MACKAY'S

Elements of Euclid

BOOKS I TO VI

PRICE 75 CENTS

THE

Elements of Euclid

Books I. to VI.

WITH
DEDUCTIONS, APPENDICES AND HISTORICAL NOTES

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PREFACE

In this text-book, compiled at the request of the publishers, a rigid adherence to Robert Simson's well-known editions of Euclid's *Elements* has not been observed; but no change has been made on Euclid's sequence of propositions, and comparatively little on his modes of proof. Here and there useful corollaries and converses have been inserted, and a few of Simson's additions have been omitted. Intimation of such insertions and omissions has been given, when it was deemed necessary, in the proper place. Several changes, mostly, however, of arrangement, have been made on the definitions.

By a slight alteration of the lettering or the construction of the figure, an attempt has been made throughout, and particularly in the Second Book, to draw the attention of the reader to the analogy which exists between certain pairs of propositions. By Euclid this analogy is well-nigh ignored.

In the naming of both congruent and similar figures, care has been taken to write the letters which denote corresponding points in a corresponding order. This is a matter of minor importance, but it does not deserve to be neglected, as is too often the case.

The deductions or exercises appended to the various propositions ('riders,' as they are sometimes termed) have been intentionally made easy and, in the First Book, numerous. It is hoped that beginners, who have little confidence in their own reasoning power, will thereby be encouraged to do more than merely learn the text of Euclid. It is hoped also that sufficient provision has been made for all classes of beginners, seeing that the questions, deductions, and corollaries to be proved number considerably over fifteen hundred. It should be stated that when a deduction is repeated once or oftener, in the same words, a different mode of proof is expected in each case.

In the appendices, much curtailed from considerations of space, a few of the more useful and interesting theorems of elementary geometry have been given. It has not been thought expedient to introduce the signs + and −, to indicate opposite directions of measurement. The important advantages which result from this use of these signs are readily apprehended by readers who advance beyond the 'elements,' and it is only of the 'elements' that the present manual treats.

The historical notes, which are not specially intended for beginners, may save time and trouble to any one who wishes to investigate more fully certain of the questions which occur throughout the work. It would perhaps be well if such notes were more frequently to be found in mathematical text-books: the names of those who have extended the boundaries, or successfully cultivated any part of the domain, of science should not be unknown to those who inherit the results of their labour.

Though the utmost pains have been taken by all concerned in the production of this volume to make it accurate and workmanlike, a few errors may have escaped notice. Corrections of these will be gratefully received.

The editor desires to express his thanks to Mr J. R. PAIRMAN for the excellence of the diagrams, and to Mr DAVID TRAILL, M.A., B.Sc., and Mr A. Y. FRASER, M.A., for valuable hints while the work was going through the press.

EDINBURGH ACADEMY,
April 1884.

CONTENTS.

BOOK I.

ANGLES, TRIANGLES, PARALLELS, PARALLELOGRAMS—

BOOK II.

RECTANGLES AND SQUARES—

BOOK III.

CIRCLES—

BOOK IV.

CIRCLES AND THEIR SCRIBED REGULAR POLYGONS, REGULAR POLYGONS AND THEIR SCRIBED CIRCLES—

APPENDIX IV.

BOOK V.

PROPORTION IN GENERAL—

BOOK VI.

PROPORTIONAL SECTION AND SIMILAR FIGURES—

APPENDIX VI.

EUCLID'S ELEMENTS.

BOOK I.

DEFINITIONS.

1. **A point** has position, but it has no magnitude.

A point is indicated by a dot with a letter attached, as the point P. · P

The dots employed to represent points are not strictly geometrical points, for they have some size, else they could not be seen. But in geometry the only thing connected with a point, or its representative a dot, which we consider, is its position.

2. **A line** has position, and it has length, but neither breadth nor thickness.

Hence the ends of a line are points, and the intersection of two lines is a point.

A line is indicated by a stroke with a letter attached, as the line *C*. C———

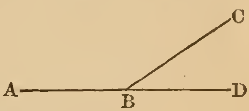
Oftener, however, a letter is placed at each end of the line, as the line *AB*. A————B

The strokes, whether of pen or pencil, employed to represent lines, are not strictly geometrical lines, for they have some breadth and some thickness. But in geometry the only things connected with a line which we consider, are its position and its length.

3. If two lines are such that they cannot coincide in any two points without coinciding altogether, each of them is called a **straight line**.

Hence two straight lines cannot inclose a space, nor can they have any part in common.

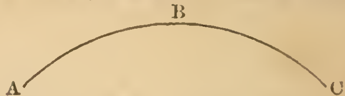
Thus the two lines *ABC* and *ABD*, which have the part *AB* in common, cannot both be straight lines.



Euclid's definition of a straight line is 'that which lies evenly to the points within itself.'

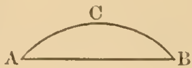
4. **A curved line**, or a **curve**, is a line of which no part is straight.

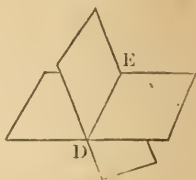
Thus ABC is a curve.



5. **A surface** (or superficies) has position, and it has length and breadth, but not thickness.

Hence the boundaries of a surface, and the intersection of two surfaces, are lines. Thus AB, ACB, and DE are lines.





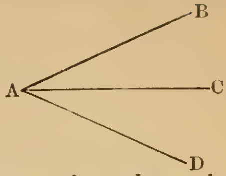
6. **A plane surface** (or a plane) is such that if any two points whatever be taken on it, the straight line joining them lies wholly in that surface.

This definition (which is not Euclid's, but is due to Heron of Alexandria) affords the practical test by which we ascertain whether a given surface is a plane or not. We take a piece of wood or iron with one of its edges straight, and apply this edge in various positions to the surface. If the straight edge fits closely to the surface in every position, we conclude that the surface is plane.

7. When two straight lines are drawn from the same point, they are said to contain a **plane angle**. The straight lines are called the **arms** of the angle, and the point is called the **vertex**.

Thus the straight lines AB, AC drawn from A are said to contain the angle BAC. AB and AC are the arms of the angle, and A is the vertex.

An angle is sometimes denoted by three letters, but these letters must be placed so that the one at the vertex shall always be between the other two. Thus the given angle is called BAC or CAB, never ABC, ACB, CBA, BCA. When only one angle is formed at a vertex it is often denoted by a single letter, that letter, namely, at the vertex. Thus the given angle may be called the angle A. But when there are several angles at the same vertex, it is necessary, in order to avoid ambiguity, to use three letters to express the angle intended. Thus, in the annexed figure, there are three angles at the vertex A, namely, BAC, CAD, BAD.



Sometimes the arms of an angle have several letters attached to them; in which case the angle may be denoted in various ways.

Fig. 1.

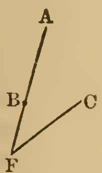


Fig. 2.

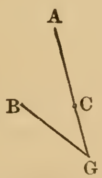
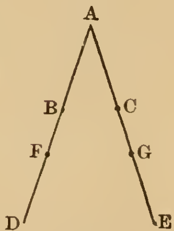
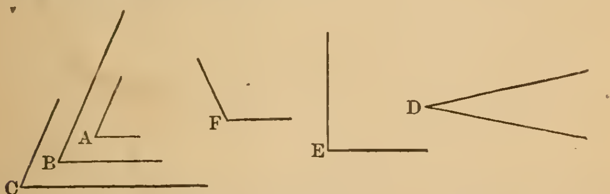


Fig. 3.



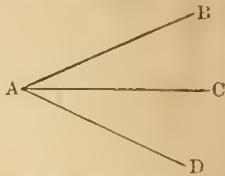
Thus the angle F (fig. 1) may be called AFC or BFC indifferently; the angle G (fig. 2) may be called AGB or CGB; the angle A (fig. 3) may be called BAC, FAG, DAE, FAC, GAB, and so on.

It is important to observe that all these ways of denoting any particular angle do not alter the angle; for example, the angle BAC (fig. 3) is not made any larger by calling it the angle FAG, or the angle DAE. In other words, the size of an angle does not depend on the length of its arms; and hence, if the two arms of one angle are respectively equal to the two arms of another angle, the angles themselves are not necessarily equal.



As a further illustration, the angles A, B, C with unequal arms are all equal; of the angles D, E, F, that with the shortest arms is the largest, and that with the longest arms is the smallest.

8. If three straight lines are drawn from the same point, three different angles are formed. Thus AB, AC, AD, drawn from A, form the three angles BAC, CAD, BAD.



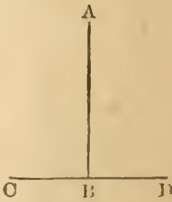
The angles BAC, CAD, which have a common arm AC, and lie on opposite sides of it, are called **adjacent** angles; and the angle BAD, which is equal to angle BAC and angle CAD added together, is called the **sum** of the angles BAC and CAD. Since the angle BAD is obtained by adding together the two angles BAC and CAD, the angle CAD will be obtained by subtracting the angle BAC from the angle BAD; and similarly the angle BAC will be obtained by subtracting the angle CAD from the angle BAD. Hence the angle CAD is called the **difference** of the angles BAD and BAC; and the angle BAC is called the **difference** of the angles BAD and CAD.

9. The **bisector** of an angle is the straight line that divides it into two equal angles.

Thus (see preceding fig.), if angle BAC is equal to angle CAD, AC is called the bisector of angle BAD.

The word *bisect*, in Mathematics, means always, to cut into two *equal* parts.

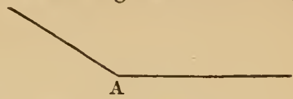
10. When a straight line stands on another straight line, and makes the adjacent angles equal to each other, each of the angles is called a **right** angle; and the straight line which stands on the other is called a **perpendicular** to it.



Thus, if AB stands on CD in such a manner that the adjacent angles ABC, ABD are equal to one another, then these angles are called right angles, and AB is said to be perpendicular to CD.

11. An **obtuse** angle is one which is greater than a right angle.

Thus A is an obtuse angle.



12. An **acute** angle is one which is less than a right angle.

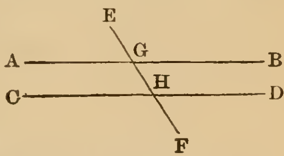
Thus B is an acute angle.

13. When two straight lines intersect each other, the opposite angles are called **vertically opposite angles.**

Thus AEC and BED are vertically opposite angles; and so are AED and BEC.

14. **Parallel** straight lines are such as are in the same plane, and being produced ever so far both ways do not meet.

Thus AB and CD are parallel straight lines.

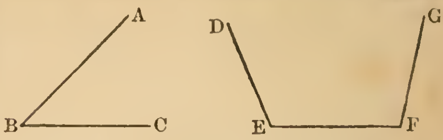


If a straight line EF intersect two parallel straight lines AB, CD, the angles AGH, GHD are called **alternate** angles, and so are angles BGH, GHC; angles AGE, BGE, CHF, DHF are called **exterior** angles, and the **interior opposite** angles corresponding to these are CHG, DHG, AGH, BGH.

15. **A figure** is that which is inclosed by one or more boundaries; and a **plane figure** is one bounded by a line or lines drawn upon a plane.

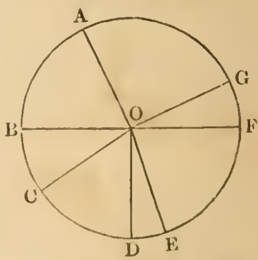
The space contained within the boundary of a plane figure is called its **surface**; and its surface in reference to that of another figure, with which it is compared, is called its **area.**

The word *figure*, as here defined, is restricted to closed figures Thus ABC, $DEFG$, according to the definition, would not be figures. The word is, however, very frequently used in a wider sense to mean any combination of points, lines, or surfaces.



16. **A circle** is a plane figure contained by one (*curved*) line which is called the **circumference**, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the **centre** of the circle.

Thus $ABCDEFG$ is a circle, if all the straight lines which can be drawn from O to the circumference, such as OA, OB, OC, &c., are equal to one another; and O is the centre of the circle.



Strictly speaking, a circle is an inclosed space or surface, and the circumference is the line which incloses it. Frequently, however, the word circle is employed instead of circumference.

It is usual to denote a circle by three letters placed at points on its circumference. The reason for this will appear later on.

17. **A radius** (plural, *radii*) of a circle is a straight line drawn from the centre to the circumference.

Thus OA, OB, OC, &c. are radii of the circle ACF.

18. **A diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

Thus in the preceding figure BF is a diameter of the circle ACF.

RECTILINEAL FIGURES.

19. **Rectilineal** figures are those which are contained by straight lines.

The straight lines are called **sides**, and the sum of all the sides is called the **perimeter** of the figure.

20. Rectilineal figures contained by three sides are called **triangles**.

21. Rectilineal figures contained by four sides are called **quadrilaterals**.

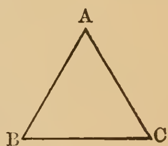
22. Rectilineal figures contained by more than four sides are called **polygons**.

Sometimes the word polygon is used to denote a rectilineal figure of any number of sides, the triangle and the quadrilateral being included.

CLASSIFICATION OF TRIANGLES.

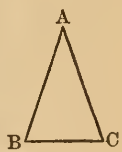
First, according to their sides—

23. An **equilateral** triangle is one that has three equal sides.



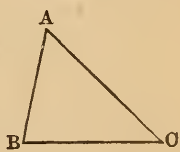
Thus, if AB, BC, CA are all equal, the triangle ABC is equilateral.

24. An **isosceles** triangle is one that has two equal sides.



Thus, if AB is equal to AC, the triangle ABC is isosceles.

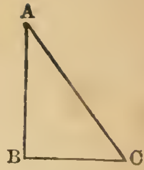
25. A **scalene** triangle is one that has three unequal sides.



Thus, if AB, BC, CA are all unequal, the triangle ABC is scalene.

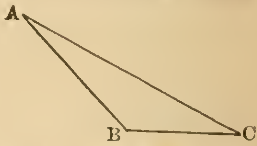
Second, according to their angles—

26. A **right-angled** triangle is one that has a right angle.



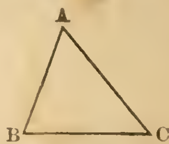
Thus, if ABC is a right angle, the triangle ABC is right-angled.

27. An **obtuse-angled** triangle is one that has an obtuse angle.



Thus, if ABC is an obtuse angle, the triangle ABC is obtuse-angled.

28. An **acute-angled** triangle is one that has three acute angles.



Thus, if angles A, B, C are each of them acute, the triangle ABC is acute-angled.

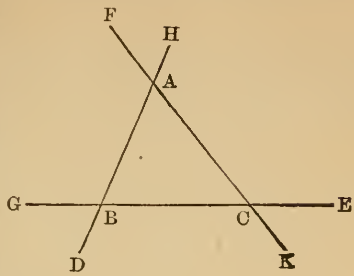
29. Any side of a triangle may be called **the base.** In an isosceles triangle, the side which is neither of the equal sides is usually called the **base.** In a right-angled triangle, one of the sides which contain the right angle is often called the **base**, and the other the **perpendicular**; the side opposite the right angle is called the **hypotenuse.**

Any of the angular points of a triangle may be called a **vertex.** If one of the sides of a triangle has been called the base, the angular point opposite that side is usually called the vertex.

Thus, if BC is called the base of a triangle ABC, A is the vertex.

30. If the sides of a triangle be prolonged both ways, nine angles are formed in addition to the angles of the triangle.

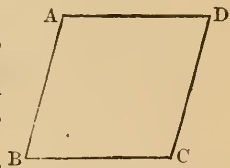
Thus at the point A there are the angles CAH, HAF, FAB; at B, the angles ABG, GBD, DBC; at C, the angles BCK, KCE, ECA.



Of these nine, six only are called **exterior** angles, the three which are not so called being HAF, GBD, KCE. Angles ABC, BCA, CAB are sometimes called the **interior** angles of the triangle.

CLASSIFICATION OF QUADRILATERALS.

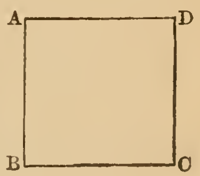
31. **A rhombus** is a quadrilateral that has all its sides equal.



Thus, if AB, BC, CD, DA are all equal, the quadrilateral $ABCD$ is a rhombus. The rhombus $ABCD$ is sometimes named by two letters placed at opposite corners, as AC or BD.

Euclid defines a rhombus to be 'a quadrilateral that has all its sides equal, but its angles not right angles.'

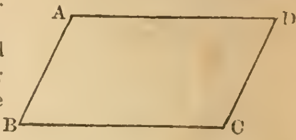
32. **A square** is a quadrilateral that has all its sides equal, and all its angles right angles.



Thus, if AB, BC, CD, DA are all equal, and the angles A, B, C, D right angles, the quadrilateral $ABCD$ is a square. The square $ABCD$ is sometimes named by two letters placed at opposite corners, as AC or BD; and it is said to be described on any one of its four sides.

33. **A parallelogram** is a quadrilateral whose opposite sides are parallel.

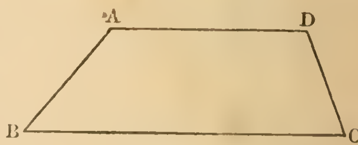
Thus, if AB is parallel to CD, and AD parallel to BC, the quadrilateral $ABCD$ is a parallelogram. The parallelogram $ABCD$ is some times named by two letters placed at opposite corners, as AC or BD; and any one of its four sides may be called the base on which it stands.



34. **A rectangle** is a quadrilateral whose opposite sides are parallel, and whose angles are right angles.

Thus, if AB is parallel to CD, AD parallel to BC, and the angles A, B, C, D right angles, the quadrilateral $ABCD$ is a rectangle. The rectangle $ABCD$ is sometimes named by two letters placed at opposite corners, as AC or BD. In books on mensuration, BC and AB would be called the length and the breadth of the rectangle. The definitions of a square and a rectangle are somewhat redundant—that is, more is said about a square and a rectangle than is absolutely necessar to distinguish them from other quadrilaterals. This will be se later on.

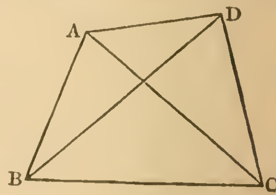
35. **A trapezium** is a quadrilateral that has two sides parallel.



Thus, if AD is parallel to BC, the quadrilateral $ABCD$ is a trapezium. The word *trapezoid* is sometimes used instead of trapezium.

36. **A diagonal** of a quadrilateral is a straight line joining any two opposite corners.

Thus AC and BD are diagonals of the quadrilateral $ABCD$.



POSTULATES.

Let it be granted :

1. That a straight line may be drawn from any one point to any other point.

2. That a terminated straight line may be produced to any length either way.

3. That a circle may be described with any centre, and at any distance from that centre.

The three postulates may be considered as stating the only instruments we are allowed to use in elementary geometry. These are the *ruler* or *straight-edge*, for drawing straight lines, and the *compasses*, for describing circles. The ruler is not to be divided at its edge (or graduated), so as to enable us to measure off particular lengths ; and the compasses are to be employed in describing circles only when the centre of the circle is at one given point, and the circumference must pass through another given point. Neither ruler nor compasses can be used to carry distances.

If two points A and B are given, and we wish to draw a straight line from A to B, it is usual to say simply 'join AB.' To *produce* a straight line, means not to make a straight line when there is none, but when there is a straight line already, to make it longer. The third postulate is sometimes expressed, 'a circle may be described with any centre and any radius.' That, however, is not to be taken as meaning with a radius equal to any given straight line, but only with a radius equal to any given straight line drawn from the centre.

[The restrictions imposed on the use of the ruler and the compasses, somewhat inconsistently on Euclid's part, are never adhered to in practice.]

AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals, the sums are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the sums are unequal, the greater sum being obtained from the greater unequal.

5. If equals be taken from unequals, the remainders are unequal, the greater remainder being obtained from the greater unequal.

6. Things which are doubles of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. The whole is greater than its part, and equal to the sum of all its parts.

9. Magnitudes which coincide with one another are equal to one another.

10. All right angles are equal to one another.

11. Two straight lines which intersect one another cannot be both parallel to the same straight line.

An axiom is a self-evident truth, or it is a statement the truth of which is admitted at once and without demonstration. Some of Euclid's axioms are general—that is, they apply to magnitudes of all kinds, and not to geometrical magnitudes only. The first axiom, which says that things which are equal to the same thing are equal to one another, applies not only to lines, angles, surfaces, and solids, but also, for example, to numbers, which are arithmetical, and to forces, which are physical, magnitudes. It will be seen that the first eight axioms are general, and that the last three are geometrical.

It ought, perhaps, to be noted that some of the axioms are often applied, not in the general form in which they are stated, but in particular cases that come under the general form. For example, under the general form of Axiom 2 would come two particular cases: If equals be added to the same thing, the sums are equal; and If the same thing be added to equals, the sums are equal. Again, a particular case coming under the general form of Axiom 4 would be: If the same thing be added to unequals, the sums are unequal.

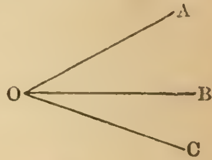
the greater sum being obtained from the greater unequal. Axioms 6 and 7, on the other hand, are only particular cases of more general ones—namely, Things which are double of equals are equal, and Things which are halves of equals are equal; and these axioms again are only particular cases of still more general ones: Similar multiples of equals (or of the same thing) are equal, and Similar fractions of equals (or of the same thing) are equal.

Axiom 9 is often called Euclid's definition or test of equality; and the method of ascertaining whether two magnitudes are equal by seeing whether they coincide—that is, by mentally applying the one to the other, is called the method of superposition. Two magnitudes (for example, two triangles) which coincide are said to be *congruent;* and this word, if it is thought desirable, may be used instead of the phrase, 'equal in every respect.' Axiom 10 is, strictly speaking, a proposition capable of proof. The proof is not given here, as at this stage it would perhaps not be fully appreciated by the pupil. After he has read and understood the definitions of the third book, he will probably be able to prove it for himself. Axiom 11, frequently referred to as Playfair's axiom (though Playfair states that it is assumed by others, particularly by Ludlam in his *Rudiments of Mathematics*), has been substituted for that given by Euclid, which is proved as a corollary to Proposition 29.

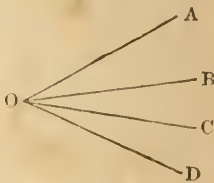
QUESTIONS ON THE DEFINITIONS, POSTULATES, AXIOMS.

1. How do we indicate a point?
2. What is the only thing that a point has? What has it not?
3. Could a number of geometrical points placed close to one another form a line? Why?
4. Draw two lines intersecting each other in two points.
5. Could two straight lines be drawn intersecting each other in two points?
6. What is Euclid's definition of a 'straight' line?
7. Could a number of geometrical lines placed close to one another form a surface? Why?
8. When two points are taken on a plane surface, and a straight line is drawn from the one to the other, where will the straight line lie?
9. If a straight line is drawn on a plane surface and then produced, where will the produced part lie?

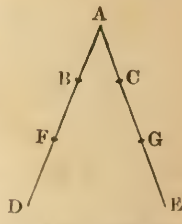
10. Would it be possible to draw a straight line upon a surface that was not plane? If so, give an example.
11. How many arms has an angle?
12. What name is given to the point where the arms meet?
13. When an angle is denoted by three letters, may the letters be arranged in any order?
14. If not, in how many ways may they be arranged, and what precaution must be observed?
15. When is it necessary to name an angle by three letters?
16. How else may an angle be named?
17. OA, OB, OC are three straight lines which meet at O. Name the three angles which they form.



18. Name the angle contained by OA and OB; by OB and OC; by OC and OA.
19. OA, OB, OC, OD are four straight lines which meet at O. Name the six angles which they form.
20. Name the angle contained by OA and OB; by OB and OC; by OC and OD; by OA and OC; by OB and OD; by OA and OD.



21. Write down all the ways in which the angle A can be named.
22. If the arms of one angle are respectively equal to the arms of another angle, what inference can we draw regarding the sizes of the angles?
23. In the figure to Question 17, if the angles AOB and BOC are added together, what angle do they form?

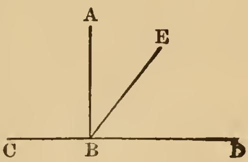


24. In the same figure, if the angle AOB is taken away from the angle AOC, what angle is left?
25. In the same figure, if the angle BOC is taken away from the angle AOC, what angle is left?
26. The following questions refer to the figure to Question 19:
    (a) Add together the angles AOB and BOC; AOB and BOD; AOC and COD; BOC and COD.
    (b) From the angle AOD subtract successively the angles COD, AOB, AOC, BOD.

(c) From the angle BOD subtract the angles COD, BOC.

(d) To the sum of the angles AOB and BOC add the difference of the angles BOD and BOC; and from the sum of AOB and BOC subtract the difference of BOD and COD.

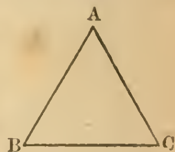
27. Draw, as well as you can, two equal angles with unequal arms.
28. " " two unequal " equal "
29. If two adjacent angles are equal, must they necessarily be right angles? Draw a figure to illustrate your answer.
30. If two adjacent angles are equal, what name could be given to the arm that is common to the two angles?
31. When an angle is greater than a right angle, what is it called?
32. " less " "
33. " equal to " "
34. In the accompanying figure, name two right angles, two acute angles, and one obtuse angle.



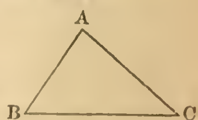
35. What are angles AEC, AED called with reference to each other? angles AEC, BED? angles AEC, BEC? angles BEC, AED? angles BEC, BED?

36. Would it be a sufficient definition of parallel straight lines to say that they never meet though produced indefinitely far either way? Illustrate your answer by reference to the edges of a book, or otherwise.
37. Draw three straight lines, every two of which are parallel.
38. Draw three straight lines, only two of which are parallel.
39. Draw three straight lines, no two of which are parallel.
40. What is the least number of lines that will inclose a space? Illustrate your answer by an example.
41. How many radii of a circle are equal to one diameter?
42. How do we know that all radii of a circle are equal?
43. Prove that all diameters of a circle are equal.
44. Are all lines drawn from the centre of a circle to the circumference equal to one another?
45. What is the distinction between a circle and a circumference?
46. Is the one word ever used for the other?

47. How many letters are generally used to denote a circle?
48. Would it be a sufficient definition of a diameter of a circle to say that it consists of two radii?
49. Prove that the distance of a point inside a circle from the centre is less than a radius of the circle.
50. Prove that the distance of a point outside a circle from the centre is greater than a radius of the circle.
51. What is the least number of straight lines that will inclose a space?
52. What name is given to figures that are contained by straight lines?
53. Could three straight lines be drawn so that, even if they were produced, they would not inclose a space?
54. What is the least number of sides that a rectilineal figure can have?

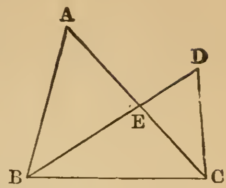


55. *ABC* is a triangle. Name it in five other ways.
56. If *AB* is equal to *AC*, what is triangle *ABC* called?
57. If *AB*, *BC*, *CA* are all equal, what is triangle *ABC* called?
58. If *AB*, *BC*, *CA* are all unequal, what is triangle *ABC* called?
59. What name is given to the sum of *AB*, *BC*, and *CA*?
60. Which side of a triangle is called the base?
61. Which side of an isosceles triangle is called the base?
62. When the hypotenuse of a triangle is mentioned, of what sort must the triangle be?
63. What names are sometimes given to those sides of a right-angled triangle which contain the right angle?
64. Would it be a sufficient definition of an acute-angled triangle to say that it had neither a right nor an obtuse angle?

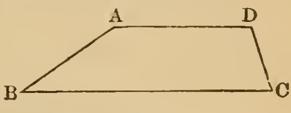


65. *ABC* is a triangle. Name by one letter the angles respectively opposite to the sides *AB*, *BC*, *CA*.
66. Name by three letters the angles respectively opposite to the sides *AB*, *BC*, *CA*.
67. Name the sides respectively opposite to the angles *A*, *B*, *C*.
68. Name by one letter and by three letters the angle contained by *AB* and *AC*; by *AB* and *BC*; by *AC* and *BC*.

69. Name all the triangles in the accompanying figure.
70. Name the additional triangles that would be formed if AD were joined.
71. Name by three letters all the angles opposite to BC; to BE; to CE.
72. Name all the sides that are opposite to angle A; to angle D.
73. Name all the angles in the figure that are called exterior angles of the triangle BEC; of the triangle AEB; of the triangle CED.



74. $ABCD$ is a quadrilateral. Name it in seven other ways.
75. If the diagonals AC, BD be drawn, and E be their point of intersection, how many triangles will there be in the diagram? Name them.



76. Name the two angles opposite to the diagonal AC.
77. " " " BD.
78. " through which the diagonal AC passes.
79. " " " " BD "
80. Could a square, with propriety, be called a rhombus?
81. Could a rhombus be called a square?
82. Could a rectangle be called a parallelogram?
83. Could a parallelogram be called a rectangle?
84. Would it be a sufficient definition of a parallelogram to say that it is a figure whose opposite sides are parallel? Why?
85. Could a parallelogram or a rectangle be called a trapezium?
86. Could a trapezium be called a parallelogram or a rectangle?
87. What is a diagonal of a quadrilateral, and how many diagonals has a quadrilateral?
88. How many sides has a polygon?
89. Which postulate allows us to join two points?
90. " " produce a straight line?
91. " " describe a circle?
92. In what sense is the word 'circle' used in the third postulate?
93. What are the only instruments that may be used in elementary plane geometry? Under what restrictions are they to be used?
94. What is an axiom? Give an example of one.
95. State Euclid's axiom about magnitudes which coincide.

96. Would it be correct to say, magnitudes which fill the same space, instead of magnitudes which coincide? Illustrate your answer by reference to straight lines, and angles.
97. What is Euclid's axiom about right angles?
98. What is the axiom about parallels?
99. Would it be correct to say, two straight lines which pass through the same point cannot be both parallel to the same straight line?
100. Could two straight lines which do not pass through the same point be both parallel to a third straight line?

EXPLANATION OF TERMS.

Propositions are divided into two classes, theorems and problems.

A **theorem** is a truth that requires to be proved by means of other truths already known. The truths already known are either axioms or theorems.

A **problem** is a construction which is to be made by means of certain instruments. The instruments allowed to be used are (see the remarks on the postulates) the ruler and the compasses.

A **corollary** is a truth which is (more or less) easily inferred from a proposition.

In the statement of a theorem there are two parts, the **hypothesis** and the **conclusion**. Thus, in the theorem, 'If two sides of a triangle be equal, the angles opposite to them shall be equal,' the part, 'if two sides of a triangle be equal,' is the hypothesis, or that which is assumed; the other part, 'the angles opposite to them shall be equal,' is the conclusion, or that which is inferred from the hypothesis.

The **converse** of a theorem is derived from the theorem by interchanging the hypothesis and the conclusion. Thus, the converse of the theorem mentioned above is, 'If in a triangle the angles opposite two sides be equal, the sides shall be equal.'

When the hypothesis of a theorem consists of several hypotheses, there may be more than one converse to the theorem.

In proving propositions, recourse is sometimes had to the following method. The proposition is supposed not to be true, and the consequences of this supposition are then examined, till at length a result is reached which is impossible or absurd. It is therefore inferred that the proposition must be true. Such a method of proof is called an **indirect** demonstration, or sometimes a **reductio ad absurdum** (a reducing to the absurd).

SYMBOLS AND ABBREVIATIONS.

+, read *plus*, is the sign of addition, and signifies that the magnitudes between which it is placed are to be added together.

−, read *minus*, is the sign of subtraction, and signifies that the magnitude written after it is to be subtracted from the magnitude written before it.

~, read *difference*, is sometimes used instead of minus, when it is not known which of the two magnitudes before and after it is the greater.

= is the sign of equality, and signifies that the magnitudes between which it is placed are equal to each other. It is used here as an abbreviation for 'is equal to,' 'are equal to,' 'be equal to,' and 'equal to.'

⊥ stands for 'perpendicular to,' or 'is perpendicular to.'

‖ " 'parallel to,' or 'is parallel to.'

∠ " 'angle.'

△ " 'triangle.'

▱^{m} " 'parallelogram.'

⊙ " 'circle.'

○^{ce} " 'circumference.'

∴ " 'therefore.' This symbol turned upside down (∵), which is sometimes used for 'because' or 'since,' I have not introduced, partly because some writers use it for 'therefore,' and partly because it is easily confounded with the other.

AB^2 stands for 'the square described on AB.'

$AB \cdot BC$, stands for 'the rectangle contained by AB and BC.'

$A : B$ stands for 'the ratio of A to B.'

$\left\{\begin{matrix} A : B \\ B : C \end{matrix}\right\}$ stands for 'the ratio compounded of the ratios of A to B and B to C.'

$A : B = C : D$ stands for the proportion 'A is to B as C is to D.'
The small letters a, b, c, m, n, p, &c. stand for numbers.

App.	stands for	'appendix.'
Ax.	"	'axiom.'
Const.	"	'construction.'
Cor.	"	'corollary.'
Def.	"	'definition.'
Hyp.	"	'hypothesis.'
Post.	"	'postulate.'
Rt.	"	'right.'

In the references given at the right-hand side of the page (Euclid gives no references), the Roman numerals indicate the number of the book, the Arabic numerals the number of the proposition. Thus, I. 47 means the forty-seventh proposition of the first book.

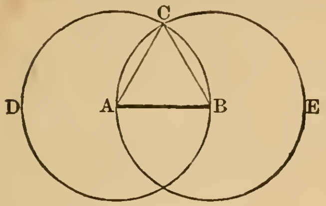
In the figures to certain of the theorems, it will be seen that some lines are *thick*, and some *dotted*. The thick lines are those which are given, the dotted lines are those which are drawn in order to prove the theorem. [In a few figures this arrangement has been neglected to attain another object.]

In the figures to certain of the problems, some lines are *thick*, some *thin*, and some *dotted*. The thick lines are those which are given, the thin lines are those which are drawn in order to effect the construction, and the dotted lines are those which are necessary for the proof that the construction is correct.

In the figures which illustrate definitions, the lines are almost invariably thin.

PROPOSITION 1. PROBLEM.

To describe an equilateral triangle on a given straight line.



Let AB be the given straight line :
it is required to describe an equilateral triangle on AB.

With centre A and radius AB, describe $\odot$ BCD. *Post.* 3
With centre B and radius BA, describe $\odot$ ACE; *Post.* 3
and let the two circles intersect at C.
Join AC, BC. *Post.* 1

ABC shall be an equilateral triangle.

For $AB = AC$, being radii of the $\odot$ BCD; *I. Def.* 16
and $AB = BC$, being radii of the $\odot$ ACE; *I. Def.* 16
$\therefore$ $AC = BC$. *I. Ax.* 1
$\therefore$ AB, AC, BC are all equal,
and ABC is an equilateral triangle. *I. Def.* 23

DEDUCTIONS.

1. If the two circles intersect also at F, and AF, BF be joined, prove that ABF is an equilateral triangle.
2. Show how to find a point which is equidistant from two given points.
3. Show how to make a rhombus having one of its diagonals equal to a given straight line.
4. Show how to make a rhombus having each of its sides equal to a given straight line.

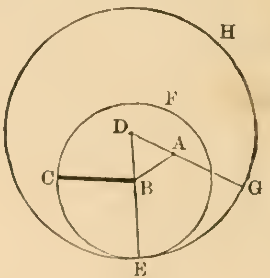
5. If AB be produced both ways to meet the two circles again at D and E, prove that the straight line DE is equal to the sum of the three sides of the triangle ABC.
6. Show how to find a straight line equal to the sum of the three sides of any triangle.

Show how to find a straight line which shall be :

7. Twice as great as a given straight line.
8. Thrice " " "
9. Four times " " "
10. Five " " " " &c.

PROPOSITION 2. Problem.

From a given point to draw a straight line equal to a given straight line.



Let A be the given point, and BC the given straight line: *it is required to draw from A a straight line = BC.*

Join AB, *Post.* 1
and on it describe the equilateral $\triangle$ DBA. *I.* 1
With centre B and radius BC, describe the $\odot$ CEF; *Post.* 3
and produce DB to meet the $\bigcirc^{ce}$ CEF in E. *Post.* 2

With centre D, and radius DE, describe the $\odot$ EGH; *Post.* 3
and produce DA to meet the $\bigcirc^{ce}$ EGH in G. *Post.* 2

AG shall $= BC$.

Because $DE = DG$, being radii of $\odot$ EGH, *I. Def.* 16
and $DB = DA$, being sides of an equilateral triangle; *I. Def.* 23
$\therefore$ remainder $BE =$ remainder AG. *I. Ax.* 3
But $BE = BC$, being radii of $\odot$ CEF; *I. Def.* 16
$\therefore$ $AG = BC$. *I. Ax.* 1

1. If the radius of the large circle be double the radius of the small circle, where will the given point be?
2. AB is a given straight line; show how to draw from A any number of straight lines equal to AB.
3. AB is a given straight line; show how to draw from B any number of straight lines equal to AB.
4. AB is a given straight line; show how to draw through A any number of straight lines double of AB.
5. AB is a given straight line; show how to draw through B any number of straight lines double of AB.
6. On a given straight line as base, describe an isosceles triangle each of whose sides shall be equal to a given straight line.

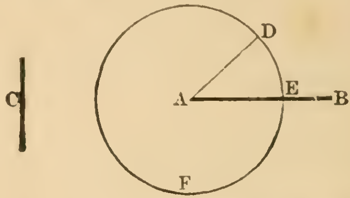
   May the second given straight line be of any size? If not, how large or how small may it be?

Give the construction and proof of the proposition—

7. When the equilateral triangle ABD is described on that side of AB opposite to the one given in the text.
8. When the equilateral triangle ABD is described on the same side of AB as in the text, but when its sides are produced through the vertex and not beyond the base.
9. When the equilateral triangle ABD is described on that side of AB opposite to the one given in the text, and when its sides are produced through the vertex.
10. When the given point A is joined to C instead of B. Make diagrams for all the cases that can arise by describing the equilateral triangle on either side of AC, and producing its sides either beyond the base or through the vertex.

PROPOSITION 3. Problem.

From the greater of two given straight lines to cut off a part equal to the less.



Let AB and C be the two given straight lines, of which AB is the greater:

it is required to cut off from AB a part $= C$.

From A draw the straight line $AD = C$; *I.* 2

with centre A and radius AD, describe the ⊙ DEF, *Post.* 3

cutting AB at E. AE shall $= C$.

For $AE = AD$, being radii of ⊙ DEF. *I. Def.* 16

But $AD = C$; *Const.*

$\therefore$ $AE = C$. *I. Ax.* 1

1. Give the construction and the proof of this proposition, using the point B instead of the point A.
2. Produce the less of two given straight lines so that it may be equal to the greater.
3. If from AB (fig. 1 and fig. 2) there be cut off AD and BE, each equal to C, prove $AE = BD$.

Fig. 1.

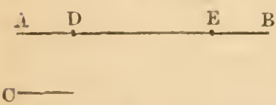
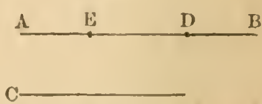


Fig. 2.



4. Show how to find a straight line equal to the sum of two given straight lines.

5. Show how to find a straight line equal to the difference of two given straight lines.
6. Show that if the difference of two straight lines be added to the sum of the two straight lines, the result will be double of the greater straight line.
7. Show that if the difference of two straight lines be taken away from the sum of the two straight lines, the result will be double of the less straight line.

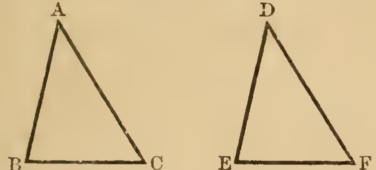
PROPOSITION 4. Theorem.

If two sides and the contained angle of one triangle be equal to two sides and the contained angle of another triangle, the two triangles shall be equal in every respect—that is,

(1) *The third sides shall be equal,*

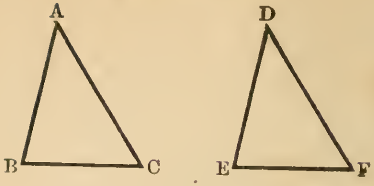
(2) *The remaining angles of the one triangle shall be equal to the remaining angles of the other triangle,*

(3) *The areas of the two triangles shall be equal.*



In $\triangle$s ABC, DEF, let $AB = DE$, $AC = DF$, $\angle A = \angle D$: *it is required to prove* $BC = EF$, $\angle B = \angle E$, $\angle C = \angle F$, $\triangle ABC = \triangle DEF$.

If $\triangle ABC$ be applied to $\triangle DEF$,
so that A falls on D, and so that AB falls on DE;
then B will coincide with E, because $AB = DE$. *Hyp.*
And because AB coincides with DE, and $\angle A = \angle D$, *Hyp.*
$\therefore$ AC will fall on DF.
And because $AC = DF$, *Hyp.*
$\therefore$ C will coincide with F.



Now, since B coincides with E, and C with F,

$\therefore$ BC will coincide with EF; *I. Def.* 3

$\therefore$ $BC = EF$. *I. Ax.* 9

Hence also $\angle B$ will coincide with $\angle E$;

$\therefore$ $\angle B = \angle E$; *I. Ax.* 9

and $\angle C$ will coincide with $\angle F$; $\therefore$ $\angle C = \angle F$; *I. Ax.* 9

and $\triangle ABC$ will coincide with $\triangle DEF$;

$\therefore$ $\triangle ABC = \triangle DEF$. *I. Ax.* 9

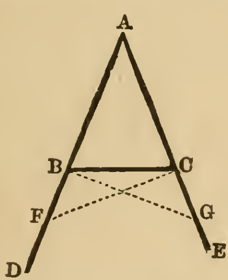
In the two $\triangle$s ABC, DEF,

1. If $AB = DE$, $AC = DF$, but $\angle A$ greater than $\angle D$, where would AC fall when ABC is applied to DEF as in the proposition?
2. If $AB = DE$, $AC = DF$, but $\angle A$ less than $\angle D$, where would AC fall?
3. If $AB = DE$, $\angle A = \angle D$, but AC greater than DF, where would C fall?
4. If $AB = DE$, $\angle A = \angle D$, but AC less than DF, where would C fall?
5. Prove the proposition beginning the superposition with the point B or the point C instead of the point A.
6. If the straight line CD bisect the straight line AB perpendicularly, prove any point in CD equidistant from A and B.
7. CA and CB are two equal straight lines drawn from the point C, and CD is the bisector of $\angle ACB$. Prove that any point in CD is equidistant from A and B.
8. The straight line that bisects the vertical angle of an isosceles triangle bisects the base and is perpendicular to the base.
9. $ABCD$ is a quadrilateral, one of whose diagonals is BD. If $AB = CB$, and BD bisects $\angle ABC$, prove that $AD = CD$, and that BD bisects also $\angle ADC$.
10. Prove that the diagonals of a square are equal.

11. $ABCD$ is a square. E, F, G, H are the middle points of AB, BC, CD, DA, and EF, FG, GH, HE are joined. Prove that $EFGH$ has all its sides equal.
12. Prove by superposition that the squares described on two equal straight lines are equal.
13. If two quadrilaterals have three consecutive sides and the two contained angles in the one respectively equal to three consecutive sides and the two contained angles in the other, the quadrilaterals shall be equal in every respect.

PROPOSITION 5. Theorem.

The angles at the base of an isosceles triangle are equal, and if the equal sides be produced, the angles on the other side of the base shall also be equal.



In $\triangle\ ABC$, let $AB = AC$, and let AB, AC be produced to D and E:

it is required to prove $\angle\ ABC = \angle\ ACB$ *and* $\angle\ DBC = \angle\ ECB$.

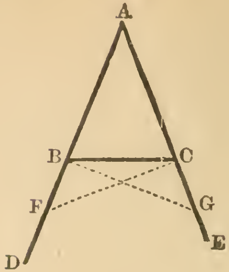
In BD take any point F,

and from AE cut off $AG = AF$; *I.* 3

join BG, CF. *Post.* 1

(1) in $\triangle$s AFC, AGB, $\left\{\begin{array}{ll} FA = GA & \textit{Const.} \\ AC = AB & \textit{Hyp.} \\ \angle\ FAC = \angle\ GAB; & \end{array}\right.$

$\therefore FC = GB, \angle\ AFC = \angle\ AGB, \angle\ ACF = \angle\ ABG.$ *I.* 4



(2) Because the whole AF = whole AG, *Const.*
and the part AB = part AC; *Hyp.*
the remainder BF = remainder CG. *I. Ax.* 3

(3) In $\triangle$s BFC, CGB, $\left\{\begin{array}{ll} BF = CG & \textit{Proved in (2)} \\ FC = GB & \textit{Proved in (1)} \\ \angle BFC = \angle CGB; & \textit{Proved in (1)} \end{array}\right.$

$\therefore$ $\angle BCF = \angle CBG$, and $\angle FBC = \angle GCB$. *I.* 4

(4) Because whole $\angle ABG$ = whole $\angle ACF$, *Proved in* (1)
and the part $\angle CBG$ = part $\angle BCF$; *Proved in* (3)
$\therefore$ the remainder $\angle ABC$ = remainder $\angle ACB$; *I. Ax.* 3
and these are the angles at the base.
But it was proved in (3) that $\angle FBC = \angle GCB$;
and these are the angles on the other side of the base.

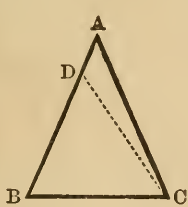
Cor.—If a triangle have all its sides equal, it will also have all its angles equal; or, in other words, if a triangle be equilateral, it will be equiangular.

1. If two angles of a triangle be unequal, the sides opposite to them will also be unequal.
2. Two isosceles triangles ABC, DBC stand on the same base BC, and on opposite sides of it; prove $\angle ABD = \angle ACD$.
3. Two isosceles triangles ABC, DBC stand on the same base BC, and on the same side of it; prove $\angle ABD = \angle ACD$.
4. In the figure to the second deduction, if AD be joined, prove that it will bisect the angles at A and D.

5. ABC is an isosceles triangle having $AB = AC$. In AB, AC, two points D, E are taken equally distant from A; prove that the triangles ABE, ACD are equal in all respects, and also the triangles DBC, ECB.
6. Prove that the opposite angles of a rhombus are equal.
7. D and E are the middle points of the sides BC and CA of a triangle; DO and EO are perpendicular to BC and CA; show that the angles OAB and OBA are equal.
8. Prove the proposition by supposing the $\triangle$ ABC, after leaving a trace or impression of itself, to be lifted up, turned over, and applied to the trace.
9. Prove the first part of the proposition by supposing the angle at the vertex to be bisected.

PROPOSITION 6. Theorem.

If two angles of a triangle be equal, the sides opposite them shall also be equal.



In $\triangle$ ABC let $\angle ABC = \angle ACB$:
it is required to prove $AC = AB$.

If AC is not $= AB$, one of them must be the greater.
Let AB be the greater;
and from it cut off $BD = AC$, *I.* 3
and join DC. *Post.* 1

In $\triangle$s DBC, ACB, $\left\{\begin{array}{ll} DB = AC & \textit{Const.} \\ BC = CB & \\ \angle DBC = \angle ACB & \textit{Hyp.} \end{array}\right.$

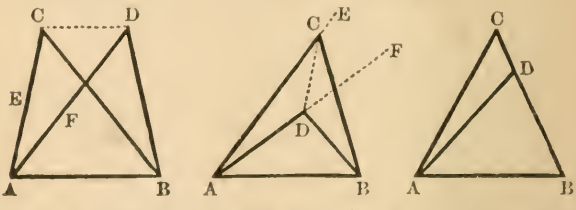
$\therefore$ area of $\triangle\ DBC$ = area of $\triangle\ ACB$; *I.* 4
which is impossible, since $\triangle\ DBC$ is a part of $\triangle\ ACB$.
Hence AC is not unequal to AB;
that is, $AC = AB$.

Cor.—If a triangle have all its angles equal, it will also have all its sides equal; or, in other words, if a triangle be equiangular, it will be equilateral.

1. If two sides of a triangle be unequal, the angles opposite to them will also be unequal.
2. If ABC be an isosceles triangle, and if the equal angles ABC, ACB be bisected by BD, CD, which meet at D; prove that DBC is also an isosceles triangle.
3. In the figure to *I.* 5, if BG, CF intersect at H, prove that HBC is an isosceles triangle.
4. Hence prove that $FH = GH$, and that AH bisects $\angle\ A$.
5. By means of what is proved in the last deduction, give a method of bisecting an angle.
6. Prove the proposition by supposing the $\triangle\ ABC$, after leaving a trace or impression of itself, to be lifted up, turned over, and applied to the trace.

PROPOSITION 7. Theorem.

Two triangles on the same base and on the same side of it cannot have their conterminous sides equal.



If it be possible, let the two $\triangle$s ABC, ABD on the same base AB, and on the same side of it, have $AC = AD$, and $BC = BD$.

Three cases may occur:

(1) The vertex of each $\triangle$ may be outside the other $\triangle$.

(2) The vertex of one $\triangle$ may be inside the other $\triangle$.

(3) The vertex of one $\triangle$ may be on a side of the other $\triangle$.

In the first case join CD; and in the second case join CD and produce AC, AD to E and F.

Because $AC = AD$, $\therefore \angle ECD = \angle FDC$. *I.* 5

But $\angle ECD$ is greater than $\angle BCD$; *I. Ax.* 5

$\therefore \angle FDC$ is greater than $\angle BCD$.

Much more then is $\angle BDC$ greater than $\angle BCD$.

But because $BC = BD$, $\therefore \angle BDC = \angle BCD$; *I.* 5

that is, $\angle BDC$ is greater than and equal to $\angle BCD$, which is impossible.

The third case needs no proof, because BC is not $= BD$.

Hence two triangles on the same base and on the same side of it cannot have their conterminous sides equal.

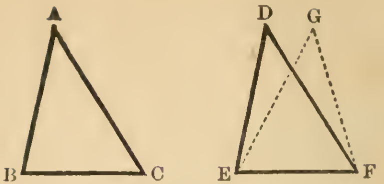
1. On the same base and on the same side of it there can be only one equilateral triangle.
2. On the same base and on the same side of it there can be only one isosceles triangle having its sides equal to a given straight line.
3. Two circles cannot cut each other at more than one point either above or below the straight line joining their centres.

PROPOSITION 8. Theorem.

If three sides of one triangle be respectively equal to three sides of another triangle, the two triangles shall be equal in every respect; that is,

(1) *The three angles of the one triangle shall be respectively equal to the three angles of the other triangle,*

(2) *The areas of the two triangles shall be equal.*



In $\triangle$s ABC, DEF, let $AB = DE$, $AC = DF$, $BC = EF$: *it is required to prove* $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, *and* $\triangle ABC = \triangle DEF$.

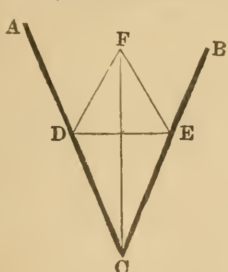
If $\triangle ABC$ be applied to $\triangle DEF$,
so that B falls on E, and so that BC falls on EF;
then C will coincide with F, because $BC = EF$. *Hyp.*
Now since BC coincides with EF,
$\therefore$ BA and AC must coincide with ED and DF.
For, if they do not, but fall otherwise as EG and GF;
then on the same base EF, and on the same side of it;
there will be two $\triangle$s DEF, GEF, having equal pairs of conterminous sides,
which is impossible. *I.* 7
$\therefore$ BA coincides with ED, and AC with DF.
Hence $\angle A$ will coincide with $\angle D$, $\therefore$ $\angle A = \angle D$; *I. Ax.* 9
and $\angle B$ will coincide with $\angle E$, $\therefore$ $\angle B = \angle E$; *I. Ax.* 9
and $\angle C$ will coincide with $\angle F$, $\therefore$ $\angle C = \angle F$; *I. Ax.* 9
and $\triangle ABC$ will coincide with $\triangle DEF$,
$\therefore$ $\triangle ABC = \triangle DEF$. *I. Ax.* 9

1. The straight line which joins the vertex of an isosceles triangle to the middle point of the base, is perpendicular to the base, and bisects the vertical angle.
2. The opposite angles of a rhombus are equal.
3. Either diagonal of a rhombus bisects the angles through which it passes.
4. $ABCD$ is a quadrilateral having $AB = BC$ and $AD = DC$; prove that the diagonal BD bisects the angles through which it passes, and that $\angle A = \angle C$.

5. Two isosceles triangles stand on the same base and on opposite sides of it; prove that the straight line joining their vertices bisects both vertical angles.
6. Two isosceles triangles stand on the same base and on the same side of it; prove that the straight line joining their vertices, being produced, bisects both vertical angles.
7. In the figures to the fifth and sixth deductions, prove that the straight line joining the vertices, or that straight line produced, bisects the common base perpendicularly.
8. Hence give a construction for bisecting a given straight line.
9. The diagonals of a rhombus or of a square bisect each other perpendicularly.
10. If any two circles cut each other, the straight line joining their points of intersection is bisected perpendicularly by the straight line joining their centres.
11. Prove the proposition by applying the triangles so that they may fall on opposite sides of a common base. Join the two vertices, and use I. 5 (Philon's method; see Friedlein's *Proclus*, p. 266).

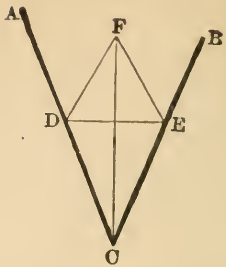
PROPOSITION 9. Problem.

To bisect a given rectilineal angle.



Let ACB be the given rectilineal angle:
it is required to bisect it.

In AC take any point D,
and from CB cut off $CE = CD$. I. 3



Join DE, and on DE, on the side remote from C, describe the equilateral $\triangle\ DEF$. *I.* 1

Join CF. CF shall bisect $\angle\ ACB$.

$$\text{In } \triangle\text{s } DCF,\ ECF, \left\{ \begin{array}{l} DC = EC \\ CF = CF \\ DF = EF; \end{array} \right. \qquad \begin{array}{r} \textit{Const.} \\ \\ \textit{I. Def. } 23 \end{array}$$

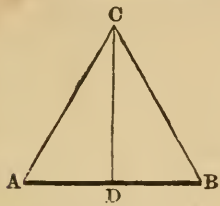
$\therefore \angle\ DCF = \angle\ ECF$; *I.* 8

that is, CF bisects $\angle\ ACB$.

1. Prove that CF bisects angle DFE.
2. If the equilateral triangle DEF were described on the same side of DE as C is, what three positions might F take?
3. Show that in one of these positions the demonstration remains the same as in the text.
4. Would an isosceles triangle DEF described on the base DE answer the purpose as well as an equilateral one? If so, why?
5. Prove the proposition and the first deduction, using I. 5 and I. 4 instead of I. 8.
6. Divide a given angle into 4 equal parts.
7. Could the number of equal parts into which an angle may be divided be extended beyond 4? If so, enumerate the numbers.
8. Prove from an equilateral triangle that if a right-angled triangle have one of the acute angles double of the other, the hypotenuse is double of the side opposite the least angle.

PROPOSITION 10. PROBLEM.

To bisect a given straight line.



Let AB be the given straight line :
it is required to bisect it.

On AB describe an equilateral $\triangle$ ABC, *I.* 1
and bisect $\angle$ ACB by CD, which meets AB at D. *I.* 9

AB shall be bisected at D.

In $\triangle$s ACD, BCD, $\left\{\begin{array}{c} AC = BC \\ CD = CD \\ \angle ACD = \angle BCD; \end{array}\right.$ *I. Def.* 23; *Const.*

$\therefore AD = BD;$ *I.* 4

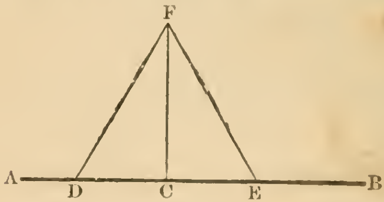
that is, AB is bisected at D.

1. Would an isosceles triangle described on AB as base, answer the purpose as well as an equilateral one? If so, why?
2. Prove that CD, besides bisecting AB, is perpendicular to AB.
3. In the figure to I. 1, suppose the two circles to cut at C and F; prove that CF bisects AB.
4. Hence give (without proof) a simple method of bisecting a given straight line.
5. In the figure to the third deduction, prove that AB and CF bisect each other perpendicularly.
6. Enunciate the preceding deduction as a property of a rhombus.
7. Divide a given straight line into 4 equal parts.
8. Could the number of equal parts into which a straight line may be divided be extended beyond 4? If so, enumerate the numbers.

9. Find a straight line half as long again as a given straight line.
10. Find a straight line equal to half the sum of two given straight lines.
11. Find a straight line equal to half the difference of two given straight lines.
12. If, in the figure to the proposition, $\angle A$ is bisected by AF, which meets BC at F, prove $BF = BD$, and $AF = CD$.

PROPOSITION 11. PROBLEM.

To draw a straight line perpendicular to a given straight line from a given point in the same.



Let AB be the given straight line, and C the given point in it:
it is required to draw from C a perpendicular to AB.

In AC take any point D,
and from CB cut off $CE = CD$. *I.* 3
On DE describe the equilateral $\triangle DEF$, *I.* 1
and join CF. CF shall be $\perp AB$.

$$\text{In } \triangle\text{s } DCF, ECF, \left\{\begin{array}{l} DC = EC \quad \textit{Const.} \\ CF = CF \\ DF = EF; \quad \textit{I. Def.}\ 23 \end{array}\right.$$

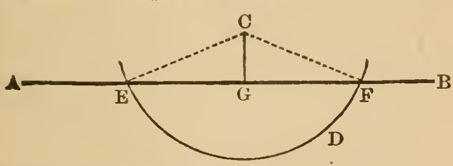
$\therefore \angle DCF = \angle ECF$; *I.* 8
$\because CF$ is $\perp AB$. *I. Def.* 10

1. Would an isosceles triangle described on DE as base answer the purpose as well as an equilateral one? If so, why?

2. If the given point were situated at either end of the given straight line, what additional construction would be necessary in order to draw a perpendicular?
3. At a given point in a given straight line make an angle equal to half of a right angle.
4. At a given point in a given straight line make an angle equal to one-fourth of a right angle.
5. Construct an isosceles right-angled triangle.
6. Construct a right-angled triangle whose base shall be equal to half the hypotenuse.
7. Find in a given straight line a point which shall be equally distant from two given points. Is this always possible? If not, when is it not?
8. ABC is any triangle; AB is bisected at L, and AC at K. From L there is drawn LO perpendicular to AB, and from K, KO perpendicular to AC, and these perpendiculars meet at O. Prove that OA, OB, OC are all equal.
9. Compare the construction and proof of I. 9 with those of I. 11, and show that the latter proposition is a particular case of the former.

PROPOSITION 12. Problem.

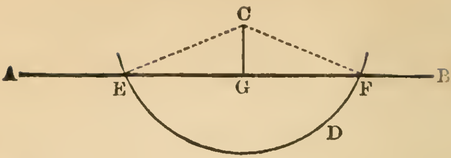
To draw a straight line perpendicular to a given straight line from a given point without it.



Let AB be the given straight line, and C the given point without it:

it is required to draw from C a perpendicular to AB.

Take any point D on the other side of AB;
with centre C and radius CD, describe the ⊙ EDF, cutting AB, or AB produced, at E and F.



Bisect EF at G; *I.* 10

and join CG. CG shall be $\perp$ AB.

Join CE, CF.

In $\triangle$s CGE, CGF, $\begin{cases} EG = FG & \textit{Const.} \\ GC = GC & \\ CE = CF; & \textit{I. Def. } 16 \end{cases}$

$\therefore \angle\ CGE = \angle\ CGF$; *I.* 8

$\therefore CG$ is $\perp$ AB. *I. Def.* 10

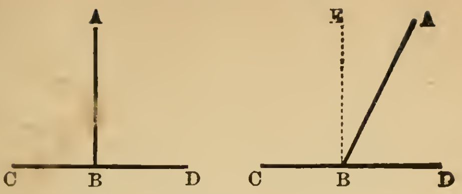
1. Is CEF an equilateral triangle?
2. Prove that CG bisects $\angle\ ECF$.
3. Instead of bisecting EF at G and joining CG, would it answer the purpose equally well to bisect $\angle\ ECF$ by CG?
4. Instead of taking D on the other side of AB, would it answer equally well to take D in AB itself?
5. Two points are situated on opposite sides of a given straight line. Find a point in the straight line such that the straight lines joining it to the two given points may make equal angles with the given straight line. Is this always possible?
6. Use the tenth deduction on I. 8 to obtain another method of drawing the perpendicular.

PROPOSITION 13. Theorem.

The angles which one straight line makes with another on one side of it are together equal to two right angles.

Let AB make with CD on one side of it the $\angle$s ABC, ABD.

it is required to prove $\angle\ ABC + \angle\ ABD = 2$ *rt.* $\angle$*s.*

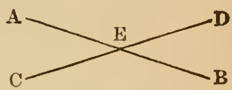


(1) If $\angle ABC = \angle ABD$,
then each of them is a right angle; *I. Def.* 10
$\therefore \angle ABC + \angle ABD = 2$ rt. $\angle$s.

(2) If $\angle ABC$ be not $= \angle ABD$,
from B draw $BE \perp CD$. *I.* 11
Then $\angle$s EBC, EBD are 2 rt. $\angle$s. *Const.*
But $\angle ABC + \angle ABD = \angle EBC + \angle EBD$; *I. Ax.* 9
$\therefore \angle ABC + \angle ABD = 2$ rt. $\angle$s. *I. Ax.* 1

Cor. 1.—Hence, if two straight lines cut one another, the four angles which they make at the point where they cut are equal to four right angles.

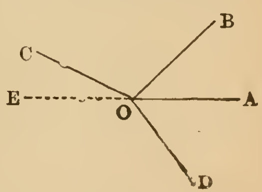
For $\angle AEC + \angle AED = 2$ rt. $\angle$s, *I.* 13
and $\angle BED + \angle BEC = 2$ rt. $\angle$s. *I.* 13



$\therefore \angle AEC + \angle AED + \angle BED + \angle BEC = 4$ rt. $\angle$s.

Cor. 2.—All the successive angles made by any number of straight lines meeting at one point are together equal to four right angles.

Let OA, OB, OC, OD, which meet at O, make the successive angles AOB, BOC, COD, DOA: *it is required to prove these* $\angle$*s* $= 4$ *rt.* $\angle$*s.*



Produce AO to E.

Then $\angle\ AOB + \angle\ BOC + \angle\ COD + \angle\ DOA$

$$= (\angle\ AOB + \angle\ BOE) + (\angle\ EOD + \angle\ DOA)$$
$$= 2 \text{ rt. } \angle\text{s} \qquad + 2 \text{ rt. } \angle\text{s.} \qquad I.\ 13$$
$$= 4 \text{ rt. } \angle\text{s.}$$

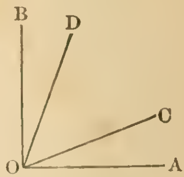
Def.—Two angles are called **supplementary** when their sum is two right angles; and either angle is called the **supplement** of the other.

Thus, in the figure to the proposition, $\angle\ ABC$ and $\angle\ ABD$ are supplementary; $\angle\ ABC$ is the supplement of $\angle\ ABD$, and $\angle\ ABD$ is the supplement of $\angle\ ABC$.

Def.—Two angles are called **complementary** when their sum is one right angle; and either angle is called the **complement** of the other.

Thus, in the figure to the proposition, $\angle\ ABD$ and $\angle\ ABE$ are complementary; $\angle\ ABD$ is the complement of $\angle\ ABE$, and $\angle\ ABE$ is the complement of $\angle\ ABD$.

1. In the figure to Cor. 1, name all the angles which are supplementary to $\angle\ AEC$, to $\angle\ AED$, to $\angle\ BED$, to $\angle\ BEC$.
2. In the figure to Cor. 2, name the angles which are supplementary to $\angle\ AOB$, $\angle\ BOE$, $\angle\ COE$, $\angle\ EOD$, $\angle\ AOD$.
3. In the figure to I. 5, name the angles which are supplementary to $\angle\ ABC$, $\angle\ ACB$, $\angle\ DBC$, $\angle\ ECB$, $\angle\ BFC$, $\angle\ CGB$, $\angle\ ABG$, $\angle\ ACF$.
4. In the accompanying figure, $\angle\ AOB$ is right. Name the angles which are complementary to $\angle\ AOC$, $\angle\ AOD$, $\angle\ BOD$, $\angle\ BOC$.

