ELEMENTS OF DYNAMICS

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THE ELEMENTS OF DYNAMICS

(MECHANICS)

WITH NUMEROUS EXAMPLES AND EXAMINATION QUESTIONS

BY

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PREFACE TO NEW EDITION

THE object of this treatise is to provide a manual of what has been generally known as the "Elements of Mechanics," but is here, in accordance with the more precise phraseology of recent works, termed "Dynamics."

As the work is primarily intended for beginners, special pains have been taken to establish the necessary propositions by proofs involving no higher mathematics than the geometry of the first two books of Euclid, and algebra as far as simple equations. At the same time, the nomenclature, definitions, and general treatment are in harmony with advanced modern works on the subject.

Examples and examination questions have been introduced into the text, in order to furnish all who make use of the book with the means of testing, as they proceed, whether each portion of the subject has been duly mastered. A selection of examination papers set by the Universities of Oxford, Cambridge, London, Edinburgh, and Glasgow, the Scotch Education Department (Leaving Certificate) and the Science and Art Department, has been added. Most of the general examples which follow each chapter are also taken from actual examinations. Answers are given in all cases, and all points likely to present difficulty to beginners are explained.

Since the first publication of this treatise I have been frequently asked to introduce additional paragraphs required for particular examinations in the English Universities and elsewhere. While complying with this request, I have not felt at liberty to introduce fresh matter into the body of the book, as it has been adopted, in its original form, by various Universities and other bodies as a text-book for Examinations. An Appendix has, however, been added, which will, it is hoped, be found useful by those who desire to continue the study of the subject. Some parts of the Appendix assume a rudimentary knowledge of Trigonometry and Conic Sections.

I gladly take this opportunity of thanking many friends who have aided me. Special acknowledgments are due to the late Professor Balfour Stewart of Owens College, at whose suggestion I undertook the work; to Professor Tait of Edinburgh University, for his ever ready encouragement and advice; and to the Rev. N. M. Ferrers, D.D., Master of Gonville and Caius College, the late Professor Clerk Maxwell of Cambridge, Professor MacGregor of Dalhousie College, Halifax, Nova Scotia, Dr. J. S. Mackay of the Edinburgh Academy, Mr. R. Tucker of University College School, Mr. E. Brook-Smith of King's College School, Professor W. Raitt of the Glasgow and West of Scotland Technical College, Mr. J. B. Clark of Heriot's Hospital School, and other friends, for valuable suggestions and corrections.

CONTENTS

CHAPTER I

PRELIMINARY

1. **Natural Philosophy.**—NATURAL PHILOSOPHY *treats of the Laws of the Material World.* It is concerned with the five fundamental ideas of Time, Space, Motion, Matter, and Force. Each of these ideas is the subject of a particular science. The five corresponding sciences are Algebra,[1] Geometry, Kinematics, Physics, and Dynamics.

The present work contains the elementary parts of Kinematics and Dynamics. It assumes a knowledge of Algebra as far as simple equations, and of the Geometry contained in the first two books of Euclid.

2. **Dynamics.**—DYNAMICS *treats of the action of Force.*

FORCE *is any cause which alters a body's state of rest, or of uniform motion in a straight line.*

KINEMATICS *treats of Motion without reference to Force.* It is a branch of pure Mathematics, and a necessary preliminary to Dynamics.

When more forces than one act on a body, their effect is either (1) to produce or change motion, or (2) to maintain rest or prevent change of motion.

Hence Dynamics is divided into two parts—Kinetics and Statics.

KINETICS *treats of the action of Force in producing or changing Motion.*

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[1] Algebra (which includes Arithmetic) deals with pure number, and hence with Time.

STATICS *treats of the action of Force in maintaining Rest or preventing change of Motion.*

HYDROSTATICS *is that branch of Statics which treats of Liquids and Gases.*

[The student reading this subject for the first time is earnestly advised to make sure that he has mastered each part before he proceeds to the next. For this purpose he is urged to write out conscientiously the answers to the examinations and examples. Definitions and other sentences printed in italics should be committed to memory. If he is preparing for examination, he will find it useful to work out both examination papers and examples a second time, and to do so *with as great attention to style, accuracy, and legibility, as if he were actually undergoing examination.*]

EXAMINATION ON CHAPTER I

1. Enumerate the fundamental ideas of Natural Philosophy, and the respective branches of Mathematics or Natural Philosophy by which they are investigated.

2. Define Force. In what two ways is it recognised as acting?

3. Define Dynamics, Kinetics, Statics, Kinematics, and Hydrostatics.

CHAPTER II

KINEMATICS

3. Motion of a Point.—MOTION *is change of Position.* In Kinematics we first consider the motion of a point. A moving point passes from one position in *space* to another during the lapse of an interval of *time.* Motion therefore connects the ideas of space and time.

A moving point traces out a continuous Line, any small part of which is either *straight* or *curved.* If the line is straight, then its *direction* is that in which the point is moving. If the line is curved, then we can draw a straight line which will have the direction of the motion of the point at any particular instant. This line is called in geometry the *tangent* to the curve.

As an example, take the case of a projectile impelled along a straight tube AB. As long as it is within the tube, its direction is that of the straight line AB. When it leaves the tube it describes the curve BCD. When the projectile is at C, its direction is that of the straight line CE, which is the tangent to the curve at C.

4. Uniform Speed.—SPEED *is rate of Motion.* It is *uniform* or *variable.*

UNIFORM SPEED *is the rate of motion of a point which passes over equal spaces in equal times.* It is greater or less as the space passed over in a given time is greater or less.

Thus a ship, which, sailing uniformly, makes 15 knots in a certain time, has three times the speed of one which makes 5 knots in the same time.

Uniform Speed is measured by the space passed over in a unit of time. It is generally convenient to take a *foot* as the unit of space, and a *second* as the unit of time. With these units, a point has unit speed when it moves one foot in each second, a speed of 12 when it moves 12 feet in each second, and so on.

When the metrieal[1] system is used for scientific purposes, a centimetre is generally taken as the unit of length, and speeds are then expressed in centimetres per second.

Examples—

1. Find the speed in feet per second of a train moving uniformly at the rate of 20 miles an hour.

We must first express the distance in feet, and the time in seconds. Employing factors,

$$20 \text{ miles} = 20 \times 1760 \times 3 \text{ feet}, \qquad 1 \text{ hour} = 60 \times 60 \text{ seconds}.$$

The data may therefore be thus stated—

In 60×60 seconds the train moves $20 \times 1760 \times 3$ feet.

$\therefore$ in 1 second it moves $\dfrac{20 \times 1760 \times 3}{60 \times 60}$ feet.

$\therefore$ the speed, or the space passed over in one second,

$$= \frac{20 \times 1760 \times 3}{60 \times 60} = \frac{88}{3} = 29\tfrac{1}{3} \text{ feet per second.}$$

2. Find the speed in centimetres per second of a man who walks 3 kilometres in 25 minutes at a steady pace.

In 25×60 seconds he walks $3 \times 100{,}000$ centimetres.

$\therefore$ in 1 second he walks $\dfrac{3 \times 100{,}000}{25 \times 60} = 200$ centimetres, or his speed is 200 centimetres a second.

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[1] A metre is 3·28 feet, or more accurately 39·37079 inches.

10 metres are called a decametre,			
100	,,	,,	hectometre,
1000	,,	,,	kilometre.
Also $\frac{1}{10}$ of a metre is called a decimetre,			
$\frac{1}{100}$	,,	,,	centimetre,
$\frac{1}{1000}$	,,	,,	millimetre.

3. A point has a speed of 32·2 feet per second, find its speed in centimetres per second.

$$\cdot 0328 \text{ feet} = 1 \text{ centimetre}, \qquad \therefore\ 1 \text{ foot} = \frac{1}{\cdot 0328} \text{ centimetres},$$

$$\therefore\ 32{\cdot}2 \text{ feet} = \frac{32{\cdot}2}{\cdot 0328} = 981 \text{ centimetres (approximately).}$$

Examples for Exercise—

1. A horse trots 2 miles in 9 minutes. Find its speed in feet per second.
2. A train has a speed of 48 kilometres an hour. Find its speed in centimetres per second.
3. Which is the greater speed, 100 yards in $11\frac{1}{2}$ seconds, or $6\frac{3}{4}$ miles in 22 minutes?
4. If the speed of sound be 1118 feet per second, how long will it take to travel 43 miles?

5. **Algebraical Formulæ of Uniform Speed.**—Let v be the speed of a point moving uniformly, that is, the number of units of space which it passes over in each unit of time. Then—

In 2 units of time it passes over $2v$ units of space,
,, 3 ,, ,, ,, $3v$,,
,, 4 ,, ,, ,, $4v$,,

and so on.

Let t be any number of units of time. During this time the point passes over $t \times v$ units of space, which may be written tv or vt. If this distance be called s, then—

$$s = vt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This equation enables us to find s when v and t are known.

Divide both sides of equation (1) by t and we obtain—

$$v = \frac{s}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which enables us to find v when s and t are known.

Again, divide both sides of (1) by v and we obtain—

$$t = \frac{s}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which enables us to find t when s and v are known.

Examples—

1. The speed of a particle is 11 feet per second. Find the distance gone in 7 seconds.

$$\text{From (1)} \qquad s = vt, \quad \text{but } v = 11, \quad t = 7,$$
$$\therefore s = 11 \times 7 = 77 \text{ feet.}$$

2. A train goes a mile in a minute. Find its speed in feet per second.

$$\text{From (2)} \qquad v = \frac{s}{t}, \quad \text{but } s = 1760 \times 3, \quad t = 60,$$
$$\therefore v = \frac{1760 \times 3}{60} = 88 \text{ feet per second.}$$

3. A race-horse has a speed of 15 metres per second. Find how long he takes to go a kilometre.

$$\text{From (3)} \quad t = \frac{s}{v} = \frac{1000}{15} = 66\tfrac{2}{3} \text{ seconds.}$$

Examples for Exercise—

1. Find the distance an eagle flies in an hour, assuming its speed to be 100 feet per second.

2. Express (in feet per second) the speed of the moon in its orbit round the earth, assuming that it describes a path of 1,511,460 miles in 27 days.

3. Find the number of units of time in which a body moving with uniform speed a passes over b units of space.

6. Variable Speed.—VARIABLE SPEED *is that of a point which passes over unequal spaces in equal times. It is measured at any instant by the space which would be passed over in a unit of time, were the speed at the instant under consideration to remain unchanged for a unit of time.* Thus when we say that a bullet leaves a rifle with a speed of 1330 feet per second, we mean that, if the speed were to remain unaltered for a second, the bullet would, during that time, describe a path 1330 feet in length.

7. Average Speed.—*When a point moves with variable speed for any interval of time, the Uniform Speed, with which the same space can be described in the same time, is called the* AVERAGE SPEED *of the point during that time.*

During part of the interval the point's speed must have been greater than the average speed, during part it must have been less, and at some one instant it must have been equal to it. The shorter the interval is, the more nearly does the average speed coincide with the actual speed at any instant of the interval.

A railway train, either increasing or decreasing its speed, is an example of variable speed. By observing how far it goes in five minutes, we may obtain the average speed during that period. A nearer approximation to its actual speed at a particular instant is found by observing the distance travelled in one minute chosen so as to include that instant, and a still nearer by observing that travelled in one second similarly chosen.

Examples—

1. A man walks 9 feet in one second, 8 in the next, 6 in the third, and 3 in the fourth; find his average speed.

Here we have to find a uniform speed with which the whole distance, 26 feet, can be described in 4 seconds.

By (2) $v = \frac{s}{t} = \frac{26}{4} = 6\frac{1}{2}$, the average speed required.

2. Find approximately the same man's speed at the end of the first second, assuming that he is constantly slackening his pace.

The smallest interval, which we can take so as to include the given instant, is the first two seconds. During this time he walked 17 feet.

$\therefore\ v = \frac{s}{t} = \frac{17}{2} = 8\frac{1}{2}$, the average speed during the first two seconds.

Examples for Exercise—

1. A stone falls 16·1 feet in the first second, 48·3 in the next, and 80·5 in the third; find its average speed.

2. With the same data, find as nearly as possible the stone's speed at the beginning of the third second.

3. A particle passes over a feet in the first second, b in the next, and c in the third. Find its average speed.

4. A point has during t_1 units of time an average speed v_1, and during the next t_2 units of time an average speed v_2. Find its average speed during the whole time.

8. Velocity.—The speed of a point is called its VELOCITY, *when the direction in which it is moving is taken into account.* Thus two points are said to have equal velocities when they are moving in parallel directions with equal speeds. Two points which have equal speeds, but are moving in different directions have different velocities. When a point moves in a straight line with uniform speed, its velocity is uniform, and the formulae of § 5 apply to this case, v standing for velocity. The *velocity* of a point at any instant is defined when its *direction* and *speed* are given. It is therefore completely represented by a straight line whose *direction* is that in which the point is moving, and whose *length* is proportional to its speed at the instant under consideration.

EXAMINATION ON SECTIONS 3-8

1. Define Motion. Show how it involves space and time.
2. Define Speed. What two kinds of speed exist? Give examples of each.
3. Define Uniform Speed. How is it measured?
4. Prove the formula $s=vt$. Deduce from this a formula which expresses the speed in terms of the space and time.
5. Define Variable Speed. How is it measured?
6. What is meant by the average speed of a point during a given time?
7. Define Velocity. How does it differ from Speed.
8. Show that a velocity may be completely represented by a straight line.

9. Velocity of a Body.—When an extended body, or a solid figure, is in motion, if all its points have the same velocity at any instant, that is if they are all moving at the same speed in parallel straight lines, that velocity is called *the velocity of the body*, and the motion is said to be one of *Translation* only. A canal boat moving along a straight canal, and the framework of a carriage drawn along a straight and level road, are examples of *translation*. A revolving fly-wheel and a spinning top are examples of bodies in which the different points have different velocities; such a motion is called one of *Rotation.* The most general form of motion is a combination of translation and rotation. The earth in its orbit, and a rifle-bullet in its flight, are examples of bodies which combine motions of translation and rotation.

10. Absolute and Relative Velocity.—It must be observed that in speaking of speed or velocity we always mean speed or velocity with regard to some body which should be either expressly stated or clearly implied. Thus, in speaking of the motions of men, horses, and trains, we understand that, unless the contrary is expressed, it is their motion relatively to the surface of the earth that is referred to.

Of course we know that the surface of the earth is not at rest. The earth rotates on its axis, revolves round the sun, accompanies the sun in his motion relatively to the stars, and accompanies sun and stars in their motion relatively to other bodies in space. It is impossible that we can ever know the absolute velocity of the earth, for even if astronomers could ascertain all the motions already mentioned, the whole visible universe might still be in motion relatively to objects invisible to us.

Thus, whenever we speak of either motion or rest, it is not absolute but relative motion or rest that is intended.

11. Component and Resultant Velocities.—*When a point has two or more simultaneous velocities, each of them is called a Component, and the combined velocity is called their Resultant.*

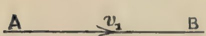
As an example, consider the case of a sailor climbing the rigging of a ship. We may in the first place find his velocity with regard to the ship. But he has at the same time the velocity of the ship with regard to the surface of the sea. If we combine these two velocities we shall find the sailor's velocity with regard to the surface of the sea.

Composition of two Uniform Velocities in the same Straight Line.—Two velocities in the same straight line may have the same direction, or they may have opposite directions. There are therefore two cases to be considered.

(1.) *If two component velocities have the same direction in the same straight line, the Resultant is their Sum.*

Let a point have two uniform velocities, v_1 and v_2 feet per second, in the same straight line. Then since v_1 and v_2 are uniform, or the same at every instant, the resultant velocity is also the same at every instant, that is to say, it is a uniform velocity.

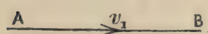
To find its magnitude let us suppose that the point moves with the velocity v_1 along the straight line AB, and that the paper is itself in motion in the same direction with the velocity v_2. (See § 9.) After one second the point will have moved v_1 feet along AB, but AB will itself have moved v_2 feet in the same direction owing to the motion of the paper. The point will therefore be now $v_1 + v_2$ feet from its original position in space, or if v be its actual velocity,—



$$v = v_1 + v_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

(2.) *If two component velocities have opposite directions in the same straight line the Resultant is their Difference, and has the direction of the greater component.*

Let a point have two uniform velocities, v_1 and v_2, in opposite directions. Then, as before, the resultant velocity is uniform. To find its magnitude, let us again suppose that the point is moving along AB with velocity v_1, but that the paper is moving in the direction of BA, that is, in the opposite direction, with velocity v_2. If v_1 be greater than v_2, the point will, after one second, be $v_1 - v_2$ feet from its original position in the direction of AB, or, if v be its actual velocity,—



$$v = v_1 - v_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If v_2 be greater than v_1, the point will, after one second, be $v_2 - v_1$ feet from its original position in the direction of

BA. It is not necessary to form a new equation in this case for equation (5) may be put into the form,—

$$v = -(v_2 - v_1),$$

and will therefore apply also to this case if we agree to consider that a *negative* velocity means a velocity in the opposite direction to that chosen as positive.

Corollary.—If a point have two equal and opposite velocities it is at rest.

For if in (5) $\quad v_1 = v_2,$

then $\quad v = v_1 - v_2 = 0,$

or the point is at rest.

Examples—

1. A ship is sailing with velocity 15, and a sailor walks along the deck in the direction of the ship's motion with velocity 6. Find the actual velocity of the sailor.

Here $\quad v_1 = 15, \quad v_2 = 6,$

∴ by (4) $\quad v = v_1 + v_2 = 21.$

2. Find the sailor's actual velocity when he walks at the same rate in the opposite direction.

Here $\quad v_1 = 15,$ and $v_2 = 6$ in the opposite direction,

∴ by (5) $\quad v = v_1 - v_2 = 9.$

3. Find the sailor's actual velocity when he runs with velocity 32 in the direction opposite to the ship's motion.

Here $\quad v_1 = 15, \quad v_2 = 32$ in the opposite direction,

∴ by (5) $\quad v = v_1 - v_2 = 15 - 32 = -17,$

or the real velocity of the sailor is 17 feet per second in the direction opposite to the ship's motion.

Examples for Exercise—

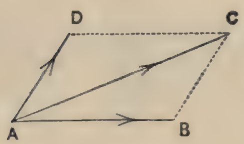
1. A stream has a velocity of 2 miles an hour, and a boat is rowed first up stream and then down at the rate of 8 miles an hour. Find its actual velocity in each case.

2. A sledge party travels northwards on an ice-floe at the rate of 12 miles a day. The floe is itself drifting southwards at the rate of 15 yards a minute. In what direction is the sledge really moving, and at what rate?

3. A river steamer can sail down stream at the rate of 15 miles an hour, and up stream at the rate of 11 miles an hour, relatively to the bank. Find the velocity of the river, and that of the steamer relatively to it.

12. Parallelogram of Velocities.—*If two component velocities be represented in magnitude and direction by two adjacent sides of a parallelogram, the resultant is represented by the diagonal passing through their intersection.*

Let a point have two *uniform* velocities represented by the straight lines AB and AD. Complete the parallelogram ABCD. Join AC. AC represents the resultant velocity.



Proof.—Since the component velocities are uniform, or the same at each instant, their resultant is also the same at each instant both in magnitude and direction, that is to say, it is a uniform velocity.

To find it, let us suppose that the point moves with velocity AB along the straight line AB, and that the paper is in motion in the direction AD with velocity AD. After one second the point will have moved along AB from A to B, but the line AB will itself have moved into the position DC, owing to the motion of the paper, and therefore the point will now be at C. It has therefore moved in the direction AC with velocity AC.

13. Triangle of Velocities.—*If a point have three component velocities, which are represented in magnitude and direction by the sides of a triangle taken in order, the point is at rest.*

Let a point have three velocities represented by the sides AB, BC, CA, of the triangle ABC.

In the parallelogram ABCD, BC is equal and parallel to AD, and thus equally represents the component velocity in

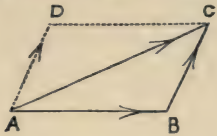


Figure 1.

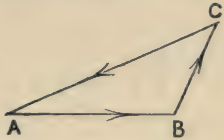


Figure 2.

that direction. Therefore in the triangle ABC (figure 1), the velocity AC is the resultant of the velocities AB and BC. If we combine with these the velocity CA (figure 2), then the two velocities AC and CA are equivalent to the three velocities AB, BC, and CA. But AC and CA are two equal and opposite velocities. Therefore the point is at rest.

14. Polygon of Velocities.—*If several component velocities be represented by all but one of the sides of a polygon taken in order, their resultant is that side taken in the opposite direction.*

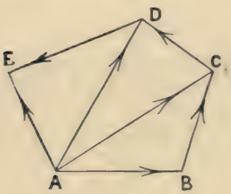


Figure 1.

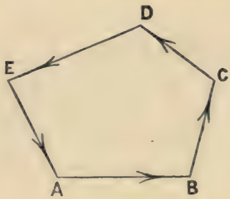


Figure 2.

Let AB, BC, CD, DE (figure 1) be the component velocities.

Then AC is the resultant of AB and BC.

AD is the resultant of AC and CD, that is, of AB, BC, and CD.

AE is the resultant of AD and DE, that is, of AB, BC, CD, and DE.

Corollary.—If a point have component velocities represented by all the sides of a polygon taken in order, it is at rest.

Let a point have velocities represented by AB, BC, CD, DE, EA.

AE is the resultant of AB, BC, CD, and DE.

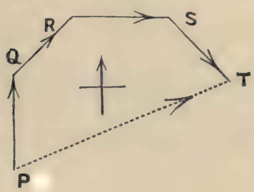
∴ AE and EA are equivalent to AB, BC, CD, DE, and EA (figure 2).

But AE and EA are two equal and opposite velocities, ∴ the point is at rest.

Note.—The above proofs hold good when the velocities, and therefore the sides of the polygon, are not all in one plane.

Example—

A point has the following simultaneous velocities, 8 north, 7 north-east, 8 east, and 7 south-east. Find the resultant velocity.



Draw a line PQ north $= 8$
QR north-east $= 7$
RS east $= 8$
ST south-east $= 7$

PT is the resultant velocity. It will be found by measurement to be rather greater than 19.

Examples for Exercise—

1. Find by means of a figure the resultant of the following velocities :—6 north, 2 east, 2 south, 5 west.

2. Find the resultant of velocities 3, 4, and 5 in the directions of the sides of an equilateral triangle taken in order.

Examination on Sections 9-14

1. When can an extended body be said to have a certain velocity? Give examples of bodies which have a motion of translation only, and of others which have a motion of translation combined with rotation.

2. Define Component and Resultant Velocities, and give examples.

3. Prove that, if two component velocities are in the same direction, the resultant is their sum.

4. Prove that, if two component velocities are in opposite directions, the resultant is their difference.

5. Prove that a point which has two equal and opposite velocities is really at rest.

6. Enunciate the proposition known as the Parallelogram of Velocities.

7. Prove that, if two velocities be represented by two adjacent sides of a parallelogram, the resultant is the diagonal through their point of intersection.

8. Enunciate the proposition known as the Triangle of Velocities.

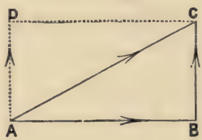
9. Prove that, if two velocities are represented by two sides of a triangle taken in order, the third side of the triangle is their resultant.

10. Prove that the resultant of three velocities, proportional and parallel to the sides of a triangle taken in order, is zero.

11. Show how to find the resultant of any number of velocities co-existing in a point.

12. When any number of simultaneous velocities exist in a point, what is the condition that their resultant vanishes?

15. Resultant of two Velocities at Right Angles.—If the component velocities AB and AD, or AB and BC, are at right angles, the parallelogram ABCD is a rectangle, and the triangle ABC is right angled.



∴ By Geometry (Euclid, I. 47),

$$AC^2 = AB^2 + BC^2,$$

or, if v_1 and v_2 be the components and v the resultant velocity,

$$v^2 = v_1^2 + v_2^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence, extracting the square root of each side,

$$v = \sqrt{v_1^2 + v_2^2}.$$

Example—

A ship sails eastwards with velocity 4, and is at the same time carried northwards by a current with velocity 3. Find its resultant velocity.

Here $v_1 = 4$, $\therefore v_1^2 = 16$. $v_2 = 3$, $\therefore v_2^2 = 9$.

And by (6) $v^2 = v_1^2 + v_2^2 = 25$ $\therefore v = 5$.

The direction of the ship's resultant motion is that of AC in the above figure, if AB = 4 and BC = 3.

Examples for Exercise—

1. A ship is sailing at the rate of 12 miles an hour. A sailor walks across the deck at the rate of 5 miles an hour. Find the resultant velocity of the sailor, showing its direction by means of a figure.

2. A man swims across a river with velocity 3·6. The velocity of the current is 10·5. Find the resultant velocity of the man.

3. If the river in the last example be 972 feet broad, find how far the man is carried down by the current.

16. Resultant of two Equal Velocities.—*If two component velocities be equal, the resultant will bisect the angle between them.*

In the parallelogram of velocities ABCD, let AB = AD, then BC = CD, and the triangles ABC, ADC have three sides of the one equal to three sides of the other,

∴ the angle DAC = the angle BAC (Euclid, I. 8).

17. Resultant of any two Velocities.—Let AB and AD, or AB and BC, be any two velocities whose directions form a given angle DAB, and let AC be their resultant.

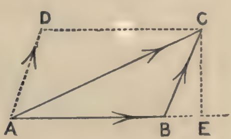


Figure 1.

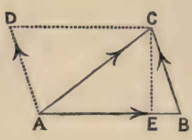


Figure 2.

From C draw CE perpendicular to AB, meeting it, produced if necessary, in E. By Geometry—

(1) If DAB be acute, as in figure 1, then ABC is obtuse.
$\therefore AC^2 = AB^2 + BC^2 + 2AB.BE$ (Euclid, II. 12). . (7)

(2) If DAB be obtuse, as in figure 2, then ABC is acute.
$\therefore AC^2 = AB^2 + BC^2 - 2AB.BE$ (Euclid, II. 13). . (8)

Equation (7), or (8), enables us to calculate the value of AC, provided we know the length of BE. This can be found by elementary Geometry when the angle DAB is one of the following— 30°, 60°, 120°, 150°, 45°, 135°.

In the first four cases the triangle BCE is the half of an equilateral triangle, and its hypotenuse BC is equal to twice its shortest side; in the last two cases CE is equal to BE, and their common value is found by Euclid I. 47. Thus in all cases BE can be found.

Graphic Method.—The value of AC may always be found approximately by construction, the angle DAB being formed with the aid of a *protractor*, or otherwise, and the lengths of AB and AD measured by means of a *scale.* AC is then found by direct measurement, using the same scale.

The trigonometrical expression for the resultant of any two velocities is given in the Appendix, § 98.

Examples—

1. Find the resultant of two velocities, 10 and 6, whose directions form an angle of 60°.

In figure 1, let AB=10, BC=6, $\angle$ DAB= $\angle$ CBE=60°. Then CBE is the half of an equilateral triangle.

$$\therefore BE = \tfrac{1}{2} BC = 3.$$

By (7) $$AC^2 = AB^2 + BC^2 + 2AB \cdot BE,$$
$$= 100 + 36 + 60 = 196.$$
$$\therefore AC = 14.$$

2. Find the resultant of velocities 7 and 4, at an angle of 135°.

In figure 2, let AB=7, BC=4, $\angle$ DAB=135°.

Then $\angle CBA = 180° - 135° = 45°.$

$$\therefore \angle BCE = 45°.$$
$$\therefore CE = BE.$$

And $BC^2 = BE^2 + CE^2 = 2BE^2.$

$$\therefore BE^2 = \tfrac{1}{2}BC^2 = 8.$$
$$\therefore BE = \sqrt{8} = 2\sqrt{2}.$$

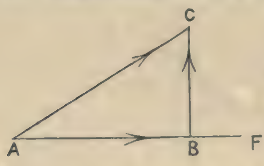
By (8) $$AC^2 = AB^2 + BC^2 - 2AB \cdot BE,$$
$$= 49 + 16 - 28\sqrt{2} = 65 - 28\sqrt{2}.$$
$$\therefore AC = \sqrt{65 - 28\sqrt{2}}.$$

Its value may be found arithmetically to any required degree of approximation by extracting the necessary roots.

Examples for Exercise—

1. Find the resultant of velocities 7 and 8 at an angle of 60°.
2. Find the resultant of velocities 8 and 5 at an angle of 120°.
3. Find the resultant of two unit velocities—(1) at 30°, (2) at 45°, (3) at 150°.

18. Resolution of a Velocity in two Directions at Right Angles.—Let AC be any velocity. Through A draw any straight line AF, and from C draw CB perpendicular to AF. Then the velocity AC is the resultant of the velocities AB and BC. These velocities are called the resolved parts or the rectangular components of AC along and perpendicular to AF.



Examples—

1. Let AC be a velocity of 5, and let AB, one of its rectangular components, be equal to 4. Find BC.

$$BC^2 = AC^2 - AB^2 = 25 - 16 = 9,$$
$$\therefore BC = 3$$

2. Let AC be a unit velocity, and let the $\angle$ CAF = 30°. Find the components of AC along and perpendicular to AF.

The angles CAB and ACB together make up a right angle.

$$\therefore \angle ACB = 60°.$$

$\therefore$ ACB is half an equilateral triangle, and $CB = \frac{1}{2}AC = \frac{1}{2}$.

$$\text{Also } AB^2 = AC^2 - CB^2 = 1 - \tfrac{1}{4} = \tfrac{3}{4}.$$
$$\therefore AB = \tfrac{1}{2}\sqrt{3}.$$

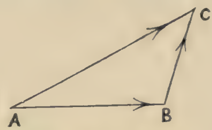
Examples for Exercise—

1. A vertical velocity is equal to 4. Find its component along a line which makes an angle of 30° with the horizon.

2. A velocity of $3\sqrt{2}$ is in a direction inclined at an angle of 45° to the horizon. Find its horizontal component.

19. Composition and Resolution of Variable Velocities. Variable velocities may be compounded and resolved in the same manner as has been shown in § 12 to hold good for uniform velocities. For the effect of a variable velocity at any instant is the same as that of an equal uniform velocity, and in compounding we are only concerned with the instantaneous effect.

20. Change of Velocity.—Let the velocity of a point at a given instant be AB. If the velocity be uniform AB will continue to represent it, but if it be variable, then after an interval of time it will be found to have changed in magnitude, or in direction, or in both.



Let AC be the new velocity. The change in the velocity is the same as would have been caused by combining with AB a velocity BC. BC is therefore called the change of velocity during the interval. The change of velocity is defined when we know its *direction* and *magnitude.*

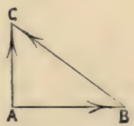
Examples—

1. A stone is thrown up with velocity 12. After an interval it is found to be descending with velocity 8. Find its change of velocity.

From A draw AB to represent the original, AC the changed velocity. BC will represent the change of velocity.

Here AB is vertically upwards $=12$
AC „ „ downwards $=8$
$\therefore$ BC „ „ „ $=20$

2. A velocity of 4 east is changed into 3 north. Find the change of velocity.



Drawing the figure we see that the triangle ABC is right angled at A.

$$\therefore BC^2=AB^2+AC^2=16+9=25.$$
$$\therefore BC=5.$$

Its direction is given by the figure.

Examples for Exercise—

1. A point revolves in a circle with unit speed. Find the *change of velocity* during an interval in which an arc of 60° is described.

2. Find the change of velocity during an interval in which the same point describes a quadrant. (A quadrant is an arc of 90°.)

21. Acceleration.—*When the velocity of a point varies, the rate of change of velocity is called the* ACCELERATION. Like Velocity it is *uniform* or *variable*. UNIFORM ACCELERATION *is measured by the change of velocity in a unit of time.*[1] A point has therefore unit acceleration when it receives in each second a change of velocity (see § 20) of one foot per second. Like velocities, accelerations are represented by straight lines in their directions, and proportional to their magnitudes. They are compounded and resolved in the same way as velocities. Thus we have the *Parallelogram of Accelerations*, the *Triangle of Accelerations*, and the *Polygon of Accelerations.* A velocity is said to be *accelerated* whenever it is changed either in magnitude or in direction, or in both; thus a velocity undergoing acceleration may have its magnitude increased or diminished, or its direction may be changed while its magnitude is left unaltered, or the velocity may be changed both in direction and magnitude.

22. Uniform Acceleration in the Direction of Motion.—Let a point be in motion with velocity V, and let it have a uniform acceleration α in the direction of motion. Then the increase of velocity in each second is equal to α.

At the beginning its velocity is V, after one second it is $V+\alpha$, after two seconds it is $V+2\alpha$, and so on; and after t seconds it is $V+t\alpha$, or as it is usually written $V+\alpha t$.

If, then, the velocity after t seconds be called v,—

$$v=V+\alpha t. \qquad (9)$$

To find the space passed over, let us first find the average velocity. Since the velocity increases uniformly, the average velocity during any interval is equal to the velocity at the middle of that interval; or, what is the same thing, half the sum of the velocities at the beginning and end of the interval.[2]

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[1] The measurement of variable acceleration and the case of uniform circular motion are discussed in the Appendix, §§ 114, 115.

[2] The same is true of any quantity which increases uniformly. Thus the average value of a man's age between his 20th and 30th year is $\frac{20+30}{2}$ or 25, and is so solely because age is a quantity which increases uniformly.

Thus, to find the average velocity during t seconds. we have—

$$\begin{aligned} \text{velocity at beginning} &= V, \\ \text{velocity at end} &= V + at, \\ \therefore \text{their sum} &= 2V + at. \end{aligned}$$

Dividing this sum by two, we find—

$$\text{the average velocity} = V + \tfrac{1}{2}\, at.$$

Also, if s be the space passed over, then by the definition of average velocity, § 7 and formula (1), we have—

$$s = (V + \tfrac{1}{2}at)t,$$

or $$s = Vt + \tfrac{1}{2}at^2. * \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equation (9) expresses the relation between v and t, and equation (10) expresses that between s and t. By combining them we may eliminate t and obtain an equation connecting v and s.

For since $$v = V + at,$$

$$\therefore v^2 = V^2 + 2aVt + a^2t^2 = V^2 + 2a\,(Vt + \tfrac{1}{2}\, at^2),$$

or $$v^2 = V^2 + 2as. \quad . \quad . \quad . \quad . \quad (11)$$

Examples—

1. A body has a uniform acceleration 12. If its initial velocity be 7, find its velocity after 5 seconds.

By (9) $$v = V + at,$$
here $$V = 7, \qquad a = 12, \qquad t = 5.$$
$$\therefore v = 7 + 12 \times 5 = 67 \text{ feet per second.}$$

2. Find the space which the same body passes over during the five seconds.

By (10) $$s = Vt + \tfrac{1}{2}at^2,$$
or $$s = 7 \times 5 + \tfrac{1}{2} \times 12 \times 25 = 35 + 150 = 185 \text{ feet.}$$

---

* This formula was first obtained by Galileo. A graphic representation of it will be found in the Appendix, § 101. Atwood's machine for illustrating uniform acceleration is described in the Appendix, § 102.

3. A body starts from rest with a uniform acceleration g. Find its velocity after t seconds, and the space passed over in that time.

Here $\quad a=g$, and $V=0$.

By (9) $\quad v=V+at=0+gt$.

By (10) $\quad s=Vt+\frac{1}{2}at^2=0+\frac{1}{2}gt^2$.

By (11) $\quad v^2=V^2+2as=0+2gs$.

Hence we obtain the equations $v=gt$, $s=\frac{1}{2}gt^2$, $v^2=2gs$.

We shall afterwards find that they apply to the case of a body falling from rest under the action of gravity.

4. A body starts from rest with a uniform acceleration of 32·2. Find its velocity after 8 seconds.

Here the equations are those of example (3).

$$v=gt, \quad s=\tfrac{1}{2}gt^2.$$

Also $\quad g=32{\cdot}2$, and $t=8$.

$$\therefore v=257{\cdot}6, \text{ and } s=1030{\cdot}4.$$

5. A body starts from rest with uniform acceleration $g=32{\cdot}2$. Find its velocity after it has passed over a distance $s=100$.

Here $\quad v^2=2gs$. (See Example 3.)

Substituting the values of g and s, we find $v^2=6440$. Extracting the root of both sides, we have $v=\pm 80{\cdot}3$ approximately.

Examples for Exercise—

1. A body starts with velocity 7, and has a uniform acceleration $\frac{1}{2}$. Find the velocity after 12 seconds, and the space passed over in that time.

2. A body starts from rest, and has a uniform acceleration 10. Find in what time it describes 125 feet, and what is its velocity at the end of that time.

3. A body starts with a velocity 48·3, and has a uniform acceleration 32·2. How far will it travel in 10 seconds, and what will be its velocity after that interval?

4. A stone falls from rest with a uniform acceleration of 9·8 metres per second per second. What will be its velocity after falling 10 metres?

23. Uniform Acceleration opposite to the Direction of Motion.—Let a point be in motion with velocity V, and let it have a uniform acceleration a opposite to the direction of motion. Then the decrease of velocity in each second is a, or the acceleration in the direction of motion is $-a$. Substituting this value in equations (9), (10), (11), we obtain—

$$\left.\begin{aligned} v &= V - at. \\ s &= Vt - \tfrac{1}{2}at^2. \\ v^2 &= V^2 - 2as. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Examples—

1. A body has a velocity 10, and an acceleration 2 opposite to its motion. Find how far it will go in 3 seconds, and its velocity at the end of that time.

Here by (12) $v = V - at$
$= 10 - 2 \times 3 = 4.$

Also $s = Vt - \frac{1}{2}at^2$
$= 10 \times 3 - \frac{1}{2} \times 2 \times 9 = 21.$

2. Find after what time the body will come to rest.
Here we have to find what value of t makes $v = 0$.

That is $V - at = 0.$

This may be written $V = at,$

$$\therefore t = \frac{V}{a} = \frac{10}{2} = 5 \text{ seconds.}$$

3. A body starts with velocity V in a certain direction, and has a uniform acceleration g in the opposite direction. Investigate its motion.

Here equations (12) become

$$\begin{aligned} v &= V - gt. \\ s &= Vt - \tfrac{1}{2}gt^2. \\ v^2 &= V^2 - 2gs. \end{aligned}$$

From these we see that v, which at first $= V$, gradually gets smaller until it is equal to nothing, when $t = \frac{V}{g}$ and $s = \frac{V^2}{2g}$. After that v is negative, or the body begins to move in the direction opposite to its former motion.

We shall afterwards find that this is the case of a body thrown upwards, and acted on by gravity.

4. A body starts with velocity 80·5, and a uniform acceleration

32·2 opposite to its direction of motion. Find its velocity and position after 4 seconds. Here the above equations give

$$v = 80{\cdot}5 - 32{\cdot}2 \times 4,$$
$$= -48{\cdot}3.$$
$$s = 80{\cdot}5 \times 4 - \tfrac{1}{2} \times 32{\cdot}2 \times 16,$$
$$= 64{\cdot}4.$$

∴ after 4 seconds the body will be returning with a speed **48·3,** and will be 64·4 feet from the starting point.

Examples for Exercises—

1. A body has a velocity 16, and a uniform acceleration 3 opposite to the direction of motion.
   (1) How far will it go in 3 seconds?
   (2) What will be its velocity after 3 seconds?
   (3) When will the body stop?
   (4) How far will it go before stopping?
   (5) What will be its velocity and distance after 8 seconds?
   (6) When will it return to its starting point, and with what velocity?
   (7) What will be its velocity at a distance 10 from the starting point?

2. A body starts with velocity ng, and has a uniform acceleration g opposite to the direction of motion.
   (1) Show that it will be at rest after n seconds.
   (2) Find the distance reached in that time.
   (3) Show that it will take n seconds to return from that distance.
   (4) Show that it will reach the starting point again with the same speed with which it left it.

Examination on Sections 15-23

1. Find the resultant of two velocities at right angles.

2. Prove that the resultant of two equal velocities bisects the angle between them.

3. Find the resultant of any two velocities whose directions form an acute angle.

4. Find the resultant of any two velocities whose directions form an obtuse angle.

5. What is meant by resolving a velocity in two directions at right angles?

6. Prove that variable velocities may be compounded and resolved in the same manner as uniform velocities.

7. Define Change of Velocity.

8. Define acceleration. How is it measured when uniform?

9. Prove that with uniform acceleration a in the direction of motion, if V be the initial velocity and v the velocity after t seconds,

$$v = V + at.$$

10. When a point moves with uniformly accelerated velocity during any interval of time, how is the average velocity obtained? At what time was the point moving with the average velocity?

11. Find the space passed over in time t by a point moving with initial velocity V and uniform acceleration in the direction of motion.

12. Form the equations of motion of a point which has uniform acceleration opposite to the direction of motion.

GENERAL EXAMPLES IN KINEMATICS

[It is not necessary that the student should solve all the general examples. He should merely write out a sufficient selection to make sure that he has mastered the principles of the chapter. The others may be useful when he comes to revise.]

1. Express the following velocities in feet per second :—(1) 12 miles an hour: (2) 2500 miles in 24 hours: (3) 9 inches in 14 minutes: (4) 25 feet a minute: (5) 15 centimetres in 2 minutes: (6) a decimetre a second: (7) 10 feet in $\frac{1}{5}$ of a second: (8) 75 feet in 2 hours: (9) 91,760,000 miles in 8 minutes and 16 seconds: (10) $\frac{1}{2}$ of $\frac{1}{3}$ of an inch in $\frac{1}{8}$ of $\frac{1}{9}$ of a second.

2. The following velocities are in feet per second. Express them in miles per hour :—

(1) channel steamer 22 :
(2) carrier pigeon 120 :
(3) sound 1122 :
(4) a rifle bullet 1342 :
(5) the earth's centre 101,090.

3. Find the number of miles travelled in 20 minutes by bodies moving with the velocities mentioned in question 2. (5 answers.)

4. Find the number of feet travelled in $\frac{1}{10}$ of a second by the same bodies. (5 answers.)

5. Find the number of seconds each of these bodies takes to pass over 22 feet. (5 answers.)

6. Find in feet per second the velocity of a point which passes over—

(1) a feet in b seconds :

(2) $2a$ feet in $3a$ seconds :

(3) a yards in b minutes :

(4) 10 centimetres in $a+b$ seconds :

(5) a^2b feet in ab^2 seconds :

(6) a^2-b^2 feet in $a+b$ seconds.

7. Find how many seconds a point takes to describe a circle with uniform speed in the following cases :—

(1) Radius 1, speed 1.

(2) „ a, „ v.

(3) Radius $\frac{1}{2}$, speed π.

(4) „ 7, „ 3π.

NOTE.—π represents the ratio of the circumference to the diameter of a circle, or $2\pi r$ is the circumference of a circle whose radius is r. $\pi = 3\frac{1}{7}$ or more approximately 3·1416.

8. A man walks 3 miles in an hour, and a fourth mile in 12 minutes. Find his average velocity—

(1) in feet per second :

(2) in miles per hour :

(3) in centimetres per second :

(4) in kilometres per hour.

9. A man walks 10 miles in $2\frac{1}{2}$ hours, walking for the first $\frac{3}{4}$ of an hour at the rate of 5 miles an hour. Find his average velocity during the remainder of the time—

(1) in feet per second :

(2) in miles per hour.

10. If the velocity of a bullet is 1250 feet per second, and that of sound 1200, how much time will elapse, on a range of 1000 yards, between the time that the bullet strikes the target and the time that the sound of the discharge reaches the target ?

11. A walks along a road at the rate of 4 miles an hour. After he has gone 7 miles B follows him, walking at the rate of 5 miles an hour. How far will B have walked when he overtakes A ?

12. Compound the following velocities in the same straight line :—

(1) 3, 5, 7, and 12, all forwards :

(2) 6, 8, and 9, all backwards :

(3) 4 and 7 forwards, 9 backwards:

(4) 7 forwards and 13 backwards:

(5) $a+b$ forwards and $a-b$ backwards:

(6) $x(x-1)(x-2)$ forwards and $x(x+1)(x+2)$ backwards.

13. A boat is rowed at the rate of 6 miles an hour in still water. How long will it take to row it 5 miles up a river against a current flowing 5 feet in a second?

14. If the boat (in last example) be rowed down the same stream, how far will it go in an hour?

15. A runs 100 yards in 12 seconds, B runs a mile in 5 minutes. Find the ratio of their velocities.

16. Find the resultant of two velocities, v_1 and v_2, whose directions are at right angles, when their values are—

(1) 6 and 8:

(2) 1, $\frac{3}{4}$:

(3) 8, 15:

(4) $5x$, $\frac{11}{12}x$:

(5) 20, 21:

(6) $\left.\begin{array}{l} v_1+v_2=17 \\ v_1-v_2=\ 7 \end{array}\right\}$:

(7) $\left.\begin{array}{l} 2av_1+bv_2=a^3 \\ av_2-2bv_1=b^3 \end{array}\right\}$.

17. Find by means of accurately drawn figures the directions and magnitudes of the resultants of the following velocities:—(1) 3 and 4 at 45°: (2) 4 and 6 at 105°: (3) 2, 5 and 7 in the directions of the sides of an equilateral triangle taken in order: (4) 3, 4, and 5 in the directions of three consecutive sides of a square: (5) 2, 3, 2, 4 and 3 in the directions of five consecutive sides of a regular octagon.

18. Find the resultants of the following pairs of velocities at an angle of 60°:—(1) 5 and 16: (2) 1, $1\frac{1}{7}$: (3) $2x+1$, $x(3x+2)$.

19. Find the resultants of the following pairs of velocities at an angle of 120°:—(1) 16 and 21: (2) $8\frac{1}{4}$, 10: (3) $2x+1$, $x(x+2)$.

20. Find the resultants of velocities in the directions of the sides of an equilateral triangle taken in order, whose values are—(1) 4, 4, and 5: (2) 9, 14, and 17.

21. Find the resultants of velocities in the directions of the sides of a square taken in order, whose values are—(1) 2, 2, 2, and 1: (2) 3, 2, 1, and 0: (3) 7, 5, −3, and −2.

22. A bullet has velocities of 400 nortnwards, of 300 eastwards, and of 1200 vertically upwards. Find its resultant velocity.

23. A ship is sailing at the rate of $11\frac{1}{4}$ miles an hour. A sailor climbs a mast 55 yards high in 24 seconds, find his rate of motion relatively to the surface of the earth.

24. Two men walk along two straight roads, which form a right angle, at the rate of 4 and $4\frac{1}{5}$ miles an hour. If they started from the intersection of the roads, how far apart will they be five hours afterwards?

25. The resultant of two velocities bisects the angle between them. Prove that the velocities are equal.

26. The resultant of two equal velocities is equal to either of them. Find the angle between the components.

27. A velocity of 12 is changed into a velocity of 5 at right angles to its former direction. Find the change of velocity.

28. A velocity of 13 has a horizontal component of 12. Find its vertical component.

29. Find the change of velocity in each of the following cases:—(1) 4 north into 5 north: (2) 4 north into 5 south: (3) 1 east into $\sqrt{2}$ north-east: (4) 1 east into 1 south.

30. A velocity represented by one side of an equilateral triangle is changed into that represented by another side taken in the opposite direction round the triangle. Find the change in magnitude and direction.

31. A velocity v has v_1 and v_2 for its rectangular components. Find v_2 in the following cases:—(1) $v=12\frac{1}{2}$, $v_1=7\frac{1}{2}$: (2) $v=29$, $v_1=21$: (3) $v=73$, $v_1=55$.

32. A velocity of 6 becomes one of 3 at an angle of 60° to its original direction. Find the change of velocity.

33. A particle moves with uniform acceleration during 3 seconds. Its initial velocity is 7 and its final velocity is 13. Find—(1) the average velocity: (2) the space passed over: (3) the acceleration.

34. A body starts with velocity 4, and has a uniform acceleration $1\frac{1}{2}$ in the direction of motion. Find—(1) the velocity after 10 seconds: (2) the average velocity during that time: (3) the space described in that time.

35. A body starts from rest with uniform acceleration 32. Find—

(1) how far it travels in one second:

(2) what distance is passed over in the second second:

(3) its velocity after $5\frac{1}{2}$ seconds:

(4) the space described in $5\frac{1}{2}$ seconds:

(5) the velocity after travelling a distance 9.

36. A body has an initial velocity $2\frac{1}{2}g$ and a uniform acceleration g in the direction of motion. Find—

(1) its velocity after 4 seconds :
(2) the space described in that time.

37. A body starts from rest with uniform acceleration $g=32\cdot2$. Find—

(1) the velocity after 5 seconds :
(2) the velocity after n seconds :
(3) the space described in n seconds :
(4) after what time the velocity will be $4g$:
(5) after what time the distance travelled will be $18g$.

38. A body has velocity 29 and a uniform acceleration 3 opposite to the direction of motion. Find—

(1) the velocity after 7 seconds :
(2) the average velocity during that time :
(3) the space described in 7 seconds :
(4) after what time the body is at rest :
(5) the whole space described before it comes to rest.

39. A body has initial velocity ng and uniform acceleration g opposite to the direction of motion. Show—

(1) that its velocity after $n+r$ seconds is equal and opposite to its velocity after $n-r$:

(2) that its distances from its starting point at the above-mentioned times are equal.

40. Show that there is only one point within a triangle such that velocities represented in magnitude and direction by the lines drawn from it to the angles would, if generated simultaneously, leave the body at rest.

41. A body is moving with velocities represented by three sides of a square taken in the same order and the fourth in the opposite. Find its real velocity.

42. How do vou express a velocity of 16 feet per second in yards per hour?

43. If a particle moves uniformly in a circle with speed v, what is its change of velocity after it has described an arc of 120°?

44. A point uniformly accelerated, moving with a velocity of 7 feet per second, passes over 3 feet, and then its velocity is 8 feet per second. What is the acceleration?

45. Find the average velocity of a postman who walks at 4 miles an hour, but loses half a minute in that time at each of 20 houses where there are no letter-boxes.

46. If a point has a velocity of 36 yards per minute in a direction making an angle of 30° with the vertical, find its horizontal component.

47. A particle whose initial velocity is 6 feet per second moves for 5 seconds under a uniform acceleration, and acquires a velocity of 21. Find the acceleration, the average velocity, and the space described.

48. A train A runs at the rate of 45 miles an hour, while another train B runs in the opposite direction at the rate of 35 miles an hour. With what velocity will the train B appear to pass a traveller in A?

49. When a train is moving with a velocity of 20 *miles an hour* alongside a station platform, the guard throws out a parcel with a horizontal velocity of 16·9 *feet per second* in a direction at right angles to the motion of the train. In what direction will the parcel move along the platform, and what will be its velocity at the beginning?

50. A local train makes its run of 13 miles and back once in every two hours, stopping half a minute at each of 14 intermediate stations, and 10 minutes at either terminus. What is its average velocity when in motion?

51. A boat is rowed in the direction of right across a river with a velocity of 8 miles an hour. The river has a velocity of 2 miles an hour, and a breadth of 800 feet. Find how far the boat will be carried down by the time it reaches the opposite bank.

52. The velocity of light is 186,000 miles per second; express this in metres per second. (Take 1 metre = 3·3 feet.)

53. What is meant by saying that an acceleration is 32, the units being feet and seconds? Express the same acceleration in terms of yards and minutes.

CHAPTER III

KINETICS

24. Preliminary.—In the preceding chapter we have considered *Motion* from a purely mathematical point of view. We have now to apply its results to the motion of *material bodies* under the action of *force.* The introduction to Dynamics is supplied by Newton in his *Principia* by means of definitions and axioms. The axioms are three in number, and are called the Laws of Motion.

25. Force.—As before, we define FORCE to be *any cause which alters a body's state of rest, or of uniform motion in a straight line.*

A force is determined when its *point of application*, its *direction*, and its *magnitude* are given. It is therefore completely represented by a straight line, drawn from the *point of application* in the *direction* in which the force acts, and of *length* proportional to the magnitude of the force.

26. First Law of Motion.—*Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it is compelled by forces to change that state.*

The meaning of this law may be illustrated by the following example:—If we propel a curling-stone along a horizontal surface of ice, its velocity gradually becomes less. This is due to two causes, the roughness of the ice and the resistance of the air. The smoother the ice and the rarer the atmosphere, the more nearly uniform is the velocity of the curling-stone. The first law of motion asserts that if we could obtain perfectly smooth ice and a perfect vacuum, the velocity of the curling-stone would be uniform.

