THE ELEMENTS OF ARITHMETIC.

BY

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FOURTH EDITION.

"Ce n'est point par la routine qu'on s'instruit, c'est par sa propre réflexion; et il est essentiel de contracter l'habitude de se rendre raison de ce qu'on fait: cette habitude s'acquiert plus facilement qu'on ne pense; et une fois acquise, elle ne se perd plus."—CONDILLAC.

LONDON:

PRINTED FOR TAYLOR AND WALTON,

BOOKSELLERS AND PUBLISHERS TO UNIVERSITY COLLEGE,

28 UPPER GOWER STREET.

M.DCCC.XL.

LONDON:

PRINTED BY MOYES AND BARCLAY,
Castle Street, Leicester Square.

PREFACE

TO

THE SECOND EDITION.

I HAVE added several new chapters in this Edition, particularly on the Square Root, Proportion, and Permutations and Combinations, with many occasional articles, principally intended to give such ideas on the subject of Algebra as a young arithmetical student may be able, with a little assistance, to comprehend. I have also added six or seven Examples to each Rule, accompanied by the answers. These would be enough for any single pupil, but may not be considered sufficient for a school. To obviate this objection, I proceed to collect some expeditious modes of forming questions, of which the answers shall be readily known. I am aware that the publication of these methods, in a preface equally open to the master and the learner, is something like calling the enemy to council; nevertheless, as the following abbreviations all contain some mathematical principle, and as some facility of computation will be necessary even to make use of them, the master may depend upon it that a pupil who discovers and applies the way to make the answer to any one rule, is fit to pass on to the next.

Addition.—Let a series of numbers be taken, each of which is the *complement* to 10^n of the preceding. Strike out one or more, and arrange the rest miscellaneously. It will be evident how to ascertain whether these have been added up correctly. Many arrangements may be made for recollecting which numbers were struck out: for example, their complements may be made to begin with a given figure, and to be the only ones which begin with that figure.

Another method, preferable perhaps to the former, is the following: Let any series of numbers be taken, such as a, b, c, &c. each of which exceeds the following; let the master form $a-b$, $b-c$, $c-d$, &c. and give the results to the pupil to add together, annexing to them the last number which he used. The answer will be a, which number cannot possibly be recovered by the pupil from the *data*, except by the very operation which he is required to perform. The continued subtractions may be done by one pupil, and the addition made by another; and thus the process may afford examples in the first two rules.

Subtraction.—In addition to the method just explained, the following may be used: Instead of giving one number to be subtracted from another *once* only, let it be required to subtract the first time after time from the second, until it can no longer be subtracted, as in the examples of article 46. This being, in point of fact, a question of division, may be proved by casting out the nines, and this after any number of steps, using the number of subtractions performed as the quotient. Or questions might be formed thus: Subtract 1259 from 12590, until this can no longer be done; or, multiplication by 25, 5, or 9, being very expeditiously done, the minuend might be $25x$, $5x$, or $9x$, and the subtrahend x.

Multiplication and Division.—For these a table of squares and cubes is amply sufficient. The most useful of the kind is "Barlow's Tables," which gives at one glance the square and cube, square and cube roots, factors, and reciprocal of any number under 10,000. From such a table, many thousands of examples in multiplication and division may be derived immediately, with the answers, and many hundreds of thousands more may be obtained from the formula,

$$ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

where a and b are both even, or both odd.

Greatest Common Measure, and Least Common Multiple.—In "Barlow's Tables" is given a list of all prime numbers under 100,000. The multiplication of any two of these by the same number, will give a question in the first rule, with its answer. For the second rule, take any low prime numbers a, b, c, d, &c. and multiply one or more of them by each of the low prime numbers e, f, g, &c. Then will $abcd$, &c. $\times\ efg$, &c. be the least common multiple of the products above mentioned. All that has been said on the first four rules, applies equally to Common and Decimal Fractions. In the former case, with a table of squares, the formula

$$\frac{a}{a+b} + \frac{b}{a-b} = \frac{a^2+b^2}{a^2-b^2}$$

will be convenient.

What has been said on Addition and Subtraction may also be applied when the quantities are compound, particularly where the examples are sums of money. The verification will be rendered more easy in the latter case, if the rule or table in article 221 be used by the master. A Ready Reckoner, or Table of Interest, will furnish examples in multiplication or division, in a manner too obvious to require any further notice.

In multiplication, different sums, making together £10 or £100, &c. may be given to two pupils to be multiplied by the same number. One result may be made to verify the other, and the answer, in this and all other cases,* should be recorded for future use.

Since the publication of the First Edition of this work, though its sale has sufficiently convinced me that there exists a disposition to introduce the *principles* of arithmetic into schools, as well as the *practice*, I have often heard it remarked that it was a hard book for children. I never dared to suppose it would be otherwise. All who have been engaged in the education of youth are aware that it is a hard thing to make them think; so hard, indeed, that masters had, within the last few years, almost universally abandoned

* The greatest difficulty which an arithmetical master finds, is that of procuring a sufficient number of examples. If he is at all acquainted with algebra, he will be able to propose to two pupils, questions of which the answers shall be simple, though not (to the pupil) obvious, verifications of one another. If in a school there were established a common book, in which the pupil who first succeeded in solving a question should have the privilege of entering the answer, with his name; besides the emulation thereby excited, a collection of examples would be obtained for future use, which would entirely do away with all anxiety on this subject.

the attempt, and taught them rules instead of principles; by authority instead of demonstration. This system is now passing away, and many preceptors may be found who are of opinion that, whatever may be the additional trouble to themselves, their pupils should always be induced to reflect upon, and know the reason of, what they are doing. Such I would advise not to be discouraged by the failure of a first attempt to make the learner understand the principle of a rule. It is no exaggeration to say, that under the present system, five years of a boy's life are partially spent in merely learning the *rules* contained in this treatise, and those, for the most part, in so imperfect a way, that he is not fit to encounter any question unless he sees the head of the book under which it falls. On a very moderate computation of the time thus bestowed, the pupil would be in no respect worse off, though he spent five hours on every page of this work. The method of proceeding which I should recommend, would be as follows: Let the pupils be taught in classes, the master explaining the article as it stands in the work. Let the former then try the demonstration on some other numbers proposed by the master, which should be as simple as possible. The very words of the book may be used, the figures being changed, and it will rarely be found that a learner is capable of making the proper alterations, without understanding the reasoning. The experience of the master will suggest to him various methods of trying this point. When the principle has been thus discussed, let the rule be distinctly stated by the master or some of the more intelligent of the pupils, and let some very simple example be worked at length. The pupils may then be dismissed, to try the more complicated exercises with which the work will furnish them, or any others which may be proposed.

A. DE MORGAN.

London,
July 30, 1832.

*** This Fourth Edition differs in nothing material from the preceding two; and I have only further to invite the attention of those who are engaged in instruction to the reprint of "Barlow's Tables," which will speedily make its appearance. The utility of this work in the construction of examples for practice, can hardly be overstated; and the specimen which will be found attached to this edition will enable every one to judge for himself.

University College,
Oct. 18, 1839.

ELEMENTS, &c.

BOOK I.

PRINCIPLES OF ARITHMETIC.

SECTION I.

NUMERATION.

1. IMAGINE a multitude of objects of the same kind assembled together; for example, a company of horsemen. One of the first things that must strike a spectator, although unused to counting, is, that to each man there is a horse. Now, though men and horses are things perfectly unlike, yet, because there is one of the first kind to every one of the second, one man to every horse, a new notion will be formed in the mind of the observer, which we express in words, by saying that there is the same *number* of men as of horses. A savage, who had no other way of counting, might remember this number by taking a pebble for each man. Out of a method as rude as this has sprung our system of calculation, by the steps which are pointed out in the following articles. Suppose that there are two companies of horsemen, and a person wishes to know in which of them is the greater number, and also to be able to recollect how many there are in each.

2. Suppose that while the first company passes by, he drops a pebble into a basket for each man whom he sees. There is no connexion between the pebbles and the horsemen but this, that for every horseman

there is a pebble; that is, in common language, the *number* of pebbles and of horsemen is the same. Suppose that while the second company passes, he drops a pebble for each man into a second basket; he will then have two baskets of pebbles, by which he will be able to convey to any other person a notion of how many horsemen there were in each company. When he wishes to know which company was the larger, or contained most horsemen, he will take a pebble out of each basket, and put them aside. He will go on doing this as often as he can, that is, until one of the baskets is emptied. Then, if he also finds the other basket empty, he says that both companies contained the same number of horsemen; if the second basket still contains some pebbles, he can tell by them how many more were in the second than in the first.

3. In this way a savage could keep an account of any numbers in which he was interested. He could thus register his children, his cattle, or the number of summers and winters which he had seen, by means of pebbles, or any other small objects which could be got in large numbers. Something of this sort is the practice of savage nations at this day, and it has in some places lasted even after the invention of better methods of reckoning. At Rome, in the time of the republic, the prætor, one of the magistrates, used to go every year in great pomp, and drive a nail into the door of the temple of Jupiter; a way of remembering the number of years which the city had been built, which probably took its rise before the introduction of writing.

4. In process of time, names would be given to those collections of pebbles which are met with most frequently. But as long as small numbers only were required, the most convenient way of reckoning them would be by means of the fingers. Any person could make with his two hands the little calculations which would be necessary for his purposes, and would name all the different collections of the fingers. He would thus get words in his own language answering to one, two, three, four, five, six, seven, eight, nine, and ten. As his wants increased, he would find it necessary to give names to larger numbers; but here he would be stopped by the immense quantity of words which

he must have, in order to express all the numbers which he would be obliged to make use of. He must, then, after giving a separate name to a few of the first numbers, manage to express all other numbers by means of those names.

5. I now shew how this has been done in our own language. The English names of numbers have been formed from the Saxon: and in the following table each number after ten is written down in one column, while another shews its connexion with those which have preceded it.

One	eleven	one left *
two	twelve	two left
three	thirteen	ten and three
four	fourteen	ten and four
five	fifteen	ten and five
six	sixteen	ten and six
seven	seventeen	ten and seven
eight	eighteen	ten and eight
nine	nineteen	ten and nine
ten	twenty	two tens
	twenty-one	two tens and one
	twenty-two	two tens and two
	&c. &c.	&c. &c.
	thirty	three tens
	&c.	&c.
	forty	four tens
	&c.	&c.
	fifty	five tens
	sixty	six tens
	seventy	seven tens
	eighty	eight tens
	ninety	nine tens
	a hundred	ten tens

* Meaning one left after ten is taken, or ten and one.

a hundred and one	ten tens and one
&c. &c.	
a thousand	ten hundreds
ten thousand	
a hundred thousand	
a million	ten hundred thousand or one thousand thousand
ten millions	
a hundred millions	
&c.	

6. Words, written down in ordinary language, would very soon be too long for such continual repetition as takes place in calculation. Short signs would then be substituted for words; but it would be impossible to have a distinct sign for every number: so that when some few signs had been chosen, it would be convenient to invent others for the rest out of those already made. The signs which we use are as follow:

0	1	2	3	4	5	6	7	8	9
nothing	one	two	three	four	five	six	seven	eight	nine

I now proceed to explain the way in which these signs are made to represent other numbers.

7. Suppose a man, first, to hold up one finger, then two, and so on, until he has held up every finger, and suppose a number of men to do the same thing. It is plain that we may thus distinguish one number from another, by causing two different sets of persons to hold up each a certain number of fingers, and that we may do this in many different ways. For example, the number fifteen might be indicated either by fifteen men each holding up one finger, or by four men each holding up two fingers and a fifth holding up seven, and so on. The question is, of all these contrivances for expressing the number, which is the most convenient? In the choice which is made for this purpose consists what is called the method of *numeration*.

8. I have used the foregoing explanation because it is very probable

that our system of numeration, and almost every other which is used in the world, sprung from the practice of reckoning on the fingers, which children usually follow when first they begin to count. The method which I have described is the rudest possible; but, by a little alteration, a system may be formed which will enable us to express enormous numbers with great ease.

9. Suppose that you are going to count some large number, for example, to measure a number of yards of cloth. Opposite to yourself suppose a man to be placed, who keeps his eye upon you, and holds up a finger for every yard which he sees you measure. When ten yards have been measured he will have held up ten fingers, and will not be able to count any further unless he begin again, holding up one finger at the eleventh yard, two at the twelfth, and so on. But to know how many have been counted, you must know, not only how many fingers he holds up, but also how many times he has begun again. You may keep this in view by placing another man on the right of the former, who directs his eye towards his companion, and holds up one finger the moment he perceives him ready to begin again, that is, as soon as ten yards have been measured. Each finger of the first man stands only for one yard, but each finger of the second stands for as many as all the fingers of the first together, that is, for ten. In this way a hundred may be counted, because the first may now reckon his ten fingers once for each finger of the second man, that is, ten times in all, and ten tens is one hundred (5).* Now place a third man at the right of the second, who shall hold up a finger whenever he perceives the second ready to begin again. One finger of the third man counts as many as all the ten fingers of the second, that is, counts one hundred. In this way we may proceed until the third has all his fingers extended, which will signify that ten hundred or one thousand have been counted (5). A fourth man would enable us to count as far as ten thousand, a fifth as far as one hundred thousand, a sixth as far as a million, and so on.

10. Each new person placed himself towards your left in the rank

* The references are to the preceding articles.

opposite to you. Now rule a number of columns, as below, and to the right of them all place in words the number which you wish to represent; in the first column on the right, place the number of fingers which the first man will be holding up when that number of yards has been measured. In the next column, place the fingers which the second man will then be holding up; and so on.

	7th.	6th.	5th.	4th.	3d.	2d.	1st.	
I.						5	7	fifty-seven.
II.					1	0	4	one hundred and four.
III.					1	1	0	one hundred and ten.
IV.				2	3	4	8	two thousand three hundred and forty-eight.
V.			1	5	9	0	6	fifteen thousand nine hundred and six.
VI.		1	8	7	0	0	4	one hundred and eighty-seven thousand and four.
VII.	3	6	9	7	2	8	5	three million, six hundred and ninety-seven thousand, two hundred and eighty-five.

11. In I. the number fifty-seven is expressed. This means (5) five tens and seven. The first has therefore counted all his fingers five times, and has counted seven fingers more. This is shewn by five fingers of the second man being held up, and seven of the first. In II. the number one hundred and four is represented. This number is (5) ten tens and four. The second person has therefore just reckoned all his fingers once, which is denoted by the third person holding up one finger; but he has not yet begun again, because he does not hold up a finger until the first has counted ten, of which ten only four are completed. When all this last-mentioned ten have been counted, he then holds up one finger, and the first being ready to begin again, has no fingers extended, and the number obtained is eleven tens, or ten tens and one ten, or one hundred and ten. This is the case in III. You will now find no difficulty with the other numbers in the table.

12. In all these numbers a figure in the first column stands for only as many yards as are written under that figure in (6). A figure

in the second column stands, not for as many yards, but for as many tens of yards: a figure in the third column stands for as many hundreds of yards; in the fourth column for as many thousands of yards; and so on: that is, if we suppose a figure to move from any column to the one on its left, it stands for ten times as many yards as before. Recollect this, and you may cease to draw the lines between the columns, because each figure will be sufficiently well known by the *place* in which it is; that is, by the number of figures which come upon the right hand of it.

13. It is important to recollect that this way of writing numbers, which has become so familiar as to seem the *natural* method, is not more natural than any other. For example, we might agree to signify one ten by the figure of one with an accent, thus, $1'$; twenty or two tens by $2'$; and so on: one hundred or ten tens by $1''$; two hundred by $2''$; one thousand by $1'''$; and so on: putting Roman figures for accents when they become too many to write with convenience. The fourth number in the table would then be written $2'''\ 3''\ 4'\ 8$, which might also be expressed by $8\ 4'\ 3''\ 2'''$, $4'\ 8\ 3''\ 2'''$; or the order of the figures might be changed in any way, because their meaning depends upon the accents which are attached to them, and not upon the place in which they stand. Hence, a cipher would never be necessary; for 104 would be distinguished from 14 by writing for the first $1''4$, and for the second $1'4$. The common method is preferred, not because it is more exact than this, but because it is more simple.

14. The distinction between our method of numeration and that of the ancients, is in the meaning of each figure depending partly upon the place in which it stands. Thus, in 44444 each four stands for four of *something;* but in the first column on the right it signifies only four of the pebbles which are counted; in the second, it means four collections of ten pebbles each; in the third, four of one hundred each; and so on.

15. The things measured in (11) were yards of cloth. In this case one yard of cloth is called the *unit.* The first figure on the right is said to be in the *units' place*, because it only stands for so many units as are in the number that is written under it in (6). The second

figure is said to be in the *tens'* place, because it stands for a number of tens of units. The third, fourth, and fifth figures are in the places of the *hundreds*, *thousands*, and *tens of thousands*, for a similar reason.

16. If the quantity measured had been acres of land, an acre of land would have been called the *unit*, for the unit is *one* of the things which are measured. Quantities are of two sorts; those which contain an exact number of units, as 47 yards, and those which do not, as 47 yards and a half. Of these, for the present, we only consider the first.

17. In most parts of arithmetic, all quantities must have the same unit. You cannot say that 2 yards and 3 feet make 5 *yards* or 5 *feet*, because 2 and 3 make 5; yet you may say that 2 *yards* and 3 *yards* make 5 *yards*, and that 2 *feet* and 3 *feet* make 5 *feet*. It would be absurd to try to measure a quantity of one kind with a unit which is a quantity of another kind; for example, to attempt to tell how many yards there are in a gallon, or how many bushels of corn there are in a barrel of wine.

18. All things which are true of some numbers of one unit are true of the same numbers of any other unit. Thus, 15 pebbles and 7 pebbles together make 22 pebbles; 15 acres and 7 acres together make 22 acres, and so on. From this we come to say that 15 and 7 make 22, meaning that 15 things of the same kind, and 7 more of the same kind as the first, together make 22 of that kind, whether the kind mentioned be pebbles, horsemen, acres of land, or any other. For these it is but necessary to say, once for all, that 15 and 7 make 22. Therefore, in future, on this part of the subject I shall cease to talk of any particular units, such as pebbles or acres, and speak of numbers only.

19. I will now repeat the principal things which have been mentioned in this chapter.

I. Ten signs are used, one to stand for nothing, the rest for the first nine numbers. They are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The first of these is called a *cipher*.

II. Higher numbers have not signs for themselves, but are signified by placing the signs already mentioned by the side of each other, and agreeing that the first figure on the right hand shall keep the value which it has when it stands alone; that the second on the right hand shall mean ten times as many as it does when it stands alone; that the third figure shall mean one hundred times as many as it does when it stands alone; the fourth, one thousand times as many; and so on.

III. The right hand figure is said to be in the *units' place*, the next to that in the *tens' place*, the third in the *hundreds' place*, and so on.

IV. When a number is itself an exact number of tens, hundreds, or thousands, &c., as many ciphers must be placed on the right of it as will bring the number into the place which is intended for it. The following are examples:

Fifty, or five tens, 50: seven hundred, 700.
Five hundred and twenty-eight thousand, 528000.

If it were not for the ciphers, these numbers would be mistaken for 5, 7, and 528.

V. A cipher in the middle of a number becomes necessary when any one of the denominations, units, tens, &c. is wanting. Thus, twenty thousand and six is 20006, two hundred and six is 206. Ciphers might be placed at the beginning of a number, but they would have no meaning. Thus 026 is the same as 26, since the cipher merely shews that there are no hundreds, which is evident from the number itself.

20. If we take out of a number, as 16785, any of those figures which come together, as 67, and ask, what does this sixty-seven mean? of what is it sixty-seven? the answer is, sixty-seven of the same collections as the 7, when it was in the number; that is, 67 hundreds. For the 6 is 6 thousands, or 6 ten hundreds, or sixty hundreds; which, with the 7, or 7 hundreds, is 67 hundreds: similarly, the 678 is 678 tens. This number may then be expressed either as

1 ten-thousand 6 thousands 7 hundreds 8 tens and 5
or 16 thousands 78 tens and 5, or 1 ten thousand 678 tens and 5;
or 167 hundreds 8 tens and 5, or 1678 tens and 5, and so on.

21. EXERCISES.

I. Write down the signs for

Four hundred and seventy-six; two thousand and ninety-seven; sixty-four thousand three hundred and fifty; two millions seven hundred and four; five hundred and seventy-eight millions of millions.

II. Write at full length 53, 1805, 1830, 66707, 180917324, 66713721, 90976390, 25000000.

III. What alteration takes place in a number made up entirely of nines, such as 99999, by adding one to it?

IV. Shew that a number which has five figures in it must be greater than one which has four, though the first have none but small figures in it, and the second none but large ones. For example, that 10111 is greater than 9879.

22. You now see that the convenience of our method of numeration arises from a few simple signs being made to change their value as they change the column in which they are placed. The same advantage arises from counting in a similar way all the articles which are used in every-day life. For example, we count money by dividing it into pounds, shillings, and pence, of which a shilling is 12 pence, and a pound 20 shillings, or 240 pence. We write a number of pounds, shillings, and pence in three columns, generally placing a point between each column. Thus, 263 pence would not be written as 263, but as £1 . 1 . 11, where £ shews that the 1 in the first column is a pound. Here is a *system of numeration* in which a number in the second column on the right means 12 times as much as the same number in the first; and one in the third column is twenty times as great as the same in the second, or 240 times as great as the same in the first. In each of the tables of measures which you will hereafter meet with, you will see a separate system of numeration, but the methods of calculation for all will be the same.

23. In order to make the language of arithmetic shorter, some other signs are used. They are as follow:

I. $15 + 38$ means that 38 is to be added to 15, and is the same thing as 53. This is the *sum* of 15 and 38, and is read fifteen *plus* thirty-eight.

II. $64-12$ means that 12 is to be taken away from 64, and is the same thing as 52. This is the difference of 64 and 12, and is read sixty-four *minus* twelve.

III. 9×8 means that 9 is to be taken 8 times, and is the same thing as 72. This is the product of 9 and 8, and is read nine *into* eight.

IV. $\frac{108}{6}$ means that 108 is to be divided by 6, or that you must find out how many sixes there are in 108; and is the same thing as 18. This is the quotient of 108 and 6; and is read a hundred and eight *by* six.

V. When two numbers, or collections of numbers, with the foregoing signs, are the same, the sign $=$ is put between them. Thus, that 7 and 5 make 12, is written in this way, $7+5=12$. This is called an *equation*, and is read, seven *plus* five *equals* twelve. It is plain that we may construct as many equations as we please. Thus:

$$7+9-3=12+1;\ \frac{12}{2}-1+3\times2=11, \text{ and so on.}$$

24. It often becomes necessary to speak of something which is true, not of any one number only, but of all numbers. For example, take 10 and 7; their sum* is 17, their difference is 3. If this sum and difference be added together, we get 20, which is twice the greater of the two numbers first chosen. If from 17 we take 3, we get 14, which is twice the less of the two numbers. The same thing will be found to hold good of any two numbers, which gives this general proposition,—If the sum and difference of two numbers be added together, the result is twice the greater of the two; if the difference be taken from the sum, the result is twice the lesser of the two. If, then, we take *any* numbers, and call them the first number and the second number, and let the first number be the greater; we have then,

(1st No. + 2d No.) + (1st No. − 2d No.) = twice 1st No.

(1st No. + 2d No.) − (1st No. − 2d No.) = twice 2d No.

* Any little computations which occur in the rest of this section may be made on the fingers, or with counters.

The brackets here enclose the things which must be first done, before the signs which join the brackets are made use of. Thus, $8-(2+1)\times(1+1)$ signifies that $2+1$ must be taken $1+1$ times, and the product must be subtracted from 8. In the same manner, any result made from two or more numbers, which is true whatever numbers are taken, may be represented by using first No., second No., &c., to stand for them, and by the signs in (23). But this may be much shortened; for as first No., second No., &c., may mean any numbers, the letters a and b may be used instead of these words; and it must now be recollected that a and b stand for two numbers, provided only that a is greater than b. Let twice a be represented by $2a$, and twice b by $2b$. The equations then become

$$(a+b)+(a-b) = 2a, \text{ and } (a+b)-(a-b) = 2b.$$

This may be explained still further, as follows:

25. Suppose a number of sealed packets, marked a, b, c, d, &c., on the outside, each of which contains a distinct but unknown number of counters. As long as we do not know how many counters each contains, we can make the letter which belongs to each stand for its number, so as to talk of *the number a*, instead of the number in the packet marked a. And because we do not know the numbers, it does not therefore follow that we know nothing whatever about them; for there are some connexions which exist between all numbers, which we call *general properties* of numbers. For example, take any number, multiply it by itself, and subtract one from the result; and then subtract one from the number itself. The first of these will always contain the second exactly as many times as the original number increased by one. Take the number 6; this multiplied by itself is 36, which diminished by one is 35: again, 6 diminished by 1 is 5; and 35 contains 5, 7 times, that is, $6+1$ times. This will be found to be true of any number, and, when proved, may be said to be true of the number contained in the packet marked a, or of the number a. If we represent a multiplied by itself by aa,* we have by (23)

* This should be (23) $a\times a$, but the sign $\times$ is unnecessary here. It is used with numbers, as in 2×7, to prevent confounding this, which is 14, with 27.

$$\frac{aa-1}{a-1} = a+1.$$

26. When, therefore, we wish to talk of a number without specifying any one in particular, we use a letter to represent it. Thus: suppose we wish to reason upon what will follow from dividing a number into three parts, without considering what the number is, or what are the parts into which it is divided. Let a stand for the number, and b, c, and d, for the parts into which it is divided. Then, by our supposition,

$$a = b+c+d.$$

On this we can reason and produce results which do not belong to any particular number, but are true of all. Thus, if one part be taken away from the number, the other two will remain, or

$$a-b = c+d.$$

If each part be doubled, the whole number will be doubled, or

$$2a = 2b+2c+2d.$$

If we diminish one of the parts, as d, by a number x, we diminish the whole number just as much, or

$$a-x = b+c+(d-x).$$

This method of reasoning on the *general properties* of numbers, belongs to the science of ALGEBRA.

27. EXERCISES.

What is $a+2b-c$, where $a = 12$, $b = 18$, $c = 7$?—*Answer*, 41.

What is $\frac{aa-bb}{a-b}$, where $a = 6$ and $b = 2$?—*Ans.* 8.

What is the difference between $(a+b)(c+d)$ and $a+bc+d$, for the following values of a, b, c, and d?

a	b	c	d	*Ans.*
1	2	3	4	10
2	12	7	1	25
1	1	1	1	1

SECTION II.

ADDITION AND SUBTRACTION.

28. There is no process in arithmetic which does not consist entirely in the increase or diminution of numbers. There is then nothing which might not be done with collections of pebbles. Probably, at first, either these or the fingers were used. Our word *calculation* is derived from the Latin word *calculus*, which means a pebble. Shorter ways of counting have been invented, by which many calculations, which would require long and tedious reckoning, if pebbles were used, are made at once with very little trouble. The four great methods are, Addition, Subtraction, Multiplication, and Division; of which, the last two are only ways of doing several of the first and second at once.

29. When one number is increased by another, the number which is as large as both the others together is called their *sum*. The process of finding the sum of two or more numbers is called ADDITION, and, as was said before, is denoted by placing a cross (+) between the numbers which are to be added together.

Suppose it required to find the sum of 1834 and 2799. In order to add these numbers, take them to pieces, dividing each into its units, tens, hundreds, and thousands:

1834 is 1 thous. 8 hund. 3 tens and 4;
2799 is 2 thous. 7 hund. 9 tens and 9.

Each number is thus broken up into four parts. If to each part of the first you add the part of the second which is under it, and then put together what you get from these additions, you will have added 1834 and 2799. In the first number are 4 units, and in the second 9: these will, when the numbers are added together, contribute 13 units to the sum. Again, the 3 tens in the first and the 9 tens in the second will contribute 12 tens to the sum. The 8 hundreds in the first and the 7 hundreds in the second will add 15 hundreds to the sum; and the thousand in the first with the 2 thousands in the second will contribute 3 thousands to the sum; therefore the sum required is

3 thousands, 15 hundreds, 12 tens, and 13 units.

To simplify this result, you must recollect that—

13 units are .... 1 ten and 3 units,
12 tens are .... 1 hund. and 2 tens
15 hund. are 1 thous. and 5 hund.
3 thous. are 3 thous.

Now collect the numbers on the right-hand side together, as was done before, and this will give, as the sum of 1834 and 2799,

4 thousands, 6 hundreds, 3 tens, and 3 units, which (19) is written 4633.

30. The former process, written with the signs of (23) is as follows:

$$1834 = 1 \times 1000 + 8 \times 100 + 3 \times 10 + 4$$
$$2799 = 2 \times 1000 + 7 \times 100 + 9 \times 10 + 9$$

Therefore,

$$1834 + 2799 = 3 \times 1000 + 15 \times 100 + 12 \times 10 + 13$$

But

$$13 = 1 \times 10 + 3$$
$$12 \times 10 = 1 \times 100 + 2 \times 10$$
$$15 \times 100 = 1 \times 1000 + 5 \times 100$$
$$3 \times 1000 = 3 \times 1000$$

Therefore,

$$1834 + 2799 = 4 \times 1000 + 6 \times 100 + 3 \times 10 + 3$$
$$= 4633.$$

31. The same process is to be followed in all cases, but not at the same length. In order to be able to go through it, you must know how to add together the simple numbers. This can only be done by memory; and to help the memory you should make the following table three or four times for yourself:

	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18

The use of this table is as follows: Suppose you want to find the sum of 8 and 7. Look in the left-hand column for either of them; 8, for example, and look in the top column for 7. On the same line as 8, and underneath 7, you find 15, their sum.

32. When this table has been thoroughly committed to memory, so that you can tell at once the sum of any two numbers, neither of which exceeds 9, you should exercise yourself in adding and subtracting two numbers, one of which is greater than 9 and the other less. You should write down a great number of such sentences as the following, which will exercise you at the same time in addition, and in the use of the signs mentioned in (23).

$12+6=18 \qquad 22+6=28 \qquad 19+8=27$

$54+9=63 \qquad 56+7=63 \qquad 22+8=30$

$100-9=91 \qquad 27-8=19 \qquad 44-6=38$, &c.

33. When the last two articles have been thoroughly studied, you will be able to find the sum of any numbers by the following process, which is the same as that in (29).

Rule I. Place the numbers under one another, units under units, tens under tens, and so on.

II. Add together the units of all, and part the *whole* number thus obtained into units and tens. Thus, if 85 is the number, part it into 8 tens and 5 units; if 136 be the number, part it into 13 tens and 6 units (20).

III. Write down the units of this number under the units of the rest, and keep in memory the number of tens.

IV. Add together all the numbers in the column of tens, remembering to take in (or carry, as it is called,) the tens which you were told to recollect in III., and divide this number of tens into tens and hundreds. Thus, if 335 tens is the number obtained, part this into 33 hundreds and 5 tens.

V. Place the number of tens under the tens, and remember the number of hundreds.

VI. Proceed in this way through every column, and at the last column, instead of separating the number you obtain into two parts, write it all down before the rest.

Example.—What is

$$1805 + 36 + 19727 + 3 + 1474 + 2008$$

1805
36
19727
3
1474
2008
25053

The addition of the units' line, or 8 + 4 + 3 + 7 + 6 + 5, gives 33, that is, 3 tens and 3 units. Put 3 in the units' place, and add together the line of tens, taking in at the beginning the 3 tens which were created by the addition of the units' line. That is, find 3 + 0 + 7 + 2 + 3 + 0, which gives 15 for the number of tens; that is, 1 hundred and 5 tens. Add the line of hundreds together, taking care to add the 1 hundred which arose in the addition of the line of tens; that is, find 1 + 0 + 4 + 7 + 8, which gives exactly 20 hundreds, or 2 thousands and no hundreds. Put a cipher in the hundreds' place (because, if you do not, the next figure will be taken for hundreds instead of thousands), and add the figures in the thousands' line together, remembering the 2 thousands which arose from the hundreds' line; that is, find 2 + 2 + 1 + 9 + 1, which gives 15 thousands, or 1 ten thousand and 5 thousand. Write 5 under the line of thousands, and collect the figures in the line of tens of thousands, remembering the ten thousand which arose out of the thousands' line;

that is, find 1 + 1 or 2 ten thousands. Write 2 under the ten thousands' line, and the operation is completed.

34. As an exercise in addition, you may satisfy yourself that what I now say of the following square is correct. The numbers in every row, whether reckoned upright, or from right to left, or from corner to corner, when added together give the number 24156.

2016	4212	1656	3852	1296	3492	936	3132	576	2772	216
252	2052	4248	1692	3888	1332	3528	972	3168	612	2412
2448	288	2088	4284	1728	3924	1368	3564	1008	2808	648
684	2484	324	2124	4320	1764	3960	1404	3204	1044	2844
2880	720	2520	360	2160	4356	1800	3600	1440	3240	1080
1116	2916	756	2556	396	2196	3996	1836	3636	1476	3276
3312	1152	2952	792	2592	36	2232	4032	1872	3672	1512
1548	3348	1188	2988	432	2628	72	2268	4068	1908	3708
3744	1584	3384	828	3024	468	2664	108	2304	4104	1944
1980	3780	1224	3420	864	3060	504	2700	144	2340	4140
4176	1620	3816	1260	3456	900	3096	540	2736	180	2376

35. If two numbers must be added together, it will not alter the sum, if you take away a part of one, provided you put on as much to the other. It is plain that you will not alter the whole number of a collection of pebbles in two baskets by taking any number out of one, and putting them into the other. Thus, 15 + 7 is the same as 12 + 10, since 12 is 3 less than 15, and 10 is three more than 7. This was the principle upon which the whole of the process in (29) was conducted.

36. Let a and b stand for two numbers, as in (24). It is impossible to tell what their sum will be until the numbers themselves are known. In the meanwhile $a+b$ stands for this sum. To say, in algebraical language, that the sum of a and b is not altered by adding c to a, provided we take away c from b, we have the following equation:

$$(a+c)+(b-c) = a+b$$

which may be written without brackets, thus

$$a+c+b-c = a+b.$$

For the meaning of these two equations will appear to be the same, on consideration.

37. If a be taken twice, three times, &c. the results are represented in algebra by $2a$, $3a$, $4a$, &c. The sum of any two of this series may be expressed in a shorter form than by writing the sign + between them; for though we do not know what number a stands for, we know that, be it what it may, $2a+2a = 4a$, $3a+2a = 5a$, $4a+9a = 13a$; and generally, if a taken m times be added to a taken n times, the result is a taken $m+n$ times, or

$$ma+na = (m+n)a.$$

38. The use of the brackets must here be noticed. They mean, that the expression contained inside them must be used exactly as a single letter would be used in the same place. Thus, pa signifies that a is taken p times, and $(m+n)a$, that a is taken $m+n$ times. It is, therefore, a different thing from $m+na$, which means that a, after being taken n times, is added to m. Thus $(3+4)\times 2$ is 7×2 or 14; while $3+4\times 2$ is $3+8$, or 11.

39. When one number is taken away from another, the number which is left is called the *difference* or *remainder*. The process of finding the difference is called SUBTRACTION. The number which is to be taken away is of course the lesser of the two.

40. The process of subtraction depends upon these two principles.

I. The difference of two numbers is not altered by adding a number to the first, if you add the same number to the second; or by subtracting a number from the first, if you subtract the same number from the second. Conceive two baskets with pebbles in them, in the first of which are 100 pebbles more than in the second. If I put 50 more pebbles into each of them, there are still only 100 more in the first than in the second, and the same if I take 50 from each. Therefore, in finding the difference of two numbers, if it should be convenient, I may add any number I please to both of them, because, though I alter the numbers themselves by so doing, I do not alter their difference.

II. Since 6 exceeds 4 by 2,
and 3 exceeds 2 by 1,
and 12 exceeds 5 by 7,

6, 3, and 12 together, or 21, exceed 4, 2, and 5 together, or 11, by 2, 1, and 7 together, or 10: the same thing may be said of any other numbers.

41. If a, b, and c be three numbers, of which a is greater than b (40), I. leads to the following.

$$(a+c)-(b+c) = a-b$$

Again, if c be less than a and b

$$(a-c)-(b-c) = a-b$$

The brackets cannot be here removed as in (36). That is, $p-(q-r)$ is not the same thing as $p-q-r$. For, in the first, the difference of q and r is subtracted from p; but in the second, first q and then r are subtracted from p, which is the same as subtracting as much as q and r together, or $q+r$. Therefore $p-q-r$ is $p-(q+r)$. In order to shew how to remove the brackets from $p-(q-r)$ without altering the value of the result, let us take the simple instance $12-(8-5)$. If we subtract 8 from 12, or form $12-8$, we subtract too much; because it is not 8 which is to be taken away, but as much of 8 as is left after diminishing it by 5. In forming $12-8$ we have therefore subtracted 5 too much. This must be set right by adding 5 to the result, which gives $12-8+5$ for the value of $12-(8-5)$. The same reasoning applies to every case, and we have therefore

$$p-(q+r) = p-q-r$$
$$p-(q-r) = p-q+r.$$

By the same kind of reasoning

$$a-(b+c-d-e) = a-b-c+d+e$$
$$2a+3b-(a-2b) = 2a+3b-a+2b = a+5b$$
$$4x+y-(17x-9y) = 4x+y-17x+9y = 10y-13x$$

42. I want to find the difference of the numbers 57762 and 34631. Take these to pieces as in (29) and

57762 is 5 ten-th. 7 th. 7 hund. 6 tens and 2 units,
34631 is 3 ten-th. 4 th. 6 hund. 3 tens and 1 unit.

Now 2 units exceed	1 unit	by 1 unit,
6 tens	3 tens	3 tens,
7 hundreds	6 hundreds	1 hundred,
7 thousands	4 thousands.........	3 thousands,
5 ten-thousands	3 ten-thous..........	2 ten-thous.

Therefore, by (40, Principle II.) all the first column *together* exceeds all the second column by all the third column, that is, by

2 ten-th. 3 th. 1 hund. 3 tens, and 1 unit,

which is 23131. Therefore the difference of 57762 and 34631 is 23131, or 57762—34631 = 23131.

43. Suppose I want to find the difference between 61274 and 39628. Write them at length, and

61274 is 6 ten-th. 1 th. 2 hund. 7 tens and 4 units,
39628 is 3 ten-th. 9 th. 6 hund. 2 tens and 8 units.

If we attempt to do the same as in the last article, there is a difficulty immediately, since 8, being greater than 4, cannot be taken from it. But from (40) it appears that we shall not alter the difference of two numbers if we add the same number to *both* of them. Add ten to the first number, that is, let there be 14 units instead of four, and add ten also to the second number, but instead of adding ten to the number of units, add one to the number of tens, which is the same thing. The numbers will then stand thus,

6 ten-thous. 1 thous. 2 hund. 7 tens and 14 *units*,*
3 ten-thous. 9 thous. 6 hund. 3 *tens* and 8 units.

You now see that the units and tens in the lower can be subtracted from those in the upper line, but that the hundreds cannot. To remedy this, add one thousand or 10 hundred to both numbers, which will not

* Those numbers which have been altered are put in italics.

alter their difference, and remember to increase the hundreds in the upper line by 10, and the thousands in the lower line by 1, which are the same things. And since the thousands in the lower cannot be subtracted from the thousands in the upper line, add 1 ten thousand or 10 thousand to both numbers, and increase the thousands in the upper line by 10, and the ten thousands in the lower line by 1, which are the same things; and at the close, the numbers which we get will be

6 ten-thous. 11 *thous.* 12 *hund.* 7 tens and 14 *units,*
4 *ten-thous.* 10 *thous.* 6 hund. 3 *tens* and 8 units.