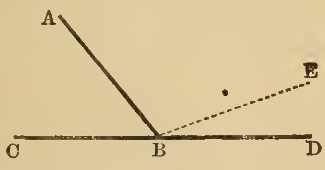


5. In the same figure, if $\angle\ AOC = \angle\ BOD$, prove $\angle\ AOD = \angle\ BOC$; and if $\angle\ AOD = \angle\ BOC$, prove $\angle\ AOC = \angle\ BOD$.
6. In the figure to the proposition, if $\angle$s ABC and ABD be bisected, prove that the bisectors are perpendicular to each other.
7. If the angles at the base of a triangle be equal, the angles on the other side of the base must also be equal.

8. If the base of an isosceles triangle be produced both ways, the exterior angles thus formed are equal.
9. ABC is a triangle, and the sides AB, AC are produced to D and E. If $\angle\ DBC = \angle\ ECB$, prove $\triangle\ ABC$ isosceles.
10. ABC is a triangle, and the base BC is produced both ways. If the exterior angles thus formed are equal, prove $\triangle\ ABC$ isosceles.

PROPOSITION 14. Theorem.

If at a point in a straight line, two other straight lines on opposite sides of it make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.



At the point B in AB, let BC and BD, on opposite sides of AB, make $\angle\ ABC + \angle\ ABD = 2$ rt. $\angle$s:

it is required to prove BD in the same straight line with BC.

If BD be not in the same straight line with BC, produce CB to E; *Post.* 2

then BE does not coincide with BD.

Now since CBE is a straight line,

$\therefore$ $\angle\ ABC + \angle\ ABE = 2$ rt. $\angle$s. *I.* 13

But $\angle\ ABC + \angle\ ABD = 2$ rt. $\angle$s; *Hyp.*

$\therefore$ $\angle\ ABC + \angle\ ABE = \angle\ ABC + \angle\ ABD$. *I. Ax.* 1

Take away from these equals $\angle\ ABC$, which is common;

$\therefore$ $\angle\ ABE = \angle\ ABD$, *I. Ax.* 3

which is impossible;

$\therefore$ BE must coincide with BD;

that is, BD is in the same straight line with BC.

1. $ABCD$, $EFGH$ are two squares. If they be placed so that F falls on C, and FE along CD, show that FG will either fall along CB, or be in the same straight line with it.
2. If in the straight line AB, a point E be taken and two straight lines EC, ED be drawn on opposite sides of AB, making $\angle\ AEC = \angle\ BED$, prove that EC and ED are in the same straight line.
3. If four straight lines, AE, CE, BE, DE, meet at a point E, so that $\angle\ AEC = \angle\ BED$ and $\angle\ AED = \angle\ BEC$, then AE and EB are in the same straight line, and also CE and ED.
4. P is any point, and AOB a right angle; PM is drawn perpendicular to OA and produced to Q, so that $QM = MP$; PN is drawn perpendicular to OB and produced to R, so that $RN = NP$. Prove that Q, O, R lie in the same straight line.
5. If in the enunciation of the proposition the words 'on opposite sides of it' be omitted, is the proposition necessarily true? Draw a figure to illustrate your answer.

PROPOSITION 15. Theorem.

If two straight lines cut one another, the vertically opposite angles shall be equal.

Let AB and CD cut one another at E:
it is required to prove $\angle\ AEC = \angle\ BED$, *and* $\angle\ BEC = \angle\ AED$.

Because CE stands on AB,

$\therefore\quad \angle\ AEC + \angle\ BEC = 2$ rt. $\angle$s. *I.* 13

Because BE stands upon CD,

$\therefore\quad \angle\ BEC + \angle\ BED = 2$ rt. $\angle$s; *I.* 13

$\therefore\quad \angle\ AEC + \angle\ BEC = \angle\ BEC + \angle\ BED$. *I. Ax.* 1

Take away from these equals $\angle\ BEC$, which is common ;

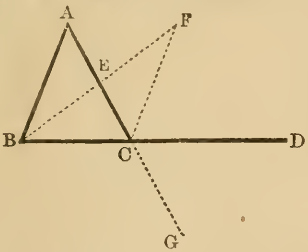
$\therefore$ $\angle\ AEC = \angle\ BED$. *I. Ax.* 3

Hence also, $\angle\ BEC = \angle\ AED$.

1. Prove $\angle\ AEC = \angle\ BED$, making $\angle\ AED$ the common angle.
2. " $\angle\ BEC = \angle\ AED$, " $\angle\ AEC$ " "
3. " $\angle\ BEC = \angle\ AED$, " $\angle\ BED$ " "
4. If $\angle\ AED$ is bisected by FE, and FE is produced to G, prove that EG bisects $\angle\ BEC$.
5. If $\angle\ AED$ is bisected by FE, and $\angle\ BEC$ bisected by GE, prove FE and GE in the same straight line.
6. If in a straight line AB, a point E be taken, and two straight lines, EC, ED, be drawn on opposite sides of AB, making $\angle\ AEC = \angle\ BED$, prove that EC and ED are in the same straight line.
7. ABC is a triangle, BD, CE straight lines drawn making equal angles with BC, and meeting the opposite sides in D and E and each other in F; prove that if $\angle\ AFE = \angle\ AFD$, the triangle is isosceles.

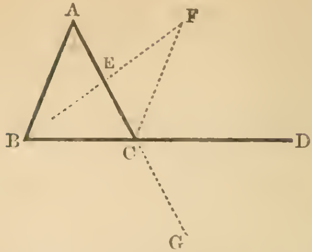
PROPOSITION 16. Theorem.

If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.



Let ABC be a triangle, and let BC be produced to D: *it is required to prove* $\angle\ ACD$ *greater than* $\angle\ BAC$, *and also greater than* $\angle\ ABC$.

Bisect AC at E; *I.* 10



join BE, and produce it to F, making $EF = BE$; *I.* 3
and join CF.

In $\triangle$s AEB, CEF, $\left\{ \begin{array}{l} AE = CE \\ EB = EF \\ \angle AEB = \angle CEF; \end{array} \right.$ Const. / Const. / *I.* 15

$\therefore \quad \angle EAB = \angle ECF.$ *I.* 4

But $\angle ACD$ is greater than $\angle ECF$; *I. Ax.* 8

$\therefore$ $\angle ACD$ is greater than $\angle EAB$.

Hence, if AC be produced to G,
$\angle BCG$ is greater than $\angle ABC$.

But $\angle ACD = \angle BCG$; *I.* 15

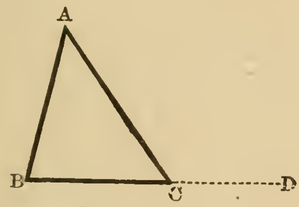
$\therefore$ $\angle ACD$ is greater than $\angle ABC$.

1. Prove $\angle A$ less than AEF, BEC, ACD, BCG.
2. " $\angle F$ " FCD, FCG, BEC, AEF.
3. " $\angle ABE$ " AEF, BEC, ACD, BCG.
4. " $\angle CBE$ " ACD, BCG, AEB, CEF.
5. " $\angle ACB$ " AEB, CEF.
6. " $\angle BEC$ " ACD, BCG.
7. " $\angle BCE$ " AEB, CEF.
8. " $\angle ECF$ " AEF, BEC.
9. Draw three figures to show that an exterior angle of a triangle may be greater than, equal to, or less than the interior adjacent angle.
10. From a point outside a given straight line, there can be drawn to the straight line only one perpendicular.
11. ABC is a triangle whose vertical $\angle A$ is bisected by a straight line which meets BC at D; prove $\angle ADC$ greater than $\angle DAC$, and $\angle ADB$ greater than $\angle BAD$

12. In the figure to the proposition, if AF be joined, prove: (1) $AF = BC$. (2) Area of $\triangle\ ABC$ = area of $\triangle\ BCF$. (3) Area of $\triangle\ ABF$ = area of $\triangle\ ACF$.
13. Hence construct on the same base a series of triangles of equal area, whose vertices are equidistant.
14. To a given straight line there cannot be drawn more than two equal straight lines from a given point without it.
15. Any two exterior angles of a triangle are together greater than two right angles.

PROPOSITION 17. Theorem.

The sum of any two angles of a triangle is less than two right angles.



Let ABC be a triangle:

it is required to prove the sum of any two of its angles less than 2 rt. $\angle$s.

Produce BC to D.

Then $\angle\ ABC$ is less than $\angle\ ACD$. *I.* 16

$\therefore\ \angle\ ABC + \angle\ ACB$ is less than $\angle\ ACD + \angle\ ACB$.

But $\angle\ ACD + \angle\ ACB = 2$ rt. $\angle$s; *I.* 13

$\therefore\ \angle\ ABC + \angle\ ACB$ is less than 2 rt. $\angle$s.

Now $\angle\ ABC$ and $\angle\ ACB$ are any two angles of the triangle;

$\therefore$ the sum of any two angles of a triangle is less than 2 rt. $\angle$s.

1. Prove that in any triangle there cannot be two right angles, or two obtuse angles, or one right and one obtuse angle.

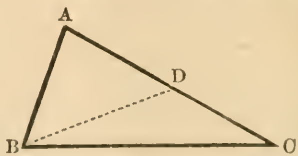
2. Prove that in any triangle there must be at least two acute angles.
3. From a point outside a straight line only one perpendicular can be drawn to the straight line.
4. Prove the proposition by joining the vertex to a point inside the base.
5. The angles at the base of an isosceles triangle are both acute.
6. All the angles of an equilateral triangle are acute.
7. If two angles of a triangle be unequal, the smaller of the two must be acute.
8. The three interior angles of a triangle are together less than three right angles.
9. The three exterior angles of a triangle made by producing the sides in succession, are together greater than three right angles.

Prove by indirect demonstrations the following theorems:

10. The perpendicular from the right angle of a right-angled triangle on the hypotenuse falls inside the triangle.
11. The perpendicular from the obtuse angle of an obtuse-angled triangle on the opposite side falls inside the triangle.
12. The perpendicular from any of the angles of an acute-angled triangle on the opposite side falls inside the triangle.
13. The perpendicular from any of the acute angles of an obtuse-angled triangle on the opposite side falls outside the triangle.

PROPOSITION 18. THEOREM.

The greater side of a triangle has the greater angle opposite to it.



Let ABC be a triangle, having AC greater than AB:
it is required to prove $\angle$ *ABC greater than* $\angle$ *C.*

From AC cut off $AD = AB$, I. 3
and join BD.

Because $\angle ADB$ is an exterior angle of $\triangle BCD$,

$\therefore \angle ADB$ is greater than $\angle C$. *I.* 16

But $\angle ADB = \angle ABD$, since $AB = AD$; *I.* 5

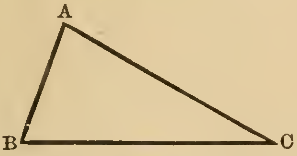
$\therefore \angle ABD$ is greater than $\angle C$.

Much more, then, is $\angle ABC$ greater than $\angle C$.

1. If two angles of a triangle be equal, the sides opposite them must also be equal.
2. A scalene triangle has all its angles unequal.
3. If one side of a triangle be less than another side, the angle opposite to it must be acute.
4. $ABCD$ is a quadrilateral whose longest side is AD, and whose shortest is BC. Prove $\angle ABC$ greater than $\angle ADC$, and $\angle BCD$ greater than $\angle BAD$.
5. Prove the proposition by producing AB to D, so that AD shall be equal to AC, and joining DC.
6. Prove the proposition from the following construction: Bisect $\angle A$ by AD, which meets BC at D; from AC out off $AE = AB$, and join DE.

PROPOSITION 19. Theorem.

The greater angle of a triangle has the greater side opposite to it.

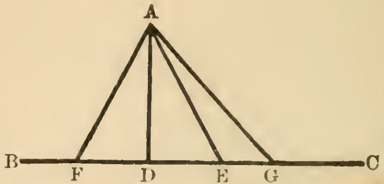


Let ABC be a triangle having $\angle B$ greater than $\angle C$:
it is required to prove AC greater than AB.

If AC be not greater than AB,
then AC must be $= AB$, or less than AB.
If $AC = AB$, then $\angle B = \angle C$. *I.* 5
But it is not;
$\therefore AC$ is not $= AB$.

If AC be less than AB, then $\angle B$ must be less than $\angle C$. *I.* 18
But it is not;
$\therefore$ AC is not less than AB.
Hence AC must be greater than AB.

COR.—The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than the more remote.



From the given point, A, let there be drawn to the given straight line, BC, (1) the perpendicular AD, (2) AE and AF equally distant from the perpendicular, that is, so that $DE = DF$, (3) AG more remote than AE or AF:

it is required to prove AD the least of these straight lines, and AG greater than AE or AF.

In $\triangle$s ADE, ADF, $\left\{\begin{array}{l} AD = AD \\ DE = DF \\ \angle ADE = \angle ADF; \end{array}\right.$ *Hyp.* *I. Ax.* 10

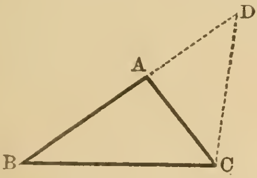
$\therefore$ $AE = AF$. *I.* 4
Because $\angle ADE$ is right, $\therefore$ $\angle AED$ is acute; *I.* 17
$\therefore$ AE is greater than AD. *I.* 19
Hence also AF is greater than AD.

Because $\angle AEG$ is greater than $\angle ADE$, *I.* 16
$\therefore$ $\angle AEG$ is obtuse;
$\therefore$ $\angle AGE$ is acute; *I.* 17
$\therefore$ AG is greater than AE. *I.* 19
Hence also AG is greater than AF, and than AD.

1. The hypotenuse of a right-angled triangle is greater than either of the other sides.
2. A diagonal of a square or of a rectangle is greater than any one of the sides.
3. In an obtuse-angled triangle the side opposite to the obtuse angle is greater than either of the other sides.
4. From A, one of the angular points of a square $ABCD$, a straight line is drawn to intersect BC and meet DC produced at E; prove that AE is greater than a diagonal of the square.
5. From a point outside not more than two equal straight lines can be drawn to a given straight line.
6. The circumference of a circle cannot cut a straight line in more than two points.
7. ABC is a triangle whose vertical angle A is bisected by a straight line which meets BC at D; prove that AB is greater than BD, and AC greater than CD.

PROPOSITION 20. Theorem.

The sum of any two sides of a triangle is greater than the third side.



Let ABC be a triangle:

it is required to prove that the sum of any two of its sides is greater than the third side.

Produce BA to D, making $AD = AC$, *I.* 3

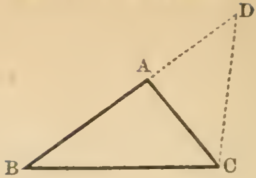
and join CD.

Then $\angle\ ACD = \angle\ D$, since $AD = AC$. *I.* 5

But $\angle\ BCD$ is greater than $\angle\ ACD$;

$\therefore\ \angle\ BCD$ is greater than $\angle\ D$;

$\therefore\ BD$ is greater than BC. *I.* 19



But $BD = BA + AC$;

$\therefore$ $BA + AC$ is greater than BC.

Now BA and AC are any two sides;

$\therefore$ the sum of any two sides of a triangle is greater than the third side.

Cor.—The difference of any two sides of a triangle is less than the third side.

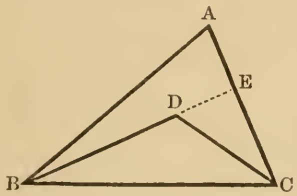
For $BA + AC$ is greater than BC. *I.* 20
Taking AC from each of these unequals,
there remains BA greater than $BC - AC$; *I. Ax.* 5
that is, the third side is greater than the difference between the other two.

1. Prove the proposition by producing CA instead of BA.
2. " " drawing a perpendicular from the vertex to the base.
3. " " bisecting the vertical angle.
4. In the first figure to I. 7, the sum of AD and BC is greater than the sum of AC and BD.
5. A diameter of a circle is greater than any other straight line in the circle which is not a diameter.
6. Any side of a quadrilateral is less than the sum of the other three sides.
7. Any side of a polygon is less than the sum of the other sides.
8. The sum of the distances of any point from the three angles of a triangle is greater than the semi-perimeter of the triangle. Discuss the three cases when the point is inside the triangle, when it is outside, and when it is on a side.
9. The semi-perimeter of a triangle is greater than any one side, and less than any two sides.
10. The sum of the two diagonals of any quadrilateral is greater than the sum of any pair of opposite sides.

11. The perimeter of a quadrilateral is greater than the su and less than twice the sum of the two diagonals.
12. The sum of the diagonals of a quadrilateral is less than the sum of the four straight lines which can be drawn to the four angles from any other point except the intersection of the diagonals.
13. The sum of any two sides of a triangle is greater than twice the median * drawn to the third side, and the excess of this sum over the third side is less than twice the median.
14. The perimeter of a triangle is greater, and the semi-perimeter is less, than the sum of the three medians.

PROPOSITION 21. THEOREM.

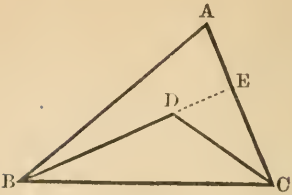
If from the ends of any side of a triangle there be drawn two straight lines to a point within the triangle, these straight lines shall be together less than the other two sides of the triangle, but shall contain a greater angle.



Let ABC be a triangle, and from B and C, the ends of BC, let BD, CD be drawn to any point D within the triangle:

it is required to prove (1) *that* $BD + CD$ *is less than* $AB + AC$; (2) *that* $\angle BDC$ *is greater than* $\angle A$.

* DEF.—A *median line*, or a *median*, is a straight line drawn from any vertex of a triangle o the middle point of the opposite side.



Produce BD to meet AC at E.

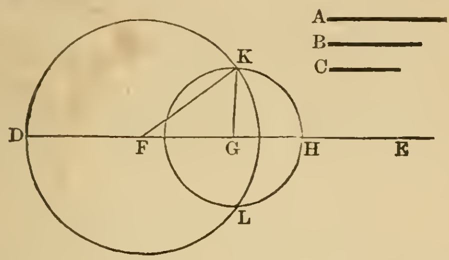
(1) Because $BA + AE$ is greater than BE; *I.* 20
add to each of these unequals EC;
$\therefore$ $BA + AC$ is greater than $BE + EC$. *I. Ax.* 4
Again, $CE + ED$ is greater than CD; *I.* 20
add to each of these unequals DB;
$\therefore$ $CE + EB$ is greater than $CD + DB$. *I. Ax.* 4
Much more, then, is $BA + AC$ greater than $CD + DB$.

(2) Because CED is a triangle,
$\therefore$ $\angle\ BDC$ is greater than $\angle\ DEC$; *I*
and because BAE is a triangle,
$\therefore$ $\angle\ DEC$ is greater than $\angle\ A$;
much more, then, is $\angle\ BDC$ greater than $\angle\ A$

1. Prove the first part of the proposition by producing CD instead of BD.
2. Prove the second part of the proposition by joining AD and producing it.
3. In the second figure to I. 7, prove that the perimeter of the triangle ACB is greater than that of ADB.
4. Prove the same thing with respect to the third figure to I. 7.
5. If a point be taken inside a triangle and joined to the three vertices, the sum of the three straight lines so drawn shall be less than the perimeter of the triangle.
6. If a triangle and a quadrilateral stand on the same base, and on the same side of it, and the one figure fall within the other, that which has the greater surface shall have the greater perimeter.

PROPOSITION 22. Problem.

To make a triangle the sides of which shall be equal to three given straight lines, but any two of these must be greater than the third.



Let A, B, C be the three given straight lines, any two of which are greater than the third:
it is required to make a triangle the sides of which shall be respectively equal to A, B, C.

Take a straight line DE terminated at D, but unlimited towards E;
and from it cut off $DF = A$, $FG = B$, $GH = C$. *I.* 3
With centre F and radius FD, describe the $\odot$ DKL;
with centre G and radius GH, describe the $\odot$ HKL,
cutting the other circle at K;
join KF, KG. KFG is the triangle required.

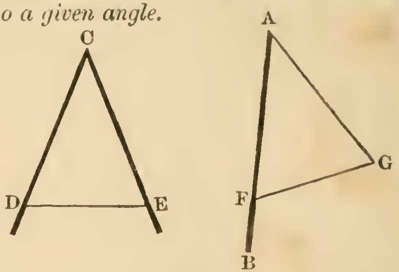
Because $FK = FD$, being radii of $\odot$ DKL, *I. Def.* 16
$\therefore$ $FK = A$.
Because $GK = GH$, being radii of $\odot$ HKL, *I. Def.* 16
$\therefore$ $GK = C$.
And FG was made $= B$;
$\therefore$ $\triangle$ KFG has its sides respectively equal to A, B, C.

1. Could any other triangle be constructed on the base FG fulfilling the given conditions?

2. If A, B, C be all equal, which preceding proposition shall we be enabled to solve?
3. Draw a figure showing what will happen when two of the given straight lines are together equal to the third.
4. Draw a figure showing what will happen when two of the given straight lines are together less than the third.
5. Since a quadrilateral can be divided into two triangles by drawing a diagonal, show how to make a quadrilateral whose sides shall be equal to those of a given quadrilateral.
6. Since any rectilineal figure may be decomposed into triangles, show how to make a rectilineal figure whose sides shall be equal to those of a given rectilineal figure.

PROPOSITION 23. Problem.

At a given point in a given straight line, to make an angle equal to a given angle.



Let AB be the given straight line, A the given point in it, and $\angle\ C$ the given angle:
it is required to make at A an angle $= \angle\ C$.

In CD, CE, take any points D, E, and join DE.
Make $\triangle\ AFG$ such that $AF = CD$, $FG = DE$, $GA = EC$. *I.* 22

A is the required angle.

In $\triangle$s AFG, CDE, $\left\{\begin{array}{l} AF = CD \\ AG = CE \\ FG = DE; \end{array}\right.$ *Const.* *Const.* *Const.*

$\therefore\ \angle\ A = \angle\ C$. *I.* 8

1. At a given point in a given straight line, to make an angle equal to the supplement of a given angle.
2. At a given point in a given straight line, to make an angle equal to the complement of a given angle.
3. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.
4. The straight line OC bisects the angle AOB; prove that if OD be any other straight line through O without the angle AOB, the sum of the angles DOA and DOB is double of the angle DOC.
5. The straight line OC bisects the angle AOB; prove that if OD be any other straight line through O within the angle AOB, the difference of the angles DOA and DOB is double of the angle DOC.

Construct an isosceles triangle, having given:

6. The vertical angle and one of the equal sides.
7. The base and one of the angles at the base.

Construct a right-angled triangle, having given:

8. The base and the perpendicular.
9. The base and the acute angle at the base.

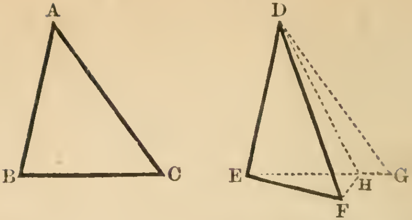
Construct a triangle, having given:

10. The base and the angles at the base.
11. Two sides and the included angle.
12. The base, an angle at the base, and the sum of the other two sides.
13. The base, an angle at the base, and the difference of the other two sides.

PROPOSITION 24. Theorem.

*If two triangles have two sides of the one respectively equal to two sides of the other, but the contained angles unequal, the base of the triangle which has the greater contained angle shall be greater than the base of the other.**

* The proof given in the text is different from Euclid's, which is defective.



Let ABC, DEF be two triangles, having $AB = DE$, $AC = DF$, but $\angle\ BAC$ greater than $\angle\ EDF$:
it is required to prove BC greater than EF.

At D make $\angle\ EDG = \angle\ BAC$; *I.* 23
cut off $DG = AC$ or DF, *I.* 3
and join EG.
Bisect $\angle\ FDG$ by DH, meeting EG at H; *I.* 9
and, if F does not lie on EG, join FH.

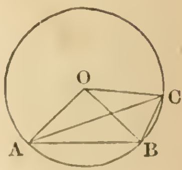
In $\triangle$s ABC, DEG, $\left\{\begin{array}{ll} BA = ED & \textit{Hyp.} \\ AC = DG & \textit{Const.} \\ \angle\ BAC = \angle\ EDG; & \textit{Const.} \end{array}\right.$

$\therefore BC = EG$. *I.* 4

In $\triangle$s FDH, GDH, $\left\{\begin{array}{ll} FD = GD & \textit{Const.} \\ DH = DH & \\ \angle\ FDH = \angle\ GDH; & \textit{Const.} \end{array}\right.$

$\therefore FH = GH$. *I.* 4

Hence $EH + FH = EH + GH = EG$.
But $EH + FH$ is greater than EF; *I.* 20
$\therefore EG$ is greater than EF;
$\therefore BC$ is greater than EF.

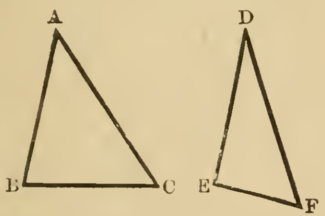


1. ABC is a circle whose centre is O. If $\angle\ AOB$ is greater than $\angle\ BOC$, prove that AB is greater than BC.
2. In the same figure, prove that AC is greater than AB or BC.
3. $ABCD$ is a quadrilateral, having $AB = CD$, but $\angle\ BCD$ greater than $\angle\ ABC$; prove that BD is greater than AC.

4. ABC is an isosceles triangle, having $AB = AC$. AD drawn to the base BC does not bisect $\angle A$; prove that D is at unequal distances from B and C.
5. Prove the proposition with the same construction as in the text, but let $\triangle DEG$ fall on the other side of DE.

PROPOSITION 25. THEOREM.

If two triangles have two sides of the one respectively equal to two sides of the other, but their bases unequal, the angle contained by the two sides of the triangle which has the greater base shall be greater than the angle contained by the two sides of the other.



Let ABC, DEF be two triangles, having $AB = DE$, $AC = DF$, but base BC greater than base EF:
it is required to prove $\angle A$ greater than $\angle D$.

If $\angle A$ be not greater than $\angle D$, it must be either equal to $\angle D$, or less than $\angle D$.

But $\angle A$ is not $= \angle D$, for then base BC would be $=$ base EF, *I.* 4
which it is not. *Hyp.*
And $\angle A$ is not less than $\angle D$, for then base BC would be less than base EF, *I.* 24
which it is not. *Hyp.*
$\therefore$ $\angle A$ must be greater than $\angle D$.

1. In the figure to the first deduction on I. 24, if AB is greater than BC, prove that $\angle AOB$ is greater than $\angle BOC$.

2. $ABCD$ is a quadrilateral, having $AB = CD$, but the diagonal BD greater than the diagonal AC; prove that $\angle\ DCB$ is greater than $\angle\ ABC$.
3. $ABCD$ is a quadrilateral, having $AB = CD$, but $\angle\ BCD$ greater than $\angle\ ABC$; prove that $\angle\ DAB$ is greater than $\angle\ ADC$.
4. $ABCD$ is a quadrilateral, having $AB = CD$, but $\angle\ DAB$ greater than $\angle\ ADC$; prove that $\angle\ BCD$ is greater than $\angle\ ABC$.
5. ABC is a triangle, having AB less than AC. D is the middle point of BC, and AD is joined; prove that $\angle\ ADB$ is acute.
6. ABC is an isosceles triangle, having $AB = AC$. D is any point such that BD is greater than DC; prove that AD does not bisect $\angle\ A$.
7. ABC is a triangle, having AB less than AC, and AD is the median drawn from A; prove that G, any point in AD, is nearer to B than to C.

PROPOSITION 26. Theorem.

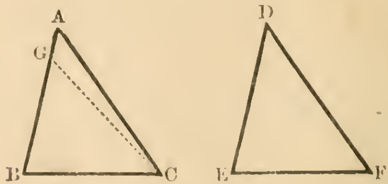
If two angles and a side in one triangle be respectively equal to two angles and the corresponding side in another triangle, the two triangles shall be equal in every respect; that is,

(1) *The remaining sides of the one triangle shall be equal to the remaining sides of the other.*

(2) *The third angles shall be equal.*

(3) *The areas of the two triangles shall be equal.*

Case 1.



In $\triangle$s ABC, DEF let $\angle\ ABC = \angle\ DEF$, $\angle\ ACB = \angle\ DFE$, and $BC = EF$:

it is required to prove $AB = DE$, $AC = DF$, $\angle A = \angle D$, $\triangle ABC = \triangle DEF$.

If AB be not $= DE$, one of them must be the greater.
Let AB be the greater, and make $BG = DE$; *I.* 3
and join GC.

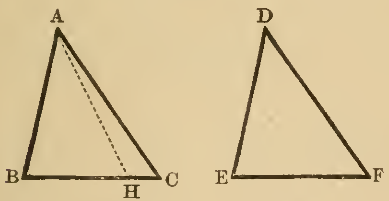
In $\triangle$s GBC, DEF, $\left\{\begin{array}{l} GB = DE \\ BC = EF \\ \angle B = \angle E; \end{array}\right.$ — *Const.*, *Hyp.*, *Hyp.*

$\therefore \angle GCB = \angle DFE$. *I.* 4.
But $\angle ACB = \angle DFE$; *Hyp.*
$\therefore \angle GCB = \angle ACB$, which is impossible.
Hence AB is not unequal to DE, that is, $AB = DE$.

Now in $\triangle$s ABC, DEF, $\left\{\begin{array}{l} AB = DE \\ BC = EF \\ \angle B = \angle E; \end{array}\right.$ — *Proved*, *Hyp.*, *Hyp.*

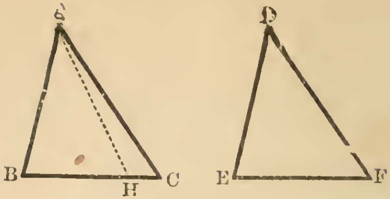
$\therefore AC = DF$, $\angle A = \angle D$, $\triangle ABC = \triangle DEF$. *I.* 4

CASE 2.



In $\triangle$s ABC, DEF let $\angle B = \angle E$, $\angle C = \angle F$, and $AB = DE$:
it is required to prove $BC = EF$, $AC = DF$, $\angle BAC = \angle EDF$, $\triangle ABC = \triangle DEF$.

If BC be not $= EF$, one of them must be the greater.
Let BC be the greater, and make $BH = EF$; *I.* 3
and join AH.



In $\triangle$s ABH, DEF, $\left\{\begin{array}{l} AB = DE \\ BH = EF \\ \angle B = \angle E; \end{array}\right.$ Hyp. / Const. / Hyp.

$\therefore\ \angle AHB = \angle DFE.$ I. 4

But $\angle ACB = \angle DFE$; Hyp.

$\therefore\ \angle AHB = \angle ACB$, which is impossible. I. 16

Hence BC is not unequal to EF, that is, $BC = EF$.

Now in $\triangle$s ABC, DEF, $\left\{\begin{array}{l} AB = DE \\ BC = EF \\ \angle B = \angle E; \end{array}\right.$ Hyp. / Proved / Hyp.

$\therefore AC = DF$, $\angle BAC = \angle EDF$, $\triangle ABC = \triangle DEF$. I. 4

1. Prove the first case of the proposition by superposition.
2. The straight line that bisects the vertical angle of an isosceles triangle bisects the base, and is perpendicular to the base.
3. The straight line drawn from the vertical angle of an isosceles triangle perpendicular to the base, bisects the base and the vertical angle.
4. Any point in the bisector of an angle is equidistant from the arms of the angle.
5. In a given straight line, find a point such that the perpendiculars drawn from it to two other straight lines may be equal.
6. Through a given point, draw a straight line which shall be equidistant from two other given points.
7. Through a given point, draw a straight line which shall form with two given intersecting straight lines an isosceles triangle.

PROPOSITION A. THEOREM.

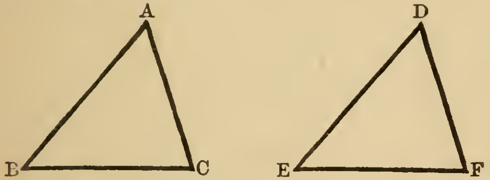
If two sides of one triangle be respectively equal to two sides of another triangle, and if the angles opposite to one pair of equal sides be equal, the angles opposite the other pair of equal sides shall either be equal or supplementary.

In $\triangle$s ABC, DEF let $AB = DE$, $AC = DF$, $\angle B = \angle E$:

it is required to prove either $\angle C = \angle F$, *or* $\angle C + \angle F = 2$ *rt.* $\angle$*s.*

$\angle A$ is either $= \angle D$, or not.

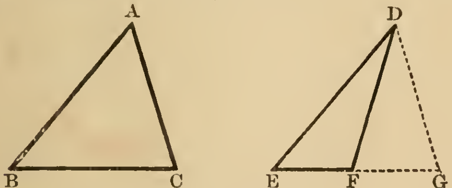
CASE 1.—When $\angle A = \angle D$.



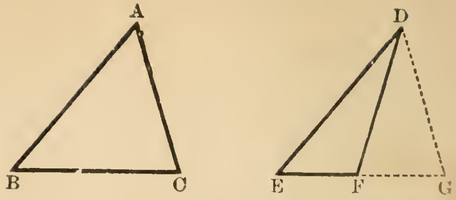
In $\triangle$s ABC, DEF, $\left\{\begin{array}{l}\angle A = \angle D \\ \angle B = \angle E \\ AB = DE;\end{array}\right.$ *Hyp.* *Hyp.*

$\therefore$ $\triangle$s ABC, DEF are equal in all respects, and $\angle C = \angle F$. *I.* 26

CASE 2.—When $\angle A$ is not $= \angle D$.



At D make $\angle EDG = \angle BAC$; *I.* 23



and let EF, produced if necessary, meet DG at G.

$$\text{In } \triangle\text{s } ABC, DEG, \left\{\begin{array}{ll} \angle BAC = \angle EDG & \textit{Const.} \\ \angle ABC = \angle DEG & \textit{Hyp.} \\ AB = DE; & \textit{Hyp.} \end{array}\right.$$

$\therefore AC = DG$, and $\angle C = \angle G$. *I.* 26

Now $AC = DF$; *Hyp.*

$\therefore DF = DG$;

$\therefore \angle DFG = \angle DGF$. *I.* 5

But $\angle DFE$ is supplementary to $\angle DFG$; *I.* 13

$\therefore \angle DFE$ is supplementary to $\angle DGF$,

and consequently to $\angle C$.

Note.—It often happens that we wish to prove two triangles equal in all respects when we know only that two sides in the one are respectively equal to two sides in the other, and that the angles opposite one pair of equal sides are equal. In such a case, since the angles opposite the other pair of equal sides may either be equal or supplementary, we must endeavour to prove that they cannot be supplementary. To do this, it will be sufficient to know

either (1) that this pair are both acute angles,

or (2) that they are both obtuse angles,

or (3) that one of them is a right angle, since the other must then be a right angle whether it be equal or supplementary to it.

We can tell that this pair of angles must be both acute in certain cases.

(*a*) When the pair of angles given equal are both right angles.

(*b*) " " " " obtuse "

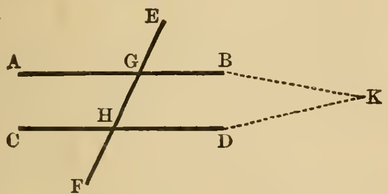
(*c*) " " equal sides opposite the given angles are greater than the other pair of equal sides.

Hence the following important Corollary :

If the hypotenuse and a side of one right-angled triangle be respectively equal to the hypotenuse and a side of another right-angled triangle, the triangles shall be equal in all respects.

PROPOSITION 27. THEOREM.

If a straight line cutting two other straight lines make the alternate angles equal to one another, the two straight lines shall be parallel.



Let EF, which cuts the two straight lines AB, CD, make $\angle\ AGH =$ the alternate $\angle\ GHD$:
it is required to prove $AB \parallel CD$.

If AB is not $\parallel CD$, AB and CD being produced will meet either towards A and C, or towards B and D.
Let them be produced, and meet towards B and D at K.

Then KGH is a triangle;
$\therefore$ exterior $\angle\ AGH$ is greater than the interior opposite $\angle\ GHD$. *I.* 16
But $\angle\ AGH = \angle\ GHD$; *Hyp.*
which is impossible.
$\therefore$ AB and CD, when produced, do not meet towards B and D.
Hence also, AB and CD, when produced, do not meet towards A and C;
$\therefore$ AB is $\parallel CD$. *I. Def.* 14

In the figure to I. 16:

1. Prove $AB \parallel CF$.
2. Join AF, and prove $AF \parallel BC$.

In the figure to I. 28:

3. If $\angle\ AGE = \angle\ DHF$, prove $AB \parallel CD$.
4. If $\angle\ BGE = \angle\ CHF$, prove $AB \parallel CD$.
5. If $\angle\ AGE + \angle\ CHF = 2$ rt. $\angle$ s, prove $AB \parallel CD$.
6. If $\angle\ BGE + \angle\ DHF = 2$ rt. $\angle$ s, prove $AB \parallel CD$.
7. The opposite sides of a square are parallel.
8. The opposite sides of a rhombus are parallel.
9. The quadrilateral whose diagonals bisect each other is a ‖m

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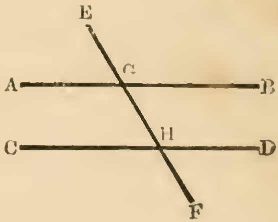
PROPOSITION 28. THEOREM.

If a straight line cutting two other straight lines make (1) *an exterior angle equal to the interior opposite angle on the same side of the cutting line, or* (2) *the two interior angles on the same side of the cutting line together equal to two right angles, the two straight lines shall be parallel.*



CASE 1.

Let EF, which cuts the two straight lines AB, CD, make the exterior $\angle\ EGB$ = the interior opposite $\angle\ GHD$: *it is required to prove* $AB \parallel CD$.

Because $\angle\ EGB = \angle\ GHD$, *Hyp.*

and $\angle\ EGB = \angle\ AGH$, being vertically opposite; *I.* 15
$\therefore\ \angle\ AGH = \angle\ GHD$;
and they are alternate angles;
$\therefore\ AB$ is $\parallel CD$. *I.* 27

Case 2.

Let EF, which cuts the two straight lines AB, CD, make $\angle\ BGH + \angle\ GHD = 2$ rt. $\angle$s:
it is required to prove $AB \parallel CD$.

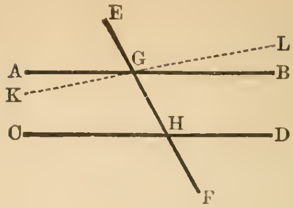
Because $\angle\ BGH + \angle\ GHD = 2$ rt. $\angle$s, *Hyp.*
and $\angle\ AGH + \angle\ BGH = 2$ rt. $\angle$s; *I.* 13
$\therefore\ \angle\ AGH + \angle\ BGH = \angle\ BGH + \angle\ GHD$.
From these equals take $\angle\ BGH$, which is common;
$\therefore\ \angle\ AGH = \angle\ GHD$; *I. Ax.* 3
and they are alternate angles;
$\therefore\ AB$ is $\parallel CD$. *I.* 27

Cor.—Straight lines which are perpendicular to the same straight line are parallel.

1. If $\angle\ BGE + \angle\ DHF = 2$ rt. $\angle$s, prove $AB \parallel CD$.
2. If $\angle\ AGE + \angle\ CHF = 2$ rt. $\angle$s, prove $AB \parallel CD$.
3. If $\angle\ AGE = \angle\ DHF$, prove $AB \parallel CD$.
4. If $\angle\ BGE = \angle\ CHF$, prove $AB \parallel CD$.
5. The opposite sides of a square are parallel.
6. $ABCD$ is a quadrilateral having $\angle\ A$ and $\angle\ B$ supplementary, as well as $\angle\ B$ and $\angle\ C$; prove that it is a ▱m.

PROPOSITION 29. Theorem.

If a straight line cut two parallel straight lines, it shall make (1) *the alternate angles equal to one another;* (2) *any exterior angle equal to the interior opposite angle on the same side of the cutting line;* (3) *the two interior angles on the same side of the cutting line equal to two right angles.*



Let EF cut the two parallel straight lines AB, CD:
it is required to prove:

(1) $\angle\ AGH$ = *alternate* $\angle\ GHD$;
(2) *exterior* $\angle\ EGB$ = *interior opposite* $\angle\ GHD$;
(3) $\angle\ BGH + \angle\ GHD = 2$ *rt.* $\angle$*s.*

(1) If $\angle\ AGH$ be not = $\angle\ GHD$, make $\angle\ KGH = \angle\ GHD$, *I.* 23
and produce KG to L.

Because $\angle\ KGH$ = alternate $\angle\ GHD$, *Const.*
$\therefore\ KL \parallel CD$. *I.* 27
But AB is also $\parallel CD$; *Hyp.*
$\therefore\ AB$ and KL, which cut one another at G, are both $\parallel CD$, which is impossible. *I. Ax.* 11
$\therefore\ \angle\ AGH$ is not unequal to $\angle\ GHD$;
$\therefore\ \angle\ AGH = \angle\ GHD$.

(2) Because $\angle\ AGH = \angle\ GHD$, *Proved*
and $\angle\ AGH = \angle\ EGB$, being vertically opposite; *I.* 15
$\therefore\ \angle\ EGB = \angle\ GHD$.

(3) Because $\angle\ AGH = \angle\ GHD$; *Proved*
to each of these equals add $\angle\ BGH$;
$\therefore\ \angle\ AGH + \angle\ BGH = \angle\ BGH + \angle\ GHD$. *I. Ax.* 2
But $\angle\ AGH + \angle\ BGH = 2$ rt. $\angle$s; *I.* 13
$\therefore\ \angle\ BGH + \angle\ GHD = 2$ rt. $\angle$s.

Cor.—If a straight line meet two others, and make with them the two interior angles on one side of it together less than two right angles, these two other straight lines will, if produced, meet on that side.

Let KL and CD meet EF and make $\angle\ KGH + \angle\ CHG$ less than 2 rt. $\angle$ s:
it is required to prove that KG and CH will, if produced, meet towards K and C.

If not, KL and CD must either be parallel, or meet towards L and D.

(1) KL and CD are not parallel;
for then $\angle\ KGH + \angle\ CHG$ would be $=$ 2 rt. $\angle$ s. *I.* 29

(2) KL and CD do not meet towards L and D;
for then $\angle$ s LGH, DHG would form angles of a triangle,
and would $\therefore$ be together less than 2 rt. $\angle$ s. *I.* 17
Now since the four $\angle$ s KGH, CHG, LGH, DHG are
together $=$ 4 rt. $\angle$ s, *I.* 13
and the first two are less than 2 rt. $\angle$ s; *Hyp.*
$\therefore$ the last two must be greater than 2 rt. $\angle$ s.
Hence KL and CD must meet towards K and C.

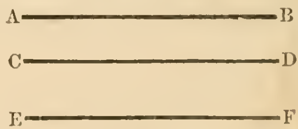
[This Cor. is the converse of I. 17.]

1. In the diagram to I. 28, if AB is $\parallel$ CD, prove $\angle\ AGE = \angle\ DHF$, and $\angle\ BGE + \angle\ DHF = 2$ rt. $\angle$ s.
2. If a straight line be perpendicular to one of two parallels, it is also perpendicular to the other.
3. A straight line drawn parallel to the base of an isosceles triangle, and meeting the sides or the sides produced, forms with them another isosceles triangle.
4. If the arms of one angle be respectively parallel to the arms of another angle, the angles are either equal or supplementary. Distinguish the cases.
5. Is it always true that if two angles be equal, and an arm of the one is parallel to an arm of the other, the other arms must be parallel?

6. If any straight line joining two parallels be bisected, any other straight line drawn through the point of bisection and terminated by the parallels will be bisected at that point.
7. The two straight lines in the last deduction will intercept equal portions of the parallels.
8. If through the vertex of an isosceles triangle a parallel be drawn to the base, it will bisect the exterior vertical angle.
9. If the bisector of the exterior vertical angle of a triangle be parallel to the base, the triangle is isosceles.
10. The diagonals of a $\|^{m}$ bisect each other.
11. Prove that by the following construction $\angle\ ACB$ is bisected: In AC take any point D; draw $DE \perp AC$, and meeting CB at E. From E draw $EF \perp DE$ and $= EC$; join CF.

PROPOSITION 30. Theorem.

Straight lines which are parallel to the same straight line are parallel to one another.



Let AB and CD be each of them $\|$ EF:
it is required to prove $AB \parallel CD$.

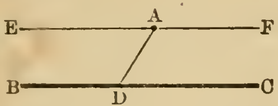
If AB and CD be not parallel, they will meet if produced; and then two straight lines which intersect each other will both be $\|$ the same straight line, which is impossible. *I. Ax.* 11

$\therefore AB$ is $\|$ CD.

1. Two $\|^{ms}$ are situated either on the same side or on different sides of a common base. Prove that the sides of the $\|^{ms}$ which are opposite the common base are $\|$ each other.
2. Prove the proposition in Euclid's manner by drawing a straight line GHK to cut AB, CD, and EF, and applying I. 29, 27.

PROPOSITION 31. Problem.

Through a given point to draw a straight line parallel to a given straight line.



Let A be the given point, and BC the given straight line: *it is required to draw through A a straight line* $\parallel BC$.

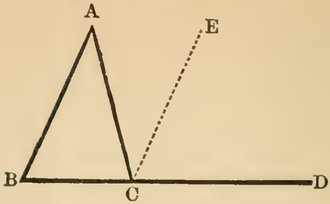
In BC take any point D, and join AD;
at A make $\angle DAE = \angle ADC$; *I.* 23
and produce EA to F. EF shall be $\parallel BC$.

Because the alternate $\angle$s EAD, ADC are equal,
$\therefore EF$ is $\parallel BC$. *I.* 27

1. Give another construction for the proposition by means of I. 12, 11, and a proof by means of I. 28.
2. Through a given point draw a straight line making with a given straight line an angle equal to a given angle.
3. Through a given point draw a straight line which shall form with two given intersecting straight lines an isosceles triangle.
4. Through a given point draw a straight line such that the part of it intercepted between two parallels may be equal to a given straight line. May there be more than one solution to this problem? Is the problem ever impossible?

PROPOSITION 32. Theorem.

If a side of a triangle be produced, the exterior angle is equal to the sum of the two interior opposite angles, and the sum of the three interior angles is equal to two right angles.



Let ABC be a triangle having BC produced to D:
it is required to prove (1) $\angle ACD = \angle A + \angle B$;
(2) $\angle A + \angle B + \angle ACB = 2$ rt. $\angle$s.

Through C draw $CE \parallel AB$. *I.* 31

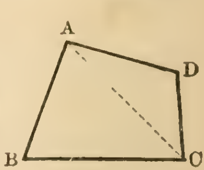
(1) Because AC meets the parallels AB, CE,
$\therefore \angle A =$ alternate $\angle ACE$. *I.* 29
Because BD meets the parallels AB, CE,
$\therefore$ interior $\angle B =$ exterior $\angle ECD$; *I.* 29
$\therefore \angle A + \angle B = \angle ACE + \angle ECD$,
$= \angle ACD$.

(2) Because $\angle A + \angle B = \angle ACD$; *Proved*
adding $\angle ACB$ to each of these equals,
$\therefore \angle A + \angle B + \angle ACB = \angle ACD + \angle ACB$,
$= 2$ rt. $\angle$s. *I.* 13

Cor. 1.—If two triangles have two angles of the one respectively equal to two angles of the other, they are mutually equiangular.

For the third angles differ from 2 rt. $\angle$s by equal amounts; $\therefore$ the third angles are equal.

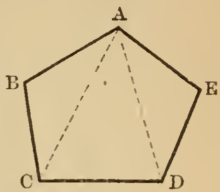
Cor. 2.—The interior angles of a quadrilateral are equal to four right angles.



For the quadrilateral $ABCD$ may be divided into two triangles by joining AC; and the six angles of the two $\triangle$s ABC, $ACD = 4$ rt. $\angle$s. *I.* 32

But the six angles of the two triangles = the interior angles of the quadrilateral;

∴ the interior angles of the quadrilateral = 4 **rt.** ∠s.



COR. 3.—A five-sided figure may be divided into three (that is, $5-2$) triangles by drawing straight lines from one of its angular points. Similarly, a six-sided figure may be divided into four (that is, $6-2$) triangles; and generally a figure of n sides may be divided into $(n-2)$ triangles.

Hence, by a proof like that for the quadrilateral,

the interior ∠s of a five-sided figure = 6 rt. ∠s;
" " six-sided " = 8 rt. ∠s; and
" " figure with n sides = $(2n-4)$ rt. ∠s.

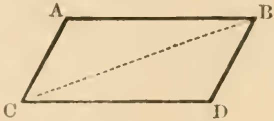
1. If an isosceles triangle be right-angled, each of the base angles is half a right angle.
2. If two isosceles triangles have their vertical angles equal, they are mutually equiangular.
3. If one angle of a triangle be equal to the sum of the other two, it must be right.
4. If one angle of a triangle be greater than the sum of the other two, it must be obtuse.
5. If one angle of a triangle be less than the sum of the other two, it must be acute.
6. Divide a right-angled triangle into two isosceles triangles.
7. Hence show that the middle point of the hypotenuse of a right-angled triangle is equidistant from the three vertices.
8. Hence also, devise a method of drawing a perpendicular to a given straight line from the end of it without producing the straight line.
9. Each angle of an equilateral triangle is two-thirds of a right angle.
10. Hence show how to trisect * a right angle.

* It is sometimes stated that the problem to trisect *any* angle is beyond the power of Geometry. This is not the case. The problem is beyond the power of Elementary Geometry, which allows the use of only the ruler and the compasses.

11. Prove the second part of the proposition by drawing through A a straight line $DAE \parallel BC$. (The Pythagorean proof.)
12. If any of the angles of an isosceles triangle be two-thirds of a right angle, the triangle must be equilateral.
13. Each of the base angles of an isosceles triangle equals half the exterior vertical angle.
14. If the exterior vertical angle of an isosceles triangle be bisected, the bisector is $\parallel$ the base.
15. Show that the space round a point can be filled up with six equilateral triangles, or four squares, or three regular hexagons.
16. Can a right angle be divided into any other number of equal parts than two or three?
17. In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the triangles on each side of it are equiangular to the whole triangle and to one another.
18. Prove the seventh deduction indirectly; and also directly by producing the median to the hypotenuse its own length.
19. If the arms of one angle be respectively perpendicular to the arms of another, the angles are either equal or supplementary.
20. Prove Cor. 3 by taking a point inside the figure and joining it to the angular points.

PROPOSITION 33. Theorem.

The straight lines which join the ends of two equal and parallel straight lines towards the same parts, are themselves equal and parallel.



Let AB and CD be equal and parallel:

it is required to prove AC and BD equal and parallel.

Join BC.

Because BC meets the parallels AB, CD,

$\therefore$ $\angle ABC$ = alternate $\angle DCB$. *I.* 29

$$\text{In } \triangle\text{s } ABC, DCB, \left\{ \begin{array}{l} AB = DC \\ BC = CB \\ \angle ABC = \angle DCB; \end{array} \right. \quad \begin{array}{r} \textit{Hyp.} \\ \\ \textit{Proved} \end{array}$$

$\therefore AC = DB$, $\angle ACB = \angle DBC$. *I.* 4

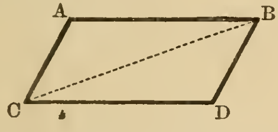
Because CB meets AC and BD, and makes the alternate $\angle$s ACB, DBC equal; *Proved*

$\therefore AC$ is $\parallel BD$. *I.* 27

1. State a converse of this proposition.
2. If a quadrilateral have one pair of opposite sides equal and parallel, it is a $\parallel^m$.
3. What statements may be made about the straight lines which join the ends of two equal and parallel straight lines towards opposite parts?

PROPOSITION 34. Theorem.

A parallelogram has its opposite sides and angles equal, and is bisected by either diagonal.



Let $ACDB$ be a $\parallel^m$ of which BC is a diagonal:
it is required to prove that the opposite sides and angles of $ACDB$ are equal, and that $\triangle ABC = \triangle DCB$.

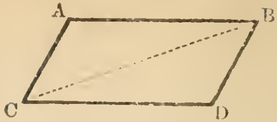
Because BC meets the parallels AB, CD,

$\therefore \angle ABC =$ alternate $\angle DCB$; *I.* 29

and because BC meets the parallels AC, BD,

$\therefore \angle ACB =$ alternate $\angle DBC$. *I.* 29

$$\text{In } \triangle\text{s } ABC, DCB, \left\{ \begin{array}{l} \angle ABC = \angle DCB \\ \angle ACB = \angle DBC \\ BC = CB; \end{array} \right. \quad \begin{array}{r} \textit{Proved} \\ \textit{Proved} \\ \end{array}$$



$\therefore$ $AB = DC$, $AC = DB$, $\angle BAC = \angle CDB$,

$\triangle ABC = \triangle DCB$. *I.* 26

Again because $\angle ABC$ was proved $= \angle DCB$, *I.* 29

and $\angle DBC$ was proved $= \angle ACB$; *I.* 29

$\therefore$ the whole $\angle ABD =$ the whole $\angle DCA$.

COR.—If the arms of one angle be respectively parallel to the arms of another, the angles are either (1) equal or (2) supplementary.

For (1) $\angle BAC$ has been proved $= \angle CDB$;

and (2) if BA be produced to E,

$\angle EAC$, which is supplementary to $\angle BAC$, *I.* 13

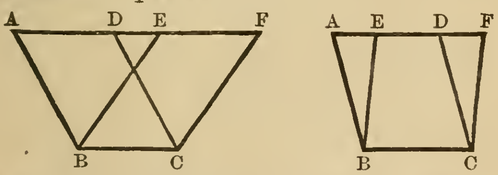
must be supplementary to $\angle CDB$.

1. If two sides of a $\|^m$ which are not opposite to each other be equal, all the sides are equal.
2. If two angles of a $\|^m$ which are not opposite to each other be equal, all the angles are right.
3. If one angle of a $\|^m$ be right, all the angles are right.
4. If two $\|^{ms}$ have one angle of the one $=$ one angle of the other, the $\|^{ms}$ are mutually equiangular.
5. If a quadrilateral have its opposite sides equal, it is a $\|^m$.
6. If a quadrilateral have its opposite angles equal, it is a $\|^m$.
7. If the diagonals of a $\|^m$ be equal to each other, the $\|^m$ is a rectangle.
8. If the diagonals of a $\|^m$ bisect the angles through which they pass, the $\|^m$ is a rhombus.
9. If the diagonals of a $\|^m$ cut each other perpendicularly, the $\|^m$ is a rhombus.
10. If the diagonals of a $\|^m$ be equal and cut each other perpendicularly, the $\|^m$ is a square.
11. Show how to bisect a straight line by means of a pair of parallel rulers.

12. Every straight line drawn through the intersection of the diagonals of a $\|^m$, and terminated by a pair of opposite sides, is bisected, and bisects the $\|^m$.
13. Bisect a given $\|^m$ by a straight line drawn through a given point either within or without the $\|^m$.
14. The straight line joining the middle points of any two sides of a triangle is $\|$ the third side and = half of it.
15. If the middle points of the three sides of a triangle be joined with each other, the four triangles thence resulting are equal.
16. Construct a triangle, having given the middle points of its three sides.

PROPOSITION 35. Theorem.

Parallelograms on the same base and between the same parallels are equal in area.



Let $ABCD$, $EBCF$ be $\|^{ms}$ on the same base BC, and between the same parallels AF, BC:
it is required to prove $\|^m ABCD = \|^m EBCF$.

Because AF meets the parallels AB, DC,
$\therefore$ interior $\angle A$ = exterior $\angle FDC$; *I.* 29
and because AF meets the parallels EB, FC,
$\therefore$ exterior $\angle AEB$ = interior $\angle F$. *I.* 29

In $\triangle$s ABE, DCF, $\left\{ \begin{array}{l} \angle EAB = \angle FDC \\ \angle AEB = \angle DFC \\ AB = DC; \end{array} \right.$ *Proved* / *Proved* / *I.* 34

$\therefore \triangle ABE = \triangle DCF$. *I.* 26

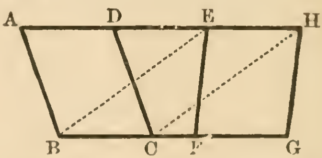
Hence quadrilateral $ABCF - \triangle ABE$
= quadrilateral $ABCF - \triangle DCF$;
$\therefore$ $\|^m EBCF = \|^m ABCD$.

NOTE.—This proposition affords a means of measuring the area of a $\parallel^{m}$; thence (by I. 34 or 41) the area of a triangle; and thence (by I. 37, Cor.) the area of any rectilineal figure. For the area of any $\parallel^{m}$ = the area of a rectangle on the same base and between the same parallels; and it is, or ought to be, explained in books on Mensuration, that the area of a rectangle is found by taking the product of its length and breadth. This phrase 'taking the product of its length and breadth,' means that the numbers, whether integral or not, which express the length and breadth in terms of the same linear unit, are to be multiplied together. Hence the method of finding the area of a $\parallel^{m}$ is to take the product of its base and altitude, the altitude being defined to be the perpendicular drawn to its base from any point in the side opposite.

1. Prove the proposition for the case when the points D and E coincide.
2. Equal $\parallel^{ms}$ on the same base and on the same side of it are between the same parallels.
3. If through the vertices of a triangle straight lines be drawn $\parallel$ the opposite sides, and produced till they meet, the resulting figure will contain three equal $\parallel^{ms}$.
4. On the same base and between the same parallels as a given $\parallel^{m}$, construct a rhombus = the $\parallel^{m}$.
5. Prove the equality of $\triangle$s ABE and DCF in the proposition by I. 4 (as Euclid does), or by I. 8, instead of by I. 26.

PROPOSITION 36. THEOREM.

Parallelograms on equal bases and between the same parallels are equal in area.



Let $ABCD$, $EFGH$ be $\parallel^{ms}$ on equal bases BC, FG, and between the same parallels, AH, BG:

it is required to prove $\parallel^{m}$ $ABCD$ = $\parallel^{m}$ $EFGH$.

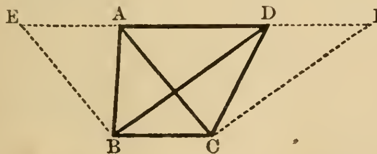
Join BE, CH.

Because $BC = FG$, and $FG = EH$, *Hyp.*, *I.* 34
$\therefore$ $BC = EH$.
And because BC is $\parallel$ EH,
$\therefore$ EB is $\parallel$ HC; *I.* 33
$\therefore$ $EBCH$ is a $\parallel^m$. *I. Def.* 33
Now $\parallel^m$ $ABCD = \parallel^m$ $EBCH$, being on the same base BC, and between the same parallels BC, AH; *I.* 35
and $\parallel^m$ $EFGH = \parallel^m$ $EBCH$, being on the same base EH, and between the same parallels EH, BG; *I.* 35
$\therefore$ $\parallel^m$ $ABCD = \parallel^m$ $EFGH$.

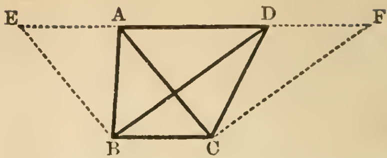
1. Prove the proposition by joining AF, DG instead of BE, CH.
2. Divide a given $\parallel^m$ into two equal $\parallel^{ms}$.
3. In how many ways may this be done?
4. Of two $\parallel^{ms}$ which are between the same parallels, that is the greater which stands on the greater base.
5. State and prove a converse of the last deduction.
6. Equal $\parallel^{ms}$ situated between the same parallels have equal bases.

PROPOSITION 37. Theorem.

Triangles on the same base and between the same parallels are equal in area.



Let ABC, DBC be triangles on the same base BC, and between the same parallels AD, BC:
it is required to prove $\triangle$ $ABC = \triangle$ DBC.



Through B draw $BE \parallel AC$, and through C draw CF $\parallel BD$; *I.* 31

and let them meet AD produced at E and F.

Then $EBCA$, $DBCF$ are $\parallel^{ms}$; *I. Def.* 33

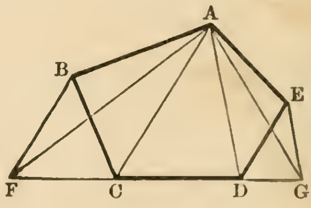
and $\parallel^m EBCA = \parallel^m DBCF$, being on the same base BC, and between the same parallels BC, EF. *I.* 35

But $\triangle ABC =$ half of $\parallel^m EBCA$, *I.* 34

and $\triangle DBC =$ half of $\parallel^m DBCF$; *I.* 34

$\therefore \triangle ABC = \triangle DBC$.

Cor.—Hence any rectilineal figure may be converted into an equivalent triangle.



Let $ABCDE$ be any rectilineal figure:
it is required to convert it into an equivalent triangle.

Join AC, AD;
through B draw $BF \parallel AC$, through E draw $EG \parallel AD$, *I.* 31
and let them meet CD produced at F and G.
Join AF, AG. AFG is the required triangle.

For $\triangle AFC = \triangle ABC$, and $\triangle AGD = \triangle AED$; *I.* 37

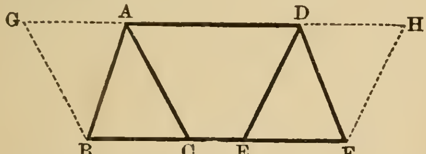
$\therefore \triangle AFC + \triangle ACD + \triangle AGD = \triangle ABC + \triangle ACD + \triangle AED.$

$\therefore \triangle AFG =$ figure $ABCDE$.

1. ABC is any triangle; DE is drawn $\parallel$ the base BC, and meets AB, AC at D and E; BE and CD are joined. Prove $\triangle DBC = \triangle EBC$, $\triangle BDE = \triangle CED$, and $\triangle ABE = \triangle ACD$.
2. $ABCD$ is a quadrilateral having $AB \parallel CD$; its diagonals AC, BD meet at O. Prove $\triangle AOD = \triangle BOC$.
3. In what case would no construction be necessary for the proof of this proposition?
4. Convert a quadrilateral into an equivalent triangle.
5. ABC is any triangle, D a point in AB; find a point E in BC produced such that $\triangle DBE = \triangle ABC$.

PROPOSITION 38. THEOREM.

Triangles on equal bases and between the same parallels are equal in area.



Let ABC, DEF be triangles on equal bases BC, EF, and between the same parallels AD, BF:

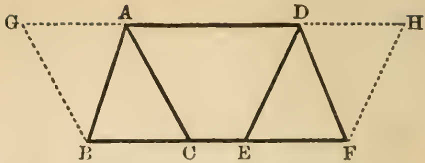
it is required to prove $\triangle ABC = \triangle DEF$.

Through B draw $BG \parallel AC$, and through F draw $FH \parallel DE$; *I.* 31

and let them meet AD produced at G and H.

Then $GBCA$, $DEFH$ are $\|^{ms}$;

and $\|^{m}$ $GBCA$ = $\|^{m}$ $DEFH$, being on equal bases BC, EF,



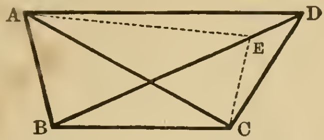
and between the same parallels BF, GH. *I.* 36
But $\triangle ABC$ = half of ||^m $GBCA$, *I.* 34
and $\triangle DEF$ = half of ||^m $DEFH$; *I.* 34
$\therefore \triangle ABC = \triangle DEF$.

Cor.—The straight line joining any vertex of a triangle to the middle point of the opposite side bisects the triangle. Hence the theorem: If two triangles have two sides of the one respectively equal to two sides of the other and the contained angles supplementary, the triangles are equal in area.

