(x)] = (x-a)^{r-1}\phi_1(x), \tag{9}$$

which is of the same form as the second of equations (3).

In like manner if $f(x)$ also has a q-fold root equal to b, and an s-fold root equal to c, and so on, then

$$f(x) \equiv (x-a)^r (x-b)^q (x-c)^s \ldots \varphi(x); \tag{10}$$

and $$f'(x) = (x-a)^{r-1}(x-b)^{q-1}(x-c)^{s-1} \ldots \varphi_1(x). \tag{11}$$

$$\therefore \quad (x-a)^{r-1}(x-b)^{q-1}(x-c)^{s-1} \ldots$$

is the G. C. D. of $f(x)$ and $f'(x)$.

Hence the multiple roots of an equation $f(x) = 0$, if there are any, can be detected by finding the G. C. D. of $f(x)$ and $f'(x)$ by the usual algebraic process.

Likewise the common roots of any two functions can be obtained by finding the G. C. D. of the two functions, and then finding the roots of this G. C. D.

Ex. If $$f(x) \equiv x^5 + x^4 - 13x^3 - x^2 + 48x - 36 = 0,$$
then $$f'(x) = 5x^4 + 4x^3 - 39x^2 - 2x + 48.$$

The G. C. D. of $f(x)$ and $f'(x)$ will be found to be
$$x^2 + x - 6 \equiv (x-2)(x+3).$$
$$\therefore \quad f(x) \equiv (x-2)^2(x+3)^2(x-1) = 0,$$
and the roots are 2, 2, -3, -3, 1.

EXAMPLES.

Solve the following equations by testing for equal roots:

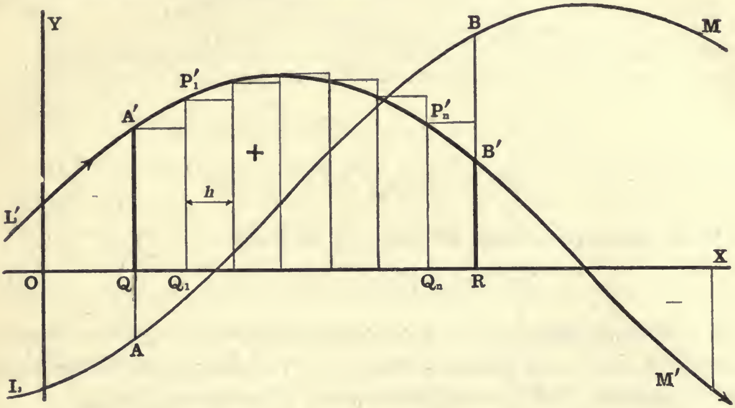
1. $x^3 + 11x^2 + 24x - 36 = 0.$
2. $x^3 - 2x^2 - 15x + 36 = 0.$
3. $x^4 - 7x^3 + 9x^2 + 27x - 54 = 0.$
4. $x^4 - 11x^3 + 44x^2 - 76x + 48 = 0.$
5. $x^4 - 5x^3 - 9x^2 + 81x - 108 = 0.$
6. $x^5 - 15x^3 + 10x^2 + 60x - 72 = 0.$
7. $x^5 - x^4 - 5x^3 + x^2 + 8x + 4 = 0.$
8. $x^4 - 2x^3 - 11x^2 + 12x + 36 = 0.$
9. $x^5 - 10x^2 + 15x - 6 = 0.$
10. $x^4 - 3x^3 - 6x^2 + 28x - 24 = 0.$
11. $x^5 - 10x^3 + 20x^2 - 15x + 4 = 0.$
12. $x^4 + 10x^3 + 24x^2 - 32x - 128 = 0.$
13. $x^5 + 19x^4 + 130x^3 + 350x^2 + 125x - 625 = 0.$
14. $x^6 - 5x^5 + 5x^4 + 9x^3 - 14x^2 - 4x + 8 = 0.$
15. $x^5 - 2x^4 - 6x^3 + 8x^2 + 9x + 2 = 0.$
16. $x^6 + 7x^5 + 4x^4 - 58x^3 - 115x^2 - 49x - 6 = 0.$
17. $x^5 - 8x^3 + 24x^2 - 28x + 16 = 0.$
18. $x^5 - 6x^3 - 28x^2 - 39x - 36 = 0.$

19. What is the condition that the cubic equation
$$x^3 + qx + r = 0$$
shall have a double root?

20. Show that in any cubic equation with rational coefficients a multiple root must be rational.

QUADRATURE.

102.* Let $y = f(x)$ and $y = f'(x)$ be the equations of the curves LM and $L'M'$ respectively.



It is required to find the area included between the curve $L'M'$, the x-axis, and the ordinates corresponding to $x = a = OQ$, and $x = b = OR$, where $b > a$. Let K denote the area $QA'B'R$.

Divide the distance QR into $(n+1)$ equal parts, each equal to $h = \delta x$. Draw ordinates at the points of division and construct rectangles as shown in the figure.

Let

$$x_1 = a + h = OQ_1, \quad x_2 = a + 2h, \; . \; . \; . \; x_n = a + nh = OQ_n.$$

Then

$$QA' = f'(a), \quad Q_1P_1' = f'(x_1), \; . \; . \; . \; Q_nP_n' = f'(x_n),$$

and the sum of the areas of the $(n+1)$ rectangles is

$$hf'(a) + hf'(x_1) + hf'(x_2) + \; . \; . \; . \; hf'(x_n).$$

$$\therefore \quad K = \lim_{n=\infty} [h\{f'(a) + f'(x_1) + f'(x_2) + \; . \; . \; . \; f'(x_n)\}]. \tag{1}$$

Now we know by § 79 that

$$f'(x) = \lim_{h=0} \frac{f(x+h) - f(x)}{h}. \tag{2}$$

Whence $$hf'(x) + h\rho = f(x+h) - f(x), \tag{3}$$

where ρ is a quantity which will vanish with h.

Therefore we may put

$$hf'(a)+h\rho_0=f(x_1)-f(a),$$

$$hf'(x_1)+h\rho_1=f(x_2)-f(x_1),$$

$$hf'(x_2)+h\rho_2=f(x_3)-f(x_2),$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$hf'(x_{n-1})+h\rho_{n-1}=f(x_n)-f(x_{n-1}),$$

$$hf'(x_n)+h\rho_n=f(b)-f(x_n).$$

From these equations we have by addition

$$\Sigma hf'(x)+\Sigma h\rho=f(b)-f(a). \tag{4}$$

The second member of (4) is independent of n, $\Sigma hf'(x)$ represents the *sum* of the areas of the $(n+1)$ rectangles however great their number, and $\Sigma h\rho$ vanishes when n becomes infinite.

$$\therefore \quad K=\lim_{n=\infty}\Sigma f'(x)\delta x=f(b)-f(a)=RB-QA. \tag{5}$$

The notation used to express this is

$$K=\int_a^b f'(x)dx=f(b)-f(a), \tag{6}$$

where the symbol $\int$ is an abbreviation of the word "sum," and means, in this case, *the summation of an infinite number of infinitesimal rectangles.*

Therefore, in order to find the required area, we must first obtain a function *which when differentiated will give* $f'(x)$; then substitute in this new function $f(x)$ the abscissas of the bounding ordinates and take the difference of the results.

Hence equation (6) may be written

$$K=\int_a^b f'(x)dx=\Big[f(x)\Big]_a^b=f(b)-f(a). \tag{7}$$

In applying the formula we must first find $f(x)$ from $f'(x)$, *i. e.* we must *reverse* the operation of differentiation. In this sense the symbol $\int$ denotes an operation which is the *inverse of differentiation.*

This inverse process is called **Integration.**

If then the symbol D be used to denote differentiation, the two symbols $\int$ and D neutralize each other, *i. e.* $\int Df(x) = f(x)$.

E. g., if $$Df(x) = f'(x) = 4x^3 - 3x^2 + 4x - 6,$$

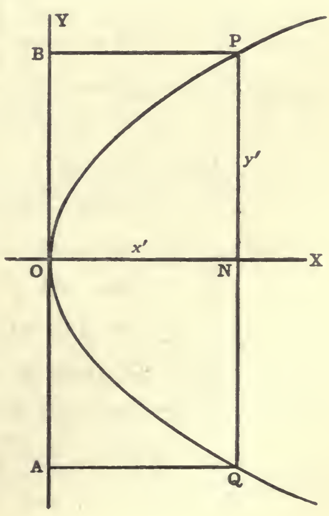
then $$\int Df(x) = \int f'(x) = f(x) = x^4 - x^3 + 2x^2 - 6x + c.$$

Hence, to *integrate* an integral function of x, increase the exponent of each power of x by unity and divide the coefficient by the increased exponent. Thus, $\int x^n = \frac{x^{n+1}}{n+1}$.

If $f'(x)$ is the derivative of $f(x)$, then $f(x)$ is called the **Integral** of $f'(x)$. The curve LM may be called the *Integral Curve* with respect to $L'M'$. Then we may say that the area bounded by the D. C., the x-axis, and two ordinates is numerically equal to the difference of the two corresponding ordinates of the I. C.

If $L'M'$ lies below the x-axis between A' and B', the slope of LM between A and B will be negative (§ 100). Hence $RB < QA$, *i. e.* $f(b) < f(a)$, and the *area is negative.* The rectangles will then lie above the curve.

Therefore the area will be *positive* or *negative* according as it lies to the *right* or *left* of the curve *viewed in the direction of x increasing.* If $L'M'$ cuts the x-axis between A' and B', the formula gives the excess (positive or negative) of the area which lies to the right over that which lies to the left.



Ex. 1. *Find the area of the segment of the parabola $y^2 = 4ax$ cut off by the double ordinate through $P(x', y')$.*

Here $y = 2\sqrt{a}\,x^{\frac{1}{2}} = f'(x)$.

$\therefore$ Area

$$ONP = \int_0^{x'} 2\sqrt{a}\,x^{\frac{1}{2}}dx = 2\sqrt{a}\int_0^{x'} x^{\frac{1}{2}}dx$$

$$= 2\sqrt{a}\left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^{x'} = 2\sqrt{a}\cdot\tfrac{2}{3}x'^{\frac{3}{2}}$$

$$= \tfrac{2}{3}x'\cdot 2\sqrt{a}\,x'^{\frac{1}{2}} = \tfrac{2}{3}x'y'$$

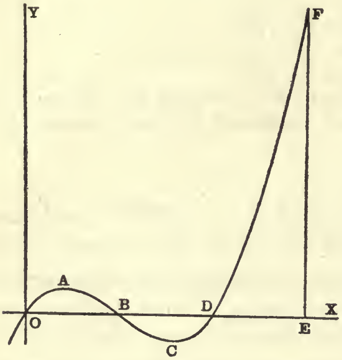
$= \frac{2}{3}$ rectangle $OBPN$.

$\therefore$ Area $OPQ = \frac{2}{3}$ rectangle $ABPQ$.

$\therefore$ Area between AB and the curve is equal to $\frac{1}{3}ABPQ$.

That is, the parabola trisects the rectangle.

Ex. 2. The curve $y = x^3 - 3x^2 + 2x$ cuts the x-axis in the points $(0, 0)$, $B(1, 0)$, $D(2, 0)$.



We now have $f'(x) = x^3 - 3x^2 + 2x$.

$$\therefore\ OAB = \int_0^1 f'(x)dx$$

$$= \int_0^1 (x^3 - 3x^2 + 2x)dx$$

$$= \left[\frac{x^4}{4} - x^3 + x^2 + c\right]_0^1 = \tfrac{1}{4}.$$

$$BCD = \left[\frac{x^4}{4} - x^3 + x^2 + c\right]_1^2$$

$$= (4 - 8 + 4 + c) - (\tfrac{1}{4} + c) = -\tfrac{1}{4}.$$

$$DEF = \left[\frac{x^4}{4} - x^3 + x^2 + c\right]_2^3$$

i.e. $DEF = (\tfrac{81}{4} - 27 + 9 + c) - (4 - 8 + 4 + c) = 2\tfrac{1}{4}.$

EXAMPLES.

1. Find the area included between the curve

$$y = x^3 - 9x^2 + 23x - 15,$$

the x-axis, and the lines $x = 1$, $x = 3$; also $x = 3$, $x = 5$; $x = 1$, $x = 5$.

2. Find the area included between the curve

$$y = x^2 - 2x - 8,$$

the x-axis, and the lines $x = -2$, $x = 4$; also between the curve $y = x^2 - 2x + 1$ and the same lines.

Find the area between the x-axis and the curve

3. $y = x^3 - 3x^2 - 9x - 27$.

4. $y = x^3 + ax$.

5. $y = x^4 - 4x^3 - 2x^2 + 12x + 9$.

Find the area between the curves

6. $y^2 = 4ax$ and $x^2 = 4ay$.

7. $y^2 = 4x$ and $y^2 = x^3$.

8. $y^3 = x^2$ and $y^2 = x^3$.

9. $y^m = x^n$ and $y^n = x^m$. Ans. $\dfrac{m-n}{m+n}$.

10. $y = x^3 - x$ and $y = x$.

11. $y = x^3 - x$ and $y^2 = x\sqrt{2}$.

12. $y^2 = 4ax$ and $y = 2x - 4a$.

13. $x^2y = a^3$, $x = b$, $x = c$, and $y = 0$. Ans. $a^3\left(\frac{b-c}{bc}\right)$.

14. $y = x^2 - 5x + 4$ and $x + y = 4$.

15. $y = x^3$ and $y = x^{\frac{1}{3}}$.

16. Show that the area included between the curve $y = Ax^n$, the x-axis and the line $x = a$ is $A\frac{ab}{n+1}$, where b is the ordinate corresponding to $x = a$. Show that the parabola is a particular case.

Maxima and Minima.

103.* Let the curves LM, $L'M'$ and $L''M''$ be the loci, respectively, of the equations

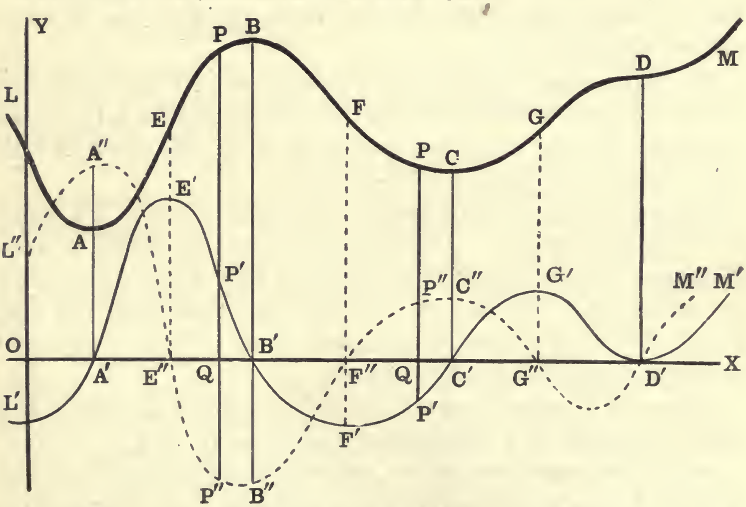
$$y = f(x), \tag{1}$$

$$y = f'(x), \tag{2}$$

and

$$y = f''(x). \tag{3}$$

Then $L''M''$ is the *Second Derivative Curve.*



Since $f''(x)$ is the first derivative of $f'(x)$, the ordinate of $L''M''$ at any point represents the slope of $L'M'$ at the corresponding point; and the intersections E'', F'', G'' of $L''M''$ with the x-axis correspond to the elbows E', F', G' of $L'M'$ (§ 100).

Let the line $x = a$ meet the curves in the corresponding points P, P', P'', and the x-axis in Q.

Then $QP = f(a)$, $QP' = f'(a)$, $QP'' = f''(a)$.

That is, QP' is the slope of LM at P, and QP'' is the slope of $L'M'$ at P'.

Suppose the point P to move along the curve LM *toward the right.* As P approaches the elbow B, the ordinate QP increases; but as P passes through B, the ordinate *ceases to increase and begins to decrease.* At such a point the ordinate, *i. e.* $f(x)$, is said to have a **Maximum Value,** or to be a **Maximum.** In like manner as P approaches the elbow A, or C, the ordinate QP decreases; but as P passes through A, or C, the ordinate *ceases to decrease and begins to increase.* At such points QP, *i. e.* $f(x)$, is said to have a **Minimum Value,** or to be a **Minimum.**

That is, a function, $f(x)$, is said to have a *maximum value* when $x = a$, if $f(a) > f(a \pm h)$; and a *minimum value,* if $f(a) < f(a \pm h)$, for *very small* values of h.

Since in these definitions the comparison is made between values of $f(x)$ in the immediate vicinity only of A, B, C, a maximum is not necessarily the *greatest,* nor a minimum the *least,* of *all* the values of the function.

Moreover, since maximum and minimum ordinates occur only at the elbows of a curve where the tangent is parallel to the x-axis, a *necessary* but not a *sufficient* condition for a maximum or minimum value of $f(x)$ is $f'(x) = 0$ (§ 100).

Suppose a tangent to be drawn to LM at any elbow, *i. e.* at any point where $f'(x) = 0$. Then the curve will lie below or above this tangent line for a short distance on both sides of the elbow, according as the ordinate of the elbow is a maximum or a minimum. If the curve crosses this tangent, as at D, the ordinate is neither a maximum nor a minimum.

Hence, as P passes (toward the right) through an elbow, as B, whose ordinate is a maximum, the slope of LM, *i. e.* $f'(x)$, changes from *positive* to *negative;* and as P passes through an elbow, such as A or C, whose ordinate is a minimum $f'(x)$ changes from *negative* to *positive.*

Therefore, the necessary and sufficient conditions that $f(x)$ shall be a maximum or a minimum when $x = a$ are as follows:

$$\left.\begin{array}{l} \textit{For max., } f'(a) = 0;\ f'(a-h), \textit{ positive; } f'(a+h), \textit{ negative.} \\ \textit{For min., } f'(a) = 0;\ f'(a-h), \textit{ negative; } f'(a+h), \textit{ positive.} \end{array}\right\} \quad (4)$$

If $f'(a+h)$ and $f'(a-h)$ have the *same* sign, $f(a)$ is neither a maximum nor a minimum value of $f(x)$.

Now suppose, as is usually the case, that a is a single root of $f'(x)=0$, so that $f''(a) \neq 0$. (§ 101.)

Then if QP passes through a maximum value, as $B'B$, when $x=a$, the slope of LM changes from $+$ to $-$. Hence the corresponding point P' crosses the x-axis from *above downwards*, and therefore the slope of $L'M'$ at B' is negative, *i. e.*

$$B'B''=f''(a) \textit{ is negative.}$$

If QP passes through a minimum value, as $C'C$, the slope of LM changes from $-$ to $+$. Hence P' crosses the x-axis from *below upwards*, and therefore the slope of $L'M'$ at C' is positive, *i. e.*

$$C'C''=f''(a) \textit{ is positive.}$$

Therefore, if $f''(a) \neq 0$, the necessary and sufficient conditions that $f(a)$ shall be a maximum or a minimum value of $f(x)$ are:

$$\left.\begin{array}{lll} \textit{For a maximum,} & f'(a)=0; & f''(a), \textit{ negative.} \\ \textit{For a minimum,} & f'(a)=0; & f''(a), \textit{ positive.} \end{array}\right\} \qquad (5)$$

If a is an r-fold root of $f'(x)=0$, then $f''(a)=0$ when $r>1$ (§ 101) and the conditions (5) fail to disclose the nature of the corresponding ordinate.

If r is an odd number the curve $L'M'$ will cut the x-axis in an odd number of coincident points, and hence will cross the x-axis at the point $(a, 0)$. Therefore the sign of $f'(x)$ will change from $+$ to $-$ for a maximum, and from $-$ to $+$ for a minimum. In this case we must use conditions (4) to determine the nature of $f(a)$.

If r is an even number, $L'M'$ will not cross the x-axis at the point $(a, 0)$, as at D'. Hence $f'(x)$ will not change sign, and therefore $f(a)$ is neither a maximum nor a minimum.

The maximum and minimum ordinates of $L'M'$ can be determined in the same manner. The points E, F, G, D on LM corresponding to the maximum and minimum ordinates of $L'M'$ are therefore, respectively, the points of maximum and minimum slope of LM. At the points where the slope of a curve ceases to increase and begins to decrease, or *vice versa*, the curve changes the direction of its curvature. Therefore E, F, G, D are the points of inflection of LM (§ 100).

Hence the position of the points of inflection of a curve are obtained by finding the position of the maximum and minimum ordinates of the D. C.

104. Illustrative Examples.

Ex. 1. The curves $y = \sin x$ and $y = \cos x$ are good examples of the relations and principles explained in § 102 and § 103.

Let $$f(x) \equiv \sin x.$$

Then $$f'(x) = \lim_{h=0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h=0}\left[\cos(x + \tfrac{1}{2}h)\frac{\sin \frac{1}{2}h}{\frac{1}{2}h}\right] \quad (1)$$

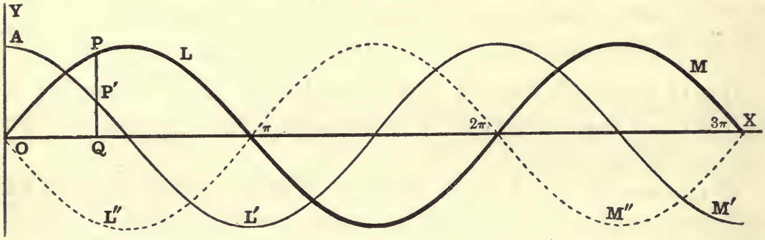
$$= \cos x. \qquad \text{(Ex. 13, p. 115.)} \quad (2)$$

In like manner it can be shown that the derivative of $\cos x$ is $-\sin x$.

Let $y = f(x) = \sin x$, equation of LM,

$y = f'(x) = \cos x$, equation of $L'M'$,

and $y = f''(x) = -\sin x$, equation of $L''M''$.



Then $f'(x) = \cos x = 0$, when $x = \frac{1}{2}\pi$, $\frac{3}{2}\pi$, $\frac{5}{2}\pi$, etc.

and $f''(\frac{1}{2}\pi) = -\sin \frac{1}{2}\pi = -1$. $\therefore$ $\sin \frac{1}{2}\pi = 1$ is a max.

$f''(\frac{3}{2}\pi) = -\sin \frac{3}{2}\pi = 1$. $\therefore$ $\sin \frac{3}{2}\pi = -1$ is a min., etc.

Also $f''(x) = -\sin x = 0$, when $x = \pi$, 2π, 3π, etc.

These values of x make $\cos x$ alternately a maximum and a minimum, and hence give the points of inflection of LM. That is, the sine curve changes the direction of its curvature as it crosses the x-axis.

Let $x = OQ$ be any line parallel to the y-axis.

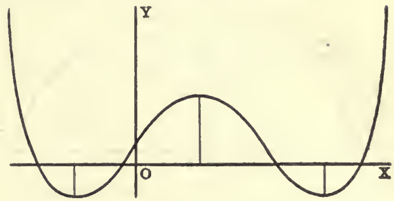
Then $f'(x) = \cos x = QP' =$ slope of LM at P.

Moreover, by § 102 we have

$$\text{Area } OAP'Q = \int_0^x f'(x)dx = \int_0^x \cos x dx = \Big[\sin x\Big]_0^x = \sin x = QP. \quad (3)$$

That is, *the ordinate of any point of the cosine curve is equal to the slope of the sine curve at the corresponding point; and the ordinate of the sine* *curve is equal to the area bounded by the ordinate, the cosine curve, and the axes of coordinates.*

Ex. 2. *Find the maximum and minimum values of the function*



$$f(x) \equiv x^4 - 4x^3 - 2x^2 + 12x + 4.$$

Here $f'(x) = 4x^3 - 12x^2 - 4x + 12$

and $f''(x) = 12x^2 - 24x - 4.$

The roots of $f'(x) = 0$

are $-1, \ 1, \ 3.$

$f''(-1) = 32.$ $\therefore$ $f(-1) = -5$ is a minimum.

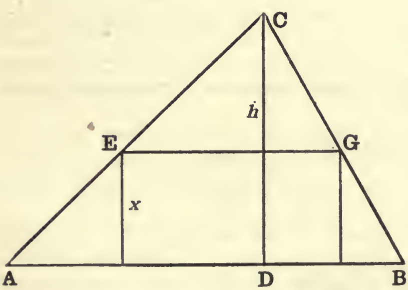
$f''(1) = -16.$ $\therefore$ $f(1) = 11$ is a maximum.

$f''(3) = 32.$ $\therefore$ $f(3) = -5$ is a minimum.

The roots of $f''(x) = 0$ are $1 \pm \frac{2}{3}\sqrt{3}$, which are the distances of the points of inflection from the y-axis.

In the solution of problems in maxima and minima, we must first obtain an algebraic expression, $f(x)$, for the quantity whose maximum or minimum is required. We may then proceed as in the preceding examples.

Ex. 3. *Find the maximum rectangle that can be inscribed in a given triangle.*



Let b = the base of the given triangle ABC, h the altitude and x the altitude of the inscribed rectangle. Then from similar triangles,

$$EG : b = (h - x) : h.$$

$$\therefore \quad EG = \frac{b}{h}(h - x).$$

Then $\frac{b}{h}(hx - x^2)$ is the area of the rectangle, which is to be made a maximum. Any value of x that will make $(hx - x^2)$ a maximum will also make $\frac{b}{h}(hx - x^2)$ a maximum. Hence we may put

$$f(x) \equiv hx - x^2.$$

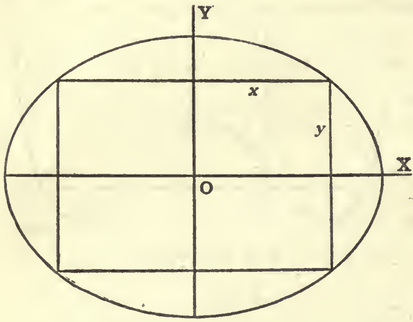
Then $\qquad f'(x) = h - 2x = 0 \quad \text{when } x = \frac{1}{2}h.$

Also $\qquad f''(x) = -2.$

$$\therefore \quad f(\tfrac{1}{2}h) = \tfrac{1}{4}h^2 \text{ is a maximum.}$$

Therefore the altitude of the maximum inscribed rectangle is one-half the altitude of the triangle.

Ex. 4. *Find the area of the largest rectangle which can be inscribed in the ellipse*



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad (1)$$

Let K denote the area of the rectangle. Then

$$K = 2x \cdot 2y = \frac{4b}{a}\sqrt{a^2x^2 - x^4} \quad (2)$$

is the function of x which is to be a maximum.

Any value of x which will make $ax^2 - x^4$ a maximum, or a minimum, will also make K a maximum, or a minimum.

Therefore, let $\qquad f(x) \equiv a\,x^2 - x^4.$

Then $\qquad f'(x) = 2a^2x - 4x^3 = 0$ when $x = 0$, or $\pm \frac{1}{2}a\sqrt{2}$,

and $\qquad f''(x) = 2a^2 - 12x^2 = -4a^2$ when $x = \frac{1}{2}a\sqrt{2}$.

$\therefore \quad x = \frac{1}{2}a\sqrt{2}$ will make K a maximum.

Therefore $K = 2ab$ is the area of the maximum rectangle, which is half the rectangle whose sides are the axes of the ellipse.

Ex. 5. *Find the dimensions of a cone of revolution which shall have the greatest volume with a given surface.*

Let $x =$ the radius of the base, $y =$ the slant height, $V =$ the volume, and $S =$ the total surface.

Then $\qquad S = \pi x^2 + \pi xy;$ whence $y = \dfrac{S}{\pi x} - x,$

and $\qquad (Altitude)^2 = y^2 - x^2 = \dfrac{S^2}{\pi^2 y^2} - \dfrac{2S}{\pi}.$

$$\therefore \quad V = \frac{\pi x^2}{3}\sqrt{\frac{S^2}{\pi^2 x^2} - \frac{2S}{\pi}} = \frac{\sqrt{S^2x^2 - 2\pi S x^4}}{3}.$$

Let $\qquad f(x) \equiv Sx^2 - 2\pi x^4.$

Then $\qquad f'(x) = 2Sx - 8\pi x^3 = 0$ when $x = 0$, or $\pm \dfrac{1}{2}\sqrt{\dfrac{S}{\pi}}$.

and $\qquad f''(x) = 2S - 24\pi x^2 = -4S$ when $x = \dfrac{1}{2}\sqrt{\dfrac{S}{\pi}}$.

$\therefore \quad V$ is a max. when $x = \dfrac{1}{2}\sqrt{\dfrac{S}{\pi}}$, and $y = \dfrac{3}{2}\sqrt{\dfrac{S}{\pi}}$.

That is, if the surface is constant, the volume of the cone is a maximum when the slant height is three times the radius of the base.

EXAMPLES.

Find the maximum and minimum ordinates and the points of inflection (points of maximum or minimum slope) of the curves

1. $y = 3x^3 - x^2 + 4.$
2. $y = x^3 - 9x^2 + 15x - 3.$
3. $y = x^3 - 3x^2 + 6x + 7.$
4. $y = x^3 - 9x^2 + 24x + 16.$

5. Find the sides of the maximum rectangle which can be inscribed in a circle; in a semi-circle.

6. Find the sides of the maximum rectangle which can be inscribed in a semi-ellipse.

7. Find the altitude of the maximum rectangle which can be inscribed in a segment of a parabola, the base of the segment being perpendicular to the axis of the parabola.

8. What is the least square that can be inscribed in a given square?

9. Find the altitude of a cylinder inscribed in a cone when the volume of the cylinder is a maximum.

10. What are the most economical proportions for a cylindrical tin can? That is, what should be the ratio of the height to the radius of the base that the capacity shall be a maximum for a given amount of tin?

11. What are the most economical proportions for a cylindrical tin cup?

12. What are the most economical proportions for an open cylindrical water tank made of iron plates, if the cost of the sides per square foot is two-thirds of the cost of the bottom per square foot?

13. An open box is to be made from a sheet of pasteboard 12 inches square by cutting equal squares from the four corners and bending up the sides. What are the dimensions of the largest box that can be made?

14. If a rectangular piece of pasteboard, whose sides are a and b, have a square cut from each corner, find the side of the square so that the remainder may form a box of maximum capacity.

15. A person being in a boat 3 miles from the nearest point of the shore, wishes to reach in the shortest possible time a place 5 miles from that point along the shore; supposing he can walk 5 miles an hour, but can row only at the rate of 4 miles an hour, required the place where he must land.

16. The cost per hour of driving a steamer through still water varies as the cube of its speed. At what rate should it be run to make a trip against a four-mile current most economically?

17. Find the altitude of the greatest cylinder that can be cut out of a given sphere.

18. Find the altitude of the greatest cone that can be inscribed in a given sphere.

19. Find the altitude of a cone inscribed in a sphere which shall make the convex surface of the cone a maximum.

20. Find the dimensions of a cone with a given convex surface and a maximum volume.

21. Find the altitude of the least cone that can be circumscribed about a given sphere.

22. Find the altitude of the maximum cylinder that can be inscribed in a given paraboloid.

23. What is the diameter of a ball which, being let fall into a conical glass of water, shall expel the most water possible from the glass; the depth of the glass being 6 inches and its diameter at the top 5 inches?

Ans. $4\frac{11}{46}$ in.

24. The sides of a rectangle are a and b. Show that the greatest rectangle that can be drawn so as to have its sides passing through the corners of the given rectangle is a square whose side is $\dfrac{a+b}{\sqrt{2}}$.

25. The strength of a beam of rectangular cross-section, if supported at the ends and loaded in the middle, varies as the product of the breadth of the cross-section by the square of its depth. Find the dimensions of the cross-section of the strongest beam that can be cut from a log 18 inches in diameter.

26. A Norman window consists of a rectangle surmounted by a semicircle. If the perimeter of the window is given, show that the quantity of light admitted is a maximum when the radius of the semicircle is equal to the height of the rectangle.

CHAPTER VII.

CONIC SECTIONS.

105. The general equation of the first degree and also some special cases of the equation of the second degree have been considered in Chapter III. We now proceed to the study of the general equation of the second degree, and the standard forms to which it can be transformed. It will presently be shown that the locus of such an equation is always a curve that can be obtained by making a plane section of a right circular cone. For this reason the locus is called a **Conic Section.***

106. *The Fundamental Property of a Plane Section of a Right Circular Cone, or a Conic Section.*

Let VO be the axis of a right circular cone, and APB any section made by a plane not passing through V.

Inscribe a sphere in the cone tangent to the plane of the section at F; then the line of contact HRK of the sphere and cone is a circle with centre C in VO, whose plane is perpendicular to VO and meets the plane of the section APB in the line ES.

Pass the plane VMN through VO perpendicular to the plane APB, meeting it in the line AB, meeting the plane HKR in HK, and the line ES in D; then the plane VMN is also perpendicular to the plane HKR, and therefore perpendicular to ES.

Let P be any point on the section.

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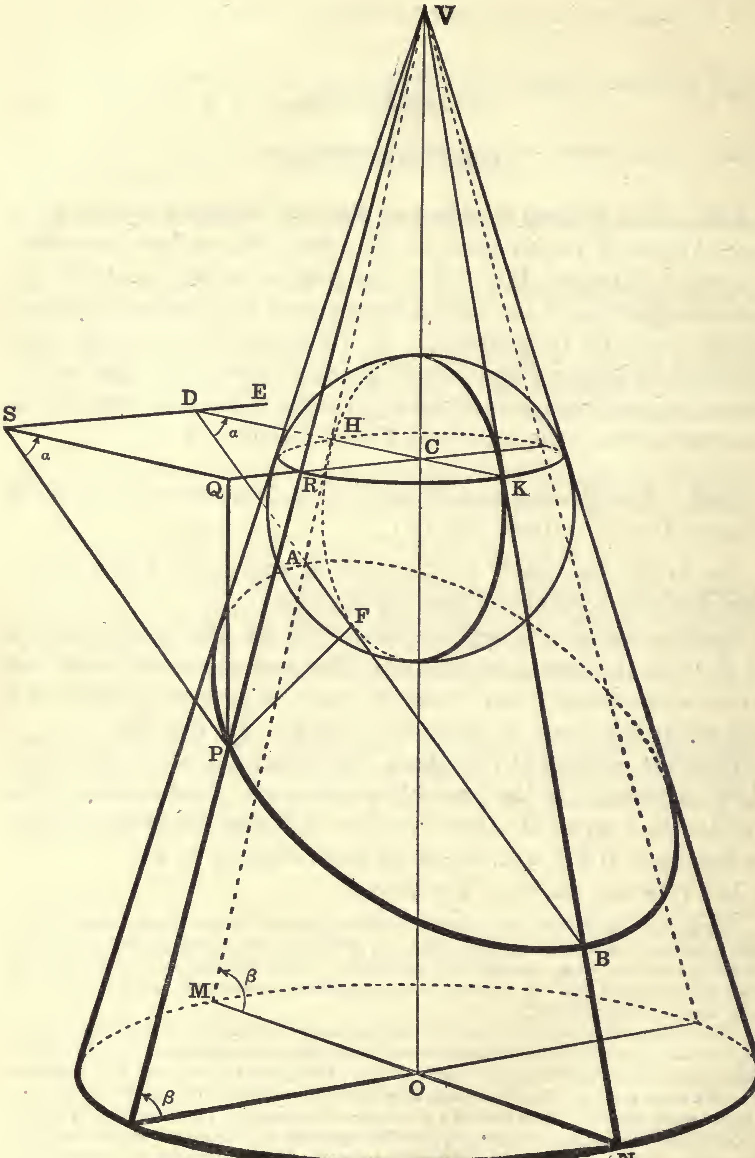
*After studying the straight line and the circle, the old Greek mathematicians turned their attention to the conic sections, and by investigating them as sections of a cone soon discovered many of their characteristic properties. The most important of these discoveries were probably made by Archimedes and Apollonius, as the latter wrote a treatise on conic sections about 200 B. C.

These curves are worthy of careful study, not only on account of their historic interest, but also on account of their importance in the physical sciences and their frequent occurrence in the experiences of everyday life. For example, the orbit of a heavenly body is a conic section. For this reason they were thoroughly studied by the astronomer, Kepler, about 1600 A. D. The path of a projectile is a parabola. The law of falling bodies, the pressure-volume law of gases, the law of moments in uniformly loaded beams, all give conic sections. The bounding line of a beam of uniform strength, the oblique section of a stove-pipe, the shadow of a circle, the apparent line dividing the dark and light parts of the moon, etc., are all conic sections. The reflectors in head-lights and search-lights are parabolic.



Draw PF, and the element PV which will be tangent to the sphere at R.

Through P draw a line perpendicular to the plane HKR, which will meet CR produced in Q; and through PQ pass a plane perpendicular to ES meeting it in S.

Let $\beta = \angle PRQ = \angle AHD$, the complement of the semi-vertical angle of the cone.

Let $$\alpha = \angle ADH = \angle PSQ.$$

Then, since tangents from an external point to a sphere are equal,

$$PF = PR.$$

From the right triangles PQR and PSQ we get

$$PQ = PR \sin \beta = PS \sin \alpha.$$

$$\therefore \quad \frac{PF}{PS} = \frac{\sin \alpha}{\sin \beta}. \tag{1}$$

So long as we consider any particular section the point F and the line ES are fixed, α is constant, and therefore the ratio of PF to PS is constant.

Equation (1) expresses the *Fundamental Property of a Conic Section*, which is used as the defining property. Moreover, all curves which have this property are plane sections of some cone; for all possible curves satisfying this condition are gotten by giving this constant ratio all possible values, and also letting the distance, FD, from the fixed point to the fixed line have all possible values. We can do this with a conic section. For any particular value of β, *i. e.* for any particular cone, the ratio can vary from zero (when $\alpha = 0$) to csc β (when $\alpha = \frac{1}{2}\pi$). For any particular value of α the ratio can vary from sin α (when $\beta = \frac{1}{2}\pi$) to ∞ (when $\beta = 0$). Thus the ratio can have any value from 0 to ∞. Also the distance of F from ES, depending as it does upon the size of the inscribed sphere, for any particular cone and any particular value of α can vary from zero to ∞. Therefore the property expressed by (1) is indeed a defining property of a conic section, that is:

*A Conic Section, or A Conic, is the locus of a point which moves in a plane so that its distance from a fixed point in the plane is in a constant ratio to its distance from a fixed line in the plane.**

*This is generally known as Boscovich's definition of a conic section, but, in the article on Analytic Geometry in the Encyclopedia Britannica, ninth edition, Cayley calls it the definition of Apollonius.

The fixed point F is called the **Focus**; the fixed line ES is called the **Directrix**; the constant ratio is called the **Eccentricity**, and is denoted by the letter e.

107. *Classification of the Conic Sections.*

Using e to denote the eccentricity, we have, by (1) of § 106,

$$\frac{PF}{PS} = \frac{\sin \alpha}{\sin \beta} = e. \tag{1}$$

When $\alpha < \beta$, $e < 1$; the plane of the section meets all the elements of the cone on the same side of the vertex; the section is a closed curve as shown in the figure § 106, and is called an **Ellipse.**

When $\alpha = 0$, $e = 0$; the plane of the section is perpendicular to the axis of the cone, VO, and the section is a **Circle**. Hence a circle is a particular case of the ellipse.

When $\alpha = \beta$, $e = 1$; the line AB (§ 106) is then parallel to VN and the point B moves off to an infinite distance; the section consists of a single branch extending to infinity, and is called a **Parabola.**

When $\alpha > \beta$, $e > 1$ and the plane APB (§ 106) meets NV produced on the other sheet of the conical surface; the section is then composed of two infinite branches, one lying on each sheet of the cone, and is called a **Hyperbola.**

Thus the parabola is the limiting case of both the ellipse and the hyperbola.

Let the plane of the section pass through the vertex of the cone.

Then if $e < 1$, the section is a point ellipse or a point circle.

If $e = 1$, the plane is tangent to the cone and the parabola reduces to two coincident straight lines.

If $e > 1$, the hyperbola becomes two intersecting straight lines, which approach in the limit two parallel lines as the vertex of the cone moves off to an infinite distance.

Hence a point, two intersecting straight lines, two parallel straight lines, and two coincident straight lines are all limiting cases of conic sections.

Under the head of conic sections we must therefore include:

(1) *The Ellipse, including the circle and the point;*

(2) *The Parabola;*

(3) *The Hyperbola;*

(4) *The Line-pair.*

EXAMPLES.

1. Inscribe a sphere* tangent to the plane *APB* (fig. § 106) on the other side and thus show that the ellipse has another focus and a corresponding directrix; and that the two directrices are parallel and equidistant from the foci.

2. By means of these two inscribed spheres, prove the property of the ellipse given in § 34.

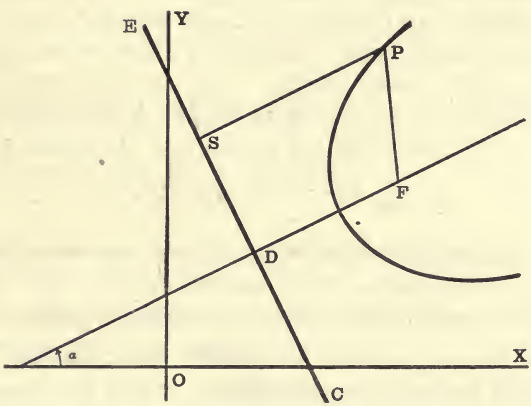
3. Inscribe spheres* in both sheets of the cone and show that the hyperbola also has two foci and two directrices.

4. Prove the property of the hyperbola stated in § 36.

5. Where are the foci and the directrices of the circle, the parabola, and two intersecting straight lines?

General Equation of the Conic Sections.

108. *To find the equation of a conic section in rectangular coordinates.*



Let the equation of the directrix EC be

$$x \cos \alpha + y \sin \alpha - p = 0. \qquad (1)$$

Let $F(k, l)$ be the corresponding focus.
Let $P(x, y)$ be any point on the conic.
Draw PS perpendicular to EC, and join P and F.
Then from equation (1) of § 107 we have

$$PF = e \cdot PS. \qquad (2)$$

* For complete diagrams see "Some Mathematical Curves and Their Graphical Construction," by F. N. Wilson, pp. 45, 46.