These numbers are not, it is true, the same as those given at the beginning of this article, but their difference is the same by (40). With the last-mentioned numbers proceed in the same way as in (42), which will give, as their difference,

2 ten-thous. 1 thous. 6 hund. 4 tens, and 6 units, which is 21646.

44. From this we deduce the following rules for subtraction:

I. Write the number which is *to be subtracted* (which is of course the lesser of the two, and is called the *subtrahend*) under the other, so that its units shall fall under the units of the other, and so on.

II. Subtract each figure of the lower line from the one above it, if that can be done. Where that cannot be done, add ten to the upper figure, and then subtract the lower figure, but recollect in this case always to increase the next figure in the lower line by 1, before you begin to subtract it from the upper one.

45. If there should not be as many figures in the lower line as in the upper one, proceed as if there were as many ciphers at the beginning of the lower line as will make the number of figures equal. You do not alter a number by placing ciphers at the beginning of it. For example, 00818 is the same number as 818, for it means

0 ten-thous. 0 thous. 8 hunds. 1 ten and 8 units;

the two first numbers are nothing, and the rest is

8 hundreds, 1 ten, and 8 units, or 818.

The second does not differ from the first, except in its being said that there are no thousands and no tens of thousands in the number, which may be known without their being mentioned at all. You may ask, perhaps, why this does not apply to a cipher placed in the middle of a number, or at the right of it, as, for example, in 28007 and 39700? But you must recollect, that if it were not for the two ciphers in the first, the 8 would be taken for 8 tens, instead of 8 thousands; and if it were not for the ciphers in the second, the 7 would be taken for 7 units, instead of 7 hundreds.

46. EXAMPLE.

What is the difference between	3708291640030174
and	30813649276188
Difference	3677477990753986

EXERCISES.

I. What is 18337 + 149263200 − 6472902?—*Answer* 142808635.

What is 1000 − 464 + 3279 − 646?—*Answer* 3169.

II. Subtract

64 + 76 + 144 − 18 from 33 − 2 + 100037.—*Answer* 99802.

III. What shorter rule might be made for subtraction when all the figures in the upper line are ciphers except the first? for example, in finding

10000000 − 2731634.

IV. Find 18362 + 2469 and 18362 − 2469, add the second result to the first, and then subtract 18362; subtract the second from the first, and then subtract 2469.—*Answer*, 18362 *and* 2469.

V. There are four places on the same line in the order A, B, C, and D. From A to D it is 1463 miles; from A to C it is 728 miles; and from B to D it is 1317 miles. How far is it from A to B, from B to C, and from C to D?—*Answer.* From A to B 146, from B to C 582, and from C to D 735 miles.

VI. In the following table subtract B from A, and B from the remainder, and so on until B can be no longer subtracted. Find how many times B can be subtracted from A, and what is the last remainder.

A	B	No. of times.	Remainder.
23604	9999	2	3606
209961	37173	5	24096
74712	6792	11	0
4802469	654321	7	222222
18849747	3141592	6	195
987654321	123456789	8	9

SECTION III.

MULTIPLICATION.

47. I have said that all questions in arithmetic require nothing but addition and subtraction. I do not mean by this that no rule should ever be used except those given in the last section, but that all other rules only shew shorter ways of finding what might be found, if we pleased, by the methods there deduced. Even the last two rules themselves are only short and convenient ways of doing what may be done with a number of pebbles or counters.

48. I want to know the sum of five seventeens, or I ask the
17 following question—There are five heaps of pebbles, and seven-
17 teen pebbles in each heap; how many are there in all? Write
17 five seventeens in a column, and make the addition, which gives
17 85. In this case 85 is called the *product* of 5 and 17, and the
17 process of finding the product is called MULTIPLICATION, which
85 is nothing more than the addition of a number of the same quantities. Here 17 is called the *multiplicand*, and 5 is called the *multiplier*.

49. If no question harder than this were ever proposed, there would be no occasion for a shorter way than the one here followed. But if there were 1367 heaps of pebbles, and 429 in each heap, the whole number is then 1367 times 429, or 429 multiplied by 1367. I should have to write 429 1367 times, and then to make an addition of enor-

mous length. To avoid this, a shorter rule is necessary, which I now proceed to explain.

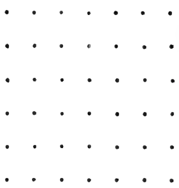
50. The student must first make himself acquainted with the products of all numbers as far as 10 times 10 by means of the following table,* which must be committed to memory.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

If from this table you wish to know what is 7 times 6, look in the first upright column on the left for either of them; 6 for example. Proceed to the right until you come into the column marked 7 at the top. You there find 42, which is the product of 6 and 7.

51. You may find, in this way, either 6 times 7, or 7 times 6, and for both you find 42. That is, six sevens is the same number as seven sixes. This may be shewn as follows: Place seven counters in a line, and repeat that line in all six times. The number of counters in the whole is 6 times 7, or six sevens, if I reckon the rows from the top to

* As it is usual to learn the products of numbers as far as 12 times 12, I have extended the table thus far. In my opinion, all pupils who shew a tolerable capacity should slowly commit the products to memory as far as 20 times 20, in the course of their progress through this work.



the bottom; but if I count the rows that stand side by side, I find seven of them, and six in each row, the whole number of which is 7 times 6, or seven sixes. And the whole number is 42, whichever way I count. The same method may be applied to any other two numbers. If the signs of (23) were used, it would be said that $7 \times 6 = 6 \times 7$.

52. To take any quantity a number of times, it will be enough to take every one of its parts the same number of times. Thus, a sack of corn will be increased fifty-fold, if each bushel which it contains be replaced by 50 bushels. A country will be doubled by doubling every acre of land, or every county, which it contains. Simple as this may appear, it is necessary to state it, because it is one of the principles on which the rule of multiplication depends.

53. In order to multiply by any number, you may multiply separately by any parts into which you choose to divide that number, and add the results. For example, 4 and 2 make 6. To multiply 7 by 6, first multiply 7 by 4, and then by 2, and add the products. This will give 42, which is the product of 7 and 6. Again, since 57 is made up of 32 and 25, 57 times 50 is made up of 32 times 50 and 25 times 50, and so on. If the signs were used, these would be written thus:

$$7 \times 6 = 7 \times 4 + 7 \times 2$$
$$50 \times 57 = 50 \times 32 + 50 \times 25$$

54. The principles in the two last articles may be expressed thus: If a be made up of the parts x, y, and z, $m\,a$ is made up of $m\,x$, $m\,y$, and $m\,z$; or,

$$\text{if} \qquad a = x + y + z$$
$$m\,a = m\,x + m\,y + m\,z$$
$$\text{or,} \qquad m(x+y+z) = m\,x + m\,y + m\,z$$

A similar result may be obtained if a, instead of being made up of x, y, and z, is made by combined additions and subtractions, such as $x+y-z$, $x-y+z$, $x-y-z$, &c. To take the first as an instance,

$$\text{let} \qquad a = x + y - z.$$
$$\text{Then} \qquad m\,a = m\,x + m\,y - m\,z.$$

For, if a had been $x+y$, $m\,a$ would have been $m\,x + m\,y$. But since a is less than $x + y$ by z, too much by z has been repeated every time that $x + y$ has been repeated;—that is, $m\,z$ too much has been taken: consequently, $m\,a$ is not $m\,x + m\,y$, but $m\,x + m\,y - m\,z$. Similar reasoning may be applied to other cases, and the following results may be obtained.

$$m\,(a + b + c - d) = m\,a + m\,b + m\,c - m\,d$$

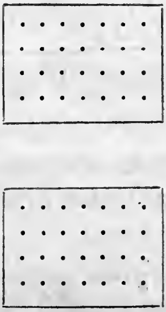
$$a\,(a-b) = a\,a - a\,b \qquad 7\,a\,(7+2\,b) = 49\,a + 14\,a\,b$$
$$b\,(a-b) = b\,a - b\,b \qquad (a\,a+a+1)\,a = a\,a\,a+a\,a+a$$
$$3\,(2\,a-4\,b) = 6\,a - 12\,b \qquad (3ab-2c)\,4abc = 12aabbc-8abcc$$

55. There is another way in which two numbers may be multiplied together. Since 8 is 4 times 2, 7 times 8 may be made by multiplying 7 and 4, and then multiplying that *product* by 2. To shew this, place 7 counters in a line, and repeat that line in all 8 times, as in figures I. and II.

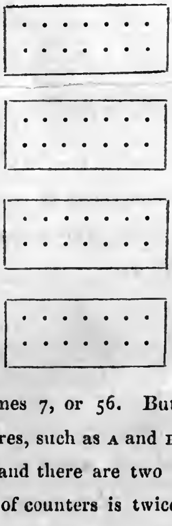
I.

A

B



II.



The number of counters in all is 8 times 7, or 56. But (as in fig. I.) enclose each four rows in oblong figures, such as A and B. The number in each oblong is 4 times 7, or 28, and there are two of those oblongs; so that in the whole the number of counters is twice 28, or

28×2, or 7 first multiplied by 4, and that product multiplied by 2. In figure II., it is shewn that 7 multiplied by 8, is also 7 first multiplied by 2, and that product multiplied by 4. The same method may be applied to other numbers. Thus, since 80 is 8 times 10, 256 times 80 is 256 multiplied by 8, and that product multiplied by 10. If we use the signs, the foregoing assertions are made thus:

$$7 \times 8 = 7 \times 4 \times 2 = 7 \times 2 \times 4$$
$$256 \times 80 = 256 \times 8 \times 10 = 256 \times 10 \times 8$$

EXERCISES.

Shew that $2 \times 3 \times 4 \times 5 = 2 \times 4 \times 3 \times 5 = 5 \times 4 \times 2 \times 3$, &c.

Shew that $18 \times 100 = 18 \times 57 + 18 \times 43$.

56. Articles (51) and (55) may be expressed in the following way, where by ab we mean a taken b times; by abc, a taken b times, and the result taken c times.

$$ab = ba$$
$$abc = acb = bca = bac, \text{ \&c.}$$
$$abc = a \times (bc) = b \times (ca) = c \times (ab).$$

If we would say that the same results are produced by multiplying by b, c, and d, one after the other, and by the product bcd at once, we write the following:

$$a \times b \times c \times d = a \times bcd.$$

The fact is, that if any numbers are to be multiplied together, the product of any two or more may be formed, and substituted instead of those two, or more; thus, the product $abcdef$ may be formed by multiplying

ab	cde	f	
abf	de	c	
abc	def		&c.

57. In order to multiply by 10, annex a cipher to the right hand of the multiplicand. Thus, 10 times 2356 is 23560. To shew this, write 2356 at length, which is

2 thousands, 3 hundreds, 5 tens, and 6 units.

Take each of these parts ten times, which, by (52), is the same as multiplying the whole number by 10, and it will then become

2 tens of thou. 3 tens of hun. 5 tens of tens, and 6 tens,

which is 2 ten-thou. 3 thous. 5 hun. and 6 tens.

This must be written 23560, because 6 is not to be 6 units, but 6 tens. Therefore $2356 \times 10 = 23560$.

In the same way you may shew, that in order to multiply by 100 you must affix two ciphers to the right; to multiply by 1000 you must affix three ciphers, and so on. The rule will be best caught from the following table.

13 ×	10 =	130	142 ×	1000 =	142000
13 ×	100 =	1300	23700 ×	10 =	237000
13 ×	1000 =	13000	3040 ×	1000 =	3040000
13 ×	10000 =	130000	10000 ×	100000 =	1000000000

58. I now shew how to multiply by one of the numbers, 2, 3, 4, 5, 6, 7, 8, or 9. I do not include 1, because multiplying by 1, or taking the number once, is what is meant by simply writing down the number. I want to multiply 1368 by 8. Write the first number at full length, which is

1 thousand, 3 hundreds, 6 tens, and 8 units.

To multiply this by 8, multiply each of these parts by 8 (50) and (52), which will give

8 thousands, 24 hundreds, 48 tens, and 64 units.

Now	64	units are written thus	64
	48	tens	480
	24	hundreds..............................	2400
	8	thousands	8000

Add these together, which gives 10944 as the product of 1368 and 8, or $1368 \times 8 = 10944$. By working a few examples in this way you will see the following rule.

59. I. Multiply the first figure of the multiplicand by the multiplier, write down the units' figure, and reserve the tens.

II. Do the same with the second figure of the multiplicand, and add to the product the number of tens from the first; put down the units' figure of this, and reserve the tens.

III. Proceed in this way till you come to the last figure, and then write down the whole number obtained from that figure.

IV. If there be a cipher in the multiplicand, treat it as if it were a number, observing that $0 \times 1 = 0$, $0 \times 2 = 0$, &c.

60. In a similar way a number can be multiplied by a figure which is accompanied by ciphers, as, for example, 8000. For 8000 is 8×1000, and therefore (55) you must first multiply by 8 and then by 1000, which last operation (57) is done by placing 3 ciphers on the right. Hence, the rule in this case is — Multiply by the simple number, and place the number of ciphers which follow it at the right of the product.

EXAMPLE.

Multiply	16794238008 72
by	60000
	1007654280523 20000

61. EXERCISES.

What is 1007360×7? *Answer*, 7051520.

$123456789 \times 9 + 10$ and $123 \times 9 + 4$? — *Ans.* 1111111111 and 1111.

What is $136 \times 3 + 129 \times 4 + 147 \times 8 + 27 \times 3000$? — *Ans.* 83100.

An army is made up of 33 regiments of infantry, each containing 800 men; 14 of cavalry, each containing 600 men; and 2 of artillery, each containing 300 men. The enemy has 6 more regiments of infantry, each containing 100 more men; 3 more regiments of cavalry, each containing 100 men less; and 4 corps of artillery of the same magnitude as those of the first: two regiments of cavalry and one of infantry desert from the former to the latter. How many men has the second army more than the first? — *Answer*, 13400.

62. Suppose it required to multiply 23707 by 4567. Since 4567 is made up of 4000, 500, 60, and 7, by (53) we must multiply 23707 by each of these, and add the products.

Now (58)	23707 × 7	is	165949
(60)	23707 × 60	is	1422420
	23707 × 500	is	11853500
	23707 × 4000	is	94828000
	The sum of these	is	108269869

which is the product required.

It will do as well if, instead of writing the ciphers at the end of each line, we keep the other figures in their places without them. If we take away the ciphers, the second line is one place to the left of the first, the third one place to the left of the second, and so on. Write the multiplier and the multiplicand over these lines, and the process will stand thus:—

```
    23707
     4567
   165949
  142242
 118535
94828
108269869
```

63. There is one more case to be noticed, that is, where there is a cipher in the middle of the multiplier. The following example will shew that in this case nothing more is necessary than to keep the first figure of each line, in the column under the figure of the multiplier from which that line arises. Suppose it required to multiply 365 by 101001. The multiplier is made up of 100000, 1000 and 1. Proceed as before, and

	365 × 1	is	365
(57)	365 × 1000	is	365000
	365 × 100000	is	36500000
	The sum of which	is	36865365

and the whole process with the ciphers struck off is

```
     365
  101001
     365
  365
365
36865365
```

64. The following is the rule in all cases.

I. Place the multiplier under the multiplicand, so that the units of one may be under those of the other.

II. Multiply the whole multiplicand by each figure of the multiplier (59), and place the unit of each line in the column under the figure of the multiplier from which it came.

III. Add together the lines obtained by II. column by column.

65. When the multiplier or multiplicand, or both, have ciphers on

the right hand, multiply the two together without the ciphers, and then place on the right of the product all the ciphers that are on the right, both of the multiplier and multiplicand. For example, what is 3200 × 13000? First, 3200 is 32 × 100, or one hundred times as great as 32. Again, 32 × 13000 is 32 × 13, with three ciphers affixed, that is 416, with three ciphers affixed, or 416000. But the product required must be 100 times as great as this, or must have two ciphers affixed. It is therefore 41600000, having as many ciphers as are in both multiplier and multiplicand.

66. When any number is multiplied by itself any number of times, the result is called a *power* of that number: Thus,

6 is called the first	power of 6
6 × 6......... second	power of 6
6 × 6 × 6 ... third	power of 6
6 × 6 × 6 × 6 fourth	power of 6
&c.	&c.

The second and third powers are usually called the *square* and *cube*, which are incorrect names, derived from certain connexions of the second and third power with the square and cube in geometry. As exercises in multiplication, the following powers are to be found.

Number proposed.	Square.	Cube.
972	944784	918330048
1008	1016064	1024192512
3142	9872164	31018339288
3163	10004569	31644451747
5555	30858025	171416328875
6789	46090521	312908547069

The fifth	power of 36	is	60466176
... fourth	 50	...	6250000
... fourth	 108	...	136048896
... fourth	 277	...	5887339441

67. It is required to multiply $a+b$ by $c+d$, that is, to take $a+b$ as many times as there are units in $c+d$. By (53) $a+b$ must be taken

c times, and d times, or the product required is $(a+b)\,c + (a+b)\,d$. But (52) $(a+b)\,c$ is $a\,c+b\,c$, and $(a+b)\,d$ is $a\,d+b\,d$; whence the product required is $a\,c+b\,c+a\,d+b\,d$; or,

$$(a+b)\,(c+d) = a\,c+b\,c+a\,d+b\,d.$$

By similar reasoning $(a-b)\,(c+d)$ is $(a-b)\,c + (a-b)\,d$; or,

$$(a-b)\,(c+d) = a\,c-b\,c+a\,d-b\,d.$$

To multiply $a-b$ by $c-d$, first take $a-b$, c times, which gives $a\,c-b\,c$. This is not correct, for in taking it c times instead of $c-d$ times, we have taken it d times too many; or have made a result which is $(a-b)\,d$ too great. The real result is therefore $a\,c-b\,c-(a-b)\,d$. But $(a-b)\,d$ is $a\,d-b\,d$, and therefore

$$\begin{aligned}(a-b)\,(c-d) &= a\,c-b\,c-(a\,d-b\,d)\\ &= a\,c-b\,c-a\,d+b\,d \qquad (41)\end{aligned}$$

From these three examples may be collected the following rule for the multiplication of algebraic quantities: Multiply each term of the multiplicand by each term of the multiplier; when the two terms have both + or both — before them, put + before their product; when one has + and the other —, put — before their product. In using the first terms, which have no sign, apply the rule as if they had the sign +.

68. For example, $(a+b)\,(a+b)$ gives $a\,a+a\,b+a\,b+b\,b$. But $a\,b+a\,b$ is $2\,a\,b$; hence the *square* of $a+b$ is $a\,a+2\,a\,b+b\,b$. Again $(a-b)\,(a-b)$ gives $a\,a-a\,b-a\,b+b\,b$. But two subtractions of $a\,b$ are equivalent to subtracting $2\,a\,b$; hence the *square* of $a-b$ is $a\,a-2\,a\,b+b\,b$. Again, $(a+b)\,(a-b)$ gives $a\,a+a\,b-a\,b-b\,b$. But the addition and subtraction of $a\,b$ makes no change; hence the product of $a+b$ and $a-b$ is $a\,a-b\,b$.

Again, the square of $a+b+c+d$, or $(a+b+c+d)\,(a+b+c+d)$ will be found to be $a\,a+2\,a\,b+2\,a\,c+2\,a\,d+b\,b+2\,b\,c+2\,b\,d+c\,c+2\,c\,d+d\,d$, or the rule for squaring such a quantity is: Square the first term, and multiply all that come *after* by twice that term: do the same with the second, and so on to the end.

SECTION IV.

DIVISION.

69. Suppose I ask whether 156 can be divided into a number of parts each of which is 13, or how many thirteens 156 contains; I propose a question, the solution of which is called DIVISION. In this case, 156 is called the *dividend*, 13 the *divisor*, and the number of parts required is the *quotient;* and when I find the quotient, I am said to divide 156 by 13.

70. The simplest method of doing this is to subtract 13 from 156, and then to subtract 13 from the remainder, and so on; or, in common language, to *tell off* 156 by thirteens. A similar process has already occurred in the exercises on subtraction, Art. (46). Do this, and mark one for every subtraction that is made, to remind you that each subtraction takes 13 once from 156, which operations will stand as follow:—

156	
13	1
143	
13	1
130	
13	1
117	
13	1
104	
13	1
91	
13	1
78	
13	1
65	
13	1
52	
13	1
39	
13	1
26	
13	1
13	
13	1
0	

Begin by subtracting 13 from 156, which leaves 143. Subtract 13 from 143, which leaves 130; and so on. At last, 13 only remains, from which when 13 is subtracted there remains nothing. Upon counting the number of times which you have subtracted 13, you find that this number is 12; or 156 contains twelve thirteens, or contains 13 twelve times.

This method is the most simple possible, and might be done with pebbles. Of these you would first count 156. You would then take thirteen from the heap, and put them into one heap by themselves. You would then take another 13 from the heap, and place them in another heap by themselves; and so on until there were none left. You would then count the number of heaps, which you would find to be 12.

71. Division is the opposite of multiplication. In multiplication you have a number of heaps, with the same number of pebbles in each, and you want to know how many *pebbles* there are in all. In division, you know how many there are

in all, and how many there are to be in each heap, and you want to know how many *heaps* there are.

72. In the last example, a number was taken which contains an exact number of thirteens. But this does not happen with every number. Take, for example, 159. Follow the process of (70), and it will appear that after having subtracted 13 twelve times, there remains 3, from which 13 cannot be subtracted. We may say then that 159 contains twelve thirteens and 3 *over*; or that 159, when divided by 13, gives a *quotient* 12, and a *remainder* 3. If we use signs,

$$159 = 13 \times 12 + 3.$$

EXERCISES.

$146 = 24 \times 6 + 2$, or 146 contains six twenty-fours and 2 over.

$146 = 6 \times 24 + 2$, or 146 contains twenty-four sixes and 2 over.

$300 = 42 \times 7 + 6$, or 300 contains seven forty-twos and 6 over.

$39624 = 7277 \times 5 + 3239.$

73. If a contain b q times, with a remainder r, a must be greater than $b\,q$ by r; that is,

$$a = b\,q + r.$$

If there be no remainder, $a = bq$. Here a is the dividend, b the divisor, q the quotient, and r the remainder. In order to say that a contains b q times, we write,

$$\frac{a}{b} = q, \text{ or } a : b = q,$$

which in old books is often found written thus:

$$a \div b = q.$$

74. If I divide 156 into several parts, and find how often 13 is contained in each of them, it is plain that 156 contains 13 as often as all its parts together. For example, 156 is made up of 91, 39, and 26. Of these,

91 contains 13, 7 times,

39 contains 13, 3 times,

26 contains 13, 2 times;

therefore, $91 + 39 + 26$ contains 13, $7 + 3 + 2$ times, or 12 times.

Again, 156 is made up of 100, 50, and 6.

Now, 100 contains 13, 7 times and 9 over,
50 contains 13, 3 times and 11 over,
6 contains 13, 0 times* and 6 over.

Therefore 100 + 50 + 6 contains 13, 7 + 3 + 0 times and 9 + 11 + 6 over; or 156 contains 13, 10 times and 26 over. But 26 is itself 2 thirteens; therefore, 156 contains 10 thirteens and 2 thirteens, or 12 thirteens.

75. The result of the last article is expressed by saying, that if

$$a = b + c + d, \text{ then } \frac{a}{m} = \frac{b}{m} + \frac{c}{m} + \frac{d}{m}$$

76. In the first example, I did not take 13 away more than once at a time, in order that the method might be as simple as possible. But if I know what is twice 13, 3 times 13, &c., I can take away as many thirteens at a time as I please, if I take care to mark at each step how many I take away. For example, take away 13 ten times at once from 156, that is, take away 130, and afterwards take away 13 twice, or take away 26, and the process is as follows:

156	
130	10 times 13
26	
26	2 times 13
0	

Therefore, 156 contains 13, 10 + 2, or 12 times.

Again, to divide 3096 by 18.

3096	
1800	100 times 18
1296	
900	50 times 18
396	
360	20 times 18
36	
36	2 times 18
0	

Therefore, 3096 contains 18, 100 + 50 + 20 + 2, or 172 times.

77. You will now understand the following sentences, and be able to make similar assertions of other numbers.

450 is 75 × 6, it therefore contains any number, as 5, 6 times as often as 75 contains it.

* To speak always in the same way, instead of saying that 6 does not contain 13, I say that it contains it 0 times and 6 over, which is merely saying that 6 is 6 more than nothing.

135	contains	3	more than	26 times; therefore,	
Twice 135	contains	3	more than	52 or twice 26	times.
10 times 135	contains	3	more than	260 or 10 times 26	times.
50 times 135	contains	3	more than	1300 or 50 times 26	times.

472 contains 18 more than 21 times; therefore,
4720 contains 18 more than 210 times,
47200 contains 18 more than 2100 times,
472000 contains 18 more than 21000 times,

32	contains	12	more than	2	times, and less than	3	times.
320	contains	12	more than	20	times, and less than	30	times.
3200	contains	12	more than	200	times, and less than	300	times.
32000	contains	12	more than	2000	times, and less than	3000	times.
&c.				&c.		&c.	

78. The foregoing articles contain the principles of division. The question now is, to apply them in the shortest and most convenient way. Suppose it required to divide 4068 by 18, or to find $\frac{4068}{18}$ (23).

If we divide 4068 into any number of parts, we may, by the process followed in (74), find how many times 18 is contained in each of these parts, and from thence how many times it is contained in the whole. Now, what separation of 4068 into parts will be most convenient? Observe that 4, the first figure of 4068, does not contain 18; but that 40, the first and second figures together, *does contain* 18 *more than twice, but less than three times.** But 4068 (20) is made up of 40 hundreds, and 68; of which, 40 hundreds (77) contains 18 more than 200 times, and less than 300 times. Therefore, 4068 also contains more than 200 times 18, since it must contain 18 more times than 4000 does. It also contains 18 less than 300 times, because 300 times 18 is 5400, a greater number than 4068. Subtract 18, 200 times from 4068; that is, subtract 3600, and there remains 468. Therefore, 4068 contains 18, 200 times, and as many more times as 468 contains 18.

It remains, then, to find how many times 468 contains 18. Proceed

* If you have any doubt as to this expression, recollect that it means "contains more than two eighteens, but not so much as three."

exactly as before. Observe that 46 contains 18 more than twice, and less than 3 times; therefore, 460 contains it more than 20, and less than 30 times (77); as does also 468. Subtract 18, 20 times from 468, that is, subtract 360; the remainder is 108. Therefore, 468 contains 18, 20 times, and as many more as 108 contains it. Now, 108 is found to contain 18 6 times exactly; therefore, 468 contains it 20 + 6 times, and 4068 contains it 200 + 20 + 6 times, or 226 times. If we write down the process that has been followed, without any explanation, putting the divisor, dividend, and quotient, in a line separated by parentheses, it will stand, as in example (A).

Let it be required to divide 36326599 by 1342 (B).

B.

```
1342)36326599(20000 + 7000 + 60 + 9
     26840000
     --------
      9486599
      9394000
      -------
        92599
        80520
        -----
        12079
        12078
        -----
            1
```

A.

```
18)4068(200 + 20 + 6
   3600
   ----
    468
    360
    ---
    108
    108
    ---
      0
```

As in the previous example, 36326599 is separated into 36320000 and 6599; the first four figures 3632 being separated from the rest, because it takes four figures from the left of the dividend to make a number which is greater than the divisor. Again, 36320000 is found to contain 1342 more than 20000, and less than 30000 times; and 1342 × 20000 is subtracted from the dividend, after which the remainder is 9486599. The same operation is repeated again and again, and the result is found to be, that there is a quotient 20000 + 7000 + 60 + 9, or 27069, and a remainder 1.

Before you proceed, you should now repeat the foregoing article at length in the solution of the following questions. What are

$$\frac{10093874}{3207}, \qquad \frac{66779922}{114433}, \qquad \frac{2718218}{13352}?$$

the quotients of which are 3147, 583, 203; and the remainders 1445, 65483, 7762.

79. In the examples of the last article, observe, 1st, that it is useless to write down the ciphers which are on the right of each subtrahend, provided that without them you keep each of the other figures in its proper place: 2d, that it is useless to put down the right-hand figures of the dividend so long as they fall over ciphers, because they do not begin to have any share in the making of the quotient, until, by continuing the process, they cease to have ciphers under them: 3d, that the quotient is only a number written at length, instead of the usual way. For example, the first quotient is 200 + 20 + 6, or 226; the second is 20000 + 7000 + 60 + 9, or 27069. Strike out, therefore, all the ciphers and the numbers which come above them, except those in the first line, and put the quotient in one line; and the two examples of the last article will stand thus:

```
18)4068(226          1342)36326599(27069
   36                     2684
   --                     ----
    46                     9486
    36                     9394
    ---                    -----
    108                      9259
    108                      8052
    ---                      -----
      0                      12079
                             12078
                             -----
                                 1
```

80. Hence the following rule is deduced:

I. Write the divisor and dividend in one line, and place parentheses on each side of the dividend.

II. Take off from the left hand of the dividend the least number of figures which make a number greater than the divisor; find what number of times the divisor is contained in these, and write this number as the first figure of the quotient.

III. Multipy the divisor by the last-mentioned figure, and subtract the product from the number which was taken off at the left of the dividend.

IV. On the right of the remainder place the figure of the dividend which comes next after those already separated in II.: if the remainder thus increased be greater than the divisor, find how many times the divisor is contained in it; put this number at the right of the first figure of the quotient, and repeat the process: if not, on the right, place the next figure of the dividend, and the next, and so on until it is greater; but remember to place a cipher in the quotient for every figure of the dividend which you are obliged to take, except the first.

V. Proceed in this way until all the figures of the dividend are exhausted.

In judging how often one large number is contained in another, a first and rough guess may be made by striking off the same number of figures from both, and using the results instead of the numbers themselves. Thus, 4,732 is contained in 14,379 about the same number of times that 4 is contained in 14, or about 3 times. The reason is, that 4 being contained in 14 as often as 4000 is in 14000, and these last only differing from the proposed numbers by lower denominations, viz. hundreds, &c. we may expect that there will not be much difference between the number of times which 14000 contains 4000, and that which 14379 contains 4732: and it generally happens so. But if the second figure of the divisor be 5, or greater than 5, it will be more accurate to increase the first figure of the divisor by 1, before trying the method just explained. Nothing but practice can give facility in this sort of guess-work.

81. This process may be made more simple when the divisor is not greater than 12, if you have sufficient knowledge of the multiplication table (50). For example, I want to divide 132976 by 4. At full length the process stands thus:

```
4)132976(33244
  12
  --
   12
   12
   --
     9
     8
     --
     17
     16
     --
      16
      16
      --
       0
```

But you will recollect, without the necessity of writing it down, that 13 contains 4 three times with a remainder 1; this 1 you will place before 2, the next figure of the dividend, and you know that 12 contains 4, 3 times exactly, and so on. It will be more convenient to write down the quotient thus:

$$4)\underline{132976}$$
$$33244$$

While on this part of the subject, we may mention, that the shortest way to multiply by 5, is to annex a cipher and divide by 2, which is equivalent to taking the half of 10 times, or 5 times. To divide by 5, multiply by 2 and strike off the last figure, which leaves the quotient; half the last figure is the remainder. To multiply by 25, annex two ciphers and divide by 4. To divide by 25, multiply by 4 and strike off the two last figures, which leaves the quotient; one fourth of the two last figures, taken as one number, is the remainder. To multiply a number by 9, annex a cipher, and subtract the number, which is equivalent to taking the number ten times, and then subtracting it once. To multiply by 99, annex two ciphers and subtract the number, &c.

In order that a number may be divisible by 2 without remainder, its unit's figure must be an even number.* That it may be divisible by 4, its two last figures must be divisible by 4. Take the example 1236: this is composed of 12 hundreds and 36, the first part of which, being hundreds, is divisible by 4, and gives 12 twenty-fives; it depends then upon 36, the last two figures, whether 1236 is divisible by 4 or not. A number is divisible by 8 if the last three figures are divisible by 8; for every digit, except the last three, is a number of thousands, and 1000 is divisible by 8; whether therefore the whole shall be divisible by 8 or not depends on the last three figures: thus, 127946 is not divisible by 8, since 946 is not so. A number is divisible by 3 or 9, only when the sum of its digits is divisible by 3 or 9. Take for example 1234; this is

* Among the even figures we include 0.

	1 thousand, or	999 and 1
	2 hundred, or	twice 99 and 2
	3 tens,	or three times 9 and 3
and	4	or 4

Now 9, 99, 999, &c. are all obviously divisible by 9 and by 3, and so will be any number made by the repetition of all or any of them any number of times. It therefore depends on $1+2+3+4$, or the sum of the digits, whether 1234 shall be divisible by 9 or 3, or not. From the above we gather, that a number is divisible by 6 when it is even, and when the sum of its digits is divisible by 3 Lastly, a number is divisible by 5 only when the last figure is 0 or 5.

82. Where the divisor is unity followed by ciphers, the rule becomes extremely simple, as you will see by the following examples:

```
100)33429(334
    300
    ---
     342
     300
     ---
      429
      400
      ---
       29

10)2717316
   -------
   271731 and rem. 6.
```

This is then the rule:—Cut off as many figures from the right hand of the dividend as there are ciphers. These figures will be the remainder, and the rest of the dividend will be the quotient.

Or we may prove these results thus: from (20), 2717316 is 271731 tens and 6; of which the first contains 10, 271731 times, and the second not at all; the quotient is therefore 271731, and the remainder 6 (72). Again (20), 33429 is 334 hundreds and 29; of which the first contains 100, 334 times, and the second not at all; the quotient is therefore 334, and the remainder 29.

83. The following examples will shew how the rule may be shortened when there are ciphers in the divisor. With each example is placed another containing the same process, all unnecessary figures being removed; and from the comparison of the two, the rule at the end of this article is derived.

```
I.  1782000)6424700000(3605          1782)6424700(3605
            5346000                       5346
            10787000                      10787
            10692000                      10692
               9500000                      9500
               8910000                      8910
                590000                       590000

II.  12300000)42176189300(3428        123)421761(3428
              36900000                    369
               52761893                    527
               49200000                    492
                35618930                    356
                24600000                    246
                110189300                   1101
                 98400000                    984
                 11789300                    11789300
```

The rule then is: Strike out as many *figures** from the right of the dividend as there are *ciphers* at the right of the divisor. Strike out all the ciphers from the divisor, and divide in the usual way; but at the end of the process, place on the right of the remainder all those figures which were struck out of the dividend.

84. **EXERCISES.**

Dividend.	Divisor.	Quotient.	Remainder.
9694	47	206	12
175618	3136	56	2
23796484	130000	183	6484
14002564	1871	7484	0
310314420	7878	39390	0
3939040647	6889	571787	4
22876792454961	43046721	531441	0

* Including both ciphers and others.

Shew that

I. $\frac{100 \times 100 \times 100 - 43 \times 43 \times 43}{100 - 43} = 100 \times 100 + 100 \times 43 + 43 \times 43.$

II. $\frac{100 \times 100 \times 100 + 43 \times 43 \times 43}{100 + 43} = 100 \times 100 - 100 \times 43 + 43 \times 43.$

III. $\frac{76 \times 76 + 2 \times 76 \times 52 + 52 \times 52}{76 + 52} = 76 + 52.$

IV. $1 + 12 + 12 \times 12 + 12 \times 12 \times 12 = \frac{12 \times 12 \times 12 \times 12 - 1}{12 - 1}.$

What is the nearest number to 1376429 which can be divided by 36300 without remainder? — *Answer*, 1379400.

If 36 oxen can eat 216 acres of grass in one year, and if a sheep eat half as much as an ox, how long will it take 49 oxen and 136 sheep together to eat 17550 acres? — *Answer*, 25 years.

85. Take any two numbers, one of which divides the other without remainder; for example, 32 and 4. Multiply both these numbers by any other number; for example, 6. The products will be 192 and 24. Now, 192 contains 24 just as often as 32 contains 4. Suppose 6 baskets, each containing 32 pebbles, the whole number of which will be 192. Take 4 from one basket, time after time, until that basket is empty. It is plain that if, instead of taking 4 from that basket, I take 4 from each, the whole 6 will be emptied together: that is, 6 times 32 contains 6 times 4 just as often as 32 contains 4. The same reasoning applies to other numbers, and therefore *we do not alter the quotient if we multiply the dividend and divisor by the same number.*

86. Again, suppose that 200 is to be divided by 50. Divide both the dividend and divisor by the same number; for example, 5. Then, 200 is 5 times 40, and 50 is 5 times 10. But by (85), 40 divided by 10 gives the same quotient as 5 times 40 divided by 5 times 10, and therefore *the quotient of two numbers is not altered by dividing both the dividend and divisor by the same number.*

87. From (55), if a number is multiplied successively by two others, it is multiplied by their product. Thus, 27, first multiplied by 5, and the product multiplied by 3, is the same as 27 multiplied by 5 times 3, or 15. Also, if a number be divided by any number, and the quotient

be divided by another, it is the same as if the first number had been divided by the product of the other two. For example, divide 60 by 4, which gives 15, and the quotient by 3, which gives 5. It is plain, that if each of the four fifteens of which 60 is composed, be divided into three equal parts, there are twelve equal parts in all; or, a division by 4, and then by 3, is equivalent to a division by 4×3, or 12.

88. The following rules will be better understood by stating them in an example. If 32 be multiplied by 24 and divided by 6, the result is the same as if 32 had been multiplied by the quotient of 24 divided by 6, that is, by 4; for, the sixth part of 24 being 4, the sixth part of any number repeated 24 times, is that number repeated 4 times; or, multiplying by 24 and dividing by 6 is equivalent to multiplying by 4.

89. Again, if 48 be multiplied by 4, and that product be divided by 24, it is the same thing as if 48 were divided at once by the quotient of 24 divided by 4, that is, by 6. For, every unit which is repeated 6 times in 48 is repeated 4 times as often, or 24 times, in 4 times 48, or the quotient of 48 and 6 is the same as the quotient of 48×4 and 6×4.

90. The results of the last five articles may be algebraically expressed thus:

$$\frac{m\,a}{m\,b} = \frac{a}{b} \qquad (85)$$

If n divides a and b without remainder,

$$\frac{\frac{a}{n}}{\frac{b}{n}} = \frac{a}{b} \qquad (86) \qquad\qquad \frac{\frac{a}{b}}{c} = \frac{a}{b\,c} \qquad (87)$$

$$\frac{a\,b}{c} = a \times \frac{b}{c} \qquad (88) \qquad\qquad \frac{a\,c}{b} = \frac{a}{\frac{b}{c}} \qquad (89)$$

It must be recollected, however, that these have only been proved in the case where all the divisions are without remainder.

91. When one number divides another without leaving any remainder, or is contained an exact number of times in it, it is said to be a *measure* of that number, or to *measure* it. Thus, 4 is a measure of 136, or measures 136; but it does not measure 137. The reason for

using the word measure is this: Suppose you have a rod 4 feet long, with nothing marked upon it, with which you want to measure some length; for example, the length of a street. If that street should happen to be 136 feet in length, you will be able to *measure* it with the rod, because, since 136 contains 4, 34 times, you will find that the street is exactly 34 times the length of the rod. But if the street should happen to be 137 feet long, you cannot measure it with the rod; for when you have measured 34 of the rods, you will find a remainder, whose length you cannot tell without some shorter measure. Hence 4 is said to measure 136, but not to measure 137. A measure, then, is a divisor which leaves no remainder.