1. Of two triangles which are between the same parallels, that is the greater which stands on the greater base.
2. State and prove a converse of the last deduction.
3. Two triangles are between the same parallels, and the base of the first is double the base of the second; prove the first triangle double the second.
4. The four triangles into which the diagonals divide a ||^m are equal.
5. If one diagonal of a quadrilateral bisects the other diagonal, it also bisects the quadrilateral.
6. $ABCD$ is a ||^m; E is any point in AD or AD produced, and F any point in BC or BC produced; AF, DF, BE, CE are joined. Prove $\triangle AFD = \triangle BEC$.
7. ABC is any triangle; L and K are the middle points of AB and AC; BK and CL are drawn intersecting at G, and AG is joined. Prove $\triangle BGC = \triangle AGC = \triangle AGB$.
8. $ABCD$ is a ||^m; P is any point in the diagonal BD or BD produced, and PA, PC are joined. Prove $\triangle PAB = \triangle PCB$, and $\triangle PAD = \triangle PCD$.
9. Bisect a triangle by a straight line drawn from a given point in one of the sides.

PROPOSITION 39. THEOREM.

Equal triangles on the same side of the same base are between the same parallels.



Let $\triangle$s ABC, DBC on the same side of the same base BC be equal, and let AD be joined :
it is required to prove $AD \parallel BC$.

If AD is not $\parallel BC$, through A draw $AE \parallel BC$, *I.* 31
meeting BD, or BD produced, at E, and join EC.

Then $\triangle\, ABC = \triangle\, EBC$. *I.* 37

But $\triangle\, ABC = \triangle\, DBC$; *Hyp.*

$\therefore$ $\triangle\, EBC = \triangle\, DBC$;

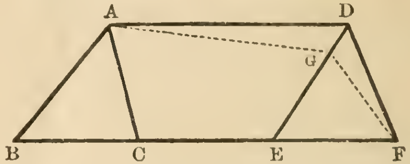
which is impossible, since the one is a part of the other.

$\therefore$ AD is $\parallel BC$.

1. The straight line joining the middle points of two sides of a triangle is $\parallel$ the third side, and = half of it.
2. Hence prove that the straight line joining the middle point of the hypotenuse of a right-angled triangle to the opposite vertex = half the hypotenuse.
3. The middle points of the sides of any quadrilateral are the vertices of a $\parallel^{m}$, whose perimeter = the sum of the diagonals of the quadrilateral. When will this $\parallel^{m}$ be a rectangle, a rhombus, a square?
4. If two equal triangles be on the same base, but on opposite sides of it, the straight line which joins their vertices will be bisected by the base.
5. Use the first deduction to solve I. 31.
6. In the figure to I. 16, prove $AF \parallel BC$.
7. If a quadrilateral be bisected by each of its diagonals, it is a $\parallel^{m}$.
8. Divide a given triangle into four triangles which shall be equal in every respect.

PROPOSITION 40. THEOREM.

Equal triangles on the same side of equal bases which are in the same straight line are between the same parallels.



Let $\triangle$s ABC, DEF, on the same side of the equal bases BC, EF, which are in the same straight line BF, be equal, and let AD be joined :
it is required to prove $AD \parallel BF$.

If AD is not $\parallel BF$, through A draw $AG \parallel BF$, *I.* 31
meeting DE, or DE produced, at G, and join GF.

Then $\triangle ABC = \triangle GEF$. *I.* 38

But $\triangle ABC = \triangle DEF$; *Hyp.*

$\therefore$ $\triangle GEF = \triangle DEF$;

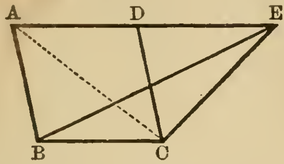
which is impossible, since the one is a part of the other.

$\therefore AD$ is $\parallel BF$.

1. Prove the proposition by joining AE and AF.
2. Prove the proposition by joining DB and DC.
3. Any number of equal triangles stand on the same side of equal b 'es. If their bases be in one straight line, their vertices will also be in one straight line.
4. Equal triangles situated between the same parallels have equal bases.
5. Trapeziums on the same base and between the same parallels are equal if the sides opposite the common base are equal.
6. The median from the vertex to the base of a triangle bisects every parallel to the base.
7. Hence devise a method of bisecting a given straight line.

PROPOSITION 41. THEOREM.

If a parallelogram and a triangle be upon the same base and between the same parallels, the parallelogram shall be double of the triangle.



Let the $\|^m$ $ABCD$ and the $\triangle$ EBC be on the same base BC, and between the same parallels AE, BC:

it is required to prove $\|^m$ $ABCD$ = *twice* $\triangle$ EBC.

Join AC.

Then $\triangle ABC = \triangle EBC$. *I.* 37

But $\|^m ABCD =$ twice $\triangle ABC$; *I.* 34

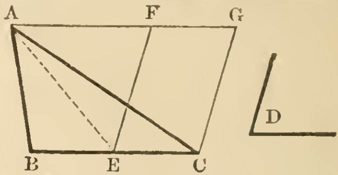
$\therefore$ $\|^m ABCD =$ twice $\triangle EBC$.

1. Prove the proposition by drawing through C a parallel to BE.
2. If a $\|^m$ and a triangle be on equal bases and between the same parallels, the $\|^m$ shall be double of the triangle.
3. A $\|^m$ and a triangle are equal if they are between the same parallels, and the base of the triangle is double that of the $\|^m$.
4. State and prove a converse of the last deduction.
5. If from any point within a $\|^m$ straight lines be drawn to the ends of two opposite sides, the sum of the triangles on these sides shall be equal to half the $\|^m$. Is the theorem true when the point is taken outside? Examine all the cases.
6. $ABCD$ is any quadrilateral, AC and BD its diagonals. A $\|^m$ $EFGH$ is formed by drawing through A, B, C, D parallels to AC and BD. Prove $ABCD$ = half of $EFGH$.
7. Hence, show that the area of a quadrilateral = the area of a triangle which has two of its sides equal to the diagonals of the quadrilateral, and the included angle equal to either of

the angles at which the diagonals intersect; and that two quadrilaterals are equal if their diagonals are equal, and also the angles at the intersection of the diagonals.

PROPOSITION 42. Problem.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given angle.



Let ABC be the given triangle, and D the given angle: *it is required to describe a* $\|^m$ *equal to* $\triangle\ ABC$, *and having one of its angles equal to* $\angle\ D$.

Bisect BC at E; *I.* 10
and at E make $\angle\ CEF = \angle\ D$. *I.* 23
Through A draw $AG \parallel BC$; through C draw $CG \parallel EF$. *I.* 31
$FECG$ is the $\|^m$ required.

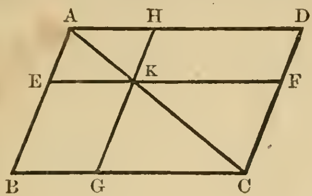
Join AE.

The figure $FECG$ is a $\|^m$; *I. Def.* 33
and $\|^m\ FECG$ = twice $\triangle\ AEC$. *I.* 41
But since $\triangle\ ABE = \triangle\ AEC$, *I.* 38
$\therefore$ $\triangle\ ABC$ = twice $\triangle\ AEC$;
$\therefore$ $\|^m\ FECG = \triangle\ ABC$,
and $\angle\ CEF$ was made $= \angle\ D$.

1. Describe a rectangle equal to a given triangle.
2. Describe a triangle that shall be equal to a given $\|^m$, and have one of its angles equal to a given angle.
3. On the same base as a $\|^m$ construct a right-angled triangle = the $\|^m$.
4. Construct a rhombus = a given triangle.

PROPOSITION 43. THEOREM.

The complements of the parallelograms which are about a diagonal of any parallelogram are equal.



Let $ABCD$ be a $\|^m$, and AC one of its diagonals;
let EH, GF be $\|^{ms}$ about AC, that is, through which AC passes, and BK, KD the other $\|^{ms}$ which fill up the figure $ABCD$, and are therefore called the complements:
it is required to prove complement BK = complement KD.

Because EH is a $\|^m$ and AK its diagonal,

$\therefore$ $\triangle AEK = \triangle AHK$. *I.* 34

Similarly $\triangle KGC = \triangle KFC$; *I.* 34

$\therefore \triangle AEK + \triangle KGC = \triangle AHK + \triangle KFC$.

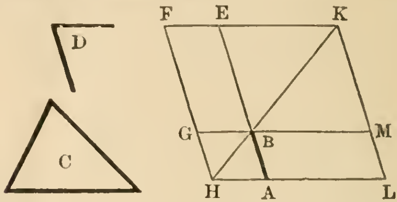
But the whole $\triangle ABC$ = whole $\triangle ADC$; *I.* 34

$\therefore$ the remainder, complement BK = the remainder, complement KD.

1. Name the eight $\|^{ms}$ into which $ABCD$ is divided by EF and GH, and prove that they are all equiangular to $\|^m$ $ABCD$.
2. Prove $\|^m$ AG = $\|^m$ ED, and $\|^m$ BF = $\|^m$ DG.
3. If a point K be taken inside a $\|^m$ $ABCD$, and through it parallels be drawn to AB and BC, and if $\|^m$ BK = $\|^m$ KD, the diagonal AC passes through K. (Converse of I. 43.)
4. Each of the $\|^{ms}$ about a diagonal of a rhombus is itself a rhombus.
5. Each of the $\|^{ms}$ about a diagonal of a square is itself a square.
6. Each of the $\|^{ms}$ about a square's diagonal produced is itself a square.
7. When are the complements of the $\|^{ms}$ about a diagonal of any $\|^m$ equal in every respect?

PROPOSITION 44. PROBLEM.

On a given straight line to describe a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given angle.



Let AB be the given straight line, C the given triangle, and D the given angle:

it is required to describe on AB a $\|^{m} = \triangle\ C$, *and having an angle* $= \angle\ D$.

Describe the $\|^{m}$ $BEFG = \triangle\ C$, and having $\angle\ EBG = \angle\ D$; and let it be so placed that BE may be in the same straight line with AB. *I.* 42

Through A draw $AH \parallel BG$ or EF, *I.* 31

and let it meet FG produced at H;

join HB.

Because HF meets the parallels AH, EF,

$\therefore\ \angle\ AHF + \angle\ HFE = 2$ rt. $\angle$s; *I.* 29

$\therefore\ \angle\ BHF + \angle\ HFE$ is less than 2 rt. $\angle$s;

$\therefore\ HB$, FE, if produced, will meet towards B, E. *I.* 29, *Cor.*

Let them be produced and meet at K;

through K draw $KL \parallel EA$ or FH, *I.* 31

and produce HA, GB to L and M.

$ABML$ is the $\|^{m}$ required.

For $FHLK$ is a $\|^{m}$, of which HK is a diagonal,

and AG, ME are $\|^{ms}$ about HK;

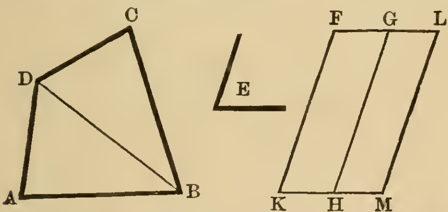
$\therefore$ complement BL = complement BF, *I.* 43
$= \triangle C$.

And $\angle ABM = \angle EBG$, *I.* 15
$= \angle D$.

1. On a given straight line describe a rectangle equal to a given triangle.
2. On a given straight line describe a triangle equal to a given $||^m$, and having one of its angles equal to a given angle.
3. On a given straight line describe an isosceles triangle equal to a given $||^m$.
4. Cut off from a triangle, by a straight line drawn from one of the vertices, a given area.

PROPOSITION 45. PROBLEM.

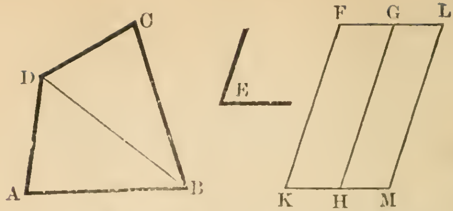
To describe a parallelogram equal to any given rectilineal figure, and having an angle equal to a given angle.



Let $ABCD$ be the given rectilineal figure, E the given angle:
it is required to describe a $||^m = ABCD$, *and having an angle* $= \angle E$.

Join BD, and describe the $||^m$ $FH = \triangle ABD$, and having $\angle K = \angle E$; *I.* 42
on GH describe the $||^m$ $GM = \triangle BCD$, and having $\angle GHM = \angle E$. *I.* 44

$FKML$ is the $||^m$ required.



Because $\angle K = \angle GHM$, since each $= \angle E$;
to each of these equals add $\angle GHK$;

$\therefore \quad \angle K + \angle GHK = \angle GHM + \angle GHK.$

But $\quad \angle K + \angle GHK = 2$ rt. $\angle$s; *I.* 29

$\therefore \angle GHM + \angle GHK = 2$ rt. $\angle$s;

$\therefore$ *KH* and *HM* are in the same straight line. *I.* 14

Again, because *FG* and *GL* drawn from *G* are both $\parallel$ *KM*:

$\therefore$ *FG* and *GL* must be in the same straight line. *I. Ax.* 11

Now because *KF* and *ML* are both $\parallel$ *HG*,

$\therefore$ *KF* is $\parallel$ *ML*; *I.* 30

and *KM* is $\parallel$ *FL*;

$\therefore$ *FKML* is a $\|^{m}$.

But $\|^{m}$ $FKML = \|^{m}$ FH + $\|^{m}$ GM,

$= \triangle ABD + \triangle BCD$, *Const.*

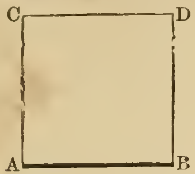
= figure *ABCD*;

and $\angle K = \angle E$. *Const.*

1. Could two $\|^{ms}$ have a common side and together not form one $\|^{m}$? Illustrate by a figure.
2. Describe a rectangle equal to a given rectilineal figure.
3. On a given straight line describe a rectangle equal to a given rectilineal figure.
4. Given one side and the area of a rectangle; find the other side.
5. Describe a $\|^{m}$ equal to a given rectilineal figure, and having an angle equal to a given angle, using I. 37, Cor.
6. Describe a $\|^{m}$ equal to the sum of two given rectilineal figures.
7. Describe a $\|^{m}$ equal to the difference of two given rectilineal figures.

PROPOSITION 46. PROBLEM.

On a given straight line to describe a square.



Let AB be the given straight line:
it is required to describe a square on AB.

From A draw $AC \perp AB$ and $= AB$; *I.* 11, 3
through C draw $CD \parallel AB$, *I.* 31
and through B draw $BD \parallel AC$. *I.* 31

$ABDC$ is the square required.

For $ABDC$ is a $\parallel^m$; *I. Def.* 33
$\therefore$ $AB = CD$ and $AC = BD$. *I.* 34
But $AB = AC$; *Const.*
$\therefore$ the four sides AB, BD, DC, CA are all equal.

Because AC meets the parallels AB, CD,
$\therefore$ $\angle A + \angle C = 2$ rt. $\angle$s. *I.* 29
But $\angle A$ is right;
$\therefore$ $\angle C$ is also right.
Now $\angle A = \angle D$ and $\angle C = \angle B$; *I.* 34
$\therefore$ the four $\angle$s A, B, D, C are right;
$\therefore$ $ABDC$ is a square. *I. Def.* 32

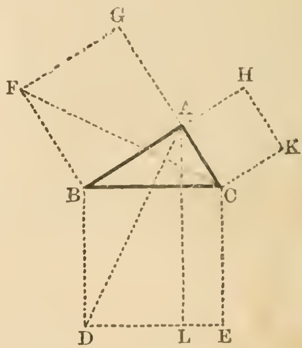
1. What is redundant in Euclid's definition of a square?
2. If two squares be equal, the sides on which they are described are equal.
3. $ABDC$ is constructed thus: At A and B draw AC and BD $\perp AB$ and $= AB$, and join CD. $ABDC$ is a square.
4. $ABDC$ is constructed thus: At A draw $AC \perp AB$ and $= AB$; with B and C as centres, and a radius $= AB$ or AC, describe

two circles intersecting at D; and join BD. DC. $ABDC$ is a square

5. Describe a square having given a diagonal.

PROPOSITION 47. THEOREM.

*The square described on the hypotenuse of a right angled triangle is equal to the squares described on the other two sides.**



Let ABC be a right-angled triangle, having the right angle BAC:

it is required to prove that the square described on BC = *square on* BA + *square on* AC.

On AB, BC, CA describe the squares GB, BE, CH; I. 46

through A draw $AL \parallel BD$ or CE; I. 31

and join AD, CF.

Because $\angle BAC + \angle BAG = 2$ rt. $\angle$s,

$\therefore$ GA and AC form one straight line. I. 14

Similarly, HA and AB form one straight line.

* This theorem is usually attributed to Pythagoras (580–510 B.C.).

Now $\angle\ DBC = \angle\ FBA$, each being right.
Add to each $\angle\ ABC$;
$\therefore\ \angle\ ABD = \angle\ FBC$.

In $\triangle$s ABD, FBC, $\left\{\begin{array}{ll} AB = FB & \textit{I. Def. } 32 \\ BD = BC & \textit{I. Def. } 32 \\ \angle\ ABD = \angle\ FBC; & \textit{Proved} \end{array}\right.$

$\therefore\ \triangle\ ABD = \triangle\ FBC$. *I.* 4

But $\|^m$ BL = twice $\triangle\ ABD$, being on the same base BD, and between the same $\|^s$ BD, AL; *I.* 41
and square BG = twice $\triangle\ FBC$, being on the same base BF, and between the same $\|^s$ BF, CG; *I.* 41
$\therefore$ $\|^m$ BL = square BG.

Similarly, if AE, BK be joined, it may be proved that $\|^m$ CL = square CH;
$\therefore$ $\|^m$ BL + $\|^m$ CL = square BG + square CH,
that is, square on BC = square on BA + square on AC.
[It is usual to write this result $BC^2 = BA^2 + AC^2$; but see p. 113.]

Cor.—The difference between the square on the hypotenuse of a right-angled triangle and the square on either of the sides is equal to the square on the other side.

For since $BC^2 = BA^2 + AC^2$,
$\therefore\ BC^2 - BA^2 = AC^2$,
and $BC^2 - AC^2 = BA^2$.

Note.—This proposition is an exceedingly important one, and numerous demonstrations of it have been given by mathematicians, some of them such as easily to afford ocular proof of the equality asserted in the enunciation. With respect to Euclid's method of proof (which is not* that of the discoverer), it may be remarked that he has chosen that position of the squares when they are all exterior to the triangle. The pupil is advised to make the seven other modifications of the figure which result from placing the squares in different positions with respect to the sides of the triangle, and to adapt Euclid's proof thereto. It will be found that AG and AC, as well as AH and AB, will always be in the same

* See Friedlein's *Proclus*, p. 426.

straight line, only, instead of being drawn in opposite directions from A as in the text, they will sometimes be drawn in the same direction, that $\angle$s ABD and FBC will sometimes be supplementary instead of equal; and that then the equality of $\triangle$s ABD and FBC will follow, not from I. 4, but from I. 38, Cor.

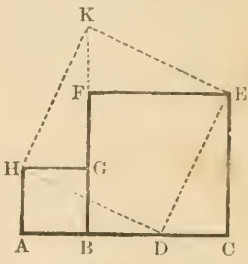
All the different varieties of figure are obtained thus:

Call X the square on the hypotenuse, Y and Z the squares on the other sides. Describe

(1)	X outwardly,	Y outwardly,	Z outwardly.
(2)	" "	" "	" inwardly.
(3)	" "	" inwardly	" outwardly.
(4)	" "	" "	" inwardly.
(5)	" inwardly,	" outwardly,	" outwardly.
(6)	" "	" "	" inwardly.
(7)	" "	" inwardly,	" outwardly.
(8)	" "	" "	" inwardly.

The following methods of exhibiting how two squares may be dissected and put together so as to form a third square, are probably the simplest and neatest ocular proofs yet given of this celebrated proposition:

FIRST METHOD.



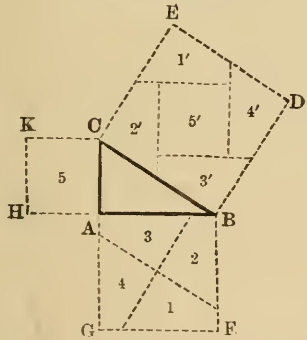
$ABGH$, $BCEF$ are two squares placed side by side, and so that AB and BC form one straight line. Cut off $CD = AB$, and join ED, DH.

(1) If, round E as a pivot, $\triangle$ ECD is rotated like the hands of a watch through a right angle, it will occupy the position EFK. If, round H as a pivot, $\triangle$ HAD is rotated in a manner opposite to the hands of a watch through a right angle, it will occupy the position HGK. The two squares $ABGH$ and $BCEF$ will then be transformed into the square $DEKH$.

(2) If $\triangle$ *ECD* be slid along the plane in such a way that *EC* always remains vertical, and *D* moves along the line *DH*, it will come to occupy the position *KGH*. If $\triangle$ *HAD* be slid along the plane in such a way that *HA* always remains vertical, and *D* moves along the line *DE*, it will come to occupy the position *KFE*. The two squares *ABGH* and *BCEF* will then be transformed into the square *DEKH*.

[This method is substantially that given by Schooten in his *Exercitationes Mathematicæ* (1657), p. 111. The first or rotational way of getting $\triangle$s *ECD*, *HAD* into their places is given by J. C. Sturm in his *Mathesis Enucleata* (1689), p. 31; the second or translational way is mentioned by De Morgan in the *Quarterly Journal of Mathematics*, vol. i. p. 236.]

SECOND METHOD.



ABC is a right-angled triangle. *BCED* is the square on the hypotenuse, *ACKH* and *ABFG* are the squares on the other sides.

Find the centre of the square *ABFG*, which may be done by drawing the two diagonals (not shown in the figure), and through it draw two straight lines, one of which is $\parallel$ *BC*, and the other $\perp$ *BC*. The square *ABFG* is then divided into four quadrilaterals equal in every respect. Through the middle points of the sides of the square *BCED* draw parallels to *AB* and *AC* as in the figure. Then the parts 1, 2, 3, 4, 5 will be found to coincide exactly with 1′, 2′, 3′, 4′, 5′.

[This method is due to Henry Perigal, F.R.A.S., and was discovered about 1830. See *The Messenger of Mathematics*, new series, vol. ii. pp. 103–106.]

1. Show how to find a square = the sum of two given squares.
2. " " = " three "
3. " " = the difference of two "
4. " " double of a given square.
5. " " half "
6. " " triple "
7. The square described on a diagonal of a given square is twice the given square.
8. Hence prove that the square on a straight line is four times the square on half the line.
9. The squares described on the two diagonals of a rectangle are together equal to the squares described on the four sides.
10. The squares described on the two diagonals of a rhombus are together equal to the squares described on the four sides.
11. If the hypotenuse and a side of one right-angled triangle be equal to the hypotenuse and a side of another right-angled triangle, the two triangles are equal in every respect.
12. If from the vertex of any triangle a perpendicular be drawn to the base, the difference of the squares on the two sides of the triangle is equal to the difference of the squares on the segments of the base.
13. The square on the side opposite an acute angle of a triangle is less than the squares on the other two sides.
14. The square on the side opposite an obtuse angle of a triangle is greater than the squares on the other two sides.
15. Five times the square on the hypotenuse of a right-angled triangle is equal to four times the sum of the squares on the medians drawn to the other two sides.
16. Three times the square on a side of an equilateral triangle is equal to four times the square on the perpendicular drawn from any vertex to the opposite side.
17. Divide a given straight line into two parts such that the sum of their squares may be equal to a given square. Is this always possible?
18. Divide a given straight line into two parts such that the square on one of them may be double the square on the other.
19. If a straight line be divided into any two parts, the square on the whole line is greater than the sum of the squares on the two parts.

20. The sum of the squares of the distances of any point from two opposite corners of a rectangle is equal to the sum of the squares of its distances from the other two corners.

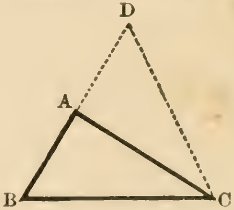
The following deductions refer to the figure of the proposition in the text. They are all, or nearly all, given in an article in *Leybourn's Mathematical Repository*, new series, vol. iii. (1814), Part II. pp. 71-80, by John Bransby, Ipswich.

21. What is the use of proving that AG and AC are in the same straight line, and also AB and AH?
22. AF and AK are in the same straight line.
23. BG is $\parallel$ CH.
24. Prove $\triangle$s ABD, FBC equal by rotating the former round B through a right angle. Similarly, prove $\triangle$s ACE, KCB equal.
25. Hence prove $AD \perp FC$, and $AE \perp KB$.
26. $\angle$s ABC and DBF are supplementary, as also are $\angle$s ACB and ECK.
27. Hence prove $\triangle$s FBD, $KCE = \triangle\ ABC$.
28. FG, KH, LA all meet at one point T.
29. $\triangle$s AGH, THG, GAT, HTA are each $= \triangle\ ABC$.
30. If from D and E, perpendiculars DU, EV be drawn to FB and KC produced, $\triangle$s UBD and VEC are each $= \triangle\ ABC$. Prove by rotating.
31. $DF^2 + EK^2 = 5\ BC^2$.
32. The squares on the sides of the polygon $DFGHKE = 8\ BC^2$.
33. If from F and K perpendiculars FM, KN be drawn to BC produced, and I be the point where AL meets BC, $\triangle\ BFM = \triangle\ ABI$, and $\triangle\ CKN = \triangle\ ACI$.
34. $FM + KN = BC$, and $BN = CM = AL$.
35. If DB and EC produced meet FG and KH at P and Q, prove by rotating $\triangle\ ABC$ that it = each of the $\triangle$s FBP, KCQ.
36. If PQ be joined, $BCQP$ is a square.
37. $ABPT$ is a $\parallel^{m}$, and = rectangle BL; $ACQT$ is a $\parallel^{m}$, and = rectangle CL.
38. $ADBT$ is a $\parallel^{m}$, and = rectangle BL; $AECT$ is a $\parallel^{m}$, and = rectangle CL.
39. $DFPU$ and $EKQV$ are $\parallel^{ms}$, and each $= 4\ \triangle\ ABC$.
40. $ADUH$ and $AEVG$ are $\parallel^{ms}$, and each $= 2\ \triangle\ ABC$.
41. BK is $\perp CT$, and $CF \perp BT$.
42. Hence prove that AL, BK, CF meet at one point O. (See App. I. 3.)

43. If BK meet AC in X, and CF meet AB in W, $\triangle$s BHX, CGW are each $= \triangle\ ABC$.
44. $AW = AX$.
45. $\triangle\ ACW = \triangle\ BCX$, and $\triangle\ ABX = \triangle\ BCW$.
46. Quadrilateral $AWOX = \triangle\ BOC$.
47. If from G and H perpendiculars GR, HS be drawn to BC or BC produced, and if these perpendiculars meet AB and AC in Y and Z, prove by rotating $\triangle\ ABC$ that it $= \triangle\ GAY$ or $\triangle\ ZAH$.
48. DU produced passes through Z, EV produced through Y, GV through W, and HU through X.
49. If through A a parallel to BC be drawn, meeting GR in G', and HS in H', $\triangle$s AGG', AZH' are $= \triangle\ ABI$, and $\triangle$s AYG' and $AHH' = \triangle\ ACI$.
50. $IR = IS$; $GR + HS = MN$; $FM + GR + HS + KN = 2\ (BC + AI)$; $GR = BS$; $HS = CR$.

PROPOSITION 48. Theorem.

If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by those two sides is a right angle.



Let ABC be a triangle, and let $BC^2 = BA^2 + AC^2$: *it is required to prove* $\angle\ BAC$ *right.*

From A draw $AD \perp AC$, and $= AB$; *I.* 11, 3
and join CD.

Because $AD = AB$; $\therefore AD^2 = AB^2$.
To each of these equals add AC^2;
$\therefore AD^2 + AC^2 = AB^2 + AC^2$.

But $AD^2 + AC^2 = CD^2$, *I.* 47
and $AB^2 + AC^2 = BC^2$; *Hyp.*
$\therefore CD^2 = BC^2$;
$\therefore CD = BC$.

In $\triangle$s BAC, DAC, $\begin{cases} BA = DA & \textit{Const.} \\ AC = AC & \\ BC = DC; & \textit{Proved} \end{cases}$

$\therefore \angle BAC = \angle DAC$, *I.* 8
$= $ a right angle.

1. In the construction it is said, draw $AD \perp AC$. Would it not be simpler, and answer the same purpose, to say, produce AB to D. Why?
2. Prove the proposition indirectly by drawing $AD \perp AC$, and on the same side of AC as AB, and using I. 7 (Proclus).
3. If the square on one side of a triangle be less than the sum of the squares on the other two sides, the angle opposite that side is acute.
4. If the square on one side of a triangle be greater than the sum of the squares on the other two sides, the angle opposite that side is obtuse.
5. Prove that the triangle whose sides are 3, 4, 5 is right-angled.*
6. Hence derive a method of drawing a perpendicular to a given straight line from a point in it.
7. Show that the following two rules,† due respectively to Pythagoras and Plato, give numbers representing the sides of right-angled triangles, and show also that the two rules are fundamentally the same.

(*a*) Take an odd number for the less side about the right angle. Subtract unity from the square of it, and halve the remainder; this will give the greater side about the right angle. Add unity to the greater side for the hypotenuse.

(*b*) Take an even number for one of the sides about the right angle. From the square of half of this number subtract unity for the other side about the right angle, and to the square of half this number add unity for the hypotenuse.

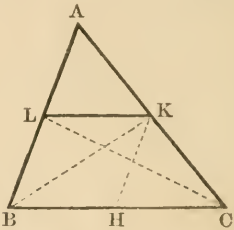
* This is said by Plutarch to have been known to the early Egyptians.

† See Friedlein's *Proclus*, p. 428, and Hultsch's *Heronis . . . reliquiæ*, pp. 56, 57.

APPENDIX I.

Proposition 1.

The straight line joining the middle points of any two sides of a triangle is parallel to the third side and equal to the half of it.



Let ABC be a triangle, and let L, K be the middle points of AB, AC:
it is required to prove $LK \parallel BC$ and $=$ half of BC.

Join BK, CL.

Because $AL = BL$, $\therefore \triangle BLC =$ half of $\triangle ABC$; *I.* 38
and because $AK = CK$, $\therefore \triangle BKC =$ half of $\triangle ABC$; *I.* 38
$\therefore \triangle BLC = \triangle BKC$.
$\therefore LK$ is $\parallel BC$. *I.* 39

Hence, if H be the middle of BC, and HK be joined, HK is $\parallel AB$;
$\therefore BHKL$ is a $\parallel^{m}$; *I. Def.* 33
$\therefore LK = BH =$ half of BC. *I.* 34

Cor. 1.—Conversely, The straight line drawn through the middle point of one side of a triangle parallel to a second side bisects the third side.*

Cor. 2.—AB is a given straight line, C and D are two points, either on the same side of AB or on opposite sides of AB, and such that AC and BD are parallel. If through E the middle point of AB, a straight line be drawn $\parallel AC$ or BD to meet CD at F, then

* The corollaries and converses given in the Appendices should be proved to be true. Many of them are not obvious.

F is the middle point of CD, and EF is equal either to half the sum of AC and BD, or to half their difference.

Proposition 2.

The straight lines drawn perpendicular to the sides of a triangle from the middle points of the sides are concurrent (that is, pass through the same point).

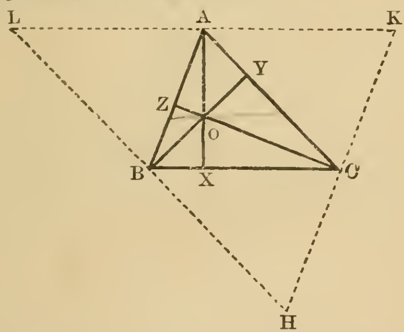
See the figure and demonstration of IV. 5.

If S be joined to H, the middle of BC, then SH is $\perp$ BC. *I.* 8

Note.—The point S is called the *circumscribed centre* of $\triangle$ ABC.

Proposition 3.

*The straight lines drawn from the vertices of a triangle perpendicular to the opposite sides are concurrent.**



Let AX, BY, CZ be the three perpendiculars from A, B, C on the opposite sides of the $\triangle$ ABC;

it is required to prove AX, BY, CZ concurrent.

Through A, B, C draw KL, LH, HK $\parallel$ BC, CA, AB. *I.* 31

Then the figures $ABCK$, $ACBL$ are $\parallel^{\text{ms}}$; *I. Def.* 33

$\therefore$ $AK = BC = AL$, *I.* 34

that is, A is the middle point of KL.

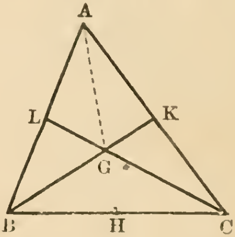
* Pappus, VII. 62. The proof here given seems to be due to F. J. Servois : see his *Solutions peu connues de différens problèmes de Géométrie-pratique* (1804), p. 15. It is attributed to Gauss by Dr R. Baltzer.

Hence also, B and C are the middle points of LH and HK.
But since AX, BY, CZ are respectively $\perp$ BC, CA, AB, *Const.*
they must be respectively $\perp$ KL, LH, HK, *I.* 29
and $\therefore$ concurrent. *App. I.* 2

NOTE.—The point O is called the *orthocentre* of the $\triangle$ ABC (an expression due to W. H. Besant), and $\triangle$ XYZ, formed by joining the feet of the perpendiculars, is called sometimes the *pedal*, sometimes the *orthocentric*, triangle.

PROPOSITION 4.

The medians of a triangle are concurrent.



Let the medians BK, CL of the $\triangle$ ABC meet at G:
it is required to prove that, if H be the middle point of BC, the median AH will pass through G.

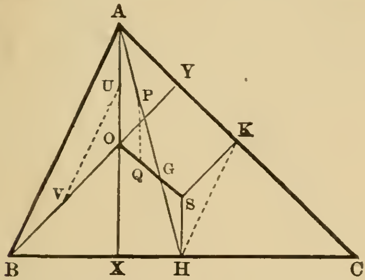
Join AG.
Because $BL = AL$ $\therefore$ $\triangle$ $BLC = \triangle$ ALC,
and $\triangle$ $BLG = \triangle$ ALG; *I.* 38
$\therefore$ $\triangle$ $BGC = \triangle$ AGC, *I. Ax.* 3
$= \text{twice } \triangle$ CKG; *I.* 38
$\therefore$ $BG = \text{twice } GK$, or $BK = \text{thrice } GK$,
that is, the median CL cuts BK at its point of trisection remote from B.
Hence also, the median AH cuts BK at its point of trisection remote from B,
that is, AH passes through G.

COR.—If the points H, K, L be joined, the medians of the $\triangle$ HKL are concurrent at G.

NOTE.—The point G is called the *centroid* of the $\triangle$ ABC (an expression due to T. S. Davies), and $\triangle$ HKL may be called the *median* triangle. The centroid of a triangle is the same point as that which in Statics is called the centre of gravity of the triangle, and may be found by drawing one median, and trisecting it.

PROPOSITION 5.

*The orthocentre, the centroid, and the circumscribed centre of a triangle are collinear (that is, lie on the same straight line), and the distance between the first two is double of the distance between the last two.**



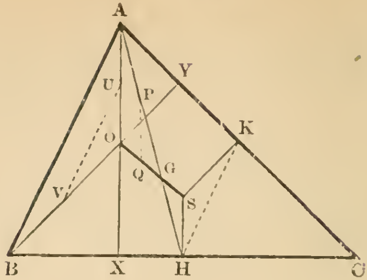
Let ABC be a triangle, O its orthocentre determined by drawin AX and $BY \perp BC$ and CA ; S its circumscribed centre determine by drawing through H and K the middle points of BC and CA, HS and $KS \perp BC$ and CA ; and AH the median from A :

it is required to prove that if SO be joined, it will cut AH at the centroid.

Let SO and AH intersect at G;
join P and Q, the middle points of GA, GO;
" U " V, " " OA, OB;
and join HK.

Because H and K are the middle points of CB, CA;
$\therefore$ HK is $\parallel$ AB and $=$ half AB. *App. I.* 1
Because U and V are the middle points of OA, OB;
$\therefore$ UV is $\parallel$ AB and $=$ half AB, *App. I.* 1
$\therefore$ HK is $\parallel$ UV and $=$ UV.

* First given by Euler in 1765. See *Novi Commentarii Academiæ Scientiarum Imperialis Petropolitanæ*, vol. xi. pp. 13, 114.



Because SH and OU are both $\perp$ BC $\therefore$ SH is $\parallel$ OU; *I.* 28, *Cor.*
" SK " OV " CA $\therefore$ SK " $\parallel$ OV. *I.* 28, *Cor.*
Hence the $\triangle$s SHK, OUV are mutually equiangular, *I.* 34, *Cor.*
and since $HK = UV$ $\therefore$ $SH = OU$ *I.* 26
$= \text{half } AO$.

Again, because P and Q are the middle points of GA, GO;
$\therefore$ PQ is $\parallel$ AO and $=$ half AO; *App. I.* 3
$\therefore$ PQ is $\parallel$ SH and $= SH$.
Hence the $\triangle$s HGS, PGQ are equal in all respects; *I.* 29, 26
$\therefore$ $HG = PG =$ half AG;
$\therefore$ G is the centroid, *App. I.* 4
and $SG = QG =$ half OG.

Cor.—The distance of the circumscribed centre from any side of a triangle is half the distance of the orthocentre from the opposite vertex.

For SH was proved $=$ half OA.

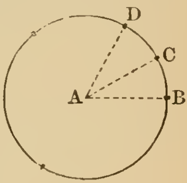
Loci.

Many of the problems which occur in geometry consist in the finding of points. Now the position of a point—and position is the only property which a point possesses—is determined by certain conditions, and if we know these conditions, we can, in general find the point which satisfies them. It will be seen that in *plane* geometry *two* conditions suffice to determine a point, provided the conditions be mutually consistent and independent. When only one of the conditions is given, though the point cannot then be determined, yet its position may be so restricted as to enable us to say that wherever the point may be, it must always lie on some one or two lines which we can describe; for example, straight lines or the circumferences of circles. The given condition may, however, be such that the point which satisfies it will lie on a line or lines which we do not as yet know how to describe. Cases where this occurs are considered as not belonging to *elementary* plane geometry.

DEF.—The line (or lines) to which a point fulfilling a given condition is restricted, that is, on which alone it can lie, is (or are) called the *locus* of the point. Instead of the phrase 'the locus of a point,' we frequently say 'the locus of points.'

For the complete establishment of a locus, it ought to be proved not only that all the points which are said to constitute the locus fulfil the given condition, but that no other points fulfil it. The latter part of the proof is generally omitted.

Ex. 1. Find the locus of a point having the property (or fulfilling the condition) of being situated at a given distance from a given point.



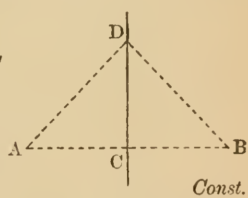
Let A be the given point, and suppose B, C, D, &c. to be points on the locus. Join AB, AC, AD, &c.

Then $AB = AC = AD =$ &c.; *Hyp.*
and hence B, C, D, &c. must be situated on the $\bigcirc^{ce}$ of a circle whose centre is A, and whose radius is the given distance.

Moreover, the distance from A of any point not situated on the $\bigcirc^{ce}$ would not be $= AB$, AC, AD. &c.

This $\bigcirc^{ce}$ $\therefore$ is the required locus.

Ex. 2. Find the locus of a point having the property (or fulfilling the condition) of being equidistant from two given points.



Let A and B be the given points.

Join AB, and bisect it at C; then C is a definite fixed point.

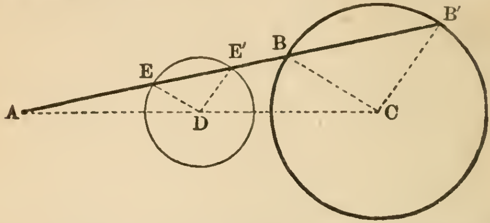
Suppose D to be any point on the locus, and join DA, DB, DC.

Then $DA = DB$; *Hyp.*
and since DC is common, and $AC = BC$, *Const.*
$\therefore$ DC is $\perp$ AB. *I.* 8, *Def.* 10

Hence, if a set of other points on the locus be taken, and joined to the definite fixed point C, a set of perpendiculars to AB will be obtained. The locus therefore consists of all the perpendiculars that can be drawn to AB through the point C; that is, CD produced indefinitely either way is the locus.

Proposition 6.

Straight lines are drawn from a given fixed point to the circumference of a given fixed circle, and are bisected: find the locus of their middle points.



Let A be the given fixed point, C the centre of the given fixed circle; let AB, one of the straight lines drawn from A to the ○ce, be bisected at E:
it is required to find the locus of E.

Join AC, and bisect it at D; *I.* 10
join DE and CB.

Because DE joins the middle points of two sides of △ ACB,
∴ $DE = \frac{1}{2} CB$. *App. I.* 1
But CB, being the radius of a fixed circle, is a fixed length;
∴ DE, its half, is also a fixed length.
Again, since A and C are fixed points,
∴ AC is a fixed straight line;
∴ D, the middle point of AC, is a fixed point;
that is, E, the middle point of AB, is situated at a fixed distance from the fixed point D.
But AB was any straight line drawn from A to the ○ce;
∴ the middle points of all other straight lines drawn from A to the ○ce must be situated at the same fixed distance from the fixed point D;
∴ the locus of the middle points is the ○ce of a circle, whose centre is D, and whose radius is half the radius of the fixed circle.

From the figure it will be seen that it is immaterial whether AB or AB' is to be considered as the straight line drawn from A to the ○ce. For if E' be the middle point of AB', then $E'D = \frac{1}{2} B'C$, that is = half the radius of the fixed circle;
∴ the locus of E' is the same ○ce as before.

[The reader is requested to make figures for the cases when the given point A is inside the given circle, and when it is on the $\bigcirc^{ce}$ of the given circle.]

INTERSECTION OF LOCI.

Since two conditions determine a point, if we can construct the locus satisfying each condition, the point or points of intersection of the two loci will be the point or points required. A familiar example of this method of determining a point, is the finding of the position of a town on a map by means of parallels of latitude and meridians of longitude. The reader is recommended to apply this method to the solution of I. 1 and 22, and to several of the problems on the construction of triangles.

DEDUCTIONS.

1. The straight line joining the middle points of the non-parallel sides of a trapezium is $\parallel$ the parallel sides and $=$ half their sum.
2. The straight line joining the middle points of the diagonals of a trapezium is $\parallel$ the parallel sides and $=$ half their difference.
3. The straight line joining the middle points of the non-parallel sides of a trapezium bisects the two diagonals.
4. The middle points of any two opposite sides of a quadrilateral and the middle points of the two diagonals are the vertices of a $\parallel^m$.
5. The straight lines which join the middle points of the opposite sides of a quadrilateral, and the straight line which joins the middle points of the diagonals, are concurrent.
6. If from the three vertices and the centroid of a triangle perpendiculars be drawn to a straight line outside the triangle, the perpendicular from the centroid $=$ one-third of the sum of the other perpendiculars. Examine the cases when the straight line cuts the triangle, and when it passes through the centroid.
7. Find a point in a given straight line such that the sum of its distances from two given points may be the least possible. Examine the two cases, when the two given points are on the same side of the given line, and when they are on different sides.

8. Find a point in a given straight line such that the difference of its distances from two given points may be the greatest possible. Examine the two cases.
9. Of all triangles having only two sides given, that is the greatest in which these sides are perpendicular.
10. The perimeter of an isosceles triangle is less than that of any other triangle of equal area standing on the same base.
11. Of all triangles having the same vertical angle, and the bases of which pass through the same given point, the least is that which has its base bisected by the given point.
12. Of all triangles formed with a given angle which is contained by two sides whose sum is constant, the isosceles triangle has the least perimeter.
13. The sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the other two sides is constant. Examine the case when the point is in the base produced.
14. The sum of the perpendiculars drawn from any point inside an equilateral triangle to the three sides is constant. Examine the case when the point is outside the triangle.
15. The sum of the perpendiculars from the vertices of a triangle on the opposite sides is greater than the semi-perimeter and less than the perimeter of the triangle.
16. If a perpendicular be drawn from the vertical angle of a triangle to the base, it will divide the vertical angle and the base into parts such that the greater is next the greater side of the triangle.
17. The bisector of the vertical angle of a triangle divides the base into segments such that the greater is next the greater side of the triangle.
18. The median from the vertical angle of a triangle divides the vertical angle into parts such that the greater is next the less side of the triangle.
19. If from the vertex of a triangle there be drawn a perpendicular to the opposite side, a bisector of the vertical angle and a median, the second of these lies in position and magnitude between the other two.
20. The sum of the three angular bisectors of a triangle is greater than the semiperimeter, and less than the perimeter of the triangle.

21. If one side of a triangle be greater than another, the perpendicular on it from the opposite angle is less than the corresponding perpendicular on the other side.
22. If one side of a triangle be greater than another, the median drawn to it is less than the median drawn to the other.
23. If one side of a triangle be greater than another, the bisector of the angle opposite to it is less than the bisector of the angle opposite to the other.
24. The hypotenuse of a right-angled triangle, together with the perpendicular on it from the right angle, is greater than the sum of the other two sides.
25. The sum of the three medians is greater than three-fourths of the perimeter of the triangle.
26. Construct an equilateral triangle, having given the perpendicular from any vertex on the opposite side.

Construct an isosceles triangle, having given :

27. The vertical angle and the perpendicular from it to the base.
28. The perimeter and the perpendicular from the vertex to the base.

Construct a right-angled triangle, having given :

29. The hypotenuse and an acute angle.
30. The hypotenuse and a side.
31. The hypotenuse and the sum of the other sides.
32. The hypotenuse and the difference of the other sides.
33. The perpendicular from the right angle on the hypotenuse and a side.
34. The median, and the perpendicular from the right angle, to the hypotenuse.
35. An acute angle and the sum of the sides about the right angle.
36. An acute angle and the difference of the sides about the right angle.

Construct a triangle, having given :

37. Two sides and an angle opposite to one of them. Examine the cases when the angle is acute, right, and obtuse.
38. One side, an angle adjacent to it, and the sum of the other two sides.
39. One side, an angle adjacent to it, and the difference of the other two sides.
40. One side, the angle opposite to it, and the sum of the other two sides.

41. One side, the angle opposite to it, and the difference of the other two sides.
42. An angle, its bisector, and the perpendicular from the angle on the opposite side.
43. The angles and the sum of two sides.
44. The angles and the difference of two sides.
45. The perimeter and the angles at the base.
46. Two sides and one median.
47. One side and two medians.
48. The three medians.

Construct a square, having given:

49. The sum of a side and a diagonal.
50. The difference of a side and a diagonal.

Construct a rectangle, having given:

51. One side and the angle of intersection of the diagonals.
52. The perimeter and a diagonal.
53. The perimeter and the angle of intersection of the diagonals.
54. The difference of two sides and the angle of intersection of the diagonals.

Construct a $\|^m$, having given:

55. The diagonals and a side.
56. The diagonals and their angle of intersection.
57. A side, an angle, and a diagonal.

58. Construct a $\|^m$ the area and perimeter of which shall = the area and perimeter of a given triangle.
59. The diagonals of all the $\|^{ms}$ inscribed* in a given $\|^m$ intersect one another at the same point.
60. In a given rhombus inscribe a square.
61. In a given right-angled isosceles triangle inscribe a square.
62. In a given square inscribe an equilateral triangle having one of its vertices coinciding with a vertex of the square.
63. AA', BB', CC' are straight lines drawn from the angular points of a triangle through any point O within the triangle, and cutting the opposite sides at A', B', C'. AP, BQ, CR are cut off from AA', BB', CC', and $= OA'$, OB', OC'. Prove $\triangle A'B'C' = \triangle PQR$.

* One figure is inscribed in another when the vertices of the first figure are on the sides of the second.

64. On AB, AC, sides of $\triangle\ ABC$, the $\|^{\text{ms}}$ $ABDE$, $ACFG$ are described; DE and FG are produced to meet at H, and AH is joined; through B and C, BL and CM are drawn $\parallel AH$, and meeting DE and FG at L and M. If LM be joined, $BCML$ is a $\|^{\text{m}}$, and $= \|^{\text{m}}\ BE + \|^{\text{m}}\ CG$. (Pappus, IV. 1.)
65. Deduce I. 47 from the preceding deduction.
66. If three concurrent straight lines be respectively perpendicular to the three sides of a triangle, they divide the sides into segments such that the sums of the squares of the alternate segments taken cyclically (that is, going round the triangle) are equal; and conversely.
67. Prove App. I. 2, 3 by the preceding deduction.
68. If from the middle point of the base of a triangle, perpendiculars be drawn to the bisectors of the interior and exterior vertical angles, these perpendiculars will intercept on the sides segments equal to half the sum or half the difference of the sides.
69. In the figure to the preceding deduction, find all the angles which are equal to half the sum or half the difference of the base angles of the triangle.
70. If the straight lines bisecting the angles at the base of a triangle, and terminated by the opposite sides, be equal, the triangle is isosceles. Examine the case when the angles below the base are bisected. [See *Nouvelles Annales de Mathématiques* (1842), pp. 138 and 311; *Lady's and Gentleman's Diary* for 1857, p. 58; for 1859, p. 87; for 1860, p. 84; *London, Edinburgh, and Dublin Philosophical Magazine*, 1852, p. 366, and 1874, p. 354.]

Loci.

1. The locus of the points situated at a given distance from a given straight line, consists of two straight lines parallel to the given straight line, and on opposite sides of it.
2. The locus of the points situated at a given distance from the $\bigcirc^{\text{ce}}$ of a given circle consists of the $\bigcirc^{\text{ces}}$ of two circles concentric with the given circle. Examine whether the locus will always consist of two $\bigcirc^{\text{ces}}$.

   [The distance of a point from the circumference of a circle is measured on the straight line joining the point to the centre of the circle.]

3. The locus of the points equidistant from two given straight lines which intersect, consists of the two bisectors of the angles made by the given straight lines.
4. What is the locus when the two given straight lines are parallel?
5. The locus of the vertices of all the triangles which have the same base, and one of their sides equal to a given length, consists of the $\bigcirc^{ces}$ of two circles. Determine their centres and the length of their radii.
6. The locus of the vertices of all the triangles which have the same base, and one of the angles at the base equal to a given angle, consists of the sides or the sides produced of a certain rhombus.
7. Find the locus of the centre of a circle which shall pass through a given point, and have its radius equal to a given straight line.
8. Find the locus of the centres of the circles which pass through two given points.
9. Find the locus of the vertices of all the isosceles triangles which stand on a given base.
10. Find the locus of the vertices of all the triangles which have the same base, and the median to that base equal to a given length.
11. Find the locus of the vertices of all the triangles which have the same base and equal altitudes.
12. Find the locus of the vertices of all the triangles which have the same base, and their areas equal.
13. Find the locus of the middle points of all the straight lines drawn from a given point to meet a given straight line.
14. A series of triangles stand on the same base and between the same parallels. Find the locus of the middle points of their sides.
15. A series of $\|^{ms}$ stand on the same base and between the same parallels. Find the locus of the intersection of their diagonals.
16. From any point in the base of a triangle straight lines are drawn parallel to the sides. Find the locus of the intersection of the diagonals of every $\|^{m}$ thus formed.
17. Straight lines are drawn parallel to the base of a triangle, to meet the sides or the sides produced. Find the locus of their middle points.

18. Find the locus of the angular point opposite to the hypotenuse of all the right-angled triangles that have the same hypotenuse.
19. A ladder stands upright against a perpendicular wall. The foot of it is gradually drawn outwards till the ladder lies on the ground. Prove that the middle point of the ladder has described part of the ○ce of a circle.
20. Find the locus of the points at which two equal segments of a straight line subtend equal angles.
21. A straight line of constant length remains always parallel to itself, while one of its extremities describes the ○ce of a circle. Find the locus of the other extremity.
22. Find the locus of the vertices of all the triangles which have the same base BC, and the median from B equal to a given length.
23. The base and the difference of the two sides of a triangle are given; find the locus of the feet of the perpendiculars drawn from the ends of the base to the bisector of the interior vertical angle.
24. The base and the sum of the two sides of a triangle are given; find the locus of the feet of the perpendiculars drawn from the ends of the base to the bisector of the exterior vertical angle.
25. Three sides and a diagonal of a quadrilateral are given: find the locus (1) of the undetermined vertex, (2) of the middle point of the second diagonal, (3) of the middle point of the straight line which joins the middle points of the two diagonals. (*Solutions raisonnées des Problèmes énoncés dans les Éléments de Géométrie de M. A. Amiot*, 7ème ed. p. 124.)

BOOK II.

DEFINITIONS.

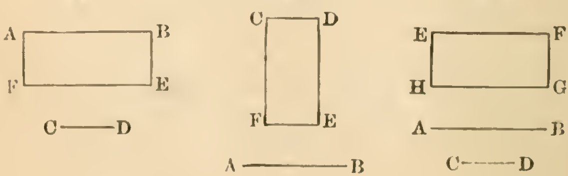
1. A **rectangle** (or rectangular parallelogram) is said to be contained by any two of its conterminous sides.

Thus the rectangle *ABCD* is said to be contained by *AB* and *BC;* or by *BC* and *CD;* or by *CD* and *DA ;* or by *DA* and *AB.*

The reason of this is, that if the lengths of any two conterminous sides of a rectangle are given, the rectangle can be constructed ; or, what comes to the same thing, that if two conterminous sides of one rectangle are respectively equal to two conterminous sides of another rectangle, the two rectangles are equal in all respects. The truth of the latter statement may be proved by applying the one rectangle to the other.

2. It is oftener the case than not, that the rectangle contained by two straight lines is spoken of when the two straight lines do not actually contain any rectangle. When this is so, the rectangle contained by the two straight lines will signify the rectangle contained by either of them, and a straight line equal to the other, or the rectangle contained by two other straight lines respectively equal to them.

Fig. 1. Fig. 2. Fig. 3.



Thus $ABEF$ (fig. 1) may be considered the rectangle contained by AB and CD, if $BE = CD$; $CDEF$ (fig. 2) may be considered the rectangle contained by AB and CD, if $DE = AB$; and $EFGH$ (fig. 3) may be considered the rectangle contained by AB and CD, if $EF = AB$ and $FG = CD$.

3. As the rectangle and the square are the figures which the Second Book of Euclid treats of, phrases such as 'the rectangle contained by AB and AC,' and 'the square described on AB,' will be of constant occurrence. It is usual, therefore, to employ abbreviations for these phrases. The abbreviation which will be made use of in the present text-book* for 'the rectangle contained by AB and BC' is $AB \cdot BC$, and for 'the square described on AB,' AB^2.

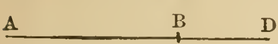
4. When a point is taken in a straight line, it is often called a point of **section**, and the distances of this point from the ends of the line are called **segments** of the line.

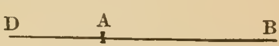
Thus the point of section D divides AB into two segments AD and BD.

In this case AB is said to be divided *internally* at D, and AD and BD are called *internal segments.*

The given straight line is equal to the sum of its internal segments; for $AB = AD + BD$.

5. When a point is taken in a straight line produced, it is also called a point of **section**, and its distances from the ends of the line are called **segments** of the line.





Thus D is called a point of section of AB, and the segments into which it is said to divide AB are AD and BD.

* In certain written examinations in England, the only abbreviation allowed for 'the rectangle contained by AB and BC' is rect. AB, BC, and for 'the square described on AB,' sq. on AB; the pupil, therefore, if preparing for these examinations, should practise himself in the use of such abbreviations.

In this case, AB is said to be divided *externally* at D, and AD, BD are called *external segments.*

The given straight line is equal to the difference of its external segments; for $AB = AD - BD$, or $BD - AD$.

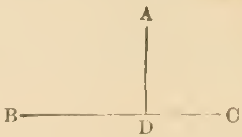
6. When a straight line is divided into two segments, such that the rectangle contained by the whole line and one of the segments is equal to the square on the other segment, the straight line is said to be divided in **medial section.***

Thus, if AB be divided at H into two segments AH and BH, such that $AB \cdot BH = AH^2$, AB is said to be divided in medial section at H.

It will be seen that AB is internally divided at H; and in general, when a straight line is said to be divided in medial section, it is understood to be internally divided. But the definition need not be restricted to internal division.

Thus, if AB be divided at H' into two segments AH' and BH', such that $AB \cdot BH' = AH'^2$, AB in this case also may be said to be divided in medial section.

7. The **projection** † of a point on a straight line is the foot of the perpendicular drawn from the point to the straight line.



Thus D is the projection of A on the straight line BC.

8. The **projection** of one straight line on another straight

* The phrase, 'medial section,' seems to be due to Leslie. See his *Elements of Geometry* (1809), p. 66.

† Sometimes the adjective 'orthogonal' is prefixed to the word projection, to distinguish this kind from others.

line is that portion of the second intercepted between perpendiculars drawn to it from the ends of the first.

Fig. 1.

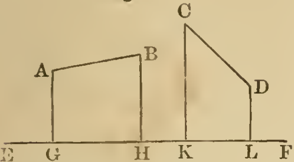
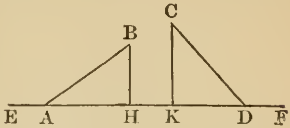


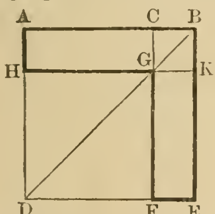
Fig. 2.

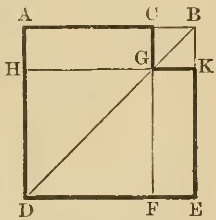


Thus the projections of AB and CD on EF are, in fig. 1, GH and KL : in fig. 2, AH and KD.

While the straight line to be projected must be limited in length, the straight line on which it is to be projected must be considered as unlimited.

9. If from a parallelogram there be taken away either of the parallelograms about one of its diagonals, the remaining figure is called a **gnomon.**





Thus if $ADEB$ is a $\|^m$, BD one of its diagonals, and HF, CK $\|^{ms}$ about the diagonal BD, the figure which remains when HF or CK is taken away from $ADEB$ is called a gnomon. In the first case, when HF is taken away, the gnomon $ABEFGH$ (inclosed within thick lines) is usually, for shortness' sake, called AKF or HCE; in the second case, when CK is taken away, the gnomon $ADEKGC$ would similarly be called AFK or CHE.

The word 'gnomon' in Greek means, among other things, a carpenter's square,* which, when the $\|^m$ $ADEB$ is a square or a

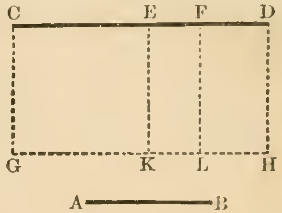
* Another less known figure was, from its shape, called by the ancient geometers, 'the shoemaker's knife.' See Pappus, IV. section 14.

rectangle, the figure AKF resembles. The only gnomons mentioned by Euclid in the second book are parts of squares.

The more general definition given by Heron of Alexandria, that a gnomon is any figure which, when added to another figure, produces a figure similar to the original one, will be partly understood after the fourth proposition has been read.

PROPOSITION 1. THEOREM.

If there be two straight lines, one of which is divided internally into any number of segments, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and the several segments of the divided line.



Let AB and CD be the two straight lines,
and let CD be divided internally into any number of segments CE, EF, FD:
it is required to prove $AB \cdot CD = AB \cdot CE + AB \cdot EF + AB \cdot FD$.

From C draw $CG \perp CD$ and $= AB$; *I.* 11, 3
through G draw $GH \parallel CD$,
and through E, F, D draw EK, FL, $DH \parallel CG$. *I.* 31

Then $CH = CK + EL + FH$; *I. Ax.* 8
that is, $GC \cdot CD = GC \cdot CE + KE \cdot EF + LF \cdot FD$.
But GC, KE, LF are each $= AB$; *Const.*, *I.* 34
$\therefore AB \cdot CD = AB \cdot CE + AB \cdot EF + AB \cdot FD$.

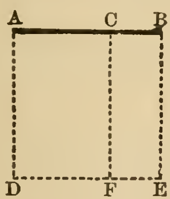
ALGEBRAICAL ILLUSTRATION.

Let $AB = a$, $CD = b$, $CE = c$, $EF = d$, $FD = e$;
then $b = c + d + e$.
Now $AB \cdot CD = ab$,
and $AB \cdot CE + AB \cdot EF + AB \cdot FD = ac + ad + ae$.
But since $b = c + d + e$,
$\therefore ab = ac + ad + ae$;
$\therefore AB \cdot CD = AB \cdot CE + AB \cdot EF + AB \cdot FD$.

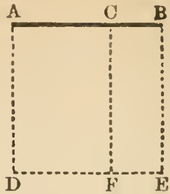
1. The rectangle contained by two straight lines is equal to twice the rectangle contained by one of them and half of the other.
2. The rectangle contained by two straight lines is equal to thrice the rectangle contained by one of them and one-third of the other.
3. The rectangle contained by two equal straight lines is equal to the square on either of them.
4. If two straight lines be each of them divided internally into any number of segments, the rectangle contained by the two straight lines is equal to the several rectangles contained by all the segments of the one taken separately with all the segments of the other.

PROPOSITION 2. THEOREM.

If a straight line be divided internally into any two segments, the square on the straight line is equal to the sum of the rectangles contained by the straight line and the two segments.



Let AB be divided internally into any two segments AC, CB:
it is required to prove $AB^2 = AB \cdot AC + AB \cdot CB$.



On AB describe the square $ADEB$, *I.* 46
and through C draw $CF \parallel AD$, meeting DE at F. *I.* 31

Then $AE = AF + CE$; *I. Ax.* 8
that is, $AB^2 = DA \cdot AC + EB \cdot CB$.
But DA and EB are each $= AB$;
$\therefore AB^2 = AB \cdot AC + AB \cdot CB$.

ALGEBRAICAL ILLUSTRATION.

Let $AC = a$, $CB = b$;
then $AB = a + b$.
Now, $AB^2 = (a + b)^2 = a^2 + 2ab + b^2$,
and $AB \cdot AC + AB \cdot CB = (a + b)\,a + (a + b)\,b = a^2 + 2ab + b^2$;
$\therefore AB^2 = AB \cdot AC + AB \cdot CB$.

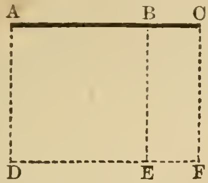
1. Prove this proposition by taking another straight line $= AB$, and using the preceding proposition.
2. If a straight line be divided internally into any three segments, the square on the straight line is equal to the sum of the rectangles contained by the straight line and the three segments.
3. If a straight line be divided internally into any number of segments, the square on the straight line is equal to the sum of the rectangles contained by the straight line and the several segments.

Show that the proposition is equivalent to either of the following:

4. The square on the sum of two straight lines is equal to the two rectangles contained by the sum and each of the straight lines.
5. The square on the greater of two straight lines is equal to the rectangle contained by the two straight lines together with the rectangle contained by the greater and the difference between the two.

PROPOSITION 3. THEOREM.

If a straight line be divided externally into any two segments, the square on the straight line is equal to the difference of the rectangles contained by the straight line and the two segments.



Let AB be divided externally into any two segments AC, CB:

it is required to prove $AB^2 = AB \cdot AC - AB \cdot CB$.

On AB describe the square $ADEB$, *I.* 46

and through C draw $CF \parallel AD$, meeting DE produced at F. *I.* 31

Then $AE = AF - CE$; *I. Ax.* 8

that is, $AB^2 = DA \cdot AC - EB \cdot CB$.

But DA and EB are each $= AB$;

$\therefore AB^2 = AB \cdot AC - AB \cdot CB$.

NOTE.—The enunciation of this proposition usually given is:

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts together with the square on the aforesaid part.

That is, in reference to the figure,

$$AC \cdot AB = AB^2 + AB \cdot BC,$$

an expression which can be easily derived from that in the text.

ALGEBRAICAL ILLUSTRATION.

Let $AC = a$, $CB = b$;

then $AB = a - b$.

Now, $AB^2 = (a - b)^2 = a^2 - 2ab + b^2$,

and $AB \cdot AC - AB \cdot CB = (a - b)\,a - (a - b)\,b = a^2 - 2ab + b^2$;
$\therefore AB^2 = AB \cdot AC - AB \cdot CB$.