Now $$PF^2 = (x-k)^2 + (y-l)^2, \qquad [(2), \S\ 7]$$

and $$PS = x\cos\alpha + y\sin\alpha - p. \qquad [(4), \S\ 50]$$

Therefore the required equation is

$$(x-k)^2 + (y-l)^2 = e^2(x\cos\alpha + y\sin\alpha - p)^2. \qquad (3)$$

Expanding (3) and collecting terms we have

$$(1 - e^2\cos^2\alpha)x^2 - 2(e^2\sin\alpha\cos\alpha)xy + (1 - e^2\sin^2\alpha)y^2$$
$$+ 2(e^2p\cos\alpha - k)x + 2(e^2p\sin\alpha - l)y + k^2 + l^2 - e^2p^2 = 0. \quad (4)$$

Since equation (4) contains five arbitrary constants, k, l, α, p, e, it may be any equation of the second degree. That is, any equation of the second degree represents a conic section.

The most general equation of a conic is, therefore, the complete equation of the second degree, and may be written (§ 53)

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad (5)$$

Equations (4) and (5) each contain five arbitrary constants. A conic section can therefore be made to satisfy five independent conditions, and no more. That is, a conic can be made to pass through any five given points.

If the directrix EC be taken for the y-axis, and FD, perpendicular to EC, for x-axis, the equation of the conic (3) reduces to

$$(x-k)^2 + y^2 = e^2x^2. \qquad (6)$$

109. *To find the parameters of the conic represented by the general equation*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad (1)$$

The equation referred to parallel axes through the point (x', y') will be found by substituting $x + x'$ for x and $y + y'$ for y [§ 66, (10)], and will therefore be

$$a(x+x')^2 + 2h(x+x')(y+y') + b(y+y')^2$$
$$+ 2g(x+x') + 2f(y+y') + c = 0,$$

or $$ax^2 + 2hxy + by^2 + 2x(ax' + hy' + g) + 2y(hx' + by' + f)$$
$$+ ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c = 0. \quad (2)$$

If, as is generally possible, x' and y' be so chosen that

$$ax' + hy' + g = 0, \qquad (3)$$

and $$hx' + by' + f = 0, \qquad (4)$$

the coefficients of x and y in (2) will both vanish, and the equation referred to (x', y') as origin will then be

$$ax^2 + 2hxy + by^2 + c' = 0,* \qquad (5)$$

where $$c' \equiv ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c. \qquad (6)$$

The locus represented by (5) is symmetrical with respect to the origin [§ 28, (9)]; *i. e.* all chords which pass through the origin are bisected at the origin.

The point (x', y') is therefore called the **Centre of the Conic**.

Hence the coordinates of the centre of the conic represented by (1) are the values of x' and y' which satisfy equations (3) and (4),

i. e. $$x' = \frac{fh - bg}{ab - h^2}, \qquad y' = \frac{gh - af}{ab - h^2}. \qquad (7)$$

Multiply equations (3) and (4) by x' and y', respectively, and subtract the sum from the right member of (6); we thus get

$$c' = gx' + fy' + c. \qquad (8)$$

$$\therefore \quad c' = g\left(\frac{fh - bg}{ab - h^2}\right) + f\left(\frac{gh - af}{ab - h^2}\right) + c \qquad \text{from (7)}$$

$$= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2} = \frac{\triangle}{ab - h^2}. \qquad (\S\ 54) \quad (9)$$

Suppose equation (4) of § 108 and equation (5) to represent the same locus; then their coefficients must be proportional, and we have the following equations for determining the parameters of the conic represented by equation (5).

$$\frac{1 - e^2\cos^2\alpha}{a} = \frac{1 - e^2\sin^2\alpha}{b} = \frac{-e^2\sin\alpha\cos\alpha}{h} = \frac{e^2p\cos\alpha - k}{0}$$

$$= \frac{e^2p\sin\alpha - l}{0} = \frac{k^2 + l^2 - e^2p^2}{c'} \equiv r, \text{ say.} \qquad (10)$$

Then $$r = \frac{1 - e^2\cos^2\alpha}{a} = \frac{1 - e^2\sin^2\alpha}{b} = \frac{2 - e^2}{a + b}, \qquad (11)$$

$$r^2 = \frac{1 - e^2 + e^4\sin^2\alpha\cos^2\alpha}{ab} = \frac{e^4\sin^2\alpha\cos^2\alpha}{h^2} = \frac{1 - e^2}{ab - h^2}. \qquad (12)$$

*Observe that the coefficients a, b, and h are not changed in this transformation, and therefore the following equations which involve only a, b, and h are the same for (1) and (5).

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$$\therefore \quad \left(\frac{2-e^2}{a+b}\right)^2 = \frac{1-e^2}{ab-h^2}. \tag{13}$$

Whence $(ab-h^2)e^4 + [(a-b)^2 + 4h^2](e^2-1) = 0.$ (14)

Considering (14) as a quadratic in $\frac{1}{e^2}$, and solving gives

$$\frac{1}{e^2} = \frac{1}{2} \pm \frac{a+b}{2\sqrt{(a-b)^2+4h^2}}. \tag{15}$$

Also $r = -\frac{e^2 \sin \alpha \cos \alpha}{h} = -\frac{e^2 \sin 2\alpha}{2h} = \frac{2-e^2}{a+b}.$ (16)

$$\left.\begin{aligned} \therefore \quad \sin 2\alpha &= -\frac{2h}{a+b}\left(\frac{2}{e^2}-1\right) \\ &= \frac{2h}{\sqrt{(a-b)^2+4h^2}}. \text{ by (15)} \\ \text{Whence} \quad \cos 2\alpha &= \frac{a-b}{\sqrt{(a-b)^2+4h^2}}, \\ \text{and} \quad \tan 2\alpha &= \frac{2h}{a-b}. \end{aligned}\right\} \tag{17}$$

From (10) and (16) we get

$$r = \frac{k^2+l^2-e^2p^2}{c'} = -\frac{e^2 \sin 2\alpha}{2h}. \tag{18}$$

Since the denominators of two fractions in (10) are zero, their numerators must also be zero.

$$\therefore \quad k = e^2 p \cos \alpha, \qquad l = e^2 p \sin \alpha. \tag{19}$$

Squaring and adding (19) gives

$$k^2 + l^2 = e^4 p^2. \tag{20}$$

Substituting (20) in (18) we have, from (9),

$$p^2 = \frac{c' \sin 2\alpha}{2h(1-e^2)} = \frac{\triangle \sin 2\alpha}{2h(1-e^2)(ab-h^2)}. \tag{21}$$

Then $k^2 + l^2 = \frac{c'\sqrt{(a-b)^2+4h^2}}{ab-h^2}.$ (22)

The value of all the parameters can now be found. Thus equations (15) and (17) give the values of e and α respectively; then the value of p can be found from (21); and lastly, the values of k and l can be obtained from (19). It must be borne in mind that the values of k, l, and p given by (19), (20), and (21) are measured from the centre of the conic.

For any particular value of e, equation (21) gives two values of p which differ only in sign. Substituting these two values of p in (19) we get two pairs of values of k and l which differ only in sign. Hence every conic has two directrices which are parallel and equidistant from the centre, and two corresponding foci which are symmetrical with respect to the centre. Moreover, since from (19)

$$l = k \tan \alpha,$$

the two foci lie on the line, $y = x \tan \alpha$, passing through the centre perpendicular to the directrices.

This line is called the **Principal Axis** of the conic section, and its equation is

$$y = x \tan \alpha, \quad \text{or} \quad y - y' = \tan \alpha (x - x'), \tag{23}$$

according as the centre is at the point (0, 0) or (x', y').

Equation (20) shows that the distance from the centre to the foci is greater or less than p according as e is greater or less than unity. That is, in the ellipse the foci lie between the centre and the directrices, while in the hyperbola the directrices pass between the centre and the foci.

The Parabola. When $e = 1$, we have, from (14), $ab - h^2 = 0$. Hence the coordinates of the centre, equation (7), are both infinite; and therefore when (1) represents a parabola the transformation from (1) to (5) becomes impossible. In this case we may obtain the equations for the determination of the parameters by putting $e = 1$ in equation (4) of § 108, and comparing the resulting coefficients with those of (1). This gives

$$\frac{\sin^2 \alpha}{a} = \frac{\cos^2 \alpha}{b} = \frac{-\sin \alpha \cos \alpha}{h} = \frac{p \cos \alpha - k}{g}$$

$$= \frac{p \sin \alpha - l}{f} = \frac{k^2 + l^2 - p^2}{c} \equiv r. \tag{24}$$

Then $$r = \frac{\sin^2 \alpha}{a} = \frac{\cos^2 \alpha}{b} = \frac{1}{a + b}. \tag{25}$$

$$\therefore \quad \sin \alpha = \sqrt{\frac{a}{a+b}}, \quad \cos \alpha = \sqrt{\frac{b}{a+b}}, \quad \tan \alpha = \sqrt{\frac{a}{b}}. \tag{26}$$

Also

$$r = \frac{p\cos\alpha - k}{g} = \frac{p\sin\alpha - l}{f} = \frac{k^2 + l^2 - p^2}{c} = \frac{1}{a+b}. \quad (27)$$

Whence $k = p\cos\alpha - \dfrac{g}{a+b}$, $l = p\sin\alpha - \dfrac{f}{a+b}$, (28)

and

$$k^2 + l^2 = p^2 + \frac{c}{a+b}. \quad (29)$$

Then from (28) and (29) we get

$$p = \frac{g^2 + f^2 - c(a+b)}{2(a+b)(g\cos\alpha + f\sin\alpha)} \quad (30)$$

$$= \frac{g^2 + f^2 - c(a+b)}{2(g\sqrt{ab+b^2} + f\sqrt{a^2+ab})}. \text{ from (26)} \quad (31)$$

Finally, substituting (26) and (31) in (28) gives

$$\left.\begin{aligned} k &= \frac{\sqrt{b}\,[g^2+f^2-c(a+b)] - g}{2(a+b)(g\sqrt{b}+f\sqrt{a})}, \\ l &= \frac{\sqrt{a}\,[g^2+f^2-c(a+b)] - f}{2(a+b)(g\sqrt{b}+f\sqrt{a})}. \end{aligned}\right\} \quad (32)$$

In deriving (30) from (28) and (29) we obtain a quadratic equation in which the coefficient of p^2 becomes zero, and therefore one root is infinite (§ 98, III.). Hence one directrix, and consequently one focus (28), of a parabola is infinitely distant from the origin.

The student should now carefully observe the correspondence between the results here found algebraically from the discussion of the general equation, and those obtained geometrically from the study of the figure in § 106.

110. *To determine by an examination of the coefficients what kind of a conic is represented by the general equation*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

From equation (12) of § 109 we have

$$\frac{e^4\sin^2\alpha\cos^2\alpha}{h^2} = \frac{1-e^2}{ab-h^2}.$$

Since the first member of this equation is always positive, $e^2 - 1$ and $h^2 - ab$ must always have the same sign, and must vanish together. We therefore have the following conditions:

For an ellipse, $e < 1$, $e^2 < 1$. $\therefore$ $h^2 < ab$.

For a parabola, $e = 1$, $e^2 = 1$. $\therefore$ $h^2 = ab$.

For a hyperbola, $e > 1$, $e^2 > 1$. $\therefore$ $h^2 > ab$.

When $e = 0$, equation (10), § 109, gives

$$\frac{1}{a} = \frac{1}{b} = \frac{0}{h}.$$

Therefore, for a circle, $e = 0$, $a = b$, $h = 0$. (*Cf.* § 32.)
When $a + b = 0$, $e = \sqrt{2}$. [(15), § 109.]
The conic is then called a **Rectangular Hyperbola.***

E. g., $x^2 - y^2 = a^2$ and $xy = K$ are rectangular hyperbolas.

When $c' = 0$, then $\triangle = 0$ [(9), § 109], and therefore equations (5) and (1) of § 109 represent two straight lines, real or imaginary. (See also § 54.)

If $\triangle = 0$, and also $ab - h^2 = 0$, then c' is not necessarily zero. The first three terms of equation (5), § 109, are then a perfect square. The equation may therefore be written

$$\sqrt{a}\,x + \sqrt{b}\,y \pm \sqrt{-c'} = 0,$$

and represents two parallel lines which coincide when $c' = 0$.

For convenience these results are collected in the following table:

Curve.	*Condition.*
Ellipse.	$(e < 1)$ $h^2 < ab$.
Parabola.	$(e = 1)$ $h^2 = ab$.
Hyperbola.	$(e > 1)$ $h^2 > ab$.
Circle.	$(e = 0)$ $a = b$, and $h = 0$.
Rectangular Hyperbola.	$(e = \sqrt{2})$ $a + b = 0$.
Two real or imaginary straight lines.	$\triangle \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.
Two parallel or coincident straight lines.	$\triangle = 0$, and $h^2 = ab$.

* See §§ 116 and 121.

EXAMPLES.

1. Show that the directrices and the foci of a parabola are infinitely distant from the centre.

2. Show that the foci of a circle coincide with the centre, while its directrices are infinitely distant from the centre.

3. Show that when the conic is two distinct intersecting lines, the foci coincide with their point of intersection; and that the directrices both pass through this point.

4. When the conic is two parallel lines (limiting case of parabola), show that the centre, the foci, and the position of the directrices are indeterminate; but the directrices are perpendicular to the two lines.

5. If the general equation represents a parabola, show that the centre is at infinity.

6. If $a = b$, show that the principal axis is equally inclined to the axes of coordinates.

7. If $a = b$ and $h = 0$, show that the direction of the principal axis is indeterminate.

8. If $h = 0$, show that the principal axis of the conic is parallel or perpendicular to the x-axis.

Find the equations and trace the conics, having given

	Directrix.	Focus.	e.
9.	$2x + y = 2$.	(2, 2)	1.
10.	$3x - y = 3$.	(2, 0)	$\frac{3}{5}$.
11.	$x - y = 2$.	(3, 1)	2.
12.	$2x - y = 1$.	(0, 0)	$\sqrt{3}$.
13.	$3x + 4y + 10 = 0$.	$(-1, 1)$	$\frac{1}{2}$.
14.	$3x + y = 5$.	$(2, -1)$	1.

Find the parameters and trace the following conics:

15. $x^2 - 16xy - 11y^2 + 20x + 50y - 35 = 0$. Ans. $e = 2$, $a = \frac{1}{2}\tan^{-1}\frac{4}{3}$.

16. $14x^2 + 2xy + 14y^2 - 32x + 32y + 29 = 0$.

17. $3x^2 - 8xy - 3y^2 - 20x + 10y - 5 = 0$.

What do the following equations represent?

18. $y^2 + ax - 2ay = 0$.

19. $xy + ax - ay = 0$.

20. $x^2 + 2x - 4y + 1 = 0$.

21. $(x - y)^2 = a(x + y)$.

22. $(x+y)^2+(x-y)^2=2a^2$. 23. $y^2-x^2+2ax=0$.

24. $3x^2+4xy+2y^2=2$. 25. $4x^2-3xy-4y^2=1$.

26. Show that if the origin is at the centre and the principal axis is taken as the x-axis, the equation of a conic may be written

$$ax^2+by^2=1.$$

Tangents.

111. *To find the equation of the tangent to the conic represented by the general equation*

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \qquad (1)$$

The equation of the tangent to any curve $f(x, y)=0$ at the point (x', y') is (§ 85)

$$y-y'=\frac{dy'}{dx'}(x-x'). \qquad (2)$$

For equation (1) we have found in § 84

$$\frac{dy}{dx}=-\frac{ax+hy+g}{hx+by+f}. \qquad (3)$$

Therefore the required equation is

$$y-y'=-\frac{ax'+hy'+g}{hx'+by'+f}(x-x'), \qquad (4)$$

or

$$axx'+h(xy'+x'y)+byy'+gx+fy$$
$$=ax'^2+2hx'y'+by'^2+gx'+fy'. \qquad (5)$$

Add $gx'+fy'+c$ to both sides of (5); then, since (x', y') is on the conic, the right member will vanish and we have the required equation,

$$axx'+h(xy'+x'y)+byy'+g(x+x')+f(y+y')+c=0. \qquad (6)$$

Observe that the equation of the tangent at (x', y') is obtained from the equation of the conic by writing xx' for x^2, $x'y+xy'$ for $2xy$, yy' for y^2, $x+x'$ for $2x$, and $y+y'$ for $2y$. Note also that putting x for x' and y for y' in (6) reproduces the equation of the curve.

E. g., the equation of the tangent to the parabola $y^2=4ax$ at (x', y') is $yy'=2a(x+x')$.

112. *Two tangents can be drawn to a conic from any point, which will be real, coincident, or imaginary, according as the point is outside, on, or within the curve.*

Let the equation of the conic be [§ 108, (6)]

$$ax^2 + y^2 + 2gx + g^2 = 0, \tag{1}$$

where $a = 1 - e^2$.

Let (h, k) be any point; then the equation of any line through this point will be (§ 46)

$$y - k = m(x - h). \tag{2}$$

Eliminating y between (1) and (2) gives

$$(a + m^2)x^2 + 2(km - hm^2 + g)x + h^2m^2 - 2hkm + k^2 + g^2 = 0. \tag{3}$$

The roots of (3) are, by § 24, the abscissas of the points of intersection of (1) and (2). If these roots are equal, the points of intersection will coincide and, by § 78, (2) will be tangent to (1). The condition that (3) shall have equal roots* is

$$(km - hm^2 + g)^2 = (a + m^2)(h^2m^2 - 2hkm + k^2 + g^2), \tag{4}$$

or $$(ah^2 + 2gh + g^2)m^2 - 2(ahk + gk)m + (ak^2 + ag^2 - g^2) = 0. \tag{5}$$

Equation (5) is a quadratic in m whose roots are the slopes of the tangents from (h, k) to the conic. Since a quadratic equation has *two* roots, *two* tangents will pass through any point (h, k).

The conic is, therefore, a curve of the *second class.*

The roots of (5) are real, equal, or imaginary, according as

$$ah^2 + k^2 + 2gh + g^2 >, =, \text{ or } < 0. \tag{6}$$

Therefore the tangents are real, coincident, or imaginary according as the point (h, k) is outside, on, or within the conic. (§ 20, II.)

Since equation (3) is a quadratic in x, any straight line meets a conic in two points, which may be real, coincident, or imaginary.

Therefore the conic is also a curve of the *second order.*

If $e = 1$ and $m = 0$, then $a + m^2 = 0$, and hence one root of (3) is infinite (§ 98, III.). Therefore a straight line parallel to the axis of the parabola meets the curve in one point at a finite distance, and in another at an infinite distance from the directrix.

---

* The two roots of $ax^2 + bx + c = 0$ will be equal, if $b^2 = 4ac$.

The method here used is worthy of special attention because of its wide application. To find the condition that any two curves shall touch we may treat their equations simultaneously and eliminate one variable, and then take the condition for equal roots.

Pole and Polar.

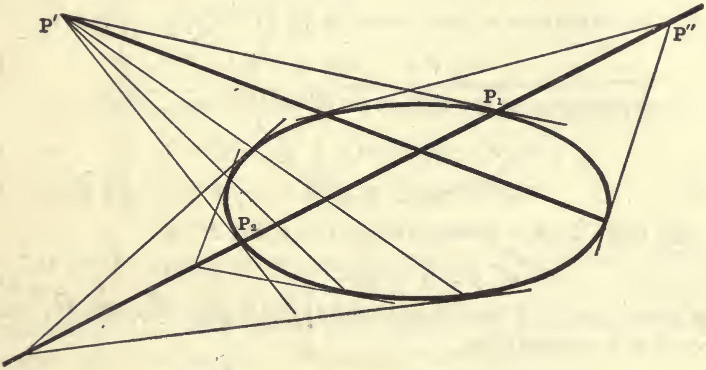
113. The equation of the tangent to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (1)$$

at the point (x', y') is (§ 111)

$$axx' + h(xy' + x'y) + byy' + g(x + x') + f(y + y') + c = 0. \qquad (2)$$

Suppose, however, that $P'(x', y')$ is not on the conic. This equation still has a meaning, still represents a straight line related in a definite way to the point (x', y') and the conic (1). This line will cut the conic in two points (§ 112).



Let these points be $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Then the equations of the tangents at these points are (§ 111)

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0, \qquad (3)$$

$$axx_2 + h(xy_2 + x_2y) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0. \qquad (4)$$

The conditions that (3) and (4) shall pass through (x', y') are

$$ax'x_1 + h(x'y_1 + x_1y') + by'y_1 + g(x' + x_1) + f(y' + y_1) + c = 0, \qquad (5)$$

$$ax'x_2 + h(x'y_2 + x_2y') + by'y_2 + g(x' + x_2) + f(y' + y_2) + c = 0. \qquad (6)$$

But (5) and (6) are also the conditions that (2) shall pass through both of the points (x_1, y_1) and (x_2, y_2).

Therefore (2) is the line passing through the points of contact of the tangents from the point $P'(x', y')$.

The point (x', y') and the line (2) are called **Pole** *and* **Polar** *with respect to the conic* (1).

The tangents from the point (x', y') will be real or imaginary according as (x', y') is outside or inside the conic (§ 112); but the line (2) is real when (x', y') is real. So that there is always a real line passing through the imaginary points of contact of the two imaginary tangents drawn from a point within a conic.

If (x', y') is on the conic, the two tangents from it will coincide, and each of the points (x_1, y_1) and (x_2, y_2) will coincide with (x', y'). *Therefore the tangent is the particular case of the polar which passes through its own pole.*

114. *If the polar of a point $P'(x', y')$ pass through $P''(x'', y'')$, then will the polar of P'' pass through P'.* (See fig. § 113.)

Let the equation of the conic be [§ 112, (1)]

$$ax^2 + y^2 + 2gx + g^2 = 0. \tag{1}$$

The equations of the polars of P' and P'' are

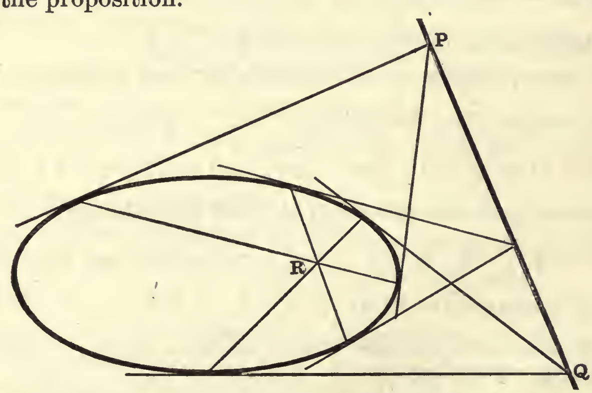
$$axx' + yy' + g(x + x') + g^2 = 0 \tag{2}$$

and $$axx'' + yy'' + g(x + x'') + g^2 = 0. \quad (\S\ 113.) \tag{3}$$

The line (2) will pass through the point P'' if

$$ax'x'' + y'y'' + g(x' + x'') + g^2 = 0; \tag{4}$$

but this is also the condition that (3) shall pass through P', which proves the proposition.



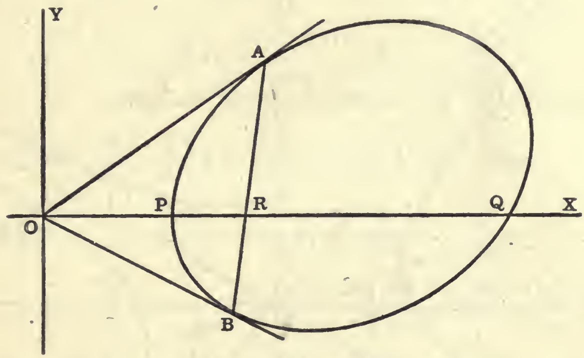
Cor. I. *The locus of the poles of all lines passing through a fixed point is a straight line; viz., the polar of the fixed point.*

Cor. II. *If the polars of two points P and Q meet in R, then R is the pole of the line PQ.*

Two straight lines are said to be *conjugate* with respect to a conic when each passes through the pole of the other.

Two points are said to be *conjugate* with respect to a conic when each lies on the polar of the other.

115.* *Any chord of a conic passing through a point O is divided harmonically by the conic and the polar of O.*



Let OQ be any line cutting the conic in P and Q and the polar of O in R. We are to prove that the line PQ is divided harmonically; *i. e.* that it is divided internally and externally in the same ratio at R and O. We must therefore prove

$$\frac{OP}{OQ} = \frac{PR}{RQ};$$

whence $$\frac{OP}{PR} = \frac{OQ}{RQ} = \frac{OP + OQ}{PR + RQ} = \frac{OP + OQ}{OQ - OP}.$$

From the first and last ratios by composition,

$$\frac{OP + PR}{OP} = \frac{2OQ}{OP + OQ}.$$

$$\therefore \quad \frac{OP + OQ}{OP \cdot OQ} = \frac{2}{OP + PR}.$$

$$\therefore \quad \frac{1}{OP} + \frac{1}{OQ} = \frac{2}{OR}.* \qquad (1)$$

*OR is called the *Harmonic Mean* between OP and OQ.

Take O for origin and the line OPQ for the x-axis; let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad (2)$$

Then the equation of AB, the polar of O, is (§ 113)

$$gx + fy + c = 0. \qquad (3)$$

Where the line $y = 0$ cuts AB we have

$$x = -\frac{c}{g} = OR, \text{ i. e. } \frac{1}{OR} = -\frac{g}{c}. \qquad (4)$$

Where the line $y = 0$ cuts the conic we have

$$ax^2 + 2gx + c = 0, \text{ or } \frac{1}{x^2} + 2\frac{g}{c}\frac{1}{x} + \frac{a}{c} = 0. \qquad (5)$$

$$\therefore \quad \frac{1}{OP} + \frac{1}{OQ} = -\frac{2g}{c}. \qquad (\S\ 91) \quad (6)$$

From (4) and (6) we get

$$\frac{1}{OP} + \frac{1}{OQ} = \frac{2}{OR}, \qquad (7)$$

which is the same as (1).

Asymptotes, Similar and Conjugate Conics.

116.* Consider the three concentric conics whose equations are

$$ax^2 + 2hxy + by^2 = c, \qquad (1)$$

$$ax^2 + 2hxy + by^2 = 0, \qquad (2)$$

and

$$ax^2 + 2hxy + by^2 = -c; \qquad (3)$$

and let $h^2 > ab$, so that all the loci are real.

Solving these equations for y gives, respectively,

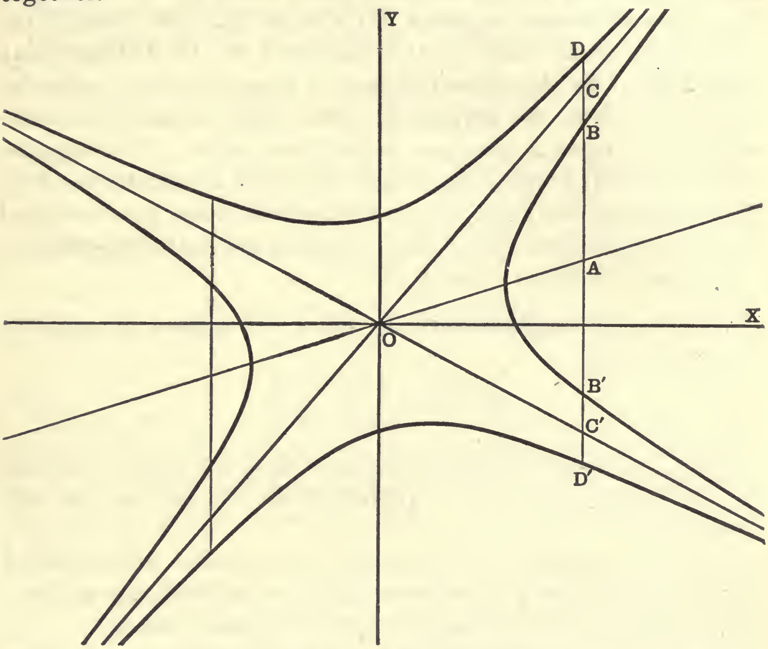
$$y = -\frac{h}{b}x \pm \frac{1}{b}\sqrt{(h^2 - ab)x^2 + bc} \equiv -\frac{h}{b}x \pm \lambda_1, \qquad (4)$$

$$y = -\frac{h}{b}x \pm \frac{1}{b}\sqrt{(h^2 - ab)x^2 + 0} \equiv -\frac{h}{b}x \pm \lambda, \qquad (5)$$

and

$$y = -\frac{h}{b}x \pm \frac{1}{b}\sqrt{(h^2 - ab)x^2 - bc} \equiv -\frac{h}{b}x \pm \lambda_2, \qquad (6)$$

where λ_1, λ, and λ_2 are put for the terms containing the radicals and are quantities such that for finite values of x, $\lambda_1 > \lambda > \lambda_2$; but for infinite values of x, $\lambda_1 = \lambda = \lambda_2$. Therefore, in the finite part of the plane, the loci represented by equations (1) and (3) lie on opposite sides of the two lines given by equation (2); but, at an infinite distance from the origin (the centre of the conics), the ordinates of the three loci are equal and the three loci come together.



The values of y are real for all values of x in equations (4) and (5), but in (6) y is imaginary when $x^2(h^2 - ab) < bc$. The three conics are as shown in the figure, where

$$D'A = AD = \lambda_1, \quad C'A = AC = \lambda, \quad B'A = AB = \lambda_2;$$

and OA is the line

$$y = -\frac{h}{b}x. \tag{7}$$

Thus OA bisects all chords, BB', CC', DD', parallel to the y-axis; *i. e.* the line (7) is a diameter of each of the three given conics. (See § 126.)

The straight lines OC and OC' which meet the conics at infinity are called **Asymptotes.***

Therefore equation (2) represents the asymptotes of the conics given by both (1) and (3), and we see that the asymptotes of a conic pass through its centre.

When we say above that the ordinates of the three loci become equal when $x = \infty$, we mean that their difference bears a vanishing ratio at last to any namable finite quantity. Two parallel lines are said to come together in the sense that the distance between them at last bears a vanishing ratio to the distance gone along them. In this sense any parallel to an asymptote meets the hyperbola where the asymptote does. This parallel, however, meets the hyperbola elsewhere in the finite region. Now suppose such a parallel, keeping its slope, to move up to coincidence with the asymptote, the finite intersection moves along the curve and goes out to infinity. Thus the asymptote meets the hyperbola in *two* points at infinity. (See § 146.)

Let 2θ be the angle between the lines represented by equation (2), then (§ 57)

$$\tan 2\theta = \frac{2\sqrt{h^2 - ab}}{a + b}. \tag{8}$$

If $h^2 < ab$, these lines are imaginary, and the loci of (1) and (3) are ellipses (§ 110). Therefore the ellipse has no real asymptote.

If $h^2 = ab$, equation (1) represents two parallel lines equidistant from the origin, (2) represents two coincident lines midway between them, and (3) represents two imaginary lines.

If $a + b = 0$, the asymptotes are perpendicular to each other.

A conic whose asymptotes are at right angles is called a **Rectangular Hyperbola** (§§ 110, 121).

Similar Conics. Two curves are similar when the one is merely a magnification of the other; i. e. when we could get the equation of the one from that of the other by merely changing rectangular axes and scale.

E. g., the equation of any circle can be reduced to $x^2 + y^2 = 1$ by moving the origin to its centre and taking its radius for the unit of the scale.

* Greek, *asymptotos*, not falling together.

Let k be the factor of magnification; then for two similar conics, we have

$$P'F' = k \cdot PF \quad \text{and} \quad P'S' = k \cdot PS.$$

$$\therefore \quad \frac{P'F'}{P'S'} = \frac{PF}{PS} = e. \qquad [(2), \S\ 108.]$$

That is, similar conics have the same eccentricity; and, conversely, conics having the same eccentricity are similar.

Hence all parabolas are similar.

From equation (8) we have by Trigonometry

$$\tan 2\theta = \frac{2 \tan\theta}{1 - \tan^2\theta} = \frac{2\sqrt{\sec^2\theta - 1}}{2 - \sec^2\theta} = \frac{2\sqrt{h^2 - ab}}{a + b}. \tag{9}$$

$$\therefore \quad \left(\frac{2 - \sec^2\theta}{a + b}\right)^2 = \frac{1 - \sec^2\theta}{ab - h^2}. \tag{10}$$

Solving (10) gives [*cf.* (13), (14), and (15), § 109]

$$\frac{1}{\sec^2\theta} = \frac{1}{2} \pm \frac{a + b}{2\sqrt{(a - b)^2 + 4h^2}} = \frac{1}{e^2}. \tag{11}$$

$$\therefore \quad \sec\theta = e. \tag{12}$$

That is, *the eccentricity of a hyperbola is equal to the secant of half the angle between its asymptotes.* Hence all hyperbolas having the same asymptotes, *and lying within the same angle,* have the same eccentricity, and are therefore similar, *i. e.* similar to the asymptote-pair.

Conjugate Hyperbolas. If $h^2 > ab$, both roots of (11) are positive. Since (11) involves only a, b, and h, and is therefore the same for equations (1), (2), and (3), these two positive roots give the eccentricities of the two hyperbolas (1) and (3); and also the secants of half the supplementary angles between their common asymptotes (2).

For the same reason, the directions of the principal axes (§ 109) of the two hyperbolas are determined by the values of α which satisfy the equation [(17), § 109]

$$\tan 2\alpha = \frac{2h}{a - b}. \tag{13}$$

But these values of α differ by 90°; therefore the principal axes of the two hyperbolas are perpendicular to each other.

Equation (13) with (5) of § 58 show that the axes are the bisectors of the angles between the asymptotes.

From equation (22), § 109, we see that the foci of the two hyperbolas are equidistant from their common centre.

Two conics having these relations will be found to satisfy the definition of **Conjugate Hyperbolas** given in § 140.

If $h^2 < ab$, equation (11) has only *one* positive root. Hence an ellipse has no real conjugate.

If c be arbitrary, equations (1) and (3) each represent a system of similar hyperbolas, such that for each conic in one system there is a corresponding conjugate in the other system. Moreover, the asymptotes are the limiting forms of both systems* corresponding to $c = 0$; *i. e.* two intersecting lines are a pair of *self-conjugate hyperbolas.* (See § 147.)

Ex. Show that the sum of the squares of the reciprocals of the eccentricities of two conjugate hyperbolas is equal to unity.

117.* *To find the equation of the asymptotes of a conic; also the equation of its conjugate.*

Let the equation of the given conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \tag{1}$$

Write down the two equations

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{\Delta}{ab - h^2} = 0, \tag{2}$$

and
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{2\Delta}{ab - h^2} = 0. \tag{3}$$

These three conics are concentric [(7), § 109]. Moving the origin to the centre without changing the direction of the axes, we get [(5) and (9), § 109], respectively,

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{ab - h^2} = 0, \tag{4}$$

$$ax^2 + 2hxy + by^2 = 0, \tag{5}$$

and
$$ax^2 + 2hxy + by^2 - \frac{\Delta}{ab - h^2} = 0. \tag{6}$$

* For the different manner in which the asymptotes are described when considered as conics belonging to the two different systems see § 159, III.

Now, if $h^2 > ab$, (4) and (6) are conjugate hyperbolas (§ 116), while (5) represents their common asymptotes. Therefore (2) and (3) are the required equations.

COR. *The lines represented by the equation*

$$ax^2 + 2hxy + by^2 = 0$$

are parallel to the asymptotes of the conic represented by the general equation (1).

EXAMPLES.

Find the equations of the tangent and normal to

1. $x^2 = 2y$, at $(-2, 2)$.
2. $y^2 = 8x$, at $(2, -4)$.
3. $x^2 + y^2 = 25$, at $(4, -3)$.
4. $x^2 - y^2 = 16$, at $(-5, 3)$.
5. $x^2 + 4y^2 = 8$, at $(-4, 3)$.
6. $2y^2 - x^2 = 4$, at $(2, -2)$.
7. $y^2 + 4x + 2y + 1 = 0$, at $(-4, 3)$.
8. $3x^2 + 5xy - 2y^2 = 0$, at $(1, 3)$ and $(-2, 1)$.
9. $2x^2 - 4xy + y^2 + 2x - 4y - 15 = 0$, at $(2, 3)$.
10. $x^2 + 3xy + 4y^2 - 3x + 5y + 9 = 0$, at $(4, -1)$.

Find the equations of the tangents to each of the following conics at the origin:

11. $2x^2 + 3y^2 + 2x = 0$.
12. $x^2 + 2x + 3y = 0$.
13. $2xy + 5x - 3y = 0$.
14. $3x^2 - 2xy + 4x - 2y = 0$.
15. $x^2 + 2xy - 3y^2 + 4y = 0$.
16. $ax^2 + 2hxy + by^2 + 2gx + 2fy = 0$.
17. State a rule for finding the tangent to a conic at the origin.

Find the polar of the point

18. $(3, 2)$ with respect to $y^2 = 6x$.
19. $(-2, -4)$ with respect to $x^2 + y^2 = 4$.
20. $(1, 1)$ with respect to $2x^2 + 3y^2 = 1$.
21. $(0, 0)$ with respect to $2x^2 - 3y^2 + 4x - 3y + 4 = 0$.
22. $(-1, 2)$ with respect to $3x^2 - 6xy + y^2 - 2x + 4y + 3 = 0$.
23. $(3, -1)$ with respect to $x^2 + 2xy + 3y^2 + 4x - 6y + 1 = 0$.
24. Give a general rule for writing the equation of the polar of the origin.

25. Find the tangent to the parabola $y^2 = 6x$ which makes an angle of 45° with the x-axis.

26. Find the equations of the tangents to the parabola $y^2 - 4x + 2y + 1 = 0$ whose slopes are 2, and $\frac{1}{2}$.

27. Find the tangents to the conic $4x^2 + y^2 = 4$ whose slope is $\frac{1}{2}$.

28. Find the equations of the lines which have the slope $(-\frac{1}{2})$, and touch the conic $2x^2 + y^2 - 4y + 1 = 0$.

29. Find the equation of the normal to the conic $(y-3)^2 + 4(x+1) = 0$, parallel to the line $2y + 3x = 0$.

30. Find the normals to the conic

$$9(2y+1)^2 - 4(3x-2)^2 = 36 \text{ whose slope is } 3.$$

Find the tangents to the following conics drawn from the given points (see § 112):

31. $y^2 = 4x$, $(2, 3)$.

32. $y^2 = 5x$, $(-3, -1)$.

33. $x^2 + y^2 = 25$, $(-1, 7)$.

34. $9x^2 + 25y^2 = 225$, $(10, -3)$.

35. $y^2 + 8x - 4y + 4 = 0$, $(-2, 2)$.

36. $x^2 + xy + y^2 = 12$, $(-3, 4)$.

37. $(x+2)^2 + 2(y-2)^2 = 27$, $(1, -1)$.

38. Show that the polar of the focus is the directrix.

What is the locus of the intersection of tangents at the ends of focal chords? (Use equation (6), § 108.)

39. Show that the line joining the focus to any point on the directrix is perpendicular to the polar of the latter point.

40. Show that tangents to a conic at the ends of a chord through the centre are parallel. (See Ex. 26, p. 167.)

41. What is the polar of the centre of a conic? Where is the pole of a line passing through the centre?

42. What is the pole of $x \cos \alpha + y \sin \alpha = p$ with respect to

$$x^2 + y^2 = r^2? \quad y^2 = 2x?$$

43. Show that if the slope of a variable chord of a conic is constant the tangents at its extremities always intersect on the same diameter. State the converse. From the definition of conjugate lines in § 114, what may we call this diameter and the diameter parallel to the variable chord?

44. If m_1 and m_2 are the slopes of the two diameters mentioned in 43, and $ax^2 + by^2 = 1$ is the equation of the conic, show that $m_1 m_2 = -\frac{a}{b}$.

45. If $S = 0$ and $S' = 0$ are the equations of two conics, what is the locus represented by the equation

$$S + \lambda S' = 0?$$

46. Find the equation of the conic passing through the point (0, 2) and the common points of

$$y^2 = 4x \text{ and } (x-4)^2 = 2(y+5).$$

What kind of a conic is it?

47. Find the equation of the conic passing through the origin and the common points of

$$x^2 - 3xy + y^2 + 10x - 10y + 21 = 0 \text{ and } 4x^2 + 9y^2 + 16x - 36y + 16 = 0.$$

48. If two conics have their principal axes parallel their points of intersection lie on a circle.

49. Find the equation of the circle passing through the common points of

$$9x^2 + 4y^2 + 18x - 24y + 9 = 0 \quad \text{and} \quad x^2 - y^2 + 2x + 4y - 4 = 0.$$

STANDARD EQUATIONS OF THE CONIC SECTIONS.

118. Let the directrix be the y-axis, the principal axis of the conic (§ 109) the x-axis, and let k denote the distance from the directrix to the corresponding focus. Then the equation of any conic takes the simple form [(6), § 108]

$$\left.\begin{aligned} &(x-k)^2 + y^2 = e^2x^2, \\ \text{or} \quad &(1-e^2)x^2 + y^2 - 2kx + k^2 = 0. \end{aligned}\right\} \quad (1)$$

If $x = 0$ in (1), then $y = \pm k\sqrt{-1}$.

Hence a conic does not intersect its directrix.

If $y = 0$, then there are two real values of x, viz.,

$$x_1 = \frac{k}{1+e}, \qquad x_2 = \frac{k}{1-e}. \quad (2)$$

Therefore a conic section cuts its principal axis in two points. These points are called the **Vertices** of the conic. The centre is midway between the vertices.

The **Latus Rectum** of a conic is the chord through either focus perpendicular to the principal axis.