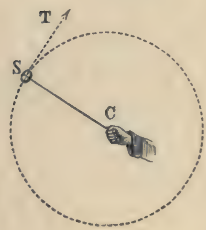
27. Inertia.—*The property of matter in virtue of which it tends to maintain its state of rest or of uniform motion in a straight line is called its* INERTIA.

The first law of motion asserts the existence of *Inertia*, and that, on account of inertia, force is required to change a body's state of rest or motion.

Thus the inertia of our bodies is the cause of the jerks we receive when sitting in a carriage which is suddenly set in motion, or suddenly stopped. In the first case our state is one of rest, and tends to remain so, in the second it is one of motion, and tends to continue so.

28. The so-called Centrifugal Force.—The first law of motion shows us that, whenever a body moves in a curve, it does so because of some force which continually causes the body to change the direction of its motion, and thus prevents it from continuing to move in a straight line.

As an example, consider the case of a stone S tied to a string, SC, and swung in a circle round the hand C. At any instant the stone is moving in the direction of the tangent ST, and would continue to move in that line were it not prevented from doing so by the tension of the string. If, however, the string break, the stone will fly off along the tangent.



In this case, the tension of the string is the force which, at each instant, prevents the stone from continuing its natural motion in a straight line. It was at one time supposed that this force was rendered necessary by a centrifugal force, called into action by the fact of the body's moving in a curve; and the name is still generally applied to the tendency such a body has to fly off at a tangent, and *thus* increase its distance from the centre. The term is doubly misleading. First, it is not an effect of *force*, but a result of the inertia of the moving body; secondly, it is not, properly speaking, *centrifugal*, for the tendency is not to motion along CS produced, but along ST.

The following are examples of bodies which move in curves owing to the action of some force :—

The earth in its annual motion round the sun has the direction of its motion continually changed by the sun's attraction. The particles of a fly-wheel are forced by cohesion to move in a circle; if the speed of its rotation be made so great as to overcome the force of cohesion, the wheel will fly asunder. When a wet mop is trundled, the adhesion of the drops of water is not sufficient to prevent their flying off at a tangent.[1]

29. The Measurement of Time.—The first law of motion gives us a means of measuring time. It tells us that every body, not acted on by force, moves uniformly, that is, passes over equal spaces in equal times. Thus we may define equal times as those in which such a body passes over equal spaces.

In the case of a rotating body, the internal forces cause the different particles of which the body is composed to move in circles, but do not alter their speeds. If, then, there are no external forces tending to alter the rate of rotation, the rotating body turns through equal angles in equal times. This condition is more nearly fulfilled in the earth than in any other body which we can easily observe.

We may therefore define equal intervals of time, as times during which the earth turns through equal angles.

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[1] For the magnitude of the force required to make a body move in a circle, see Appendix, § 115.

Examination on Sections 24-29

1. How does Kinetics differ from Kinematics?

2. Define Force. What are the three elements which specify a force? Show that a straight line completely represents a force.

3. Enunciate the first law of motion, and illustrate it by an example.

4. Define Inertia. Give examples which prove its existence.

5. What is meant by Centrifugal Force? Show that the term is a misleading one.

6. Show that the first law of motion enables us to define equal portions of time.

30. Mass, Momentum.—We have seen that every body possesses *Inertia*, in virtue of which it tends to maintain its state of rest or motion. When, in speaking of a body, we put all other differences out of account, and consider it only as possessing more or less inertia, we say that the body consists of a certain *Mass*.

MASS is another name for *Quantity of Matter*, the quantity being measured as shown in § 33. The British unit of mass is the imperial pound. The standard pound is a mass of platinum deposited in the Exchequer Office.

In the metrical system of units the unit of mass is a gramme, which is the thousandth part of a platinum kilogramme[1] deposited in the Archives of France.

The product of the mass and the velocity of a moving body is called its MOMENTUM.

Thus if a body containing m units of mass be in motion with velocity v, and its momentum be called M,—

$$M = mv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Examples—

1. Masses of 10 and 15 pounds have velocities 7 and 5 respectively. Compare their momenta.

Let m_1, m_2 be the masses, v_1, v_2 their velocities, and M_1, M_2 their momenta.

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[1] A kilogramme = 2·20462 lbs.

Then by (13) $M_1 = m_1 v_1 = 70, \qquad M_2 = m_2 v_2 = 75.$

$$\therefore \frac{M_1}{M_2} = \frac{70}{75} = \frac{14}{15}.$$

2. The momentum of a mass m, moving with velocity v is communicated to a mass m'. Find the velocity produced.

Let v' be the velocity produced.

Then by (13) $M = mv = m'v', \qquad \therefore v' = \frac{mv}{m'}.$

Examples for Exercise—

1. A mass of 4 pounds has a velocity of 1000 feet per second. A mass of 100 has the same momentum, find its velocity.

2. A mass of 6 lbs. has velocity 12. If the mass were divided into 3 portions of 1, 2, and 3 lbs., what must be their respective velocities, in order that each portion may have $\frac{1}{3}$ of the original momentum of the whole mass?

31. Second Law of Motion.—*Change of Momentum is proportional to the force which causes it, and is in the direction of the straight line in which the force acts.*

In other words, a given force acting on a body for a given time always produces the same change of momentum, whatever be the mass of the body on which it acts, and whether that body be at rest or in motion in any direction with any velocity.

32. The Measurement of Force.—The second law of motion gives us a means of comparing different forces. Apply different forces to the *same* body and observe the accelerations produced. Since the mass is the same in each case, the momenta generated in equal times are proportional to the accelerations, but by the second law of motion the forces are proportional to the momenta, and therefore in this case to the accelerations.

A uniform force is measured by the momentum which it generates in a unit of time.

Let f be a uniform force, and let it act on a mass m, and produce an acceleration a. Then a is the change of velocity in a unit of time (§ 21). Therefore ma is the change of momentum, or the momentum generated in a unit of time.

$$\therefore f = ma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

It follows that our *Unit of Force is that which acting on unit mass produces unit acceleration.*[1]

33. The Measurement of Mass.—The second law also enables us to compare masses. For if equal forces act on different masses, the accelerations produced are inversely as the masses.

Again, if different forces applied to these masses give them the same acceleration, the forces must be proportional to the masses. Thus we see that the mass of a body may be measured by the force required to give it a certain acceleration. In other words—*the mass of a body is measured by its inertia.*

Example—

The same force is applied to masses of 4, 6, and 8 pounds. When applied to 4, it causes an acceleration 12. Find the force. What acceleration will it give the other masses?

Here	$f = ma = 48.$	
Therefore, when	$m = 6,$	$a = 8.$
And when	$m = 8,$	$a = 6.$

Example for Exercise—

Two forces f_1 and f_2, applied for equal time to masses of 9 and 12 pounds, cause the same acceleration. Find f_1, if $f_2 = 8$.

34. The Force of Gravity.—The great Law of Gravitation announced by Sir Isaac Newton is *Every particle in the universe attracts every other with a force whose direction is that of the line joining the two, and whose magnitude is proportional to the product of their masses divided by the square of their distance from each other.*

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[1] The British unit of force is that which, acting on one pound, gives it an acceleration of one foot per second per second. It is called a Poundal. The Metrical unit is that which, acting on one gramme, gives it an acceleration of one centimetre per second. It is called a Dyne.

A particle of matter near the surface of the earth attracts, and is attracted by, all the particles of which the earth is composed. The resultant (see § 39) of all these forces is a single force, tending to move the particle towards the centre of the earth, and called the force of gravity or the *weight* of the particle.

Let g represent *the weight of unit mass* at a certain place. It follows from the law of attraction that if w be the weight of a mass m at the same place

$$w = mg.^* \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Comparing this equation with equation (14), we see that g also represents *the acceleration caused by gravity* in a falling body at that place.

Owing to the figure, density, and rotation of the earth, the value of g differs at different parts of the earth's surface. At the equator it is 32·09. It gradually increases with the latitude, and is 32·25 at the pole. The average value for the British Isles at the level of the sea is 32·2.

Examples—

1. Two heavy particles, whose masses are m and m', when placed at a distance d, attract each other with a force f. Find their attraction at a distance d'.

Since the attraction is proportional to the product of the masses divided by the square of the distance, we have

$$f = k\frac{mm'}{d^2}$$

where k is a constant quantity.

Let f' be the attraction at the distance d', then similarly

$$f' = k\frac{mm'}{d'^2}.$$

Multiplying both sides of the first equation by d^2, and of the second by d'^2, we get

$$d^2 f = d'^2 f' \text{ or } f' = \frac{d^2}{d'^2} f.$$

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* It was experimentally demonstrated by Newton that pendulums of the same length vibrate in equal times whatever be the material of which their bobs are formed. Hence he concluded that the attraction of the earth on any body is proportional to the mass of the body.

2. Heavy particles, whose masses are 3 and 4, and whose distance is 5, attract each other with a force A. Find the attraction of particles whose masses are 43 and 50 at a distance 100.

$$A=k\frac{mm'}{d^2}=\frac{12}{25}k, \qquad \therefore k=\frac{25}{12}A.$$

$$A'=k\frac{43\times 50}{10000}, \qquad \therefore A'=\frac{25\times 43\times 50}{12\times 10000}A=\frac{43}{96}A.$$

Examples for Exercise—

1. Two masses at a distance of 20 feet exert an attraction f. Find the attraction at the distance of a mile.

2. The mass of the planet Jupiter is 340 times that of the earth. Taking its distance from the sun as 5 times that of the earth, compare the force with which the sun attracts the earth and Jupiter.

3. The diameter of Jupiter is 11·4 times that of the earth, while its mass is 340 times that of the earth. A man weighs (see § 35) 12 stones on the surface of the earth. Find his weight[1] on the surface of Jupiter.

35. Mass and Weight.—It has been pointed out that an imperial pound, a kilogramme, and their subdivisions are, properly speaking, standards, not of *weight*, but of *mass*.

In an ordinary balance or *pair of scales* (see § 61), the substance to be "weighed" is placed in one scale and the standard mass in the other. The balance shows when the *weights* are equal. If m and m' represent the masses, w and w' the weights, we have

$$w=mg, \qquad w'=m'g;$$

and since $$w=w', \qquad \therefore m=m',$$

or the *mass* of the substance is the same as that of the standard.

Thus when we speak of the weight of a substance being so many pounds, we mean that its mass, as ascertained by common "weighing," is so many pounds. This is quite independent of the value of g, and the same result will therefore be obtained wherever the substance be weighed.

[1] It is here required to find what *mass* is attracted by the earth at its surface with the force with which Jupiter attracts a mass of 12 stones at the surface of Jupiter.

A *spring-balance*, on the other hand, measures the *weight* of a body, and a spring balance, graduated correctly for a certain place, will give wrong readings when used in a different latitude or at a different height. A spring balance may be used to measure other forces besides that of gravity, and it has thus become common to speak of a force of one pound, etc., where what is meant is a force equal to the weight of one pound, and therefore to g poundals.

Example—

A spring balance graduated in London is used at the equator. Find the error in weighing out one pound.

The weight of a pound at the equator is 32 units of force, but the weight in London is 32·2. It will thus require more than a pound at the equator to draw out the spring as far as it was drawn by one pound in London, the excess being $\frac{\cdot 2}{32}$ or $\frac{1}{160}$ of a pound.

Example for Exercise—

A spring balance graduated at the equator is used in London. Find the error in weighing out one pound.

36. Falling Bodies.—We are now able to investigate the motion of bodies falling under the action of gravity.

The earth attracts a mass m, with a force mg, and therefore, as is shown in § 34, causes a uniform[1] acceleration g in the direction of motion. This is the case discussed in § 22; substitute g for a in equations (9), (10), and (11), and we obtain—

$$v = V + gt \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$
$$s = Vt + \tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad . \quad (17)$$
$$v^2 = V^2 + 2gs. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

the equations of motion of a falling body, with an initial

[1] Strictly speaking the acceleration is not uniform, as it varies with the body's distance from the centre of the earth. This variation, however, is very small for ordinary heights, and is therefore here neglected.

velocity V, downwards. If the body fall from rest, $V=0$, and the equations become—

$$\left.\begin{aligned} v&=gt \\ s&=\tfrac{1}{2}gt^2 \\ v^2&=2gs \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Examples—

1. A mass of 6 lbs. falls from rest. How far will it fall in 10 seconds, and what will be its momentum after that time?

By (19) $s=\frac{1}{2}gt^2=\frac{1}{2}\times 32{\cdot}2\times 100=1610$ feet.
Also $v=gt=32{\cdot}2\times 10=322.$
∴ its momentum $mv=6\times 322=1932.$

2. A stone is thrown down a well with velocity 50. If it reach the bottom in 2 seconds, how deep is the well?

By (17) $s=Vt+\frac{1}{2}gt^2=50\times 2+\frac{1}{2}\times 32{\cdot}2\times 4=164{\cdot}4$ feet.

3. A body falls from rest. In what time will it fall 100 feet?

By (19) $s=\frac{1}{2}gt^2$, $\therefore\ t^2=\frac{2s}{g}=\frac{200}{32{\cdot}2}=6{\cdot}21$, approximately.
$\therefore$ $t=\sqrt{6{\cdot}21}=2{\cdot}5$ seconds, approximately.

Examples for Exercise—

1. A stone weighing 4 lbs. is dropped from a balloon at rest, and reaches the earth after 18 seconds. Find the momentum with which it strikes the earth, and the height of the balloon.

2. With what momentum must a bullet of 2 ounces weight be propelled downwards, in order that it may fall 1000 feet in two seconds?

3. A well is 559 feet deep. Find approximately the time which will elapse between dropping a stone into the well and hearing the sound of the splash. (Take the velocity of sound as 1118.)

37. Bodies thrown up.—If a body be thrown vertically upwards, gravity causes a uniform acceleration opposite to the direction of motion, the case discussed in § 23. Substituting g for a in equations (12) we obtain—

$$\left.\begin{aligned} v&=V-gt \\ s&=Vt-\tfrac{1}{2}gt^2 \\ v^2&=V^2-2gs \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Example—

A ball is thrown vertically upwards with velocity 50. Find how high it will rise, and when, and with what velocity, it will return to the ground.

(1.) To find how high the ball will rise.

At the highest point $v=0$, and therefore by the third equation of (20)—

$$V^2-2gs=0,$$

$$\therefore\ s=\frac{V^2}{2g}=\frac{2500}{64{\cdot}4}=38{\cdot}7 \text{ feet, approximately.}$$

(2.) To find when the ball will return to the ground.

As the ball falls s diminishes, and when it reaches the ground $s=0$, and therefore by the second equation of (20)—

$$Vt-\tfrac{1}{2}gt^2=0. \quad \therefore\ V-\tfrac{1}{2}gt=0. \quad \therefore\ V=\tfrac{1}{2}gt.$$

$$\therefore\ t=\frac{2V}{g}=\frac{100}{32{\cdot}2}=3{\cdot}1 \text{ seconds, approximately.}$$

(3.) To find the velocity with which the ball reaches the ground. As before $s=0$, and therefore by the third equation of (20)—

$$v^2=V^2,\ \therefore\ v=+V \text{ or } -V.$$

Since the direction is opposite to that of projection, the answer is—

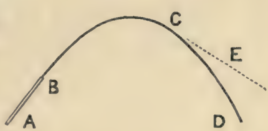
$$v=-V.$$

Thus the ball returns to the ground in the same time that it took to rise to its highest position, and reaches the ground with an equal and opposite velocity to that with which it was thrown up.

Example for Exercise—

With what velocity must a body be thrown vertically upwards that it may remain in the air for 4 seconds, and how high will it be at the end of each second?

38. Parabolic path of a Projectile.—When a body is projected in any direction other than vertical, its position at any moment is found by considering its velocity as resolved in two directions, vertical and horizontal. The vertical velocity is altered by gravity in the manner described in the last two paragraphs, but the horizontal velocity remains constant, as the body is not acted on by any horizontal force. Combining these results, it may be



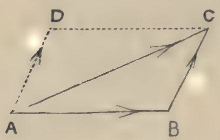
proved that the body describes a parabola, the curve represented in the figure. (See also Appendix, §§ 104, 105, 106.)

39. The Composition of Forces.—It follows from the second law of motion that, when any number of forces act simultaneously on a particle, each of them produces the same acceleration that it would have produced if it had acted singly on the particle at rest. The resultant acceleration is therefore the resultant of these accelerations.

A force which, acting singly, would have produced this acceleration in the particle is called the *Resultant Force*, and the several forces are called *Components.*

Parallelogram of Forces.—*If two forces acting simultaneously on a particle be represented by adjacent sides of a parallelogram, the Resultant Force is represented by the diagonal passing through their intersection.*

Proof.—Let AB, AD represent two forces acting simultaneously on a particle at A. Then since forces are measured by the momenta they produce in unit of time, AB and AD are proportional to the momen and therefore to the accelerations, which these forces acting separately would produce in the particle at A.



But by the parallelogram of accelerations, if AB and AD are the component accelerations, AC is the resultant acceleration, and therefore AC represents the resultant force.

In the same manner, by substituting "force" for "velocity," all the propositions regarding the composition and resolution of velocities, proved in §§ 13, 14, 15, 16, 17, and 18, are proved to be equally true of forces.

Examples—

1. Forces, which are represented in magnitude and direction by all but one of the sides of a polygon taken in order, act simultaneously on a particle ; find their resultant.

Since the sides of the polygon represent the forces, they also represent the accelerations which the forces would produce acting separately. By § 14 the resultant of these accelerations is the remaining side taken in the opposite direction. The resultant force is therefore also represented by that side.

2. Forces 5 and 12 act on a particle at right angles; find their resultant.

Let f be the resultant force. Then, applying § 15 to forces, we have—

$$f^2 = 5^2 + 12^2 = 25 + 144 = 169, \quad \therefore f = 13.$$

Examples for Exercise—

1. Prove that if three forces acting on a particle are represented by three sides of a triangle taken in order, they will be in equilibrium.

2. Find the resultant of forces $3\frac{1}{2}$ and 4 at an angle of 60°.

3. A vertical force is equal to 12. Find its component along a line which makes an angle of 30° with the horizon.

Examination on Sections 30-39

1. What is the British unit of mass?
2. Define Momentum.
3. Enunciate the second law of motion.
4. Show that if different forces are applied to equal masses the accelerations generated are proportional to the forces.
5. How is mass measured—(1) directly, (2) indirectly?
6. Give the law of universal gravitation. What is the average acceleration caused by gravity in the British Islands?
7. Show that a pair of scales determines the mass, and a spring balance determines the weight, of a body.
8. Find the equations of motion of a body falling from rest under the action of gravity.
9. If a body is thrown up with velocity V, show that its velocity after t seconds is $V - gt$, and that its distance from the ground at the same time is $Vt - \frac{1}{2}gt^2$.
10. What is meant by the resultant of any number of forces?
11. Assuming the parallelogram of accelerations, prove the parallelogram of forces.

40. The Third Law of Motion.—*To every action there is always an equal and contrary reaction.*

When one body presses another, it is pressed by that other with an equal force in the opposite direction. A stone resting on a table causes pressure on the table in consequence of its weight, but the table exerts an equal and opposite pressure on the stone, and this pressure, acting on the stone, balances the force of gravity, and the stone remains at rest relatively to the table.

Again, when one body attracts another, it is attracted by that other with equal force in the opposite direction. Thus the moon exercises on the earth the same attraction that the earth exercises on the moon, but in the opposite direction.

Stress.—The mutual action between two portions of matter is called a *Stress.* The third law of motion shows us that all force is of the nature of stress. A force is, in fact, a stress regarded from the point of view of one only of the two bodies between which the stress takes place. Action and reaction are different aspects of the same stress.[1]

41. Recoil.—The third law of motion enables us to investigate the recoil of a gun when a projectile leaves it. Since action and reaction are equal and opposite, the momentum acquired by the gun is equal and opposite to that acquired by the projectile, or if M, m be their masses, and V, v their initial velocities,

$$MV = -mv. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Example—

A gun weighing 100 tons discharges a shot weighing 1000 lbs. If the initial velocity of the shot be 1250, find that of the recoil of the gun.

From (21) $$V = -\frac{mv}{M} = -\frac{1000 \times 1250}{100 \times 20 \times 112} = -5\tfrac{65}{112}.$$

The *minus* sign shows that the direction of the motion of the gun is opposite to that of the bullet.

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[1] See Clerk Maxwell's *Matter and Motion*, Arts. xxxvii. cvii.

Example for Exercise—

A bullet weighing 2 ounces is discharged from a rifle weighing 10 lbs., and the initial velocity of the recoil is observed to be 14; find the velocity with which the bullet leaves the rifle.

42. **Work.**—*A force does* WORK *when the body on which it acts moves in the direction of the force.* Thus when a stone falls vertically downwards, gravity does work.

When a stone slides or rolls down a hill, its velocity may be resolved into two components, a horizontal and a vertical. The vertical component is in the direction of gravity, and therefore gravity does work in this case also.

When a body is raised, it is acted on by a force equal and opposite to that of gravity. This force does work, and the work done is said to be done against gravity.

The work done by a uniform force acting on a body is measured by the product of the force and the distance through which the body moves in the direction of the force.

The scientific unit of work is that done by unit force acting through unit space.[1]

The unit practically used in this country is the *foot-pound*, or the work required to raise a mass of one pound through one foot against gravity. As was shown in § 34, the weight of one pound is equal to g British units of force, and hence a foot-pound is equal to g British units of work. Thus a foot-pound at London is not the same as a foot-pound at Edinburgh, as the value of g is different.

The *rate at which work is done* by an agent is measured by the quantity of work done in a unit of time.

The rate at which an agent works when doing 550 foot-pounds of work per second, is the unit practically used by engineers, and is called a *horse-power*.

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[1] The British unit of work is that done by a Poundal acting through a foot, and is called a Foot-poundal; the Metrical unit is that done by a Dyne acting through a centimetre, and is called an Erg. See Note, § 32.

Examples—

1. A mass of 4 pounds is allowed to slide down an inclined plane whose height is three feet. Express in foot-pounds the work done by gravity.

Since gravity acts vertically downwards, we must multiply the weight of the body in pounds by the distance which the body moves vertically downwards. The work done is therefore 12 foot-pounds.

Similarly if the mass be drawn up the plane 12 foot-pounds of work will be done against gravity.

2. How many scientific units of work are done in one second by an engine of one horse-power?

550 foot-pounds are done in one second.

$\therefore$ $550g$ scientific units of work are done in one second.

Thus the number $550g$ represents the rate of doing work, or, as it is sometimes called, the *power* of the engine.

3. In a waterfall 30 tons of water fall from a height of 50 feet in each minute, and are employed to turn a turbine which transforms $\frac{6}{10}$ of the energy of the water into useful work. Find the horse-power of the turbine.

Here the number of foot-pounds of work done by gravity in each second is $\frac{30 \times 2240 \times 50}{60} = 56{,}000$. $\therefore$ the turbine works at the rate of $\frac{6}{10} \times 56{,}000 = 33{,}600$ foot-pounds a second,

$$\therefore \text{ its horse-power is } \frac{33{,}600}{550} = 61\tfrac{1}{11}.$$

Examples for Exercise—

1. How much work is done against gravity by a man who weighs 14 stones, in climbing a mountain 3000 feet high?

2. An engine of one horse-power is employed to raise a weight of one ton. How high will it raise it in five minutes?

3. A turbine of 10 horse-power transforms 55 per cent. of the energy of a waterfall into useful work. If the height of the fall be 100 feet, find what weight of water falls in each second.

43. Energy.—ENERGY *is capacity of doing work.* It is of two kinds, *Kinetic* and *Potential.*

Kinetic Energy is the capacity which a moving body has of doing work in virtue of its motion. (See Appendix, § 117.)

If, on the other hand, a body is constrained to move against a force, the work done takes the form of *Potential Energy.*[1]

Thus, a mass requires work to raise it to a height, but the raised mass is a store of energy which can be made use of at pleasure.

Bodies have, therefore, Potential Energy in virtue of their position with regard to other bodies which exert force on them.

44. Conservation of Energy.—The third law of motion has a second interpretation. It applies to energy as well as to momentum. This was pointed out by Newton, as has been shown by Thomson and Tait (*Natural Philosophy*, § 269).

The principle of the Conservation of Energy asserts that the amount of energy in the universe is constant, that it may be transmitted from one body to another, or transformed from one form of energy to another, but can neither be increased nor diminished in quantity.

Besides energy of visible motion, Heat, Light, Sound, and Current Electricity are all recognised as forms of Kinetic Energy. Gravity, Chemical Attraction, Electric and Magnetic Attraction depend on Potential Energy.

The behaviour of a common pendulum illustrates the principle of Conservation of Energy. When the pendulum is in the middle of its swing, it possesses a certain amount of kinetic energy. As it approaches its turning point its energy of motion gradually diminishes, and finally disappears; but the bob of the pendulum has been raised a small distance against the force of gravity, and has thus acquired a certain amount of potential energy, which is in its turn reconverted into kinetic energy. These quantities of potential and kinetic energy would be exact equivalents to each other, and the pendulum would continue to swing

[1] This is only true in the case of forces which act equally and in the same direction, if the motion of the body be reversed. Thus it is true if the force be gravity, but not true if the force be friction.

through equal distances for ever, were it not that during each swing a certain amount of the energy is required to set the air in motion, and to overcome friction. Thus the energy of the pendulum is gradually transmitted to the air around, and in course of time is all transformed into that form of kinetic energy which we call heat.

Examination on Sections 40-44

1. Give Newton's third law of motion. Define Stress.

2. Show how the third law of motion applies to a book resting on a table, and to a horse pulling a canal boat.

3. Given the masses of a gun and a projectile, and the initial velocity with which the gun recoils, find the initial velocity of the projectile.

4. When is a force said to do Work? How is work measured?

5. How is rate of doing work measured? What is a horse-power?

6. Define Energy. What are the two kinds of Energy?

7. Explain what is meant by Conservation of Energy.

8. Illustrate the principle of Conservation of Energy by means of (1), the swinging of a pendulum; (2), a stone thrown on to the roof of a house.

9. Name several forms of energy, and give instances of transformation of energy from one form to another.

GENERAL EXAMPLES IN KINETICS.

1. Find the momentum of each of the following systems :—
   (1) Masses of 4, 7, and 10 pounds moving in the same direction, with velocities 12, 10, and 6 respectively.
   (2) A mass of 10 pounds, with velocity 6, and a mass of 12 pounds moving in the opposite direction with velocity 5.

2. A mass of 30 pounds has the velocity of 100 in a direction making an angle of 60° with the horizon. Find its horizontal momentum.

3. How long must gravity act on a falling body to give it a velocity of 161?

4. How much work is done against gravity by a man of 12 stones in climbing a mountain 4000 feet high?

5. How much work is done by the same man in walking a mile along a level road?

6. In Atwood's machine[1] the weight of one ounce is employed to move a mass of two pounds. Find the velocity of the mass after one second.

7. How far will a stone fall from rest in 5 seconds, and what velocity will it have at the end of that time?

8. A rifle ball is discharged horizontally from a position 1000 feet above the level of the sea. What will be its height 4 seconds after the discharge?

9. How long will a body falling from rest take to acquire a velocity of 80 feet per second?

10. How far will a body fall from rest in $3\frac{1}{2}$ seconds?

11. A stone dropped from the top of a cliff is observed to reach the bottom in $6\frac{1}{2}$ seconds. Find the height.

12. A body which has fallen from rest reaches the ground with a velocity of 1127 feet. Find how long the body has been falling, and from what height.

13. A mass of 10 lbs., resting without friction on a horizontal table, has a constantly acting horizontal force of 2 lbs. (see § 35) applied to it. What will be the velocity after 1 second?

14. With what velocity must a ball be thrown up, in order to return to the hand after the lapse of 1 second?

15. A body is projected upwards with a velocity of 161 feet per second. How high will it rise?

16. A stone thrown vertically upwards strikes the ground after an interval of 10 seconds. With what velocity was it projected, and to what height did it rise?

17. A body is projected vertically upwards with a velocity of 80 feet. Where will it be after 2, 4, 6 seconds respectively?

18. A body is thrown upwards with a velocity of 250. Find its velocity after 7 seconds.

19. (1) An arrow is shot vertically upwards with a velocity of 100 feet in a second. How long will it be before it reaches the ground again?

(2) Another arrow, half the mass of the former, is shot vertically upwards with the same momentum. How long will it be before it reaches the ground?

20. A projectile is discharged in a slanting direction upwards. If there had been no gravity it would have reached an altitude of 1000

[1] See Appendix, § 102.

feet above the earth's surface at the end of 1 second. Find its actual altitude at that moment.

21. Let AB and AC represent in magnitude and direction two equal forces acting at A, the angle between their directions being 50°. Draw a line which shall represent in direction and magnitude a force in equilibrium with these forces; and state the angle which its direction makes with AB.

22. Find the resultant of the following forces:—5 northwards, 5 eastwards, 1 westwards, and 2 southwards.

23. Two forces, in magnitude 3 and 4, act at the same point in directions at right angles to each other. Draw their resultant and find its magnitude.

24. If a particle be urged towards the east with a force of 10 lbs., and towards the north with an equal force, what is the joint effect of the forces, and in what direction will the particle be urged?

25. A body is pushed simultaneously by a force of 3 lbs. acting from east to west, a force of 4 lbs. acting from north to south, and a force of 12 lbs. acting vertically downwards. What is the resultant force that acts upon the body?

26. A gun weighing 8 lbs. avoirdupois discharges a one-ounce ball with a velocity of 1000 feet per second. What will be the velocity of the gun's recoil?

27. A shot weighing 30 lbs. is fired from a gun weighing 3 tons, and leaves the gun with a velocity of 1120 feet per second. Find the velocity of the gun's recoil.

28. A shell bursts into two fragments, whose weights are 12 and 20 lbs. The former travels onwards with a velocity of 700, and the latter with a velocity of 380. What was the momentum of the shell when the explosion took place?

29. A shell weighing 20 lbs. explodes when in motion with a velocity of 600; at the moment of explosion one-third of the shell is reduced to rest. Find the momentum of the other two-thirds.

30. A shell in motion, with a velocity of 80 feet per second, bursts into two equal pieces. If one of them continue its path with a velocity of 120, what will be the motion of the other?

31. A force of 97 lbs. is resolved into two component forces at right angles. If one component be 65 lbs., find the other.

32. Find the magnitude of the resultant of two forces of 13 and 35 lbs. when the angle between their directions is 60°.

33. Find the velocity which a heavy body will acquire by falling freely through a height of $3\frac{1}{23}$ feet.

34. Two forces act at right angles, find their resultant when their values are—

(1) 48 and 55 ; (2) a and b ; (3) $a^2 - b^2$ and $2ab$.

35. A body is projected with a velocity of 10 feet per second, and is brought to rest by a uniform force after passing over 10 feet in a straight line. Find the magnitude of the force.

36. Two forces act on a particle. When their direction is such that their resultant is greatest, their resultant is 12 ; when their resultant is least, it is 6. Find the forces.

37. R is the resultant of two forces P and Q. If R be at right angles to P and equal to half Q, find the angle at which P and Q act.

38. The resultant of two forces is represented by the diameter of a circle. If one force be represented by a chord through one extremity of the diameter, prove that the other force will be represented by the chord drawn through the same extremity at right angles to the first chord.

39. How far must a body fall to acquire a velocity of 1 foot per second, and how long will it take to acquire that velocity? (To simplify the work let $g=32$, as at the equator.)

40. Find the average velocity and the space described by a body which, thrown vertically upwards, returns to its starting-place in 10 seconds ($g=32$).

41. Find the angle between two forces of 3 and 5 lbs., so that they may be kept in equilibrium by a force of 7 lbs.

42. Find the horse-power of an engine which can raise 15,000 gallons of water per hour from the bottom of a shaft 1100 feet deep, assuming that a gallon of water weighs 10 lbs.

43. If a train is moving uniformly, what may you infer with respect to the relation between the driving-power and the resistance?

44. A string is hung over a pulley with masses of 3 and 4 lbs. at either end. How much work is done by gravity when the mass of 4 lbs. is allowed to descend a foot?

45. A force equal to the weight of one ounce acts on a pound weight for 10 seconds. Find the velocity generated, and the space through which the weight will be moved in 10 seconds, supposing it to start from rest at the beginning of the time.

46. $ABCD$ is a quadrilateral. Find the resultant of the forces represented in direction and magnitude by the straight lines AC, DB, AD, and BC.

47. A particle is projected horizontally, with a velocity of 32 feet per second, from a point 128 above the ground. Find the direction

of motion of the particle when it has fallen half-way to the ground, taking $g=32$.

48. An Arctic explorer uses a spring balance which was graduated at the equator. What is the greatest correction he may require to apply to his readings? (The value of g at the equator is 32·09, and at either pole 32·25.)

49. How much work is done by means of a crane in raising from the ground the material required to build a stone wall 100 feet long, 36 feet high, and 2 feet thick, the density of the stone being 153 pounds per cubic foot, and the height of a course = 1 foot?

50. Calculate the amount of work done (independently of that lost through friction) in drawing a car of 2 tons weight, laden with 30 passengers averaging 10 stones each in weight, up a slope, the ends of which differ in level by 50 feet.

51. A stone is let fall, and another is, at the same instant, projected upwards from a point 500 feet lower in the same vertical. With what speed must it be projected so that the two may meet half-way?

52. A certain force acting on a mass of 10 lbs for 5 seconds produces in it a velocity of 100 feet per second. Compare the force with the weight of 1 lb., and find the acceleration it would produce if it acted on a ton.

53. Show how to resolve a force into two forces, each equal to a given force. With what limitation is this always possible?

54. Resolve a force of 9 lbs. into two others, each making an angle of 60° with it.

55. Resolve a force of 2 lbs. into two others, one making an angle of 30° with it, and the other an angle of 60°.

56. If the work done against friction, etc., in moving a train along a railway be $\frac{1}{280}$ of the work required to raise the train the same distance against gravity, find—

(1) the horse-power required to draw a train of 150 tons, at the rate of 20 miles an hour, along a level line;

(2) the same, if the line have a rise of 1 in 560;

(3) the same, if the line have a fall of 1 in 560.

CHAPTER IV

STATICS

45. Equilibrium of a Particle.—When any number of forces acting on a body exactly balance or neutralise each other, the velocity of the body is unchanged, and the forces are said to be *in equilibrium* with each other.

The body is also said to be *in equilibrium* under the action of the forces (see § 56).

Two Forces.—*Two equal and opposite forces, acting on a particle, are in equilibrium.*

By the second law of motion (see § 31) the two forces give the particle two equal and opposite accelerations, and therefore its velocity is unchanged (§§ 11, 21).

Three Forces.—*Three forces, acting on a particle, which are represented by the three sides of a triangle taken in order, are in equilibrium.*

By the second law of motion the three forces give the particle three accelerations, which are represented by the sides of the same triangle, taken in order, and therefore its velocity is unchanged (§§ 13, 21).

This proposition has two important converse propositions, which we proceed to prove.

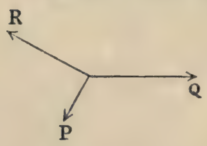
(i.) *If three forces, acting on a particle, are in equilibrium, they may be represented by three sides of a triangle taken in order.*

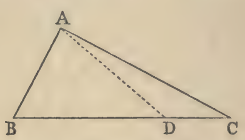
For, if lines representing two of the forces be placed so as to form two sides of a triangle taken in order, and the triangle be completed, the third side of the triangle must, by the preceding proposition, represent a force in equilibrium with the first two. It must therefore represent the third force.

(ii.) *If three forces, acting on a particle, are in equilibrium, and if their directions are those of the sides of a triangle taken in order, then, if one of the sides of the triangle be taken to represent the force to which it is parallel, the other sides will, on the same scale, represent the other forces.*

Let the forces P, Q, and R be in equilibrium, and let the sides of the triangle ABC be parallel to their directions. Then if AB be taken to represent P, BC will represent Q, and CA will represent R, on the same scale.

For, if BC does not represent Q, let BD represent it. Join





AD. Because ABD is a triangle, the force represented by DA will be in equilibrium with those represented by AB and BD. Therefore DA will represent R. But this is impossible, since CA is parallel to the direction of R.

Therefore BC represents Q.

Similarly CA represents R.

Any number of Forces.—*If any number of forces, acting on a particle, be represented in magnitude and direction by the sides of a polygon taken in order*, then, by the polygon of accelerations (§§ 14, 21), the velocity of the particle is unchanged, and therefore *the forces are in equilibrium.*

Examples—

1. Forces of 6, 8, and 10, acting on a particle, are in equilibrium, prove that the directions of the first two form a right angle.

Since the three forces are in equilibrium, a triangle can be formed whose sides represent the forces in magnitude and direction. The sides of this triangle will therefore be 6, 8, and 10.

But $10^2 = 100 = 64 + 36 = 8^2 + 6^2$.

$\therefore$ the triangle is right angled (Euclid I. 48).

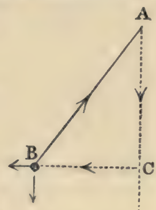
$\therefore$ the forces 6 and 8 act at right angles.

2. Three forces, acting on a particle, are in equilibrium; prove that their lines of action are in one plane.

As before, a triangle can be formed whose sides represent the forces in magnitude and direction.

∴ the three forces are in a plane passing through the particle, and parallel to the plane of the triangle.

3. A mass of 8 lbs. hangs by a chain 20 feet long, and is pulled out by a horizontal force to a distance of 12 feet from the vertical through the point of support. Find the tension of the chain.



Let AB be the chain. Draw AC vertically downwards, and from B draw BC at right angles to AC. Then AB=20, BC=12, and ABC is a right-angled triangle.

$$\therefore AC^2 = AB^2 - BC^2 = 256.$$

$$\therefore AC = 16.$$

The mass at B may be considered as a particle. It is in equilibrium under the action of three forces. These forces are the tension of the chain acting in the direction BA, the weight, which acts vertically downwards, and the horizontal force. They are in the direction of the sides of the triangle BAC taken in order. If we take AC to represent the weight, CB will represent the horizontal force, and BA the tension of the chain.

Thus a line of	16 feet	represents a force	=	the weight of	8 lbs.,
∴	,,	2 feet	,,	,, =	,,	1 lb.,
and	,,	20 feet	,,	,, =	,,	10 lbs.

But AB, which represents the tension of the chain, is 20 feet, and therefore the tension is the same as if the chain hung vertically downwards, and supported a weight of 10 lbs.

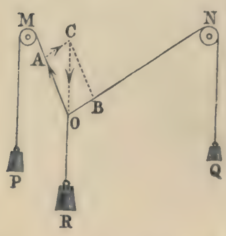
Examples for Exercise—

1. Three equal forces are in equilibrium, find the angles between them.

2. A mass of 100 lbs. is suspended by two strings, each of which makes an angle of 30° with the horizon. Find the tension of each string.

3. Show that if three forces acting at a point are in equilibrium, no two of these forces can be together less than the third.

46.—Experimental verification of the Parallelogram and Triangle of Forces.—The laws which regulate the equilibrium of three forces acting on a particle may be verified in the following way. Three masses, P, Q, and R, are attached by three strings knotted together at O. Two of the strings, OP and OQ, are placed over two pulleys M and N, and the third string, OR, is allowed to hang vertically downwards. If the weights are such that equilibrium is possible (see § 45, Example for Exercise 3), the system will come to rest in some such position as that shown in the figure.



The effect of the pulleys at M and N is simply to change the direction of the strings without altering their tensions (§ 65). We have therefore three forces in equilibrium acting at O, namely, the weights of P and Q, in the directions OM and ON, and the weight of R in the direction OR.

Mark off, on OM and ON, lengths OA and OB, to represent the weights of P and Q. Complete the parallelogram OACB. Then OC is the resultant of OA and OB. It will be found that OC is vertical, and that its length represents the weight of R on the same scale as that on which OA and OB represent those of P and Q. Thus the parallelogram of forces is experimentally verified.

We also observe that, in the triangle OAC, OA, AC, and CO represent the three forces in magnitude and direction, and thus the triangle of forces also is experimentally verified.

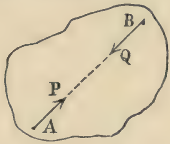
47. A Rigid Body.—*A Rigid Body is a collection of material particles so united that their relative positions do not change.*

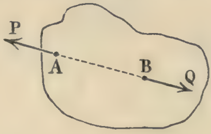
No body is perfectly rigid, but all solids are approximately rigid, when the forces which act on them are not too great.

48. Equilibrium of a Rigid Body.

Two Forces.—*Two equal forces, acting on a rigid body in opposite directions along the same straight line, are in equilibrium.*

Let P act at A and Q at B in opposite directions along the straight line AB. The force P tends to move the particle at A, and to move with it the whole mass of the body. The





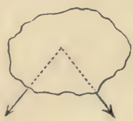
force Q tends to move the particle at B in the opposite direction, and along with it the whole mass of the body. But owing to the rigid connection of A and B these two forces neutralise each other, and equilibrium is maintained.

This fact may be verified by experiment.

If P act at A in the direction AB, then it is neutralised by the equal and opposite force Q, at whatever point in the line AB, Q is supposed to act. Hence, if at B two equal and opposite forces, each equal to P and in the direction of the line AB, be applied, one of these forces neutralises P, and the other acting at B is equivalent to P at A. (This is often called the principle of the *transmission of force.*)

Since the two forces P and Q are in equilibrium, if any number of forces act on a body, we may suppose such a pair of forces as P and Q to act along with them, and the result will be unaltered.

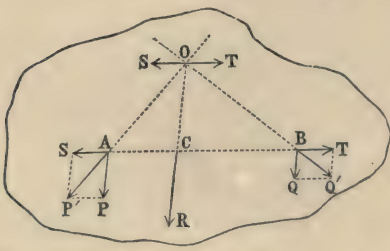
Forces whose Lines meet.—If the lines of action of two or more forces meet in a point, we may suppose the forces to act at that point, and the conditions of equilibrium are the same as those already investigated for a particle.[1]



When the directions of the forces do not all pass through a point, the problem is more complicated, and the only case of this kind which will be considered in this treatise is that of parallel forces.

[1] For the equilibrium of three non-parallel forces acting on a rigid body, see Appendix, § 110.

49. Resultant of Two Parallel Forces.—Let P and Q be two parallel forces acting on a body, and let A and B be two points in their lines of action. Join AB. At A and B apply two equal and opposite forces, S and T, acting in opposite directions along AB. Then by § 48 the effect of P and Q will be unaltered.



Thus the four forces P, Q, S, and T are equivalent to the two forces P and Q. Let P′ be the resultant of S and P, and let Q′ be the resultant of Q and T, and let the directions of P′ and Q′ meet in O; P′ and Q′ may be considered as acting at O.

From O draw OC parallel to the direction of P and Q, and meeting AB in C.

Substitute for P′ at O the equivalent forces P and S, and for Q′ the equivalent forces Q and T.

Then S and T, being equal and opposite, may be removed, and there remain P and Q acting in the direction OC.

Call this resultant R. Then—

$$R = P + Q. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and acts at C, in a direction parallel to that of P and Q.

To find the position of C.

The three forces P, P′, and S are parallel to the sides of the triangle OAC. If, for a moment, we consider OC to represent P, AC will, on the same scale, represent S.

$$\therefore \frac{OC}{AC} = \frac{P}{S}.$$

Multiplying both sides of this equation by $S \times AC$, we obtain—

$$S \times OC = P \times AC. \quad . \quad . \quad . \quad . \quad . \quad (i.)$$

Also the three forces Q′, Q, and T are parallel to the sides of the triangle OCB. If, therefore, OC represent Q, BC will represent T.

$$\therefore \frac{OC}{CB} = \frac{Q}{T}.$$

Multiplying both sides by T × BC we obtain

$$T \times OC = Q \times BC. \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

Combining the equations (i.) and (ii.), and remembering that $S = T$, and therefore $S \times OC = T \times OC$, we obtain the equation—

$$P \times AC = Q \times BC. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If we divide both sides of this equation by $Q \times AC$, it takes the form—

$$\frac{P}{Q} = \frac{BC}{AC}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

We see from this equation that the forces P and Q are inversely proportional to their distances from C.

Examples—

1. Parallel forces 5 and 3 act at two points 8 inches apart. Find their resultant and its point of application.

Let P and Q be the forces, A and B their points of application.

By (22) $R = P + Q = 8$.

By (23) $P \times AC = Q \times BC$.

Let $AC = x$ inches, then $BC = 8 - x$ inches; substituting these values, equation (23) becomes—

$$5x = 3\,(8 - x), \quad \text{or } 5x = 24 - 3x.$$

$$\therefore 8x = 24. \quad \therefore x = 3.$$

Thus the resultant force is 8, and acts at a point C, 3 inches from A along AB.

2. P and Q are equal parallel forces. Find their resultant and its point of application.

By (22) $R = P + Q = 2P$.

By (23) $P \times AC = Q \times BC$,

but $P = Q$,

$\therefore \quad AC = BC = \frac{1}{2} AB$.

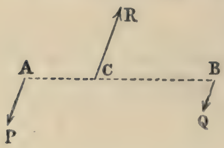
Examples for Exercise—

1. Weights of 5 and 7 lbs. are hung at the extremities of a rod 2 feet long. Find their resultant and its point of application.

2. Forces of 3 and 9 lbs. (see § 35) act in parallel directions at two points 20 inches apart. Find their resultant and its point of application.

3. Three forces in one plane are in equilibrium. Show that they must either be parallel forces, or forces whose directions meet in a point. *(This is an important theorem.)*

50. Equilibrium of Three Parallel Forces.—Since in the figure of § 49, R, acting at C, is the resultant of the parallel forces P and Q, if we apply at C a force equal and opposite to R, it will be in equilibrium with P and Q.



Thus the three forces P, Q, and R, in the adjoining figure, are in equilibrium.

Since any one of these three forces is equal and opposite to the resultant of the other two, we may apply the equations of § 49 to find the resultant of parallel forces acting in *opposite* directions. The following examples will serve to explain how this is done.

Examples—

1. P and R are parallel forces of 3 and 4 lbs. acting in opposite directions at two points, A and C, 3 inches apart. Find a third force which will produce equilibrium.

Let Q be the third force, and B its point of application.

Then by (22) $R = P + Q$. $\therefore Q = R - P = 4 - 3 = 1$.

Also by (23) $P \times AC = Q \times BC$.

Let $BC = x$, and substitute their values for P, Q and AC. This equation becomes—

$$3 \times 3 = 1 \times x, \qquad \text{or } x = 9.$$

Thus Q is a force of 1 lb., acting parallel to P at a point B, 9 inches from C, in AC produced.

The *resultant* of P and R is therefore a force of 1 lb. acting at B in the direction of the greater force R.

2. Find the resultant of two opposite parallel forces 7 and 4, acting at two points 6 inches apart.

As before, let $P = 4$, $R = 7$, $AC = 6$, $BC = x$.

Then by (22) $Q = R - P = 3$.

And by (23) $P \times AC = Q \times BC$.

Or $4 \times 6 = 3x$. $\therefore x = 8$.

The resultant is therefore a force 3, acting in the direction of the greater force R at a point 8 inches from C in AC produced.

Examples for Exercise—

1. Two parallel forces, 7 and 10, act in opposite directions at points 6 inches apart. Find a third force which will be in equilibrium with them.

2. Parallel forces, 5 and 2, act in opposite directions at a distance of 1 foot. Find their resultant.

Examination on Sections 45-50

1. What is the condition that two forces acting on a particle are in equilibrium?

2. Prove that three forces, acting on a particle, which are represented in magnitude and direction by the three sides of a triangle taken in order, are in equilibrium.

3. Prove that, if the sides of a triangle, taken in order, are parallel to the directions of three forces in equilibrium, they may be taken to represent these forces in magnitude.

4. Enunciate the condition that any number of forces acting on a particle are in equilibrium.

5. Describe an experiment by means of which the parallelogram and triangle of forces may be verified.

6. Define a rigid body.

7. What is the condition that two forces, acting on a rigid body, are in equilibrium? Deduce the principle of the transmission of force.

8. Any two parallel forces, P and Q, act on a body. Find the direction and magnitude of their resultant.

9. Prove that the resultant of two parallel forces acts at a point dividing the distance between their points of application inversely as the forces.

10. Three parallel forces, P, Q, and R, acting on a rigid body at three points, A, B, and C, in the same straight line, are in equilibrium; prove that if R be the greatest force—

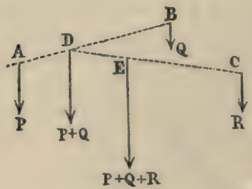
 (1) $R = P + Q$:

 (2) R acts in a direction opposite to that of P and Q, and lies between them.

 (3) $P \times AC = Q \times BC$.

51. Resultant of more than Two Parallel Forces.—Let P, Q, and R be three parallel forces acting on a rigid body at the points A, B, and C.

The resultant of P at A, and Q at B has been shown to be equal to P + Q, and to act at a point D in AB, taken so as to satisfy equation (23), which here takes the form—



$$P \times AD = Q \times BD.$$

Similarly the resultant of P + Q at D and R at C is equal to P + Q + R acting at a point E in CD, taken so as to satisfy the equation—

$$(P + Q) \times DE = R \times CE.$$

Thus the resultant of P, Q, and R is a force P + Q + R acting at E.

In the same way, we may prove that the resultant of any number of parallel forces is equal to the sum of the forces, and acts at a point which may be found by the above method. From the above proof we see that the position of this point depends on the magnitudes and points of application of the forces, but is independent of the direction in which the parallel forces act.

This point is called the *Centre of the Parallel Forces.*

52. Centre of Inertia.—*If a set of parallel forces act on all the particles of a body, and if each force be proportional to the mass of the particle on which it acts, then the centre of these parallel forces is called the Centre of Mass or the Centre of Inertia of the body.*[1]

The attraction of a distant star for the earth or for any terrestrial object forms a set of forces, which very approximately fulfils the condition of the above definition.

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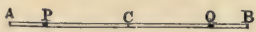
[1] The centre of mass (or inertia) may also be defined without reference to force. If A and B (figure § 51) are two particles, then their centre of mass divides AB inversely as their masses. To find the centre of mass of a system of particles we substitute for A and B the sum of their masses at their centre of mass, and find the centre of mass of this sum along with the next particle of the system, and so on. See Clerk Maxwell's *Matter and Motion*, Articles lx. lxi.

The earth's attraction for any object also approximately forms such a set of forces, especially when the object is small or at a considerable distance from the centre of the earth.

It follows, that the resultant of the weights of the different particles of a body either passes through, or very near to, its centre of inertia, and on this account it is common, though not strictly accurate, to call the centre of inertia the *centre of gravity*.

In the next two sections, we assume that the bodies spoken of are so small that the directions of the weights of their various parts may be assumed to be parallel, and that therefore centres of gravity exist which are identical in position with their centres of inertia.

53. Centres of Inertia (Gravity) found geometrically.— (1.) *To find the centre of gravity of a uniform material straight line.*—

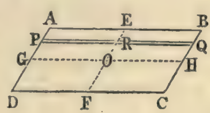


Let AB be a straight line or thin rod, composed of uniformly heavy matter. Bisect AB in C. Suppose AB to be divided into any number of small equal portions, and let P and Q be two of these portions, and let CP=CQ. Then the weights of the portions P and Q are two equal parallel forces, and therefore (§ 49, Ex. 2) their resultant acts at C.

In the same way we may take the other portions in pairs equally distant from C, and the resultant of the weights of each pair acts at C.

Therefore the resultant of the weights of all the portions acts at C, and C is the centre of gravity of the straight line AB.

(2.) *To find the centre of gravity of a parallelogram.*—Let ABCD be a parallelogram composed of uniformly heavy matter. Bisect AB in E and draw EF parallel to AD. Suppose the whole parallelogram divided into thin strips by lines drawn parallel to AB, and let PRQ be one of those strips. Then,



since AERP and EBQR are parallelograms, PR = AE and RQ = EB.

$$\therefore PR = RQ.$$

∴ R is the centre of gravity of PQ.

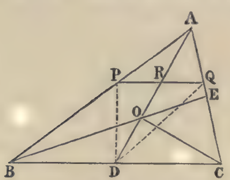
Similarly it can be shown that EF contains the centre of gravity of each strip, and that it therefore contains the centre of gravity of the whole parallelogram.

Bisect AD in G and draw GH parallel to AB, meeting EF in O.

Then, as before, by drawing lines parallel to AD, we may divide the parallelogram into thin strips; and the centre of gravity of each strip, and therefore that of the whole parallelogram, lies in GH.