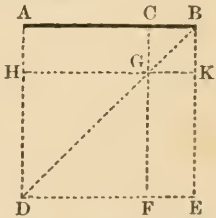
1. Prove this proposition by taking another straight line $= AB$, and using the first proposition.

   Show that the proposition is equivalent to either of the following:

2. The rectangle contained by the sum of two straight lines and one of them is equal to the square on that one together with the rectangle contained by the two straight lines.
3. The rectangle contained by two straight lines is equal to the square on the less together with the rectangle contained by the less and the difference of the two straight lines.

PROPOSITION 4. THEOREM.

If a straight line be divided internally into any two segments, the square on the straight line is equal to the squares on the two segments increased by twice the rectangle contained by the segments.



Let AB be divided internally into any two segments AC, CB:

it is required to prove $AB^2 = AC^2 + CB^2 + 2\,AC \cdot CB$.

On AB describe the square $ADEB$, and join BD. *I.* 46
Through C draw $CF \parallel AD$, meeting DB at G;
and through G draw $HK \parallel AB$, meeting DA and EB at H and K. *I.* 31

Because $CG \parallel AD$, $\therefore \angle CGB = \angle ADB$; *I.* 29
and because $AD = AB$, $\therefore \angle ADB = \angle ABD$; *I.* 5

$\therefore \quad \angle\, CGB = \angle\, ABD,$
$\qquad\qquad = \angle\, CBG;$
$\therefore \qquad CB = CG.$ *I.* 6

Hence the $\|^{\text{m}}$ CK, having two adjacent sides equal, has all its sides equal. *I.* 34

But the $\|^{\text{m}}$ CK has one of its angles, KBC, right, since $\angle\, KBC$ is the same as $\angle\, ABE$;

$\therefore$ it has all its angles right; *I.* 34

$\therefore$ the $\|^{\text{m}}$ CK is a square, and $= CB^2$. *I. Def.* 32

Similarly, the $\|^{\text{m}}$ HF is a square, and $= HG^2 = AC^2$.

Again, the $\|^{\text{m}}$ $AG = AC \cdot CG = AC \cdot CB$;

$\therefore \qquad GE = AC \cdot CB;$ *I.* 43

$\therefore \qquad AG + GE = 2\, AC \cdot CB.$

Now $\qquad AB^2 = ADEB,$

$\qquad = HF + CK + AG + GE,$ *I. Ax.* 8

$\qquad = AC^2 + CB^2 + 2\, AC \cdot CB.$

Cor. 1.—The square on the sum of two straight lines is equal to the sum of the squares on the two straight lines, increased by twice the rectangle contained by the two straight lines.

For if AC and CB be the two straight lines, then their sum $= AC + CB = AB$.

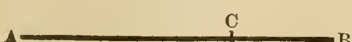
Now since $AB^2 = AC^2 + CB^2 + 2\, AC \cdot CB,$ *II.* 4

$\therefore (AC + CB)^2 = AC^2 + CB^2 + 2\, AC \cdot CB.$

Cor. 2.—The $\|^{\text{ms}}$ about a diagonal of a square are themselves squares.

[It is recommended that II. 7 be read immediately after II. 4.]

OTHERWISE:



$AB^2 = \quad AB \cdot AC \quad + \quad AB \cdot BC,$ *II.* 2

$\quad = (AC \cdot AC + BC \cdot AC) + (AC \cdot BC + BC \cdot BC),$ *II.* 3

$\quad = AC^2 + BC^2 + 2\, AC \cdot BC.$

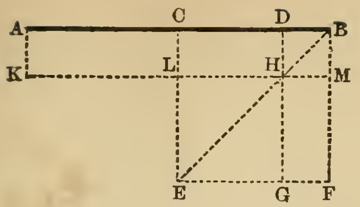
ALGEBRAICAL ILLUSTRATION.

Let $AC = a$, $CB = b$;
then $AB = a + b$.
Now $AB^2 = (a + b)^2 = a^2 + 2ab + b^2$.
and $AC^2 + CB^2 + 2\,AC \cdot CB = a^2 + b^2 + 2ab$;
$\therefore\ AB^2 = AC^2 + CB^2 + 2\,AC \cdot CB$.

1. Name the two figures which form the sum of the squares on AC and CB.
2. Name the figure which is the square on the sum of AC and CB.
3. Name the figure which is the difference of the squares on AB and AC.
4. Name the figure which is the difference of the squares on AB and BC.
5. Name the figure which is the square on the difference of AB and AC.
6. Name the figure which is the square on the difference of AB and BC.
7. By how much does the square on the sum of AC and CB exceed the sum of the squares on AC and CB?
8. Show that the proposition may be enunciated: The square on the sum of two straight lines is greater than the sum of the squares on the two straight lines by twice the rectangle contained by the two straight lines.
9. The square on any straight line is equal to four times the square on half of the line.
10. If a straight line be divided internally into any three segments, the square on the whole line is equal to the squares on the three segments, together with twice the rectangles contained by every two of the segments.
11. Illustrate the preceding deduction algebraically.

PROPOSITION 5. THEOREM.

If a straight line be divided into two equal, and also internally into two unequal segments, the rectangle contained by the unequal segments is equal to the difference between the square on half the line and the square on the line between the points of section.



Let AB be divided into two equal segments AC, CB, and also internally into two unequal segments AD, DB: *it is required to prove* $AD \cdot DB = CB^2 - CD^2$.

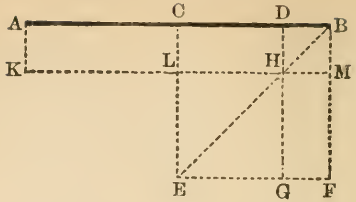
On CB describe the square $CEFB$, and join BE. *I.* 46
Through D draw $DHG \parallel CE$, meeting EB and EF at H and G;
through H draw $MHLK \parallel AB$, meeting FB and EC at M and L;
and through A draw $AK \parallel CL$. *I.* 31

Then	$AD \cdot DB = AD \cdot DH,$	*II.* 4. *Cor.* 2
	$= AH,$	
	$= AL + CH,$	*I. Ax.* 8
	$= CM + HF,$	*I.* 36, 43
	$=$ gnomon CMG.	*I. Ax.* 8
But	$CB^2 - CD^2 = CB^2 - LH^2,$	*I.* 34
	$= CEFB - LEGH,$	
	$=$ gnomon CMG.	*I. Ax.* 8

$\therefore AD \cdot DB = CB^2 - CD^2$.

Cor.—The difference of the squares on two straight lines is equal to the rectangle contained by the sum and the difference of the two straight lines.

Let AC and CD be the two straight lines:
it is required to prove
$AC^2 - CD^2 = (AC + CD) \cdot (AC - CD)$.



$$
\begin{aligned}
AC + CD &= AD, \\
\text{and } AC - CD = CB - CD &= DB; \\
\therefore (AC + CD) \cdot (AC - CD) &= AD \cdot DB, \\
&= CB^2 - CD^2, \qquad \text{II. 5} \\
&= AC^2 - CD^2.
\end{aligned}
$$

ALGEBRAICAL ILLUSTRATION.

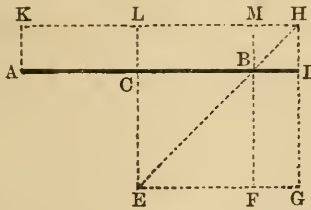
Let $AC = CB = a$, $CD = b$;
then $AD = a + b$, and $DB = a - b$.
Now $AD \cdot DB = (a + b)(a - b) = a^2 - b^2$,
and $CB^2 - CD^2 = a^2 - b^2$;
$\therefore AD \cdot DB = CB^2 - CD^2$.

1. By how much does the rectangle $AC \cdot CB$ exceed the rectangle $AD \cdot DB$? The rectangle contained by the two internal segments of a straight line is the greatest possible when the segments are equal. (Pappus, VII. 13.)
2. The rectangle contained by the two internal segments of a straight line grows less according as the point of section is removed farther from the middle point of the straight line. (Pappus, VII. 14.)
3. Prove that $AC =$ half the sum and $CD =$ half the difference of AD and DB.
4. Name two figures in the diagram, each of which = the rectangle contained by half the sum, and half the difference of AD and DB.
5. Name that figure in the diagram which is the square on half the sum of AD and DB.
6. Name that figure in the diagram which is the square on half the difference of AD and DB.

7. Hence show that the proposition may be enunciated: The rectangle contained by any two straight lines is equal to the square on half their sum diminished by the square on half their difference.
8. The perimeter of the rectangle $AD \cdot DB$ = the perimeter of the square on CB.
9. Hence show that if a square and a rectangle have equal perimeters, the square has the greater area.
10. Construct a rectangle equal to the difference of two given squares.
11. By means of the first deduction above, and II. 4, show that the sum of the squares on the two segments of a straight line is least when the segments are equal.
12. The square on either of the sides about the right angle of a right-angled triangle, is equal to the rectangle contained by the sum and the difference of the hypotenuse and the other side.

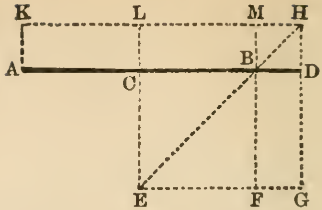
PROPOSITION 6. Theorem.

If a straight line be divided into two equal, and also externally into two unequal segments, the rectangle contained by the unequal segments is equal to the difference between the square on the line between the points of section and the square on half the line.



Let AB be divided into two equal segments AC, CB, and also externally into two unequal segments AD, BD:
it is required to prove $AD \cdot DB = CD^2 - CB^2$.

On CB describe the square $CEFB$, and join BE. *I.* 46



Through D draw $HDG \parallel CE$, meeting EB and EF produced at H and G;
through H draw $HMLK \parallel AB$, meeting FB and EC produced at M and L;
and through A draw $AK \parallel CL$. *I.* 31

Then	$AD \cdot DB = AD \cdot DH,$	*II.* 7, *Cor.* 2
	$= AH,$	
	$= AL + CH,$	*I. Ax.* 8
	$= CM + HF,$	*I.* 36, 43
	$=$ gnomon CMG.	*I. Ax.* 8
But	$CD^2 - CB^2 = LH^2 - CB^2,$	*I.* 34
	$= LEGH - CEFB,$	
	$=$ gnomon CMG.	*I. Ax.* 8

$\therefore AD \cdot DB = CD^2 - CB^2.$

Cor.—The difference of the squares on two straight lines is equal to the rectangle contained by the sum and the difference of the two straight lines.

Let AC and CD be the two straight lines:
it is required to prove
$CD^2 - AC^2 = (CD + AC) \cdot (CD - AC)$.

$CD + AC = AD,$
and $CD - AC = CD - CB = DB;$
$\therefore (CD + AC) \cdot (CD - AC) = AD \cdot DB,$
$= CD^2 - CB^2,$ *II.* 6
$= CD^2 - AC^2.$

OTHERWISE: *

Let AB be divided into two equal segments AC, CB, and also externally into two unequal segments AD, DB:
it is required to prove $AD \cdot DB = CD^2 - CB^2$.

Produce BA to E, making $AE = BD$. *I.* 3
Then $EC = CD$, and $EB = AD$.
Now, because ED is divided into two equal segments EC, CD, and also internally into two unequal segments EB, BD,
$\therefore\ EB \cdot BD = CD^2 - CB^2$; *II.* 5
$\therefore\ AD \cdot BD = CD^2 - CB^2$.

ALGEBRAICAL ILLUSTRATION.

Let $AC = CB = a$, $CD = b$;
then $AD = b + a$, and $DB = b - a$.
Now $AD \cdot DB = (b + a)(b - a) = b^2 - a^2$,
and $CD^2 - CB^2 = b^2 - a^2$;
$\therefore\ AD \cdot DB = CD^2 - CB^2$.

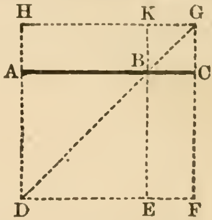
1. Does the rectangle $AD \cdot DB$ exceed the rectangle $AC \cdot CB$? Examine the various cases.
2. The rectangle contained by the two external segments of a straight line grows greater according as the point of section is removed farther from the middle point of the straight line.
3. Prove that AC = half the difference, and CD = half the sum of AD and DB.
4. Name two figures in the diagram each of which = the rectangle contained by half the sum and half the difference of AD and DB.
5. Name that figure in the diagram which is the square on half the sum of AD and DB.
6. Name that figure in the diagram which is the square on half the difference of AD and DB.

* Due to Mauricius Brescius (of Grenoble), a professor of Mathematics in Paris (probably about the end of the sixteenth century).

7. Hence, show that the proposition may be enunciated: The rectangle contained by any two straight lines is equal to the square on half their sum diminished by the square on half their difference.
8. The perimeter of the rectangle $AD \cdot DB$ = the perimeter of the square on CD.

PROPOSITION 7. THEOREM.

If a straight line be divided externally into any two segments, the square on the straight line is equal to the squares on the two segments diminished by twice the rectangle contained by the segments.



Let AB be divided externally into any two segments AC, CB:

it is required to prove $AB^2 = AC^2 + CB^2 - 2\,AC \cdot CB$.

On AB describe the square $ADEB$, and join BD. *I.* 46
Through C draw $CF \parallel AD$, meeting DB produced at G; and through G draw $HK \parallel AB$, meeting DA and EB produced at H and K. *I.* 31

Because $CG \parallel AD$, $\therefore\ \angle\ CGB = \angle\ ADB$; *I.* 29
and because $AD = AB$, $\therefore\ \angle\ ADB = \angle\ ABD$; *I.* 5
$\therefore\ \angle\ CGB = \angle\ ABD$,
$= \angle\ CBG$; *I.* 15
$\therefore\quad CB = CG$. *I.* 6

Hence the $\|^{m}$ CK, having two adjacent sides equal, has all its sides equal. *I.* 34

But the $\|^{m}$ CK has one of its angles, KBC, right,
since $\angle KBC = \angle ABE$; *I.* 15
$\therefore$ it has all its angles right; *I.* 34
$\therefore$ the $\|^{m}$ CK is a square, and $= CB^2$. *I. Def.* 32
Similarly, the $\|^{m}$ HF is a square, and $= HG^2 = AC^2$.

Again, the $\|^{m}$ $AG = AC \cdot CG = AC \cdot CB$;

$\therefore \quad GE = AC \cdot CB$; *I.* 43

$\therefore \quad AG + GE = 2\, AC \cdot CB$.

Now $\quad AB^2 = ADEB$,

$= HF + CK - AG - GE$, *I. Ax.* 8

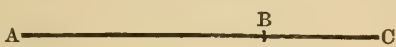
$= AC^2 + CB^2 - 2\, AC \cdot CB$.

Cor. 1.—The square on the difference of two straight lines is equal to the sum of the squares on the two straight lines diminished by twice the rectangle contained by the two straight lines.

For if AC and CB be the two straight lines,
then their difference $= AC - CB = AB$.
Now since $\quad AB^2 = AC^2 + CB^2 - 2\, AC \cdot CB$, *II.* 7
$\therefore \quad (AC - CB)^2 = AC^2 + CB^2 - 2\, AC \cdot CB$.

Cor. 2.—The $\|^{ms}$ about a square's diagonal produced are themselves squares.

OTHERWISE:



$AB^2 = \quad AB \cdot AC \quad - \quad AB \cdot BC$, *II.* 3
$= (AC \cdot AC - BC \cdot AC) - (AC \cdot BC - BC \cdot BC)$, *II.* 2, 3
$= AC^2 + BC^2 - 2\, AC \cdot BC$.

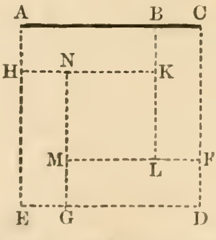
ALGEBRAICAL ILLUSTRATION.

Let $AC = a$, $CB = b$;
then $AB = a - b$.
Now $AB^2 = (a - b)^2 = a^2 - 2ab + b^2$,
and $AC^2 + CB^2 - 2\, AC \cdot CB = a^2 + b^2 - 2ab$;
$\therefore AB^2 = AC^2 + CB^2 - 2\, AC \cdot CB$.

1. Name the two figures which form the sum of the squares on AC and CB.
2. Name the figure which is the square on the difference of AC and CB.
3. Name the figure which is the difference of the squares on AB and AC.
4. Name the figure which is the square on the difference of AB and AC.
5. By how much is the square on the difference of AC and CB exceeded by the sum of the squares on AC and CB?
6. Show that the proposition may be enunciated: The square on the difference of two straight lines is less than the sum of the squares on the two straight lines by twice the rectangle contained by the two straight lines.
7. The sum of the squares on two straight lines is never less than twice the rectangle contained by the two straight lines.
8. If a straight line be divided internally into two segments, and if twice the rectangle contained by the segments be equal to the sum of the squares on the segments, the straight line is bisected.

PROPOSITION 8. Theorem.

The square on the sum of two straight lines diminished by the square on their difference, is equal to four times the rectangle contained by the two straight lines.



Let AB and BC be two straight lines:

it is required to prove $(AB + BC)^2 - (AB - BC)^2$ $4\, AB \cdot BC$.

Place AB and BC in the same straight line,
and on AC describe the square $ACDE$. *I.* 46
From CD, DE, EA cut off CF, DG, EH each $= AB$; *I.* 3
through B and G draw BL, $GN \parallel AE$,
and through F and H draw FM, $HK \parallel AC$. *I.* 31

Then all the $\parallel^{\text{ms}}$ in the figure are rectangles. *I.* 34, *Cor.*
Now because CD, DE, EA are each $= AC$, *I. Def.* 32
and CF, DG, EH are each $= AB$; *Const.*
$\therefore$ DF, EG, AH are each $= BC$;
$\therefore$ the four rectangles AK, CL, DM, EN are each $= AB \cdot BC$.

Because $AC = AB + BC$,
$\therefore ACDE = AC^2 = (AB + BC)^2$.

Because BL, FM, GN, HK are each $= AB$, *I.* 34
and BK, FL, GM, HN are each $= BC$; *I.* 34
$\therefore$ KL, LM, MN, NK are each $= AB - BC$;
$\therefore$ the rectangle $KLMN$ is a square, and $= (AB - BC)^2$.
Hence $(AB + BC)^2 - (AB - BC)^2 = ACDE - KLMN$,
$= AK + CL + DM + EN$,
$= 4\ AB \cdot BC$.

OTHERWISE:

$(AB + BC)^2 = AB^2 + BC^2 + 2\ AB \cdot BC$, *II.* 4, *Cor.* 1
$(AB - BC)^2 = AB^2 + BC^2 - 2\ AB \cdot BC$. *II.* 7, *Cor.* 1
Subtract the second equality from the first;
then $(AB + BC)^2 - (AB - BC)^2 = 4\ AB \cdot BC$.

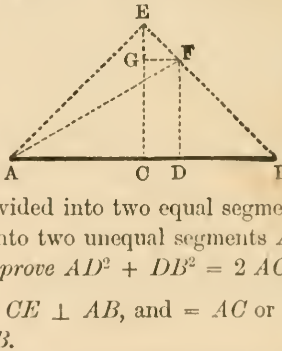
ALGEBRAICAL ILLUSTRATION.

Let $AB = a$, $BC = b$;
then $AB + BC = a + b$, and $AB - BC = a - b$.
Now $(AB + BC)^2 - (AB - BC)^2 = (a + b)^2 - (a - b)^2 = 4ab$,
and $4\ AB \cdot BC = 4ab$;
$\therefore (AB + BC)^2 - (AB - BC)^2 = 4\ AB \cdot BC$.

1. Name the figure which is the square on the sum of AB and BC.
2. Name the figure which is the square on the difference of AB and BC.
3. Name the figures by which the square on the sum of AB and BC exceeds the square on the difference of AB and BC.
4. By how much does the square on the sum of AB and BC exceed the sum of the squares on AB and BC?
5. By how much does the sum of the squares on AB and BC exceed the square on the difference of AB and BC?

PROPOSITION 9. THEOREM.

If a straight line be divided into two equal, and also internally into two unequal segments, the sum of the squares on the two unequal segments is double the sum of the squares on half the line and on the line between the points of section.



Let AB be divided into two equal segments AC, CB, and also internally into two unequal segments AD, DB:
it is required to prove $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$.

From C draw $CE \perp AB$, and $= AC$ or CB, *I.* 11, 3
and join AE, EB.
Through D draw $DF \parallel CE$, meeting EB at F; *I.* 31
through F draw $FG \parallel AB$, meeting EC at G; *I.* 31
and join AF.

(1) To prove $\angle\ AEB$ right.
Because $\angle\ ACE$ is right,
$\therefore\ \angle\ CAE + \angle\ CEA =$ a right angle. *I.* 32

But $\angle\ CAE = \angle\ CEA$; *I.* 5

$\therefore$ each of them is half a right angle.

Similarly, $\angle\ CBE$ and $\angle\ CEB$ are each half a right angle;

$\therefore\ \angle\ AEB$ is right.

(2) To prove $EG = GF$.

$\angle\ EGF$ is right, because it $= \angle\ ECB$; *I.* 29

and $\angle\ GEF$ was proved to be half a right angle;

$\therefore\ \angle\ GFE$ is half a right angle; *I.* 32

$\therefore\ \angle\ GEF = \angle\ GFE$;

$\therefore\quad EG = GF$. *I.* 6

(3) To prove $DF = DB$.

$\angle\ FDB$ is right, because it $= \angle\ ECB$; *I.* 29

and $\angle\ DBF$ is half a right angle, being the same as $\angle\ CBE$;

$\therefore\ \angle\ DFB$ is half a right angle; *I.* 32

$\therefore\ \angle\ DBF = \angle\ DFB$;

$\therefore\quad DF = DB$. *I.* 6

$$\begin{aligned}
\text{Now } AD^2 + DB^2 &= AD^2 + DF^2, && (3)\\
&= AF^2, && I.\ 47\\
&= AE^2 + EF^2, && I.\ 47,\ (1)\\
&= AC^2 + CE^2 + EG^2 + GF^2, && I.\ 47\\
&= 2\,AC^2 + 2\,GF^2, && Const.,\ (2)\\
&= 2\,AC^2 + 2\,CD^2. && I.\ 34
\end{aligned}$$

OTHERWISE:

Consider AC and CD as two straight lines;

then $\quad AD = AC + CD$,

and $\quad DB = CB - CD = AC - CD$.

Hence $AD^2 = (AC + CD)^2 = AC^2 + CD^2 + 2\,AC \cdot CD$, *II.* 4, *Cor.* 1

and $\quad DB^2 = (AC - CD)^2 = AC^2 + CD^2 - 2\,AC \cdot CD$. *II.* 7, *Cor.* 1

Add the second equality to the first;

then $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$.

ALGEBRAICAL ILLUSTRATION.

Let $AC = CB = a$, $CD = b$;

then $AD = a + b$, and $DB = a - b$.

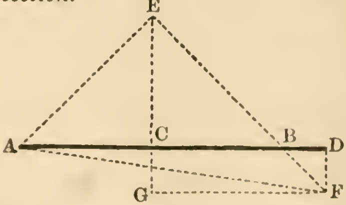
Now $AD^2 + DB^2 = (a + b)^2 + (a - b)^2 = 2a^2 + 2b^2$,

and $2AC^2 + 2CD^2 = 2a^2 + 2b^2$;
$\therefore AD^2 + DB^2 = 2AC^2 + 2CD^2$.

1. Show that the proposition may be enunciated: The square on the sum together with the square on the difference of two straight lines = twice the sum of the squares on the two straight lines. Or, The sum of the squares on two straight lines = twice the square on half their sum together with twice the square on half their difference.
2. By how much does $AD^2 + DB^2$ exceed $AC^2 + CB^2$?
3. The sum of the squares on two internal segments of a straight line is the least possible when the straight line is bisected.
4. The sum of the squares on two internal segments of a straight line becomes greater and greater the nearer the point of section approaches either end of the line. (Euclid, x. Lemma before Prop. 43.)
5. Prove that $AD^2 + DB^2 = 4CD^2 + 2AD \cdot DB$.
6. In the hypotenuse of an isosceles right-angled triangle any point is taken and joined to the opposite vertex; prove that twice the square on this straight line is equal to the sum of the squares on the segments of the hypotenuse.

PROPOSITION 10. THEOREM.

If a straight line be divided into two equal, and also externally into two unequal segments, the sum of the squares on the two unequal segments is double the sum of the squares on half the line and on the line between the points of section.



Let AB be divided into two equal segments AC, CB, and also externally into two unequal segments AD, DB:

it is required to prove $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$.

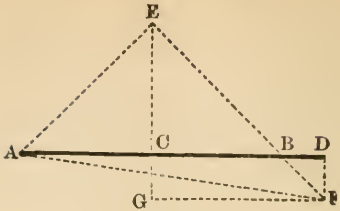
From C draw $CE \perp AB$, and $= AC$ or CB, *I.* 11, 3
and join AE, EB.
Through D draw $DF \parallel CE$, meeting EB produced at F; *I.* 31
through F draw $FG \parallel AB$, meeting EC produced at G; *I.* 31
and join AF.

(1) To prove $\angle\ AEB$ right.
Because $\angle\ ACE$ is right,
$\therefore\ \angle\ CAE + \angle\ CEA =$ a right angle. *I.* 32
But $\angle\ CAE = \angle\ CEA$; *I.* 5
$\therefore$ each of them is half a right angle.
Similarly, $\angle\ CBE$ and $\angle\ CEB$ are each half a right angle;
$\therefore\ \angle\ AEB$ is right.

(2) To prove $EG = GF$.
$\angle\ EGF$ is right, because it $= \angle\ ECB$; *I.* 29
and $\angle\ GEF$ was proved to be half a right angle;
$\therefore\ \angle\ GFE$ is half a right angle; *I.* 32
$\therefore\ \angle\ GEF = \angle\ GFE$;
$\therefore\quad EG = GF$. *I.* 6

(3) To prove $DF = DB$.
$\angle\ FDB$ is right, because it $= \angle\ ECB$; *I.* 29
and $\angle\ DBF$ is half a right angle, being $= \angle\ CBE$; *I.* 15
$\therefore\ \angle\ DFB$ is half a right angle; *I.* 32
$\therefore\ \angle\ DBF = \angle\ DFB$;
$\therefore\quad DF = DB$. *I.* 6

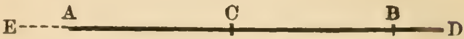
Now $AD^2 + DB^2 = AD^2 + DF^2$, (3)
$= AF^2$, *I.* 47
$= AE^2 + EF^2$, *I.* 47, (1)
$= AC^2 + CE^2 + EG^2 + GF^2$, *I.* 47
$= 2\,AC^2 + 2\,GF^2$, *Const.*, (2)
$= 2\,AC^2 + 2\,CD^2$. *I.* 34



OTHERWISE:

Consider AC and CD as two straight lines;
then $AD = CD + AC$,
and $DB = CD - CB = CD - AC$.
Hence $AD^2 = (CD + AC)^2 = CD^2 + AC^2 + 2\,CD \cdot AC$; *II.* 4, *Cor.* 1
and $DB^2 = (CD - AC)^2 = CD^2 + AC^2 - 2\,CD \cdot AC$. *II.* 7, *Cor.* 1
Add the second equality to the first;
then $AD^2 + DB^2 = 2\,CD^2 + 2\,AC^2$.

OR: *



Let AB be divided into two equal segments AC, CB, and also externally into two unequal segments AD, DB:
it is required to prove $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$.

Produce BA to E, making $AE = BD$. *I.* 3
Then $EC = CD$, and $EB = AD$.
Now because ED is divided into two equal segments EC, CD, and also internally into two unequal segments EB, BD;
$\therefore\ EB^2 + BD^2 = 2\,EC^2 + 2\,CB^2$; ***II.*** 9
$\therefore\ AD^2 + BD^2 = 2\,CD^2 + 2\,AC^2$.

ALGEBRAICAL ILLUSTRATION.

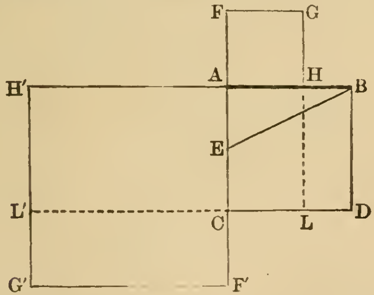
Let $AC = CB = a$, $CD = b$;
then $AD = b + a$, and $DB = b - a$.
Now $AD^2 + DB^2 = (b + a)^2 + (b - a)^2 = 2b^2 + 2a^2$,
and $2\,AC^2 + 2\,CD^2 = 2\,a^2 + 2b^2$;
$\therefore\ AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$.

* *Clavii Commentaria in Euclidis Elementa Geometrica* (1612), p. 93.

1. Show that the proposition may be enunciated: The square on the sum together with the square on the difference of two straight lines = twice the sum of the squares on the two straight lines. Or, The sum of the squares on two straight lines = twice the square on half their sum together with twice the square on half their difference.
2. By how much does $AD^2 + DB^2$ exceed $AC^2 + CB^2$?
3. The sum of the squares on two external segments of a straight line becomes less and less the nearer the point of section approaches either end of the line.
4. Prove that $AD^2 + DB^2 = 4\,CD^2 - 2\,AD \cdot DB$.
5. In the hypotenuse produced of an isosceles right-angled triangle, any point is taken and joined to the opposite vertex; prove that twice the square on this straight line is equal to the sum of the squares on the segments of the hypotenuse.

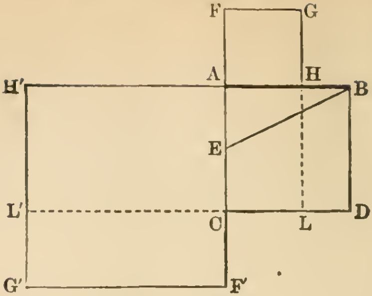
PROPOSITION 11. Problem.

To divide a given straight line internally and externally in medial section.*



Let AB be the given straight line:
It is required to divide it in medial section.

* The second part of this proposition is not given by Euclid.



(1) Internally:

On AB describe the square $ABDC$. *I.* 46

Bisect AC at E; *I.* 10

join EB, and produce CA to F, making $EF = EB$. *I.* 3

On AF (the difference of EF and EA) describe the square $AFGH$. *I.* 46

H is the point required.

Complete the rectangle FL.

Because CA is divided into two equal segments CE, EA, and also externally into two unequal segments CF, FA;

$\therefore$ $CF \cdot FA = EF^2 - EA^2$, *II.* 6

$= EB^2 - EA^2$,

$= AB^2$; *I.* 47, *Cor.*

that is, $CF \cdot FG = AB^2$;

that is, $CG = AD$.

From each of these equals take AL;

$\therefore$ $FH = HD$;

that is, $AH^2 = DB \cdot BH$,

$= AB \cdot BH$.

(2) Externally:

On AB describe the square $ABDC$. *I.* 46

Bisect AC at E; *I.* 10

join EB, and produce AC to F', making $EF' = EB$. *I.* 3

On AF' (the sum of EF' and EA) describe the square $AF'G'H'$, *I.* 46

H' is the point required.

Complete the rectangle $F'L'$.

Because CA is divided into two equal segments CE, EA, and also externally into two unequal segments CF', $F'A$;

$\therefore \quad CF' \cdot F'A = EF'^2 - EA^2,$ *II.* 6

$= EB^2 - EA^2,$

$= AB^2;$ *I.* 47, *Cor.*

that is, $\quad CF' \cdot F'G' = AB^2;$

that is, $\quad CG' = AD.$

To each of these equals add AL';

$\therefore \quad F'H' = H'D;$

that is, $\quad AH'^2 = DB \cdot BH',$

$= AB \cdot BH'.$

Cor. 1.—If a straight line be divided internally in medial section, and from the greater segment a part be cut off equal to the less segment, the greater segment will be divided in medial section.

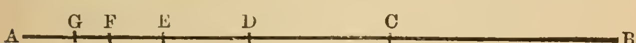
For in the proof of the proposition it has been shown that $CF \cdot FA = AB^2$, that is $= AC^2$;

$\therefore$ CF is divided internally in medial section at A.

Now, from AB, which $= AC$, the greater segment of CF, a part AH has been cut off $= AF$, the less segment of CF;

and AB has been shown to be divided in medial section at H.

Let AB be divided internally in medial section at C, so that AC is the greater segment.



From AC cut off $AD = BC$; then AC is divided in medial section at D, and AD is the greater segment.

From AD cut off $AE = CD$; then AD is divided in medial section at E, and AE is the greater segment.

From AE cut off $AF = DE$; then AE is divided in medial section at F, and AF is the greater segment.

From AF cut off $AG = EF$; then AF is divided in medial section at G, and AG is the greater segment.

This process may evidently be continued as long as we please, and it will be seen on comparison that it is equivalent to the arithmetical method of finding the greatest common measure. That method, if applied to two integers, always, however, comes to an end; unity, in default of any other number, being always a common measure of any two integers. In like manner any two fractions, whether vulgar or decimal, have always some common measure, for instance, unity divided by their least common denominator. From these considerations, therefore, it will appear that the segments of a straight line divided in medial section cannot both be expressed exactly either in integers or fractions; in other words, these segments are incommensurable.

Cor. 2.—If a straight line be divided internally in medial section, and to the given straight line a part be added equal to the greater segment, the whole straight line will be divided in medial section.

For this process is just the reversal of that described in Cor. 1, as will be evident from the following. (See fig. to Cor. 1.)

Let AF be divided in medial section at G, so that AG is the greater segment.

To AF add FE, which $= AG$; then AE is divided in medial section at F, and AF is the greater segment.

To AE add ED, which $= AF$; then AD is divided in medial section at E, and AE is the greater segment.

To AD add DC, which $= AE$; then AC is divided in medial section at D, and AD is the greater segment.

To AC add CB, which $= AD$; then AB is divided in medial section at C, and AC is the greater segment.

ALGEBRAICAL APPLICATION.

Let $AB = a$; to find the length of AH or AH'.

Denote AH by x; then $BH = a - x$.

Now, since $AB \cdot BH = AH^2$

$\therefore a(a - x) = x^2$, a quadratic equation, which being solved gives

$$x = \frac{a(\sqrt{5} - 1)}{2} \text{ or } \frac{-a(\sqrt{5} + 1)}{2}.$$

The first value of x, which is less than a, since $\frac{\sqrt{5}-1}{2}$ is less than unity, corresponds to AH; and the second value of x, which is numerically greater than a, since $\frac{\sqrt{5}+1}{2}$ is greater than unity, corresponds to AH'. The significance of the $-$ in the second value cannot be explained here; it will be enough to say that it indicates that AH and AH' are measured in opposite directions from A.

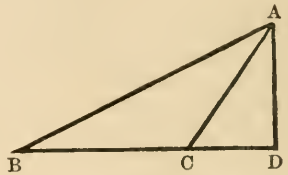
The following approximation to the values of the segments of a straight line divided internally in medial section, is given in Leslie's *Elements of Geometry* (4th edition, p. 312), and attributed to Girard, a Flemish mathematician (17th cent.).

Take the series 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, &c., where each term is got by taking the sum of the preceding two. If any term be considered as denoting the length of the straight line, the two preceding terms will approximately denote the lengths of its segments when it is divided internally in medial section. Thus, if 89 be the length of the line, its segments will be nearly 34 and 55; because $89 \times 34 = 3026$, and $55^2 = 3025$. If 144 be the length of the line, its segments will be nearly 55 and 89; because $144 \times 55 = 7920$, and $89^2 = 7921$.

1. It is assumed in the construction that a side of the square described on AF will coincide with AB. Prove this.
2. If $AB \cdot BH = AH^2$, prove that AH is greater than BH.
3. If CH be produced, it will cut BF at right angles.
4. The point of intersection of BE and CH is the projection of A on CH.
5. It is assumed in the proof of the second part that a side of the square described on AF' will be in the same straight line with AB. Prove this.
6. If $AB \cdot BH' = AH'^2$, prove that AH' is greater than AB.
7. If CH' be produced, it will cut BF' at right angles.
8. The point of intersection of BE and CH' is the projection of A on CH'.
9. Prove that HB is divided externally in medial section at A, and $H'B$ internally at A.
10. Hence name all the straight lines in the figure that are divided internally or externally in medial section.

PROPOSITION 12. Theorem.

In obtuse-angled triangles, the square on the side opposite the obtuse angle is equal to the sum of the squares on the other two sides increased by twice the rectangle contained by either of those sides and the projection on it of the other side.



Let ABC be an obtuse-angled triangle, having the obtuse angle ACB; and let CD be the projection of CA on BC: *it is required to prove* $AB^2 = BC^2 + CA^2 + 2\,BC \cdot CD$.

Because BD is divided internally into any two segments BC, CD,

$\therefore\ BD^2 = BC^2 + CD^2 + 2\,BC \cdot CD.$ *II.* 4

Adding DA^2 to both sides,

$$BD^2 + DA^2 = BC^2 + CD^2 + DA^2 + 2\,BC \cdot CD;$$

$\therefore\quad AB^2 = BC^2 + CA^2 + 2\,BC \cdot CD.$ *I.* 47

ALGEBRAICAL APPLICATION.

Let the sides opposite the $\angle$s A, B, C be denoted by a, b, c, so that $AB = c$, $BC = a$, $CA = b$;

then, since $AB^2 = BC^2 + CA^2 + 2\,BC \cdot CD$, *II.* 12

$$c^2 = a^2 + b^2 + 2a \cdot CD;$$

$$\therefore \qquad CD = \frac{c^2 - a^2 - b^2}{2a};$$

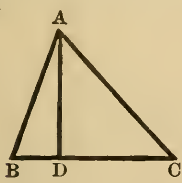
$$\therefore \qquad BD = BC + CD = a + \frac{c^2 - a^2 - b^2}{2a} = \frac{a^2 - b^2 + c^2}{2a}.$$

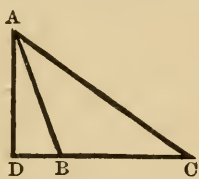
Hence, if the three sides of an obtuse-angled triangle are known, we can calculate the lengths of the segments into which either side about the obtuse angle is divided by a perpendicular from one of the acute angles.

1. If from B there be drawn $BE \perp AC$ produced, then $BC \cdot CD = AC \cdot CE$.
2. $ABCD$ is a $\|^m$ having $\angle ABC$ equal to an angle of an equilateral triangle; prove $BD^2 = BC^2 + CD^2 + BC \cdot CD$.
3. If $AB^2 = AC^2 + 3\,CD^2$ (figure to proposition), how will the perpendicular AD divide BC?
4. If $\angle ACB$ become more and more obtuse, till at length A falls on BC produced, what does the proposition become?

PROPOSITION 13. Theorem.

In every triangle the square on the side opposite an acute angle is equal to the sum of the squares on the other two sides diminished by twice the rectangle contained by either of those sides and the projection on it of the other side.





Let ABC be any triangle, having the acute angle ACB; and let CD be the projection of CA on BC:

it is required to prove $AB^2 = BC^2 + CA^2 - 2\,BC \cdot CD$.

Because BD is divided externally into any two segments BC, CD,

$$\therefore \quad BD^2 = BC^2 + CD^2 - 2\,BC \cdot CD. \qquad II.\ 7$$

Adding DA^2 to both sides,

$$BD^2 + DA^2 = BC^2 + CD^2 + DA^2 - 2\,BC \cdot CD;$$

$$\therefore \quad AB^2 = BC^2 + CA^2 - 2\,BC \cdot CD. \qquad I.\ 47$$

ALGEBRAICAL APPLICATION.

As before, let $AB = c$, $BC = a$, $CA = b$;

then, since $AB^2 = BC^2 + CA^2 - 2\,BC \cdot CD$, *II.* 13

$$c^2 = a^2 + b^2 - 2a \cdot CD;$$

$\therefore$ $$CD = \frac{a^2 + b^2 - c^2}{2a};$$

$\therefore$ (fig. 1) $$BD = BC - CD = a - \frac{a^2 + b^2 - c^2}{2a} = \frac{a^2 - b^2 + c^2}{2a};$$

and (fig. 2) $$BD = CD - BC = \frac{a^2 + b^2 - c^2}{2a} - a = \frac{b^2 - c^2 - a^2}{2a}.$$

Hence, from the results of this proposition and the preceding, if the three sides of any triangle are known, we can calculate the lengths of the segments into which any side is divided by a perpendicular from the opposite angle.

Hence, again, if the three sides of any triangle are known, we can calculate the length of the perpendicular drawn from any angle of a triangle to the opposite side.

For example (fig. 1), to find the length of AD.

$$AD^2 = AC^2 - CD^2, \qquad \textit{I. 47, Cor}$$

$$= b^2 - \left(\frac{a^2 + b^2 - c^2}{2a}\right)^2$$

$$= \frac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{4a^2},$$

$$= \frac{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)}{4a^2},$$

$$= \frac{\{(a^2 + 2ab + b^2) - c^2\}\{c^2 - (a^2 - 2ab + b^2)\}}{4a^2}$$

$$= \frac{\{(a + b)^2 - c^2\}\{c^2 - (a - b)^2\}}{4a^2},$$

$$= \frac{(a + b + c)(a + b - c)(c + a - b)(c - a + b)}{4a^2};$$

$$\therefore AD = \frac{1}{2a}\sqrt{(a + b + c)(a + b - c)(a - b + c)(b + c - a)}.$$

This expression for the length of AD may be put into a shorter and more convenient form, thus:

Denote the semi-perimeter of the $\triangle\ ABC$ by s;

then $a + b + c =$ the perimeter $= 2s$;

$\therefore a + b - c = a + b + c - 2c = 2s - 2c = 2(s - c),$

$a - b + c = a + b + c - 2b = 2s - 2b = 2(s - b),$

and $b + c - a = a + b + c - 2a = 2s - 2a = 2(s - a).$

Hence $AD = \frac{1}{2a}\sqrt{2s \cdot 2(s-c) \cdot 2(s-b) \cdot 2(s-a)}$,

$$= \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$$

Similarly, the perpendicular from B on $CA = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$

and " " C on $AB = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$.

Hence, lastly, if the three sides of a triangle are known, we can calculate the area of the triangle.

For the area of $\triangle\ ABC = \frac{1}{2}\ BC \cdot AD$, *I.* 41, 35

$$= \frac{a}{2} \cdot \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)},$$

$$= \sqrt{s(s-a)(s-b)(s-c)};$$

which expression may be put into the form of a rule, thus :

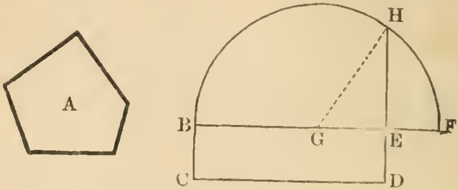
From half the sum of the three sides, subtract each side separately ; multiply the half sum and the three remainders together, and the square root of the product will be the area.*

1. If from B there be drawn $BE \perp AC$ or AC produced, then $BC \cdot CD = AC \cdot CE$.
2. $ABCD$ is a ||m having $\angle\ ABC$ double of an angle of an equilateral triangle ; prove $BD^2 = BC^2 + CD^2 - BC \cdot CD$.
3. If $AB^2 = AC^2 + 3\,CD^2$ (fig. 1 to proposition), how will the perpendicular AD divide BC?
4. If $\angle\ ACB$ become more and more acute till at length A falls on CB or CB produced, what does the proposition become?
5. If the square on one side of a triangle be greater than the sum of the squares on the other two sides, the angle contained by these two sides is obtuse. (Converse of II. 12.)
6. If the square on one side of a triangle be less than the sum of the squares on the other two sides, the angle contained by these two sides is acute. (Converse of II. 13.)
7. The square on the base of an isosceles triangle is equal to twice the rectangle contained by either of the equal sides and the projection on it of the base.

* The discovery of this expression for the area of a triangle is due to Heron of Alexandria. See Hultsch's *Heronis Alexandrini . . . reliquiæ* (Berlin, 1864), pp. 235–237.

PROPOSITION 14. PROBLEM.

To describe a square that shall be equal to a given rectilineal figure.



Let A be the given rectilineal figure:
it is required to describe a square $= A$.

Describe the rectangle $BCDE = A$. *I.* 45

Then, if $BE = ED$, the rectangle is a square, and what was required is done.

But if not, produce BE to F, making $EF = ED$. *I.* 3

Bisect BF in G; *I.* 10

with centre G and radius GF describe the semicircle BHF;
and produce DE to H. $EH^2 = A$.

Join GH.

Because BF is divided into two equal segments BG, GF, and also internally into two unequal segments BE, EF;

$$\therefore \quad BE \cdot EF = GF^2 - GE^2, \qquad II.\ 5$$
$$= GH^2 - GE^2,$$
$$= EH^2. \qquad I.\ 47,\ Cor.$$
$$\therefore \quad BD = EH^2;$$
$$\therefore \quad A = EH^2.$$

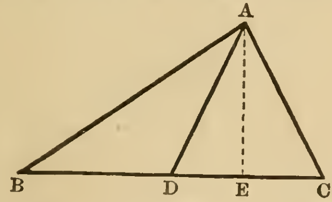
1. From any point in the arc of a semicircle, a perpendicular is drawn to the diameter. Prove that the square on this perpendicular = the rectangle contained by the segments into which it divides the diameter.

2. Divide a given straight line internally into two segments, such that the rectangle contained by them may be equal to the square on another given straight line. What limits are there to the length of the second straight line?
3. Divide a given straight line externally into two segments, such that the rectangle contained by them may be equal to the square on another given straight line. Are there any limits to the length of the second straight line?
4. Describe a rectangle equal to a given square, and having one of its sides equal to a given straight line.

APPENDIX II.

Proposition 1.

*The sum of the squares on two sides of a triangle is double the sum of the squares on half the base and on the median to the base.**



Let ABC be a triangle, AD the median to the base BC:
it is required to prove $AB^2 + AC^2 = 2\,BD^2 + 2\,AD^2$.

Draw $AE \perp BC$. *I.* 12

Then $AB^2 = BD^2 + AD^2 + 2\,BD \cdot DE$, *II.* 12

and $AC^2 = CD^2 + AD^2 - 2\,CD \cdot DE$. *II.* 13

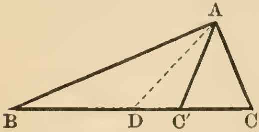
But $BD^2 = CD^2$, and $BD \cdot DE = CD \cdot DE$, since $BD = CD$;

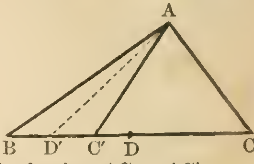
$\therefore\ AB^2 + AC^2 = 2\,BD^2 + 2\,AD^2$.

Cor.—The theorem is true, however near the vertex A may be to the base BC. When A falls on BC, the theorem becomes II. 9; when A falls on BC produced, the theorem becomes II. 10.

* Pappus, VII. 122.

NOTE.—It may be well to remark that the converse of the theorem, 'If ABC be a triangle, and from the vertex A a straight line AD be drawn to the base BC, so that $AB^2 + AC^2 = 2\,BD^2 + 2\,AD^2$, then D is the middle point of BC,' is not always true.





For, let ABC, ABC' be two triangles having $AC = AC'$.

Find D, the middle point of BC. D must fall either between B and C', between C and C', or on C'.

In the first case, join AD.

Then $AB^2 + AC^2 = 2\,BD^2 + 2\,AD^2$; *App. II.* 1

$\therefore$ $AB^2 + AC'^2 = 2\,BD^2 + 2\,AD^2$;

and we know that D is not the middle point of BC'.

In the second case, find D' the middle point of BC', and join AD'.

Then $AB^2 + AC'^2 = 2\,BD'^2 + 2\,AD'^2$; *App. II.* 1

$\therefore$ $AB^2 + AC^2 = 2\,BD'^2 + 2\,AD'^2$;

and we know that D' is not the middle point of BC.

The third case needs no discussion.

PROPOSITION 2.

*The difference of the squares on two sides of a triangle is double the rectangle contained by the base and the distance of its middle point from the perpendicular on it from the vertex.**

Fig. 1.

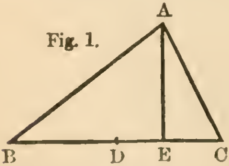
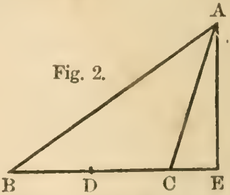


Fig. 2.



Let ABC be a triangle, D the middle point of the base BC, and AE the perpendicular from A on BC:

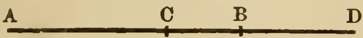
it is required to prove $AB^2 - AC^2 = 2\,BC \cdot DE$.

* Pappus, VII. 120.

For $$AB^2 - AC^2 = (BE^2 + AE^2) - (EC^2 + AE^2), \quad I.\ 47$$
$$= BE^2 - EC^2,$$
$$= (BE + EC)(BE - EC), \quad II.\ 5,\ 6,\ Corr.$$
$$= BC \cdot 2\,DE \text{ in fig. 1};$$
$$\text{or} = 2\,DE \cdot BC \text{ in fig. 2},$$
$$= 2\,BC \cdot DE.$$

PROPOSITION 3.

*If the straight line AD be divided internally at any two points C and B, then $AC \cdot BD + AD \cdot BC = AB \cdot CD$.**

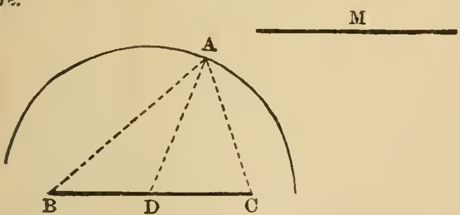


For $$AC \cdot BD + AD \cdot BC = AC \cdot BD + (BD + AB) \cdot BC,$$
$$= AC \cdot BD + BD \cdot BC + AB \cdot BC, \quad II.\ 1$$
$$= BD \cdot (AC + BC) + AB \cdot BC, \quad II.\ 1$$
$$= BD \cdot AB + AB \cdot BC,$$
$$= AB \cdot (BD + BC), \quad II.\ 1$$
$$= AB \cdot CD.$$

LOCI.

PROPOSITION 4.

Find the locus of the vertices of all the triangles which have the same base and the sum of the squares of their sides equal to a given square.



Let BC be the given base, M^2 the given square.

Suppose A to be a point situated on the required locus.

Join AB, AC;

bisect BC in D, and join AD. *I.* 10

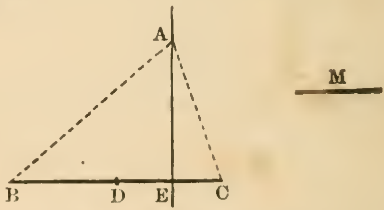
* Euler, *Novi Comm. Petrop.*, vol. i. p. 49.

Then, since A is a point on the locus, $AB^2 + AC^2 = M^2$. *Hyp.*
But $AB^2 + AC^2 = 2\,BD^2 + 2\,AD^2$; *App. II.* 1
$\therefore\ 2\,BD^2 + 2\,AD^2 = M^2$;
$\therefore\ AD^2 = \frac{1}{2}M^2 - BD^2$.
Now $\frac{1}{2}M^2$ is a constant magnitude, and so is BD^2, being the square on half the given base;
$\therefore\ \frac{1}{2}M^2 - BD^2$ must be constant;
$\therefore\ AD^2$ must be constant.
And since AD^2 is constant, AD must be equal to a fixed length; that is, the vertex of any triangle fulfilling the given conditions is always at a constant distance from a fixed point D, the middle of the given base. Hence, the locus required is the ○ce of a circle whose centre is the middle point of the base.

To determine the locus completely, it would be necessary to find the length of the radius of the circle. This may be left to the reader.

Proposition 5.

Find the locus of the vertices of all the triangles which have the same base, and the difference of the squares of their sides equal to a given square.



Let BC be the given base, M^2 the given square.
Suppose A to be a point situated on the required locus.
Join AB, AC;
bisect BC in D, and draw $AE \perp BC$ or BC produced. *I.* 10, 12
Then, since A is a point on the locus $AB^2 - AC^2 = M^2$. *Hyp.*
But $AB^2 - AC^2 = 2\,BC \cdot DE$; *App. II.* 2
$\therefore\ 2\,BC \cdot DE = M^2$.
Now M^2 is a constant magnitude, and so is $2\,BC$;
$\therefore\ DE$ must be constant;
$\therefore$ a perpendicular drawn to BC from the vertex of any triangle fulfilling the given conditions will cut BC at a fixed point.

If $AC^2 - AB^2 = M^2$, the perpendicular from A on BC will cut BC at a point E' on the other side of D, such that $DE' = DE$.

Hence, the locus consists of two straight lines drawn perpendicular to the base and equally distant from the middle point of the base.

DEDUCTIONS.

1. If from the vertex of an isosceles triangle a straight line be drawn to cut the base either internally or externally, the difference between the squares on this line and either side is equal to the rectangle contained by the segments of the base. (Pappus, III. 5.)
2. The sum of the squares on the diagonals of a $\|^m$ is equal to the sum of the squares on the four sides.
3. The sum of the squares on the diagonals of any quadrilateral is equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.
4. The sum of the squares on the four sides of any quadrilateral exceeds the sum of the squares on the two diagonals by four times the square on the straight line which joins the middle points of the diagonals. (Euler, *Novi Comm. Petrop.*, i. p. 66.)
5. The centre of a fixed circle is the middle point of the base of a triangle. If the vertex of the triangle be on the $\bigcirc^{ce}$, the sum of the squares on the two sides of the triangle is constant.
6. The centre of a fixed circle is the point of intersection of the diagonals of a $\|^m$. Prove that the sum of the squares on the straight lines drawn from any point on the $\bigcirc^{ce}$ to the four vertices of the $\|^m$ is constant.
7. Two circles are concentric. Prove that the sum of the squares of the distances from any point on the $\bigcirc^{ce}$ of one of the circles to the ends of a diameter of the other is constant.
8. The middle point of the hypotenuse of a right-angled triangle is equidistant from the three vertices.
9. Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the three medians, or equal to nine times the sum of the squares on the straight lines which join the centroid to the three vertices.
10. If $ABCD$ be a quadrilateral, and P, Q, R, S be the middle points of AB, BC, CD, DA respectively, then $2PR^2 + AB^2 + CD^2 = 2QS^2 + BC^2 + DA^2$.

11. Thrice the sum of the squares on the sides of any pentagon = the sum of the squares on the diagonals together with four times the sum of the squares on the five straight lines joining, in order, the middle points of those diagonals.
12. If A, B be fixed points, and O any other point, the sum of the squares on OA and OB is least when O is the middle point of AB.
13. Prove II. 9, 10 by the following construction: On AD describe a rectangle $AEFD$ whose sides AE, DF are each $= AC$ or CB. According as D is in AB, or in AB produced, from DF, or DF produced, cut off $FG = DB$; and join EC, CG, GE. Show how these figures may be derived from those in the text.
14. If from the vertex of the right angle of a right-angled triangle a perpendicular be drawn to the hypotenuse, then (1) the square on this perpendicular is equal to the rectangle contained by the segments of the hypotenuse; (2) the square on either side is equal to the rectangle contained by the hypotenuse and the segment of it adjacent to that side.
15. The sum of the squares on two unequal straight lines is greater than twice the rectangle contained by the straight lines.
16. The sum of the squares on three unequal straight lines is greater than the sum of the rectangles contained by every two of the straight lines.
17. The square on the sum of three unequal straight lines is greater than three times the sum of the rectangles contained by every two of the straight lines.
18. The sum of the squares on the sides of a triangle is less than twice the sum of the rectangles contained by every two of the sides.
19. If one side of a triangle be greater than another, the median drawn to it is less than the median drawn to the other.
20. If a straight line AB be bisected in C, and divided internally at D and E, D being nearer the middle than E, then $AD \cdot DB = AE \cdot EB + CD \cdot DE + CE \cdot ED$.
21. ABC is an isosceles triangle having each of the angles B and $C = 2\,A$. BD is drawn $\perp AC$; prove $AD^2 + DC^2 = 2\,BD^2$
22. Divide a given straight line internally so that the squares on the whole and on one of the segments may be double of the square on the other segment.

23. Given that AB is divided internally at H, and externally at H', in medial section, prove the following:

(1) $AH \cdot BH = (AH + BH) \cdot (AH - BH)$;
$AH' \cdot BH' = (BH' + AH') \cdot (BH' - AH')$.

(2) $AH \cdot (AH - BH) = BH^2$; $AH' \cdot (AH' + BH') = BH'^2$.

(3) $AB^2 + BH^2 = 3\,AH^2$; $AB^2 + BH'^2 = 3\,AH'^2$.

(4) $(AB + BH)^2 = 5\,AH^2$; $(AB + BH')^2 = 5\,AH'^2$.

(5) $(AH - BH)^2 = 3\,BH^2 - AH^2$; $(BH' - AH')^2 = 3\,AH'^2 - BH'^2$.

(6) $(AH + BH)^2 = 3\,AH^2 - BH^2$; $(AH' + BH')^2 = 3\,BH'^2 - AH'^2$.

(7) $(AB + AH)^2 = 8\,AH^2 - 3\,BH^2$; $(AH' - AB)^2 = 8\,AH'^2 - 3\,BH'^2$.

(8) $AB^2 + AH^2 = 4\,AH^2 - BH^2$; $AB^2 + AH'^2 = 4\,AH'^2 - BH'^2$.

24. In any triangle ABC, if BP, CQ be drawn $\perp$ CA, BA, produced if necessary, then shall $BC^2 = AB \cdot BQ + AC \cdot CP$.

25. If from the hypotenuse of a right-angled triangle segments be cut off equal to the adjacent sides, the square of the middle segment thus formed = twice the rectangle contained by the extreme segments. Show how this theorem may be used to find numbers expressing the sides of a right-angled triangle. (Leslie's *Elements of Geometry*, 1820, p. 315.)

Loci.

1. Given a $\triangle\, ABC$; find the locus of the points the sum of the squares of whose distances from B and C, the ends of the base, is equal to the sum of the squares of the sides AB, AC.
2. Given a $\triangle\, ABC$; find the locus of the points the difference of the squares of whose distances from B and C, the ends of the base, is equal to the difference of the squares of the sides AB, AC.
3. Of the $\triangle\, ABC$, the base BC is given, and the sum of the sides AB, AC; find the locus of the point where the perpendicular from C to AC meets the bisector of the exterior vertical angle at A.
4. Of the $\triangle\, ABC$, the base BC is given, and the difference of the sides AB, AC; find the locus of the point where the perpendicular from C to AC meets the bisector of the interior vertical angle at A.
5. A variable chord of a given circle subtends a right angle at a fixed point; find the locus of the middle point of the chord. Examine the cases when the fixed point is inside the circle, outside the circle, and on the $\bigcirc^{ce}$

BOOK III.

DEFINITIONS.

1. **A circle** is a plane figure contained by one line which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal. This point is called the **centre** of the circle, and the straight lines drawn from the centre to the circumference are called **radii.**

Cor. 1.—If a point be situated inside a circle its distance from the centre is less than a radius; and if it be situated outside, its distance from the centre is greater than a radius

Thus, in fig. 1, OP, the distance of the point P from the centre O, is less than the radius OA; in fig. 2, OP is greater than the radius OA.

Fig. 1.

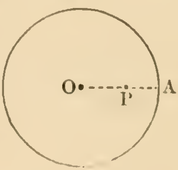
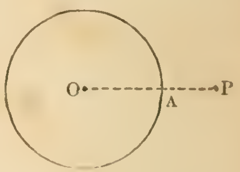


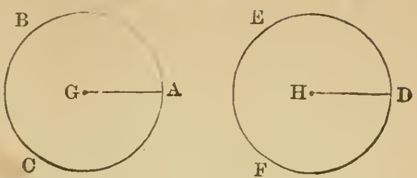
Fig. 2.



Cor. 2.—Conversely, if the distance of a point from the centre of a circle be less than a radius, the point must be situated inside the circle; if its distance from the centre be greater than a radius, it must be situated outside the circle.

Cor. 3.—If the radii of two circles be equal, the circumferences are equal, and so are the circles themselves.

This may be rendered evident by applying the one circle to the other, so that their centres shall coincide. Since the radii of the one circle are equal to those of the other, every point in the circumference of the one circle will coincide with a point in the circumference of the other; therefore, the two circumferences coincide and are equal. Consequently also the two circles coincide and are equal.



Cor. 4.—Conversely, if two circles be equal, their radii are equal, and also their circumferences.

This may be proved indirectly, by supposing the radii unequal.

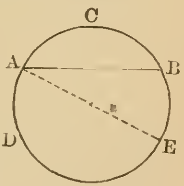
Cor. 5.—A circle is given in magnitude when the length of its radius is given, and a circle is given in position and magnitude when the position of its centre and the length of its radius are given. (Euclid's *Data*, Definitions 5 and 6.)

Cor. 6.—The two parts into which a diameter divides a circle are equal.

This may be proved, like Cor. 3, by superposition.

The two parts are therefore called *semicircles.*

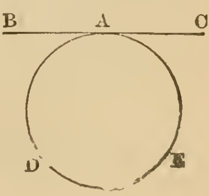
Cor. 7.—The two parts into which a straight line not a diameter divides a circle are unequal.



Thus if AB is not a diameter of the circle ABC, the two parts ACB and ADB into which AB divides the circle are unequal.

For if a diameter AE be drawn, the part ACB is less than the semicircle ABE, and the part ADB is greater than the semicircle ADE.

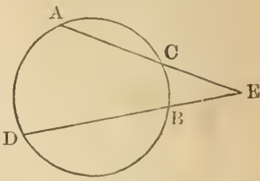
2. **Concentric** circles are those which have a common centre.



3. A straight line is said to **touch** a circle, or to be a **tangent** to it, when it meets the circle, but being produced does not cut it.

Thus BC is a tangent to the circle ADE.

4. A straight line drawn from a point outside a circle, and cutting the circumference, is called a **secant.**



Thus ECA and EBD are secants of the circle ABC.

If the secant ECA were, like one of the hands of a watch, to revolve round E as a pivot, the points A and C would approach one another, and at length coincide. When the points A and C coincided, the secant would have become a tangent. Hence a tangent to a circle may be defined to be a secant in its limiting position, or a secant which meets the circle in two coincident points.

This way of regarding a tangent straight line may be applied also to a tangent circle.

5. Circles which meet but do not cut one another, are said to **touch** one another.

Fig. 1.

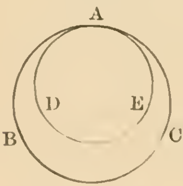
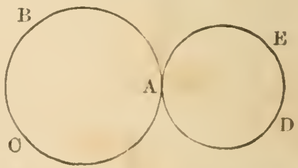


Fig. 2.



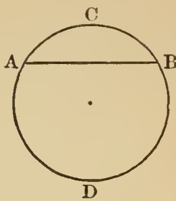
Thus the circles ABC, ADE, which meet but do not intersect, are said to touch each other. In fig. 1, the circles are said to touch one another *internally*, although in strictness only one of them touches the other internally; in fig. 2, they are said to touch one another *externally*.

6. The points at which circles touch each other, or at which straight lines touch circles, are called **points of contact.**

Thus in the figures to definitions 3 and 5, the points A are points of contact.

7. A **chord** of a circle is the straight line joining any two points on the circumference.

Thus AB is a chord of the circle ABC.



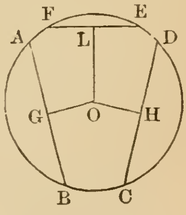
8. An **arc** of a circle is any part of the circumference.

Thus ACB is an arc of the circle ABC; so is ADB.

9. A chord of a circle which does not pass through the centre divides the circumference into two unequal arcs. These arcs are called the **major** and the **minor** arcs, and they are said to be **conjugate** to each other.

Thus the chord AB divides the circumference of the circle ABC into the conjugate arcs ADB, ACB, of which ADB is a major arc, and ACB a minor arc.

10. Chords of a circle are said to be **equidistant** from the centre when the perpendiculars drawn to them from the centre are equal; and one chord is farther from the centre than another, when the perpendicular on it from the centre is greater than the perpendicular on the other.