To find its length, let $x = k$ in (1), then

$$y = \pm ek, \quad \text{and} \quad 2y = \textit{Latus Rectum} = 2ek.$$

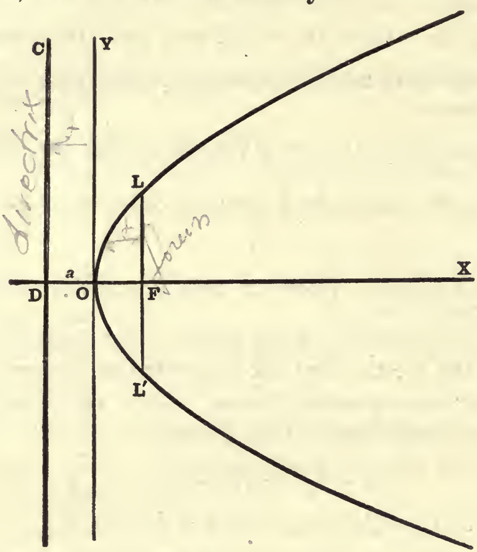
The different cases corresponding to the different values of e will now be separately considered.

119. The Parabola. $e = 1$.

When $e = 1$, equations (2) of § 118 give

$$x_1 = \tfrac{1}{2}k = DO, \qquad x_2 = \frac{k}{0} = \infty.$$

Hence the parabola has one vertex midway between the focus and directrix, and the other at infinity.*



When $e = 1$, equation (1) of § 118 gives for the equation of the parabola referred to its axis and directrix

$$y^2 = 2k(x - \tfrac{1}{2}k). \tag{1}$$

Let $a = \frac{1}{2}k = DO = OF$; then this equation becomes

$$y^2 = 4a(x - a). \tag{2}$$

Now write $x + a$ in the place of x; this moves the origin to the vertex $O(a, 0)$ [§ 66, (10)], and the equation becomes

$$y^2 = 4ax, \tag{3}$$

which is the *standard form* of the equation of the parabola.

When $x = a$ in (3), $y = \pm 2a$.

$$\therefore \quad L'L = 4a = \textit{Latus Rectum}.$$

Ex. 1. Trace the parabolas $y^2 = -4ax$ and $x^2 = \pm 4ay$.

Ex. 2. Construct the parabola, having given the focus and the directrix.

* Compare this result with the position of B in the figure of § 106 when $\alpha = \beta$.

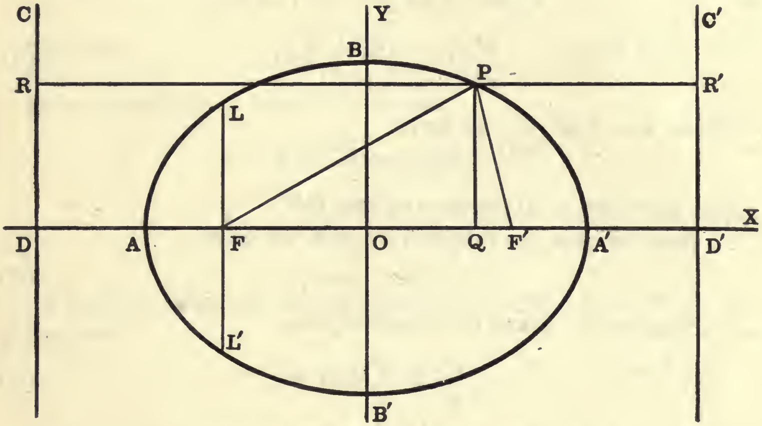
120. THE ELLIPSE. $e < 1$.

When $e < 1$, the two x-intercepts [(2), § 118] are both finite and positive; that is,

$$x_1 = \frac{k}{1+e} = DA < k,$$

$$x_2 = \frac{k}{1-e} = DA' > k.$$

Hence the ellipse has two vertices lying on the same side of the directrix, but on opposite sides of the focus.



Let O be the centre, and let $AA' = 2a$.

Then $$2a = x_2 - x_1 = \frac{k}{1-e} - \frac{k}{1+e} = \frac{2ek}{1-e^2}. \quad (1)$$

$$\therefore \quad \frac{a}{e} = \frac{k}{1-e^2};$$

whence $$k = \frac{a}{e} - ae. \quad (2)$$

Also $$DO = \tfrac{1}{2}(x_1 + x_2) = \tfrac{1}{2}\left(\frac{k}{1+e} + \frac{k}{1-e}\right)$$

$$= \frac{k}{1-e^2} = \frac{a}{e}. \quad (3)$$

$$\therefore \quad FO = DO - DF = \frac{a}{e} - k = ae. \quad (4)$$

Substituting in equation (1) of § 118 the value of k given by (2) gives for the equation of the ellipse referred to DC and DX

$$\left(x - \frac{a}{e} + ae\right)^2 + y^2 = e^2x^2. \tag{5}$$

The origin may be transferred to the centre, $O\left(\frac{a}{e}, 0\right)$, by writing $x + \frac{a}{e}$ in the place of x [§ 66, (10)]; this gives

$$(x + ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2,$$

or

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2).$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1. \tag{6}$$

When $x = 0$ in (6), we have

$$y = \pm a\sqrt{1 - e^2};$$

which gives the y-intercepts OB and OB'.

If these lengths are denoted by $\pm b$, we have

$$b^2 = a^2(1 - e^2), \tag{7}$$

and equation (6) takes the *standard form*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.* \tag{8}$$

Since $e < 1$, $b < a$ from (7); therefore

$$B'B < AA'.$$

Hence the line AA' is called the **Major Axis**, and BB' is called the **Minor Axis** of the ellipse.

Take $OF' = FO$ and $OD' = DO$; draw $D'C'$ perpendicular to OX. Then F' is the other focus, and $D'C'$ the corresponding directrix (§ 109). Hence the foci are the points $F'(ae, 0)$ and and $F(-ae, 0)$ from (4); and the equations of the directrices are, from (3),

$$x = \pm \frac{a}{e}. \tag{9}$$

Let $P(x, y)$ be any point on the ellipse; draw a line through P parallel to AA' meeting the directrices in R and R', and draw PQ perpendicular to AA'.

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* For a discussion of this equation see § 35.

Then $$FP = e \cdot RP,$$

and $$F'P = e \cdot R'P. \qquad [(2), \S\ 108]$$

$$\therefore \quad FP = e \cdot DQ = e(DO + OQ)$$

$$= e\left(\frac{a}{e} + x\right) = a + ex, \qquad (10)$$

and $$F'P = e \cdot QD' = e(OD' - OQ)$$

$$= e\left(\frac{a}{e} - x\right) = a - ex. \qquad (11)$$

Whence $$FP + F'P = 2a. \qquad (Cf. \S\ 34.) \qquad (12)$$

From equations (7) and (4) we get

$$ae = \sqrt{a^2 - b^2} = FO = OF'.$$

$$\therefore \quad e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{FF'}{AA'}.* \qquad (13)$$

To find the length of the latus rectum we put $x = \pm ae$ in (8); this gives

$$y^2 = b^2(1 - e^2) = \frac{b^4}{a^2}. \qquad \text{from (7)}$$

$$\therefore \quad L'L = \frac{2b^2}{a}. \qquad (14)$$

If $a = b$, equation (8) reduces to

$$x^2 + y^2 = a^2,$$

and equations (13), (4), and (3), respectively, give

$$e = 0, \quad FO = OF' = 0, \quad DO = OD' = \infty.$$

That is, the circle is the limiting form of the ellipse as the eccentricity approaches zero, and the directrices recede to infinity.

Ex. Construct an ellipse, having given the foci and the length of the major axis.

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* In all conics $e = \dfrac{\textit{distance between foci}}{\textit{distance between vertices}}$; both distances become infinite in the parabola, and both become zero in the case of two intersecting lines. (See also (11), § 121.)

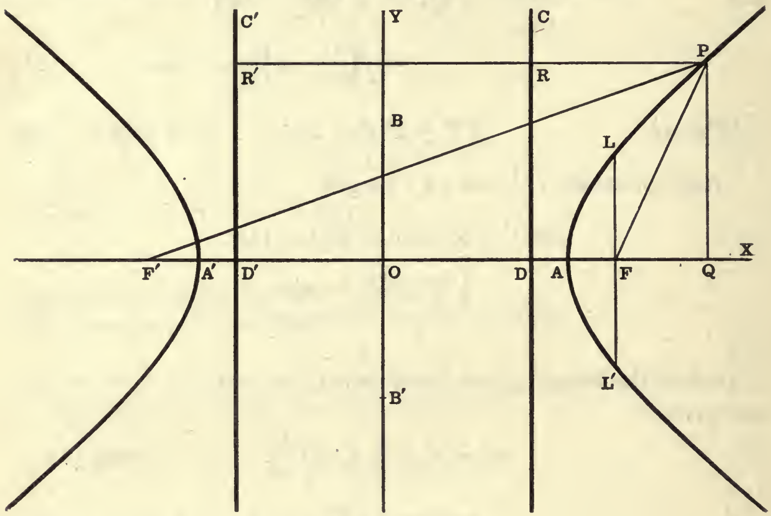
121. THE HYPERBOLA. $e > 1$.

From equations (2) of § 118 we have for the vertices

$$x_1 = \frac{k}{1+e} \quad \text{and} \quad x_2 = \frac{k}{1-e}.$$

Since $e > 1$, $x_1 = DA < k$, and $x_2 = DA'$ is negative.

Therefore, the hyperbola has two vertices lying on the same side of the focus but on opposite sides of the directrix.



Let O be the centre, and let $A'A = 2a$.

Then
$$2a = A'D + DA = -x_2 + x_1$$

$$= \frac{k}{e-1} + \frac{k}{e+1} = \frac{2ek}{e^2-1}. \tag{1}$$

$$\therefore \quad \frac{a}{e} = \frac{k}{e^2-1} \quad \text{and} \quad k = ae - \frac{a}{e}. \tag{2}$$

$$DO = \tfrac{1}{2}(x_1 + x_2) = \tfrac{1}{2}\left(\frac{k}{1+e} + \frac{k}{1-e}\right)$$

$$= \frac{k}{1-e^2} = -\frac{a}{e}. \tag{3}$$

$$FO = FD + DO = -\left(k + \frac{a}{e}\right) = -ae. \tag{4}$$

The equation of the hyperbola referred to DC and DX is, from (2), and (1) of § 118,

$$\left(x - ae + \frac{a}{e}\right)^2 + y^2 = e^2x^2. \tag{5}$$

Moving the origin to the centre $O\left(-\frac{a}{e}, 0\right)$ gives

$$(x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2,$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1. \tag{6}$$

Since $e > 1$, the quantity $a^2(1 - e^2)$ is *negative;* if we put $-b^2 = a^2(1-e^2)$, or

$$b^2 = a^2(e^2 - 1), \tag{7}$$

equation (6) reduces to the *standard form*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{8}$$

When $x = 0$, $y = \pm b\sqrt{-1}$. Since these values of y are both imaginary, the hyperbola does not meet the line through its centre perpendicular to its principal axis in *real* points; but, if B, B' are points on this line such that $B'O = OB = b$, the line BB' is called the **Conjugate Axis.** The line AA' joining the vertices is called the **Transverse Axis.**

On the line OX take $OF' = FO$, and $OD' = DO$; then F' is the other focus and $D'C'$, perpendicular to OX, is the corresponding directrix (§ 109). Hence the coordinates of the foci are $(\pm ae, 0)$, from (4), and the equations of the directrices are, from (3),

$$x = \pm \frac{a}{e}. \tag{9}$$

As in the ellipse, we find the latus rectum

$$LL' = \frac{2b^2}{a}. \tag{10}$$

Equations (7) and (4) give

$$ae = \sqrt{a^2 + b^2} = OF.$$

$$\therefore \quad e = \frac{\sqrt{a^2+b^2}}{a} = \frac{OF}{OA} = \frac{F'F}{A'A}. \tag{11}$$

Let $P(x, y)$ be any point on the hyperbola; draw a line through P parallel to AA' meeting the directrices in R and R', and draw PQ perpendicular to AA'.

Then $\qquad FP = e \cdot RP, \qquad F'P = e \cdot R'P.$ [(2), § 108]

$$\therefore \quad FP = e \cdot D'Q = e(OQ - OD) = e\left(x - \frac{a}{e}\right) = ex - a; \quad (12)$$

and $$F'P = e \cdot D'Q = e(OQ + D'O) = e\left(x + \frac{a}{e}\right) = ex + a. \quad (13)$$

Whence $$F'P - FP = 2a. \qquad (Cf. \text{ § } 36.) \quad (14)$$

If $a = b$, the equation of the hyperbola becomes

$$x^2 - y^2 = a^2. \quad (15)$$

This is called the *Equilateral* or *Rectangular Hyperbola*. (See §§ 110, 116.)

Then from (11), (3), and (4) we have, respectively,

$$e = \sqrt{2}, \quad OD = \tfrac{1}{2}a\sqrt{2}, \quad OF = a\sqrt{2}.$$

Ex. Construct a hyperbola, having given the foci and the distance between the vertices.

122. *Limiting cases of conic sections.*

If $k = 0$, equation (1) of § 118 reduces to

$$y^2 = x^2(e^2 - 1). \quad (1)$$

This equation represents *two straight lines*, which are real if $e > 1$, coincident if $e = 1$, and imaginary, but with a real point of intersection, if $e < 1$.

From (2) of § 118 we then have $x_1 = x_2 = 0$. Hence the foci, the vertices, and the centre of two intersecting lines all coincide on the directrix. The two directrices also coincide.

When $e = \infty$ (a being finite), the equation of the hyperbola [(8), § 121] reduces to $x^2 = a^2$, which represents two parallel lines. Equations (3) and (4) of § 121 then show that the foci of two parallel lines (considered as the limiting case of a hyperbola) are at infinity while their directrices coincide and are equidistant from the two lines.

Hence we must consider two intersecting lines, real or imaginary (*i. e.* a real point), two coincident lines, and two parallel lines as limiting cases of conic sections. (*Cf.* § 107.)

CHAPTER VIII.

THE PARABOLA.

123. *Standard equations of the tangent, polar, and normal to the parabola.*

In studying the properties of the parabola in this chapter we shall use the standard form of the equation found in § 119, viz.,

$$y^2 = 4ax. \qquad (1)$$

Then the focus is the point $(a, 0)$, the directrix is the line $x = -a$, and the latus rectum is $4a$.

Equation (6), § 111, applied to (1) gives

$$yy' = 2a(x + x'), \qquad (2)$$

as the equation of the tangent at the point (x', y'), if (x', y') is on the curve; but always the equation of the polar of (x', y'), (§ 113), with respect to the parabola (1).

The equation of the normal at the point (x', y') on the curve is [(2), § 85]

$$y - y' = -\frac{y'}{2a}(x - x'), \qquad (3)$$

or

$$2a(y - y') + y'(x - x') = 0. \qquad (4)$$

The tangent at the vertex $(0, 0)$ is the line $x = 0$; and the normal at the same point is $y = 0$, *i. e.* the axis of the curve.

Ex. 1. Show that the equation of the parabola is

$$y^2 = 4a(x \pm a),$$

according as the origin is at the focus or on the directrix.

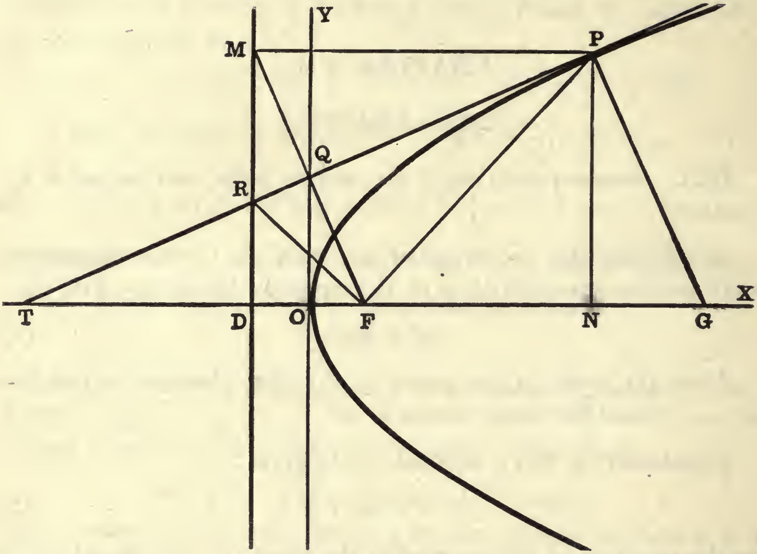
Ex. 2. Change the equations of the parabolas

$$(y - k)^2 = 4a(x - h) \quad \text{and} \quad (x - h)^2 = 4a(y - k)$$

to the standard form, and show that their vertices are at the point (h, k).

Ex. 3. What relation does the line (3) have to the parabola when the point (x', y') is not on the curve?

124. *Geometric properties of the parabola.*



Let the tangent at the point $P(x', y')$ meet the axis in T, the directrix in R, and the tangent at the vertex in Q. Let PM and PN be the perpendiculars from P to the directrix and axis, respectively.

Let the normal at P meet the axis in G.

Then we have the following properties:

$TO = ON = x'$. [(2), § 123.] (1)

$\therefore$ *Subtangent* $= TN = 2\,ON = 2x'$. (2)

$OQ = \frac{1}{2}NP = \frac{1}{2}y'$. (3)

$TF = FP = FG = a + x'$. (4)

$\angle FPR = \angle MPR$. (5)

$\angle RFP = \angle RMP = \frac{1}{2}\pi$. (See Ex. 39, p. 178.) (6)

FM is perpendicular to TP. (7)

FM, PT, and OY meet in a point. (8)

$OG = 2a + x'$. [(4), § 123.] (9)

$\therefore$ *Subnormal* $= NG = 2a$, *a constant.* (10)

The use of parabolic reflectors depends on the property expressed in (5). Let the student explain.

Properties (5) and (7) suggest a method of drawing tangents from an exterior point. Show how this can be done.

125. *Equations of the tangent and normal in terms of the slope m.*

The equation of the tangent [(2), § 123] may be written

$$y = \frac{2a}{y'}x + \frac{2ax'}{y'} = \frac{2a}{y'}x + \frac{4ax'}{2y'}, \tag{1}$$

or

$$y = \frac{2a}{y'}x + \frac{y'}{2}. \tag{2}$$

Let $\frac{2a}{y'} = m$; then $\frac{y'}{2} = \frac{a}{m}$, and (2) may be written

$$y = mx + \frac{a}{m}, \tag{3}$$

which is the required equation. That is, the line (3) will touch the parabola $y^2 = 4ax$, whatever the value of m may be.

In a similar manner it can be shown from (3), § 123, that the equation of the normal expressed in terms of its slope is

$$y = mx - 2am - am^3. \tag{4}$$

Ex. Show that the tangents from the point (x', y') to the parabola will be real, coincident, or imaginary according as $y'^2 - 4ax' >, =,$ or < 0.

EXAMPLES.

1. Find the equations of the tangents, and the normals at the ends of the latus rectum.

2. Find the value of a if the parabola $y^2 = 4ax$ goes through (3, 2); (9, — 12). How many conditions can the curve $y^2 = 4ax$ be made to satisfy?

3. Show that the line $y = 3x + \frac{a}{3}$ touches the parabola $y^2 = 4ax$; and also that $y = 4x + \frac{a}{2}$ touches $y^2 = 8ax$.

4. Find the equation of the tangent to $y^2 = 12x$ which makes an angle of 60° with the x-axis.

5. Find the equations of the tangents drawn from the point (— 2, 2) to the parabola $y^2 = 6x$.

Find the coordinates of the vertex, of the focus, the length of the latus rectum, and the equation of the directrix of each of the following parabolas:

6. $y^2 = 3x + 6$. 7. $x^2 + 4x + 2y = 0$. 8. $(y - 4)^2 = 6(x + 2)$.

9. $4(x - 3)^2 = 3(y + 1)$. 10. $y^2 + 8x - 6y + 1 = 0$.

11. For what point on the parabola $y^2 = 4ax$ is (1) the subtangent equal to the subnormal, and (2) the normal equal to the difference between the subtangent and the subnormal?

12. Show that the lines $y = \pm(x + 2a)$ touch both the parabola $y^2 = 8ax$ and the circle $x^2 + y^2 = 2a^2$.

13. Find the equation of the common tangent to the parabolas $y^2 = 4ax$ and $x^2 = 4by$. Show also that if $a = b$, the line touches both at the end of the latus rectum.

14. Find the equations of the tangents to the parabolas in examples 6, 7, 8, 9, 10 whose slope is -2.

15. Show that for all values of m the line

$$y = m(x + a) + \frac{a}{m} \text{ will touch } y^2 = 4a(x + a);$$

$$y = m(x - a) + \frac{a}{m} \text{ will touch } y^2 = 4a(x - a);$$

and $$(y - k) = m(x - h) + \frac{a}{m} \text{ will touch } (y - k)^2 = 4a(x - h).$$

16. If (x', y') and (x'', y'') are the points of contact of two tangents to $y^2 = 4ax$, show that the coordinates of their point of intersection are

$$x = \sqrt{x'x''}, \quad y = \tfrac{1}{2}(y' + y'').$$

17. Show that the directrix is the locus of the vertex of a right angle whose sides slide upon a parabola. (§ 125.)

18. Two lines are perpendicular to one another; one of them is tangent to $y^2 = 4a(x + a)$, and the other is tangent to $y^2 = 4b(x + b)$; show that these lines intersect on the line $x + a + b = 0$.

19. Show that the line $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$, if $ln = am^2$.

20. If the chord PQR passes through a fixed point Q on the axis of the parabola, show that the product of the ordinates, and also the product of the abscissas of the points P and R, is constant.

21. Find the coordinates of the point of intersection of $y = mx + \frac{a}{m}$ and $y = m'x + \frac{a}{m'}$. Show that the locus of this point is a straight line if mm' is constant. What is the locus when $mm' = -1$?

22. If perpendiculars be let fall on any tangent to a parabola from two points on the axis which are equidistant from the focus, the difference of their squares will be constant.

23. All chords of a parabola which subtend a right angle at the vertex meet the axis in the same point.

24. The vertex A of a parabola is joined to any point P on the curve, and PQ is drawn at right angles to AP to meet the axis in Q. Prove that the projection of PQ on the axis is always equal to the latus rectum.

25. If P, Q, and R be three points on a parabola whose ordinates are in geometrical progression, the tangents at P and R will meet on the ordinate of Q.

26. Show that the locus of the intersection of two tangents to a parabola at points on the curve whose ordinates are in a constant ratio is a parabola.

27. Prove that the circle described on a focal radius as diameter touches the tangent drawn through the vertex.

28. Prove that the circle described on a focal chord as diameter touches the directrix.

29. Find the locus of the point of intersection of two tangents to a parabola which make a given angle α with one another.

If $\alpha = 45°$, show that the locus is

$$y^2 - 4ax = (x + a)^2.$$

If $\alpha = 60°$, show that the locus is

$$y^2 - 3x^2 - 10ax - 3a^2 = 0.$$

[*Suggestion.* The line $y = mx + \frac{a}{m}$ will go through (x', y') if $m^2x' - my' + a = 0$. The roots of this equation are the slopes of the two tangents which meet in (x', y'). Let m_1, m_2 be these roots, then see § 91.]

30. The two tangents from a point P to the parabola $y^2 = 4ax$ make angles $\tan^{-1} m_1$ and $\tan^{-1} m_2$ with the x-axis. Find the locus of P, (1) when $m_1 + m_2$ is constant, (2) when $m_1^2 + m_2^2$ is constant, and (3) when m_1m_2 is constant.

31. If K is the area of a triangle inscribed in the parabola $y^2 = 4ax$, and K' is the area of the triangle formed by the tangents at the vertices of the inscribed triangle, prove that

$$8aK = 16aK' = (y_1 \sim y_2)(y_2 \sim y_3)(y_3 \sim y_1),$$

where y_1, y_2, y_3 are the ordinates of the vertices of the inscribed triangle. (See Ex. 16.)

Find the locus of the middle points

32. Of all ordinates of a parabola.

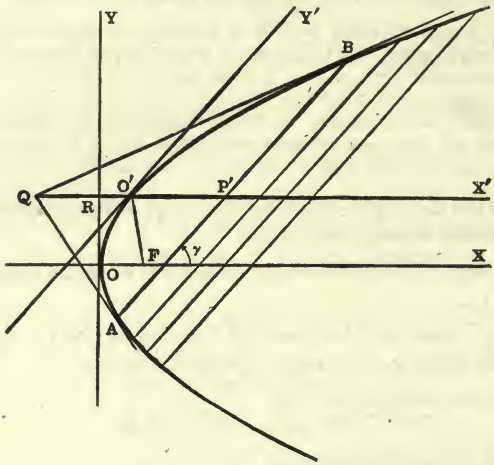
33. Of all focal radii.

34. Of all chords through the fixed point (h, k).

As special cases, let (h, k) be (1) the focus, (2) the vertex, (3) the point $(4a, 0)$, and (4) the point $(-a, 0)$.

35. Show that the parabola is concave towards its axis.

126. *The locus of the middle points of a system of parallel chords of a parabola is a straight line parallel to the axis of the parabola.*



Let AB be any one of the chords, let $P'(x', y')$ be its middle point, and let γ be the angle it makes with the axis of the parabola.

Then the equation of AB may be written [(4), § 46]

$$\frac{x-x'}{\cos\gamma}=\frac{y-y'}{\sin\gamma}=r, \tag{1}$$

or $$x=x'+r\cos\gamma, \quad y=y'+r\sin\gamma. \tag{2}$$

Let the equation of the parabola be

$$y^2=4ax. \tag{3}$$

Substituting in (3) the values of x and y given by (2), we have for the points common to the chord and the curve

$$(y'+r\sin\gamma)^2=4a(x'+r\cos\gamma),$$

or $$r^2\sin^2\gamma+2(y'\sin\gamma-2a\cos\gamma)r+y'^2-4ax'=0, \tag{4}$$

a quadratic equation in r, whose roots are represented by the distances $P'B$ and $P'A$. Since P' is the middle point of AB, the sum of these roots is zero. That is,

$$y'\sin\gamma-2a\cos\gamma=0. \qquad (\S\ 91.)$$

Whence $$y'=2a\cot\gamma=\frac{2a}{m}, \tag{5}$$

where m is the constant slope of the chords.

The coordinates of P' therefore satisfy the equation

$$y = \frac{2a}{m} = 2a \cot \gamma. \tag{6}$$

Hence the locus of P', as AB moves keeping m constant, is a straight line $O'X'$ parallel to the axis of the parabola.

DEFINITION. The locus of the middle points of a system of parallel chords of a conic is called a **Diameter**; and the chords it bisects are oblique double ordinates to that diameter considered as an axis of abscissas.

We have seen in § 112 that a diameter of a parabola meets the curve in only *one* point at a finite distance from the directrix. This point is called the **Extremity** of the diameter.

COR. The line (6) meets the curve in O' where

$$x = \frac{a}{m^2} = RO', \quad y = \frac{2a}{m}. \tag{7}$$

The equation of the tangent at O' is, therefore [(2), 123],

$$y = mx + \frac{a}{m}. \tag{8}$$

Hence the tangent at the extremity of a diameter is parallel to the chords bisected by that diameter.

127. *To find the equation of a parabola when the axes are any diameter and the tangent at its extremity.*

Using the figure of § 126, and keeping the same notation, we will let $O'P' = x$, the new abscissa, and $P'B = y$, the new ordinate.

Then y is always the same as r of equation (4), § 126. And since the coefficient of the first power of r in this equation is zero, we have

$$y^2 = \frac{4ax' - y'^2}{\sin^2 \gamma}, \tag{1}$$

where $$y' = \frac{2a}{m}, \qquad [(5), \S\ 126]$$

and $$x' = RO' + O'P' = \frac{a}{m^2} + x. \qquad [(7), \S\ 126]$$

$$\therefore \quad y^2 = \frac{4a}{\sin^2 \gamma} x. \tag{2}$$

Now $FO' = a + RO'$ [(4), § 124]

$$= \frac{a(1+m^2)}{m^2} = a\frac{1+\tan^2\gamma}{\tan^2\gamma} = \frac{a}{\sin^2\gamma}. \tag{3}$$

Therefore, if $a' = \dfrac{a}{\sin^2\gamma} = FO'$, the required equation is

$$y^2 = 4a'x. \tag{4}$$

Hence the equation $y^2 = 4ax$ always represents a parabola, the x-axis being a diameter, the y-axis the tangent at its extremity, a the distance from the focus to the origin, and $4a$ the length of the focal chord parallel to the y-axis.

Formula (6), § 111, by means of which equation (2), § 123, was obtained, and also the derivation of equation (3), § 125, from equation (2), § 123, hold good equally whether the axes are rectangular or not. That is, if the equation of a parabola is $y^2 = 4ax$, the line

$$yy' = 2a(x + x') \tag{5}$$

will be the tangent at the point (x', y') if the point is on the curve; but always the polar of (x', y') with respect to the parabola. And the line

$$y = mx + \frac{a}{m} \tag{6}$$

will also touch the parabola for all values of m, the meaning of m being that given in § 59.

Cor. *The polar of any point with respect to a parabola is parallel to the chords bisected by the diameter through the point.*

Conversely, the locus of the poles of parallel chords is the bisecting diameter.

For the polar of any point $(x', 0)$ is, by (5), $x = -x'$.

128. *Through any point three normals can be drawn to a parabola.*

The equation of the normal at any point (x', y') of the parabola $y^2 = 4ax$ is [(4), § 123]

$$2a(y - y') + y'(x - x') = 0. \tag{1}$$

If the line (1) goes through the point (h, k), then, since

$$x' = \frac{y'^2}{4a},$$

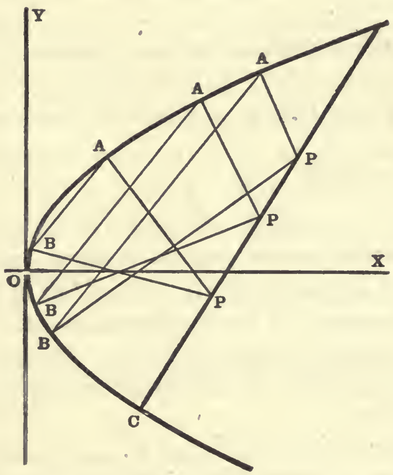
we have
$$2a(k - y') + y'\left(h - \frac{y'^2}{4a}\right) = 0,$$

or
$$y'^3 + 4a(2ak - h)y' - 8a^2k = 0. \qquad (2)$$

The *three* roots of equation (2) are the ordinates of the *three* points the normals at which pass through any given point (h, k).

Let y_1, y_2, y_3 be the three roots of (2); then, since the coefficient of y'^2 is zero (§ 91),

$$y_1 + y_2 + y_3 = 0. \qquad (3)$$



Let y_1 and y_2 be the ordinates of the ends of any one, AB, of a system of parallel chords whose slope is m; then

$$\tfrac{1}{2}(y_1 + y_2) = \frac{2a}{m}. \qquad [(5), \S\, 126.] \quad (4)$$

$$\therefore \quad y_3 = -\frac{4a}{m}, \text{ a constant.} \qquad (5)$$

Therefore, the normals at A and B, as AB moves keeping m constant, always meet on the fixed normal at C whose ordinate is $-\dfrac{4a}{m}$.

That is, *the locus of the intersection of normals at the ends of a system of parallel chords of a parabola is the normal to the curve at the point whose ordinate is minus twice the ordinate of the middle points of the chords.*

EXAMPLES ON CHAPTER VIII.

1. Find the equation of that chord of the parabola $y^2 = 6x$ which is bisected by the point (4, 3).

2. Find the equation of the chord of $x^2 = -8y$ whose middle point is $(-3, -2)$.

3. Find the equation of a normal to $y^2 = 4x$ which shall pass through $(2, -8)$.

4. Find the equations of the normals to the parabola $y^2 = 8x$ which pass through the point (8, 2).

5. Find the equations of the normals to $y^2 = 4ax$ which meet in the point $(b, 0)$. For what values of b are the three normals real?

6. Show that the axis of the parabola $y^2 = 8x$ divides each of the chords whose equations are $\frac{y}{\sin 30°} = \frac{x \pm 2}{\cos 30°}$ into two segments whose product is 64.

7. Any tangent to a parabola will meet the directrix and the latus rectum (produced) in two points equidistant from the focus.

8. The angle between two tangents to a parabola is equal to half the angle between the focal radii of their points of contact.

9. If a line is a normal to a parabola at one end of the latus rectum, its pole with respect to the parabola lies on the diameter through the other end of the latus rectum.

10. Show that the locus of the centre of a circle which intercepts a chord of given length $2a$ on the x-axis and passes through the fixed point $(0, b)$ is the curve

$$x^2 - 2by + b^2 = a^2.$$

11. Find the locus of the centre of a circle which touches a given circle and also a given straight line.

12. The perpendicular from a point Q on its polar with respect to a parabola meets the polar in M and the axis in G; the polar cuts the axis in T, and the ordinate through Q meets the curve in P and P'. Show that the points T, P, M, G, P' are all on a circle whose centre is F.

13. Prove that the two parabolas $y^2 = ax$ and $x^2 = by$ cut one another an angle

$$\tan^{-1} \frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2(a^{\frac{2}{3}} + b^{\frac{2}{3}})}$$

14. Show that the two parabolas

$$x^2 + 4a(y - 2b - a) = 0 \quad \text{and} \quad y^2 = 4b(x - 2a + b)$$

intersect at right angles at a common end of a latus rectum of each.

15. If EFG is a focal chord of a parabola whose vertex is A, and GA meets the directrix in B, show that BE is parallel to the axis of the parabola.

16. Show that the equation of the chord of $y^2 = 4ax$ which is bisected at the point (h, k) is

$$k(y - k) = 2a(x - h).$$

17. Show that if three normals meet in a point, the sum of their slopes is zero. Show also that if the sum of the slopes of two normals is constant, the locus of their intersection is a third normal to the parabola. [(4), § 125 and § 128.]

18. Show that the locus of the point of intersection of two perpendicular normals to the parabola $y^2 = 4ax$ is the parabola $y^2 = a(x - 3a)$.

19. Show that if two tangents to a parabola intercept a fixed length on the tangent at the vertex, the locus of their point of intersection is another equal parabola.

20. The tangents and the normals at the ends of any focal chord intersect on the circle whose diameter is the chord.

21. Show that two tangents to a parabola which make complementary angles with the axis, but are not at right angles, meet on the latus rectum.

22. A perpendicular drawn from the vertex of the parabola $y^2 = 4ax$ to the tangent at any point P meets the diameter through P in Q, the tangent in R and the ordinate through P in S. Show that the loci of Q, R, and S are, respectively,

$$x + 2a = 0, \quad x^3 + y^2(x + a) = 0, \quad \text{and} \quad x^3 = ay^2.$$

(Draw the normal at P and the ordinate of Q.)

23. From any point on the latus rectum of a parabola perpendiculars are drawn to the tangents at its extremities. Show that the line joining the feet of these perpendiculars touches the parabola.

24. Show that the locus of the foot of the perpendicular drawn from the focus to any normal to the parabola $y^2 = 4ax$ is the parabola

$$y^2 = a(x - a).$$

25. Show that if tangents are drawn to the parabola $y^2 = 4ax$ from any point on the line $x + 4a = 0$, their chord of contact will subtend a right angle at the vertex.

26. Prove that the chord of the parabola $y^2 = 4ax$, whose equation is $y - x\sqrt{2} + 4a\sqrt{2} = 0$, is a normal to the curve and that its length is $6\sqrt{3a}$.

27. The perpendicular TN from any point T on its polar with respect to a parabola meets the axis in M. Show that if $TN \cdot TM$ is constant, or if the ratio $TN : TM$ is constant, the locus of T is a parabola.

28. Two equal parabolas have their axes parallel and a common tangent at their vertices; a straight line is drawn parallel to their axes meeting the parabolas in P and Q. Show that the locus of the middle point of PQ is an equal parabola.

29. Two parabolas have a common axis and concavities in opposite directions; if any line parallel to the common axis meets the curves in P and Q, prove that the locus of the middle point of PQ is another parabola, provided the given parabolas are not equal.

30. Two parabolas touch one another and have their axes parallel. Show that, if the tangents at two points of these parabolas meet in any point on their common tangent, the line joining the points of contact will be parallel to their axes.

31. Two parabolas have the same axis. Find the locus of the middle points of chords of one which touch the other.

32. Two parabolas have the same axis; tangents are drawn from points on the first to the second; prove that the middle points of the chords of contact with the second lie on a fixed parabola.

33. Two parabolas have a common focus, and their axes in opposite directions. Prove that the locus of the middle points of chords of either which touch the other is another parabola.

34. Two equal parabolas, A and B, have the same vertex and their axes in opposite directions. Prove that the locus of the poles with respect to B of tangents to A is the parabola A.

35. Show that the locus of the poles of tangents to the parabola $y^2 = 4ax$ with respect to the parabola $y^2 = 4bx$ is the parabola $ay^2 = 4b^2x$.

36. The locus of the poles of tangents to either of the parabolas $y^2 = 4ax$ or $x^2 = -4ay$ with respect to the other is $xy = 2a^2$.

37. If a line touches the circle $x^2 + y^2 = 4a^2$, its pole with respect to the parabola $y^2 = 4ax$ lies on the rectangular hyperbola $x^2 - y^2 = 4a^2$.

38. The middle point of a chord PQ is on a fixed straight line perpendicular to the axis of a parabola; show that the locus of the pole of the chord is another parabola.

39. The base of a triangle is $2a$, and the sum of the tangents of the base angles is k. Show that the locus of the vertex is a parabola whose latus rectum is $\frac{2a}{k}$.

40. If, in the triangle ABC, AB is constant and $\tan A \tan \frac{1}{2}B = 2$, the locus of C is a parabola of which A is the vertex and B is the focus.

41. If θ is the angle which a focal chord makes with the axis, prove that the length of the chord is $4a \csc^2\theta$, and the length of the perpendicular on it from the vertex is $a \sin\theta$.

42. Two parallel chords of a parabola meet the axis in points equidistant from the vertex. Show that the axis divides each chord into two segments whose products are equal. (Use (4), § 46.)

43. PQ is any one of a system of parallel chords of a parabola; O is any point on PQ such that the product $PO \cdot OQ$ is constant. Show that the locus of O is a parabola.

44. Prove that the locus of the middle point of that portion of the normal intercepted between the curve $y^2 = 4ax$ and its axis is a parabola whose vertex is the focus and whose latus rectum is a.

45. The locus of the middle points of normal chords of the parabola $y^2 = 4ax$ is

$$y^4 - 2ay^2(x - 2a) + 8a^4 = 0.$$

46. Prove that the distance between a tangent to the parabola $y^2 = 4ax$ and the parallel normal is $a \csc\theta \sec^2\theta$, where θ is the angle that either makes with the axis.

47. If the normals at two points of a parabola are inclined to the axis at angles θ and ϕ such that $\tan\theta \tan\phi = 2$, show that they intersect on the parabola.

48. The locus of a point from which two normals can be drawn making complementary angles with the axis is a parabola.

49. Two equal parabolas have the same focus and their axes are at right angles; a normal to the one is perpendicular to a normal to the other; prove that the locus of the intersection of these normals is another parabola.

50. If a normal to a parabola makes an angle ϕ with the axis, show that it will cut the curve again at an angle $\tan^{-1}(\frac{1}{2}\tan\phi)$.

51. If TP and TQ are tangents to a parabola whose vertex is A, and if the lines PA, QA, TA, produced if necessary, meet the directrix in P', Q', T', respectively, show that $P'T' = T'Q'$.

52. Prove that there is a fixed point K on the axis of any parabola such that

$$\frac{1}{PK^2} + \frac{1}{QK^2}$$

is the same for all positions of the chord PKQ.

53. If the diameter through any point O on a paraoola meets any chord in P, and the tangents at the ends of that chord in Q and R, show that $OP^2 = OQ \cdot OR$.

54. A chord is normal to a parabola and makes an angle θ with the axis. Prove that the area of the triangle formed by it and the tangents at its extremities is $4a^2 \sec^3\theta \csc^3\theta$.

55. The vertex of a triangle is fixed, the base is of constant length and moves along a fixed straight line. Show that the locus of the centre of its circumscribing circle is a parabola.

56. A chord of the parabola $y^2 = 4ax$ passes through the fixed point $(-2a, 0)$. Prove that the normals at its extremities meet on the curve.

57. If from any point on a focal chord of a parabola two tangents are drawn, these two tangents are equally inclined to the tangents at the ends of the chord.

58. If r_1 and r_2 are the lengths of radii vectores of a parabola which are drawn at right angles to one another from the vertex, prove that

$$r_1^{\frac{4}{3}} r_2^{\frac{4}{3}} = 16a^2(r_1^{\frac{2}{3}} + r_2^{\frac{2}{3}}).$$

59. On the diameter through a point O on a parabola two points P and Q are taken such that $OP \cdot OQ$ is constant; prove that the four points of intersection of the tangents drawn from P and Q will lie on two fixed straight lines parallel to the tangent at O and equidistant from it.

60. T is the pole of the chord PQ; prove that the perpendiculars from P, T, and Q on any tangent to the parabola are in geometrical progression.

61. PFQ is a focal chord of a parabola; R is the middle point of PQ, and RO is perpendicular to PQ and meets the axis in O; prove that FO and RO are the arithmetic and geometric means between FP and FQ.

62. Prove that the locus of the point of intersection of two tangents, which with the tangents at the vertex form a triangle of constant area c^2, is the curve

$$x^2(y^2 - 4ax) = 4a^2c^2.$$

63. Parallel chords are drawn to a parabola; the locus of the intersection of tangents at the ends of these chords is a straight line (Cor. § 127); and the locus of the intersection of normals at these points is also a straight line (§ 128). Show that the locus of the intersection of these two lines, as the chords change direction, is a parabola.

64. Show that the locus of the poles of chords which subtend a constant angle α at the vertex is

$$(x + 4a)^2 = 4 \cot^2\alpha(y^2 - 4ax).$$

65. Prove that three tangents to a parabola, which are such that the tangents of their inclinations to the axis are in a given harmonical progression, form a triangle whose area is constant.