The centre of gravity of the parallelogram is therefore the point O, in which EF and GH intersect.

(3) *To find the centre of gravity of a triangle.*—Let ABC be a triangle, bisect BC in D, and join AD, and suppose the triangle divided into thin strips by lines drawn parallel to BC. Let PRQ be one of these strips:



$$\because BD = DC,$$

$\therefore \triangle BAD = \triangle CAD$ (Euclid I. 38).

Similarly $\triangle BPD = \triangle DQC$,

$\therefore$ the remaining $\triangle APD = \triangle AQD$, and $PR = RQ$.

For if not, let $PR > RQ$.
Then $\triangle PAR > \triangle RAQ$,[1]
and $\triangle PDR > \triangle RDQ$,
∴ by addition $\triangle APD > \triangle AQD$.
But the $\triangle$ APD has been shown to be equal to $\triangle$ AQD,
∴ PR is not $>$ RQ.
Similarly PR is not $<$ RQ.
And therefore $PR = RQ$.

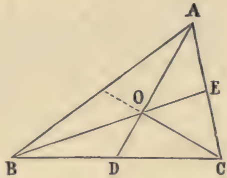
∴ R is the centre of gravity of PQ.

[1] By Euclid i. 35, any parallelogram is equal to the rectangle on the same base and between the same parallels, and therefore its area = base × altitude. Therefore by Euclid i. 41, any triangle = $\frac{1}{2}$ base × altitude. If, therefore, two triangles have equal altitudes, but unequal bases, the greater triangle is that which has the greater base.

In the same way it may be shown that AD contains the centre of gravity of each of the parallel strips, therefore AD must contain the centre of gravity of the whole triangle. Bisect AC in E. Join BE, meeting AD in O. By dividing the triangle into strips parallel to AC, we may prove, exactly as above, that BE contains the centre of gravity of the triangle. Therefore the point O, in which AD and BE intersect, is the centre of gravity.

The point O is a point of trisection of each of the lines joining an angular point with the middle point of the opposite side.

Join OC.
∵ AE = EC,
Δ BAE = Δ BCE.
Also Δ AOE = Δ EOC.
∴ the remaining Δ AOB = Δ BOC.
Again, ∵ BD = DC.
Δ BOD = Δ DOC.
∴ BOC = 2 Δ BOD,
∴ Δ AOB = 2 Δ BOD,



and these triangles are between the same parallels.

∴ The base AO must be equal to twice OD,* or OD is the third part of AD.

Similarly it may be shown that O is a point of trisection of the lines joining B and C with the middle points of the opposite sides.

(4.) *To find the centre of gravity of a circle.*—A circle may be divided into pairs of equal particles, so that each pair consists of two particles on opposite sides of the centre and equally distant from it. Thus, the centre of the circle is the centre of gravity of each pair of particles, and therefore of the whole circle. In the same manner the centre of a sphere may be shown to be its centre of gravity.

(5.) *To find the centre of gravity of a system of bodies whose separate masses and centres of gravity are known.*—Suppose the weight of each body to act at its own centre of gravity and find the centre of the set of parallel forces thus formed.

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* As before, this follows from the fact that any triangle = ½ base × altitude. If, therefore, one triangle be double another triangle, and have the same altitude, the base of the first triangle must be double the base of the second.

Example—

Three equal masses are placed at the angular points of a triangle. Find the centre of gravity of the system.

Let the weight of each mass be W, and let them be placed at the points A, B, and C, in the last figure. The weights at B and C are equivalent to a weight 2W at D, the middle point of BC.

We have thus to find the resultant of W at A and 2W at D. We must, therefore, divide AD at O, so that by equation (23)—

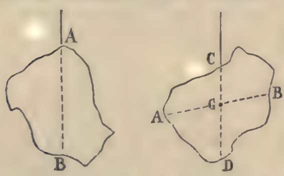
$$W \times AO = 2W \times OD,$$
$$\text{or} \qquad AO = 2OD.$$

The centre of gravity of the three masses is therefore the same point as the centre of gravity of the triangle.

Examples for Exercise—

Masses of 1, 3, 9, and 3 lbs. are placed at the angles of a square taken in order round the figure. Find the centre of gravity of the system.

54. Experimental Method of finding the Centre of Gravity of a Body.—The centre of gravity of a small body may be found experimentally in the following manner. Tie a string to any part of the body and suspend the body by the string. The only forces which act on the body are the tension of the string, acting vertically upwards at the point of suspension, and the weight of the body acting vertically downwards at its centre of gravity. These forces must be equal and opposite, and therefore the centre of gravity must be in the vertical line through the point of suspension.



Now let the body be hung up by any other part. As before, the vertical line through the point of suspension will pass through the centre of gravity. The point where the two lines intersect is therefore the centre of gravity.

In many bodies, as for example a flat plate, the centre of gravity may be readily found by the method described above.

Examination on Sections 51-54

1. Show how to find the resultant of three parallel forces acting in the same direction.

2. Prove that the resultant of any number of parallel forces is equal to the sum of the forces, and acts at a point whose position is independent of the direction of the forces.

3. Explain what is meant by the centre of parallel forces.

4. Define the Centre of Inertia.

5. Find the centre of gravity of a uniform material straight line.

6. Show how to find the centre of gravity of a parallelogram.

7. Show that the centre of gravity of a triangle lies in the line joining any angular point to the middle point of the opposite side.

8. Find the centre of gravity of a uniform thin flat circular disc.

9. Show how to find the centre of mass of a system of bodies whose separate masses and centres of mass are known.

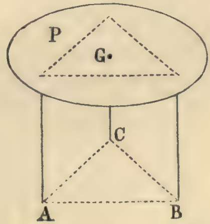
10. Explain a method of finding the centre of gravity of a small body experimentally.

55. Equilibrium of a body resting on a Surface.—If a body be placed on a surface, the weight of the body causes a pressure on the surface. To this action there is, by the third law of motion, an equal and contrary reaction.

If the body rest on more points than one, there will be a pressure, and therefore a reaction at each point on which it rests.

When the resultant of all the reactions is equal and opposite to the weight of the body, the body is *in equilibrium.*

As an example, take the case of a three-legged table, resting on three legs at the points A, B, and C. The resultant of the reactions at A, B, and C acts upwards through the centre of gravity G, and is equal to the weight of the body. If, however, a heavy weight be placed on some point, P, outside the triangle formed by joining the three legs, so that the centre of gravity of the loaded table is

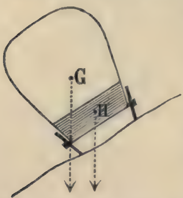


removed to some point beyond the triangle, it is clear that the resultant of the reactions at A, B, and C can no longer act vertically through the centre of gravity, and the table will consequently fall over.

Thus a body resting on a surface is in equilibrium when the vertical line through its centre of gravity passes through the area formed by joining its points of support.

Example—

A cart laden with hay is driven along a rough road. Show that it is more likely to upset than an unladen cart.



Let G be the centre of gravity of the laden, H that of the unladen cart; then if, as in the figure, one wheel is considerably higher than the other, the vertical line through G falls outside the base, and equilibrium is impossible for the laden cart. Under the same circumstances the vertical line through H may, as in the figure, fall within the base, and the unladen cart may therefore remain in equilibrium.

Example for Exercise—

Sketch a leaning tower, and show, by reference to its centre of gravity, that it will fall when made to exceed a certain height, unless it be rigidly connected with the ground.

56. Stable, Unstable, and Neutral Equilibrium.

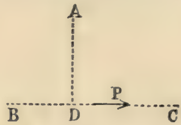
(1.) Stable Equilibrium.—*A body is said to be in stable equilibrium, when, after receiving a small displacement, it tends to return to its former position.* A rod suspended by a string, a cone standing on its base, and a loaded sphere placed on a flat surface with the loaded part lowest, are examples of bodies in stable equilibrium. When any one of these bodies receives a small displacement, the centre of gravity is raised, and therefore gravity tends to bring the body back to its former position.

(2.) Unstable Equilibrium.—*A body is in unstable equilibrium, when, after receiving a small displacement, it tends to fall away from its former position.* A rod balanced on one end, a cone standing on its apex, and a loaded sphere with the loaded part upwards, are examples of unstable equilibrium. When any one of these bodies receives a small displacement, the centre of gravity is lowered, and therefore gravity tends to draw the body away from its first position.

(3.) **Neutral Equilibrium.**—*A body is in neutral equilibrium, when, after receiving a small displacement, it remains in the neu position.* A sphere, and a cone or cylinder resting on its side on a flat surface, are examples of neutral equilibrium. In this case the centre of gravity is neither raised nor lowered by a small displacement.

57. The Moment of a Force about a Point.—*The Moment of a force about a point is the product of the force and the perpendicular from the point on its line of action.*

Let P be a force 2 acting along the line BC, and from A let AD be drawn perpendicular to BC, then, if AD = 3 feet, the moment of P about A is $P \times AD = 2 \times 3 = 6$.



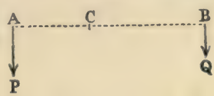
The moment of P about A represents the power which P has to turn a body about the point A, if we suppose that point in the body fixed.

Moments are regarded as *positive* or *negative* according to the direction of their rotation.

It is important to notice that, in the case of parallel forces, the moments of the component forces about any point in the line of action of the resultant are equal, and tend to turn the body in opposite directions.

[This is a special case of the Principle of Moments :—The moment of the resultant is equal to the sum of the moments of the components.]

Let P and Q be the two forces, and let the line AB be drawn perpendicular to their direction. Then, if C be a point in the line of action of the resultant, by (23)—

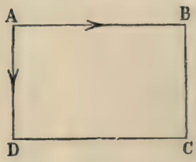


$$P \times AC = Q \times BC,$$

or the moments of P and Q about C are equal, and it is evident that they tend to turn the body in opposite directions about C.

Examples—

1. ABCD is a rectangle. Forces represented by AB and AD act at A. Prove that their moments about C are equal.



Since ABC is a right angle, the moment of AB about C is $AB \times BC$. It is therefore represented by the area of the rectangle. Similarly, the moment of AD about C is $AD \times CD$, and is also represented by the rectangle. Therefore they are equal.

2. ACB is a straight line, AB = 10 in., and AC = 4 in. If a force of 10 lbs. act at C in a direction perpendicular to ACB, what force must act at B in order that their moments about A may be equal?

Let P and Q be the forces at B and C, then—

$$P \times AB = Q \times AC.$$

Substituting their values for AB, AC, and Q, this equation becomes—

$$10P = 10 \times 4.$$
$$\therefore P = 4.$$

Examples for Exercise—

1. ABCD is a straight line 10 feet long, and is divided into three equal parts at B and C. Forces of 1, 2, 3, and 4 lbs. act at A, B, C, and D perpendicularly to AD. Find their moments about A.

2. Prove that, if two forces act at a point in a body, their moments about any point in their resultant are equal, and tend to turn the body in opposite directions. (*This involves an important theorem.*)

Examination on Sections 55-57

1. What is the condition of equilibrium of a body resting on a surface?

2. Show that a coach laden with passengers on the outside is more liable to be upset than when it is laden in the inside.

3. What are the three kinds of equilibrium?

4. Define the three kinds of equilibrium, giving examples of each.

5. Define the moment of a force about a point.

6. Prove that the moments of two parallel forces about any point in the line of their resultant are equal and opposite.

GENERAL EXAMPLES IN STATICS

1. Three forces, P, Q, and R, acting at a point, are in equilibrium, the angle between P and Q being a right angle :—

(1) given $P=15$ and $Q=8$; find R :

(2) given $P=15$ and $R=17$; find Q :

(3) given the angle between Q and $R=150°$ and $P=5$; find R :

(4) given the angle between Q and $R=135°$ and $P=7$; find Q.

2. Three forces P, Q, and R, acting at a point, are in equilibrium. If $P>Q$, and $Q>R$, prove that the angle between P and Q is the greatest, and that the angle between Q and R is the least.

3. Two equal forces in the same plane act at an angle of 120°. Find a third which will maintain equilibrium.

4. Masses of 4 and 6 lbs. are hung at either extremity of a light rod 15 inches long. Find at what point the rod should be supported.

5. P and Q are two parallel forces acting in similar directions at two points A and B. Their resultant R acts at C. Find R and AC in the following cases :—

(1) $P=7$ lbs., $Q=2$ lbs., $AB=3$ feet :

(2) $P=5$ lbs., $Q=6$ lbs., $AB=1$ foot 10 inches :

(3) $P=15$ lbs., $Q=5$ lbs., $AB=8$ inches.

6. P and Q are two parallel forces acting in dissimilar directions at two points A and B. Their resultant R acts at C. Find R and AC in the following cases :—

(1) $P=7$ lbs., $Q=12$ lbs., $AB=10$ inches :

(2) $P=4$ lbs., $Q=6$ lbs., $AB=1$ foot :

(3) $P=15$ lbs., $Q=5$ lbs., $AB=6$ inches.

7. Two men, A and B, carry a mass weighing 200 lbs. hanging from a pole between them. If the men be 5 feet apart, and the mass be 2 feet from A, what part of the weight will be borne by each ?

8. A well-balanced dog-cart, laden with 9 cwt., is found on a level road to exert a pressure of 7 lbs. on the horse's back. If the horizontal distance between the pad and the axle be 6 feet, find how far before the axle the centre of gravity of the laden dog-cart is.

9. Three uniform rods form three sides of a square, find the centre of gravity of the whole.

10. A uniform rod, 2 feet long, and weighing 5 lbs., has a weight of 1 lb. placed at one extremity. Find the centre of gravity of the whole.

11. ABC is a straight line, $AB=5$ inches, $BC=3$ inches. Weights

of 3, 2, and 1 lbs. are laid at A, B, and C respectively. Find the centre of gravity of the system.

12. ABC is a straight line, AB=7 inches, and BC=9 inches. Weights of 3, 4, and 5 lbs. are placed at A, B, and C. Find the centre of gravity of the system.

13. Find the position of the centre of gravity of a hat, the crown and rim of which are of equal mass, and both flat.

14. Prove that the centre of gravity of a parallelogram is the point of bisection of its diagonals.

15. Prove that if the outline of a parallelogram be formed of wire, its centre of gravity is the same as that of the whole figure cut out of paper.

16. Show that the three lines which join the angular points of a triangle to the middle points of the opposite sides, meet in a point.

17. Find by a geometrical construction the centres of gravity of—(1) a quadrilateral; (2) any rectilineal figure.

18. What is the condition of stability of a given distribution of load on a table?

19. ABC is an equilateral triangle, and AC is bisected in D. Prove that the moment about A of a force represented by BC is twice the moment of a force represented by BD.

20. ABCD is a square. Prove that the moments about A of the forces represented by BC, CD, and BD are equal.

21. Parallel forces P and Q, acting in similar directions at L and M, are balanced by a force R acting at N:—

(1) prove that the moments of P and R about M are equal and opposite;

(2) prove that the sum of the moments of P and Q about any point in LM produced is equal and opposite to the moment of R about the point.

22. Two forces act at a point. Prove that the algebraical sum of their moments about any point in their plane is equal to the moment of their resultant. (*This and the previous example involve important theorems.*)

23. Prove that the centre of gravity of a regular polygon is the centre of the inscribed circle.

24. A wire is in the form of a circle on which are strung equal beads separated by equal intervals. Find the centre of gravity of the whole.

25. Determine the character of the equilibrium in the following cases:—

(1) a thin book lying on its side;

(2) the same standing on its end ;
(3) the same balanced on a corner ;
(4) a body of any shape supported at its centre of gravity ;
(5) a hemisphere on a horizontal surface, with the flat side turned down ;
(6) a hemisphere on a horizontal surface, with the flat side turned up ;
(7) a thin circular disc standing on its edge.

26. One end of a uniform heavy rod of weight W rests on a smooth horizontal plane, and a string tied to the other end of the rod is fastened to a fixed point above the plane. Find the tension of the string.

27. From a square a portion is cut off by a line passing through the middle points of two adjacent sides. Find the centre of gravity of the remainder.

28. Forces act along the sides AB, CB, CD, AD of a square ABCD, their magnitudes being P, 2P, 3P, 4P respectively. Prove that their resultant acts along the straight line which bisects AB and AD, and that its magnitude is $2\sqrt{2}\cdot P$.

29. Find in what directions two forces of 21 and 28 lbs. respectively must act on a particle so as to be kept in equilibrium by a force of 35 lbs.

30. D, E, F are the middle points of the sides of a triangle ABC ; show that the three forces represented by AD, BE, CF are in equilibrium.

31. Weights 5, 4, 6, 2, 7, 3 are placed at the corners of a regular hexagon taken in order. Prove that the centre of the hexagon is the centre of gravity.

32. Prove that the centre of gravity of a triangle, formed by bending a uniform thin heavy rod at two points till its ends meet, is the centre of the circle inscribed in the triangle formed by joining the middle points of its sides.

33. Two flexible strings, one of which is horizontal, and the other inclined to the vertical at an angle of 30°, support a weight of 10 lbs. Find the tension of each string.

34. A sinker or small leaden bullet is attached to a fishing-line, which is then thrown into a running stream. Show, by means of a diagram, the forces which act on the sinker so as to maintain its equilibrium, and give the relation existing between the forces.

35. Three bodies, whose masses are as 3, 4, and 5, are placed in a

straight line, with their centres of inertia, 8, 7, and 6 feet respectively on one side of a given point in that line. Find their centre of inertia.

36. Find the centre of gravity of four particles of weight, 2, 3, 4, and 5 lbs. respectively, placed at the corners of a horizontal square. Does the construction hold if the square be inclined to the horizon? Give a reason for your answer.

37. Three men are to carry a beam which is of uniform size and density, and has a length of 12 feet. If one of the three lifts at one end, and the other two lift by means of a bar, where ought the bar to be applied in order that each man may bear one-third of the weight?

38. Three ropes are tied together, and a man pulls at each. If, when their efforts are in equilibrium, the angle between the first and second ropes is 90°, and that between the first and third 150°, what are the relative strengths of the men as regards pulling?

39. A uniform plate, 10 inches square, has a hole 3 inches square cut out of it, the centre of the hole being $2\frac{1}{2}$ inches distant from the centre of the plate. Find the centre of gravity.

40. Three forces keep a particle in equilibrium; one acts towards the east, another towards the north-west, and the third towards the south. If the first force be 3, find the others.

41. A uniform iron rail, which weighs 100 lbs., is supported by two posts 10 feet apart, the posts being 12 and 8 feet respectively from the ends of the rail. Find the pressures on the posts.

42. A carpenter's rule 2 feet in length is bent into two parts at right angles, of which the shorter is 6 inches. If the shorter part be placed on a smooth horizontal table, what is the length of the least portion of it which must be on the table that it may remain in equilibrium?

43. AC and BC are two cords, 4 and 5 feet long respectively, fastened to two fixed points, A and B, which are 6 feet apart in the same horizontal line, and a weight of 50 lbs. is fastened to C. Find by means of a construction drawn to scale the reactions at A and B.

44. A and B are two points 10 feet apart, and such that the line joining them makes an angle of 30° with the horizon, and a cord 15 feet long has its ends fastened to A and B. Find the tension when a smooth ring weighing 5 lbs. is threaded on it.

45. If one point of a body is fixed, what is the condition of equilibrium?

A square plate ABCD, weighing 10 lbs., can turn freely in its plane, which is vertical, about A. What force acting along BD will keep AB vertical?

46. The lengths of the bars in a triangular frame ABC are 3, 4, and 5 feet respectively; the longest bar, BC, is horizontal, and rests on supports at its ends; a load of 25 lbs. is hung at the overjoint A. Find the stresses in the three bars, assuming that they act along the bars, and the reactions at the supports.

47. Find the tension in a cord supporting a picture — weight 32 lbs., distance between rings 6 ft., whole length of cord 10 ft. from ring to ring. What effect has lengthening the cord?

48. Pegs are stuck into a wall at the corners of an equilateral triangle ABC, BC is horizontal and A is above BC; a string whose length is four times a side is hung over the pegs, and has its ends fastened to a mass of 20 lbs. Find the pressures on the pegs.

CHAPTER V

MACHINES

58. Preliminary.—*A Machine is an instrument by means of which a force, acting in a given direction at a given point, is made to counterbalance or overcome a resistance, generally acting at a different point and in a different direction.* We have already considered cases in which forces are transmitted by such simple machines as the leg of a table and a stretched cord.

The machines to be considered in this chapter are (1) the Lever and its modifications, namely, the Balance, the Steelyard, and the Wheel and Axle; (2) the Pulley; (3) the Inclined Plane. The principles of Statics show us under what conditions the applied force is in equilibrium with the resistance. In accordance with the usual custom in elementary treatises on this subject, the applied force is called the Power, and the resistance the Weight, and they are designated by the letters P and W.

59. The Lever.—*The Lever is a rod moveable about a fixed point, which is called the Fulcrum.* Two forces P and W act on a lever at two points A and B. (See figure, § 60.) This causes a pressure on the fulcrum C, and, by the third law of motion, the fulcrum C exerts an equal and contrary pressure on the lever. This pressure is called the Reaction. Thus we have a case of three forces acting on a rigid body. When there is equilibrium these forces must (§ 49, *Example for Exercise* 3) meet in a point or be parallel.

In either case the conditions of equilibrium are most easily found by the following method. P is a force tending to turn the lever in one direction about C, W tends to turn it in the other. Therefore their moments about C (§ 57) must be equal.

Assuming, for simplicity, that P and W each act at right angles[1] to the line AB, it follows that—

$$P \times AC = W \times BC.$$

This equation is at once seen to be identical with (23).

If we divide both sides of this equation by $W \times AC$, it takes the form

$$\frac{P}{W} = \frac{BC}{AC},$$

which is identical with (24), and shows us that P bears to W the same proportion that BC bears to AC.

Thus P is less than W when AC is greater than BC, P is equal to W when AC is equal to BC, and P is greater than W when AC is less than BC.

When P is less than W, the arrangement is said to be one of *mechanical advantage.*

Examples—

1. Two bodies, weighing 20 lbs. and 4 lbs., balance at the extremities of a lever 2 feet long. Find the position of the fulcrum.

By (23) $P \times AC = W \times BC.$

Here $P = 20g$, $W = 4g$.

Let $AC = x$ inches, then $BC = 24 - x$ inches.

$$\therefore 20xg = 4(24 - x)g = (96 - 4x)g.$$

$$\therefore 20x = 96 - 4x.$$

$$\therefore x = \frac{96}{24} = 4 \text{ inches.}$$

2. A rod is attached to a wall by a hinge, and a mass of 3 lbs. is suspended at a point 2 inches from the hinge. At what distance must a force equal to the weight of 2 ounces act upwards to produce equilibrium?

By (24) $P \times AC = W \times BC$

In this case $P = \frac{1}{8}g$, $W = 3g$, $BC = 2$ in.

∴ If $AC = x$ inches, the equation becomes—

$$\tfrac{1}{8}x = 3 \times 2.$$

$$\therefore x = 48 \text{ inches or 4 feet.}$$ [2]

---

[1] If either or both forces are not at right angles to the lever, the moments may be found by drawing perpendiculars from the fulcrum on their lines of action, and the condition of equilibrium is the same, viz.: that the moments are equal.

[2] Here, and in subsequent cases, the factor g, occurring on both sides of the equation, may be omitted.

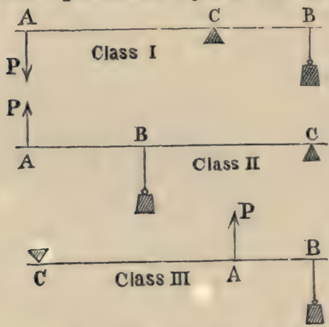
Examples for Exercise—

1. Masses of 3 lbs. and 9 lbs. balance each other at the extremities of a lever AB. Find the position of the fulcrum C.

2. A straight rod, 6 feet long, projects horizontally from a wall, to which it is attached by a horizontal hinge, and a mass of 10 lbs. is hung from its extremity. What pressure upwards must be exerted at 2 feet from the hinge in order to maintain the rod in its horizontal position?

3. In a lever, AB is 4 inches and the power is twice the weight. Find AC.

60. Popular Classification of Levers.—Although the problem presented by all levers is dynamically the same, *i.e.* the equilibrium of three forces acting on a rigid body, nevertheless for practical purposes it is useful to discriminate between certain classes of levers.



It has long been customary to divide levers into three classes, according to the position of the fulcrum with respect to the points of application of the Power and Weight.

In levers of the first class, C is between A and B; in those of the second class, B is between A and C; and in those of the third class, A is between B and C.

Examples—

Class I.—(1) A poker as usually employed for raising coals. Here the bar is the fulcrum, the coals are the weight, and the power is applied by the hand.

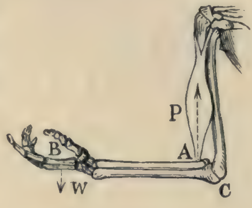
(2) The common balance.

Class II.—(1) A loaded wheel-barrow supported by the handles. The fulcrum in this case is the point where the wheel rests on the ground.

(2) A chipping-knife, in which one end is fixed to the block by a hinge.

Class III.—(1) The human fore-arm supporting a weight. Here the elbow is the fulcrum, the power is applied close to the elbow by the biceps muscle, while the weight is held in the hand.

(2) A treadle, when the foot is applied between the hinge and the connecting-rod.

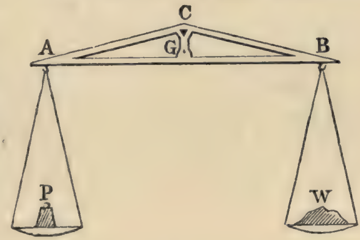


Other machines are formed of two similar levers connected by a joint, which acts as the double fulcrum.

Examples—

Class I. Scissors, Pincers.
Class II. Nut-crackers.
Class III. Sugar-tongs.

61. The Balance—As has been already pointed out, the common balance is a machine for ascertaining the mass of any given body by comparison with known masses. It is a lever of the first order, in which AC=BC, and therefore P=W. AB is called the beam. Its centre of gravity G is a short distance vertically below the point of support. The scales, which are of equal mass, are hung from A and B. The substance to be "weighed" is placed in one scale, and known masses are placed in the other until the beam rests in a horizontal position.



The qualities of a good balance are *stability* and *sensibility*.

A balance is *stable* if after being disturbed it tends to return quickly to its position of equilibrium. By lengthening CG the stability of a balance is increased, as we thus increase the moments of the forces which tend to bring AB back to a horizontal position.

A balance is *sensible* if AB deviates perceptibly from its horizontal position when P and W differ by a very small quantity. The sensibility of a balance is increased by lengthening AB, or by shortening CG.

62. False Balances.—A balance gives false results when its arms are of unequal length. In this case there are two methods by which the true mass of a body may be determined.

The method of double weighing consists in placing the body in one scale, and filling the other with shot, or other substances, until the beam is horizontal. The body is then removed, and known masses are placed in its stead until the beam is again horizontal.

It is clear that the true mass is thus obtained, as the same force must have been exerted in each case in order to maintain equilibrium.

The other method consists in weighing the body in each scale, and calculating its true mass from the results.

Let M be the true mass of the body, P and Q its apparent masses in the two scales. Let a and b be the lengths of the arms.

Then by (23) $Ma=Pb$,

And $Mb=Qa$.

Multiplying together the corresponding sides of these two equations, we obtain—

$$M^2ab=PQab.$$

$$\therefore M^2=PQ.$$

$$\therefore M=\sqrt{PQ}.$$

The true mass is therefore found by multiplying together the apparent masses and extracting the square root of this product.

Example—

The mass of a body appears to be 10 lbs. in one scale, and $12\frac{1}{10}$ lbs. in the other. Find its true mass.

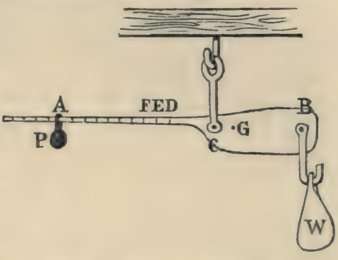
Here $M^2=PQ=10\times 12\frac{1}{10}=121.$

$\therefore M=11$ lbs.

Example for Exercise—

A body appears to be 8 lbs. in one scale, and 12 lbs. in the other. Find its true mass, and the ratio of the lengths of the arms.

63. The Steelyard.—The common or Roman Steelyard is a lever of the first class, and consists of a rod suspended from a point C, which is the fulcrum. The substance to be weighed is suspended from a fixed point B, while a moveable mass of weight P is made to slide along the graduated bar AC until there is equilibrium. The reading opposite P, when equilibrium is obtained, indicates the weight of the substance.



AC is graduated in the following manner. First, nothing is hung at B, and P is made to slide along AC until a point D is found, such that the rod is in equilibrium. Then if Q be the weight of the bar and G its centre of gravity, $P \times CD = Q \times CG$, and the zero mark is placed at D.

A mass of 1 lb. is now hung at B, and P is moved along the rod until a point E is found, such that the rod is in equilibrium.

The moment of P is now $P \times (CD + DE)$, or $P \times CD + P \times DE$. $P \times CD$ has been shown to be equal to the moment of the weight of the bar, and therefore $P \times DE$ is equal to the moment of the weight of 1 lb. at B. The mark, 1 lb., is therefore placed at E.

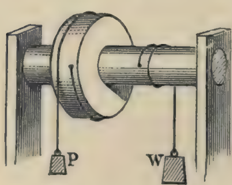
Next mark off $EF = DE$. Then $P \times DF = 2P \times DE =$ the moment of the weight of 2 lbs. at B. The mark, 2 lbs., is therefore placed at F, and by marking off equal distances along AC the remainder of the steelyard is graduated.

The *Danish Steelyard* has a moveable fulcrum, and is thus able to dispense with a moveable weight. The object to be weighed, W, is hung on a hook at one extremity B of the yard, and is balanced by the weight of the steelyard P, acting at its centre of gravity G. The fulcrum F is moved along the bar to a point between B and G. If there is equilibrium, then $W \times BF = P \times FG$. Let $BG = a$, $BF = x$, and let $W = n$ lbs. Then—

$$nx = P(a - x), \qquad \therefore x = \frac{aP}{n + P}.$$

By giving n different values in this formula we are able to graduate the steelyard.

64. The Wheel and Axle.—The simplest form of the wheel and axle consists of two cylinders with the same axis. The larger cylinder is called the wheel and the smaller the axle. The Weight is applied to the axle by a cord, which is wrapped round its circumference, and the Power is applied to the wheel in a similar way; the two cords being wrapped in opposite directions.

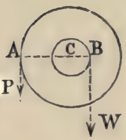


Another form of the wheel and axle is the windlass, in which the power is applied by means of a handle.

A third form is the capstan, in which the axis is vertical, and the power is applied horizontally by means of spokes.

Another common modification is the toothed wheel.

To find the condition of equilibrium of the wheel and axle, we must consider a cross section, and suppose both P and W to act in the plane of the paper. Let C be a point in the common axis, and let P act at A, and W at B. Then since there is equilibrium, the moments of P and W about C must be equal, and therefore



$$P \times AC = W \times BC.$$

The same equation is obtained at whatever point in the circumference P acts.

It is evident that we may regard the wheel and axle as a lever whose fulcrum is C.

Example—

The radii of a wheel and axle are 4 feet, and 6 inches, respectively. What force must be applied to raise a mass of 56 lbs?

As above $P \times AC = W \times BC$.

In this case $AC = 48$ inches, $BC = 6$ inches, $W = 56$.

$$\therefore 48P = 6 \times 56.$$

$$\therefore P = \frac{336}{48} = 7.$$

Thus the force required is the weight of 7 lbs.

Example for Exercise—

In a capstan the length of the spoke is 6 feet, measured from the axis, and the radius of the drum is 1 foot. Find the mass of an anchor which can be raised by 6 men, each of whom exerts a force equal to the weight of 100 lbs.

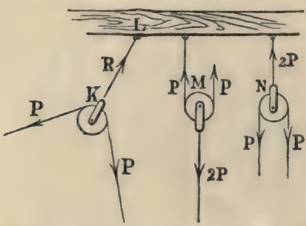
Examination on Sections 58-64

1. Define a Machine, and explain what is meant by the terms Power and Weight in connection with it.
2. Define a Lever. Find the condition of equilibrium of the lever.
3. In what three ways may the points of application of the Power and Weight be situated with regard to the fulcrum of a lever? Illustrate by a diagram, and give examples of each kind of lever.
4. What useful machines are formed by joining two similar levers at the fulcrum?
5. Describe the construction of the common balance. What are the characteristics of a good balance, and how may they be obtained?
6. Show how the true mass of a body may be ascertained by means of a balance whose arms are unequal.
7. Describe the common steelyard, and show how to graduate it.
8. Find the condition of equilibrium of the wheel and axle. Name any modifications of the wheel and axle.

65. The Pulley.—A pulley consists of a small wheel in the rim of which a uniform groove has been cut. It is easily moveable about its axis, which is fixed in a block, the block being either fixed to a beam or suspended by a cord. A cord passes round part of the circumference of the pulley, and is kept stretched by means of forces acting at both ends of the cord.[1] When there is equilibrium these forces must be equal, as otherwise the cord would be drawn round the pulley in the direction of the greater force. Thus if a force P act on one end of a cord which passes round one or more pulleys, an equal force P must act on the other end of the cord. Since this force P keeps the cord stretched, it is called the tension of the cord.

[1] In this and the following sections the weights of both pulleys and cords are left out of account. The cords are supposed to be perfectly flexible and the pulleys perfectly smooth.

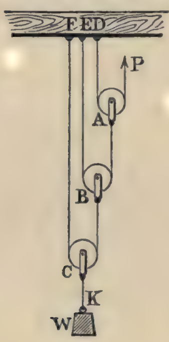
Let K be a pulley supported by a cord KL, and acted on by a second cord whose tension is P. If R be the tension of the cord KL, then the three forces, P, P, and R, must be in equilibrium, or R is equal and opposite to the resultant of P and P.



In the case of the pulleys M and N, the forces P and P are parallel, and therefore the resultant is in each case equal to 2P.

There are various combinations of pulleys, and in each we can find the relation of the Power and the Weight, by considering the equilibrium of each pulley separately. The three following cases have been called the first, second, and third systems of pulleys. There is no special reason for this order, but it is retained here for convenience of reference.

66. First System of Pulleys.—In this system a number of cords are used, each of which is attached to the beam and supports one moveable pulley. In the figure the power P acts on the cord AD, which is attached to the beam at D and supports a pulley at A. To this pulley is attached a cord ABE, which supports a second pulley B, and so on.



The last pulley supports the Weight W.
Since the tension of AD is P,
∴ the tension of ABE is 2P (§ 65),
„ „ BCF is 4P,
„ „ CK is 8P,
$\therefore W = 8P.$

This may be written $W = 2^3P$. If 4 strings are used, we have in like manner $W = 2^4P$, and so on. Therefore, if n strings are used,—

$$W = 2^nP. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

W is supported by P and by the beam DEF. Therefore the part supported by the beam is—

$$W-P=7P.$$

This may also be proved by adding the tensions of the three strings which are attached to the beam.

Thus the tension of AD=P,
" " BE=2P,
" " CF=4P.

∴ the weight supported by the beam=7P.

In the same manner, when there are n strings, the weight supported by the beam is—

$$2^nP-P \quad \text{or} \quad (2^n-1)\,P.$$

Example—

The weight of a mass of 1 lb. is sustained by that of a mass of 1 oz. How many cords and pulleys are employed?

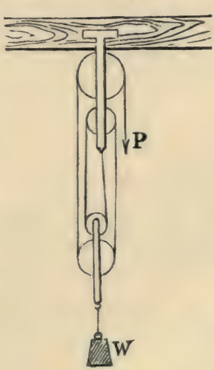
By (25)	$W=2^nP$.	
But	$W=16P$,	$\therefore 16=2^n$.
But	$16=2^4$,	$\therefore n=4$,

or the system has 4 cords supporting 4 moveable pulleys.

Example for Exercise—

If 5 pulleys are used, and the Power is 3 lbs., find the Weight.

67. **Second System of Pulleys.**—Here there are two blocks of pulleys, one of which is fixed to the beam, while the other supports the weight. Only one cord is used, which passes round a pulley in the upper and lower block alternately, and is finally fixed to one of the blocks.



In finding the condition of equilibrium, we have only to observe how many times the tension of the string acts on the lower block. In the figure there are 4 strings at the lower block,[1] and therefore—

$$W=4P.$$

[1] The number of strings at the lower block is always equal to the whole number of pulleys in both blocks.

If there were n strings at the lower block we should have in like manner—

$$W = nP. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

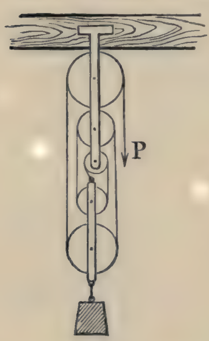
In this case the beam supports both W and P. Thus the weight supported by the beam is $W+P=5P$. This result may also be obtained by observing that there are 5 strings acting on the upper block.

In the same manner, if there are n strings at the lower block, the weight supported by the beam is—

$$nP+P \quad \text{or} \quad (n+1)P.$$

This system of pulleys is of much practical use. In the common form of block all the wheels (or sheaves) are on the same axle.

In the other systems of pulleys too much space is taken up by the motion of the pulleys.



Example—

The upper block contains 3 sheaves and the string is attached to the lower block. What mass is sustained by a Power equal to the Weight of 1 lb.?

Drawing the figure we can see that there are 5 strings at the lower block, therefore—

$$W = 5P,$$

or a mass of 5 lbs. is sustained.

Example for Exercise—

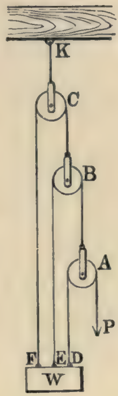
1 lb. supports 6 lbs. Draw the diagram, and find how many sheaves there are in each block.

68. Third System of Pulleys.—The third system of pulleys is simply the first system inverted, the beam and weight changing places.

As before, the tension of AD is P, that of ABE is 2P, of BCF, 4P, and of CK, 8P.

Thus the beam supports a weight of 8P, and since this is made up of W and P, we have

$$W = 8P - P = 7P.$$



Similarly when there are n strings the beam supports a tension of 2^nP, and therefore—

$$W = 2^nP - P.$$

Or $$W = (2^n - 1)\,P. \quad . \quad . \quad . \quad (27)$$

Example—

If there are 4 pulleys, and the Power is the weight of 3 lbs., find the Weight.

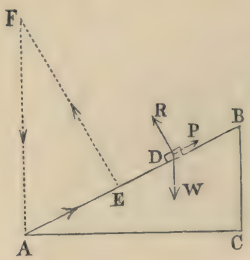
By (27) $$\begin{aligned} W &= (2^n - 1)\,P. \\ &= (2^4 - 1) \times P. \\ &= (16 - 1) \times 3. \\ &= 15 \times 3 = \text{the weight of 45 lbs.} \end{aligned}$$

Example for Exercise—

If there are 5 pulleys, and the Weight is 62 lbs., find the Power.

69. The Inclined Plane.—When a body is in contact with a smooth plane, it generally exerts some pressure upon it, and this pressure is at right angles to the surface of the plane. By the third law of motion the plane exerts on the body an equal and opposite pressure which is called the reaction of the plane (§ 55). In the case of a body lying on a horizontal plane, the reaction of the plane supports the weight of the body, but when the plane is inclined to the horizon, the direction of the reaction is no longer opposite to the direction of gravity, and a third force is necessary to maintain equilibrium. We shall consider the conditions of equilibrium in two simple cases.

70. Equilibrium on an Inclined Plane when the Power acts up the Plane.—Let AB be a section of an inclined plane, made by a vertical plane perpendicular to it. Let the plane of the paper represent the vertical plane, and let AC be a horizontal line, and BC be perpendicular to AC. Let D be a body resting on the inclined plane AB. Then D is kept in equilibrium by three forces, its weight W acting downwards, the power P acting in the direction AB, and the reaction R acting at right angles to the plane.



From AB cut off AE=BC. Draw EF at right angles to AB and equal to AC. Join FA.

The △FAE=△ABC in every respect (Euclid I. 4), and therefore the angle FAE=the angle ABC.

But the angles ABC and BAC are together equal to a right angle (Euclid I. 32).

∴ FAE and BAC together make up a right angle.

∴ FA is parallel to BC, and the three sides FA, AE, EF are parallel to the directions of the forces W, P, R.

If then FA be taken to represent W, AE will represent P, and EF will represent R on the same scale (§ 45).

But FA=AB, AE=BC, and EF=AC.

∴ If AB be taken to represent W in magnitude, BC will represent P, and AC will represent R.

Thus the Power is to the Weight as the *height* of the plane is to its *length*.

Examples—

1. If the height of the plane be to its length as 3 to 5, what Power will support a mass of 20 lbs.?

In this case 5 represents $20g$,
∴ 1 „ $4g$,
∴ 3 „ $12g$, the Power.

2. A railway train, whose mass is 126 tons, rests on an incline, and is kept from moving downwards by a force equal to the weight of 15 cwt. What is the slope of the incline?

Here $\frac{W}{P}=\frac{126\times 20}{15}=168.$ ∴ $W=168$ P.

Thus the Weight is 168 times the Power, and therefore the length of the plane is 168 times its height.

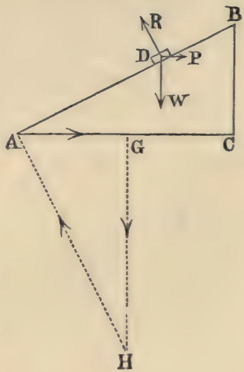
That is, the incline rises 1 in 168.

Examples for Exercise—

1. If the length of the plane be 13 feet and the base 12 feet, what Weight will be supported by a Power of $5g$?

2. The height of a plane is to its base as 12 to 35. If a mass of 3 tons 14 cwt. be placed on the plane, find what force up the plane is required to keep it at rest. Find also the pressure on the plane.

71. Equilibrium on an Inclined Plane when the Power acts Horizontally.—Let AB represent the inclined plane, D the body resting on it. Let W be the weight of the body, R the reaction of the plane, and P the Power acting horizontally. Let AC be horizontal, BC vertical.



From AC cut off AG = BC.

From G draw GH at right angles to AC and equal to AC. Join AH.

Then $\triangle$ AGH = $\triangle$ ABC in every respect (Euclid I. 4).

$\therefore \angle GAH = \angle ABC$.

But $\angle ABC + \angle BAC$ = a right angle (Euclid I. 32).

$\therefore \angle GAH + \angle BAC$ = a right angle.

$\therefore$ AH is at right angles to AB, and the three sides AG, GH, HA are parallel to the directions of the forces P, W, and R.

If then AG or BC be taken to represent P in magnitude, GH or AC will, on the same scale, represent W, and AH or AB will represent R.

Therefore the Power is to the Weight as the *height* of the plane is to its *base*.

Example—

If the height of the plane be to its base as 3 to 8, and the Power be $12g$, that is, a force equal to the weight of 12 lbs., find the Weight.

Here 3 represents $12g$,
$\therefore$ 1 ,, $4g$,
$\therefore$ 8 ,, $32g$, the Weight.

Example for Exercise—

If the length be to the base as 25 to 24, and the Power be $14g$, find the Weight.

Examination on Sections 65-71

1. What is a pulley? Explain what is meant by the tension of a cord.

2. A rope passes round a pulley, prove that its tension on both sides of the pulley must be equal if equilibrium is maintained.

3. Find the condition of equilibrium in a system of pulleys in which all the cords are fastened to the beam. Find also the weight supported by the beam.

4. Find the condition of equilibrium in a system of pulleys in which the same cord passes round all the pulleys.

5. Find the condition of equilibrium in a system of pulleys in which all the cords are fastened to the weight.

6. Explain what is meant by the reaction of a plane.

7. Find the condition of equilibrium of a body on a smooth inclined plane acted on by a force parallel to the plane.

8. Find the condition of equilibrium of a body resting on a smooth inclined plane, and acted on by a horizontal force.

72. **Friction.**—When two bodies are pressed together, and we try to make one of them slide over the other, a resistance is experienced which is called *Statical Friction.*

As long as the body is not moved, the amount of friction called into play is just sufficient to maintain equilibrium, and acts in the direction opposite to that in which the forces tend to produce motion.

When the body is on the point of being moved, the *whole amount* of statical friction is called into play.

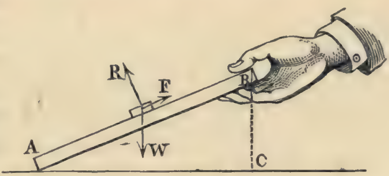
The whole amount of statical friction between two given bodies bears a fixed proportion to the normal[1] pressure between the surfaces, if other circumstances remain the same.

Thus if R be the normal pressure, the greatest possible value of the friction is μR, where μ is a fraction depending on the nature of the surfaces, and varying from $\frac{1}{50}$ in such bodies as oiled and polished metals, to $\frac{4}{5}$ in such bodies as wet leather.

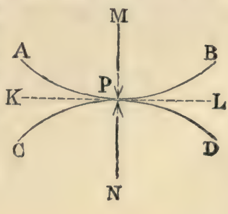
μ is called the *co-efficient* of statical friction.

When the statical friction is overcome and motion takes place, a force of friction continues to act, which is proportional to the pressure and independent of the rate of motion, but the co-efficient of *kinetic friction* is less than that of statical friction.

73. Angle of Repose.—When a body is laid on a rough plane, and the plane slightly tilted, the force of gravity tends to make the body slide down the plane. This calls into play the force of friction acting up the plane. If W be the weight of the body, R the reaction of the plane, and F the friction, then the three forces W, R, and F are in equilibrium, and are therefore (§ 70) proportional to the lengths AB, AC, and BC.



[1] By *normal pressure* is meant the pressure at right angles to both surfaces. If APB and CPD be sections of two surfaces which are pressed together at P, then the portion in contact is in each case approximately a part of a plane surface KPL, and the direction of the straight line MPN drawn through P at right angles to KL is said to be normal to both surfaces.



If the plane be now tilted until the body is on the point of sliding, it is easily seen that BC is increased, and therefore F, which is proportional to BC, is also increased.

In the limiting position, by § 72,—

$$F=\mu R;$$

but F and R are proportional to BC and AC, and therefore also

$$BC=\mu AC.$$

Or, dividing both sides of this equation by AC,

$$\mu=\frac{BC^*}{AC}. \qquad \qquad (28)$$

When the plane is in this position the angle BAC is called the *Angle of Repose*, or the *Angle of Friction.*

Examples—

1. What must the co-efficient of friction be in order that a body may rest on a plane inclined at an angle of 45°?

By (28) $\mu=\frac{BC}{AC}$, but if $\angle BAC=45°$, $AC=BC$,

$$\therefore \mu=1.$$

This case is imaginary, since μ is less than unity for all known substances.

2. A body just rests on a plane in which the height is one-half the length. Find the angle of repose and the co-efficient of friction.

Here ABC is one-half of an equilateral triangle. Therefore the angle of repose is 30°.

Also if BC=1, $AC=\sqrt{AB^2-BC^2}=\sqrt{4-1}=\sqrt{3}$.

$$\therefore \mu=\frac{BC}{AC}=\frac{1}{\sqrt{3}}.$$

Examples for Exercise—

1. A mass of 80 lbs. rests on a horizontal plane, and it is found that it can be moved by a horizontal force of $10g$. Find the co-efficient of friction.

2. A body lies on a horizontal slab 10 inches long. If the co-efficient of friction be $\frac{3}{4}$, how high may one end of the slab be raised before the body will begin to move?

---

* This may be trigonometrically expressed thus:—

μ=tangent $\angle$ BAC.

74. Machines in Motion.—We have investigated the relation of the Power to the Weight necessary in each case to keep the machine in equilibrium. If the Power be made to exceed this value by a very small amount, the machine will be set in motion, and the force P will do work (§ 42).

In the cases we have considered no part of the work is spent in giving energy to the machine, and therefore in each case *the work done by the Power is equal to the work done against the Weight.* (See § 44.)

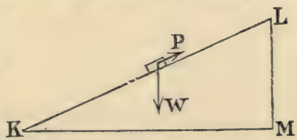
In the lever and the pulley the points of application of both P and W move in the line in which these forces act, and therefore in these cases the work done by each force is measured by that force multiplied by the distance moved by its point of application.

$$\therefore P \times P\text{'s path} = W \times W\text{'s path}.$$

Thus, if P be less than W, its path is greater than that of W in the same proportion, or, as it is sometimes stated, *what is gained in power is lost in space.*

In the case of the inclined plane, in measuring the work done by P and against W, we must consider the distance moved by the body in the directions of these forces respectively.

Thus, if the force be parallel to the plane, and the body be moved from K to L, the work done by P is $P \times KL$, while the work done against W is $W \times LM$.



Thus the path of the body is increased in the same proportion as the power is diminished, and again we see that what is gained in power is lost in space.

The fact that the work done by the Power is equal to the work done against the Weight may be proved in each machine from the conditions of equilibrium. Thus the above equation follows from the result of § 70, which states that

P is to W in the same proportion as the height of the plane is to its length, or—

$$\frac{P}{W}=\frac{LM}{KL}.$$

If both sides of this equation be multiplied by $W \times KL$, it takes the form—

$$P \times KL = W \times LM.$$

As another example of the application of this principle we may take the screw. Here the Power P acts by a lever or otherwise on the screw, and its point of application describes a circle of radius r about the axis of the screw, while the end of the screw is pressed forward by an amount equal to the distance d between two of its threads. Thus if W be the resistance overcome, we have $P \times 2\pi r = Wd$. (See General Example 46.)

Example—

Show that, in the second system of pulleys, the work done by P is equal to that done against W.

Since there are n strings connecting the two blocks, if W be raised x feet, each of the n strings is shortened by x feet, and therefore P descends nx feet.

But by the conditions of equilibrium,—

$$W = nP,$$
$$\therefore\ xW = nxP;$$

or the amount of work done by P is equal to that done against W.

Example for Exercise—

Prove that the work done by the Power is equal to that done against the Weight in (1) the lever, (2) the first system of pulleys, (3) the wheel and axle.

Examination on Sections 72-74

1. Define Statical Friction.
2. When is the *whole amount* of statical friction called into play, and in what direction does it act?
3. What is meant by the co-efficient of statical friction?
4. What is Kinetic Friction? In what particulars do its laws resemble those of statical friction, and in what do they differ?
5. Explain what is meant by the Angle of Repose.

6. Show how we may find the co-efficient of friction, when the angle of repose is given.

7. When a machine is in motion, what is the connection between the work done by the power and that done against the weight?

8. Explain by reference to any simple machine the meaning of the phrase, "What is gained in power is lost in space."

GENERAL EXAMPLES IN MACHINES

The Lever

1. Where must I place the fulcrum in order that P may balance W at the extremities of a straight lever AB, in the following cases?—

(1) P = 5, W = 3, AB = 3 feet.
(2) P = 6, W = 4, AB = 1 foot.
(3) P = 10, W = $\frac{1}{2}$, AB = $3\frac{1}{2}$ feet.

2. The force at one end of a lever is $3g$, and its distance from the fulcrum 2 inches. If the lever be 5 inches long, find the parallel force which will maintain equilibrium at the other end.

3. In a pair of nut-crackers the nut is placed 1 inch from the hinge, and the hand is applied at a distance of 6 inches from the hinge. If the nut requires a force[1] of $2\frac{1}{4}$ lbs. to break it, how much pressure must be exerted by the hand?

4. If a force of 3 lbs. 10 oz. act on a lever at 1 foot 8 inches from the fulcrum, where must a force of 5 lbs. act to produce equilibrium?

5. A uniform rod, 2 feet long, weighs 1 lb. What weight must be hung at one end in order that the rod may balance on a point 3 inches from that end?

6. Two men carry a mass M by means of a pole AB, one end of which is held by each. The mass is placed at the point C. Find how much of the mass is supported by each man in the following cases:—

(1) AB = 5 feet, AC = 2 feet, M = 200 lbs.
(2) AB = 7 feet, AC = $1\frac{1}{2}$ feet, M = 350 lbs.
(3) AC = 4 feet, BC = 6 feet, M = 300 lbs.

7. A lever, 5 feet long, rests on a fulcrum at one end; a weight of 10 lbs. is suspended 1 foot $10\frac{1}{2}$ inches from the fulcrum. What force acting perpendicularly at the other end will preserve equilibrium?

---

[1] As previously explained, this phrase is used to denote the force equal to the weight of the mass named.

8. Two forces, which are as 3 to 2, balance on a lever of the second class. If the distance between their points of application be 2 feet, find the whole length of the lever.