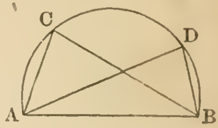
Thus in the circle ABC, whose centre is O, if the perpendiculars OG, OH on the chords AB, CD are equal, AB and CD are said to be equidistant from O; if the perpendicular OL on the chord EF is greater than OG or OH, the chord EF is said to be farther from the centre than AB or CD.

11. A **segment** of a circle is the figure contained by a chord, and either of the arcs into which the chord divides the circumference. The segments are called **major** or **minor segments**, according as their arcs are major or minor arcs.

Thus (see figure to definition 7) the figure contained by the minor arc ACB and the chord AB is a minor segment; the figure contained by the major arc ADB and the chord AB is a major segment.

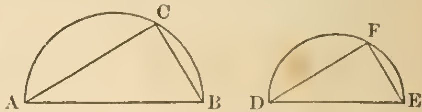
It is worthy of observation that a segment, like a circle, is generally named by three letters; but the letters may not be arranged anyhow. The letters at the ends of the chord must be placed either first or last.

12. An **angle in a segment** of a circle is the angle contained by two straight lines drawn from any point in the arc of the segment to the ends of the chord.



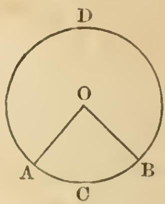
Thus ACB and ADB are angles in the segment ACB.

13. **Similar segments** of circles are those which contain equal angles.



Thus if the angles C and F are equal, the segment ACB is said to be similar to the segment DFE.

14. A **sector** of a circle is the figure contained by an arc and the two radii drawn to the ends of the arc.



Thus if O be the centre of the circle ABD, the figure $OACB$ is a sector; so is $OADB$.

It is obvious that, when the radii are in the same straight line, the sector becomes a semicircle.

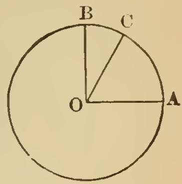
15. The **angle of a sector** is the angle contained by the two radii.

Thus the angle of the sector $OACB$ is the angle AOB.

16. Two radii of a circle not in the same straight line divide the circle into two sectors, one of which is greater and the other less than a semicircle; the former may be called a **major**, and the latter a **minor sector**.

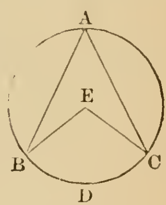
Thus $OADB$ is a major sector, and $OACB$ is a minor sector.

17. Sectors have received particular names according to the size of the angle contained by the radii. When the contained angle is a right angle, the sector is called a **quadrant**; when the contained angle is equal to one of the angles of an equilateral triangle, the sector is called a **sextant**.



Thus if AOB is a right angle, or one-fourth of four right angles, the sector OAB is a quadrant; if AOC is two-thirds of one right angle (see p. 71, deduction 9), or one-sixth of four right angles, the sector OAC is a sextant.

18. An angle is said to be at the centre, or at the circumference of a circle, when its vertex is at the centre, or on the circumference of the circle.



Thus BEC is an angle at the centre. and BAC an angle at the circumference of the circle ABC.

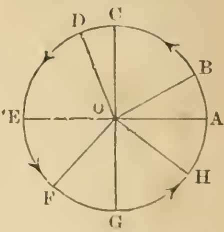
19. An angle either at the centre or at the circumference of a circle is said to stand on the arc intercepted between the arms of the angle.

Thus the angle BEC at the centre and the angle BAC at the circumference both stand on the same arc BDC.

In respect to the angle BEC at the centre of the circle ABC, it may readily occur to the reader to inquire whether the minor arc BDC is the only arc intercepted by EB and EC, the arms of the angle. Obviously enough EB and EC intercept also the major arc BAC. What, then, is the angle which stands on the major arc BAC? This inquiry leads us naturally to reconsider our definition of an angle.

20. An angle may be regarded as generated (or described) by a straight line which revolves round one of its end points, the size of the angle depending on the amount of revolution.

Thus if the straight line OB occupy at first the position OA, and then revolve round O in a manner opposite to that of the hands of a watch, till it comes into the position OB, it will have generated or described the angle AOB. If OB continue its revolution round O till it occupies the position OD, it will have generated the angle AOD; if OB still continue its revolution round O till it occupies successively the positions OF, OH, it will have generated the angles AOF, AOH. The angles AOB, AOD, AOF, AOH, being successively generated by the revolution of OB, are therefore arranged in order of magnitude, AOD being greater than AOB, AOF greater than AOD, and AOH greater than AOF.



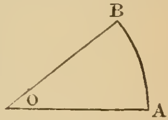
It is plain enough that OB, after reaching the position OH, may continue its revolution till it occupies the position it started from, when it will coincide again with OA. OB will then have described a complete revolution. If the revolution be supposed to continue, the angle generated by OB will grow greater and greater (since its size depends on the amount of revolution), but OB itself will return to the positions it occupied before; and therefore in its second revolution OB will not indicate any new direction relatively to OA, which it did not indicate in its first. Hence there is no need at present to consider angles greater than those generated by a straight line in one complete revolution.

21. In the course of the revolution of OB from the position of OA round to OA again, OB will at some time or other occupy the position OE, which is in a straight line with OA; the angle AOE thus generated is called a **straight** (or sometimes a **flat**) **angle**.

When OB occupies the position OC midway between that of OA and OE (that is, when the angles AOC and COE are equal), the angle AOC thus generated is called a right angle. Hence a straight angle is equal to two right angles.

When OB occupies the position OG which is in a straight line with OC, the angle AOG thus generated is an angle of three right angles; when OB again coincides with OA, it has generated an angle of four right angles. Hence angle AOB is less than a right angle; angle AOD is greater than one right angle, and less than two; angle AOF is greater than two and less than three right angles; angle AOH is greater than three and less than four right angles.

22. It has been explained how OB, starting from the position OA, and revolving in a manner opposite to that of the hands of a watch, generates the angle AOB, less than a right angle when it reaches the position OB. But we may suppose



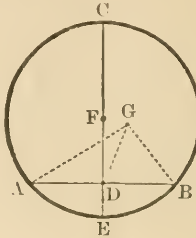
that OB, starting from OA, reaches the position OB by revolving round O in the same manner as the hands of a watch; it will then have generated another angle AOB, greater than three right angles. Thus it appears that two straight lines drawn from a point contain two angles having common arms and a common vertex. Such angles are said to be **conjugate**, the greater being called the **major conjugate**, and the less the **minor conjugate** angle. When, however, the angle contained by two straight lines is spoken of, the minor conjugate angle is understood to be meant.

23. It will be apparent from the preceding that the sum of two conjugate angles is equal to four right angles; and that when two conjugate angles are unequal, the minor conjugate must be less than two right angles, and the major conjugate greater than two right angles. When two conjugate angles are equal, each of them must be a straight angle.

Major conjugate angles are often called **reflex** angles, and to prevent obtuse angles from being confounded with reflex angles, obtuse angles may now be defined to be angles greater than one right angle, and less than two right angles.

PROPOSITION 1. Problem.

To find the centre of a given circle.



Let ABC be the given circle:
it is required to find its centre.

Draw any chord AB, and bisect it at D; I. 10
from D draw $DC \perp AB$, I. 11
and let DC, produced if necessary, meet the $\bigcirc^{ce}$ at C and E.
Bisect CE at F. I. 10

F is the centre of $\odot ABC$.

For if F be not the centre, let G be the centre;
and join GA, GD, GB.

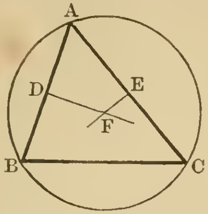
In $\triangle$s ADG, BDG, $\left\{\begin{array}{l} AD = BD \\ DG = DG \\ GA = GB; \end{array}\right.$ — *Const.*; *III. Def.* 1

$\therefore \angle ADG = \angle BDG$; I. 8
$\therefore \angle ADG$ is right. *I. Def.* 10
But $\angle ADC$ is right; *Const.*
$\therefore \angle ADG = \angle ADC$, which is impossible;
$\therefore G$ is not the centre.

Now G is any point out of CE;
$\therefore$ the centre is in CE.
But, since the centre is in CE, it must be at F, the middle point of CE.

Cor. 1.—The straight line which bisects any chord of a circle perpendicularly, passes through the centre of the circle.

Cor. 2.—Hence a circle may be described which shall pass through the three vertices of a triangle.



For if a circle could be described to pass through A, B, C, the vertices of the triangle ABC, AB and AC would be chords of this circle;

$\therefore$ DF, which bisects AB perpendicularly, would pass through the centre. *III.* 1, *Cor.* 1

Similarly EF, which bisects AC perpendicularly, would pass through the centre. *III.* 1, *Cor.* 1

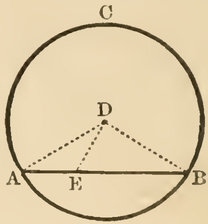
Hence F will be the centre, and FA, FB, or FC the radius.

1. Show how, by twice applying Cor. 1, to find the centre of a given circle.
2. Similarly, show how to find the centre of a circle, an arc only of which is given.
3. Describe a circle to pass through three given points. When is this impossible?
4. Describe a circle to pass through two given points, and have its centre in a given straight line. When is this impossible?
5. Describe a circle to pass through two given points, and have its radius equal to a given straight line. When is this impossible?
6. A quadrilateral has its vertices situated on the ○ce of a circle. Prove that the straight lines which bisect the sides perpendicularly are concurrent.
7. From a point outside a circle two equal straight lines are drawn to the ○ce. Prove that the bisector of the angle they contain passes through the centre of the circle.

8. Show also that the same thing is true when the point is taken either within the circle or on the ○ce.
9. Hence give another method of finding the centre of a given circle.

PROPOSITION 2. Theorem.

*If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.**



Let ABC be a circle, A and B any two points in the ○ce: *it is required to prove that AB shall fall within the circle.*

Find D the centre of the ⊙ ABC; *III.* 1
take any point E in AB, and join DA, DE, DB.

Because $DA = DB$, $\therefore \angle A = \angle B$. *I.* 5
But $\angle DEB$ is greater than $\angle A$; *I.* 16
$\therefore$ $\angle DEB$ is greater than $\angle B$;
$\therefore$ DB is greater than DE. *I.* 19
Now since DE drawn from the centre of the ⊙ ABC is less than a radius, E must be within the circle. *III. Def.* 1, *Cor.* 1
But E is any point in AB, except the end points A and B;
$\therefore$ AB itself is within the circle.

1. Prove that a straight line cannot cut the ○ce of a circle in more than two points.

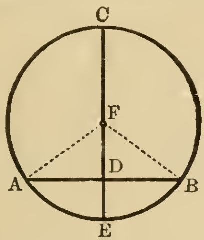
* Euclid's proof is indirect. The one in the text is found in *Clavii Commentaria in Euclidis Elementa* (1612), p. 109.

2. Describe a circle whose ○ce shall pass through a given point, whose centre shall be in one given straight line, and whose radius shall be equal to another given straight line. May more than one circle be so drawn? If so, how many? When will there be only one, and when none at all?

PROPOSITION 3. Theorems.

If a straight line drawn through the centre of a circle bisect a chord which does not pass through the centre, it shall cut it at right angles.

Conversely: If it cut it at right angles, it shall bisect it.



(1) Let ABC be a circle, F its centre; and let CE, which passes through F, bisect the chord AB which does not pass through F:

it is required to prove $CE \perp AB$.

Join FA, FB.

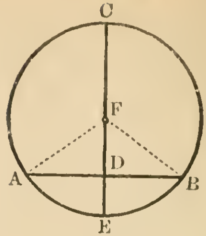
In $\triangle$s ADF, BDF, $\left\{\begin{array}{l} AD = BD \\ DF = DF \\ FA = FB; \end{array}\right.$ — *Hyp.*; ; *III. Def.* 1

$\therefore \angle ADF = \angle BDF$; — *I.* 8

$\therefore CE$ is $\perp AB$. — *I. Def.* 10

(2) In $\odot$ ABC let CE be $\perp AB$:

it is required to prove $AD = BD$.



Join FA, FB.

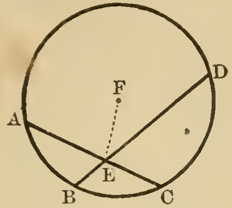
$$\text{In } \triangle\text{s } ADF,\ BDF, \left\{\begin{array}{ll} \angle ADF = \angle BDF & \textit{Hyp.} \\ \angle FAD = \angle FBD & \textit{I. } 5 \\ DF = DF; & \end{array}\right.$$

$\therefore AD = BD.$ *I.* 26

1. In the figure to the proposition, C and E are on the ○ce. Need they be so?
2. The ○ce of a circle passes through the vertices of a triangle. Prove that the straight lines drawn from the centre of the circle perpendicular to the sides will bisect those sides.
3. Two concentric circles intercept between their ○ces two equal portions of a straight line cutting them both.
4. Through a given point within a circle draw a chord which shall be bisected at that point.
5. If two chords in a circle be parallel, their middle points will lie on the same diameter.
6. Hence give a method of finding the centre of a given circle.
7. If the vertex of an isosceles triangle be taken as centre, and a circle be described cutting the base or the base produced, the segments of the base intercepted between the ○ce and the ends of the base will be equal.
8. If two circles cut each other, any two parallel straight lines drawn through the points of intersection to the ○ces will be equal.
9. If two circles cut each other, any two straight lines drawn through one of the points of intersection to the ○ces and making equal angles with the line of centres will be equal.

PROPOSITION 4. THEOREM.

If two chords of a circle cut one another and do not both pass through the centre, they do not bisect one another.



Let ABC be a circle, AC, BD two chords which cut one another at E, but do not both pass through the centre: *it is required to prove that AC, BD do not bisect one another.*

(1) If one of them pass through the centre, it may bisect the other which does not pass through the centre; but it cannot be itself bisected by that other.

(2) If neither of them pass through the centre, let $AE = EC$, and $BE = ED$.

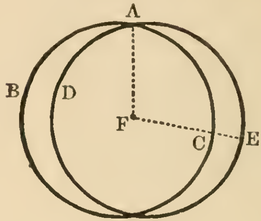
Find F the centre of $\odot$ ABC, *III.* 1
and join FE.

Because FE passes through the centre, and bisects AC,
$\therefore$ $\angle$ FEA is right. *III.* 3
Because FE passes through the centre, and bisects BD,
$\therefore$ $\angle$ FEB is right; *III.* 3
$\therefore$ $\angle$ FEA = $\angle$ FEB, which is impossible.
$\therefore$ AC, BD do not bisect one another.

1. If two chords of a circle bisect each other, what must both of them be?
2. No $\|^m$ whose diagonals are unequal can have its vertices on the $\bigcirc^{ce}$ of a circle.
3. No $\|^m$ except a rectangle can have its vertices on the $\bigcirc^{ce}$ of a circle.

PROPOSITION 5. Theorem.

If two circles cut one another, they cannot have the same centre.



Let the ⊙s ABC, ADE cut one another at A:
it is required to prove that they cannot have the same centre.

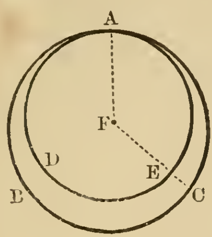
If they can, let F be the common centre.
Join FA, and draw any other straight line FCE to meet the two ○ces.

Then $FA = FC$, being radii of ⊙ ABC, *III. Def.* 1
and $FA = FE$, being radii of ⊙ ADE; *III. Def.* 1
∴ $FC = FE$, which is impossible.
∴ ⊙s ABC, ADE cannot have the same centre.

1. If two circles do not cut one another, can they have the same centre?
2. If two circles cut one another, can their common chord be a diameter of either of them? Can it be a diameter of both?
3. If the common chord of two intersecting circles is the diameter of one of them, prove that it is ⊥ the straight line joining the centres.
4. If two circles cut one another, the distance between their centres is less than the sum, and greater than the difference of their radii.
5. Prove the converse of the preceding deduction.

PROPOSITION 6. Theorem.

If two circles touch one another internally, they cannot have the same centre.



Let the ⊙s ABC, ADE touch one another internally at A: *it is required to prove that they cannot have the same centre.*

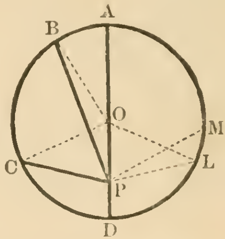
If they can, let F be the common centre.
Join FA; and draw any other straight line FEC to meet the two ○ces.

Then $FA = FC$, being radii of ⊙ ABC, *III. Def.* 1
and $FA = FE$, being radii of ⊙ ADE; *III. Def.* 1
∴ $FC = FE$, which is impossible.
∴ ⊙s ABC, ADE cannot have the same centre.

1. If two circles touch one another externally, can they have the same centre?
2. Enunciate III. 5, 6, and the preceding deduction in one statement.
3. If one circle be inside another, and do not touch it, the distance between their centres is less than the difference of their radii.
4. If one circle be outside another and do not touch it, the distance between their centres is greater than the sum of their radii.
5. Prove the converses of the two preceding deductions.

PROPOSITION 7. THEOREM.

If from any point within a circle which is not the centre, straight lines be drawn to the circumference, the greatest is that which passes through the centre, and the remaining part of that diameter is the least; of the others, that which is nearer to the greatest is greater than the more remote; and from the given point straight lines which are equal to one another can be drawn to the circumference only in pairs, one on each side of the diameter.



Let ABC be a circle, and P any point within it which is not the centre; from P let there be drawn to the $\bigcirc^{ce}$ DPA, PB, PC, of which DPA passes through the centre O:

it is required to prove (1) *that PA is greater than PB;*
(2) *that PB is greater than PC;*
(3) *that PD is less than PC;*
(4) *that only one straight line can be drawn from P to the $\bigcirc^{ce}$ $= PC$*

Join OB, OC.

(1) Because $OB = OA$, being radii of the same circle;

$\therefore$ $PO + OB = PO + OA$, or PA.

But $PO + OB$ is greater than PB; *I.* 20

$\therefore$ PA is greater than PB.

(2) In $\triangle$s POB, POC, $\begin{cases} PO = PO \\ OB = OC & \textit{III. Def. 1} \\ \angle POB \text{ is greater than } \angle POC; \end{cases}$

$\therefore PB$ is greater than PC. *I.* 24

(3) Because $OC - OP$ is less than PC, *I.* 20, *Cor.*
and $OC = OD$, being radii of the same circle;
$\therefore OD - OP$ is less than PC;
$\therefore PD$ is less than PC.

(4) At O make $\angle POL = \angle POC$, *I.* 23
and join PL.

In $\triangle$s POL, POC, $\begin{cases} PO = PO \\ OL = OC & \textit{III. Def. 1} \\ \angle POL = \angle POC; & \textit{Const.} \end{cases}$

$\therefore PL = PC$. *I.* 4

And besides PL no other straight line can be drawn from P to the $\bigcirc^{ce} = PC$.
For if PM were also $= PC$,
then $PM = PL$, which is impossible.

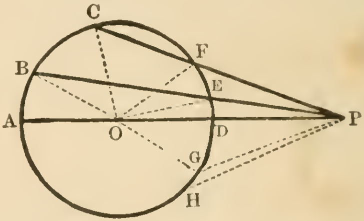
Cor.—If from a point inside a circle more than two equal straight lines can be drawn to the $\bigcirc^{ce}$, that point must be the centre.

For another proof of this Cor., see III. 9.

1. Prove PC greater than PD, using I. 20 instead of I. 20, Cor.
2. Wherever the point P be taken, provided it be inside the circle ABC, the sum of the greatest and the least straight lines that can be drawn from it to the $\bigcirc^{ce}$ is constant.
3. Find another point whose greatest and least distances from the $\bigcirc^{ce}$ are respectively = those of P from the $\bigcirc^{ce}$. How many such points are there? Where do they lie?
4. Prove, by considering POA and POD as infinitely thin triangles, that PA is greater than PB, and PC greater than PD by I. 24.

PROPOSITION 8. Theorem.

If from any point without a circle straight lines be drawn to the circumference, of those which fall upon the concave part of the circumference the greatest is that which passes through the centre, and of the others that which is nearer to the greatest is greater than the more remote: but of those which fall on the convex part of the circumference the least is that which, when produced, passes through the centre, and of the others that which is nearer to the least is less than the more remote; and from the given point straight lines which are equal to one another can be drawn to the circumference only in pairs, one on each side of the diameter.



Let ABC be a circle, and P any point without it; from P let there be drawn to the $\bigcirc^{ce}$ PDA, PEB, PFC, of which PDA passes through the centre O:

it is required to prove (1) *that PA is greater than PB;*

(2) *that PB is greater than PC;*

(3) *that PD is less than PE;*

(4) *that PE is less than PF;*

(5) *that only one straight line can be drawn from P to the $\bigcirc^{ce} = PF$.*

Join OB, OC, OE, OF.

(1) Because $OB = OA$, being radii of the same circle;

$\therefore$ $PO + OB = PO + OA$, or PA.

But $PO + OB$ is greater than PB; *I.* 20

$\therefore$ PA is greater than PB.

(2) In $\triangle$s POB, POC, $\begin{cases} PO = PO \\ OB = OC & \textit{III. Def. } 1 \\ \angle\ POB \text{ is greater than } \angle\ POC, \end{cases}$

$\therefore$ PB is greater than PC. *I.* 24

(3) Because $OP - OE$ is less than PE, *I.* 20, *Cor.*
and $OE = OD$, being radii of the same circle;
$\therefore$ $OP - OD$ is less than PE;
$\therefore$ PD is less than PE.

(4) In $\triangle$s POE, POF, $\begin{cases} PO = PO \\ OE = OF & \textit{III. Def. } 1 \\ \angle\ POE \text{ is less than } \angle\ POF; \end{cases}$

$\therefore$ PE is less than PF. *I.* 24

(5) At O make $\angle\ POG = \angle\ POF$, *I.* 23
and join PG.

In $\triangle$s POG, POF, $\begin{cases} PO = PO \\ OG = OF & \textit{III. Def. } 1 \\ \angle\ POG = \angle\ POF; & \textit{Const.} \end{cases}$

$\therefore$ $PG = PF$. *I.* 4

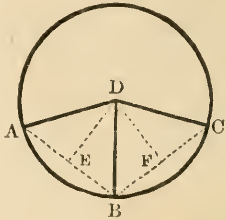
And besides PG no other straight line can be drawn from P to the ○ce $= PF$.
For if PH were also $= PF$,
then $PH = PG$, which is impossible.

1. Prove PE greater than PD, using I. 20 instead of I. 20, Cor.
2. Prove that PE is less than PF, using I. 21 instead of I. 24.
3. Wherever the point P be taken, provided it be outside the circle ABC, the difference of the greatest and the least straight lines that can be drawn from it to the ○ce is constant.
4. Compare the enunciations of the last deduction and of the analogous one from III. 7, and state and prove the corresponding theorem when the point P is on the ○ce of the ⊙ ABC.
5. Prove that AD is greater than BE, and BE greater than CF.
6. If the straight line PFC be supposed to revolve round P as a pivot, till the points F and C coincide, what would the straight line PFC become?
7. The tangent to a circle from any external point is less than any secant to the circle from that point, and greater than the external segment of the secant.

8. Could a line be drawn to separate the concave from the convex part of the $\bigcirc^{ce}$ of the $\odot$ ABC viewed from the point P? How?

PROPOSITION 9. THEOREM.

*If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre.**



Let ABC be a circle, and let three equal straight lines DA, DB, DC be drawn from the point D to the $\bigcirc^{ce}$:
it is required to prove that D is the centre of the circle.

Join AB, BC, and bisect them at E, F; *I.* 10
and join DE, DF.

$$\text{In } \triangle\text{s } AED,\ BED, \left\{\begin{array}{l} AE = BE \\ ED = ED \\ DA = DB; \end{array}\right. \qquad \begin{array}{r} \textit{Const.} \\ \\ \textit{Hyp.} \end{array}$$

$\therefore\ \angle AED = \angle BED$; *I.* 8

$\therefore\ DE$ is $\perp AB$;

$\therefore\ DE$, since it bisects AB perpendicularly, must pass through the centre of the circle. *III.* 1, *Cor.* 1

Hence also DF must pass through the centre;

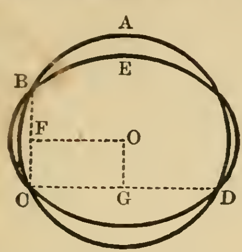
$\therefore\ D$, the only point common to DE and DF, is the centre.

Prove the proposition by using the eighth deduction from III. 1.

* In the MSS. of Euclid, two proofs of this proposition occur, only the second of which Simson inserted in his edition. The one given in the text is the first.

PROPOSITION 10. Theorem.

*One circle cannot cut another at more than two points.**



If it be possible, let the ⊙ ABC cut the ⊙ EBC at more than two points—namely, at B, C, D.

Join BC, CD, and bisect them at F and G; *I.* 10
through F and G draw FO, $GO \perp BC$, CD, *I.* 11
and let FO, GO intersect at O.

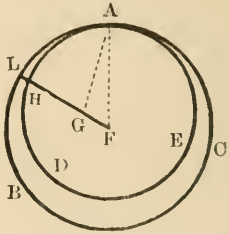
Because BC is a chord in both circles, and FO bisects it perpendicularly,
∴ the centres of both circles lie in FO. *III.* 1, *Cor.* 1
Hence also the centres of both circles lie in GO;
∴ O is the centre of both circles,
which is impossible, since they cut one another. *III.* 5
∴ one circle cannot cut another at more than two points.

1. Two circles cannot meet each other in more than two points.
2. If two circles have three points in common, how must they be situated?
3. Show, by supposing the radius of one of the circles to increase indefinitely in length, that the first deduction from III. 2 is a particular case of this proposition.

* In the MSS. of Euclid, two proofs of this proposition occur, only the second of which Simson inserted in his edition. The one given in the text is the first.

PROPOSITION 11. Theorem.

If two circles touch one another internally at any point, the straight line which joins their centres, being produced, shall pass through that point.



Let the two $\odot$s ABC, ADE, whose centres are F and G, touch one another internally at the point A:
it is required to prove that FG produced passes through A.

If not, let it pass otherwise, as $FGHL$.
Join FA, GA.

Because $FA = FL$, being radii of $\odot$ ABC, *III. Def.* 1
and $GA = GH$, being radii of $\odot$ ADE; *III. Def.* 1

$\therefore FA - GA = FL - GH,$
$= FG + HL;$

$\therefore$ $FA - GA$ is greater than FG by HL.
But $FA - GA$ is less than FG; *I.* 20, *Cor.*

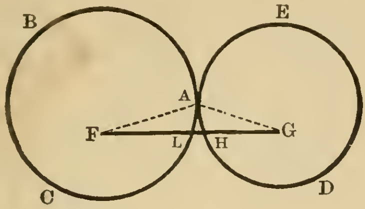
$\therefore$ $FA - GA$ is both greater and less than FG, which is impossible;

$\therefore$ FG produced must pass through A.

1. If two circles touch internally, the distance between their centres is equal to the difference of their radii.
2. Two circles touch internally at a point, and through that point a straight line is drawn to cut the $\bigcirc^{ces}$ of the two circles. If the points of intersection be joined with the respective centres, the two straight lines will be parallel.
3. This proposition is a particular case of the tenth deduction from I. 8.

PROPOSITION 12. Theorem.

If two circles touch one another externally at any point, the straight line which joins their centres shall pass through that point.



Let the two ⊙s ABC, ADE, whose centres are F and G, touch one another externally at the point A:

it is required to prove that FG passes through A.

If not, let it pass otherwise, as $FLHG$.

Join FA, GA.

Because $FA = FL$, being radii of ⊙ ABC, *III. Def.* 1

and $GA = GH$, being radii of ⊙ ADE; *III. Def.* 1

$\therefore FA + GA = FL + GH,$

$= FG - HL;$

$\therefore FA + GA$ is less than FG by HL.

But $FA + GA$ is greater than FG; *I.* 20

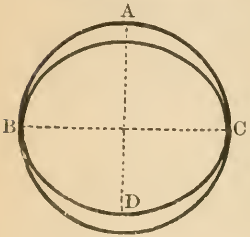
$\therefore FA + GA$ is both less and greater than FG, which is impossible;

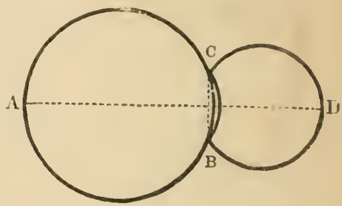
$\therefore FG$ must pass through A.

1. If two circles touch externally, the distance between their centres is equal to the sum of their radii.
2. Two circles touch externally at a point, and through that point a straight line is drawn to cut the ○^{ces} of the two circles. If the points of intersection be joined with the respective centres, the two straight lines will be parallel.
3. This proposition is a particular case of the tenth deduction from I. 8.

PROPOSITION 13. Theorem.

Two circles cannot touch each other at more points than one, whether internally or externally.





For, if it be possible, let the two ⊙s ABC, BDC touch each other at the points B and C.

Join BC, and draw AD bisecting BC perpendicularly. *I.* 10, 11

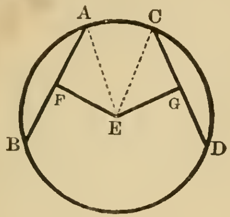
Because B and C are points in the ⊙ces of both circles,
∴ BC is a chord of both circles.
And because AD bisects BC perpendicularly, *Const.*
∴ AD passes through the centres of both circles; *III.* 1, *Cor.* 1.
∴ AD passes also through the points of contact B and C, *III.* 11, 12
which is impossible.
Hence the two ⊙s ABC, BDC cannot touch each other at more points than one, whether internally or externally.

1. If the distance between the centres of two circles be equal to the sum of their radii, the two circles touch each other externally.
2. If the distance between the centres of two circles be equal to the difference of their radii, the two circles touch each other internally.

PROPOSITION 14. Theorems.

Equal chords in a circle are equidistant from the centre.

Conversely: Chords in a circle which are equidistant from the centre are equal.



(1) Let AB, CD be equal chords in the $\odot$ ABC, and EF, EG their distances from the centre E:
it is required to prove $EF = EG$.

Join EA, EC.

Because EF drawn through the centre E is $\perp$ AB,
$\therefore$ EF bisects AB, that is, AB is double of AF. *III. 3*
Hence also CD is double of CG.
Now since $AB = CD$, $\therefore$ $AF = CG$, and $AF^2 = CG^2$.
But because $EA = EC$, $\therefore$ $EA^2 = EC^2$;
$\therefore$ $AF^2 + FE^2 = CG^2 + GE^2$. *I. 47*
Take away AF^2 and CG^2 which are equal;
$\therefore$ $FE^2 = GE^2$, and $FE = GE$.

(2) Let AB, CD be chords in the $\odot$ ABC, and let EF, EG, their distances from the centre E, be equal:
it is required to prove $AB = CD$.

Join EA, EC.

It may be proved as before that $AB = 2\,AF$, $CD = 2\,CG$, and that $AF^2 + FE^2 = CG^2 + GE^2$.

Now $FE^2 = GE^2$, since $FE = GE$; *Hyp.*

$\therefore$ $AF^2 = CG^2$, and $AF = CG$;

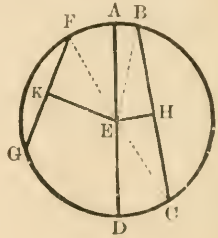
$\therefore$ $2\,AF = 2\,CG$, that is, $AB = CD$.

1. If a series of equal chords be placed in a circle, their middle points will lie on the $\bigcirc^{ce}$ of another circle.
2. Two parallel chords in a circle whose diameter is 10 inches, are 8 inches and 6 inches; find the distance between them.
3. If two chords of a circle intersect each other and make equal angles with the diameter drawn through their point of intersection, they are equal.
4. If two secants of a circle intersect, and make equal angles with the diameter drawn through their point of intersection, those parts of the secants intercepted by the $\bigcirc^{ce}$ are equal.
5. If in a given circle a chord of given length be placed, the distance of the chord from the centre will be fixed.
6. Prove the converse of the preceding deduction.
7. If two equal chords intersect either within or without a circle, the segments of the one are equal to the segments of the other.

PROPOSITION 15. Theorems.

The diameter is the greatest chord in a circle; and of all others that which is nearer to the centre is greater than one more remote.

Conversely: The greater chord is nearer to the centre than the less.



Let ABC be a circle of which AD is a diameter, and BC, FG two other chords whose distances from the centre E are EH, EK:

it is required to prove:

(1) *that AD is greater than BC or FG;*

(2) *that, if EH is less than EK, BC must be greater than FG;*

(3) *that, if BC is greater than FG, EH must be less than EK.*

(1) Join EB, EC.

Because $AE = BE$, and $ED = EC$; *III. Def.* 1

$\therefore AD = BE + EC$.

But $BE + EC$ is greater than BC; *I.* 20

$\therefore AD$ is greater than BC.

(2) Join EB, EC, EF.

It may be proved, as in the preceding proposition, that BC is double of BH, that FG is double of FK, and that $EH^2 + HB^2 = EK^2 + KF^2$.

Now EH^2 is less than EK^2, since EH is less than EK; *Hyp.*

$\therefore HB^2$ is greater than KF^2, and HB greater than KF.

$\therefore$ twice HB is greater than twice KF,

that is, BC is greater than FG.

(3) Join EB, EC, EF.

It may be proved, as before, that $BC = 2\,BH$, $FG = 2\,FK$, and that $EH^2 + HB^2 = EK^2 + KF^2$.

Now, since BC is greater than FG, *Hyp.*

$\therefore BH$ is greater than FK, and BH^2 greater than FK^2.

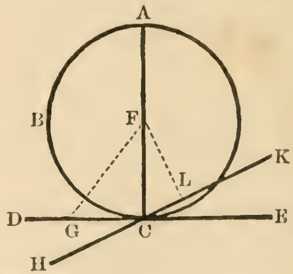
Hence EH^2 must be less than EK^2, and EH less than EK

1. The shortest chord that can be drawn through a given point within a circle is that which is perpendicular to the diameter through the point.

2. Of two chords of a circle which intersect each other, and make unequal angles with the diameter drawn through their point of intersection, that which makes the less angle is the greater.
3. If two secants of a circle intersect each other, and make unequal angles with the diameter drawn through their point of intersection, that part which is intercepted by the ○ce on the secant making the less angle is greater than the corresponding part on the other.
4. Through either of the points of intersection of two circles draw the greatest possible straight line terminated both ways by the ○ces. Draw also the least possible, and show that the two are at right angles to each other.

PROPOSITION 16. THEOREM.

*The straight line drawn perpendicular to a diameter of a circle from either end of it, is a tangent to the circle; and every other straight line drawn through the same point cuts the circle.**



* Euclid's proof of this proposition is indirect. The one in the text is given by Orontius Finæus (1544), the second part, however, being somewhat simplified.

Let ABC be a circle, of which F is the centre and AC a diameter; through C let there be drawn $DE \perp AC$, and any other straight line HK:
it is required to prove that DE is a tangent to the ⊙ *ABC, and that HK cuts the circle.*

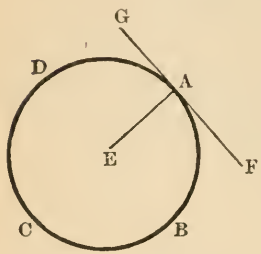
Take any point G in DE, and join FG;
from F draw $FL \perp HK$. *I.* 12

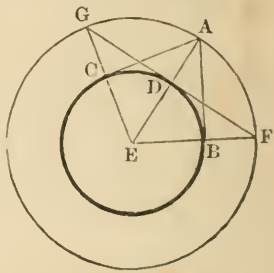
Because $\angle\ FCG$ is right, *Hyp.*
$\therefore$ FG is greater than FC, a radius of the circle; *I.* 19 *Cor.*
$\therefore$ the point G must be outside the circle. *III. Def.* 1, *Cor.* 2
Now G is any point in DE, except the point C;
$\therefore$ DE is a tangent to the circle. *III. Def.* 3
Again, because $\angle\ FLC$ is right, *Const.*
$\therefore$ FL is less than FC, a radius of the circle; *I.* 19 *Cor.*
$\therefore$ the point L must be inside the circle. *III. Def.* 1, *Cor.* 2
Now L is a point in HK;
$\therefore$ HK cuts the circle.

1. Draw a tangent to a circle at a given point on the ○ce.
2. Only one tangent can be drawn to a circle at a given point on its ○ce.
3. Two (or a series of) circles touch each other, externally or internally, at the same point. Prove that they have the same tangent at that point.
4. If a series of equal chords be placed in a circle, they will be tangents to another circle concentric with the former.
5. A straight line will cut, touch, or lie entirely outside a circle, according as its distance from the centre is less than, equal to, or greater than a radius.
6. Draw a tangent to a circle which shall be ∥ a given straight line.
7. Draw a tangent to a circle which shall be ⊥ a given straight line.
8. Draw a tangent to a circle which shall make a given angle with a given straight line. How many tangents can be drawn in each of the three cases?

PROPOSITION 17. Problem.

To draw a tangent to a circle from a given point.





Let BDC be the given circle, and A the given point:
it is required to draw a tangent to the ⊙ BDC from A.

Case 1.—When the given point A is inside the ⊙ BDC, the problem is impossible.

Case 2.—When the given point A is on the ○^{ce} of the ⊙ BDC.

Find E the centre of the ⊙ BDC; *III.* 1
join EA, and through A draw $FG \perp EA$. *I.* 11
Then FG is a tangent to the ⊙ BDC. *III.* 16

Case 3.—When the given point A is outside the ⊙ BDC.

Find E the centre of the ⊙ BDC; *III.* 1
and join AE, cutting the ○^{ce} of ⊙ BDC at D.
With centre E and radius EA, describe ⊙ AGF;
through D draw $FDG \perp AE$, and meeting the ○^{ce} of ⊙ AGF at F and G. *I.* 11
Join EF, EG, cutting the ○^{ce} of ⊙ BDC at B and C,
and join AB, AC. AB or AC is the required tangent.

In $\triangle$s ABE, FDE, $\left\{\begin{array}{ll} AE = FE & \text{III. Def. 1} \\ EB = ED & \text{III. Def. 1} \\ \angle E = \angle E; & \end{array}\right.$

$\therefore \angle ABE = \angle FDE$, *I.* 4

$= $ a right angle. *Const.*

$\therefore AB$ is a tangent to the $\odot$ BDC. *III.* 16

Hence also, AC is a tangent to the $\odot$ BDC.

Cor.—The two tangents that can be drawn to a circle from an external point are equal.

By comparing $\triangle$s ABE, FDE it may be proved that $AB = FD$; *I.* 4

and by comparing $\triangle$s ACE, GDE, it may be proved that $AC = GD$. *I.* 4

Now, since FG is a chord of the $\odot$ AFG, and FD drawn through the centre is $\perp FG$; *Const.*

$\therefore FD = GD$. *III.* 3

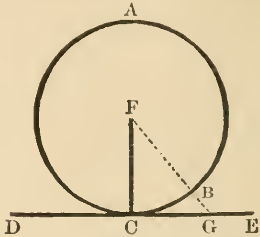
Hence $AB = AC$.

1. Prove $AB = AC$ by (*a*) I. 47, (*b*) I. 5, 6.
2. The tangents AB, AC make equal angles with the diameter through A.
3. Prove $\angle BAC$ supplementary to $\angle BEC$. State this result in words.
4. No more than two tangents can be drawn to a circle from an external point.
5. If a quadrilateral be circumscribed* about a circle, the sum of two opposite sides is equal to the sum of the other two.
6. Generalise the preceding deduction.
7. If a $\|^m$ be circumscribed about a circle, it must be a rhombus.
8. From a point outside a circle two tangents are drawn. The straight line joining the point with the centre bisects perpendicularly the chord of contact. (In fig. 2, BC is the chord of contact.)

* A figure is circumscribed about a circle when its sides touch the circle.

PROPOSITION 18. Theorem.

The radius of a circle drawn to the point of contact of a tangent is perpendicular to the tangent.



Let ABC be a circle whose centre is F, and DE a tangent to it at the point C:
it is required to prove that the radius FC is $\perp$ DE.

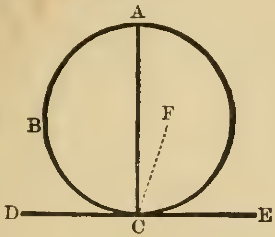
If not, from F draw $FG \perp DE$, and meeting the $\bigcirc^{ce}$ at B. *I.* 12

Because $\angle\ FGC$ is a right angle, *Const.*
$\therefore$ FG is less than FC. *I.* 19 *Cor.*
But $FC = FB$; *III. Def.* 1
$\therefore$ FG is less than FB,
which is impossible;
$\therefore$ FC must be $\perp$ DE.

1. Tangents at the ends of a diameter of a circle are parallel.
2. If a series of chords in a circle be tangents to another concentric circle, the chords are all equal.
3. If two circles be concentric, and a chord of the greater be a tangent to the less, it is bisected at the point of contact.
4. Through a given point within a circle draw a chord which shall be equal to a given length. May the given point be outside the circle? What are the limits to the given length?
5. Deduce this proposition from I. 5, by supposing the tangent DE to be at first a secant.
6. Two circles, whose centres are A and B, have a common tangent CD; prove $AC \parallel BD$.

PROPOSITION 19. THEOREM.

The straight line drawn from the point of contact of a tangent to a circle perpendicular to the tangent passes through the centre of the circle.



Let DE be a tangent to the ⊙ ABC at the point C, and let CA be $\perp$ DE:

it is required to prove that CA passes through the centre.

If not, let F be the centre,
and join FC.

Then $\angle$ FCE is right. *III.* 18
But $\angle$ ACE is right; *Hyp.*
$\therefore$ $\angle$ FCE = $\angle$ ACE, which is impossible;
$\therefore$ CA must pass through the centre of the circle.

1. In the figure, A is on the ○ᶜᵉ. Need it be so?
2. This proposition is a particular case of III. 1, Cor. 1.
3. A series of circles touch a given straight line at a given po. Where will their centres all lie?
4. Describe a circle to touch two given straight lines at two given points. When is this problem possible?
5. If two tangents be drawn to a circle from any point, the angle contained by the tangents is double the angle contained by the chord of contact and the diameter drawn through either point of contact.

PROPOSITION 20. THEOREM.

An angle at the centre of a circle is double of an angle at the circumference which stands on the same arc.

Fig. 1.

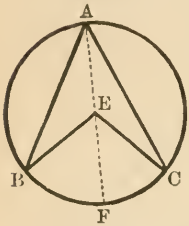


Fig. 2.

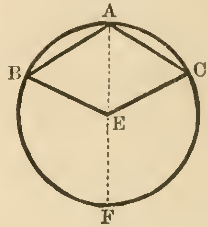
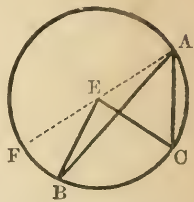


Fig. 3.



In the ⊙ ABC let $\angle\ BEC$ at the centre and $\angle\ BAC$ at the ○ᶜᵉ stand on the same arc BC:

it is required to prove $\angle\ BEC$ = *twice* $\angle\ BAC$.

Join AE and produce it to F.

Because $EA = EC$, $\therefore\ \angle\ EAC = \angle\ ECA$; *I.* 5

$\therefore\ \angle\ EAC + \angle\ ECA$ = twice $\angle\ EAC$.

But $\angle\ FEC = \angle\ EAC + \angle\ ECA$; *I.* 32

$\therefore\ \angle\ FEC$ = twice $\angle\ EAC$.

Similarly $\angle\ FEB$ = twice $\angle\ EAB$.

Hence, in figs. 1 and 2,

$\angle\ FEC + \angle\ FEB$ = twice $\angle\ EAC$ + twice $\angle\ EAB$,

that is, $\angle\ BEC$ = twice $\angle\ BAC$;

and in fig. 3,

$\angle\ FEC - \angle\ FEB$ = twice $\angle\ EAC$ − twice $\angle\ EAB$,

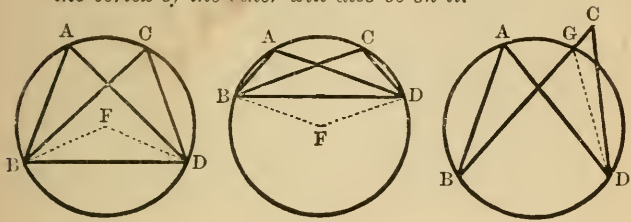
that is, $\angle\ BEC$ = twice $\angle\ BAC$.

1. In the figures to the proposition, F is on the ○ᶜᵉ. Need it be so?
2. The angle in a semicircle is a right angle.
3. B and C are two fixed points in the ○ᶜᵉ of the circle ABC. Prove that wherever A be taken on the arc BAC, the magnitude of the angle BAC is constant.

PROPOSITION 21. Theorems.

Angles in the same segment of a circle are equal.

*Conversely: If two equal angles stand on the same arc, and the vertex of one of them be on the conjugate arc, the vertex of the other will also be on it.**



(1) Let ABD be a circle, and $\angle$s A and C in the same segment BCD:

it is required to prove $\angle\ A = \angle\ C$.

Find F the centre of the $\odot\ ABD$, *III.* 1

and join BF, DF.

Then $\angle\ BFD$ = twice $\angle\ A$, *III.* 20

and $\angle\ BFD$ = twice $\angle\ C$; *III.* 20

$\therefore$ $\angle\ A = \angle\ C$.

(2) Let $\angle$s A and C, which are equal, stand on the same arc BD, and let the vertex A be on the conjugate arc BAD:

it is required to prove that the vertex C will also be on it.

If not, let the arc BAD cut BC or BC produced at G; join DG.

Then $\angle\ A = \angle\ BGD$. *III.* 21

But $\angle\ A = \angle\ C$; *Hyp.*

$\therefore$ $\angle\ BGD = \angle\ C$, which is impossible. *I* 16

Hence C must be on the circle which passes through B, A, D.

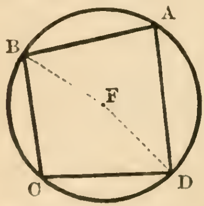
* The second part of this proposition is not given by Euclid.

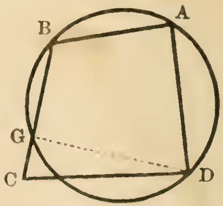
1. In the figure to III. 4, if AB, CD be joined, $\triangle\ AEB$ is equiangular to $\triangle\ DEC$.
2. If from a point E outside a circle, two secants ECA, EBD be drawn, and AB, CD be joined, $\triangle\ AEB$ is equiangular to $\triangle\ DEC$.
3. Given three points on the $\bigcirc^{ce}$ of a circle; find any number of other points on the $\bigcirc^{ce}$ without knowing the centre.
4. Two tangents AB, AC are drawn to a circle from an external point A; D is any point on the $\bigcirc^{ce}$ outside the $\triangle\ ABC$. Show that the sum of $\angle$s ABD, ACD is constant.
5. Is the last theorem true when D lies elsewhere on the $\bigcirc^{ce}$?
6. Segments of two circles stand upon a common chord AB. Through C, any point in one segment, are drawn the straight lines ACE, BCD meeting the other segment in E, D. Prove that the length of the arc DE is invariable wherever the point C be taken.

PROPOSITION 22. THEOREMS.

The opposite angles of a quadrilateral inscribed in a circle are supplementary.

*Conversely: If the opposite angles of a quadrilateral be supplementary, a circle may be circumscribed about the quadrilateral.**





(1) Let the quadrilateral $ABCD$ be inscribed in the $\odot\ ABC$: *it is required to prove that* $\angle\ A + \angle\ C = 2$ *rt.* $\angle$*s.*

Find F the centre of the $\odot\ ABD$, *III.* 1

and join BF, DF.

* The second part of this proposition is not given by Euclid, and he proves the first part by joining AC, BD.

Then $\angle BFD$ = twice $\angle A$, *III.* 20
and the reflex $\angle BFD$ = twice $\angle C$; *III.* 20
$\therefore$ the sum of the two conjugate $\angle$s BFD
= twice $\angle A$ + twice $\angle C$.
But the sum of the two conjugate $\angle$s BFD
= 4 rt. $\angle$s; *III. Def.* 23
$\therefore$ $\angle A + \angle C$ = 2 rt. $\angle$s.

(2) Let $\angle$s A and C, which are supplementary, be opposite angles of the quadrilateral $ABCD$,
and the vertex A be on an arc BAD which passes also through B and D:
it is required to prove that the vertex C will be on the conjugate arc.

If not, let the arc conjugate to BAD cut BC or BC produced at G; *III.* 1, *Cor.* 2
join DG.

Then $\angle A$ is supplementary to $\angle BGD$. *III.* 22
But $\angle A$ is supplementary to $\angle C$; *Hyp.*
$\therefore$ $\angle BGD = \angle C$, which is impossible. *I.* 16
Hence C must be on the circle which passes through B, A, D.

Cor.—If one side of a quadrilateral inscribed in a circle be produced, the exterior angle is equal to the remote interior angle of the quadrilateral.

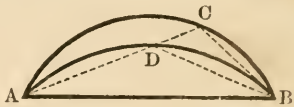
For each is supplementary to the interior adjacent angle. *I.* 13, *III.* 22

1. If a $\parallel^m$ be inscribed in a circle, it must be a rectangle.
2. If, from a point E outside a circle, two secants ECA, EBD be drawn, and AD, BC be joined, $\triangle AED$ is equiangular to $\triangle BEC$.
3. If a polygon of an even number of sides (a hexagon, for example) be inscribed in a circle, the sum of its alternate angles is half the sum of all its angles.
4. If an arc be divided into any two parts, the sum of the angles in the two segments is constant.

5. Divide a circle into two segments, such that the angle in the one segment shall be (*a*) twice, (*b*) thrice, (*c*) five times, (*d*) seven times the angle in the other segment.
6. ACB is a right-angled triangle, right-angled at C, and O is the point of intersection of the diagonals of the square described on AB outwardly to the triangle; prove that CO bisects $\angle ACB$.
7. What modification must be made on the last theorem when the square is described on AB inwardly to the triangle?
8. If two chords cut off one pair of similar segments from two circles, the other pair of segments they cut off are also similar.
9. Given three points on the ○ce of a circle: find any number of other points on the ○ce without knowing the centre.
10. ABC is a triangle; AX, BY, CZ are the three perpendiculars from the vertices on the opposite sides, intersecting at O. Prove the following sets of four points concyclic (that is, situated on the ○ce of a circle): A, Z, O, Y; B, X, O, Z; C, Y, O, X; A, B, X, Y; B, C, Y, Z; C, A, Z, X.

PROPOSITION 23. Theorem.

On the same chord and on the same side of it there cannot be two similar segments of circles not coinciding with one another.



If it be possible, on the same chord AB, and on the same side of it, let there be two similar segments of ⊙s ACB, ADB not coinciding with one another.

Draw any straight line ADC cutting the arcs of the segments at D and C;
and join BC, BD.

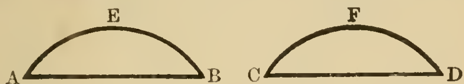
Because segment ADB is similar to segment ACB, *Hyp.*
$\therefore \angle ADB = \angle ACB$, *III. Def.* 13
which is impossible. *I.* 16

Hence two similar segments on the same chord and on the same side of it must coincide.

1. Of all the segments of circles on the same side of the same chord, that which is the greatest contains the least angle.
2. Prove by this proposition the second part of III. 21.

PROPOSITION 24. THEOREM.

Similar segments of circles on equal chords are equal.

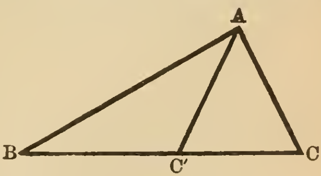


Let AEB, CFD be similar segments on equal chords AB, CD:

it is required to prove segment AEB = *segment* CFD.

If segment AEB be applied to segment CFD,
so that A falls on C, and so that AB falls on CD;
then B will coincide with D, because $AB = CD$. *Hyp.*
Hence the segment AEB being similar to the segment CFD, must coincide with it; *III.* 23
$\therefore$ segment AEB = segment CFD.

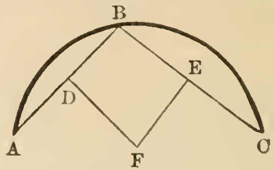
1. Similar segments of circles on equal chords are parts of equal circles.
2. ABC, ABC' are two $\triangle$s such that $AC = AC'$. Prove that the circle which passes through A, B, C is equal to the circle which passes through A, B, C'.



3. If $ABCD$ is a $\|^m$, and BE makes with AB, $\angle ABE = \angle BAD$, and meets DC produced in E, the circles described about $\triangle$s BCD, BED will be equal.

PROPOSITION 25. Problem.

An arc of a circle being given, to complete the circle.



Let ABC be the given arc of a circle:
it is required to complete the circle.

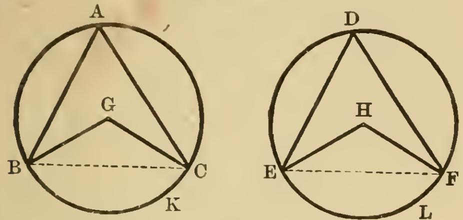
Take any point B in the arc, and join AB, BC.
Bisect AB and BC at D and E; *I.* 10
draw DF and EF respectively $\perp$ AB and BC, *I.* 11
and let them meet at F.

Because DF bisects the chord AB perpendicularly,
$\therefore$ DF passes through the centre. *III.* 1, *Cor.* 1
Hence also, EF passes through the centre;
$\therefore$ F is the centre.
Hence, with F as centre, and FA, FB, or FC as radius, the circle may be completed.

1. Prove that DF and EF must meet.
2. Prove the proposition with Euclid's construction, which is: Bisect the chord AC at D, draw $DB \perp AC$, meeting the arc at B, and join AB. At A make $\angle BAE = \angle ABD$, and let AE meet BD or BD produced at E. E shall be the centre.
3. Find a point equidistant from three given points. When is the problem impossible?
4. The straight lines bisecting perpendicularly the three sides of a triangle are concurrent.
5. Find a point equidistant from four given points. When is the problem possible?

PROPOSITION 26. Theorem.

In equal circles, or in the same circle, if two angles, whether at the centre or at the circumference, be equal, the arcs on which they stand are equal.



Let ABC, DEF be equal circles, and let $\angle$s G and H at the centres be equal, as also $\angle$s A and D at the $\bigcirc^{ces}$: *it is required to prove that arc BKC = arc ELF.*

Join BC, EF.

Because $\odot$s ABC, DEF are equal, *Hyp.*
$\therefore$ their radii are equal. *III. Def.* 1, *Cor.* 4

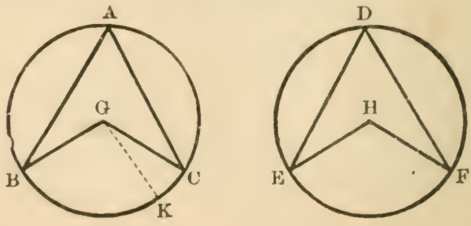
In $\triangle$s BGC, EHF, $\left\{\begin{array}{l} BG = EH \\ GC = HF \\ \angle G = \angle H; \end{array}\right.$ *Hyp.*
$\therefore BC = EF$. *I.* 4
But because $\angle A = \angle D$,
$\therefore$ segment BAC is similar to segment EDF; *III. Def.* 13
and they are on equal chords BC, EF,
$\therefore$ segment BAC = segment EDF. *III.* 24
Now $\odot ABC = \odot DEF$; *Hyp.*
$\therefore$ remaining segment BKC = remaining segment ELF;
$\therefore$ arc BKC = arc ELF.

Cor.—In equal circles, or in the same circle, those sectors are equal which have equal angles.

1. If AB and CD be two parallel chords in a circle $ACDB$, prove arc $AC =$ arc BD, and arc $AD =$ arc BC.
2. In equal circles, or in the same circle, if two angles, whether at the centre or at the $\bigcirc^{ce}$ be unequal, that which is the greater stands on the greater arc.
3. If two opposite angles of a quadrilateral inscribed in a circle be equal, the diagonal which does not join their vertices is a diameter of the circle.
4. Any segment of a circle containing a right angle is a semicircle.
5. Any segment of a circle containing an acute angle is greater than a semicircle, and one containing an obtuse angle is less than a semicircle.
6. If two angles at the $\bigcirc^{ce}$ of a circle are supplementary, the sum of the arcs on which they stand = the whole $\bigcirc^{ce}$.
7. Prove the proposition by superposition.
8. If two chords intersect within a circle, the angle they contain is equal to an angle at the centre standing on half the sum of the intercepted arcs.
9. If two chords produced intersect without a circle, the angle they contain is equal to an angle at the centre standing on half the difference of the intercepted arcs.
10. Show how to divide the $\bigcirc^{ce}$ of a circle into 3, 4, 6, 8 equal parts.

PROPOSITION 27. Theorem.

In equal circles, or in the same circle, if two arcs be equal, the angles, whether at the centre or at the circumference, which stand on them are equal.



Let ABC, DEF be equal circles, and let arc BC = arc EF: *it is required to prove that* $\angle\ BGC = \angle\ EHF$, *and* $\angle\ A = \angle\ D$.

If $\angle\ BGC$ be not = $\angle\ EHF$, one of them must be the greater.

Let $\angle\ BGC$ be the greater, and make $\angle\ BGK = \angle\ EHF$. *I.* 23

Because the circles are equal, and $\angle\ BGK = \angle\ EHF$,

$\therefore$ arc BK = arc EF. *III.* 26

But arc BC = arc EF; *Hyp.*

$\therefore$ arc BK = arc BC, which is impossible.

Hence $\angle\ BGC$ must be = $\angle\ EHF$.

Now, since $\angle\ A$ = half of $\angle\ BGC$, *III.* 20

and $\angle\ D$ = half of $\angle\ EHF$, *III.* 20

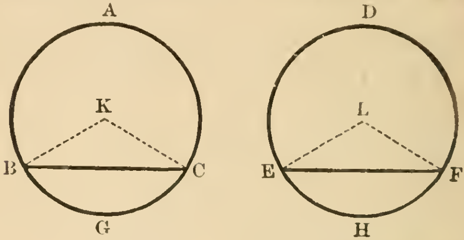
$\therefore \angle\ A = \angle\ D$.

Cor.—In equal circles, or in the same circle, those sectors are equal which have equal arcs.

1. If AC and BD be two equal arcs in a circle $ACDB$, prove chord $AB \parallel$ chord CD.
2. In equal circles, or in the same circle, if two arcs be unequal, that angle, whether at the centre or at the $\bigcirc^{ce}$, is the greater which stands on the greater arc.
3. The angle in a semicircle is a right angle.
4. The angle in a segment greater than a semicircle is less than a right angle, and the angle in a segment less than a semicircle is greater than a right angle.
5. If the sum of two arcs of a circle be equal to the whole $\bigcirc^{ce}$, the angles at the $\bigcirc^{ce}$ which stand on them are supplementary.
6. Prove the proposition by superposition.
7. Two circles touch each other internally, and a chord of the greater circle is a tangent to the less. Prove that the chord is divided at its point of contact into segments which subtend equal angles at the point of contact of the circles.

PROPOSITION 28. THEOREM.

In equal circles, or in the same circle, if two chords be equal, the arcs they cut off are equal, the major arc equal to the major arc, and the minor equal to the minor.



Let ABC, DEF be equal circles, and let chord BC = chord EF:

it is required to prove that major arc BAC = major arc EDF, and minor arc BGC = minor arc EHF.

Find K and L the centres of the circles, *III.* 1
and join BK, KC, EL, LF.

Because ⊙s ABC, DEF are equal, *Hyp.*
∴ their radii are equal. *III. Def.* 1, *Cor.* 4

In △s BKC, ELF, $\begin{cases} BK = EL \\ KC = LF \\ BC = EF; \end{cases}$ *Hyp.*

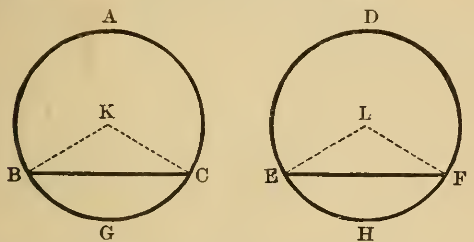
∴ $\angle K = \angle L$; *I.* 8
∴ arc BGC = arc EHF. *III.* 26
But ○ᶜᵉ ABC = ○ᶜᵉ DEF; *III. Def.* 1, *Cor.* 4
∴ remaining arc BAC = remaining arc EDF.

1. If AC and BD be two equal chords in a circle $ACDB$, prove chord $AB \parallel$ chord CD.
2. Hence devise a method of drawing through a given point a straight line parallel to a given straight line.

3. If two equal circles cut one another, any straight line drawn through one of the points of intersection will meet the circles again in two points which are equidistant from the other point of intersection.

PROPOSITION 29. Theorem.

In equal circles, or in the same circle, if two arcs be equal, the chords which cut them off are equal.



Let ABC, DEF be equal circles, and let arc BGC = arc EHF:

it is required to prove that chord BC = chord EF.

Find K and L the centres of the circles, *III.* 1
and join BK, KC, EL, LF.

Because the circles are equal, *Hyp.*
∴ their radii are equal. *III. Def.* 1, *Cor.* 4
And because the circles are equal, and arc BGC = arc EHF,
∴ $\angle K = \angle L$. *III.* 27

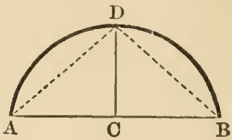
$$\text{In } \triangle\text{s } BKC, ELF, \left\{ \begin{array}{l} BK = EL \\ KC = LF \\ \angle K = \angle L; \end{array} \right.$$

∴ $BC = EF$. *I.* 4

1. If AC and BD be two equal arcs in a circle $ACDB$, prove chord AD = chord BC.
2. Prove the proposition by superposition.

PROPOSITION 30. PROBLEM.

To bisect a given arc.



Let ADB be the given arc:
it is required to bisect it.

Draw the chord AB, and bisect it at C; *I.* 10
from C draw $CD \perp AB$, and meeting the arc at D. *I.* 11
D is the point of bisection.

Join AD, BD.

In $\triangle$s ACD, BCD, $\left\{\begin{array}{c} AC = BC \\ CD = CD \\ \angle ACD = \angle BCD; \end{array}\right.$ *Const.*

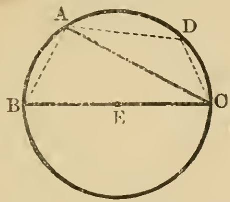
$\therefore AD = BD$. *I.* 4

But in the same circle equal chords cut off equal arcs, the major arc being = the major arc, and the minor = the minor;
and AD and BD are both minor arcs, since DC if produced would be a diameter; *III.* 1, *Cor.* 1
$\therefore$ arc AD = arc BD. *III.* 28

1. If two circles cut one another, the straight line joining their centres, being produced, bisects all the four arcs.
2. A diameter of a circle bisects the arcs cut off by all the chords to which it is perpendicular.
3. Bisect the arc ADB without joining AB.
4. Prove $\triangle$ DAB greater than any other triangle on the same base AB, and having its vertex on the arc ADB.

PROPOSITION 31. THEOREM.

An angle in a semicircle is a right angle; an angle in a segment greater than a semicircle is less than a right angle; and an angle in a segment less than a semicircle is greater than a right angle.



Let ABC be a circle, of which E is the centre and BC a diameter; and let any chord AC be drawn dividing the circle into the segment ABC which is greater than a semicircle, and the segment ADC which is less than a semicircle:

it is required to prove

(1) $\angle$ *in semicircle* BAC = *a rt.* $\angle$;

(2) $\angle$ *in segment* ABC *less than a rt.* $\angle$;

(3) $\angle$ *in segment* ADC *greater than a rt.* $\angle$.

Join AB;

take any point D in arc ADC, and join AD, CD.

(1) Because an angle at the $\bigcirc^{ce}$ of a circle is half of the angle at the centre which stands on the same arc; *III.* 20

$\therefore \angle BAC$ = half of the straight $\angle BEC$,

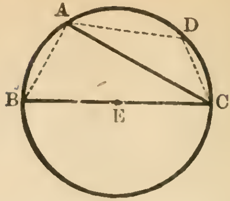
= half of two rt. $\angle$s, *III. Def.* 21

= a rt. $\angle$.