66. Find the equation of the parabola when the axes are the tangents at the ends of the latus rectum.

CHAPTER IX.

THE CIRCLE.

129. *Equations of the circle, and the corresponding equations of the tangent, polar, and normal.*

We have seen in § 32 that the equation of the circle whose radius is r takes the simple form

$$x^2 + y^2 = r^2, \tag{1}$$

when the origin is at the centre; while if the centre is at the point (a, b) the equation may be written

$$(x - a)^2 + (y - b)^2 = r^2. \tag{2}$$

Moreover, it was further shown in § 110 that the general equation of the second degree will represent a circle if $a = b$, and $h = 0$; so that the most general equation of a circle in rectangular coordinates is

$$x^2 + y^2 + 2gx + 2fy + c = 0. \tag{3}$$

Equation (3) may be put in the form (2), which gives

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c. \tag{4}$$

Hence the centre of the circle represented by (3) is the point $(-g, -f)$, and the radius is equal to $\sqrt{g^2 + f^2 - c}$.

The circle will therefore be real, a point, or imaginary according as $g^2 + f^2 - c >, \ =, \text{ or } < 0$.

By applying the rule of § 111 to equations (1), (2), and (3), respectively, we obtain

$$xx' + yy' = r^2, \tag{5}$$

$$(x - a)(x' - a) + (y - b)(y' - b) = r^2, \tag{6}$$

and

$$xx' + yy' + g(x + x') + f(y + y') + c = 0. \tag{7}$$

These are the equations of the tangent to the circles (1), (2), (3), respectively, at the point (x', y') if this point is on the curve; but, by § 113, they are always the equations of the polar of the point (x', y') with respect to the circles represented by (1), (2), (3).

Since the normal (§ 78) at any point (x', y') of the circle $x^2 + y^2 = r^2$ is perpendicular to (5), its equation is [(2), § 85]

$$y - y' = \frac{y'}{x'}(x - x'),$$

or
$$xy' - x'y = 0. \qquad (8)$$

That is, the normal at any point of a circle passes through the centre.

The equations of the normals to the circles (2) and (3) at the point (x', y') are, respectively [(2), § 85],

$$y - y' = \frac{y' - b}{x' - a}(x - x'), \qquad (9)$$

and
$$y - y' = \frac{y' + f}{x' + g}(x - x'); \qquad (10)$$

or
$$xy' - x'y - b(x - x') + a(y - y') = 0, \qquad (11)$$

and
$$xy' - x'y + f(x - x') - g(y - y') = 0. \qquad (12)$$

The general equation of the circle (3), or (2), contains three parameters, or constants. Therefore a circle can be made to satisfy three conditions, and no more. If we wish to find the equation of a circle which satisfies three given conditions, we assume the equation to be of the form (3), or (2), and then determine the values of the constants g, f, c, or a, b, r, from the given conditions.

Ex. Find the equation of the circle passing through the three points (0, 1), (2, 0), and (0, −3).

Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \qquad (1)$$

Since the given points are on the circle, their coordinates must satisfy equation (1).

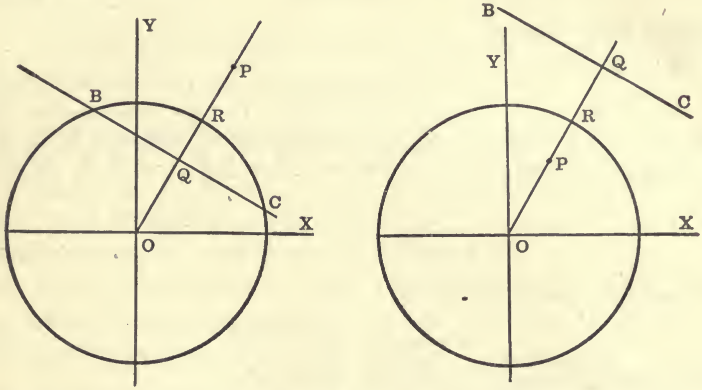
$$\therefore\ 1 + 2f + c = 0,\quad 4 + 4g + c = 0,\quad 9 - 6f + c = 0.$$

Whence we find $g = -\frac{1}{4}$, $f = 1$, and $c = -3$. Substituting these values in (1) the required equation becomes

$$x^2 + y^2 - \tfrac{1}{2}x + 2y - 3 = 0.$$

The centre is the point $(\frac{1}{4}, -1)$, and the radius is $\frac{1}{4}\sqrt{65}$.

130. *A geometrical construction for the polar of a point with respect to a circle.*



Let the equation of the circle be

$$x^2 + y^2 = r^2. \tag{1}$$

Let $P(x', y')$ be any point, BC its polar, and let OP and BC intersect in Q. Then the equation of BC is [(5), § 129]

$$xx' + yy' = r^2, \tag{2}$$

and the equation of the line OP is (§ 47)

$$xy' - x'y = 0. \tag{3}$$

Hence BC is perpendicular to OP (§ 48), and therefore

$$OQ = \frac{r^2}{\sqrt{x'^2 + y'^2}}. \qquad [(5), § 50.] \quad (4)$$

Also $$OP = \sqrt{x'^2 + y'^2}. \qquad [(4), § 7.] \quad (5)$$

$$\therefore \quad OP \cdot OQ = r^2. \tag{6}$$

We therefore have the following construction for the polar of a point P. Draw OP and let it cut the circle in R; then construct a third proportional, OQ, to OP and r, *i. e.* take Q on the line OP, such that $OP:OR = OR:OQ$, and draw a line through Q perpendicular to OP.

Ex. 1. Construct the pole of a given line.

Ex. 2. Prove the theorem of § 115 for the circle.

131. *To find the equation of the tangent to the circle*

$$x^2 + y^2 = r^2 \qquad (1)$$

in terms of its slope m.

The line

$$y = mx + b \qquad (2)$$

will touch the circle (1) if the perpendicular distance from it to the origin is equal to the radius r of the circle; that is, (§ 50) if

$$r = \frac{b}{\sqrt{1+m^2}}, \quad \text{or} \quad b = r\sqrt{1+m^2}. \qquad (3)$$

Therefore the straight line

$$y = mx + r\sqrt{1+m^2} \qquad (4)$$

will touch the circle (1) for all values of m.

Since either sign may be given to the radical $\sqrt{1+m^2}$ in (3), it follows that there are *two* tangents to the circle for every value of m; *i. e.* there are two tangents parallel to any given straight line.

Ex. 1. Derive equation (3) by treating (1) and (2) simultaneously and taking the condition for equal roots.

Ex. 2. What is the equation of the normal to (1) in terms of its slope?

Ex. 3. How many normals can be drawn from a point to a circle?

EXAMPLES.

Find the centres and radii of the following circles:

1. $x^2 + y^2 \pm 4x = 0$.
2. $x^2 + y^2 \pm 6y = 0$.
3. $x^2 + y^2 + 2x - 4y = 0$.
4. $x^2 + y^2 - 3x + 5y = 0$.
5. $x^2 + y^2 + 6x - 4y + 9 = 0$.
6. $4(x^2 + y^2) - 12x + 8y + 23 = 0$.

Find the equation of the circle passing through the three points

7. (0, 0), (6, 0), (0, 4).
8. (0, 0), (1, 1), (4, 0).
9. (2, —3), (3, —4), (—2, —1).
10. (1, 2), (3, —4), (5, 6).
11. (0, 0), $(a, 0)$, $(0, b)$.
12. $(a, 0)$, $(-a, 0)$, $(0, -b)$.

13. Find the equation of a circle passing through (0, 4) and (6, 0), and having $\sqrt{13}$ for radius.

14. Find the equation of a circle whose centre is (3, 4) and which touches the line $4x - 3y + 20 = 0$.

15. Find the general equation of the circle which touches both axes.

16. Find the equation of the circle passing through the point $(-3, 6)$ and touching both axes.

17. Find the equation of the circle touching the line $y = c$ and both axes.

Write down the equation of the tangent to the circle

18. $x^2 + y^2 - 2x + 3y - 4 = 0$ at the point $(2, 1)$.

19. $x^2 + y^2 + 4x - 6y - 13 = 0$ at the point $(-3, -2)$.

20. Show that the lines $y = m(x - r) \pm r\sqrt{1 + m^2}$ touch the circle $x^2 + y^2 = 2rx$, whatever the value of m may be.

Find the equations of the tangents to the circle

21. $x^2 + y^2 = 4$ parallel to $2x + 3y + 1 = 0$.

22. $x^2 + y^2 = 6x$ parallel to $3x - 2y + 2 = 0$.

23. $9(x^2 + y^2) - 9(6x - 8y) + 125 = 0$ parallel to $3x + 4y = 0$.

24. Show that the line $x - 2y = 0$ touches the circle

$$x^2 + y^2 - 4x + 8y = 0.$$

25. The line $y = 3x - 9$ touches the circle

$$x^2 + y^2 + 2x + 4y - 5 = 0.$$

Find the coordinates of the point of contact.

26. Find the equation of the tangent to $x^2 + y^2 = r^2$ (1) which is perpendicular to $y = mx + b$, (2) which passes through the point $(c, 0)$, (3) which makes with the axes a triangle whose area is r^2.

Find the polar of the point

27. $(1, 2)$ with respect to $x^2 + y^2 = 5$.

28. $(3, -2)$ with respect to $3(x^2 + y^2) = 14$.

29. $(-4, 1)$ with respect to $x^2 + y^2 - 2x + 6y + 7 = 0$.

30. $(2, -\frac{1}{2})$ with respect to $x^2 + y^2 + 3x - 5y + 3 = 0$.

31. $(-a, b)$ with respect to $x^2 + y^2 - 2ax + 2by + a^2 - b^2 = 0$.

Find the pole of the line

32. $2x + y = 1$ and $x - 3y = 1$ with respect to $x^2 + y^2 = 2$.

33. $x - 2y = 3$ and $2x + y = 4$ with respect to $x^2 + y^2 = 6$.

34. $x + y + 1 = 0$ with respect to $x^2 + y^2 + 4x - 6y + 11 = 0$.

35. $2x + 14y = 15$ with respect to $2(x^2 + y^2) - 3x + 5y - 2 = 0$.

36. $3(ax - by) = a^2 + b^2$ with respect to $x^2 + y^2 - 2ax + 2by = a^2 + b^2$.

37. Show that the circles $x^2 + y^2 - 4x + 2y = 15$ and $x^2 + y^2 = 5$ touch one another at the point $(-2, 1)$.

38. Show that the polars of the point (1, 0) with respect to the two circles $x^2+y^2+4x-14=0$ and $x^2+y^2=4$ are the same line; show that the same is true of the point (4, 0).

39. Find two points such that the polars of each with respect to the two circles $x^2+y^2-2x-3=0$ and $x^2+y^2+2x-11=0$ coincide.

40. A certain point has the same polar with respect to two circles; prove that any common tangent subtends a right angle at that point. Show also that there are two such points for any two circles.

41. Find the locus of the intersection of two tangents to $x^2+y^2=r^2$ which are at right angles to one another.

42. Find the locus of the intersection of two tangents to $x^2+y^2=r^2$ which intersect at an angle α.

43. Show that if the coordinates of the extremities of a diameter of a circle are (x_1, y_1) and (x_2, y_2), respectively, the equation of the circle will be

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

[*Suggestion.* Lines joining any point (x, y) on the circle to (x_1, y_1) and (x_2, y_2) are at right angles to one another.]

Find the equation of the circle which touches

44. the lines $x=0$, $x=a$, and $3y=4x+3a$.

One Ans. $4(x^2+y^2)-4a(x+5y)+25a^2=0$.

45. both axes and the line $\dfrac{x}{a}+\dfrac{y}{b}=1$.

46. Prove analytically that the locus of the middle points of a system of parallel chords of a circle is the diameter perpendicular to the chords. (See § 126.)

47. Show that as α varies the locus of the intersection of the lines $x\cos\alpha+y\sin\alpha=a$ and $x\sin\alpha-y\cos\alpha=b$ is a circle.

48. A circle touches the y-axis and cuts off a constant length $(2a)$ from the x-axis; show that the locus of its centre is $x^2-y^2=a^2$.

49. Two lines are drawn through the points $(a, 0)$ and $(-a, 0)$ and make an angle α with one another. Show that the locus of their point of intersection is

$$x^2+y^2\pm 2ay\cot\alpha=a^2.$$

50. If the polar of the point (x', y') with respect to the circle $x^2+y^2=a^2$ touches the circle $x^2+y^2=2ax$, show that $y'^2+2ax'=a^2$.

51. Show that if the axes are inclined at an angle ω, the equation of the circle is (§ 8)

$$(x-a)^2+(y-b)^2+2(x-a)(y-b)\cos\omega=r^2,$$

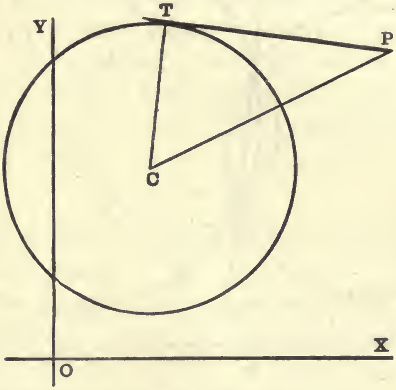
where (a, b) is the centre and r the radius.

132. *To find the length of a tangent drawn from a given point $P(x', y')$ to a given circle.*

Let the equation of the circle be

$$(x-a)^2+(y-b)^2-r^2=0. \qquad (1)$$

Let C be the centre and PT one tangent from P.



Then, since CPT is a right triangle,

$$PT^2 = CP^2 - CT^2. \qquad (2)$$

But $\quad CP^2 = (x'-a)^2+(y'-b)^2, \qquad$ [§ 7, (2)]

and $\quad CT^2 = r^2.$

$$\therefore \quad PT^2 = (x'-a)^2+(y'-b)^2-r^2. \qquad (3)$$

That is, the square of the tangent is found by substituting the coordinates x', y' of the given point in the left member of equation (1).

Since the general equation of the circle,

$$x^2+y^2+2gx+2fy+c=0, \qquad (4)$$

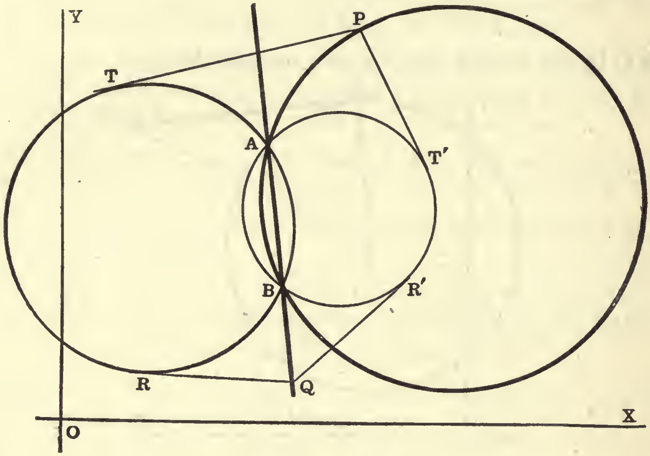
can be put in the form of (1) by merely adding and subtracting g^2 and f^2 in the first member, it follows that if the coordinates of any point are substituted in the first member of (4) the result will be equal to the square of the length of the tangent drawn from the point to the circle; or *the product of the segments of any chord (or secant) drawn through the point.*

Ex. 1. What is the meaning of (3) when the second member is negative?

Ex. 2. What is represented by c in equation (4)?

Ex. 3. Where is the origin if c is positive? if c is zero? if c is negative?

133. *If a circle passes through the common points of two given circles, tangents drawn from any point on it to the two given circles are in a constant ratio.*



Let $$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

and $$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0, \quad (2)$$

be the equations of the two given circles.

Then the locus of $S = \lambda S'$, *i. e.* (See Ex. 10, p. 71.)

$$x^2 + y^2 + 2gx + 2fy + c = \lambda(x^2 + y^2 + 2g'x + 2f'y + c'), \quad (3)$$

for all values of λ, will pass through the common points, A, B, of (1) and (2). Moreover, (3) is a circle (§ 110), and therefore, for different values of λ, represents all circles through the intersection of (1) and (2).

Let $P(x', y')$ be any point on (3); let PT and PT' be the tangents to (1) and (2) respectively. Then the coordinates x', y' must satisfy (3), and we therefore have

$$x'^2 + y'^2 + 2gx' + 2fy' + c = \lambda(x'^2 + y'^2 + 2g'x' + 2f'y' + c'). \quad (4)$$

Therefore $$PT^2 = \lambda \cdot PT'^2, \qquad (§ 132) \quad (5)$$

which proves the proposition, since λ is constant for any particular circle.

When $\lambda = 1$, it is easy to show that the radius and the coordinates of the centre (§ 129) of the circle represented by equation (3) all become infinite. In this case the equation reduces to

$$2(g-g')x + 2(f-f')y + c - c' = 0, \qquad (6)$$

which is of the first degree, and therefore represents the straight line AB through the common points of the two given circles.

Let QR and QR' be tangents to $S=0$ and $S'=0$, respectively, from any point Q on AB; then, since ABQ is the circle through the common points of (1) and (2) corresponding to $\lambda = 1$, it follows from (5) that

$$QR = QR'. \qquad (7)$$

That is, tangents drawn to the two given circles from any point on the line (6) are equal.

It is to be noticed that the straight line given by (6) is in all cases real, provided g, f, c, g', f', c' are real, although the circles $S=0$ and $S'=0$ may not intersect in real points; in fact one or both of the circles may be wholly imaginary. We have here, therefore, the case of a real straight line passing through the imaginary points of intersection of two real or imaginary circles. (*Cf.* § 113.)

Definition. The straight line through the points of intersection (real or imaginary) of two circles is called the **Radical Axis** of the two circles.

From equation (7) it follows that the radical axis may also be defined as the locus of the points from which tangents drawn to the two circles are equal to one another.

Cor. *If the coefficients of x^2 in S and S' are unity, the equation of the radical axis of the two circles $S=0$ and $S'=0$ is $S-S'=0$.*

Ex. 1. Show that the radical axis of two circles is perpendicular to the line joining their centres.

Ex. 2. If tangents are drawn to two circles from any point on a line parallel to their radical axis, show that the difference of the squares of these tangents is constant.

Ex. 3. Show that the radical axis of two circles divides the line joining their centres into two segments, such that the difference of their squares is equal to the difference of the squares of the radii.

134. *The radical axes of three circles, taken in pairs, meet in a point.*

Let $S_1=0$, $S_2=0$, $S_3=0$ be the equations of three circles, in each of which the coefficient of x^2 is unity.

Then the equations of their three radical axes are (§ 133, Cor.)

$$S_1-S_2=0, \quad S_2-S_3=0, \quad S_3-S_1=0.$$

The sum of any two of these equations is equivalent to the third. Hence they form a consistent system, and therefore their loci meet in a point. (See § 53, Ex.)

This point is called the **Radical Centre** of the three circles.

EXAMPLES.

Find the length of the tangents (or the product of the segments of the chords) drawn from the points

1. $(3, 2)$, $(5, -4)$ to the circle $x^2+y^2=4$.
2. $(-3, 2)$, $(4, -4)$ to the circle $x^2+y^2=25$.
3. $(3, -2)$, $(1, 3)$ to the circle $x^2+y^2-2x-4y=0$.
4. $(2, 1)$, $(0, 0)$ to the circle $2(x^2+y^2)-12x-4y+15=0$.
5. $(0, 0)$, $(-2, -5)$ to the circle $x^2+y^2-6x+4y+4=0$.
6. $(0, 0)$, $(6, -3)$ to the circle $x^2+y^2+6x-8y-11=0$.

Find the radical axis of the circles

7. $x^2+y^2+6x-4y-3=0$ and $x^2+y^2-4x+8y-5=0$.
8. $x^2+y^2-8x-10y+25=0$ and $x^2+y^2+8x-2y+8=0$.
9. $x^2+y^2+ax+by-c=0$ and $ax^2+ay^2+a^2x+b^2y=0$.

10. Find the radical axis and the length of the common chord of the circles

$$x^2+y^2+ax+by+c=0 \text{ and } x^2+y^2+bx+ay+c=0.$$

11. Show that the three circles

$$x^2+y^2-2x-4y=0, \qquad x^2+y^2-6x+4y+4=0,$$
$$x^2+y^2-8x+8y+6=0$$

have a common radical axis. Find the equation of a fourth circle such that the four shall have a common radical axis.

Find the radical centre of the three circles

12. $x^2+y^2-4x+8y-5=0, \qquad x^2+y^2-8x-10y+25=0,$
$$x^2+y^2+8x+11y-10=0.$$

13. $x^2+y^2+6x-8y+9=0, \qquad x^2+y^2+8x+2y+9=0,$
$$2(x^2+y^2)-5(3x+y)+18=0.$$

14. What is the analytic condition that the origin shall be the radical centre of three given circles?

15. Find the equation of the circle through the origin and the points of intersection of the circles

$$x^2+y^2-5x-7y+6=0 \text{ and } x^2+y^2+4x+6y-12=0.$$

What is the ratio of the tangents drawn from any point on it to the two given circles?

16. Find the equation of the circle which touches the line $4y=3x$ and passes through the common points of

$$x^2+y^2=9 \text{ and } x^2+y^2+x+2y=14.$$

17. What is the ratio of the tangents drawn from any point on the third circle in Ex. 11 to the other two circles?

18. Find the equations of the straight lines which touch both of the circles $x^2+y^2=4$ and $(x-4)^2+y^2=1$.

Ans. $3x \pm \sqrt{7}\,y=8$ and $x \pm \sqrt{15}y=8$.

19. Find the equations of the common tangents to the circles

$$x^2+y^2+6y+5=0 \quad \text{and} \quad x^2+y^2-12y+20=0.$$

20. If the length of the tangent from the point (x', y') to the circle $x^2+y^2=9$ is twice the length of the tangent from the same point to $x^2+y^2+3x-6y=0$, show that

$$x'^2+y'^2+4x'-8y'+3=0.$$

21. If the tangent from P to the circle $x^2+y^2+3y=0$ is four times as long as the tangent from P to the circle $x^2+y^2=9$, show that the locus of P is

$$5(x^2+y^2)=y+48.$$

22. The length of a tangent drawn from a point P to the circle $x^2+y^2+4x-6y+4=0$ is three times the length of the tangent from P to the circle $x^2+y^2-6x+2y+6=0$. Find the locus of P.

23. Find the locus of a point whose distance from the origin is equal to the length of the tangent drawn from it to the circle

$$x^2+y^2-8x-4y+4=0.$$

24. Find the locus of a point P whose distance from a fixed point is in a constant ratio to the tangent drawn from P to a given circle. Under what condition is the locus a straight line?

25. Show that the polar of any point on the circle

$$x^2+y^2-2ax-3a^2=0,$$

with respect to the circle $x^2+y^2+2ax-3a^2=0$, will touch the parabola $y^2+4ax=0$.

SYSTEMS OF CIRCLES.

135. *Coaxial circles.*

Let $S=0$ and $S'=0$ represent two circles.

Then
$$S+\lambda S'=0, \tag{1}$$

for any value of λ, represents a circle through the points of intersection of $S=0$ and $S'=0$ (§ 133). Moreover, by assigning to λ all possible values, it can be made to represent all the circles which pass through these points; that is, it represents the entire system of circles such that the radical axis (§ 133) of any pair of circles of the system is the same as the radical axis of the circles $S=0$ and $S'=0$.

Circles which have a common radical axis are called a **System of Coaxial Circles.**

If $S=0$ is the equation of a circle, and $L=0$ is the equation of a straight line, then will

$$S+\lambda L=0 \tag{2}$$

be the equation of a system of coaxial circles whose radical axis is the line $L=0$.

Let $S=x^2+y^2+c, \quad L=x, \quad \text{and} \quad \lambda=2g.$

Then equation (2) becomes

$$x^2+y^2+2gx+c=0, \tag{3}$$

which represents a system of coaxial circles whose centres are $(-g, 0)$, and whose radical axis is the y-axis.

For, whatever value g may have, the circle (3) cuts the y-axis in the same two points $(0, \pm\sqrt{-c})$.

Therefore, the radical axis meets these circles in *real* points if c is *negative*, in *imaginary* points if c is *positive*.

Equation (3) may be written

$$(x+g)^2+y^2=g^2-c. \tag{4}$$

Hence, if g is taken equal to $\pm\sqrt{c}$, the circle will reduce to one of the points $(\pm\sqrt{c}, 0)$.

These two points are called the **Limiting Points** of the system of coaxial circles. When the circles intersect in imaginary points, that is, when c is positive, the limiting points are real; and conversely, when the circles intersect in real points, that is, when c is negative, the limiting points are imaginary.

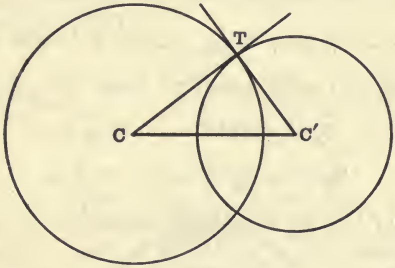
136. *To find the condition that two circles shall intersect orthogonally.*

The two circles

$$x^2 + y^2 + 2gx + 2fy + c = 0, \tag{1}$$

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0, \tag{2}$$

will intersect at right angles if the distance between their centres



is equal to the sum of the squares of their radii; *i. e.* if [§§ 7 and 129]

$$(g - g')^2 + (f - f')^2 = (g^2 + f^2 - c) + (g'^2 + f'^2 - c'),$$

or

$$2gg' + 2ff' - c - c' = 0. \tag{3}$$

Ex. 1. Show that the circles

$$x^2 + y^2 - 2ax - 2by - 2ab = 0 \text{ and } x^2 + y^2 + 2bx + 2ay - 2ab = 0$$

cut one another at right angles.

Ex. 2. Show that the two circles

$$x^2 + y^2 - 6x + 2y - 22 = 0 \text{ and } x^2 + y^2 + 4x - 6y + 4 = 0$$

intersect orthogonally. Find the equation of a third circle which shall cut each of the given circles at right angles.

Ex. 3. Find the equation of a circle which passes through the origin and cuts orthogonally each of the circles

$$x^2 + y^2 - 6x + 8 = 0 \text{ and } x^2 + y^2 - 2x - 2y = 7.$$

Ex. 4. Find the equation of the circle which intersects orthogonally each of the circles

$$x^2 + y^2 - 6x - 6y + 14 = 0,$$

$$x^2 + y^2 + 6x - 4y + 4 = 0,$$

$$x^2 + y^2 - 2x + 6y = 6.$$

Ans. $17(x^2 + y^2) - 10(2x + 5y) = 28.$

137. *If a circle cuts orthogonally each of the circles*

$$x^2 + y^2 + 2gx + 2fy + c = 0, \tag{1}$$

and

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0, \tag{2}$$

it will cut orthogonally every circle of the coaxial system

$$x^2 + y^2 + 2gx + 2fy + c + \lambda(x^2 + y^2 + 2g'x + 2f'y + c') = 0,$$

or

$$x^2 + y^2 + 2\left(\frac{g + \lambda g'}{1 + \lambda}\right)x + 2\left(\frac{f + \lambda f'}{1 + \lambda}\right)y + \frac{c + \lambda c'}{1 + \lambda} = 0. \tag{3}$$

Let the circle

$$x^2 + y^2 + 2Gx + 2Fy + C = 0 \tag{4}$$

cut both (1) and (2) orthogonally; then [(3), § 136]

$$2gG + 2fF - c - C = 0, \tag{5}$$

and

$$2g'G + 2f'F - c' - C = 0. \tag{6}$$

If we multiply (6) by λ, add the result to (5), and then divide by $1 + \lambda$, we obtain

$$2\left(\frac{g + \lambda g'}{1 + \lambda}\right)G + 2\left(\frac{f + \lambda f'}{1 + \lambda}\right)F - \frac{c + \lambda c'}{1 + \lambda} - C = 0. \tag{7}$$

Therefore (3) and (4) intersect orthogonally [§ 136, (3)].

COR. I. *If a circle cuts orthogonally a system of coaxial circles, its centre lies on the radical axis of the system.*

The radical axis of the system of coaxial circles represented by (3) is the line [(6), § 133]

$$2(g - g')x + 2(f - f')y + c - c' = 0. \tag{8}$$

Subtracting (6) from (5) gives

$$2(g - g')G + 2(f - f')F - c + c' = 0, \tag{9}$$

which shows that the point $(-G, -F)$, the centre of (4), lies on the line (8).

COR. II. *If the circles $S_1 = 0$ and $S_2 = 0$ cut the circles $S_3 = 0$ and $S_4 = 0$ orthogonally, then all the circles of the coaxial system $S_1 + \lambda S_2 = 0$ will cut orthogonally all the circles of the coaxial system $S_2 + \lambda_1 S_3 = 0$, and the locus of the centres of each system is the radical axis of the other system.*

The equations of two systems of coaxial and orthogonal circles referred to their radical axes as axes of coordinates (§ 133, Ex. 1) may be written

$$x^2 + y^2 + 2gx - c = 0, \tag{10}$$

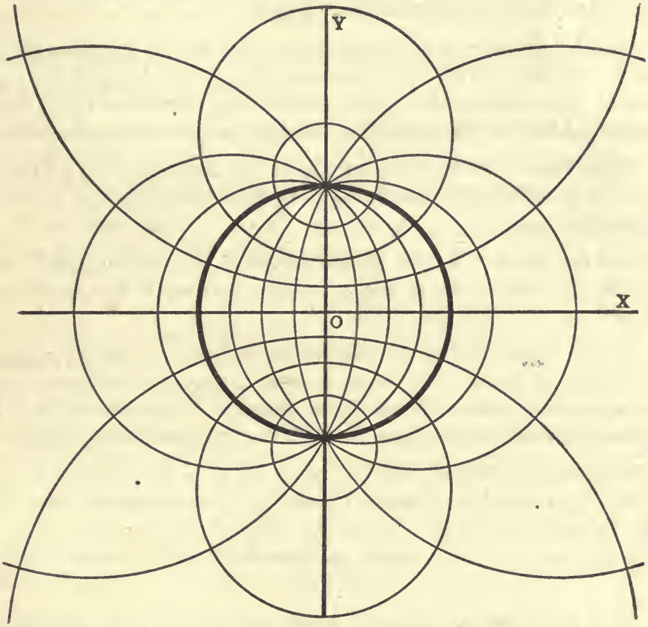
and

$$x^2 + y^2 + 2fy + c = 0, \tag{11}$$

where c is constant, and g and f are arbitrary.

For these equations represent two coaxial systems [§ 135, (3)], and satisfy the condition (3) of § 136 for all values of g and f.

The system of circles represented by (10) intersect in the points $(0, \pm\sqrt{c})$, which are the limiting points (§ 135) of the system represented by (11).



The heavy circle in the figure is the smallest one of the system represented by (10) and corresponds to the value $g = 0$. If this circle is considered as the boundary of the map of a hemisphere, the other circles of the same system are the meridians, while the circles of the other system are the parallels of latitude on this map, thrown on the usual *stereographic projection.*

Hence, f and g may be said to represent the latitude and longitude of any point on the map. (See § 73.)

EXAMPLES ON CHAPTER IX.

1. Show that the two circles

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ and } x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

are tangent to each other if

$$\sqrt{(g-g')^2 + (f-f')^2} = \sqrt{g^2 + f^2 - c} \pm \sqrt{g'^2 + f'^2 - c'}.$$

2. Find the equation of the circle whose diameter is the common chord of the circles

$$x^2 + y^2 + 2x - 4y - 4 = 0 \text{ and } x^2 + y^2 - 6x + 4y + 4 = 0.$$

3. Find the equation of the straight lines joining the origin to the points of intersection of the line $2x + 3y = 7$ and the circle $x^2 + y^2 - 4x + 2y = 0$, and show that these lines are at right angles.

4. Find the equation of the straight lines joining the origin to the points in which the line $y = mx + c$ intersects the circle $x^2 + y^2 = 2(ax + by)$. Hence find the condition that these points may subtend a right angle at the origin. Also find the condition that the line may touch the circle.

5. A point moves so that the square of its distance from a fixed point varies as its perpendicular distance from a fixed straight line; show that it describes a circle.

6. Find the locus of a point which moves so that the square of its distance from the base of an isosceles triangle is equal to the product of its distances from the other two sides.

7. A point moves so that the sum of the squares of the perpendiculars let fall from it on the sides of an equilateral triangle is constant (k); prove that its locus is a circle. Where is its centre? For what value of k will the circle touch the sides? pass through the vertices of the triangle?

8. Tangents are drawn from the point (h, k) to the circle $x^2 + y^2 = r^2$; prove that the area of the triangle formed by them and their chord of contact is

$$\frac{r(h^2 + k^2 - r^2)^{\frac{3}{2}}}{h^2 + k^2}.$$

9. Find the polar equation of a circle, the initial line being a tangent.

10. Prove that the equations

$$\rho = a \cos(\theta - \alpha) \text{ and } \rho = b \sin(\theta - \alpha)$$

represent two circles which intersect orthogonally.

11. Find the polar equation of a circle whose centre is the point (a, b), (in rectangular coordinates) and whose radius is r.

12. Find the polar equation of the tangent to a circle at a given point, the centre of the circle being at the pole.

13. What curve is represented by the equation

$$\rho^2 - \rho a \cos 2\theta \sec \theta - 2a^2 = 0?$$

14. Determine the locus of the equation

$$\rho = a \cos(\theta - \alpha) + b \cos(\theta - \beta) + c \cos(\theta - \gamma) + \ldots$$

15. The axes being inclined at an angle ω, find the centre and radius of the circle

$$x^2 + 2xy \cos \omega + y^2 - 2gx - 2fy = 0.$$

16. The axes being inclined at 60°, find the equation of the circle whose centre is $(-3, -5)$ and radius 6.

17. Find the locus of a point which moves so that the square of the tangent drawn from it to the circle $x^2 + y^2 = r^2$ is equal to a times its distance from the line $lx + my + n = 0$.

18. Find the locus of a point such that tangents from it to two given circles are inversely as their radii.

19. Find the locus of the vertex of a triangle, having given (1) its base and the sum of the squares of its sides, (2) its base and the sum of m times the square of one side and n times the square of the other.

20. Show that the equation of the circle circumscribing the triangle formed by the lines $x + y = 6$, $2x + y = 4$, and $x + 2y = 5$ is

$$x^2 + y^2 - 17x - 19y + 50 = 0.$$

21. Show that the radical axis of two circles bisects their four common tangents.

22. If Q is one of the limiting points of a system of coaxial circles, show that the polar of Q with respect to any circle of the system passes through the other limiting point, and is the same for all circles of the system.

23. If Q is one of the limiting points of a system of coaxial circles, show that a common tangent to any two circles of the system will subtend a right angle at Q.

24. Tangents are drawn to a circle from any point on a given line; prove that the locus of the middle point of the chord of contact is another circle.

25. Find the locus of the middle points of chords of the circle $x^2 + y^2 = r^2$ which subtend a right angle at the point $(a, 0)$.

26. Prove that the square of the tangent drawn from any point on one circle to another circle is equal to twice the product of the distance between the centres and the perpendicular distance of the point from the radical axis of the two circles.

27. Find the equations of the straight lines joining the origin to the points of intersection of

$$x^2+y^2-4x-2y=4 \text{ and } x^2+y^2+2x+4y=10.$$

28. The distances of two points from the centre of a circle are proportional to the distances of each from the polar of the other.

29. If the circle $x^2+y^2+2gx+2fy+c=0$ cuts the parabola $y^2=4ax$ in four points, the algebraic sum of the ordinates of those points will be zero. (See § 91.)

30. If the normals at the three points P, Q, and R of a parabola meet in a point, then the circle through P, Q, and R will pass through the vertex of the parabola. (§ 128 and Ex. 29 above.)

31. The distances from the origin to the centres of three circles $x^2+y^2-2\lambda x=a^2$ (where a is constant and λ variable) are in geometrical progression; prove that the tangents drawn to them from any point on $x^2+y^2=r^2$ are also in geometrical progression.

32. The polar of P with respect to the circle $x^2+y^2=r^2$ touches the circle $(x-a)^2+(y-b)^2=r_1^2$; prove that the locus of P is the curve given by the equation

$$r_1^2(x^2+y^2)=(ax+by-r^2)^2.$$

33. A tangent is drawn to the circle $(x-a)^2+y^2=b^2$ and a perpendicular tangent to the circle $(x+a)^2+y^2=c^2$; find the locus of their point of intersection, and prove that the bisector of the angle between them always touches one or other of two circles.

34. Show that the equation of the circle whose diameter is the common chord of the two circles

$$x^2+y^2=2ax \text{ and } x^2+y^2=2by$$

is

$$(a^2+b^2)(x^2+y^2)=2ab(bx+ay).$$

35. Prove that the length of the common chord of the two circles whose equations are

$$(x-a)^2+(y-b)^2=r^2 \text{ and } (x-b)^2+(y-a)^2=r^2,$$

is

$$\sqrt{4r^2-2(a-b)^2}.$$

Hence find the condition that the two circles may touch.

36. The polar equation of the circle on the line joining the points (a, α) and (b, β) as diameter is

$$\rho^2-\rho[a\cos(\theta-\alpha)+b\cos(\theta-\beta)]+ab\cos(\alpha-\beta)=0.$$

37. The polars of a point P with respect to two fixed circles meet in the point Q. Prove that the circle on PQ as diameter passes through two fixed points, and cuts both of the given circles orthogonally.

38. Prove that the two circles, which pass through the two points $(0, a)$ and $(0, -a)$ and touch the line $y = mx + b$, will cut orthogonally if

$$b^2 = a^2(2 + m^2).$$

39. Find the coordinates of the centre of the circle inscribed in the triangle the equations of whose sides are

$$3x = 4y, \quad 7x = 24y, \quad \text{and} \quad 5x - 12y = 36.$$

40. O is any point in the plane and OP_1P_2 any chord of a circle which meets the circle in P_1 and P_2. On this chord a point Q is taken such that OQ is equal to (1) the arithmetic, (2) the geometric, and (3) the harmonic mean between OP_1 and OP_2. In each case find the locus of Q.

41. Find the locus of the intersection of the tangent to any circle and the perpendicular let fall on this tangent from a fixed point on the circle.

42. A point moves so that the sum of the squares of its distances from the sides of a regular polygon is constant. Show that its locus is a circle.

43. Show that the locus of the poles of tangents to the parabola $y^2 = 4ax$ with respect to the circle $x^2 + y^2 = 2ax$ is the circle

$$x^2 + y^2 = ax.$$

44. A straight line moves so that the product of the perpendiculars on it from two fixed points is constant. Prove that the locus of the feet of these perpendiculars is a circle, the same for each.

45. A straight line moves so that the sum of the perpendiculars on it from two fixed points is constant. Find the locus of the point midway between the feet of these perpendiculars.

46. O is a fixed point and AP and BQ are two fixed parallel lines; BOA is perpendicular to both and POQ is a right angle. Prove that the locus of the foot of the perpendicular drawn from O on PQ is the circle on AB as diameter.

47. Find the locus of a point from which two circles subtend the same angle.

48. A, B, C, and D are four points in a straight line. Prove that the locus of a point P, such that the angles APB and CPD are equal, is a circle.

49. If two points A and B are harmonic conjugates with respect to C and D, the circles on AB and CD as diameters cut orthogonally.

50. In any circle prove that the perpendicular from any point of it on the line joining the points of contact of two tangents is a mean proportional between the perpendiculars from the point upon the two tangents.

51. From any point on one given circle tangents are drawn to another given circle. Prove that the locus of the middle point of the chord of contact is a third circle.

52. If ABC is an acute-angled triangle, P any point in the plane, the three circular loci,

$$PA^2 = PB^2 + PC^2, \quad PB^2 = PC^2 + PA^2, \quad PC^2 = PA^2 + PB^2,$$

will have their radical centre at the centre of the circle circumscribing the triangle.

53. Prove that all circles touching two fixed circles are orthogonal to one of two other fixed circles.

54. If $S = 0$ and $S' = 0$ are the equations of two circles whose radii are r and r', then the circles

$$\frac{S}{r} \pm \frac{S'}{r'} = 0$$

will intersect at right angles.

55. If two circles cut orthogonally, prove that an indefinite number of pairs of points can be found on their common diameter such that either point has the same polar with respect to one circle that the other has with respect to the other. Also show that the distance between such pairs of points subtends a right angle at one of the points of intersection of the two circles.

56. Show that the equation of the orthogonal circle of three given circles is

$$\begin{vmatrix} g_1, & f_1, & 1 \\ g_2, & f_2, & 1 \\ g_3, & f_3, & 1 \end{vmatrix} (x^2+y^2) + \begin{vmatrix} c_1, & f_1, & 1 \\ c_2, & f_2, & 1 \\ c_3, & f_3, & 1 \end{vmatrix} x + \begin{vmatrix} g_1, & c_1, & 1 \\ g_2, & c_2, & 1 \\ g_3, & c_3, & 1 \end{vmatrix} y - \begin{vmatrix} g_1, & f_1, & c_1 \\ g_2, & f_2, & c_2 \\ g_3, & f_3, & c_3 \end{vmatrix} = 0.$$

57. If AB is a diameter of a given circle, the polar of A with respect to any circle which cuts the given circle orthogonally will pass through B.

58. From the preceding example, considering the orthogonal circle of three given circles as the locus of a point such that its polars with respect to the circles meet in a point, prove that the equation of the circle orthogonal to each of three circles is

$$\begin{vmatrix} x+g_1, & y+f_1, & g_1x+f_1y+c_1 \\ x+g_2, & y+f_2, & g_2x+f_2y+c_2 \\ x+g_3, & y+f_3, & g_3x+f_3y+c_3 \end{vmatrix} = 0.$$

Show that this equation is the same as that given in Ex. 56.

CHAPTER X.

THE ELLIPSE AND HYPERBOLA.