9. Two forces of 5 and 6 lbs. act perpendicularly on a lever at a distance of $5\frac{1}{2}$ inches. Find the position of the fulcrum when the directions in which the forces act are—(1) similar, (2) dissimilar.

10. A lever, 2 feet long, is supported in a horizontal position by 2 pegs, one below it at 3 inches from its extremity, the other above it at its extremity. A weight of 5 lbs. is suspended from the other end. Find the pressure on each peg.

11. A lever, 2 feet long, has the fulcrum at its middle point. Weights of 3, 6, and 12 lbs. are suspended at distances of 4, 8, and 10 inches on one side of the fulcrum. What force must act at the extremity of the other half of the rod to produce equilibrium?

12. Two weights of 4 lbs. are hung on one side of a lever, at distances of 5 and 9 inches from the fulcrum. What vertical force must act at 8 inches from the fulcrum to balance them?

13. A heavy beam, 3 feet long, weighs 4 lbs., and has a weight of 3 lbs. hung at one end. If the fulcrum be 1 foot from the 3 lb. weight, what weight must be hung at the other end to produce equilibrium?

14. A heavy uniform rod has a weight of 3 lbs. hung from one end, and balances about a point 4 feet from that end. If a weight of 5 lbs. be substituted for the 3 lbs. weight, the rod balances about a point 3 feet from the end. Find the length and weight of the rod.

The Balance

15. A body weighs 4 lbs. in one scale of a false balance and $6\frac{1}{4}$ lbs. in the other. Find its real weight.

16. The arms of a false balance are 9 and 10 inches long. If the real weight of a body is 5 lbs. 10 oz., how much will it appear to weigh in each scale?

17. A body which really weighs 3 lbs., appears to weigh $3\frac{1}{4}$ lbs. in a false balance. Find the proportion of the lengths of the arms.

18. A body, whose real weight is 4 lbs. 8 oz., appears to weigh 4 lbs. in one scale of a balance; find—

(1) what it will seem to weigh in the other scale;
(2) what is the proportion of the lengths of the arms.

The Wheel and Axle

19. The radius of the wheel is 4 feet, and that of the axle 6 inches. Find what Power will sustain a Weight of 120 lbs.

20. The radius of the wheel is five times that of the axle. What weight can be raised by a force of 10 lbs.?

21. Twelve sailors, each exerting a force of 96 lbs., work a capstan with levers 7 feet 6 inches long. The radius of the capstan is 16 inches. What weight can they sustain?

Pulleys

22. Sketch two systems of pulleys, in each of which the Weight is four times the Power.

23. Sketch two systems of pulleys, in each of which the Weight is seven times the Power.

24. In the first system of pulleys, if there are five pulleys, and the Weight is 64 lbs., find the Power.

25. In the second system of pulleys, if the Power is 7 lbs. and the Weight 56 lbs., draw the diagram.

26. In the first system of pulleys three pulleys are used, each of which weighs 1 lb. Find what Weight will be sustained by a Power of 10 lbs.

27. In the second system of pulleys each block contains three sheaves, and the moveable block weighs 3 lbs. What Weight will be sustained by a Power of 10 lbs.?

28. In the third system of pulleys three pulleys are used, each weighing 1 lb. Find the Weight, if the Power be 10 lbs.

The Inclined Plane

29. On a smooth plane, rising 3 in 5, a mass of 10 lbs. is kept from sliding by a force in the direction of the plane. Find—

(1) this force;

(2) the pressure on the plane.

30. An inclined plane rises 16 in 65. Find what Weight will be supported by a horizontal force of 80 lbs.

31. A horizontal force of 35 lbs. supports a Weight of 120 lbs. upon an inclined plane. Find the pressure on the plane.

32. A plane is inclined at an angle of 45°. Find the horizontal force necessary to support a Weight of 10 lbs.

33. A body is supported on an inclined plane by a force parallel to the plane. If the Power be half the Weight, find the inclination of the plane.

34. A body weighing 20 lbs. rests on a plane, and is supported by a horizontal force of 15 lbs. Find—

(1) the pressure on the plane;

(2) what power parallel to the plane would support it;

(3) the pressure on the plane in this second case.

Miscellaneous

35. Two levers, each 1 foot long, are combined so that the Weight of the first is the Power of the second. In the first the Power is 9 inches from the fulcrum, and in the second 8 inches. Find the ratio of the Power to the Weight.

36. Two inclined planes of the same height slope in opposite directions, and two weights rest, one on each plane, connected with each other by a cord passing over a pulley at the common vertex. If the lengths of the planes are 5 feet and 6 feet, find the relation of the weights that equilibrium may be possible.

37. In a lever the Weight is four times the Power. What will be the velocity of the Power when that of the Weight is 7?

38. Find the force which, acting up an inclined plane, will keep a weight of 10 lbs. in equilibrium; it being given that the force, the pressure on the plane, and the weight are in arithmetical progression.

39. Compare the greatest and least weights which can be supported by a given force acting on a wheel with a square axle.

40. Two weights, 6 lbs. and 9 lbs., balance on a straight lever; if they be transposed, which weight will prevail, and what weight must be added to restore the balance?

41. If the less of two forces acting on a lever be 16 lbs. and be at a distance of 15 feet from the fulcrum, the pressure on which is 4 lbs.; find the other force and its distance from the fulcrum when there is equilibrium.

42. A weight is to be raised by means of a rope passing round a horizontal cylinder 9 inches in diameter, turned by a winch, with an arm 2 feet 3 inches long. Find the weight a man can raise by exerting a pressure of 74 lbs. on the handle.

43. What force can a man weighing 165 lbs. exert on a stone by (1) pressing on one end of a *horizontal* crowbar 6 feet long, propped at a distance of 5 inches from the point in contact with the stone; and (2) if the crowbar were inclined at an angle of 60° to the horizon?

44. The wire-rope working a railway signal passes over and has its end attached to the rim of a fixed grooved pulley; the pulley is 1 foot in diameter, and is turned by a lever 3 feet in length. Find

the tension transmitted along the rope when a man pulls with a force of 75 lbs. at right angles to the lever and at its extremity.

45. A steelyard takes a horizontal position when no body is attached to it, and the hook is at a distance of 6 inches from the point of suspension. To balance a certain body a 14 lb. weight must be placed at 2·5 feet from the point of suspension ; what is the weight of the body ?

46. What is the gain in power in the case of a screw-jack which is moved by a lever 2 feet in length, and the screw of which has a distance of half an inch between its threads ?

47. AB is a Roman steelyard 12 feet long ; its centre of gravity is 11 inches from A, and the fulcrum 8 inches from A. If the weight of the steelyard be 4 lbs. and that of the moveable weight 3 lbs., find how many inches from B the graduation marked 15 lbs. will be.

48. A particle weighing $8\sqrt{3}$ lbs. is in equilibrium under the action of a force of 12 lbs. acting parallel to the plane. Find the inclination of the plane and the pressure upon it.

49. What weight can be supported by a force of 2 lbs., by means of a system of eight pulleys arranged in two blocks, the weight of the lower block being 1 lb. ?

CHAPTER VI

HYDROSTATICS

75. Fluids.—It has been already stated that all solid bodies possess a certain amount of rigidity. Liquids and gases have no rigidity. They are considered together under the name of Fluids.

In Hydrostatics we consider only fluids at rest. *When at rest* all fluids satisfy the following definition.

A FLUID *is incapable of resisting a change of shape.*

In other words, fluids do not exert statical friction. When *in motion* all known fluids exert kinetic friction, and therefore do not satisfy this definition. If a fluid could be found incapable of exerting kinetic friction, it would be called a *Perfect Fluid.*

The different portions of a fluid exert pressure on each other, and on the sides of the vessels which contain them.

In order to investigate fluid pressure we may consider the equilibrium of any small portion of the fluid by itself. The surfaces by which this portion of the fluid is bounded are pressed by the surfaces (whether solid or fluid) with which they are in contact, and, by the third law of motion, they exert on them an equal and opposite pressure.

The pressure exerted by any portion of a fluid at rest is perpendicular to the surface of that portion.

For, if the pressure were not perpendicular to the surface, it could be resolved into two components, one perpendicular, and one parallel to the surface. This latter component would tend to produce sliding motion, and since, by the definition, it could not be resisted, motion would take place, and therefore the fluid would not be at rest.

It is assumed throughout this chapter that *no change of temperature* takes place in the bodies treated of.

76. Liquids and Gases.—Fluids are divided into *Liquids* and *Gases*, which differ in the following respects :—

Liquids are nearly incompressible, that is, the volume of a given mass of a liquid remains nearly constant, to whatever pressure it is subjected. The volume of a liquid contained in a closed vessel must be either equal to, or less than, the volume of the vessel. In the latter case the liquid only fills part of the vessel, and has, therefore, a free surface. A liquid can also be contained in an open vessel.

Gases are easily compressed, that is, the volume of a given mass of a gas is not constant, but becomes smaller when the pressure to which it is subjected is increased, and larger when the pressure is diminished. (See § 90.) If a quantity of gas is placed in a closed vessel, it expands so as to fill the whole vessel, and then presses on and is pressed by, the sides of the vessel. A gas cannot be contained in an open vessel, unless the space outside the vessel contains some fluid which exerts pressure on the gas. For example, if an open vessel containing a gas be placed in a space exhausted of air, the gas will expand and fill the whole space. On the other hand, if a vessel containing gas be held mouth downwards in water, the gas will remain in the vessel.

It is to be observed that changes due to such causes as chemical reactions, absorption of gases by liquids, etc., are not considered in this treatise.

77. Density.—If we take equal volumes of two bodies, and find that the mass of one is greater than the mass of the other, the body which has the greater mass is said to be *denser* than the other.

Thus, if we compare a cubic foot of iron with a cubic foot of wood, we find that the iron has the greater mass, and is therefore denser than the wood. Similarly mercury is denser than iron, iron than water, water than air.

The DENSITY *of a body is the mass contained in unit volume.* If we take a foot as the unit of length, a *cubic foot* will be the corresponding unit of volume. Let ρ be the density of a body of mass M, whose volume is V cubic feet.

Then the mass of 1 cubic foot is ρ,
,, ,, 2 ,, feet ,, 2ρ,
,, ,, 3 ,, feet ,, 3ρ,

and so on. Therefore the mass of V cubic feet is $V\rho$, or

$$M = V\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Since the volume of a given mass of a *liquid* is constant, its density is also constant, but different liquids have different densities.

In a *gas* the volume varies with the pressure, and therefore the density also varies. If we divide both sides of (29) by (1) ρ, (2) V, it takes the forms

$$V = \frac{M}{\rho} \text{ and } \rho = \frac{M}{V}, \quad . \quad . \quad . \quad . \quad . \quad (29')$$

which show us that the volume of a given mass varies inversely as the density.

Examples—

1. A cubic inch of iron weighs $\frac{1}{4}$ of a lb.,[1] find its density.

By (29') $\rho = \dfrac{M}{V}$.

Here $M = \frac{1}{4}$, $V = \frac{1}{1728}$.

$\therefore \rho = \frac{1}{4} \times 1728 = 432$,

or the mass of a cubic foot of iron is 432 lbs.

2. A cubic foot of water weighs 1000 ounces, find its density.

Here $M = \frac{1000}{16}$, $V = 1$.

$\therefore \rho = \frac{1000}{16} = 62\frac{1}{2}$.

Examples for Exercise—

1. Three cubic feet of green oak weigh 219 lbs., find its density.
2. The density of mercury is 812, find the mass of one cubic inch.

---

[1] That is, its *mass*, as ascertained by "weighing," is $\frac{1}{4}$ of a lb.

78. Pressure at a point in a Fluid.—We have seen that the pressure on the surface of any portion of a fluid at rest is perpendicular to the surface. This pressure may be *uniform*, or may *vary* from point to point of the surface. The pressure at a given point in a given direction is found by considering the pressure on a plane area including the point and perpendicular to the direction.

When the pressure is uniform over this area, then the pressure at the point is measured by the pressure on a unit of area taken so as to include the point in question. When the pressure is variable, then the pressure at the point is the pressure which would be exerted on a unit of area if the pressure all over the unit of area were the same as at the point under consideration.

As we have taken a foot as the unit of length, a *square foot* will be the unit of area. It is, however, very common in measuring fluid pressure to take a *square inch* as the unit of area. As a square foot contains 144 square inches, the pressure on a square inch is $\frac{1}{144}$ of the pressure on a square foot under the same circumstances.

Examples—

1. An area of 20 square feet sustains a uniform atmospheric pressure altogether amounting to 43,200 lbs. (that is, a pressure equal to the weight of 43,200 lbs.). Find the pressure at a point in the surface.

If a square foot be the unit,—

$$\text{pressure} = \frac{43200}{20} = 2160 \text{ lbs.}$$

If a square inch be the unit,—

$$\text{pressure} = \frac{2160}{144} = 15 \text{ lbs.}$$

2. A rectangle measuring 9 inches by 4 undergoes a uniform pressure of 8 lbs. on the square inch; find the pressure on the whole surface.

The surface contains 36 square inches,—

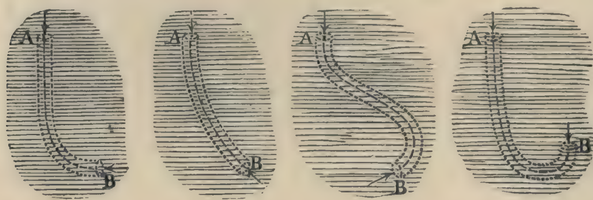
$$\therefore \text{pressure} = 36 \times 8 = 288 \text{ lbs.}$$

Examples for Exercise—

If a square area measuring 4 inches each way sustain a uniform pressure altogether amounting to 72 lbs., find the pressure (1) on a square foot, (2) on a square inch.

79. Fluid not acted on by External Forces.—*When a fluid is not acted on by external forces, the pressure is the same at all points and in all directions.*

Suppose the fluid to be contained in a closed vessel, and subject only to the pressure of the sides of the vessel.



Let A and B be any two points in the fluid and draw any curved line AB.

Consider the equilibrium of a portion of the fluid in the shape of a tube of uniform small section described about AB as axis. The tubular portion of the fluid is subjected to the pressure of the adjoining fluid on its curved surface and on its ends.

Since the pressures on the curved surface are all at right angles to that surface, they do not tend to prevent the fluid from flowing through the tube. Therefore the pressures on the two ends at A and B must be equal, as otherwise the fluid would begin to flow in the direction of the greater pressure. Now the areas of the two ends are equal, and they may be taken as small as we please, therefore the pressures at the points A and B perpendicular to the ends of the cylinder are equal.

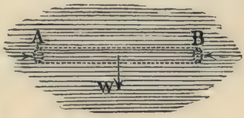
Let us now, as in the figure, vary the direction of the line AB, so that the direction of the pressure at A remains the same, while that at B is changed. In each case the pressure at B is equal to the pressure at A, and thus we see that the pressure at B is the same in all directions. And since B is any point in the fluid, it follows that the pressure is the same at all points and in all directions throughout the fluid.

80. Heavy Fluids.—(1) *When external forces, such as gravity, act on a fluid, the pressure at a point is the same in all directions.*

For the preceding demonstration will still apply when the points A and B are taken very near each other, as in that case the weight of the tubular portion of the fluid may be neglected when compared with the pressures on its ends.[1]

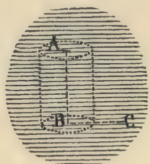
(2) *The pressures at any two points in the same horizontal plane are equal to one another.*

Let A and B be two points in the same horizontal plane, and consider a small cylinder of the fluid of which AB is the axis. Since the weight of the fluid acts vertically, it does not prevent horizontal motion. Therefore the pressures at A and B must be equal to each other.



(3) *The pressures at any two points in the fluid differ by the weight of a cylinder of the fluid whose base is unit of area, and whose height is the vertical distance between the points.*

Let A and C be any two points in the fluid. Through C draw a horizontal plane, and through A draw AB perpendicular to this plane.



About AB as axis describe a cylinder whose base is unit of area, and consider the equilibrium of the fluid within the cylinder.

The pressure from above on the end A and the weight of the fluid tend to make the fluid within the cylinder flow downwards. This is prevented solely by the pressure from below on the end B, since all the pressures on the curved surface are horizontal.

---

[1] The pressures on the ends are proportional to their areas, while the weight is proportional to the volume. If the distance AB is reduced to $\frac{1}{10}$ of its former length, the proportions of the tube being unaltered, the pressures on the ends are reduced to $\frac{1}{100}$ of their former values and the weight to $\frac{1}{1000}$. Thus the weight may finally be left out of account in comparison with the pressures.

∴ pressure on end A + weight of cylinder of fluid = pressure on end B.

But the end A is part of a horizontal plane, and, as proved above, the pressure on it is uniform, and is therefore equal to the pressure *at the point* A (§ 78):—

∴ pressure at A + weight of cylinder = pressure at B.

And since C is in the same horizontal plane as B,—

pressure at C = pressure at B.

∴ pressure at A + weight of cylinder = pressure at C.

In liquids the density is the same throughout, and hence the weight of the cylinder AB is the same, from whatever part of the liquid it is taken, as long as its size is unaltered. In gases, however, it is important to notice that the ends of the cylinder must be in the planes of A and C.

The fact that the pressure at any point of a fluid is equal to the pressure at its highest point, together with the pressure due to the weight of the fluid, is often called the principle of the *Transmissibility of Fluid Pressure*.

Examination on Sections 75-80

1. Define a Fluid. Prove that in a fluid at rest the pressure on any surface is at right angles to that surface.

2. Prove that fluids cannot exert statical friction.

3. What is meant by a perfect fluid?

4. Distinguish between liquids and gases.

5. Define Density. Prove the formula $M = V\rho$.

6. Show that the density of a gas is inversely proportional to its volume.

7. Explain what is meant by the pressure of a fluid at a given point in a given direction. How is it measured (1) when uniform, (2) when variable?

8. Prove that, in a fluid not acted on by external forces, the pressure is the same at all points and in all directions.

9. Prove that the pressure at any point in a heavy fluid is the same in all directions.

10. Prove that in a heavy fluid the pressures at all points in the same horizontal plane are equal.

11. Prove that in a heavy fluid the pressures at any two points differ by the weight of a column of the fluid, whose base is unit of area and whose height is the perpendicular distance between the points.

12. What is meant by the Transmissibility of Fluid Pressure?

81. Heavy Liquids.—We have seen that in all heavy fluids, and therefore in liquids, the pressures at any two points differ by the weight of a cylinder of the fluid, whose base is unit of area and whose height is the vertical distance between the points.

When a liquid fills a closed vessel, the whole surface of the liquid is generally subjected to pressure by the inner surface of the vessel.

When a liquid has a free surface, the space above this free surface is generally occupied by a gas (as, for example, air or steam), which exerts a nearly uniform pressure on the surface.

Let p be the pressure at the highest point (or at the free surface) of the liquid, and let q be the weight of a cubic foot of the liquid. Then at a depth of 1 foot the pressure is equal to $p+q$, at a depth of 2 feet the pressure is $p+2q$, and so on.

Therefore at a point x feet below the surface the pressure is $p+xq$.

When liquids are subjected to great pressure, as, for example, in Bramah's press (§ 97), the applied pressure is so great that the pressure caused by the weight of the liquid may be left out of account in comparison with it.

That is, in the expression $p+xq$, q is very small compared with p, and we may therefore neglect xq and consider the pressure to be uniform and equal to p.

If, on the other hand, the pressure p at the surface of a liquid is very small, as, for example, in the barometric column (§ 91), we may leave it out of account in comparison with the pressure caused by the weight of the liquid, and the pressure at any point in the liquid is therefore xq. That is to say:—

When there is no pressure at the surface of a heavy liquid, then the pressure at any point is proportional to its depth below the surface.

Examples—

1. If the pressure of the air on the surface of a lake is 15 lbs. on the square inch, and if a cubic foot of water weighs 1000 oz., find the pressure at a depth of 100 feet.

Take a foot as the unit of length. Then $x=100$.

$\therefore$ the pressure at 100 feet is $p+100\,q$.

$p=$ the pressure on a square foot at the surface,

$\therefore\ p=144\times 15=2160$ lbs.,

$q=1000$ oz. $=62\frac{1}{2}$ lbs.

$\therefore\ p+100\,q=2160+6250=8410$ lbs.

This is the pressure on a square foot; the pressure on a square inch is found by dividing this result by 144.

2. The pressure at the surface of mercury being zero, find the pressure at a depth of 6 feet, assuming mercury to be $13\frac{1}{2}$ times as dense as water.

Here $p=0$, and $q=13\frac{1}{2}\times 62\frac{1}{2}=843\frac{3}{4}$ lbs. Also $x=6$

$\therefore$ the pressure $=6q=5062\frac{1}{2}$ lbs.

As before, this is the pressure on a square foot; that on a square inch is $35\frac{5}{32}$ lbs.

Examples for Exercise—

1. The pressure at the surface of a liquid is 10 lbs. on the square inch; that at a depth of 4 feet is 20 lbs. on the square inch. Find the weight of a cubic foot of the liquid.

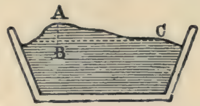
2. If there be no pressure on the surface of water, at what depth will the pressure be 15 lbs. on the square inch?

3. The pressure on the surface of a liquid is 2 lbs. on the square inch, that at a depth of 5 feet is $4\frac{1}{2}$ lbs. Find the pressure at a depth of 400 yards.

82. **Surface of Heavy Liquids.**—We have seen that in heavy fluids, the pressures at all points in the same horizontal plane are equal. It follows that *the surface of a heavy*

liquid is horizontal if all parts of that surface are subjected to the same external pressure.

For, if possible, let A and C be two points in the surface of a heavy liquid, which are subjected to the same external pressure, but are not in the same horizontal plane, and let AB be the perpendicular from A on the horizontal plane through C. Then by § 80 the pressure at C exceeds that at A by the weight of a cylinder of the liquid whose base is unit of area and whose height is AB. Therefore the pressures at A and C cannot be the same unless they are points in the same horizontal plane.



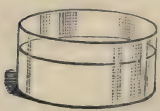
The proposition as above enunciated is only true in the case of small[1] surfaces. In the case of such large areas as lakes, canals, and seas, the direction in which gravity acts at distant points in the surface are not even approximately parallel. Each small part of the surface is very approximately a plane perpendicular to the direction of gravity at that part, but the whole surface is approximately spherical.

83. Resultant Pressures on the Sides and Base of a Vessel containing a Heavy Liquid.—We have seen that the pressure at any point of a heavy liquid is equal to $p+xq$. p represents the atmospheric (or other) pressure on the surface of the liquid, and may be left out of account, in considering the pressure due to the liquid.

Thus the pressure on the interior surface of the vessel at any point is xq, where x is the depth of that point below the surface of the liquid. Suppose the vessels represented in the following diagrams to be filled to the same depth with water, then the pressures at all points in their bases (supposed horizontal) are equal and parallel, and their resultant is found by adding them together. (See § 51.) Thus the resultant pressure sustained by the base of each vessel is

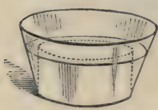
[1] In very fine tubes and near the sides of vessels the phenomenon known as *capillarity* occurs. It is described and explained in works on Physics.

equal to the weight of a cylinder of water standing on that base and terminated by the surface of the liquid.

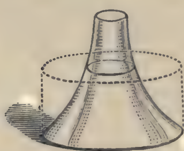


In the first figure the vessel is cylindrical and the pressure on the base is equal to the weight of all the liquid in the vessel, the pressure on the sides being horizontal.

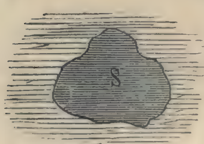
In the second figure, the pressure on the base is less than the weight of the liquid in the vessel, by the weight of the portion of liquid contained in the wedge-shaped ring which surrounds the dotted cylinder. The pressure on the sides may in this case be resolved into two components, one horizontal and the other vertically downwards. The latter is equal to the weight of the liquid in the wedge-shaped ring.



In the third figure the pressure on the base is more than the weight of all the liquid in the vessel, and the pressure on the sides has a component acting vertically upwards. This component is equal to the weight of the liquid which would be vertically over the sides if the liquid stood at the same level outside and inside the vessel. Thus the *resultant* downward pressure on the whole vessel is found by subtracting the upward pressure on the sides from the downward pressure on the base, and is, as in the other cases, equal to the weight of the liquid in the vessel.[1]



84. Resultant Pressure on a Body immersed in a Fluid.—Let S be a solid body immersed in a fluid. Suppose S removed, and the space occupied by it filled by the fluid.



Then the portion of the fluid occupying the space S is acted on only by its weight, which acts vertically downwards through its centre of gravity, and by the pressure of the

[1] For the definition of Centre of Pressure, see the Appendix, § 118.

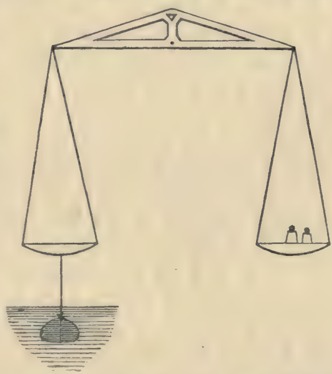
surrounding fluid, and, since there is equilibrium, these must be equal and opposite.

But the pressures of the surrounding portions of the fluid depend only on the depth, and are therefore the same, whether the space S is occupied by a liquid or a solid.

Therefore the *resultant* pressure on the solid acts vertically upwards through the centre of gravity of the displaced fluid, and is equal to the weight of that part of the fluid which would fill the same space.

85. **Weighing in Air and in Water.**—When a body is immersed in any fluid, we have seen that the resultant pressure of the fluid on the body is upwards, and equal to the weight of that portion of the fluid which the body displaces.

Thus when any body is surrounded by air, the air exerts an upward pressure on it equal to the weight of the same volume of air. If, as in the case of a balloon or a mass of heated air, the density of the body is less than that of the surrounding atmosphere, the upward pressure is greater than the weight of the body, and therefore causes the body to ascend. Similarly the downward pressure which any body surrounded by air exerts on the hand, the ground, or the scale of a balance, is not, strictly speaking, equal to the weight of the body, but to the weight of the body diminished by the weight of the air which it displaces.



To find the exact weight of a body, it must therefore be weighed in a vacuum. In the case of ordinary solids, however, the loss of apparent weight due to atmospheric pressure is so small that it may be neglected when compared with the whole weight of the body.

On the other hand, when a solid body is immersed in water, oil, or any other heavy liquid, the apparent loss of weight is generally considerable, and we shall presently see that this process, combined with ordinary weighing, enables us to ascertain the density of the body.

The diagram shows how this loss of weight may be ascertained. Bodies soluble in water may be weighed in oil, or they may be encased in a thin film of wax.

Examination on Sections 81-85

1. Prove that the pressure at any point in a liquid is equal to the pressure on the free surface together with the pressure caused by the weight of the liquid.

2. Explain under what circumstances the pressure in a liquid is approximately uniform, and under what circumstances it may be considered to be proportional to the depth below the surface.

3. Prove that the surface of a liquid at rest is a horizontal plane.

4. Explain why the proof of the preceding proposition applies only to surfaces of moderate dimensions.

5. Prove that the pressure on any portion of the horizontal base of a vessel containing liquid is equal to the weight of the liquid which would be contained in a cylindrical figure erected on that base and terminated by the surface of the liquid.

6. Mention, and illustrate by a diagram, cases of vessels in which the whole pressure on the base is (1) greater, and (2) less than the weight of all the fluid in the vessel.

7. Show that in both cases of the last question the resultant downward pressure on the base and sides of the vessel is equal to the weight of the fluid contained in the vessel.

8. A body is immersed in a fluid. Find the resultant of all the pressures exerted by the fluid on the surface of the body.

9. Prove that, when a body surrounded by any fluid is weighed, the result is not the true weight of the body.

10. Show by examples that fluids exert an upward pressure on bodies immersed in them.

86. Specific Gravity.—The ratio of the weight of a given volume of any substance to the weight of an equal volume of a standard substance is called the *Specific Gravity* of the substance.

The standard substance employed is distilled water at the temperature of its maximum density (4° centigrade).

The specific gravity of water is therefore unity.

It follows that *the* SPECIFIC GRAVITY *of any body is equal to the weight of the body divided by the weight of an equal volume of water.*

Let W be the weight of the body, W' the weight of an equal volume of water, then if s be the specific gravity of the body,—

$$s=\frac{W}{W'}; \quad . \quad . \quad . \quad . \quad . \quad (30)$$

or, multiplying both sides by W',—

$$W=sW'. \quad . \quad . \quad . \quad . \quad . \quad (30')$$

Since the weights of bodies are proportional to their masses (see § 34), and the masses of equal volumes of different bodies are proportional to their densities (see § 77), it follows that the specific gravities of bodies are proportional to their densities.

Thus a table of densities may be converted into a table of specific gravities by dividing the density of each substance by that of water.

Examples—

1. The densities of water, green oak, iron, and mercury, were found in § 77 to be $62\frac{1}{2}$, 73, 432, and 812 respectively, find their specific gravities.

Dividing the density of each by that of water we obtain—

$$\text{Specific gravity of water}=\frac{62\frac{1}{2}}{62\frac{1}{2}}=1.$$

$$\text{,,} \quad \text{,,} \quad \text{green oak}=\frac{73}{62\frac{1}{2}}=1{\cdot}2.$$

$$\text{,,} \quad \text{,,} \quad \text{iron}=\frac{432}{62\frac{1}{2}}=6{\cdot}9.$$

$$\text{,,} \quad \text{,,} \quad \text{mercury}=\frac{812}{62\frac{1}{2}}=13.$$

2. The specific gravity of silver is $10\frac{1}{2}$, find the weight of a cubic inch. The weight of a cubic inch of water is $\frac{62\frac{1}{2}}{1728}$ lbs.

By (30′) $$W=sW',$$

but $s=10\frac{1}{2}$, $W'=\frac{62\frac{1}{2}}{1728}$, $\therefore W=10\frac{1}{2}\times\frac{62\frac{1}{2}}{1728}=\frac{875}{2304}$ lbs.

3. A piece of wood weighs 23 lbs., and the same volume of water weighs 25 lbs. Find the specific gravity of the wood.

By (30) $$s=\frac{W}{W'}=\frac{23}{25}.$$

4. Thirty-one cubic centimetres of gold weigh 599 grammes. Find the specific gravity of gold.

In using the metrical system it is important to remember that *a cubic centimetre of distilled water at its maximum density weighs 1 gramme.*[1]

∴ 31 cubic centimetres of water weigh 31 grammes.

∴ by (30) $$s=\frac{W}{W'}=\frac{599}{31}=19\cdot3.$$

Examples for Exercise—

1. A cubic foot of copper weighs 540 lbs. ; find its specific gravity.

2. The specific gravity of ice is $\frac{9}{10}$. Find (1) the volume occupied by one ton of ice ; (2) the weight of a cubic yard of ice.

3. Sixteen cubic centimetres of zinc weigh 108·8 grammes ; find its specific gravity.

87. Specific Gravity found by Weighing in Water.—When the volume and weight of a body are known, its specific gravity may be found by the methods of § 86. When the volume is not known, we may find the weight of an equal volume of water, by weighing the substance in water, and observing by how much the weight in water is less than the true weight.

Thus if W be the weight of the body, w be its weight when weighed in water, $W-w$ is the portion of its weight sustained by the pressure of the water, and is therefore equal to W' the weight of an equal volume of water (§ 84).

$$\therefore\ s=\frac{W}{W'}=\frac{W}{W-w}. \quad . \quad . \quad . \quad . \quad (31)$$

Examples—

1. A piece of ivory weighs 3 lbs. in air and 1·4 lb. in water. Find its specific gravity.

By (31) $$s=\frac{W}{W-w}=\frac{3}{3-1\cdot4}=\frac{3}{1\cdot6}=\frac{30}{16}=1\tfrac{7}{8}.$$

---

[1] Thus the specific gravity of any substance is equal to the weight in grammes of a cubic centimetre of that substance.

2. A piece of wood weighing 5 lbs. is attached to a piece of metal weighing 10 lbs., and the whole weighs 7 lbs. in water. The metal alone weighs 9 lbs. in water. Find the specific gravity of the metal and of the wood.

For the metal we have by (31)

$$s = \frac{W}{W - w} = \frac{10}{10 - 9} = 10.$$

To find the specific gravity of the wood, we observe that the whole weight of both wood and metal is 15 lbs., and therefore the upward pressure of the water sustains a weight of 8 lbs. But the water displaced by the metal weighs 1 lb. Therefore the water displaced by the wood weighs 7 lbs.

$\therefore$ by (30) $$s = \frac{W}{W'} = \frac{5}{7}.$$

Examples for Exercise—

1. A certain substance weighs 14 lbs. in air and 10 lbs. in water. Find its specific gravity.

2. A body A, weighing 10 lbs., is attached to a body B, which weighs 6 lbs., and the whole is found to weigh 3 lbs. in water. If A weigh 5 lbs. in water, find the specific gravity of each.

3. A piece of cork, weighing 1 lb., is attached to 21 lbs. of silver, and the whole weighs 16 lbs. in water. If the specific gravity of cork be $\frac{1}{4}$, find that of silver.

88. **Specific Gravity of Fluids.**—The specific gravity of any given fluid may be ascertained by weighing the same body both in water and in the fluid, and comparing the apparent losses of weight.

Thus if W be the real weight of the body, w its weight in water, and w_1 its weight in the fluid, $W - w$ is the weight of the water displaced by the body, and $W - w_1$ is the weight of the given fluid displaced.

Thus the weight of a fixed volume of the fluid, divided by the weight of an equal volume of water is—

$$s = \frac{W - w_1}{W - w}. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

A more direct method of finding the specific gravity of a fluid is given on next page in Example 4.[1]

[1] For the method of Hydrometers, see the Appendix, § 119.

Examples—

1. A body weighs 1000 grains in air, 300 in water, and 420 in another liquid. Find the specific gravity of the latter liquid.

By (32) $$s=\frac{W-w_1}{W-w}=\frac{1000-420}{1000-300}=\frac{580}{700}=\frac{29}{35}.$$

2. A mass of 10 lbs. weighs 6 lbs. in a fluid A, and 8 lbs. in a fluid B. Compare the specific gravities of A and B.

Here A supports 4 lbs., B 2 lbs. Thus the weights of equal volumes of A and B are as 4 to 2, that is, as 2 to 1.

Therefore the specific gravity of A is twice that of B.

3. A body weighs 5120 grains in a vacuum, 4120 in distilled water, 4095 in salt water, find the specific gravities of the body, and of the salt water.

By (31) the specific gravity of the body is—

$$s=\frac{W}{W-w}=\frac{5120}{5120-4120}=5\cdot12.$$

By (32) the specific gravity of the salt water is—

$$s=\frac{W-w_1}{W-w}=\frac{5120-4095}{5120-4120}=1\cdot025.$$

4. A bottle weighs 3 oz. It is filled with water and found to weigh 12 oz.; it is then filled with oil, and found to weigh 11 oz. Find the specific gravity of the oil.

Here the weights of equal volumes of the water and oil are found, by subtracting the weight of the bottle, to be 9 oz. and 8 oz. respectively.

$\therefore$ by (30) $$s=\frac{W}{W'}=\frac{8}{9}.$$

5. Three lbs. of a fluid A, whose specific gravity is $1\frac{1}{4}$, are mixed with 2 lbs. of a fluid B, whose specific gravity is $\frac{3}{4}$. Find the specific gravity of the mixture, assuming that its volume remains unaltered.

Here the whole weight is 5 lbs., and we must find the weight of the water which would have the same volume.

In fluid A, $\frac{5}{4}$ lb. occupies the space of 1 lb. of water

	$\therefore$ 1 lb.	,,	,, $\frac{4}{5}$ lb.	,,
	$\therefore$ 3 lbs.	,,	,, $2\frac{2}{5}$ lbs.	,,
In fluid B,	$\frac{3}{4}$ lb.	,,	,, 1 lb.	,,
	$\therefore$ 1 lb.	,,	,, $\frac{4}{3}$ lb.	,,
	$\therefore$ 2 lbs.	,,	,, $2\frac{2}{3}$ lbs.	,,

$\therefore$ 3 lbs. of A + 2 lbs. of B occupy the space of $2\frac{2}{5}+2\frac{2}{3}=5\frac{1}{15}$ lbs. of water.

$\therefore$ by (30) $$s=\frac{5}{5\frac{1}{15}}=\frac{75}{76}.$$

Examples for Exercise—

1. A body weighs 5 oz. in a vacuum, 4 in water, and 3 in another liquid. Find the specific gravities of the body and of the second liquid.

2. A mass of 4 lbs. is found to weigh 1 lb. in a liquid (*s.g.* $\frac{1}{2}$). Find the specific gravity of the body.

3. A bottle weighs 10 grains when exhausted of air. When air is admitted it weighs $10\frac{1}{2}$ grains. When filled with water it weighs 510 grains. Find the specific gravity of air.

4. Find the specific gravity of a mixture containing 4 cubic inches of copper (*s.g.* 9), and 6 cubic inches of zinc (*s.g.* 7), assuming that the volume of the mixture is 10 cubic inches.

89. Floating Bodies.—When a body is specifically lighter than a liquid in which it is immersed, the upward pressure of the liquid (§ 84) causes the body to rise to the surface, and to float partly above and partly below the surface.

When in equilibrium, the weight of the body, acting vertically downwards through its centre of gravity, is supported by the upward pressure of the liquid which is equal to the weight of the liquid displaced, and acts vertically upwards through the centre of gravity of the liquid displaced. *Thus the weight of the liquid displaced is equal to the weight of the floating body, and the two centres of gravity are in the same vertical line.*[1]

If, then, the body is of uniform density, and we know what fractional part of the body is immersed, we can find its specific gravity.

Let W be the weight of the body, and let $\frac{x}{y}$ of the body be immersed. Then the weight of the portion immersed is $\frac{x}{y}$ of the weight of the whole body, and is therefore equal to $\frac{x}{y}W$. Dividing the weight of this portion by W, the weight of the water which it displaces, we obtain—

$$s=\frac{x}{y}. \quad . \quad . \quad . \quad . \quad (33).$$

[1] For the properties of the Metacentre, see Appendix, § 120.

When this fraction cannot be ascertained, the specific gravity may be found by attaching a sinker, and proceeding in the manner shown in § 87, Example 2.

Examples—

1. A body floats with three-fourths of its volume immersed. Find its specific gravity.

By (33) $s=\frac{x}{y}=\frac{3}{4}$.

2. A rod of wood floats with one-third of its volume out of the water. Find its specific gravity.

Here the portion immersed $=1-\frac{1}{3}=\frac{2}{3}$ of the rod. $\therefore\ s=\frac{2}{3}$.

3. The specific gravity of mercury is $13\frac{1}{2}$. A piece of iron containing 27 cubic inches floats in mercury with 15 cubic inches below the surface. Find the specific gravity.

If the specific gravity of mercury were taken as unity, that of the iron would be $\frac{15}{27}$; but since the specific gravity of mercury is $13\frac{1}{2}$, that of the iron is $\frac{15}{27}$ of $13\frac{1}{2}=7\frac{1}{2}$.

Examples for Exercise—

1. What is the specific gravity of a substance which floats with $\frac{3}{40}$ of its bulk above the surface of water?

2. What is the specific gravity of a body which, when placed in water, is wholly submerged, but does not sink?

3. A body, which weighs 6 lbs., floats in a liquid with one-third above the surface. What pressure must be applied to keep the body wholly submerged?

Examination on Sections 86-89

1. Define Specific Gravity. How is it measured?

2. What substance is taken as a standard in determining specific gravities?

3. Show that the specific gravities of any number of substances are proportional to their densities, and explain how a table of specific gravities may be formed from a table of densities.

4. Explain how the specific gravity of a substance may be approximately obtained by weighing it in air and in water, and prove that, if W be the weight of the body, and w its weight in water, the specific gravity is equal to $\frac{W}{W-w}$.

5. Explain two methods by which the specific gravity of a fluid can be ascertained.

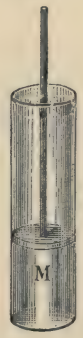
6. What are the conditions of equilibrium of a floating body ?

7. Mention two methods of finding the specific gravity of a body which floats in water.

90. Boyle's Law.—This law, *which connects the pressure and volume of gases*, was discovered by the Honourable Robert Boyle, one of the founders of the Royal Society of London. It is sometimes called *Mariotte's Law*, after a French physician Mariotte, who is believed to have discovered it independently ; but there can be no doubt as to Boyle's priority, as his work on gases was published in 1662, while Mariotte's was published fifteen or eighteen years later.

It is as follows: *The pressure of a given quantity of gas is inversely proportional to the volume, the temperature remaining constant.*

Let M be a quantity of gas enclosed in a cylinder, occupying a volume V, and subjected to a pressure p by means of a piston. By § 81 the pressure at any point in the gas is equal to $p+$the pressure due to the weight of the gas. But in this case the pressure due to the weight is so small, that we may neglect it, and consider the pressure at any point in the cylinder to be equal to p. If now the pressure on the piston be increased to $2p$, the piston will be forced down until the volume of M is reduced to $\frac{1}{2}$V, when the pressure at every point throughout the gas will be equal to $2p$. Similarly, if the pressure be made $3p$, the volume will be reduced to $\frac{1}{3}$V. If, on the other hand, the pressure on the cylinder be diminished to $\frac{1}{2}p$, the gas will expand, forcing up the piston until the volume is 2V. If the pressure be diminished to $\frac{1}{3}p$, the volume will increase to 3V, and so on.



This may be expressed by the equation,—

$$p=\frac{C}{V}, \quad . \quad . \quad . \quad . \quad . \quad (34)$$

which may also be written, $pV=C$ or $V=\frac{C}{p}$, C being constant and equal to the pressure of the gas at unit volume. Comparing (34) with equation (29′)—

$$\rho=\frac{M}{V},$$

we see that Boyle's Law may be expressed in this form:—

The pressure is proportional to the density.

Examples—

1. A bladder contains 24 cubic inches of air when exposed to the pressure of the atmosphere. Find its volume when (1) the pressure is increased to three atmospheres; (2) diminished to $\frac{1}{4}$ of an atmosphere.

(1) Since the pressure is tripled, the volume will be reduced to $\frac{1}{3}$, and is therefore $\frac{1}{3}$ of $24=8$ cubic inches.

(2) Since the pressure is diminished to $\frac{1}{4}$, the volume will be increased to four times its former value, and is therefore $4\times 24=96$ cubic inches.

2. The pressure on the surface of the sea is 2160 lbs. on the square foot, and the weight of a cubic foot of water is $62\frac{1}{2}$ lbs. If a diving-bell contain 179 cubic feet of air at the surface, into what space will the air be compressed when the bell is lowered to a depth of 80 feet?

Using the formula $p=\frac{C}{V}$, at the surface of the sea—

$$p=2160, \text{ and } V=179.$$

$$\therefore\ C=pV=2160\times 179.$$

By § 81 the pressure at a depth of 80 feet is—

$$p+xq=2160+80\times 62\tfrac{1}{2}=7160.$$

Call this pressure p', and the volume of the air in the diving-bell at this depth V'. Then by (34)—

$$V'=\frac{C}{p'}=\frac{2160\times 179}{7160}=\frac{54\times 40\times 179}{40\times 179}=54.$$

Examples for Exercise—

1. A cubic foot of air weighs 570 grains at a pressure of 15 lbs. on the square inch. What will it weigh at a pressure of 16 lbs.?

2. If the specific gravity of air at the surface of the sea be ·001, at

what depth will the air in a bladder be compressed to the density of water, assuming the pressure at a depth of 100 feet to be four times the pressure of the atmosphere?

91. The Barometer.—The pressure of the atmosphere is ascertained by the barometer. This instrument is constructed in the following manner:—

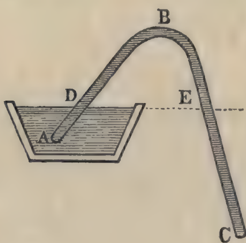
A glass tube AB, about 32 inches long, sealed at one end A, is filled with mercury. The end B is then closed and placed under the surface of some mercury in a vessel, and the tube is held in a vertical position. When B is opened the mercury in the tube falls a short distance, leaving an empty space above, but the surface remains about 29 or 30 inches higher than that of the mercury in the vessel.

The space above the mercury in the barometer contains a little vapour of mercury, but is practically a perfect vacuum. There is therefore no pressure on the surface of the mercury within the tube, and the pressure at any point below is proportional to its depth below this surface (§ 81). The pressure at the level CD is equal to the pressure of the atmosphere. Thus the pressure of the atmosphere is proportional to the height of the mercury in the tube of the barometer. A barometer may be constructed of any liquid, but mercury is nearly always employed on account of its great density. If water be employed, since mercury is $13\frac{1}{2}$ times as dense as water, the column of water will be $13\frac{1}{2}$ times the length of the column of mercury, that is, about 33 feet. At any point above the surface of the earth, the pressure is, by § 80, less than that on the surface of the earth, and therefore the mercury in the barometer is lower. Thus the height of a mountain, or of the ascent of a balloon, may be approximately found by means of a barometer.

Since by Boyle's law the density is proportional to the pressure, the atmosphere becomes less dense the further it is removed from the surface of the earth, thus differing altogether from the water of the sea, which retains very nearly the same density at all depths, although the pressure at the lowest depths is very great indeed.

92. The Syphon.—The syphon is a bent tube open at both ends. It is used to bring a liquid from a higher to a lower level over an obstacle such as the side of a vessel.

Let ABC be a bent tube, and let it be filled with the same liquid as that which is contained in the vessel to be emptied.



Let both ends be closed, and one end A placed below the surface of the liquid in the vessel, the end C being outside the vessel and also below the surface plane.

Let the end A be now opened, and let D and E be the points where the syphon meets the plane of the surface of the liquid. Then since the pressure is the same at all points in the same horizontal plane, the pressures within syphon at D and E are each equal to the pressure of the atmosphere on the surface of the liquid in the vessel. This pressure will support the portion of the liquid DBE, unless the height of B above the surface is greater than the height of a barometric column of the liquid.

At the end C the pressure of the liquid within the syphon is, by § 81, greater than the pressure at E, and therefore greater than the pressure of the atmosphere.

It follows that when C is opened, the pressure of the atmosphere will be unable to sustain the liquid in the tube, and it will therefore flow through the tube until the level of the liquid in the vessel falls below either A or C.

93. The Air Pump.—This is a machine for diminishing the quantity of air in a vessel. It consists of a cylinder AB, in which a piston P is moved alternately up and down, and a tube leading from the cylinder to the vessel V from which air is to be removed. In the piston P, and at the bottom of the cylinder AB, there are valves opening upwards, that is, apertures which are closed if the pressure of the air above them is greater than that below, but which immediately open and allow air to pass through them if the pressure below is greater than the pressure above.

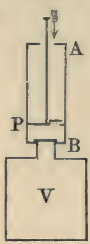


Figure 1.

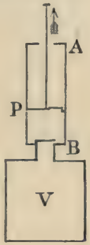
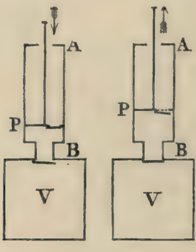


Figure 2.

When the piston is pushed down (see figure 1), the air in PB is condensed, and therefore, by Boyle's Law, its pressure is increased. The effect of this is to close the valve at B, and open the valve at P. As the piston descends, the air escapes through P, and finally a very small quantity of air remains in the cylinder. The piston is now drawn up (see figure 2), which causes the air in PB to expand, aud therefore to exert less pressure than the outside atmosphere. The valve P is therefore closed and that at B opened. As the piston rises the air in V expands, and fills both the vessel and the cylinder. When the motion of the piston is reversed, the valve B is immediately closed, and as soon as the density of the air in PB exceeds that of the outer atmosphere, the valve P is opened (as in figure 1), and the process continues as before. In this way more and more air can be removed from V, but, as only a part of what remains can be removed by each stroke, we can never, by this process, obtain an absolute vacuum.

In Sprengel's air pump the vessel from which air is to be taken is placed in communication with the vacuum of a barometric tube, and the air is removed by a continuous flow of mercury from a vessel at the top of the tube. By its use the density of air may be reduced to $\frac{1}{1000000}$ of its ordinary value.

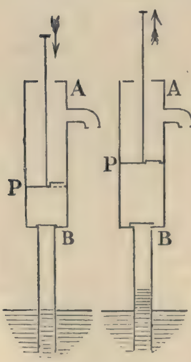
94. The Condenser.—If both valves of an air pump are constructed to open downwards instead of upwards, it becomes a condenser, an instrument for forcing additional air into a vessel. When the piston is lowered (as in figure 1), the air in PB is condensed, the valve P is closed, and, as soon as the density of the air in PB exceeds that of the air in V, the valve B is opened and air enters V.



When the piston is raised (figure 2) the air in PB expands, and, its pressure being diminished, the valve at B is closed, and as soon as the density of the air in PB is less than that of the atmosphere, the valve P is opened and air enters the cylinder. The alternate motion of the piston is continued, and more air is forced into V with each stroke.

The efficiency of both the air pump and the condenser is increased by making the piston rod work in an air-tight collar and adding a valve at A, as in that case the density of the air in PB need only exceed the density of that in AP in order to open (or close) the valve at P.

95. The Common Pump.—This pump (often called the *Suction Pump* or the *Lifting Pump*) is used for raising water and other liquids. Its construction is the same as that of the air pump, except that the tube at the bottom of the cylinder, instead of leading into a closed vessel, dips into the liquid to be raised. The working of the pump removes air from this tube. The pressure of the atmosphere on the surface of the liquid outside causes the liquid in the tube to rise, until the weight of the column of liquid in the tube, together with the diminished pressure of air, is equal to the atmospheric pressure.

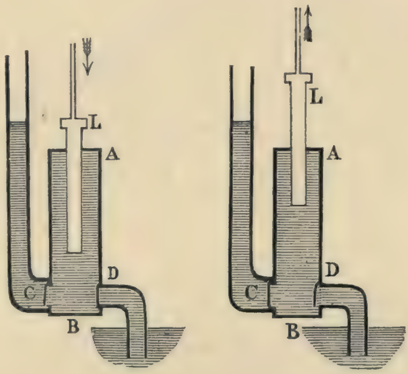


Thus water cannot be raised by a common pump higher than the column of water in a water barometer (say 33 feet). Similarly, mercury cannot be raised more than about 29 inches.

When the length of the tube does not exceed this limit, the exhaustion of air is continued until the liquid passes through B, then through the valve in the piston. Each stroke of the piston now lifts the water above it, and finally causes it to flow out at the spout.

96. **The Force Pump.**—In the force pump a solid piston or plunger L is moved alternately up and down[1] within a vessel AB, in which there are two valves, one at C, opening outwards, through which the liquid is forced, the other at D, opening inwards, for the admission of the liquid.

When the plunger is forced in, the pressure in AB is increased, the valve D is closed, and some of the liquid is forced through C; when the plunger is withdrawn the pressure in AB is diminished, C is closed, and additional liquid from the reservoir enters at D. It is, of course, necessary that the height of A above the liquid in the reservoir must not exceed the height of the column which can be supported by the pressure of the atmosphere.



A common pump may be transformed into a force pump

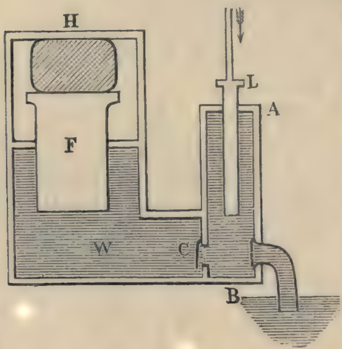
[1] In practice the motion is often horizontal.

by making the piston work in an air-tight collar, and converting the spout into a tube leading upwards.

The fire-engine is an example of a double force pump, a continuous stream of water being produced by the aid of an air-chamber.

97. Bramah's Press.— This machine (sometimes called the *hydraulic press*) is used for exerting great pressure.

It consists of a force pump AB, which drives water though C into a vessel W. The pressure thus caused forces up a solid cylinder F, and the substance to be compressed is placed between F and a fixed plate H.



To find the pressure which this machine can exert, let us suppose that the area of the extremity of the plunger L is one square inch, and that of the extremity of F is 400 square inches. While the plunger descends the valve C is open, and therefore the pressure is the same at any two points in the same horizontal plane throughout the liquid.