(2) Because $\angle BAC + \angle B$ is less than two rt. $\angle$s, *I.* 17

and $\angle BAC$ = a rt. $\angle$;

$\therefore \angle B$ is less than a rt. $\angle$.



(3) Because $ABCD$ is a quadrilateral inscribed in the circle,

$\therefore \angle B + \angle D =$ two rt. $\angle$s. *III.* 22

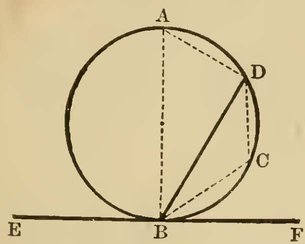
But $\angle B$ is less than a rt. $\angle$;

$\therefore \angle D$ is greater than a rt. $\angle$.

1. Circles described on the equal sides of an isosceles triangle as diameters intersect at the middle point of the base.
2. Circles described on any two sides of a triangle as diameters intersect on the third side or the third side produced.
3. Use the first part of the proposition to solve I. 11, and I. 12.
4. Solve III. 1 by means of a set square.
5. Solve III. 17, Case 3, by the following construction: Join AE, and on it as diameter describe a circle cutting the given circle at B and C. B and C are the points of contact of the tangents from A.
6. If one circle pass through the centre of another, the angle in the exterior segment of the latter circle is acute.
7. If one circle be described on the radius of another circle, any chord in the latter drawn from the point in which the circles meet is bisected by the former.
8. If two circles cut one another, and from one of the points of intersection two diameters be drawn, their extremities and the other point of intersection will be in one straight line.
9. Use the first part of the proposition to find a square equal to the difference of two given squares.
10. The middle point of the hypotenuse of a right-angled triangle is equidistant from the three vertices.
11. State and prove a converse of the preceding deduction.
12. Two circles touch externally at A; B and C are points of contact of a common tangent to the two circles. Prove $\angle BAC$ right.

PROPOSITION 32. Theorem.

If a straight line be a tangent to a circle, and from the point of contact a chord be drawn, the angles which the chord makes with the tangent shall be equal to the angles in the alternate segments of the circle.



Let ABC be a circle, EF a tangent to it at the point B, and from B let the chord BD be drawn:

it is required to prove $\angle\ DBF =$ *the* $\angle$ *in the segment* BAD,

and $\angle\ DBE =$ *the* $\angle$ *in the segment* BCD.

From B draw $BA \perp EF$; *I.* 11

take any point C in the arc BD, and join BC, CD, DA.

Because BA is drawn $\perp$ the tangent EF from the point of contact,

$\therefore$ BA passes through the centre of the circle; *III.* 19

$\therefore$ $\angle\ ADB$, being in a semicircle, = a rt. $\angle$; *III.* 31

$\therefore$ $\angle\ BAD + \angle\ ABD =$ a rt. $\angle$, *I.* 32

$= \angle\ ABF$.

From these equals take away the common $\angle\ ABD$;

$\therefore$ $\angle\ BAD = \angle\ DBF$.

Again, because $ABCD$ is a quadrilateral in a circle,

$\therefore$ $\angle\ A + \angle\ C = 2$ rt. $\angle$s. *III.* 22

But $\angle\ DBF + \angle\ DBE = 2$ rt. $\angle$s; *I.* 13

$\therefore$ $\angle\ A + \angle\ C = \angle\ DBF + \angle\ DBE$.

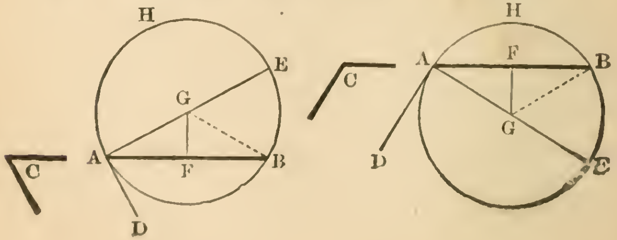
Now $\angle A = \angle DBF$;

$\therefore \quad \angle C = \angle DBE$.

1. The chord which joins the points of contact of parallel tangents to a circle is a diameter.
2. If two circles touch each other externally or internally, any straight line passing through the point of contact cuts off pairs of similar segments.
3. If two circles touch each other externally or internally, and two straight lines be drawn through the point of contact, the chords joining their extremities are parallel.
4. If two tangents be drawn to a circle from any point, the angle contained by the tangents is double the angle contained by the chord of contact, and the diameter drawn through either point of contact.
5. Enunciate and prove the converse of the proposition.
6. A and B are two points on the $\bigcirc^{ce}$ of a given circle. With B as centre and BA as radius describe a circle cutting the given circle at C and AB produced at D. Make arc $DE =$ arc DC, and join AE. AE is a tangent to the given circle.
7. Show that this proposition is a particular case either of III. 21, or of III. 22, Cor.

PROPOSITION 33. Problem.

On a given straight line to describe a segment of a circle which shall contain an angle equal to a given angle.



Let AB be the given straight line, $\angle\ C$ the given angle: *it is required to describe on AB a segment of a circle which shall contain an angle $= \angle\ C$.*

At A make $\angle\ BAD = \angle\ C$. *I.* 23
From A draw $AE \perp AD$; *I.* 11
bisect AB at F, *I.* 10
and draw $FG \perp AB$. *I.* 11
Join BG.

In $\triangle$s AFG, BFG, $\left\{\begin{array}{c} AF = BF \\ FG = FG \\ \angle\ AFG = \angle\ BFG; \end{array}\right.$ *Const.*

$\therefore\ AG = BG$; *I.* 4
$\therefore$ a circle described with centre G and radius AG will pass through B.
Let this circle be described, and let it be AHB.
The segment AHB is the required segment.

Because AD is $\perp$ AE, a diameter of the $\odot$ AHB,
$\therefore$ AD is a tangent to the circle. *III.* 16
Because AB is a chord of the circle drawn from the point of contact A,
$\therefore$ the angle in the segment $AHB = \angle\ BAD$, *III.* 32
$= \angle\ C$.

1. Show that the point G could be found equally well by making at B an angle $= \angle\ BAE$, instead of bisecting AB perpendicularly.

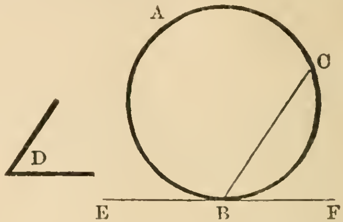
Construct a triangle, having given:

2. The base, the vertical angle, and one side.
3. The base, the vertical angle, and the altitude.
4. The base, the vertical angle, and the perpendicular from one end of the base on the opposite side.
5. The base, the vertical angle, and the sum of the sides.
6. The base, the vertical angle, and the difference of the sides.

[Several other methods of solving this proposition will be found in T. S. Davies's edition (12th) of Hutton's *Course of Mathematics*, vol. i. pp. 389, 390.]

PROPOSITION 34. PROBLEM.

From a given circle to cut off a segment which shall contain an angle equal to a given angle.



Let ABC be the given circle, and $\angle\ D$ the given angle: *it is required to cut off from $\odot\ ABC$ a segment which shall contain an angle $= \angle\ D$.*

Take any point B on the $\bigcirc^{ce}$, and at B draw the tangent EF. *III.* 17

At B make $\angle\ FBC = \angle\ D$. *I.* 23

The segment BAC is the required segment.

Because EF is a tangent to the circle, and the chord BC is drawn from the point of contact B,

$\therefore$ the angle in the segment $BAC = \angle\ FBC$, *III.* 32

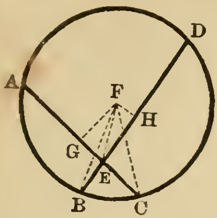
$= \angle\ D$.

Through a given point either within or without a given circle, draw a straight line cutting off a segment containing a given angle. Is the problem always possible?

PROPOSITION 35. THEOREMS.

If two chords of a circle cut one another, the rectangle contained by the segments of the one shall be equal to the rectangle contained by the segments of the other.

*Conversely: If two straight lines cut one another so that the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other, the four extremities of the two straight lines are concyclic.**



(1) Let AC, BD two chords of the circle ABC cut one another at E:

it is required to prove $AE \cdot EC = BE \cdot ED$.

Find F the centre of the $\odot\, ABC$, *III.* 1

and from it draw $FG \perp AC$, and $FH \perp BD$. *I.* 12

Join FB, FC, FE.

Because FG drawn from the centre is $\perp AC$,

$\therefore$ AC is bisected at G. *III.* 3

Because AC is divided into two equal segments AG, GC, and also internally into two unequal segments AE, EC,

$\therefore \quad AE \cdot EC = GC^2 - GE^2,$ *II.* 5

$= (FC^2 - FG^2) - (FE^2 - FG^2),$ *I.* 47, *Cor.*

$= FC^2 - FE^2.$

Similarly, $BE \cdot ED = FB^2 - FE^2.$

But $FC^2 = FB^2$;

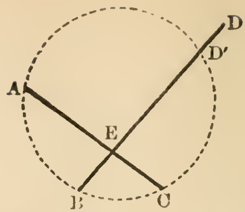
$\therefore FC^2 - FE^2 = FB^2 - FE^2$;

$\therefore \quad AE \cdot EC = BE \cdot ED.$

(2) Let the two straight lines AC, BD cut one another at E, so that $AE \cdot EC = BE \cdot ED$:

it is required to prove the four points A, B, C, D concyclic.

*** The second part of this proposition is not given by Euclid.**



Since a circle can always be described through three points which are not in the same straight line,

let a circle be described through A, B, C. *III.* 1, *Cor.* 2

If this circle do not pass also through D, let it cut BD or BD produced at the point D';

then $AE \cdot EC = BE \cdot ED'$. *III.* 35

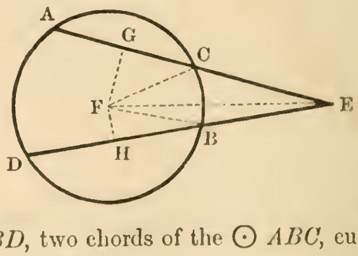
But $AE \cdot EC = BE \cdot ED$; *Hyp.*

$\therefore\ BE \cdot ED' = BE \cdot ED$;

$\therefore\ ED' = ED$, which is impossible;

$\therefore$ the circle which passes through A, B, C must pass also through D.

Cor.—If two chords of a circle when produced cut one another, the rectangle contained by the segments of the one shall be equal to the rectangle contained by the segments of the other; and conversely.



Let AC, BD, two chords of the $\odot\ ABC$, cut one another when produced at E:

it is required to prove $AE \cdot EC = BE \cdot ED$.

Find F the centre of the $\odot\, ABC$, *III.* 1
and from it draw $FG \perp AC$, and $FH \perp BD$. *I.* 12
Join FB, FC, FE.

Because FG drawn from the centre is $\perp AC$,
$\therefore AC$ is bisected at G. *III.* 3
Because AC is divided into two equal segments AG, GC, and also externally into two unequal segments AE, EC,

$$\begin{aligned}\therefore AE \cdot EC &= GE^2 - GC^2; && \textit{II. } 6\\ &= (FE^2 - FG^2) - (FC^2 - FG^2), && \textit{I. } 47, \textit{Cor.}\\ &= FE^2 - FC^2.\end{aligned}$$

Similarly, $BE \cdot ED = FE^2 - FB^2$.
But $FC^2 = FB^2$;
$\therefore FE^2 - FC^2 = FE^2 - FB^2$;
$\therefore AE \cdot EC = BE \cdot ED$.

The converse is proved in exactly the same way as the converse of the proposition.

NOTE.—It was proved in the proposition that
$AE \cdot EC = FC^2 - FE^2$.
Now, if the $\odot\, ABC$ and the point E be fixed, FC and FE are constant lengths, and $\therefore FC^2 - FE^2$ is a constant magnitude.
Hence $AE \cdot EC$ is constant.
But AC is any chord through E;
$\therefore$ the rectangles contained by the segments of all the chords that can be drawn through E are constant;
or, in other words, if a variable chord pass through a fixed point inside a circle, the rectangle contained by the segments which the point makes on it is constant.
This constant value may be called the *internal potency* of the point with respect to the circle.
It was proved in the cor. that $AE \cdot EC = FE^2 - FC^2$.
Hence, as before, if the $\odot\, ABC$ and the point E be fixed, $AE \cdot EC$ is constant;
that is, if a variable chord pass through a fixed point outside a circle, the rectangle contained by the segments which the point makes on it is constant.

This constant value may be called the *external potency* of the point with respect to the circle.

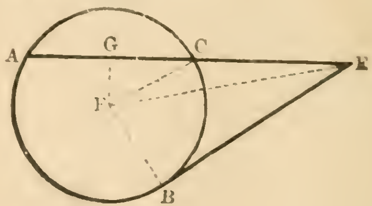
When the point is situated on the ○ce of the circle, its potency with respect to the circle is zero.

[The phrase 'potency of a point with respect to a circle' is due to Steiner. See Jacob Steiner's *Gesammelte Werke*, vol. i. p. 22.]

1. If two circles intersect, and through any point in their common chord two other chords be drawn, one in each circle, their four extremities are concyclic.
2. ABC is a triangle, AX, BY, CZ the perpendiculars from its vertices on the opposite sides, intersecting at O. Prove $AO \cdot OX = BO \cdot OY = CO \cdot OZ$.
3. ABC is a triangle, right-angled at C; from any point D in AB, or AB produced, a perpendicular to AB is drawn, meeting AC, or AC produced, in E. Prove $AB \cdot AD = AC \cdot AE$.
4. ABC is any triangle; D and E are two points on AB and AC, or on AB and AC produced either through the vertex or below the base, such that $\angle ADE = \angle ACB$. Prove $AB \cdot AD = AC \cdot AE$.
5. Through a point P within a circle a chord APB is drawn such that $AP \cdot PB \doteq$ a given square. Determine the square.
6. Prove VI. B, and VI. C.

PROPOSITION 36. THEOREM.

If from a point without a circle a secant and a tangent be drawn to the circle, the rectangle contained by the secant and its external segment shall be equal to the square on the tangent.



Let ABC be a circle, and from the point E without it let there be drawn a secant ECA and a tangent EB:
it is required to prove $AE \cdot EC = EB^2$.

Find F the centre of the ⊙ ABC, *III.* 1
and from it draw $FG \perp AC$. *I.* 12
Join FB, FC, FE.

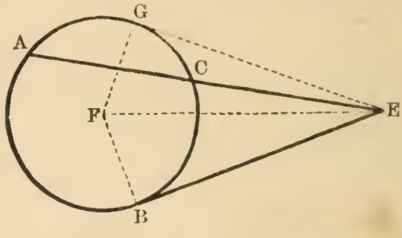
Because FB is drawn from the centre of the circle to B, the point of contact of the tangent EB,
$\therefore \angle FBE$ is right. *III.* 18
Because FG, drawn from the centre, is $\perp AC$,
$\therefore AC$ is bisected at G. *III.* 3
Because AC is divided into two equal segments AG, GC, and also externally into two unequal segments AE, EC,

$$\begin{aligned} \therefore \quad AE \cdot EC &= GE^2 - GC^2, && \textit{II. } 6 \\ &= (FE^2 - FG^2) - (FC^2 - FG^2), && \textit{I. } 47, \textit{ Cor.} \\ &= FE^2 - FC^2, \\ &= FE^2 - FB^2, \\ &= EB^2. && \textit{I. } 47, \textit{ Cor.} \end{aligned}$$

1. Prove the proposition when the secant passes through the centre of the circle. (Euclid gives this particular case.)
2. If two circles intersect, their common chord produced bisects their common tangents.
3. If two circles intersect, the tangents drawn to them from any point in their common chord produced are equal.
4. ABC is a triangle, AX, BY, CZ the perpendiculars from its vertices on the opposite sides. Prove $AC \cdot AY = AB \cdot AZ$, $BC \cdot BX = BA \cdot BZ$, $CA \cdot CY = CB \cdot CX$.
5. From a given point as centre describe a circle to cut a given straight line in two points, so that the rectangle contained by their distances from a fixed point in the straight line may be equal to a given square.
6. Show, by revolving the secant EBD (fig. to III. 35, Cor.) round E, that this proposition is a particular case of III. 35, Cor.

PROPOSITION 37. THEOREM.

If from a point without a circle two straight lines be drawn, one of which cuts the circle, and the other meets it, and if the rectangle contained by the secant and its external segment be equal to the square on the line which meets the circle, that line shall be a tangent.



Let ABC be a circle, and from the point E without it let there be drawn a secant ECA and a straight line EB to meet the circle; also, let $AE \cdot EC = EB^2$:
it is required to prove that EB is a tangent to the $\odot$ ABC.

Draw EG touching the circle at G, *III.* 17
and join the centre F to B, G, and E.

Then $\angle FGE =$ a rt. $\angle$. *III.* 18

Now, since EG is a tangent, and ECA a secant,

$\therefore \quad EG^2 = AE \cdot EC,$ *III.* 36

$\qquad = EB^2;$ *Hyp.*

$\therefore \quad EG = EB.$

In $\triangle$s EBF, EGF, $\begin{cases} EB = EG \\ BF = GF \\ EF = EF; \end{cases}$

$\therefore \angle EBF = \angle EGF,$ *I.* 8

$\qquad = $ a rt. $\angle$;

$\therefore$ EB is a tangent to the $\odot$ ABC. *III.* 16

1. Prove the proposition indirectly by supposing EB to meet the circle again at D.
2. Prove the proposition indirectly by drawing the tangent EG on the other side of EF, and using I. 7.
3. Describe a circle to pass through two given points, and touch a given straight line.
4. Describe a circle to pass through one given point, and touch two given straight lines. Show that to this and the previous problem there are in general two solutions.
5. Describe a circle to touch two given straight lines and a given circle. Show that to this problem there are in general four solutions.
6. Describe a circle to pass through two given points, and touch a given circle. Show that to this problem there are in general two solutions.
7. AB is a straight line, C and D two points on the same side of it; find the point in AB at which the distance CD subtends the greatest angle.

[The third, fourth, fifth, and sixth deductions, along with IV. 4, 5, are cases of the general problem of the Tangencies, a subject on which Apollonius of Perga (about 222 B.C.) composed a treatise, now lost. This problem consists in describing a circle to pass through or touch any three of the following nine data: three points, three straight lines, three circles. It comprises ten cases, which, denoting a point by P, a straight line by L, and a circle by C, may be symbolised thus: PPP, PPL, PPC, PLL, PLC, PCC, LLL, LLC, LCC, CCC. An excellent historical account of the solutions given to this problem in its various cases will be found in an article by T. T. Wilkinson, 'De Tactionibus,' in the *Transactions of the Historic Society of Lancashire and Cheshire* (1872). To the authorities there mentioned should be added *Das Problem des Apollonius*, by C. Hellwig (1856); *Das Problem des Pappus von den Berührungen*, by W. Berkhan (1857); 'The Tangencies of Circles and of Spheres,' by Benjamin Alvord, published in 1855 in the 8th vol. of the *Smithsonian Contributions*, and 'The Intersection of Circles and the Intersection of Spheres,' by the same author in the *American Journal of Mathematics*, vol. v., pp. 25–44.]

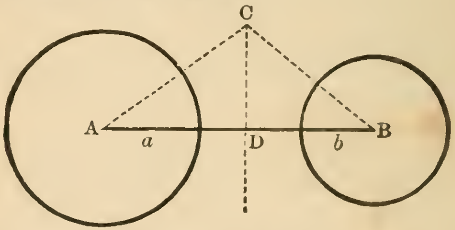
APPENDIX III.

Radical Axis.

Def. 1.—The locus of a point whose potencies (both external or both internal) with respect to two circles are equal, is called the *radical axis** of the two circles.

Proposition 1.

The radical axis of two circles is a straight line perpendicular to the line of centres of the two circles.



Let A and B be the centres of the given circles, whose radii are a and b, and suppose C to be any point on the required locus.

Join CA, CB, and from C draw $CD \perp AB$ the line of centres.

Since the potency of C with respect to circle $A = AC^2 - a^2$, *Def.*
and since the potency of C with respect to circle $B = BC^2 - b^2$; *Def.*

$\therefore AC^2 - a^2 = BC^2 - b^2$;

$\therefore AC^2 - BC^2 = a^2 - b^2$.

But since the circles A and B are given, their radii (a and b) are constant;

$\therefore$ the squares on the radii (a^2 and b^2) are constant;

$\therefore$ the difference of the squares on the radii ($a^2 - b^2$) is constant;

$\therefore AC^2 - BC^2$ is constant.

Hence the locus of C is a straight line $\perp AB$. *App. II.* 5

* This name, as well as that of 'radical centre,' was introduced by L. Gaultier de Tours. See *Journal de l'École polytechnique*, 16e cahier, tome ix. (1813), pp. 139, 143.

Cor. 1.—Tangents drawn to the two circles from any point in their radical axis are equal.

Cor. 2.—The radical axis of two circles bisects their common tangents. Hence may be derived a method of drawing the radical axis of two circles.

Cor. 3.—If the two circles are exterior to each other and have no common point, the radical axis is situated outside both circles.

Cor. 4.—If the two circles touch each other either externally or internally, their radical axis consists of the common tangent at the point of contact.

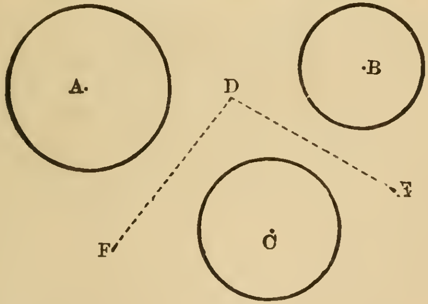
Cor. 5.—If the two circles intersect each other, their radical axis consists of their common chord produced.

Cor. 6.—If one circle is inside the other and does not touch it, their radical axis is situated outside both circles.

Cor. 7.—The radical axis of two unequal circles is nearer to the centre of the small circle than to the centre of the large one, but nearer to the ○ce of the large circle than to the ○ce of the small one.

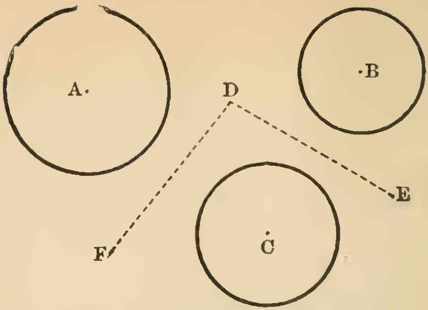
Proposition 2.

*The radical axes of three circles taken in pairs are concurrent.**



Let A, B, C be three circles, whose radii are a, b, c: *it is required to prove that the radical axis of A and B, that of B and C, and that of C and A all meet at one point.*

* This theorem, in one of its cases, is attributed to Monge (1746–1818), in Poncelet's *Propriétés Projectives des Figures*, § 71.



Suppose the centres of the three circles not to be in the same straight line.

Then DE, the radical axis of B and C, and DF, the radical axis of C and A, will meet at some point D;

for they are respectively $\perp$ BC and CA, and BC and CA are not in the same straight line.

Since D is a point on the radical axis of B and C;

$\therefore\ BD^2 - b^2 = CD^2 - c^2$.

Since D is a point on the radical axis of C and A;

$\therefore\ CD^2 - c^2 = AD^2 - a^2$;

$\therefore\ AD^2 - a^2 = BD^2 - b^2$;

$\therefore$ D is a point on the radical axis of A and B,

that is, the radical axis of A and B passes through D.

Def. 2.—The point of concourse of the radical axes of three circles taken in pairs, is called the *radical centre* of the three circles.

Cor. 1.—When the three circles all cut one another, the radical centre lies either within or without all the three circles.

Cor. 2.—When the centres of the three circles are in one straight line, the radical axes are all parallel, and the radical centre therefore is infinitely distant.

Cor. 3.—When the three circles all touch one another at the same point, the common tangent at that point is the radical axis of all three, and the radical centre therefore is indeterminate—that is, any point on the common radical axis will be a radical centre.

Cor. 4.—In all other cases the radical centre is outside the three circles.

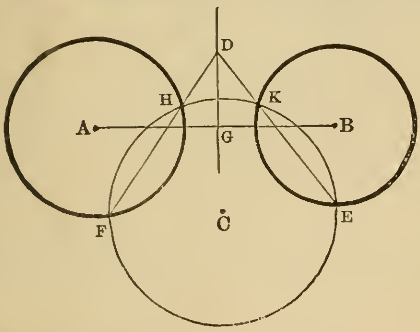
Cor. 5.—If from the radical centre tangents be drawn to the three circles, their points of contact will be concyclic.

Cor. 6.—If there be several points from which equal tangents can be drawn to three circles, these three circles must have the same radical axis, and the several points must be situated on it.

Cor. 7.—The orthocentre of a triangle is the radical centre of the circles whose diameters are the sides of the triangle, and also the radical centre of the circles whose diameters are the segments of the perpendiculars between the orthocentre and the vertices.

Proposition 3.

To find the radical axis of two circles which have no common point.



Let A and B be the two circles.

Describe any third circle C so as to cut the circles A and B.

Draw FH the common chord of A and C, and EK the common chord of B and C, and let them meet at D.

From D draw $DG \perp AB$.

Then FD is the radical axis of A and C, and ED the radical axis of B and C; *App. III.* 1, *Cor.* 5

$\therefore$ D is the radical centre of A, B, and C; *App. III.* 2

$\therefore$ D is a point on the radical axis of A and B;

$\therefore$ DG is the radical axis of A and B. *App. III.* 1

Cor. 1.—The radical axis of A and B may also be obtained thus: After finding D, draw a fourth circle to intersect A and B. A second pair of common chords will thus be obtained whose intersection will determine another point on the radical axis of A and B. Join D with this other point.

COR. 2.—The radical centre of three circles which have no common point may be found by describing two circles each of which shall cut all the three given circles.

DEDUCTIONS.

1. Find a point inside a triangle at which the three sides shall subtend equal angles. Is this always possible?
2. Given two intersecting circles, to draw, through one of the points of intersection, a straight line terminated by the circles, and such that (*a*) the sum, (*b*) the difference, of the two chords may = a given length.
3. Of all the straight lines which can be drawn from two given points to meet on the convex $\bigcirc^{ce}$ of a circle, the sum of those two will be the least, which make equal angles with the tangent at the point of concourse.
4. With the extremities of the diameter of a semicircle as centres, any two other semicircles are drawn touching each other externally, and a straight line is drawn to touch them both. Prove that this straight line will also touch the original semicircle.
5. Find a point in the diameter produced of a given circle, such that a tangent drawn from it to the circle shall be of given length.
6. ABC is a triangle having $\angle\ BAC$ acute; prove BC^2 less than $AB^2 + AC^2$ by twice the square on the tangent drawn from A to the circle of which BC is a diameter.
7. ABC is a triangle, AX, BY, CZ, the perpendiculars from its vertices on the opposite sides. Prove that these perpendiculars bisect the angles of $\triangle\ XYZ$, and that $\triangle$s AYZ, XBZ, XYC, ABC are mutually equiangular.
8. If the perpendiculars of a triangle be produced to meet the circle circumscribed about the triangle, the segments of these perpendiculars between the orthocentre and the $\bigcirc^{ce}$ are bisected by the sides of the triangle.
9. If O be the orthocentre of $\triangle\ ABC$, the circles circumscribed about $\triangle$s ABC, AOB, BOC, COA are equal.
10. If D, E, F be situated respectively on BC, CA, AB, the sides of $\triangle\ ABC$, the $\bigcirc^{ces}$ of the circles circumscribed about the three $\triangle$s AEF, BFD, CDE will pass through the same point.

11. If on the three sides of any triangle equilateral triangles be described outwardly, the straight lines joining the circumscribed centres of these triangles will form an equilateral triangle.

Construct a triangle, having given the base, the vertical angle, and

12. The perpendicular from the vertex to the base.
13. The median to the base.
14. The projection of the vertex on the base.
15. The point where the bisector of the vertical angle meets the base.
16. The sum or difference of the other sides.

17. Construct a triangle, having given its orthocentric triangle.
18. Draw all the common tangents to two circles. Examine the various cases. (One pair are called *direct*, the other pair *transverse*, common tangents.)
19. Of the chords drawn from any point on the $\bigcirc^{ce}$ of a circle to the vertices of an equilateral triangle inscribed in the circle, the greatest = the sum of the other two.
20. If two chords in a circle intersect each other perpendicularly, the sum of the squares on their four segments = the square on the diameter. (This is the 11th of the *Lemmas* ascribed to Archimedes, 287–212 B.C.)
21. A quadrilateral is inscribed in a circle, and its sides form chords of four other circles. Prove that the second points of intersection of these four circles are concyclic.
22. If four circles be described, either all inside or all outside of any quadrilateral, each of them touching three of the sides or the sides produced, their centres will be concyclic.
23. The opposite sides of a quadrilateral inscribed in a circle are produced to meet. Prove that the bisectors of the two angles thus formed are $\perp$ each other.
24. If the opposite sides of a quadrilateral inscribed in a circle be produced to meet, the square on the straight line joining the points of concourse = the sum of the squares on the two tangents from these points. (A converse of this is given in Matthew Stewart's *Propositiones Geometricæ*, 1763, Book i., Prop. 39.)
25. If a circle be circumscribed about a triangle, and from the ends of the diameter $\perp$ the base, perpendiculars be drawn to the other two sides, these perpendiculars will intercept on the sides segments = half the sum or half the difference of the sides.

26. In the figure to the preceding deduction, find all the angles which are = half the sum or half the difference of the base angles of the triangle.

27. If from any point in the ○ce of the circle circumscribed about a triangle, perpendiculars be drawn to the sides of the triangle, the feet of these perpendiculars are collinear. (This theorem is frequently attributed to Robert Simson, 1687–1768. I have not been able to find it in his works.)

28. If from any point in the ○ce of the circle circumscribed about a triangle, straight lines be drawn, making with the sides, in cyclical order, equal angles, the feet of these straight lines are collinear.

29. If P be any point in the ○ce of the circle circumscribed about $\triangle\ ABC$, X, Y, Z, its projections on the sides BC, CA, AB, the circle which passes through the centres of the circles circumscribed about $\triangle$s AZY, BXZ, CYX is constant in magnitude.

30. If a straight line cut the three sides of a triangle, and circles be circumscribed about the new triangles thus formed, these circles will all pass through one point; and this point will be concyclic with the vertices of the original triangle. (Steiner's *Gesammelte Werke*, vol. i. p. 223.)

31. If any number of circles intersect a given circle, and pass through two given points, the straight lines joining the intersections of each circle with the given one will all meet in the same point.

32. A series of circles touch a fixed straight line at a fixed point; show that the tangents at the points where they cut a parallel fixed straight line all touch a fixed circle.

33. $ABCD$ is a quadrilateral having $AB = AD$, and $\angle\ C = \angle\ B + \angle\ D$; prove $AC = AB$ or AD.

34. From C two tangents CD, CE are drawn to a semicircle whose diameter is AB; the chords AE, BD intersect at F. Prove that CF produced is $\perp\ AB$. (This is the 12th of the *Lemmas* ascribed to Archimedes, and the preceding deduction is assumed in the proof of it.)

35. On the same supposition, prove that if the chords AD, BE intersect at F', $F'C$ produced is $\perp\ AB$.

36. A series of circles intersect each other, and are such that the tangents to them from a fixed point are equal; prove that the common chords of each pair pass through this point.

37. Find a point in the $\bigcirc^{ce}$ of a given circle, the sum of whose distances from two given straight lines at right angles to each other, which do not cut the circle, is the greatest, or the least possible.
38. From a given point in the $\bigcirc^{ce}$ of a circle draw a chord which shall be bisected by a given chord in the circle.
39. From a point P outside a circle two secants PAB, PDC are drawn to the circle $ABCD$; AC, BD are joined and intersect at O. Prove that O lies on the chord of contact of the tangents drawn from P to the circle. (See Poudra's *Œuvres de Desargues*, tome i., pp. 189–192, 273, 274.)
40. Hence devise a method of drawing tangents to a circle from an external point by means of a ruler only.

Loci.

Find the locus of the centres of the circles which touch

1. A given straight line at a given point.
2. A given circle at a given point.
3. A given straight line, and have a given radius.
4. A given circle, and have a given radius.
5. Two given straight lines.
6. Two given equal circles.

7. A series of parallel chords are placed in a circle; find the locus of their middle points.
8. A series of equal chords are placed in a circle; find the locus of their middle points.
9. A series of right-angled triangles are described on the same hypotenuse; find the locus of the vertices of the right angles.
10. A variable chord of a given circle passes through a fixed point; find the locus of the middle point of the chord. Examine the cases when the fixed point is inside the circle, outside the circle, and on the $\bigcirc^{ce}$.
11. Find the locus of the vertices of all the triangles which have the same base, and their vertical angles equal to a given angle.
12. Of the $\triangle\ ABC$, the base BC is given, and the vertical angle A; find the locus of the point D, such that $BD =$ the sum of the sides BA, AC.
13. Of the $\triangle\ ABC$, the base BC is given, and the vertical angle A; find the locus of the point D, such that $BD =$ the difference of the sides BA, AC.

14. AB is a fixed chord in a given circle, and from any point C in the arc ACB, a perpendicular CD is drawn to AB. With C as centre and CD as radius a circle is described, and from A and B tangents are drawn to this circle which meet at P; find the locus of P.

15. A quadrilateral inscribed in a circle has one side fixed, and the opposite side constant; find the locus of the intersection of the other two sides, and of the intersection of the diagonals.

16. Two circles touch a given straight line at two given points, and also touch one another; find the locus of their point of contact.

17. Find the locus of the points from which tangents drawn to a given circle may be perpendicular to each other.

18. Find the locus of the points from which tangents drawn to a given circle may contain a given angle.

19. Find the locus of the points from which tangents drawn to a given circle may be of a given length.

20. From any point on the $\bigcirc^{ce}$ of a given circle, secants are drawn such that the rectangle contained by each secant and its exterior segment is constant; find the locus of the ends of the secants.

21. A is a given point and BC a given straight line; any point P is taken on BC, and AP is joined. Find the locus of a point Q taken on AP such that $AP \cdot AQ$ is constant.

22. The hypotenuse of a right-angled triangle is given; find the loci of the corners of the squares described outwardly on the sides of the triangle.

23. A variable chord of a given circle passes through a fixed point, and tangents to the circle are drawn at its extremities; prove that the locus of the intersection of the tangents is a straight line. (This straight line is called the *polar* of the given fixed point, and the given fixed point is called the *pole*, with reference to the given circle. See the reference to Desargues on p. 221.)

24. Examine the case when the fixed point is outside the circle.

BOOK IV.

DEFINITIONS.

1. Any closed rectilineal figure may be called a **polygon.** Thus triangles and quadrilaterals are polygons of three and four sides.

Polygons of five sides are called pentagons; of six sides, hexagons; of seven, heptagons; of eight, octagons; of nine, nonagons or enneagons; of ten, decagons; of eleven, undecagons or hendecagons; of twelve, dodecagons; of fifteen, quindecagons or pentedecagons; of twenty, icosagons.

Sometimes a polygon having n sides is called an n-gon.

2. A polygon is said to be **regular** when all its sides are equal, and all its angles equal.

It is important to observe that the triangle is unique among polygons. For if a triangle have all its sides equal, it must have all its angles equal (I. 5, Cor.); if it have all its angles equal, it must have all its sides equal (I. 6, Cor.)

Polygons with more than three sides may have all their sides equal without having their angles equal; or they may have all their angles equal without having their sides equal. A rhombus and a rectangle are illustrations of the preceding remark.

Hence in order to prove a polygon (other than a triangle) regular, it must be proved to be both equilateral and equiangular.

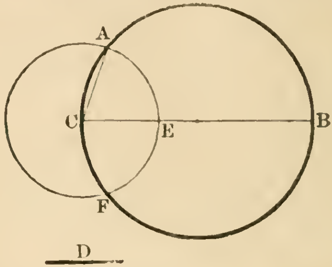
3. When each of the angular points of a polygon lies on the circumference of a circle, the polygon is **inscribed** in the circle, or the circle is **circumscribed** about the polygon.

4. When each of the sides of a polygon touches the circumference of a circle, the polygon is **circumscribed** about the circle, or the circle is **inscribed** in the polygon.

5. The **diagonals** of a polygon are the straight lines which join those vertices of the polygon which are not consecutive.

PROPOSITION 1. Problem.

In a given circle to place a chord equal to a given straight line which is not greater than the diameter of the circle.



Let D be the given straight line which is not greater than the diameter of the given $\odot$ ABC:
it is required to place in the $\odot$ ABC *a chord* $= D$.

Draw BC any diameter of the $\odot$ ABC. *III.* 1
Then if $BC = D$, what was required is done.
But if not, BC is greater than D. *Hyp.*
Make $CE = D$; *I.* 3
with centre C and radius CE, describe the $\odot$ AEF;
join CA.

Then $CA = CE$, being radii of the $\odot$ AEF,
$= D$. *Const.*

1. How many chords can be placed in the circle equal to the given straight line D?
2. Place a chord in the $\odot$ ABC equal to the given straight line D, and so that one of its extremities shall be at a given point in the $\bigcirc^{ce}$. How many chords can be so placed?
3. About a given chord to circumscribe a circle. How many circles can be so circumscribed? Where will their centres all lie? What limits are there to the lengths of the diameters of all such circles?

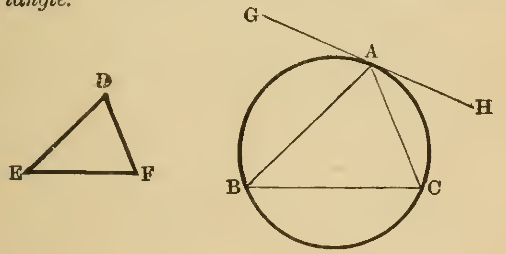
4. About a given chord to circumscribe a circle having a given radius. How many circles can be so circumscribed?

Place a chord in the ⊙ ABC equal to the given straight line D, and so that it shall

5. Pass through a given point within the circle.
6. " " " without "
7. Be parallel to another given straight line.
8. Be perpendicular " " "

PROPOSITION 2. Problem.

In a given circle to inscribe a triangle equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle: *it is required to inscribe in ABC a triangle equiangular to △ DEF.*

Take any point A on the ○^{ce} of ABC,
and at A draw the tangent GAH. *III.* 17
Make ∠ HAC = ∠ E, and ∠ GAB = ∠ F; *I.* 23
join BC. ABC is the required triangle.

Because the chord AC is drawn from A, the point of contact of the tangent GAH,

∴ ∠ B = ∠ HAC, *III.* 32
= ∠ E. *Const.*
Similarly, ∠ C = ∠ GAB, *III.* 32
= ∠ F; *Const.*

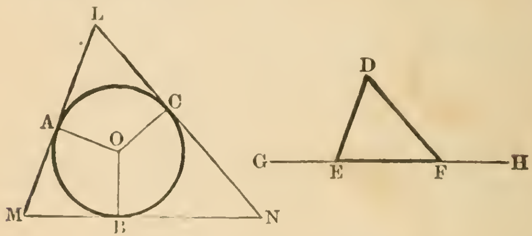
∴ remaining ∠ BAC = remaining ∠ D; *I.* 32, *Cor.* 1

∴ △ ABC is equiangular to △ DEF.

1. Show that there may be innumerable triangles inscribed in the ⊙ ABC equiangular to the given △ DEF.
2. If the problem were, In a given circle to inscribe a triangle equiangular to a given △ DEF, and having one of its vertices at a given point A on the ○^{ce}, show that six different positions of the inscribed triangle would be possible.
3. Given a ⊙ ABC; inscribe in it an equilateral triangle.
4. Two △s ABC, LMN are inscribed in the ⊙ ABC, each of them equiangular to the △ DEF; prove △s ABC, LMN equal in all respects.

PROPOSITION III. Problem.

About a given circle to circumscribe a triangle equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle: *it is required to circumscribe about ABC a triangle equiangular to △ DEF.*

Produce EF both ways to G and H.

Find O the centre of the ⊙ ABC, *III.* 1

and draw any radius OB.

Make ∠ BOA = ∠ DEG, and ∠ BOC = ∠ DFH; *I.* 23

and at A, B, C, draw tangents to the circle intersecting each other at L, M, N. LMN is the required triangle.

Because $OAMB$ is a quadrilateral,

$\therefore$ the sum of its four $\angle$s = 4 rt. $\angle$s. *I.* 32, *Cor.* 2

But $\angle OAM + \angle OBM$ = 2 rt. $\angle$s; *III.* 18

$\therefore \angle M$ is supplementary to $\angle BOA$.

But $\angle DEF$ is supplementary to $\angle DEG$, *I.* 13

and $\angle BOA = \angle DEG$; *Const.*

$\therefore \angle M = \angle DEF$.

Similarly, $\angle N = \angle DFE$;

$\therefore$ remaining $\angle L$ = remaining $\angle D$; *I.* 32, *Cor.* 1

$\therefore \triangle LMN$ is equiangular to $\triangle DEF$.

1. It is assumed in the proposition that the tangents at A, B, C will meet and form a triangle. Prove this.
2. Show that there may be innumerable triangles circumscribed about the $\odot ABC$ equiangular to the given $\triangle DEF$.
3. Given a $\odot ABC$; circumscribe about it an equilateral triangle.
4. If the points of contact of the sides of the circumscribed equilateral triangle be joined, an inscribed equilateral triangle will be obtained.
5. A side of the circumscribed equilateral triangle is double of a side of the inscribed equilateral triangle, and the area of the circumscribed equilateral triangle is four times the area of the inscribed equilateral triangle.

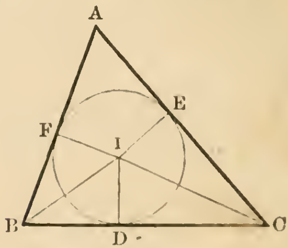
Supply the demonstration of the proposition from the following constructions, which do not require EF to be produced:

6. In the given circle, whose centre is O, draw any diameter BOG. Make $\angle GOA = \angle E$, $\angle GOC = \angle F$, and at A, B, C draw tangents intersecting at L, M, N. LMN is the required triangle.
7. At any point B on the $\bigcirc^{ce}$ of the given circle draw a tangent PBQ, and on the tangent take any points P, Q, on opposite sides of B. At P make $\angle QPR = \angle E$, and at Q make $\angle PQR = \angle F$. Assuming that PR, QR do not touch the given circle, from O the centre draw perpendiculars to PR, QR, and let these perpendiculars, produced if necessary, meet the circle at A and C. At A and C draw tangents LM, LN to the circle. LMN is the required triangle.
8. In the given circle inscribe a $\triangle ABC$ equiangular to $\triangle DEF$. Bisect the arcs AB, BC, CA, and at the points of bisection draw tangents.

9. Any rectilineal figure $ABCDE$ is inscribed in a circle. Bisect the arcs AB, BC, CD, DE, EA, and at the points of bisection draw tangents. The resulting figure is equiangular to $ABCDE$.
10. Two triangles are circumscribed about the $\odot$ ABC, each of them equiangular to $\triangle$ DEF; prove that they are equal in all respects.
11. Describe a triangle equiangular to a given triangle, and such that a given circle shall be touched by one of its sides, and by the other two produced. Show that there are three solutions of this deduction.

PROPOSITION 4. Problem.

To inscribe a circle in a given triangle.



Let ABC be the given triangle :
it is required to inscribe a circle in $\triangle$ ABC.

Bisect $\angle$s ABC, ACB by BI, CI, which intersect at I; *I.* 9
from I draw ID, IE, IF $\perp$ BC, CA, AB. *I.* 12

In $\triangle$s IDB, IFB, $\left\{\begin{array}{l} \angle\ IDB = \angle\ IFB \\ \angle\ IBD = \angle\ IBF \\ IB = IB; \end{array}\right.$ *Const.*

$\therefore$ $ID = IF$. *I.* 26
Similarly, $ID = IE$;
$\therefore$ ID, IE, IF are all equal.

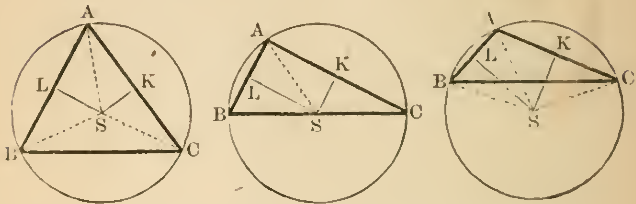
With centre I and radius ID describe a circle, which will pass through the points D, E, F.
Of this circle, ID, IE, IF will be radii;
and since BC, CA, AB are $\perp$ ID, IE, IF, *Const.*
$\therefore$ BC, CA, AB will be tangents to the $\odot$ DEF; *III.* 16
$\therefore$ the $\odot$ DEF is inscribed in the $\triangle$ ABC.

NOTE.—This proposition is included in the more general one, to describe a circle which shall touch three given straight lines. See Appendix IV. 1, p. 250.

1. It is assumed in the proposition that the bisectors BI, CI will meet at some point I. Prove this.
2. If IA be joined, it will bisect $\angle$ BAC.
3. The centre of the circle inscribed in an equilateral triangle is equidistant from the three vertices.
4. The centre of the circle inscribed in an isosceles triangle is equidistant from the ends of the base.
5. Prove $AF + BD + CE = FB + DC + EA =$ semi-perimeter of $\triangle$ ABC.
6. Prove $AF + BC = BD + CA = CE + AB =$ semi-perimeter of $\triangle$ ABC.
7. With A, B, C, the vertices of $\triangle$ ABC as centres, describe three circles, each of which shall touch the other two.
8. Find the centre of a circle which shall cut off equal chords from the three sides of a triangle.
9. If through I a straight line be drawn $\parallel$ BC, and terminated by AB, AC, this parallel will be equal to the sum of the segments of AB, AC between it and BC. Examine the cases for I_1, I_2, I_3, in Appendix IV. I.
10. If D, E, F, the points of contact of the inscribed circle, be joined, $\triangle$ DEF is acute-angled.
11. The angles of $\triangle$ DEF are respectively complementary to half the opposite angles of $\triangle$ ABC.
12. ABC is a triangle. D and E are points in AB and AC, or in AB and AC produced. Prove that the vertex A, and the centres of the circles inscribed in $\triangle$s ABC, ADE, are collinear.
13. Draw a straight line which would bisect the angle between two straight lines which are not parallel, but which cannot be produced to meet.

PROPOSITION 5. PROBLEM.

To circumscribe a circle about a given triangle.



Let ABC be the given triangle:
it is required to circumscribe a circle about $\triangle\ ABC$.

Bisect AB at L and AC at K; *I.* 10
from L and K draw $LS \perp AB$ and $KS \perp AC$, *I.* 11
and let LS, KS intersect at S.
Join SA; and if S be not in BC, join SB, SC.

$$\text{In } \triangle\text{s } ALS,\ BLS, \left\{\begin{array}{l} AL = BL \\ LS = LS \\ \angle ALS = \angle BLS; \end{array}\right.$$ *Const.*

$\therefore\ SA = SB$. *I.* 4
Similarly, $SA = SC$;
$\therefore\ SA$, SB, SC are all equal.

With centre S and radius SA, describe a circle;
this circle will pass through the points A, B, C, and will be circumscribed about the $\triangle\ ABC$.

COR.—From the three figures it appears that S, the centre of the circumscribed circle, may occupy three positions:

(1) It may be inside the triangle.
(2) It may be on one of the sides.
(3) It may be outside the triangle.

In the first case, when S is inside the triangle, the $\angle$s ABC, BCA, CAB, being in segments greater than a semicircle, are each less than a right angle; *III.* 31

$\therefore$ the triangle is acute-angled.

In the second case, when S is on one of the sides as BC, $\angle\ BAC$, being in a semicircle, is right; *III.* 31

$\therefore$ the triangle is right-angled.

In the third case, when S is outside the triangle, $\angle\ BAC$, being in a segment less than a semicircle, is greater than a right angle; *III.* 31

$\therefore$ the triangle is obtuse-angled.

And conversely, if the given triangle be acute-angled, the centre of the circumscribed circle will fall within the triangle; if the triangle be right-angled, the centre will fall on the hypotenuse; if the triangle be obtuse-angled, the centre will fall without the triangle beyond the side opposite the obtuse angle.

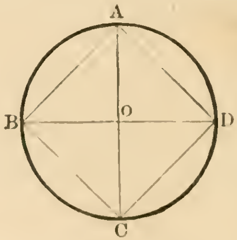
1. It is assumed in the proposition that the perpendiculars at L and K will intersect. Prove this.
2. With which proposition in the Third Book may this proposition be regarded as identical?
3. Give an easy construction for circumscribing a circle about a right-angled triangle.
4. An isosceles triangle has its vertical angle double of each of the base angles. Prove that the diameter of its circumscribed circle is equal to the base of the triangle.
5. A quadrilateral has one pair of opposite angles supplementary. Show how to circumscribe a circle about it.
6. If a perpendicular SH be drawn from S to BC, it will bisect BC.
7. If the perpendicular in the preceding deduction meet the circle below the base at D, and above the base at E, prove

   (*a*) $\angle\ BSD = \angle\ CSD = \angle\ BAC$;
   (*b*) $\angle\ BSE = \angle\ CSE = \angle\ ABC + \angle\ ACB$;
   (*c*) $\angle\ ASE = \angle\ ABC - \angle\ ACB$;
   (*d*) that AD and AE bisect the interior and exterior vertical angles at A.

8. The angle between the circumscribed radius drawn to the vertex of a triangle, and the perpendicular from the vertex on the opposite side, is equal to the difference of the angles at the base of the triangle.
9. The centre of the circle circumscribed about an equilateral triangle is equidistant from the three sides.
10. The centre of the circle circumscribed about an isosceles triangle is equidistant from the equal sides.
11. When the inscribed and circumscribed centres of a triangle coincide, the triangle is equilateral.
12. When the straight line joining the inscribed and circumscribed centres of a triangle passes through one of the vertices, the triangle is isosceles.
13. If H be the middle point of BC, what will the point S be in reference to $\triangle\ HKL$?
14. SA, SB, SC are respectively $\perp$ the sides of the orthocentric triangle of $\triangle\ ABC$.
15. The straight line joining the inscribed centre of a triangle to any vertex bisects the angle between the circumscribed radius to that vertex, and the perpendicular from that vertex on the opposite side.

PROPOSITION 6. PROBLEM.

To inscribe a square in a given circle.



Let ABC be the given circle:
it is required to inscribe a square in ABC.

Find O the centre of the $\odot\ ABC$, *III.* 1

and through O draw two diameters AC, $BD \perp$ each other; *I.* 11
join AB, BC, CD, DA. $ABCD$ is the required square.

(1) To prove $ABCD$ equilateral.

$$\text{In } \triangle\text{s } AOB, AOD, \left\{\begin{array}{c} AO = AO \\ OB = OD \\ \angle AOB = \angle AOD; \end{array}\right.$$

$\therefore AB = AD$. *I.* 4

Hence also $AB = BC$, $BC = CD$;

$\therefore ABCD$ is equilateral.

(2) To prove $ABCD$ rectangular.

Because $\angle$s ABC, BCD, CDA, DAB are right, being angles in semicircles; *III.* 31

$\therefore ABCD$ is rectangular;

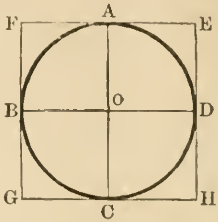
$\therefore ABCD$ is a square. *I. Def.* 32

Cor.—If the arcs AB, BC, CD, DA be bisected, the points of bisection along with A, B, C, D will form the vertices of a regular octagon inscribed in the circle. If the arcs cut off by the sides of the octagon be bisected, the vertices of a regular figure of 16 sides inscribed in the circle will be obtained. Repeated bisections will give regular figures of 32, 64, 128, 256, &c. sides inscribed in the circle. All these numbers 4, 8, 16, 32, 64, &c. are comprised in the formula 2^n, where n is any positive integer greater than 1.

1. Prove that $ABCD$ is equilateral by using III. 26, 29.
2. The square inscribed in a circle is double of the square on the radius, and half of the square on the diameter.
3. All the squares inscribed in a circle are equal.
4. If the ends of any two diameters of a circle be joined consecutively, the figure thus inscribed is a rectangle.
5. What is the magnitude of the angle at the centre of a circle subtended by a side of the inscribed square?
6. If r denote the radius of the given circle, then the side of the inscribed square will be denoted by $r\sqrt{2}$.

PROPOSITION 7. Problem.

To circumscribe a square about a given circle.



Let ABC be the given circle:
it is required to circumscribe a square about ABC.

Find O the centre of the $\odot$ ABC, *III.* 1
and through O draw two diameters AC, BD $\perp$ each other. *I.* 11
At A, B, C, D, draw EF, FG, GH, HE, tangents to the circle. *III.* 17

$EFGH$ is the required square.

(1) To prove $EFGH$ equilateral.
Because EF and GH are both $\perp$ AC, *III.* 18
and BD is also $\perp$ AC; *Const.*
$\therefore$ EF, BD, and GH are all parallel. *I.* 28, *Cor.*
Hence also FG, AC, and HE are all parallel;
$\therefore$ all the quadrilaterals in the figure are $\|^{ms}$. *I. Def.* 33
Hence EF and GH are each $= BD$, *I.* 34
and FG and HE are each $= AC$. *I.* 34
But $AC = BD$;
$\therefore$ EF, FG, GH, HE are all equal.

(2) To prove $EFGH$ rectangular.
Because OE is a $\|^{m}$, $\therefore$ $\angle E = \angle AOD$; *I.* 34
$\therefore$ $\angle E$ is right.

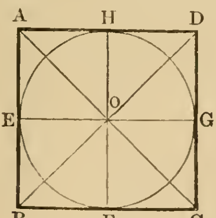
Hence also $\angle$s F, G, H are right.

$\therefore$ $EFGH$ is a square. *I. Def.* 32

1. It is assumed in the proposition that the four tangents at A, B, C, D will form a closed figure. Prove this.
2. The square circumscribed about a circle is double of the square inscribed in the circle.
3. All the squares circumscribed about a circle are equal.
4. If a rectangle be circumscribed about a circle, it must be a square.
5. If tangents be drawn at the ends of any two diameters of a circle and produced to meet, the figure thus circumscribed is a rhombus.
6. What is the magnitude of the angle at the centre of a circle, subtended by a side of the circumscribed square?
7. If r denote the radius of the given circle, then the side of the circumscribed square will be denoted by $2r$.

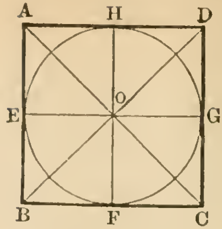
PROPOSITION 8. Problem.

To inscribe a circle in a given square.



Let $ABCD$ be the given square:
it is required to inscribe a $\odot$ *in* $ABCD$.

Join AC, BD intersecting at O;
and from O draw OE, OF, OG, OH $\perp$ the sides of the square.



$$\text{In } \triangle\text{s } BAC,\ DAC, \left\{\begin{array}{ll} BA = DA & \textit{I. Def. } 32 \\ AC = AC & \\ BC = DC; & \textit{I. Def. } 32 \end{array}\right.$$

$\therefore\ \angle\ BAC = \angle\ DAC$, and $\angle\ BCA = \angle\ DCA$; *I.* 8

$\therefore$ the diagonal AC bisects $\angle$ s BAD, BCD.

Hence also, the diagonal BD bisects $\angle$ s ABC, ADC.

$$\text{In } \triangle\text{s } OEB,\ OFB, \left\{\begin{array}{l} \angle\ OEB = \angle\ OFB \\ \angle\ OBE = \angle\ OBF \\ \quad OB = OB; \end{array}\right.$$

$\therefore\ OE = OF$. *I.* 26

Hence also $OF = OG$, $OG = OH$;

$\therefore\ OE$, OF, OG, OH are all equal.

With centre O and radius OE, describe a circle which will pass through the points E, F, G, H.

Of this circle, OE, OF, OG, OH will be radii;

and since AB, BC, CD, DA are $\perp$ OE, OF, OG, OH, *Const.*

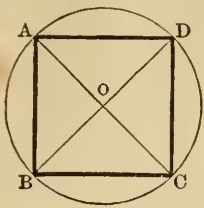
$\therefore\ AB$, BC, CD, DA will be tangents to $\odot\ EFGH$; *III.* 16

$\therefore\ \odot\ EFGH$ is inscribed in the square $ABCD$.

1. Could O, the centre of the inscribed circle, be found in any other way than by joining AC, BD?
2. Show that a circle cannot be inscribed in a rectangle unless it be a square.
3. Inscribe a circle in a given rhombus.
4. Enumerate the $\|^{ms}$ in which circles can be inscribed.
5. If a denote a side of the given square, then the radius of the inscribed circle will be denoted by $\frac{1}{2}a$.

PROPOSITION 9. PROBLEM.

To circumscribe a circle about a given square.



Let $ABCD$ be the given square :
it is required to circumscribe a circle about $ABCD$.

Join AC, BD intersecting at O.

In $\triangle$s BAC, DAC, $\begin{cases} BA = DA & \textit{I. Def. } 32 \\ AC = AC \\ BC = DC; & \textit{I. Def. } 32 \end{cases}$

$\therefore \angle BAC = \angle DAC$, and $\angle BCA = \angle DCA$, *I.* 8

$\therefore$ the diagonal AC bisects $\angle$s BAD, BCD.

Hence also, the diagonal BD bisects $\angle$s ABC, ADC.

Because $\angle OAB = \angle OBA$, each being half a rt. $\angle$,

$\therefore OA = OB$. *I.* 6

Hence also $OB = OC$, and $OC = OD$;

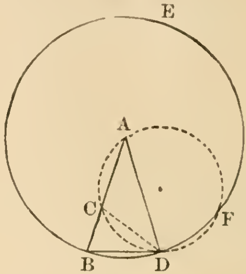
$\therefore OA$, OB, OC, OD are all equal.

With centre O and radius OA, describe a circle which will pass through the points A, B, C, D,
and $\therefore$ will be circumscribed about the square $ABCD$.

1. Show that a circle cannot be circumscribed about a rhombus unless it be a square.
2. Circumscribe a circle about a given rectangle.
3. Enumerate the $\|^{\text{ms}}$ about which circles can be circumscribed.
4. If a denote a side of the given square, then the radius of the circumscribed circle will be denoted by $\frac{1}{2}a\sqrt{2}$.

PROPOSITION 10. Problem.

To describe an isosceles triangle having each of the angles at the base double of the third angle.



Take any straight line AB,
and divide it internally at C so that $AB \cdot BC = AC^2$. *II.* 11
With centre A and radius AB, describe the $\odot$ BDE,
in which place the chord $BD = AC$. *IV.* 1
Join AD. ABD is the required isosceles triangle.

Join CD, and about $\triangle$ ACD circumscribe the $\odot$ ACD. *IV.* 5
Because $AB \cdot BC = AC^2$, *Const.*
$= BD^2$;
$\therefore$ BD is a tangent to the $\odot$ ACD. *III.* 37
Because the chord DC is drawn from D, the point of contact of the tangent BD;
$\therefore$ $\angle BDC = \angle A$. *III.* 32
Add to each the $\angle CDA$;
$\therefore$ $\angle BDA = \angle A + \angle CDA$;
$\therefore$ $\angle DBA = \angle A + \angle CDA$. *I.* 5
But $\angle DCB = \angle A + \angle CDA$; *I.* 32
$\therefore$ $\angle DBA$ or $\angle DBC = \angle DCB$;

$\therefore \qquad DC = DB, \qquad$ *I.* 6
$\qquad\qquad = AC.$

$\therefore \angle A = \angle CDA$, or $\angle A + \angle CDA =$ twice $\angle A$. *I.* 5

But $\qquad \angle BDA = \angle A + \angle CDA$;

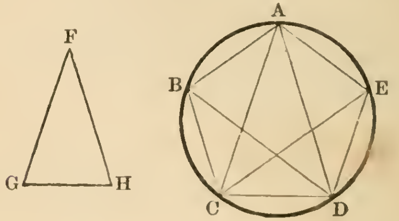
$\therefore \angle BDA$, and consequently $\angle DBA =$ twice $\angle A$.

1. The $\triangle DBC$ is equiangular to $\triangle ABD$.
2. Angle A = one-fifth of two right angles.
3. Divide a right angle into five equal parts.
4. The $\triangle CAD$ has one of its angles thrice each of the other two.
5. On a given base, construct an isosceles triangle having each of its base angles double of the vertical angle.
6. On a given base, construct an isosceles triangle having each of its base angles one-third of the vertical angle.
7. The small circle in the figure to the proposition must cut the large one. (Campanus.)
8. If the small circle cut the large one at F, and DF be joined, $DF = BD$. (Campanus.)
9. BD is a side of a regular decagon inscribed in the large circle.
10. AC and CD are sides of a regular pentagon inscribed in the small circle.
11. The small circle = the circle circumscribed about $\triangle ABD$.
12. If BF be joined, BF is a side of a regular pentagon inscribed in the large circle.
13. If AF and FC be joined, $\triangle$s ADF, FAC possess the property required in the proposition.
14. If DC be produced to meet the large circle at G, and BG be joined, BG is a side of a regular pentagon inscribed in the large circle.
15. If FG be joined, FG bisects AC perpendicularly.
16. Divide a right angle into fifteen equal parts.
17. The square on a side of a regular pentagon inscribed in a circle is greater than the square on a side of the regular decagon inscribed in the same circle by the square on the radius. (Euclid, XIII. 10.)
18. Show, by referring to I. 22, that the large circle could be omitted from the figure of the proposition.
19. Show that the proposition could be proved without describing the small circle, by drawing a perpendicular from D to BC.

20. Show that the centre of the circle circumscribed about $\triangle$ BCD is the middle point of the arc CD.
21. What is the magnitude of the angle at the centre of a circle subtended by a side of the inscribed regular decagon?
22. If r denote the radius of the circle, then the side of the inscribed regular decagon will be denoted by $\frac{1}{2}r(\sqrt{5} - 1)$.

PROPOSITION 11. PROBLEM.

To inscribe a regular pentagon in a given circle.



Let ABC be the given circle:
it is required to inscribe a regular pentagon in ABC.

Describe an isosceles $\triangle$ FGH, having each of its $\angle$s G, H double of $\angle$ F; *IV.* 10
in the $\odot$ ABC inscribe a $\triangle$ ACD equiangular to $\triangle$ FGH, so that $\angle$s ACD, ADC may each be double of $\angle$ CAD. *IV.* 2
Bisect $\angle$s ACD, ADC by CE, DB; *I.* 9
and join AB, BC, DE, EA.

$ABCDE$ is the required regular pentagon.

(1) To prove the pentagon equilateral.

Because $\angle$s ACD, ADC are each double of $\angle$ CAD, *Const.*
and they are bisected by CE, DB;
$\therefore$ the five $\angle$s ADB, BDC, CAD, DCE, ECA are all equal;
$\therefore$ the five arcs AB, BC, CD, DE, EA are all equal; *III.* 26
$\therefore$ the five chords AB, BC, CD, DE, EA are all equal. *III.* 29

(2) To prove the pentagon equiangular.

Since the five arcs AB, BC, CD, DE, EA are all equal,

$\therefore$ each is one-fifth of the whole $\bigcirc^{ce}$;

$\therefore$ any three of them = three-fifths of the $\bigcirc^{ce}$.

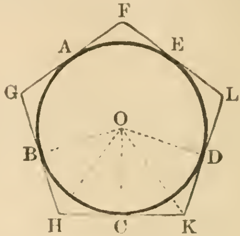
Now the five $\angle$s ABC, BCD, CDE, DEA, EAB stand each on an arc = three-fifths of the $\bigcirc^{ce}$;

$\therefore$ these five angles are all equal. *III.* 27

1. How many diagonals can be drawn in a regular pentagon?
2. Prove that each diagonal is $\parallel$ a side of the regular pentagon.
3. All the diagonals of a regular pentagon are equal.
4. The diagonals of a regular pentagon cut each other in medial section.
5. The intersections of the diagonals of a regular pentagon are the vertices of another regular pentagon.
6. The intersections of the alternate sides of a regular pentagon are the vertices of another regular pentagon.
7. If BE be joined, show that there will be in the figure five pentagons, each of which is equilateral but not equiangular.
8. Prove $\triangle$ ABC less than one-third, but greater than one-fourth of $ABCDE$.
9. Prove $\triangle$ ACD less than one-half, but greater than one-third of $ABCDE$.
10. Use the twelfth deduction from IV. 10, to obtain another method of inscribing a regular pentagon in a given circle.
11. What is the magnitude of an angle of a regular pentagon?
12. Knowing the magnitude of an angle of a regular pentagon, how can we construct a regular pentagon on a given straight line?
13. Construct a regular pentagon on a given straight line, by any other method.
14. If the alternate sides of a regular pentagon be produced to meet, the sum of the five angles at the points of intersection is equal to two right angles. (Campanus.)
15. What is the magnitude of the angle at the centre of a circle subtended by a side of the inscribed regular pentagon?
16. If r denote the radius of the circle, then the side of the inscribed regular pentagon will be denoted by $\frac{1}{2}r\sqrt{10 - 2\sqrt{5}}$.