138. *Standard equations of the tangent, polar, and normal to the ellipse and hyperbola.*

It has been shown in § 120 and § 121* that, if the axes of the curve are taken as coordinate axes, the equations of the *central conics* may be written in the standard form

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1.\dagger \qquad (1)$$

Then the coordinates of the foci are $(\pm ae, 0)$; the equations of the directrices are $x = \pm \frac{a}{e}$; the length of the latus rectum is $\frac{2b^2}{a}$; and $e = \frac{\sqrt{a^2 \mp b^2}}{a}$.

For equation (1) formula (6), § 111, gives

$$\frac{xx'}{a^2} \pm \frac{yy'}{b^2} = 1. \qquad (2)$$

Equation (2) is the equation of the polar (§ 113) of the point (x', y') with respect to the central conic (1), which polar is a tangent at the point (x', y') when (x', y') is on the conic.

The equation of the normal at any point (x', y') on the conic (1) is

$$y - y' = \frac{a^2 y'}{\pm b^2 x'}(x - x'), \qquad [(2), \S 85]$$

or

$$\frac{x - x'}{\frac{x'}{a^2}} = \frac{y - y'}{\frac{y'}{\pm b^2}}. \qquad (3)$$

Ex. 1. Find the equations of the central conics when the origin is at either focus; at either vertex; at the point (h, k), the coordinate axes being parallel to the axes of the conic.

Ex. 2. What relation does the line (3) have to the conic when (x', y') is not on the curve?

---

* These sections should now be carefully reviewed.

† We shall use this form of the equation, although the simpler form $ax^2 + by^2 = 1$ is sometimes more convenient. When the double sign $\pm$ or $\mp$ is prefixed to b^2, the upper sign holds for the ellipse and the lower for the hyperbola. All results are true for both curves unless the contrary is expressly stated. Furthermore, results for the ellipse include those for the circle as the special case when $a = b$.

139. *To find the equation of the tangent to the conic*

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1 \tag{1}$$

in terms of its slope m.

Assume the equation of the tangent to be

$$y = mx + c, \tag{2}$$

where m is known, and *c is to be determined* so that (1) and (2) shall intersect in two coincident points (§ 78).

Eliminating y between (1) and (2) gives

$$\frac{x^2}{a^2} \pm \frac{(mx + c)^2}{b^2} = 1,$$

or $$x^2(a^2m^2 \pm b^2) + 2a^2cmx + a^2(c^2 \mp b^2) = 0. \tag{3}$$

The roots of equation (3) will be equal if

$$a^2(c^2 \mp b^2)(a^2m^2 \pm b^2) = a^4c^2m^2.$$

Whence $$c^2 = a^2m^2 \pm b^2. \tag{4}$$

That is, the points of intersection of the straight line and the conic will coincide if

$$c = \pm \sqrt{a^2m^2 \pm b^2}. \tag{5}$$

Hence the line whose equation is

$$y = mx \pm \sqrt{a^2m^2 \pm b^2}, \tag{6}$$

will touch the conic (1) for all values of m.

The double sign before the radical in (6) shows that there are two tangents for every value of m; *i. e.* there are two tangents to a central conic parallel to any given straight line; and these two parallel tangents are equidistant from the center of the conic.

Ex. 1. Derive equation (6) by the method used in § 125.

Ex. 2. In a similar manner show that the equation of the normal to (1) expressed in terms of its slope is

$$y = mx - \frac{m(a^2 \mp b^2)}{\sqrt{a^2 \pm b^2m^2}}.$$

Ex. 3. How many normals can be drawn from a given point to a central conic?

EXAMPLES.

Find the eccentricity, foci, and latus rectum of each of the following conics:

1. $x^2 + 2y^2 = 4$.
2. $4x^2 - 9y^2 = 36$.
3. $4x^2 + y^2 = 8$.
4. $3x^2 - y^2 = 9$.
5. $3(x-1)^2 + 4(y+2)^2 = 1$.
6. $3(y-1)^2 - 4(x+1)^2 = 1$.

Find the equation of an ellipse referred to its axes

7. if the latus rectum is 6 and the eccentricity $\frac{1}{2}$.

8. if the latus rectum is 4 and the minor axis is equal to the distance between the foci.

9. Find the equation of the hyperbola whose foci are the points $(\pm 4, 0)$ and whose eccentricity is $\sqrt{2}$.

10. Find the eccentricity and the equation of the ellipse, if the latus rectum is equal to half the minor axis.

11. Find the equation of the hyperbola with eccentricity 2 which passes through $(-4, 6)$.

12. Find the equation of the ellipse passing through the points $(-2, 2)$ and $(3, -1)$; also the equation of the hyperbola through $(1, -3)$ and $(2, 4)$.

Through how many points can a central conic be made to pass if its axes are given? Why?

13. Find the eccentricity and the equation of a central conic if the foci lie midway between the centre and the vertices; if the vertices lie midway between the centre and the foci.

14. Show that the tangents at the ends of either axis of a central conic are parallel to the other axis; and also that tangents at the ends of any chord through the centre are parallel.

15. Find the equations of the tangents and normals at the ends of the latera recta. Where do they meet the x-axis? One Ans. $y + ex = a$.

16. Show that the line $y = 2x - \sqrt{\frac{7}{6}}$ touches the conic

$$3x^2 - 6y^2 = 1.$$

17. Find the equations of the tangents to the ellipse $x^2 + 4y^2 = 16$ which make angles of 45° and 60° with the x-axis.

18. Show that the directrix is the polar of the focus.

19. If the slope of a moving line remains constant, the locus of its pole with respect to a central conic is a straight line through the centre of the conic.

20. Show that the minor axis is a mean proportional between the major axis and the latus rectum.

21. Show that the ellipse is concave towards both axes, while the hyperbola is concave only towards its transverse axis

22. Show that the line $lx + my = n$ will touch

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1 \quad \text{if} \quad a^2l^2 \pm b^2m^2 = n^2.$$

The line $x \cos \alpha + y \sin \alpha = p$ will touch the same curves if

$$a^2 \cos^2 \alpha \pm b^2 \sin^2 \alpha = p^2.$$

23. Show that the point (x_1, y_1) is inside, on, or outside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

according as

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 <, \ =, \text{ or } > 0.$$

24. Show that the point (x_1, y_1) is inside, on, or outside the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

according as

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 >, \ =, \text{ or } < 0.$$

25. Are the points $(3, \frac{2}{3})$, $(-\frac{7}{2}, -\sqrt{2})$, $(5, -2)$ inside or outside of the curves $5x^2 \pm 9y^2 = 50$?

26. Find the equations and the coordinates of the points of contact of tangents to $b^2x^2 \pm a^2y^2 = a^2b^2$ which make equal intercepts on the axes.

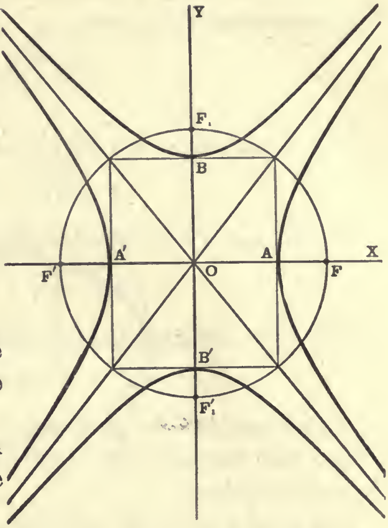
27. If the normal at the end of the latus rectum of an ellipse passes through the extremity of the minor axis, show that the eccentricity is given by the equation $e^4 + e^2 = 1$. Find the corresponding equation for the hyperbola and interpret the result.

28. If any ordinate MP of a central conic is produced to meet the tangent at the end of the latus rectum through the focus F in Q, show that $FP = MQ$.

29. Find the product of the segments into which a focal chord of a central conic is divided by the focus.

30. Two tangents can be drawn to a central conic from any point, which will be real, coincident, or imaginary according as the point is outside, on, or inside the conic. Thus determine which is the *inside* of a hyperbola.

140. *Conjugate Hyperbolas.*



The two hyperbolas whose equations are

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \qquad (1)$$

and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1,$

or $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1, \qquad (2)$

are so related that the transverse axis of the one is the conjugate axis of the other.

The two hyperbolas are then said to be be **conjugate** to one another.

The eccentricity of the *Conjugate Hyperbola** is $e_1=\frac{\sqrt{b^2+a^2}}{b}$; the coordinates of its foci are $(0, \pm be_1)$; the equations of its directrices are $y=\pm\frac{b}{e_1}$; and its latus rectum is $\frac{2a^2}{b}$.

When $a=b$ equations (1) and (2) become, respectively,

$$\left.\begin{aligned} x^2-y^2&=a^2,\\ \text{and} \quad y^2-x^2&=a^2. \end{aligned}\right\} \qquad (3)$$

Hence if a hyperbola is equilateral or rectangular [§ 121, (15)], its conjugate is also rectangular.

Two conjugate hyperbolas are not, in general, similar (§ 116), *i. e.* of the same shape, but two conjugate rectangular hyperbolas are similar and equal.

* The hyperbola (2) is usually called the ***Conjugate Hyperbola***, while (1) is called the ***Original***, or ***Primary Hyperbola***. It is to be noticed that the equation of the conjugate hyperbola is found by ***changing the sign of one member*** of the equation of the primary hyperbola. Likewise the equation of the conjugate ellipse is found to be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1.$$

Hence the conjugate of an ellipse is imaginary.

The student should compare the results given above with the conjugate properties discussed in § 116.

141. *To find the locus of the point of intersection of two perpendicular tangents to the conic*

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1. \tag{1}$$

The equation of any tangent to (1) may be written (§ 139)

$$y = mx + \sqrt{a^2m^2 \pm b^2}. \tag{2}$$

If this line (2) passes through (x_1, y_1), we shall have

$$y_1 = mx_1 + \sqrt{a^2m^2 \pm b^2},$$

which when rationalized becomes

$$(x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 \mp b^2) = 0. \tag{3}$$

This equation is a quadratic in m whose two roots are the slopes of the two tangents which pass through the point (x_1, y_1), whose locus is required.

Let m_1 and m_2 be the two roots of (3); then (§ 91)

$$m_1m_2 = \frac{y_1^2 \mp b^2}{x_1^2 - a^2}.$$

The two tangents will be at right angles if $m_1m_2 = -1$ (§ 48); *i. e.* if

$$\frac{y_1^2 \mp b^2}{x_1^2 - a^2} = -1,$$

or

$$x_1^2 + y_1^2 = a^2 \pm b^2. \tag{4}$$

The required locus is, therefore, the circle

$$x^2 + y^2 = a^2 \pm b^2, \tag{5}$$

which is called the **Director Circle** of the conic.

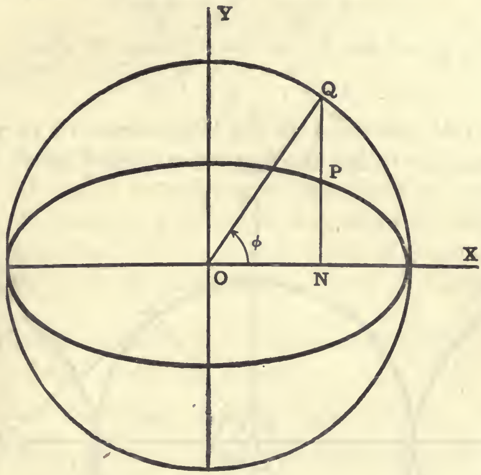
COR. I. *If $a < b$, the director circle of a hyperbola is imaginary. Hence one of the director circles of two conjugate hyperbolas is always imaginary.*

COR. II. *The director circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the foci of the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$, and vice versa.*

What does this mean when $a = b$?

142. *Auxiliary Circle, and Eccentric Angle.*

I. The circle described on the major axis of an ellipse as diameter is called the **Auxiliary Circle.**



If the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1}$$

the equation of the auxiliary circle will be

$$x^2 + y^2 = a^2. \tag{2}$$

If the ordinate NP of any point P on the ellipse is produced to meet the auxiliary circle in Q, then P and Q are called **Corresponding Points.**

Let $P(x_1, y_1)$ and $Q(x_1, y_2)$ be any two corresponding points; then, since these points are on (1) and (2), respectively,

$$y_1 = \frac{b}{a}\sqrt{a^2 - x_1^2}, \tag{3}$$

and

$$y_2 = \sqrt{a^2 - x_1^2}. \tag{4}$$

$$\therefore \quad \frac{y_1}{y_2} = \frac{b}{a}. \tag{5}$$

That is, the ordinates of corresponding points are in a constant ratio.

Ex. Show that the area of the ellipse is πab.

The angle XOQ is called the **Eccentric Angle** of the point P. It will be denoted by φ.

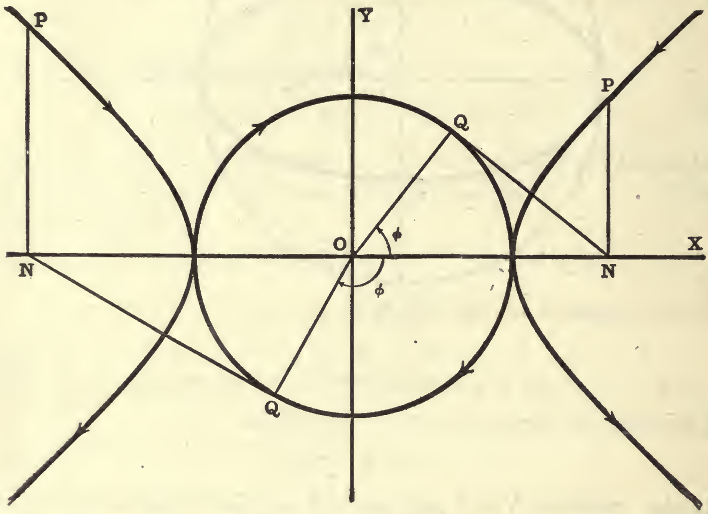
Then the coordinates of the point Q are

$$x_1 = a \cos \varphi, \quad y_2 = a \sin \varphi.$$

Since $y_1 = \frac{b}{a} y_2 = b \sin \varphi$, the coordinates of P are

$$x_1 = a \cos \varphi, \quad y_1 = b \sin \varphi. \qquad (6)$$

II. The circle described on the transverse axis of a hyperbola as diameter may be called the *Auxiliary Circle* * of the hyperbola.



Let $P(x, y)$ be any point on the hyperbola and NP its ordinate. Draw NQ tangent to the auxiliary circle at Q, so that P and Q are on the same side of the transverse axis when P is on the right branch, and on opposite sides when P is on the left branch of the curve. Then, as P describes the complete hyperbola in the direction indicated by the arrows, Q will move consecutively around the circle in the direction indicated. Thus, for every position of P on the hyperbola, there is one and only one corresponding position of Q on the circle.

Hence P and Q may be called *Corresponding Points*, and the angle $XOQ \equiv \varphi$ may be called the *Eccentric Angle* * of the point P.

* The terms "Auxiliary Circle" and "Eccentric Angle" are not generally used with reference to the hyperbola, but are here employed in order to express the coordinates of any point on the curve in terms of a single variable ϕ.

Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{7}$$

Then

$$ON = x = a \sec \varphi, \tag{8}$$

which substituted in (7) gives

$$y = b \tan \varphi. \tag{9}$$

That is, P is the point $(a \sec \varphi, b \tan \varphi)$.

Similarly, $x^2 + y^2 = b^2$ is the auxiliary circle of the conjugate hyperbola, and $(a \tan \varphi, b \sec \varphi)$ is any point on the curve if φ is measured *clockwise* from the positive end of the y-axis; if φ is measured from the x-axis the point is $(a \cot \varphi, b \csc \varphi)$.

143. *To find the equation of the straight line joining two points on a conic whose eccentric angles are φ and φ'.*

If the conic is an ellipse, the points are (§ 142)

$$(a \cos \varphi, b \sin \varphi) \quad \text{and} \quad (a \cos \varphi', b \sin \varphi').$$

The equation of the line through these points is [(3), § 47]

$$\frac{x - a \cos \varphi}{a \cos \varphi - a \cos \varphi'} = \frac{y - b \sin \varphi}{b \sin \varphi - b \sin \varphi'}. \tag{1}$$

Since $\cos \varphi - \cos \varphi' = -2 \sin \tfrac{1}{2}(\varphi + \varphi') \sin \tfrac{1}{2}(\varphi - \varphi')$

and $\sin \varphi - \sin \varphi' = 2 \cos \tfrac{1}{2}(\varphi + \varphi') \sin \tfrac{1}{2}(\varphi - \varphi')$,

equation (1) reduces to

$$\frac{\dfrac{x}{a} - \cos \varphi}{-2 \sin \tfrac{1}{2}(\varphi + \varphi')} = \frac{\dfrac{y}{b} - \sin \varphi}{2 \cos \tfrac{1}{2}(\varphi + \varphi')} \tag{2}$$

$$\therefore \quad \frac{x}{a} \cos \tfrac{1}{2}(\varphi + \varphi') + \frac{y}{b} \sin \tfrac{1}{2}(\varphi + \varphi') = \cos \tfrac{1}{2}(\varphi - \varphi'), \tag{3}$$

which is the required equation.

In like manner the equation of the line joining the points $(a \sec \varphi, b \tan \varphi)$ and $(a \sec \varphi', b \tan \varphi')$ on the hyperbola can be shown to be

$$\frac{x}{a} \cos \tfrac{1}{2}(\varphi - \varphi') - \frac{y}{b} \sin \tfrac{1}{2}(\varphi + \varphi') = \cos \tfrac{1}{2}(\varphi + \varphi'). \tag{4}$$

To find the equation of the tangent at the point φ, we put $\varphi' = \varphi$ in equations (3) and (4), and we obtain for the ellipse

$$\frac{x}{a}\cos\varphi + \frac{y}{b}\sin\varphi = 1, \tag{5}$$

and for the hyperbola

$$\frac{x}{a}\sec\varphi - \frac{y}{b}\tan\varphi = 1. \tag{6}$$

From equation (3) we see that if the sum of the eccentric angles of two points on an ellipse is constant and equal to 2α, the equation of the line joining them is

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = \cos\tfrac{1}{2}(\varphi - \varphi'). \tag{7}$$

Hence the chord is always parallel to the tangent

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1. \tag{8}$$

Conversely, in a system of parallel chords of an ellipse, the sum of the eccentric angles of the extremities of any chord is constant.

Similarly from equation (4) we see that if the sum of the eccentric angles of two points on a hyperbola is constant and equal to 2α, the equation of the chord through these points is

$$\frac{x}{a}\cos\tfrac{1}{2}(\varphi - \varphi') - \frac{y}{b}\sin\alpha = \cos\alpha, \tag{9}$$

and therefore the chord, and the tangent at the point α, viz.,

$$\frac{x}{a} - \frac{y}{b}\sin\alpha = \cos\alpha, \tag{10}$$

always meet the y-axis in the same fixed point.

144. *To find the equation of the normal at any point in terms of the eccentric angle of the point.*

Let $(a\cos\varphi, b\sin\varphi)$ (§ 142) be any point on the ellipse; then the slope of the tangent at the point φ is $-\dfrac{b\cos\varphi}{a\sin\varphi}$. [§ 143, (5).]

Hence the equation of the normal at φ is [(2), § 85]

$$y - b \sin \varphi = \frac{a \sin \varphi}{b \cos \varphi}(x - a \cos \varphi), \qquad (1)$$

or

$$\frac{ax}{\cos \varphi} - \frac{by}{\sin \varphi} = a^2 - b^2. \qquad (2)$$

Similarly we find the equation of the normal to the hyperbola at the point $(a \sec \varphi, b \tan \varphi)$ to be

$$\frac{ax}{\sec \varphi} + \frac{by}{\tan \varphi} = a^2 + b^2. \qquad (3)$$

EXAMPLES.

1. Show that the equation of the locus of the foot of the perpendicular from the centre of a conic on a tangent is $\rho^2 = a^2 \cos^2 \theta \pm b \sin^2 \theta$. (See Ex. 22, p. 224.)

2. An ellipse slides between two perpendicular lines; show that the locus of the centre is a circle. (§ 141.)

3. Show that, for all values of b, tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at points having the same abscissa meet the x-axis in the same point. Hence show how a tangent can be drawn to an ellipse from any point on the x-axis.

4. Two tangents are drawn to a conic from any point on the auxiliary circle; prove that the sum of the squares of the chords which the auxiliary circle intercepts on them is equal to the square of the line joining the foci. (See (9), § 145.)

5. If the points Q and Q' are taken on the minor axis of a conic such that $QO = OQ' = OF$, where O is the centre and F a focus, show that the sum of the squares of the perpendiculars from Q and Q' on any tangent to the conic is constant.

6. If p is the length of the perpendicular from the centre on the chord joining the extremities of two perpendicular diameters of an ellipse, show that

$$p = \frac{ab}{\sqrt{a^2 + b^2}}.$$

7. A line is drawn through the centre of a conic parallel to the focal radius of a point P and meeting the tangent at P in Q. Find the locus of Q.

8. From one focus of an ellipse a perpendicular is drawn to any tangent and produced to an equal distance on the other side. Show that its terminus Q is in the straight line through the other focus and the point of tangency. Also find the locus of Q.

9. Show that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angles differ by a constant is an ellipse.

[If the tangents at $\phi + \alpha$ and $\phi - \alpha$ meet at (x', y'), then $\frac{x'}{a} = \cos\phi \sec\alpha$, $\frac{y'}{b} = \sin\phi \sec\alpha$. Eliminate ϕ for the locus.]

What is the corresponding theorem for the hyperbola?

10. The point $P(-3, -1)$ is on the ellipse $x^2 + 3y^2 = 12$; find the corresponding point on the auxiliary circle, and also find the eccentric angle of P.

11. The polar of a point P with respect to an ellipse cuts the minor axis in A; and the perpendicular from P to its polar cuts the polar in B and the minor axis in C. Show that the circle through A, B, and C will pass through the foci.

[Prove $AO \cdot OC = F'O \cdot OF$, where O is the centre.]

12. Prove that the circle on any focal radius as diameter touches the auxiliary circle.

13. Prove that the line $lx + my + n = 0$ is normal to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{if} \quad \frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}.$$

[Compare $lx + my + n = 0$ with $\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2$. (See § 63.)]

14. Prove that a circle can be drawn through the foci of a hyperbola and the points in which any tangent meets the tangents at the vertices.

15. The perpendicular from the focus of an ellipse upon any tangent and the line joining the centre to the point of contact meet on the corresponding directrix.

16. If Q is the point on the auxiliary circle corresponding to the point P on the ellipse, the normals at P and Q will meet on the circle

$$x^2 + y^2 = (a + b)^2.$$

17. Prove that the focal radius of any point on a central conic and the perpendicular from the centre on the tangent at that point meet on a circle whose centre is the focus and whose radius is the semi-major axis.

18. If $P(x', y')$ is a point on an ellipse, prove that the angle between the tangent at P and the focal radius of P is $\tan^{-1}\frac{b^2}{aey'}$.

19. If Q is the point on the auxiliary circle corresponding to the point P on the ellipse, show that the perpendicular distances of the foci F, F' from the tangent at Q are equal to FP and $F'P$ respectively.

20. If a polar with respect to a central conic touches the circle $x^2+y^2=b^2$, what is the locus of the pole?

21. Show that the polar of any point on either of the curves

$$\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1$$

with respect to the other touches the first curve.

22. The polar of any point P on either of the curves

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\pm 1$$

with respect to the other touches the first curve at the opposite extremity of the diameter through P.

23. The polars of any point with respect to the two conics

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\pm 1$$

are parallel and equidistant from the centre.

24. The product of the focal radii of any point on a rectangular hyperbola is equal to the square of the distance from the centre to that point.

25. The distance of any point Q from the centre of a rectangular hyperbola varies inversely as the perpendicular from the centre upon the polar of Q.

26. If the normal at any point P of a rectangular hyperbola meets the axes in N and N', and O is the centre, then $PN=PN'=OP$.

27. Chords are drawn through the end of an axis of an ellipse. Find the locus of their middle points.

28. Find the locus of the pole of a chord of

$$\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1,$$

which subtends a right angle (1) at the centre, (2) at the vertex, and (3) at the focus of the curve.

29. Show that the area of a triangle inscribed in an ellipse is

$$\tfrac{1}{2}ab[\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)],$$

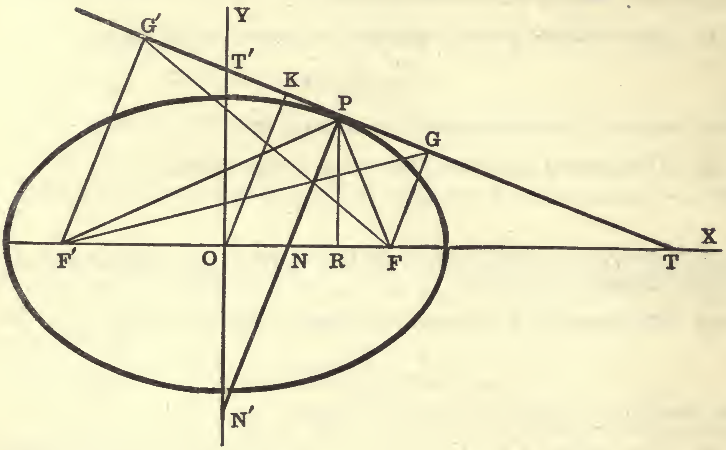
where α, β, γ are the eccentric angles of the vertices.

Prove also that its area is to the area of the triangle formed by the corresponding points on the auxiliary circle as $b:a$; and hence its area is a maximum when the latter is equilateral; *i. e.* when

$$\alpha\sim\beta=\beta\sim\gamma=\gamma\sim\alpha=\tfrac{2}{3}\pi.$$

30. If P is a point on the director circle of an ellipse, and O the centre, the product of the distances of O and P from the polar of P with respect to the ellipse is constant.

145. *Geometric properties of the ellipse and hyperbola.*



Let the tangent at $P(x', y')$ meet the axes in T and T'; let the normal at P meet the axes in N and N'; let RP be the ordinate of P and F, F' the foci of the conic.

Draw FG, $F'G'$, and OK perpendicular to the tangent PT.

Then $OT = \frac{a^2}{x'}$, $\quad OT' = \frac{\pm b^2}{y'}$. [(2), § 138.] (1)

$\therefore$ *Subtangent* $= RT = \frac{a^2 - x'^2}{x'}$. (2)

$ON = e^2 x'$, $\quad ON' = \frac{e^2}{e^2 - 1} y'$. [(3), § 138.] (3)

$\therefore$ *Subnormal* $= NR = (1 - e^2)x'$. (4)

$OK \cdot NP = FG \cdot F'G' = \pm b^2$. (5)

$ON \cdot OT = a^2 e^2 = OF^2$. (6)

$PN \cdot PN' = FP \cdot F'P = \pm(a^2 - e^2 x'^2)$. (§§ 120–1.) (7)

$F'G$ and FG' bisect PN. (8)

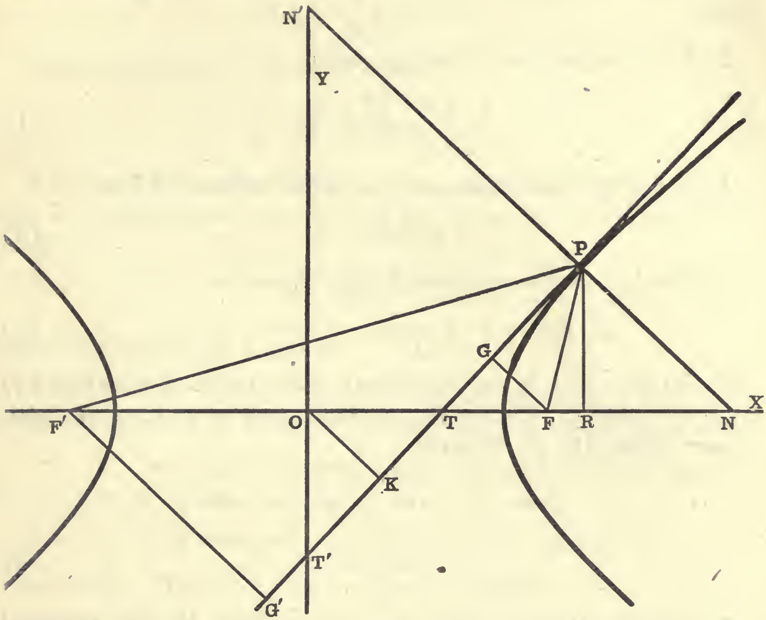
The locus of the points G and G' is the circle

$x^2 + y^2 = a^2$. [Use (6), § 139.] (9)

$$\frac{F'N}{NF} = \frac{F'O + ON}{OF - ON} = \frac{ae + e^2 x'}{ae - e^2 x'} = \frac{a + ex'}{a - ex'}.$$

$\therefore \frac{F'N}{NF} = \frac{F'P}{\pm FP}$ (§§ 120–1.) (10)

Therefore the tangent and the normal bisect the angles between the focal radii FP and $F'P$.



Hence, if an ellipse and a hyperbola have the same foci, the *tangent* and the *normal* to one of the curves at any one of their four common points are, respectively, the *normal* and the *tangent* to the other. That is, the two conics intersect orthogonally.

Conics having the same foci are called **Confocal Conics.**

Ex. 1. Explain what would happen if a light were placed at one focus of an ellipse; a hyperbola.

Ex. 2. What is the limit of ON, ON', and NR as $x' = a$? as $x' = 0$?

Ex. 3. Show that equations (1), (3), and $OK \cdot NP = b^2$ are also true when P is any point, TT' the polar of P, and PN is perpendicular to TT'.

Ex. 4. Show that the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

represents a system of confocal conics, where λ is the arbitrary parameter; prove analytically that confocal conics intersect at right angles.

146. Def. An **Asymptote*** is a line which meets a curve in *two* points at infinity, but which is itself not altogether at infinity.

To find the asymptotes of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \tag{1}$$

As in § 139, the abscissas of the points where the line

$$y=mx+c \tag{2}$$

meets the hyperbola are given by the equation

$$x^2(a^2m^2-b^2)+2a^2cmx+a^2(c^2+b^2)=0. \tag{3}$$

If the line (2) is an asymptote, both roots of equation (3) must be infinite. Hence the coefficients of x^2 and x must both be zero (§ 98, III.). That is,

$$a^2cm=0, \quad \text{and} \quad a^2m^2-b^2=0.$$

$$\therefore \quad c=0, \quad \text{and} \quad m=\pm\frac{b}{a}. \tag{4}$$

Substituting these values in (2), we have for the required equations of the asymptotes

$$y=\pm\frac{b}{a}x, \tag{5}$$

or expressed in one equation (§ 53)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0. \tag{6}$$

Therefore the hyperbola has two asymptotes, both passing through the centre and equally inclined to the transverse axis.

The equations of the asymptotes to a hyperbola can also be found by considering them the limiting positions of the tangent as the point of contact moves off to infinity.

The equation of the tangent to (1) at (x', y') is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1. \tag{7}$$

* See note under § 116.

Since the point (x', y') is on the conic (1), we have

$$y' = \pm \frac{b}{a}\sqrt{x'^2 - a^2}.$$

Hence equation (7) may be written

$$\frac{x}{a^2} \pm \frac{y}{ab}\sqrt{1 - \frac{a^2}{x'}} = \frac{1}{x'}. \tag{8}$$

If now the point of contact (x', y') moves off to infinity, so that $x' = \infty$, the limiting position of the line (8) is given by the equation

$$\frac{x}{a} \pm \frac{y}{b} = 0, \tag{9}$$

which is the same as equation (5) above.

COR. I. *Two conjugate hyperbolas have the same asymptotes, which are the diagonals of the rectangle formed by the tangents at their vertices.*

COR. II. *A straight line parallel to an asymptote will meet the conic in one point at infinity.*

For, if c is not zero, only *one* root of (3) is infinite.

COR. III. *The line $y = mx$ will cut the hyperbola in real or imaginary points according as $m <$ or $> \frac{b}{a}$. It will meet either the hyperbola or its conjugate in real points for all values of m.*

COR. IV. *The asymptotes of an ellipse are imaginary.*

For, if we change the sign of b^2, the values of m for infinite roots in (3) become imaginary.

It is to be noticed that the equations of two conjugate hyperbolas and the equation of their common asymptotes, viz.,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1 \quad \text{and} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

differ only in their constant terms. Moreover, this must always be true; for any transformation of coordinates will affect the first members of these equations in precisely the same way. Hence the new equations will differ only in their constant terms (not usually by unity); and the value of the constant in the equation of the asymptotes will be equal to half the sum of the constants in the equations of the two hyperbolas. (*Cf.* § 117.)

147. *Similar and Coaxial Conics.*

Since $a\sqrt{K}$ and $b\sqrt{K}$ are the semi-axes of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=K, \tag{1}$$

its eccentricity is given by the equation

$$e=\frac{\sqrt{a^2K-b^2K}}{a\sqrt{K}}=\frac{\sqrt{a^2-b^2}}{a}. \qquad (\S\ 138.)$$

That is, the eccentricity of (1) is the same as the eccentricity of the ellipse represented by the standard equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{2}$$

Therefore the two ellipses (1) and (2) are similar (§ 116) whatever value may be assigned to K.

Conics having their axes on the same lines are said to be *Coaxial.*

Hence if K is an arbitrary parameter, (1) will represent an infinite system of similar and coaxial ellipses.

For any particular value of K the equations

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\pm K \tag{3}$$

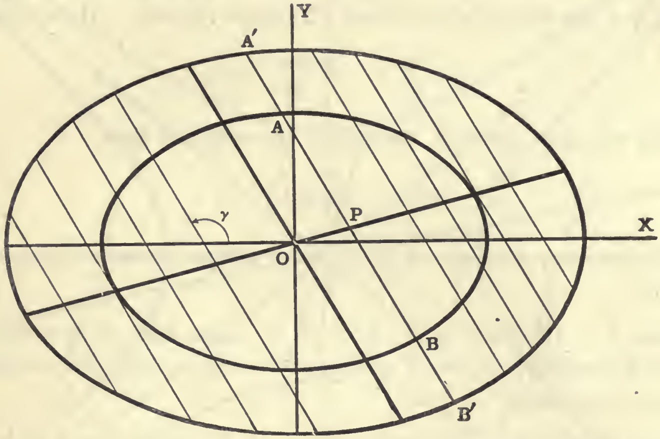
represent a pair of conjugate hyperbolas (§ 140).

If, however, K is arbitrary, equations (3) will give (as in the case of the ellipse) a system of similar and coaxial hyperbolas, together with their corresponding conjugate hyperbolas, which are also similar. It follows from § 146 that these two infinite systems of hyperbolas all have the same asymptotes. Moreover, the asymptotes are the limit which both systems approach as K becomes zero. Thus two intersecting lines are not only one of a system of similar and coaxial hyperbolas, but may also be regarded as a pair of *self-conjugate hyperbolas.*

It is also to be noticed that although both axes of two intersecting lines are zero, their ratio in the limit is the tangent of half the angle between the lines.

Cor. *The axes of similar conics are proportional.*

148. *To find the locus of the middle points of a system of parallel chords of a central conic.*



I. Let AB be any one of a system of parallel chords of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=K. \qquad (1)$$

Let $P(x', y')$ be the middle point of AB, and γ its inclination to the x-axis.

Then the equation of AB may be written [§ 46, (4)]

$$\frac{x-x'}{\cos\gamma}=\frac{y-y'}{\sin\gamma}=r,$$

or
$$x=x'+r\cos\gamma, \quad y=y'+r\sin\gamma, \qquad (2)$$

where r is the distance from (x', y') to any point (x, y) on the line.

If the point (x, y) is on the ellipse, these values (2) may be substituted in equation (1); this gives

$$\frac{(x'+r\cos\gamma)^2}{a^2}+\frac{(y'+r\sin\gamma)^2}{b^2}=K,$$

or
$$\left(\frac{\cos^2\gamma}{a^2}+\frac{\sin^2\gamma}{b^2}\right)r^2+2\left(\frac{x'\cos\gamma}{a^2}+\frac{y'\sin\gamma}{b^2}\right)r$$
$$+\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-K=0. \qquad (3)$$

The values of r found by solving this quadratic equation are the lengths of the lines PA and PB, which can be drawn from P

along AB to the ellipse. Since P is the *middle* point of the chord, these two values of r must be *equal in magnitude and opposite in sign; i. e.* the sum of the roots of (3) must be zero. Hence (§ 91)

$$\frac{x' \cos\gamma}{a^2} + \frac{y' \sin\gamma}{b^2} = 0. \tag{4}$$

The required locus is, therefore, the straight line

$$y = -\frac{b^2}{a^2} \cot\gamma \cdot x. \tag{5}$$

Hence every *diameter* (§ 126) of an ellipse passes through the centre.

COR. I. *All chords intercepted on the same line, or on a series of parallel lines, by a system of similar and coaxial ellipses are bisected by the same diameter.*

Since equation (5) is independent of K, the locus of P is the same whatever value may be given to K in (1). (§ 147.)

COR. II. *If a straight line meets each of two similar and coaxial ellipses in two real points, the two portions of the line intercepted between them are equal; i. e.* $A'A = BB'$.

COR. III. *Chords of an ellipse which are tangent to a similar and coaxial ellipse are bisected at the point of contact.*

COR. IV. *The tangent at either extremity of any diameter is parallel to the chords bisected by that diameter.*

II. In like manner, if γ is the inclination to the x-axis of a system of parallel chords of the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm K, \tag{6}$$

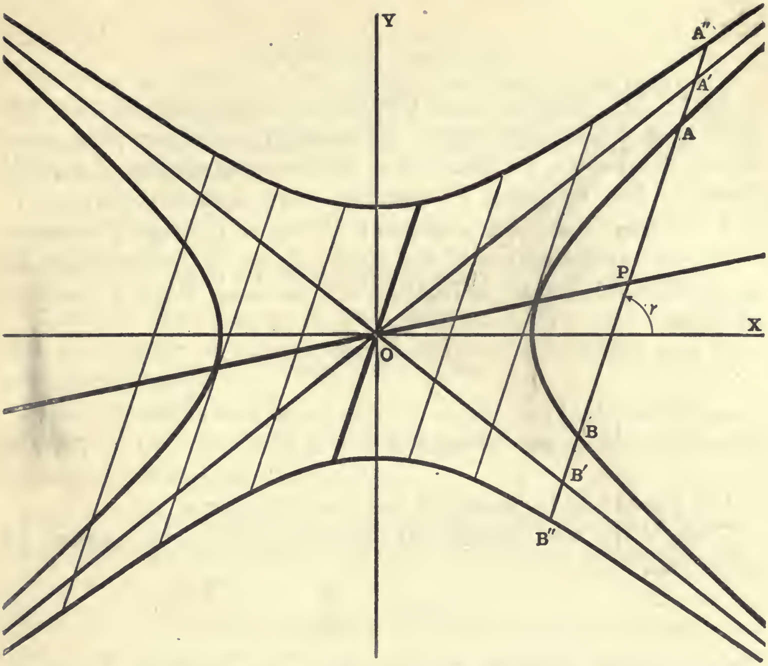
we find the locus of the middle points of the chords to be the straight line

$$y = \frac{b^2}{a^2} \cot\gamma \cdot x, \tag{7}$$

for all values of K, *including the case* $K = 0$.

Hence all diameters of a hyperbola pass through the centre.

The preceding corollaries apply also to similar and coaxial hyperbolas.



COR. V. *Chords intercepted on the same line, or on a system of parallel lines, by two conjugate hyperbolas, and their asymptotes, are bisected by the same diameter.*

COR. VI. *If a straight line meets each of two conjugate hyperbolas in real points, the two portions of the line intercepted between the curves are equal. The portions intercepted between either hyperbola and the asymptotes are also equal; i. e.* $A''A = BB''$ *and* $A'A = BB'$. *Hence the part of a tangent to a hyperbola included between the two branches of its conjugate, and also the part included between its asymptotes, are bisected at the point of contact.*

Ex. 1. Find the locus of the middle points of chords of the ellipse

$$4x^2 + 9y^2 = 36$$

parallel to $3x - 2y = 1$.

Ex. 2. Find the equation of the chord of the hyperbola

$$25x^2 - 16y^2 = 400$$

which is bisected at the point $(5, -3)$.

Ex. 3. Find the equation of the chord of the ellipse $4x^2 + 8y^2 = 32$ which is bisected at the point $(-2, 1)$.

Conjugate Diameters.

149. We have seen in § 148 that all diameters of a central conic pass through the centre. Conversely, *every chord which passes through the centre is a diameter, i. e.* bisects some system of parallel chords. For, by giving γ a suitable value, equations (5) and (7) of § 148 may be made to represent any chord through the centre.

If γ' is the inclination to the x-axis of the diameter which bisects all chords whose inclination is γ, we have, from (5) and (7) of § 148,

$$\tan \gamma' = \mp \frac{b^2}{a^2} \cot \gamma,$$

or

$$\tan \gamma \tan \gamma' = \mp \frac{b^2}{a^2}. \tag{1}$$

Let $y = mx$ and $y = m'x$ be any two diameters.

Then, if the first bisects all chords parallel to the second, we have from (1)

$$mm' = \mp \frac{b^2}{a^2}. \tag{2}$$

Since this is the only condition that must hold in order that the second may bisect all chords parallel to the first, it follows that, *if one diameter of a conic bisects all chords parallel to a second, the second diameter will also bisect all chords parallel to the first.*

Def. Two diameters, so related that each bisects every chord parallel to the other, are called **Conjugate Diameters.***

For example, the axes are a pair of conjugate diameters.

From equation (2) we see that the slopes of two conjugate diameters of an ellipse have *opposite* signs, whereas in the hyperbola the signs are the *same.* (See figures under § 148.)

If $m < \frac{b}{a}$, then $m' > \frac{b}{a}$, *numerically.*

Hence conjugate diameters of an ellipse are separated by the axes, and also by the lines $ay = \pm bx$; while conjugate diameters of a hyperbola are separated by the asymptotes, but not by the axes.

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* It is evident that none but central conics can have conjugate diameters, since in the parabola all diameters have the same direction (§ 126).

If $m = \frac{b}{a}$, then $m' = -\frac{b}{a}$ in the ellipse.