If therefore L is pressed downwards with a force of 100 lbs. the extremity of F will be pressed upwards by a force of 100 lbs. on the square inch (the difference of pressure due to any difference of level being so small that we may disregard it). Thus the whole upward pressure exerted on F is $100 \times 400 = 40{,}000$ lbs.

Example for Exercise—

Find the pressure exerted by a Bramah's Press in which the diameters of the pistons are as 3 to 50, and the applied force is 72 lbs.

EXAMINATION ON SECTIONS 90-97

1. Give Boyle's Law. By what other name is it known?

2. Assuming Boyle's Law connecting pressure and volume in a gas, deduce the relation between pressure and density.

3. Describe the construction and use of the barometer.

4. Explain the action of the syphon.

5. Describe the common air pump, illustrating its actions by means of diagrams.

6. What is the principle of Sprengel's air pump?

7. Describe the condenser.

8. Describe the common pump, and explain why it cannot raise water above a certain height.

9. Describe the force pump, and Bramah's press.

GENERAL EXAMPLES IN HYDROSTATICS

1. If 319 cubic inches of cork weigh as much as 11 cubic inches of iron, compare their densities.

2. The atmospheric pressure being 15 lbs. on the square inch and the weight of a cubic foot of water being 1000 ounces, find the pressures exerted on a square inch at the following depths below the surface of water :—

(1) one foot : (2) eighty-five feet : (3) three miles.

3. With the data of last question, if the density of mercury be $13\frac{1}{2}$ times that of water, find the pressure on a square foot at the following depths below the surface of mercury :—

(1) one inch : (2) six feet : (3) one mile.

4. Neglecting the pressure of the atmosphere, compare the pressures at a depth of 17 feet under water and at a depth of 15 inches under mercury, assuming that their densities are in the proportion of 3 to 34.

5. A vessel in the form of a cube measuring 3 feet each way, is filled with water. Find—

(1) the pressure on its base in vacuo :

(2) the same in air :

(3) whether the vessel, if just strong enough to contain the water in vacuo, will be able to contain it when air is admitted.

6. A cubic inch of mercury weighs 8 ounces, and the atmospheric pressure is $14\frac{1}{2}$ lbs. on the square inch. Find the pressure—

(1) on a square inch 4 inches below the surface of mercury :

(2) on the base of a vessel of mercury one foot square, the mercury being 7 inches deep.

7. A vessel contains mercury, and over the mercury 10 inches depth of water. Find (with the data of examples 2 and 3) to one decimal place the pressures on a square inch at the following depths below the surface of the water :—

(1) six inches : (2) one foot : (3) three feet.

8. Find the specific gravities of the following substances, the first number expressing the mass of a portion of the substance, the second that of an equal volume of water :—

(1) gold ; 145 oz., $7\frac{1}{2}$ ounces :
(2) diamond ; $5\frac{1}{4}$ grains, $1\frac{1}{2}$ grains :
(3) ether ; 18 drachms, 25 drachms.

9. A lump of lead weighing 1 lb. 6 oz. (specific gravity 11) is suspended in water by a string. Find the tension of the string.

10. Find the specific gravities of the following substances, the first number being the apparent weight of a portion weighed in air, the second that of the same portion weighed in water :—

(1) lead ; 12 lbs., $10\frac{16}{17}$ lbs. :
(2) brass ; $16\frac{2}{3}$ lbs., $14\frac{2}{3}$ lbs. :
(3) copper ; $31\frac{1}{2}$ grains, 28 grains :
(4) ivory ; 6·08 oz., 2·81 oz.

11. Find, to two decimal places, the specific gravities of the following substances which float in water, the first number being the number of cubic inches above, the second that below the surface :—

(1) beech ; 12, 71 : (2) pine ; 112, 217 : (3) ice ; 62, 713.

12. A piece of cork floats in water with $40\frac{1}{2}$ cubic inches above the surface. If the specific gravity of cork be $\frac{1}{4}$, find—

(1) the whole volume of the cork :

(2) how many cubic inches would be above the surface if the cork were caused to float in mercury (*s.g.* $13\frac{1}{2}$).

13. The specific gravity of sea-water is 1·025. A piece of metal, weighing 36·51 lbs., weighs 32·11 lbs. in distilled water. What does it weigh in sea-water ?

14. A piece of gold, weighing 5 lbs. in air and 4·74 in water, has the following weights :—

(1) in sulphuric acid, 4·52 : (2) in hydrochloric acid, 4·68 : (3) in alcohol, 4·79.

Find the specific gravities of these liquids to two decimal places.

15. Find the specific gravity of a body, the volume of which is 16 cubic centimetres, and the weight 100 grammes.

16. A body, weighing 26 grammes, floats in water with two-thirds of its bulk submerged. Find its volume.

17. A certain body A is observed to float in water with half its volume submerged, and when attached to another B of twice its own volume, the combined mass is just submerged. Find the specific gravities of A and B.

18. A piece of platinum weighing 15 lbs. is attached to a piece of iron weighing 10 lbs., and the whole is found to weigh 1 lb. in mercury (*s.g.* $13\frac{1}{2}$). The platinum by itself weighs 6 lbs. in mercury. Find—

(1) the specific gravity of platinum :
(2) „ „ iron :
(3) the weight in water of the piece of platinum :
(4) „ „ „ iron.

19. A body A, whose specific gravity is $\frac{3}{4}$, is attached to a body B of equal volume, and the two together just float in water. Find the specific gravity of B.

20. A cubic foot of cork is made to float in water with its top horizontal, and the base is then found to be 2·9 inches below the surface. Find its specific gravity.

21. A cubic foot of iron (*s.g.* $7\frac{1}{4}$) floats in mercury (*s.g.* $13\frac{1}{2}$). How many cubic inches are above the surface ?

22. A cubic foot of water weighs 1000 oz., and a cubic inch of zinc weighs 4 oz. Find the specific gravity of zinc.

23. A small vessel, exactly filled with distilled water, weighs 530 grains, a small stone weighing 26 grains is thrown in, and an equal bulk of water is thus forced out. The vessel now weighs 546 grains. Find the specific gravity of the stone.

24. A stone of specific gravity 2·5, weighing 5 cwt., is sunk in water. What force is required to raise it ?

25. A body weighs 25 lbs. in air, 20 in water, 21 in alcohol. Find the specific gravity—

(1) of the body : (2) of the alcohol.

26. The weight of a cubic inch of mercury being half a lb., find the pressure on a square inch at the following depths below the surface of the mercury in the tube of a barometer :—

(1) 5 inches : (2) 27 inches : (3) 31 inches.

27. If the height of the mercurial barometer be 29·6 incnes, what is the height of the water barometer?

28. A piece of cork (*s.g.* $\frac{1}{4}$) weighing 1 lb. is attached to 34 lbs. of lead. The whole weighs 28 lbs. in water. Find the specific gravity of the lead.

29. A block of iron weighs 4·65 lbs. in air, 4 lbs. in water, and 3·85 lbs. in oil. Find the specific gravity of the oil.

30. A piece of wood weighs 6 lbs. A block of lead weighing 17 lbs. is attached to it, and the whole mass weighs 14 lbs. in water. Find the specific gravity of the wood, that of the lead being $11\frac{1}{3}$.

31. A cylindrical cork floats in water with one inch above the surface and $\frac{3}{10}$ of an inch below. What is its specific gravity?

32. A piece of oak (*s.g.* ·74) of 32 cubic inches floats in water. How much water does it displace?

33. If the weight of a cubic inch of mercury be $7\frac{4}{5}$ oz., what is the pressure of the air on a square inch when the barometer stands at $29\frac{1}{2}$ inches?

34. Two bodies, which weigh 9 lbs. and 10 lbs. in air, weigh each 4 lbs. in water. Compare their densities.

35. A piece of silver weighing 1 lb. appears to lose ·095 lb. weight when surrounded by water. Find its specific gravity to two decimal places.

36. An iceberg floats with 1000 cubic feet above the surface of the sea. Find its volume, assuming its specific gravity to be ·925, and that of the sea 1·025.

37. A mercurial barometer is constructed under 27 feet of water. How high will the mercury stand in it when the reading of a barometer at the surface is 29 inches?

38. A barometer, in which a little air has got into the upper part, is found to record 28 inches when it should record 30. If the volume of the space above the mercury be $7\frac{1}{2}$ cubic inches, what would be the volume of the air within it at the atmospheric pressure?

39. In a Bramah's press the piston of the force pump is pressed down by means of a lever, the length from the fulcrum to the end of the piston-rod being 3 inches, and from the fulcrum to the hand 21 inches. The diameter of the small piston is 1 inch, that of the large piston 8 inches. How many tons will the machine raise if a force of 20 lbs. be applied to the handle of the lever?

40. A man finds that he can just float in fresh water (a cubic foot of which weighs 1000 oz.), and that his body displaces 3 cubic feet of water; find—

(1) how many cubic feet he would displace when floating in sea-water of specific gravity 1·025:

(2) what weight he could now bear (clear of the water) so as again to have his whole body just immersed.

41. Assuming the specific gravity of mercury to be 13·566, find the height of a barometer formed of a liquid whose specific gravity is 1·5, the height of the mercurial barometer being 30 inches.

42. If the diameter of the smaller piston of a Bramah's press be diminished one-half, what effect is produced on the magnitude of the force which the engine is capable of exerting?

43. The diameter of the two cylinders of a hydraulic press are 25 inches and 1 inch respectively; the mechanical advantage of the lever employed is 14. Calculate the mechanical advantage of the press.

44. A cylindrical jar is immersed mouth downwards in water. At what depth will it be $\frac{1}{4}$ full of water?

45. If a cylinder containing air at the ordinary atmospheric pressure have its volume increased from 20 to 25 cubic inches, the temperature remaining the same, what will be the new pressure?

46. The capacity of a balloon is 30,000 cubic feet. The weight of the empty balloon and its appendages in air is half a ton, and the specific gravity of the gas within it is $\frac{9}{20}$ that of air. What will be the force tending to make it ascend, if a cubic foot of air weigh 1·2 oz.?

47. A cubic foot of air weighs 418 grains at a pressure of 15 lbs. on the square inch. What will a cubic foot of air weigh at a pressure of 13 lbs. on the square inch, and at the same temperature?

48. One pound of lead (*s.g.* 11) is tied by a thin string to a piece of cork (*s.g.* ·22), and when they are put in water one-half of the cork is immersed. How much does the cork weigh?

49. A sunken vessel with a bulk of 20,000 cubic feet and weighing in air 700 tons has to be raised by attaching to it barrels of air, each weighing in air 56 lbs. and having a bulk of 30 cubic feet. Find how many such barrels will be needed, taking a cubic foot of water to weigh $62\frac{1}{2}$ lbs.

50. A lump of iron (*s.g.* $7\frac{1}{4}$) floats on the surface of mercury (*s.g.* $13\frac{1}{2}$). Water is poured on the mercury till the iron is covered. What proportion of the iron will now be immersed in the mercury?

51. The mercury barometer stands at 75·5 centimetres. What would be the height of an alcohol barometer at the same place, the specific gravity of these liquids being 13·598 and ·803 respectively?

52. What is the pressure at a depth of 3 miles under the surface of the ocean? (Weight of 1 cubic foot of sea-water = 64·11 lbs.)

53. A body floats with one-tenth of its volume above the surface of pure water. What fraction of its volume would project above the surface if it were floating in liquid of specific gravity 1·25.

54. A hollow cylinder contains air at a pressure of 15 lbs. on the square inch when the piston is 12 inches from the bottom. If more air be now forced in, until the cylinder contain 3 times as much air as at first, and if the piston be allowed to rise 4 inches higher, find the new pressure on the square inch. (Temperature constant.)

55. In a common suction pump 1 foot of the barrel holds a gallon of water (10 lbs.), the piston works through 4 inches at each stroke, and the spout is 24 feet above the surface of the water in the well. Neglecting friction, find—

(1) what force is required to raise the piston:

(2) how many foot-pounds of work are done at each stroke.

56. A force pump is used to raise water from a well to a tank. If the pump be 18 feet above the well, and 54 feet below the tank, and if the area of a section of the plunger be 6 square inches, find—

(1) the force required to work the pump when sucking:

(2) the force required to work the pump when lifting:

(3) the work done during a double stroke of 4 inches each way.

APPENDIX

98. To express the Resultant of two Components[1] in Terms of the two Components and the Cosine of the Angle between them.

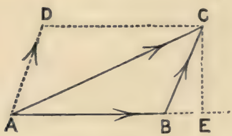


Figure 1.

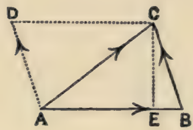


Figure 2.

In other words, it is required to express the diagonal of a parallelogram in terms of the two sides adjacent to its extremity, and the cosine of the angle formed by them.

As shown in § 17, when ∠ DAB is acute (figure 1)—

$$AC^2 = AB^2 + BC^2 + 2AB \cdot BE. \quad \text{(Euclid, II. 12.)}$$

Let ∠ DAB $= \theta$, then $\cos\theta = \cos CBE = \frac{BE}{BC}$, or $BE = BC \cos\theta$.

Substituting this value for BE, we have—

$$AC^2 = AB^2 + BC^2 + 2AB \cdot BC \cos\theta. \quad . \quad . \quad . \quad . \quad (35)$$

Similarly, when ∠ DAB is obtuse (figure 2)—

$$AC^2 = AB^2 + BC^2 - 2AB \cdot BE. \quad \text{(Euclid, II. 13.)}$$

But $\cos\theta = -\cos \angle CBE = -\frac{BE}{BC}$, or $-BE = BC\cos\theta$.

Substituting this value for $-$ BE, we again obtain (35). Thus this equation holds good for all values of θ.

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[1] The same proofs apply to Velocities, Accelerations, and Forces. *See* §§ 12, 21, and 39.

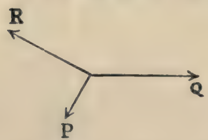
If we write P and Q for the components, and R for the resultant, and extract the root of both sides, equation (35) takes the form—

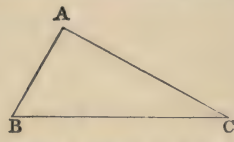
$$R = \sqrt{P^2+Q^2+2PQ\cos\theta}. \quad . \quad . \quad . \quad . \quad (36)$$

[Examples and Examination Questions are given at the end of the Appendix.]

99. When three Components are in Equilibrium, each of them is proportional to the Sine of the Angle formed by the other two.

Let P, Q, and R, be the components, and let them be





parallel to the sides of the triangle ABC. Then it has been proved (§§ 13, 21, and 45) that P, Q, and R, are proportional to the sides of ABC or—

$$P : Q : R = AB : BC : CA.$$

But, by trigonometry—

$$AB : BC : CA = \sin C : \sin A : \sin B.$$

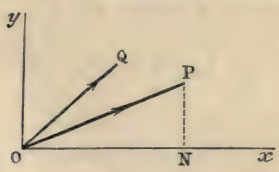
$$\therefore P : Q : R = \sin C : \sin A : \sin B.$$

But the angle QR formed by the directions of Q and R is the supplement of the angle C, etc.

$\therefore$ by trigonometry, $\sin QR = \sin C$, &c. = &c.

$$\therefore P : Q : R = \sin QR : \sin RP : \sin PQ. \quad . \quad . \quad (37)$$

100. Resultant of any Number of Components.—Let OP, OQ, &c., be any number of components whose directions are in one plane.



From P draw PN perpendicular to Ox. Then, by § 18, ON and NP are the components of OP along and perpendicular to Ox.

Call PON α, QON β, and so on.

By trigonometry, ON = OP cos α, PN = OP sin α.

Similarly for OQ, etc.

Then, if X be the *sum* of the components along Ox, and Y the sum of the components perpendicular to Ox,

$$X = OP \cos\alpha + OQ \cos\beta + ..$$
$$Y = OP \sin\alpha + OQ \sin\beta + ..$$

Let R be the resultant of X and Y, then, by § 15,

$$R = \sqrt{X^2 + Y^2} \quad . \quad . \quad . \quad . \quad . \quad (38)$$

If R makes an angle θ with OX, then $X = R\cos\theta$, $Y = R\sin\theta$.

If the components are in equilibrium, R = 0, and therefore—

$$X = 0,\ Y = 0. \quad . \quad . \quad . \quad . \quad . \quad (39)$$

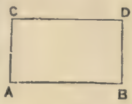
[When the directions of the components are not all in one plane, we may first resolve each component along and perpendicular to the plane xOy. Draw through O a line Oz perpendicular to the plane xOy, and let Z = the sum of the components perpendicular to the paper. Then—

$$R = \sqrt{X^2 + Y^2 + Z^2},$$

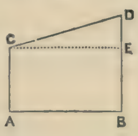
and for equilibrium,

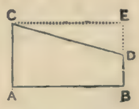
$$X = 0,\ Y = 0,\ Z = 0.]$$

101. Graphic Representation of the Distance travelled by a Point—(1) Uniform Speed.—Let a point move for t seconds with uniform speed V. Draw a straight line AB of length $= t$ units. At A draw AC perpendicular to AB and $= V$ units, and complete the rectangle. ABDC = AC × AB = Vt. But by formula (1) $s = Vt$. ∴ The area ABDC contains s units of area, and thus represents the distance travelled by the point.



(2) **Uniform Acceleration.**—Let a point move with initial speed V and uniform acceleration a in the direction of motion (§ 22). Then the speed after t seconds is $V + at$. Draw a straight line AB of length $= t$ units. At A draw AC perpendicular to AB and $= V$





units. At B draw BD perpendicular to AB and $= V + at$ units. Join CD and draw CE parallel to AB, meeting BD in E.

$$BE = AC = V. \qquad \therefore ED = at.$$

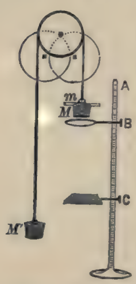
$$ABDC = ABEC + CED = AC \times AB + \tfrac{1}{2}ED \times CE = Vt + \tfrac{1}{2}at^2.$$

But, by formula (10), $s = Vt + \frac{1}{2}at^2$.

$\therefore$ the area ABDC contains s units, and thus represents the distance travelled by the point.

The second figure represents the case of a negative acceleration. The proof is the same as the above, substituting $-$ for $+$ throughout.

102. Atwood's Machine.—This machine was invented in 1780 by Mr. George Atwood, who graduated as Third Wrangler at Cambridge in 1769. By its means the laws of uniformly accelerated motion may be approximately verified. In it, two equal heavy masses, M and M', are joined by a string which passes over a pulley resting on friction wheels. A small bar m is now laid on M, in consequence of which M falls and M' rises. If we neglect the effect of friction and leave out of account the mass of the string, and the inertia of the wheelwork, then the mass moved is—



$$M + m + M' = 2M + m,$$

while the resultant force causing motion is—

$$(M + m)g - M'g = mg.$$

Let a be the acceleration produced.

Then by (14), $mg = (2M + m)a$, or $a = \dfrac{m}{2M + m}g$.

By making m small in comparison with $2M$, a is made small, and the motion becomes capable of easy observation.

As an example, let—

$$m = \frac{1}{15 \cdot 6} M.$$

Then, $$a = \frac{1}{32 \cdot 2} g = 1.$$

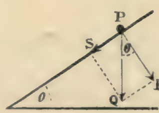
Thus the equations of § 22 become—

$$V = t, \quad s = \tfrac{1}{2}t^2, \quad V^2 = 2s.$$

From these we see that during the first second the masses move half a foot, and that at the end of that second their velocity is one foot per second. To verify this, a scale ABC may be attached to the machine, having a moveable ring B large enough to intercept m but allow M to pass through it, and a stage C below. Then if AB = 6 inches, and BC = 1 foot, and if the masses be in the proportion given above, M and m will move with uniformly accelerated velocity from A to B during the first second, and, m remaining at B, M will move with uniform velocity from B to C in the second second. A clock beating seconds enables us to verify this approximate result. (The tension of the string may be found as in the answer to Ex. 8, p. 151.)

103. Motion of a Particle on an Inclined Plane.—(1) A particle slides down a smooth inclined plane; find the space described and the velocity acquired in t seconds.

Let θ be the inclination of the plane. Then, if P represent the particle, draw PQ $= g$ vertically downwards, QS and PR perpendicular to the surface of the plane, and QR perpendicular to PR. The acceleration PQ may be replaced by its components PR ($= g \cos \theta$), and PS ($= g \sin \theta$). PR is neutralised by the resistance of the plane, and PS forms a uniform acceleration in the direction of motion (§ 22). Thus equations (9) and (10) become—



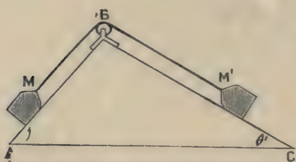
$$v = V + g \sin \theta \cdot t, \qquad s = Vt + \tfrac{1}{2}g \sin \theta \cdot t^2.$$

(2) Find the same when the plane is rough.

Let μ' be the co-efficient of kinetic friction.

By § 72, the force of friction acts up the plane and is $\mu' \times$ the normal pressure. But, by the previous example, the normal pressure $= g \cos\theta$. $\therefore$ the acceleration in the direction of motion is $g \sin\theta - \mu' g \cos\theta$. We therefore substitute this expression for a in the equations of § 22.

(3) Two masses M and M′ are connected by a cord which passes over a pulley at B and allows them to slide on the smooth planes AB, BC. To determine the acceleration, let $\angle BAC = \theta$, $BCA = \theta'$. The component of the weight of M which acts along the plane AB is $Mg \sin\theta$, and the component of the weight of M′, which acts along BC is $M'g \sin\theta'$.



Since these forces tend to move the masses in opposite directions, the resultant force tending to move M + M′ is $Mg \sin\theta - M'g \sin\theta'$, and if a be the acceleration in the directions CB and BA,

$$(M + M')a = Mg \sin\theta - M'g \sin\theta'.$$

If $M' = 0$, $a = g \sin\theta$, the case discussed in (1).

If $M' = 0$ and $\theta = 90°$, $a = g$, the case discussed in § 36.

If $\theta' = 90°$, $a = \frac{M \sin\theta - M'}{M + M'} g$, this is the case of one mass hanging freely, while the other slides on a smooth inclined plane.

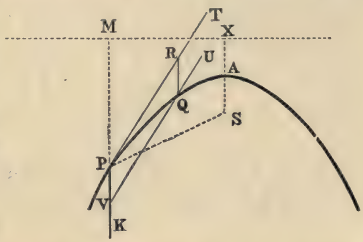
If $\theta' = 0°$, and $\theta = 90°$, $a = \frac{M}{M + M'} g$, the case of one mass hanging freely, while the other slides along a smooth horizontal table.

If $\theta = 90°$, and $\theta' = 90°$, we have $a = \frac{M - M'}{M + M'} g$, the case of Atwood's machine.

The equations of motion in these various cases may be obtained by substituting the corresponding values for a in equations (9), (10), (11), and (12).

104. Parabolic Motion.—Let a point P be in motion with velocity V in the direction PT, and let it have a uniform acceleration g in the direction PK, it will describe a parabola.

To combine the effects of the velocity V and the acceleration g, let us suppose (as in § 12) that the point is in motion along the line PT with uniform velocity V, but that the page, on which PT is drawn, is in motion in the direction PK with uniform acceleration g.



Then, after t seconds the point will have moved along PT a distance—

$$PR = Vt,$$

but owing to the uniform acceleration of the page, the line PT will have moved into the position VU, each point in it having moved through a distance—

$$PV = RQ = \tfrac{1}{2}gt^2.$$

Thus the point will be found at Q.

Now $QV = PR = Vt, \quad \therefore QV^2 = V^2t^2.$

And $PV = \frac{1}{2}gt^2, \quad \therefore t^2 = \dfrac{2PV}{g}, \quad \therefore QV^2 = \dfrac{2V^2}{g} PV.$

But, in the parabola, $QV^2 = 4SP.PV$. These equations are therefore identical if $4SP = \dfrac{2V^2}{g}$.

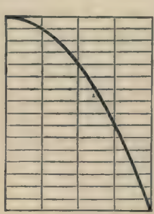
$\therefore$ Q lies on a parabola of which the axis is parallel to the direction of acceleration.

To find the focus and directrix, produce VP to M, so that $PM = SP = \dfrac{V^2}{2g}$, and make the $\angle$ TPS = TPM, and PS = PM.

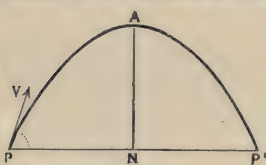
Then S is the focus, and a straight line MX, perpendicular to PM, is the directrix.

The path of a particle projected under the action of gravity is therefore a parabola, when the resistance of the atmosphere is left out of account.

105. Morin's Machine.—This machine illustrates the parabolic path of a projectile. A mass to which a pencil is attached falls freely down the side of a uniformly revolving vertical cylinder covered with paper. If the cylinder were at rest the pencil would describe a vertical straight line with uniformly accelerated velocity. If the pencil were at rest, a horizontal circle would be described with uniform speed. The composition of these two motions gives the parabola shown in the figure, when the paper is flattened out.



106. Greatest Height, Time of Flight, and Horizontal Range of a Projectile in a Vacuum.—Let a heavy particle be projected from P, with velocity V, in a direction making an angle θ with the horizontal line PP′ drawn in the same vertical plane. It has been shown that it will describe the parabola PAP′. The initial vertical velocity is $V \sin \theta$, and the initial horizontal velocity is $V \cos \theta$. (See § 18.) The latter remains constant, but the former is altered by gravity. If v_1 be the vertical velocity, and h the height of the particle after t seconds, then by equation (20)—



$$v_1 = V \sin \theta - gt,$$

$$\text{and } h = V \sin \theta \cdot t - \tfrac{1}{2}gt^2.$$

When the highest point A is reached, $v_1 = 0$. ∴ from the first equation, $t = \dfrac{V \sin \theta}{g}$

Substituting this value of t in the second equation, we find—

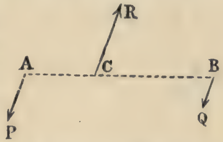
$$h = \frac{V^2 \sin^2 \theta}{2g} = \textit{Greatest Height.}$$

When the particle reaches P′, $h = 0$. ∴ from the second equation, $t = \dfrac{2V \sin \theta}{g} = \textit{Time of Flight.}$

But during this time the particle has had a uniform horizontal velocity, $V \cos \theta$.

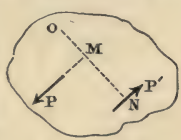
$$\therefore \ PP' = \frac{2V^2 \sin\theta \cos\theta}{g} = \textit{Horizontal Range.}$$

107. Couples.—It was shown in § 50 that, if three parallel forces are in equilibrium, certain relations hold between them. Thus Q=R−P and Q × BC=P × AC. If the force R acting at C be decreased so as to be made nearly equal to the force P acting at A, then, to maintain equilibrium, the force Q must be decreased by the same amount, but its arm BC must be increased in order that its moment about C may remain equal to that of P. Thus when R is very nearly equal to P, the system of forces R and P will be balanced by a very small force Q acting at a very great distance BC.



It is therefore clear that *two equal forces acting* on a finite rigid body *in opposite directions along parallel lines* cannot be balanced by a single force, and cannot therefore have a single force as their resultant. Such a system is called *a Couple.*

Let P and P′ constitute a couple, and consider their moments about any point O. From O draw OMN perpendicular to the direction of P and P′. Then the moment of P′ is P′ × ON, while the moment of P is P × OM in the opposite direction of rotation. The resultant moment is therefore—

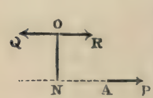


$$P' \times ON - P \times OM = P\,(ON - OM) = P \times MN,*$$

and tends to turn the body in the direction opposite to that of the hands of a watch. This direction is taken as the *positive* direction of rotation.

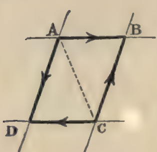
* It should be observed that if a straight line be taken to represent a force, then a moment, the product of a line and a force, will be represented by a plane area.

108. A Force acting at any point in a rigid body is equivalent to an equal parallel Force acting at any other point and a Couple whose Moment is the Moment of the given Force about that point.—Let a force P act at any point A of a rigid body, and let O be any other point in the body. Suppose two forces Q and R, equal and parallel to P, but acting in opposite directions, applied at O. The equilibrium of the body will be unaltered (§ 48). Draw ON from O perpendicular to the line of action of P. P and Q constitute a couple whose moment is P × ON.



Thus P acting at A is equivalent to R acting at O and the couple formed by P and Q

109. Two Couples in the same plane whose Moments are equal in Magnitude and opposite in Direction of Rotation neutralise each other.—Let there be equal and opposite couples acting in the same plane, and let the lines of action of their forces form the parallelogram ABCD. Since the moments of the couples are equal, let each be represented in magnitude by the parallelogram ABCD (§ 107, footnote). The component forces will then be represented on the same scale by the sides of the parallelogram AB, CD, AD, CB. But the forces AB and AD are equivalent to their resultant AC, and the forces CB and CD are equivalent to their resultant CA. Thus the four forces neutralise each other.



Similarly the couple formed by AD and BC will neutralise any couple in the same plane, of moment equal to that of AB and CD.

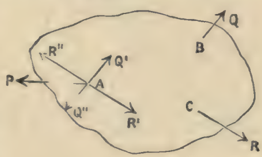
It follows that a couple is equivalent to any couple in the same plane of equal moment.

It also follows that two couples in the same plane are equivalent to a single couple whose moment is the algebraical sum of the moments of the component couples.

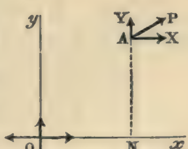
These theorems may be extended to couples in parallel planes.

110. Equilibrium of a Rigid Body acted on by three Non-parallel Forces.—*If a rigid body be in equilibrium under the action of three non-parallel forces, these forces must lie in one plane, pass through one point, and be represented by the sides of a triangle taken in order.*

Let P, Q, and R be the three forces acting at the points A, B, and C. At A apply two equal and opposite forces Q′ and Q″, so that Q′ is equal and parallel to Q, and also two equal and opposite forces R′ and R″, so that R′ is equal and parallel to R. The original system of forces is equivalent to the three forces P, Q′, R′, at A, and to the couples Q, Q″, and R, R″. Since there is equilibrium, P, Q′, and R′ must neutralise each other and therefore be in one plane, and represented by the sides of a triangle taken in order (§ 45). Also the couples Q, Q″, and R, R″, must neutralise each other, and therefore tend to turn the body in opposite directions. Therefore they must act in the same or parallel planes. But their planes both pass through A, and must therefore coincide. Therefore P, Q, and R lie in the plane ABC. They must also pass through one point, for the sum of their moments about any point must vanish. Thus by taking moments about the point where P and Q meet, we see that R must also pass through that point.



111. To find the Resultant of any number of Forces acting on a rigid body in one Plane.—Let the plane of the paper be the plane in which the forces act, and let P be one of the forces acting on the body at A. From a point O draw Ox, Oy, at right angles to each other, and let X and Y be the components of P parallel to Ox and Oy. Also draw AN perpendicular to Ox, and let ON $= x$, AN $= y$.



Then, by § 108, X at A is equivalent to X at O acting along

Ox and a couple Xy in the negative direction. Y at A is equivalent to Y at O, acting along Oy, and a couple Yx in the positive direction. By § 109 the resultant couple is therefore $Yx-Xy$ in the positive direction.

If there be any other forces P′, P″, etc., they can be resolved in the same way, and we obtain—

$$X+X'+X''+\ldots=\Sigma X \text{ acting along } Ox,$$
$$Y+Y'+Y''+\ldots=\Sigma Y \text{ acting along } Oy,$$

and a couple—

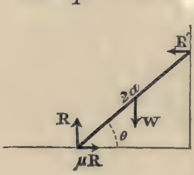
$$(Yx-Xy)+(Y'x'-X'y')+\ \ldots\ =\Sigma\,(Yx-Xy).$$

For equilibrium we must therefore have—

$$\left.\begin{array}{r}\Sigma X=0\\ \Sigma Y=0\\ \text{and } \Sigma(Yx-Xy)=0\end{array}\right\}\quad\ldots\ldots\quad(45)$$

or the sums of the rectangular components of the forces in any two directions at right angles must vanish, and the sum of their moments about any point in their plane must vanish.

Example.—A ladder weighing 30 lbs. leans against a very smooth wall, and is on the point of slipping, the friction of the ground being just sufficient to prevent motion. Find the pressure against the wall and the ground, and the angle at which the ladder rests, the co-efficient of friction between the ladder and the ground being $\frac{2}{3}$.



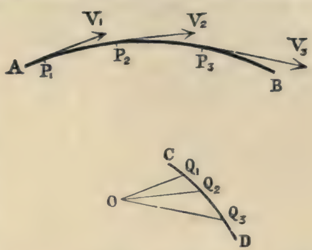
Using the letters indicated by the figure, and taking the foot of the ladder for the origin of co-ordinates,

$$\text{by (45),}\qquad \Sigma X=\mu R-R'=0,\quad \Sigma Y=R-W=0,$$
$$\Sigma\,(Yx-Xy)=-Wa\cos\theta+R'\,2a\sin\theta=0.$$

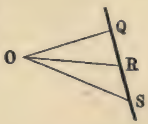
$$\therefore\ R'=\mu R,\quad R=W,\quad \tan\theta=\frac{Wa}{2R'a}=\frac{1}{2\mu},$$

$$\text{or,}\quad R=30g,\quad R'=20g,\quad \theta=\tan^{-1}\tfrac{3}{4}.$$

112. The Hodograph.—Let a point P move in any manner along a line AB, and let P_1, P_2, and P_3 be its positions at different times. Draw V_1, V_2, and V_3 to represent its velocities at these times, and from a fixed point O draw OQ_1, OQ_2, and OQ_3 equal and parallel to V_1, V_2, and V_3. Then as P moves along AB, Q will in general trace out a curve CD. The curve CD is called the *Hodograph* of the motion of P.



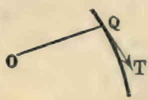
113. When a point moves with Uniform Acceleration its Hodograph is a Straight Line.—Let OQ be the velocity of P at any instant, OR its velocity after a unit of time. Then QR is the change of velocity during a unit of time, that is, QR represents the acceleration of P. Similarly if OS be the velocity of P after another unit of time, RS also represents the acceleration of P. Therefore RS is equal and parallel to QR, or the hodograph QRS is a straight line. If QS be vertical, this is the case of a body moving in a vacuum under the action of gravity.



114. The Acceleration of P is equal to the Velocity with which Q moves in the Hodograph.

(1.) **Uniform Acceleration.**—It was shown in § 113 that QR is the acceleration of P, but QR is the space described by Q in one second, that is, QR is the velocity of Q.

(2.) **Variable Acceleration.** — *Variable Acceleration is measured by the Change of Velocity which would take place in a unit of time, were the Acceleration at the instant under consideration to remainu nchanged for a unit of time.* As before, let OQ be the velocity of P at any instant. Draw QT = the acceleration of P at that instant, that is, equal to the change of velocity

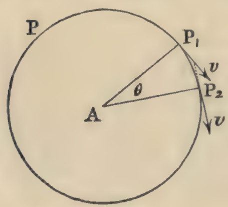


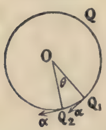
which would take place in a unit of time, were the acceleration to continue uniform for a unit of time. We have already seen that if this acceleration were uniform, Q would describe a straight line with uniform velocity equal to the acceleration. Thus QT is the space which would be described by Q in a unit of time, were the velocity of Q to remain uniform for a unit of time.

In other words, QT is the velocity of Q. (See § 6.)

$\therefore$ the velocity of Q = the acceleration of P.

115. Uniform Circular Motion.—Let P move uniformly in a circle of radius r about A as centre. The speed of P is constant in magnitude and equal to v, but its direction, being that of the tangent to the circle, continually changes. Let Q trace out the hodograph of P's motion.





Then $OQ = OQ_1 = OQ_2 = v$, or QQ_1Q_2 is a circle.

Let P move from P_1 to P_2, and let $P_1AP_2 = \theta$. Then by geometry the angle formed by the tangents at P_1 and P_2 is also θ, and $\therefore \angle Q_1OQ_2 = \theta$, or P and Q describe similar arcs.

Therefore their velocities are proportional to the radii of their circles. But the velocity of Q = the acceleration of P = α.

$$\therefore v : \alpha = AP : OQ, \quad \text{or } v : \alpha = r : v.$$

$$\therefore \alpha = \frac{v^2}{r}. \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Since Q moves in a circle, the direction of its velocity is at right angles to OQ. It is therefore parallel to PA, or the direction of acceleration of P is towards the centre of the circle.

If a mass m move in a circle of radius r with velocity v under the influence of a force f tending to the centre of the circle, then, by (14)—

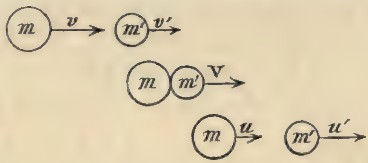
$$f = m\alpha = \frac{mv^2}{r}. \quad . \quad . \quad . \quad . \quad . \quad (41)$$

116. Impact.—A spherical body of mass m moving with velocity v overtakes another spherical body of mass m' moving with velocity v' in the same straight line. By the third law of motion the momentum lost by m is equal to that gained by m', and therefore the whole momentum of the two masses is unchanged by the impact; or, if V be the common velocity of the two masses when their velocities have been equalised by impact,—

$$(m+m')\,V=mv+m'v',$$

$$\therefore\ V=\frac{mv+m'v'}{m+m'}. \qquad (42)$$

In all natural bodies, unless prevented by coherence or artificial appliances, an action between the two bodies is called into play by the impact, which causes them to separate.



Newton found by experiment that the relative velocity of the two bodies after impact bears a fixed ratio to their relative velocity before impact, or, if u be the velocity of m, and u' that of m', after separation,—

$$u'-u=e\,(v-v'), \qquad (43)$$

where e is always less than unity and is constant for the same bodies. It is called the *co-efficient of restitution.*

As before, the whole momentum is unchanged.

$$\therefore\ mu+m'u'=(m+m')\,V=mv+m'v',$$

or $$V=\frac{mv+m'v'}{m+m'}=\frac{mu+m'u'}{m+m'}.$$

It follows that $$V-u=\frac{mu+m'u'}{m+m'}-u=\frac{m'}{m+m'}\,(u'-u),$$

and $$v-V=v-\frac{mv+m'v'}{m+m'}=\frac{m'}{m+m'}\,(v-v').$$

$$\therefore\ \frac{V-u}{v-V}=\frac{u'-u}{v-v'}=e,$$

or $$V-u=e\,(v-V). \qquad (44)$$

Similarly, or by subtracting this equation from equation (43), we have $u' - V = e\,(V - v')$.

In very elastic bodies the value of e may be very nearly equal to unity. If any bodies existed for which the coefficient of restitution was *unity*, these bodies would necessarily be *perfectly elastic.*

117. Kinetic Energy.—A particle of mass m is in motion with velocity V, it is required to find its Kinetic Energy, or the quantity of work which it is capable of doing in virtue of its motion.

Let us suppose that the body is acted on by a uniform force f, in a direction opposite to that of motion, and that it is brought to rest after moving a distance S against this force. Then the work done is fS. (§ 42.)

Let a be the acceleration caused by f, and let v be the velocity, and s the space travelled at any time, then by (12)

$$v^2 = V^2 - 2as.$$

But, when $v=0$, $s=S$, $\quad \therefore\ V^2 = 2aS$, $\quad$ or $aS = \frac{1}{2}V^2$.

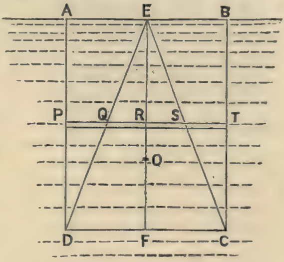
Also by (14), $f = ma$, $\quad \therefore\ fS = maS = \frac{1}{2}mV^2$.

Thus the kinetic energy of a mass m moving with velocity V is $\frac{1}{2}mV^2$.

118. Centre of Pressure.—When a plane surface is exposed to the pressure of a fluid, the pressures of the different portions of the fluid are all perpendicular to the surface, and thus form a system of parallel forces. The centre of these parallel forces (see § 51) is called the *Centre of Pressure.*

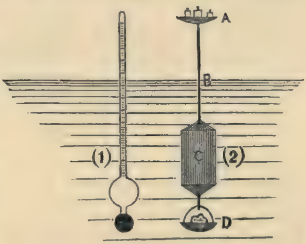
To find the centre of pressure of a rectangle immersed in a liquid, one edge of the rectangle being in the surface of the liquid.

Let ABCD be the rectangle, and let AB be on the surface. Bisect AB in E, CD in F, join ED, EF, EC, and suppose the rectangle divided into horizontal strips of which PQRST is one. The pressure on the strip varies with its depth below the surface. But this is proportional to ER and therefore to QS. Therefore the resultant pressure on PT acts at R and is proportional to QS. It follows that the problem of finding the centre of pressure of ABCD is the same as that of finding the centre of gravity of the triangle EDC. It is therefore at O, if $EO = \frac{2}{3}EF$.



119. Hydrometers—(1.) **The Common Hydrometer.**—This consists of a glass stem ending in two bulbs, the lower of which is loaded to cause the instrument to float in a vertical position. More or less of the stem projects above the surface according to the density of the liquid. The stem contains a graduated scale from which the specific gravity of the liquid is read off.

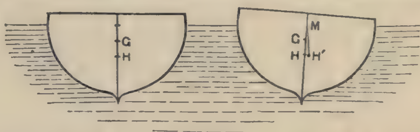
(2.) **Nicholson's Hydrometer.**—This consists of a metal cylinder C, surmounted by a thin stiff wire bearing a pan, A. A mark B shows how far the hydrometer is to be immersed. (Hence this hydrometer is called a hydrometer of constant immersion). A heavy pan D is suspended below. To find the specific gravity of a liquid, immerse the hydrometer in water, and observe what weight w_1 must be placed in A to bring B down to the surface. If w be the weight of the instrument, $w + w_1$ is the weight of the water displaced. Then immerse the hydrometer in the liquid whose specific gravity is required, and let w_2 be the weight which must now be placed in A. If s be the specific gravity of the liquid,



$$s = \frac{w + w_2}{w + w_1}.$$

This instrument is also used to find the specific gravity of a small solid substance. Place the instrument in water and load A as before. Now place the substance in A and let w' be the weight which must be removed from A to bring B to the surface. w' is the weight of the substance. Remove the substance from A to D, and let w'' be the weight which must now be added to A. w'' is the weight of the water displaced by the substance. Therefore, if s be the specific gravity of the substance, $s=\frac{w'}{w''}$.

120. Metacentre.—When a body floats in a liquid it is often important to determine whether its equilibrium is stable, unstable, or neutral (§ 56). As already shown (§ 89), two equal and opposite forces act on such a body; its



weight, acting vertically downwards through G its centre of gravity, and the resultant pressure of the liquid, which is equal to the weight of the liquid displaced, and acts vertically upwards through H, the centre of gravity of the liquid displaced. If now the body be slightly tilted to one side, the weight still acts downwards through G, but the upwards pressure acts through H′, the centre of gravity of the portion of the liquid displaced by the body in its new position. Let the direction of the upward pressure through H′ meet HG in M. The limiting position of M when the body is very slightly tilted is called the *Metacentre.* If M be above G, the couple formed by the two forces tends to restore the body to its original position, or the equilibrium is stable. If M be below G, the couple tends to remove the body farther from its original position, or the equilibrium is unstable. If M coincide with G the equilibrium is neutral. To secure proper stability in ships the metacentre should be high.

ADDITIONAL EXAMPLES AND EXAMINATION QUESTIONS

Composition and Resolution

1. Find the resultant of two velocities V and V', whose directions are inclined at an angle θ.

The resultant of two forces of 5 lbs. and 8 lbs. acting at a point is a force of 7 lbs.; at what angle do those forces act?

2. Three forces, P, Q, R, keep a particle A in equilibrium. Prove that—

$$\frac{P}{\sin QAR} = \frac{Q}{\sin RAP} = \frac{R}{\sin PAQ}.$$

Find the ratios of the three forces, when one is half the second and at right angles to the third.

3. Show that the condition $X=0$, $Y=0$ (§ 100) is the same as the condition that the components are proportional and parallel to the sides of a polygon taken in order (§ 14).

4. A swinging pendulum is, at the instant under consideration, inclined at an angle θ to the vertical. Find its acceleration in the direction of motion.

Uniformly Accelerated Motion

5. A particle starts from rest and moves with uniform acceleration in the direction of motion. Prove that the space described in the m^{th} second is half the sum of the spaces described in the $(m+n)^{th}$ and $(m-n)^{th}$ seconds.

6. Calculate the time required and the velocity acquired by a stone sliding 1000 feet down a smooth slope of 1 in 3.

7. A particle moving with a horizontal velocity of 87 receives in each second a change of velocity of 32 vertically downwards. Find its velocity after 13 seconds.

8. Describe Atwood's Machine.

If the weights on an Atwood's Machine be 3 lbs. and 5 lbs. respectively, what will be the velocity at the end of 3 seconds from rest, and what will be the distance through which each weight will have moved? What will be the velocity of their centre of gravity? Find also the tension of the string.

9. A mass of 5 lbs. is placed on a smooth inclined plane of slope 2 in 5. A mass of 1 lb. is attached to this by means of a cord, which passes over a pulley at the top of the plane and hangs freely downwards. The masses of the cord and pulley are left out of account. Investigate the motion.

10. Prove that the time of descent down all chords of a vertical circle, starting from the highest point, is constant.

11. If a mass M′ resting on a smooth horizontal table be attached by a string to a mass M hanging over the edge, find (1) the acceleration; (2) the tension in the string during the motion.

12. If, in the previous question, M′ be 6 lbs. and M 2 lbs., and the string be 24 feet long, and if at first M′ be 12 feet from the edge of the table, and M 4 feet from the ground, find when M′ reaches the ground.

Parabolic Motion

13. Show that an unresisted projectile describes a parabola.

14. Describe Morin's Machine, and show how it may be employed to determine the value of g.

15. Find the range, greatest height, and time of flight of a projectile on a horizontal plane.

16. A projectile has initial velocity V in a given vertical plane. Show that the position of the directrix of the parabola described is independent of the direction of projection, and find the locus of the foci of the different parabolas which may be described.

17. Find the direction of projection when the greatest height reached by a projectile is one-fourth of its range on a horizontal plane.

Couples

18. Define a couple.

Prove that the moment of a couple about any point in its plane is constant.

19. Show that a force acting at a point is equivalent to an equal force acting at another point together with a couple.

20. Prove that two couples in the same plane, whose moments are equal and opposite, neutralise each other.

21. Prove that equal and opposite couples in parallel planes neutralise each other.

Forces acting on a Rigid Body

22. Find the resultant of any number of forces acting on a rigid body in one plane.

23. Prove that when forces in one plane acting on a rigid body are in equilibrium, three relations exist between the forces and the co-ordinates of their points of application.

24. Find the resultant of any number of forces in different planes acting on a rigid body.

25. Three forces are proportional to the sides of a triangle along which they act. Are they in equilibrium? If not, find their resultant.

26. If three non-parallel forces keep a body at rest, what conditions must they satisfy?

A uniform rod, weighing 10 lbs., hangs by two strings at its ends, making angles with it of 60° and 30°. Find the tension in the strings.

27. If two forces balance each other on a weightless rod capable of turning freely round a fixed point, what relation must exist between the forces?

C is a point in a weightless rod AB, round which it is capable of turning freely; AC is one-third of AB; a force of 10 units acts at A perpendicularly to AB, and is balanced by an equal force acting at B. Find how the second force must act, and the magnitude and direction of the pressure on the fulcrum.

28. A uniform beam weighing 50 lbs. is hinged at one end, and rests in a horizontal position, with the other end against a smooth plane, rising 1 in 2. Find the pressure on the hinge and the reaction of the plane.

The Hodograph

29. Define the Hodograph.

Prove that, when a body moves with constant acceleration in a fixed direction, the hodograph of its motion is a straight line.

30. Prove that the acceleration of a point's motion is equal to the velocity with which the hodograph is traced out.

31. Prove that when a point moves with uniform velocity its hodograph is a point.

32. Find the hodograph of a point which moves at a uniform rate along the thread of a screw.

Uniform Circular Motion

33. A point moves uniformly in a circle. Prove that the acceleration is directed toward the centre and is equal to $\frac{v^2}{r}$.

34. Calculate the tension of a string 7 feet long, at one end of which a mass of 10 lbs. revolves about the other end once in two seconds.

35. At what rate must a stone be projected horizontally so as (if unresisted by the air) just to revolve round the earth?

36. Explain what is meant by the term Centrifugal Force.

A square framework, of a yard in the side, rotates once per second about one of the sides which is vertical. What must be the coefficient of friction to keep a ring from slipping down the opposite side?

Impact

37. Spheres of 10 and 20 lbs. moving with velocities 5 and 3 in the same direction, impinge. Find the velocities after rebound [$e=\frac{1}{2}$].

38. Two masses of 5 and 10 lbs. respectively impinge directly, moving with velocities of 8 and 10 feet per second. Find the common velocity after impact, and show that there has been a transformation of kinetic energy.

39. A ball moving at the rate of ten feet per second impinges on an equal ball at rest. Find the subsequent velocity of each, the coefficient of elasticity (restitution) being $\frac{1}{2}$.

40. Two perfectly elastic equal spheres impinge directly. Prove that they exchange velocities.

41. A ball dropped from the top of a tower strikes a rigid projecting ledge and rebounds. Show that if it be perfectly elastic it will reach the ground with the same velocity as if it had not struck the ledge, but will do so only after a longer interval.

42. If, in the last example, instead of striking a ledge, the ball be struck directly by another ball moving horizontally, show that the time of reaching the ground will be the same as if it had fallen unstruck, but the velocity, with which it reaches the ground, will be increased whether the balls be elastic or not.

Energy and Work

43. What is work, and how is it measured?

A truck weighing 12 tons is drawn from rest by a horse through a distance of 20 ft., and is then moving at the rate of 5 ft. per second. If the resistances are $8\frac{1}{3}$ lbs. per ton, how many units of work must the horse have done on the truck?

44. Define a foot-pound and a horse-power.

What is the H.-P. of a waterfall of 18 ft. when the stream above passes through a section of 6 sq. ft. at $2\frac{1}{2}$ miles an hour? (1 cub. ft. water weighs $62\frac{1}{2}$ lbs.)

45. A train of 100-ton mass exposed to resistance of 8 lbs. a ton is driven with constant speed by an engine of 64 H.-P. Find its speed. If the train is to be stopped in $\frac{1}{3}$ mile by shutting off the steam and putting on the brakes, find the brake resistance in lbs. per ton.

46. Prove that the kinetic energy of a mass m moving with velocity v is $\frac{1}{2}mv^2$.

47. Give Newton's third law of motion, pointing out his two interpretations of it.

Apply these two interpretations to find the initial velocity of a shot of 1000 lbs. discharged from a 100-ton gun: supposing none of the 30,000 foot-tons of energy given out by the explosion to be wasted in heat, light, or sound.

48. How is the force of gravity measured in absolute units?

Find the work done by gravity on a mass m falling through a space s. Prove that it may be measured by the increase of kinetic energy of the body in passing over that distance.

49. What constant horizontal force is required to stop a train of 100-ton mass, running at 50 miles an hour: (1) in one minute, (2) in 200 yards?

50. Find the horizontal pressure on the rails when a 20-ton engine runs at 50 miles per hour on a curve of a mile radius.

51. Employ the principle of conservation of energy to show that the velocity of the bob of a pendulum at its lowest point is equal to the velocity which it would have acquired by falling vertically a distance equal to the difference of the heights of its highest and lowest points.

52. A bullet weighing three ounces strikes a target with a velocity of 1200 feet per second, and falls dead. How much kinetic energy is lost?

53. Calculate the kinetic energy of a tram-car weighing 2·5 tons, when it is moving at the rate of 6 miles an hour, and is laden with 36 passengers averaging 9 stones each in weight.

54. If the co-efficient of kinetic friction for a tram-car moving on its rails is $\frac{1}{28}$, find how much work is done when the above car, loaded as stated, is pulled 3 miles along a level road.

55 Compare the amounts of momentum and of kinetic energy in (*a*) a pillow of 20 lbs. which has fallen through one foot vertically, and (*b*) an ounce bullet moving at 200 feet per second.

56. What is the amount of kinetic energy in a train of 25 tons moving with a velocity of 20 miles per hour? What force (measured in foot-pound-second units) acting for ten seconds is sufficient to stop the train?