PROPOSITION 12. Problem.

To circumscribe a regular pentagon about a given circle.



Let ABC be the given circle:
it is required to circumscribe a regular pentagon about ABC.

Find A, B, C, D, E the vertices of a regular pentagon inscribed in the circle; *IV.* 11
at these points draw FG, GH, HK, KL, LF tangents to the circle. *III.* 17

$FGHKL$ is the required regular pentagon.

Find O the centre of the circle, *III.* 1
and join OB, OH, OC, OK, OD.

(1) To prove the pentagon equiangular.
Because $OBHC$ is a quadrilateral;
∴ the sum of its four $\angle$s = 4 rt. $\angle$s. *I.* 32, *Cor.* 2
But $\angle\ OBH + \angle\ OCH$ = 2 rt. $\angle$s; *III.* 18
∴ $\angle\ BHC$ is supplementary to $\angle\ BOC$.
Hence also, $\angle\ CKD$ is supplementary to $\angle\ COD$.
But since B, C, D are consecutive vertices of an inscribed regular pentagon;
∴ arc BC = arc CD; *III.* 28
∴ $\angle\ BOC = \angle\ COD$. *III.* 27
Hence $\angle\ BHC = \angle\ CKD$.

Now $\angle$ *BHC* and $\angle$ *CKD* are any two consecutive angles of the pentagon;

$\therefore$ all the angles of the pentagon are equal.

(2) To prove the pentagon equilateral.

$$\text{In } \triangle\text{s } BOH, COH, \left\{\begin{array}{l} BO = CO \\ OH = OH \\ BH = CH; \end{array}\right. \qquad \textit{III. 17 Cor.}$$

$\therefore \angle BOH = \angle COH;$ *I.* 8

$\therefore \angle BOC$ is double of $\angle HOC$.

Hence also, $\angle DOC$ is double of $\angle KOC$.

But because $\angle BOC = \angle DOC$, $\therefore \angle HOC = \angle KOC$.

$$\text{In } \triangle\text{s } HOC, KOC, \left\{\begin{array}{l} \angle HOC = \angle KOC \\ \angle OCH = \angle OCK \\ \quad OC = OC; \end{array}\right. \qquad \textit{III. 18}$$

$\therefore HC = KC;$ *I.* 26

$\therefore HK$ is double of HC.

Similarly, GH is double of HB.

But since $HB = HC$, $\therefore GH = HK$.

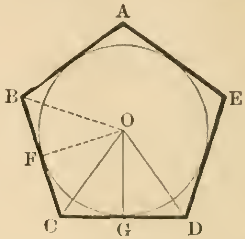
Now GH and HK are any two consecutive sides of the pentagon;

$\therefore$ all the sides of the pentagon are equal.

1. It is assumed in the proposition that the five tangents at A, B, C, D, E will form a closed figure. Prove this.
2. Prove that the regular pentagon circumscribed about a circle might be obtained thus: Inscribe a regular pentagon $ABCDE$ in the circle; bisect the arcs AB, BC, CD, DE, EA, and at the points of bisection draw parallels to the sides of the inscribed pentagon.
3. What is the magnitude of the angle at the centre of a circle subtended by a side of the circumscribed regular pentagon?
4. If any regular polygon be inscribed in a circle, tangents at its vertices will form another regular polygon of the same number of sides circumscribed about the circle.

PROPOSITION 13. Problem.

To inscribe a circle in a given regular pentagon.



Let $ABCDE$ be the given regular pentagon:
it is required to inscribe a circle in $ABCDE$.

Bisect $\angle$ s BCD, CDE by CO, DO intersecting at O; *I.* 9
join OB, and draw OF, $OG \perp BC$, CD. *I.* 12

$$\text{In } \triangle\text{s } BCO,\ DCO, \left\{\begin{array}{ll} BC = DC & \textit{Hyp.} \\ CO = CO & \\ \angle BCO = \angle DCO; & \textit{Const.} \end{array}\right.$$

$\therefore \angle CBO = \angle CDO$. *I.* 4
But $\angle CDO$ is half of the angle of a regular pentagon; *Const.*
$\therefore \angle CBO$ is half of the angle of a regular pentagon;
$\therefore OB$ bisects $\angle CBA$.
Hence also, OA would bisect $\angle BAE$, and OE, $\angle AED$.

$$\text{In } \triangle\text{s } OFC,\ OGC, \left\{\begin{array}{ll} \angle OFC = \angle OGC & \\ \angle OCF = \angle OCG & \textit{Const.} \\ OC = OC: & \end{array}\right.$$

$\therefore OF = OG$. *I.* 26
Now since O is the point where the bisectors of all the angles of the pentagon meet, and OF, OG are perpendiculars on any two consecutive sides;
$\therefore$ the perpendiculars from O on all the sides are equal.
Hence the circle described with O as centre and OF as

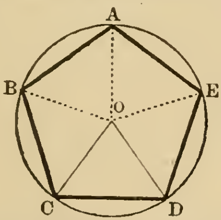
radius, will pass through the feet of all the perpendiculars from O;

and will touch AB, BC, CD, DE, EA. *III.* 16

1. Find the centre and radius of the circle inscribed in a regular pentagon by means of a square. (A set square or T square is meant.)
2. The area of a regular pentagon is equal to the rectangle contained by its semi-perimeter and the radius of the inscribed circle.

PROPOSITION 14. Problem.

To circumscribe a circle about a given regular pentagon.



Let $ABCDE$ be the given regular pentagon :
it is required to circumscribe a circle about $ABCDE$.

Bisect $\angle$s BCD, CDE by CO, DO intersecting at O; *I.* 9
and join OB, OA, OE.

OB, OA, OE bisect the $\angle$s CBA, BAE, AED. *IV.* 13
Because $\angle\ OCD = \angle\ ODC$, each being half of the angle of a regular pentagon;

$\therefore\ OC = OD$. *I.* 6

Hence also, $OD = OE$, $OE = OA$, $OA = OB$;

$\therefore$ the circle described with O as centre and OA as radius, will pass through A, B, C, D, E,

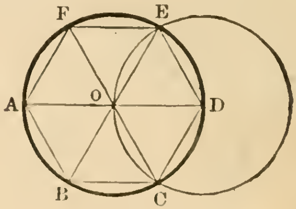
and will be circumscribed about the pentagon $ABCDE$.

1. Find the centre and radius of the circle circumscribed about a regular pentagon by means of a square.

2. The square on the diameter of the circle circumscribed about a regular pentagon = the square on one of the sides of the pentagon together with the square on the diameter of the inscribed circle.
3. If a denote a side of the given regular pentagon, then the radius of the circumscribed circle will be denoted by $\frac{1}{10} a\sqrt{50 + 10\sqrt{5}}$.

PROPOSITION 15. PROBLEM.

To inscribe a regular hexagon in a given circle.



Let ABC be the given circle:
it is required to inscribe a regular hexagon in ABC.

Find O the centre of the circle, *III.* 1
and draw a diameter AOD.
With centre D and radius DO, describe the $\odot$ EOC;
join EO, CO, and produce them to B and F.
Join AB, BC, CD, DE, EF, FA.
$ABCDEF$ is the required regular hexagon.

(1) To prove the hexagon equilateral.
$\triangle$s DOE, DOC are equilateral; *I.* 1
$\therefore$ $\angle$s DOE, DOC are each one-third of two rt. $\angle$s. *I.* 32
But $\angle$ DOE + $\angle$ DOC + $\angle$ COB = two rt. $\angle$s; *I.* 13
$\therefore$ $\angle$ COB = one-third of two rt. $\angle$s.
Hence $\angle$s BOA, AOF, FOE are each = one-third of two rt. $\angle$s; *I.* 15

∴ the six $\angle$s AOB, BOC, COD, DOE, EOF, FOA are all equal;

∴ the six arcs AB, BC, CD, DE, EF, FA are all equal; *III.* 26

∴ the six chords AB, BC, CD, DE, EF, FA are all equal. *III.* 29

(2) To prove the hexagon equiangular.

Since the six arcs AB, BC, CD, DE, EF, FA are all equal,

∴ each is one-sixth of the whole $\bigcirc^{ce}$;

∴ any four of them = four-sixths of the whole $\bigcirc^{ce}$.

Now the six $\angle$s FAB, ABC, BCD, CDE, DEF, EFA stand each on an arc = four-sixths of the $\bigcirc^{ce}$;

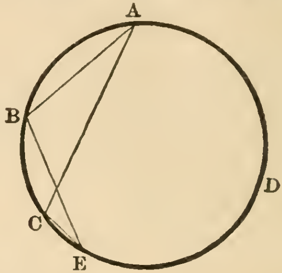
∴ these six angles are all equal. *III.* 27

Cor.—The side of a regular hexagon inscribed in a circle is equal to the radius.

1. If the points A, C, E be joined, $\triangle$ ACE is equilateral.
2. The area of an inscribed equilateral triangle is half that of a regular hexagon inscribed in the same circle.
3. Construct a regular hexagon on a given straight line.
4. The area of an equilateral triangle described on a given straight line is one-sixth of the area of a regular hexagon described on the same straight line.
5. The opposite sides of a regular hexagon are parallel.
6. The straight lines which join the opposite vertices of a regular hexagon are concurrent, and are each $\parallel$ one of the sides.
7. How many diagonals can be drawn in a regular hexagon?
8. Prove that six of them are parallel in pairs.
9. The area of a regular hexagon inscribed in a circle is half of the area of an equilateral triangle circumscribed about the circle.
10. The square on a side of an inscribed regular hexagon is one-third of the square on a side of the equilateral triangle inscribed in the same circle.
11. What is the magnitude of the angle at the centre of a circle subtended by a side of an inscribed regular hexagon?
12. Give the constructions for inscribing a circle in a regular hexagon; and for circumscribing a regular hexagon about a circle, and a circle about a regular hexagon.

PROPOSITION 16. PROBLEM.

To inscribe a regular quindecagon in a given circle.



Let ABC be the given circle:
it is required to inscribe a regular quindecagon in ABC.

Find AC a side of an equilateral triangle inscribed in the circle; *IV.* 2
and find AB, BE two consecutive sides of a regular pentagon inscribed in the circle. *IV.* 11

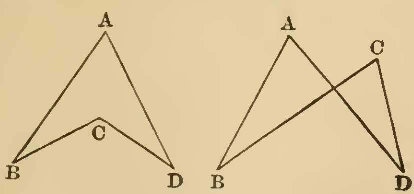
Then arc $ABE = \frac{2}{5}$ of the $\bigcirc^{ce}$,
and arc $AC = \frac{1}{3}$ of the $\bigcirc^{ce}$;
$\therefore$ arc $CE = (\frac{2}{5} - \frac{1}{3})$, or $\frac{1}{15}$, of the $\bigcirc^{ce}$.
Hence, if CE be joined, CE will be a side of a regular quindecagon inscribed in the $\odot$ ABC.
Place consecutively in the $\bigcirc^{ce}$ chords equal to CE; *IV.* 1
then a regular quindecagon will be inscribed in the circle.

1. How could the regular quindecagon be obtained, if, besides AC, a side of an equilateral triangle, only one side AB of the regular pentagon be drawn?
2. How could the regular quindecagon be obtained by making use of the sides of the regular inscribed hexagon and decagon?

3. In a given circle inscribe a triangle whose angles are as the numbers 2, 5, 8 ; and another whose angles are as the numbers 4, 5, 6.
4. Give the constructions for inscribing a circle in a regular quindecagon ; and for circumscribing a regular quindecagon about a circle, and a circle about a regular quindecagon.
5. How many diagonals can be drawn in a regular quindecagon ?
6. Show that if a polygon have n sides, it will have $\frac{1}{2}n(n-3)$ diagonals.
7. Show that the centres of the circles inscribed in, and circumscribed about, any regular figure coincide, and are obtained by bisecting any two consecutive angles of the figure.

NOTE 1.—The regular polygons of 3, 4, 5, and 15 sides, and such as may be derived from them by continued arcual bisection, were, till the time of Gauss, the only ones discovered by the ancient Greek, and known to the modern European, geometers to be inscriptible in a circle by the methods of elementary geometry. Gauss, in 1796, found that a regular polygon of 17 sides was inscriptible, and in his *Disquisitiones Arithmeticæ*, published in 1801, he showed that any regular polygon was inscriptible, provided the number of its sides was a prime number, and expressible by $2^n + 1$. (A good account of Gauss and his works is given in *Nature*, vol. xv. pp. 533–537.)

NOTE 2.—The polygons of which Euclid treats are all of one kind, namely, *convex* polygons, that is to say, polygons each of whose angles is less than two right angles. There are others, however, called *re-entrant*, and *intersectant* (or *concave*, and *crossed*), such as $ABCD$ in the accompanying figures. The reader will find



it instructive to inquire how far the properties of convex polygons (for example, quadrilaterals) are true for the others. Among the intersectant polygons there is a class called *stellate* or *star*, which are obtained thus: Suppose A, B, C, D, E (see fig. to IV. 11) to be five points in order on the $\bigcirc^{ce}$ of a circle. Join AC, CE, EB, BD, DA; then $ACEBD$ is a star pentagon. If the arcs AB, BC, &c. are all equal, the star pentagon $ACEBD$ is regular. Similarly, if 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 denote the vertices of a regular decagon inscribed in a circle, the regular star decagon (there can be only one) is got by joining consecutively 1, 4, 7, 10, 3, 6, 9, 2, 5, 8, 1. It will be found that if a regular polygon have n sides, the number of regular star polygons that may be derived from it is equal to the number of integers prime to n contained in the series 2, 3, 4, ... $\frac{1}{2}(n - 1)$. (For more information on the subject of star polygons, see Chasles, *Aperçu Historique sur l'Origine et le Développement des Méthodes en Géométrie*, sec. éd. pp. 476–487, and Georges Dostor, *Théorie Générale des Polygones Étoilés*, 1880.)

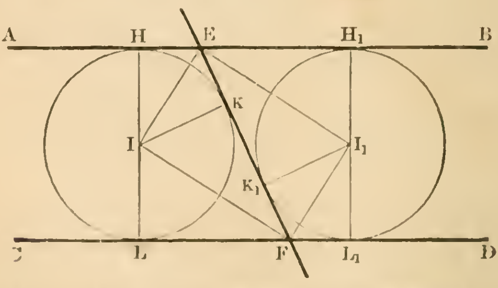
APPENDIX IV.

Proposition 1.

To describe a circle which shall touch three given straight lines.

(1) If the three straight lines be so situated that *every two are parallel*, the solution is impossible.

(2) If they be so situated that *only two are parallel*, there can be two solutions, as will appear from the following figure:

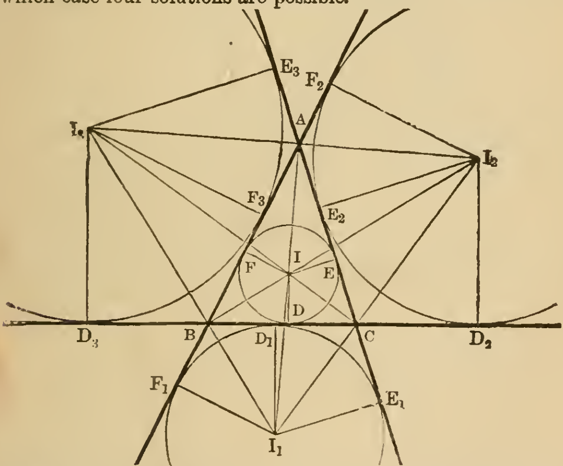


Let AB, CD, EF be the three straight lines of which AB is $\parallel$ CD.

Bisect $\angle$s AEF, CFE by EI and FI, which meet at I; *I.* 9
from I draw IH, IK, IL respectively $\perp$ AB, EF, CD. *I.* 12

Then $\triangle$s IEH, IEK are equal in all respects; *I.* 26
$\therefore$ $IH = IK$.
Similarly, $IK = IL$; *I.* 26
$\therefore$ $IH = IK = IL$.
Now since $\angle$s at H, K, L are right,
$\therefore$ the circle described with I as centre and IH as radius will touch AB, EF, CD. *III.* 16
A similar construction on the other side of EF will give another circle touching the three given straight lines.

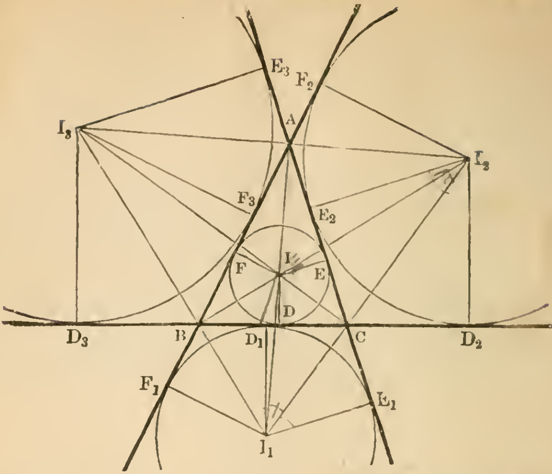
(3) If they be so situated that *no two are parallel*, then they will either all pass through the same point, in which case the solution is impossible; or they will form a triangle with its sides produced, in which case four solutions are possible.



Let AB, BC, CA produced be the three given straight lines forming by their intersection the $\triangle$ ABC.

If the interior $\angle$s B and C be bisected, the bisectors will meet at some point I, which is the centre of the circle inscribed in the triangle, as may be proved by drawing perpendiculars ID, IE, IF to the sides BC, CA, AB of the triangle. *IV.* 4

If the exterior angles at B and C be bisected by BI_1, CI_1 which



meet at I_1, and perpendiculars I_1D_1, I_1E_1, I_1F_1 be drawn to the sides BC, and AC, AB produced,

it may be proved that I_1D_1, I_1E_1, I_1F_1 are all equal, and

$\therefore$ that I_1 is the centre of a circle touching BC, and AC, AB produced.

Hence also, I_2, the point of intersection of the bisectors of the exterior angles at C and A, will be the centre of a circle touching CA, and BA, BC produced; I_3, the point of intersection of the bisectors of the exterior angles at A and B, will be the centre of a circle touching AB, and CB, CA produced.

Cor.—The following sets of points are collinear:

A, I, I_1; B, I, I_2; C, I, I_3; I_2, A, I_3; I_3, B, I_1; I_1, C, I_2.

In other words, the six bisectors of the interior and exterior angles at A, B, C meet three and three in four points, I, I_1, I_2, I_3, which are the centres of the four circles touching the three given straight lines. Or, the six straight lines joining two and two the centres of the four circles which touch AB, BC, CA, pass each through a vertex of the $\triangle$ ABC.

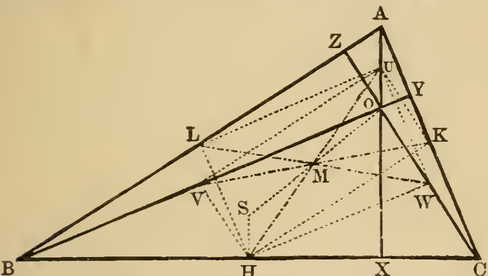
The circles whose centres are I_1, I_2, I_3 are called *escribed* or *exscribed* circles of the $\triangle$ ABC, an expression which, in its French form (*ex-inscrit*), is said to be due to Simon Lhuilier. See his *Élémens d'Analyse Géométrique et d'Analyse Algébrique* (1809), p. 198.

It is usual to denote the radius of the circle inscribed in a triangle by r, the radii of the three escribed circles by r_1, r_2, r_3, and the radius of the circumscribed circle by R.

THE MEDIOSCRIBED CIRCLE.

PROPOSITION 2.

The circle which passes through the middle points of the sides of a triangle passes also through the feet of the perpendiculars from the vertices to the opposite sides, and bisects the segments of the perpendiculars between the orthocentre and the vertices.



Let ABC be a triangle; H, K, L the middle points of its sides; X, Y, Z the feet of its perpendiculars; U, V, W the middle points of AO, BO, CO:

it is required to prove that one circle will pass through these nine points.

Join HK, HL, HU, HV, HW, KU, KV, UV, UW, WL, UL.

In $\triangle$ ABO, LU is $\parallel$ BO, and in $\triangle$ CBO, HW is $\parallel$ BO; *App. I.* 1

$\therefore$ LU is $\parallel$ HW. *I.* 30

Similarly, in $\triangle$s ABC, AOC, LH and UW are $\parallel$ AC; *App. I.* 1

$\therefore$ $LHWU$ is a $\parallel^{m}$.

But since BO is $\perp$ AC, $\therefore$ LU, HW are $\perp$ LH, UW;

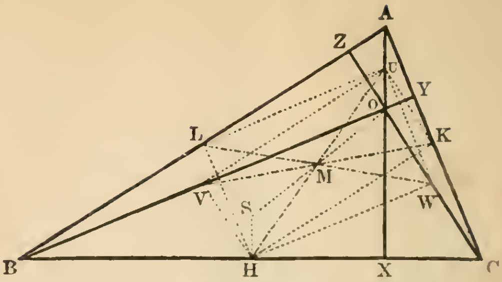
$\therefore$ $LHWU$ is a rectangle.

$\therefore$ the four points H, W, U, L lie on the circle described with HU or LW as diameter. *III.* 31

Similarly, HK, UV are $\parallel$ AB, and HV, KU $\parallel$ CO; *App. I.* 1

and since AB is $\perp$ CO, $\therefore$ $HKUV$ is a rectangle.

$\therefore$ the four points H, K, U, V lie on the circle described with HU or KV as diameter. *III.* 31



Hence the six points H, K, L, U, V, W lie on the same circle. and HU, KV, LW are diameters of it.

But since the angles at X, Y, Z are right;

$\therefore$ X lies on the circle whose diameter is HU,

Y " " KV,

Z " " LW; *III.* 31

$\therefore$ the nine specified points are concyclic.

Cor. 1.—Since HU, KV, LW are diameters of the same circle, their common point of intersection M is the centre.

Cor. 2.—M is midway between the orthocentre and the circumscribed centre.

Let S be the circumscribed centre, and SH be joined.

Then SH is $\perp$ BC (*III.* 3); and $\therefore \parallel OU$.

But $SH = OU$ (*App. I.* 5, *Cor.*); $\therefore$ $SHOU$ is a $\parallel^m$;

$\therefore$ the diagonal SO bisects HU, that is, passes through M, and is itself bisected at M.

Cor. 3.—The medioscribed diameter = the circumscribed radius.

For $SHUA$ is a $\parallel^m$; and $\therefore$ $HU = SA$.

Cor. 4.—$\triangle$s ABC, AOB, BOC, COA have the same medioscribed circle.

Since the medioscribed circle of $\triangle$ ABC passes through U, L, V, the middle points of the sides of $\triangle$ AOB, and since a circle is determined by three points; $\therefore$ the medioscribed circle of $\triangle$ ABC must also be the medioscribed circle of $\triangle$ AOB. Similarly for the other triangles.

Cor. 5.—By reference to Cor. 3, it will be seen that the circles circumscribed about $\triangle$s ABC, AOB, BOC, COA must be equal. (Carnot, *Géométrie de Position*, 1803, § 130.)

Cor. 6.—The medioscribed circle of $\triangle ABC$ is also the medioscribed circle of an infinite series of triangles.

For H, K, L, the middle points of the sides of $\triangle ABC$, may be taken as the feet of the perpendiculars of another $\triangle A'B'C'$; the middle points of the sides of $\triangle A'B'C'$ may be taken as the feet of the perpendiculars of a $\triangle A''B''C''$; and so on.

Or, instead of the median $\triangle HKL$, $\triangle$s XKL, YLH, ZHK may be taken as median triangles, and the triangles formed of which they are the median triangles; and so on.

[The circle HKL is generally called the *nine-point* circle of $\triangle ABC$, a name given by Terquem, 'le cercle des neuf points.' Following, however, the suggestion of an Italian geometer, Marsano, who calls it 'il circolo medioscritto,' I have adopted the name *medioscribed*. The property that one circle does pass through these nine points was first published in Gergonne's *Annales de Mathématiques*, vol. xi. p. 215 (1821), in an article by Brianchon and Poncelet. See this reference, or Poncelet's *Applications d'Analyse et de Géométrie*, vol. ii. p. 512. It is probable that K. W. Feuerbach of Erlangen, and T. S. Davies of Woolwich, also discovered the property independently, though they were later in publication. See Feuerbach's *Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks* (1822), and a paper by Davies on 'Symmetrical Properties of Plane Triangles,' in the *Philosophical Magazine* for July 1827. For other proofs, see Rev. Joseph Wolstenholme in *Quarterly Journal of Pure and Applied Mathematics*, vol. ii. pp. 138, 139 (1858); W. H. Besant in the *Messenger of Mathematics* (old series), vol. iii. pp. 222, 223 (1866); William Godward in the Mathematical Reprint from the *Educational Times*, vol. vii. p. 86 (1867); Desboves' *Questions de Géométrie Élémentaire*, 2ème éd. p. 146 (1875); and Casey's *Elements of Euclid*, p. 153 (1882).

The proof in the text was given by T. T. Wilkinson of Burnley in the *Lady's and Gentleman's Diary* for 1855, p. 67.

It may be mentioned that it was discovered by Feuerbach (see his *Eigenschaften*, &c. § 57) that the medioscribed circle touches the inscribed and escribed circles of $\triangle ABC$. The proofs that have been given of this theorem by elementary geometry are rather complicated: see *Lady's and Gentleman's Diary* for 1854, p. 56; *Quarterly Journal of Pure and Applied Mathematics*, vol. iv. (1861), p. 245, and vol. v. (1862), p. 270; Baltzer, *Die Elemente der Mathematik*, vol. ii. pp. 92, 93. It is also proved by J. J. Robinson in the *Lady's and Gentleman's Diary* for 1857 (and it seems to have been first noted by T. T. Wilkinson), that the medioscribed circle touches an infinite series of circles.]

DEDUCTIONS.

1. Every equilateral figure inscribed in a circle is equiangular.
2. In a given circle inscribe (*a*) three, (*b*) four, (*c*) five, (*d*) six equal circles touching each other and the given circle.
3. The perpendicular from the vertex to the base of an equilateral triangle = the side of an equilateral triangle inscribed in a circle whose diameter is the base.
4. The area of an inscribed regular hexagon = three-fourths of the area of the regular hexagon circumscribed about the same circle.
5. Inscribe a regular hexagon in a given equilateral triangle, and compare its area with that of the triangle.
6. Inscribe a regular dodecagon in a given circle, and prove that its area = that of a square described on the side of an equilateral triangle inscribed in the same circle.
7. Construct a regular octagon on a given straight line.
8. A regular octagon inscribed in a circle = the rectangle contained by the sides of the inscribed and circumscribed squares.
9. The following construction is given by Ptolemy (about 130 A.D.) in the first book of his *Almagest*, for inscribing a regular pentagon and decagon in a circle: Draw any diameter AB, and from C the centre draw $CD \perp AB$, meeting the $\bigcirc^{ce}$ at D; bisect AC at E, and join ED. From EB cut off $EF = ED$, and join DF. CF will be a side of the inscribed regular decagon, and DF a side of the inscribed regular pentagon. Prove this.
10. A ribbon or strip of paper whose edges are parallel, is folded up into a flat knot of five edges. Prove that the sides of the knot form a regular pentagon.
11. Construct a regular decagon on a given straight line.
12. In a given square inscribe an equilateral triangle one of whose vertices may be (*a*) on the middle of a side, (*b*) on one of the angular points, of the square.

Construct a triangle having given

13. The inscribed circle, and an escribed circle.
14. Two escribed circles.
15. Any three of the centres of the four contact circles.

16. The base, the vertical angle, and the inscribed radius.
17. The perimeter, the vertical angle, and the inscribed radius.
18. The base, the sum or difference of the other two sides, and the inscribed radius.
19. Prove the following properties with respect to $\triangle\ ABC$ (see fig. on p. 251):

(1) $s = AE_1 = AF_1 = BD_2 = BF_2 = CD_3 = CE_3.$

(2) $\left\{\begin{array}{l} s-a = AE = AF = BD_3 = BF_3 = CD_2 = CE_2. \\ s-b = AE_3 = AF_3 = BD = BF = CD_1 = CE_1. \\ s-c = AE_2 = AF_2 = BD_1 = BF_1 = CD = CE. \end{array}\right.$

(3) $\left\{\begin{array}{l} a = EE_1 = E_2E_3 = FF_1 = F_2F_3. \\ b = DD_2 = D_1D_3 = FF_2 = F_1F_3. \\ c = DD_3 = D_1D_2 = EE_3 = E_1E_2. \end{array}\right.$

(4) $\left\{\begin{array}{l} a + b = F_1F_2. \\ b + c = D_2D_3. \\ c + a = E_1E_3. \end{array}\right.$ (5) $\left\{\begin{array}{l} a \sim b = FF_3. \\ b \sim c = DD_1. \\ c \sim a = EE_2. \end{array}\right.$

(6) $$\begin{aligned} a + b + c &= AE + AE_1 + AE_2 + AE_3 \\ &= AF + AF_1 + AF_2 + AF_3 \\ &= BD + BD_1 + BD_2 + BD_3 \\ &= BF + BF_1 + BF_2 + BF_3 \\ &= CD + CD_1 + CD_2 + CD_3 \\ &= CE + CE_1 + CE_2 + CE_3. \end{aligned}$$

(7) $$\begin{aligned} a^2+b^2+c^2 &= AE^2 + AE_1^2 + AE_2^2 + AE_3^2 \\ &= AF^2 + AF_1^2 + AF_2^2 + AF_3^2 \\ &= BD^2 + BD_1^2 + BD_2^2 + BD_3^2 \\ &= BF^2 + BF_1^2 + BF_2^2 + BF_3^2 \\ &= CD^2 + CD_1^2 + CD_2^2 + CD_3^2 \\ &= CE^2 + CE_1^2 + CE_2^2 + CE_3^2. \end{aligned}$$

(8) $$\left.\begin{array}{l} AI^2 + AI_1^2 + AI_2^2 + AI_3^2 \\ + BI^2 + BI_1^2 + BI_2^2 + BI_3^2 \\ + CI^2 + CI_1^2 + CI_2^2 + CI_3^2 \end{array}\right\} = \begin{array}{l} 3\,(a^2 + b^2 + c^2) + \\ 3\,(r^2 + r_1^2 + r_2^2 + r_3^2). \end{array}$$

(9) $$\left.\begin{array}{l} AD^2 + AD_1^2 + AD_2^2 + AD_3^2 \\ + BE^2 + BE_1^2 + BE_2^2 + BE_3^2 \\ + CF^2 + CF_1^2 + CF_2^2 + CF_3^2 \end{array}\right\} = 5\,(a^2 + b^2 + c^2).^*$$

* The last four sets of expressions may be written more shortly by using the Greek letter Σ (sigma) as equivalent to 'the sum of all such terms as.' Thus (6) would be $a + b + c = \Sigma\,(AE) = \Sigma\,(AF) =$ &c. (9) would be $\Sigma\,(AD^2) + \Sigma\,(BE^2) + \Sigma\,(CF^2) = 5\,(a^2 + b^2 + c^2)$. This property is due to W. H. Levy; see *Lady's and Gentleman's Diary* for 1852, p. 71.

(10) Triangles mutually equiangular in sets of four are :
AIE, AI_1E_1, AI_2E_2, AI_3E_3 ; BIF, BI_1F_1, BI_2F_2, BI_3F_3 ;
CID, CI_1D_1, CI_2D_2, CI_3D_3.

(11) Mention other twelve triangles which are mutually equiangular in sets of four.

(12) Triangles mutually equiangular in sets of three are :
AIB, ACI_1, I_2CB ; BIC, BAI_2, I_3AC ; CIA, CBI_3, I_1BA.

(13) Triangles mutually equiangular in sets of four are :
I_1BI_2, I_1CI_3, IBI_3, ICI_2 ; I_2CI_3, I_2AI_1, ICI_1, IAI_3 ;
I_3AI_1, I_3BI_2, IAI_2, IBI_1.

(14) Express in terms of $\angle$s A, B, C,

(*a*) The angles of $\triangle$s $I_1I_2I_3$, DEF ; I_1BC, I_2CA, I_3AB ;
AEF, BFD, CDE.

(*b*) " subtended by AB, BC, CA at I, I_1, I_2, I_3.

(*c*) " " DE, EF, FD : I_1I_2, I_2I_3, I_3I_1 ;
I_1D, I_2E, I_3F at I.

(15) D and D_1 are equidistant from the middle point of BC ; so are D_2 and D_3. Similar relations hold for the E points and the F points.

20. Of the four points I, I_1, I_2, I_3, any one is the orthocentre of the triangle formed by joining the other three, and in each case ABC is the orthocentric triangle.

21. The orthocentre and vertices of a triangle are the inscribed and escribed centres of its orthocentric triangle. Verify in the four cases.

22. Six straight lines join the inscribed and escribed centres ; the circles described on these as diameters pass each through two vertices of the triangle, and the centres of these six circles lie on the $\bigcirc^{ce}$ of the circle circumscribed about the triangle.

23. Prove the second part of the last deduction without assuming the property of the medioscribed circle.

24. Prove the following properties (see fig. on p. 251) :

(1) The radii I_1D_1, I_2E_2, I_3F_3 are concurrent at S_1 ; ID, I_3E_3, I_2F_2 at A_1 ; I_3D_3, IE, I_1F_1 at B_1 ; I_2D_2, I_1E_1, IF at C_1.

(2) The figures $A_1I_3S_1I_2$, $B_1I_1S_1I_3$, $C_1I_2S_1I_1$ are rhombi, and $A_1I_3B_1I_1C_1I_2$ is an equilateral hexagon whose opposite sides are parallel.

(3) $\triangle$s $A_1B_1C_1$, $I_1I_2I_3$ are congruent, and their corresponding sides are parallel.

(4) The points S_1, A_1, B_1, C_1, are the circumscribed centres of $\triangle$s $I_1I_2I_3$, II_2I_3, II_3I_1, II_1I_2.

(5) The figures $A_1IB_1I_3$, $B_1IC_1I_1$, $C_1IA_1I_2$ are rhombi, and I is the circumscribed centre of $\triangle\ A_1B_1C_1$.

(6) The circumscribed circle of $\triangle\ ABC$ is the medioscribed circle of $\triangle$s $I_1I_2I_3$, II_2I_3, II_3I_1, II_1I_2; $A_1B_1C_1$, $S_1B_1C_1$, $S_1C_1A_1$, $S_1A_1B_1$; and its centre is the middle point of IS_1.

[See Davies' *Symmetrical Properties*, &c. quoted on p. 255.]

25. The area of $\triangle\ ABC = rs = r_1(s-a) = r_2(s-b) = r_3(s-c)$.

26. The bisector of the vertical angle of a triangle cuts the $\bigcirc^{ce}$ of the circumscribed circle at a point which is equidistant from the ends of the base and from the centre of the inscribed circle.

27. The diameter of the circle inscribed in a right-angled triangle together with the hypotenuse = the sum of the other two sides.

28. The rectangle under the two segments of the hypotenuse of a right-angled triangle made by the point of contact of the inscribed circle = the area of the triangle.

29. Twice the circumscribed diameter = the sum of the three escribed radii diminished by the inscribed radius.

30. The sum of the distances of the circumscribed centre from the sides of a triangle = the sum of the inscribed and circumscribed radii; and the sum of the distances of the orthocentre from the vertices = the sum of the inscribed and circumscribed diameters. (Carnot's *Géométrie de Position*, § 137.)

31. Examine the case when the circumscribed centre and orthocentre are outside the triangle.

32. If $A_1B_1C_1$ be the triangle formed by joining the escribed centres of $\triangle\ ABC$; $A_2B_2C_2$ the triangle formed by joining the escribed centres of $\triangle\ A_1B_1C_1$; $A_3B_3C_3$ the triangle formed by joining the escribed centres of $\triangle\ A_2B_2C_2$; and this process of construction be continued, the successive triangles will approximate to an equilateral triangle. (Booth's *New Geometrical Methods*, vol. ii. p. 315.)

33. If an equilateral polygon be circumscribed about a circle, it will be equiangular if the number of sides be odd. Examine the case when the number of sides is even.

34. AB, CD, two alternate sides of a regular polygon, are produced to meet at E, and O is the centre of the polygon. Prove A, E, C, O concyclic, and also D, E, B, O.

35. The sum of the perpendiculars on the sides of a regular n-gon from any point inside $= n$ times the radius of the inscribed circle. Examine the case when the point is outside.

Loci.

The base and the vertical angle of a triangle are given; find the locus of

1. The orthocentre of the triangle.
2. The centre of the inscribed circle.
3. The centres of the three escribed circles.
4. The centroid of the triangle.
5. ABC is a triangle, and E is any point in AC. Through E a straight line DEF is drawn cutting AB at F and BC produced at D; circles are circumscribed about $\triangle$s AEF, CDE. Find the locus of the other point of intersection of the circles.
6. AB and AC are two straight lines containing a fixed angle; and between AB and AC there is moved a straight line DE of given length. The perpendiculars from D and E to AB and AC meet at P, and the perpendiculars from D and E to AC and AB meet at O; find the loci of O and P.
7. Given the vertical angle of a triangle, and the sum of the sides containing it; find the locus of the centre of the circle circumscribed about the triangle.
8. A circle is given, and in it are inscribed triangles, two of whose sides are respectively parallel to two fixed straight lines. Find the locus of the centres of the circles inscribed in these triangles.
9. A circle is given, and from any point P on another given concentric circle of greater radius, tangents are drawn touching the first circle at Q and R; find the loci of the centres of the inscribed and circumscribed circles of the triangle PQR.
10. A point is taken outside a square such that of the straight lines drawn from it to the vertices of the square, the two inner ones trisect the angle between the two outer ones; show that the locus of the point is the $\bigcirc^{ce}$ of the circle circumscribed about the square.

BOOK V.

DEFINITIONS.

1. A less magnitude is said to be a **submultiple** of a greater magnitude, when the less **measures** the greater; that is, when the less is contained a certain number of times exactly in the greater.

2. A greater magnitude is said to be a **multiple** of a less, when the greater is measured by the less; that is, when the greater contains the less a certain number of times exactly.

3. **Equimultiples** of magnitudes are multiples that contain these magnitudes, respectively, the same number of times.

4. **Ratio** is a relation of two magnitudes of the same kind to one another, in respect of quantuplicity (a word which refers to the number of times or parts of a time that the one is contained in the other). The two magnitudes of a ratio are called its **terms**. The first term is called the **antecedent**; the latter, the **consequent**.

The ratio of A to B is usually expressed $A : B$. Of the two terms A and B, A is the antecedent, B the consequent.

5. If there be four magnitudes, such that if any equimultiples whatsoever be taken of the first and third, and any equimultiples whatsoever of the second and fourth, and if, according as the multiple of the first is greater than the multiple of the second, equal to it, or less, so is the multiple of the third greater than the multiple of the fourth, equal

to it, or less; then the first of the magnitudes has to the second the **same ratio** that the third has to the fourth.

COR.—Conversely, if the first of four magnitudes have to the second the same ratio that the third has to the fourth, and if any equimultiples whatsoever be taken of the first and third, and any whatsoever of the second and fourth; then according as the multiple of the first is greater than the multiple of the second, equal to it, or less, the multiple of the third shall be greater than the multiple of the fourth, equal to it, or less.

6. Magnitudes are said to be **proportionals** when the first has the same ratio to the second that the third has to the fourth; and the third to the fourth the same ratio which the fifth has to the sixth; and so on, whatever be their number.

When four magnitudes, A, B, C, D, are proportionals, it is usual to say that A is to B as C to D, and to write them thus—$A : B :: C : D$, or thus, $A : B = C : D$.

7. In proportionals, the antecedent terms of the ratios are called **homologous** to one another; so also are the consequents.

8. When four magnitudes are proportional, they constitute a **proportion**. The first and last terms of the proportion are called the **extremes**; the second and third, the **means**.

9. When of the equimultiples of four magnitudes, taken as in the fifth definition, the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first has to the second a **greater ratio** than the third magnitude has to the fourth; and the third has to the fourth a **less ratio** than the first has to the second.

COR.—Conversely, if the first of four magnitudes have to the second a greater ratio than the third has to the fourth, two numbers m and n may be found, such that, while m times the first magnitude is greater than n times the second, m times the third shall not be greater than n times the fourth.

10. When there is any number of magnitudes greater than two, of which the first has to the second the same ratio that the second has to the third, and the second to the third the same ratio which the third has to the fourth, and so on, the magnitudes are said to be **continual proportionals, or in continued proportion.**

11. When three magnitudes are in continued proportion, the second is said to be a **mean proportional** between the other two.

Three magnitudes in continued proportion are sometimes said to be in geometrical progression, and the mean proportional is then called a geometric mean between the other two.

12. When there is any number of magnitudes of the same kind, the first is said to have to the last of them the **ratio compounded** of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on to the last magnitude. Thus:

If A, B, C, D be four magnitudes of the same kind, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D. This is expressed $A : D = \left\{ \begin{array}{c} A : B \\ B : C \\ C : D \end{array} \right\}$.

13. A ratio which is compounded of two equal ratios is said to be **duplicate** of either of these ratios.

Cor.—If the three magnitudes A, B, and C are continual proportionals, the ratio of A to C is duplicate of that of A to B, or of B to C. For, by the last definition, the ratio of A to C is compounded of the ratios of A to B, and of B to C; but the ratio of A to B = the ratio of B to C, because A, B, C are continual proportionals; therefore the ratio of A to C, by this definition, is duplicate of the ratio of A to B, or of B to C.

14. A ratio which is compounded of three equal ratios is said to be **triplicate** of any one of these ratios.

Cor.—If four magnitudes A, B, C, D be continual proportionals, the ratio of A to D is triplicate of the ratio of A to B, or of B to C, or of C to D. For the ratio of A to D is compounded of the three ratios of A to B, B to C, C to D; and these three ratios are equal to one another, because A, B, C, D are continual proportionals; therefore the ratio of A to D is triplicate of the ratio of A to B, or of B to C, or of C to D.

The following technical words may be used to signify certain ways of changing either the order or the magnitude of the terms of a proportion, so that they continue still to be proportionals:

15. By **alternation**, when the first is to the third, as the second is to the fourth. (V. 16.)

16. By **inversion**, when the second is to the first, as the fourth is to the third. (V. A.)

17. By **addition**, when the sum of the first and the second is to the second, as the sum of the third and the fourth is to the fourth. (V. 18.)

18. By **subtraction**, when the difference of the first and the second is to the second, as the difference of the third and the fourth is to the fourth. (V. 17.)

19. By **equality**, when there is any number of magnitudes more than two, and as many others, so that they are proportionals when taken two and two of each rank, and it is inferred—that the first is to the last of the first rank of magnitudes, as the first is to the last of the others. Of this there are the two following kinds, which arise from the different order in which the magnitudes are taken two and two:

20. By **direct equality**, when the first magnitude is to the second of the first rank, as the first to the second of

the other rank; and as the second is to the third of the first rank, so is the second to the third of the other; and so on in a direct order. (V. 22.)

21. By **transverse equality**, when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the last but three to the last but two of the second rank; and so on in a transverse order. (V. 23.)

AXIOMS.

1. Equimultiples of the same, or of equal magnitudes, are equal to one another.

2. Those magnitudes of which the same, or equal magnitudes, are equimultiples, are equal to one another.

3. A multiple of a greater magnitude is greater than the same multiple of a less.

4. That magnitude of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

PROPOSITION 1. THEOREM.

If any number of magnitudes be equimultiples of as many others, each of each, what multiple soever any one of the first is of its submultiple, the same multiple is the sum of all the first of the sum of all the rest.

Let any number of magnitudes A, B, and C be equimultiples of as many others D, E, and F, each of each:
it is required to prove that $A + B + C$ is the same multiple of $D + E + F$ that A is of D.

Let A contain D, B contain E, and C contain F, each any number of times, as, for instance, three times;

then $\qquad A = D + D + D.$

Similarly, $\qquad B = E + E + E,$

and $\qquad C = F + F + F;$

$\therefore A + B + C = D + E + F$ taken three times. *I. Ax.* 2

Hence also, if A, B, and C were each any other equimultiple of D, E, and F, $A + B + C$ would be the same multiple of $D + E + F$.

Cor.—Hence, if m be any number, $mD + mE + mF = m(D + E + F)$.

PROPOSITION 2. Theorem.

If to a multiple of a magnitude by any number, a multiple of the same magnitude by any number be added, the sum will be the same multiple of that magnitude that the sum of the two numbers is of unity.

Let $A = mC$, and $B = nC$:
it is required to prove $A + B = (m + n)\, C$.

Since $A = mC$, $A = C + C + C + \ldots\ldots$ repeated m times.

Similarly, $\qquad B = C + C + C + \ldots\ldots$ repeated n times;

$\therefore \qquad A + B = C + C + C + \ldots\ldots$ repeated $m + n$ times,

that is, $\qquad A + B = (m + n)C;$

$\therefore A + B$ contains C as often as there are units in $m + n$.

Cor. 1.—If there be any number of multiples whatsoever, as $A = mE$, $B = nE$, $C = pE$, then $A + B + C = (m + n + p)E$.

Cor. 2.—Since $A + B + C = (m + n + p)E$,
and since $A = mE$, $B = nE$, and $C = pE$,
$\therefore mE + nE + pE = (m + n + p)E$.

PROPOSITION 3. THEOREM.

If the first of three magnitudes contain the second as often as there are units in a certain number, and if the second contain the third as often as there are units in a certain number, the first will contain the third as often as there are units in the product of these two numbers.

Let $A = mB$, and $B = nC$:
it is required to prove $A = mnC$.

Since $B = nC$,
$\therefore \quad mB = nC + nC + nC + \ldots\ldots$ repeated m times.
But $nC + nC + nC + \ldots\ldots$ repeated m times $= C$ multiplied by $\quad n + n + n + \ldots\ldots$ repeated m times. *V. 2, Cor. 2*
Now $\quad n + n + n + \ldots\ldots$ repeated m times $= mn$:
$\therefore\ mB = mnC$.
But $\quad A = mB$; *Hyp.*
$\therefore \quad A = mnC$.

PROPOSITION 4. THEOREM.

If any equimultiples be taken of the antecedents of a proportion and any equimultiples of the consequents, these multiples taken in the order of the terms are proportional.

Let $A : B = C : D$, and let m and n be any two numbers:
it is required to prove $mA : nB = mC : nD$.

Of mA and mC take equimultiples by any number p; and of nB and nD take equimultiples by any number q. Then the equimultiples of mA and mC by p are equimultiples also of A and C, for they contain A and C as

often as there are units in pm; *V.* 3
and they are equal to pmA and pmC.
Similarly the multiples of nB and nD by q are qnB, qnD.

Now since $A : B = C : D$, and of A and C there are taken any equimultiples pmA and pmC, and of B and D there are taken any equimultiples qnB, qnD;
if pmA be equal to, greater, or less than qnB, then pmC is equal to, greater, or less than qnD. *V. Def.* 5, *Cor.*
But pmA, pmC are also equimultiples of mA and mC by p;
and qnB, qnD are also equimultiples of nB and nD by q;
$\therefore\ mA : nB = mC : nD$. *V. Def.* 5

PROPOSITION 5. THEOREM.

If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other, the remainder is the same multiple of the remainder that the whole is of the whole.

Let A and B be two magnitudes of which A is greater than B, and let mA and mB be any equimultiples of them: *it is required to prove that $mA - mB$ is the same multiple of $A - B$ that mA is of A; that is, that $mA - mB = m(A - B)$.*

Let D be the excess of A above B;
then $A - B = D$.
Adding B to both, $A = D + B$;
$\therefore\ mA = mD + mB$. *V.* 1
Taking mB from both, $mA - mB = mD$.
Now $D = A - B$; $\therefore\ mA - mB = m(A - B)$.

PROPOSITION 6. THEOREM.

If from a multiple of a magnitude by any number a multiple of the same magnitude by a less number be taken away, the remainder will be the same multiple of that magnitude that the difference of the numbers is of unity.

Let mA and nA be multiples of the magnitude A by the numbers m and n, and let m be greater than n:
it is required to prove $mA - nA = (m - n)\,A$.

Let $m - n = q$; then $m = n + q$;
$\therefore\ mA = nA + qA$. V. 2
Taking nA from both, $mA - nA = qA$;
$\therefore\ mA - nA$ contains A as often as there are units in q, that is, as often as there are units in $m - n$;
$\therefore\ mA - nA = (m - n)\,A$.

PROPOSITION A. THEOREM.

The terms of a proportion are proportional by inversion.

Let $A : B = C : D$:
it is required to prove $B : A = D : C$.

Let mA and mC be any equimultiples of A and C, nB and nD any equimultiples of B and D.
Then, because $A : B = C : D$,
if mA be less than nB, mC will be less than nD; *V. Def.* 5, *Cor.*
$\therefore$ if nB be greater than mA, nD will be greater than mC.
For the same reason, if $nB = mA$, $nD = mC$,
and if nB be less than mA, nD will be less than mC.
But nB, nD are any equimultiples of B and D,
and mA, mC are any equimultiples of A and C;
$\therefore\ B : A = D : C$. *V. Def.* 5

PROPOSITION B. Theorem.

If the first be the same multiple or submultiple of the second that the third is of the fourth, the first is to the second as the third to the fourth.

Let mA, mB be equimultiples of the magnitudes A and B: *it is required to prove* $mA : A = mB : B$, and $A : mA = B : mB$:

Of mA and mB take equimultiples by any number n, and of A and B take equimultiples by any number p; these will be nmA, pA, nmB, pB. *V.* 3
Now if nmA be greater than pA, nm is greater than p; and if nm is greater than p, nmB is greater than pB;
$\therefore$ when nmA is greater than pA, nmB is greater than pB.
Similarly, if $nmA = pA$, $nmB = pB$,
and if nmA is less than pA, nmB is less than pB.
But nmA, nmB are any equimultiples of mA and mB,
and pA, pB are any equimultiples of A and B;
$\therefore$ $mA : A = mB : B$. *V. Def.* 5

Again, since $mA : A = mB : B$,
$\therefore$ $A : mA = B : mB$, by inversion. *V.* A

PROPOSITION C. Theorem.

If the first term of a proportion be a multiple or a submultiple of the second, the third is the same multiple or submultiple of the fourth.

Let $A : B = C : D$, and first let $A = mB$:
it is required to prove $C = mD$.

Of A and C take equimultiples by any number as 2,

and of B and D take equimultiples by the number $2m$;
these will be $2A$, $2C$, $2mB$, $2mD$. *V.* 3
Now since $A = mB$, $2A = 2mB$;
and since $A : B = C : D$, $\therefore$ $2C = 2mD$; *V. Def.* 5
$\therefore$ $C = mD$.

Next let A be a submultiple of B:
it is required to prove that C *is the same submultiple of* D.

Since $A : B = C : D$, *Hyp.*
$\therefore$ $B : A = D : C$, by inversion. *V.* A
But A being a submultiple of B, B is a multiple of A;
$\therefore$ D is the same multiple of C;
$\therefore$ C is the same submultiple of D that A is of B.

PROPOSITION 7. Theorem.

Equal magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.

Let A and B be equal magnitudes, and C any other:
it is required to prove $A : C = B : C$, and $C : A = C : B$.

Let mA, mB be any equimultiples of A and B,
and nC any multiple of C.

Because $A = B$, $mA = mB$; *V. Ax.* 1
$\therefore$ if mA be greater than nC, mB is greater than nC;
and if $mA = nC$, $mB = nC$;
and if mA be less than nC, mB is less than nC.
But mA and mB are any equimultiples of A and B, and nC is any multiple of C;
$\therefore$ $A : C = B : C$. *V. Def.* 5

Hence also $C : A = C : B$, by inversion. *V.* A

PROPOSITION 8. Theorem.

Of unequal magnitudes, the greater has a greater ratio to any other magnitude than the less has; and the same magnitude has a greater ratio to the less of two magnitudes than it has to the greater.

Let $A + B$ be a magnitude greater than A, and C a third magnitude:

it is required to prove $A + B : C$ *greater than* $A : C$,
and $C : A$ *greater than* $C : A + B$.

Let m be such a number that mA and mB are each of them greater than C, and let nC be the least multiple of C that exceeds $mA + mB$;

then $nC - C$ will be less than $mA + mB$,
that is, $(n - 1)C$ will be less than $m(A + B)$;
$\therefore$ $m(A + B)$ is greater than $(n - 1)C$.
But because nC is greater than $mA + mB$,
and C is less than mB;
$\therefore$ $nC - C$ is greater than mA,
that is, mA is less than $nC - C$, or $(n - 1)C$.
Hence the multiple of $A + B$ by m exceeds the multiple of C by $n - 1$, but the multiple of A by m does not exceed the multiple of C by $n - 1$,
$\therefore$ $A + B : C$ is greater than $A : C$. *V. Def.* 9

Again, because the multiple of C by $n - 1$ exceeds the multiple of A by m, but does not exceed the multiple of $A + B$ by m;
$\therefore$ $C : A$ is greater than $C : A + B$. *V. Def* 9

PROPOSITION 9. THEOREMS.

Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

First let $A : C = B : C$:
it is required to prove $A = B$.

For if A be greater than B,
then $A : C$ is greater than $B : C$. *V.* 8
And if B be greater than A,
then $B : C$ is greater than $A : C$. *V.* 8
Hence $A = B$.

Next let $C : A = C : B$:
it is required to prove $A = B$.

For $A : C = B : C$, by inversion; *V.* A
$\therefore A = B$.

PROPOSITION 10. THEOREMS.

That magnitude which has a greater ratio than another has to the same magnitude is the greater of the two; and that magnitude to which the same has a greater ratio than it has to another magnitude is the less of the two.

Let $A : C$ be greater than $B : C$:
it is required to prove A greater than B.

Because $A : C$ is greater than $B : C$,
two numbers m and n may be found such that mA is greater than nC, and mB not greater than nC; *V. Def.* 9, *Cor.*
$\therefore mA$ is greater than mB,
$\therefore A$ is greater than B. *V. Ax.* 4

Next let $C : B$ be greater than $C : A$:
it is required to prove B less than A.

For two numbers m and n may be found such that nC is greater than mB, and nC not greater than mA;

$\therefore$ mB is less than mA;

$\therefore$ B is less than A. *V. Ax.* 4

PROPOSITION 11. THEOREM.

Ratios that are equal to the same ratio are equal to one another.

Let $A : B = C : D$ and $C : D = E : F$:
it is required to prove $A : B = E : F$.

Take mA, mC, mE any equimultiples of A, C, and E,
and nB, nD, nF any equimultiples of B, D, and F.

Because $A : B = C : D$, if mA be greater than nB,
mC must be greater than nD. *V. Def.* 5, *Cor.*
But because $C : D = E : F$, if mC be greater than nD,
mE must be greater than nF; *V. Def.* 5, *Cor.*
$\therefore$ if mA be greater than nB, mE is greater than nF.
Similarly, if $mA = nB$, $mE = nF$,
and if mA be less than nB, mE is less than nF;
$\therefore$ $A : B = E : F$. *V. Def.* 5

PROPOSITION 12. THEOREM.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let $A : B = C : D$ and $C : D = E : F$:
it is required to prove $A : B = A + C + E : B + D + F$.

Take mA, mC, mE any equimultiples of A, C, and E,
and nB, nD, nF any equimultiples of B, D, and F.

Because $A : B = C : D$, if mA be greater than nB, mC must be greater than nD; *V. Def.* 5, *Cor.*
and because $C : D = E : F$, when mC is greater than nD, mE is greater than nF. *V. Def.* 5, *Cor.*
$\therefore$ if mA be greater than nB, $mA + mC + mE$ is greater than $nB + nD + nF$.
Similarly, if $mA = nB$, $mA + mC + mE = nB + nD + nF$; and if mA be less than nB, $mA + mC + mE$ is less than $nB + nD + nF$.
Now $mA + mC + mE = m(A + C + E)$; *V.* 1 *Cor.*
so that mA and $mA + mC + mE$ are any equimultiples of A and $A + C + E$.
Similarly, nB and $nB + nD + nF$ are any equimultiples of B and $B + D + F$;
$\therefore$ $A : B = A + C + E : B + D + F$. *V. Def.* 5

PROPOSITION 13. Theorem.

If the first have to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth, the first shall also have to the second a greater ratio than the fifth has to the sixth.

Let $A : B = C : D$, but $C : D$ greater than $E : F$:
it is required to prove $A : B$ *greater than* $E : F$.

Because $C : D$ is greater than $E : F$,
there are two numbers m and n such that mC is greater than nD, but mE is not greater than nF. *V. Def.* 9
But because $A : B = C : D$, if mC is greater than nD, mA is greater than nB; *V. Def.* 5, *Cor.*
$\therefore$ mA is greater than nB, and mE is not greater than nF;
$\therefore$ $A : B$ is greater than $E : F$. *V. Def.* 9

PROPOSITION 14. Theorem.

If the first term of a proportion be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

Let $A : B = C : D$:
it is required to prove that if A be greater than C, B is greater than D; if $A = C$, $B = D$; if A be less than C, B is less than D.

First, let A be greater than C;
then $A : B$ is greater than $C : B$. *V.* 8
But $A : B = C : D$; *Hyp.*
$\therefore$ $C : D$ is greater than $C : B$; *V.* 13
$\therefore$ B is greater than D. *V.* 10
Similarly, it may be proved that if $A = C$, $B = D$;
and if A be less than C, B is less than D.

PROPOSITION 15. Theorem.

Magnitudes have the same ratio to one another which their equimultiples have.

Let A and B be two magnitudes, and m any number:
it is required to prove $A : B = mA : mB$.

Because $A : B = A : B$; *V.* 7
$\therefore$ $A : B = A + A : B + B$, *V.* 12
$= 2A : 2B$.
Again, since $A : B = 2A : 2B$;
$\therefore$ $A : B = A + 2A : B + 2B$, *V.* 12
$= 3A : 3B$;
and so on for all the equimultiples of A and B.

PROPOSITION 16. Theorem.

The terms of a proportion, if they be all of the same kind, are proportional by alternation.

Let $A : B = C : D$:
it is required to prove $A : C = B : D$.

Take mA, mB any equimultiples of A and B,
and nC, nD any equimultiples of C and D.

Then	$A : B = mA : mB$.	*V.* 15
But	$A : B = C : D$;	*Hyp.*
$\therefore$	$C : D = mA : mB$.	*V.* 11
Again,	$C : D = nC : nD$;	*V.* 15
$\therefore$	$mA : mB = nC : nD$.	*V.* 11

Now, if mA be greater than nC, mB is greater than nD;
if $mA = nC$, $mB = nD$; and if mA be less than nC,
mB is less than nD; *V.* 14
$\therefore A : C = B : D$. *V. Def.* 5

PROPOSITION 17. Theorem.

The terms of a proportion are proportional by subtraction.

Let $A + B : B = C + D : D$:
it is required to prove $A : B = C : D$.

Take mA and nB any multiples of A and B by the numbers m and n; and first let mA be greater than nB.
To each of these unequals add mB;
then $mA + mB$ is greater than $mB + nB$. *I. Ax.* 4
But $mA + mB = m(A + B)$, *V.* 1, *Cor.*
and $mB + nB = (m + n)B$; *V.* 2
$\therefore m(A + B)$ is greater than $(m + n)B$.

Now because $A + B : B = C + D : D$,
if $m(A + B)$ be greater than $(m + n)B$, $m(C + D)$ is greater than $(m + n) D$; *V. Def.* 5, *Cor.*
or $mC + mD$ is greater than $mD + nD$;
that is, taking mD from both, mC is greater than nD.
Hence when mA is greater than nB, mC is greater than nD.
Similarly it may be proved that if $mA = nB$, $mC = nD$;
and if mA be less than nB, mC is less than nD;
$\therefore$ $A : B = C : D$. *V. Def.* 5

COR.—The proposition is equivalent to the following:
If $A : B = C : D$, then $A - B : B = C - D : D$.
Hence also, on the same hypothesis, it may be proved that $A - B : A = C - D : C$; that $A : A - B = C : C - D$; and that $B : A - B = D : C - D$.

[If it be thought desirable, any one of these changes on the proportion $A : B = C : D$ may be denoted by the word *subtraction*.]

PROPOSITION 18. THEOREM.

The terms of a proportion are proportional by addition.

Let $A : B = C : D$:
it is required to prove $A + B : B = C + D : D$.

Take $m(A + B)$ and nB any multiples of $A + B$ and B.

First, let m be greater than n.
Because $A + B$ is greater than B;
$\therefore$ $m(A + B)$ is greater than nB.
Similarly $m(C + D)$ is greater than nD;
$\therefore$ when m is greater than n, $m(A + B)$ is greater than nB, and $m(C + D)$ is greater than nD.

Second, let $m = n$.
In the same manner it may be proved that in this case $m(A + B)$ is greater than nB, and $m(C + D)$ greater than nD.

Third, let m be less than n.
Then $m(A + B)$ may be greater than nB, or may be equal to it, or may be less than it.

First, let $m(A + B)$ be greater than nB;
then $mA + mB$ is greater than nB. *V.* 1
Take mB, which is less than nB, from both;
$\therefore$ mA is greater than $nB - mB$,
or mA is greater than $(n - m)B$. *V.* 6
But because $A : B = C : D$; *Hyp.*
$\therefore$ if mA is greater than $(n - m)B$, mC is greater than $(n - m)D$,
that is, mC is greater than $nD - mD$. *V.* 6
Add mD to each of these unequals;
then $mC + mD$ is greater than nD,
that is, $m(C + D)$ is greater than nD. *V.* 1
If therefore $m(A + B)$ is greater than nB, $m(C + D)$ is greater than nD.

In the same manner it may be proved that,
if $m(A + B) = nB$, $m(C + D) = nD$;
if $m(A + B)$ be less than nB, $m(C + D)$ is less than nD.
Hence $A + B : B = C + D : D$. *V. Def.* 5

Cor.—Hence also, on the same hypothesis, it may be proved
that $A + B : A = C + D : C$; that $A : A + B = C : C + D$;
and that $B : A + B = D : C + D$.

[If it be thought desirable, any one of these changes on the proportion $A : B = C : D$ may be denoted by the word *addition*. The words *addition* and *subtraction*, as being more significant of the operations performed on the terms of the proportion, have been substituted for *composition* (*componendo*) and *division* (*dividendo*), which are the translations of the words (*σύνθεσις*, *διαίρεσις*) used by the Greek geometers.]

PROPOSITION 19. THEOREM.

If a whole magnitude be to a whole as a magnitude taken from the first is to a magnitude taken from the other, the remainder shall be to the remainder as the whole to the whole.

Let $A : B = C : D$, and let C be less than A:
it is required to prove $A - C : B - D = A : B$.

Because $A : B = C : D$;		*Hyp.*
$\therefore \quad A : C = B : D$,	by alternation;	*V.* 16
$\therefore \quad A - C : C = B - D : D$,	by subtraction;	*V.* 17
$\therefore \quad A - C : B - D = C : D$,	by alternation;	*V.* 16
$\therefore \quad A - C : B - D = A : B$.		*V.* 11

PROPOSITION 20. THEOREM.

If there be three magnitudes, and other three, which, taken two and two in direct order, have the same ratio; if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Let A, B, C be three magnitudes, and D, E, F other three, such that $A : B = D : E$, and $B : C = E : F$:
it is required to prove that if A be greater than C, D will be greater than F; if $A = C$, D will $= F$; if A be less than C, D will be less than F.