The two diameters are then equally inclined to the major axis, and, from the symmetry of the curve, the two diameters are equal in length. The equations of the *equal conjugate diameters* of an ellipse are, therefore,

$$y = \pm \frac{b}{a}x. \tag{3}$$

If $m = \pm \frac{b}{a}$, then in the hyperbola $m' = \pm \frac{b}{a}$, respectively.

Therefore equi-conjugate diameters of a hyperbola coincide with *an asymptote*, so that an asymptote may be regarded as a self-conjugate diameter.

The equi-conjugate diameters of a conic, therefore, in all cases coincide in direction with the diagonals of the rectangle formed by tangents at the ends of its axes.

Cor. I. *If two diameters are conjugate with respect to one of two conjugate hyperbolas, they will be conjugate with respect to the other also.* [(6) and (7), § 148.]

Cor. II. *One of two conjugate diameters of a hyperbola meets the curve in real points, and the other meets the conjugate hyperbola in real points.* (Cor. III., § 146.)

For this reason we will call the extremities of any diameter of a hyperbola the points in which it cuts either the primary or the conjugate hyperbola, as the case may be; and the length of the diameter will be the distance between these points.

Cor. III. *Tangents at the ends of any diameter are parallel to the conjugate diameter.*

Ex. 1. Write down the equations of the diameters conjugate to

$$x - y = 0, \quad x + y = 0, \quad by = ax, \quad ay = bx.$$

Ex. 2. In the ellipse $2x^2 + 4y^2 = 8$, find two conjugate diameters, one of which bisects the chord $x + 2y = 2$.

Ex. 3. Find the equation of the diameter of the hyperbola

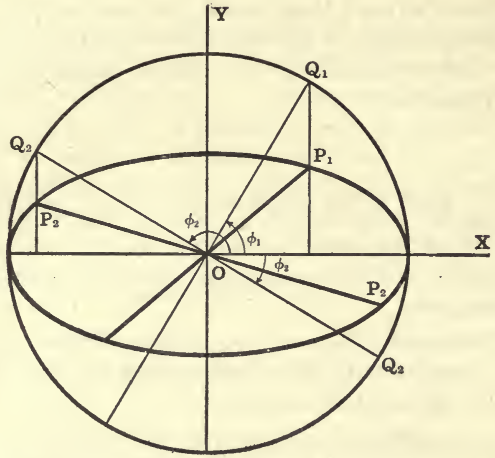
$$16x^2 - 9y^2 = 144$$

conjugate to $x + 2y = 0$.

Ex. 4. Find two conjugate diameters of the ellipse $4x^2 + 25x^2 = 100$, one of which passes through the point $(3, -1)$.

Ex. 5. Find the equation of the chord of the hyperbola $x^2 - y^2 = 16$, whose middle point is $(-2, 3)$.

150. *Given the extremity of any diameter, to find the extremities of the conjugate diameter.*



I. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be the extremities of two conjugate diameters of an ellipse.

Then the equations of OP_1 and OP_2 are

$$y = \frac{y_1}{x_1}x \quad \text{and} \quad y = \frac{y_2}{x_2}x.$$

$$\therefore \quad \frac{y_1 y_2}{x_1 x_2} = -\frac{b^2}{a^2}, \qquad [(1), \S\, 149]$$

or $$\frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} = 0. \qquad (1)$$

Let φ_1, φ_2 be the eccentric angles of P_1, P_2, respectively.

Then $$x_1 = a \cos\varphi_1, \qquad y_1 = b \sin\varphi_1,$$

$$x_2 = a \cos\varphi_2, \qquad y_2 = b \sin\varphi_2. \qquad (\S\, 142,\ \text{I.})$$

Substituting these values in (1), we have

$$\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2 \equiv \cos(\varphi_1 \sim \varphi_2) = 0. \qquad (2)$$

$$\therefore \quad \varphi_1 \sim \varphi_2 = 90°.$$

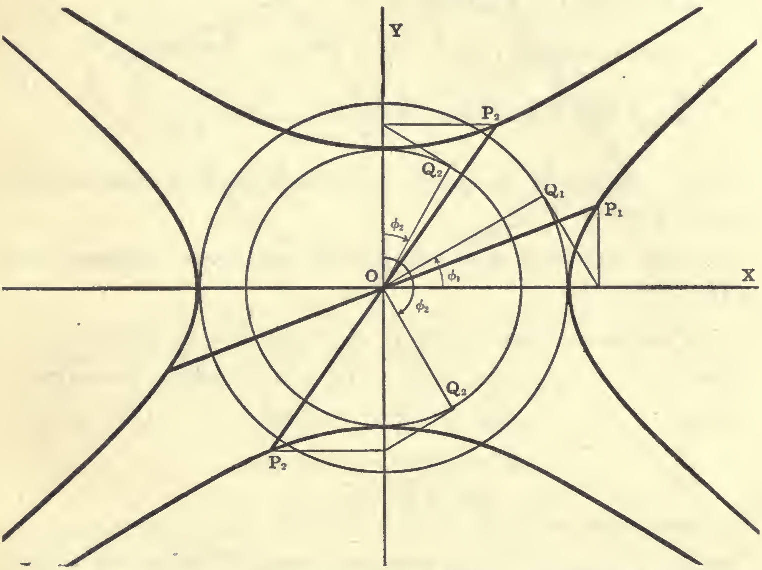
That is, the eccentric angles of the extremities of two conjugate diameters of an ellipse differ by a right angle. Hence the corresponding diameters OQ_1, OQ_2 of the auxiliary circle are perpendicular to one another.

Since $$\varphi_2 = \varphi_1 \pm 90°,$$

$$\sin\varphi_2 = \pm\cos\varphi_1, \qquad \cos\varphi_2 = \mp\sin\varphi_1.$$

Therefore the extremities of two conjugate diameters of an ellipse may be written

$$\left.\begin{array}{l} P_1(a\cos\varphi_1, b\sin\varphi_1) \text{ and } P_2(\mp a\sin\varphi_1, \pm b\cos\varphi_1), \\ \text{or} \quad P_1(x_1, y_1) \text{ and } P_2\left(\mp\frac{a}{b}y_1, \pm\frac{b}{a}x_1\right). \end{array}\right\} \quad (3)$$



II. If P_1, P_2 are the extremities of two conjugate diameters of a hyperbola, equation (1) becomes

$$\frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2} = 0. \quad (4)$$

Then from § 143, II., and § 149, Cor. II., we also have

$$x_1 = a\sec\varphi_1, \quad y_1 = b\tan\varphi_1,$$

$$x_2 = a\tan\varphi_2, \quad y_2 = b\sec\varphi_2.$$

Substituting these values in (4) gives

$$\sec\varphi_1\tan\varphi_2 - \tan\varphi_1\sec\varphi_2 = 0, \quad (5)$$

or $$\frac{\sin\varphi_2}{\cos\varphi_1\cos\varphi_2} - \frac{\sin\varphi_1}{\cos\varphi_1\cos\varphi_2} = 0. \quad (6)$$

$$\therefore \quad \varphi_2 = \varphi_1, \quad \text{or} \quad \pi - \varphi_1. \quad (7)$$

That is, the eccentric angles of the ends of two conjugate diameters of a hyperbola are either equal or supplementary. Therefore the corresponding diameters OQ_1, OQ_2 of the auxiliary circles are equally inclined to the transverse axes of the two conjugate hyperbolas.

Since $\tan \varphi_2 = \pm \tan \varphi_1$. and $\sec \varphi_2 = \pm \sec \varphi_1$,

the extremities of any two conjugate diameters of a hyperbola may be expressed in the form

$$\left.\begin{array}{ll} & P_1(a \sec \varphi_1, b \tan \varphi_1) \text{ and } P_2(\pm a \tan \varphi_1, \pm b \sec \varphi_1), \\ \text{or} & P_1(x_1, y_1) \text{ and } P_2\left(\pm \frac{a}{b} y_1, \pm \frac{b}{a} x_1\right). \end{array}\right\} \quad (8)$$

151. *The sum of the squares of two conjugate semi-diameters of an ellipse is constant.*

Let the extremities of any two conjugate diameters be [§ 150, (3)]

$$P_1(a \cos \varphi, b \sin \varphi) \quad \text{and} \quad P_2(\mp a \sin \varphi, \pm b \sin \varphi).$$

Let $OP_1 = a'$, $OP_2 = b'$, O being the centre.

Then $$a'^2 = a^2 \cos^2 \varphi + b^2 \sin^2 \varphi, \qquad [(4), \S 7]$$

$$b'^2 = a^2 \sin^2 \varphi + b^2 \cos^2 \varphi.$$

$$\therefore \quad a'^2 + b'^2 = a^2 + b^2.$$

152. *The area of the parallelogram formed by tangents at the ends of conjugate diameters of an ellipse is constant.*

Let $P_1(a \cos \varphi, b \sin \varphi)$ and $P_2(\mp a \sin \varphi, \pm b \cos \varphi)$

be the extremities of any two conjugate diameters, and let $ABCD$ be the parallelogram formed by tangents at the ends of these diameters.

Draw P_1N perpendicular to OP_2; then

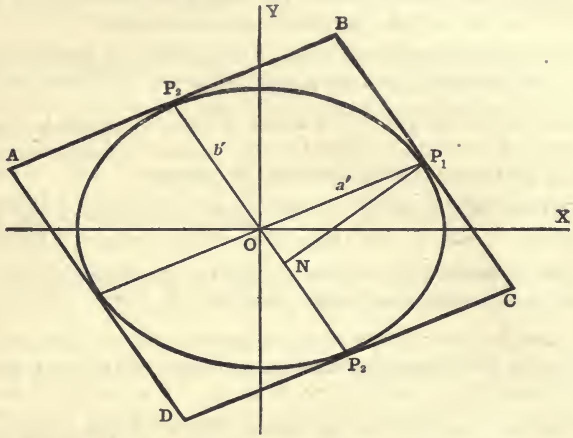
$$\text{Area } ABCD = 4\, OP_2 \cdot P_1N = 4b' \cdot P_1N.$$

Since OP_2 is parallel to the tangent at P_1 [§ 149, Cor. III.], the equation of OP_2 may be written [(5), § 143]

$$\frac{x}{a} \cos \varphi + \frac{y}{b} \sin \varphi = 0.$$

$$\therefore \quad P_1N = \frac{\cos^2\varphi + \sin^2\varphi}{\sqrt{\dfrac{\cos^2\varphi}{a^2} + \dfrac{\sin^2\varphi}{b^2}}} = \frac{ab}{\sqrt{b^2\cos^2\varphi + a^2\sin^2\varphi}} = \frac{ab}{b'}.$$

$$\therefore \quad \text{Area } ABCD = 4ab.$$



Cor. If angle $P_1OP_2 = \omega$, then

$$\sin\omega = \frac{P_1N}{a'} = \frac{ab}{a'b'}.$$

EXAMPLES.

1. The difference of the squares of two conjugate semi-diameters of a hyperbola is constant.

2. The area of the parallelogram formed by tangents to two conjugate hyperbolas at the ends of two conjugate diameters is equal to $4ab$.

3. If ω denotes the angle between two conjugate diameters of a hyperbola, then $\sin\omega = \dfrac{ab}{a'b'}$.

4. Show that the acute angle between two conjugate diameters of an ellipse is least when the diameters are equal.

5. Show that the eccentric angles of the extremities of the equi-conjugate diameters of an ellipse are 45° and 135°.

6. Conjugate diameters of a rectangular hyperbola are equal, and equally inclined to the asymptotes.

7. Tangents to two conjugate hyperbolas at the extremities of two conjugate diameters meet on the asymptotes.

8. The area of the triangle formed by two semi-conjugate diameters and the chord joining their ends is constant.

9. Prove that for all values of m the line

$$y = mx \pm \sqrt{\tfrac{1}{2}(a^2m^2 + b^2)}$$

passes through the extremities of two conjugate diameters of an ellipse. What is the corresponding equation for the hyperbola?

10. The product of the focal radii of a point P is equal to the square of the semi-diameter parallel to the tangent at P.

11. The locus of the poles of a series of parallel chords is the diameter which bisects the chords. Hence the line joining the intersection of two tangents to the centre bisects the chord of contact.

12. Find the equations of two conjugate diameters of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, one of which bisects the chord through $(0, b)$ and $(ae, 0)$.

13. In the hyperbola $4x^2 - 5y^2 = 20$ find the equations of two conjugate diameters, one of which bisects the chord $2x + 3y = 6$.

14. If straight lines drawn through any point of an ellipse to the ends of any diameter POP' meet the conjugate diameter P_1OP_1' in Q and R, show that $OQ \cdot OR = OP_1^2$.

15. Show that the locus of the intersection of the perpendiculars from the foci upon a pair of conjugate diameters of an ellipse is a similar concentric ellipse.

16. Two conjugate diameters of an ellipse are drawn, and their four extremities are joined to any point on a given circle whose centre is at the centre of the ellipse. Show that the sum of the squares of these four lines is constant.

17. Any tangent to an ellipse meets the director circle in P and Q. Prove that OP and OQ are semi-conjugate diameters of the ellipse.

18. P_1 is a point on a *branch* of a hyperbola, P_2 is a point on a *branch* of its conjugate, OP_1 and OP_2 being semi-conjugate diameters. If F_1 and F_2 are the interior foci of these two *branches*, respectively, show that

$$F_2P_2 \sim F_1P_1 = a \sim b.$$

19. Find the equation of the chord passing through the extremities of two conjugate diameters.

20. The lengths of the chords joining the extremities of two conjugate diameters of an ellipse are

$$\sqrt{a^2 + b^2 \pm a^2e^2 \sin 2\phi}.$$

For what value of ϕ are these chords, one a maximum and the other a minimum?

Show that the corresponding result for the hyperbola is

$$ae(\sec\phi \pm \tan\phi).$$

21. If the eccentric angles of two points P, Q on an ellipse are ϕ_1, ϕ_2, prove that the area of the parallelogram formed by tangents at the ends of diameters through P and Q is

$$4ab \csc(\phi_1 - \phi_2);$$

and hence show that this area is least when P and Q are the ends of conjugate diameters.

22. The sides of a parallelogram circumscribing an ellipse are parallel to conjugate diameters; prove that the product of the perpendiculars from two opposite vertices on any tangent is equal to the product of those from the other two vertices.

23. The radius of a circle which touches a hyperbola and its asymptotes is equal to that part of the latus rectum intercepted between the curve and the asymptotes.

24. A line parallel to the y-axis meets two conjugate hyperbolas and one of their asymptotes in P, Q, R. Show that the normals at P, Q, R meet on the x-axis.

25. If a tangent drawn at any point P of the inner of two similar coaxial conics meets the outer in the points T and T', then any chord of the inner through P is half the algebraic sum of the parallel chords of the outer through T and T'.

26. Def. The two chords of a central conic which join any point on the curve to the extremities of any diameter are called *Supplemental Chords.*

Show that two supplemental chords are parallel to a pair of conjugate diameters.

153. *To find the equation of a conic referred to a pair of conjugate diameters as axes.*

We may assume the required equation to be (§ 67)

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad (1)$$

By supposition each axis bisects all chords parallel to the other (§ 149). Hence for every value of either x or y the two values of the other must be *equal in magnitude and opposite in sign.*

Solving (1) for x and y respectively gives

$$ax = -(hy + g) \pm \lambda, \quad (2)$$

and

$$by = -(hx + f) \pm \lambda'; \quad (3)$$

so that for the values of x to be opposite $hy + g$ must vanish for all values of y, while for the values of y to be opposite $hx + f$ must vanish for all values of x. That is, $f=g=h=0$, and we have

$$ax^2+by^2+c=0, \tag{4}$$

or

$$\frac{x^2}{-\frac{c}{a}}+\frac{y^2}{-\frac{c}{b}}=1. \tag{5}$$

Since $(a',0)$ and $(0,b')$ are points on the curve

$$a'^2=-\frac{c}{a} \quad \text{and} \quad b'^2=-\frac{c}{b}.$$

Therefore the equation of the ellipse referred to conjugate diameters is

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1, \tag{6}$$

where a', b' are the lengths of the semi-diameters.

In the case of the hyperbola, the conic meets *one* of the axes in imaginary points; let this be the y-axis. Then the equation is

$$\frac{x^2}{a'^2}-\frac{y^2}{b'^2}=1. \tag{7}$$

The conjugate hyperbola will therefore meet the x-axis in imaginary points (§ 149, Cor. III.), so that its equation is

$$\frac{y^2}{b'^2}-\frac{x^2}{a'^2}=1, \tag{8}$$

where a', b' are the *real* intercepts of the two curves (7) and (8) on the axes.

If the ellipse is referred to its equi-conjugate diameters, so that $a'=b'$, its equation will be

$$x^2+y^2=a'^2. \tag{9}$$

We thus see that the equation of a conic referred to a pair of conjugate diameters is of the same *form* as when referred to its own axes. Moreover, the proofs in § 111, § 113, and § 139 hold good whether the axes are rectangular or oblique. Therefore, when the equation of the conic is referred to a pair of conjugate diameters, the equations of the polar and tangent will be of the same *form* as those given in §§ 138, 139.

154. *To find the equation of a hyperbola when referred to its asymptotes as axes of coordinates.*

The equations of two conjugate hyperbolas referred to their own axes are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \pm 1. \tag{1}$$

Let ω be the angle between the asymptotes such that

$$\tan\frac{\omega}{2} = \frac{b}{a}. \qquad (\S\ 146.)$$

Then $$\sin\frac{\omega}{2} = \frac{b}{\sqrt{a^2+b^2}}, \qquad \cos\frac{\omega}{2} = \frac{a}{\sqrt{a^2+b^2}}. \tag{2}$$

Since the axes bisect the angles between the asymptotes, the formulæ for effecting the required transformation are, from (2) and Ex. 20, p. 98,

$$\left.\begin{aligned} x &= (y'+x')\frac{a}{\sqrt{a^2+b^2}}, \\ y &= (y'-x')\frac{b}{\sqrt{a^2+b^2}}. \end{aligned}\right\} \tag{3}$$

Substituting these values in (1), reducing and dropping the accents, we obtain

$$4xy = \pm (a^2+b^2), \tag{4}$$

which is the required equation.

If the hyperbola is rectangular, so that $a = b$, its equation referred to its asymptotes will be

$$2xy = \pm a^2. \tag{5}$$

Otherwise. The equation of the asymptotes, *referred to themselves as axes of coordinates,* is $xy = 0$.

Therefore the equations of any two conjugate hyperbolas referred to them is of the form (§ 146)

$$xy = \pm K. \tag{6}$$

Hence the equation $xy = K$, where K is any constant, always represents a hyperbola referred to its asymptotes as axes of coordinates; so that, if the axes of coordinates are at right angles, the hyperbola $xy = K$ is rectangular.

155. *To find the equation of the tangent at any point (x', y') of the hyperbola $xy = K$.*

Since equation (6), § 111, holds good for oblique axes, the required equation may be written

$$xy' + x'y = 2K,$$

or
$$\frac{x}{x'} + \frac{y}{y'} = \frac{2K}{x'y'}. \tag{1}$$

Since (x', y') is on the curve, $x'y' = K$, and therefore the equation of the tangent at (x', y') is

$$\frac{x}{x'} + \frac{y}{y'} = 2. \tag{2}$$

Cor. *The area of the triangle formed by the asymptotes and any tangent to a hyperbola is constant.*

From equation (2) we see that the intercepts of any tangent on the asymptotes are $2x'$ and $2y'$.

Hence if A denotes the area, we have

$$A = 2x'y' \sin\omega.$$

But
$$\sin\omega = 2\sin\frac{\omega}{2}\cos\frac{\omega}{2} = \frac{2ab}{a^2+b^2}, \quad [\S\,154, (2)]$$

and
$$2x'y' = \tfrac{1}{2}(a^2+b^2). \quad [\S\,154, (4)]$$

$$\therefore\ A = ab.$$

Ex. 1. Show that the tangent to the interior of any two similar and coaxial conics cuts off a constant area from the exterior conic.

Ex. 2. Show that the equation of the normal to the rectangular hyperbola $x^2 - y^2 = a^2$ at the point (x', y') may be written

$$\frac{x}{x'} + \frac{y}{y'} = 2. \qquad (Cf.\ \text{Ex. 22, p. 122.})$$

156. *The parabola is a limiting form of each of the central conics.*

The equation of the central conic referred to the left vertex as origin is found by writing $(x - a)$ for x in the standard equation. The equation thus obtained will be

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} - \frac{2x}{a} = 0, \tag{1}$$

or
$$\frac{x^2}{a} \pm \frac{ay^2}{b^2} - 2x = 0. \tag{2}$$

Let $\frac{b^2}{a}=l=$ half the latus rectum (§ 138), then

$$\frac{x^2}{a} \pm \frac{y^2}{l} - 2x = 0. \tag{3}$$

Now if a becomes infinite, while l remains finite, the equation (3) becomes in the limit

$$y^2 = \pm 2lx, \tag{4}$$

which is the equation of a parabola, whether we use the plus or minus sign in the second member.

Since $l=\frac{b^2}{a}$, it follows that if l is not zero when a is infinite, b must also be infinite. Therefore the parabola is the limiting form of the central conic whose latus rectum is finite, but whose axes are both infinite, the centre and one focus being at infinity.

Notice that, although both a and b are infinite, a is infinite compared to b, in fact $\frac{a}{b}=\frac{b}{l}$. Thus the *width* of a parabola is nothing compared to its *length*, and we see how coincident or parallel lines are limiting cases of parabolas. The shape is the same, only the scale has been changed.

157. *To find the polar equation of a central conic, the pole being at the centre.*

The formulæ for changing from rectangular to polar coordinates are (§ 6)

$$x = \rho \cos\theta, \quad y = \rho \sin\theta.$$

These values substituted in

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

give

$$\rho^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{\pm b^2}\right) = 1,$$

or

$$\rho^2 = \frac{\pm a^2 b^2}{a^2\sin^2\theta \pm b^2\cos\theta} = \frac{\pm a^2b^2}{a^2-(a^2\mp b^2)\cos^2\theta}.$$

$$\therefore \quad \rho^2 = \frac{\pm b^2}{1-e^2\cos^2\theta},$$

which is the required equation.

Examples on Chapter X.

1. Q is the point on the auxiliary circle corresponding to P on the ellipse; PLM is drawn parallel to OQ* to meet the axes in L and M. Prove that $PL = b$ and $PM = a$.

2. Show that the sum of the squares of the reciprocals of two perpendicular diameters of an ellipse is constant. (See § 156.)

3. If an equilateral triangle is inscribed in an ellipse, the sum of the squares of the reciprocals of the diameters parallel to the sides is constant.

4. Find the inclination to the major axis of the diameter of an ellipse the square of whose length is (1) the arithmetical mean, (2) the geometrical mean, and (3) the harmonical mean between the squares on the major and minor axes. (§ 156.) Ans. to (3). 45°.

5. In a rectangular hyperbola the angles subtended at its vertices by any chord parallel to its conjugate axis are supplementary.

6. In a rectangular hyperbola the angle subtended by any chord at the centre is the supplement of the angle between the tangents at the ends of the chord.

7. Any point P of an ellipse is joined to the extremities of the major axis; prove that the portion of a directrix intercepted between the joining lines subtends a right angle at the corresponding focus.

8. The normal to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ meets the axes in M, N, and MP and NP are drawn perpendicular to the axes; prove that the locus of P is the similar hyperbola

$$a^2x^2 - b^2y^2 = (a^2 + b^2)^2.$$

9. The tangent at any point P of an ellipse meets the tangent at one vertex in Q; show that OQ is parallel to the line joining P to the other vertex.

10. Prove that the foci of the hyperbola $4xy = a^2 + b^2$ are given by

$$x = y = \pm \frac{a^2 + b^2}{2a}.$$

11. Show that two concentric rectangular hyperbolas whose axes meet at an angle of 45° cut orthogonally.

12. A point moves so that the sum of the squares of its distances from two intersecting straight lines is constant. Prove that its locus is an ellipse, and find the eccentricity in terms of the angle between the lines.

13. PNP' is a double ordinate of an ellipse, and Q is any point on the curve; show that if QP and QP' meet the major axis in M and M', respectively,

$$OM \cdot OM' = a^2.$$

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* In this set of examples O is always at the centre of the conic.

14. A straight line always passes through a fixed point; prove that the locus of the middle point of the portion of it, which is intercepted between two fixed straight lines, is a hyperbola whose asymptotes are parallel to the fixed lines.

15. A straight line has its extremities on two fixed straight lines and cuts off from them a triangle of constant area. Find the locus of the middle point of the line.

16. The tangent to an ellipse at P meets the axes in T, T', and OQ is the perpendicular on it from the centre. Prove (1) that $TT' \cdot PQ = a^2 - b^2$, and (2) that the least value of TT' is $a + b$.

17. Show that the four lines which join the foci to two points P and Q on an ellipse all touch a circle whose centre is the pole of PQ.

18. If the ordinate NP of any point P of any ellipse is produced to Q, so that NQ is equal to the subtangent at P, prove that the locus of Q is a hyperbola.

19. The rectangular coordinates of a point are $a \tan(\theta + \alpha)$ and $b \tan(\theta + \beta)$, where θ is variable; prove that the locus of the point is a hyperbola.

20. The tangent at any point P of an ellipse meets the equi-conjugate diameters in Q, Q'; show that the triangles POQ and POQ' are in the ratio of $OQ^2 : OQ'^2$.

21. If in an ellipse OQ is conjugate to the normal at P, then will OP be conjugate to the normal at Q.

22. OA and OB are fixed straight lines, P any point, PM and PN the perpendiculars from P on OA and OB. If the area of the quadrilateral $OMPN$ is constant, show that by a proper choice of axes the locus of P is

$$x^2 - y^2 = K \csc AOB.$$

23. A series of circles touches a given straight line at a given point. Prove that the locus of the poles of another fixed straight line with regard to these circles is a hyperbola whose asymptotes are, respectively, a parallel to the first given line and a perpendicular to the second.

24. A point is such that the perpendicular from the centre on its polar with respect to the ellipse is constant and equal to c. Show that its locus is the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}.$$

25. If any pair of conjugate diameters of an ellipse cut the tangent at a point P in T and T', show that $TP \cdot PT' = OP'^2$, where OP' is the diameter conjugate to OP. (§ 153.)

26. If a number of hyperbolas have the same transverse axis, show that tangents at points having a common abscissa meet the x-axis in the same

point; and that tangents to the conjugate hyperbolas at points having the *same* abscissa meet the same axis in a point equally distant from the centre.

27. If the focus of an ellipse is the common focus of two parabolas, whose vertices are at the ends of the major axis, these parabolas will intersect at right angles in points P and Q such that

$$PQ = 4b.$$

28. If P, P' are the extremities of conjugate diameters of an ellipse, and PQ, PQ' are chords parallel to an axis of the ellipse, show that PQ' and $P'Q$ are parallel to the equi-conjugate diameters.

29. The two straight lines joining the points in which two tangents to a hyperbola meet the asymptotes are parallel to the chord of contact of the tangents and equidistant from it. (§ 154.)

30. If through any point P a line PQR is drawn parallel to an asymptote of a hyperbola, cutting the curve in Q and the polar of P in R, show that

$$PQ = QR. \qquad \text{[Use § 46, (4).]}$$

31. Prove that the ends of the latera recta of all ellipses, having a given major axis $2a$, lie on the parabolas $x^2 \pm ay = a^2$.

32. The tangent at any point P of a circle meets the tangent at a fixed point A in T, and T is joined to B, the other end of the diameter through A. Prove that the locus of the intersection of AP and BT is an ellipse whose eccentricity is $\frac{1}{2}\sqrt{2}$.

33. Prove that the part of the tangent at any point of a hyperbola intercepted between the point of contact and the transverse axis is a harmonic mean between the lengths of the perpendiculars drawn from the foci on the normal at the same point. (See footnote to § 115.)

34. If a right triangle is inscribed in a rectangular hyperbola, the tangent at the vertex of the right angle is perpendicular to the hypotenuse.

35. If P, P' are the extremities of conjugate diameters, and the tangent at P cuts the major axis in T, and the tangent at P' cuts the minor axis in T', show that TT' will be parallel to one of the equi-conjugates.

36. QQ' is any chord of an ellipse parallel to one of the equi-conjugates, and the tangents at Q, Q' meet in T. Show that the circle QTQ' passes through the centre. (§ 153.)

37. If one axis of a variable central conic is fixed in magnitude and position, prove that the locus of the point of contact of a tangent drawn from a fixed point on the other axis is a parabola.

38. Find the coordinates of the points of contact of the common tangents to the two hyperbolas

$$x^2 - y^2 = 3a^2 \quad \text{and} \quad xy = 2a^2.$$

39. PNP' is a double ordinate of an ellipse, and the normal at P meets OP' in Q. Show that the locus of Q is a similar ellipse.

40. In an ellipse the normal at P meets the major axis in N and the minor axis in N'. Show that the loci of the middle points of PN and PN' are, respectively, the ellipses

$$\frac{4x^2}{a^2(1+e^2)^2}+\frac{4y^2}{b^2}=1, \quad 4(a^2x^2+b^2y^2)=(a^2-b^2)^2.$$

41. Show that the line $y=mx+2c\sqrt{-m}$ always touches the hyperbola $xy=c^2$, and that its point of contact is

$$\left(\frac{c}{\sqrt{-m}},\ c\sqrt{-m}\right).$$

42. A point P moves along the fixed line $y=mx$; prove that the locus of the foot of the perpendicular from P on its polar with respect to the ellipse $b^2x^2+a^2y^2=a^2b^2$ is a rectangular hyperbola, one of whose asymptotes is the diameter of the ellipse conjugate to the given fixed line.

43. The tangents drawn from a point P to an ellipse make angles γ_1 and γ_2 with the major axis. Find the locus of P when (1) $\tan\gamma_1+\tan\gamma_2$ is constant, (2) $\cot\gamma_1+\cot\gamma_2$ is constant, and (3) $\tan\gamma_1\tan\gamma_2$ is constant.

44. The line joining two extremities of any two diameters of an ellipse is either parallel or conjugate to the line joining two extremities of their conjugate diameters.

45. A, A' are the vertices of a rectangular hyperbola, and P is any point on the curve; show that the internal and external bisectors of the angle APA' are parallel to the asymptotes.

46. A, A' are the ends of a fixed diameter of a circle, and PP' is any chord perpendicular to AA'. Show that the locus of the intersection of AP and $A'P'$ is a rectangular hyperbola. Show also that the words *circle* and *rectangular hyperbola* can be interchanged.

47. If P and P' are the ends of conjugate diameters of an ellipse, show that (1) the tangents at P and P' meet on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$, (2) the locus of the middle point of PP' is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$, and (3) the locus of the foot of the perpendicular from the centre upon PP' is

$$2(x^2+y^2)^2=a^2x^2+b^2y^2.$$

48. A line is drawn parallel to the minor axis of an ellipse midway between a focus and the corresponding directrix; prove that the product of the perpendiculars upon it from the ends of any chord passing through that focus is constant.

49. Show that the coordinates of the point of intersection of two tangents to the hyperbola $xy=K$ are harmonic means between the coordinates of the points of contact. (See footnote to § 115.)

50. From any point on one hyperbola tangents are drawn to another hyperbola which has the same asymptotes. Show that the chord of contact cuts off a constant area from the asymptotes. (See Ex. 22, p. 233.)

51. If the chord joining two points whose eccentric angles are α and β cuts the major axis of an ellipse at a distance d from the centre, show that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{d-a}{d+a}.$$

52. If any two chords are drawn through two points on the major axis of an ellipse equidistant from the centre, show that $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=1$, where α, β, γ, δ are the eccentric angles of the ends of the chords. (Use Ex. 51.)

53. If F, F' are the foci of an ellipse and any point A is taken on the curve and chords AFB, $BF'C$, CFD, $DF'E$ . . . are drawn, and the eccentric angles of A, B, C, D . . . are θ_1, θ_2, θ_3, θ_4 . . . , prove that

$$\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}=\cot\frac{\theta_2}{2}\cot\frac{\theta_3}{2}=\tan\frac{\theta_3}{2}\tan\frac{\theta_4}{2}= \ldots \text{ (Use Ex. 52.)}$$

54. Show that the locus of the poles with respect to the parabola $y^2=4ax$ of tangents to the hyperbola $x^2-y^2=a^2$ is the ellipse $4x^2+y^2=4a^2$.

55. Show that the locus of the pole, with respect to the auxiliary circle, of a tangent to the ellipse is a similar concentric ellipse, whose major axis is perpendicular to that of the original ellipse.

56. Prove that the locus of the pole, with respect to the ellipse, of any tangent to the auxiliary circle is the curve

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}.$$

57. Chords of the ellipse touch the parabola $ay^2=-2b^2x$. Prove that the locus of their poles is the parabola $ay^2=2b^2x$.

58. Show that the pole of any tangent to the rectangular hyperbola $xy=c^2$, with respect to the circle $x^2+y^2=a^2$, lies on a concentric and similarly placed rectangular hyperbola.

59. Tangents are drawn from any point on the ellipse $b^2x^2+a^2y^2=a^2b^2$ to the circle $x^2+y^2=r^2$; prove that the chords of contact are tangents to the ellipse $a^2x^2+b^2y^2=r^2$.

If $\frac{1}{r^2}=\frac{1}{a^2}+\frac{1}{b^2}$, prove that the lines joining the centre to the points of contact with the circle are conjugate diameters of the second ellipse.

60. Prove that the locus of the pole with respect to the hyperbola $b^2x^2-a^2y^2=a^2b^2$ of any tangent to the circle, whose diameter is the line joining the foci, is the ellipse

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}.$$

61. In the two ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{and} \quad \frac{x^2}{a}+\frac{y^2}{b}=a+b$$

tangents to the former meet the latter in P and Q. Prove that the tangents at P and Q are at right angles to each other.

62. A tangent to the parabola $x^2=4ay$ meets the hyperbola $xy=c^2$ in two points P and Q. Prove that the middle point of PQ lies on a parabola.

63. From points on the circle $x^2+y^2=a^2$ tangents are drawn to the hyperbola $x^2-y^2=a^2$; prove that the locus of the middle points of the chords of contact is the curve

$$(x^2-y^2)^2=a^2(x^2+y^2).$$

64. Show that the locus of the poles of normal chords of an ellipse is the curve

$$(a^2-b^2)^2x^2y^2=a^6y^2+b^6x^2.$$

65. Prove that the locus of the poles of all normal chords of the rectangular hyperbola $xy=c^2$ is the curve

$$(x^2-y^2)^2+4c^2xy=0.$$

66. P, Q are fixed points on an ellipse and R any other point on the curve; V, V' are the middle points of PR, QR; and VG, $V'G'$ are perpendicular to PR and QR, respectively, and meet the axis in G, G'. Show that GG' is constant.

67. On the focal radii of any point of an ellipse as diameters two circles are described; prove that the eccentric angle of the point is equal to the angle which a common tangent to the circles makes with the minor axis. (See Ex. 43, p. 206).

68. A, A' are the vertices of an ellipse, and P any point on the curve; show that, if PN is perpendicular to AP and PM perpendicular to $A'P$, M and N being on the axis AA', then will MN be equal to the latus rectum.

69. The equation of the line joining the two points (x_1, y_1) and (x_2, y_2) on the hyperbola $xy=c^2$ is

$$\frac{x}{x_1+x_2}+\frac{y}{y_1+y_2}=1.$$

70. Show that the area of the triangle formed by the tangents to an ellipse at the points whose eccentric angles are α, β, γ, respectively, is

$$ab \tan \tfrac{1}{2}(\alpha-\beta) \tan \tfrac{1}{2}(\beta-\gamma) \tan \tfrac{1}{2}(\gamma-\alpha).$$

71. If P, P' are the points of contact of perpendicular tangents to an ellipse, and Q, Q' are the corresponding points on the auxiliary circle, show that OQ and OQ' are conjugate diameters of the ellipse.

72. The locus of the point from which can be drawn two straight lines at right angles to each other, each of which touches one of the rectangular

hyperbolas $xy = \pm c^2$, is also the locus of the feet of the perpendiculars from the origin on the tangents to the rectangular hyperbolas

$$x^2 - y^2 = \pm 4c^2.$$

73. Show that the locus of the intersection of two tangents to an ellipse, if the sum of the eccentric angles of the points of contact is equal to the constant 2α, is the line $ay = bx \tan \alpha$.

74. If the difference of the eccentric angles of the points of contact of two tangents to an ellipse is 120°, show that the tangents will intersect on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4.$$

75. Show that the locus of the middle points of chords of an ellipse which pass through the point (h, k) is

$$b^2x(x-h) + a^2y(y-k) = 0.$$

76. A parallelogram is constructed with its sides parallel to the asymptotes of a hyperbola, and one of its diagonals is a chord of the hyperbola; show that the other diagonal will pass through the centre.

77. Two equal circles touch one onother; find the locus of a point which moves so that the sum of the tangents from it to the two circles is constant.

78. Prove that the sum of the products of the perpendiculars from the two extremities of each of two conjugate diameters on any tangent to an ellipse is equal to the square of the perpendicular from the centre on that tangent.

79. The straight lines drawn from any point on an equilateral hyperbola to the extremities of any diameter are equally inclined to the asymptotes.

80. If the product of the perpendiculars from the foci upon the polar of P, with respect to the ellipse, is constant and equal to c^2, prove that the locus of P is the ellipse

$$b^4x^2(c^2 + a^2e^2) + a^4c^2y^2 = a^4b^4.$$

81. Prove that the locus of the middle points of the portions of tangents to an ellipse included between the axes is the curve

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = 4.$$

82. The locus of the middle points of normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is the curve

$$(y^2 - x^2)^3 = 4a^2x^2y^2.$$

83. From the centre O of two concentric circles two radii OQ, OR are drawn equally inclined to a fixed straight line, the first to the outer circle, the second to the inner. Prove that the locus of the middle point P of QR is an ellipse, that PQ is the normal at P to this ellipse, and that QR is equal to the diameter conjugate to OP.

84. The locus of the intersection of normals to an ellipse at the ends of conjugate diameters is the curve

$$2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2.$$

85. If two tangents to a conic are at right angles to one another, the product of the perpendiculars from the centre and the intersection of the tangents on the chord of contact is constant.

86. A circle intersects a hyperbola in four points. Prove that the product of the distances of the four points of intersection from one asymptote is equal to the product of their distances from the other.

87. Show that if a rectangular hyperbola cuts a circle in four points the centre of mean position of the four points is midway between the centres of the two curves. (See Ex. 33, p. 17.)

88. Q is a point on the normal at any point P of an ellipse such that the lines OP and OQ make equal angles with the axis of the ellipse. Show that PQ varies as the diameter conjugate to OP.

89. If P is any point on an ellipse and any chord PQ cuts the diameter conjugate to OP in R, then will $PQ\cdot PR$ be equal to half the square on the diameter parallel to PQ.

90. Show that the locus of the middle points of all chords of an ellipse which are of constant length, $2c$, is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)=\frac{c^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right). \text{ (Use (4), § 46.)}$$

91. Tangents at right angles are drawn to an ellipse. Show that the locus of the middle point of the chord of contact is the curve

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2+y^2}{a^2+b^2}.$$

92. The area of the parallelogram formed by the tangents at the ends of any two diameters of an ellipse varies inversely as the area of the parallelogram formed by joining the points of contact.

93. POP' is a diameter of an ellipse, QOQ' the corresponding diameter of the auxiliary circle, and ϕ the eccentric angle of P. Show that the area of the parallelogram formed by tangents at P, P', Q, Q' is

$$\frac{8a^2b}{(a-b)\sin 2\phi}.$$

94. If from any point R in the plane of an ellipse the perpendiculars RS, RQ are drawn on the equal conjugate diameters, the diagonal PR of the parallelogram $PQRS$ will be perpendicular to the polar of R.

95. Normals to an ellipse are drawn at the extremities of a chord parallel to one of the equi-conjugate diameters; prove that they intersect on a diameter perpendicular to the other equi-conjugate.

96. If normals are drawn at the ends of any focal chord of an ellipse, a line through their intersection parallel to the major axis will bisect the chord.

97. If circles are described on two semi-conjugate diameters of an ellipse as diameters, the locus of their second point of intersection will be the curve

$$2(x^2+y^2)^2=a^2x^2+b^2y^2.$$

98. Show that the angle which a diameter of an ellipse subtends at either end of the major axis is supplementary to that which the conjugate diameter subtends at the end of the minor axis.

99. If θ, θ' are the angles subtended by the major axis of an ellipse at the extremities of a pair of conjugate diameters, show that $\cot^2\theta+\cot^2\theta'$ is constant.

100. If the line joining the foci of an ellipse subtends angles 2θ, $2\theta'$ at the ends of a pair of conjugate diameters, show that $\tan^2\theta+\tan^2\theta'$ is constant.

101. If θ, θ' are the angles which any two conjugate diameters subtend at a fixed point on an ellipse, show that $\cot^2\theta+\cot^2\theta'$ is constant.

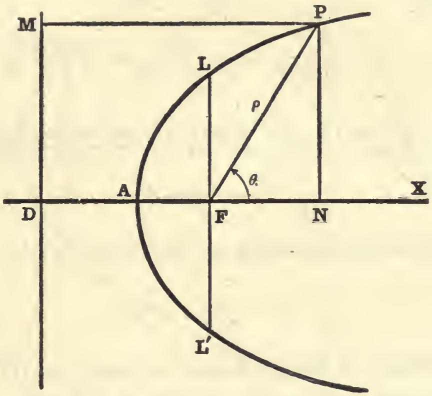
102. The normals at three points of an ellipse whose eccentric angles are α, β, γ, will meet in a point if

$$\sin 2\alpha\sin(\beta-\gamma)+\sin 2\beta\sin(\gamma-\alpha)+\sin 2\gamma\sin(\alpha-\beta)=0.$$

CHAPTER XI.

POLAR EQUATION OF A CONIC, THE FOCUS BEING THE POLE.

158. *To find the polar equation of a conic, the focus being the pole.*



Let F be the focus, DM the directrix, and e the eccentricity of the conic.