92.* When one number is a measure of two others, it is called a *common measure* of the two. Thus, 15 is a common measure of 180 and 75. Two numbers may have several common measures. For example, 360 and 168 have the common measures 2, 3, 4, 6, 24, and several others. Now, this question may be asked: Of all the common measures of 360 and 168, which is the greatest? The answer to this question is derived from a rule of arithmetic, called the rule for finding the GREATEST COMMON MEASURE, which we proceed to consider.

93. If one quantity measures two others, it measures their sum and difference. Thus, 7 measures 21 and 56. It therefore measures $56+21$ and $56-21$, or 77 and 35. This is only another way of saying what was said in (74).

94. If one number measures a second, it measures every number which the second measures. Thus, 5 measures 15, and 15 measures 30, 45, 60, 75, &c.; all which numbers are measured by 5. It is plain that if

15 contains 5 3 times,
30, or 15 + 15 contains 5 3 + 3 times, or 6 times,
45, or 15 + 15 + 15 contains 5 3 + 3 + 3 or 9 times;

and so on.

95. Every number which measures both the dividend and divisor measures the remainder also. To shew this, divide 360 by 112. The quotient is 3, and the remainder 24, that is (72), 360 is three times 112

and 24, or $360 = 112 \times 3 + 24$. From this it follows, that 24 is the difference between 360 and 3 times 112, or $24 = 360 - 112 \times 3$. Take any number which measures both 360 and 112; for example, 4. Then

4 measures 360,

4 measures 112, and therefore (94) measures 112×3,

or $112 + 112 + 112$.

Therefore (93) it measures $360 - 112 \times 3$, which is the remainder 24. The same reasoning may be applied to all other measures of 360 and 112; and the result is, that every quantity which measures both the dividend and divisor also measures the remainder. Hence, every *common measure* of a dividend and divisor is also a *common measure* of the divisor and remainder.

96. Every common measure of the remainder and divisor is also a common measure of the dividend and divisor. Take the same example, and recollect that $360 = 112 \times 3 + 24$. Take any common measure of the remainder 24 and the divisor 112; for example, 8. Then

8 measures 24;

and 8 measures 112, and therefore (94) measures 112×3.

Therefore (93) 8 measures $112 \times 3 + 24$, or measures the dividend 360. Then every common measure of the remainder and divisor is also a common measure of the divisor and dividend, or there is no common measure of the remainder and divisor which is not also a common measure of the divisor and dividend.

97. I. It is proved in (95) that the remainder and divisor have all the common measures which are in the dividend and divisor.

II. It is proved in (96) that they have no others.

It therefore follows, that the greatest of the common measures of the first two is the greatest of those of the second two, which shews how to find the greatest common measure of any two numbers,* as follows:

98. Take the preceding example, and let it be required to find the g. c. m. of 360 and 112, and observe that

* For shortness, I abbreviate the words *greatest common measure* into their initial letters, g. c. m.

360 divided by 112 gives the remainder 24,
112 divided by 24 gives the remainder 16,
24 divided by 16 gives the remainder 8,
16 divided by 8 gives no remainder.

Now, since 8 divides 16 without remainder, and since it also divides itself without remainder, 8 is the g. c. m. of 8 and 16, because it is impossible to divide 8 by any number greater than 8; so that, even if 16 had a greater measure than 8, it could not be *common* to 16 and 8.

Therefore 8 is g. c. m. of 16 and 8
(97) g. c. m. of 16 and 8 is g. c. m. of 24 and 16
g. c. m. of 24 and 16 is g. c. m. of 112 and 24
g. c. m. of 112 and 24 is g. c. m. of 360 and 112
Therefore 8 is g. c. m. of 360 and 112.

The process carried on may be written down in either of the following ways:

```
112)360(3
    336
    ---
     24)112(4
         96
         --
         16)24(1
            16
            --
             8)16(2
               16
               --
                0
```

112	360	3
96	336	4
16	24	1
16	16	2
0	8	

The rule for finding the greatest common measure of two numbers is,

I. Divide the greater of the two by the less.

II. Make the remainder a divisor, and the divisor a dividend, and find another remainder.

III. Proceed in this way until there is no remainder, and the last divisor is the greatest common measure required.

99. You may perhaps ask how the rule is to shew when the two numbers have no common measure. The fact is, that there are, strictly speaking, no such numbers, because all numbers are measured by 1; that is, contain an exact number of units, and therefore 1 is a common measure of every two numbers. If they have no other common measure, the last divisor will be 1, as in the following example, where the greatest common measure of 87 and 25 is found.

```
25)87(3
   75
   --
   12)25(2
      24
      --
       1)12(12
         12
         --
          0
```

EXERCISES.

Numbers.		g. c. m.
6197	9521	1
58363	2602	1
5547	147008443	1849
6281	326041	571
28915	31495	5
1509	300309	3

What are $36 \times 36 + 2 \times 36 \times 72 + 72 \times 72$

and $36 \times 36 \times 36 + 72 \times 72 \times 72$;

and what is their greatest common measure? — *Answer*, 11664.

100. If two numbers be divisible by a third, and if the quotients be again divisible by a fourth, that third is not the greatest common measure. For example, 360 and 504 are both divisible by 4. The quotients are 90 and 126. Now 90 and 126 are both divisible by 9, the quotients of which division are 10 and 14. By (87), dividing a number by 4, and then dividing the quotient by 9, is the same thing as dividing the number itself by 4×9, or by 36. Then, since 36 is a common measure of 360 and 504, and is greater than 4, 4 is not the greatest common measure. Again, since 10 and 14 are both divisible by 2, 36 is not the greatest common measure. It therefore follows, that when two numbers are divided by their greatest common measure, the quotients have no common measure except 1 (99). Otherwise, the number which was called the greatest common measure in the last sentence is not so in reality.

101. To find the greatest common measure of three numbers, find the g. c. m. of the first and second, and of this and the third. For since all common divisors of the first and second are contained in their g. c. m., and no others; whatever is common to the first, second, and third, is common also to the third, and the g. c. m. of the first and second, and no others. Similarly, to find the g. c. m. of four numbers, find the g. c. m. of the first, second, and third, and of that and the fourth.

102. When a first number contains a second, or is divisible by it without remainder, the first is called a multiple of the second. The

words *multiple* and *measure* are thus connected: Since 4 is a measure of 24, 24 is a multiple of 4. The number 96 is a multiple of 8, 12, 24, 48, and several others. It is therefore called a *common multiple* of 8, 12, 24, 48, &c. The product of any two numbers is evidently a common multiple of both. Thus, 36×8, or 288, is a common multiple of 36 and 8. But there are common multiples of 36 and 8 less than 288; and because it is convenient, when a common multiple of two quantities is wanted, to use the least of them, I now shew how to find the least common multiple of two numbers.

103. Take, for example, 36 and 8. Find their greatest common measure, which is 4, and observe that 36 is 9×4, and 8 is 2×4. The quotients of 36 and 8, when divided by their greatest common measure, are therefore 9 and 2. Multiply these quotients together, and multiply the product by the greatest common measure, 4, which gives $9 \times 2 \times 4$, or 72. This is a multiple of 8, or of 4×2, by (55); and also of 36, or of 4×9. It is also the least common multiple; but this cannot be proved to you, because the demonstration cannot be thoroughly understood without some knowledge of algebra. But you may satisfy yourself that it is the least in this case, and that the same process will give the least common multiple in any other case which you may take. It is not even necessary that you should know it is the least. Whenever a common multiple is to be used, any one will do as well as the least. It is only to avoid large numbers that the least is used in preference to any other.

When the greatest common measure is 1, the least common multiple of the two numbers is their product.

The rule then is: To find the least common multiple of two numbers, find their greatest common measure, and multiply one of the numbers by the quotient which the other gives when divided by the greatest common measure. To find the least common multiple of three numbers, find the least common multiple of the first two, and find the least common multiple of that multiple, and the third, and so on.

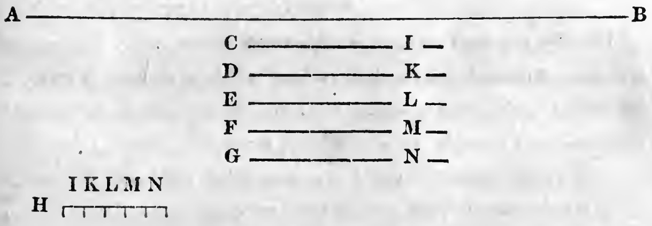
EXERCISES.

Numbers proposed.	Least common multiple.
14, 21	42
16, 5, 24	240
1, 2, 3, 4, 5, 6, 7, 8, 9, 10	2520
6, 8, 11, 16, 20	2640
876, 864	63072
868, 854	52948

SECTION V.

FRACTIONS.

104. Suppose it required to divide 49 yards into five equal parts, or, as it is called, to find the fifth part of 49 yards. If we divide 49 by 5, the quotient is 9, and the remainder is 4; that is (72), 49 is made up of 5 times 9 and 4. Let the line A B represent 49 yards:



take 5 lines, C, D, E, F, and G, each 9 yards in length, and the line H, 4 yards in length. Then, since 49 is 5 nines and 4, C, D, E, F, G, and H, are together equal to A B. Divide H, which is 4 yards, into five equal parts, I, K, L, M, and N, and place one of these parts opposite to each of the lines C, D, E, F, and G. It follows that the ten lines, C, D, E, F, G, I, K, L, M, N, are together equal to A B, or 49 yards. Now D and K together are of the same length as C and I together, and so are E and L, F and M, and G and N. Therefore C and I together, repeated 5 times, will be 49 yards; that is, c and 1 together make up the fifth part of 49 yards.

105. c is a certain number of yards, viz. 9; but 1 is a new sort of quantity, to which hitherto we have never come. It is not an exact number of yards, for it arises from dividing 4 yards into 5 parts, and taking one of those parts. It is the fifth part of 4 yards, and is called a FRACTION of a yard. It is written thus, $\frac{4}{5}$ (23), and is what we must add to 9 yards in order to make up the fifth part of 49 yards.

The same reasoning would apply to dividing 49 bushels of corn, or 49 acres of land, into 5 equal parts. We should find for the fifth part of the first 9 bushels, and the fifth part of 4 bushels; and for the second 9 acres, and the fifth part of 4 acres.

We say then, once for all, that the fifth part of 49 is 9 and $\frac{4}{5}$, or $9+\frac{4}{5}$; which is usually written $9\frac{4}{5}$, or if we use signs, $\frac{49}{5} = 9\frac{4}{5}$.

EXERCISES.

What is the seventeenth part of 1237? — *Answer*, $72\frac{13}{17}$.

What are $\frac{10032}{1974}$, $\frac{663819}{23710}$, and $\frac{22773399}{2424}$?

Answer, $5\frac{162}{1974}$, $27\frac{23649}{23710}$, $9394\frac{2343}{2424}$.

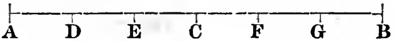
106. By the term fraction is understood a part of any number, or the sum of any of the equal parts into which a number is divided. Thus, $\frac{49}{5}$, $\frac{4}{5}$, $\frac{20}{7}$, are fractions. The term fraction even includes whole numbers:* for example, 17 is $\frac{17}{1}$, $\frac{34}{2}$, $\frac{51}{3}$, &c.

The upper number is called the *numerator*, the lower number is called the *denominator*, and both of these are called *terms* of the fraction. As long as the numerator is less than the denominator, the fraction is less than a unit: thus, $\frac{6}{17}$ is less than a unit; for 6 divided into 6 parts gives 1 for each part, and must give less when divided into 17 parts. Similarly, the fraction is equal to a unit when the numerator and denominator are equal, and greater than a unit when the numerator is greater than the denominator.

* Numbers which contain an exact number of units, such as 5, 7, 100, &c., are called *whole numbers* or *integers*, when we wish to distinguish them from fractions.

107. By $\frac{2}{3}$ is meant the third part of 2. This is the same as twice the third part of 1.

To prove this, let A B be two yards, and divide each of the yards A C and C B into three equal parts.



Then, because A E, E F, and F B, are all equal to one another, A E is the third part of 2. It is therefore $\frac{2}{3}$. But A E is twice A D, and A D is the third part of one yard, or $\frac{1}{3}$; therefore $\frac{2}{3}$ is twice $\frac{1}{3}$, that is, in order to get the length $\frac{2}{3}$, it makes no difference whether we divide *two yards* at once into three parts, and take *one* of them, or whether we divide *one* yard into three parts, and take *two* of them. By the same reasoning, $\frac{5}{8}$ may be found either by dividing 5 into 8 parts, and taking one of them, or by dividing 1 into 8 parts, and taking 5 of them. In future, of these two meanings, I shall use that which is most convenient at the time, as it is proved that they are the same thing. This principle is the same as the following: The third part of any number may be obtained by adding together the thirds of all the units of which it consists. Thus, the third part of 2, or of two units, is made by taking one-third out of each of the units, that is,

$$\frac{2}{3} = \frac{1}{3} \times 2.$$

This meaning appears ambiguous, when the numerator is greater than the denominator: thus, $\frac{15}{7}$ would mean that 1 is to be divided into 7 parts, and 15 of them are to be taken. We should here let as many units be each divided into 7 parts as will give more than 15 of those parts, and take 15 of them.

108. The value of a fraction is not altered by multiplying the numerator and denominator by the same quantity. Take the fraction $\frac{3}{4}$, multiply its numerator and denominator by 5, and it becomes $\frac{15}{20}$, which is the same thing as $\frac{3}{4}$; that is, one-twentieth part of 15 yards is the same thing as one-fourth of 3 yards: or, if our second meaning of the word fraction be used, you get the same length by dividing a yard into 20 parts and taking 15 of them, as you get by dividing it into 4 parts and taking 3 of them. To prove this,

let A B represent a yard; divide it into 4 equal parts, A C, C D, D E, and E B, and divide each of these parts into 5 equal parts. Then A E is $\frac{3}{4}$. But the second division cuts the line into 20 equal parts, of which A E contains 15. It is therefore $\frac{15}{20}$. Therefore $\frac{15}{20}$ and $\frac{3}{4}$ are the same thing.

Again, since $\frac{3}{4}$ is made from $\frac{15}{20}$ by dividing both the numerator and denominator by 5, the value of a fraction is not altered by dividing both its numerator and denominator by the same quantity. This principle, which is of so much importance in every part of arithmetic, is often used in common language, as when we say that 14 out of 21 is 2 out of 3, &c.

109. Though the two fractions $\frac{3}{4}$ and $\frac{15}{20}$ are the same in value, and either of them may be used for the other without error, yet the first is more convenient than the second, not only because you have a clearer idea of the fourth of three yards than of the twentieth part of fifteen yards, but because the numbers in the first, being smaller, are more convenient for multiplication and division. It is therefore useful, when a fraction is given, to find out whether its numerator and denominator have any common divisors or common measures. In (98) was given a rule for finding the greatest common measure of any two numbers; and it was shewn, that when the two numbers are divided by their greatest common measure, the quotients have no common measure except 1. Find the greatest common measure of the terms of the fraction, and divide them by that number. The fraction is then said to be *reduced to its lowest terms*, and is in the state in which the best notion can be formed of its magnitude.

EXERCISES.

With each fraction is written the same reduced to its lowest terms.

$$\frac{2794}{2921} = \frac{22 \times 127}{23 \times 127} = \frac{22}{23}$$

$$\frac{2788}{4920} = \frac{17 \times 164}{30 \times 164} = \frac{17}{30}$$

$$\frac{93208}{13786} = \frac{764 \times 122}{113 \times 122} = \frac{764}{113}$$

$$\frac{888800}{40359600} = \frac{22 \times 40400}{999 \times 40400} = \frac{22}{999}$$

$$\frac{95469}{359784} = \frac{121 \times 789}{456 \times 789} = \frac{121}{456}$$

110. When the terms of the fraction given are already in factors,* any one factor in the numerator may be divided by a number, provided some one factor in the denominator is divided by the same. This follows from (88) and (108). In the following examples the figures altered by division are accented.

$$\frac{12 \times 11 \times 10}{2 \times 3 \times 4} = \frac{3' \times 11 \times 10}{2 \times 3 \times 1'} = \frac{1' \times 11 \times 5'}{1' \times 1' \times 1'} = 55.$$

$$\frac{18 \times 15 \times 13}{20 \times 54 \times 52} = \frac{2' \times 3' \times 1'}{4' \times 6' \times 4'} = \frac{1' \times 1' \times 1'}{2' \times 2' \times 4'} = \frac{1}{16}.$$

$$\frac{27 \times 28}{9 \times 70} = \frac{3' \times 4'}{1' \times 10'} = \frac{3' \times 2'}{1' \times 5'} = \frac{6}{5}.$$

111. As we can, by (108), multiply the numerator and denominator of a fraction by any number, without altering its value, we can now readily reduce two fractions to two others, which shall have the same value as the first two, and which shall have the same denominator. Take, for example, $\frac{2}{3}$ and $\frac{4}{7}$; multiply both terms of $\frac{2}{3}$ by 7, and both terms of $\frac{4}{7}$ by 3. It then appears that

$$\frac{2}{3} \text{ is } \frac{2 \times 7}{3 \times 7} \text{ or } \frac{14}{21}$$

$$\frac{4}{7} \text{ is } \frac{4 \times 3}{7 \times 3} \text{ or } \frac{12}{21}$$

Here are then two fractions $\frac{14}{21}$ and $\frac{12}{21}$, equal to $\frac{2}{3}$ and $\frac{4}{7}$, and having the same denominator, 21; in this case, $\frac{2}{3}$ and $\frac{4}{7}$ are said to be *reduced to a common denominator.*

It is required to reduce $\frac{1}{10}$, $\frac{5}{6}$, and $\frac{7}{9}$ to a common denominator. Multiply both terms of the first by the product of 6 and 9; of the

* A factor of a number is a number which divides it without remainder: thus, 4, 6, 8, are factors of 24, and 6×4, 8×3, $2\times2\times2\times3$, are several ways of decomposing 24 into factors.

second by the product of 10 and 9; and of the third by the product of 10 and 6. Then it appears (108) that

$$\frac{1}{10} \text{ is } \frac{1\times6\times9}{10\times6\times9} \text{ or } \frac{54}{540}$$

$$\frac{5}{6} \text{ is } \frac{5\times10\times9}{6\times10\times9} \text{ or } \frac{450}{540}$$

$$\frac{7}{9} \text{ is } \frac{7\times10\times6}{9\times10\times6} \text{ or } \frac{420}{540}$$

On looking at these last fractions, we see that all the numerators and the common denominator are divisible by 6, and (108) this division will not alter their values. On dividing the numerators and denominators of $\frac{54}{540}$, $\frac{450}{540}$, and $\frac{420}{540}$ by 6, the resulting fractions are, $\frac{9}{90}$, $\frac{75}{90}$, and $\frac{70}{90}$. These are fractions with a common denominator, and which are the same as $\frac{1}{10}$, $\frac{5}{6}$, and $\frac{7}{9}$; and, therefore, these are a more simple answer to the question than the first fractions. Observe also that 540 is one common multiple of 10, 6, and 9, namely, $10\times6\times9$; but that 90 is *the least* common multiple of 10, 6, and 9 (103). The following process, therefore, is shorter. To reduce the fractions $\frac{1}{10}$, $\frac{5}{6}$, and $\frac{7}{9}$, to others having the same value and a common denominator, begin by finding the least common multiple of 10, 6, and 9, by the rule in (103), which is 90. Observe that 10, 6, and 9 are contained in 90, 9, 15, and 10 times. Multiply both terms of the first by 9, of the second by 15, and of the third by 10, and the fractions thus produced are $\frac{9}{90}$, $\frac{75}{90}$, and $\frac{70}{90}$, the same as before.

If one of the numbers be a whole number, it may be reduced to a fraction having the common denominator of the rest, by (106).

EXERCISES.

Fractions proposed	reduced to a common denominator.
$\frac{2}{3}$ $\frac{1}{5}$ $\frac{1}{6}$	$\frac{20}{30}$ $\frac{6}{30}$ $\frac{5}{30}$
$\frac{1}{3}$ $\frac{2}{7}$ $\frac{3}{14}$ $\frac{12}{21}$ $\frac{3}{4}$	$\frac{28}{84}$ $\frac{24}{84}$ $\frac{18}{84}$ $\frac{48}{84}$ $\frac{63}{84}$
3 $\frac{4}{10}$ $\frac{5}{100}$ $\frac{6}{1000}$	$\frac{3000}{1000}$ $\frac{400}{1000}$ $\frac{50}{1000}$ $\frac{6}{1000}$
$\frac{33}{379}$ $\frac{281}{677}$	$\frac{22341}{256583}$ $\frac{106499}{256583}$

112. By reducing two fractions to a common denominator, we are able to compare them, that is, to tell which is the greater and which the less of the two. For example, take $\frac{1}{2}$ and $\frac{7}{15}$. These fractions reduced, without alteration of their value, to a common denominator, are $\frac{15}{30}$ and $\frac{14}{30}$. Of these the first must be the greater, because (107) it may be obtained by dividing 1 into 30 equal parts and taking 15 of them, but the second is made by taking 14 of those parts.

It is evident that of two fractions which have the same denominator, the greater has the greater numerator; and also that of two fractions which have the same numerator, the greater has the less denominator. Thus, $\frac{8}{7}$ is greater than $\frac{8}{9}$, since the first is a 7th, and the last only a 9th part of 8. Also, any numerator may be made to belong to as small a fraction as we please, by sufficiently increasing the denominator. Thus, $\frac{10}{100}$ is $\frac{1}{10}$, $\frac{10}{1000}$ is $\frac{1}{100}$, and $\frac{10}{1000000}$ is $\frac{1}{100000}$ (108).

We can now also increase and diminish the first fraction by the second. For the first fraction is made up of 15 of the 30 equal parts into which 1 is divided. The second fraction is 14 of those parts. The sum of the two, therefore, must be 15 + 14, or 29 of those parts; that is, $\frac{1}{2}+\frac{7}{15}$ is $\frac{29}{30}$. The difference of the two must be 15 − 14, or 1 of those parts; that is, $\frac{1}{2}-\frac{7}{15}=\frac{1}{30}$.

113. From the two last articles the following rules are obtained:

1. To compare, to add, or to subtract fractions, first reduce them to a common denominator. When this has been done, that is the greatest of the fractions which has the greatest numerator.

Their sum has the sum of the numerators for its numerator, and the common denominator for its denominator.

Their difference has the difference of the numerators for its numerator, and the common denominator for its denominator.

EXERCISES.

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}=\frac{53}{60} \qquad \frac{44}{3}-\frac{153}{427}=\frac{18329}{1281}$$

$$1+\frac{8}{10}+\frac{3}{100}+\frac{4}{1000}=\frac{1834}{1000} \qquad 2-\frac{1}{7}+\frac{12}{13}=\frac{253}{91}$$

$$\frac{1}{2}+\frac{8}{16}+\frac{94}{188}=\frac{3}{2} \qquad \frac{163}{521}-\frac{97}{881}=\frac{93066}{459001}$$

114. Suppose it required to add a whole number to a fraction, for example, 6 to $\frac{4}{9}$. By (106) 6 is $\frac{54}{9}$, and $\frac{54}{9}+\frac{4}{9}$ is $\frac{58}{9}$; that is, $6+\frac{4}{9}$, or, as it is usually written, $6\frac{4}{9}$, is $\frac{58}{9}$. The rule in this case is, Multiply the whole number by the denominator of the fraction; and to the product add the numerator of the fraction, the sum will be the numerator of the result, and the denominator of the fraction will be its denominator. Thus, $3\frac{1}{4}=\frac{13}{4}$, $22\frac{5}{9}=\frac{203}{9}$, $74\frac{2}{55}=\frac{4072}{55}$. This rule is the opposite of that in (105).

115. From the last rule it appears that $1723\frac{907}{10000}$ is $\frac{17230907}{10000}$, $667\frac{225}{1000}$ is $\frac{667225}{1000}$, and $23\frac{99}{100000}$ is $\frac{2300099}{100000}$. Hence, when a whole number is to be added to a fraction whose denominator is 1 followed by *ciphers*, the number of which is not less than the number of *figures* in the numerator, the rule is, Write the whole number first, and then the numerator of the fraction, with as many ciphers between them as the number of *ciphers* in the denominator exceeds the number of *figures* in the numerator. This is the numerator of the result, and the denominator of the fraction is its denominator. If the number of ciphers in the denominator be equal to the number of figures in the numerator, write no ciphers between the whole number and the numerator.

EXERCISES.

Reduce the following mixed quantities to fractions: $1\frac{23707}{100000}$, $2457\frac{6}{10}$, $1207\frac{299}{10000000}$, and $233\frac{2210}{10000}$.

116. Suppose it required to multiply $\frac{2}{3}$ by 4. This by (48) is taking $\frac{2}{3}$ four times; that is, finding $\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}$. This by (112) is $\frac{8}{3}$; so that to multiply a fraction by a whole number, the rule is: Multiply the numerator by the whole number, and let the denominator remain.

117. If the denominator of the fraction be divisible by the whole number, the rule may be stated thus: Divide the denominator of the fraction by the whole number, and let the numerator remain. For example, multiply $\frac{7}{36}$ by 6. This (116) is $\frac{42}{36}$, which, since the numerator and denominator are now divisible by 6, is (108) the same as $\frac{7}{6}$. It is plain that $\frac{7}{6}$ is made from $\frac{7}{36}$ in the manner stated in the rule.

118. Multiplication has been defined to be the taking as many of one number as there are units in another. Thus, to multiply 12 by 7 is to take as many twelves as there are units in 7, or to take 12 as many times as you must take 1 in order to make 7. Thus, what is done with 1 in order to make 7, is done with 12 to make 7 times 12. For example,

$$7 \text{ is } 1+1+1+1+1+1+1$$
$$7 \text{ times } 12 \text{ is } 12+12+12+12+12+12+12$$

When the same thing is done with two fractions, the result is still called their product, and the process is still called multiplication. There is this difference, that whereas a whole number is made by adding 1 to itself a number of times, a fraction is made by dividing 1 into a number of equal parts, and adding *one of these parts* to itself a number of times. This being the meaning of the word multiplication, as applied to fractions, what is $\frac{3}{4}$ multiplied by $\frac{7}{8}$? Whatever is done with 1 in order to make $\frac{7}{8}$ must now be done with $\frac{3}{4}$; but to make $\frac{7}{8}$, 1 is divided into 8 parts, and 7 of them are taken. Therefore to make $\frac{3}{4} \times \frac{7}{8}$, $\frac{3}{4}$ must be divided into 8 parts, and 7 of them must be taken. Now $\frac{3}{4}$ is, by (108), the same thing as $\frac{24}{32}$. Since $\frac{24}{32}$ is made by dividing 1 into 32 parts and taking 24 of them, or, which is the same thing, taking 3 of them 8 times, if $\frac{24}{32}$ be divided into 8 equal parts, each of them is $\frac{3}{32}$; and if 7 of these parts be taken, the result is $\frac{21}{32}$ (116): therefore $\frac{3}{4}$ multiplied by $\frac{7}{8}$ is $\frac{21}{32}$; and the same reasoning may be applied to any other fractions. But $\frac{21}{32}$ is made from $\frac{3}{4}$ and $\frac{7}{8}$ by multiplying the two numerators together for the numerator, and the two denominators for the denominator; which furnishes a rule for the multiplication of fractions.

119. If this product $\frac{21}{32}$ is to be multiplied by a third fraction, for example by $\frac{5}{9}$, the result is, by the same rule, $\frac{105}{288}$; and so on. The general rule for multiplying any number of fractions together is therefore:

Multiply all the numerators together for the numerator of the product, and all the denominators together for its denominator.

120. Suppose it required to multiply together $\frac{15}{16}$ and $\frac{8}{10}$. The pro-

duct may be written thus: $\frac{15 \times 8}{16 \times 10}$, and is $\frac{120}{160}$, which reduced to its lowest terms (109) is $\frac{3}{4}$. This result might have been obtained directly, by observing that 15 and 10 are both measured by 5, and 8 and 16 are both measured by 8, and that the fraction may be written thus: $\frac{3 \times 5 \times 8}{2 \times 8 \times 2 \times 5}$. Divide both its numerator and denominator by 5×8 (108) and (87), and the result is at once $\frac{3}{4}$; therefore, before proceeding to multiply any number of fractions together, if there be any numerator and any denominator, whether belonging to the same fraction or not, which have a common measure, divide them both by that common measure, and use the quotients instead of the dividends.

A whole number may be considered as a fraction whose denominator is 1; thus, 16 is $\frac{16}{1}$ (106); and the same rule will apply when one or more of the quantities are whole numbers.

EXERCISES.

$$\frac{136}{7470} \times \frac{268}{919} = \frac{36448}{6864930} = \frac{18224}{3432465}$$

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{1}{5}, \qquad \frac{2}{17} \times \frac{17}{45} = \frac{2}{45}$$

$$\frac{2}{59} \times \frac{13}{7} \times \frac{241}{19} = \frac{6266}{7847}, \qquad \frac{13}{461} \times \frac{601}{11} = \frac{7813}{5071}$$

Fraction proposed.	Square.	Cube.
$\frac{701}{158}$	$\frac{491401}{24964}$	$\frac{344472101}{3944312}$
$\frac{140}{141}$	$\frac{19600}{19881}$	$\frac{2744000}{2803221}$
$\frac{355}{113}$	$\frac{126025}{12769}$	$\frac{44738875}{1442897}$

From 100 acres of ground, two-thirds of them are taken away; 50 acres are then added to the result, and $\frac{5}{7}$ of the whole is taken; what number of acres does this produce?—*Answer*, $59\frac{11}{21}$.

121. In dividing one whole number by another, for example 108 by 9, this question is asked,—Can we, by the addition of any number of nines, produce 108? and, if so, how many nines will be sufficient for that purpose?

Suppose we take two fractions, for example $\frac{2}{3}$ and $\frac{4}{5}$, and ask, Can we, by dividing $\frac{4}{5}$ into some number of equal parts, and adding a number of these parts together, produce $\frac{2}{3}$? if so, into *how many parts* must we divide $\frac{4}{5}$, and *how many of them* must we add together? The solution of this question is still called the division of $\frac{2}{3}$ by $\frac{4}{5}$; and the fraction whose denominator is the number of parts into which $\frac{4}{5}$ is divided, and whose numerator is the number of them which is taken, is called the quotient. The solution of this question is as follows: Reduce both these fractions to a common denominator (111), which does not alter their value (108); they then become $\frac{10}{15}$ and $\frac{12}{15}$. The question now is, to divide $\frac{12}{15}$ into a number of parts, and to produce $\frac{10}{15}$ by taking a number of these parts. Since $\frac{12}{15}$ is made by dividing 1 into 15 parts and taking 12 of them, if we divide $\frac{12}{15}$ into 12 equal parts, each of these parts is $\frac{1}{15}$; if we take 10 of these parts, the result is $\frac{10}{15}$. Therefore, in order to produce $\frac{10}{15}$ or $\frac{2}{3}$ (108), we must divide $\frac{12}{15}$ or $\frac{4}{5}$ into 12 parts, and take 10 of them; that is, the quotient is $\frac{10}{12}$. If we call $\frac{2}{3}$ the dividend, and $\frac{4}{5}$ the divisor, as before, the quotient in this case is derived from the following rule, which the same reasoning will shew to apply to other cases:

The numerator of the quotient is the numerator of the dividend multiplied by the denominator of the divisor. The denominator of the quotient is the denominator of the dividend multiplied by the numerator of the divisor. This rule is the reverse of multiplication, as will be seen by comparing what is required in both cases. In multiplying $\frac{4}{5}$ by $\frac{10}{12}$, I ask, if from $\frac{4}{5}$ be taken 10 parts out of 12, how much *of a unit* is taken, and the answer is $\frac{40}{60}$, or $\frac{2}{3}$. Again, in dividing $\frac{2}{3}$ by $\frac{4}{5}$, I ask, what part of $\frac{4}{5}$ is $\frac{2}{3}$, the answer to which is, $\frac{10}{12}$.

122. By taking the following instance, we shall see that this rule can be sometimes simplified. Divide $\frac{16}{33}$ by $\frac{28}{15}$. Observe that 16 is 4×4, and 28 is 4×7; 33 is 3×11, and 15 is 3×5; therefore the two fractions are $\frac{4 \times 4}{3 \times 11}$ and $\frac{4 \times 7}{3 \times 5}$, and their quotient, according to the rule, is $\frac{4 \times 4 \times 3 \times 5}{3 \times 11 \times 4 \times 7}$, in which 4×3 is found both in the numerator and denominator. The fraction is therefore (108) the same as $\frac{4 \times 5}{11 \times 7}$, or $\frac{20}{77}$. The rule of the last article therefore admits of this modification:

If the two numerators or the two denominators have a common measure, divide by that common measure, and use the quotients instead of the dividends.

123. In dividing a fraction by a whole number, for example, $\frac{2}{3}$ by 15, consider 15 as the fraction $\frac{15}{1}$. The rule gives $\frac{2}{45}$ as the quotient. Therefore, to divide a fraction by a whole number, multiply the denominator by that whole number.

EXERCISES.

Dividend.	Divisor.	Quotient.
$\frac{41}{33}$	$\frac{63}{11}$	$\frac{41}{189}$
$\frac{467}{151}$	$\frac{907}{101}$	$\frac{47167}{136957}$
$\frac{7813}{5071}$	$\frac{601}{11}$	$\frac{13}{461}$

What are $\dfrac{\frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}-\frac{2}{17}\times\frac{2}{17}\times\frac{2}{17}}{\frac{1}{5}-\frac{2}{17}}$, and $\dfrac{\frac{8}{11}\times\frac{8}{11}-\frac{3}{11}\times\frac{3}{11}}{\frac{8}{11}-\frac{3}{11}}$?

Answer $\frac{559}{7225}$, and 1.

A can reap a field in 12 days, B in 6, and C in 4 days; in what time can they all do it together ?*—*Answer,* 2 days.

In what time would a cistern be filled by cocks which would separately fill it in 12, 11, 10, and 9 hours ?—*Answer,* $2\frac{454}{763}$ hours.

124. The principal results of this section may be exhibited algebraically as follows ; let a, b, c, &c. stand for any whole numbers. Then,

(107) $\frac{a}{b} = \frac{1}{b}\times a$ (108) $\frac{a}{b} = \frac{ma}{mb}$

(111) $\frac{a}{b}$ and $\frac{c}{d}$ are the same as $\frac{ad}{bd}$ and $\frac{bc}{bd}$

* The method of solving this and the following question may be shewn thus : If the number of days in which each could reap the field is given, the part which each could do in a day by himself can be found, and thence the part which all could do together; this being known, the number of days which it would take all to do the whole can be found.

$$(112)\quad \frac{a}{c}+\frac{b}{c} = \frac{a+b}{c} \qquad \frac{a}{c}-\frac{b}{c} = \frac{a-b}{c}$$

$$(113)\quad \frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd} \qquad \frac{a}{b}-\frac{c}{d} = \frac{ad-bc}{bd}$$

$$(118)\quad \frac{a}{b}\times\frac{c}{d} = \frac{ac}{bd} \qquad (121)\ \frac{a}{b}\text{ div}^{\text{d}}\text{ by }\frac{c}{d}\text{ or }\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}.$$

125. These results are true even when the letters themselves represent fractions. For example, take the fraction $\frac{\frac{a}{b}}{\frac{c}{d}}$ whose numerator and denominator are fractional, and multiply its numerator and denominator by the fraction $\frac{e}{f}$, which gives $\frac{\frac{ae}{bf}}{\frac{ce}{df}}$, which (121) is $\frac{aedf}{bfce}$, which dividing the numerator and denominator by ef (108), is $\frac{ad}{bc}$. But the original fraction itself is $\frac{ad}{bc}$; hence $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}\times\frac{e}{f}}{\frac{c}{d}\times\frac{e}{f}}$, which corresponds to the second formula* in (124). In a similar manner it may be shewn, that the other formulæ of the same article are true, when the letters there used either represent fractions, or are removed and fractions introduced in their place. All formulæ established throughout this work are equally true when fractions are substituted for whole numbers. For example (54), $(m+n)a = ma+na$. Let m, n, and a be respectively the fractions $\frac{p}{q}$, $\frac{r}{s}$, and $\frac{b}{c}$. Then $m+n$ is $\frac{p}{q}+\frac{r}{s}$ or $\frac{ps+qr}{qs}$, and $(m+n)a$ is $\frac{ps+qr}{qs}\times\frac{b}{c}$, or $\frac{(ps+qr)b}{qsc}$ or $\frac{psb+qrb}{qsc}$. But this (112) is $\frac{psb}{qsc}+\frac{qrb}{qsc}$, which is $\frac{pb}{qc}+\frac{rb}{sc}$, since $\frac{psb}{qsc}=\frac{pb}{qc}$, and $\frac{qrb}{qsc}=\frac{rb}{sc}$ (108). But $\frac{pb}{qc}=\frac{p}{q}\times\frac{b}{c}$, and $\frac{rb}{sc}=\frac{r}{s}\times\frac{b}{c}$. Therefore $(m+n)\,a$, or $\left(\frac{p}{q}+\frac{r}{s}\right)\frac{b}{c} = \frac{p}{q}\times\frac{b}{c}+\frac{r}{s}\times\frac{b}{c}$. In a similar manner, the same may be proved of any other formula.

The following examples may be useful.

$$\frac{\frac{a}{b}\times\frac{c}{d}+\frac{e}{f}\times\frac{g}{h}}{\frac{a}{b}\times\frac{e}{f}+\frac{c}{d}\times\frac{g}{h}} = \frac{acfh+bdeg}{aedh+bcfg}$$

* A formula is a name given to any algebraical expression which is commonly used.

$$\frac{1}{a+\frac{1}{b}} = \frac{b}{ab+1}$$

$$\frac{1}{a+\frac{1}{b+\frac{1}{c}}} = \frac{1}{a+\frac{c}{bc+1}} = \frac{bc+1}{abc+a+c}$$

Thus, $$\frac{1}{6+\frac{1}{7+\frac{1}{8}}} = \frac{1}{6+\frac{8}{57}} = \frac{57}{350}$$

The rules that have been proved to hold good for all numbers, may be applied when the numbers are represented by letters.

SECTION VI.

DECIMAL FRACTIONS.

126. We have seen (112) (121) the necessity of reducing fractions to a common denominator, in order to compare their magnitudes. We have seen also how much more readily operations are performed upon fractions which have the same, than upon those which have different denominators. On this account it has long been customary, in all those parts of mathematics where fractions are often required, to use none but such as either have, or can be easily reduced to others having, the same denominators. Now, of all numbers, those which can be most easily managed are such as 10, 100, 1000, &c., where 1 is followed by ciphers. These are called DECIMAL NUMBERS; and a fraction, whose denominator is any one of them, is called a DECIMAL FRACTION, or, more commonly, a DECIMAL.

127. A whole number may be reduced to a decimal fraction, or one decimal fraction to another, with the greatest ease. For example, 94 is $\frac{940}{10}$, or $\frac{9400}{100}$, or $\frac{94000}{1000}$ (106); $\frac{3}{10}$ is $\frac{30}{100}$, or $\frac{300}{1000}$, or $\frac{3000}{10000}$ (108). The placing of a cipher on the right-hand of any number is the same thing as multiplying that number by 10 (57), and this may be done as

often as we please in the numerator of a fraction, provided it is done as often in the denominator (108).