Centre of Pressure

57. What is the centre of fluid pressure on a plane area? If the area pressed be a rectangle with one edge on the surface of the fluid, where is the centre of pressure?

A rectangular hole ABCD, whose lower side CD is horizontal, is made in a side of a reservoir, and is closed by a door whose plane is vertical; the door can turn freely outward round a hinge coinciding with CD. Calculate the force that must be applied to AB to keep the door shut, assuming that AB is 1 ft. and AD 12 ft. long, and that the water rises to the level of AB.

58. Where is the centre of pressure situated in the following:—(1) triangle with base in surface; (2) triangle with vertex in surface and base horizontal?

ABCD is a rectangle. AB (2 ft.) is on the surface, and BC (6 ft.) points vertically downwards. Find the centre of pressure of the part below the horizontal line through the centre of the rectangle.

Hydrometers and Specific Gravity

59. Describe briefly Nicholson's hydrometer, and how it is used to find the specific gravity of a small substance.

The standard weight in a Nicholson's hydrometer is 1250 grains. What is the specific gravity of a body which requires 530 or 620 grains to sink the instrument to the marked point, according as it is placed in the upper or under pan?

60. Explain the principle of the hydrometer of variable immersion.

A common hydrometer sinks in water to a point on its stem A, in a liquid whose specific gravity is 0·9 to C, and in a third liquid to B. If B be midway between A and C, find the specific gravity of the third liquid.

61. The two parallel legs of a bent tube are held vertically, and liquids which do not mix are poured slowly into the tube, the heavier being poured first. Find in what position they come to rest.

If the two liquids be water and oil (sp. gr. 0·88), and the water rise 11 in. above the common surface, to what height does the oil rise?

A bent tube containing water has a uniform section of a square inch. Into one branch are now poured 3 cub. in. of oil (sp. gr. 0·8). Find how much the surface of the water in that branch is depressed.

62. The specific gravity of a thin rod whose mass is 324 grains is 0·81. Find the limits to the mass of a particle which can be fixed to one end so as to make the rod float upright in water.

Miscellaneous

63. Show that the numerical measure of a force depends on the square of the unit employed for time.

64. Prove that a particle within a uniform homogeneous spherical shell is attracted equally in all directions.

65. Show that a hollow sphere cannot rest on a rough inclined plane if the outer and inner surfaces be concentric; and that it can rest in two positions on a horizontal plane if the outer and inner surfaces be not concentric; one position being one of stable and the other of unstable equilibrium.

Note.—The student will observe that examples 10, 21, 24, 25, 40, 58, 61, 63, and 64, involve important theorems.

EXAMINATION PAPERS

I

UNIVERSITY OF CAMBRIDGE

General Examination for the Ordinary B.A. Degree

Mechanics and Hydrostatics

1. Show that forces may be represented in magnitude and direction by straight lines, and give one illustration.

2. Find the magnitude and line of direction of the resultant of two parallel forces acting in the same direction, and deduce from your results the corresponding proposition when the forces act in opposite directions.

3. Determine the force which, acting parallel to an inclined plane, will just support a given weight placed upon it. Find also the pressure on the plane.

Two weights, W and $2\ W$, are in equilibrium when connected by a string passing over the common vertex of two inclined planes. Given the inclination of the plane on which W rests, show how the inclination of the other plane may be determined by a geometrical construction.

4. What is meant by the moment of a force about a point? Show how a moment may be represented geometrically.

5. Find the ratio of the power to the weight in that system of pulleys in which each string is attached to the weight.

6. Define the term centre of gravity; and find that of a uniform straight rod.

A pencil rests on a table with five-twelfths of its length projecting beyond the edge. A beetle, whose weight is one-fourth of that of the pencil, crawls along it. How far may it crawl without upsetting it?

7. Describe the common balance, stating the requisites of a good balance, and how they are obtained.

How may the true weight of a body be found with a balance in which the arms are unequal?

8. Define a fluid, and distinguish between a liquid and a gas.

9. Show that the surface of a liquid at rest is a horizontal plane.

10. A cylinder open at the top is inverted and immersed vertically in water. Find the depth of the vessel under water when half full.

11. Define specific gravity, and show how to determine the specific gravity of a solid body by means of the hydrostatic balance, the specific gravity of the solid being greater than that of the liquid in which it is weighed.

A ball of gutta-percha, 2 inches in diameter, encloses a ball of cork 1 inch in diameter, and floats in water. Specific gravity of gutta-percha is ·98, of cork ·24. Find what proportion of the volume of the ball will float above the surface of the water.

12. Describe the common barometer. If the tube be not exactly vertical, will the indications of the instrument be incorrect?

II

UNIVERSITY OF OXFORD

Second Public Examination—Pass School

Elements of Mechanics

1. Explain how uniform velocities are measured.

Two men, A and B, run a race, the former having a start of a yards. If the speed of B be to that of A as 3 to 2, find how many yards A will have run when B overtakes him.

2. Enunciate and prove the principle of the parallelogram of forces.

3. How is the parallelogram of forces verified by experiment?

At what angle must two forces, $a^2 - b^2$, $2ab$ act so that their resultant may be $a^2 + b^2$?

4. Find the magnitude of the resultant of two forces of 13 and 35 lbs., when the angle between their directions is 60°.

5. Define the moment of a force about a point.

Prove that the moments of two component forces about any point situated upon the line of action of their resultant are equal.

6. Find the velocity which a heavy body will acquire by falling freely through a height of $3\frac{1}{23}$ feet ($g = 32{\cdot}2$).

7. Enunciate Newton's laws of motion.

Two bodies start together from the same point, and move uniformly in directions at right angles to each other: one body moves at the rate of $2m$ feet per second, and at the end of t seconds the distance between them is $(1 + m^2)t$ feet. Find the velocity of the other body.

8. Define specific gravity, and show how to find the specific gravity of a mixture of two fluids whose specific gravities are known.

Equal volumes of two fluids, whose specific gravities are 1 and 1·7, contract when mixed by one-tenth of their original volume. Find the specific gravity of the mixture.

9. Explain the action of a syphon and of a force pump.

10. Explain the principle and construction of the hydrostatic press.

If the diameters of the pistons or plungers be 1 and 9 inches, if the lever be jointed to the rod of the smaller plunger at 6 inches distance from the lever's fulcrum, and if the workman press on the lever with the weight of his body, 140 lbs., applied at 32 inches distance from the fulcrum, find the pressure in tons with which the larger plunger is forced upwards.

III

UNIVERSITY OF EDINBURGH

Medical Degree—Preliminary Examination—*Mechanics*

(A)

1. Define uniform velocity.

A railway train moving uniformly, travels $37\frac{1}{2}$ miles in an hour. Find its velocity in feet per second.

2. A body falls from rest. What is its velocity, after falling for two seconds and a half, and what distance will it fall in that time?

3. Define mass and momentum.

A cricket ball weighs 6 ounces, and is moving with a velocity of 40 feet per second. Compare its momentum with that of a cannon ball which weighs 20 lbs., but is moving with a velocity of 18 feet per second.

4. Enunciate the second law of motion. Show that it enables us to investigate the motion of a falling body.

5. Find the resultant of the following forces:—5 northwards, 5 eastwards, 1 westwards, and 2 southwards.

6. Parallel forces of 4 and 6 pounds act in the same directions at two points, A and B, in a body. Find at what point in AB a single force must act to maintain equilibrium.

7. State the three kinds of equilibrium, and give an example of each.

8. A horizontal force of 5 lbs. supports a weight of 12 lbs. on an inclined plane. Find the pressure on the plane.

9. Define a fluid.

What is the law of pressure at different depths below the surface of a liquid?

10. The specific gravity of lead is 11. What is the real weight of a mass of lead which, when suspended in water, seems to weigh 1 lb.?

(B)

1. Express a velocity of 60 miles per hour in terms of kilometres per second (1 millimetre = 0·0394 inch).

2. A mass of snow, 28 lbs. in weight, falls from the roof of a house to the ground, a distance of 40 feet. Calculate the velocity, momentum, and kinetic energy acquired at the time of impact.

3. Enunciate the law of gravitation.

The earth attracts as if it were condensed at its centre. Suppose that the earth shrank until its diameter were 6000 miles, what would be the effect on the weight of an inhabitant? (The diameter of the earth is approximately 8000 miles.)

4. Define the term rigid body.

A man and a boy are to carry a natural magnet (weight 120 lbs.) by means of a stout bar. Show how to arrange the matter so that the boy may have only one-third of the weight to carry.

5. Describe how the true weight of a body may be found by means of a common balance which is faulty in construction.

6. Explain how it is that a skater can assume an inclined attitude without falling.

7. Distinguish between pressure at a point and the resultant pressure in a body immersed in a fluid.

How is it that a diver under water can move about large stones with comparative ease?

8. Define density and specific gravity.

A lump of solid cast-iron floats in the molten liquid with $\frac{1}{21}$ of its volume out of the liquid. What is the specific gravity of the solid cast-iron relatively to the molten cast-iron?

9. Define centre of inertia, and centre of gravity.

Find the centre of inertia of a T-square, the two pieces being of the same material, and equal in length, breadth, and thickness.

10. Define a machine.

Explain and illustrate the principle *What is gained in power is lost in space.*

IV

UNIVERSITY OF LONDON

MATRICULATION EXAMINATION—*Natural Philosophy*

1. State the parallelogram of forces. Explain the meaning of the terms employed in your statement.

Apply it to show that if four forces acting on a point be represented by the sides of a rectangle taken in order, they will be in equilibrium.

2. Explain the meaning of the words composition and resolution of forces, and show how forces may be compounded and resolved.

A particle is acted upon by a force whose magnitude is unknown, and whose direction makes an angle of 60° with the horizon. The horizontal component of the force is known to be 1·35. Determine the total force and also its vertical component.

3. A substance is weighed from both arms of an unequal balance, and its apparent weights are 9 lbs. and 4 lbs. Find the ratio between the arms.

4. A balloon is carried along by a current of air moving from east to west at the rate of 60 miles an hour, having no motion of its own through the air, and a feather is dropped from the balloon. What sort of a path will it appear to describe as seen by a man in the balloon?

5. Suppose that at the equator a straight hollow tube were thrust vertically down towards the centre of the earth, and that a heavy body were dropped through the centre of such a tube. It would soon strike one side. Find which, giving a reason for your reply.

6. A body weighs in air 80 grains, in water 56 grains, and in another liquid 46 grains. Find the specific gravity of this liquid.

7. What becomes of the weight which the body, in the last question, appears to lose?

8. A cylindrical bell 4 feet deep, whose content is 20 cubic feet, is lowered into water until its top is 14 feet below the surface of the water, and air is forced into it until it is three-quarters full. What volume would the air occupy under the atmospheric pressure—the water barometer being at 34 feet.

State the principles on which your answer is based.

V

UNIVERSITY OF GLASGOW

Preliminary Medical Examination—*Elements of Mechanics*

1. Explain how a force may be represented by a straight line.

Two forces of 30 and 40 lbs. respectively act upon a particle in directions at right angles to each other. Find the magnitude of their resultant in pounds.

2. What is a Lever?

Distinguish between the three kinds of simple levers, and point out (with reasons) to which class belongs a pair of nut-crackers, a pair of tongs, a pair of scissors.

3. Explain the terms Velocity and Acceleration. When the acceleration is constant what is the connection between it and velocity?

A body under a constant acceleration moves from rest over 10 feet in the first second. How far has it moved at the end of the second second?

4. Find the ratio of the Power to the Weight in a system of pulleys, in which each pulley hangs by a separate string.

5. When a cannon is fired, explain carefully what takes place.

If the cannon be free to move, find its velocity on being fired, supposing it to weigh 200 lbs., the bullet weighing 5 lbs. and being projected horizontally with a velocity of a mile in 5 seconds.

VI

UNIVERSITY OF CAMBRIDGE

Local Examination—*Mechanics*

1. Show how a straight line may be made to represent the several elements which completely determine a force.

Three forces of 3, 7, and 10 lbs. respectively, are in equilibrium at a point. How are they directed?

2. Enunciate the proposition known as the parallelogram of forces.

The resultant of two forces acting at an angle of 120° is perpendicular to the smaller component. The greater component is a force of 100 lbs. Find the other component and the resultant.

3. A heavy rod hangs vertically and is free to turn about its upper end. A horizontal force equal to half the weight of the rod is applied at the lower end. Find the inclination to the horizon when there is equilibrium.

4. Find the resultant in magnitude and direction of two parallel forces acting in the same direction.

Weights of 6, 8, and 10 lbs. respectively, are placed on the middle points of the edges of a triangular table. Find how much the pressures on each of the three legs are increased thereby.

5. Define the "centre of gravity" of a body, and find that of a plane triangular area.

Weights of 1, 2, 3, 4 lbs. respectively are attached to the corners of a light square frame. Determine the point by which the frame may be suspended in any position.

6. Find the relation of the Power to the Weight in the wheel and axle.

Find the radius of the axle when a weight of 16 lbs. just balances a weight of 1 cwt., the radius of the wheel being 21 inches.

7. A horse, in drawing a load up a steep hill, takes a zig-zag course, and a man going down the same on horseback leans backwards. Explain the advantage in each case.

VII

OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION BOARD

Examination for Certificates—(1) *Statics and Dynamics*

1. Assuming the Parallelogram of Forces for the direction, prove it for the magnitude of the resultant.

Three forces are represented by OA, OB, and OC. P is the middle point of OC, Q is the middle point of AB, and R is the middle point of PQ. Prove that the resultant force will be represented in magnitude and direction by 4 OR.

Extend this to any number of forces.

2. Find the resultant of two parallel forces in the same direction.

The three angles of the triangle ABC are joined to any two points P and Q, and six forces act on a rigid body completely represented by the lines PA, AQ, PB, BQ, PC, and CQ. Prove that the resultant is parallel and equal to 3PQ, and always passes through a fixed point for all positions of P and Q.

3. Find the centre of gravity of a triangle.

If the centre of gravity of the uniform four-sided figure ABCD coincide with one of the angles, as A, prove that the distances of A and C from the line BD are as 1 to 2.

4. State the principal laws of friction, and find the angle of inclination of a plane in order that a body may just slide down.

5. Find the ratio between the power and the weight in a system of weightless pulleys in which each string is attached to the weight.

Prove that the principle of virtual velocities holds good in this case.

6. How is velocity measured, (1) when uniform, (2) when variable?

What is meant by the mean velocity of a moving point, while moving over a given space?

7. A railway train is running at the rate of 28 miles an hour, when a bullet, moving horizontally, enters a compartment at the corner furthest from the engine and passes out at the diagonally opposite corner.

If the compartment be 8 feet long and 6 wide, and the sine of the angle of inclination of the bullet's path to the train be $\frac{3}{5}$, prove that the bullet's velocity is 80 miles an hour.

8. Prove the formula $s=\frac{1}{2}ft^2$ in uniformly accelerated motion.

9. Prove that the path of a projectile in a vacuum is a parabola.

10. Find the velocities of two elastic spheres after direct impact, and prove that kinetic energy is lost when the elasticity is imperfect.

What becomes of that energy?

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(2) *Natural Philosophy* (*Mechanical Division*)

1. What are the essential differences between a solid and a fluid, and between liquid water and steam? What are the effects of compressing steam in a cylinder with a close-fitting piston?

2. Explain the meanings of the terms weight, specific gravity, acceleration, pressure of a fluid at a point, and horse-power.

Find the weight of a cylindrical stone column 2 feet in diameter and 16 feet in height, if the specific gravity of the stone be 2·8 and a cubic yard of water weigh 15 cwt.

3. A truck of coals weighing 12 tons is to be drawn up a smooth incline of 1 in 10 by means of a rope coiled round a cylinder two feet in diameter, which forms the "axle" of a wheel and axle, the diameter of the wheel being 5 feet. Find the force which must be applied at the circumference of the wheel.

4. State the experimental laws of friction between two solid bodies.

Show that the work done in dragging a body up a rough inclined plane is the same as that done in dragging the body along a horizontal plane of like roughness through a distance equal to the base of the plane, and then lifting it up the height of the plane.

5. A force equal to the weight of one ounce acts on a pound weight for 10 seconds. Find the velocity generated and the space through which the weight will be moved in the 10 seconds, supposing it to start from rest at the beginning of the time.

6. Describe the construction and action of a force pump.

Explain the action of the hydraulic press.

If the diameter of the plunger of the pump is $\frac{1}{3}$ in., and that of the piston in the large barrel three inches, find the pressure which can be exerted by the latter when the former is pressed down by a force equal to the weight of 2 cwt.

7. What are the conditions of equilibrium of a floating body?

If the weight of a balloon and its appendages be 3200 lbs., and that of the air it displaces 3400 lbs., with what acceleration will it begin to ascend when set free?

8. State the relation between the pressure and volume of a given mass of air at constant temperature

How would you proceed to demonstrate experimentally the truth of your statement?

VIII

UNIVERSITY OF EDINBURGH

M.A. Pass Examination—*Natural Philosophy*

(A)

1. Define velocity.

Prove that we cannot know the *absolute* velocity of a body.

A and B are two points in motion on a horizontal plane. At the moment under consideration A has a velocity of 1 northwards, and B has a velocity of $\sqrt{3}$ eastwards. Find the velocity of B with respect to A.

2. A point has an initial velocity V, and receives uniform acceleration g in a direction opposite to the direction of motion. Show that if v be the velocity after t seconds. and s the space passed over in that time

$$(1)\ v = V - gt. \quad (2)\ s = Vt - gt^2.$$

3. State Newton's Second Law of Motion, and show that it involves the following facts :—

(1) A constant force produces uniformly accelerated motion.

(2) Forces are to be combined and resolved by the parallelogram law.

4. Assuming the acceleration due to gravity to be uniform, prove that when a mass M falls from rest through a distance s,—

$$Ws = \tfrac{1}{2}MV^2,$$

where W is the weight of the mass and V its final velocity.

Interpret this equation in terms of energy and work.

5. A 50 lb. shell, moving at the rate of 1000 feet per second, explodes symmetrically into 2 lb. fragments. One of these has its velocity increased in magnitude without change in direction. Find its new velocity, assuming that 250,000 foot-pounds of kinetic energy are generated by the explosion.

6. A mass of 10 lbs. moving northwards with velocity 20, impinges on a mass of 15 lbs. moving eastward with velocity 10. At the instant of impact the two masses cohere. Find the velocity of the combined mass, and indicate its direction by a figure.

Calculate the kinetic energy before and after impact, and account for the apparent loss.

[*This Examination also contains Questions in Physics.*]

(B)

1. Define Velocity and Acceleration. What is the relation of Acceleration to Force?

A lift ascends under a uniform acceleration during the former part of its course, and with a uniform velocity during the latter part of its course, and is stopped suddenly at the end. Describe the sensations of a passenger with respect to weight.

2. Enunciate Newton's Second Law of Motion, and show how to derive from it the resultant of two forces acting on a particle.

3. Write down the expressions for the momentum and the kinetic energy of a body in terms of its mass and velocity.

Calculate the momentum and the kinetic energy of a hammer of one ton weight let fall half a foot, expressing the results in terms of the pound, foot, and second.

4. Assuming 32·2 as the foot-second measure of the acceleration produced by gravity, express the same quantity numerically in yard-minute, and in mile-hour, units.

5. State in kinematical terms the nature of the motion of a boat propelled by a couple of oars; (*a*) when worked simultaneously; (*b*) when worked alternately.

6. A balloon is 400 feet from the ground, and ascending at the rate of 10 feet per second. What time would a sand-bag take to fall to the ground from it?

7. Define Work and Horse-power.

Calculate the amount of work done against gravity in drawing a car of 2·5 tons weight, laden with 30 passengers averaging 9 stones each in weight, up an incline the ends of which differ 120 feet in level.

Find the horse-power sufficient to do that work in half an hour.

8. Show that when a ladder, resting against a wall, is about to slip, a man standing on one of the lower steps will make it more secure; but that, if he mounts higher than its centre of gravity, he will bring it down.

IX

SCIENCE AND ART DEPARTMENT, SOUTH KENSINGTON

Theoretical Mechanics—(1) *Elementary Examination*

1. State how to find the position of the centre of gravity in the following cases:—(*a*) a square lamina; (*b*) a triangular lamina; (*c*) a cone; each body being of uniform density.

If a body is held suspended by a thread fastened to a point of it, in what position does it come to rest?

2. Forces of 10, 13, and 16 units act at a point and are in equilibrium; show by a diagram how they act, and note the angles between their directions.

3. A rod (AB) can turn freely round a hinge at A, and rests in an inclined position with its end B against a smooth vertical wall. Show in a diagram how the forces act which keep the rod at rest, and name them.

4. Define the moment of a force with respect to an assigned point.

A uniform rod rests in a horizontal position on two supports 8 ft. apart, one under each end; it weighs 6 lbs.; a weight of 24 lbs. is hung to it from a point distant 3 ft. from one of the points of support. Find the pressure on each point of support.

5. State what is meant by the sensibility of a balance, and explain why the sensibility is great when the arms are long, and also when the centre of gravity of the beam is near the point of support.

6. Define a foot-pound of work.

A man weighing 140 lbs. puts a load of 100 lbs. on his back and carries it up a ladder to a height of 50 ft.; how many foot-pounds of work does he do altogether, and what part of his work is done usefully?

7. If the velocity of a body is increased uniformly in each second by 32 ft. a second, by how many feet a second is its velocity increased in one minute?

If a velocity is 1920 ft. a second, what is it in yards a minute?

8. Write down the formula for the distance described in a given time by a body whose velocity is uniformly accelerated.

A body thrown upwards against gravity reaches a greatest height of 121 ft. Find the velocity with which it is thrown up, and the number of seconds that will elapse before it returns to the point of projection. $(g=32.)$

9. A body whose mass is 10 lbs. is moving at the rate of 50 ft. a second. What is the numerical value of its kinetic energy at that instant? If from that instant it moves against a constant resistance equal to one-twentieth of its weight, how far does it go before being brought to rest? $(g=32.)$

10. When a body is wholly or partly immersed in a liquid, what is the magnitude of the resultant pressure of the liquid on the body?

A body, whose specific gravity is 1·4 and volume 3 cubic ft., is placed in a vessel in which there is water enough to cover it. What pressure does the body produce on the points of the bottom at which it is supported?

11. Explain briefly the method of finding the specific gravity of an insoluble body by means of the balance.

If the body weighs 732 grains in vacuo, and 252 grains in water, what is its specific gravity?

12. A tube filled with water is inverted with its open end in water, no air having got in; the top of the tube is 20 ft. above the surface of the external water. If the water-barometer stands at 34 ft., what is the pressure, in pounds per square foot, at a point of the inside of the top of the tube? (1 cubic ft. of water weighs 1000 oz.)

What would be the consequence of making a hole through the top of the tube, and why would the consequence follow?

(2) *Advanced Examination*

1. State and prove the rule for finding the resultant of two parallel forces acting towards the same part on a rigid body.

Parallel forces of 10 and 20 units act towards the same part at A and B; a force of 15 units acts from A to B. Find the resultant of the three forces, and show in a diagram how it acts.

2. Show that two couples, whose moments are equal and of opposite signs, are in equilibrium when they act in the same plane on a rigid body.

3. ABCD is a quadrilateral figure, P and Q are the middle points of the opposite sides AB and CD; O is the middle point of PQ. Show that four forces, represented by OA, OB, OC, OD respectively, are in equilibrium.

4. ABC is a rigid equilateral triangle (whose weight is put out of the question); the vertex B is fastened by a hinge to a wall, while the vertex C rests against the wall, under B. If a given weight is hung from A, find the reactions at B and C.

What are the magnitudes and directions of the forces exerted by the weight on the wall at B and C?

5. Define the co-efficient of friction.

A weight of 500 lbs. is placed on a table, and is just not made to slide by a horizontal pull of 155 lbs. Find the co-efficient of friction, and the number of degrees in the angle of friction by drawing it to scale; or, if you have no instruments, explain how to calculate the number of degrees.

6. Find the relation between the power and the weight in the screw press, taking into account the friction between the threads of the screw and of the companion screw.

7. Define a foot-pound of work, and a horse-power.

A steam-crane, working with 3 horse-powers, is found to raise a weight of 10 tons to a height of 50 ft. in 20 minutes. What part of the work is done against friction? If the crane is kept at similar work for 8 hours, how many foot-pounds of work are wasted on friction?

8. Find the position of a body at the end of a given time from the instant at which it is thrown with a given velocity in a given direction, the motion being supposed to take place in vacuo.

A body is thrown in a direction making an angle of 30° with the horizon, and passes through a point whose horizontal distance from the point of projection is $400\sqrt{3}$ ft., and vertical height above the point of projection 76 ft. Find the velocity of projection. $(g=32.)$

9. A particle, whose mass is 10, moving with a velocity 5, meets and impinges directly on another particle whose mass is 20 and velocity 3; the co-efficient of restitution is 0·125. Find from first principles the velocities of the particles at the end of the impact.

State the dynamical principles employed in answering this question, and define the co-efficient of restitution.

10. A fly-wheel weighs 10,000 lbs., and is of such a size that the matter composing it may be treated as if concentrated on the circumference of a circle 12 feet in radius. What is its kinetic energy when moving at the rate of 15 revolutions a minute? How many turns would it make before coming to rest, if the steam were cut off and it moved against a friction of 400 lbs. exerted on the circumference of an axle 1 ft. in diameter? ($g=32$.)

11. Define the centre of pressure of a fluid on a plane area, and find its position in the case of a rectangular area, one edge being on the surface of the fluid.

Find where the centre of pressure of the rectangle would be, if its plane were vertical and its upper edge (which is horizontal) below the surface of the water at a distance equal to the height of the rectangle.

12. Given the specific gravities of two liquids, show how to calculate the specific gravity of a mixture of given volumes of the two liquids, assuming that the mixture takes place without change of volume.

Three pints of a liquid, whose specific gravity is 0·8, are mixed with five pints of another liquid, whose specific gravity is 1·04. Find the specific gravity of the mixture, (*a*) if there is no contraction, (*b*) if on mixture there is a contraction of 5 per cent. of the joint volumes.

ANSWERS

CHAPTER I.—PRELIMINARY

Examinations on Chapter I

[The Answers to the Examination Questions will all be found in the text of the book. The forms of the questions have been occasionally varied in order to prepare the student for the various ways in which the same question may be set by different examiners.]

CHAPTER II.—KINEMATICS

Examples for Exercise

4

1. $\frac{2 \times 1760 \times 3}{9 \times 60} = 19\frac{5}{9}$. 2. $1333\frac{1}{3}$.

3. The former velocity is $26\frac{2}{23}$ feet per second, the latter is 27, and is therefore the greater.

4. To travel 1118 feet sound requires 1 second.

$\therefore$ to travel $43 \times 1760 \times 3$ ft. requires $\frac{43 \times 1760 \times 3}{1118} = 203\frac{1}{13}$ sec.

5

1. $s = 100 \times 60 \times 60$ ft. $= 360,000$ ft. $= 68\frac{2}{11}$ miles.

2. $v = \frac{1,511,460 \times 1760 \times 3}{27 \times 24 \times 60 \times 60} = 3421$ feet per second.

3. $t = \frac{b}{a}$.

7

1. 48·3.

2. Find the average velocity for the last two seconds. 64·4.

3. $\frac{a+b+c}{3}$ 4. $v = \frac{s}{t} = \frac{s_1 + s_2}{t_1 + t_2} = \frac{v_1 t_1 + v_2 t_2}{t_1 + t_2}$.

11

1. It is convenient to take one mile per hour as the unit of velocity in this example.

Velocity up stream $=8-2=6$ miles an hour.

" down " $=8+2=10$ " "

2. Take one foot per *minute* as the unity of velocity.

$$v_1 = \text{velocity of sledge} = \frac{12 \times 1760 \times 3}{24 \times 60} = 44.$$

$$v_2 = \text{velocity of floe} = 45. \qquad \therefore v = 44 - 45 = -1.$$

Thus the sledge moves *southward* at the average rate of one foot per minute.

3. Let v_1 = velocity of steamer in miles per hour.

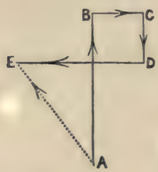
v_2 = " river " "

Then we obtain the simultaneous equations—

$$v_1 + v_2 = 15, \quad v_1 - v_2 = 11.$$

Solving these we find $v_1 = 13$, $v_2 = 2$.

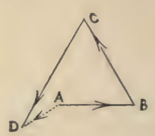
14



1. Drawing the velocities AB, BC, CD, DE we obtain the resultant AE $=5$. (See § 15).

The student will observe that two of the velocities intersect, hence the figure is not strictly speaking a polygon. Nevertheless the proof in § 14 includes such cases.

2. Drawing the velocities AB, BC, CD, we obtain the resultant AD. It is a little greater than $1\frac{2}{3}$. The advanced student will be able to prove geometrically that AD $=\sqrt{3}$, and the angle DAB $=150°$. (See § 17).



15

1. $v^2 = v_1{}^2 + v_2{}^2 = 169. \qquad \therefore v = 13.$

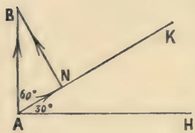
2. $v^2 = (10{\cdot}5)^2 + (3{\cdot}6)^2 = 123{\cdot}21. \qquad \therefore v = 11{\cdot}1.$

3. Since the stream is 972 feet broad, and the man swims across 3·6 feet in each second, he will arrive at the other side in t seconds if $t = \dfrac{972}{3{\cdot}6} = 270$, and during that time he will have been carried down $270 \times 10{\cdot}5 = 2835$ feet.

17

1. In figure 1, let AB = 7, BC = 8. Then BE = 4.
$\therefore AC^2 = 49 + 64 + 2 \times 4 \times 7 = 169$. $\therefore AC = 13$.
2. In figure 2, let AB = 5, BC = 8. Then BE = 4.
$AC^2 = 25 + 64 - 2 \times 4 \times 5 = 49$. $\therefore AC = 7$.
3. (1) $\sqrt{2+\sqrt{3}} = \sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}}$, (2) $\sqrt{2+\sqrt{2}}$, (3) $\sqrt{2-\sqrt{3}} = \sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}}$.

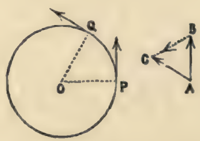
18



1. AN is the component required, and is equal to 2, since BAN is the half of an equilateral triangle.

2. Taking the figure in § 18, let the angle CAB = 45°. Then $AC^2 = AB^2 + BC^2 = 2\,AB^2$.
$\therefore AB = \frac{1}{\sqrt{2}} AC$, and if $AC = 3\sqrt{2}$, AB = 3.

20



1. From any point A draw AB, AC to represent the velocities at P and Q, complete the Δ ABC. The third side BC is the change in velocity, and is equal to *one*. Its direction makes an angle of 120° with that of the first velocity.

2. By a similar process we find the change of velocity $= \sqrt{2}$ in a direction at an angle of 135° to the first velocity.

22

1. $v = 13$, $s = 120$. 2. $t = 5$, $v = 50$. 3. $s = 2093$, $v = 370{\cdot}3$.
4. $v^2 = 2gs = 196$, $\therefore v = 14$ metres per second.

23

1. (1) $34\frac{1}{2}$ feet. (2) 7. (3) In $5\frac{1}{3}$ seconds. (4) $42\frac{2}{3}$ feet. (5) $v = -8$, $s = 32$. (6) In $10\frac{2}{3}$ seconds, $v = -16$. (7) $v^2 = V^2 - 2as$, $\therefore v = \pm 14$.

2. (1) $v = V - gt = ng - ng = 0$.

(2) $v^2 = V^2 - 2gs$, $\therefore$ when $v = 0$, $s = \frac{V^2}{2g} = \frac{1}{2}n^2 g$.

(3) Putting $t = 2n$, $s = Vt - \frac{1}{2}gt^2 = 2n^2 g - 2n^2 g = 0$.

(4) $v^2 = V^2 - 2gs$, $\therefore$ when $s = 0$, $v = V$.

General Examples in Kinematics

1. (1) $17\frac{3}{5}$. (2) $152\frac{7}{9}$. (3) $\frac{1}{1120}$. (4) $\frac{5}{12}$. (5) ·0041. (6) ·328.
(7) 50. (8) $\frac{1}{96}$. (9) 976,800,000. (10) 1.

2. (1) 15. (2) $81\frac{9}{11}$. (3) 765. (4) 915. (5) 68,925.

3. (1) 5. (2) $27\frac{3}{11}$. (3) 255. (4) 305. (5) 22,975.

4. (1) 2·2. (2) 12. (3) 112·2. (4) 134·2. (5) 10,109.

5. (1) 1. (2) $\frac{11}{60}$. (3) $\frac{1}{51}$. (4) $\frac{1}{61}$. (5) $\frac{1}{4595}$.

6. (1) $\frac{a}{b}$. (2) $\frac{2}{3}$. (3) $\frac{a}{20b}$. (4) $\frac{\cdot 328}{a+b}$. (5) $\frac{a^2b}{ab^2}=\frac{a}{b}$.
(6) $\frac{a^2-b^2}{a+b}=a-b$.

7. (1) 2π. (2) $\frac{2\pi a}{v}$. (3) 1. (4) $4\frac{2}{3}$.

8. (1) $4\frac{8}{9}$. (2) $3\frac{1}{3}$. (3) 149 approx. (4) 5·364 approx.

9. (1) $5\frac{5}{21}$. (2) $3\frac{4}{7}$.

10. One-tenth of a second. 11. 35 miles.

12. (1) 27 forwards. (2) 23 backwards. (3) 2 forwards. (4) 6 backwards. (5) $2b$ forwards. (6) $6x^2$ backwards.

13. $1\frac{53}{57}$ hour = 1 hour 55 min. $47\frac{7}{19}$ sec.

14. $9\frac{9}{22}$ miles = 9 miles 2160 feet. 15. 125 : 88.

16. (1) 10. (2) $1\frac{1}{4}$. (3) 17. (4) $5\frac{1}{12}x$. (5) 29. (6) Solving the simultaneous equations, we get $v_1=12$, $v_2=5$. Hence $v=13$. (7) $v_1=\frac{1}{2}(a^2-b^2)$, $v_2=ab$; $v=\frac{1}{2}(a^2+b^2)$.

17. The answers found by measurement are approximately :—
(1) 6·75. (2) 6·33. (3) 4·33. (4) 4·5. (5) 7·12.

18. (1) 19. (2) $1\frac{6}{7}$. (3) $3x^2+3x+1$.

19. (1) 19. (2) $9\frac{1}{4}$. (3) x^2+x+1.

20. (1) 1. (2) 7. 21. (1) 1. (2) $\sqrt{8}=2\sqrt{2}$. (3) $\sqrt{149}$.

22. 1300. 23. $17\frac{7}{8}$ feet per second. 24. 29 miles.

25. This is the converse of § 16. 26. 120°. 27. 13. 28. 5.

29. (1) 1 north. (2) 9 south. (3) 1 north. (4) $\sqrt{2}$ south-west.

30. The third side of a triangle. 31. (1) 10. (2) 20. (3) 48.

32. $\sqrt{27}=3\sqrt{3}$. 33. (1) 10. (2) 30 feet. (3) 2.

34. (1) 19. (2) $11\frac{1}{2}$. (3) 115 feet.

35. (1) 16. (2) 48. (3) 176. (4) 484. (5) 24.

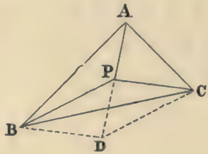
36. (1) $6\frac{1}{2}g$. (2) $18g$. 37. (1) $5g=161$. (2) $ng=32{\cdot}2n$. (3) $\frac{1}{2}n^2g=16{\cdot}1n^2$. (4) 4 seconds. (5) 6 seconds.

38. (1) 8. (2) $18\frac{1}{4}$. (3) $129\frac{1}{2}$. (4) In $9\frac{2}{3}$ seconds. (5) $140\frac{1}{6}$ feet.

39. (1) After $n-r$ seconds $v=V-gt=ng-(n-r)g=rg$.
" $n+r$ " $v=V-gt=ng-(n+r)g=-rg$.

(2) $s=Vt-\frac{1}{2}gt^2$, which reduces to $\frac{1}{2}(n^2-r^2)g$ whether we substitute for t the value of $n-r$ or $n+r$.

40. Let ABC be a triangle and P the point. Join PA, PB, PC. Complete the parallelogram BPCD. Then AP produced must be the diagonal of this parallelogram, and must therefore bisect BC. Similarly BP must bisect AC.



∴ P is the intersection of the lines drawn from the angular points to bisect the opposite sides.

41. Twice that represented by the fourth side of the square.

42. 19200.

43. $\sqrt{3}\cdot v$ forming an angle of 150° with first velocity.

44. $2\frac{1}{2}$. 45. $3\frac{1}{3}$ miles an hour. 46. 18 yards per minute.

47. Acceleration $=3$, average velocity $=13\frac{1}{2}$, space $=67\frac{1}{2}$ feet.

48. 80 miles an hour.

49. At an angle of 30° with the direction of the train. Initial velocity 33·8 approximately.

50. $18\frac{6}{43}$ miles per hour. 51. 200 feet. 52. 297,600,000.

53. (1) That the rate of change of velocity is such that if it were to remain uniform for one second, the velocity would be increased during that second by 32 feet per second. (2) 38,400 yards per minute per minute.

CHAPTER III—KINETICS

Examples for Exercise

30

1. $\frac{4\times 1000}{100}=40$ feet per second.

2. The whole momentum is 72. ∴ the momentum of each part is 24.
∴ the velocities are 24, $\frac{24}{2}$, and $\frac{24}{3}$, or 24, 12, and 8.

33

Let $v=$ velocity generated in one second in either mass.

Then $f_2=12v=8$. ∴ $v=\frac{8}{12}=\frac{2}{3}$, and $f'=9v=6$.

34

1. $f=k\,\dfrac{mm}{400}$, $\quad f'=k\,\dfrac{mm'}{5280\times 5280}$.

$$\therefore\ f'=\frac{400}{5280\times 5280}f=\frac{1}{69{,}696}f.$$

2. Let m and m' represent the masses of the Sun and Earth, and let d be the distance of the earth from the sun.

Then the mutual attraction between sun and earth is $f=k\,\dfrac{mm'}{d'}$,

and that between the Sun and Jupiter is $f'=k\,\dfrac{m\times 340m'}{(5d)^2}$,

$$\therefore\ f'=\frac{340}{25}f=13{\cdot}6f.$$

3. Let d be the radius of the earth. Then $11{\cdot}4d$ is the radius of Jupiter. Also let $m=$ mass of earth, and $m'=$ mass of man.

Then, on the earth, $w=k\,\dfrac{mm'}{d^2}$. On Jupiter, $w'=k\,\dfrac{340mm'}{(11{\cdot}4d)^2}$.

$$\therefore\ w'=\frac{340}{129{\cdot}96}w.$$

Thus his *weight* as ascertained with a spring balance on the surface of Jupiter is $\dfrac{340\times 12}{129{\cdot}96}$ stones, or more than the weight of 31 stones on the surface of the earth.

35

The quantity weighed is too little, the deficiency being $\dfrac{1}{161}$ of a lb

36

1. $mv=mgt=2318{\cdot}4$. $\quad s=\frac{1}{2}gt^2=5216{\cdot}4$.
2. $s=Vt+\frac{1}{2}gt^2$. $\quad\therefore\ 1000=2V+\frac{1}{2}\times 32{\cdot}2\times 4=2V+64{\cdot}4$.

$\therefore\ V=467{\cdot}8$. $\quad\therefore$ the initial momentum $mV=\frac{1}{8}\times 467{\cdot}8=58{\cdot}5$.[1]

3. To find the time of falling—

$s=\frac{1}{2}gt^2$. $\quad\therefore\ t^2=\dfrac{2s}{g}=\dfrac{1118}{32{\cdot}2}=34{\cdot}7$. $\quad\therefore\ t=\sqrt{34{\cdot}7}=5{\cdot}9$ seconds.

---

[1] Since the value of g varies at different heights in any one place, the value 32·2 is only approximate; and in all calculations founded on that value figures beyond the first decimal place should be omitted. In the present case the result of dividing gives 58·475, and as this number is nearer in value to 58·5 than to 58·4, the former is the approximate answer. The answers to subsequent questions are formed on this principle.

To find the time the sound of the splash takes to reach the ear, we employ formula (3) of uniform velocity (*see* § 5) $t = \frac{s}{v} = \frac{559}{1118} = \cdot 5$.

$$\therefore \text{ time required} = 5\cdot 9 + \cdot 5 = 6\cdot 4 \text{ seconds.}$$

37

From (20) $s = Vt - \frac{1}{2}gt^2$. But when $t = 4$, $s = 0$.

$$\therefore\ 4V - 8g = 0. \qquad \therefore\ V = 2g = 64\cdot 4.$$

Substituting this value for V, when $t = 1$, $s = 48\cdot 3$; $t = 2$, $s = 64\cdot 4$; $t = 3$, $s = 48\cdot 3$.

39

1. By § 13 the resultant acceleration is zero. Therefore the forces are in equilibrium.

2. By § 17—

$$f^2 = (3\tfrac{1}{2})^2 + 4^2 + 2 \times 3\tfrac{1}{2} \times 2 = 12\tfrac{1}{4} + 16 + 14 = 42\tfrac{1}{4} = \frac{169}{4}.$$

$$\therefore\ f = \frac{13}{2} = 6\tfrac{1}{2}.$$

3. Here the angle between the two directions is 60°. The figure is the same as that in the answer to Example 1, § 18.

AB = 12, $\therefore$ AN, the resolved force, is 6.

41

Here $M = 10$, $V = 14$, $m = \frac{1}{8}$. $\therefore\ v = 1120$.

42

1. Here a weight of 196 lbs. has to be raised 3000 feet.

$$\therefore\ \text{work} = 196 \times 3000 = 588{,}000 \text{ foot-pounds.}$$

2. In five minutes the work done is $550 \times 300 = 165{,}000$ foot-pounds. Also one ton = 2240 pounds.

$$\therefore \text{ the height} = \frac{165{,}000}{2240} = 73\tfrac{37}{56} \text{ feet.}$$

3. Turbine does 5500 foot-pounds in each second, and this is $\frac{55}{100}$ of the work done by gravity. $\therefore$ work done by gravity = 10,000 foot-pounds a second; or 100 lbs. of water falls 100 feet in each second.

General Examples in Kinetics

1. (1) 178; (2) 0. 2. 1500. 3. 5 seconds.
4. 672,000 foot-pounds. 5. No work is done against gravity.
6. $f = ma$, but $f = \frac{1}{16}g$, $m = 2$, $\therefore\ a = \frac{1}{32}g = 1$. $\therefore\ v = at = 1$.

7. $s=402 \cdot 5$, $v=161$. 8. 742·4 feet. 9. 2·5 seconds. 10. 197·2 feet.

11. 680·2 feet. 12. 35 seconds, 19722·5 feet.

13. $f=ma$, but $f=2g$, $m=10$. $\therefore a=\frac{1}{5}g$, $v=at=\frac{1}{5}g=6 \cdot 4$.

14. 16·1. 15. 402·5 feet.

16. 161 ; $402\frac{1}{2}$ feet. (Compare Example 7.)

17. After 2 seconds at a height of 95·6 feet.
After 4 ,, ,, 62·4 ,,
After 6 ,, 99·6 feet below the point of projection.

18. 24·6. 19. (1) 6·2 seconds. (2) Since the initial momentum is the same as in the last case, the initial velocity will be 200 feet. Answer, 12·4 seconds.

20. 983·9 feet.

21. Complete the parallelogram ABDC, and produce the diagonal DA its own length (backwards) to E. AE is the line required, and makes an angle of 155° with AB.

22. The resultant force is 5. 23. 5.

24. $10\sqrt{2}$ or 14·14 approximately. Direction NE.

25. 13 lbs. 26. 7·8125 or $7\frac{13}{16}$. 27. 5.

28. Since action and reaction are equal and opposite, the momentum given to one part forwards is equal to that given to the other part backwards. Thus the whole momentum is unchanged, and is equal to that of the two parts.

$$\therefore mv = 12 \times 700 + 20 \times 380 = 16{,}000.$$

29. The whole momentum is unchanged, and since one part is reduced to rest, the other part moves on with the original momentum of the shell $= 20 \times 600 = 12{,}000$.

30. The two equal parts receive equal and opposite changes of momentum, and therefore of velocity. The change of velocity given to the first part is 40, therefore that given to the second is -40. Its new velocity is therefore $80-40=40$.

31. 72 lbs. 32. 43 lbs.

33. The formula $v^2=2gs$ gives us—

$$v^2 = 2 \times 32 \cdot 2 \times 3\tfrac{1}{23}. \qquad \therefore v = 14.$$

34. (1) 73. (2) $\sqrt{(a^2+b^2)}$. (3) a^2+b^2.

35. Substituting in the third of equations (12), § 23, we get

$$0 = 100 - 20a. \qquad \therefore a = 5. \qquad \therefore \text{by (14), } f = 5m.$$

36. It follows from the triangle of forces that the resultant of two forces is greatest when they act in the same direction along the same straight line, and least when in opposite directions.

$$\therefore f_1 + f_2 = 12,\ f_1 - f_2 = 6, \text{ from which we get } f_1 = 9,\ f_2 = 3.$$

37. Forming the triangle of forces, we find that it is half an equilateral triangle, and the angle at which P and Q act is its exterior angle$=150°$.

38. This follows from the fact that the angle in a semicircle is a right angle.

39. $s=\frac{1}{64}$ of a foot. $t=\frac{1}{32}$ of a second.

40. $V=160$. When $t=5$, $v=0$ and $s=400$.

When $t=10$, $v=-160$.

Average velocity during first 5 seconds 80.

" " " 10 " 0.

41. 60. 42. $83\frac{1}{3}$.

43. That the driving power is just sufficient to overcome the resistance, and thus causes no acceleration.

43. 1 foot-pound. (The mass of 3 lbs. moves 1 foot in the direction opposite to that of gravity.)

45. $a=\frac{1}{16}g$. $\therefore v=at=\frac{5}{8}g$. $s=\frac{1}{2}at^2=3\frac{1}{8}g$.

(If $g=32$, $v=20$ and $s=100$.)

46. 2 AC.

47. Consider the vertical velocity acquired by the fall. When $s=64$, $t=2$ and $v=64$. If we combine this with the horizontal velocity of 32, which remains unaltered, we find that the resultant velocity makes an angle with the horizontal whose tangent is 2.

48. He must reduce his readings by $\frac{16}{3225}$.

49. $100\times 36\times 2\times 153\times 17\frac{1}{2}$ foot-pounds.

50. 434,000 foot-pounds.

51. For the falling stone $s=\frac{1}{2}gt^2=250$.

For the stone thrown up—

$s=Vt-\frac{1}{2}gt^2$, or $Vt=500$; but $t=\sqrt{\frac{500}{g}}$, $\therefore V=\sqrt{500g}$.

52. $v=at=100$. $\therefore a=20$. $f=ma=200$, while weight of 1 lb.$=32{\cdot}2$. Acceleration given to one ton$=\frac{5}{56}$.

53. The given force must be greater than half the first force. Then we can erect on the line representing the first force an isosceles triangle whose sides are equal and parallel to the components required.

54. Each force$=9$ lbs. 55. $\sqrt{3}$ lbs. and 1 lb.

56. (1) 64. (2) 96. (3) 32.

CHAPTER IV.—STATICS

Examples for Exercise

45

1. The triangle of forces is equilateral. Its exterior angles are each 120°.

2. The tensions of the strings and the weight form 3 forces in equilibrium. Their directions are at 120° to each other. Therefore the three forces are equal. (Compare last example.)

3. No two sides of a triangle can together be less than the third.

49

1. 12 lbs., 10 inches from the greater weight.

2. 12 lbs., 5 inches from the greater force.

3. If they are not parallel two of them meet in a point, and the third must, to be in equilibrium with their resultant, pass through that point.

50

1. A force 3, 14 inches beyond the greater force.

2. A force 3, 8 inches beyond the greater force.

53

Let ABCD be the square, the weights of 1 and 3 lbs. are equivalent to a weight of 4 lbs. acting at E in AB, $EB = \frac{1}{4}$ AB, the weights of 9 and 3 lbs. are equivalent to a weight of 12 lbs. at a point F in CD, $CF = \frac{1}{4}$ CD. Join EF, the centre of gravity of 4 lbs. at E and 12 lbs. at F is G in EF, $FG = \frac{1}{4}$ EF. By combining the weights in other ways we may show G to be on the diagonal.

55

The vertical line drawn through the centre of gravity must fall within the base.

57.

1. Moment of force at $A = 0$, since perpendicular $= 0$.
   " " $B = 2 \times \frac{1}{3}$ of $10 = 6\frac{2}{3}$.
   " " $C = 3 \times \frac{2}{3}$ of $10 = 20$.
   " " $D = 4 \times 10 = 40$.

2. Let AB, AD be the forces, and let E be any point in their resultant. Then the moment of AB about E is twice the triangle ABE, and is therefore equal to the parallelogram on AB, as base, formed by drawing a parallel to AB through E. Similarly the

moment of AD about E is equal to the parallelogram on AD as base, formed by drawing a parallel to AD through E. These two parallelograms are easily proved to be equal by the help of Euclid, I. 43.

General Examples in Statics

1. (1) R=17; (2) Q=8; (3) R=10; (4) Q=7.

2. In the triangle of forces the angle opposite P is the greatest, and that opposite R is the least. But the angles mentioned in the question are the exterior angles of the triangle.

3. An equal force, making angles of 120° with each of the given forces. (See Examples, § 45.)

4. Six inches from the greater weight.

5. (1) R=9 lbs., AC=8 in.; (2) R=11 lbs., AC=1 ft.; (3) R=20 lbs., AC=2 in.

6. (1) R=5 lbs., AC=2 ft.; (2) R=2 lbs., AC=3 ft.; (3) R=10 lbs., AC=3 in.

7. A bears 120 lbs., B bears 80 lbs. 8. Half an inch.

9. One third along the line drawn from the centre of the square to bisect the middle rod.

10. Ten inches from the extremity on which the weight is placed.

11. Three inches from A. 12. Nine inches from A.

13. The middle point of the axis of the hat.

14. Since the diagonals bisect each other, each diagonal contains the centres of gravity of the two triangles into which the other divides the parallelogram. Therefore the centre of gravity of the whole parallelogram, lies in each diagonal. It must therefore be their point of intersection.

15. In the wire parallelogram the centre of gravity of either pair of parallel sides is the point O found in § 53 (2).

16. Each has been shown to pass through the centre of gravity.

17. (1) In a quadrilateral, the centre of gravity of the whole is in the line joining the centres of gravity of the two triangles into which it is divided. But this may be done in two ways. Thus we obtain two lines, each of which contains the centre of gravity. Therefore the centre of gravity is at their intersection.

(2) Any rectilineal figure may be divided into triangles. Find the centre of gravity of each triangle, and conceive a heavy particle equal to its mass placed at that point. Then find the centre of gravity of the system of heavy particles. § 53 (5).

18. The centre of gravity of the loaded table must be vertically over a point within the area formed by joining the extremities of the legs. (This area must have no re-entrant angles.)

19. The first moment is represented by twice the area of the triangle ABC, the second by twice ABD.

20. Each of these moments is represented by the area of the square.

21. (1) From M draw MC $\perp$ R and meeting its line of action at C, and produce it to meet P's line of action at A. Then by § 48, P may be supposed to act at A and R at C. $\therefore$ by (25) $P \times AC = Q \times MC$. Add to each side $P \times MC$ and we get $P \times AM = R \times MC$.

(2) Let O be a point in LM produced. From O draw a line $\perp$ the line of action of the forces, and let it meet P in A, Q in B, R in C. Then $(P+Q) \times OB = R \times OB$, but by last exercise $P \times AB = R \times BC$. Adding these two equations, we get $P \times OA + Q \times OB = R \times OC$.

22. Take the figure of § 39, and let O be a point in the plane, and first let O be above AB and to the left of AD.

Then $OAC + ABC = OAB + OBC$.

But $OBC = OAD + ABC$ (see § 53, foot-note).

$$\therefore OAC = OAB + OAD.$$

$$\therefore \text{moment of } AC = \text{moment of } AB + \text{moment of } AD.$$

In a similar way the proposition is proved for any other position of O. It is important to observe that when the forces tend to turn the body in opposite directions about O, the algebraical sum of the moments is the difference of the figures which represent the moments.

23. The diameter drawn through *any* angular point, bisects the figure, and therefore by symmetry contains the centre of gravity.