Draw FD perpendicular to the directrix, and let DFX be taken for the positive direction of the initial line.

Let $L'FL$ be the latus rectum, and let

$$l = FL = e \cdot DF. \tag{1}$$

Let $P(\rho, \theta)$ be any point on the curve, and let PM and PN be perpendicular, respectively, to DM and DX.

Then we have

$$FP = e \cdot MP = e \cdot DN = e \cdot DF + e \cdot FN. \tag{2}$$

Whence $$\rho = l + e\rho \cos \theta. \tag{3}$$

$$\therefore \quad \frac{l}{\rho} = 1 - e \cos \theta, \quad \text{or} \quad \rho = \frac{l}{1 - e \cos \theta}. \tag{4}$$

If FD is taken as the positive direction of the initial line and θ measured clockwise, the equation of the curve is

$$\frac{l}{\rho} = 1 + e \cos \theta. \tag{5}$$

If the axis of the conic makes an angle γ with the initial line, the equation of the curve will be (§ 68)

$$\frac{l}{\rho} = 1 - e \cos(\theta - \gamma). \tag{6}$$

If the conic is a parabola, we have $e = 1$, and the equation may then be written

$$\rho = \frac{l}{1 - \cos \theta} = \frac{l}{2 \sin^2 \frac{1}{2}\theta} = \frac{1}{2} l \csc^2 \frac{\theta}{2}. \tag{7}$$

Since $DF = \frac{l}{e}$, from (1), *the equation of the directrix, DM*, is (§ 44)

$$\frac{l}{\rho} = - e \cos \theta. \tag{8}$$

The equation of the directrix of the conic (6) is

$$\frac{l}{\rho} = - \cos(\theta - \gamma). \tag{9}$$

The perpendicular distance from the focus $(ae, 0)$ to the asymptote whose equation is $bx - ay = 0$ is [(5), § 50]

$$\frac{abe}{\sqrt{a^2 + b^2}} = b.$$

Also the angle which this asymptote makes with the principal axis of the conic is

$$\tan^{-1} \frac{b}{a} = \cos^{-1} \frac{a}{\sqrt{a^2 + b^2}} = \cos^{-1} \frac{1}{e}.$$

The perpendicular on this asymptote from the focus therefore makes an angle $\frac{\pi}{2} + \cos^{-1} \frac{1}{e}$.

Hence, by § 44, the polar equation of the asymptote is

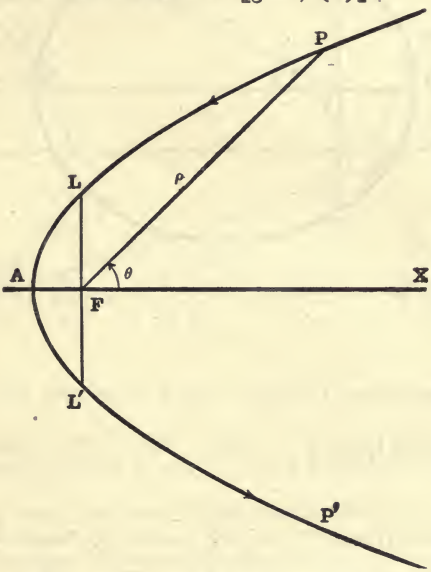
$$\rho \sin\left(\theta - \cos^{-1} \frac{1}{e}\right) = b. \tag{10}$$

Similarly, the polar equation of the other asymptote is

$$\rho \sin\left(\theta + \cos^{-1} \frac{1}{e}\right) = - b. \tag{11}$$

159. *To trace the curve* $\rho = \dfrac{l}{1 - e \cos \theta}$.

Since, for any value of θ, $\cos \theta = \cos(-\theta)$, the curve is always symmetrical about the initial line [§ 29, (1)].



I. Let $e = 1$, then the curve is a parabola, and the equation becomes

$$\rho = \frac{l}{1 - \cos \theta}.$$

When $\theta = 0$, then $\rho = \frac{l}{0} = \infty$. As θ increases *continuously* from 0 to 90°, $\cos \theta$ decreases from 1 to 0, and therefore ρ decreases *continuously* from infinity to l.

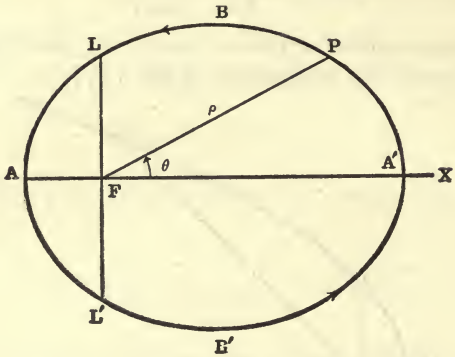
As θ increases from 90° to 180°, $\cos \theta$ decreases from 0 to -1, and therefore ρ decreases from l to $\frac{1}{2}l$.

Similarly, as θ varies from 180° to 270°, ρ increases from $\frac{1}{2}l$ to l; and as θ increases beyond 270°, ρ continues to increase until $\theta = 360°$, when it again becomes infinite.

Hence, for θ increasing, the parabola is described in the direction $\infty PLAL'P' \infty$, as shown in the figure, going to infinity in the direction FX.

It is to be noticed that ρ is always positive, *not changing its sign when it passes through infinity.*

II. Let $e < 1$, then the curve is an ellipse.



At the point A', $\theta = 0$ and $\rho = \frac{l}{1 - e}$, which is positive, since $e < 1$.

As θ increases from 0 to 90°, $\cos\theta$ decreases from 1 to 0, and hence ρ decreases from $\frac{l}{1 - e}$ to l. We thus obtain the portion $A'PBL$.

As θ increases from 90° to 180°, $\cos\theta$ decreases from 0 to -1, and therefore ρ decreases from l to $\frac{l}{1 + e}$. The curve therefore cuts the initial line again at some point A, such that $FA = \frac{l}{1 + e}$, and we thus obtain the portion LA.

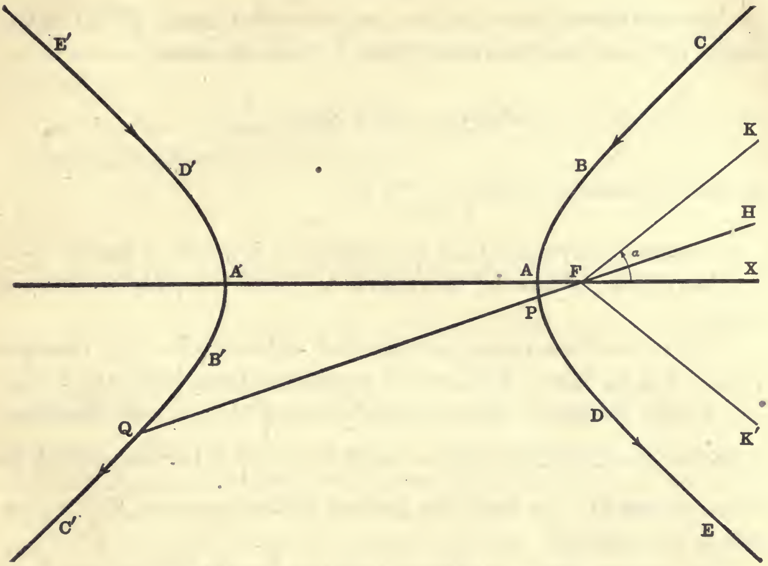
Similarly, as θ continues to increase from 180° to 270°, and then to 360°, ρ increases from $\frac{l}{1 + e}$ to l, and then to $\frac{l}{1 - e}$, and we have the portions AL' and $L'B'A'$.

Since ρ is finite for all values of θ when $e < 1$, the locus is a closed curve symmetrical about the initial line.

III. Let $e > 1$, then the curve is a hyperbola.

When $\theta = 0$, $1 - e\cos\theta = 1 - e = -(e - 1)$.

Hence $\rho = -\frac{l}{e - 1}$, which is a *negative* quantity since $e > 1$. The corresponding point is A' in the figure.



Let θ increase from 0 to $\cos^{-1}\frac{1}{e} \equiv a = XFK$ in the figure.

Then $1 - e\cos\theta$ increases *algebraically* from $-(e-1)$ to -0.

Hence ρ decreases *algebraically* from $-\frac{l}{e-1}$ to $-\infty$.

Therefore, as θ varies from 0 to a, ρ is negative and increases in *magnitude* from FA' to infinity. We thus obtain the portion of the curve $A'B'C'\infty$.

If θ is very slightly greater than a, then $\cos\theta$ is just a little less than $\frac{1}{e}$, so that $1 - e\cos\theta$ is very small and positive, and therefore ρ is very great and positive.

Hence, as θ *increases* through the angle a, the value of ρ changes from $-\infty$ to $+\infty$.

As θ increases from a to π, $1 - e\cos\theta$ increases from 0 to $1+e$, and therefore ρ decreases from ∞ to $\frac{l}{1+e}$.

Now $\frac{l}{1+e} < \frac{l}{1-e}$. Hence the point A, which corresponds to $\theta = \pi$, is such that $FA < FA'$.

As θ increases from π to $2\pi - \alpha$ (the reflex angle XFK' in the figure), $1 - e\cos\theta$ decreases from $1 + e$ to 0, since

$$\cos(2\pi - \alpha) = \cos\alpha = \frac{1}{e},$$

so that ρ increases from $\dfrac{l}{1+e}$ to ∞.

Therefore, corresponding to values of θ between α and $2\pi - \alpha$, we have the portion of the curve $\infty\, CBADE\, \infty$, for which ρ *is positive.*

Finally, as θ increases through the value $(2\pi - \alpha)$, ρ changes from $+\infty$ to $-\infty$; and as θ increases from $2\pi - \alpha$ to 2π, $1 - e\cos\theta$ decreases *algebraically* from 0 to $1 - e$, and therefore ρ varies continuously from $-\infty$ to $-\dfrac{l}{e-1}$. Corresponding to these values of θ we have the portion of the curve $\infty\, E'D'A'$, for which ρ *is negative.*

Thus we see that ρ is always positive for the right branch of the curve, and negative for the left branch; and furthermore, that as θ varies from 0 to 2π, the complete curve is described in the order

$$A'B'C'\infty\ \infty\, CBADE\, \infty\ \infty\, E'D'A'.$$

The lines FK and FK' are parallel to the asymptotes, for

$$\cos\alpha = \cos(2\pi - \alpha) = \frac{1}{e} = \frac{a}{\sqrt{a^2 + b^2}}.$$

$$\therefore\quad \tan\alpha = \frac{b}{a}, \quad \text{and} \quad \tan(2\pi - \alpha) = -\frac{b}{a}.$$

Therefore the radius vector is infinite when it is parallel to either of the asymptotes.

If a line FPQ is drawn cutting the curve in two points P and Q, which are on different branches, these points must not be regarded as having the same vectorial angle. The radius vector FQ is *negative,* that is to say, FQ is drawn in the direction *opposite* to that which bounds its vectorial angle; and, therefore, if QF be produced to H, the vectorial angle of Q is XFH. So that, if θ is the vectorial angle of Q, that of P is $(\theta + \pi)$.

160. *To find the polar equation of the straight line through two given points on a conic, and to find the polar equation of the tangent at any point.*

Let $P(\rho_1, \alpha - \beta)$ and $Q(\rho_2, \alpha + \beta)$ be the two given points.

Let the equation of the conic be

$$\frac{l}{\rho} = 1 - e \cos \theta. \qquad (1)$$

If A and B are arbitrary parameters, the equation

$$\frac{l}{\rho} = A \cos \theta + B \cos (\theta - \alpha) \qquad (2)$$

will represent a straight line; for by changing to rectangular coordinates the result will be found to be of the first degree. Moreover, this line can be made to pass through *any* two points, since its equation contains two independent parameters, A and B. It will pass through P and Q if the values of A and B are so determined that ρ will have the same value in (2) as in (1) when $\theta = \alpha - \beta$, and also when $\theta = \alpha + \beta$.

Hence, by substituting $(\alpha - \beta)$ and $(\alpha + \beta)$ for θ in (1) and (2), we have, for the determination of the proper values of A and B, the two identities

$$\frac{l}{\rho_1} = 1 - e \cos(\alpha - \beta) \equiv A \cos (\alpha - \beta) + B \cos \beta, \qquad (3)$$

and $$\frac{l}{\rho_2} = 1 - e \cos (\alpha + \beta) \equiv A \cos (\alpha + \beta) + B \cos \beta. \qquad (4)$$

$$\therefore \quad A = -e, \quad \text{and} \quad B = \sec \beta. \qquad (5)$$

Substituting these values of A and B in (2) gives

$$\frac{l}{\rho} = - e \cos \theta + \sec \beta \cos (\theta - \alpha), \qquad (6)$$

which is the required equation.

If $\beta = 0$, the two points P and Q will coincide in the point whose vectorial angle is α, and the line (6) will become the tangent at that point. Hence, to find the equation of the tangent at the point whose vectorial angle is α, we put $\beta = 0$ in (6), and thus obtain

$$\frac{l}{\rho} = \cos (\theta - \alpha) - e \cos \theta. \qquad (7)$$

If the equation of the conic is [§ 158, (6)]

$$\frac{l}{\rho} = 1 - \cos(\theta - \gamma),$$

the equation of the chord joining the points $(\alpha - \beta)$ and $(\alpha + \beta)$ will be (§ 68)

$$\frac{l}{\rho} = \sec\beta \cos(\theta - \alpha) - e\cos(\theta - \gamma); \qquad (8)$$

and the equation of the tangent at the point α will be

$$\frac{l}{\rho} = \cos(\theta - \alpha) - e\cos(\theta - \gamma). \qquad (9)$$

161. *To find the polar equation of the polar of a point (ρ', θ') with respect to the conic*

$$\frac{l}{\rho} = 1 - e\cos\theta. \qquad (1)$$

Let $P(\rho_1, \alpha - \beta)$ and $Q(\rho_2, \alpha + \beta)$ be the points of contact of the two tangents drawn from the point (ρ', θ') to the conic.

Then the equation of the line PQ, the polar of (ρ', θ'), in terms of α and β will be [(6), § 160]

$$\frac{l}{\rho} = \sec\beta \cos(\theta - \alpha) - e\cos\theta. \qquad (2)$$

The values of α and β must now be determined in terms of l, e, ρ', and θ'.

The equations of the tangents at P, Q, respectively, are

$$\frac{l}{\rho} = \cos(\theta - \alpha + \beta) - e\cos\theta \quad [(7), \S\ 160] \qquad (3)$$

$$\frac{l}{\rho} = \cos(\theta - \alpha - \beta) - e\cos\theta. \qquad (4)$$

Since these tangents pass through (ρ', θ'), we have

$$\frac{l}{\rho'} = \cos(\theta' - \alpha + \beta) - e\cos\theta' \qquad (5)$$

$$\frac{l}{\rho'} = \cos(\theta' - \alpha - \beta) - e\cos\theta'. \qquad (6)$$

Subtracting (6) from (5) gives

$$\cos(\theta' - a + \beta) = \cos(\theta' - a - \beta). \tag{7}$$

$$\therefore \quad \theta' - a + \beta = -(\theta' - a - \beta), \text{ since } \beta \neq 0. \tag{8}$$

$$\therefore \quad a = \theta'. \tag{9}$$

From (9) and (5) we now get

$$\cos\beta = \frac{l}{\rho'} + e\cos\theta'. \tag{10}$$

Substitute these values of a and $\cos\beta$ in (2), and we have

$$\left(\frac{l}{\rho} + e\cos\theta\right)\left(\frac{l}{\rho'} + e\cos\theta'\right) = \cos(\theta - \theta'), \tag{11}$$

which is the required equation.

Examples on Chapter XI.

1. In a parabola, prove that the length of a focal chord which is inclined at 30° to the axis is four times the length of the latus rectum.

2. In any conic section the semi-latus rectum is a harmonic mean between the segments of any focal chord. (See note under § 115.)

3. The sum of the reciprocals of two perpendicular focal chords of any conic is constant.

4. If PFP' and QFQ' are any two focal chords of a conic at right angles to one another, show that $\dfrac{1}{PF \cdot FP'} + \dfrac{1}{QF \cdot FQ'}$ is constant.*

5. The tangents at two points P and Q of a conic meet in T. If F is the pole, show that the vectorial angle of T is half the sum of the vectorial angles of P and Q.

Hence, prove that if P and Q are on different branches of a hyperbola, then FT bisects the supplement of the angle PFQ, while in all other cases FT bisects the angle PFQ.

6. If the conic is a parabola and a, β are the vectorial angles of P, Q, show that the coordinates of T are given by the equations

$$\theta = \tfrac{1}{2}(a + \beta), \qquad \frac{l}{\rho} = 2\sin\frac{a}{2}\sin\frac{\beta}{2}.$$

7. If the conic is a parabola, then

$$FT^2 = FP \cdot FQ.$$

* In this list of examples F will always be the focus.

8. If the conic is central and b is the semi-minor axis, then

$$\frac{1}{PF \cdot FQ} - \frac{1}{FT^2} = \frac{1}{b^2}.$$

9. If a tangent at any point P of a conic meets the directrix in K, then the angle KFP is a right angle. (*Cf.* Ex. 39, p. 178.)

10. If the tangents at the two points P, Q of a conic meet in T, and if the straight line PQ meets the directrix corresponding to F in K, then the angle KFT is a right angle.

11. Prove that perpendicular focal chords of a rectangular hyperbola are equal.

12. If a straight line drawn through the focus of a hyperbola, parallel to an asymptote, meets the curve in P, prove that FP is one-fourth of the latus rectum.

13. Show that the length of any focal chord of a conic is a third proportional to the transverse axis and the diameter parallel to the chord.

14. If chords of a conic subtend a constant angle 2β at the focus, the tangents at the ends of the chords will meet on a fixed conic whose focus is F, and the chords will touch another conic whose focus is F. Consider the cases in which $\cos\beta >, =,$ and $< e$.

15. The exterior angle between any two tangents to a parabola is equal to half the difference of the vectorial angles of the points of contact.

16. The locus of the point of intersection of two tangents to a parabola which intersect at a constant angle is a hyperbola having the same focus and directrix as the given parabola. (Exs. 14 and 15.)

17. If PQ and PR are two chords of a conic subtending equal angles at the focus, the tangent at P and the chord QR will intersect on the directrix. Hence, if the sum of the vectorial angles of two points is constant, the chord through those points will meet the directrix in a fixed point.

18. PQ is a chord of a conic which subtends a right angle at a focus. Show that the locus of the pole of PQ and the locus enveloped by PQ are each conics whose latera recta are to that of the original conic as $\sqrt{2}:1$ and $1:\sqrt{2}$, respectively.

19. PFQ is a focal chord of a conic, and a parallel chord AP' through the vertex A meets the latus rectum in Q'. Prove that the ratio $PF \cdot FQ : AP' \cdot AQ'$ is constant.

20. If A, B, C are any three points on a parabola, and the tangents at these points form a triangle $A'B'C'$, show that $FA \cdot FB \cdot FC = FA' \cdot FB' \cdot FC'$.

21. Show that the equations $\frac{l}{\rho} = \pm 1 - e\cos\theta$ represent the same conic. If (ρ, θ) is a point on the curve when the upper sign is taken, what will be its coordinates when the lower sign is used?

22. The product of the segments of any focal chord of any conic is equal to the product of one-fourth the latus rectum and the whole chord.

23. Show that the polar equation of the normal at the point whose vectorial angle is α is

$$\frac{e\sin\alpha}{1 - e\cos\alpha}\cdot\frac{l}{\rho} = e\sin\theta - \sin(\theta - \alpha).$$

24. If a focal chord of an ellipse makes an angle α with the axis, the angle between the tangents at its extremities is

$$\tan^{-1}\frac{2e\sin\alpha}{1 - e^2}.$$

25. By means of the equation $\frac{l}{\rho} = 1 - e\cos\theta$, show that the ellipse might be generated by a point which moves so that the sum of its distances from two fixed points is constant.

26. If a chord of a conic subtends a constant angle 2β at the focus, prove that the locus of the point where it intersects the internal bisector of the angle 2β is the conic (*cf.* Ex. 14)

$$\frac{l\cos\beta}{\rho} = 1 - e\cos\beta\cos\theta.$$

27. Show that the locus of the middle points of focal chords of a conic section is a conic whose equation is

$$\rho = \frac{le\cos\theta}{1 - e^2\cos^2\theta}.$$

Show that in Cartesian coordinates this equation becomes

$$x^2(1 - e^2) + y^2 = elx,$$

and discuss this result for different values of e.

28. Given the focus and directrix of a conic, show that the polar of a given point with respect to it passes through a fixed point.

29. Two conics have a common focus. Prove that two of their common chords pass through the intersection of their directrices.

Sug. If the conics are $\frac{l}{\rho} = 1 - e\cos\theta$ and $\frac{l'}{\rho} = 1 - e'\cos(\theta - \gamma)$, the common chords are $\frac{l}{\rho} + e\cos\theta = \pm\left[\frac{l'}{\rho} + e'\cos(\theta - \gamma)\right]$.

30. P is any point on a conic and a straight line is drawn through F at a given angle with FP to meet the tangent at P in T. Prove that the locus of T is a conic whose focus and directrix are the same as those of the original conic.

31. A circle whose diameter is d passes through the focus F of a conic and meets it in four points whose distances from F are ρ_1, ρ_2, ρ_3, and ρ_4. Prove that

$$(1)\quad \rho_1\rho_2\rho_3\rho_4 = \frac{d^2l^2}{e^2};$$

and

$$(2)\quad \frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}+\frac{1}{\rho_4}=\frac{2}{l}.$$

32. A circle whose centre is at a fixed point on the principal axis intersects the conic in four points. Prove that the sum of their distances from the focus is constant.

33. If PFQ and $RF'R$ are two chords of an ellipse through the foci F and F', then will $\frac{PF}{FQ}+\frac{PF'}{F'R}$ be independent of the position of P.

34. Two conics have the same focus, and the distance of this focus from the corresponding directrix of each is the same; if the conics touch one another, prove that twice the sine of half the angle between their principal axes is equal to the difference of the reciprocals of their eccentricities.

[Write the equations of the common tangent at β, compare the coefficients of $\sin\theta$ and $\cos\theta$, and eliminate β.]

35. Show that the equation of the circle circumscribing the triangle formed by three tangents to the parabola $\frac{l}{\rho}=1-\cos\theta$ drawn at points whose vectorial angles are α, β, and γ is

$$\rho = \frac{l}{2}\csc\frac{\alpha}{2}\csc\frac{\beta}{2}\csc\frac{\gamma}{2}\sin\left(\frac{\alpha+\beta+\gamma}{2}-\theta\right);$$

and therefore it always passes through the focus. (See Ex. 6.)

36. A comet is moving in a parabolic orbit around the sun at its focus and when at 100,000,000 miles from the sun, the radius vector makes an angle of 60° with the axis of the orbit. What is the polar equation of the comet's orbit? How near does it approach to the sun?

Ans. $\rho = \frac{50,000,000}{1-\cos\theta}$.

37. Two straight lines bisect each other at right angles. Prove that the locus of the points at which they subtend equal angles is

$$\frac{\rho^2}{ab} = \frac{a\cos\theta - b\sin\theta}{b\cos\theta - a\sin\theta},$$

$2a$ and $2b$ being the lengths of the lines, their point of intersection the pole. Is the locus a conic?

CHAPTER XII.

THE GENERAL EQUATION OF THE SECOND DEGREE.

CONSTRUCTION OF CURVES OF THE SECOND ORDER.

162. The most general equation of the second degree in Cartesian coordinates is of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad (1)$$

From the investigations of § 108 and § 67 it follows that this equation always represents a conic section, whether the axes are rectangular or oblique. In order to determine the general nature of this conic, and to draw the curve in its proper position with ıeference to the axes of coordinates, solve the equation with respect to one of the variables, y say.

There are two cases to be considered, according as y appears in the equation to the second, or only to the first degree; *i. e.* according as $b \neq 0$, or $b = 0$.

First suppose that $b \neq 0$, and solve (1) with respect to y; we thus obtain

$$y = -\frac{hx + f}{b} \pm \frac{1}{b}\sqrt{(h^2 - ab)x^2 + 2(fh - bg)x + (f^2 - bc)}, \qquad (2)$$

or

$$y = -\frac{hx + f}{b} \pm \frac{1}{b}\sqrt{Lx^2 + 2Mx + N}, \qquad (3)$$

where $\quad L \equiv h^2 - ab, \quad M \equiv fh - bg, \quad N \equiv f^2 - bc.$

Thus for any given value of x there are two values of y.

Let

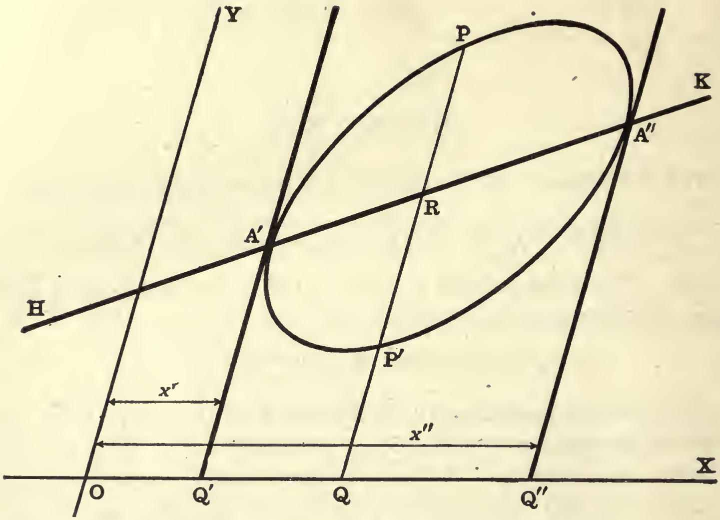
$$y' = -\frac{hx + f}{b}, \qquad (4)$$

and

$$Y = \frac{1}{b}\sqrt{Lx^2 + 2Mx + N}. \qquad (5)$$

Then equation (3) takes the form

$$y = y' \pm Y. \qquad (6)$$



Draw the line HK represented by equation (4).

Then equation (6) shows that for any given value of x, say OQ, the ordinates of the curve, QP, QP', may be found by adding to and subtracting from the corresponding ordinate, $y' = QR$, of the line HK, the same quantity Y.

$$\therefore \quad P'R = RP = Y, \quad \text{and} \quad P'P = 2Y. \tag{7}$$

Hence the chord joining two points of the curve which have the same abscissa, *i. e.* any chord parallel to the y-axis, is bisected by the line HK. This line is, therefore, a diameter of the curve (§ 126), and Y is the length of the ordinate measured from this diameter. The form of the curve, therefore, is determined by the nature of the function Y; and the construction of the locus is reduced to the study of the trinomial

$$Lx^2 + 2Mx + N.$$

Let x' and x'' be the roots of this trinomial, x' being the smaller; then we may write (§ 89)

$$Y = \frac{1}{b}\sqrt{L(x - x')(x - x'')}. \tag{8}$$

Draw the two lines $Q'A'$ and $Q''A''$, whose equations are, respectively,

$$x = x' \quad \text{and} \quad x = x''.$$

Now when $x = x' = OQ'$, and also when $x = x'' = OQ''$ in equation (3), and hence in (8), we have $Y = 0$.

$$\therefore \quad P'P = 0, \text{ from (7).}$$

That is, the curve intersects each of the lines $Q'A'$, $Q''A''$ in two coincident points at A', A'', respectively. Therefore the lines $Q'A'$ and $Q''A''$ are tangents to the curve; and since they are parallel to the chords bisected by HK, the points of contact are the extremities of the diameter HK. (§ 148, Cor. IV.)

As the *form* of the conic depends mainly on the *sign* of the coefficient L, there are three principal cases to be considered, each of which may be subdivided into several others, according to the nature of the roots x', x'' of the trinomial.

The Ellipse. $L \equiv h^2 - ab < 0$.

163. Consider the case when the coefficient L has a negative value, say $-K$. We may then write [(8), § 162]

$$Y = \frac{1}{b}\sqrt{-K(x - x')(x - x'')},$$

or

$$= \frac{1}{b}\sqrt{K(x' - x)(x - x'')}.$$

I. Let $M^2 - LN \equiv M^2 + KN > 0$.

Then x' and x'' are real and unequal.

When x is either less than x', or greater than x'', the two factors $(x' - x)$ and $(x - x'')$ have different signs; hence their product is negative, and therefore Y is imaginary. That is, there are no real points on the locus to the left of $Q'A'$ or to the right of $Q''A''$. (Fig. § 162.)

For every value of x taken between the limits x' and x'' (*i. e.* such that $x' < x < x''$) the factors $(x' - x)$ and $(x - x'')$ are both negative; hence their product is positive, and therefore Y is real. Moreover, as x varies from x' to x'', Y starts with the value zero, is always finite, and returns to zero. The locus is, therefore, a closed curve lying between the two tangent lines $Q'A'$ and $Q''A''$, and passing through the points A', A''.

Therefore the conic is an ellipse.

II. Let $M^2 - LN \equiv M^2 + KN = 0$.

Then the two roots x', x'' are equal, and we have

$$Y = \frac{1}{b}(x - x')\sqrt{-K}.$$

The quantity Y is, therefore, imaginary for all values of x, except $x = x'$, and then $Y = 0$. The two tangents $Q'A$ and $Q''A''$ then coincide, A'' coincides with A', the locus reduces to a single point on the diameter HK, and is called a *point ellipse.*

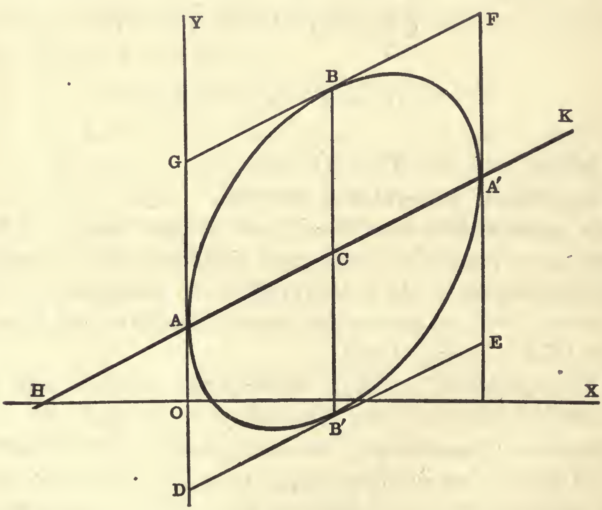
III. Let $M^2 - LN \equiv M^2 + KN < 0$.

The trinomial may be written

$$-Kx^2 + 2Mx + N \equiv -K\left[\left(x - \frac{M}{K}\right)^2 - \frac{M^2 + KN}{K^2}\right].$$

The quantity within the bracket is positive for all values of x, since $M^2 + KN$ is negative. Hence Y is imaginary for every value of x. The given equation, therefore, has no real solution, and consequently does not represent a real geometrical locus.

Ex. *Trace the conic* $3x^2 - 2xy + 2y^2 - 16x - 8y + 8 = 0$.



Here $L \equiv h^2 - ab = 1 - 6 = -5$.

Hence the curve is an ellipse. Solving the equation with respect to y gives

$$y = \tfrac{1}{2}x + 2 \pm \tfrac{1}{2}\sqrt{-5x(x-8)}.$$

Therefore we have the diameter HK represented by the equation

$$y = \tfrac{1}{2}x + 2;$$

and

$$Y = \tfrac{1}{2}\sqrt{-5x(x-8)}.$$

Placing the quantity under the radical sign equal to zero gives

$$5x(x-8) = 0.$$

The roots of this equation are $x' = 0$ and $x'' = 8$.

When x is less than 0, or greater than 8, Y is imaginary; while for all values of x between 0 and 8, Y is real and finite, but reduces to 0 both when $x = 0$ and when $x = 8$.

Therefore the curve lies between the tangents $x = 0$ and $x = 8$.

The equation of the diameter BB' parallel to the y-axis, and therefore conjugate to HK (§ 149), is

$$x = \tfrac{1}{2}(x' + x'') = 4.$$

When $x = 4$, then $Y = 2\sqrt{5} = 4.4+$.

Measure $CB = CB' = 4.4$, and through B, B' draw lines parallel to HK. The curve lies within the parallelogram $DEFG$ thus formed, and is tangent to its sides at the points A, A', B, B'.

The Parabola. $L \equiv h^2 - ab = 0$.

164. Suppose next that the coefficient L is zero. Equation (5), § 162, then takes the form

$$Y = \frac{1}{b}\sqrt{2Mx + N}.$$

One root x'' of the trinomial is now infinite (§ 98, III.), and consequently the tangent $Q''A''$ has moved off to an infinite distance.

I. If $M \neq 0$, the equation of the other tangent $Q'A'$ is

$$x = -\frac{N}{2M}.$$

As x varies from $-\frac{N}{2M}$ to infinity on the *positive side* of this line, the values of Y are real and vary from 0 to ∞; while for all values of x on the *negative side* of this tangent Y will be imaginary. The conic is therefore a parabola, since it consists of a single infinite branch. It passes through the point A', lies on the *positive side* of the tangent $2Mx + N = 0$, and HK is the diameter which bisects all chords parallel to the y-axis.

II. If $M = 0$, both tangents are at infinity (§ 98, III.), and the given equation (3), § 162, reduces to

$$y = -\frac{hx + f}{b} \pm \frac{1}{b}\sqrt{N}.$$

If N is positive, this equation represents two real straight lines which are parallel to the diameter HK and equidistant from it. If $N = 0$, these two parallel lines coincide with HK. If N is negative, the equation has no real solution, and therefore the two lines are imaginary.

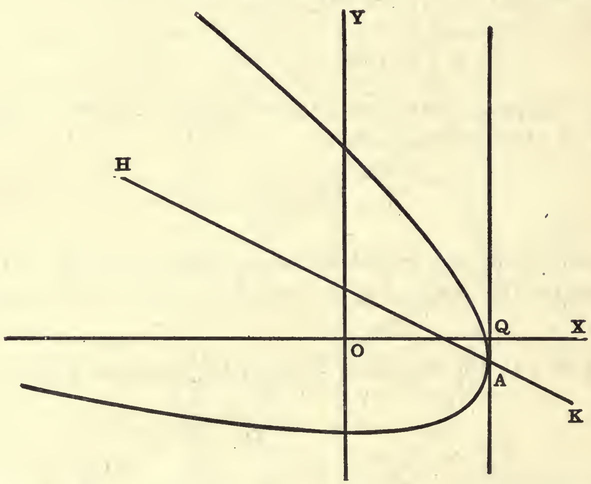
Ex. *Let the given equation be*

$$x^2 + 4xy + 4y^2 + 8x - 8y - 32 = 0.$$

Solving for y we have

$$y = -\tfrac{1}{2}x + 1 \pm \tfrac{1}{2}\sqrt{-12x + 36}.$$

$\therefore$ $L = 0$, and the curve is a parabola.



The equation of HK is

$$y = -\tfrac{1}{2}x + 1,$$

and $$Y = \tfrac{1}{2}\sqrt{-12x + 36}.$$

When $x = 3$, $Y = 0$; and when $x > 3$, Y is imaginary.

Therefore the curve lies to the left of the line $x = 3$.

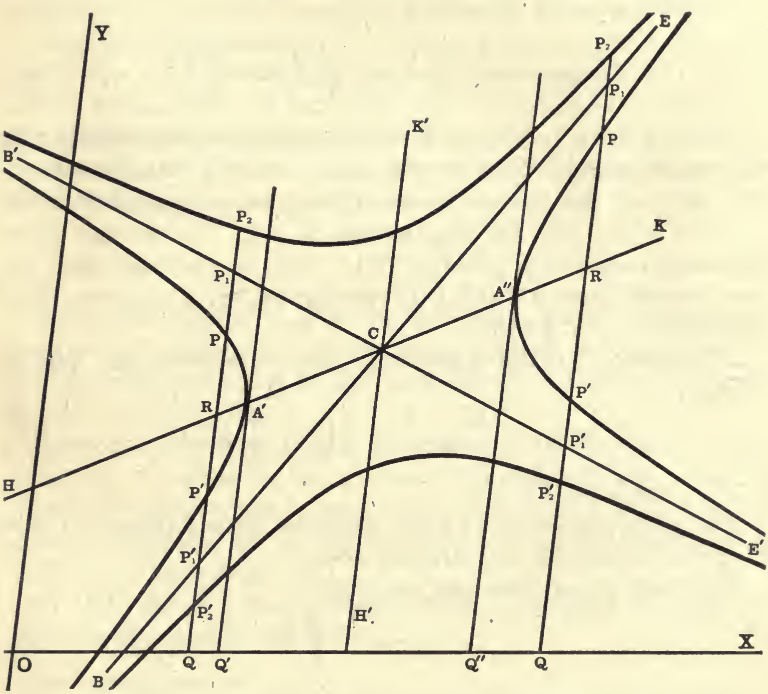
When $x = 0$, $y = 1 \pm 3$. Hence the curve passes through the points (0, 4) and (0, -2).

THE HYPERBOLA. $L \equiv h^2 - ab > 0$.

165. Finally, suppose that L is positive.

Then $$Lx^2 + 2Mx + N \equiv L(x - x')(x - x''),$$

and $$Y = \frac{1}{b}\sqrt{L(x - x')(x - x'')}.$$



I. Let $M^2 - LN > 0$.

Then x' and x'' are real and unequal.

When x is greater than x' and less than x'' (*i. e.* $x' < x < x''$), Y is imaginary. That is, there are no points on the curve lying between the two tangents $Q'A'$ and $Q''A''$. As x varies from x'' to $+\infty$, or from x' to $-\infty$, Y is real and varies from 0 to ∞. Hence the curve consists of two distinct infinite branches, the one lying to the right of $Q''A''$, the other to the left of $Q'A'$. That is, the conic is a hyperbola.

The Asymptotes.

Since $$Lx^2 + 2Mx + N \equiv L\left(x + \frac{M}{L}\right)^2 - \frac{M^2 - LN}{L},$$

the given equation [(3), § 162] may be written

$$y = -\frac{hx + f}{b} \pm \frac{1}{b}\sqrt{L\left(x + \frac{M}{L}\right)^2 - \frac{M^2 - LN}{L}}. \tag{1}$$

Consider now the equation

$$y_1 = -\frac{hx + f}{b} \pm \frac{1}{b}\sqrt{L\left(x + \frac{M}{L}\right)^2}. \tag{2}$$

When x has a very large *numerical* value, the first term under the radical sign in (1) is very large as compared with the numerical value of the second term. Hence, as x approaches either $+\infty$ or $-\infty$, the limiting values of y in (1) are the corresponding values of y_1 given by (2). That is, the conic and the two straight lines BE and $B'E'$ represented by (2) come together at infinity. (*Cf.* § 116.)

Therefore (2) is the equation of the asymptotes, and may be written

$$y = -\frac{(hx + f)}{b} \pm \frac{1}{b}\left(x + \frac{M}{L}\right)\sqrt{L}. \tag{3}$$

II. Let $M^2 - LN = 0$.

The given equation (1) then takes the same form as (2), and therefore represents two straight lines.

The roots of the trinomial are then

$$x' = x'' = -\frac{M}{L}.$$

Hence the two lines intersect on the diameter HK in the point for which $x = -\frac{M}{L}$, since this value of x makes $Y = 0$.

III. Let $M^2 - LN < 0$.

Then the two roots x' and x'' of the trinomial are imaginary. In this case the curve has no tangents parallel to the y-axis.

The trinomial

$$Lx^2 + 2Mx + N \equiv L\left(x + \frac{M}{L}\right)^2 + \frac{LN - M^2}{L}$$

is now the sum of two positive quantities, and therefore *the value of Y is real for all values of x and never becomes zero.* Hence the curve does not cut the diameter HK. The given equation may now be written

$$y = -\frac{hx+f}{b} \pm \frac{1}{b}\sqrt{L\left(x+\frac{M}{L}\right)^2 + \frac{LN-M^2}{L}}, \tag{4}$$

which shows that the asymptotes are the two lines given by equation (3), as under condition I.

Comparing equations (1), (2), and (4), we see that for the same value of x the value of Y is least in (1) and greatest in (4); *i. e.* $RP < RP_1 < RP_2$. Therefore the loci of (1) and (4) lie in different angles of the asymptotes.

The value of y given by (4) is least when the first term under the radical sign is zero; *i. e.* when

$$x = -\frac{M}{L}. \tag{5}$$

Since this line (5), $H'K'$ in the figure, passes through the intersection of the asymptotes, and is parallel to the y-axis, it is the diameter conjugate to HK, and therefore bisects all chords parallel to HK (§ 149).

It should be noticed that if the change in the sign of $M^2 - LN$ is due to a change in the value of the constant term c of the given equation (which cannot affect L and M, § 162), the two conics represented by (1) and (4) will have the same asymptotes. (*Cf.* § 116, II., and § 146.)

If $M^2 - LN < 0$, it is generally more convenient to solve the given equation with respect to x.

Ex. *Trace the curve*

$$4y^2 - 7xy - 2x^2 - 4x - 7y + 18\tfrac{1}{4} = 0. \tag{1}$$

Solving for y gives

$$y = \tfrac{1}{8}\left\{7x + 7 \pm 9\sqrt{x^2+2x-3}\right\}. \tag{2}$$

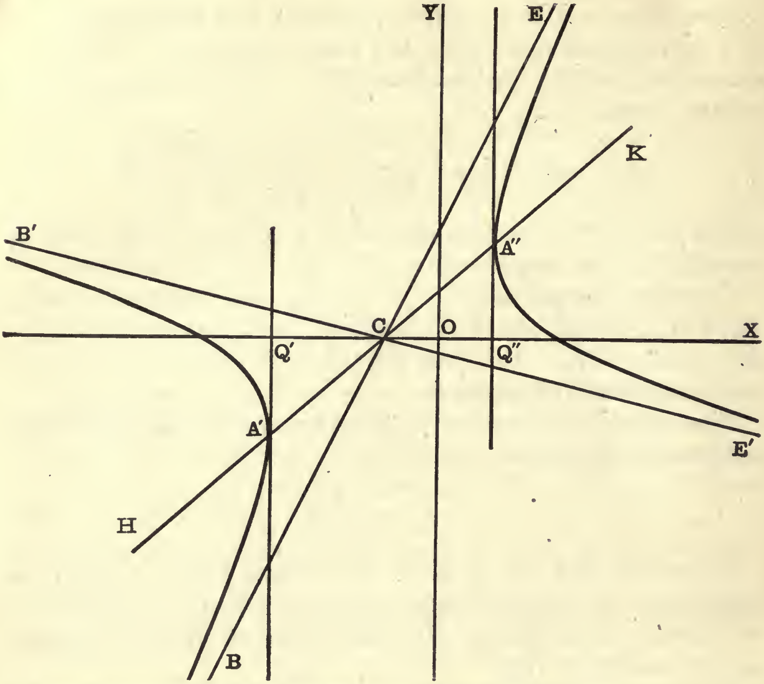
Since L is now positive, the curve is a hyperbola.