128. The next question is, How can we reduce a fraction which is not decimal to another which is, without altering its value? Take, for example, the fraction $\frac{7}{16}$, multiply both the numerator and denominator successively by 10, 100, 1000, &c., which will give a series of fractions, each of which is equal to $\frac{7}{16}$ (108), viz. $\frac{70}{160}$, $\frac{700}{1600}$, $\frac{7000}{16000}$, $\frac{70000}{160000}$, &c. The denominator of each of these fractions can be divided without remainder by 16, the quotients of which divisions form the series of decimal numbers 10, 100, 1000, 10000, &c. If, therefore, one of the numerators is divisible by 16, the fraction to which that numerator belongs has a numerator and denominator both divisible by 16. When that division has been made, which (108) does not alter the value of the fraction, we shall have a fraction whose denominator is one of the series 10, 100, 1000, &c., and which is equal in value to $\frac{7}{16}$. The question is then reduced to finding the first of the numbers 70, 700, 7000, 70000, &c., which can be divided by 16 without remainder.

Divide these numbers, one after the other, by 16, as follows.

```
16)70(4     16)700(43     16)7000(437     16)70000(4375
   64          64            64              64
   --          --            --              --
    6           60            60              60
                48            48              48
                --            --              --
                12            120             120
                              112             112
                              ---             ---
                                8              80
                                               80
                                               --
                                                0
```

It appears, then, that 70000 is the first of the numerators which is divisible by 16. But it is not necessary to write down each of these divisions, since it is plain that the last contains all which came before. It will do then to proceed at once as if the number of ciphers were without end, to stop when the remainder is nothing, and then count the number of ciphers which have been used. In this case, since 70000 is 16×4375, $\frac{70000}{160000}$, which is $\frac{16 \times 4375}{16 \times 10000}$, or $\frac{4375}{10000}$, is the fraction required.

Therefore, to reduce a fraction to a decimal fraction, annex ciphers to the numerator, and divide by the denominator until there is no remainder. The quotient will be the numerator of the required fraction, and the denominator will be unity, followed by as many ciphers as were used in obtaining the quotient.

EXERCISES.

Reduce to decimal fractions

$$\frac{1}{2},\ \frac{1}{4},\ \frac{2}{25},\ \frac{1}{50},\ \frac{3927}{1250},\ \text{and}\ \frac{453}{625}.$$

Answer, $\frac{5}{10}$, $\frac{25}{100}$, $\frac{8}{100}$, $\frac{2}{100}$, $\frac{31416}{10000}$, and $\frac{7248}{10000}$.

120. It will happen in most cases that the annexing of ciphers to the numerator will never make it divisible by the denominator without remainder. For example, try to reduce $\frac{1}{7}$ to a decimal fraction.

$$7)\underline{100000000000000000000,\ \&c.}$$
$$142857142857142857,\ \&c.$$

The quotient here is a continual repetition of the figures 1, 4, 2, 8, 5, 7, in the same order; therefore $\frac{1}{7}$ cannot be reduced to a decimal fraction. But, nevertheless, if we take as a numerator any number of figures from the quotient 142857142857, &c., and as a denominator, 1 followed by as many ciphers as were used in making that part of the quotient, we shall get a fraction which differs very little from $\frac{1}{7}$, and which will differ still less from it if we put more figures in the numerator, and more ciphers in the denominator.

Thus,	$\frac{1}{10}$	is less than	$\frac{1}{7}$	by	$\frac{3}{70}$	which is not so much as	$\frac{1}{10}$
	$\frac{14}{100}$	....	$\frac{1}{7}$	......	$\frac{2}{700}$	....	$\frac{1}{100}$
	$\frac{142}{1000}$	....	$\frac{1}{7}$	......	$\frac{6}{7000}$	....	$\frac{1}{1000}$
	$\frac{1428}{10000}$	....	$\frac{1}{7}$	...	$\frac{4}{70000}$	....	$\frac{1}{10000}$
	$\frac{14285}{100000}$	....	$\frac{1}{7}$	...	$\frac{5}{700000}$	....	$\frac{1}{100000}$
	$\frac{142857}{1000000}$	....	$\frac{1}{7}$	...	$\frac{1}{7000000}$	....	$\frac{1}{1000000}$
	&c.		&c.		&c.		&c.

In the first column is a series of decimal fractions, which come nearer and nearer to $\frac{1}{7}$, as the third column shews. Therefore, though we cannot find a decimal fraction which is exactly $\frac{1}{7}$, we can find one which differs from it as little as we please.

EXERCISES.

Make similar tables with these fractions $\frac{3}{91}$, $\frac{17}{143}$, and $\frac{1}{247}$.

The recurring quotient of $\frac{3}{91}$ is 329670,329670, &c.

.... $\frac{17}{143}$ ... 118881,118881, &c.

.... $\frac{1}{247}$ ... 4048582995951417000,4048582, &c.

130. The reason for the *recurrence* of the figures of the quotient in the same order is as follows: If 1000, &c. be divided by the number 247, the remainder at each step of the division is less than 247, being either 0, or one of the first 246 numbers. If, then, the remainder never becomes nothing, by carrying the division far enough, one remainder will occur a second time. If possible, let the first 246 remainders be all different, that is, let them be 1, 2, 3, &c. up to 246, variously distributed. As the 247th remainder cannot be so great as 247, it must be one of these which have preceded. From the step where the remainder becomes the same as a former remainder, it is evident that former figures of the quotient must be repeated in the same order.

131. You will here naturally ask, What is the use of decimal fractions, if the greater number of fractions cannot be reduced at all to decimals? The answer is this: The addition, subtraction, multiplication, and division, of decimal fractions, are much easier than those of common fractions; and though we cannot reduce all common fractions to decimals, yet we can find decimal fractions so near to each of them, that the error arising from using the decimal instead of the common fraction will not be perceptible. For example, if we suppose an inch to be divided into ten million of equal parts, one of those parts by itself will not be visible to the eye. Therefore, in finding a length, an error

of a ten-millionth part of an inch is of no consequence, even where the finest measurement is necessary. Now, by carrying on the table in (129), we shall see that $\frac{1428571}{10000000}$ does not differ from $\frac{1}{7}$ by $\frac{1}{10000000}$; and if these fractions represented parts of an inch, the first might be used for the second, since the difference is not perceptible. In applying arithmetic to practice, nothing can be measured so accurately as to be represented in numbers without any error whatever, whether it be length, weight, or any other species of magnitude. It is therefore unnecessary to use any other than decimal fractions; since, by means of them, any quantity may be represented with as much correctness as by any other method.

EXERCISES.

Find decimal fractions which do not differ from the following fractions by $\frac{1}{100000000}$.

$\frac{1}{3}$ — *Answer*,	$\frac{33333333}{100000000}$.	$\frac{113}{355}$ — *Answer*,	$\frac{31830985}{100000000}$.
$\frac{4}{7}$	$\frac{57142857}{100000000}$.	$\frac{355}{113}$	$\frac{314159292}{100000000}$.

132. Every decimal may be immediately reduced to a quantity consisting either of a whole number and more simple decimals, or of more simple decimals alone, having one figure only in each of the numerators. Take, for example, $\frac{147326}{1000}$. By (115) $\frac{147326}{1000}$ is $147\frac{326}{1000}$; and since 326 is made up of 300, and 20, and 6; by (112) $\frac{326}{1000} = \frac{300}{1000} + \frac{20}{1000} + \frac{6}{1000}$. But (108) $\frac{300}{1000}$ is $\frac{3}{10}$, and $\frac{20}{1000}$ is $\frac{2}{100}$. Therefore $\frac{147326}{1000}$ is made up of $147 + \frac{3}{10} + \frac{2}{100} + \frac{6}{1000}$. Now, take any number, for example, 147326, and form a number of fractions, having for their numerators this number, and for their denominators 1, 10, 100, 1000, 10000, &c., and reduce these fractions into numbers and more simple decimals, in the foregoing manner, which will give the table on the following page.

EXERCISES.

Reduce the following fractions into a series of numbers and more simple fractions :

$$\frac{31415926}{10}, \quad \frac{31415926}{100}, \quad \text{\&c.}$$

$$\frac{2700031}{10}, \quad \frac{2700031}{100}, \quad \text{\&c.}$$

$$\frac{2073000}{10}, \quad \frac{2073000}{100}, \quad \text{\&c.}$$

$$\frac{3331303}{1000}, \quad \frac{3331303}{10000}, \quad \text{\&c.}$$

DECOMPOSITION OF A DECIMAL FRACTION.

$$\frac{147326}{1} = 147326$$

$$\frac{147326}{10} = 14732 + \frac{6}{10}$$

$$\frac{147326}{100} = 1473 + \frac{2}{10} + \frac{6}{100}$$

$$\frac{147326}{1000} = \ldots\ 147 + \frac{3}{10} + \frac{2}{100} + \frac{6}{1000}$$

$$\frac{147326}{10000} = \ldots\ldots\ 14 + \frac{7}{10} + \frac{3}{100} + \frac{2}{1000} + \frac{6}{10000}$$

$$\frac{147326}{100000} = \ldots\ldots\ 1 + \frac{4}{10} + \frac{7}{100} + \frac{3}{1000} + \frac{2}{10000} + \frac{6}{100000}$$

$$\frac{147326}{1000000} = \ldots\ldots\ldots\ldots \frac{1}{10} + \frac{4}{100} + \frac{7}{1000} + \frac{3}{10000} + \frac{2}{100000} + \frac{6}{1000000}$$

$$\frac{147326}{10000000} = \ldots\ldots\ldots\ldots\ldots\ldots \frac{1}{100} + \frac{4}{1000} + \frac{7}{10000} + \frac{3}{100000} + \frac{2}{1000000} + \frac{6}{10000000}$$

N.B. The student should write this table himself, and then proceed to make similar tables from the exercises in (132).

133. If, in this table, and others made in the same manner, you look at those fractions which contain a whole number, you will see that they be made thus: Mark off, from the right hand of the numerator, as many *figures* as there are *ciphers* in the denominator, by a point, or any other convenient mark.

This will give 14732·6 when the fraction is $\frac{147326}{10}$

.... 1473·26 .... $\frac{147326}{100}$

.... 147·326 .... $\frac{147326}{1000}$

&c. &c.

The figures on the left of the point, by themselves make the whole number which the fraction contains. Of those on its right, the first is the numerator of the fraction whose denominator is 10, the second of that whose denominator is 100, and so on. We now come to those fractions which do not contain a whole number.

134. The first of these is $\frac{147326}{1000000}$ in which the number of *ciphers* in the denominator is the same as the number of *figures* in the numerator. If we still follow the same rule, and mark off all the figures, by placing the point before them all, thus, ·147326, the observation in (133) still holds good; for, on looking at $\frac{147326}{1000000}$ in the table, we find it is

$$\frac{1}{10} + \frac{4}{100} + \frac{7}{1000} + \frac{3}{10000} + \frac{2}{100000} + \frac{6}{1000000}$$

The next fraction is $\frac{147326}{10000000}$, which we find by the table to be

$$\frac{1}{100} + \frac{4}{1000} + \frac{7}{10000} + \frac{3}{100000} + \frac{2}{1000000} + \frac{6}{10000000},$$

In this, 1 is not divided by 10 but by 100; if, therefore, we put a point before the whole, the rule is not true, for the first figure on the left of the point has the denominator which, according to the rule, the second ought to have, the second that which the third ought to have, and so on. In order to keep the same rule for this case, we must contrive to make 1 the second figure on the right of the point instead of the first. This may be done by placing a cipher between it and the point, thus, ·0147326. Here the rule holds good, for by that rule this fraction is

$$\frac{0}{10} + \frac{1}{100} + \frac{4}{1000} + \frac{7}{10000} + \frac{3}{100000} + \frac{2}{1000000} + \frac{6}{10000000},$$

which is the same as the preceding line, since $\frac{0}{10}$ is 0, and need not be reckoned.

Similarly, when there are two ciphers more in the denominator than there are figures in the numerator, the rule will be true if we place two ciphers between the point and the numerator. The rule, therefore, stated fully is this:

To reduce a decimal fraction to a whole number and more simple decimals, or to more simple decimals alone if it does not contain a whole number, mark off by a point as many figures from the numerator as there are ciphers in the denominator. If the numerator has not places enough for this, write as many ciphers before it as it wants places, and put the point before these ciphers. Then, if there are any figures before the point, they make the *whole number* which the fraction contains. The first figure after the point with the denominator 10, the second with the denominator 100, and so on, are the *fractions* of which the first fraction is composed.

135. Decimal fractions are not usually written at full length. It is more convenient to write the numerator only, and to cut off from the numerator as many figures as there are ciphers in the denominator, when that is possible, by a point. When there are more ciphers in the denominator than figures in the numerator, as many ciphers are placed before the numerator as will supply the deficiency, and the point is placed before the ciphers. Thus, ·7 will be used in future to denote $\frac{7}{10}$, ·07 for $\frac{7}{100}$, and so on. The following tables will give the whole of this notation at one view, and will shew its connexion with the decimal notation explained in the first section. You will observe that the numbers on the right of the units' place stand for units *divided* by 10, 100, 1000, &c. while those on the left, are units *multiplied* by 10, 100, 1000, &c.

The student is recommended always to write the decimal point in a line with the top of the figures, or in the middle, as is done here, and never at the bottom. The reason is, that it is usual in the higher branches of mathematics to use a point placed between two numbers or letters which are multiplied together; thus, 15.16, $a.b$, $\overline{a+b}.\overline{c+d}$ stand for the products of those numbers or letters.

I. $123{\cdot}4$ stands for $\frac{1234}{10}$ or $123\frac{4}{10}$ or $123+\frac{4}{10}$

$12{\cdot}34$ .... $\frac{1234}{100}$ or $12\frac{34}{100}$ or $12+\frac{3}{10}+\frac{4}{100}$

$1{\cdot}234$ .... $\frac{1234}{1000}$ or $1\frac{234}{1000}$ or $1+\frac{2}{10}+\frac{3}{100}+\frac{4}{1000}$

${\cdot}1234$ .... $\frac{1234}{10000}$ .... or ...... $\frac{1}{10}+\frac{2}{100}+\frac{3}{1000}+\frac{4}{10000}$

${\cdot}01234$......... $\frac{1234}{100000}$ .... or .... $\frac{1}{100}+\frac{2}{1000}+\frac{3}{10000}+\frac{4}{100000}$

${\cdot}001234$ ...... $\frac{1234}{1000000}$ .... or .... $\frac{1}{1000}+\frac{2}{10000}+\frac{3}{100000}+\frac{4}{1000000}$

II. ${\cdot}1003$ is $\frac{1003}{10000}$ or $\frac{1}{10}+\frac{3}{10000}$.

$10{\cdot}03$ is $\frac{1003}{100}$ or $10+\frac{3}{100}$

$100{\cdot}3$ is $\frac{1003}{10}$ or $100+\frac{3}{10}$

III. ${\cdot}1283 = \frac{1}{10}+\frac{2}{100}+\frac{8}{1000}+\frac{3}{10000}$

$= {\cdot}1 + {\cdot}02 + {\cdot}008 + {\cdot}0003$

$= {\cdot}1 + {\cdot}0283 = {\cdot}12 + {\cdot}0083$

IV. In 1234·56789 inches, the

- 1 is 1000 inches
- 2 is 200 ......
- 3 is 30 ......
- 4 is 4 ......
- 5 is $\frac{5}{10}$ of an inch
- 6 is $\frac{6}{100}$ ......
- 7 is $\frac{7}{1000}$ ......
- 8 is $\frac{8}{10000}$ ......
- 9 is $\frac{9}{100000}$ ......

136. The ciphers on the right hand of the decimal point serve the same purpose as the ciphers in (10). They are not counted as any thing themselves, but serve to shew the place in which the accompanying numbers stand. They might be dispensed with by writing the numbers in ruled columns, as in the first section. They are distinguished from the numbers which accompany them by calling the latter *significant figures.* Thus, ·0003747 is a decimal of seven places with four significant figures, ·346 is a decimal of three places with three significant figures, &c.

137. The value of a decimal is not altered by putting any number of ciphers on its right. Take, for example, ·3 and ·300. The first (135) is $\frac{3}{10}$, and the second $\frac{300}{1000}$, which is made from the first by multiplying both its numerator and denominator by 100, and (108) is the same quantity.

138. To reduce two decimals to a common denominator, put as many ciphers on the right of that which has the smaller number of places as will make the number of places in both fractions the same. Take, for example, ·54 and 4·3297. The first is $\frac{54}{100}$ and the second $\frac{43297}{10000}$. Multiply the numerator and denominator of the first by 100 (108), which reduces it to $\frac{5400}{10000}$, which has the same denominator as $\frac{43297}{10000}$. But $\frac{5400}{10000}$ is ·5400 (135). In whole numbers, the decimal point should

H

be placed at the end: thus, 129 should be written 129·. It is, however, usual to omit the point, but you must recollect that 129 and 129·000 are the same thing, since the first is 129 and the second $\frac{129000}{1000}$.

139. The rules which were given in the last chapter for addition, subtraction, multiplication, and division, apply to all fractions, and therefore to decimal fractions among the rest. But the way of writing decimal fractions, which is explained in this chapter, makes the application of these rules more simple. We proceed to the different cases.

Suppose it required to add 42·634, 45·2806, 2·001, and 54. By (112) these must be reduced to a common denominator, which is done (138) by writing them as follows: 42·6340, 45·2806, 2·0010, and 54·0000. These are decimal fractions, whose numerators are 426340, 452806, 20010, and 540000, and whose common denominator is 10000. By (112) their sum is $\frac{426340+452806+20010+540000}{10000}$, which is $\frac{1439156}{10000}$ or 143·9156. The simplest way of doing this is as follows: write the decimals down under one another, so that the decimal points may fall under one another, thus:

```
 42·634
 45·2806
  2·001
 54
--------
143·9156
```

Add the different columns together as in common addition, and place the decimal point under the other decimal points.

EXERCISES.

What is 1527 + 64·732094 + 2·0013 + ·00001974;

2276·3 + ·107 + ·9 + 26·3172 + 56732·001;

and 1·11 + 7·7 + ·0039 + ·00142 + ·8838?

Answer, 1593·73341374, 59035·6252, 9·69912.

140. Suppose it required to subtract 91·07324 from 137·321. These fractions when reduced to a common denominator are 91·07324 and 137·32100 (138). Their difference is therefore $\frac{13732100-9107324}{100000}$, which is $\frac{4624776}{100000}$ or 46·24776. This may be most simply done as

follows: write the less number under the greater, so that its decimal point may fall under that of the greater, thus:

$$\begin{array}{r} 137 \cdot 321 \\ 91 \cdot 07324 \\ \hline 46 \cdot 24776 \end{array}$$

Subtract the lower from the upper line, and wherever there is a figure in one line and not in the other, proceed as if there were a cipher in the vacant place.

EXERCISES.

What is $12362 - 274 \cdot 22107 + \cdot 5$;

$9976 \cdot 2073942 - \cdot 00143976728$;

and $1 \cdot 2 + \cdot 03 + \cdot 004 - \cdot 0005$?

Answer, 12088·27893, 9976·20595443272, and 1·2335.

141. The multiplication of a decimal by 10, 100, 1000, &c., is performed by merely moving the decimal point to the right. Suppose, for example, 13·2079 is to be multiplied by 100. The decimal is $\frac{132079}{10000}$, which multiplied by 100 is (117) $\frac{132079}{100}$, or 1320·79. Again, $1 \cdot 309 \times 100000$ is $\frac{1309}{1000} \times 100000$, or (116) $\frac{130900000}{1000}$ or 130900. From these and other instances we get the following rule:—To multiply a decimal fraction by a decimal number (126), move the decimal point as many places to the right as there are ciphers in the decimal number. When this cannot be done, add ciphers to the right of the decimal (137) until it can.

142. Suppose it required to multiply 17·036 by 4·27. The first of these decimals is $\frac{17036}{1000}$, and the second $\frac{427}{100}$. By (118) the product of these fractions has for its numerator the product of 17036 and 427, and for its denominator the product of 1000 and 100; therefore this product is $\frac{7274372}{100000}$ or 72·74372. This may be done more shortly by multiplying the two numbers 17036 and 427, and cutting off by the decimal point as many places as there are decimal places both in 17·036 and 4·27, because the product of two decimal numbers will contain as many ciphers as there are ciphers in both.

143. This question now arises: What if there should not be as

many figures in the product as there are decimal places in the multiplier and multiplicand together? To see what must be done in this case, multiply ·172 by ·101, or $\frac{172}{1000}$ by $\frac{101}{1000}$. The product of these two is $\frac{17372}{1000000}$ or ·017372 (135). Therefore, when the number of places in the product is not sufficient to allow the rule of the last article to be followed, as many ciphers must be placed at the beginning as will make up the deficiency.

ADDITIONAL EXAMPLES.

·001 × ·01 is ·00001

56 × ·0001 is ·0056.

EXERCISES.

Shew that

3·002 × 3·002 = 3 × 3 + 2 × 3 × ·002 + ·002 × ·002

11·5609 × 5·3191 = 8·44 × 8·44 − 3·1209 × 3·1209

8·217 × 10·001 = 8 × 10 + 8 × ·001 + 10 × ·217 + ·001 × ·217.

Fraction.	Square.	Cube.
82·92	6875·7264	570135·233088
·0173	·00029929	·000005177717
1·43	2·0449	2·924207
·009	·000081	·000000729

15·625 × 64 = 1000 ·15625 × ·64 = ·1

1·5625 × ·64 = 1 1562·5 × ·064 = 100

·015625 × ·0064 = ·0001 15625000 × ·064 = 1000000

144. The division of a decimal by a decimal number, such as 10, 100, 1000, &c., is performed by moving the decimal point as many places to the left as there are ciphers in the decimal number. If there are not places enough in the dividend to allow of this, annex ciphers to the beginning of it until there are. For example, divide 1734·229 by 1000: the decimal fraction is $\frac{1734229}{1000}$, which divided by 1000 (123) is $\frac{1734229}{1000000}$ or 1·734229. If, in the same way, 1·2106 be divided by 10000, the result is ·00012106.

145. Before proceeding to shorten the rule for the division of one decimal fraction by another, it will be necessary to resume what was

said in (128) upon the reduction of any fraction to a decimal fraction. It was there shewn that $\frac{7}{16}$ is the same fraction as $\frac{4375}{10000}$ or ·4375. As another example, convert $\frac{3}{128}$ into a decimal fraction. Follow the same process as in (108), thus:

```
128)300000000000(234375          480
    256                          384
    ---                          ---
     440                         960
     384                         896
     ---                         ---
      560                         640
      512                         640
      ---                         ---
       480                          0
```

Since 7 ciphers are used, it appears that 30000000 is the first of the series 30, 300, &c., which is divisible by 128; and therefore $\frac{3}{128}$, or, which is the same thing (108), $\frac{30000000}{1280000000}$ is equal to $\frac{234375}{10000000}$ or ·0234375 (135).

From these examples the rule for reducing a fraction to a decimal is:—Annex ciphers to the numerator; divide by the denominator, and annex a cipher to each remainder, after the figures of the numerator are all used, proceeding exactly as if the numerator had an unlimited number of ciphers annexed to it, and was to be divided by the denominator. Continue this process until there is no remainder, and observe how many ciphers have been used. Place the decimal point in the quotient so as to cut off as many figures as you have used ciphers; and if there be not figures enough for this, annex ciphers to the beginning until there are places enough.

146. From what was shewn in (129) it appears that it is not every fraction which can be reduced to a decimal fraction. It was there shewn, however, that there is no fraction to which we may not find a decimal fraction as near as we please. Thus, $\frac{1}{10}$, $\frac{14}{100}$, $\frac{142}{1000}$, $\frac{1428}{10000}$, $\frac{14285}{100000}$, &c., or ·1, ·14, ·142, ·1428, ·14285, were shewn to be fractions which approach nearer and nearer to $\frac{1}{7}$. To find either of these fractions, the rule is the same as that in the last article, with this exception, that, I. instead of stopping when there is no remainder, which never

happens, stop at any part of the process, and make as many decimal places in the quotient as are equal in number to the number of ciphers which have been used, annexing ciphers to the beginning when this cannot be done, as before. II. instead of obtaining a fraction which is exactly equal to the fraction from which we set out, we get a fraction which is very near to it, and may get one still nearer, by using more of the quotient. Thus, ·1428 is very near to $\frac{1}{7}$, but not so near as ·142857; nor is this last, in its turn, so near as ·142857142857, &c.

147. If there should be ciphers in the numerator of a fraction, these must not be reckoned with the number of ciphers which are necessary in order to follow the rule for changing it into a decimal fraction. Take, for example, $\frac{100}{125}$; annex ciphers to the numerator, and divide by the denominator. It appears that 1000 is divisible by 125, and that the quotient is 8. One cipher only has been annexed to the numerator, and therefore 100 divided by 125, is ·8. Had the fraction been $\frac{1}{125}$, since 1000 divided by 125 gives 8, and three ciphers would have been annexed to the numerator, the fraction would have been ·008.

148. Suppose that the given fraction has ciphers at the right of its denominator; for example, $\frac{31}{2500}$. The annexing a cipher to the numerator is the same thing as taking one away from the denominator; for, (108) $\frac{310}{2500}$ is the same thing as $\frac{31}{250}$, and $\frac{3100}{2500}$ as $\frac{31}{25}$. The rule therefore is, in this case: Take away the ciphers from the denominator; annex ciphers to the numerator; proceed as before; and in counting how many ciphers have been used, reckon not only the ciphers which have been annexed to the numerator, but also those which have been taken away from the denominator.

EXERCISES.

Reduce the following fractions to decimal fractions:

$$\frac{1}{800},\quad \frac{36}{1250},\quad \frac{297}{64},\ \text{and}\ \frac{1}{128}.$$

Answer, ·00125, ·0288, 4·640625, and ·078125.

Find decimals of 6 places very near to the following fractions:

$$\frac{27}{49}, \frac{156}{33}, \frac{22}{37000}, \frac{194}{13}, \frac{2637}{9907}, \frac{1}{2908}, \frac{1}{466}, \text{and}\ \frac{3}{277}.$$

Answer, ·551020, 4·727272, ·000594, 14·923076, ·266175, ·0003438, ·0021459, and ·0108303.

149. From (121), it appears, that if two fractions have the same denominator, the first may be divided by the second by dividing the numerator of the first by the numerator of the second. Suppose it required to divide 17·762 by 6·25. These fractions (138), when reduced to a common denominator, are 17·762 and 6·250, or $\frac{17762}{1000}$ and $\frac{6250}{1000}$. Their quotient is therefore $\frac{17762}{6250}$, which must now be reduced to a decimal fraction by the last rule. The process at full length is as follows: Leave out the cipher in the denominator, and annex ciphers to the numerator, or, which will do as well, to the remainders, when it becomes necessary, and divide as in (145).

```
625)17762(284192
    1250
    ----
     5262
     5000
     ----
      2620
      2500
      ----
       1200
        625
       ----
        5750
        5625
        ----
         1250
         1250
         ----
            0
```

Here four ciphers have been annexed to the numerator, and one has been taken from the denominator. Make five decimal places in the quotient, which then becomes 2·84192, and this is the quotient of 17·762 divided by 6·25.

150. The rule for division of one decimal by another is as follows: Equalise the number of decimal places in the dividend and divisor, by annexing ciphers to that which has fewest places. Then, further, annex as many ciphers to the dividend * as it is required to have decimal places, throw away the decimal point, and operate as in common division. Make the required number of decimal places in the quotient.

Thus, to divide 6·7173 by ·014 to three decimal places, I first write 6·7173 and ·0140, with four places in each. Having to provide for three decimal places, I should annex three ciphers to 6·7173; but, observing that the divisor ·0140 has one cipher, I strike that one out and annex two ciphers to 6·7173. Throwing away the decimal points, then divide

* Or remove ciphers from the divisor; or make up the number of ciphers partly by removing from the divisor, and annexing to the dividend, if there be not a sufficient number in the divisor.

6717300 by 014 or 14 in the usual way, which gives the quotient 479807 and the remainder 2. Hence 479·807 is the answer.

Or the following rule may be used: Expunge the decimal point of the divisor, and move that of the dividend as many places to the right as there were places in the divisor, using ciphers if necessary. Then proceed as in common division, making one decimal place in the quotient for every decimal place of the final divisor which is used. Thus 17·314 divided by 61·2 is 173·14 divided by 612, and the decimal point must precede the first figure of the quotient. But 17·314 divided by 6617·5 is 173·14 by 66175; and since three decimal places of 173·14000 .... must be used before a quotient figure can be found, that quotient figure is the third decimal place, or the quotient is ·002 .....

EXAMPLES.

$$\frac{3\cdot1}{\cdot0025} = 1240, \quad \frac{\cdot00062}{\cdot64} = \cdot00096875.$$

If the remainder never becomes 0, the same rule will apply, if we stop at any part of the quotient, and count the number of ciphers, and proceed afterwards in the same manner.

EXERCISES.

Shew that $\frac{15\cdot006 \times 15\cdot006 - \cdot004 \times \cdot004}{15\cdot01} = 15\cdot002$, and that

$$\frac{\cdot01 \times \cdot01 \times \cdot01 + 2\cdot9 \times 2\cdot9 \times 2\cdot9}{2\cdot91} = 2\cdot9 \times 2\cdot9 - 2\cdot9 \times \cdot01 + \cdot01 \times \cdot01.$$

What are $\frac{1}{3\cdot14159}$, $\frac{1}{2\cdot7182818}$, and $\frac{365}{\cdot18349}$, as far as 6 places of decimals? — *Answer*, ·318310, ·367879, and 1989·209221

Calculate 10 terms of each of the following series, as far as 5 places of decimals.

$$1 + \frac{1}{2} + \frac{1}{2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{2 \times 3 \times 4 \times 5} + \&c. = 1\cdot71824.$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c. = 2\cdot92895.$$

$$\frac{80}{81} + \frac{81}{82} + \frac{82}{83} + \frac{83}{84} + \frac{84}{85} + \&c. = 9\cdot88286.$$

151. We now enter upon methods by which unnecessary trouble is

saved in the computation of decimal quantities. And first, suppose a number of miles has been measured, and found to be 17·846217 miles. If you were asked how many miles there are in this distance, and a rough answer were required which should give miles only, and not parts of miles, you would probably say 17. But this, though the number of whole miles contained in the distance, is not the nearest number of miles; for, since the distance is more than 17 miles and 8 tenths, and therefore more than 17 miles and a half, it is nearer the truth to say, it is 18 miles. This, though too great, is not so much too great as the other was too little, and the error is not so great as half a mile. Again, if the same were required within a tenth of a mile, the correct answer is 17·8; for though this is too little by ·046217, yet it is not so much too little as 17·9 is too great; and the error is less than half a tenth, or $\frac{1}{20}$. Again, the same distance, within a hundredth of a mile, is more correctly 17·85 than 17·84, since the last is too little by ·006217, which is greater than the half of ·01; and therefore 17·84+·01 is nearer the truth than 17·84. Hence this general rule: When a certain number of the decimals given is sufficiently accurate for the purpose, strike off the rest from the right hand, observing, if the first figure struck off is equal to or greater than 5, to increase the last remaining figure by 1.

The following are examples of a decimal abbreviated by one place at a time.

3·14159, 3·1416, 3·142, 3·14, 3·1, 3·0

2·7182818, 2·718282, 2·71828, 2·7183, 2·718, 2·72, 2·7, 3·0

1·9919, 1·992, 1·99, 2·00, 2·0

152. In multiplication and division it is useless to retain more places of decimals in the result than were certainly correct in the multiplier, &c., which gave that result. Suppose, for example, that 9·98 and 8·96 are distances in inches which have been measured correctly to two places of decimals, that is, within half a hundredth of an inch each way. The real value of that which we call 9·98 may be any where between 9·975 and 9·985, and that of 8·96 may be any where between 8·955 and 8·965. The product, therefore, of the numbers which represent the correct distances will lie between 9·975 × 8·955 and 9·985 × 8·965, that is, taking

three decimal places in the products, between 89·326 and 89·516. The product of the actual numbers given is 89·4208. It appears, then, that in this case no more than the whole number 89 can be depended upon in the product, or, at most, the first place of decimals. The reason is, that the error made in measuring 8·96, though only in the third place of decimals, is in the multiplication increased at least 9·975, or nearly 10 times; and therefore affects the second place. The following simple rule will enable us to judge how far a product is to be depended upon. Let a be the multiplier, and b the multiplicand; if these be true only to the first decimal place, the product is within $\frac{a+b}{20}$* of the truth; if to two decimal places, within $\frac{a+b}{200}$; if to three, within $\frac{a+b}{2000}$; and so on. Thus, in the above example, we have 9·98 and 8·96, which are true to two decimal places: their sum divided by 200 is ·0947, and their product is 89·4208, which is therefore within ·0947 of the truth. If, in fact, we increase and diminish 89·4208 by ·0947, we get 89·5155 and 89·3261, which are very nearly the limits found, within which the product must lie. We see, then, that we cannot in this case depend upon the first place of decimals, as (151) an error of ·05 cannot exist, if this place be correct; and here is a possible error of ·09, and upwards. It is hardly necessary to say, that if the numbers given be exact, their product is exact also, and that this article applies where the numbers given are correct only to a certain number of decimal places. The rule is: Take half the sum of the multiplier and multiplicand, remove the decimal point as many places to the left as there are correct places of decimals in either the multiplier or multiplicand; the result is the quantity within which the product can be depended upon. In division, the rule is: Proceed as in the last rule, putting the dividend and divisor in place of the multiplier and multiplicand, and divide by the *square* of the divisor; the quotient will be the quantity within which the division of the first dividend and divisor may be depended upon. Thus, if 17·324 be divided by 53·809, both being correct to the third place, their half sum will be 35·566, which, by the last rule, is made ·035566, and

* These are not quite correct, but sufficiently so for every practical purpose.

is to be divided by the square of 53·809, or, which will do as well for our purpose, the square of 50, or 2500. The result is something less than ·00002, so that the quotient of 17·324 and 53·809 can be depended on to four places of decimals.

153. It is required to multiply two decimal fractions together, so as to retain in the product only a given number of decimal places, and dispense with the trouble of finding the rest. First, it is evident that we may write the figures of any multiplier in a contrary order (for example, 4321 instead of 1234), provided that in the operation we move each line one place to the right instead of to the left, as in the following example:

```
   2221              2221
   1234              4321
   ----              ----
   8884              2221
  6663                4442
 4442                  6663
2221                    8884
-------           -------
2740714           2740714
```

Suppose now we wish to multiply 348·8414 by 51·30742, reserving only four decimal places in the product. If we reverse the multiplier, and proceed in the manner just pointed out, we have the following:

```
  3488414
  2470315  |
-----------|
 17442070  |
    3488414|
   10465242|2
       24418|898
        1395|3656
          69|76828
------------|-----
 17898·1522|23188
```

Cut off, by a vertical line, the first four places of decimals, and the columns which produced them. It is plain that in forming our abbreviated rule, we have to consider only, I. all that is on the left of the vertical line; II. all that is carried from the first column on the right of the line. On looking at the first column to the left of the line, we see 4, 4, 8, 5, 9, of which the first 4 comes from* $4 \times 1'$, the second 4 from $1 \times 3'$, the 8 from $8 \times 7'$, the 5 from $8 \times 4'$, and the 9 from $4 \times 2'$. If, then, we arrange the multiplicand and the reversed multiplier thus,

```
3488414
 2470315
```

* The 1′ here means that the 1 is in the multiplier.

each figure of the multiplier is placed under the first figure of the multiplicand which is used with it in forming the first *four* places of decimals. And here observe, that the units' figure in the multiplier 51·30742, viz. 1, comes under 4, the *fourth* decimal place in the multiplicand. If there had been no carrying from the right of the vertical line, the rule would have been: Reverse the multiplier, and place it under the multiplicand, so that the figure which was the units' figure in the multiplier may stand under the last place of decimals in the multiplicand, which is to be preserved; place ciphers over those figures of the multiplier which have none of the multiplicand above them, if there be any: proceed to multiply in the usual way, but begin each figure of the multiplier with the figure of the multiplicand which comes above it, taking no account of those on the right: place the first figures of all the lines under one another. To correct this rule, so as to allow for what is carried from the right of the vertical line, observe that this consists of two parts, 1st, what is carried directly in the formation of the different lines, and 2dly, what is carried from the addition of the first column on the right. The first of these may be taken into account by beginning each figure of the multiplier with the one which comes on its right in the multiplicand, and carrying the tens to the next figure as usual, but without writing down the units. But both may be allowed for at once, with sufficient correctness, on the principle of (151), by carrying 1 from 5 up to 15, 2 from 15 up to 25, &c.; that is, by carrying the nearest ten. Thus, for 37, 4 would be carried, 37 being nearer to 40 than to 30. This will not always give the last place quite correctly, but the error may be avoided by setting out so as to keep one more place of decimals in the product that is absolutely required to be correct. The rule, then, is as follows:

154. To multiply two decimals together, retaining only n decimal places.

I. Reverse the multiplier, strike out the decimal points, and place the multiplier under the multiplicand, so that what was its units' figure shall fall under the n^{th} decimal place of the multiplicand, placing ciphers, if necessary, so that every place of the multiplier shall have a figure or cipher above it.

II. Proceed to multiply as usual, beginning each figure of the multiplier with the one which is in the place to its right in the multiplicand: do not set down this first figure, but carry its *nearest* ten to the next, and proceed.

III. Place the first figures of all the lines under one another; add as usual; and mark off n places from the right for decimals.

It is required to multiply 136·4072 by 1·30609, retaining 7 decimal places.

```
 1364072000
     906031
 ----------
 1364072000
  409221600
    8184432
     122766
 ----------
178·1600798
```

In the following examples the first two lines are the multiplicand and multiplier; and the number of decimals to be retained will be seen from the results.

·4471618	33·166248	3·4641016
3·7719214	1·4142136	1732·508
37719214	033166248	346410160
8161744	63124141	8052371
15087686	3316625	346410160
1508768	1326650	242487112
264034	33166	10392305
3772	13266	692820
2263	663	173205
38	33	2771
30	10	6001·58373
1·6866591	2	
	46·90415	

Exercises may be got from article (143).

155. With regard to division, take any two numbers, for example, 16·80437921 and 3·142, and divide the first by the second, as far as any required number of decimal places, for example, five. This gives the following:

```
                     3·142)16·80437921(5·34830
                           15 710
                           ------|
                           1 0943|
                             9426|
                            -----|
                            15177|
   (A)                      12568|
  ----                      -----|-
  2609                       2609|9
  2514                       2513|6
  ----                       ----|--
    95                         96|32
    94                         94|26
    --                         --|---
     1                          2|061
```

Now, cut off by a vertical line, as in (153), all the figures which come on the right of the first figure 2, in the last remainder 2061. As in multiplication, we may obtain all that is on the left of the vertical line by an abbreviated method, as represented at (A). After what has been said on multiplication, it is useless to go further into the detail; the following rule will be sufficient: To divide one decimal by another, retaining only n places: Proceed one step in the ordinary division, and determine, by (150), in what place is the quotient so obtained; proceed in the ordinary way, until the number of figures remaining to be found in the quotient is less than the number of figures in the divisor: if this should be already the case, proceed no further in the ordinary way. Instead of annexing a figure or cipher to the remainder, cut off a figure from the divisor, and proceed one step with this curtailed divisor as usual, remembering, however, in multiplying this divisor, to carry the *nearest ten*, as in (154), from the figure which was struck off; repeat this, striking off another figure of the divisor, and so on, until no figures are left. Since we know from the beginning in what place the first figure of the quotient is, and also how many decimals are required, we can tell from the beginning how many figures there will be in the whole quotient. If the divisor contains more figures than the quotient, it will be unnecessary to use them: and they may be rejected, the rest being corrected as in (151): if there are ciphers at the beginning of the divisor, if it be, for example, ·003178, since this is $\frac{\cdot 3178}{100}$, divide by

·3178 in the usual way, and afterwards multiply the quotient by 100, or remove the decimal point two places to the right. If, therefore, six decimals be required, eight places must be taken in dividing by ·3178, for an obvious reason. In finding the last figure of the quotient, the nearest should be taken, as in the second of the subjoined examples.