24. The diameter drawn through any bead must pass through the centre of gravity, which must therefore be the centre of the circle.

25. (1) Stable; (2) stable in one direction, unstable in another; (3) unstable; (4) neutral; (5) stable; (6) stable; (7) neutral in one direction, unstable in another.

26. Since the plane is (perfectly) smooth the rod will slide along it until its centre of gravity is as low as possible. The string will then be vertical. The weight is supported by two parallel forces, the resistance of the table, and the tension of the string. These must be each equal to $\frac{1}{2}$ W.

27. Let ABCD be the square, O its centre of gravity, G the centre of gravity of remainder. Then if the corner A is cut off, $OG = \frac{2}{21}$ OC.

28. Bisect AB in E, AD in F, and let FE produced meet CB in G. $GB = \frac{1}{2}$ BC.

The given system of forces is equivalent to forces P, 2P, and 3P along CB, CD, and AD respectively. Let CD represent a force of 2P, then the resultant of the first two forces is represented by CF, therefore the resultant of the three forces passes through F.

But GC and CF represent the forces at F.

$\therefore$ GF represents their resultant $= \text{CD}\sqrt{2}$. $\therefore$ resultant $= 2\sqrt{2}\cdot\text{P}$.

29. They must form an angle of 90°.

30. Substitute for AD, AB and BD. Similarly for the others. Thus the three forces are equivalent to three forces parallel and proportional to the three sides of the triangle ABC.

31. Instead of 5 and 2 we can take 3 and 0, with 4 at the centre, instead of 4 and 7 we can take 0 and 3, with 8 at the centre, and instead of 6 and 3 we can take 3 and 0, with 6 at the centre. Thus the system is equivalent to 3 equal weights of 3 at the alternate corners of the hexagon, and 18 at the centre. Therefore the centre of the hexagon is the centre of gravity of the whole.

32. Substitute for the sides heavy particles, at their middle points, of weights proportional to the sides. Join the middle points. The bisectors of the angles of the triangle thus formed divide the opposite sides inversely as the masses of the heavy particles. The point where these bisectors meet is therefore the centre of gravity of the heavy particles.

33. The tension of the horizontal string $= \frac{10}{\sqrt{3}}$lbs., that of the other $\frac{20}{\sqrt{3}}$ lbs.

34. This is the case of § 45, Example 3. AC represents the weight, CB the force of the current, BA the tension of the string.

35. 6 ft. 10 in. from the point.

36. Let ABCD be the square, and let AE $= \frac{5}{7}$ AD, BF $= \frac{4}{7}$ BC. Bisect EF in G. G is the centre of inertia in whatever way the square be held, as the proof in § 49 applies at whatever angle the forces act.

37. Three feet from the end. Each of the two men holds one end of a bar which is passed under the beam.

38. As $\sqrt{3} : 1 : 2$.

39. Produce the line joining the two centres $\frac{45}{182}$ in.

40. $3\sqrt{2}$, 3. 41. 70 lbs., 30 lbs. 42. $5\frac{1}{4}$ inches.

43. 37·8 and 28·35 lbs. approximately. 44. 3·06 lbs. approximately.

45. (1) The resultant of the forces acting on the body must be a

single force passing through the fixed point, or the algebraical sum of the moments of all the forces acting on the body, taken with respect to the fixed point, must be zero.

(2) $5\sqrt{2}$ lbs.

46. The stresses are as follows: on AB, 20 lbs., on AC, 15 lbs., on BC, 12 lbs. The reactions are: at B, 16 lbs., at C, 9 lbs.

47. (1) 20 lbs. (2) The tension is diminished, but it will always exceed 16 lbs.

48. On A, 20 lbs., on B and C, $\frac{20}{\sqrt{3}}$ lbs.

CHAPTER V.—MACHINES

Examples for Exercise

59

1. If the 3 lb. mass is at A, $AC = \frac{3}{4}AB$. 2. 30 lbs.
3. $AC = -4$ in., or $1\frac{1}{3}$ in.

62

$W^2 = PQ = 96. \quad \therefore W = \sqrt{96} = 9$ lbs. $1\frac{1}{2}$ oz. *nearly*,

$A : B = 8 : \sqrt{96} = \sqrt{2} : \sqrt{3}.$

64

Here P is 600 lbs., AC = 6 BC = 1. ∴ W = 3600 lbs.

66

$$W = 2^n P = 2^5 \times 3 = 32 \times 3 = 96 \text{ lbs.}$$

67

The diagram resembles the first figure of § 67, but has 3 pulleys in each block.

68

$$W = 62 = (2^5 - 1)P = 31P. \quad \therefore P = 2 \text{ lbs.}$$

70

1. To find the height of the plane we have

$$BC = \sqrt{AB^2 - AC^2} = 5 \text{ ft.}$$

If, therefore, 5 ft. represents 5 lbs., 13 feet represents 13 lbs.

2. Length of plane $= \sqrt{12^2 + 35^2} = 37$.

Thus 37 represents 74 cwt.
12 „ 24 cwt. = 1 ton 4 cwt. = P.
35 „ 70 cwt. = 3 tons 10 cwt. = R.

71

Here the height is $\sqrt{25^2-24^2}=7$.

Thus 7 represents 14 lbs. $\therefore$ 24 represents 48 lbs.

73

1. Since $F=\mu R$, we have $10=\mu 80$. $\therefore \mu=\frac{10}{80}=\frac{1}{8}$.

2. $\mu=\frac{3}{4}=\frac{BC}{AC}$. If then, BC=3, AC=4, and $AB=\sqrt{3^2+4^2}=5$.

But AB is 10 inches. $\therefore$ BC is 6 inches.

74

(1) In the lever the points A and B describe arcs of circles, whose radii are AC and BC. Since the arcs are proportional to their radii, if we take kAC to represent the arc described by A, kBC will be the arc described by B, and since by (23)—

$$P \times AC = W \times BC, \quad \therefore P \times kAC = W \times kBC,$$

or the work done by P is equal to that done against W.

(2) If, in the figure of § 66, P be raised 1 foot, the strings DA and AP will each be shortened half a foot, and A will be raised half a foot. Similarly if A be raised half a foot, B will be raised $\frac{1}{4}$ of a foot, and C $\frac{1}{8}$ of a foot.

Thus the path of P is 8 times that of W, or if k represents P's path, 2^3k represents W's.

Now $$W=2^3P, \quad \therefore kW=2^3kP,$$

or the work done by P is equal to that done against W.

The theorem is proved similarly for any number of pulleys.

(3) This follows directly from (1).

General Examples in Machines

The Lever

1. (1) AC=1 ft. $1\frac{1}{2}$ in. (2) AC=$4\frac{4}{5}$ in. (3) AC=2 in.

2. $2g$. 3. 6 oz.

4. 1 ft. $2\frac{1}{2}$ in. from the fulcrum. 5. 3 lbs.

6. (1) A 120 lbs., B 80 lbs. (2) A 275 lbs., B 75 lbs.
(3) A 180 lbs., B 120 lbs.

7. $3\frac{3}{4}$ lbs. 8. 6 feet.

9. If the force of 5 lbs. act at A, and that of 6 lbs. at B, then the fulcrum is at C.

(1) AC=3 in. C is between A and B.

(2) AC=33 in. B is between A and C.

10. 40 lbs. and 35 lbs. Regard each peg in turn as the fulcrum.

11. 15 lbs. 12. 7 lbs. 13. $\frac{1}{2}$ a lb.

14. Let x be the weight of the rod, y the distance of its centre of gravity from the first position of the fulcrum. Then we have—

$$(1)\ 3 \times 4 = xy, \quad (2)\ 5 \times 3 = x(y+1).$$

Subtracting the first equation from the second, we get $x = 3$ lbs., and therefore $y = 4$ ft. Thus the rod weighs 3 lbs., and is 16 feet long.

The Balance

15. 5 lbs. 16. 5 lbs. 1 oz. and 6 lbs. 4 oz.

17. 13 to 12. 18. (1) 5 lbs. 1 oz. (2) 9 to 8.

The Wheel and Axle

19. 15 lbs. 20. 50 lbs. 21. 6480 lbs.

Pulleys

22. The first system with 2 pulleys, and the second system with 4 pulleys.

23. The second system with 7 pulleys, and the third system with 3 pulleys. Observe that in the second system of pulleys the number of sheaves in the fixed block is either equal to that in the moveable block or exceeds it by one.

24. 2 lbs. 25. 4 pulleys in each block.

26. See figure of § 66. Tension of AD is 10 lbs. ∴ weight of A + tension of ABE = 20 lbs. ∴ tension of ABE = 19 lbs. ∴ weight of B + tension of BCE = 38 lbs. ∴ tension of BCE = 37 lbs. ∴ weight of C + W = 74 lbs. ∴ W = 73 lbs.

27. Weight of block + W = 60 lbs. ∴ W = 57 lbs.

28. See figure of § 68. Tension of AD = 10 lbs. ∴ tension of ABE = 20 lbs. + weight of A = 21 lbs. ∴ tension of BCF = 42 lbs. + weight of B = 43 lbs. ∴ tension of CK = 86 lbs. + weight of C = 87 lbs. This tension is equal to W + P + the weight of the 3 pulleys. ∴ W = 87 − 10 − 3 = 74 lbs. (or W may be obtained by adding the three tensions 10 + 21 + 43 = 74 lbs.).

The Inclined Plane

29. Here the length is to the height as 5 to 3.

∴ the weight is to the power as 5 to 3.

But the weight is 10 lbs.

∴ (1) the force is 6 lbs. (2) the pressure is 8 lbs.

30. Here the base is 63. If 16 represents 80 lbs., 63 represents 315 lbs.

31. $R^2 = P^2 + W^2 = 35^2 + 120^2 = 125^2$. $\therefore R = 125$ lbs.

32. 10 lbs. 33. 30° to the horizon.

34. (1) 25 lbs. (2) 12 lbs. (3) 16 lbs.

MISCELLANEOUS

35. The power is $\frac{1}{6}$ of the weight.

36. Let h be the common height, W and W′ the weights, P the tension of the cord. Then P is a force parallel to the plane in each case. If therefore h represents P, 5 represents W, and in the same way 6 represents W′. Thus W is to W′ as 5 to 6.

37. The power moves through an arc of a circle four times that travelled by the weight in the same time. $\therefore$ its velocity is 28.

38. 6 lbs. 39. $\frac{W}{W'} = \sqrt{2}$.

40. The greater weight will prevail. $7\frac{1}{2}$ lbs. must be added to the smaller.

41. 20 lbs., 12 feet. 42. 444 lbs.

43. (1) 2211 lbs. (2) $1105\frac{1}{2}$ lbs. In the latter case the perpendicular from the fulcrum on the direction of the man's weight is one-half its former value, and the moment of the force is reduced proportionately.

44. 450 lbs. 45. 70 lbs.

46. While the end of the lever describes a circle whose circumference is approximately 12 ft. 7 in. the end of the screw is moved $\frac{1}{2}$ in. The former path is 302 times the latter. Therefore 302 represents the gain in power.

47. 100 inches. 48. (1) 60°. (2) $4\sqrt{3}$. 49. 15 lbs.

CHAPTER VI.—HYDROSTATICS

EXAMPLES FOR EXERCISE

77

1. $\frac{219}{3} = 73$. 2. $\frac{812}{1728} = \frac{203}{432}$ of a lb.

78

1. 648 lbs. 2. $4\frac{1}{2}$ lbs.

81

1. $p = 10 \times 144 = 1440$ lbs.

$p + xq = 20 \times 144 = 2880$ lbs. $\therefore xq = 1440$ lbs.

But $x = 4$, $\therefore q = 360$ lbs.

2. $xq = 15 \times 144$. But $q = 62\frac{1}{2}$. $\therefore x = \frac{15 \times 144}{62\frac{1}{2}} = 34{\cdot}56$ ft.

This is the height of the water barometer for a pressure of 15 lbs. on the square inch.

3. Here the additional pressure due to 5 ft. is $2\frac{1}{2}$ lbs.

" " 1200 ft. is 600 lbs.

$\therefore$ the whole pressure at 400 yds. is $2 + 600 = 602$ lbs.

86

1. $\frac{540}{62\frac{1}{2}} = 8{\cdot}64$.

2. (1) A ton of water occupies $\frac{20 \times 112}{62\frac{1}{2}} = 35{\cdot}84$ cubic feet.

$\therefore$ a ton of ice occupies $\frac{10 \times 35{\cdot}84}{9} = 39{\cdot}82$ cubic feet.

(2) A cubic yard of water weighs $27 \times 62\frac{1}{2} = 1687\frac{1}{2}$ lbs.

$\therefore$ a cubic yard of ice weighs $\frac{9 \times 1687\frac{1}{2}}{10} = 1518\frac{3}{4}$ lbs.

3. The weight of 16 cubic centimetres of water is 16 grammes.

$$\therefore s = \frac{108{\cdot}8}{16} = 6{\cdot}8.$$

87

1. $s = \frac{14}{4} = 3\frac{1}{2}$. 2. For A, $s = \frac{10}{5} = 2$.

A and B displace 13 lbs. of water.

A alone displaces 5 lbs. of water. $\therefore$ B displaces 8 lbs. of water.

$$\therefore \text{for B, } s = \frac{6}{8} = \frac{3}{4}.$$

3. The water displaced by the cork and silver weighs 6 lbs.

" " cork alone " 4 lbs.

$\therefore$ " " silver " 2 lbs.

$$\therefore s = \frac{21}{2} = 10\frac{1}{2}.$$

88

1. For the body, $s = 5$. For the liquid, $s = \frac{5-3}{5-4} = 2$.

2. The weight of the liquid displaced by the body is 3 lbs. The same bulk of water would weigh 6 lbs.

$$\therefore s = \frac{4}{6} = \frac{2}{3}.$$

3. $s=\frac{\frac{1}{2}}{500}=\frac{1}{1000}$.

4. Call the weight of a cubic inch of water 1.

Then the copper weighs 36, and the zinc weighs 42.

∴ each cubic inch of mixture weighs 7·8. ∴ $s=7{\cdot}8$.

89

1. The portion immersed is $1-\frac{3}{40}=\frac{37}{40}$. ∴ $s=\frac{37}{40}$.

2. The same as that of water = 1.

3. The water displaced by $\frac{2}{3}$ of the body weighs 6 lbs., ∴ the water displaced by the whole body weighs 9 lbs. A force of 3 lbs. downwards must therefore be applied to counterbalance the upward pressure of the water.

90

1. By Boyle's Law the density is proportional to the pressure.

But when the pressure is 15 lbs., 570 measures the density.

∴ " " 1 lb., 38 " "

∴ " " 16 lbs., 608 " "

Thus a cubic foot of air weighs 608 grains.

2. When the pressure is 1 the density is ·001.

∴ " " 1000 " 1.

At 100 feet there is an *additional* pressure of 3 atmospheres.

∴ at 33,300 feet there is an additional pressure of 999 atmospheres.

Or the whole pressure at 33,300 feet is 1000 atmospheres. (This result is only approximately true, as Boyle's law is only approximately true, the discrepancy increasing with the pressure.)

97

The areas of the pistons are proportional to the squares of their diameters, or are as 9 to 2500.

If, then, $P=72$, $W=\frac{2500\times 72}{9}=20{,}000$ lbs.

General Examples in Hydrostatics

1. Iron is 29 times as dense as cork.

2. (1) 15·4 lbs. (2) 51·9 lbs. (3) 6890 lbs.

3. (1) 2230·3 lbs. (2) 7222·5 lbs. (3) 4,457,160 lbs.

4. As 6 to 5.

5. (1) 1687½ lbs. (2) 21,127½ lbs. (3) Yes, for the additional pressure is the same outside as inside.

6. (1) $16\frac{1}{2}$ lbs. (2) 2592 lbs.
7. (1) 15·2 lbs. (2) 16·3 lbs. (3) 28·1 lbs. approximately.
8. (1) $19\frac{1}{3}$. (2) $3\frac{1}{2}$. (3) $\frac{18}{25}$ or ·72. 9. 1 lb. 4 oz.
10. (1) $11\frac{1}{3}$. (2) $8\frac{1}{3}$. (3) 9. (4) 1·86.
11. (1) ·86. (2) ·66. (3) ·92.
12. (1) 54 cubic inches. (2) 53 cubic inches. 13. 32 lbs.
14. (1) 1·85. (2) 1·23. (3) ·81. 15. $6\frac{1}{4}$.
16. 39 cubic centimetres. 17. A $\frac{1}{2}$, B $1\frac{1}{4}$.
18. (1) $22\frac{1}{2}$. (2) 9. (3) $14\frac{1}{3}$. (4) $8\frac{8}{9}$. 19. $1\frac{1}{4}$.
20. $\frac{29}{120}$. 21. 800. 22. 6·912. 23. 2·6. 24. 3 cwt.
25. (1) 5. (2) $\frac{4}{5}$. 26. (1) $2\frac{1}{2}$ lbs. (2) $13\frac{1}{2}$ lbs. (3) $15\frac{1}{2}$ lbs.
27. 33·3 feet. 28. $11\frac{1}{3}$. 29. $1\frac{3}{13}$. 30. $\frac{4}{5}$. 31. $\frac{3}{13}$.
32. 23·68 cubic inches. 33. 14 lbs. 6·1 oz.
34. As $\frac{9}{5}$ to $\frac{10}{6}$, or 27 to 25. 35. 10·53. 36. 10,250 cubic feet
37. 53 inches. 38. $\frac{1}{2}$ a cubic inch. 39. 4 tons.
40. (1) 2·92 cubic feet. (2) 75 ounces. 41. 271·32 inches.
42. It will be four times as great. 43. 8750.

44. Taking the surface pressure as 15 lbs. on the square inch, the depth required is, by Boyle's law, that sufficient to give an additional pressure of 5 lbs. Answer, $11\frac{1}{2}$ feet approximately.

45. $\frac{4}{5}$ of the first pressure.

46. In measuring the specific gravity of a gas, air at a given pressure and density is taken as the unit of specific gravity.

$$\therefore \text{weight of 30,000 cubic feet of gas} = 16{,}200 \text{ oz.}$$
$$\text{whole weight of balloon} = 34{,}120 \text{ oz.}$$
$$\text{weight of air displaced} = 36{,}000 \text{ oz.}$$
$$\therefore \text{force required} = \text{weight of 1880 oz.} = 117\tfrac{1}{2} \text{ lbs.}$$

47. $362\frac{4}{15}$ grains. 48. $\frac{5}{7}$ lbs.

49. (1) § 85.

(2) Weight to be raised = 318,000 lbs., and each barrel raises 1819 lbs. ∴ number required $= \frac{318{,}000}{1819} = 175$ nearly.

50. One-half. 51. 12·785 metres.
52. 7052·1 lbs. on the sq. in. + atmospheric pressure. 53. $\frac{7}{25}$.
54. $33\frac{3}{4}$ lbs. 55. (1) 240 lbs. (2) 80 foot-pounds.
56. (1) $46\frac{7}{8}$ lbs. (2) $140\frac{5}{8}$ lbs. (3) $62\frac{1}{2}$ foot-pounds.

APPENDIX

Additional Examples

Composition and Resolution

1. (1) $\sqrt{V^2+V'^2+2\,VV'\cos\theta}$. (2) Cos $\theta=-\frac{1}{2}$, $\therefore\ \theta=120°$.

2. (1) § 99. (2) As $1:2:\sqrt{3}$.

3. X is equal to the sum of the projections[1] of all the sides of the polygon on the axis of x, taken with the proper signs. Thus if the polygon be closed, the sum of the projections is zero, or X = 0. Similarly Y = 0.

[If the polygon be not closed, then the remaining side taken in the opposite direction is R, and if R make an angle θ with ox, we can prove that R cos θ = X, R sin θ = Y.

$$\therefore\ R=\sqrt{X^2+Y^2}.]$$

4. $g\sin\theta$, or $-g\sin\theta$.

Uniformly Accelerated Motion

5. Space described in $(m-1)$ seconds $=\frac{1}{2}at^2=\frac{1}{2}a\,(m^2-2m+1)$.

" " m " $=$ " $=\frac{1}{2}am^2$.

$\therefore$ " " mth second $=\frac{1}{2}a\ (2m-1)$.

Similarly " " $(m+n)$th " $=\frac{1}{2}a\ (2m+2n-1)$.

" " $(m-n)$th " $=\frac{1}{2}a\ (2m-2n-1)$.

But—

$\frac{1}{2}a(2m+2n-1)+\frac{1}{2}a(2m-2n-1)=\frac{1}{2}a(4m-2)=2\times\frac{1}{2}a(2m-1)$.

6. From § 70 we find that the acceleration in the direction of motion is $\frac{1}{3}g$. Substituting $\frac{1}{3}g$ for g in equations (19) we get—

$$t^2=\frac{2s}{\frac{1}{3}g}=\frac{6000}{g},\quad v^2=\frac{2}{3}gs=666\tfrac{2}{3}g.$$

Extracting the roots we find t and v.

7. $\sqrt{87^2+(13\times 32)^2}=425$.

8. (1) § 102. (2) Let T = the tension of the string. Then the acceleration of the mass of 5 lbs. is $\dfrac{5g-T}{5}$ while that of the mass of 3 lbs. is $\dfrac{T-3g}{3}$. But these accelerations are equal since the one weight rises at the same rate as the other falls. Therefore $T=3\frac{3}{4}g$,

---

[1] Let AB be any line in the plane xy. From A draw AM, and from B draw BN perpendicular to ox. MN is called the (orthogonal) projection of AB on ox.

and the common acceleration in $\frac{1}{4}g=8$, approximately. [The value of a may also be found by the method of § 103 (3).] After t seconds, $v=at$, $s=\frac{1}{2}at^2$. If $t=3$, $v=24$, $s=36$.

To find the velocity of the centre of gravity, suppose all the forces applied there. Then $f=8g-2T$. $\therefore a=\frac{8g-2T}{8}=2$, approximately.

$\therefore$ after three seconds the velocity of the centre of gravity is 6.

9. By § 103 (3) $a=\frac{1}{8}g$. After t seconds $v=at$, $s=\frac{1}{2}at^2$.

10. In § 103, $V=0$, $\therefore s=\frac{1}{2}g\sin\theta\cdot t^2$, but $s=2r\sin\theta$.

$\therefore t=2\sqrt{\frac{r}{g}}$. Thus t is the same for all values of θ.

11. As in § 103 (3) $a=\frac{M}{M+M'}g$, and $Ma=Mg-T$.

$$\therefore T=\frac{MM'}{M+M'}g.$$

12. Until M reaches the ground, acceleration $=\frac{1}{4}g=8$. Then, until M′ reaches the edge of the table, acceleration $=0$, $v=8$. Lastly, M′ falls 16 feet under gravity. Each part of the motion occupies one second.

Parabolic Motion

13. § 104.

14. (1) § 105. (2) Since we know the horizontal velocity we are able to determine the time spent in falling down any part of the parabola, and may therefore calculate g from the formula $s=\frac{1}{2}gt^2$, where s is the distance fallen vertically. 15. § 106.

16. As in § 104, $PM=\frac{V^2}{2g}$ which is independent of θ. Since $SP=PM$, the locus of S is a circle, of which P is the centre.

17. 45°.

Couples

18. § 107. 19. § 108. 20. § 109.

21. By § 108 a couple is equivalent to another couple in the same plane and with the same moment. Let the first couple consist of two equal forces P at A and Q at B, the line AB being perpendicular to the direction of P, and let the equal and opposite couple in the parallel plane be represented by R at C and S at D, the forces R and S being equal and opposite to P and Q, while the line CD is equal and parallel to AB. Then ABDC is a parallelogram. Let AD and

BC meet at O. P and S are equivalent to 2P at O, and Q and R are equivalent to 2P at O, acting in the opposite direction. Thus the four forces neutralise each other.

Forces Acting on a Rigid Body

22. § 111. The resultant is a single force $R=\sqrt{(\Sigma X)^2+(\Sigma Y)^2}$ and the couple which is equal to $\Sigma\,(Yx-Xy)$.

23. The three equations given in § 111.

24. Applying § 111 to three dimensions by introducing a third axis Oz perpendicular to the plane of the paper, we find that the forces can be reduced to three forces acting along the axes and three couples in each of the co-ordinate planes. The three forces have a single resultant, and the three couples may also be compounded into a single resultant. The proof of this last statement is given in Thomson and Tait's *Elements of Natural Philosophy*, §§ 565, 566.

25. Let P, Q, and R be the forces acting along BC, CA, and AB. At A apply two forces in opposite directions, equal and parallel to P. Then we have 3 forces P, Q, and R, at A parallel and proportional to the sides of a triangle, and also a couple consisting of P at A and P at B. Since the three forces are in equilibrium, the resultant is the couple. Draw AN perpendicular to BC, $P\times AN$ is the moment of the resultant couple.

26. (1) § 110. (2) $5\sqrt{3}$ lbs. in the string making ∠ of 30°, and 5 lbs. in the other. The rod makes an ∠ of 60° with the vertical.

27. (1) The resultant of the two forces must pass through the fixed point. (2) At an ∠ of 30° with AB. (3) $10\sqrt{3}$ at an ∠ of 60° with CB.

28. $P=R=\frac{W}{\sqrt{3}}$.

Hodograph

29. §§ 112, 113. 30. § 114.

31. Since the velocity is constant, the line OQ in § 105 remains unchanged, or the locus of Q is a point.

32. OQ traces out a cone, the circular base of which is the Hodograph required.

Uniform Circular Motion

33. § 115.

34. $\text{Tension}=\frac{mv^2}{r}=\frac{10}{7}\left(\frac{14\pi}{2}\right)^2=70\pi^2=691$ approximately.

This result is measured in poundals or absolute units of force. If we wish to express it as a force equal to the weight of so many pounds we must divide by g. If $g=32\cdot2$, tension $=21\frac{1}{2}$ lbs. approximately.

35. From § 115, $g=\frac{v^2}{r}$, $\therefore v=\sqrt{gr}$. Taking r as about 4000 miles. $v=26{,}000$ feet per second, approximately.

36. (1) See § 28 and § 115 last paragraph.

(2) Pressure $=\frac{mv^2}{r}$. $\therefore$ friction $=\mu\times\frac{mv^2}{r}=mg$.

$$\therefore \mu=\frac{gr}{v^2}=\frac{32\cdot2\times3}{(6\pi)^2}=\cdot27 \text{ approximately}$$

Impact

37. $V=3\frac{2}{3}$, $u=V-e(v-V)=3$, $u'=V+e(V-v')=4$.

38. (1) $V=9\frac{1}{3}$ or 4, according to whether the masses were moving in the same or in opposite directions before impact.

(2) K.E. before impact $=\frac{1}{2}\times5\times8^2+\frac{1}{2}\times10\times10^2=660$.

K.E. after impact $=653\frac{1}{3}$, or 120.

39. $V=\frac{mv+m'v'}{m+m'}$. But $m'=m$, $v=10$, $v'=0$, $\therefore V=5$.

$\therefore u=V-e(v-V)=2\frac{1}{2}$. $\therefore$ by (44) $u'=V+e(V-v')=7\frac{1}{2}$.

40. By (42) $V=\frac{1}{2}(v+v')$.

By (44) $u=2V-v=v'$, $u'=2V-v'=v$.

41. Since the ball is perfectly elastic, the co-efficient of restitution is 1, $\therefore$ the ball leaves the ledge with the same velocity as that with which it reaches it. Let this velocity be V, and let v be the velocity with which the ball reaches the ground. Then by the principle of kinetic energy, if h be the distance of the ledge from the ground, $mgh=\frac{1}{2}mv^2-\frac{1}{2}mV^2$ or $v^2=V^2+2gh$. This is the same velocity as the ball would have had if it had continued its fall. If the ball strike the ledge the path will be longer than if it had fallen directly, and since the velocity at each point is the same as that which it would have had at the same height if it had fallen directly, the time taken must be longer.

42. In this case the vertical velocity is unaltered, but a horizontal velocity is given. The time of falling depends only on the vertical velocity. When the ball reaches the ground it will have the same vertical velocity as if it had not been struck, but we must compound with this the horizontal velocity communicated to it. The whole velocity will therefore be greater.

Energy and Work

43. (1) § 42. (2) 2000 foot-pounds against friction + 33,600 foot-poundals in giving kinetic energy = 12,500 foot-pounds, if $g=32$.

44. (1) § 42. (2) 45 H.-P.

45. (1) 30 miles per hour. (2) 30·5 lbs. per ton. [$Fs=\frac{1}{2}mv^2$, $F=100\times 38{\cdot}5g$. Take g as 32.] 46. § 117.

47. (1) §§ 40, 44.

(2) By the first interpretation $MV+mv=0$. $\therefore V=-\frac{mv}{M}$.

By the second interpretation $\frac{1}{2}MV^2+\frac{1}{2}mv^2$ = energy of explosion.

$$\therefore \tfrac{1}{2}MV^2+\tfrac{1}{2}mv^2=30{,}000\times 20\times 112\times g.$$

$$\therefore v^2=\frac{30{,}000\times 20\times 112\times g\times M}{\frac{1}{2}(Mm+m^2)}.$$

$\therefore v=2069$ feet per second, approximately, if g be 32.

48. (1) By § 34 the force of gravity is g ($=32{\cdot}2$) absolute units of force. Compare § 32. (2) Work $=mgs=\frac{1}{2}mv^2$. For proof see equation (19) in § 36. Compare § 117.

49. (1) Let V = velocity of train. Then $v=V-at$, but when $t=60$, $v=0$. $\therefore a=\frac{1}{60}V$ and $f=ma=273{,}777\frac{7}{9}$ in absolute units = $8555\frac{5}{9}$ lbs., if g be 32.

(2) $fs=\frac{1}{2}mV^2$. But $s=600$ feet. $\therefore f=\frac{mV^2}{1200}=1{,}003{,}851\frac{23}{27}$ poundals $=31{,}370\frac{10}{27}$ lbs.

50. $\frac{mv^2}{r}=\frac{20\times 20\times 112}{5280}\times\left(\frac{50\times 5280}{60\times 60}\right)^2=45629\frac{17}{27}$ absolute units of force = 1426 lbs. approximately, if g be 32.

51. The kinetic energy is $\frac{1}{2}mv^2$, and this must be equal to the work done, that is (§§ 42, 117) the product of the weight and the distance through which it has fallen measured vertically. Let this distance be h, then $\frac{1}{2}mv^2=mgh$, $\therefore v^2=2gh$.

But this last equation is the same as the third of equations (19) in § 36, which gives the velocity in terms of the distance fallen.

52. 135,000. 53. 392,465·92 foot-poundals.

54. 5,734,080 foot-pounds. 55. (a) $fS=20g=644$. (b) $\frac{1}{2}mv^2=1250$.

56. (1) 24,092,444$\frac{4}{9}$ foot-poundals. (2) 164,266$\frac{2}{3}$.

Centre of Pressure

57. (1) § 118. (2) The average pressure = pressure half-way down the rectangle $=6\times 62\frac{1}{2}$ lbs. in the sq. ft. (Compare average velocity, § 22). $\therefore$ Whole pressure $=12\times 6\times 62\frac{1}{2}=4500$ lbs.

This acts at the centre of pressure, which is 4 ft. from CD. Call the required force P. Then by taking moments about CD we find P = 1500 lbs.

58. (1) The middle point of the line which joins the vertex to the middle point of the base. (2) Since both pressure and area increase as the distance from the vertex, this problem is similar to that of finding the centre of gravity of a cone. It is a point on the axis at a distance from the vertex $= \frac{3}{4}$ of the axis. (3) $4\frac{2}{3}$ ft. below the middle point of AB. (Compare answer to question 11, Science and Art Department, Advanced Examination).

Hydrometers and Specific Gravity

59. (1) § 119. (2) 8.

60. (1) § 119, Common Hydrometer. (2) Let V = volume of hydrometer below A, and let v = volume of AB or BC. Then if w be the weight of unit volume of water, s_1, s_2 the specific gravities of the two fluids, W the weight of the hydrometer—

$$W = Vw = (V + 2v)\, s_1 w. \quad \text{Now } s_1 = \frac{9}{10}. \qquad \therefore v = \frac{1}{18} V.$$

$$\text{Again } Vw = (V + v)s_2 w.$$

Substituting for v we find $s_2 = \frac{18}{19} = \cdot947$ approximately.

61. (1) The pressures at the level of the common surface in the two tubes are equal. Let h_1, h_2 be the heights of the free surfaces above this, s_1, s_2 the specific gravities, and w the weight of unit volume of water. Then (§ 81) $h_1 s_1 w = h_2 s_2 w$. $\therefore h_1 : h_2, \therefore s_2 : s_1$, or the heights are inversely as the specific gravities. (2) $12\frac{1}{2}$ inches. (3) 1·2 inches.

62. Between 36 and 76 grains. If the weight exceed 76 grains the rod will sink. If the weight be less than 36 the resultant fluid pressure will act at a point below the centre of gravity of the rod, and the equilibrium in the vertical position will therefore be unstable. § 120.

(The weight of the particle must lie between

$$W\left(\frac{1}{\sqrt{s}} - 1\right) \text{ and } W\left(\frac{1}{s} - 1\right)$$

where W is the weight and s the specific gravity of the rod).

Miscellaneous

63. A force is proportional to the acceleration it produces. (§ 32.)

An acceleration of one (in feet and seconds) means that in each second there is a change of velocity of one foot per second.

Let a minute be taken as the new unit of time, and consider the value of the acceleration just mentioned. A velocity of one foot per second is a velocity of 60 feet per minute. Thus in each *second* the change of velocity is 60 feet per minute, and therefore in each *minute* the change is 60 times as great, or 60^2 feet per minute per minute. Thus the measure of the acceleration depends on the square of the unit of time.

To make the above demonstration *general*, the student should assume that the new unit of time $= x$ old units, and prove, as above, that the new acceleration $= x^2$ old units of acceleration.

64. Let P be the particle, and through P draw any line AP meeting the sphere in A, and consider the attraction of a *small* triangular element ABC of the sphere. Produce AP, BP, CP, to meet the sphere again in A′, B′, C′. Then the attractions of the triangles ABC, A′B′C′ on P are equal and opposite. Since the arcs AB, BC, etc., are small, we may consider them to be straight lines. Then, by similar triangles, AB : AP = A′B′ : A′P, and so on.

In this way we prove the triangles ABC, A′B′C′ to be similar, and therefore in the duplicate ratio of AB to A′B′, that is, of AP to A′P.

$$\therefore \text{ABC} : \text{A}'\text{B}'\text{C}' = \text{AP}^2 : \text{A}'\text{P}^2, \text{ or } \frac{\text{ABC}}{\text{AP}^2} = \frac{\text{A}'\text{B}'\text{C}'}{\text{A}'\text{P}^2}$$

But the triangle ABC attracts P with a force proportional to $\frac{\text{ABC}}{\text{AP}^2}$, and A′B′C′ attracts P with a force proportional to $\frac{\text{A}'\text{B}'\text{C}^1}{\text{A}'\text{P}^2}$, or the two triangles attract P equally in opposite directions.

In the same way the whole sphere may be divided into small triangles, which attract P equally in opposite directions. Thus all the attractions neutralise one another.

65. (1) The centre of the sphere is the centre of gravity. The sphere is acted on by three forces which are not parallel and do not pass through one point. They cannot therefore be in equilibrium. (See § 49, Example for Exercise 3.)

(2) There are two ways in which the centre of gravity can be placed over the base.

EXAMINATION PAPERS

I. Cambridge—General Examination

1. § 25. For an illustration see § 55 Example, where a straight line represents the weight of a cart. 2. §§ 49, 50.

3. (1) Prove as in § 70 that if AB represent the given weight, BC represents the force, and AC the pressure on the plane.

(2) Compare Machines, General Example 36, where in a

similar case it is proved that the lengths are proportional to the weights. Therefore the length of the second plane is to be made twice that of the first. The geometrical construction follows easily. See also § 103 (3).

4. § 57. Compare also § 107, foot-note. 5. § 68.

6. (1) § 52, § 53 (1). (2) By § 55 the pencil will fall as soon as the centre of gravity is beyond the base. The centre of gravity of the pencil is $\frac{1}{12}$ of its length within the table. The beetle may therefore crawl $\frac{4}{12}$ of the pencil's length beyond the table, when it will be $\frac{1}{12}$ from the projecting end.

7. (1) § 61. (2) § 62. 8. §§ 75, 76. 9. § 82.

10. The cylinder contains air which has been compressed into half its volume, and therefore, by Boyle's law, the pressure has been doubled. The additional pressure is therefore that of one atmosphere, and the depth of the centre of the cylinder is the same as the height of the column of water in the water barometer (32 to 34 feet).

11. (1) §§ 86, 87, and 85. (2) Since spheres are proportional to the cubes of their radii, the volume of the whole ball is eight times that of the cork, or we have 7 volumes of gutta-percha to 1 of cork. Call the weight of 1 volume of water unity, then 1 volume of cork weighs ·24.

$\therefore$ 7 volumes of gutta-percha weigh $7 \times {\cdot}98 = 6{\cdot}86$.

$\therefore$ the 8 volumes weigh 7·1, and 8 volumes of water weigh 8.

$\therefore$ the specific gravity of the whole is $\frac{71}{80}$, and $\frac{9}{80}$, of the whole floats above the surface.

Compare § 88, Ex. 5, and Ex. for Exercise 4. Also see § 89.

12. (1) § 91. (2) By the height of the column of mercury is meant the perpendicular distance between the level of the mercury in the tube and the level in the cistern. If this be measured in each case, it is of no importance whether the tube be vertical or not. But, if the height be read from a scale attached to the tube, it is evident that this scale will only give correct results when the tube is vertical.

II. Oxford—Second Examination

1. (1) § 4. (2) While A runs $2x$ yards, B runs $3x$. If at this point B overtake A, then $3x = 2x + a$. $\therefore x = a$, and $2x = 2a$. 2. § 39.

3. (1) § 46. (2) At a right angle: see Kinetics, General Example 34 (3). 4. 43 lbs. Compare § 17, Ex. 1.

5. (1) § 57. (2) § 57, Example for Exercise 2.

6. 14: see Kinetics, General Example 33.

7. (1) §§ 26, 31, 40. (2) $1 - m^2$.

8. (1) § 86: § 88, Example 5. (2) Let the weight of a volume of

water be 1. Then the weight of one volume of the first fluid is 1 and of the second fluid 1·7. But the two volumes when mixed occupy only $\frac{9}{10}$ of two volumes. Therefore the same volume of water weighs 1·8. $\therefore s = \frac{1+1\cdot7}{1\cdot8} = 1\frac{1}{2}$.

9. §§ 92, 96. 10. (1) § 97. (2) 27 tons.

III. Edinburgh—Preliminary Examination

(A)

1. (1) § 4. (2) 55. 2. $v = 80\frac{1}{2}$, $s = 100\frac{5}{8}$.

3. (1) § 30. (2) As 1 to 24. 4. §§ 31, 36.

5. The forces are equivalent to 3 northwards and 4 eastwards. The resultant is 5 in the direction of the diagonal of the parallelogram. See § 14, Ex. 1.

6. At C, if AC$=\frac{3}{5}$ of AB. See § 49, Ex. 1. 7. § 56.

8. 13 lbs. See § 71, and Machines, Gen. Ex. 31.

9. (1) § 75. (2) § 81.

10. The water supports $\frac{1}{11}$ of the weight. $\therefore \frac{10}{11}$ of the weight$=1$ lb. $\therefore$ the whole weight$=\frac{11}{10}$ lbs.$=1\frac{1}{10}$ lbs.

(B)

1. ·027 kilometres approximately.

2. $v^2 = 2gs$, $v = 50\cdot75$, $mv = 1421$, $\frac{1}{2}mv^2 = 36{,}064$.

3. (1) § 34. (2) It would be increased in the proportion of 16 to 9. Compare § 34, Example 1.

4. (1) § 47. (2) Hang the magnet at $\frac{1}{3}$ of the bar's length from the man. 5. § 62.

6. Gravity may be resolved into two components g_1 acting along his body and neutralised by the reaction of the ice, and g_2 acting horizontally and causing him to move in a curve. See §§ 28, 115.

7. (1) §§ 78, 84. (2) The water supports part of the weight. See § 84. 8. (1) §§ 77, 86. (2) $\frac{20}{21}$. See § 89.

9. (1) § 52. (2) $\frac{1}{4}$ down the central part. 10. (1) § 58. (2) § 74.

IV. London—Matriculation Examination

1. (1) § 39. (2) In figure of § 39, the resultant of AB and BC is AC, that of AC and CD is AD, that of AD and DA is 0. Similarly for a rectangle.

2. (1) § 39, first paragraph: also § 18 applied to forces.

(2) Compare § 18, Ex. 2. Total force$=2\cdot7$, vertical component$= \sqrt{(2\cdot7)^2 - (1\cdot35)^2} = 2\cdot34$.

3. See § 62. Here $Wa = 9b$, $4a = Wb$. $\therefore 4Wa^2 = 9Wb^2$.
$\therefore 4a^2 = 9b^2$. $\therefore 2a = 3b$. Or $a : b = 3 : 2$.

4. If the feather be carried along at the same rate as the balloon, it will appear to the man to fall vertically downwards. If, on the other hand, the feather fall into a stratum of still air, its horizontal velocity will be gradually checked by the resistance of the air, and it will thus have, relatively to the balloon, an increasing velocity from west to east gradually approaching 60 miles an hour, while at the same time it will acquire under the action of gravity an increasing vertical velocity. Its path relatively to the balloon will thus be a curved line in a vertical plane, passing through the balloon.

5. The earth rotates from west to east. The stone, though falling retains its eastward motion, which is quicker than that of those parts of the earth which are nearer the centre. Hence it strikes the east side.

6. $1\frac{5}{12}$. See § 88, Ex. 1. 7. § 84.

8. (1) The surface of the water in the cylinder is 17 feet deep. The pressure is therefore that due to $34 + 17 = 51$ feet of water. At this pressure the air occupies 15 cubic feet. At a pressure due to one foot, the volume would be 51×15. $\therefore$ At a pressure due to 34 feet, it would be $\frac{51 \times 15}{34} = 22\frac{1}{2}$ cubic feet.

(2) State Boyle's law, and the results of § 81.

v. Glasgow—Preliminary Examination

1. (1) § 25. (2) 50 lbs. 2. § 59.

3. (1) §§ 4, 21. (2) In equation (10) $V = 0$, $\therefore s = \frac{1}{2}at^2$. If $t = 1$, $s = \frac{1}{2}a = 10$, $\therefore a = 20$. If $t = 2$, $s = 2a = 40$.

4. § 66. 5. (1) See § 41. (2) Here $v = \frac{5280}{5} = 1056$.

$\therefore V = -\frac{5 \times 1056}{200} = -26{\cdot}4$ feet per second.

vi. Cambridge—Local Examination

1. (1) § 25. (2) The forces of 3 and 7 are in the same direction, that of 10 in the opposite direction.

2. (1) § 39. (2) The figure on p. 18 will represent this case if the direction of the arrowhead on BC be reversed. AC and CB are the forces, AB the resultant. Since ACB is half an equilateral triangle, $CB = \frac{1}{2}$, $AC = 50$, $AB = CB\sqrt{3} = 86{\cdot}6$.

3. Let ABC be the rod in its position of equilibrium, B being its middle point. From A and B draw vertical lines AD, BE, meeting a horizontal line through C at D and E ; join AE. Three forces act

on the rod, its weight at B, the horizontal force at C, and the reaction at A. These forces must pass through one point (§ 49, Ex. for Exercise 3), and, as the directions of the first two meet in E, AE is the direction of the reaction. The three forces are therefore in the direction of the three sides of the triangle ADE. Therefore DE $= \frac{1}{2}$ of AD. But DE is $\frac{1}{2}$ of DC. (From B draw BF perpendicular to AD, and compare the $\triangle$s ABF, BEC. Prove DE = BF = EC.) $\therefore$ DC = AD, or the rod is at an angle of 45° to the horizon.

4. (1) § 49. (2) Replace each weight by two weights of half its value at the extremities of the sides. Thus the pressures on the legs are increased by $3+4=7$ lbs., $4+5=9$ lbs., and $5+3=8$ lbs.

5. (1) § 52 and § 53 (3). (2) Call the square ABCD. The weights at A and D are equivalent to 5 lbs., at a point E, $\frac{4}{5}$ along AD. The weights at B and C are equivalent to 5 lbs., at a point F, $\frac{3}{5}$ along BC. Therefore the centre of gravity of the whole is the middle point of EF.

6. (1) § 64. (2) 3 inches.

7. In the first case the slope of the plane is diminished, and therefore the force exerted is less (see § 70). In the second case, the centre of gravity of horse and man is kept above the base.

VII. Oxford and Cambridge—(1) Statics and Dynamics

1. (1) This form of question is not likely to recur. It refers to Duchayla's proof of the Parallelogram of Forces, formerly given in treatises where Statics was considered independently of Kinetics. Should it be again asked, then, *after answering the rest of the paper*, the candidate should state that he has learned to derive the Parallelogram of Forces from that of velocities, and write out §§ 12, 39.[1]

(2) OQ is $\frac{1}{2}$ the diagonal of the parallelogram formed by OA and OB. $\therefore$ Resultant of OA and OB = 2 OQ. Similarly, resultant of OQ and OP = 2 OR.

But resultant of OA, OB, and OC = resultant of 2 OQ and 2 OP. $\therefore$ Resultant of OA, OB, and OC = 4 OR or 2^2 OR.

If a fourth force OD be added, and a point S be taken on OD, so that OS $= \frac{1}{4}$ OD, and if T be the middle point of RS, then 8 OT (or 2^3 OT) is the resultant of OA, OB, OC, and OD. And so on for any number of forces. Thus if there be n forces and X be the middle point finally obtained, the resultant is 2^{n-1}OX.

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[1] According to the latest regulations for the Cambridge Previous Examination, either the Statical or the Kinetical proof of the Parallelogram of Forces will in future be accepted.

2. (1) § 49. (2) By the triangle of forces, the resultant of PA and AQ is PQ acting at A. Similarly the resultant of PB and BQ is PQ at B, and that of PC and CQ is PQ at C. Thus we have to find the resultant of 3 parallel forces. This by § 51 is 3 PQ at the centre of gravity of the triangle.

3. (1) § 53 (3). (2) To draw the figure, draw any two lines CB, CD, take a point A within the triangle BCD, and join BA, AD ; ABCD is the figure. Bisect CB in E, CD in F. Join AE, AF. The centre of gravity of ACB is in AE, and that of ACD is in AF. But A is the centre of gravity of the whole figure, ∴ EAF is a straight line. The rest follows easily by geometry.

4. §§ 72, 73. 5. (1) § 66. (2) Prove that the work done by the power in a small displacement is equal to that done against the weight. See § 74, Example for Exercise (2).

6. (1) §§ 4, 6. (2) Mean velocity = average velocity. See § 7.

7. Let v be the velocity of the bullet, V that of the train, θ the angle the bullet makes with the train. Then since $\sin\theta=\frac{3}{5}$, by trigonometry, $\cos\theta=\frac{4}{5}$. The velocity of the bullet across the train is therefore $\frac{3}{5}v$, while its velocity along the train is, relatively to the train, $\frac{4}{5}v-V$. But these are in the proportion of the length and width of the compartment, or—

$$\tfrac{3}{5}v : \tfrac{4}{5}v-V=8 : 6. \quad \therefore v=\tfrac{20}{7}V.$$

But V is a velocity of 28 miles an hour, ∴ v is a velocity of 80 miles an hour.

8. § 22, omitting V throughout. Compare Ex. 3.

9. Appendix, § 104. 10. (1) Appendix, § 116. (2) Let $\mathrm{A}=\frac{1}{2}mv^2+\frac{1}{2}m'v'^2$, $\mathrm{B}=\frac{1}{2}(m+m')V^2$, $\mathrm{C}=\frac{1}{2}mu^2+\frac{1}{2}m'u'^2$. Since $V=\frac{mv+m'v'}{m+m'}$,

$\mathrm{A}-\mathrm{B}=\frac{mm'(v-v')^2}{2(m+m')}$. Similarly $\mathrm{C}-\mathrm{B}=\frac{mm(u-u')^2}{2(m+m')}$. ∴ by (43)

$\mathrm{A}-\mathrm{C}=\frac{mm'(v-v')^2}{2(m+m')}\times(1-e^2)$. Hence A, the kinetic energy before impact, is greater than C, the kinetic energy after impact, if $e<1$, but equal to it, if $e=1$.

(3) It takes different forms of vibratory motion, *e.g.* heat.

(2) Natural Philosophy

1. (1) §§ 75, 76. (2) At first the pressure is increased following Boyle's Law, § 90. For the subsequent behaviour of the steam, see Clerk Maxwell's *Theory of Heat*, Chapter VI., page 113 in the second edition.

2. (1) §§ 34, 86, 21, 78, 42. (2) Volume $=16\pi$ cubic feet. Weight of a cubic foot of water $=\frac{15}{27}$ cwt.

$\therefore$ weight of cylinder $=16\pi \times 2{\cdot}8 \times \frac{15}{27}g$ cwt. $=78{\cdot}19$ cwt.

3. See §§ 70, 64. The force = weight of $\frac{12}{25}$ tons.

4. (1) § 72. (2) See figure, § 74. In this figure P = the resolved part of the weight acting down the plane. In the problem the force to be resisted is $P+\mu R$.

$$\therefore \text{work done} = (P+\mu R)\ KL = P \times KL + \mu R \times KL.$$
$$= W \times LM + \mu W \times KM.$$
$$(\text{From § 70, } R \times KL = W \times KM.)$$

5. Here $v=at$, $s=\frac{1}{2}\,at$, but $a=\frac{1}{16}g=2$, $t=10$,

$\therefore$ $v=20$ feet per second, $s=100$ feet, approximately.

6. (1) § 96. (2) § 97. (3) 162 cwt.

7. (1) § 89. (2) $\frac{200}{3200}g=\frac{1}{16}g=$ per second per second, approximately.

8. § 90.

VIII. Edinburgh—M.A. Examination—(A)

1. (1) § 4. (2) § 10. (3) Velocity = 2. Direction 30° S. of E.

2. § 23. 3. §§ 31, 32, 39.

4. Combine equations (15) and (19). See also § 117.

5. The fragment receives an additional kinetic energy of $10{,}000g$. If this $=\frac{1}{2}mv^2$, then v the additional velocity $=100\sqrt{g}$.

6. (1) Combining the moments by the parallelogram law, we find their resultant to be 250. But the new mass is 25, and therefore the velocity is 10 in the direction of the diagonal of the parallelogram. (2) Before impact the kinetic energy is 2750, after impact it is 1250. The greater part of it has therefore been transformed into heat or other forms of vibratory motion.

(B)

1. (1) §§ 4, 21, 32. (2) While the lift ascends with uniform acceleration, the mutual pressure between the man and the lift will be increased, and his sensation will be that of motion upwards. As soon as the acceleration ceases and the lift ascends with uniform velocity, the mutual pressure being that due to gravity alone, the man will cease to have any sensation of motion. When the lift stops, the man will have an upward velocity, and will feel for an instant as if the lift were suddenly moving downwards from beneath him.

2. §§ 31, 39. 3. (1) §§ 30, 117. (2) $mv=12{,}711$ approximately, $\frac{1}{2}mv^2=36{,}064$ if g be $32{\cdot}2$.

4. Acceleration of 32·2 ft. per second per second = acceleration of $\frac{32 \cdot 2}{3} \times 60^2$ yds. per minute per minute = acceleration of $\frac{32 \cdot 2}{3 \times 1760} \times 60^4$ miles per hour per hour. (See Additional Example 63.)

5. We may consider that during each stroke of the oar a force is applied at the rowlock parallel to the axis of the boat. This is equivalent (§ 108) to an equal force at the centre of gravity and to a couple. The force will give the boat an acceleration in the line of its action, while the couple will tend to make it rotate in a horizontal plane. When the oars are worked simultaneously on each side, the two couples neutralise each other and no rotation is produced. When they are worked alternately, each stroke will set up a rotation, but in opposite directions. The boat's direction of motion will therefore be changed by each stroke but brought back by the alternate stroke.

6. By (12) $v^2 = V^2 - 2gs$. If $g = 32 \cdot 2$, $V = 10$, $s = -400$, we find $v = -161$ approx. Substituting in $v = V - gt$, we get $t = 5 \cdot 3$ approx.

7. § 42. 1,125,600 foot-pounds. 1·14 horse-power, nearly.