The equation of HK is

$$y = \tfrac{7}{8}(x+1);$$

and $$Y = \tfrac{9}{8}\sqrt{(x+3)(x-1)}.$$

$$\therefore\ x' = -3,\quad x'' = 1.$$



Hence the equations of the tangents $Q'A'$ and $Q''A''$ are

$$x+3=0 \quad \text{and} \quad x-1=0. \tag{3}$$

When x takes any value between the limits -3 and $+1$, the value of Y is imaginary, so that no part of the curve lies between the lines $x+3=0$ and $x=1$.

Completing the square under the radical sign in equation (2), we have

$$y=\tfrac{1}{8}\left\{7x+7\pm 9\sqrt{(x+1)^2-4}\right\}. \tag{4}$$

Dropping the constant -4, we finally have

$$y=\tfrac{7}{8}(x+1)\pm\tfrac{9}{8}(x+1). \tag{5}$$

Therefore the equations of the asymptotes BE, $B'E'$ are

$$y=2x+2 \quad \text{and} \quad 4y+x+1=0. \tag{6}$$

The curve is the hyperbola shown in the figure.

166. $b=0$. It has so far been assumed that the coefficient b was not zero. If $b=0$, and $a\neq 0$, the equation can be solved with respect to x and the curve can be traced by the methods already given. When either a or b is zero, the conic is a hyperbola, for h^2-ab is then positive, unless h is also zero, when the locus is a parabola. In case, however, a variable appears only in the first power, it is preferable to solve the equation with respect to that variable.

Suppose the given equation to be

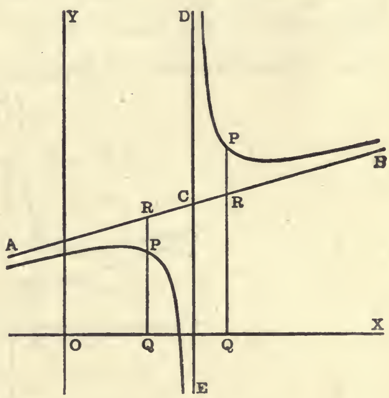
$$ax^2 + 2hxy + 2gx + 2fy + c = 0. \qquad (1)$$

Solving with respect to y gives

$$y = -\frac{ax^2 + 2gx + c}{2(hx + f)}. \qquad (2)$$

If we perform the division in the second member of (2) until a remainder is found that does not contain x, the equation will reduce to the form

$$y = a'x + b' + \frac{c'}{x - d}. \qquad (3)$$



Let AB be the line

$$y = a'x + b',$$

and DE the line

$$x - d = 0.$$

Let

$$Y = \frac{c'}{x - d} = RP,$$

and suppose $c' > 0$, in order to fix the ideas.

For any given value of x, say OQ, there is only *one* value of Y, and only one value of y given by (1), which is found by adding the quantity $Y = RP$ to the ordinate QR of the line AB. Moreover, Y is positive or negative according as $x >$ or $< d$ (since $c' > 0$); *i. e.* according as QP is to the right or left of ED. When x is very slightly less than d, Y is very great and negative; when x is very slightly greater than d, Y is very great and positive. As x, increasing, approaches d, *i. e.* as QR approaches ED from the left, $Y = RP = -\infty$; while as QR approaches ED from the right, x decreasing to d, $Y = RP = +\infty$. Furthermore, when $x = +\infty$ or $-\infty$, $Y = 0$.

Therefore the two lines AB and ED are the asymptotes, and the curve lies in the angles ACE and BCD.

If $c' < 0$, then Y will be positive when $x < d$, and negative when $x > d$. Hence the curve will lie in the angles ACD and BCE.

If $c' = 0$, the numerator of the second member of (2) is divisible by the denominator, and the given equation may be written

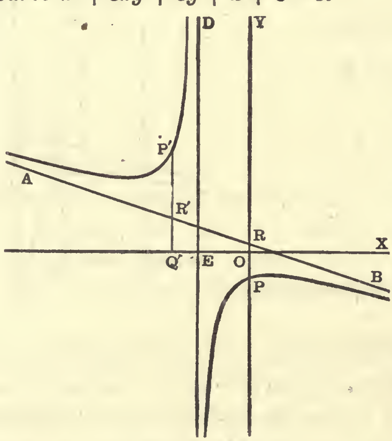
$$(y - a'x - b')(x - d) = 0,$$

which represents two intersecting lines, one of which is parallel to the y-axis.

If $a = b = 0$, the conic can be traced in the same way. We then have $a' = 0$, and therefore the asymptotes are parallel to the axes of coordinates.

If b and h are both zero, the general equation takes the form $y = a'x^2 + b'x + c'$. The curve is then a parabola with axis parallel to the y-axis, and is easily constructed.

Ex. *Trace the curve* $x^2 + 3xy + 6y + x + 6 = 0$.



Solving for y gives $\qquad y = -\frac{x-1}{3} - \frac{1}{3}\left(\frac{8}{x+2}\right).$

$$\therefore \quad Y = -\frac{1}{3}\left(\frac{8}{x+2}\right).$$

When $x < -2$, Y is positive. When $x = -2$, Y is infinite.

When $x > -2$, Y is negative. When $x = +\infty$, or $-\infty$, $Y = 0$.

$$\therefore \quad y = -\frac{x-1}{3} \quad \text{and} \quad x + 2 = 0$$

are the equations of the asymptotes AB and ED.

The curve passes through the points $P(0, -1)$ and $P'(-3, 4)$; its position with reference to the axes is shown in the figure.

EXAMPLES.

Trace the conics given by the following equations:

1. $x^2 - 4xy + 4y^2 - 8y = 0.$
2. $5x^2 + 4xy + 4y^2 - 12x - 24y = 0.$
3. $2x^2 + 2xy - 4y^2 - 6x + 6y \pm 9 = 0.$
4. $10x^2 + 6xy + y^2 + 8x + 4y + 8 = 0.$
5. $4x^2 - 12xy + 9y^2 + 6x - 9y - 4 = 0.$
6. $4x^2 - 6xy - 4y^2 + 28x + 4y + 49 = 0.$
7. $4x^2 - 4xy + y^2 + 12x + 6y + 25 = 0.$
8. $16x^2 - 16xy + 12y^2 + 96x - 58y + 81 = 0.$
9. $13x^2 + 6xy + y^2 + 4 = 0.$
10. $xy + 3x - 2y = 0.$
11. $45xy - 85x - 63y + 119 = 0.$
12. $4xy - x^2 + 4x - 8y - 8 \pm 24 = 0.$
13. $x^2 + 4xy + 4y^2 + 4x + 8y - 12 = 0.$
14. $y^2 - 2xy + 2x + y - 2 \pm 2 = 0.$
15. $3x^2 - 4xy + 2y^2 + 2x - 8y + 17 = 0.$
16. $9x^2 - 24xy + 16y^2 + 48x - 64y + 64 = 0.$
17. $2x^2 + 7xy - 4y^2 + 4x + 7y + 22\frac{1}{4} = 0.$
18. $4x^2 - 4xy + y^2 + 4x - 2y + 5 = 0.$
19. $x^2 + 3xy + 9x + 3y + 18 \pm 18 = 0.$
20. $9x^2 + 12xy + 4y^2 - 7x - 8y - 11 = 0.$
21. $6x^2 + 7xy - 3y^2 + 5x + 13y - 4 = 0.$
22. $3x^2 - 2xy + 2y^2 - 22x - 6y + 27 = 0.$
23. $4x^2 + 7xy - 2y^2 + 15x + 3y + 90 = 0.$
24. $2x^2 - 2xy + y^2 - 2x + 2y + 3 = 0.$
25. $16x^2 + 24xy + 9y^2 - 16x - 12y + 6 = 0.$
26. $2y^2 + 2xy + 5x - y - 20 = 0.$
27. $12xy - 33x + 8y - 66 = 0.$
28. $91x^2 - 117xy - 21x + 27y - 33 = 0.$

To Transform the General Equation of the Second Degree to One of the Standard Forms.

167. *When* $h^2 - ab \neq 0$.

It has been shown in § 109 that the equations for finding the centre (x', y') of the conic represented by the general equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad (1)$$

are $\quad ax + hy + g = 0$ and $hx + by + f = 0$;*

whence
$$x' = \frac{bg - fh}{h^2 - ab}, \quad y' = \frac{af - gh}{h^2 - ab}; \quad (2)$$

and when the origin is moved to the centre without changing the direction of the axes, the equation (1) reduces to

$$ax^2 + 2hxy + by^2 + c' = 0, \quad (3)$$

where
$$c' = gx' + fy' + c.$$

Hence, if $h^2 - ab \neq 0$, the coordinates of the centre are both finite, and this transformation is possible.

In order to reduce (3) to any one of the standard forms (§§ 119–121) we must remove the term $2hxy$. For this purpose we turn the axes through a certain angle θ, keeping the origin fixed.

To turn the axes through an angle θ we substitute for x and y, respectively [§ 66, (11)],

$$x \cos\theta - y \sin\theta \quad \text{and} \quad x \sin\theta + y \cos\theta.$$

Substituting these values in (3), expanding and collecting terms, we have

$$\begin{aligned}(a \cos^2\theta + 2h \sin\theta \cos\theta + b \sin^2\theta)x^2 \\ + 2[(b - a) \sin\theta \cos\theta + h(\cos^2\theta - \sin^2\theta)]xy \\ + (a \sin^2\theta - 2h \sin\theta \cos\theta + b \cos^2\theta)y^2 + c' = 0. \quad (4)\end{aligned}$$

The coefficient of xy in equation (4) will vanish if θ be so chosen that

$$2(b - a) \sin\theta \cos\theta + 2h(\cos^2\theta - \sin^2\theta) = 0.$$

---

* It should be noticed here that the line $hx + by + f = 0$ bisects all chords parallel to the y-axis. (See (4), § 164.) By solving equation (1) for x, it can also be shown that the line $ax + hy + g = 0$ bisects all chords parallel to the x-axis. (*Cf.* also § 84.)

This equation is equivalent to

$$(a-b)\sin 2\theta = 2h\cos 2\theta. \tag{5}$$

$$\therefore \quad \tan 2\theta = \frac{2h}{a-b}.* \tag{6}$$

Whence $$\sin 2\theta = \pm\frac{2h}{\sqrt{(a-b)^2+4h^2}}, \tag{7}$$

and $$\cos 2\theta = \pm\frac{a-b}{\sqrt{(a-b)^2+4h^2}}. \tag{8}$$

Using this value of θ equation (4) takes the form

or $$\left.\begin{aligned} a'x^2+b'y^2+c'=0,\\ \frac{x^2}{-\frac{c'}{a'}}+\frac{y^2}{-\frac{c'}{b'}}=1,\end{aligned}\right\} \tag{9}$$

where $$a' \equiv a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta, \tag{10}$$

and $$b' \equiv a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta. \tag{11}$$

Equation (9) is therefore the required result.

The values of a' and b' may be expressed in terms of a, b, and h as follows:

From (10) and (11), by addition and subtraction, we obtain

$$a'+b'=a+b, \tag{12}$$

and $$a'-b'=(a-b)\cos 2\theta + 2h\sin 2\theta. \tag{13}$$

Substituting (7) and (8) in (13) gives

$$a'-b' = \pm\sqrt{(a-b)^2+4h^2} = \frac{2h}{\sin 2\theta}. \tag{14}$$

Whence, from (12) and (14),

$$a' = \tfrac{1}{2}\left\{a+b\pm\sqrt{(a-b)^2+4h^2}\right\}, \tag{15}$$

and $$b' = \tfrac{1}{2}\left\{a+b\mp\sqrt{(a-b)^2+4h^2}\right\}. \tag{16}$$

The ambiguity in the values of a' and b' given by (15) and (16) may be removed by (14). From the many values of θ which satisfy (6) we will agree always to choose that one which lies between 0° and 180°. Then θ will always be an *acute* angle, and

*Cf. equations (17), § 109.

sin 2θ will always be *positive*. Therefore it follows from (14) that $a' - b'$ will always have the *same sign as h*.

It is also worthy of notice that the values of a' and b' given by (15) and (16) are the two roots of the equation

$$x^2 - (a + b)x - (h^2 - ab) = 0. \qquad (17)$$

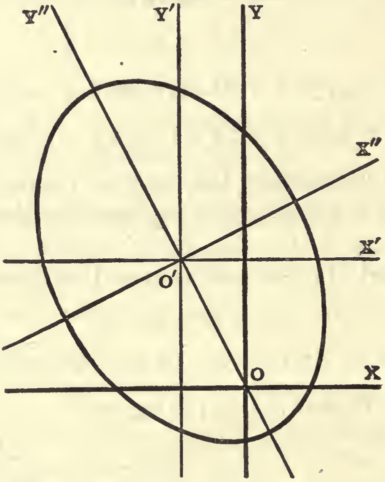
Hence a' and b' will have the *same sign* or *opposite* signs according as $h^2 - ab < \text{or} > 0$; *i. e.* according as the curve is an ellipse or a hyperbola (§ 110).

If $h^2 - ab = 0$, *i. e.* if the curve is a parabola, the roots of (17) are 0 and $a + b$.

Ex. *Transform the equation*

$$8x^2 + 4xy + 5y^2 + 8x - 16y - 16 = 0$$

to the standard form, and construct the conic.



The equations for finding the centre are

$$4x + y + 2 = 0 \quad \text{and} \quad 2x + 5y = 8.$$

$$\therefore \quad x' = -1, \quad y' = 2.$$

Then $$c' \equiv gx' + fy' + c = -36.$$

Therefore the equation referred to parallel axes $O'X'$, $O'Y'$ through the centre is

$$8x^2 + 4xy + 5y^2 = 36.$$

Also $$a' = \tfrac{1}{2}\left\{a + b \pm \sqrt{(a - b)^2 + 4h^2}\right\} = \tfrac{1}{2}(13 \pm 5) = 9 \text{ or } 4,$$

and $$b' = \tfrac{1}{2}\left\{a + b \mp \sqrt{(a - b)^2 + 4h^2}\right\} = \tfrac{1}{2}(13 \mp 5) = 4 \text{ or } 9.$$

Since h is positive, we take $a'=9$ and $b'=4$.

Hence the equation of the curve referred to its own axes $O'X''$, $O'Y''$ as axes of coordinates is

$$\frac{x^2}{4}+\frac{y^2}{9}=1.$$

Also, $$\tan 2\theta=\frac{2h}{a-b}=\frac{4}{3}.$$

Therefore the line $O'X''$ must be drawn so that

$$\angle X'O'X''=\tfrac{1}{2}\tan^{-1}\tfrac{4}{3}.$$

168. *When* $h^2-ab=0$.

In this case the coordinates of the centre [(2), § 167] are both infinite, and therefore the first degree terms can not be removed by changing to a new system of axes parallel to the old.

Since the second degree terms now form a perfect square, the general equation may be written

$$(\beta y+\alpha x)^2+2gx+2fy+c=0, \qquad (1)$$

where $\alpha\equiv\sqrt{a}$, $\beta\equiv\sqrt{b}$, α has the same sign as h, and β is always positive.

$$\therefore\quad h=\alpha\beta. \qquad (2)$$

First Method. From equation (6), § 167, we have

$$\tan 2\theta=\frac{2h}{a-b}=\frac{2\alpha\beta}{\alpha^2-\beta^2}=\frac{2\tan\theta}{1-\tan^2\theta}. \qquad (3)$$

$$\therefore\quad \tan\theta=\frac{\beta}{\alpha},\quad\text{or}\quad -\frac{\alpha}{\beta}. \qquad (4)$$

If we turn the axes through an angle given by either of these values of $\tan\theta$, the coefficient of xy in the new equation will vanish. If we take $\theta=\tan^{-1}\left(-\frac{\alpha}{\beta}\right)$, the equation of the new x-axis will be

$$\alpha x+\beta y=0. \qquad (5)$$

We will use this value, and will agree always to take the positive direction of the new x-axis so that θ shall be *numerically less than* 90°. Then θ will be positive or negative according as h (or α) is negative or positive, and we have from (4)

$$\sin\theta=\frac{-\alpha}{\sqrt{\alpha^2+\beta^2}},\qquad \cos\theta=\frac{\beta}{\sqrt{\alpha^2+\beta^2}}.$$

Hence, to turn the axes through an angle θ thus chosen, we must substitute for x and y, respectively [§ 66, (11)],

$$\frac{\beta x + \alpha y}{\sqrt{\alpha^2 + \beta^2}} \quad \text{and} \quad \frac{-\alpha x + \beta y}{\sqrt{\alpha^2 + \beta^2}}. \tag{6}$$

Substituting the expressions (6) in (1) gives

$$(\alpha^2 + \beta^2)y^2 + 2\frac{\beta g - \alpha f}{\sqrt{\alpha^2 + \beta^2}}x + 2\frac{\alpha g + \beta f}{\sqrt{\alpha^2 + \beta^2}}y + c = 0. \tag{7}$$

Completing the square in the terms containing y, equation (7) may be reduced to the form

$$(y - K)^2 = 2\frac{\alpha f - \beta g}{\sqrt{(\alpha^2 + \beta^2)^3}}(x - H), \tag{8}$$

where $$H \equiv \frac{c(\alpha^2 + \beta^2)^2 - (\alpha g + \beta f)^2}{2(\alpha f - \beta g)\sqrt{(\alpha^2 + \beta^2)^3}},$$

and $$K \equiv -\frac{\alpha g + \beta f}{\sqrt{(\alpha^2 + \beta^2)^3}}.$$

If now the origin be moved to the point (H, K), equation (8) will take the standard form

$$y^2 = 2\frac{\alpha f - \beta g}{\sqrt{(\alpha^2 + \beta^2)^3}}x. \tag{9}$$

Therefore equation (1) represents a parabola whose axis is parallel to the line (5), and whose latus rectum is

$$\frac{2(\alpha f - \beta g)}{\sqrt{(\alpha^2 + \beta^2)^3}}.$$

Second Method. Equation (1) may be written

$$(\alpha x + \beta y + \lambda)^2 = 2(\alpha\lambda - g)x + 2(\beta\gamma - f)y + \lambda^2 - c, \tag{10}$$

where λ is any constant, for which a particular value will now be determined.

We observe that the line whose equation is

$$\alpha x + \beta y + \lambda = 0 \tag{11}$$

is parallel to the axis of the parabola [see (5) above] for all values of λ. Hence we will choose λ so that the straight line

$$2(\alpha\lambda - g)x + 2(\beta\lambda - f)y + \lambda^2 - c = 0 \tag{12}$$

shall be perpendicular to the line (11).

The lines (11) and (12) will be at right angles (§ 48) if

$$\alpha(\alpha\lambda - g) + \beta(\beta\lambda - f) = 0,$$

i. e. if

$$\lambda = \frac{\alpha g + \beta f}{\alpha^2 + \beta^2}. \tag{13}$$

With this value of λ equation (10) may be written

$$(\alpha x + \beta y + \lambda)^2 = 2\frac{\alpha f - \beta g}{\alpha^2 + \beta^2}(\beta x - \alpha y + K), \tag{14}$$

where

$$K \equiv \frac{\alpha^2 + \beta^2}{\alpha f - \beta g}\left(\frac{\lambda^2 - c}{2}\right). \tag{15}$$

Changing the linear expressions in (14) to the distance form gives

$$\left(\frac{\alpha x + \beta y + \lambda}{\sqrt{\alpha^2 + \beta^2}}\right)^2 = 2\frac{\alpha f - \beta g}{\sqrt{(\alpha^2 + \beta^2)^3}}\left(\frac{\beta x - \alpha y + K}{\sqrt{\alpha^2 + \beta^2}}\right). \tag{16}$$

If now we take the lines

$$\alpha x + \beta y + \lambda = 0 \tag{17}$$

and

$$\beta x - \alpha y + K = 0 \tag{18}$$

for new axes of x and y, respectively, the new equation will be (§ 70 and § 71)

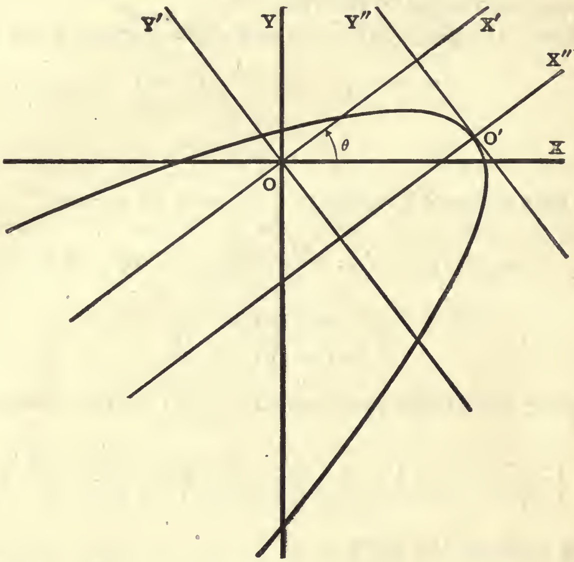
$$y^2 = 2\frac{\alpha f - \beta g}{\sqrt{(\alpha^2 + \beta^2)^3}}x. \tag{19}$$

Therefore, if we assign to λ and K the values given by (13) and (15), then (17) will represent the axis of the parabola and (18) the tangent at the vertex. For any other value of λ, (17) is a diameter and (18) is the tangent at its extremity; the equation of the curve (19) will then be expressed in terms of the perpendiculars upon the oblique axes. (See Ex. 3, § 71.) In all cases the curve will lie on the *positive* or *negative side* of the line (18) according as $(\alpha f - \beta g)$ is *positive* or *negative.**

* In the investigations of §§ 162–166, and again in §§ 167, 168, we have really shown by independent methods that the general equation of the second degree always represents a conic section. In the latter the conditions for the limiting cases have not been pointed out. The student should do this, and compare the results of both these discussions with the table given in § 110.

Ex. *Find the standard form of the equation*

$$(4y-3x)^2-20x+110y-75=0. \tag{1}$$



First Method. Take $4y-3x=0$ as the new x-axis; *i. e.* turn the axes through an angle θ, such that $\tan\theta=\frac{3}{4}$, and therefore $\sin\theta=\frac{3}{5}$, $\cos\theta=\frac{4}{5}$.

Then the formulas of transformation are

$$x=x'\cos\theta-y'\sin\theta=\frac{4x'-3y'}{5},$$

and
$$y=x'\sin\theta+y'\cos\theta=\frac{3x'+4y'}{5}.$$

Substituting these values in equation (1), it becomes

$$y'^2+2x'+4y'-3=0,$$

or
$$(y'+2)^2=-2(x'-\tfrac{7}{2}), \tag{2}$$

which is the equation of the curve referred to the new axes OX', OY'.

Moving the origin to the point $O'(\frac{7}{2}, -2)$, with respect to the new axes, we obtain from (2) the required equation

$$y''^2=-2x''. \tag{3}$$

Hence the curve is on the negative side of the y-axis $O'Y''$.

Second Method. The given equation (1) may be written

$$(4y-3x+\lambda)^2=(20-6\lambda)x+(8\lambda-110)y+\lambda^2+75. \tag{4}$$

We will now determine λ so that the two lines

$$4y-3x+\lambda=0 \tag{5}$$

and
$$(20-6\lambda)x+(8\lambda-110)y+\lambda^2+75=0 \tag{6}$$

shall be at right angles.

The required value of λ is given by the equation (§ 48)

$$-3(20-6\lambda)+4(8\lambda-110)=0.$$

$$\therefore \quad \lambda = 10.$$

With this value of λ equation (4) becomes

$$(4y-3x+10)^2=-10(4x+3y-17\tfrac{1}{2}),$$

or

$$\left(\frac{4y-3x+10}{5}\right)^2=-2\left(\frac{4x+3y-17\frac{1}{2}}{5}\right). \qquad (7)$$

Draw the lines

$$4y-3x+10=0,\ O'X'', \qquad (8)$$

and

$$4x+3y-17\tfrac{1}{2}=0,\ O'Y''. \qquad (9)$$

These lines are at right angles. If we take (8) as the new x-axis and (9) as the new y-axis, the equation of the curve will be (§ 70)

$$y^2=-2x. \qquad (10)$$

Therefore the locus is a parabola whose latus rectum is 2, and lies on the *negative side* of the line (9).

EXAMPLES.

Construct the following conics by transforming the equations to their standard forms:

1. $(4y-3x)^2+4(4x+3y)=0.$
2. $3x^2+2xy+3y^2=8.$
3. $x^2-6xy+y^2=16.$
4. $(3x-4y-12)^2=15(4x+3y).$
5. $4x^2-24xy+11y^2-16x-2y-89=0.$
6. $5x^2-4xy+8y^2-24x+16y-4=0.$
7. $9x^2-12xy+4y^2=10(2x+3y+5).$
8. $3x^2-2xy+2y^2-16x-8y+8=0.$ (See Ex., § 163.)
9. $2x^2+4xy+5y^2=36.$
10. $6x^2+24xy-y^2+50y-55=0.$
11. $x^2-2xy+y^2-5x-y-2=0.$
12. $8x^2-5xy-4y^2=34.$
13. $x^2-6xy+9y^2-2x+6y+1=0.$
14. $4x^2+4xy+y^2+4x-3y+4=0.$
15. $2x^2+xy+3y^2=23.$
16. $24xy+7y^2-6(6x-10y-9)=0.$

17. $25x^2 - 20xy + 4y^2 + 5x - 2y - 6 = 0.$

18. $x^2 - 2xy - y^2 = 20.$

19. $(5y + 12x)^2 = 102x.$

20. $2x^2 + xy - 6y^2 - 5x + 11y - 3 = 0.$

21. $x^2 + 2xy + y^2 - 12x + 2y - 3 = 0.$

22. $xy + 3x - 5y + 5 = 0.$

23. $2x^2 + 7xy - 4y^2 + 4x + 7y - 18\frac{1}{4} = 0.$ (See Ex., § 165.)

169. The standard forms of the equations of the conic sections are

$$y^2 = 4ax,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and

$$xy = K.$$

If the axes be changed in any manner whatever, the new equations are obtained by substituting for x and y expressions of the form (§ 66)

$$lx + my + n, \quad \text{and} \quad l'x + m'y + n'.$$

The new equations may therefore be written

$$(l'x + m'y + n')^2 - 4a(lx + my + n) = 0,$$

$$b^2(lx + my + n)^2 + a^2(l'x + m'y + n')^2 - a^2b^2 = 0,$$

$$b^2(lx + my + n)^2 - a^2(l'x + m'y + n')^2 - a^2b^2 = 0,$$

and

$$(lx + my + n)(l'x + m'y + n') - K = 0.$$

Hence, if the left member of an equation, the right member being zero, is the square of one real linear expression plus some multiple of the first power of another real linear expression, the equation represents a parabola; if the left member is the sum of any multiples of the squares of two real linear expressions plus a constant, the equation represents an ellipse; if the left member is the difference of some multiples of the squares, or (what is the same thing) the product, of two real linear expressions plus a constant, the equation represents a hyperbola.

EXAMPLES.

Transform the following equations, taking as new axes the lines represented by the linear expressions which the equations contain. (See §§ 70, 71.) Construct the curves:

1. $(x+2)^2+3(y-4)=0.$
2. $9(x-3)^2+4(y+5)^2=36.$
3. $9(x+5)^2-16(y-4)^2=144.$
4. $(x+3)(y-5)+9=0.$
5. $(3x-4y+6)^2=10(4x+3y-5).$
6. $4(3x-4y-8)^2+(4x+3y-3)^2=20.$
7. $3(y+2x-2)^2-2(x-2y-6)^2=30.$
8. $(12x+5y-9)^2+52(5x-12y)=0.$
9. $3(2x+3y+6)^2+4(3x-2y+6)^2=156.$
10. $(2y-4x+3)^2-4(x+2y+9)^2=20.$
11. $(7x-24y)^2=40(24x+7y-21).$
12. $(x-3y+12)^2+(3x+y+2)^2=40.$
13. $(y-\sqrt{3}x+1)^2+(y+\sqrt{3}x+5)^2=36.$
14. $(y-2x+4)^2+(3y+4x-3)^2=100.$
15. $2(x+y+1)^2-4(x-y+4)^2=32.$
16. $(x-2y+4)^2+5(3x-y+6)=0.$
17. $5(x+2y-8)^2-4(4y-3x)^2=100.$
18. $(y+2x+3)^2+2(x+3y-6)^2=40.$
19. $(5x+12y-36)(12x-5y-15)=676.$
20. $(y+\sqrt{3}x+3)^2-12(y-\sqrt{3}x-2)=0.$
21. $2(y-3x+9)^2-9(2x+y+1)^2=180.$
22. $(x+4y-10)(4x-y-4)+34=0.$
23. $(x+\sqrt{3}y+\sqrt{3})(x-\sqrt{3}y+2\sqrt{3})+16=0.$
24. $2(x+2y+6)^2+(x-3y-6)^2=40.$
25. $(x-3y+6)^2-2(7y-24x+28)=0.$
26. $(x-2y-4)(x+3y+9)+20\sqrt{2}=0.$
27. $(2x+3y+6)^2+(4x-y-4)^2=221.$
28. $(3x-y+9)^2-2(2x+y-2)^2=90.$

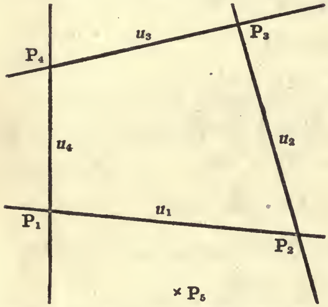
170. *The equation of a conic through given points.*

The general equation of a conic contains five independent constants (§ 108). When the equation is written

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \qquad (1)$$

these constants are the five independent ratios between the six coefficients a, b, c, f, g, h. It follows, therefore, that, *in general, one and only one conic can be made to pass through any five given points in the plane.* For, if we substitute for x and y in (1) the coordinates of the five given points, we shall have five linear equations from which we may determine *uniquely* the values of the five ratios. These values substituted in (1) will give the required equation.

A more convenient method for finding the equation of a conic through five given points is as follows:



Take any four of the given points and connect them so as to form the quadrilateral $P_1P_2P_3P_4$.

Let u_1, u_2, u_3, u_4 be the equations of the lines P_1P_2. P_2P_3, P_3P_4, P_4P_1, respectively.

Then

$$u_1u_3 = 0 \quad \text{and} \quad u_2u_4 = 0$$

are the equations of two conics whose common points are P_1, P_2, P_3, P_4.

$$\therefore \quad u_1u_3 + \lambda u_2u_4 = 0, \qquad (3)$$

whatever the value of λ may be, is the equation of a conic through the four points P_1, P_2, P_3, P_4.

Equation (3) involves one arbitrary parameter λ. Its locus can therefore be made to satisfy a single condition; *e. g.*, it can be made to pass through any other point in the the plane. If we substitute for x and y in (3) the coordinates of P_5, we will get an equation of the first degree in λ. The *one* value of λ thus determined substituted in (3) will give the required equation.

Since equation (3) contains one arbitrary parameter, it follows that *an infinite number of conics can be made to pass through any four given points.*

Four points, however, are sufficient to determine a parabola. This follows from the fact that when equation (1) represents a parabola, one condition connecting the coefficients is always given, viz. $h^2 - ab = 0$.

In order to find the equation of a parabola determined by four given points, we may form equation (3) as before. From the coefficients of the equation thus found we then form the equation

$$h^2 - ab = 0. \qquad (4)$$

Substituting in (3) the value of λ given by (4) will give the equation of the parabola required. Furthermore, (4) will be an equation of the *second degree in* λ. *Therefore two parabolas can be made to pass through any four given points in the plane.*

Let the student discuss the cases when three or more of the given points lie on the same straight line.

171. *To find the equation of a conic having two given lines for its asymptotes.*

Let the equations of the asymptotes be

$$lx + my + n = 0 \quad \text{and} \quad l'x + m'y + n' = 0.$$

Then, since the equation of the conic differs from the equation of its asymptotes only by a constant (§ 117 and § 146), the required equation is

$$(lx + my + n)(l'x + m'y + n') + \lambda = 0, \qquad (1)$$

where λ may have any value whatever.

The conic, therefore, is not uniquely determined, but can still be made to satisfy *one* more condition; that is, it can be made to pass through a given point, or touch a given line, etc.

Since equation (1) involves only *one* arbitrary parameter, having given the asymptotes of a conic is equivalent to having given *four* of the conditions which a conic can be made to satisfy.

How many conditions can a conic be made to satisfy,

(1) if the centre is given?

(2) if the two foci are given?

(3) if one focus and the corresponding directrix are given?

(4) if the position of the axes is given?

(5) if the two directrices are given?

Examples on Chapter XII.

Find the equation of the conics through the points

1. $(4, 3), (2, 5), (-2, 3), (0, -2), (2, -1)$.

2. $(4, 3), (0, 0), (-2, 3), (0, -2), (2, -1)$.

3. $(a, a), (\pm a, 0), (0, \pm a)$.

4. $(1, 2), (-2, 4), (-3, -1), (1, -2), (2, -1)$.

5. $(-1, -1), (2, 3), (-3, 5), (-3, -2), (1, -7)$.

Find the equations of the parabolas through the points

6. $(4, 6), (2, -4), (-2, 0), (-3, 6)$.

7. $(1, 5), (4, 2), (-3, -1), (1, -1)$.

8. $(2, 0), (-4, -2), (2, 7), (0, -3)$.

9. Find the equation of a conic through the origin and having for its asymptotes the lines

$$x-2y+4=0 \quad \text{and} \quad 3x+y-2=0.$$

10. Find the equation of a conic through the point (1, 3), if the equations of its asymptotes are

$$2x-3y-7=0 \quad \text{and} \quad 5y+3x-8=0.$$

11. What is the equation of a conic touching the x-axis, if its asymptotes are the lines

$$2x+y-2=0 \quad \text{and} \quad x-3y+5=0?$$

12. What is the equation of the conic touching the line $y+2x+1=0$, and having for asymptotes

$$x-y-1=0 \quad \text{and} \quad x+3y-6=0?$$

13. Find the equation of the conic which passes through the point $(2, -2)$ and has the same asymptotes as

$$5x^2-2xy-3y^2+4x+12y=0.$$

14. Find the asymptotes of the hyperbola

$$3x^2-7xy-6y^2-8x+2y-4=0;$$

find also the equation of the conjugate hyperbola.

15. The equation of a given hyperbola is

$$3x^2-8xy-3y^2-13x-y-2=0.$$

Find the equation of its conjugate.

16. Show that the product of the semi-axes of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is

$$\frac{-c'}{\sqrt{a'b'}} = \frac{-c'}{\sqrt{ab - h^2}} = \frac{-\triangle}{(ab - h^2)^{\frac{3}{2}}},$$

where $\triangle$ is the discriminant, a', b' are given by (15) and (16), § 167, and c' by (9), § 109.

17. Show that $\frac{5}{\sqrt{-4}}$ is the product of the semi-axes of the conic

$$x^2 - 2xy - 3y^2 - 2x + 10y - 8 = 0.$$

Show also that the equation of the axes of this conic is

$$x - y^2 + xy - 5x + 5 = 0.$$

18. Show that the squares of the semi-axes of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

are

$$2\triangle \div (ab - h^2) \left\{ a + b \pm \sqrt{(a - b)^2 + 4h^2} \right\},$$

where $\triangle$ is the discriminant.

19. Show that the lengths of the semi-axes of the conic

$$ax^2 + 2hxy + ay^2 = K,$$

are

$$\sqrt{\frac{K}{a + h}} \quad \text{and} \quad \sqrt{\frac{K}{a - h}},$$

respectively, and that their equation is $x^2 - y^2 = 0$.

20. Show that of all the conics which pass through the points of intersection of two conics only two are parabolas.

21. Find the equations of the two parabolas which pass through the common points of

$$x^2 - y^2 = 1 \quad \text{and} \quad x^2 + y^2 - 2x = 4.$$

22. Show that all conics passing through the intersections of two rectangular hyperbolas are rectangular hyperbolas.

23. If two rectangular hyperbolas intersect in four real points, the line joining any two of the points of intersection is perpendicular to the line joining the other two. (Ex. 22.)

24. Show that if

$$ax^2 + 2hxy + by^2 = 1 \quad \text{and} \quad a'x^2 + 2h'xy + b'y^2 = 1$$

represent the same conic, and the axes are rectangular, then

$$(a - b)^2 + 4h^2 = (a' - b')^2 + 4h'^2.$$

25. Show that for all positions of the axes, so long as they remain rectangular and the origin is unchanged, the value of $g^2 + f^2$ in the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is constant.

26. If $ax^2 + 2hxy + by^2 = 1$ and $a'x^2 + 2h'xy + b'y^2 = 1$ are the equations of two conics, then will $aa' + 2hh' + bb'$ be unaltered by any change of rectangular axes.

27. The polars of a point P with respect to two given circles meet in Q; if P moves along a given straight line, show that the locus of Q is a hyperbola whose asymptotes are perpendicular to the given line and to the line joining the centres of the circles.

28. A variable circle always passes through a fixed point O and cuts a conic in P, Q, R, S; show that

$$\frac{OP \cdot OQ \cdot OR \cdot OS}{(\text{radius of circle})^2}$$

is constant.

29. If OPQ and $OP'Q'$ are two straight lines which are always parallel to two fixed straight lines, and meet a given conic in P, Q and P', Q', respectively, then will the ratio $\dfrac{OP \cdot OQ}{OP' \cdot OQ'}$ be the same for all positions of O. Find the value of this ratio by putting O at the centre of the conic.

30. Show that the conic $S + \lambda u^2 = 0$ touches the conic $S = 0$ where the latter is cut by the straight line $u = 0$.

31. If two conics have their axes parallel, a circle will pass through their points of intersection.

32. Two conics are said to be similar and similarly placed when their axes are parallel and they have the same eccentricity. (§ 116.)

Hence show that the two conics

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and

$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$$

will be similar if

$$\frac{(a+b)^2}{h^2 - ab} = \frac{(a'+b')^2}{h'^2 - a'b'}.$$

Show also that they will be similarly placed if

$$\frac{h}{a-b} = \frac{h'}{a'-b'}.$$

Then show that they will be both similar and similarly placed if

$$\frac{a}{a'} = \frac{h}{h'} = \frac{b}{b'}.$$

[Use equations (15) and (17) of § 109.]

APPENDIX.

TRIGONOMETRICAL FORMULÆ.

1. $\sin\theta\csc\theta = 1.$

2. $\cos\theta\sec\theta = 1.$

3. $\tan\theta\cot\theta = 1.$

4. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}.$

5. $\sin^2\theta + \cos^2\theta = 1.$

6. $\sec^2\theta - \tan^2\theta = 1.$

7. $\csc^2\theta - \cot^2\theta = 1.$

8. $\sin(-\theta) = -\sin\theta.$

9. $\cos(-\theta) = \cos\theta.$

10. $\sin(90^\circ \pm \theta) = \cos\theta.$

11. $\cos(90^\circ \pm \theta) = \mp\sin\theta.$

12. $\sin(180^\circ \pm \theta) = \mp\sin\theta.$

13. $\cos(180^\circ \pm \theta) = -\cos\theta.$

14. $\sin(270^\circ \pm \theta) = -\cos\theta.$

15. $\cos(270^\circ \pm \theta) = \pm\sin\theta.$

16. $\sin(\theta \pm \theta') = \sin\theta\cos\theta' \pm \cos\theta\sin\theta'.$

17. $\cos(\theta \pm \theta') = \cos\theta\cos\theta' \mp \sin\theta\sin\theta'.$

18. $\tan(\theta \pm \theta') = \dfrac{\tan\theta \pm \tan\theta'}{1 \mp \tan\theta\tan\theta'}.$

19. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}.$

20. $\cot(\theta \pm \theta') = \dfrac{\cot\theta\cot\theta' \mp 1}{\cot\theta' \pm \cot\theta}.$

21. $\cot 2\theta = \dfrac{\cot^2\theta - 1}{2\cot\theta}.$

22. $\sin 2\theta = 2\sin\theta\cos\theta.$

23. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$

24. $\sin\frac{1}{2}\theta = \sqrt{\frac{1}{2}(1 - \cos\theta)}.$

25. $\cos\frac{1}{2}\theta = \sqrt{\frac{1}{2}(1 + \cos\theta)}$

26. $\sin\theta + \sin\theta' = 2\sin\frac{1}{2}(\theta + \theta')\cos\frac{1}{2}(\theta - \theta').$

27. $\sin\theta - \sin\theta' = 2\cos\frac{1}{2}(\theta + \theta')\sin\frac{1}{2}(\theta - \theta').$

28. $\cos\theta + \cos\theta' = 2\cos\frac{1}{2}(\theta + \theta')\cos\frac{1}{2}(\theta - \theta').$

29. $\cos\theta - \cos\theta' = -2\sin\frac{1}{2}(\theta + \theta')\sin\frac{1}{2}(\theta - \theta').$

In any plane triangle

30. $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}.$

31. $\dfrac{a+b}{a-b} = \dfrac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.$

32. $a^2 = b^2 + c^2 - 2bc\cos A.$

33. $\text{Area} = \frac{1}{2}bc\sin A.$