Places required,	2	8
Divisor,	·41432	3·1415927
Dividend,	673·1489	2·71828180
	414 32	2·51327416
	258 828	20500764
	248 592	18849556
	10 237*	1651208
	8 286	1570796
	1 951	80412
	1 657	62832
	294	17580
	290	15708
	4	1872
	4	1571
	0	301
		283
		18
		19
Quotient,	1624·71	·86525596

Examples may be obtained from (143) and (150)

SECTION VII.

ON THE EXTRACTION OF THE SQUARE ROOT.

156. We have already remarked (66), that a number multiplied by itself produces what is called the *square* of that number. Thus, 169, or 13 × 13, is the square of 13. Conversely, 13 is called the *square root*

* This is written 7 instead of 6, because the figure which is abandoned in the dividend is 9 (151).

of 169, and 5 is the square root of 25; and any number is the square root of another, which when multiplied by itself will produce that other. The square root is signified by the sign $\surd$ or $\sqrt{}$; thus, $\surd 25$ means the square root of 25, or 5; $\sqrt{16+9}$ means the square root of 16 + 9, and is 5, and must not be confounded with $\surd 16 + \surd 9$, which is 4 + 3, or 7.

157. The following equations are evident from the definition:

$$\surd a \times \surd a = a$$

$$\sqrt{a\,a} = a$$

$$\sqrt{a\,b} \times \sqrt{a\,b} = a\,b$$

$$(\surd a \times \surd b) \times (\surd a \times \surd b) = \surd a \times \surd a \times \surd b \times \surd b = a\,b$$

whence $$\surd a \times \surd b = \sqrt{a\,b}$$

158. It does not follow that a number has a square root because it has a square; thus, though 5 can be multiplied by itself, there is no number which multiplied by itself will produce 5. It is proved in algebra, that no fraction * multiplied by itself can produce a whole number, which may be found true in any number of instances; therefore 5 has neither a whole nor a fractional square root; that is, it has no square root at all. Nevertheless, there are methods of finding fractions whose squares shall be as *near* to 5 as we please, though not exactly equal to it. One of these methods gives $\frac{15127}{6765}$, whose square, viz. $\frac{15127}{6765} \times \frac{15127}{6765}$ or $\frac{228826129}{45765225}$, differs from 5 by only $\frac{4}{45765225}$, which is less than ·0000001: hence we are enabled to use $\surd 5$ in arithmetical and algebraical reasoning: but when we come to the practice of any problem, we must substitute for $\surd 5$ one of the fractions whose square is nearly 5, and on the degree of accuracy we want, depends what fraction is to be used. For some purposes, $\frac{123}{55}$ may be sufficient, as its square only differs from 5 by $\frac{4}{3025}$; for others, the fraction first given might be necessary, or one whose square is even nearer to 5. We proceed to shew how to find the square root of a number, when it has

* Meaning of course a really fractional number, such as $\frac{7}{8}$ or $\frac{15}{7}$, not one which, though fractional in form, is whole in reality, such as $\frac{10}{5}$ or $\frac{27}{3}$.

one, and from thence how to find fractions whose squares shall be as near as we please to the number, when it has not. We premise, what is sufficiently evident, that of two numbers, the greater has the greater square; and that if one number lies between two others, its square lies between the squares of those others.

159. Let x be a number consisting of any number of parts, for example, four, viz. a, b, c, and d; that is, let

$$x = a+b+c+d$$

The square of this number, found as in (68), will be

$$\begin{aligned} & aa+2a(b+c+d) \\ & +bb+2b(c+d) \\ & +cc+2cd \\ & +dd \end{aligned}$$

The rule there found for squaring a number consisting of parts was: Square each part, and multiply all that come after by twice that part, the sum of all the results so obtained will be the square of the whole number. In the expression above obtained, instead of multiplying $2a$ by each of the succeeding parts, b, c, and d, and adding the results, we multiplied $2a$ by the sum of all the succeeding parts, which (52) is the same thing; and as the parts, however disposed, make up the number, we may reverse their order, putting the last first, &c.; and the rule for squaring will be: Square each part, and multiply all that come before by twice that part. Hence a reverse rule for extracting the square root presents itself with more than usual simplicity. It is: To extract the square root of a number N, choose a number A, and see if N will bear the subtraction of the square of A; if so, take the remainder, choose a second number B, and see if the remainder will bear the subtraction of the square of B, and twice B multiplied by the preceding part A: if it will, there is a second remainder. Choose a third number C, and see if the second remainder will bear the subtraction of the square of C, and twice C multiplied by A + B: go on in this way either until there is no remainder, or the remainder will not bear the subtraction arising from any new part, even though that part were the least number, which is 1.

In the first case, the square root is the sum of A, B, C, &c.; in the second, there is no square root.

160. For example, I wish to know if 2025 has a square root. I choose 20 as the first part, and find that 400, the square of 20, subtracted from 2025, gives 1625, the first remainder. I again choose 20, whose square, together with twice itself, multiplied by the preceding part, is $20 \times 20 + 2 \times 20 \times 20$, or 1200; which subtracted from 1625, the first remainder, gives 425, the second remainder. I choose 7 for the third part, which appears to be too great, since 7×7, increased by 2×7 multiplied by the sum of the preceding parts $20 + 20$, gives 609, which is more than 425. I therefore choose 5, which closes the process, since 5×5, together with 2×5 multiplied by $20 + 20$, gives exactly 425. The square root of 2025 is therefore $20 + 20 + 5$, or 45, which will be found, by trial, to be correct; since $45 \times 45 = 2025$. Again, I ask if 13340 has, or has not, a square root. Let 100 be the first part, whose square is 10000, and the first remainder is 3340. Let 10 be the second part. Here $10 \times 10 + 2 \times 10 \times 100$ is 2100, and the second remainder, or $3340 - 2100$, is 1240. Let 5 be the third part; then $5 \times 5 + 2 \times 5 \times (100 + 10)$ is 1125, which, subtracted from 1240, leaves 115. There is, then, no square root; for a single additional unit will give a subtraction of $1 \times 1 + 2 \times 1 \times (100 + 10 + 5)$, or 231, which is greater than 115. But if the number proposed had been less by 115, each of the remainders would have been 115 less, and the last remainder would have been nothing. Therefore $13340 - 115$, or 13225, has the square root $100 + 10 + 5$, or 115; and the answer is, that 13340 has no square root, and that 13225 is the next number below it which has one, namely, 115.

161. It only remains to put the rule in such a shape as will guide us to those parts which it is most convenient to choose. It is evident (57) that any number which terminates with ciphers, as 4000, has double the number of ciphers in its square. Thus, $4000 \times 4000 = 16000000$; therefore, any square number,* as 49, with an even number

* By square number, I mean a number which has a square root. Thus, 25 is a square number, but 26 is not.

of ciphers annexed, as 490000, is a square number. The root* of 490000 is 700. This being premised, take any number, for example 76176; setting out from the right hand towards the left, cut off two figures; then two more, and so on, until one or two figures only are left; thus, 7,61,76. This number is greater than 7,00,00, of which the first figure is not a square number, the nearest square below it being 4. Hence, 4,00,00 is the nearest square number below 7,00,00, which has four ciphers, and its square root is 200. Let this be the first part chosen: its square subtracted from 76176 leaves 36176, the first remainder, and it is evident that we have obtained the highest number of the highest denomination which is to be found in the square root of 76176; for 300 is too great, its square 9,00,00 being greater than 76176: and any denomination higher than hundreds has a square still greater. It remains, then, to choose a second part, as in the examples of (160), with the remainder 36176. This part cannot be as great as 100, by what has just been said; its highest denomination is therefore a number of tens. Let N stand for a number of tens, which is one of the simple numbers 1, 2, 3, &c.; that is, let the new part be 10 N, whose square is 10 N × 10 N, or 100 N N, and whose double multiplied by the former part is 20 N × 200, or 4000 N; the two together are 4000 N + 100 N N. Now, N must be so taken that this may not be greater than 36176: still more, 4000 N must not be greater than 36176. We may therefore try, for N, the number of times which 36176 contains 4000, or that which 36 contains 4. The remark in (81) applies here. Let us try 9 tens, or 90. Then, 2 × 90 × 200 + 90 × 90, or 44100, is to be subtracted, which is too great, since the whole remainder is 36176. We then try 8 tens, or 80, which gives 2 × 80 × 200 + 80 × 80, or 38400, which is likewise too great. On trying 7 tens, or 70, we find 2 × 70 × 200 + 70 × 70, or 32900, which subtracted from 36176 gives 3276, the second remainder. The rest of the square root can only be units. As before, let N be this number of units. Then, the sum of the preceding parts being 200 + 70, or 270, the number to be subtracted

* The term root is frequently used as an abbreviation of square root.

is $270 \times 2 N + N N$, or $540 N + N N$. Hence, as before, $540 N$ must be less than 3276, or N must not be greater than the number of times which 3276 contains 540, or (81) which 327 contains 54. We therefore try if 6 will do, which gives $2 \times 6 \times 270 + 6 \times 6$, or 3276, to be subtracted. This being exactly the second remainder, the third remainder is nothing, and the process is finished. The square root required is therefore $200 + 70 + 6$, or 276.

The process of forming the numbers to be subtracted may be shortened thus. Let A be the sum of the parts already found, and N a new part: there must then be subtracted $2 A N + N N$, or (54) $2 A + N$ multiplied by N. The rule, therefore, for forming it is: Double the sum of all the preceding parts, add the new part, and multiply the result by the new part.

162. The process of the last article is as follows:

```
        7,61,76(200              7,61,76(276
        4 00 00   70             4
                   6             ---
   400)3,61,76               47)361
    70)3 29 00                  329
                                ---
   400)  32 76              546) 3276
   140)  32 76                   3276
     6)  -----                   ----
             0                      0
```

In the first of these, the numbers are written at length, as we found them; in the second, as in (79), unnecessary ciphers are struck off, and the periods 61, 76, are not brought down, until, by the continuance of the process, they cease to have ciphers under them. The following is another example, to which the reasoning of the last article may be applied.

```
          34,86,78,44,01(50000          34,86,78,44,01(59049
          25 00 00 00 00   9000         25
                             40         --
  100000) 9 86 78 44 01       9     109) 986
    9000) 9 81 00 00 00                  981
                                         ---
  100000)   5 78 44 01          11804)  57844
   18000)   4 72 16 00                  47216
      40)                               -----
  100000)   1 06 28 01        118089)1062801
   18000)   1 06 28 01               1062801
      80)                            -------
       9)            0                     0
```

163. The rule is as follows: To extract the square root of a number;—

I. Beginning from the right hand, cut off periods of two figures each, until not more than two are left.

II. Find the root of the nearest square number next below the number in the first period. This root is the first figure of the required root; subtract its square from the first period, which gives the first remainder.

III. Annex the second period to the right of the remainder, which gives the first dividend.

IV. Double the first figure of the root; see how often this is contained in the number made by cutting one figure from the right of the first dividend, attending to IX., if necessary; use the quotient as the second figure of the root; annex it to the right of the double of the first figure, and call this the first divisor.

V. Multiply the first divisor by the second figure of the root; if the product be greater than the first dividend, use a lower number for the second figure of the root, and for the last figure of the divisor, until the multiplication just mentioned gives the product less than the first dividend; subtract this from the first dividend, which gives the second remainder.

VI. Annex the third period to the second remainder, which gives the second dividend.

VII. Double the first two figures of the root;* see how often the result is contained in the number made by cutting one figure from the right of the second dividend; use the quotient as the third figure of the root; annex it to the right of the double of the two first figures, and call this the second divisor.

VIII. Get a new remainder, as in V., and repeat the process until all the periods are exhausted; if there be then no remainder, the square root is found; if there be a remainder, the proposed number has no

* Or, more simply, add the second figure of the root to the first divisor.

square root, and the number found as its square root is the square root of the proposed number diminished by the remainder.

IX. When it happens that the double of the figures of the root is not contained at all in all the dividend except the last figure, or when, being contained once, 1 is found to give more than the dividend, put a cipher in the square root, and bring down the next period; should the same thing still happen, put another cipher in the root, and bring down another period; and so on.

EXERCISES.

Numbers proposed.	Square roots.
73441	271
2992900	1730
6414247921	80089
903687890625	950625
42420747482776576	205962976
13422659310152401	115856201

164. Since the square of a fraction is obtained by squaring the numerator and the denominator, the square root of a fraction is found by taking the square root of both. Thus, the square root of $\frac{25}{64}$ is $\frac{5}{8}$, since 5×5 is 25, and 8×8 is 64. If the numerator or denominator, or both, are not square numbers, it does not therefore follow that the fraction has no square root; for it may happen that multiplication or division by the same number may convert both the numerator and denominator into square numbers (108). Thus, $\frac{27}{48}$, which appears at first to have no square root, has one in reality, since it is the same as $\frac{9}{16}$, whose square root is $\frac{3}{4}$.

165. We now proceed from (158), where it was stated that any number or fraction being given, a second may be found, whose square is as near to the first as we please. Thus, though we cannot solve the problem, "Find a fraction whose square is 2," we can solve the following, "Find a fraction whose square shall not differ from 2 by so much as ·00000001." Instead of this last, a still smaller fraction may be substituted; in fact, any one however small: and in this process we

are said to approximate to the square root of 2. This can be done to any extent, as follows: Suppose we wish to find the square root of 2 within $\frac{1}{57}$ of the truth; by which I mean, to find a fraction $\frac{a}{b}$ whose square is less than 2, but such that the square of $\frac{a}{b}+\frac{1}{57}$ is greater than 2. Multiply the numerator and denominator of $\frac{2}{1}$ by the square of 57, or 3249, which gives $\frac{6498}{3249}$. On attempting to extract the square root of the numerator, I find (163) that there is a remainder 98, and that the square number next below 6498 is 6400, whose root is 80. Hence, the square of 80 is less than 6498, while that of 81 is greater. The square root of the denominator is of course 57. Hence, the square of $\frac{80}{57}$ is less than $\frac{6498}{3249}$, or 2, while that of $\frac{81}{57}$ is greater, and these two fractions only differ by $\frac{1}{57}$; which was required to be done.

166. In practice, it is usual to find the square root true to a certain number of places of decimals. Thus, 1·4142 is the square root of 2 true to four places of decimals, since the square of 1·4142, or 1·99996164, is less than 2, while an increase of only 1 in the fourth decimal place, giving 1·4143, gives the square 2·00024449, which is greater than 2. To take a more general case: Suppose it required to find the square root of 1·637 true to four places of decimals. The fraction is $\frac{1637}{1000}$, whose square root is to be found within ·0001, or $\frac{1}{10000}$. Annex ciphers to the numerator and denominator, until the denominator becomes the square of $\frac{1}{10000}$, which gives $\frac{163700000}{100000000}$: extract the square root of the numerator, as in (163), which shews that the square number nearest to it is 163700000 − 13564, whose root is 12794. Hence, $\frac{12794}{10000}$, or 1·2794, gives a square less than 1·637, while 1·2795 gives a square greater. In fact, these two squares are 1·63686436 and 1·63712025.

167. The rule, then, for extracting the square root of a number, or decimal to any number of places, is: Annex ciphers until there are twice as many places following the units' place as there are to be decimal places in the root; extract the nearest square root of this number, and mark off the given number of decimals. Or, more simply: Divide the number into periods, so that the units' figure shall be the last of a period; proceed in the usual way; and if, when decimals follow the

units' place, there is one figure on the right, in a period by itself, annex a cipher in bringing down that period, and afterwards let each new period consist of two ciphers. Place the decimal point after that figure, in forming which the period containing the units was used.

168. For example, what is the square root of $1\frac{3}{8}$ to five places of decimals? This is (145) 1·375, and the process is the first example below. The second example is the extraction of the root of ·081 to seven places, the first period being 08, from which the cipher is omitted as useless.

```
        1,37,5(1·17260                  08,1(·2846049
        1                                4
21)     37                     48)      410
        21                              384
227)   1650                    564)    2600
       1589                            2256
2342)  6100                    5686)  34400
       4684                           34116
23446)141600                   569204) 2840000
      140676                           2276816
23452)  92400                  569208) 56318400
```

169. When more than half the decimals required have been found, the remainder may be simply found by dividing the dividend by the divisor, as in (155). The extraction of the square root of 12 to ten places, which will be found in the next page, is an example. It must, however, be observed in this process, as in all others where decimals are obtained by approximation, that the last place cannot always be depended upon; on which account it is advisable to carry the process so far, that one or even two more decimals shall be obtained than are absolutely required to be correct.

A

```
12(3·46410161513
   9
64) 300
    256
686) 4400
     4116
6924) 28400
      27696
69281) 70400
       69281
6928201) 11190000
         6928201
69282026) 4261799|00
          4156921|56
692820321) 104877|4400
            69282|0321
6928203225) 35595|407900
            34641|016125
69282032301) 954|39177500
             692|82032301
692820323023) 261|5714519900
              207|8460969069
               53|7253550831
```

B

```
692820323026)537253550831(77545870549
             484974226118
              52279324713
              48497422611
               3781902102
               3464101615
                317800487
                277128129
                 40672358
                 34641016
                  6031342
                  5542562
                   488780
                   484974
                     3806
                     3464
                      342
                      277
                       65
                       62
                        3
```

If from any remainder we cut off the ciphers, and all figures which would come under, or on the right of these ciphers, by a vertical line, we find on the left of that line a contracted division, such as those in (155). Thus, after having found the root as far as 3·464101, we have the remainder 4261799, and the divisor 6928202. The figures on the left of the line are nothing more than the contracted division of this remainder by the divisor, with this difference, however, that we have to begin by striking a figure off the divisor, instead of using the whole divisor once, and then striking off the first figure. By this alone we might have doubled our number of decimal places, and got the additional figures 615137, the last 7 being obtained by carrying the contracted division one step further with the remainder 53. We have then, this rule: When half the number of decimal places have been obtained, instead of annexing two ciphers to the remainder, strike off a figure from what would be the divisor if the process were continued at length, and divide the remainder by this contracted divisor, as in (155).

As an example, let us double the number of decimal places already obtained, which are contained in 3·46410161513. The remainder is 537253550831, the divisor 692820323026, and the process is as in (B). Hence the square root of 12 is,

3·4641016151377545870549;

which is true to the last figure, and a little too great; but the substitution of 8 instead of 9 on the right hand would make it too small.

EXERCISES.

Numbers.	Square roots.
·001728	·0415692194
64·34	8·02122185
8074	89·8554394
10	3·16227766
1·57	1·2529964086141667788495

SECTION VIII.

ON THE PROPORTION OF NUMBERS.

170. When two numbers are named in any problem, it is usually necessary, in some way or other, to compare the two; that is, by considering the two together, to establish some connexion between them, which may be useful in future operations. The first method which suggests itself, and the most simple, is to observe which is the greater, and by how much it differs from the other. The connexion thus established between two numbers may also hold good of two other numbers; for example, 8 differs from 19 by 11, and 100 differs from 111 by the

same number. In this point of view, 8 stands to 19 in the same situation in which 100 stands to 111, the first of both couples differing in the same degree from the second. The four numbers thus noticed, viz.:

8, 19, 100, 111,

are said to be in *arithmetical* proportion*. When four numbers are thus placed, the first and last are called the *extremes*, and the second and third the *means*. It is obvious that $111+8 = 100+19$, that is, the sum of the extremes is equal to the sum of the means. And this is not accidental, arising from the particular numbers we have taken, but must be the case in every arithmetical proportion; for in $111+8$, by (35), any diminution of 111 will not affect the sum, provided a corresponding increase be given to 8; and, by the definition just given, one mean is as much less than 111 as the other is greater than 8.

171. A set or series of numbers is said to be in *continued* arithmetical proportion, or in arithmetical *progression*, when the difference between every two succeeding terms of the series is the same. This is the case in the following series:

1,	2,	3,	4,	5,	&c.
3,	6,	9,	12,	15,	&c.
$1\frac{1}{2}$,	2,	$2\frac{1}{2}$,	3,	$3\frac{1}{2}$,	&c.

The difference between two succeeding terms is called the common difference. In the three series just given, the common differences are, 1, 3, and $\frac{1}{2}$.

172. If a certain number of terms of any arithmetical series be taken, the sum of the first and last terms is the same as that of any other two terms, provided one is as distant from the beginning of the series as the other is from the end. For example, let there be 7 terms, and let them be,

$$a \quad b \quad c \quad d \quad e \quad f \quad g$$

Then, since, by the nature of the series, b is as much above a as f is

* This is a very incorrect name, since the term arithmetical applies equally to every notion in this book. It is necessary, however, that the pupil should use words in the sense in which they will be used in his succeeding studies.

below g (170), $a+g = b+f$. Again, since c is as much above b as e is below f (170), $b+f = c+e$. But $a+g = b+f$; therefore, $a+g = c+e$, and so on. Again, twice the middle term, or the term equally distant from the beginning and the end (which exists only when the number of terms is odd), is equal to the sum of the first and last terms; for since c is as much below d as e is above it, we have $c+e = d+d = 2d$. But $c+e = a+g$; therefore, $a+g = 2d$. This will give a short rule for finding the sum of any number of terms of an arithmetical series. Let there be 7, viz. those just given. Since $a+g$, $b+f$, and $c+e$, are the same, their sum is three times $(a+g)$, which with d, the middle term, or half $a+g$, is three times and a half $\overline{a+g}$, or the sum of the first and last terms multiplied by $3\frac{1}{2}$, or $\frac{7}{2}$, or half the number of terms. If there had been an even number of terms, for example, six, viz. a, b, c, d, e, and f, we know now that $a+f$, $b+e$, and $c+d$, are the same, whence the sum is three times $\overline{a+f}$, or the sum of the first and last terms multiplied by half the number of terms, as before. The rule, then, is: To sum any number of terms of an arithmetical progression, multiply the sum of the first and last terms by half the number of terms. For example, what are 99 terms of the series 1, 2, 3, &c.? The 99th term is 99, and the sum is $(99+1)\frac{99}{2}$, or $\frac{100 \times 99}{2}$, or 4950. The sum of 50 terms of the series $\frac{1}{3}$, $\frac{2}{3}$, 1, $\frac{4}{3}$, $\frac{5}{3}$, 2, &c. is $\left(\frac{1}{3}+\frac{50}{3}\right)\frac{50}{2}$, or 17×25, or 425.

173. The first term being given, and also the common difference and number of terms, the last term may be found by adding to the first term the common difference multiplied by one less than the number of terms. For it is evident that the second term differs from the first by the common difference, the *third* term by *twice*, the *fourth* term by *three* times the common difference; and so on. Or, the passage from the first to the nth term is made by $n-1$ steps, at each of which the common difference is added.

EXERCISES.

Given.		To find.	
Series.	No. of terms.	Last term.	Sum.
4, $6\frac{1}{2}$, 9, &c.	33	84	1452
1, 3, 5, &c.	28	55	784
2, 20, 38, &c.	100,000	1799984	89999300000

174. The sum being given, the number of terms, and the first term, we can thence find the common difference. Suppose, for example, the first term of a series to be 1, the number of terms 100, and the sum 10,000. Since 10,000 was made by multiplying the sum of the first and last terms by $\frac{100}{2}$, if we divide by this, we shall recover the sum of the first and last terms. Now, $\frac{10,000}{1}$ divided by $\frac{100}{2}$ is (122) 200, and the first term being 1, the last term is 199. We have then to pass from 1 to 199, or through 198, by 99 equal steps. Each step is, therefore, $\frac{198}{99}$, or 2, which is the common difference; or the series is 1, 3, 5, &c. up to 199.

Given.			To find.	
Sum.	No. of terms.	First term.	Last term.	Common diff.
1809025	1345	1	2689	2
44	10	3	$\frac{29}{5}$	$\frac{14}{45}$
7075600	1330	4	10636	8

175. We now return to (170), in which we compared two numbers together by their difference. This, however, is not the method of comparison which we employ in common life, as any single familiar instance will shew. For example, we say of A, who has 10 thousand pounds, that he is much richer than B, who has only 3 thousand; but we do not say that C, who has 107 thousand pounds, is much richer than D, who has 100 thousand, though the difference of fortune is the same in both cases, viz. 7 thousand pounds. In comparing numbers, we take into our reckoning not only the differences, but the numbers themselves. Thus, if B and D both received 7 thousand pounds, B would receive 233 pounds and a third for every 100 pounds which he had before, while D for every 100 pounds would receive only 7 pounds.

And though, in the view taken in (170), 3 is as near to 10 as 100 is to 107, yet, in the light in which we now regard them, 3 is not so near to 10 as 100 is to 107, for 3 differs from 10 by more than twice itself, while 100 does not differ from 107 by so much as one-fifth of itself. This is expressed in mathematical language by saying, that the *ratio* or *proportion* of 10 to 3 is greater than the *ratio* or *proportion* of 107 to 100. We proceed to define these terms more accurately.

176. When we use the term *part* of a number or fraction, in the remainder of this section, we mean one of the various sets of *equal* parts into which it may be divided, either the half, the third, the fourth, &c.: the term multiple has been already explained (102). By the term *multiple-part* of a number, we mean the abbreviation of the words *multiple of a part*. Thus, 1, 2, 3, 4, and 6, are parts of 12; $\frac{1}{2}$ is also a part of 12, being contained in it 24 times; 12, 24, 36, &c., are multiples of 12; and 8, 9, $\frac{5}{2}$, &c. are multiple-parts of 12, being multiples of some of its parts. And when multiple-parts generally are spoken of, the parts themselves are supposed to be included, on the same principle that 12 is counted among the multiples of 12, the multiplier being 1. The multiples themselves are also included in this term; for 24 is also 48 halves, and is therefore among the multiple-parts of 12. Each part is also in various ways a multiple-part; for one-fourth is two-eighths, and three-twelfths, &c.

177. Every number or fraction is a multiple-part of every other number or fraction. If, for example, we ask what part 12 is of 7, we see that on dividing 7 into 7 parts, and repeating one of these parts 12 times, we obtain 12; or, on dividing seven into 14 parts, each of which is one half, and repeating one of these parts 24 times, we obtain 24 halves, or 12. Hence, 12 is $\frac{12}{7}$, or $\frac{24}{14}$, or $\frac{36}{21}$ of 7, and so on. Generally when a and b are two whole numbers, $\frac{a}{b}$ expresses the multiple-part which a is of b, and $\frac{b}{a}$ that which b is of a. Again, suppose it required to determine what multiple-part $2\frac{1}{7}$ is of $3\frac{1}{5}$, or $\frac{15}{7}$ of $\frac{16}{5}$. These fractions reduced to a common denominator, are $\frac{75}{35}$ and $\frac{112}{35}$, of which the second divided into 112 parts gives $\frac{1}{35}$, which repeated 75 times

gives $\frac{75}{35}$, the first. Hence, the multiple-part which the first is of the second is $\frac{75}{112}$, which being obtained by the rule given in (121), shews that $\frac{a}{b}$, or a divided by b, according to the notion of division there given, expresses the multiple-part which a is of b in every case.

178. When the first of four numbers is the same multiple-part of the second which the third is of the fourth, the four are said to be *geometrically* proportional*, or simply *proportional.* This is a word in common use; and it remains to shew that our mathematical definition of it, just given, is in fact the common notion attached to it. For example, suppose a picture is copied on a smaller scale, so that a line of two inches long in the original is represented by a line of one inch and a half in the copy; we say that the copy is not correct unless all the parts of the original are reduced in the same proportion, namely, that of 2 to $1\frac{1}{2}$. Since, on dividing two inches into 4 parts, and taking 3 of them, we get $1\frac{1}{2}$; the same must be done with all the lines in the original, that is, the length of any line in the copy must be three parts out of four of its length in the original. Again, interest being at 5 per cent, that is, £5 being given for the use of £100, a similar proportion of every other sum would be given; the interest of £70, for example, would be just such a part of £70 as £5 is of £100.

Since, then, the part which a is of b is expressed by the fraction $\frac{a}{b}$, or any other fraction which is equivalent to it, and that which c is of d by $\frac{c}{d}$, it follows, that when a, b, c, and d, are proportional, $\frac{a}{b} = \frac{c}{d}$. This equation will be the foundation of all our reasoning on proportional quantities; and in considering proportionals, it is necessary to observe not only the quantities themselves, but also the order in which they come. Thus, a, b, c, and d, being proportionals, that is, a being the same multiple-part of b which c is of d, it does not follow that a, d, b, and c, are proportionals, that is, that a is the same multiple-part of d

* The same remark may be made here as was made in the note on the term arithmetical proportion, page 99. The word geometrical is, generally speaking, dropped, except when we wish to distinguish between this kind of proportion, and that which has been called arithmetical.

which b is of c. It is plain that a is greater than, equal to, or less than b, according as c is greater than, equal to, or less than d.

179. Four numbers, a, b, c, and d, being proportional in the order written, a and d are called the *extremes*, and b and c the *means* of the proportion. For convenience, we will call the two extremes, or the two means, *similar* terms, and an extreme and a mean, *dissimilar* terms. Thus, a and d are similar, and so are b and c; while a and b, a and c, d and b, d and c, are dissimilar. It is customary to express the proportion by placing dots between the numbers, thus,

$$a : b :: c : d$$

180. Equal numbers will still remain equal when they have been increased, diminished, multiplied, or divided, by equal quantities. This amounts to saying, that if $a = b$ and $p = q$, $a+p = b+q$, $a-p = b-q$, $a\,p = b\,q$, and $\frac{a}{p} = \frac{b}{p}$. It is also evident, that $a+p-p$, $a-p+p$, $\frac{a\,p}{p}$, and $\frac{a}{p} \times p$, are all equal to a.

181. The product of the extremes is equal to the product of the means. Let $\frac{a}{b} = \frac{c}{d}$, and multiply these equal numbers by the product $b\,d$. Then, $\frac{a}{b} \times b\,d = \frac{a\,b\,d}{b}$ (116) $= a\,d$, and $\frac{c}{d} \times b\,d = \frac{c\,b\,d}{d} = c\,b$: hence (180), $a\,d = b\,c$. Thus, 6, 8, 21, and 28, are proportional, since $\frac{6}{8} = \frac{3}{4} = \frac{3 \times 7}{4 \times 7} = \frac{21}{28}$ (180); and it appears that $6 \times 28 = 8 \times 21$, since both products are 168.

182. If the product of two numbers be equal to the product of two others, these numbers are proportional in any order whatever, provided the numbers in the same product are so placed as to be similar terms; that is, if $a\,b = p\,q$, we have the following proportions:—

$a : p :: q : b$	$p : a :: b : q$
$a : q :: p : b$	$p : b :: a : q$
$b : p :: q : a$	$q : a :: b : p$
$b : q :: p : a$	$q : b :: a : p$

To prove any one of these, divide both $a\,b$ and $p\,q$ by the product of its second and fourth terms; for example, to shew the truth of $a : q :: p : b$, divide both ab and pq by bq. Then, $\frac{ab}{bq} = \frac{a}{q}$, and $\frac{pq}{bq} = \frac{p}{b}$; hence (180), $\frac{a}{q} = \frac{p}{b}$, or $a : q :: p : b$. The pupil should not fail to prove every one of the eight cases, and to verify them by some simple examples, such as $1 \times 6 = 2 \times 3$, which gives $1 : 2 :: 3 : 6$, $3 : 1 :: 6 : 2$, &c.

183. Hence, if four numbers are proportional, they are also proportional in any other order, provided it be such that similar terms still remain similar. For since, when $\frac{a}{b} = \frac{c}{d}$, it follows (181) that $ad = bc$; all the proportions which follow from $ad = bc$, by the last article, follow also from $\frac{a}{b} = \frac{c}{d}$.

184. From (114) it follows that $1 + \frac{a}{b} = \frac{b+a}{b}$, and if $\frac{a}{b}$ be less than 1, $1 - \frac{a}{b} = \frac{b-a}{b}$, while if $\frac{a}{b}$ be greater than one, $\frac{a}{b} - 1 = \frac{a-b}{b}$. Also (122), if $\frac{a+b}{b}$ be divided by $\frac{a-b}{b}$ the result is $\frac{a+b}{a-b}$. Hence, a, b, c, and d, being proportionals, we may obtain other proportions, thus:

Let $\frac{a}{b} = \frac{c}{d}$

Then (114) $1 + \frac{a}{b} = 1 + \frac{c}{d}$

or $\frac{a+b}{b} = \frac{c+d}{d}$

or $a+b : b :: c+d : d$

That is, the sum of the first and second is to the second, as the sum of the third and fourth is to the fourth. For brevity, we shall not state in words any more of these proportions, since the pupil will easily supply what is wanting.

Resuming the proportion $a : b :: c : d$

or $\frac{a}{b} = \frac{c}{d}$

$1 - \frac{a}{b} = 1 - \frac{c}{d}$, if $\frac{a}{b}$ is less than 1,

or $\frac{b-a}{b} = \frac{d-c}{d}$

that is, $b-a : b :: d-c : d$

or, $a-b : b :: c-d : d$, if $\frac{a}{b}$ is greater than 1.

Again, since $\frac{a+b}{b} = \frac{c+d}{d}$ and $\frac{a-b}{b} = \frac{c-d}{d}$ $\left(\frac{a}{b}\right.$ being greater than 1), dividing the first by the second we have $\frac{a+b}{a-b} = \frac{c+d}{c-d}$,

$$\text{or} \quad a+b : a-b :: c+d : c-d$$

and also $a+b : b-a :: c+d : d-c$, if $\frac{a}{b}$ is less than 1.

185. Many other proportions might be obtained in the same manner. We will, however, content ourselves with writing down a few which can be obtained by combining the preceding articles.

$$a+b : a :: c+d : c$$
$$a : a-b :: c : c-d$$
$$a+c : a-c :: b+d : b-d.$$

In these and all others it must be observed, that when such expressions as $a-b$ and $c-d$ occur, it is supposed that a is greater than b, and c greater than d.

186. If four numbers be proportional, and any two dissimilar terms be both multiplied, or both divided by the same quantity, the results are proportional. Thus, if $a : b :: c : d$, and m and n be any two numbers, we have also the following:

$$ma : b :: mc : d \qquad ma : nb :: mc : nd$$
$$a : mb :: c : md \qquad \frac{a}{m} : \frac{b}{m} :: \frac{c}{m} : \frac{d}{m}$$
$$\frac{a}{n} : mb :: \frac{c}{n} : md \qquad \frac{a}{m} : \frac{b}{m} :: \frac{c}{n} : \frac{d}{n}$$

and various others. To prove any one of these, recollect that nothing more is necessary to make four numbers proportional, except that the product of the extremes should be equal to that of the means. Take the third of those just given; the product of its extremes is $\frac{a}{n} \times md$, or $\frac{mad}{n}$, while that of the means is $mb \times \frac{c}{n}$, or $\frac{mbc}{n}$. But since $a : b :: c : d$, by (181) $ad = bc$, whence, by (180) $mad = mbc$, and $\frac{mad}{n} = \frac{mbc}{n}$. Hence, $\frac{a}{n}$, mb, $\frac{c}{n}$, and md, are proportionals.

187. If the terms of one proportion be multiplied by the terms of a second, the products are proportional; that is, if $a : b :: c : d$, and $p : q :: r : s$, it follows that $ap : bq :: cr : ds$. For, since

$ad = bc$, and $ps = qr$, by (180) $adps = bcqr$, or $ap \times ds = bq \times cr$, whence (182) $ap : bq :: cr : ds$.

188. If four numbers are proportional, any similar powers of these numbers are also proportional; that is, if

$$a : b :: c : d$$

Then $$aa : bb :: cc : dd$$

$$aaa : bbb :: ccc : ddd$$

&c. &c.

For, if we write the proportion twice, thus,

$$a : b :: c : d$$

$$a : b :: c : d$$

by (187) $$aa : bb :: cc : dd$$

But $$a : b :: c : d,$$

Whence (187) $aaa : bbb :: ccc : ddd$; and so on.

189. An expression is said to be homogeneous with respect to any two or more letters, for instance, a, b, and c, when every term of it contains the same number of letters, counting a, b, and c only. Thus, $maab+nabc+rccc$ is homogeneous with respect to a, b, and c; and of the third degree, since in each term there is either a, b, and c, or one of these repeated alone, or with another, so as to make three in all. Thus, $8aaabc$, $12abccc$, $maaaaa$, $naabbc$, are all homogeneous, and of the fifth degree, with respect to a, b, and c only; and any expression made by adding or subtracting these from one another, will be homogeneous and of the fifth degree. Again, $ma+mnb$ is homogeneous with respect to a and b, and of the first degree; but it is not homogeneous with respect to m and n, though it is so with respect to a and n. This being premised, we proceed to a theorem,* which will contain all the results of (184), (185), and (188).

190. If any four numbers be proportional, and if from the first two, a and b, any two homogeneous expressions of the same degree be formed;

* A theorem is a general mathematical fact: thus, that every number is divisible by four when its last two figures are divisible by four, is a theorem; that in every proportion the product of the extremes is equal to the product of the means, is another.

and if from the last two, two other expressions be formed, in precisely the same manner, the four results will be proportional. For example, if $a : b :: c : d$, and if $2aaa+3aab$ and $bbb+abb$ be chosen, which are both homogeneous with respect to a and b, and both of the third degree; and if the corresponding expressions $2ccc+3ccd$, and $ddd+cdd$ be formed, which are made from c and d precisely in the same manner as the two former ones from a and b, then will

$$2aaa+3aab : bbb+abb :: 2ccc+3ccd : ddd+cdd$$

To prove this, let $\frac{a}{b}$ be called x. Then, since $\frac{a}{b} = x$, and $\frac{a}{b} = \frac{c}{d}$, it follows that $\frac{c}{d} = x$. But since a divided by b gives x, x multiplied by b will give a, or $a = bx$. For a similar reason, $c = dx$. Put bx and dx instead of a and c in the four expressions just given, recollecting that when quantities are multiplied together, the result is the same, in whatever order the multiplications are made; that, for example, $bxbxbx$ is the same as $bbbxxx$.

Hence, $2aaa+3aab = 2bxbxbx+3bxbxb$

$= 2bbbxxx+3bbbxx$

which is bbb multiplied by $2xxx+3xx$

or $bbb\,(2xxx+3xx)$*

Similarly, $2ccc+3ccd = ddd\,(2xxx+3xx)$

Also, $bbb+abb = bbb+bxbb$

$= bbb$ multiplied by $1+x$

or $bbb\,(1+x)$

Similarly, $ddd+cdd = ddd\,(1+x)$

Now, $bbb : ddd :: bbb : ddd$

Whence (186), $bbb\,(2xxx+3xx) : ddd\,(1+x) :: bbb\,(2xxx+3xx) : ddd\,(1+x)$, which, when instead of these expressions, their equals just found are substituted, becomes $2aaa+3aab : bbb+abb :: 2ccc+3ccd : ddd+cdd$. The same reasoning may be

* If bx be substituted for a in any expression which is homogeneous with respect to a and b, the pupil may easily see that b must occur in every term as often as there are units in the degree of the expression: thus, $aa+ab$ becomes $bxbx+bxb$ or $bb\,(xx+x)$; $aaa+bbb$ becomes $bxbxbx+bbb$ or $bbb(xxx+1)$; and so on.

applied to any other case, and the pupil may in this way prove the following theorems:

If $a : b :: c : d$

$$2a+3b : b :: 2c+3d : d$$

$$aa+bb : aa-bb :: cc+dd : cc-dd$$

$$mab : 2aa+bb :: mcd : 2cc+dd$$

191. If the two means of a proportion be the same, that is, if $a : b :: b : c$, the three numbers, a, b, and c, are said to be in *continued* proportion, or in *geometrical progression*. The same terms are applied to a series of numbers, of which any three that follow one another are in continued proportion, such as

$$1 \quad 2 \quad 4 \quad 8 \quad 16 \quad 32 \quad 64 \quad \&c.$$

$$2 \quad \frac{2}{3} \quad \frac{2}{9} \quad \frac{2}{27} \quad \frac{2}{81} \quad \frac{2}{243} \quad \frac{2}{729} \quad \&c.$$

Which are in continued proportion, since

$$1 : 2 :: 2 : 4 \qquad 2 : \frac{2}{3} :: \frac{2}{3} : \frac{2}{9}$$

$$2 : 4 :: 4 : 8 \qquad \frac{2}{3} : \frac{2}{9} :: \frac{2}{9} : \frac{2}{27}$$

&c. &c.