8. The man's weight is equivalent to an equal weight at the ladder's centre of gravity and to a couple. The additional weight will increase all the pressures and frictions proportionately, and thus will not affect the equations of equilibrium. The couple, on the other hand, will in the first case tend to prevent slipping, and in the latter to cause it.

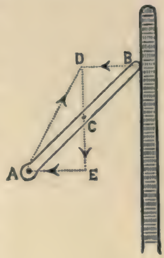
IX. Science and Art Department—(1) Elementary

1. (1) (*a*) and (*b*) § 53. (*c*) The *c.g.* of a cone is a point in its axis whose distance from the vertex is $\frac{3}{4}$ of the axis. It may be found experimentally as in § 54. (2) § 54.

2. Describe a △ ABC whose sides AB, BC, CA are 10, 13, and 16 units respectively, and place arrow-heads on the sides pointing the same way round the △. From a point O draw OP in the same direction as AB and equal to it, OQ in the same direction as BC and equal to it, and OR in the same direction as CA and equal to it. (For a similar diagram see § 99.) The second figure shows how the forces act. The angles may be measured by a protractor, and will be found to be POQ = 92°, 52′. QOR = 141°, 22½′. ROP = 125°, 45½′.

3. Let AB be the rod. The three forces are the weights acting vertically downwards through the *c.g.* along DE, the reaction of the

wall acting horizontally along BD and the reaction of the hinge which must be in equilibrium with these two, and therefore acts along AD (§ 112). If AE be drawn ∥ BD, these three forces are parallel to and may therefore be represented by the sides of the △ ADE.



4. (1) § 57. (2) Let R and R′ be the reactions of the supports A and B, and let the weight be hung 3 feet from B. Then, since the forces which act on the rod are in equilibrium, the sum of their moments about any point is nothing. Take moments about B. Then $8R = 4 \times 6 + 3 \times 24 = 96$. $\therefore R = 12$. Similarly $R' = 18$.

5. (1) § 61. (2) If G be near C, and the weights in the scales are unequal, the balance must be tilted through a large angle before the moment about C of the weight of the balance at G is equal to the moment of the difference of the weights.

6. (1) § 42. (2) 12,000 foot-pounds, of which 5000, or $\frac{5}{12}$ of the whole are done usefully.

7. (1) 1920. (2) 38,400.

8. (1) Equation (10) § 22. (2) In the third of equations (20) § 36, substitute $v = 0$, $g = 32$, $s = 121$, and we find $V = 88$. In the second of the same equations substitute $s = 0$, $V = 88$, and we find $t = 5\frac{1}{2}$.

9. (1) K.E. $= \frac{1}{2}mv^2 = 12{,}500$ foot-poundals (§ 117 and note § 42) $= fs$. If $f = \frac{1}{20}mg = 16$, $s = 781\frac{1}{4}$ ft.

10. (1) §§ 84, 89. (2) Since $s.g. = 1{\cdot}4$, a cubic foot weighs 1400 oz., of which the water supports 1000 (§ 77). ∴ the bottom supports 3×400 oz. $= 75$ lbs.

11. (1) § 87. (2) By (31) $s = \frac{W}{W - w} = \frac{732}{480} = 1{\cdot}525$.

12. (1) Pressure = weight of 14 cubic feet of water = 14,000 oz. = 875 lbs. on the sq. ft. (2) If a hole were made, the pressure of the atmosphere would act on the water in the tube, which would then fall to the level of the external water.

(2) Advanced

1. (1) § 49. (2) The parallel forces are equivalent to a force of 30 units, having the same direction and acting at C, the point of trisection of AB nearer to B. The resultant of this force and of the force 15 units along CB, is found as in § 39. It will vary with the angle which AB makes with the parallel forces.

2. § 109.

3. The resultant of OA and OB = 2 OP. The resultant of OC and OD = 2 OQ. But OP and OQ are equal and opposite. ∴ the forces are in equilibrium.

4. (1) The reaction R at C is perpendicular to the wall, and meets AD, the line of action of the weight W, at D. ∴ the reaction S at B acts along DB, and these forces are proportional to the sides of the △ BCD. ∴ $R = \frac{1}{2}\sqrt{3}\cdot W$. $S = \frac{1}{2}\sqrt{7}\cdot W$. (2) The forces exerted by the triangle on the wall at B and C are equal and opposite to R and S.

5. (1) § 72.

$$(2)\ \mu = \frac{155}{500} = \frac{31}{100} = \tan\theta \text{ or } \frac{BC}{AC} \text{ (see § 73). } \theta = 17^\circ \text{ approximately.}$$

6. Let P and W denote the power and the weight, a the arm at the extremity of which the power acts, b the radius of the screw, c the vertical distance between two threads of the screw, and μ the coefficient of friction. Then by the principle of work (§§ 42, 74) $2\pi aP = Wc + 2\pi b\mu W$ approximately.

7. (1) § 42. (2) Work done = 1650 foot-pounds per second, of which $933\frac{1}{3}$ are effective, and $716\frac{2}{3}$, or $\frac{43}{99}$ of the whole, are spent in overcoming friction, etc.

8. (1) Let the body be projected from A with velocity V in a direction making an $\angle\theta$ with AN, the horizontal line in the plane of motion, and let P be its position after t seconds. Draw $PN \perp AN$. Then, as in § 26, $PN = V\sin\theta\cdot t - \frac{1}{2}gt^2$, $AN = V\cos\theta\cdot t$.

$$(2)\ \text{Here } \cos\theta = \frac{\sqrt{3}}{2},\ \sin\theta = \tfrac{1}{2}\quad PN = 76,\ AN = 400\sqrt{3},\ g = 32.$$

Substituting these values, we find $t = 4\frac{1}{2}$ and $V = 177\frac{7}{9}$.

9. See § 116, and observe that here v' is negative.

$$V = \frac{mv + m'v'}{m + m'} = -\tfrac{1}{3}.\quad u = V - e(v - V) = -1.\quad u' = V + e(V - v') = 0.$$

10. $$\text{K.E.} = \tfrac{1}{2}mv^2.\qquad \text{But } v = \frac{12 \times 2\pi \times 15}{60} = 6\pi.$$

$$\therefore \text{K.E.} = 180{,}000\pi^2 = fS.\quad f = 400g = 12{,}800.\quad \therefore S = 14\tfrac{1}{16}\pi^2.$$

Now S is the length of path described by a point on the circumference of the axle. ∴ number of turns $= \dfrac{S}{\pi} = 14\frac{1}{16}\pi = 44\cdot 2$ approximately.

11. (1) and (2) § 118. (3) The problem is the same as that of finding the centre of gravity of the quadrilateral QSCD in the figure of § 118, if EQ = QD. The point is in EF, and its distance from E is $\frac{4}{9}$ EF.

12. (1) Let V_1, V_2 be the two volumes, W_1, W_2 their weights, s_1, s_2 their specific gravities, and let V be the volume, W the weight, and s the specific gravity of the whole. Then if w represent the weight of unit volume of water,

$$W = W_1 + W_2, \text{ or, } Vsw = V_1 s_1 w + V_2 s_2 w. \quad \therefore s = \frac{V_1 s_1 + V_2 s_2}{V_1 + V_2}.$$

(2) (*a*) ·95. (*b*) 1.

EXAMINATION PAPERS—*Continued.*

X

UNIVERSITIES OF EDINBURGH AND ST. ANDREWS

Degrees in Arts and Science.—Preliminary Examination

Dynamics

1. State the law by which velocities are compounded and resolved. Mention other kinematical or physical quantities which are compounded and resolved according to the same law.

A point has a velocity of 10 units in a direction making an angle of 60° with the horizon. Find the vertical and horizontal components of the velocity.

2. What is Acceleration ? What is the unit in terms of which it is measured ?

If a point be moving with uniform acceleration, and move through 61 feet in the 14th second from the commencement of its motion, and through 113 feet in the 27th second, find the initial velocity and also the acceleration of the point.

3. Define momentum, and show that force may be defined as the time-rate at which momentum varies.

Find the uniform force under which a mass of $\frac{1}{12}$th of a lb. will acquire a speed of 1600 feet per second in 4 seconds. Find also through what space this force must have been exerted.

4. Explain the terms Work, Power, Horse-Power.

A well, which is in form a square prism of $3\frac{1}{2}$ feet side and 50 feet deep, is filled with water. What is the horse-power of an engine which can lift the water to the level of the top of the well in $1\frac{1}{2}$ hours ?

5. What is the centre of a system of parallel forces?

ABC is an equilateral triangle, each of whose sides measures 1 foot. At A, B, and C parallel forces of 7, 5, and 4 lbs. act. Show how to find the centre of these parallel forces.

6. State the laws of statical friction, and define the coefficient of friction.

A body is placed on an inclined plane of 1 in 5, and is just about to slip down. Determine the coefficient of friction between the body and the plane.

7. Define density and specific gravity.

If the specific gravity of silver be 10·5, find the weight of 7·2 cubic inches.

8. Prove that the pressure in a heavy liquid at rest is the same at all points of a horizontal plane.

A tank, 4 feet square and 10 feet deep, is filled with water. Find the pressure on the bottom of the tank and on one side. (The weight of a cubic foot of water is 62·5 lbs.)

9. Explain, by the aid of a diagram, the action of the common pump, and also of the force-pump.

10. State the relation among the pressure, volume, and temperature of a given mass of air. If the pressure be increased by one-tenth and the temperature raised from 50° to 100° C., in what ratio will the volume be altered?

XI

UNIVERSITY OF LONDON

Matriculation Examination —*Mechanics*

1. A boy in a toboggan slides down a perfectly smooth hill whose inclination is 1 in 20. At what rate will he be going (in miles per hour) when he has travelled 100 yards from the start?

2. What force must be applied for one-tenth of a second to a mass of 10 tons in order to produce in it a velocity of 3840 feet per minute? What would be the momentum of the mass so moving? (*N.B.*—A numerical answer is meaningless unless the unit intended is also stated.)

3. A 3-ton cage, descending a shaft with a speed of 9 yards a second, is brought to a stop by a uniform force in the space of 18 feet. What is the tension in the rope while the stoppage is occurring? (Express it in tons' weight.)

4. State the third law of motion, and explain clearly its application to the case of a horse starting a cart into motion. If the pull back of the cart be exactly equal to the pull forward of the horse, why do they begin moving?

5. A weightless rod, 3 feet long, is supported, horizontally, one end being hinged to a vertical wall and the other attached by a string to a point 4 feet above the hinge; a weight of 180 lbs. is hung from the end supported by the string. Calculate the tension in the string, and the pressure along the rod.

6. An artificial lake, $\frac{1}{4}$ mile long and 100 yards broad, with a gradually shelving bottom varying from nothing at one end to 88 feet at the other, is dammed at the deep end by a masonry wall across its entire breadth. Find the total pressure on the embankment when the lake is full of water weighing three-quarters of a ton to the cubic yard. Find also the total weight of water in the lake.

7. What do you know about the density of gases in relation to temperature and pressure? Describe experiments which show that the density of a gas at constant temperature is proportional to its pressure. A uniform tube, closed at top, open at bottom, is plunged into mercury, so that it contains 25 cm. of gas at atmospheric pressure of 76 cm.; the tube is now raised until the gas occupies 50 cm.: how much has it been raised?

8. How can the atmospheric pressure be measured and expressed in grammes' weight per square centimetre?

XII

UNIVERSITY OF GLASGOW

Degrees in Arts and Science.—Preliminary Examination

Dynamics

1. Define resultant of two velocities, and show how to resolve a velocity into two components in given directions.

ABCD is a square of which AC and BD are the diagonals. If AC represents a velocity and BD represents one of its components, find a line which represents the other component.

2. State and explain the Second Law of Motion.

A balloon is moving horizontally at the rate of 10 feet per second at a height of 100 feet above the ground. If a stone is let fall from the balloon, with what speed will it be moving when it strikes the ground?

How will the stone appear to a spectator in the balloon to move during its fall?

(Take $g=32$ foot-second units.)

3. When is a force said to do work? What units of work are in use in dynamics?

The height and length of an inclined plane are 100 feet and 200 feet respectively. Find the work done against gravity in dragging 10 lbs. 50 feet up the plane.

4. If 30 per cent. of the work done by an engine is wasted by friction, what must be the horse-power of the engine if it can project 30,000 lbs. of water per minute with a speed of 40 feet per second?

5. State the "Parallelogram of Forces," and show, with the aid of a figure, how it may be verified experimentally.

Find the magnitude of the resultant of the following three forces acting in one plane at a point :—

2 poundals east.
$\sqrt{2}$ poundals north-east.
3 poundals north.

6. Distinguish, with examples, between stable, unstable, and neutral equilibrium.

A sphere whose centre of gravity is not at its centre, and whose weight is equal to the weight of its own volume of water, is wholly immersed in water, and then left free to move. Show what will be the relative positions of its centre and of its centre of gravity when it has come to rest.

7. Define the centre of gravity of a body, and show how to find the centre of gravity of a body made up of two parts.

ABCD is a square, and O is its centre. If the triangle AOB is cut away, find the centre of gravity of the remainder of the figure.

8. Show that if three forces acting in one plane on a body keep the body at rest, the forces must either be parallel or must meet in a point.

AB is a uniform rod, 4 feet long, which weighs 10 lbs. It is free to turn about the end A, which is hinged to a fixed point, and it is supported in a horizontal position by a string which is attached to the end B, and fastened to a fixed point C, 3 feet vertically above A. Find the tension of the string.

9. State the principle of Archimedes.

A solid weighs 30 grammes in water, and 40 grammes in a liquid whose specific gravity is ·8. What is the volume of the body in cubic centimetres, and what is its mass in grammes?

10. Show that the difference of the pressures at two points of a liquid of uniform density is proportional to the difference of level of the two points.

Why is this proposition not true in the case of the atmosphere?

A diving-bell is sunk in a lake of fresh water until the volume of the air in the bell is reduced to $\frac{3}{4}$ of the volume at atmospheric pressure. What is the depth below the surface of the lake of the surface of the water in the bell, the water barometer standing at 33 feet?

11. Describe the simple mercury barometer.

What would be the effect on the height of a barometer (a) if the barometer were taken up in a balloon, (b) if it were taken down under water in a diving-bell? Justify your answers.

Give a numerical result for the case in which the diving-bell is sunk to a depth of 25 feet in water. [Specific gravity of mercury = 13·6.]

XIII

SCOTCH EDUCATION DEPARTMENT

Leaving Certificate—(1) *Elements of Dynamics*

1. Define speed and velocity. Give an instance of a body whose speed is uniform though its velocity constantly varies.

A man walks along a road with velocity v. After a certain time the direction of the road is so altered as to make an angle of 60° with its original direction, but the man's speed remains unaltered. Find his change of velocity.

2. A balloon is rising, and just as it reaches the height of 1000 feet, a stone is dropped from it. If the stone reach the ground in ten seconds, find the velocity with which the balloon was rising, assuming the acceleration due to gravity to be 32 feet per second in a second, and neglecting the resistance of the air.

3. State the third law of motion, and give two illustrations of its action.

A bomb-shell, moving with a velocity of 300 feet per second, bursts into two equal fragments. By the explosion one part is reduced to rest. Find the initial velocity of the other fragment.

4. Find the centre of gravity of a triangle.

One diagonal of a quadrilateral bisects the other. Show that the former diagonal passes through the centre of gravity of the quadrilateral.

5. When is a force said to do work?

In a waterfall 20 tons of water pass over in each minute, and the potential energy lost by the fall amounts to 22,000 foot-pounds per second. Find the height of the fall.

6. State the laws of statical friction. How does kinetic friction differ from statical friction?

Find the work done in dragging a mass of 10 lbs. up an inclined plane 10 feet long, rising 6 feet, if the coefficient of friction be $\frac{7}{10}$.

7. A sphere, whose weight is 10 lbs., rests on two planes which are at right angles to each other and are inclined to the horizontal plane at angles of 30° and 60° respectively. Find the pressure exerted on each plane.

8. Show why a body suspended in water appears to lose part of its weight.

The specific gravity of mercury is $13\frac{1}{2}$. A piece of iron floats in mercury with $\frac{4}{9}$ of its bulk above the surface. Find the specific gravity of the iron.

9. State Boyle's Law.

A cubic foot of air weighs 434 grains, at a pressure of $15\frac{1}{2}$ lbs. on the square inch. What will it weigh at a pressure of 7 lbs. on the square inch, the temperature being unchanged?

(2) *Higher Dynamics*

1. Enunciate the proposition called the polygon of forces.

If forces acting at a point are represented by AB, 2BC, 3CD, 4DE, 5EA, when ABCDE is a pentagon, show that their resultant is the same as that of forces represented by BA, CA, DA, EA, acting at the same point.

2. Find the relation between the power P and the weight W in a system of three pulleys, each of weight w, each attached to the weight, and one only suspended from the beam.

3. Find the centre of mass of a thin uniform triangular lamina, and prove that it coincides with that of three equal particles placed one at each vertex.

Prove that a similar proposition is only true of a quadrilateral when it is a parallelogram.

4. Show how to find the resultant of any number of forces acting on a rigid body in one plane, and deduce the conditions of equilibrium.

5. Prove that the path of a projectile is, neglecting the resistance of the air, a parabola.

From the top of a cliff a shot fired horizontally is seen to strike a point below, whose angle of depression is a; show that if another shot is discharged with the same initial velocity at an angle of elevation $\frac{1}{2}\pi - a$, it will hit the same mark.

6. Explain, carefully, what happens when two elastic spheres impinge directly, and show how to determine the velocity of each after impact.

7. Prove the formula giving the acceleration towards the centre when a particle is moving in a circle of radius r, with velocity v.

If the moon, at a distance of 240,000 miles, be supposed to revolve uniformly round the earth in 28 days, find the acceleration towards the earth's centre.

8. Show how to find the centre of pressure on a vertical rectangle, with one side horizontal, exposed to the action of a heavy liquid.

A flood-gate in a lock is in the form of a square, with horizontal and vertical sides of length $2a$, and with its centre at a depth h below the surface. Prove that the centre of pressure is at a depth $h + \frac{a^2}{3h}$.

XIV

SCIENCE AND ART DEPARTMENT

Theoretical Mechanics—Elementary Examination

(1) *Solids*

1. State what units of time, distance, and mass are commonly employed in questions of dynamics.

Define momentum. If the mass of a body is 12 lbs., and it is moving at the rate of 10 feet a second, what is the numerical value of its momentum, the units being pounds, feet, and seconds? What would be the numerical value of the momentum if the units were hundredweights, yards, and hours?

2. State the rule for the composition of two velocities.

A ball moving at the rate of 10 feet a second is struck in such a way that its velocity is increased to 12 feet a second, and the direction of the new velocity makes an angle of 45° with that of the old velocity; find by construction the velocity imparted by the blow and its direction.

3. The velocity of a moving body is increased by 12,000 yards a minute in each minute of its motion; by how many feet a second is its velocity increased in each second of its motion?

4. Taking for granted the formula $s = Vt + \frac{1}{2}ft^2$, state the meaning of each letter in it.

If a body falling freely describes a distance of 30 yards in half a second, what was its velocity at the beginning of the half-second? ($g = 32$.)

5. Define (1) an absolute unit of force; (2) a poundal.

A body, whose mass is twelve pounds, is found to gain a velocity of 15 feet a second, when acted on by a constant force (P) for 3 seconds; find the number of poundals (or British absolute units of force) in P. What ratio does P bear to the force exerted by gravity on the body? ($g = 32$.)

6. What is meant by the moment of a force with respect to a point? If there are two moments, and one is marked with a positive sign and the other with a negative sign, what do these signs show?

ABCD is a square, and AC a diagonal; in AC take a point O, such that AO is a third of AC; forces of 20, 7, and 5 units act from A to B, C to B, and D to C respectively; if the side of the square is 6 units long, write down the moment about O of each force, with its proper sign, and find their sum.

7. The kinetic energy of a particle being given by the expression $\frac{1}{2}mv^2$, state what is meant by m and v, and in what units these quantities must be measured if the result is to be obtained in foot-poundals.

A body, whose mass is 10 lbs., is capable of doing 605 foot-poundals of work in virtue of its mass and velocity; at the rate of how many feet per second is it moving?

8. Draw two lines AB and AC, containing an angle of 120°, and suppose a force of 7 units to act from A to B, and a force of 10 units from A to C; find by construction the resultant of the forces, and the number of degrees in the angle its direction makes with AB.

9. State how to find by construction the centre of gravity of three particles.

ABC is an equilateral triangle, each side being 2 inches long; particles whose masses are 1, 2, 3 are placed at A, B, C respectively;

find their centre of gravity by construction, and note its distance from A.

10. If a body is pressed against a smooth surface, in what direction does the mutual action take place?

Draw a circle, suppose that its plane is vertical and that A is its lowest point; draw a chord AB equal in length to the radius. If we suppose AB to represent a rod placed inside a circle, and that there is no friction, name the forces that act on the rod, and show them in a diagram. Explain why the forces cannot keep the rod at rest.

11. A body slides down a smooth inclined plane, the height of which is 10 feet and the length 100 feet; find (*a*) the acceleration of the body's velocity while sliding; (*b*) the velocity which the body acquires in sliding from the top to the bottom of the plane; (*c*) the time it takes, starting without initial velocity, to get from the top to the bottom. ($g=32$.)

12. In the case of a simple pendulum, the time of a small oscillation is given by the formula $2\pi\sqrt{\frac{l}{g}}$; what is meant by an oscillation? what by a small oscillation? what do the letters π, l, and g, in the formula stand for?

A pendulum 8 feet long makes 38 beats a minute; what is the acceleration due to gravity at the place of observation?

(2) *Fluids*

1. Mention any one property of the centre of gravity of a body. Where is the centre of gravity situated (*a*) in a square lamina, (*b*) in a cylinder?

2. The masses of two bodies (P and Q) are in the ratio of 3 to 2; the former is moving at the rate of $7\frac{1}{2}$ miles an hour, the latter at the rate of 200 yards a minute; find the ratio of P's momentum to Q's momentum.

What is meant by the momentum of a body?

3. Define the British absolute unit of force, or poundal.

The masses of two bodies are 5 lbs. and 7 lbs. respectively; at any instant the former is moving at the rate of 12 feet a second, the latter at the rate of 900 yards a minute. The former has acquired its velocity by the action of a constant force (P) in three seconds, the latter by the action of a constant force (Q) in a quarter of a minute; find the ratio of P to Q, and express each of them in poundals.

4. Define kinetic energy. Write down the formula for the kinetic energy of a particle whose mass is M and velocity V.

The mass of a particle is 10 lbs.; at a given instant it is moving at the rate of 24 feet a second; it moves against a constant resistance of 4 poundals; what distance would it describe from that instant before coming to rest?

5. How is the pressure at any point within a liquid measured?

A circle, whose radius is one foot, is described on a vertical wall of a reservoir; the surface of the water is $2\frac{1}{2}$ feet above the centre of the circle; find the ratio of the fluid pressure at the highest point to the fluid pressure at the lowest point of the area.

Find also the magnitude of the resultant fluid pressure on the area of the circle. (A cubic foot of water weighs 1000 oz.)

6. State how to find the resultant fluid pressure on a body partly immersed.

Suppose the plane of the paper to be vertical; draw a square ABCD, take AE a third of AB and DF a third of the parallel side DC, and draw a line of indefinite length through E, F; let that line represent the surface of water, and let the square represent a cube (whose edge is a foot long) held in it, with AD under water; find the resultant of the fluid pressure on the cube. If now we suppose the cube turned round E, so that D comes into the surface of the water, find the resultant fluid pressure in this case, and show it in the diagram.

7. A cylinder is 2 feet high, and the radius of its base is 3 feet; its specific gravity is 0·7; it floats with its axis vertical; find (*a*) how much of its axis will be under water, (*b*) the force required to sustain it one inch higher.

8. State how the specific gravity of a body heavier than water can be found by the balance. State also the principle in hydrostatics on which the method is based.

The specific gravity of a substance is 7·5, a portion of it weighs 390 grains in water; what is its weight in air?

9. When is the temperature of one body said to be higher than that of another, and when are they said to have equal temperatures?

What is the boiling-point of a thermometer, and in what sense can it be called a fixed point?

10. State exactly what is meant by each of the letters in the formula for gases, viz.:—

$$\frac{vp}{1+at}=\frac{\mathrm{VP}}{1+a\mathrm{T}}.$$

State carefully the properties of gases embodied in the formula.

11. A certain quantity of gas is contained in a vessel whose volume

is one cubic foot, and its temperature is 20° C. If in any way (*e.g.* by pressing down a piston) its volume is changed to 1000 cubic inches, and its temperature raised to 30° C., find the ratio of the pressure of the gas in its former state to its pressure in its latter state. (*N.B.*—The coefficient of expansion is 0·00366.)

12. Describe briefly the suction-pump and its action.

The diameter of the piston is 4 inches, and the spout is 20 feet above the surface of the water in the well; find the force along the piston-rod required to work the pump. Explain how the calculation of the force is justified.

XV

ADDITIONAL EXAMINATION PAPERS.

(A)

1. Prove the triangle of velocities.

A point is moving due east with a speed of 10 units; find the direction and magnitude of the additional velocity which must be given to it that it may move north-east with the same speed.

2. Define uniform velocity and uniform acceleration.

With what velocity must a stone be projected vertically upwards so as to return to the ground after six seconds? How high will it rise?

3. State Newton's second law of motion, and explain how it enables us to measure force.

How many units of force, measured on this system, are there in 12 pounds weight?

4. When is a force said to "do work"?

A bullet, weighing 1 oz., is discharged from a rifle with a velocity of 1200 feet per second. If the length of the barrel be 3 feet, find, in pounds weight, the average force acting on the bullet in the barrel.

5. Two parallel forces, of 3 and 4 units, act on a body in opposite directions, in lines 1 foot apart; specify the force required to balance them, and show by a diagram how the three forces act.

6. What are the qualities of a good balance?

If M be the true mass of a body, P and Q its apparent masses in the two scales of a balance, whose arms are of unequal length, show that $M^2 = PQ$.

7. A body rests on a rough inclined plane. Represent by a figure the forces acting on the body, and find the relation between them.

If the body is on the point of slipping when the inclination of the plane is 30°, what is the coefficient of friction?

8. State the conditions for equilibrium, and for stable equilibrium, of a floating solid.

9. Describe briefly the action of the force-pump.

If the plunger of a force-pump has a cross section of 4 square inches, and works 60 feet below a cistern, with what pressure must it be forced down in order to raise water to the cistern? (1 cubic foot of water = 62·5 lbs.)

(B)

1. State the proposition known as the parallelogram of velocities. Give a diagram representing the resultant of the velocities 16 north, 7 east, 4 south, and 2 west.

2. A train is observed to pass a station at a speed of 30 miles an hour; if it increase its speed uniformly so that in every minute it adds on a speed of 20 yards per minute, how far will it be beyond the station at the end of 20 minutes?

3. How is the change in the momentum of a moving body connected with the force that causes it? A mass of 10 lbs. is being pushed along a smooth table by a force equal to the weight of 1 lb.: how far will it go from rest in 2 seconds?

4. State the law of gravitation given by Newton.

If the force of attraction between two bodies be measured by 20 when they are 5 miles apart, what number will measure the force when they are 20 miles apart?

5. Define "work" and "horse-power." Find the horse-power of an engine which can raise 20,000 gallons of water per hour from the bottom of a shaft 500 feet deep, assuming that a gallon of water weighs 10 lbs.

6. The ends of a string are tied to the rings of a picture and the string is then passed over a nail from which the picture hangs. Will the tension of the string be the same whatever its length may be? Give reasons for your answer.

7. When is the whole amount of statical friction called into play, and in what direction does it act?

A mass of 2 cwt. rests on a horizontal plane; find the horizontal force which will just be sufficient to move it if the coefficient of friction be $\frac{1}{4}$.

8. Find the condition of equilibrium of a system of pulleys in which each string is attached to the weight, and each, except the last, supports one moveable pulley.

9. The pressure in the water-pipe at the basement of a building is 34 lbs. to the square inch, whereas at the third floor it is only 18 lbs. to the square inch. Find the height of the third floor. (A cubic foot of water weighs 1000 oz.)

10. State Boyle's Law. The pressure on the surface of the sea is 15 lbs. on the square inch, and the weight of a cubic foot of water is $62\frac{1}{2}$ lbs. If a diving-bell contain 200 cubic feet of air at the surface, into what space will the air be compressed when the bell is lowered to a depth of 60 feet?

SOLUTIONS TO EXAMINATION PAPERS X. TO XV.

X. Edinburgh and St. Andrews—Preliminary Examination in Arts and Science

1. (1) §§ 12, 14.

(2) Accelerations, § 21; Momenta, § 30; Forces, § 39.

(3) Vertical component $=5\sqrt{3}$ units; horizontal component $=5$ units. See § 18.

2. (1) § 21.

(2) $61=14V+\frac{1}{2}\times 14^2a-(13V+\frac{1}{2}\times 13^2a)=V+13\frac{1}{2}a.$

Similarly $113=V+26\frac{1}{2}a.$

Hence $a=4$ feet per second per second; and $V=7$ feet per second.

3. (1) § 30.

(2) Force $=$ mass $\times$ acceleration.

$$=\frac{\text{mass}\times\text{change of velocity}}{\text{change of time}}=\frac{\text{change of momentum}}{\text{change of time}}.$$

(3) $4f=\frac{1}{12}\times 1600$, $\therefore f=33\frac{1}{3}$ poundals; $33\frac{1}{3}\,s=\frac{1}{2}\times\frac{1}{12}\times 1600^2$, $\therefore s=3200$ feet. (See § 117.)

4. (1) § 42. The *Power* of an agent is the rate at which it does work.

(2) ·32 horse-power approximately. ($612\frac{1}{2}$ cubic feet of water weighing $62\frac{1}{2}$ lbs. per cubic foot has to be lifted 25 feet on an average.)

5. (1) § 51.

(2) See § 53, (5) example. In this case $BD = \frac{4}{9}BC$ and $AO = \frac{9}{16}AD$.

6. (1) § 72.

(2) $\mu = \frac{1}{\sqrt{24}} = \cdot 204$ approximately.

7. (1) §§ 77, 86.

(2) $43\frac{3}{4}$ oz.

8. (1) § 80.

(2) Pressure on bottom = whole weight of liquid = 10,000 lbs.

" " side = average pressure × area = pressure on centre of gravity of side × area = 12,500 lbs.

9. §§ 95, 96.

10. This question does not belong to pure dynamics as it involves change of temperature.

Boyle's Law (§ 90) gives the relation $pV = C$ when there is no change in temperature. When the temperature changes the relation is $\frac{pV}{1+at} = C$, where t is the temperature in centigrade degrees and $a = \frac{1}{273}$. In this case, if V be the original and V′ the final volume, we have $\frac{pV}{1+\frac{50}{273}} = \frac{\frac{11}{10}pV'}{1+\frac{100}{273}}$, or $\frac{V'}{V} = \frac{3730}{3553}$.

XI. University of London—Matriculation Examination

1. § 103. $\sin\theta = \frac{1}{20}$. 21·12 miles per hour approximately, if $g = 32$.

2. § 32. Mass = 10 × 2240 pounds. Velocity = 64 feet per second. ∴ momentum = 10 × 2240 × 64 in foot-pound-second units. This would be generated by a force of 10 × 2240 × 64 poundals acting for one second, but since the force is only to act for one-tenth of a second it must equal 100 × 2240 × 64 poundals = weight of 200 tons, if $g = 32$.

3. Suppose the force applied by means of the rope, and let T be resulting tension. Then resultant force on cage $= T - mg$, ∴ $T - mg = ma$, by § 32. Also since the acceleration is uniform, and a velocity v is destroyed in passing over a space s, $v^2 = 2as$, by § 22,

$$\therefore a = \frac{v^2}{2s} = \frac{27^2}{36} = \frac{81}{4},$$

and $T = m(a+g) = 3 \times 2240 \times (\frac{81}{4} + 32)$ poundals = weight of 4·9 tons, nearly.

4. (1) § 40.

(2) The pull back of the cart and the pull forward of the horse, though equal and opposite forces, do not produce equilibrium because they act on different bodies. The horse begins to move, because the reaction of the ground caused by his muscular action on it is greater than the pull back of the cart, and his acceleration is proportional to the difference between these forces. Similarly, the cart begins to move because the pull forward of the horse is greater than the resistance of the ground to its motion.

5. Compare § 45, Ex. 3. Tension = weight of 225 lbs.
Pressure = weight of 135 lbs.

6. See § 81. Pressure = area of surface × depth of C.G. of surface × weight of unit volume of fluid.

(1) $32,266\frac{2}{3}$ tons weight.

(2) 484,000 tons.

7. (1) Density is proportional to pressure when temperature is constant.

Density is inversely proportional to absolute temperature when pressure is constant. The zero point of absolute temperature is $-273°$ centigrade.

(2) § 90.

(3) The volume is doubled ∴ pressure is halved, and is ∴ 38 centimetres. Hence mercury must stand 38 centimetres up the tube, and ∴ since the whole length of the tube above the level of the mercury is now $50 + 38 = 88$ centimetres, the tube has been raised by $88 - 25 = 63$ centimetres (leaving out of account the slight fall in the level of the open surface of mercury).

8. (1) § 91.

(2) Let barometric column, for example, be 76 centimetres. Hence, pressure of atmosphere per square centimetre = weight of mercury column 76 centimetres high and 1 square centimetre section = weight of 76 cubic centimetres of mercury = $76 \times 13{\cdot}6$ grammes = $1033{\cdot}6$ grammes.

XII. Glasgow—Preliminary Examination in Arts and Science

1. (1) § 11.

(2) See § 18. In this case the two directions are not given at right angles, so that CB need not be at right angles to AB. AB

must be drawn parallel to one given direction and CB parallel to the other.

(3) Through A draw AE equal and parallel to BD. Join ED, and show that EC is a straight line, and that it represents the other component.

2. (1) § 31.

(2) §§ 36, 15. Let v = vertical speed of stone when it reaches ground. Hence $v=\sqrt{2gs}=80$ feet per second. Therefore actual speed of stone $=\sqrt{10^2+80^2}=10\sqrt{65}$ feet per second.

(3) It would appear to fall vertically.

3. (1) § 42. (2) § 42 (with note).

(3) The mass is raised through $\frac{1}{4}$ of the length of the plane, and therefore through $\frac{1}{4}$ of its height, *i.e.* through 25 feet.

$$W=fs=10\times 25=250 \text{ foot-pounds,}$$
$$\text{or } 250g \text{ foot-poundals.}$$

4. §§ 42, 117.

$$\text{H.P.}=\frac{\frac{10}{7}\times\frac{1}{2}mv^2}{g\times 33000}=32{\cdot}47 \text{ approximately.}$$

5. (1) §§ 39, 46.

(2) Show that $\sqrt{2}$ poundals NE is equivalent to 1 poundal N and 1 poundal E, and hence that the resultant is 5 poundals (§ 15); or use graphic method (§ 14).

6. (1) § 56.

(2) Show that there can only be stable equilibrium when the centre of gravity of the sphere is vertically under the centre.

7. (1) § 52. (2) § 53, section 5, and § 49.

(3) The three triangles AOD, DOC, COB are equal in area. The C.G. of the figure made up of AOD and BOC is at O. Bisect DC at E, join OE. Take $OG=\frac{2}{3}$ OE; then G is the C.G. of DOC. Divide OG at F, so that $OF=\frac{1}{3}$ OG; then F is the required C.G.

$$OF=\tfrac{1}{3}\,OG=\tfrac{1}{3}\times\tfrac{2}{3}\,OE=\tfrac{2}{9}\,OE.$$

8. (1) § 49. Example for Exercise 3. See Solution, p. 180.

(2) Let D and E be the mid-points of AB and BC. The weight of the rod may be considered to act vertically at D. The rod is acted on by three forces,—its weight, the tension of the string, and the reaction of the hinge. The lines of action of the two former forces pass through E, and therefore the line of action of the third force is AE. Thus the three forces are parallel to the sides of the triangle CAE, and if CA be taken to represent the weight, EC will on the same scale represent the tension. (§ 45.) But $EC=\frac{1}{2}BC=\frac{5}{6}$ CA, ∴ tension = weight of $8\frac{1}{3}$ lbs.

9. (1) §§ 84, 89. The resultant pressure on a body totally or partially immersed in a fluid is equal to the weight of the displaced fluid, and acts vertically upwards through the C.G. of the displaced fluid.

(2) § 88. Let W = weight of solid,

$$\therefore \cdot 8 = \frac{W - 40 \text{ gms.}}{W - 30 \text{ gms.}} \quad \therefore W = 80 \text{ grammes.}$$

It follows that the weight of the water displaced by the solid is 50 grammes. ∴ the volume of the water displaced, which is equal to the volume of the solid, is 50 cubic centimetres. (§ 86.)

10. (1) § 81. At depth x, pressure $= p + xq$, at depth x', pressure $= p + x'q$; therefore difference of pressures $= (x - x')q$, and is proportional to $x - x'$, the difference of level.

(2) Because the atmosphere is not of uniform density. See § 91.

(3 Let x feet = difference of level.

Now $p_1 v_1 = p_2 v_2$. (Boyle's Law.)

$\therefore 33 \times 1 = (33 + x) \times \frac{3}{4}$.

$\therefore x = 11$ feet.

11. (1) § 91.

(2) (a) it would be diminished, (b) it would be increased. See § 80.

(3) Height of barometer in diving-bell exceeds height at sea-level by $\frac{25 \times 12}{13 \cdot 6} = 22 \cdot 06$ inches nearly.

XIII.—Leaving Certificate—(1) Elementary

1. (1) §§ 4, 8. (2) Part of the rim of a fly-wheel which is revolving uniformly. (3) § 20. A velocity v making an angle of 120° with original velocity.

2. Find V from equations (20) in § 37 when $s = -1000$ and $t = 10$. V = 60 feet per second.

3. (1) § 40. (2) § 41. 600 feet per second.

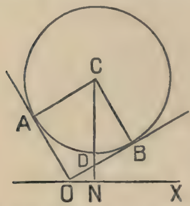
4. (1) § 53, (3).

(2) Show that this diagonal passes through the centres of gravity of the triangles into which the other diagonal divides the quadrilateral.

5. (1) § 42. (2) $\frac{20 \times 2240}{60} x = 22{,}000$, $\therefore x = 29\frac{13}{28}$ feet.

6. (1) and (2) § 72. (3) w = work done against gravity + work done against friction $= 10 \times 6 + 10 \times 5 \cdot 6 = 116$ foot-lbs. See §§ 70, 72.

7. The sphere is acted upon by three forces, viz. its own weight acting vertically downwards, the reaction of the one plane at A, and the reaction of the other plane at B. These three forces are in equilibrium, and the sides of the triangle BCD are parallel to their directions. Hence if CD represent the weight (10 lbs.), DB and BC will represent the reactions and therefore the pressures on the planes. Now angle BCD = angle BOX = 30°, therefore DB = $\frac{1}{2}$ CD. Hence DB represents 5 lbs. weight, and BC, $5\sqrt{3} = 8 \cdot 66$ lbs. weight.



8. (1) §§ 84, 85. (2) $s = \frac{5}{9} \times 13\frac{1}{2} = 7\frac{1}{2}$. See § 89.

9. (1) § 90. (2) $\rho_2 = \rho_1 \dfrac{p_2}{p_1} = 434 \times \dfrac{7}{15\frac{1}{2}} = 196$ grains.

(2) Higher Dynamics

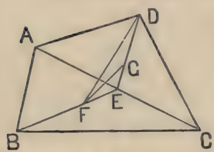
1. (1) §§ 14, 39.

(2) See figure 1, § 14. AB + 2BC + 3CD + 4DE + 5EA = (AB + BC + CD + DE + EA) + (BC + CD + DE + EA) + (CD + DE + EA) + (DE + EA) + EA = 0 + BA + CA + DA + EA.

2. See figure, § 68. W = tensions of three cords AD, BE, CF = P + (2P + w) + {2(2P + w) + w} = 7P + 4w.

3. (1) § 53 (3) and (5) *Example.*

(2) F is C.G. of triangle ABC, G is C.G. of triangle ADC, ∴ C.G. of quadrilateral lies in FG. But C.G. of four particles lies in FD, ∴ if the two centres of gravity coincide, FD coincides with FG, or BD passes through E, the mid-point of AC. Similarly AC passes through mid-point of BD, and it easily follows that ABCD is a parallelogram.



4. § 111.

5. (1) § 104.

(2) Let the point struck by the shot be a feet below the top of the cliff and b feet horizontally distant. Then $\tan \alpha = \dfrac{a}{b}$.

Also, if v be the initial velocity and t the time of flight of the first shot, $b = vt$ and $a = \frac{1}{2}gt^2 = \dfrac{b^2 g}{2v^2}$.

In the case of the second shot the initial horizontal velocity is

$v \sin a$ and the initial vertical velocity $v \cos a$. The shot will therefore travel a horizontal distance b in time $t' = \dfrac{b}{v \sin a}$, and during the same time it will fall vertically through a distance

$$x = \tfrac{1}{2}gt'^2 - v \cos a . t' = \frac{b^2 g}{2v^2 \sin^2 a} - \frac{b \cos a}{\sin a} = \frac{a}{\sin^2 a} - \frac{b}{\tan a}.$$

But $\sin^2 a = \dfrac{\tan^2 a}{1 + \tan^2 a} = \dfrac{a^2}{a^2 + b^2}$, $\therefore\ x = \dfrac{a^2 + b^2}{a} - \dfrac{b^2}{a} = a$.

Thus the second shot will strike the same point as the first.

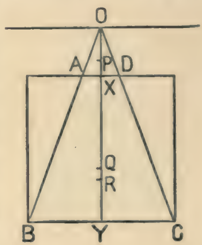
Another proof may be obtained from the fact that both parabolas have the same directrix (since the initial velocity is the same in both cases) and that the tangent to the parabola bisects the angle between the focal radius and the perpendicular on the directrix.

6. § 116.

7. (1) § 115. (2) Nearly 21 miles per hour per hour.

8. (1) § 118. Show that the centre of pressure of any part of the rectangle cut off by two horizontal lines is the centre of gravity of the part of the triangle cut off by the same horizontal lines.

(2) The centre of pressure is the centre of gravity of the figure ABCD. Now $OX = h - a$, and $OY = h + a$. Hence if P be the C.G. of triangle OAD and Q the C.G. of triangle OBC, we have $OP = \frac{2}{3}(h - a)$ and $OQ = \frac{2}{3}(h + a)$. Also the three areas OAD, OBC, and ABCD are in the ratios $(h-a)^2 : (h+a)^2 : (h+a)^2 - (h-a)^2$. Let R be the C.G. of ABCD. Taking moments about O, we have :—$OR\{(h+a)^2 - (h-a)^2\} + \frac{2}{3}(h-a)(h-a)^2 = \frac{2}{3}(h+a)(h+a)^2$, from which it easily follows that $OR = h + \dfrac{a^2}{3h}$.



XIV. Science and Art Department—Elementary

(1) *Solids*

1. (1) §§ 4, 30. (2) § 30. (3) $M = mv = 120$. (4) $1285\frac{5}{7}$.

2. (1) § 12.

(2) Draw a line AB representing 10 feet; draw AC, 12 feet to same scale, making 45° with AB. Join BC, which represents the imparted velocity in magnitude and direction. BC represents a velocity of $8\frac{1}{2}$ feet per second, and makes an angle of 100° with AB produced.

3. 10 feet per second per second.

4. (1) § 22 ($f=a$). (2) 172 feet per second.

5. (1) An absolute unit of force depends only on the fundamental units of length, mass, and time, and not at all upon the locality in which the unit is employed. See § 32.

(2) § 32. Note. (3) 60 poundals. (4) 60 : 12g or 1 : 6·4.

6. (1) and (2) § 57. (3) Taking the moment of the force 20 as positive, the moments are 40, – 28, – 20. (4) Sum = – 8.

7. (1) § 117. (2) Pounds, and feet per second. (3) 11 feet per second.

8. § 39 and § 17 (Graphic method).

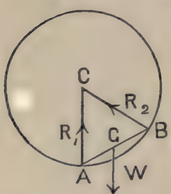
Resultant = nearly 9 units, making with AB an angle of about 77°.

9. (1) Find the centre of three parallel forces whose magnitudes are proportional to the three masses. § 51.

(2) Divide AB in D so that AD = 2BD ; bisect CD in G. G is the centre of gravity. G is nearly $1\frac{1}{2}$ inches from A.

10. (1) In a direction at right angles to the smooth surface.

(2) The three forces are—1°, the weight of the rod acting vertically through its C.G., and, 2° and 3°, the reactions at A and B whose directions are normal to the circle, and therefore, for all positions of the rod, pass through its centre.



(3) The condition for equilibrium of the rod is that W, R_1, and R_2 shall have their directions concurrent. See § 110. This will be the case only when G lies vertically under C, *i.e.* when the rod is horizontal.

11. (a) 3·2 feet per second per second. (b) 25·3 feet per second nearly. (c) 7·9 seconds nearly.

12. (1) An oscillation is a complete swing, to and fro, of the pendulum. (2) A small oscillation is one in which θ, the angle of maximum deviation from the vertical, is so small that it may be considered equal to sin θ. (3) π = ratio of circumference to diameter of a circle = 3·1416. l = length of pendulum. g = acceleration, due to gravity. (4) $\pi\sqrt{\frac{l}{g}}=\frac{60}{38}$, whence g = 31·7 feet per second per second approximately.

(2) *Fluids*

1. If the body be supported at that point it will remain in any position. (a) At the intersection of the diagonals. (b) At the mid-point of the axis of the cylinder. See §§ 52, 53, 54.

2. (1) $\frac{M_1}{M_2}=\frac{m_1v_1}{m_2v_2}=\frac{33}{20}$. (2) § 30.

3. (1) § 32. Note.

(2) $P = ma = \frac{mv}{t} = \frac{5 \times 12}{3} = 20$ poundals.

Similarly $Q = 21$ poundals.

$\therefore P : Q = 20 : 21$.

4. (1) § 43. (2) $\frac{1}{2}MV^2$. See § 117.

(3) The particle has an amount of energy $= \frac{1}{2}mv^2 = \frac{1}{2} \times 10 \times 24^2$ foot-poundals. In order that the particle may be brought to rest, this energy must be wholly expended in the form of work done, therefore if the particle exert a force of 4 poundals it must be through a space of $\frac{\frac{1}{2} \times 10 \times 24^2}{4}$ feet $= 720$ feet. Or, briefly, $Sf = \frac{1}{2}mv^2$

$\therefore S = \frac{\frac{1}{2}mv^2}{f}$.

5. (1) § 78.

(2) 3 : 7 (neglecting pressure of atmosphere). See § 81.

(3) The resultant fluid pressure is the weight of a column of water whose base is the area of the circle, and whose height is the depth of the centre of the circle. This gives pressure = weight of $\frac{22}{7} \times 1^2 \times 2\frac{1}{2} \times 1000$ oz. = weight of $491\frac{1}{14}$ lbs.

6. (1) §§ 84, 89.

(2) $\frac{1}{3}$ of cube, *i.e.* $\frac{1}{3}$ cubic foot is immersed $\therefore$ resultant pressure $= \frac{1}{3} \times 1000$ oz. = weight of $333\frac{1}{3}$ oz.

(3) $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ cubic foot is now immersed $\therefore$ resultant pressure = weight of $10\frac{5}{12}$ lbs., and it acts upwards through C.G. of immersed portion.

7. Read § 89 carefully.

(*a*) Let x = the length in feet of the part of the axis which is immersed, $\therefore \frac{x}{\text{axis}} = \cdot 7$, $\therefore x = \cdot 7 \times \text{axis} = 1\cdot 4$ feet.

(*b*) The force required to sustain the cylinder one inch higher is equal to the amount by which the upward pressure of the water is diminished, viz.: weight of $\frac{22}{7} \times 3^2 \times \frac{1}{12} \times 1000$ oz. = weight of $147\frac{9}{28}$ lbs.

8. (1) § 87. (2) § 84.

(3) $\frac{W}{W - 390} = \frac{15}{2}$ $\therefore W = 450$ grains.

9. and 10. See some text-book on Heat.

11. Using the formula given in Question 10, we get $\frac{p}{P}=\frac{335375}{599292}$ $=\cdot 56$ approximately.

12. (1) § 95.

(2) Force along piston-rod is equal to weight of column of water in pump. Hence force $=\frac{22}{7}\times\frac{1}{36}\times 20\times 1000$ oz. $=109$ lbs. nearly.

XV. Additional Examination Papers.

(A)

1. (1) § 13.

(2) § 17, Graphic Method. $V=7\cdot 65$ units approximately, making an angle of $112\frac{1}{2}°$ with the original velocity, or N.N.W.

2. (1) §§ 4, 8, 21.

(2) Apply equations (20) on p. 40. Putting $s=0$, and $t=6$, we find $V=3g=96$ feet per second approximately. Hence, when $v=0$, $s=4\frac{1}{2}g=144$ feet approximately.

3. (1) §§ 31, 32.

(2) 12 pounds weight $=12g$ units of force $=$ (putting $g=32$) 384 units of force. See § 34, equation (15).

4. (1) § 42.

(2) § 117. $fs=\frac{1}{2}mv^2$, $\therefore f=\frac{mv^2}{2s}$ absolute units $=\frac{mv^2}{2gs}$ lbs. $=$ weight of $468\frac{3}{4}$ lbs.

5. § 50. Ex. 1. The equilibrant is a force of 1 unit, acting parallel to, and in the same direction as, the less force, and at a point 3 feet beyond the greater. The figure is similar to the figure on page 60 if CB be made $=3$AC.

6. (1) § 61. (2) § 62.

7. (1) § 73.

(2) $\mu=\frac{\text{height}}{\text{base}}$ (when plane is in limiting position) $=\frac{1}{\sqrt{3}}=\cdot 577$ nearly.

8. (1) § 89. (2) § 120

9. (1) § 96.

(2) The force exerted on the plunger when in action, and hence the force which must be exerted by it, is the weight of a column of water 60 feet high, and 4 square inches in cross section.

$$f=60\times\frac{1}{36}\times 62\frac{1}{2}=\text{weight of } 104\tfrac{1}{6}\text{ lbs.}$$

(B)

1. (1) § 12. (2) See Example at end of § 14.

2. § 22. $s = Vt + \frac{1}{2}at^2 = 21{,}600$ yards $= 12\frac{3}{11}$ miles. It is convenient in this example to use a yard and a minute as the units of space and time.

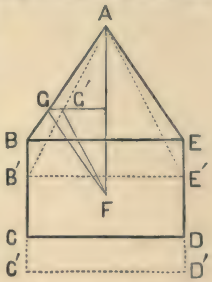
3. (1) § 31. (2) $a = \frac{f}{m} = \frac{g}{10} = 3{\cdot}2$ feet per second per second.

Now $s = \frac{1}{2}at^2 = \frac{1}{2} \times 3{\cdot}2 \times 2^2 = 6{\cdot}4$ feet.

4. (1) § 34. (2) Force of attraction $= \frac{20}{16} = 1\frac{1}{4}$.

5. (1) § 42. (2) $50\frac{50}{99}$ horse-power.

6. This rigid body BCDE is in equilibrium under the action of three forces (§§ 45, 110). These are the tensions along BA, EA, and the weight which must act through A. Draw AF vertically through A to represent the weight. Through F draw FG parallel to EA meeting AB in G. The three forces are parallel to, and therefore proportional to, the sides of the triangle AGF. Thus AG represents the tension of the cord on the same scale on which AF represents the weight of the picture. Now let the cord be lengthened, so that the picture takes the position B′C′D′E′. Then with the same construction AG′F is the triangle to whose sides the forces are proportional. Thus AG′, which is by geometry less than AG, represents the new tension. It is therefore proved that when the string is lengthened its tension is diminished. Compare Statics, general Ex. 47.



7. (1) § 72.

(2) $f = \mu R = \frac{1}{4} \times 2 \times 112g = 56g$ poundals = weight of 56 lbs.

8. § 68.

9. Let x feet = height of third floor. Then the weight of a column of water, whose height is x feet and whose base is a square inch, is 16 lbs.,

$$\therefore\ 62\tfrac{1}{2} \times \frac{x}{144} = 16 \quad \therefore\ x = 36{\cdot}864 \text{ feet.}$$

10. (1) § 90.

(2) See § 90, Ex. 2. Volume = 73·1 cubic feet approximately.

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The numbers refer to the sections.