192. Let a, b, c, d, &c. be in continued proportion; we have then

$$a : b :: b : c \quad \text{or} \quad \frac{a}{b} = \frac{b}{c} \quad \text{or} \quad ac = bb$$

$$b : c :: c : d \quad \ldots \quad \frac{b}{c} = \frac{c}{d} \quad \ldots \quad bd = cc$$

$$c : d :: d : e \quad \ldots \quad \frac{c}{d} = \frac{d}{e} \quad \ldots \quad ce = dd$$

Each term is formed from the preceding, by multiplying it by the same number. Thus, $b = \frac{b}{a} \times a$ (180); $c = \frac{c}{b} \times b$; and since $\frac{a}{b} = \frac{b}{c}, \frac{b}{a} = \frac{c}{b}$, or $c = \frac{b}{a} \times b$. Again, $d = \frac{d}{c} \times c$, but $\frac{d}{c} = \frac{c}{b}$, which is $= \frac{b}{a}$; therefore, $d = \frac{b}{a} \times c$, and so on. If, then, $\frac{b}{a}$ (which is called the *common ratio* of the series) be denoted by r, we have

$$b = ar \qquad c = br = arr \qquad d = cr = arrr$$

and so on; whence the series

	a	b	c	d	&c.
is	a	ar	arr	$arrr$	&c.

Hence $a : c :: a : arr$

(186) $:: aa : aarr$

$:: aa : bb$

because, b being ar, bb is $arar$ or $aarr$. Again,

$a : d :: a : arrr$

(186) $:: aaa : aaarrr$

$:: aaa : bbb$

Also, $a : e :: aaaa : bbbb$, and so on; that is, the first bears to the n^{th} term from the first, the same proportion as the n^{th} power of the first to the n^{th} power of the second.

193. A short rule may be found for adding together any number of terms of a continued proportion. Let it be first required to add together the terms 1, r, rr, &c. where r is greater than unity. It is evident that we do not alter any expression by adding or subtracting any numbers, provided we afterwards subtract or add the same. For example,

$$p = p-q+q-r+r-s+s$$

Let us take four terms of the series, 1, r, rr, &c. or,

$$1+r+rr+rrr$$

It is plain that

$$rrrr-1 = rrrr-rrr+rrr-rr+rr-r+r-1$$

Now (54), $rr-r = r\ (r-1)$, $rrr-rr = rr\ (r-1)$, $rrrr-rrr = rrr\ (r-1)$, and the above equation becomes $rrrr-1 = rrr\,(r-1) + rr\ (r-1)+r\ (r-1)+r-1$; which is (54) $rrr+rr+r+1$, taken $r-1$ times. Hence, $rrrr-1$ divided by $r-1$ will give $1+r+rr+rrr$, the sum of the terms required. In this way may be proved the following series of equations:

$$1+r = \frac{rr-1}{r-1}$$

$$1+r+rr = \frac{rrr-1}{r-1}$$

$$1+r+rr+rrr = \frac{rrrr-1}{r-1}$$

$$1+r+rr+rrr+rrrr = \frac{rrrrr-1}{r-1}$$

If r be less than unity, in order to find $1+r+rr+rrr$, observe that,

$$1-rrrr = 1-r+r-rr+rr-rrr+rrr-rrrr$$
$$= 1-r+r(1-r)+rr(1-r)+rrr(1-r);$$

whence, by similar reasoning, $1+r+rr+rrr$ is found by dividing $1-rrrr$ by $1-r$; and equations similar to these just given may be found, which are,

$$1+r = \frac{1-rr}{1-r}$$

$$1+r+rr = \frac{1-rrr}{1-r}$$

$$1+r+rr+rrr = \frac{1-rrrr}{1-r}$$

$$1+r+rr+rrr+rrrr = \frac{1-rrrrr}{1-r}$$

The rule is: To find the sum of n terms of the series, $1+r+rr+$&c. divide the difference between 1 and the $(n+1)^{\text{th}}$ term, by the difference between 1 and r.

194. This may be applied to finding the sum of any number of terms of a continued proportion. Let a, b, c, &c. be the terms of which it is required to sum four, that is, to find $a+b+c+d$, or (192) $a+ar+arr+arrr$, or (54) $a\,(1+r+rr+rrr)$, which (193) is $\dfrac{rrrr-1}{r-1}\times a$, or $\dfrac{1-rrrr}{1-r}\times a$, according as r is greater or less than unity. The first fraction is $\dfrac{arrrr-a}{r-1}$, or (192) $\dfrac{e-a}{r-1}$. Similarly, the second is $\dfrac{a-e}{1-r}$. The rule, therefore, is: To sum n terms of a continued proportion, divide the difference of the $\overline{n+1}^{\text{th}}$ and first terms by the difference between unity and the common measure. For example, the sum of 10 terms of the series $1+3+9+27+$&c. is required. The eleventh term is 59049 and $\dfrac{59049-1}{3-1}$ is 29524. Again, the sum of 18 terms of the series $2+1+\dfrac{1}{2}+\dfrac{1}{4}+$&c. of which the nineteenth term is $\dfrac{1}{131072}$, is $\dfrac{2-\dfrac{1}{131072}}{1-\dfrac{1}{2}} = 3\dfrac{131070}{131072}$.

EXAMPLES.

9	terms of $1 + 4 + 16 + \&c.$	are	87381
10	.... $3 + \frac{6}{7} + \frac{12}{49} + \&c.$	...	$\frac{847422675}{201768035}$
20	.... $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c.$	...	$\frac{1048575}{1048576}$

195. The powers of a number or fraction greater than unity increase; for since $2\frac{1}{2}$ is greater than 1, $2\frac{1}{2} \times 2\frac{1}{2}$ is $2\frac{1}{2}$ taken more than once, that is, is greater than $2\frac{1}{2}$, and so on. This increase goes on without limit, that is, there is no quantity so great but that some power of $2\frac{1}{2}$ is greater. To prove this, observe that every power of $2\frac{1}{2}$ is made by multiplying the preceding power by $2\frac{1}{2}$, or by $1 + 1\frac{1}{2}$, that is, by adding to the former power that power itself and its half. There will, therefore, be more added to the 10th power to form the 11th, than was added to the 9th power to form the 10th. But it is evident that if any quantity, however small, be continually added to $2\frac{1}{2}$, the result will come in time to exceed any other quantity, however great; much more, then, will it do so, if the quantity added to $2\frac{1}{2}$ be increased at each step, which is the case when the successive powers of $2\frac{1}{2}$ are formed. It is evident, also, that the powers of 1 never increase, being always 1; thus, $1 \times 1 = 1$, &c. Also, if a be greater than m times b, the square of a is greater than mm times the square of b. Thus, if $a = 2b + c$, where a is greater than $2b$, the square of a, or aa, which is (68) $4bb + 4bc + cc$, is greater than $4bb$, and so on.

196. The powers of a fraction less than unity continually decrease; thus, the square of $\frac{2}{5}$, or $\frac{2}{5} \times \frac{2}{5}$ is less than $\frac{2}{5}$, being only two-fifths of it. This decrease continues without limit, that is, there is no quantity so small but that some power of $\frac{2}{5}$ is less. For if $\frac{5}{2} = x$, $\frac{2}{5} = \frac{1}{x}$, and the powers of $\frac{2}{5}$ are $\frac{1}{xx}$, $\frac{1}{xxx}$, and so on. Since x is greater than 1 (195), some power of x may be found which shall be greater than a given quantity. Let this be called m; then $\frac{1}{m}$ is the corresponding power of $\frac{2}{5}$; and a fraction whose denominator can be made as great as we please, can itself be made as small as we please (112).

197. We have then, in the series

$$1 \qquad r \qquad rr \qquad rrr \qquad rrrr \qquad \&c.$$

I. A series of increasing terms, if r be greater than 1. II. Of terms having the same value, if r be equal to 1. III. A series of decreasing terms, if r be less than 1. In the first two cases, the sum

$$1+r+rr+rrr+\&c.$$

may evidently be made as great as we please, by sufficiently increasing the number of terms. But in the third, this may or may not be the case; for though something is added at each step, yet, as that augmentation diminishes at every step, we may not certainly say, that we can, by any number of such augmentations, make the result as great as we please. To shew the contrary in a simple instance, consider the series,

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\&c.$$

Carry this series to what extent we may, it will always be necessary to add the last term, in order to make as much as 2. Thus,

$$\left(1+\frac{1}{2}+\frac{1}{4}\right)+\frac{1}{4}=1+\frac{1}{2}+\frac{1}{2}=1+1=2.$$

$$\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)+\frac{1}{8}=2.$$

$$\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\right)+\frac{1}{16}=2, \&c.$$

But in the series, every term is only the half of the preceding; consequently, no number of terms, however great, can be made as great as 2 by adding one more. The sum, therefore, of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. continually approaches to 2, diminishing its distance from 2 at every step, but never reaching it. Hence, 2 is called the *limit* of $1+\frac{1}{2}+\frac{1}{4}$ + &c. We are not, therefore, to conclude that *every* series of decreasing terms has a limit. The contrary may be shewn in the very simple series, $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ &c. which may be written thus:

$$1+\frac{1}{2}\times\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\ldots\text{ up to }\frac{1}{8}\right)+\left(\frac{1}{9}+\ldots\text{ up to }\frac{1}{16}\right)+\left(\frac{1}{17}+\ldots\text{ up to }\frac{1}{32}\right)+\&c.$$

We have thus divided all the series, except the first two terms, into lots, each containing half as many terms as there are units in the denominator of its last term. Thus, the fourth lot contains 16 or $\frac{32}{2}$ terms. Each of these lots may be shewn to be greater than $\frac{1}{2}$. Take the third for example, consisting of $\frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}$, and $\frac{1}{16}$. All except $\frac{1}{16}$, the last, are greater than $\frac{1}{16}$; consequently, by substituting $\frac{1}{16}$ for each of them, the amount of the whole lot would be lessened; and as it would then become $\frac{8}{16}$, or $\frac{1}{2}$, the lot itself is greater than $\frac{1}{2}$. Now, if to $1+\frac{1}{2}, \frac{1}{2}$ be continually added, the result will in time exceed any given number. Still more will this be the case, if, instead of $\frac{1}{2}$, the several lots written above be added one after the other. But it is thus that the series $1+\frac{1}{2}+\frac{1}{3}$, &c. is composed, which proves what was said, that this series has no limit.

198. The series $1+r+rr+rrr+$ &c. always has a limit, when r is less than 1. To prove this, let the term succeeding that at which we stop be a, whence (194) the sum is $\frac{1-a}{1-r}$ or (112) $\frac{1}{1-r} - \frac{a}{1-r}$. The terms decrease without limit (196), whence we may take a term so far distant from the beginning, that a, and therefore $\frac{a}{1-r}$ shall be as small as we please. But it is evident, that in this case $\frac{1}{1-r} - \frac{a}{1-r}$, though always less than $\frac{1}{1-r}$, may be brought as near to $\frac{1}{1-r}$ as we please, that is, the series $1+r+rr+$ &c. continually approaches to the limit $\frac{1}{1-r}$. Thus, $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$ &c. where $r=\frac{1}{2}$, continually approaches to $\frac{1}{1-\frac{1}{2}}$ or 2, was shewn in the last article.

EXERCISES.

The limit of $2+\frac{2}{3}+\frac{2}{9}+$ &c.

or $2\left(1+\frac{1}{3}+\frac{1}{9}+\text{\&c.}\right)$ is 3

.... $1+\frac{9}{10}+\frac{81}{100}+$ &c. ... 10

.... $5+\frac{15}{7}+\frac{45}{49}+$ &c. ... $8\frac{3}{4}$

199. When the fraction $\frac{a}{b}$ is not equal to $\frac{c}{d}$, but greater, a is said to

have to b a greater ratio than c has to d; and when $\frac{a}{b}$ is less than $\frac{c}{d}$, a is said to have to b a less ratio than c has to d. We propose the following questions as exercises, since they follow very simply from this definition.

I. If a is greater than b, and c less than, or equal to d; a will have a greater ratio to b than c has to d.

II. If a is less than b, and c greater than, or equal to d; a has a less ratio to b than c has to d.

III. If a is to b as c is to d, and if a has a greater ratio to b than c has to x; d is less than x; and if a has a less ratio to b than c to x, d is greater than x.

IV. a has to b a greater ratio than ax to $bx+y$, and a less ratio than ax to $bx-y$.

200. If a has to b a greater ratio than c has to d, $a+c$ has to $b+d$ a less ratio than a has to b, but a greater ratio than c has to d; or, in other words, if $\frac{a}{b}$ be the greater of the two fractions $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a+c}{b+d}$ will be greater than $\frac{c}{d}$, but less than $\frac{a}{b}$. To shew this, let $\frac{c}{a}=x$, or let $c=ax$: then, since $\frac{a}{b}=\frac{ax}{bx}$, it is $=\frac{c}{bx}$, and therefore $\frac{c}{bx}$ is greater than $\frac{c}{d}$, that is (112) d is greater than bx. Let $d=bx+y$. Then the fraction $\frac{a}{b}$, being (108) the same as $\frac{a(1+x)}{b(1+x)}$ or $\frac{a+ax}{b+bx}$, is the same as $\frac{a+c}{b+bx}$. But this last is greater than $\frac{a+c}{b+bx+y}$ or $\frac{a+c}{b+d}$, because, of two fractions having the same numerator, that is the greater which has the less denominator. Therefore $\frac{a}{b}$ is greater than $\frac{a+c}{b+d}$. Again, let $\frac{a}{c}=z$ or $a=cz$: then $\frac{c}{d}$ being $\frac{cz}{dz}$ or $\frac{a}{dz}$, this last is less than $\frac{a}{b}$, that is, b is less than dz. Let $b=dz-w$. Then $\frac{c}{d}$ is $\frac{c(z+1)}{d(z+1)}$ or $\frac{cz+c}{dz+d}$ or $\frac{a+c}{dz+d}$. This last is less than $\frac{a+c}{dz-w+d}$ or $\frac{a+c}{b+d}$ (112): therefore $\frac{c}{d}$ is less than $\frac{a+c}{b+d}$. Thus, $\frac{2+3}{5+7}$ or $\frac{5}{12}$ lies between $\frac{2}{5}$ and $\frac{3}{7}$. Again, since $\frac{a}{b}$ and $\frac{c}{d}$ are respectively equal to $\frac{ap}{bp}$ and $\frac{cq}{dq}$, and since, as has just been proved, $\frac{ap+cq}{bp+dq}$ lies between the two last, it also lies between the two first; that is, if p and q be any numbers or fractions whatsoever, $\frac{ap+cq}{bp+dq}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$.

201. By the last article we may often form some notion of the value of an expression, too complicated to be easily calculated. Thus, $\frac{1+x}{1+xx}$ lies between $\frac{1}{1}$ and $\frac{x}{xx}$ or 1 and $\frac{1}{x}$; $\frac{ax+by}{axx+bbyy}$ lies between $\frac{ax}{axx}$ and $\frac{by}{bbyy}$; that is, between $\frac{1}{x}$ and $\frac{1}{by}$. In a similar way it may be shewn that $\frac{a+b}{2}$ lies between a and b, the denominator being considered as $1+1$.

202. It may also be proved that a fraction, such as $\frac{a+b+c+d}{p+q+r+s}$ always lies among $\frac{a}{p}, \frac{b}{q}, \frac{c}{r}$, and $\frac{d}{s}$, that is, is less than the greatest of them, and greater than the least. Let these fractions be arranged in order of magnitude; that is, let $\frac{a}{p}$ be greater than $\frac{b}{q}$, $\frac{b}{q}$ be greater than $\frac{c}{r}$, and $\frac{c}{r}$ greater than $\frac{d}{s}$. Then by (200)

$$\begin{array}{lcllcl} \frac{a+b}{p+q} & & \frac{a}{p} & & & \frac{b}{q} \text{ and } \frac{c}{r} \\ \frac{a+b+c}{p+q+r} & \text{is less than} & \frac{a+b}{p+q} & \text{and } \frac{a}{p} & \text{and greater than} & \frac{c}{r} \text{ and } \frac{d}{s} \\ \frac{a+b+c+d}{p+q+r+s} & & \frac{a+b+c}{p+q+r} & \text{and } \frac{a}{p} & & \frac{d}{s} \end{array}$$

whence the proposition is evident.

203. It is usual to signify "a is greater than b" by $a > b$, and "a is less than b" by $a < b$; the opening of $\wedge$ being turned towards the greater quantity. The pupil is recommended to make himself familiar with these signs.

SECTION IX.

ON PERMUTATIONS AND COMBINATIONS.

204. If a number of counters, distinguished by different letters, be placed on the table, and any number of them, say four, be taken away, the question is to determine in how many different ways this can be done. Each way of doing it gives what is called a *combination* of four, but which might with more propriety be called a *selection* of four. Two

combinations or selections are called different, which differ in any way whatever; thus, *a b c d* and *a b c e* are different, *d* being in one and *e* in the other, the remaining parts being the same. Let there be six counters, *a*, *b*, *c*, *d*, *e*, and *f*; the combinations of three which can be made out of them are twenty in number, as follow:

a b c	*a c e*	*b c d*	*b e f*
a b d	*a c f*	*b c e*	*c d e*
a b e	*a d e*	*b c f*	*c d f*
a b f	*a d f*	*b d e*	*c e f*
a c d	*a e f*	*b d f*	*d e f*

The combinations of four are fifteen in number, namely,

a b c d	*a b d e*	*a c d e*	*a d e f*	*b c e f*
a b c e	*a b d f*	*a c d f*	*b c d e*	*b d e f*
a b c f	*a b e f*	*a c e f*	*b c d f*	*c d e f*

and so on.

205. Each of these combinations may be written in several different orders; thus, *a b c d* may be disposed in any of the following ways:

a b c d	*a c b d*	*a c d b*	*a b d c*	*a d b c*	*a d c b*
b a c d	*c a b d*	*c a d b*	*b a d c*	*d a b c*	*d a c b*
b c a d	*c b a d*	*c d a b*	*b d a c*	*d b a c*	*d c a b*
b c d a	*c b d a*	*c d b a*	*b d c a*	*d b c a*	*d c b a*

of which no two are entirely in the same order. Each of these is said to be a distinct *permutation* of *a b c d*. Considered as a *combination*, they are all the same, as each contains *a*, *b*, *c*, and *d*.

206. We now proceed to find how many permutations, each containing one given number, can be made from the counters in another given number, six, for example. If we knew how to find all the permutations containing four counters, we might make those which contain five thus: Take any one which contains four, for example, *a b c f*, in which *d* and *e* are omitted; write *d* and *e* successively at the end, which gives *a b c f d*, *a b c f e*, and repeat the same process with every other permutation of four; thus, *d a b c* gives *d a b c e* and *d a b c f*. No permutation of five can escape us, if we proceed in this manner,

provided only we know those of four; for any given permutation of five, as *d b f e a*, will arise in the course of the process from *d b f e*, which according to our rule, furnishes *d b f e a*. Neither will any permutation be repeated twice, for *d b f e a*, if the rule be followed, can only arise from the permutation *d b f e*. If we begin in this way to find the permutations of two out of the six,

$$a \quad b \quad c \quad d \quad e \quad f$$

each of these gives five; thus,

a	gives	*ab*	*ac*	*ad*	*ae*	*af*
b	......	*ba*	*bc*	*bd*	*be*	*bf*

and the whole number is 6 × 5, or 30.

Again,	*ab*	gives	*abc*	*abd*	*abe*	*abf*
	ac	......	*acb*	*acd*	*ace*	*acf*

and here are 30, or 6 × 5 permutations of 2, each of which gives 4 permutations of 3; the whole number of the last is therefore 6 × 5 × 4, or 120.

Again,	*abc*	gives	*abcd*	*abce*	*abcf*
	abd	......	*abdc*	*abde*	*abdf*

and here are 120, or 6 × 5 × 4 permutations of three, each of which gives 3 permutations of four; the whole number of the last is therefore 6 × 5 × 4 × 3, or 360.

In the same way, the number of permutations of 5 is 6 × 5 × 4 × 3 × 2, and the number of permutations of six, or the number of different ways in which the whole six can be arranged, is 6 × 5 × 4 × 3 × 2 × 1. The last two results are the same, which must be, for since a permutation of five only omits one, it can only furnish one permutation of six. If instead of six we choose any other number, x, the number of permutations of two, will be $x(x-1)$, that of three will be $x(x-1)(x-2)$, that of four $x(x-1)(x-2)(x-3)$, the rule being,—Multiply the whole number of counters by the next less number, and the result by the next less, and so on, until as many numbers have been multiplied together as there are to be counters in each permutation, the product will be the whole number of permutations of the sort required. Thus, out of 12 counters, permutations of four may be made to the number of $12 \times 11 \times 10 \times 9$, or 11880.

EXERCISES.

207. In how many different ways can eight persons be arranged on eight seats? *Answer*, 40320.

In how many ways can eight persons be seated at a round table, so that all shall not have the same neighbours in any two arrangements?*
Answer, 5040.

If the hundredth part of a farthing be given for every different arrangement which can be made of fifteen persons, to how much will the whole amount? *Answer*, £13621608.

Out of seventeen consonants and five vowels, how many words can be made, having two consonants and one vowel in each? *Answer*, 4080.

208. If two or more of the counters have the same letter upon them, the number of distinct permutations is less than that given by the last rule. Let there be a, a, a, b, c, d, and, for a moment, let us distinguish between the three a's thus, a, a', a''. Then, $a\,b\,c\,a'\,a''\,d$, and $a''\,b\,c\,a\,a'\,d$, are reckoned as distinct permutations in the rule, whereas they would not have been so, had it not been for the accents. To compute the number of distinct permutations, let us make one with b, c, and d, leaving places for the a's thus ( ) $b\,c$ ( ) ( ) d. If the a's had been distinguished as a, a', a'', we might have made $3 \times 2 \times 1$ distinct permutations, by filling up the vacant places in the above, all which six are the same when the a's are not distinguished. Hence, to deduce the number of permutations of a, a, a, b, c, d, from that of $a\,a'\,a''\,b\,c\,d$, we must divide the latter by $3 \times 2 \times 1$, or 6, which gives $\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$ or 120. Similarly, the number of permutations of $a\,a\,a\,a\,b\,b\,b\,c\,c$ is $\frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1 \times 2 \times 1}$.

EXERCISE.

How many variations can be made of the order of the letters in the word antitrinitarian? *Answer*, 126126000.

* The difference between this problem and the last is left to the ingenuity of the pupil.

209. From the number of permutations we can easily deduce the number of combinations. But, in order to form these combinations independently, we will shew a method similar to that in (206). If we know the combinations of two which can be made out of *a*, *b*, *c*, *d*, *e*, we can find the combinations of three, by writing successively at the end of each combination of two, the letters which come after the last contained in it. Thus, *a b* gives *a b c*, *a b d*, *a b e*; *a d* gives *a d e* only. No combination of three can escape us if we proceed in this manner, provided only we know the combinations of two; for any given combination of three, as *a c d*, will arise in the course of the process, from *a c*, which, according to our rule, furnishes *a c d*. Neither will any combination be repeated twice, for *a c d*, if the rule be followed, can only arise from *a c*, since neither *a d* nor *c d* furnishes it. If we begin in this way to find the combinations of the five,

	a	*b*	*c*	*d*	*e*
a	gives	*a b*	*a c*	*a d*	*a e*
b	......		*b c*	*b d*	*b e*
c	......			*c d*	*c e*
d	......				*d e*

Of these	*a b*	gives	*a b c*	*a b d*	*a b e*
	a c	......		*a c d*	*a c e*
	a d	......			*a d e*
	b c	......		*b c d*	*b c e*
	b d	......			*b d e*
	c d	......			*c d e*

a e *b e* *c e* and *d e* give none.

Of these	*a b c*	gives	*a b c d*	*a b c e*
	a b d	......		*a b d e*
	a c d	......		*a c d e*
	b c d	......		*b c d e*

Those which contain *e* give none, as before.

Of the last, *a b c d* gives *a b c d e*, and the others none, which is evidently true, since only one selection of five can be made out of five things.

210. The rule for calculating the number of combinations is derived directly from that for the number of permutations. Take 7 counters; then, since the number of permutations of two is 7×6, and since two permutations, *b a* and *a b*, are in any combination *a b*, the number of combinations is half that of the permutations, or $\frac{7 \times 6}{2}$. Since the number of permutations of three is $7 \times 6 \times 5$, and as each combination *a b c* has $3 \times 2 \times 1$ permutations, the number of combinations of three is $\frac{7 \times 6 \times 5}{1 \times 2 \times 3}$. Also, since any combination of four, *a b c d*, contains $4 \times 3 \times 2 \times 1$ permutations, the number of combinations of four is $\frac{7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4}$, and so on. The rule is: To find the number of combinations, each containing *n* counters, divide the corresponding number of permutations by the product of 1, 2, 3, &c. up to *n*. If *x* be the whole number, the number of combinations of two is $\frac{x(x-1)}{1 \times 2}$; that of three is $\frac{x(x-1)(x-2)}{1 \times 2 \times 3}$; that of four is $\frac{x(x-1)(x-2)(x-3)}{1 \times 2 \times 3 \times 4}$; and so on.

211. The rule may in half the cases be simplified, as follows. Out of ten counters, for every distinct selection of seven which is taken, a distinct combination of 3 is left. Hence, the number of combinations of seven is as many as that of three. We may, therefore, find the combinations of three instead of those of seven; and we must moreover expect, and may even assert, that the two formulæ for finding these two numbers of combinations, are the same in result, though different in form. And so it proves; for the number of combinations of seven out of ten is $\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}$, in which the product $7 \times 6 \times 5 \times 4$ occurs in both terms, and therefore may be removed from both (108), leaving $\frac{10 \times 9 \times 8}{1 \times 2 \times 3}$, which is the number of combinations of three out of ten. The same may be shewn in other cases.

EXERCISES.

How many combinations of four can be made out of twelve things?

Answer, 495.

What number of combinations can be made of $\left\{\begin{matrix} 6 \\ 4 \\ 26 \\ 6 \end{matrix}\right\}$ out of $\left\{\begin{matrix} 8 \\ 11 \\ 28 \\ 15 \end{matrix}\right\}$ *Answer*, $\left\{\begin{matrix} 28 \\ 330 \\ 378 \\ 5005 \end{matrix}\right.$

How many combinations can be made of 13 out of 52; or how many different hands may a person hold at the game of whist?

Answer, 635013559600.

BOOK II.

COMMERCIAL ARITHMETIC.

SECTION I.

WEIGHTS, MEASURES, &c.

212. In making the calculations which are necessary in commercial affairs, no more processes are required than those which have been explained in the preceding book. But there is still one thing wanted —not to ensure the accuracy of our calculations, but to enable us to compare and judge of their results. We have hitherto made use of a single unit (15), and have treated of other quantities which are made up of a number of units, in Sections II., III., and IV., and of those which contain parts of that unit in Sections V. and VI. Thus, if we are talking of distances, and take a mile as the unit, any other length may be represented,* either by a certain number of miles, or a certain

* It is not true that if we choose any quantity as a unit, *any* other quantity of the same kind can be exactly represented either by a certain number of units, or of parts of a unit. To understand how this is proved, the pupil would require more knowledge than he can be supposed to have; but we can shew him that, for any thing he knows to the contrary, there may be quantities which are neither units nor parts of the unit. Take a mathematical line of one foot in length, divide it into ten parts, each of those parts into ten parts, and so on continually. If a point A be taken at hazard in the line, it does not appear self-evident that if the decimal division be continued ever so far, one of the points of division must at last fall exactly on A : neither would the same appear necessarily true if the division were made into sevenths, or elevenths, or in any other way. There may then possibly be a part of a foot which is no exact numerical fraction whatever of the foot; and this, in a higher branch of mathematics, is found to be the case times without number. What is meant in the words on which this note is written is, that any part of a foot can be represented as nearly as we please by a numerical fraction of it, and this is sufficient fo practical purposes.

number of parts of a mile, and (1 meaning one mile) may be expressed either by a whole number or a fraction. But we can easily see that in many cases inconveniences would arise. Suppose, for example, I say, that the length of one room is $\frac{1}{180}$ of a mile, and of another $\frac{1}{174}$ of a mile, what idea can we form as to how much the second is longer than the first? It is necessary to have some smaller measure; and if we divide a mile into 1760 equal parts, and call each of these parts a yard, we shall find that the length of the first room is 9 yards and $\frac{7}{9}$ of a yard, and that of the second 10 yards and $\frac{10}{87}$ of a yard. From this we form a much better notion of these different lengths, but still not a very perfect one, on account of the fractions $\frac{7}{9}$ and $\frac{10}{87}$. To get a clearer idea of these, suppose the yard to be divided into 3 equal parts, and each of these parts to be called a foot; then $\frac{7}{9}$ of a yard contains $2\frac{1}{3}$ feet, and $\frac{10}{87}$ of a yard contains $\frac{30}{87}$ of a foot, or a little more than $\frac{1}{3}$ of a foot. Therefore the length of the first room is now 9 yards, 2 feet, and $\frac{1}{3}$ of a foot; that of the second is 10 yards and a little more than $\frac{1}{3}$ of a foot. We see, then, the convenience of having large measures for large quantities, and smaller measures for small ones; but this is done for convenience only, for it is *possible* to perform calculations upon any sort of quantity, with one measure alone, as certainly as with more than one.

The measures which are used in this country are not those which would have been chosen had they been made all at one time, and by a people well acquainted with arithmetic and natural philosophy. We proceed to shew how the results of the latter science are made useful in our system of measures. Whether the circumstances introduced are sufficiently well known to render the following methods exact enough for the recovery of *astronomical* standards, may be matter of opinion; but no doubt can be entertained of their being amply correct for commercial purposes.

It is evidently desirable that weights and measures should always continue the same, and that posterity should be able to replace any one of them when the original measure is lost. It is true that a yard, which is now exact, is kept by the public authorities; but if this were burnt by accident,* how are those who shall live 500 years hence to know what was the length which their ancestors called a yard? To ensure them this knowledge, the measure must be derived from something which cannot be altered by man, either from design or accident. We find such a quantity in the time of the daily revolution of the earth, and also in the length of the year, both of which, as is shewn in astronomy, will remain the same, at least for an enormous number of centuries, unless some great and totally unknown change takes place in the solar system. So long as astronomy is cultivated, it is impossible to suppose that either of these will be lost, and it is known that the latter is 365·24224 mean solar days, or about $365\frac{1}{4}$ of the average interval which elapses between noon and noon, that is, between the times when the sun is highest in the heavens. Our year is made to consist of 365 days, and the odd quarter is allowed for by adding one day to every fourth year, which gives what we call leap-year. This is the same as adding $\frac{1}{4}$ of a day to each year, and is rather too much, since the excess of the year above 365 days is not ·25 but ·24224 of a day. The difference is ·00776 of a day, which is the quantity by which our average year is too long. This amounts to a day in about 128 years, or to about 3 days in 4 centuries. The error is corrected by allowing only one out of four of the years which close the centuries to be leap-years. Thus, A.D. 1800 and 1900 are not leap-years, but 2000 is so.

213. The day is therefore the first measure obtained, and is divided into 24 parts or hours, each of which is divided into 60 parts or minutes, and each of these again into 60 parts or seconds. One second, marked thus,† 1^s, is therefore the 86400^{th} part of a day, and the following is the

* Within the last few years, this accident has happened. The *standard yard* was so injured as to be rendered useless, by the fire at the Houses of Parliament.

† The minute and second are often marked thus 1′, 1″, but this notation is now almost entirely appropriated to the minute and second of *angular* measure.

MEASURE OF TIME.*

60 *seconds*	are	1 *minute*	1^m
60 *minutes*	......	1 *hour*	1 h.
24 *hours*	......	1 *day*	1 d.
7 *days*	......	1 *week*	1 wk.
365 *days*	......	1 *year*	1 yr.

214. The *second* having been obtained, a pendulum can be constructed which shall, when put in motion, perform one vibration in exactly one second, in the latitude of Greenwich.† If we were inventing measures, it would be convenient to call the length of this pendulum a yard, and make it the standard of all our measures of length. But as there is a yard already established, it will do equally well to tell the length of the pendulum in yards. It was found by commissioners appointed for the purpose, that this pendulum in London was 39·1393 inches, or about one yard, three inches, and $\frac{5}{36}$ of an inch. The following is the division of the yard.

MEASURES OF LENGTH.

The lowest measure is a barleycorn.‡

3 *barleycorns*	are	1 *inch*	1 in.
12 *inches*	......	1 *foot*	1 ft.
3 *feet*	......	1 *yard*	1 yd.
$5\frac{1}{2}$ *yards*	......	1 *pole*	1 po.
40 *poles* or 220 *yards*	......	1 *furlong*	1 fur.
8 *furlongs* or 1760 *yards*		1 *mile*	1 mi.
3 *miles*	......	1 *league*	1 lea.
Also, 6 feet	are	1 fathom	1 fth.
$69\frac{1}{3}$ miles	......	1 deg.	1 deg. or 1°.

And a geographical mile is $\frac{1}{60}$ of a degree.

* The measures in italics are those which it is most necessary that the student should learn by heart.

† The lengths of the pendulums which will vibrate in one second are different in different latitudes. Greenwich is chosen as the station of the Royal Observatory.

‡ The inch was originally obtained by putting together three grains of barley.

In the measurement of cloth or linen the following are also used:

$2\frac{1}{4}$ inches	are	1 nail	1 nl.
4 nails		1 quarter (of a yard)	1 qr.
3 quarters		1 Flemish ell	1 Fl. e.
5 quarters		1 English ell	1 E. e.
6 quarters		1 French ell	1 Fr. e.

215. MEASURES OF SURFACE, OR SUPERFICIES.

All surfaces are measured by square inches, square feet, &c.; the square inch being a square whose side is an inch in length, and so on. The following measures may be deduced from the last, as will afterwards appear.

144 *square inches*	are	1 *square foot*	1 sq. ft.
9 *square feet*		1 *square yard*	1 sq. yd.
$30\frac{1}{4}$ square yards		1 square pole	1 sq. p.
40 square poles		1 rood	1 rd.
4 roods		1 acre	1 ac.

Thus the acre contains 4840 square yards, which is ten times a square of 22 yards in length and breadth. This 22 yards is the length which land-surveyors' chains are made to have, and the chain is divided into 100 links, each ·22 of a yard or 7·92 inches. An acre is then 10 square chains. It may also be noticed that a square whose side is $69\frac{4}{7}$ yards is nearly an acre, not exceeding it by $\frac{1}{5}$ of a square foot.

216. MEASURES OF SOLIDITY OR CAPACITY.*

Cubes are solids having the figure of dice. A cubic inch is a cube, each of whose sides is an inch, and so on.

1728 cubic inches	are	1 cubic foot	1 c. ft.
27 cubic feet		1 cubic yard	1 c. yd.

This measure is not much used, except in purely mathematical

* Capacity is a term which cannot be better explained than by its use. When one measure holds more than another, it is said to be more capacious, or to have a greater capacity.

questions. In the measurements of different commodities various measures were used, which are now reduced, by act of parliament, to one. This is commonly called the imperial measure, and is as follows:

MEASURE OF LIQUIDS AND OF ALL DRY GOODS.

4 *gills*	are	1 *pint*	1 pt.
2 *pints*		1 *quart*	1 qt.
4 *quarts*		1 *gallon* ...	1 gall.
2 *gallons*		1 *peck** ...	1 pk.
4 *pecks*		1 *bushel* ...	1 bu.
8 *bushels*		1 *quarter*..	1 qr.
5 *quarters*		1 *load*	1 ld.

The gallon in this measure is about 277·274 cubic inches; that is, very nearly $277\frac{1}{4}$ cubic inches.†

217. The smallest weight in use is the grain, which is thus determined. A vessel whose interior is a cubic inch, when filled with water,‡ has its weight increased by 252·458 grains. Of the grains so determined, 7000 are a pound *averdupois*, and 5760 a pound *troy*. The first pound is always used, except in weighing precious metals and stones, and also medicines. It is divided as follows:

* This measure, and those which follow, are used for dry goods only.

† Since the publication of the last edition, the *heaped* measure, which was part of the new system, has been abolished. The following paragraph from the last edition, will serve for reference to it:

"The other imperial measure is applied to goods which it is customary to sell by *heaped measure*, and is as follows:

2 gallons	1 peck
4 pecks	1 bushel
3 bushels	1 sack
12 sacks	1 chaldron.

The gallon and bushel in this measure hold the same, when only just filled, as in the last. The bushel, however, heaped up as directed by the act of parliament, is a little more than one-fourth greater than before."

‡ Pure water, cleared from foreign substances by distillation, at a temperature of 62° Fahr.

AVERDUPOIS WEIGHT.

$27\frac{11}{32}$ grains	are	1 dram	1 dr.
16 *drams*, or *drachms*		1 *ounce**	1 oz.
16 *ounces*		1 *pound*	1 lb.
28 *pounds*		1 *quarter*	1 qr.
4 *quarters*		1 *hundred-weight*	1 cwt.
20 *hundred-weight*		1 *ton*	1 ton.

The pound averdupois contains 7000 grains. A cubic foot of water weighs 62·3210606 pounds averdupois, or 997·1369691 ounces.

For the precious metals and for medicines, the pound troy, containing 5760 grains, is used, but is differently divided in the two cases. The measures are as follow:

TROY WEIGHT.

24 *grains*	are	1 *pennyweight*	1 dwt.
20 *pennyweights*		1 *ounce*	1 oz.
12 *ounces*		1 *pound*	1 lb.

The pound troy contain 5760 grains. A cubic foot of water weighs 75·7374 pounds troy, or 908·8488 ounces.

APOTHECARIES' WEIGHT.

20 *grains*	are	1 *scruple*	℈
3 *scruples*		1 *dram*	ʒ
8 *drams*		1 *ounce*	℥
12 *ounces*		1 *pound*	℔

218. The standard coins of copper, silver, and gold, are,—the penny, which is $10\frac{2}{3}$ drams of copper; the shilling, which weighs 3 pennyweights 15 grains, of which 3 parts out of 40 are alloy, and the rest pure silver; and the sovereign, weighing 5 pennyweights, and $3\frac{1}{4}$ grains, of which one part out of 12 is copper, and the rest pure gold.

* It is more common to divide the ounce into four quarters than into sixteen drams.

MEASURES OF MONEY.

The lowest coin is a farthing, which is marked thus, $\frac{1}{4}$, being one fourth of a penny.

2 *farthings*	are	1 *halfpenny*	....	$\frac{1}{2}$*d.*
2 *halfpence*	....	1 *penny*	....	1*d.*
12 *pence*	....	1 *shilling*	....	1*s.*
20 *shillings*	....	1 *pound** or *sovereign*	...	£1
21 *shillings*	....	1 *guinea.*†		

219. When any quantity is made up of several others, expressed in different units, such as £1 . 14 . 6, or 2 cwt. 1 qr. 3 lbs., it is called a *compound quantity.* From these tables it is evident that any compound quantity of any substance can be measured in several different ways. For example, the sum of money which we call five pounds four shillings is also 104 shillings, or 1248 pence, or 4992 farthings. It is easy to reduce any quantity from one of these measurements to another; and the following examples will be sufficient to shew how to apply the same process, usually called REDUCTION, to all sorts of quantities.

I. How many farthings are there in £18 . 12 . $6\frac{3}{4}$‡?

Since there are 20 shillings in a pound, there are, in £18, 18×20, or 360 shillings; therefore, £18 . 12 is $360 + 12$, or 372 shillings. Since there are 12 pence in a shilling, in 372 shillings there are 372×12, or 4464 pence; and, therefore, in £18 . 12 . 6 there are $4464 + 6$, or 4470 pence.

Since there are 4 farthings in a penny, in 4470 pence there are 4470×4, or 17880 farthings; and, therefore, in £18 . 12 . $6\frac{3}{4}$, there are $17880 + 3$, or 17883 farthings. The whole of this process may be written as follows:

* The English pound is generally called a *pound sterling*, which distinguishes it from the weight called a pound, and also from foreign coins.

† The coin called a guinea is now no longer in use, but the name is still given, from custom, to 21 shillings. The pound, which was not a coin, but a note promising to pay 20 shillings to the bearer, is also disused for the present, and the sovereign supplies its place; but the name pound is still given to 20 shillings.

‡ Farthings are never written but as parts of a penny. Thus, three farthings, being $\frac{3}{4}$ of a penny, is written $\frac{3}{4}$, or $^3/_4$. One halfpenny may be written either as $\frac{2}{4}$ or $\frac{1}{2}$; the latter is most common.

$$
\begin{array}{l}
£18 \,.\, 12 \,.\, 6\frac{3}{4} \\
\underline{20} \\
360+12 = 372 \\
\qquad\qquad \underline{12} \\
\qquad\qquad 4464+6 = 4470 \\
\qquad\qquad\qquad\qquad \underline{4} \\
\qquad\qquad\qquad\qquad 17880+3 = 17883
\end{array}
$$

II. In 17883 farthings, how many pounds, shillings, pence, and farthings are there?

Since 17883, divided by 4, gives the quotient 4470, and the remainder 3, 17883 farthings are 4470 pence and 3 farthings (218).

Since 4470, divided by 12, gives the quotient 372, and the remainder 6, 4470 pence is 372 shillings and 6 pence.

Since 372, divided by 20, gives the quotient 18, and the remainder 12, 372 shillings is 18 pounds and 12 shillings.

Therefore, 17883 farthings is $4470\frac{3}{4}$*d.*, which is 372*s.* $6\frac{3}{4}$*d.*, which is £18 . 12 . $6\frac{3}{4}$.

The process may be written as follows:

$$
\begin{array}{l}
4)\underline{17883} \\
12)\underline{4470} \ldots 3 \\
20)\underline{372} \ldots 6 \\
\quad £18 \,.\, 12 \,.\, 6\frac{3}{4}
\end{array}
$$

EXERCISES.

A has £100 . 4 . $11\frac{1}{2}$, and B has 64392 farthings. If A receive 1492 farthings, and B £1 . 2 . $3\frac{1}{2}$, which will then have the most, and by how much?—*Answer*, A will have £33 . 12 . 3 more than B.

In the following table, the quantities written opposite to each other are the same: each line furnishes two exercises.

£15 . 18 . $9\frac{1}{2}$	15302 farthings.
$115^{lbs}\ 1^{oz}\ 8^{dwt}$	663072 grains.
$3^{lbs}\ 14^{oz}\ 9^{dr}$	1001 drams.
$3^{m}\ 149^{yds}\ 2^{ft}\ 9^{in}$	195477 inches.
$19^{bu}\ 2^{pks}\ 1^{gall}\ 2^{qts}$	1260 pints.
$16^{h}\ 23^{m}\ 47^{s}$	59027 seconds.

220. The same may be done where the number first expressed is fractional. For example, how many shillings and pence are there in $\frac{4}{15}$ of a pound? Now, $\frac{4}{15}$ of a pound is $\frac{4}{15}$ of 20 shillings; $\frac{4}{15}$ of 20 is $\frac{4 \times 20}{15}$, or $\frac{4 \times 4}{3}$ (110), or $\frac{16}{3}$, or (105) $5\frac{1}{3}$ of a shilling. Again, $\frac{1}{3}$ of a shilling is $\frac{1}{3}$ of 12 pence, or 4 pence. Therefore £$\frac{4}{15}$ = 5*s.* 4*d.*

Also, ·23 of a day is ·23 × 24 in hours, or 5^{h}·52; and ·52 of an hour is ·52 × 60 in minutes, or 31^{m}·2; and ·2 of a minute is ·2 × 60 in seconds, or 12^{s}; whence ·23 of a day is $5^{h}\ 31^{m}\ 12^{s}$.

Again, suppose it required to find what part of a pound 6*s.* 8*d.* is. Since 6*s.* 8*d.* is 80 pence, and since the whole pound contains 20 × 12, or 240 pence, 6*s.* 8*d.* is made by dividing the pound into 240 parts, and taking 80 of them. It is therefore £$\frac{80}{240}$ (107), but $\frac{80}{240} = \frac{1}{3}$ (108); therefore, 6*s.* 8*d.* = £$\frac{1}{3}$.

EXERCISES.

$\frac{2}{5}$ of a day ... is ... $9^{h}\ 36^{m}$

·12841 of a day ...... $3^{h}\ 4^{m}\ 54^{s}$·624*

·257 of a cwt. ...... $28^{lbs}\ 12^{oz}\ 8^{dr}$·704

£·14936 .... $2^{s}\ 11^{d}\ 3^{f}$·3856.

221. The most necessary species of calculation is that which is made in pounds, shillings, and pence; since the merchant generally uses the value of a commodity in money, in preference to the measure of its quantity. It is often convenient to reduce shillings, pence, and farthings, into an equivalent decimal of a pound. With the shillings there is no difficulty, since 1*s.* is £$\frac{5}{100}$. Hence, the rule for reducing any number of shillings is,—Multiply the number of shillings by 5, and make two decimal places. Thus, 17*s.* is £·85. On finding what decimal one farthing is of a pound, we have ·001041666, &c. (220). The occurrence of a cipher between the 1 and the 4 furnishes a very simple rule for converting any number of farthings under a shilling into the nearest less decimal of three places. For, ·001041666 is

* When a decimal follows a whole number, the decimal is always of the same unit as the whole number. Thus, 5″·5 is five *seconds* and five-tenths of a *second.* Thus, 0″·5 means five-tenths of a second; 0^{h}·3, three-tenths of an hour.

·001 + ·0000417 very nearly (151); hence, as far as three decimal places, we may call one farthing the thousandth part, two farthings two thousandth parts, and so on, until the number of farthings is so great that the multiplication of ·0000417 puts a figure in the third place of decimals, which can never happen until 24 farthings are taken, in which case (·001 + ·0000417) × 24 gives ·024 + ·001, very nearly, or ·025. From this up to 48 farthings, only one is carried to the third place, which gives the following rule: To convert any number of pence and farthings under a shilling into the equivalent decimal of a pound, convert the shillings and pence into farthings, add one, if their number be 24 or upwards, and make three decimal places. Thus $2\frac{1}{4}$*d.* is £·009; $10\frac{3}{4}$ is £·044.

EXAMPLE.

What decimal of a pound is 15*s.* $7\frac{1}{2}$*d.*? Here, 15*s.* is ·75, and $7\frac{1}{2}$*d.* is ·031; therefore, 15*s.* $7\frac{1}{2}$*d.* is ·75 + ·031, or £·781.

The reverse rule, to convert a decimal of a pound into shillings, pence, and farthings, is as follows: Double the first decimal place, and add one if the second place be 5 or upwards; the result is the number of shillings. Form a number out of the second decimal figure, or what is left of it, after subtracting 5, and the third; subtract one, if this number be 24 or upwards, and the result is the number of farthings, which must be turned into pence and farthings.

EXAMPLE.

What is £·376? Divide this into ·35 and ·026. The first place doubled, and increased by unity, gives 7 for the number of shillings; the number in the second, 26, diminished by 1, gives 25 for the number of farthings.

Hence,	£·376	is	7*s.*	$6\frac{1}{4}$*d.*
	£·064	......	1*s.*	$3\frac{1}{2}$*d.*
	£·998	......	19*s.*	$11\frac{3}{4}$*d.*
	£·774	......	15*s.*	$5\frac{3}{4}$*d.*
	£·357	......	7*s.*	$1\frac{3}{4}$*d.*

222. For calculations which require more correctness, we subjoin the opposite table: it is, however, tolerably easy to convert any number of shillings, pence, and farthings, into the *exact* decimal of a pound, by allowing 5000 for every shilling, 2500 for sixpence, 1250 for three pence, &c., and $104\frac{1}{6}$ for every farthing; and finally, by making 5 decimal places. Thus, to find 17*s.* $10\frac{3}{4}$*d.*, we have,

For	16*s.*	allow	80,000
...	1*s.*	......	5,000
...	6*d.*	......	2,500
...	3*d.*	......	1,250
...	$1\frac{3}{4}$	......	$729\frac{1}{6}$

17*s.* $10\frac{3}{4}$*d.* is £$\cdot 89479\frac{1}{6} = \cdot 894791666$ .....

	0	3	6	9	Com. Part.
	0000	0125	0250	0375	000
$\frac{1}{4}$	0010	0135	0260	0385	416
$\frac{1}{2}$	0020	0145	0270	0395	833
$\frac{3}{4}$	0031	0156	0281	0406	250
	1	4	7	10	
	0041	0166	0291	0416	666
$\frac{1}{4}$	0052	0177	0302	0427	083
$\frac{1}{2}$	0062	0187	0312	0437	500
$\frac{3}{4}$	0072	0197	0322	0447	916
	2	5	8	11	
	0083	0208	0333	0458	333
$\frac{1}{4}$	0093	0218	0343	0468	750
$\frac{1}{2}$	0104	0229	0354	0479	166
$\frac{3}{4}$	0114	0239	0364	0489	583

To turn any number of pence and farthings into a decimal of a pound, look between the thick lines for the pence, and then look down the column until you come opposite the farthings: the first four figures of the decimal are there; the rest are on the same line, in the column marked *common part*. If more places be wanted, the last figure of the common part must be repeated as often as may be necessary. For example, $2\frac{1}{2}$*d.* is £·010416666, &c. This table will be found convenient, when, as often happens, the price of one hundred, one thousand, &c., is required, the price of one being given. For example, what do ten thousand cost, where one costs £1 . 15 . $6\frac{3}{4}$? Here, 15*s.*

is ·75; by the table $6\frac{3}{4}d.$ is ·028125. The price of one is therefore 1·778125; and that of ten thousand is 1·778125 × 10,000, or (141) 17781·25, or £17781 . 5.

223. The rule of addition* of two compound quantities of the same sort will be evident from the following example. Suppose it required to add £192 . 14 . $2\frac{1}{2}$ to £64 . 13 . $11\frac{3}{4}$, the sum of these two is the whole of that which arises from adding their several parts. Now

$\frac{3}{4}d.$ +	$\frac{1}{2}d.$ =	$\frac{5}{4}d.$ =	£0 . 0 . $1\frac{1}{4}$	(219)
11*d.* +	2*d.* =	13*d.* =	0 . 1 . 1	
13*s.* +	14*s.* =	27*s.* =	1 . 7 . 0	
£64 +	£192 =		256 . 0 . 0	
The sum of all of which is,			£257 . 8 . $2\frac{1}{4}$	

This may be done at once, and written as follows:

£192 .	14 .	$2\frac{1}{2}$
64 .	13 .	$11\frac{3}{4}$
£257 .	8 .	$2\frac{1}{4}$

Begin by adding together the farthings, and reduce the result to pence and farthings. Set down the last only, carry the first to the line of pence, and add the pence in both lines to it. Reduce the sum to shillings and pence; set down the last only, and carry the first to the line of shillings, and so on. The same method must be followed when the quantities are of any other sort; and if the tables are kept in memory, the process will be easy.

224. Subtraction is performed on the same principle as in (40), namely, that the difference of two quantities is not altered by adding the same quantity to both. Suppose it required to subtract £19 . 13 . $10\frac{3}{4}$ from £24 . 5 . $7\frac{1}{2}$. Write these quantities under one another thus:—

* Before reading this article and the next, articles (29) and (42) should be read again carefully.

$$£24 \cdot 5 \cdot 7\frac{1}{2}$$

$$19 \cdot 13 \cdot 10\frac{3}{4}$$

Since $\frac{3}{4}$ cannot be taken from $\frac{1}{2}$ or $\frac{2}{4}$, add 1*d.* to both quantities, which will not alter their difference; or, which is the same thing, add 4 farthings to the first, and 1*d.* to the second. The pence and farthings in the two lines then stand thus: $7\frac{6}{4}d.$ and $11\frac{3}{4}d.$ Now subtract $\frac{3}{4}$ from $\frac{6}{4}$, and the difference is $\frac{3}{4}$, which must be written under the farthings. Again, since 11*d.* cannot be subtracted from 7*d.*, add 1*s.* to both quantities by adding 12*d.* to the first, and 1*s.* to the second. The pence in the first line are then 19, and in the second 11, and the difference is 8, which write under the pence. Since the shillings in the lower line were increased by 1, there are now 14*s.* in the lower, and 5*s.* in the upper one. Add 20*s.* to the upper and £1 to the lower line, and the subtraction of the shillings in the second from those in the first leaves 11*s.* Again, there are now £20 in the lower, and £24 in the upper line, the difference of which is £4; therefore the whole difference of the two sums is £4 . 11 . $8\frac{3}{4}$. If we write down the two sums with all the additions which have been made, the process will stand thus:—

$$£24 \cdot 25 \cdot 19\frac{6}{4}$$

$$20 \cdot 14 \cdot 11\frac{3}{4}$$

$$\text{Difference } £4 \cdot 11 \cdot 8\frac{3}{4}$$

225. The same method may be applied to any of the quantities in the tables. The following is another example:

From 7 cwt. 2 qrs. 21 lbs. 14 oz.
Subtract 2 cwt. 3 qrs. 27 lbs. 12 oz.

After alterations have been made similar to those in the last article, the question becomes —

From 7 cwt. 6 qrs. 49 lbs. 14 oz.
Subtract 3 cwt. 4 qrs. 27 lbs. 12 oz.

The difference is 4 cwt. 2 qrs. 22 lbs. 2 oz.

In this example, and almost every other, the process may be a little shortened in the following way. Here we do not subtract 27 lbs. from 21 lbs., which is impossible, but we increase 21 lbs. by 1 qr. or 28 lbs. and then subtract 27 lbs. from the sum. It would be shorter, and lead to the same result, first to subtract 27 lbs. from 1 qr. or 28 lbs. and add the difference to 21 lbs.

226. EXERCISES.

A man has the following sums to receive — £193 . 14 . $11\frac{1}{4}$, £22 . 0 . $6\frac{3}{4}$, £6473 . 0 . 0, and £49 . 14 . $4\frac{1}{2}$; and the following debts to pay — £200 . 19 . $6\frac{1}{4}$, £305 . 16 . 11, £22, and £19 . 6 . $0\frac{1}{2}$. How much will remain after paying the debts?

Answer, £6190 . 7 . $4\frac{3}{4}$.

There are four towns, in the order A, B, C, and D. If a man can go from A to B in $5^h\,20^m\,33^s$, from B to C in $6^h\,49^m\,2^s$, and from A to D in $19^h\,0^m\,17^s$, how long will he be in going from B to D, and from C to D?

Answer, $13^h\,39^m\,44^s$, and $6^h\,50^m\,42^s$.

227. In order to perform the process of MULTIPLICATION, it must be recollected that, as in (52), if a quantity be divided into several parts, and each of these parts be multiplied by a number, and the products be added, the result is the same as would arise from multiplying the whole quantity by that number.

It is required to multiply £7 . 13 . $6\frac{1}{4}$ by 13. The first quantity is made up of 7 pounds, 13 shillings, 6 pence, and 1 farthing. And

1 farth.	× 13	is	13 farth.	or	£0 . 0 . $3\frac{1}{4}$ (219)
6 pence	× 13	is	78 pence,	or	0 . 6 . 6
13 shill.	× 13	is	169 shill.	or	8 . 9 . 0
7 pounds	× 13	is	91 pounds,	or	91 . 0 . 0

The sum of all these is £99 . 15 . $9\frac{1}{4}$

which is therefore £7 . 13 . $6\frac{1}{4}$ × 13.

This process is usually written as follows:—

$$\begin{array}{r} £7 \;.\; 13 \;.\; 6\frac{1}{4} \\ 13 \\ \hline £99 \;.\; 15 \;.\; 9\frac{1}{4} \end{array}$$

228. Division is performed upon the same principle as in (74), viz. that if a quantity be divided into any number of parts, and each part be divided by any number, the different quotients added together will make up the quotient of the whole quantity divided by that number. Suppose it required to divide £99 . 15 . $9\frac{1}{4}$ by 13. Since 99 divided by 13 gives the quotient 7, and the remainder 8, the quantity is made up of £13 × 7, or £91, and £8 . 15 . $9\frac{1}{4}$. The quotient of the first, 13 being the divisor, is £7: it remains to find that of the second. Since £8 is 160*s*., £8 . 15 . $9\frac{1}{4}$ is 175*s*. $9\frac{1}{4}$*d*., and 175 divided by 13 gives the quotient 13, and the remainder 6; that is, 175*s*. $9\frac{1}{4}$*d*. is made up of 169*s*. and 6*s*. $9\frac{1}{4}$*d*., the quotient of the first of which is 13*s*., and it remains to find that of the second. Since 6*s*. is 72*d*., 6*s*. $9\frac{1}{4}$*d*. is $81\frac{1}{4}$*d*., and 81 divided by 13 gives the quotient 6 and remainder 3; that is, $81\frac{1}{4}$*d*. is 78*d*. and $3\frac{1}{4}$*d*., of the first of which the quotient is 6*d*. Again, since 3*d*. is $\frac{12}{4}$, or 12 farthings, $3\frac{1}{4}$*d*. is 13 farthings, the quotient of which is 1 farthing, or $\frac{1}{4}$, without remainder. We have then divided £99 . 15 . $9\frac{1}{4}$*d*. into four parts, each of which is divisible by 13, viz. £91, 169*s*., 78*d*., and 13 farthings; so that the thirteenth part of this quantity is £7 . 13 . $6\frac{1}{4}$. The whole process may be written down as follows; and the same sort of process may be applied to the exercises which follow:

$$
\begin{array}{l}
\qquad\quad \text{£.}\ \ s.\ \ d.\ \ \text{£.}\ \ s.\ \ d.\\
13)99\ \ 15\ \ 9\frac{1}{4}(7\ \ 13\ \ 6\frac{1}{4}\\
\quad\ \underline{91}\\
\qquad 8\\
\quad\ \underline{\ 20}\\
\quad 160+15=175\\
\qquad\qquad\qquad \underline{13}\\
\qquad\qquad\qquad\ 45\\
\qquad\qquad\qquad\ \underline{39}\\
\qquad\qquad\qquad\quad 6\\
\qquad\qquad\qquad\ \underline{12}\\
\qquad\qquad\qquad 72+9=81\\
\qquad\qquad\qquad\qquad\qquad \underline{78}\\
\qquad\qquad\qquad\qquad\qquad\ \ 3\\
\qquad\qquad\qquad\qquad\qquad\ \ \underline{4}\\
\qquad\qquad\qquad\qquad\qquad 12+1=13\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{13}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

Here, each of the numbers 99, 175, 81, and 13, is divided by 13 in the usual way, though the divisor is only written before the first of them.

EXERCISES.

$$2 \text{ cwt. } 1 \text{ qr. } 21 \text{ lbs. } 7 \text{ oz.} \times 53 = 129 \text{ cwt. } 1 \text{ qr. } 16 \text{ lbs. } 3 \text{ oz.}$$
$$2^d\,4^h\,3^m\,27^s \times 109 = 236^d\,10^h\,16^m\,3^s$$
$$£27 . 10 . 8 \times 569 = £15666 . 9 . 4$$
$$£7 . 4 . 8 \times 123 = £889 . 14$$
$$£166 \times \frac{8}{33} = £40 . 4 . 10\frac{6}{33}$$
$$£187 . 6 . 7 \times \frac{3}{100} = £5 . 12 . 4\frac{3}{4}\frac{2}{25}$$
$$4s.\ 6\frac{1}{2}d. \times 1121 = £254 . 11 . 2\frac{1}{2}$$
$$4s.\ 4d. \times 4260 = £6s.\ 6d. \times 2840$$

229. Suppose it required to find how many times 1*s.* $4\frac{1}{4}$*d.* is contained in £3 . 19 . $10\frac{3}{4}$. The way to do this is to find the number of farthings in each. By (219), in the first there are 65, and in the second, 3835 farthings. Now, 3835 contains 65, 59 times; and, therefore, the second quantity is 59 times as great as the first. In the case, however, of pounds, shillings, and pence, it would be best to use decimals of a pound, which will give a sufficiently exact answer. Thus 1*s.* $4\frac{1}{4}$*d.* is ·067, and £3 . 19 . $10\frac{3}{4}$ is £3·994, and 3·994 divided by ·067 is 3994 by 67, or $59\frac{41}{67}$. This is an extreme case, for the smaller the divisor, the greater the effect of an error in a given place of decimals.

EXERCISES.

How many times does 6 cwt. 2 qrs. contain 1 qr. 14 lbs. 1 oz? and $1^d\,2^h\,0^m\,47^s$ contain $3^m\,46^s$? *Answer*, 17·30758 and 414·367257.

If 2 cwt. 3 qrs. 1 lb. cost £150 . 13 . 10, how much does 1 lb. cost? *Answer*, 9*s.* 9*d.* $\frac{13}{309}$.

A grocer mixes 2 cwt. 15 lbs. of sugar at 11*d.* per pound with 14 cwt. 3 lbs at 5*d.* per pound. At how much per pound must he sell the mixture so as not to lose by mixing them? *Answer*, 5*d.* $\frac{3}{4}\frac{153}{905}$.

230. There is a convenient method of multiplication called PRACTICE. Suppose I ask, How much do 153 tons cost, if each ton cost

£2 . 15 . $7\frac{1}{2}$? It is plain that if this sum be multiplied by 153, the product is the price of the whole. But this is also evident, that, if I buy 153 tons at £2 . 15 . $7\frac{1}{2}$ each ton, payment may be made by first putting down £2 for each ton, then 10*s.* for each, then 5*s.*, then 6*d.*, and then $1\frac{1}{2}d.$ These sums together make up £2 . 15 . $7\frac{1}{2}d.$ and the reason for this separation of £2 . 15 . $7\frac{1}{2}$ into different parts will be soon apparent. The process may be carried on as follows:

	£	s.	d.
1. 153 tons, at £2 each ton, will cost	£306	0	0
2. Since 10*s.* is £$\frac{1}{2}$, 153 tons, at 10*s.* each, will cost £$\frac{153}{2}$, which is	76	10	0
3. Since 5*s.* is $\frac{1}{2}$ of 10*s.*, 153 tons, at 5*s.*, will cost half as much as the same number at 10*s.* each, that is, $\frac{1}{2}$ of £76 . 10*s.*, which is	38	5	0
4. Since 6*d.* is $\frac{1}{10}$ of 5*s.*, 153 tons, at 6*d.* each, will cost $\frac{1}{10}$ of what the same number costs at 5*s.* each, that is, $\frac{1}{10}$ of £38 . 5, which is..............	3	16	6
5. Since $1\frac{1}{2}$ or 3 halfpence is $\frac{1}{4}$ of 6*d.* or 12 halfpence, 153 tons, at $1\frac{1}{2}d.$ each, will cost $\frac{1}{4}$ of what the same number costs at 6*d.* each, that is, $\frac{1}{4}$ of £3 . 16 . 6, which is	0	19	$1\frac{1}{2}$
The sum of all these quantities is	425	10	$7\frac{1}{2}$

which, is, therefore, £2 . 15 . $7\frac{1}{2}$ × 153.

The whole process may be written down as follows:

	£153 0 0	or what 153 tons would cost at	£1 per ton.
£2 is 2 × £1	306 0 0		2 0 0
10*s.* is $\frac{1}{2}$ of £1	76 10 0		0 10 0
5*s.* is $\frac{1}{2}$ of 10*s.*	38 5 0		0 5 0
6*d.* is $\frac{1}{10}$ of 5*s.*	3 16 6		0 0 6
$1\frac{1}{2}d.$ is $\frac{1}{4}$ of 6*d.*	0 19 $1\frac{1}{2}$		0 0 $1\frac{1}{2}$
Sum......	£425 10 $7\frac{1}{2}$		£2 15 $7\frac{1}{2}$

ANOTHER EXAMPLE.

What do 1735 lbs. cost at 9*s*. $10\frac{3}{4}$*d*. per lb.? The price 9*s*. $10\frac{3}{4}$*d*. is made up of 5*s*., 4*s*., 10*d*., $\frac{1}{2}$*d*., and $\frac{1}{4}$*d*.; of which 5*s*. is $\frac{1}{4}$ of £1, 4*s*. is $\frac{1}{5}$ of £1, 10*d*. is $\frac{1}{6}$ of 5*s*., $\frac{1}{2}$*d*. is $\frac{1}{20}$ of 10*d*., and $\frac{1}{4}$*d*. is $\frac{1}{2}$ of $\frac{1}{2}$*d*. Follow the same method as in the last example, which gives the following:

	£	s.	d.		£	s.	d.
	£1735	0	0		£1 per lb.		
5*s*. is $\frac{1}{4}$ of £1	433	15	0	or what 1735 lbs. would cost at	0	5	0
4*s*. is $\frac{1}{5}$ of £1	347	0	0		0	4	0
10*d*. is $\frac{1}{6}$ of 5*s*.	72	5	10		0	0	10
$\frac{1}{2}$*d*. is $\frac{1}{20}$ of 10*d*.	3	12	$3\frac{1}{2}$		0	0	$0\frac{1}{2}$
$\frac{1}{4}$*d*. is $\frac{1}{2}$ of $\frac{1}{2}$*d*.	1	16	$1\frac{3}{4}$		0	0	$0\frac{1}{4}$
by addition ...	£ 858	9	$3\frac{1}{4}$		£0	9	$10\frac{3}{4}$

In all cases, the price must first be divided into a number of parts, each of which is a simple fraction* of some one which goes before. No rule can be given for doing this, but practice will enable the student immediately to find out the best method for each case. When that is done, he must find how much the whole quantity would cost if each of these parts were the price, and then add the results together.

EXERCISES.

What is the cost of

243 cwt. at £14 . 18 . $8\frac{1}{4}$ per cwt.?—*Answer*, £3629 . 1 . $0\frac{3}{4}$.

169 bushels at £2 . 1 . $3\frac{1}{4}$ per bushel?—*Answer*, £348 . 14 . $9\frac{1}{4}$.

273 qrs. at 19*s*. 2*d*. per quarter?—*Answer*, £261 . 12 . 6.

2627 sacks at 7*s*. $8\frac{1}{2}$*d*. per sack?—*Answer*, £1012 . 9 . $9\frac{1}{2}$.

* Any fraction of a unit, whose numerator is unity, is generally called an ***aliquot part*** of that unit. Thus, 2*s*. and 10*s*. are both aliquot parts of a pound, being £$\frac{1}{10}$ and £$\frac{1}{2}$.

231. Throughout this section, it must be observed, that the rules can be applied to cases where the quantities given are expressed in common or decimal fractions, instead of the measures in the tables. The following are examples:

What is the price of 272·3479 cwt. at £2 . 1 . $3\frac{1}{2}$ per cwt.?

Answer, £562·2849, or £562 . 5 . $8\frac{1}{4}$.

$66\frac{1}{2}$ lbs. at 1*s*. $4\frac{1}{2}$*d*. per lb. cost £4 . 11 . $5\frac{1}{4}$.

How many pounds, shillings, and pence, will 279·301 acres let for, if each acre lets for £3·1076?—*Answer*, £867·9558, or £867 . 19 . $1\frac{1}{4}$.

What does $\frac{1}{4}$ of $\frac{3}{13}$ of 17 bush. cost, at $\frac{1}{5}$ of $\frac{2}{3}$ of £17 . 14 per bushel?

Answer, £2·3146, or £2 . 6 . $3\frac{1}{2}$.

What is the cost of 19 lbs. 8 oz. 12 dwt. 8 gr. at £4 . 4 . 6 per ounce?—*Answer*, £999 . 14 . $1\frac{1}{4}\frac{1}{5}$.

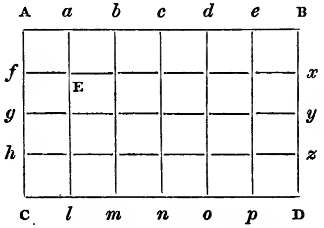
232. It is often required to find to how much a certain sum per day will amount in a year. This may be shortly done, since it happens that the number of days in a year is 240 + 120 + 5; so that a penny per day is a pound, half a pound, and 5 pence per year. Hence the following rule: To find how much any sum per day amounts to in a year, turn it into pence, and fractions of a penny; to this add the half of itself, and let the pence be pounds, and each farthing five shillings; then add five times the daily sum, and the total is the yearly amount. For example, what does 12*s*. $3\frac{3}{4}$*d*. amount to in a year? This is $147\frac{3}{4}$*d*., and its half is $73\frac{7}{8}$*d*., which added to $147\frac{3}{4}$*d*. gives $221\frac{5}{8}$*d*., which turned into pounds is £221 . 12 . 6. Also, 12*s*. $3\frac{3}{4}$*d*. × 5 is £3 . 1 . $6\frac{3}{4}$, which added to the former sum gives £224 . 14 . $0\frac{3}{4}$ for the yearly amount. In the same way, the yearly amount of 2*s*. $3\frac{1}{2}$*d*. is £41 . 16 . $5\frac{1}{2}$; that of $6\frac{3}{4}$*d*. is £10 . 5 . $3\frac{3}{4}$; and that of 11*d*. is £16 . 14 . 7.

233. An inverse rule may be formed, sufficiently correct for every purpose, in the following way. If the year consisted of 360 days, or $\frac{3}{2}$ of 240, the subtraction of one-third from any sum per year would give the proportion which belongs to 240 days; and every pound so obtained would be one penny per day. But as the year is not 360, but 365 days, if we divide each day's share into 365 parts, and take 5 away, the whole of the subtracted sum, or 360 × 5 such parts, will give 360 parts for each

of the 5 days which we neglected at first. But 360 such parts are left behind for each of the 360 first days; therefore, this additional process divides the whole annual amount equally among the 365 days. Now, 5 parts out of 365 is one out of 73; or the 73d part of the first result must be subtracted from it to produce the true result. Unless the daily sum be very large, the 72d part will do equally well, which, as 72 farthings are 18 pence, is equivalent to subtracting at the rate of one farthing for 18*d.*, or $\frac{1}{2}$*d.* for 3*s.*, or 10*d.* for £3. The rule, then, is as follows: To find how much per day will produce a given sum per year, turn the shillings, &c. in the given sum into decimals of a pound (221); subtract one-third; consider the result as pence; and diminish it by one farthing for every eighteen pence, or ten pence for every £3. For example, how much per day will give £224 . 14 . $0\frac{3}{4}$ per year? This is 224·703, and its third is 74·901, which subtracted from 224·703, gives 149·802, which, if they be pence, amounts to 12*s.* 5·802*d.*, in which 1*s.* 6*d.* is contained 8 times. Subtract 8 farthings, or 2*d.*, and we have 12*s.* 3·802*d.*, which differs from the truth only about $\frac{1}{20}$ of a farthing. In the same way, £100 per year is 5*s.* $5\frac{3}{4}$*d.* per day.

234. The following connexion between the measures of length and the measures of surface, is the foundation of the application of arithmetic to geometry.

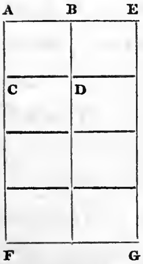
Suppose an oblong figure, A, B, C, D, as here drawn (which is called a *rectangle* in geometry), with the side A B 6 inches, and the side A C 4



inches. Divide A B and C D (which are equal) each into 6 inches by the points *a*, *b*, *c*, *l*, *m*, &c.; and A C and B D (which are also equal) into 4 inches by the points *f*, *g*, *h*, *x*, *y*, and *z*. Join *a* and *l*, *b* and *m*,

&c., and f and x, &c. Then, the figure A B C D is divided into a number of squares; for a square is a rectangle whose sides are equal, and therefore AafE is square, since Aa is of the same length as Af, both being 1 inch. There are also four rows of these squares, with six squares in each row; that is, there are 6×4, or 24 squares altogether. Each of these squares has its sides 1 inch in length, and is what was called in (215) *a square inch.* By the same reasoning, if one side had contained 6 *yards*, and the other 4 *yards*, the surface would have contained 6×4 *square yards*; and so on.

235. Let us now suppose that the sides of A B C D, instead of being a whole number of inches, contain some inches and a fraction. For example, let A B be $3\frac{1}{2}$ inches, or (114) $\frac{7}{2}$ of an inch, and let A C contain $2\frac{1}{4}$ inches, or $\frac{9}{4}$ of an inch. Draw A E twice as long as A B, and A F four times as long as A C, and complete the rectangle A E F G. The rest of the figure needs no description. Then, since A E is twice A B, or twice $\frac{7}{2}$ inches, it is 7 inches. And since A F is four times A C, or four times $\frac{9}{4}$ inches, it is 9 inches. Therefore, the whole rectangle A E F G contains, by (234), 7×9, or 63 square inches. But the rectangle A E F G contains 8 rectangles, all of the same figure as A B C D; and therefore A B C D is one-eighth part of A E F G, and contains $\frac{63}{8}$ square inches. But $\frac{63}{8}$ is made by multiplying $\frac{9}{4}$ and $\frac{7}{2}$ together (118). From this and the last article it appears, that, whether the sides of a rectangle be a whole or a fractional number of inches, the number of square inches in its surface is the product of the number of inches in its sides. The square itself is a rectangle whose sides are all equal, and therefore the number of square inches which a square contains is found by multiplying the number of inches in its side by itself. For example, a square whose side is 13 inches in length contains 13×13, or 169 square inches.



236. EXERCISES.

What is the content, in square feet and inches, of a room whose sides are 42 ft. 5 inch. and 31 ft. 9 inch. ? and supposing the piece from which its carpet is taken to be three quarters of a yard in breadth, what length of it must be cut off ?—*Answer*, The content is 1346 square feet 105 square inches, and the length of carpet required is 598 feet $6\frac{5}{9}$ inches.

The sides of a rectangular field are 253 yards and a quarter of a mile ; how many acres does it contain ?—*Answer*, 23.

What is the difference between 18 *square miles*, and a square of 18 miles long, or 18 *miles square* ?—*Answer*, 306 square miles.

237. It is by this rule that the measure in (215) is deduced from that in (214); for it is evident that twelve inches being a foot, the square foot is 12×12, or 144 square inches, and so on. In a similar way it may be shewn that the content in cubic inches of a cube, or parallelopiped,* may be found by multiplying together the number of inches in those three sides which meet in a point. Thus, a cube of 6 inches contains $6 \times 6 \times 6$, or 216 cubic inches ; a chest whose sides are 6, 8, and 5 feet, contains $6 \times 8 \times 5$, or 240 cubic feet. By this rule the measure in (216) was deduced from that in (214).

SECTION II.

RULE OF THREE.

238. Suppose it required to find what 156 yards will cost, if 22 yards cost 17*s.* 4*d.* This quantity, reduced to pence, is 208*d.*; and if 22 yards cost 208*d.*, each yard costs $\frac{208}{22}d.$ But 156 yards cost 156 times the price of one yard, and therefore cost $\frac{208}{22} \times 156$ pence, or $\frac{208 \times 156}{22}$ pence (117). Again, if $25\frac{1}{2}$ French francs are 20 shillings sterling, how many francs are in £20 15*s.*? Since $25\frac{1}{2}$ francs are 20 shillings, twice the number of francs must be twice the number of

* A parallelopiped, or more properly, a *rectangular* parallelopiped, is a figure of the form of a brick ; its sides, however, may be of any length ; thus, the figure of a plank has the same name. A cube is a parallelopiped with equal sides, such as is a die.

shillings; that is, 51 francs is 40 shillings, and one shilling is the fortieth part of 51 francs, or $\frac{51}{40}$ francs. But £20 15*s.* contain 415 shillings (219); and since 1 shilling is $\frac{51}{40}$ francs, 415 shillings is $\frac{51}{40} \times 415$ francs, or (117) $\frac{51 \times 415}{40}$ francs.

239. Such questions as the last two belong to the most extensive rule in Commercial Arithmetic, which is called the RULE OF THREE, because in it three quantities are given, and a fourth is required to be found. From both the preceding examples the following rule may be deduced, which the same reasoning will shew to apply to all similar cases.

It must be observed, that in these questions there are two quantities which are of the same sort, and a third of another sort, of which last the answer must be. Thus, in the first question there are 22 and 156 yards and 208 pence, and the thing required to be found is a number of pence. In the second question there are 20 and 415 shillings and $25\frac{1}{2}$ francs, and what is to be found is a number of francs. Write the three quantities in a line, putting that one last which is the only one of its kind, and that one first which is connected with the last in the question.* Put the third quantity in the middle. In the first question the quantities will be placed thus:

22 yds. 156 yds. 17*s.* 4*d.*

In the second question they will be placed thus:

20*s.* £20 15*s.* $25\frac{1}{2}$ francs.

Reduce the first and second quantities, if necessary, to quantities of the same denomination. Thus, in the second question, £20 15*s.* must be reduced to shillings (219). The third quantity may also be reduced to any other denomination, if convenient; or the first and third may

* This generally comes in the same member of the sentence. In some cases the ingenuity of the student must be employed in detecting it. The reasoning of (238) is the best guide. The following may be very often applied. If it is evident that the answer must be less than the given quantity of its kind, multiply that given quantity by the less of the other two; if greater, by the greater. Thus, in the first question, 156 yards must cost more than 22; multiply, therefore, by 156.

be multiplied by any quantity we please, as was done in the second question; and, on looking at the answer in (238), and at (108), it will be seen that no change is made by that multiplication. Multiply the second and third quantities together, and divide by the first. The result is a quantity of the same sort as the third in the line, and is the answer required. Thus, to the first question the answer is (238) $\frac{208 \times 156}{22}$ pence, or which is the same thing, $\frac{17s.\ 4d. \times 156}{22}$.

240. The whole process in the first question is as follows:*

```
Yds.    Yds.     s.  d.
 22  :  156  ::  17 . 4.
                 12
                ----
                208 pence.
                156
               -----
               1248
              1040
              208
             ------
          22) 32448 (1474¾d. and 14/22, or 7/11 of a farthing,
              22                or (219) £6 . 2 . 10¾ 7/11.
              ---
              104
               88
              ---
              164
              154
              ---
               108
                88
                --
                20
       (228)     4
                --
                80
                66
                --
                14
```

The question might have been solved without reducing 17*s.* 4*d.* to pence, thus:

* It is usual to place points, in the manner here shewn, between the quantities. Those who have read Section VIII. will see that the Rule of Three is no more than the process for finding the fourth term of a proportion from the other three.

```
Yds.   Yds.        s.  d.
 22  :  156  ::  17 . 4
                    156                          (227)
        22)£135 . 4 . 0(£6 . 2 . 10 3/4 7/11     (228)
           132
             3 × 20 + 4 = 64
                          44
                          20 × 12 = 240
                                    220
                                     20 × 4 = 80
                                              66
                                              14
```

The student must learn by practice which is the most convenient method for any particular case, as no rule can be given.

241. It may happen that the three given quantities are all of one denomination; nevertheless, it will be found that two of them are of one, and the third of another sort. For example: What must an income of £400 pay towards an income-tax of 4*s.* 6*d.* in the pound? Here the three given quantities are, £400, 4*s.* 6*d.*, and £1, which are all of the same species, viz. money. Nevertheless, the first and third are income; the second is a tax, and the answer is also a tax; and therefore, by (152), the quantities must be placed thus:

£1 : £400 :: 4*s.* 6*d.*

242. The following exercises either depend directly upon this rule, or can be shewn to do so by a little consideration. There are many questions of the sort, which will require some exercise of ingenuity before the method of applying the rule can be found.

EXERCISES.

If 15 cwt. 2 qrs. cost £198 . 15 . 4, what does 1 qr. 22 lbs. cost?
Answer, £5 . 14 . $5\frac{3}{4}\frac{185}{217}$.

If a horse go 14 m. 3 fur. 27 yds. in $3^h\ 26^m\ 12^s$, how long will he be in going 23 miles? *Answer*, $5^h\ 29^m\ 34^s\ \frac{2462}{25327}$.

Two persons, A and B, are bankrupts, and owe exactly the same sum; A can pay 15*s*. $4\frac{1}{2}$*d*. in the pound, and B only 7*s*. $6\frac{3}{4}$*d*. At the same time, A has in his possession £1304 . 17 more than B; what do the debts of each amount to? *Answer*, £3340 . 8 . $3\frac{3}{4}\frac{9}{25}$.

For every $12\frac{1}{2}$ acres which one country contains, a second contains $56\frac{1}{4}$. The second country contains 17,300 square miles. How much does the first contain? Again, for every 3 people in the first, there are 5 in the second; and there are in the first 27 people on every 20 acres. How many are there in each country?—*Answer*, The number of square miles in the first is $3844\frac{4}{9}$, and its population 3,321,600; and the population of the second is 5,536,000.

If $42\frac{1}{2}$ yds. of cloth, 18 in. wide, cost £59 . 14 . 2, how much will $118\frac{1}{4}$ yds. cost, if the width be 1 yd.? *Answer*, £332 . 5 . $2\frac{4}{17}$.

If £9 . 3 . 6 last six weeks, how long will £100 last?

Answer, $65\frac{145}{367}$ weeks.

How much sugar, worth $9\frac{3}{4}$*d*. a pound, must be given for 2 cwt. of tea, worth 10*d*. an ounce? *Answer*, 32 cwt. 3 qrs. 7 lbs. $\frac{35}{39}$.

243. Suppose the following question asked: How long will it take 15 men to do that which 45 men can finish in 10 days? It is evident that one man would take 45×10, or 450 days, to do the same thing, and that 15 men would do it in one-fifteenth part of the time which it employs one man, that is, in $\frac{450}{15}$, or 30 days. By this and similar reasoning the following questions can be solved.

EXERCISES.

If 15 oxen eat an acre of grass in 12 days, how long will it take 26 oxen to eat 14 acres? *Answer*, $96\frac{12}{13}$ days.

If 22 masons build a wall 5 feet high in 6 days, how long will it take 43 masons to build 10 feet? *Answer*, $6\frac{6}{43}$ days.

244. The questions in the preceding article form part of a more general class of questions, whose solution is called the Double Rule of Three, but which might, with more correctness, be called the Rule of *Five*, since five quantities are given, and a sixth is to be found. The following is an example: If 5 men can make 30 yards of cloth in 3 days, how long will it take 4 men to make 68 yards? The first

thing to be done is to find out, from the first part of the question, the time it will take one man to make one yard. Now, since one man, in three days, will do the fifth part of what 5 men can do, he will in 3 days make $\frac{30}{5}$, or 6 yards. He will, therefore, make one yard in $\frac{3}{6}$ or in $\frac{3 \times 5}{30}$ of a day. From this we are to find how long it will take 4 men to make 68 yards. Since one man makes a yard in $\frac{3 \times 5}{30}$ of a day, he will make 68 yards in $\frac{3 \times 5}{30} \times 68$ days, or (116) in $\frac{3 \times 5 \times 68}{30}$ days; and 4 men will do this in one-fourth of the time, that is (123), in $\frac{3 \times 5 \times 68}{30 \times 4}$ days, or in $8\frac{1}{2}$ days.

Again, suppose the question to be:—If 5 men can make 30 yards in 3 days, how much can 6 men do in 12 days? Here we must first find the quantity one man can do in one day, which appears, on reasoning similar to that in the last example, to be $\frac{30}{3 \times 5}$ yards. Hence, 6 men, in one day, will make $\frac{6 \times 30}{5 \times 3}$ yards, and in 12 days will make $\frac{12 \times 6 \times 30}{5 \times 3}$, or 144 yards.

From these examples the following rule may be drawn. Write the given quantities in two lines, keeping quantities of the same sort under one another, and those which are connected with each other in the same line. In the two examples above given, the quantities must be written thus:

5 men 30 yds. 3 days.

4 men. 68 yds.

SECOND EXAMPLE.

5 men. 30 yds. 3 days.

6 men. 12 days.

Draw a curve through the middle of each line, and the extremities of the other. There will be three quantities on one curve, and two on the other. Divide the product of the three by the product of the two, and the quotient is the answer to the question.

If necessary, the quantities in each line must be reduced to more simple denominations (219), as was done in the common Rule of Three (238).

EXERCISES.

If 6 horses can, in 2 days, plough 17 acres, how many acres will 93 horses plough in $4\frac{1}{2}$ days? *Answer*, $592\frac{7}{8}$.

If 20 men, in $3\frac{1}{4}$ days, can dig 7 rectangular fields, the sides of each of which are 40 and 50 yards, how long will 37 men be in digging 53 fields, the sides of each of which are 90 and $125\frac{1}{2}$ yards?

Answer, $75\frac{2451}{20720}$ days.

If the carriage of 60 cwt. through 20 miles, cost £14 10*s.* what weight ought to be carried 30 miles for £5 . 8 . 9? *Answer*, 15 cwt.

If £100 gain £5 in a year, how much will £850 gain in 3 years and 8 months? *Answer*, £155 . 16 . 8.

SECTION III.

INTEREST, ETC.

245. In the questions contained in this Section, almost the only process which will be employed, is the taking a fractional part of a sum of money, which has been done before in several cases. Suppose it required to take 7 parts out of 40 from £16, that is, to divide £16 into 40 equal parts, and take 7 of them. Each of these parts is £$\frac{16}{40}$, and 7 of them make $\frac{16}{40} \times 7$, or $\frac{16 \times 7}{40}$ pounds (116). The process may be written as in the accompanying example:

$$
\begin{array}{l}
\pounds16 \\
\underline{\quad 7} \\
40)112(\pounds2\ .\ 16s. \\
\quad \underline{80} \\
\quad 32 \\
\quad \underline{\ 20} \\
\quad 640 \\
\quad \underline{40} \\
\quad 240 \\
\quad \underline{240} \\
\quad\ 0
\end{array}
$$

Suppose it required to take 13 parts out of a hundred from £56 . 13 . $7\frac{1}{2}$.

$$
\begin{array}{l}
\quad 56\ .\ 13\ .\ 7\frac{1}{2} \\
\quad \underline{\qquad\qquad\quad 13} \\
100)736\ .\ 17\ .\ 1\frac{1}{2}(\pounds7\ .\ 7\ .\ 4\frac{1}{4}\ \frac{41}{50} \\
\quad \underline{700} \\
\quad 36 \times 20 + 17 = 737 \\
\qquad\qquad\qquad\quad \underline{700} \\
\qquad\qquad\qquad\quad 37 \times 12 + 1 = 445 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{400} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 45 \times 4 + 2 = 182 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{100} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 82
\end{array}
$$

Let it be required to take $2\frac{1}{2}$ parts out of a hundred from £3 12*s.* The result, by the same rule, is $\frac{\pounds3\ 12s. \times 2\frac{1}{2}}{100}$, or (123) $\frac{\pounds3\ 12s. \times 5}{200}$; so that taking $2\frac{1}{2}$ out of a hundred is the same as taking 5 parts out of 200.

EXERCISES.

Take $7\frac{1}{3}$ parts out of 53 from £1 10*s.* *Answer*, 4*s.* $1\frac{129}{159}$*d.*

Take 5 parts out of 100 from £107 13*s.* $4\frac{3}{4}$*d.*

Answer, £5 . 7 . 8 and $\frac{3}{20}$ of a farthing.

£56 3*s.* 2*d.* is equally divided among 32 persons. How much does the share of 23 of them exceed that of the rest?

Answer, £24 . 11 . $4\frac{1}{2}\frac{1}{2}$.

246. It is usual, in mercantile business, to mention the fraction which one sum is of another, by saying how many parts out of a hundred must be taken from the second in order to make the first. Thus, instead of saying that £16 12*s*. is the half of £33 4*s*., it is said that the first is 50 per cent of the second. Thus, £5 is $2\frac{1}{2}$ per cent of 200; because, if £200 be divided into 100 parts, $2\frac{1}{2}$ of those parts are £5. Also, £13 is 150 per cent of £8 . 13 . 4, since the first is the second, and half the second. Suppose it asked, How much per cent is 23 parts out of 56 of any sum? The question amounts to this: If he who has £56 gets £100 for them, how much will he who has 23 receive? This, by (238), is $\frac{23 \times 100}{56}$, or $\frac{2300}{56}$, or $41\frac{1}{14}$. Hence, 23 out of 56 is $41\frac{1}{14}$ per cent.

Similarly, 16 parts out of 18 is $\frac{16 \times 100}{18}$, or $88\frac{8}{9}$ per cent, and 2 parts out of 5 is $\frac{2 \times 100}{5}$, or 40 per cent.

From which the method of reducing other fractions to the rate per cent is evident.

Suppose it asked, How much per cent is £6 . 12 . 2 of £12 . 3? Since the first contains 1586*d*., and the second 2916*d*., the first is 1586 out of 2916 parts of the second; that is, by the last rule, it is $\frac{158600}{2916}$, or $54\frac{1136}{2916}$, or £54 . 7 . $9\frac{1}{2}$ per cent, very nearly. The more expeditious way of doing this is to reduce the shillings, &c. to decimals of a pound. The rule in (221) will usually give the rate per cent to the nearest shilling, which is near enough for all practical purposes. For instance, in the last example, which is to find how much £6·608 is of £12·15, 6·608 × 100 is 660·8, which divided by 12·15 gives £54·38, or £54 . 7. Greater correctness may be had, if necessary, by using the table in (222).

EXERCISES.

How much per cent is $198\frac{1}{4}$ out of 233 parts?—*Ans.* £85 . 1 . $8\frac{3}{4}$.

Goods which are bought for £193 . 12, are sold for £216 . 13 . 4; how much per cent has been gained by them?

Answer, A little less than £11 . 18 . 6.

A sells goods for B to the amount of £230 . 12, and is allowed a commission* of 3 per cent; what does that amount to?

Answer, £6 . 18 . $4\frac{1}{4}\frac{7}{25}$.

A stockbroker buys £1700 stock, brokerage being at £$\frac{1}{8}$ per cent; what does he receive? — *Answer*, £2 . 2 . 6.

A ship whose value is £15,423 is insured at $19\frac{2}{3}$ per cent; what does the insurance amount to? — *Answer*, £3033 . 3 . $9\frac{1}{2}\frac{2}{5}$.

247. In reckoning how much a bankrupt is able to pay his creditors, as also to how much a tax or rate amounts, it is usual to find how many shillings in the pound is paid. Thus, if a person who owes £100 can only pay £50, he is said to pay 10*s*. in the pound. The rule is easily derived from the same reasoning, as in (246). For example, £50 out of £82 is £$\frac{50}{82}$ out of £1, or $\frac{50 \times 20}{82}$ shillings, or 12*s*. $2\frac{1}{4}\frac{15}{41}$ in the pound.

248. Interest is money paid for the use of other money, and is always a per-centage upon the sum lent. It may be paid either yearly, half-yearly, or quarterly; but when it is said that £100 is lent at 4 per cent, it must be understood to mean 4 per cent per annum; that is, that 4 pounds are paid every year for the use of £100.

The sum lent is called the *principal*, and the interest upon it is of two kinds. If the borrower pay the interest as soon as, from the agreement, it becomes due, it is evident that he has to pay the same sum every year; and that the whole of the interest which he has to pay in any number of years, is one year's interest multiplied by the number of years. But if he does not pay the interest at once, but keeps it in his hands until he returns the principal, he will then have more of his

* Commission is what is allowed by one merchant to another for buying or selling goods for him, and is usually a per-centage on the whole sum employed. Brokerage is an allowance similar to commission, under a different name, principally used in the buying and selling of stock in the funds.

Insurance is a per-centage paid to those who engage to make good to the payers any loss they may sustain by accidents from fire, or storms, according to the agreement, up to a certain amount which is named, and is a per-centage upon this amount. Tare, tret, and cloff, are allowances made in selling goods by wholesale, for the weight of the boxes or barrels which contain them, waste, &c.; and are usually either the price of a certain number of pounds of the goods for each box or barrel, or a certain allowance on each cwt.

creditor's money in his hands every year, and (if it were so agreed) will have to pay interest upon each year's interest for the time during which he keeps it after it becomes due. In the first case, the interest is called *simple*, and in the second *compound*. The interest and principal together are called the *amount*.

249. What is the simple interest of £1049 . 16 . 6 for 6 years and one-third, at $4\frac{1}{2}$ per cent? This interest must be $6\frac{1}{3}$ times the interest of the same sum for one year, which (245) is found by multiplying the sum by $4\frac{1}{2}$, and dividing by 100. The process is as in the first column following:

(230)	(a)	£1049 . 16 . 6
	$a \times 4$	4199 . 6 . 0
	$a \times \frac{1}{2}$	524 . 18 . 3
(82)		100)47,24 . 4 . 3 (£47 . 4 . 10 $\frac{11}{100}$
		20
(228)		4,84*
		12
		10,11 †
(b)		£47 . 4 . 10 $\frac{11}{100}$ Int. for one yr.
	$b \times 6$	283 . 9 . 0 $\frac{66}{100}$
	$b \times \frac{1}{3}$	15 . 14 . 11 $\frac{37}{100}$
		£299 . 4 . 0 $\frac{3}{100}$ Int. for $6\frac{1}{3}$ yrs.

Second column:

1049·825
4
4199·300
524·913
100)4724·213
47·2421
6
283·4526
15·7474
299·2000
= £299 . 4 . 0

The process may be shortened by using decimals, as in the second column.

EXERCISES.

What is the interest of £105 . 6 . 2, for 19 years and 7 weeks, at 3 per cent? *Answer*, £60 . 9, very nearly.

* Here the 4*s*. from the dividend is taken in.

† Here the 3*d*. from the dividend is taken in.

What is the difference between the interest of £50 . 19, for 7 years, at 3 per cent, and for 8 years, at $2\frac{1}{2}$ per cent? *Answer*, 10*s*. $2\frac{1}{2}$*d*.

What is the interest of £157 . 17 . 6 for one year, at 5 per cent? *Answer*, £7 . 17 . $10\frac{1}{2}$.

Shew that the interest of any sum for 9 years, at 4 per cent, is the same as that of the same sum for 4 years at 9 per cent.

250. In order to find the interest of any sum at compound interest, it is necessary to find the amount of the principal and interest at the end of every year; because in this case (248) it is the amount of both principal and interest, at the end of the first year, for which interest accumulates during the second year. Suppose, for example, it is required to find the interest, for 3 years, on £100, at 5 per cent, compound interest. The following is the process:

	£100	First principal.
	5	First year's interest.
	105	Amount at the end of the first year.
(249)	5 . 5	Interest for the second year on £105.
	110 . 5	Amount at the end of two years.
	5 . 10 . 3	Interest due for the third year.
	115 . 15 . 3	Amount at the end of three years.
	100 . 0 . 0	First principal.
	15 . 15 . 3	Interest gained in the three years.

When the number of years is great, and the sum considerable, this process is very troublesome; on which account tables* are constructed to shew the amount of one pound, for different numbers of years, at different rates of interest. To make use of these tables in the present example, look into the column headed "5 per cent;" and opposite to the number 3, in the column headed "Number of years," is found 1·157625; meaning that £1 will become £1·157625 in 3 years. Now, £100 must become 100 times as great; and 1·157625 × 100 is 115·7625

* Sufficient tables for all common purposes are contained in the article on Interest, in the Penny Cyclopædia; and ample ones in the Treatise on Annuities and Reversions, in the Library of Useful Knowledge.

(141); but (221) £·7625 is 15*s.* 3*d.*; therefore the whole amount of £100 is £115 . 15 . 3, as before.

251. Suppose that a sum of money has lain at simple interest 4 years, at 5 per cent, and has, with its interest, amounted to £350; it is required to find what the sum was at first. Whatever the sum was, if we suppose it divided into 100 parts, 5 of those parts were added every year for 4 years, as interest; that is, 20 of those parts have been added to the first sum to make £350. If, therefore, £350 be divided into 120 parts, 100 of those parts are the principal which we want to find, and 20 parts are interest upon it; that is, the principal is £$\frac{350 \times 100}{120}$, or £291 . 13 . 4.

252. Suppose that A was engaged to pay B £350 at the end of four years from this time, and that it is agreed between them that the debt shall be paid immediately; suppose, also, that money can be employed at 5 per cent, simple interest; it is plain that A ought not to pay the whole sum, £350, because if he did, he would lose 4 years' interest of the money, and B would gain it. It is fair, therefore, that he should only pay to B as much as will, *with interest*, amount in four years to £350, that is (251), £291 . 13 . 4. Therefore, £58 . 6 . 8 must be struck off the debt in consideration of its being paid before the time. This is called DISCOUNT; and £291 . 13 . 4 is called the *present value* of £350 due four years hence, discount being at 5 per cent. The rule for finding the present value of a sum of money (251) is: Multiply the sum by 100, and divide the product by 100, increased by the product of the rate per cent and number of years. If the time that the debt has yet to run is expressed in years and months, or months only, the months must be reduced to the equivalent fraction of a year.

EXERCISES.

What is the discount on a bill of £138 . 14 . 4 due 2 years hence, discount being at $4\frac{1}{2}$ per cent? *Answer*, £11 . 9 . 1.

What is the present value of £1031 . 17 due 6 months hence, interest being at 3 per cent? *Answer*, £1016 . 12.

253. If we multiply by $a+b$, or by $a-b$, when we should multiply

by a, the result is wrong by the fraction $\frac{b}{a+b}$, or $\frac{b}{a-b}$, of itself: being too great in the first case and too small in the second. Again, if we divide by $a+b$, where we should have divided by a, the result is too small by the fraction $\frac{b}{a}$ of itself; while, if we divide by $a-b$ instead of a, the result is too great by the same fraction of itself. Thus, if we divide by 20 instead of 17, the result is $\frac{3}{17}$ of itself too small; and if we divide by 360 instead of 365, the result is too great by $\frac{5}{365}$, or $\frac{1}{73}$ of itself.

If then we wish to find the interest of a sum of money for a portion of a year, and have not the assistance of tables, it will be found convenient to suppose the year to contain only 360 days, in which case its 73rd part (the 72nd part will generally do) must be subtracted from the result, to make the alteration of 360 into 365. The number 360 has so large a number of divisors, that the rule of Practice (230) may always be readily applied. Thus, it is required to find the portion which belongs to 274 days, the yearly interest being £18 . 9 . 10, or 18·491.

274	18·491
180 is $\frac{1}{2}$ of 360	9·246
94	
90 is $\frac{1}{2}$ of 180	4·623
4 is $\frac{1}{90}$ of 360	·205
	9)14·074
	8)1·564
	·196
	13·878 = £13 . 17 . 7 *Answer.*

But if the nearest farthing be wanted, the best way is to take 2-tenths of the number of days as a multiplier, and 73 as a divisor; since $m \div 365$ is $2m \div 730$, or $\frac{2}{10}m \div 73$. Thus, in the preceding instance, we multiply by 54·8 and divide by 73; and $54{\cdot}8 \times 18{\cdot}491 =$ 1013·3068, which divided by 73 gives 13·881, very nearly agreeing with the former, and giving £13 . 17 . $7\frac{1}{2}$, which is certainly within a farthing of the truth.

254. Suppose it required to divide £100 among three persons in

such a way that their shares may be as 6, 5, and 9; that is, so that for every £6 which the first has, the second may have £5, and the third £9. It is plain that if we divide the £100 into 6+5+9, or 20 parts, the first must have 6 of those parts, the second 5, and the third 9. Therefore (245) their shares are respectively, £$\frac{100 \times 6}{20}$, £$\frac{100 \times 5}{20}$, and £$\frac{100 \times 9}{20}$, or £30, £25, and £45.

EXERCISES.

Divide £394 . 12 among four persons, so that their shares may be as 1, 6, 7, and 18.—*Answer,* £12 . 6 . 7$\frac{1}{2}$; £73 . 19 . 9; £86 . 6 . 4$\frac{1}{2}$; £221 . 19 . 3.

Divide £20 among 6 persons, so that the share of each may be as much as those of all who come before put together.—*Answer,* The first two have 12*s.* 6*d.*; the third £1 . 5; the fourth £2 . 10; the fifth £5; and the sixth £10.

255. When two or more persons employ their money together, and gain or lose a certain sum, it is evidently not fair that the gain or loss should be equally divided among them all, unless each contributed the same sum. Suppose, for example, A contributes twice as much as B, and they gain £15, A ought to gain twice as much as B; that is, if the whole gain be divided into 3 parts, A ought to have two of them and B one, or A should gain £10 and B £5. Suppose that A, B, and C engage in an adventure, in which A embarks £250, B £130, and C £45. They gain £1000. How much of it ought each to have? Each one ought to gain as much for £1 as the others. Now, since there are 250+130+45, or 425 pounds, embarked, which gain £1000, for each pound there is a gain of £$\frac{1000}{425}$. Therefore A should gain $\frac{1000 \times 250}{425}$ pounds, B should gain $\frac{1000 \times 130}{425}$ pounds, and C $\frac{1000 \times 45}{425}$ pounds. On these principles, by the process in (245), the following questions may be answered.

A ship is to be insured, in which A has ventured £1928, and B £4963. The expense of insurance is £474 . 10 . 2. How much ought each to pay of it? *Answer,* A must pay £132 . 15 . 2$\frac{1}{2}$.

A loss of £149 is to be made good by three persons, A, B, and C.

Had there been a gain, A would have gained 4 times as much as B, and C as much as A and B together. How much of the loss must each bear? *Answer*, A pays £59 . 12, B £14 . 18, and C £74 . 10.

256. It may happen that several individuals employ several sums of money together for different times. In such a case, unless there is a special agreement to the contrary, it is right that the more time a sum is employed, the more profit should be made upon it. If, for example, A and B employ the same sum for the same purpose, but A's money is employed twice as long as B's, A ought to gain twice as much as B. The principle is, that one pound employed for one month, or one year, ought to give the same return to each. Suppose, for example, that A employs £3 for 6 months, B £4 for 7 months, and C £12 for 2 months, and the gain is £100; how much ought each to have of it? Now, since A employs £3 for 6 months, he must gain 6 times as much as if he employed it one month only; that is, as much as if he employed £6×3, or 18, for one month: also, B gains as much as if he had employed £4×7 for one month, and C as if he had employed £12×2 for one month. If, then, we divide £100 into $6 \times 3 + 4 \times 7 + 12 \times 2$, or 70 parts, A must have 6×3, or 18, B must have 4×7, or 28, and C 12×2, or 24 of those parts. The shares of the three are, therefore,

$$£\frac{6 \times 3 \times 100}{6 \times 3 + 4 \times 7 + 12 \times 2},\ £\frac{4 \times 7 \times 100}{6 \times 3 + 4 \times 7 + 12 \times 2},\ \text{and } £\frac{12 \times 2 \times 100}{6 \times 3 + 4 \times 7 + 12 \times 2}.$$

EXERCISES.

A, B, and C embark in an undertaking; A placing £3 . 6 for 2 years, B £100 for 1 year, and C £12 for $1\frac{1}{2}$ years. They gain £4276 . 7. How much must each receive of the gain?

Answer, A £226 . 10 . 4; B £3432 . 1 . 3; C £617 . 15 . 5.

A, B, and C rent a house together for two years, at £150 per annum. A remains in it the whole time, B, 16 months, and C, $4\frac{1}{2}$ months, during the occupancy of B. How much must each pay of the rent? *

Answer, A should pay £190 . 12 . 6; B £90 . 12 . 6; C £18 . 15.

* This question does not at first appear to fall under the rule. A little thought will serve to shew, that what probably will be the first idea of the proper method of solution is erroneous.

257. These are the principal rules employed in the application of arithmetic to commerce. There are others, which, as no one who understands the principles here laid down can fail to see, are virtually contained in those which have been given. Such is what is commonly called the Rule of Exchange, for such questions as the following: If 20 shillings be worth $25\frac{1}{2}$ francs, in France, what is £160 worth? This may evidently be done by the Rule of Three. The rules here given are those which are most useful in common life; and the student who understands them need not fear that any ordinary question will be above his reach.

APPENDIX.

RULES FOR THE APPLICATION OF ARITHMETIC TO GEOMETRY.

258. The student should make himself familiar with the most common terms of Geometry, after which the following rules will present no difficulty. In them all, it must be understood, that when we talk of multiplying one line by another, we mean the repetition of one line as often as there are units of a given kind, as feet or inches, in another. In any other sense, it is absurd to talk of multiplying a quantity by another quantity. All quantities of the same kind should be represented in numbers of the same unit; thus, all the lines should be either feet and decimals of a foot, or inches and decimals of an inch, &c. And in whatever unit a length is represented, a surface is expressed in the corresponding square units, and a solid in the corresponding cubic units. This being understood, the rules apply to all sorts of units.

259. *To find the area of a rectangle.* Multiply together the units in two sides which meet, or multiply together two sides which meet; the product is the number of square units in the area. Thus, if 6 feet and 5 feet be the sides, the area is 6×5, or 30 square feet. Similarly, the area of a square of six feet long is 6×6, or 36 square feet (234).

To find the area of a parallelogram. Multiply one side by the perpendicular distance between it and the opposite side; the product is the area required in square units.

*To find the area of a trapezium.** Multiply either of the two sides

* A four-sided figure, which has two sides parallel, and two sides not parallel.

which are not parallel, by the perpendicular let fall upon it from the middle point of the other.

To find the area of a triangle. Multiply any side by the perpendicular let fall upon it from the opposite vertex, and take half the product. Or, halve the sum of the three sides, subtract the three sides severally from this half sum, multiply the four results together, and find the square root of the product. The result is the number of square units in the area; and twice this, divided by either side, is the perpendicular distance of that side from its opposite vertex.

To find the radius of the internal circle which touches the three sides of a triangle. Divide the area, found in the last paragraph, by half the sum of the sides.

260. *Given the two sides of a right-angled triangle, to find the hypothenuse.* Add the squares of the sides, and extract the square root of the sum.

Given the hypothenuse and one of the sides, to find the other side. Multiply the sum of the given lines by their difference, and extract the square root of the product.

261. *To find the circumference of a circle from its radius, very nearly.* Multiply twice the radius, or the diameter, by 3·1415927, taking as many decimal places as may be thought necessary. For a rough computation, multiply by 22 and divide by 7. For a very exact computation, in which decimals shall be avoided, multiply by 355 and divide by 113. See (131, last example).

To find the arc of a circular sector, very nearly, knowing the radius and the angle. Turn the angle into seconds,* multiply by the radius, and divide the product by 206265. The result will be the number of units in the arc.

To find the area of a circle from its radius, very nearly. Multiply the square of the radius by 3·1415927.

* The right angle is divided into 90 equal parts called *degrees*, each degree into 60 equal parts called *minutes*, and each minute into 60 equal parts called *seconds*. Thus, $2^\circ\ 15'\ 40''$ means 2 degrees, 15 minutes, and 40 seconds.

To find the area of a sector, very nearly, knowing the radius and the angle. Turn the angle into seconds, multiply by the square of the radius, and divide by 206265×2, or 412530.

262. *To find the solid content of a rectangular parallelopiped.* Multiply together three sides which meet: the result is the number of cubic units required. If the figure be not rectangular, multiply the area of one of its planes by the perpendicular distance between it and its opposite plane.

To find the solid content of a pyramid. Multiply the area of the base by the perpendicular let fall from the vertex upon the base, and divide by 3.

To find the solid content of a prism. Multiply the area of the base by the perpendicular distance between the opposite bases.

263. *To find the surface of a sphere.* Multiply 4 times the square of the radius by 3·1415927.

To find the solid content of a sphere. Multiply the cube of the radius by $3{\cdot}1415927 \times \frac{4}{3}$, or 4·18879.

To find the surface of a right cone. Take half the product of the circumference of the base and slanting side. *To find the solid content,* take one-third of the product of the base and the altitude.

To find the surface of a right cylinder. Multiply the circumference of the base by the altitude. *To find the solid content,* multiply the area of the base by the altitude.

264. The weight of a body may be found, when its solid content is known, if the weight of one cubic inch or foot of the body is known. But it is usual to form tables, not of the weights of a cubic unit of different bodies, but of the proportion which these weights bear to some one amongst them. The one chosen is usually distilled water, and the proportion just mentioned is called the *specific gravity.* Thus, the specific gravity of gold is 19·362, or a cubic foot of gold is 19·362 times as heavy as a cubic foot of distilled water. Suppose now the weight of a sphere of gold is required, whose radius is 4 inches. The content of this sphere is $4 \times 4 \times 4 \times 4{\cdot}1888$, or 268·0832 cubic inches; and since,

by (217), each cubic inch of water weighs 252·458 grains, each cubic inch of gold weighs 252·458 × 19·362, or 4888·091 grains; so that 268·0832 cubic inches of gold weigh 268·0832 × 4888·091 grains, or $227\frac{1}{2}$ pounds troy nearly. Tables of specific gravities may be found in most works of chemistry and practical mechanics.

The cubic foot of water is 908·8488 troy ounces, 75·7374 troy pounds, 997·1369691 averdupois ounces, and 62·3210606 averdupois pounds. For all rough purposes it will do to consider the cubic foot of water as being 1000 common ounces, which reduces tables of specific gravities to common terms in an obvious way. Thus, when we read of a substance which has the specific gravity 4·1172, we may take it that a cubic foot of the substance weighs 4117 ounces. For greater correctness, diminish this result by 3 parts out of a thousand.

THE END.

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The work is accompanied by an explanation of the mode of using the tables. The subjoined specimen will shew the extent of the parts and the arrangement of the whole.

Num.	Square.	Cube.	Squ. Root.	Cube Root.	Recipr. ·000
1901	3 61 38 01	6 869 835 701	43·6004587	12·3877959	5260389
1902	3 61 76 04	6 880 682 808	43·6119249	12·3899676	5257624
1903	3 62 14 09	6 891 541 327	43·6233882	12·3921386	5254861
1904	3 62 52 16	6 902 411 264	43·6348485	12·3943089	5252101
1905	3 62 90 25	6 913 292 625	43·6463057	12·3964784	5249344
1906	3 63 28 36	6 924 185 416	43·6577599	12·3986471	5246590
1907	3 63 66 49	6 935 089 643	43·6692111	12·4008151	5243838
1908	3 64 04 64	6 946 005 312	43·6806593	12·4029823	5241090
1909	3 64 42 81	6 956 932 429	43·6921045	12·4051488	5238345
1910	3 64 81 00	6 967 871 000	43·7035467	12·4073145	5235602
1911	3 65 19 21	6 978 821 031	43·7149860	12·4094794	5232862
1912	3 65 57 44	6 989 782 528	43·7264222	12·4116436	5230126
1913	3 65 95 69	7 000 755 497	43·7378554	12·4138070	5227392
1914	3 66 33 96	7 011 739 944	43·7492857	12·4159697	5224660
1915	3 66 72 25	7 022 735 875	43·7607129	12·4181316	5221932
1916	3 67 10 56	7 033 743 296	43·7721373	12·4202928	5219207
1917	3 67 48 89	7 044 762 213	43·7835585	12·4224533	5216484
1918	3 67 87 24	7 055 792 632	43·7949768	12·4246129	5213764
1919	3 68 25 61	7 066 834 559	43·8063922	12·4267719	5211047
1920	3 68 64 00	7 077 888 000	43·8178046	12·4289300	5208333
1921	3 69 02 41	7 088 952 961	43·8292140	12·4310875	5205622
1922	3 69 40 84	7 100 029 448	43·8406204	12·4332441	5202914
1923	3 69 79 29	7 111 117 467	43·8520239	12·4354001	5200208
1924	3 70 17 76	7 122 217 024	43·8634244	12·4375552	5197505
1925	3 70 56 25	7 133 328 125	43·8748219	12·4397097	5194805
1926	3 70 94 76	7 144 450 776	43·8862165	12·4418634	5192108
1927	3 71 33 29	7 155 584 983	43·8976081	12·4440163	5189414
1928	3 71 71 84	7 166 730 752	43·9089968	12·4461685	5186722
1929	3 72 10 41	7 177 888 089	43·9203825	12·4483200	5184033
1930	3 72 49 00	7 189 057 000	43·9317652	12·4504707	5181347
1931	3 72 87 61	7 200 237 491	43·9431451	12·4526206	5178664
1932	3 73 26 24	7 211 429 568	43·9545220	12·4547699	5175983
1933	3 73 64 89	7 222 633 237	43·9658959	12·4569184	5173306
1934	3 74 03 56	7 233 848 504	43·9772668	12·4590661	5170631
1935	3 74 42 25	7 245 075 375	43·9886349	12·4612131	5167959
1936	3 74 80 96	7 256 313 856	44·0000000	12·4633594	5165289
1937	3 75 19 69	7 267 563 953	44·0113622	12·4655049	5162623
1938	3 75 58 44	7 278 825 672	44·0227214	12·4676497	5159959
1939	3 75 97 21	7 290 099 019	44·0340777	12·4697937	5157298
1940	3 76 36 00	7 301 384 000	44·0454311	12·4719370	5154639
1941	3 76 74 81	7 312 680 621	44·0567815	12·4740796	5151984
1942	3 77 13 64	7 323 988 888	44·0681291	12·4762214	5149331
1943	3 77 52 49	7 335 308 807	44·0794737	12·4783625	5146680
1944	3 77 91 36	7 346 640 384	44·0908154	12·4805029	5144033
1945	3 78 30 25	7 357 983 625	44·1021541	12·4826426	5141388
1946	3 78 69 16	7 369 338 536	44·1134900	12·4847815	5138746
1947	3 79 08 09	7 380 705 123	44·1248229	12·4869197	5136107
1948	3 79 47 04	7 392 083 392	44·1361530	12·4890571	5133470
1949	3 79 86 01	7 403 473 349	44·1474801	12·4911938	5130836
1950	3 80 25 00	7 414 875 000	44·1588043	12·4933298	5128205

Num.	Square.	Cube.	Squ. Root.	Cube Root.	Recipr. ·000
1951	3 80 64 01	7 426 288 351	44·1701256	12·4954651	5125577
1952	3 81 03 04	7 437 713 408	44·1814441	12·4975995	5122951
1953	3 81 42 09	7 449 150 177	44·1927596	12·4997333	5120328
1954	3 81 81 16	7 460 598 664	44·2040722	12·5018664	5117707
1955	3 82 20 25	7 472 058 875	44·2153819	12·5039988	5115090
1956	3 82 59 36	7 483 530 816	44·2266888	12·5061304	5112474
1957	3 82 98 49	7 495 014 493	44·2379927	12·5082612	5109862
1958	3 83 37 64	7 506 509 912	44·2492938	12·5103914	5107252
1959	3 83 76 81	7 518 017 079	44·2605919	12·5125208	5104645
1960	3 84 16 00	7 529 536 000	44·2718872	12·5146495	5102041
1961	3 84 55 21	7 541 066 681	44·2831797	12·5167775	5099439
1962	3 84 94 44	7 552 609 128	44·2944692	12·5189047	5096840
1963	3 85 33 69	7 564 163 347	44·3057558	12·5210313	5094244
1964	3 85 72 96	7 575 729 344	44·3170396	12·5231571	5091650
1965	3 86 12 25	7 587 307 125	44·3283205	12·5252822	5089059
1966	3 86 51 56	7 598 896 696	44·3395985	12·5274065	5086470
1967	3 86 90 89	7 610 498 063	44·3508737	12·5295302	5083884
1968	3 87 30 24	7 622 111 232	44·3621460	12·5316531	5081301
1969	3 87 69 61	7 633 736 209	44·3734155	12·5337753	5078720
1970	3 88 09 00	7 645 373 000	44·3846820	12·5358968	5076142
1971	3 88 48 41	7 657 021 611	44·3959457	12·5380176	5073567
1972	3 88 87 84	7 668 682 048	44·4072066	12·5401377	5070994
1973	3 89 27 29	7 680 354 317	44·4184646	12·5422570	5068424
1974	3 89 66 76	7 692 038 424	44·4297198	12·5443757	5065856
1975	3 90 06 25	7 703 734 375	44·4409720	12·5464936	5063291
1976	3 90 45 76	7 715 442 176	44·4522215	12·5486107	5060729
1977	3 90 85 29	7 727 161 833	44·4634681	12·5507272	5058169
1978	3 91 24 84	7 738 893 352	44·4747119	12·5528430	5055612
1979	3 91 64 41	7 750 636 739	44·4859528	12·5549580	5053057
1980	3 92 04 00	7 762 392 000	44·4971909	12·5570723	5050505
1981	3 92 43 61	7 774 159 141	44·5084262	12·5591860	5047956
1982	3 92 83 24	7 785 938 168	44·5196586	12·5612989	5045409
1983	3 93 22 89	7 797 729 087	44·5308881	12·5634111	5042864
1984	3 93 62 56	7 809 531 904	44·5421149	12·5655226	5040323
1985	3 94 02 25	7 821 346 625	44·5533388	12·5676334	5037783
1986	3 94 41 96	7 833 173 256	44·5645599	12·5697435	5035247
1987	3 94 81 69	7 845 011 803	44·5757781	12·5718529	5032713
1988	3 95 21 44	7 856 862 272	44·5869936	12·5739615	5030181
1989	3 95 61 21	7 868 724 669	44·5982062	12·5760695	5027652
1990	3 96 01 00	7 880 599 000	44·6094160	12·5781767	5025126
1991	3 96 40 81	7 892 485 271	44·6206230	12·5802832	5022602
1992	3 96 80 64	7 904 383 488	44·6318272	12·5823891	5020080
1993	3 97 20 49	7 916 293 657	44·6430286	12·5844942	5017561
1994	3 97 60 36	7 928 215 784	44·6542271	12·5865987	5015045
1995	3 98 00 25	7 940 149 875	44·6654228	12·5887024	5012531
1996	3 98 40 16	7 952 095 936	44·6766158	12·5908054	5010020
1997	3 98 80 09	7 964 053 973	44·6878059	12·5929078	5007511
1998	3 99 20 04	7 976 023 992	44·6989933	12·5950094	5005005
1999	3 99 60 01	7 988 005 999	44·7101778	12·5971103	5002501
2000	4 00 00 00	8 000 000 000	44·7213596	12·5992105	5000000

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