First, let A be greater than C;
then $A : B$ is greater than $C : B$. *V.* 8
But $A : B = D : E$; *Hyp.*
$\therefore$ $D : E$ is greater than $C : B$. *V.* 13

Now $B : C = E : F$; *Hyp.*

$\therefore$ $C : B = F : E$, by inversion; V. A

$\therefore$ $D : E$ is greater than $F : E$; V. 13

$\therefore$ D is greater than F. V. 10

Second, let $A = C$;

then $A : B = C : B$. V. 7

But $A : B = D : E$; *Hyp.*

$\therefore$ $C : B = D : E$. V. 11

Now $C : B = F : E$;

$\therefore$ $D : E = F : E$; V. 11

$\therefore$ $D = F$. V. 9

Third, let A be less than C;
then C is greater than A; and, as was shown in case first, $C : B = F : E$, and $B : A = E : D$.
$\therefore$ by case first, if C be greater than A, F is greater than D. that is, if A be less than C, D is less than F.

PROPOSITION 21. Theorem.

If there be three magnitudes, and other three, which, taken two and two in transverse order, have the same ratio; if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Let A, B, C be three magnitudes, and D, E, F other three, such that $A : B = E : F$, and $B : C = D : E$:
it is required to prove that if A be greater than C, D will be greater than F; if $A = C$, D will $= F$; if A be less than C, D will be less than F.

First, let A be greater than C;
then $A : B$ is greater than $C : B$. V. 8

But $A : B = E : F$; *Hyp.*
$\therefore$ $E : F$ is greater than $C : B$. V. 13
Now $B : C = D : E$; *Hyp.*
$\therefore$ $C : B = E : D$, by inversion; V. A
$\therefore$ $E : F$ is greater than $E : D$: V. 13
$\therefore$ D is greater than F. V. 10

Second, let $A = C$;
then $A : B = C : B$. V. 7
But $A : B = E : F$; *Hyp.*
$\therefore$ $C : B = E : F$. V. 11
Now $C : B = E : D$;
$\therefore$ $E : F = E : D$; V. 11
$\therefore$ $D = F$. V. 9

Third, let A be less than C;
then $A : B$ is less than $C : B$. V. 8
But $A : B = E : F$; *Hyp.*
$\therefore$ $E : F$ is less than $C : B$ V. 13
Now $C : B = E : D$;
$\therefore$ $E : F$ is less than $E : D$, V. 13
$\therefore$ D is less than F. V. 10

PROPOSITION 22. THEOREM.

If there be any number of magnitudes, and as many others, which taken two and two in direct order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.

First, let there be three magnitudes A, B, C, and other three D, E, F, such that $A : B = D : E$, and $B : C = E : F$: *it is required to prove* $A : C = D : F$.

Of A and D take any equimultiples whatever mA, mD

of B and E any whatever nB, nE; and of C and F any whatever qC, qF.

Because $A : B = D : E$; *Hyp.*

$\therefore$ $mA : nB = mD : nE$. *V.* 4

Similarly $nB : qC = nE : qF$; *V.* 4

$\therefore$ according as mA is greater than qC, equal to it, or less, mD is greater than qF, equal to it, or less, *V.* 20

$\therefore$ $A : C = D : F$. *V. Def.* 5

Second, let there be four magnitudes A, B, C, D, and other four E, F, G, H, such that $A : B = E : F$, $B : C = F : G$, $C : D = G : H$:
it is required to prove $A : D = E : H$.

Since A, B, C are three magnitudes, and E, F, G, other three, which, taken two and two in direct order, have the same ratio,

$\therefore$ $A : C = E : G$, by the first case.

But because $C : D = G : H$; *Hyp.*

$\therefore$ $A : D = E : H$, by the first case.

Similarly the demonstration may be extended to any number of magnitudes.

PROPOSITION 23. Theorem.

If there be any number of magnitudes, and as many others, which taken two and two in transverse order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.

First, let there be three magnitudes A, B, C, and other three D, E, F, such that $A : B = E : F$, and $B : C = D : E$
it is required to prove $A : C = D : F$.

Of A, B, and D take any equimultiples mA, mB, mD; and of C, E, and F take any equimultiples nC, nE, nF.

Because $A : B = mA : mB$, V. 15

and $E : F = nE : nF$, V. 15

and because $A : B = E : F$; *Hyp.*

$\therefore$ $mA : mB = nE : nF$. V. 11

Again, because $B : C = D : E$; *Hyp.*

$\therefore$ $mB : nC = mD : nE$; V. 4

$\therefore$ according as mA is greater than nC, equal to it, or less, mD is greater than nF, equal to it, or less; V. 21

$\therefore$ $A : C = D : F$. V. *Def.* 5

Second, let there be four magnitudes A, B, C, D, and other four E, F, G, H, such that $A : B = G : H$, $B : C = F : G$, $C : D = E : F$: *it is required to prove* $A : D = E : H$.

Since A, B, C are three magnitudes, and F, G, H other three, which, taken two and two in transverse order, have the same ratio,

$\therefore$ $A : C = F : H$, by the first case.

But because $C : D = E : F$; *Hyp.*

$\therefore$ $A : D = E : H$ by the first case.

Similarly the demonstration may be extended to any number of magnitudes.

Cor.—From this proposition and the preceding it may be inferred that ratios which are compounded of equal ratios are equal to one another.

For $A : C = \left\{ \begin{array}{l} A : B \\ B : C \end{array} \right\}$, and $D : F = \left\{ \begin{array}{l} D : E \\ E : F \end{array} \right\}$:

and it has been shown that $A : C = D : F$.

PROPOSITION 24. THEOREM.

If the first has to the second the same ratio which the third has to the fourth, and the fifth to the second the same ratio which the sixth has to the fourth; the first and fifth together shall have to the second the same ratio which the third and sixth together have to the fourth.

Let $A : B = C : D$, and $E : B = F : D$:
it is required to prove $A + E : B = C + F : D$.

Because	$E : B = F : D$;		
$\therefore$	$B : E = D : F$,	by inversion.	*V.* A
But	$A : B = C : D$;		*Hyp.*
$\therefore$	$A : E = C : F$,	by direct equality;	*V.* 22
$\therefore$	$A + E : E = C + F : F$,	by addition.	*V.* 18
But again,	$E : B = F : D$;		*Hyp.*
$\therefore$	$A + E : B = C + F : D$,	by direct equality.	*V.* 22

PROPOSITION D. THEOREM.

The terms of a proportion are proportional by addition and subtraction.

Let $A : B = C : D$:
it is required to prove $A + B : A - B = C + D : C - D$.

Because	$A - B : B = C - D : D$,	by subtraction;	*V.* 17
$\therefore$	$B : A - B = D : C - D$,	by inversion;	*V.* A
But	$A + B : B = C + D : D$,	by addition;	*V.* 18
$\therefore$	$A + B : A - B = C + D : C - D$,	by direct equality.	*V.* 22

[Proposition 25 has been omitted, as being of little use.]

BOOK VI.

DEFINITIONS.

1. **Similar** rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportional.

Of the two requisites for similarity among figures, namely, equiangularity and proportionality of sides, it will be seen from VI. 4, 5, that if two triangles possess the one, they also possess the other. In this respect triangles are unique. Hence, in order to prove two rectilineal figures (other than triangles) similar, it must be shown that they possess both requisites.

2. When any proportion is stated among the sides of two similar figures, those pairs of sides which form antecedents or consequents of the ratios are called **homologous** sides.

3. Similar figures are said to be **similarly described upon given straight lines** when the given straight lines are homologous sides of the figures.

4. When two similar figures have their homologous sides parallel and drawn in the same direction, they are said to be **similarly situated**; when they have them parallel and drawn in opposite directions, they are said to be **oppositely situated.**

5. Triangles and parallelograms which have their sides about two of their angles proportional in such a manner that a side of the first figure is to a side of the second, as the other side of the second is to the other side of the first, are said to have these sides **reciprocally proportional.**

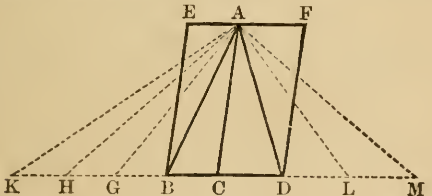
6. The **altitude** of a triangle is the perpendicular drawn from the vertex to the base, or the base produced; the **altitude** of a parallelogram is the perpendicular drawn from any point in one of its sides to the opposite side, or that side produced.

7. A straight line is said to be cut in **extreme and mean ratio** when the whole line is to one segment as that segment is to the other.

As in the case of medial section, a straight line might be cut in extreme and mean ratio both internally and externally; but internal division only is generally implied by the phrase.

PROPOSITION 1. Theorem.

Triangles and parallelograms of the same altitude are to one another as their bases.

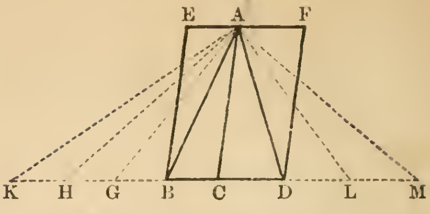


Let $\triangle$s ABC, ACD, and $\|^{ms}$ EC, CF have the same altitude, namely, the perpendicular drawn from A to BD, or BD produced:

it is required to prove $BC : CD = \triangle\ ABC : \triangle\ ACD$,

and $BC : CD = \|^{m}\ EC : \|^{m}\ CF$.

Produce BD both ways, and take any number of straight lines BG, GH, HK each $= BC$, *I.* 3
and DL, LM, any number of them, each $= CD$; *I.* 3
and join A with the points K, H, G, L, M.



Because KH, HG, GB, BC are all equal, *Const.*
∴ △s AKH, AHG, AGB, ABC are all equal. *I.* 38
∴ whatever multiple the base KC is of the base BC, the same multiple is △ AKC of △ ABC.
Similarly, whatever multiple the base CM is of the base CD, the same multiple is △ ACM of △ ACD.
And if the base KC be equal to, greater, or less than the base CM, △ AKC will be equal to, greater, or less than △ ACM. *I.* 38

Now since there are four magnitudes BC, CD, △ ABC, △ ACD;
and of BC and △ ABC (the first and third) any equimultiples whatever have been taken, namely, KC and △ AKC,
and of CD and △ ACD (the second and fourth) any equimultiples whatever have been taken, namely, CM and △ ACM;
and since it has been shown that if KC be equal to, greater, or less than CM,
△ AKC is equal to, greater, or less than △ ACM;
∴ $BC : CD = \triangle\, ABC : \triangle\, ACD$. *V. Def.* 5

Again, because $BC : CD = \triangle\, ABC : \triangle\, ACD$;
∴ $BC : CD = 2 \triangle\, ABC : 2 \triangle\, ACD$ *V.* 15, 11
$= \|^{m}\, EC : \|^{m}\, CF$. *I.* 41

Cor. 1.—Triangles and parallelograms that have equal altitudes are to one another as their bases.

Cor. 2.—Triangles and parallelograms that have equal bases are to one another as their altitudes.

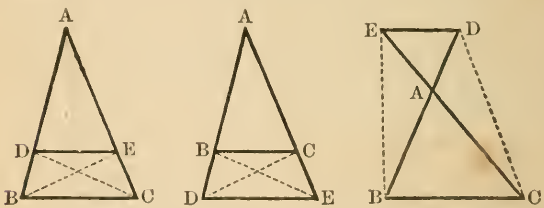
For each triangle or $\|^m$ may be converted into an equivalent right-angled triangle or rectangle with base and altitude = its base and altitude; and in these latter figures the bases and altitudes may be interchanged.

1. If two triangles or $\|^{ms}$ have the same ratio as their bases, they must have equal altitudes; if they have the same ratio as their altitudes, they must have equal bases.
2. The rectangle contained by two straight lines is a mean proportional between their squares.
3. A, B, and C are three straight lines; prove that A has to B the same ratio as the rectangle contained by A and C has to the rectangle contained by B and C.
4. A quadrilateral is such that the perpendiculars on a diagonal from the opposite vertices are equal. Show that the quadrilateral can be divided into four equal triangles by straight lines drawn from the middle point of the diagonal.
5. AB is $\|$ CD, and AD, BC are joined, intersecting at E; prove $AE : ED = BE : EC$.
6. Triangles ABC, DEF have $\angle A = \angle D$, and $AB = DE$; prove $\triangle ABC : \triangle DEF = AC : DF$.
7. AD, BE, CF drawn from the vertices of $\triangle ABC$ to the opposite sides are concurrent at O; prove $BD : DC = \triangle AOB : \triangle AOC$, $CE : EA = \triangle BOC : \triangle BOA$, $AF : FB = \triangle COA : \triangle COB$.
8. E is the middle point of AD, a median of $\triangle ABC$; BE is joined and produced to meet AC at F. Prove $CF = 2\,AF$.
9. ABC is any triangle; from BC and CA are cut off BD = one-fourth of BC, and CE = one-fourth of CA. If AD, BE intersect at O, prove that CO produced will divide AB into two segments in the ratio of 9 to 1.
10. Perpendiculars are drawn from any point within an equilateral triangle to the three sides. Prove that their sum is constant.
11. Triangles and $\|^{ms}$ are to one another in the ratio compounded of the ratios of their bases and altitudes.

PROPOSITION 2. THEOREMS.

If a straight line be drawn parallel to one side of a triangle, it shall cut the other sides, or those sides produced proportionally.*

Conversely: If the sides or the sides produced be cut proportionally, the straight line joining the points of section shall be parallel to the remaining side of the triangle.



(1) Let DE be drawn $\parallel BC$, one of the sides of $\triangle ABC$: *it is required to prove that* $BD : DA = CE : EA$.

Join BE, CD.

Then $\triangle BDE = \triangle CDE$, being on the same base DE, and between the same parallels DE, BC: *I.* 37

$\therefore \triangle BDE : \triangle ADE = \triangle CDE : \triangle ADE$. *V.* 7

But $\triangle BDE : \triangle ADE = BD : DA$, *VI.* 1

and $\triangle CDE : \triangle ADE = CE : EA$; *VI.* 1

$\therefore BD : DA = CE : EA$. *V.* 11

(2) Let $BD : DA = CE : EA$, and DE be joined: *it is required to prove* $DE \parallel BC$.

Join BE, CD.

Because $BD : DA = CE : EA$, *Hyp.*

and $BD : DA = \triangle BDE : \triangle ADE$, *VI.* 1

and $CE : EA = \triangle CDE : \triangle ADE$: *VI.* 1

* This useful extension was introduced by Robert Simson.

$\therefore \quad \triangle BDE : \triangle ADE = \triangle CDE : \triangle ADE;$ *V.* 11

$\therefore \quad \triangle BDE = \triangle CDE.$ *V.* 9

Now these triangles are on the same base DE and on the same side of it;

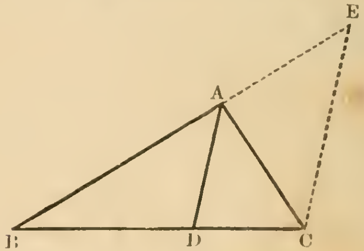
$\therefore$ DE is $\parallel$ BC. *I.* 39

1. The straight line which joins the middle points of two sides of a triangle is $\parallel$ the third side.
2. The straight line drawn through the middle point of one of the sides of a triangle and $\parallel$ another side will bisect the third side.
3. Any two straight lines cut by three parallel straight lines are cut proportionally. (Euclid, *Data*, Prop. 38.)
4. Any straight line drawn $\parallel$ the parallel sides of a trapezium divides the non-parallel sides, or those sides produced proportionally.
5. In the figures to the proposition, if DE be $\parallel$ BC, prove $BA : AD = CA : AE$, and conversely.
6. ABC is any angle, and P a given point within it; draw through P a straight line terminated by BA, BC, and bisected at P.
7. In the base BC of $\triangle ABC$ any point D is taken, and DE, DF, drawn $\parallel$ AB, AC respectively, meet the other sides at E, F: prove $\triangle AFE$ a mean proportional between $\triangle$s FBD, EDC. Examine the case when D is taken in BC produced.
8. ABC, DBC are two triangles either on the same side, or on opposite sides of a common base BC; from any point E in BC there are drawn EF, EG respectively $\parallel$ BA, BD, and meeting the other sides in F, G. Prove $FG \parallel AD$. Examine the case when E is taken in BC produced.
9. ABC is any triangle; D and E are points on AB and AC such that DE is $\parallel$ BC; BE and CD intersect at F. Prove that $\triangle ADF = \triangle AEF$, and that AF produced bisects BC. Examine also the cases when D and E are on AB and AC produced.
10. Prove the following construction for trisecting a straight line AB in G and H: On AB as diagonal construct a $\parallel^m$ $ACBD$; bisect AC, BD in E and F. Join DE, FC cutting AB in G and H.
11. AB is a straight line, and C is any point in it; find in AB produced a point D such that $AD : DB = AC : CB$.

PROPOSITION 3. Theorems.

If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the internal segments of the base shall have to one another the same ratio as the other sides of the triangle have.

Conversely: If the internal segments of the base have to one another the same ratio as the other sides of the triangle have, the straight line drawn from the vertex to the point of section shall bisect the vertical angle.



(1) Let the vertical $\angle\ BAC$ of the $\triangle\ ABC$ be bisected by AD, which meets the base at D:
it is required to prove that $BD : DC = BA : AC$.

Through C draw $CE \parallel DA$, *I.* 31
and let CE meet BA produced at E.

Because DA and CE are parallel,

$\therefore\ \angle\ BAD = \angle\ AEC$, and $\angle\ DAC = \angle\ ACE$. *I.* 29

But $\angle\ BAD = \angle\ DAC$; *Hyp.*

$\therefore\ \angle\ AEC = \angle\ ACE$;

$\therefore\ \quad AC = AE$. *I.* 6

Because DA is $\parallel$ CE, a side of the $\triangle\ BCE$,

$\therefore\ BD : DC = BA : AE$; *VI.* 2

$\therefore\ BD : DC = BA : AC$. *V.* 7

(2) Let $BD : DC = BA : AC$, and AD be joined:
it is required to prove $\angle\ BAD = \angle\ DAC$.

Through C draw $CE \parallel DA$, I. 31
and let CE meet BA produced at E.

Because DA is $\parallel$ CE, a side of the $\triangle\ BCE$,

$\therefore\ BD : DC = BA : AE$. VI. 2

But $BD : DC = BA : AC$; Hyp.

$\therefore\ BA : AE = BA : AC$; V. 11

$\therefore\ AE = AC$, V. 9

and $\angle\ AEC = \angle\ ACE$. I. 5

But because DA and CE are parallel,

$\therefore\ \angle\ AEC = \angle\ BAD$, and $\angle\ ACE = \angle\ DAC$; I. 29

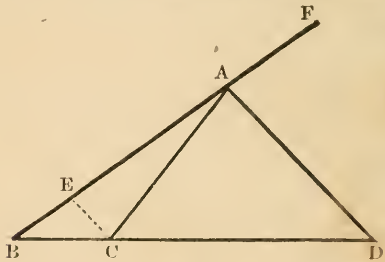
$\therefore\ \angle\ BAD = \angle\ DAC$.

1. With the same figure and construction as in I. 10, prove that AB is bisected.
2. If a straight line bisect both the base and the vertical angle of a triangle, the triangle must be isosceles.
3. The bisector of an angle of a triangle divides the triangle into two others, which are proportional to the sides of the bisected angle.
4. ABC is a triangle whose base BC is bisected at D; $\angle$s ADB, ADC are bisected by DE, DF meeting AB, AC at E, F. Prove $EF \parallel BC$.
5. Trisect a given straight line.
6. Divide a given straight line into parts which shall be to one another as 3 to 2.
7. Divide a given straight line into n equal parts.
8. The bisectors of the angles of a triangle are concurrent.
9. Express BD and DC (fig. to the proposition) in terms of a, b, c, the three sides of the triangle.
10. AB is a diameter of a circle, CD a chord at right angles to it, and E any point in CD; AE, BE produced cut the circle at F and G. Prove that the quadrilateral $CFDG$ has any two of its adjacent sides in the same ratio as the other two.
11. H is the middle point of BC (fig. to the proposition): prove $HC : HD = BA + AC : BA - AC$.
12. The straight lines which trisect an angle of a triangle do not trisect the opposite side.

PROPOSITION A.* Theorems.

If the exterior vertical angle of a triangle be bisected by a straight line which also cuts the base produced, the external segments of the base shall have to one another the same ratio as the other sides of the triangle have.

Conversely: If the external segments of the base have to one another the same ratio as the other sides of the triangle have, the straight line drawn from the vertex to the point of section shall bisect the exterior vertical angle.



(1) Let the exterior vertical $\angle\ CAF$ of the $\triangle\ ABC$ be bisected by AD, which meets the base produced at D:
it is required to prove that $BD : DC = BA : AC$.

Through C draw $CE \parallel DA$, — I. 31
and let CE meet BA at E.

Because DA and CE are parallel,
$\therefore\ \angle\ FAD = \angle\ AEC$, and $\angle\ DAC = \angle\ ACE$. — I. 29
But $\angle\ FAD = \angle\ DAC$; — *Hyp.*
$\therefore\ \angle\ AEC = \angle\ ACE$;
$\therefore\ AC = AE$. — I. 6
Because DA is $\parallel$ CE, a side of the $\triangle\ BCE$,

* Assumed in Pappus, VII. 39, second proof.

$\therefore \quad BD : DC = BA : AE;$ VI. 2

$\therefore \quad BD : DC = BA : AC.$ V. 7

(2) Let $BD : DC = BA : AC$, and AD be joined :
it is required to prove $\angle\ FAD = \angle\ DAC$.

Through C draw $CE \parallel DA$, I. 31
and let CE meet BA at E.

Because DA is $\parallel$ CE, a side of the $\triangle\ BCE$,

$\therefore\ BD : DC = BA : AE.$ VI. 2

But $BD : DC = BA : AC;$ Hyp.

$\therefore\ BA : AE = BA : AC;$ V. 11

$\therefore \quad AE = AC,$ V. 9

and $\angle\ AEC = \angle\ ACE.$ I. 5

But because DA and CE are parallel,

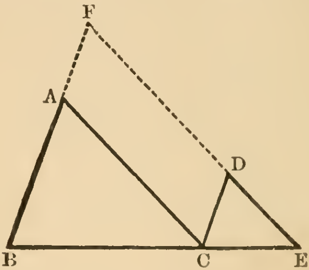
$\therefore \quad \angle\ AEC = \angle\ FAD$, and $\angle\ ACE = \angle\ DAC;$ I. 29

$\therefore \quad \angle\ FAD = \angle\ DAC.$

1. What does the proposition become when the triangle is isosceles ?
2. The bisector of the vertical angle of a triangle, and the bisectors of the exterior angles below the base, are concurrent.
3. Express BD and DC (fig. to the proposition) in terms of a, b, c, the three sides of the triangle.
4. Prove the tenth deduction from VI. 3 when E is taken in CD produced.
5. P is any point in the $\bigcirc^{ce}$ of the circle of which AB is a diameter; PC, PD drawn on opposite sides of AP, and making equal angles with it, meet AB at C and D. Prove $AC : CB = AD : DB$.
6. AB is a straight line, and C is any point in it; find in AB produced a point D such that $AD : DB = AC : CB$.
7. Prove the proposition by cutting off from BA produced, $AE = AC$, and joining DE.
8. If in any $\triangle\ ABC$ there be inscribed a $\triangle\ XYZ$ (X being on BC, Y on CA, Z on AB), such that every two of its sides make equal angles with that side of $\triangle\ ABC$ on which they meet, then AX, BY, CZ are respectively $\perp$ BC, CA, AB.
   Examine the case when X and Y are on BC and AC produced.

PROPOSITION 4. THEOREM.

*If two triangles be mutually equiangular, they shall be similar, those sides being homologous which are opposite to equal angles.**



In $\triangle$s ABC, DCE, let $\angle\ ABC = \angle\ DCE$, $\angle\ BCA = \angle\ CED$, $\angle\ BAC = \angle\ CDE$:
it is required to prove $\triangle$*s* ABC, DCE *similar.*

Place $\triangle\ DCE$ so that CE may be contiguous to BC, and in the same straight line with it. *I.* 22

Because $\angle\ ABC + \angle\ ACB$ is less than 2 rt. $\angle$s; *I.* 17
and $\angle\ ACB = \angle\ DEC$; *Hyp.*
$\therefore$ $\angle\ ABC + \angle\ DEC$ is less than 2 rt. $\angle$s;
$\therefore$ BA and ED if produced will meet. *I.* 29, *Cor.*
Let them be produced and meet at F.

Because $\angle\ DCE = \angle\ ABC$, *Hyp.*
$\therefore$ BF is $\parallel$ CD; *I.* 28
and because $\angle\ BCA = \angle\ CED$, *Hyp.*
$\therefore$ AC is $\parallel$ FE; *I.* 28
$\therefore$ $FACD$ is a $\parallel^{m}$;
$\therefore$ $AF = CD$, and $AC = FD$. *I.* 34

* This theorem is usually attributed to Thales (640–546 B.C.).

Now because AC is $\parallel$ FE, a side of the $\triangle$ FBE,

$\therefore$ $BA : AF = BC : CE$; *VI.* 2

$\therefore$ $BA : CD = BC : CE$; *V.* 7

$\therefore$ $BA : BC = CD : CE$, by alternation. *V.* 16

Again, because CD is $\parallel$ BF, a side of the $\triangle$ FBE,

$\therefore$ $BC : CE = FD : DE$; *VI.* 2

$\therefore$ $BC : CE = AC : DE$; *V.* 7

$\therefore$ $BC : CA = CE : DE$, by alternation. *V.* 16

Lastly, because AB, BC, CA are three magnitudes,

and DC, CE, ED other three;

and since it has been proved that $AB : BC = DC : CE$,

and $BC : CA = CE : ED$;

$\therefore$ $AB : AC = DC : DE$, by direct equality. *V.* 22

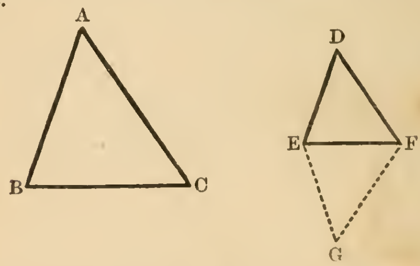
Hence $\triangle$s ABC, DCE are similar. *VI. Def.* 1

1. From a given triangle another is cut off by a parallel to the base; prove the two triangles similar.
2. Two right-angled triangles are similar if an acute angle of the one be equal to an acute angle of the other.
3. Two isosceles triangles are similar if their vertical angles are equal.
4. $ABCD$ is a rhombus; through D a straight line is drawn so as to cut BA and BC produced at E and F. Prove $\triangle$s EAD, DCF similar.
5. Two chords AC, BD of a circle ABC intersect at E, either within or without the circle; prove $\triangle$s AEB, CED similar, and also $\triangle$s AED, BEC.
6. The straight line which joins the middle points of two sides of a triangle is half of the third side.
7. A straight line which is $\parallel$ one of the sides of a triangle and = half of it must bisect each of the other sides.
8. If one of the two parallel sides of a trapezium be double of the other, the diagonals intersect at a point of trisection.
9. In mutually equiangular triangles the perpendiculars drawn from corresponding vertices to the opposite sides are proportional to those sides.
10. The median to the base of a triangle bisects all the parallels to the base intercepted by the sides.

11. Three straight lines AB, AC, AD are drawn through one point A, and are cut by two parallels at the points E, F, G and B, C, D respectively: prove $BC : CD = EF : FG$.
12. Hence devise a method of dividing a given straight line into any number of equal parts.
13. Prove the proposition from VI. 2, by superposing the one triangle on the other.

PROPOSITION 5. THEOREM.

If two triangles have the sides taken in order about each of their angles proportional, they shall be similar, those angles being equal which are opposite to the homologous sides.



In $\triangle$s ABC, DEF, let $AB : BC = DE : EF$, $BC : CA = EF : FD$, and $BA : AC = ED : DF$:
it is required to prove $\triangle$s ABC, DEF similar.

At E make $\angle\ FEG = \angle\ ABC$, and at F make $\angle\ EFG = \angle\ ACB$. *I.* 23

Then $\angle\ G = \angle\ A$, *I.* 32, *Cor.* 1
and $\triangle\ ABC$ is equiangular to $\triangle\ GEF$;

$\therefore\ AB : BC = GE : EF$. *VI.* 4

But $AB : BC = DE : EF$; *Hyp.*

$\therefore\ DE : EF = GE : EF$; *V.* 11

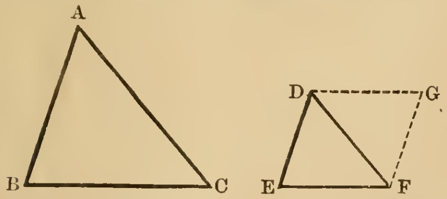
$\therefore\ DE = GE$. *V.* 9

Similarly, $DF = GF$.
Now $\triangle$s DEF, GEF have the three sides of the one respectively equal to the three sides of the other;
$\therefore$ they are mutually equiangular. *I.* 8
But $\triangle$ ABC is equiangular to $\triangle$ GEF;
$\therefore$ $\triangle$ ABC is equiangular to $\triangle$ DEF.
Hence $\triangle$s ABC, DEF are similar. *VI. Def.* 1

1. What is the analogous proposition in the First Book proving the equality of two triangles?
2. The triangle formed by joining the middle points of the sides of another triangle is similar to that other.
3. Prove the proposition from the following construction: From AB cut off $AG = DE$, and through G draw $GH \parallel BC$, meeting AC at H.

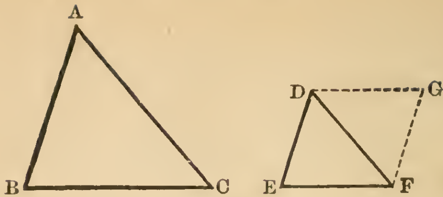
PROPOSITION 6. Theorem.

If two triangles have one angle of the one equal to one angle of the other, and the sides about these angles proportional, they shall be similar, those angles being equal which are opposite to the homologous sides.



In $\triangle$s ABC, DEF, let $\angle\ BAC = \angle\ EDF$, and $BA : AC = ED : DF$:
it is required to prove $\triangle$s ABC, DEF *similar.*

At D make $\angle\ FDG = \angle\ BAC$, or $\angle\ EDF$, *I.* 23
and at F make $\angle\ DFG = \angle\ ACB$. *I.* 23



Then $\angle G = \angle B$, *I.* 32, *Cor.* 1
and $\triangle ABC$ is equiangular to $\triangle DGF$;
$\therefore \quad BA : AC = GD : DF$. *VI.* 4
But $BA : AC = ED : DF$; *Hyp.*
$\therefore \quad ED : DF = GD : DF$; *V.* 11
$\therefore \qquad ED = GD$. *V.* 9

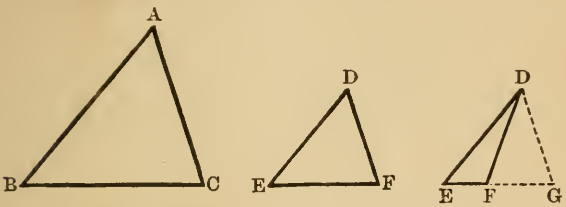
Now in $\triangle$s EDF, GDF, $\left\{ \begin{array}{l} ED = GD \\ DF = DF \\ \angle EDF = \angle GDF; \end{array} \right.$ *Const.*

$\therefore \quad \angle E = \angle G$, and $\angle DFE = \angle DFG$. *I.* 4
But $\angle B = \angle G$, and $\angle ACB = \angle DFG$;
$\therefore \quad \angle B = \angle E$, and $\angle ACB = \angle DFE$.
Hence $\triangle$s ABC, DEF are similar. *VI. Def.* 1

1. What is the analogous proposition in the First Book proving the equality of two triangles?
2. Prove the proposition with the same construction as in the third deduction from VI. 5.
3. ABC is a triangle, and the perpendicular AD drawn from A to BC falls within the triangle. Prove that if AD is a mean proportional between BD and DC, $\angle BAC$ is right, and that if AB is a mean proportional between BC and BD, $\angle BAC$ is right.
4. AB is a straight line, D and E two points on it; DF and EG are parallel, and proportional to AD and AE. Prove A, F, and G to be in one straight line.
5. AB is divided internally at C and D so that $AB : AC = AC : AD$. From A any other straight line AE is drawn $= AC$. Prove $\triangle$s ABE, AED similar, and that EC bisects $\angle BED$.

PROPOSITION 7. Theorem.

If two triangles have two sides of the one proportional to two sides of the other, and the angles opposite to one pair of homologous sides equal, the angles opposite to the other pair of homologous sides shall be either equal or supplementary.



In $\triangle$s ABC, DEF, let $BA : AC = ED : DF$, and $\angle B = \angle E$:

it is required to prove either $\angle C = \angle F$, *or* $\angle C + \angle F = 2$ *rt.* $\angle$*s.*

(1) $\angle A$ is either $= \angle D$, or not.

If $\angle A = \angle D$, then since $\angle B = \angle E$, *Hyp.*

$\therefore \angle C = \angle F$. *I.* 32, *Cor.* 1

(2) If $\angle A$ is not $= \angle D$,

at D make $\angle EDG = \angle A$; *I.* 23

and, if necessary, produce EF to meet DG.

Because $\angle B = \angle DEG$, *Hyp.*

and $\angle A = \angle EDG$; *Const.*

$\therefore \triangle ABC$ is equiangular to $\triangle DEG$; *I.* 32, *Cor.* 1

$\therefore BA : AC = ED : DG$. *VI.* 4

But $BA : AC = ED : DF$; *Hyp.*

$\therefore ED : DF = ED : DG$; *V.* 11

$\therefore DF = DG$; *V.* 9

$\therefore \angle DFG = \angle G$. *I.* 5

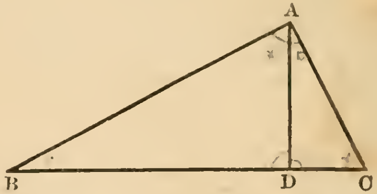
Now $\angle\ DFE$ is supplementary to $\angle\ DFG$; *I.* 13
$\therefore$ $\angle\ DFE$ is supplementary to $\angle\ G$;
$\therefore$ $\angle\ DFE$ is supplementary to $\angle\ C$.

NOTE.—See the note appended to I. A., p. 62.

1. What is the analogous proposition in the First Book proving, under certain conditions, the equality of two triangles?
2. ABC is a triangle, and AD is drawn $\perp$ BC. If $BC : CA = AB : AD$, then $\triangle\ ABC$ is right-angled.

PROPOSITION 8. THEOREM.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle and to one another.



Let $\triangle\ ABC$ be right-angled at A, and let AD be drawn perpendicular to the hypotenuse BC:
it is required to prove $\triangle$*s* DBA *and* DAC *similar to* $\triangle\ ABC$, *and to one another.*

In $\triangle$s DBA, ABC, $\left\{ \begin{array}{c} \angle\ ADB = \angle\ CAB \\ \angle\ B = \angle\ B; \end{array} \right.$

$\therefore$ these triangles are mutually equiangular; *I.* 32, *Cor.* 1
$\therefore$ they are similar. *VI.* 4
In the same way, $\triangle$s DAC and ABC may be proved similar.
Now since $\triangle$s DBA and DAC are similar to $\triangle\ ABC$, they are similar to one another.

Cor.—From the similarity of $\triangle$s DBA, DAC it follows that $$BD : DA = AD : DC. \quad (1)$$
From the similarity of $\triangle$s ABC, DBA it follows that
$$CB : BA = AB : BD. \quad (2)$$
From the similarity of $\triangle$s ABC, DAC it follows that
$$BC : CA = AC : CD, \quad (3)$$
and $$BC : BA = AC : AD. \quad (4)$$

These results expressed in words are:

(1) The perpendicular from the right angle on the hypotenuse is a mean proportional between the two segments into which it divides the hypotenuse.

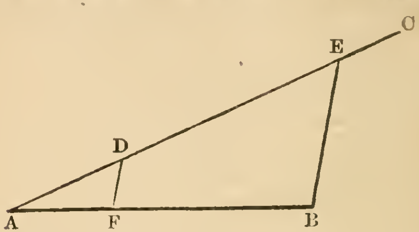
(2) and (3) Either of the sides is a mean proportional between the hypotenuse and its projection on the hypotenuse.

(4) The hypotenuse is to either side as the other side is to the perpendicular.

1. If from any point in the $\bigcirc^{ce}$ of a circle a perpendicular be drawn to any radius, and a tangent from the same point to meet the radius produced, the radius will be a mean proportional between the segments intercepted between the centre and the points of concourse.
2. That part of a tangent to a circle intercepted by tangents at the extremities of any diameter is divided at the point of contact so that the radius is a mean proportional between the segments.
3. Prove $BD : DC =$ duplicate of $BA : AC$.
4. ABC is a triangle; AD and AE are drawn to the base BC so as to make $\angle$s ADB, AEC each = the vertical $\angle$ BAC; prove
   (1) $BD : AD = AE : CE$, (2) $CB : BA = AB : BD$,
   (3) $BC : CA = AC : CE$, (4) $BC : BA = AC : AE$.
   Draw figures for the cases when $\angle$ BAC is acute and obtuse, and deduce from this theorem the results given in the Cor. to the proposition.
5. Examine the converses of the results (1), (2), (3), (4) of the Cor. to the proposition, and of the preceding deduction.

PROPOSITION 9. PROBLEM.

From a given straight line to cut off any aliquot part.



Let AB be the given straight line:
it is required to cut off from AB any aliquot part.

From A draw AC, making any angle with AB;
in AC take any point D;
and from AC cut off AE, containing AD as many times as AB contains the part required. *I.* 3

Join EB, and through D draw $DF \parallel EB$. *I.* 31

AF is the part required.

Because DF is $\parallel EB$, a side of $\triangle ABE$,

$\therefore \quad ED : DA = BF : FA$; *VI.* 2

$\therefore \quad EA : DA = BA : FA$, by addition. *V.* 18

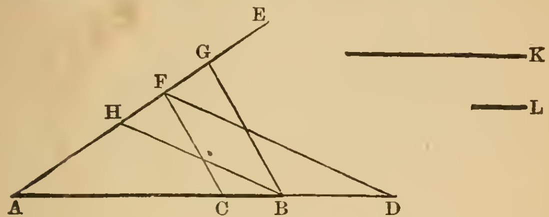
But EA contains DA a certain number of times;

$\therefore$ BA contains FA the same number of times. *V.* C

1. Which proposition in the First Book is a particular case of this?
2. Trisect a given straight line.
3. Show how to find three-fifths of a given straight line.
4. From a given triangle or $\parallel^m$ cut off any aliquot part.
5. Show how to find four-sevenths of a given $\parallel^m$.

PROPOSITION 10. PROBLEM.

*To divide a given straight line internally and externally in a given ratio.**



Let AB be the given straight line, $K : L$ the given ratio: *it is required to divide AB internally and externally in the ratio $K : L$.*

Draw a straight line AE making an angle with AB; cut off $AF = K$, and FG, FH on opposite sides of F, each $= L$. *I.* 3

Join BG, BH;

and through F draw $FC \parallel BG$, and $FD \parallel BH$, meeting AB produced at D. C and D are the required points.

Because FC is $\parallel$ BG, a side of the $\triangle$ ABG,

$\therefore$ $AC : CB = AF : FG$, *VI.* 2

$= K : L$. *V.* 11

Again, because FD is $\parallel$ BH, a side of the $\triangle$ ABH,

$\therefore$ $AD : DB = AF : FH$, *VI.* 2

$= K : L$. *V.* 11

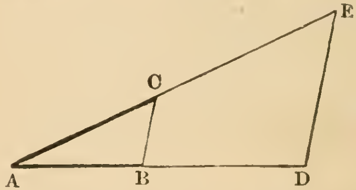
1. AB and AC are two straight lines, and AC is divided internally at the points D and E. Divide AB similarly to AC.

* **This proposition has been inserted instead of Euclid's tenth, which is given as the first deduction.**

2. Make the figure and prove the proposition when K is less than L. What becomes of the external section when $K = L$.
3. Divide a given triangle or $\|^{m}$ into two parts which shall have to each other a given ratio.
4. Given two points on the $\bigcirc^{ce}$ of a circle, to find a third point on the $\bigcirc^{ce}$ such that the ratio of its distances from the two given points may be equal to a given ratio.

PROPOSITION 11. PROBLEM.

To find a third proportional to two given straight lines.



Let AB, AC be the two given straight lines:
it is required to find a third proportional to AB, AC.

Place AB, AC so as to contain any angle;
produce AB, AC, making $BD = AC$; *I.* 3
join BC, and through D draw $DE \parallel BC$. *I.* 31

CE is the third proportional.

Because BC is $\parallel$ DE, a side of $\triangle$ ADE,

$\therefore$ $AB : BD = AC : CE$; *VI.* 2

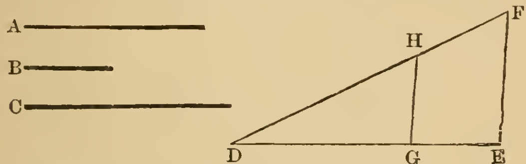
$\therefore$ $AB : AC = AC : CE$, since $BD = AC$. *V.* 7

1. Does the magnitude of the third proportional to two straight lines depend on the order in which the straight lines are taken? How many third proportionals can be found to two straight lines?
2. To AB and AC obtain the third proportional measured from A.
3. By VI. 8, Cor., find a third proportional to two straight lines in two other ways.

4. AB and AC are two straight lines drawn from A. Produce CA to D, making $AD = AC$; describe a circle through the three points B, C, D, and produce BA to meet it at E. AE is a third proportional to AB, AC.
5. Use the fourth deduction from VI. 4 to find a third proportional to two given straight lines.
6. Use the fourth deduction from VI. 8 for the same purpose.

PROPOSITION 12. PROBLEM.

To find a fourth proportional to three given straight lines.



Let A, B, C be the three given straight lines:
it is required to find a fourth proportional to A, B, C.

Take two straight lines DE, DF containing any angle;
from these cut off $DG = A$, $GE = B$, and $DH = C$; *I.* 3
join GH, and through E draw $EF \parallel GH$. *I.* 31

HF is the fourth proportional.

Because GH is $\parallel$ EF, a side of $\triangle$ DEF,

$\therefore$ $DG : GE = DH : HF$; *VI.* 2

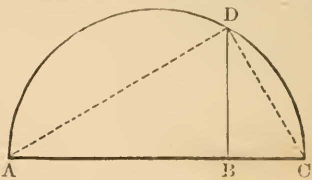
$\therefore$ $A : B = C : HF$. *V.* 7

1. Which previous proposition is a particular case of this? VI 1
2. Does the magnitude of the fourth proportional to three straight lines depend on the order in which they are taken? How many fourth proportionals can be found to three straight lines?
3. To A, B, C obtain the fourth proportional measured from D.
4. By a method similar to that of the fourth deduction from VI. 11, find a fourth proportional to three given straight lines.

5. Given a triangle or ||^m; construct another triangle or ||^m which shall have to it a given ratio.
6. AB and AC are two straight lines, and D is a point between them. Draw through D a straight line such that the parts of it intercepted between D and the two given straight lines may be in a given ratio.

PROPOSITION 13. PROBLEM.

To find a mean proportional between two given straight lines.



Let AB, BC be the two given straight lines:
it is required to find a mean proportional between AB, BC.

Place AB, BC in the same straight line,
and on AC describe the semicircle ADC; *I.* 10
from B draw $BD \perp AC$. *I.* 11

BD is the mean proportional.

Join AD, CD.

Then $\triangle$ ADC is right-angled, and AC is the hypotenuse; *III.* 31
$\therefore$ BD is a mean proportional between AB, BC. *VI.* 8, *Cor.*

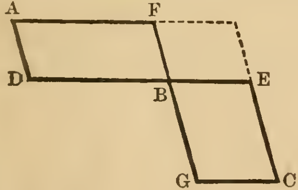
1. If the given straight lines were AC, BC, placed as in the figure to the proposition, show how to find a mean proportional between them.
2. To find a mean proportional between AB, BC placed as in the figure to the proposition. Describe any circle passing through A and C; join B to the centre O, and draw $DBE \perp OB$, meeting the $\bigcirc^{ce}$ at D and E. BD or BE is the mean proportional.

3. To find a mean proportional between AC, BC, placed as in the figure to the proposition. Describe any segment of a circle on AC, make $\angle\ CBD =$ the angle in the segment, and join CD. CD is the mean proportional.
4. Half the sum of two straight lines is greater than the mean proportional between them.
5. A point E is taken in the side AB of a $\|^{m}$ $ABCD$; DE meets BC produced in F. Prove $\triangle\ AEF$ a mean proportional between $\triangle$s AED and BEF.
6. By repetitions of the process of finding a mean proportional, what numbers of mean proportionals could be found between two given straight lines so as to form a continued proportion? Devise an algebraical expression which will include all these numbers.

PROPOSITION 14. THEOREMS.

Equal parallelograms, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional.

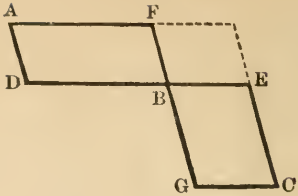
Conversely: Parallelograms which have one angle of the one equal to one angle of the other, and the sides about the equal angles reciprocally proportional, are equal.



(1) Let AB and BC be equal $\|^{ms}$, having $\angle\ DBF = \angle\ GBE$:

it is required to prove that $DB : BE = GB : BF$.

Place the $\|^{ms}$ so that DB and BE may be in one straight line.



Then since $\angle\ GBE = \angle\ DBF$; *Hyp.*

$\therefore\ \angle\ GBE + \angle\ FBE = \angle\ DBF + \angle\ FBE,$

$= 2$ rt. $\angle$s; *I.* 13

$\therefore$ GB and BF are in one straight line. *I.* 14

Complete the $\|^m$ FE.

Because $\|^m\ AB = \|^m\ BC,$ *Hyp.*

$\therefore\ \|^m\ AB : \|^m\ FE = \|^m\ BC : \|^m\ FE.$ *V.* 7.

But $\|^m\ AB : \|^m\ FE = DB : BE,$ *VI.* 1

and $\|^m\ BC : \|^m\ FE = GB : BF;$ *VI.* 1

$\therefore\ DB : BE = GB : BF.$ *V.* 11

(2) Let $\angle\ DBF = \angle\ GBE$, and $DB : BE = GB : BF$: *it is required to prove* $\|^m\ AB = \|^m\ BC$.

Make the same construction as before.

Because $DB : BE = GB : BF,$ *Hyp.*

and $DB : BE = \|^m\ AB : \|^m\ FE,$ *VI.* 1

and $GB : BF = \|^m\ BC : \|^m\ FE;$ *VI.* 1

$\therefore\ \|^m\ AB : \|^m\ FE = \|^m\ BC : \|^m\ FE;$ *V.* 11

$\therefore\ \|^m\ AB = \|^m\ BC.$ *V.* 9

X 1. Prove the proposition by joining EF and DG, and using the fifth deduction from VI. 2.

2. Prove AD, CG and the diagonal of the $\|^m$ FE drawn through B concurrent.

3. Prove $AC \parallel EF$.

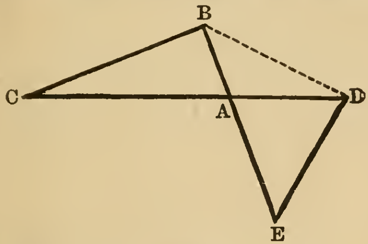
4. Equal rectangles have their bases and altitudes reciprocally proportional, and conversely.

5. Equal $\|^{ms}$ that have their sides reciprocally proportional are mutually equiangular.

PROPOSITION 15. THEOREMS.

Equal triangles which have one angle of the one equal to one angle of the other have their sides about the equal angles reciprocally proportional.

Conversely: Triangles which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal.



(1) Let BAC, DAE be equal triangles having $\angle\ BAC = \angle\ DAE$:

it is required to prove that $AC : AD = AE : AB$.

Place the triangles so that AC and AD may be in one straight line.

Then since $\angle\ DAE = \angle\ BAC$, *Hyp.*

$\therefore\ \angle\ DAE + \angle\ BAD = \angle\ BAC + \angle\ BAD$,

$= 2$ rt. $\angle$s; *I.* 13

$\therefore$ EA and AB are in one straight line. *I.* 14

Join BD.

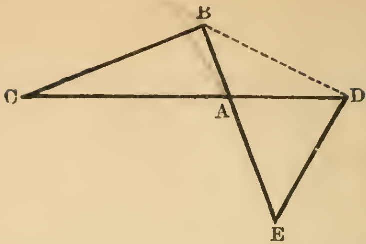
Because $\triangle\ BAC = \triangle\ DAE$, *Hyp.*

$\therefore\ \triangle\ BAC : \triangle\ BAD = \triangle\ DAE : \triangle\ BAD$. *V.* 7

But $\triangle\ BAC : \triangle\ BAD = AC : AD$, *VI.* 1

and $\triangle\ DAE : \triangle\ BAD = AE : AB$; *VI.* 1

$\therefore\ AC : AD = AE : AB$. *V.* 11



(2) Let $\angle\ BAC = \angle\ DAE$, and $AC : AD = AE : AB$: *it is required to prove* $\triangle\ BAC = \triangle\ DAE$.

Make the same construction as before.

Because	$AC : AD = AE : AB$,	*Hyp.*
and	$AC : AD = \triangle\ BAC : \triangle\ BAD$,	*VI.* 1
and	$AE : AB = \triangle\ DAE : \triangle\ BAD$;	*VI.* 1
$\therefore$	$\triangle\ BAC : \triangle\ BAD = \triangle\ DAE : \triangle\ BAD$;	*V.* 11
$\therefore$	$\triangle\ BAC = \triangle\ DAE$.	*V.* 9

1. Could this proposition have been inferred from VI. 14?

X 2. Prove the proposition by joining CE, and using the fifth deduction from VI. 2.

3. If in the figure to VI. 14, AB and EG be joined, what modification of this proposition should we be enabled to prove?

4. If $\triangle\ ABC$ is right-angled at B, and BD, the perpendicular on AC, is produced to E so that DE is a third proportional to BD and DC, $\triangle\ ADE = \triangle\ BDC$.

5. Equal triangles which have the sides about one pair of angles reciprocally proportional have those angles either equal or supplementary.

6. If, in the fig. to VI. 8, BE be drawn $\perp$ BA, and meet AD produced at E, then $\triangle\ ABD = \triangle\ ECD$.

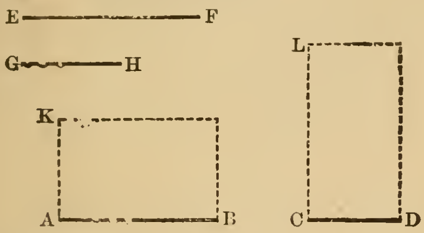
X 7. Find a point in a side of a triangle, from which two straight lines drawn, one to the opposite angle, and the other $\parallel$ the base, shall cut off towards the vertex and towards the base, equal triangles.

Examine the case for a point in a side produced.

PROPOSITION 16. Theorems.

If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means.

Conversely: If the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportional.



(1) Let $AB : CD = EF : GH$:
it is required to prove $AB \cdot GH = CD \cdot EF$.

From A draw $AK \perp AB$, and $= GH$,		*I.* 11, 3
from C draw $CL \perp CD$, and $= EF$;		*I.* 11, 3
and complete the rectangles KB, LD.		*I.* 31
Because	$AB : CD = EF : GH$,	*Hyp.*
and	$CL = EF$, and $AK = GH$;	*Const.*
$\therefore$	$AB : CD = CL : AK$,	*V.* 7

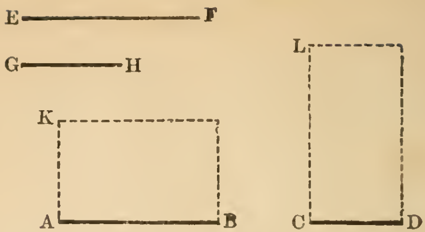
that is, the sides about the equal angles of the $\|^{ms}$ KB, LD are reciprocally proportional.

$\therefore \quad KB = LD$; *VI.* 14

$\therefore \; AB \cdot AK = CD \cdot CL$;

$\therefore \; AB \cdot GH = CD \cdot EF$.

(2) Let $AB \cdot GH = CD \cdot EF$:
it is required to prove $AB : CD = EF : GH$.



Make the same construction as before.

Because $AB \cdot GH = CD \cdot EF$, *Hyp.*

and $AK = GH$, and $CL = EF$; *Const.*

$\therefore AB \cdot AK = CD \cdot CL$,

that is, the $\|^{ms}$ KB, LD which have $\angle A = \angle C$, are equal.

$\therefore AB : CD = CL : AK$; *VI.* 14

$\therefore AB : CD = EF : GH$. *V.* 7

1. In the figure to VI. 8, prove (1) $BD \cdot DC = AD^2$, (2) $CB \cdot BD = AB^2$, (3) $BC \cdot CD = AC^2$, (4) $BC \cdot AD = BA \cdot AC$.
2. Using the results (2) and (3) of the preceding deduction, prove I. 47.
3. Show that these results are established in Euclid's proof of I. 47.
4. Two chords AC, BD of a circle ABC intersect at E, either within or without the circle; prove $AE \cdot EC = BE \cdot ED$.
5. In the figure to the fourth deduction from VI. 8, prove
   (1) $BD \cdot CE = AD \cdot AE$, (2) $CB \cdot BD = AB^2$,
   (3) $BC \cdot CE = AC^2$, (4) $BC \cdot AE = BA \cdot AC$.
6. Using the results (2) and (3) of the preceding deduction, show that when $\angle BAC$ is acute, $AB^2 + AC^2$ is greater than BC^2 by $BC \cdot DE$; when $\angle BAC$ is obtuse, $AB^2 + AC^2$ is less than BC^2 by $BC \cdot DE$.
7. What becomes of the rectangle $BC \cdot DE$ when $\angle BAC$ is right?
8. Give another proof of III. 35 and its Cor.
9. A square is inscribed in a right-angled triangle, one side of the square coinciding with the hypotenuse; prove that the area of the square = the rectangle contained by the extreme segments of the hypotenuse.

PROPOSITION 17. THEOREMS.

If three straight lines be proportional, the rectangle contained by the extremes is equal to the square on the mean.

Conversely: If the rectangle contained by the extremes be equal to the square on the mean, the three straight lines are proportional.

(1) Let $A : B = B : C$:

it is required to prove $A \cdot C = B^2$.

Make $D = B$.

Because $A : B = B : C$, *Hyp.*

$\therefore$ $A : B = D : C$; *V.* 7

$\therefore$ $A \cdot C = B \cdot D$, ***VI.*** 16

$= B^2$.

(2) Let $A \cdot C = B^2$:

it is required to prove $A : B = B : C$.

Make the same construction as before.

Because $A \cdot C = B^2$, *Hyp.*

$\therefore$ $A \cdot C = B \cdot D$;

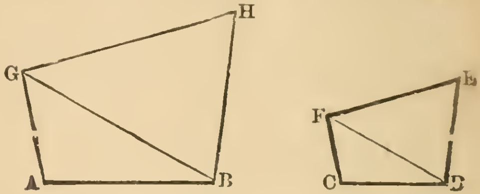
$\therefore$ $A : B = D : C$; ***VI.*** 16

$\therefore$ $A : B = B : C$. *V.* 7

1. Of which proposition is this merely a particular case?
2. Prove that a straight line divided in extreme and mean ratio is divided in medial section, and conversely.
3. From B, one of the vertices of $\|^{m}$ $ABCD$, a straight line is drawn cutting the diagonal AC at E, CD at F, and AD produced at G: prove $GE \cdot EF = BE^2$.

PROPOSITION 18. PROBLEM.

*On a given straight line to describe a rectilineal figure which shall be similar to a given rectilineal figure.**



Let AB be the given straight line, $CDEF$ the given rectilineal figure:

it is required to describe on AB a rectilineal figure which shall be similar to $CDEF$.

Join DF.

At A make $\angle\ BAG = \angle\ DCF$, and at B make $\angle\ ABG = \angle\ CDF$; *I.* 23

then $\triangle\ GAB$ is equiangular to $\triangle\ FCD$. *I.* 32, *Cor.* 1

At G make $\angle\ BGH = \angle\ DFE$, and at B make $\angle\ GBH = \angle\ FDE$; *I.* 23

then $\triangle\ HGB$ is equiangular to $\triangle\ EFD$. *I.* 32, *Cor.* 1

$ABHG$ is the figure required.

(1) To prove $ABHG$ and $CDEF$ mutually equiangular.

Because $\angle\ AGB = \angle\ CFD$, and $\angle\ BGH = \angle\ DFE$, *Const.*

$\therefore$ the whole $\angle\ AGH =$ the whole $\angle\ CFE$.

Similarly, $\angle\ ABH = \angle\ CDE$.

But $\angle\ A = \angle\ C$, and $\angle\ H = \angle\ E$; *Const.*, *I.* 32, *Cor.* 1

$\therefore$ $ABHG$ and $CDEF$ are mutually equiangular.

* The second case added by Simson has been omitted as unnecessary.

(2) To prove that $ABHG$ and $CDEF$ have their sides proportional.

Because $\triangle$s AGB, CFD are mutually equiangular,

$\therefore\ AG : GB = CF : FD.$ *VI.* 4

Because $\triangle$s BGH, DFE are mutually equiangular,

$\therefore\ BG : GH = DF : FE.$ *VI.* 4

Now since AG, GB, GH are three magnitudes,

and CF, FD, FE other three;

and since it has been proved that $AG : GB = CF : FD$,

and $GB : GH = FD : FE$;

$\therefore\ AG : GH = CF : FE$, by direct equality. *V.* 22

Similarly, $AB : BH = CD : DE.$

But $BA : AG = DC : CF$, *VI.* 4

and $GH : HB = FE : ED$; *VI.* 4

$\therefore$ $ABHG$ and $CDEF$ have their sides about the equal angles proportional.

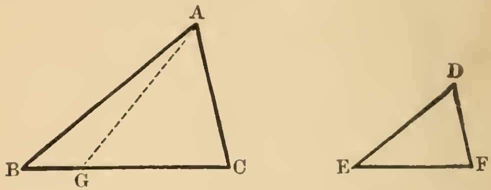
Hence $ABHG$ and $CDEF$ are similar. *VI. Def.* 1

The method of construction and proof would be similar if the given rectilineal figure had more than four sides.

1. How many polygons could be described on AB similar to the polygon $CDEF$?
2. Would the following constructions answer the same purpose as that given in the text? (*a*) Place AB and CD either parallel or in the same straight line; through A and B draw AG, BG respectively $\parallel$ CF, DF; through G and B draw GH, BH respectively $\parallel$ FE, DE. (*b*) Place $AB \parallel CD$, and let AC, BD meet at O. Join OE, OF, and let AG, BH drawn respectively $\parallel$ CF, DE meet OF, OE at G, H. Join GH.
3. If on BA, BG, BH, or on these lines produced, there be taken points L, M, N, such that $BL : BA = BM : BG = BN : BH$, the figure $BLMN$ is similar and similarly situated to the figure $BAGH$.
4. How could a figure $BLMN$ similar and oppositely situated to the figure $BAGH$ be obtained?

PROPOSITION 19. Theorem.

Similar triangles are to one another in the duplicate ratio of their homologous sides.



Let ABC and DEF be similar triangles, having $\angle B = \angle E$, and $\angle C = \angle F$, so that BC and EF are homologous sides:

it is required to prove $\triangle ABC : \triangle DEF =$ duplicate of $BC : EF$.

Take BG a third proportional to BC and EF,
so that $BC : EF = EF : BG$; *VI.* 11
and join AG.

Because $AB : BC = DE : EF$, *Hyp.*
$\therefore$ $AB : DE = BC : EF$, by alternation, *V.* 15
$= EF : BG$; *V.* 11

that is, the sides of $\triangle$s ABG, DEF about their equal angles B and E are reciprocally proportional.

$\therefore$ $\triangle ABG = \triangle DEF$. *VI.* 15

Again, because $BC : EF = EF : BG$, *Const.*
$\therefore$ $BC : BG =$ duplicate of $BC : EF$. *V. Def.* 13, *Cor*

But $BC : BG = \triangle ABC : \triangle ABG$, *VI.* 1
$\therefore$ $\triangle ABC : \triangle ABG =$ duplicate of $BC : EF$; *V.* 11
$\therefore$ $\triangle ABC : \triangle DEF =$ duplicate of $BC : EF$. *V.* 7

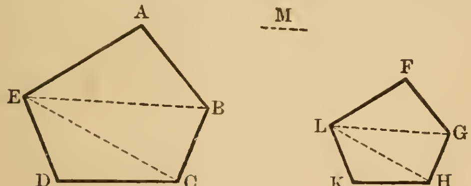
1. If three straight lines be proportional, as the first is to the third so is any triangle described on the first to a similar and similarly described triangle on the second.

Prove the proposition with either of the following constructions:

2. Take EG, measured along EF produced, a third proportional to EF and BC, and join DG.
3. From BC cut off $BG = EF$; join AG, and through G draw $GH \parallel AC$.
4. Similar triangles are to one another in the duplicate ratio of (1) their corresponding medians, (2) their corresponding altitudes, (3) the radii of their inscribed circles, (4) the radii of their circumscribed circles. (Assume, what can be easily proved from V. 23, Cor., that if two ratios be equal, their duplicates are equal.)

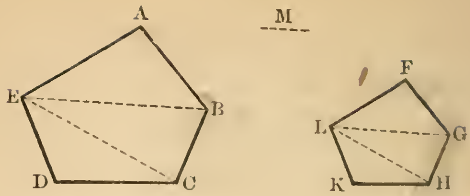
PROPOSITION 20. THEOREM.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons are to one another in the duplicate ratio of their homologous sides.



Let $ABCDE$, $FGHKL$ be similar polygons, and let AB and FG be homologous sides:

it is required to prove that $ABCDE$ and $FGHKL$ may be divided into the same number of similar triangles; that these triangles have each to each the same ratio which the polygons have; and that the polygons are to one another in the duplicate ratio of their homologous sides.



Join BE, EC, GL, LH.

Because the polygon $ABCDE$ is similar to the polygon $FGHKL$, *Hyp.*

$\therefore \angle A = \angle F$, and $BA : AE = GF : FL$; *VI. Def.* 1

$\therefore \triangle ABE$ is similar to $\triangle FGL$. *VI.* 6

Because the polygons are similar,

$\therefore \angle ABC = \angle FGH$; *VI. Def.* 1

and because $\triangle$s ABE, FGL are similar,

$\therefore \angle ABE = \angle FGL$; *VI. Def.* 1

$\therefore$ the remainder, $\angle EBC$ = remainder, $\angle LGH$.

And because $\triangle$s ABE, FGL are similar,

$\therefore EB : BA = LG : GF$; *VI. Def.* 1

and because the polygons are similar,

$\therefore BA : BC = GF : GH$. *VI. Def.* 1

$\therefore EB : BC = LG : GH$, by direct equality; *V.* 22

that is, the sides about the equal $\angle$s EBC, LGH are proportional;

$\therefore \triangle EBC$ is similar to $\triangle LGH$. *VI.* 6

For the same reason, $\triangle EDC$ is similar to $\triangle LKH$.

Because $\triangle ABE$ is similar to $\triangle FGL$,

$\therefore \triangle ABE : \triangle FGL$ = duplicate of $BE : GL$. *VI.* 19

Similarly, $\triangle EBC : \triangle LGH$ = duplicate of $BE : GL$;

$\therefore \triangle ABE : \triangle FGL = \triangle EBC : \triangle LGH$. *V.* 11

Because $\triangle EBC$ is similar to $\triangle LGH$,

$\therefore \triangle EBC : \triangle LGH$ = duplicate of $EC : LH$. *VI.* 19

Similarly, $\triangle ECD : \triangle LHK$ = duplicate of $EC : LH$;
$\therefore \triangle EBC : \triangle LGH = \triangle ECD : \triangle LHK$. *V.* 11
Hence $\triangle ABE : \triangle FGL = \triangle EBC : \triangle LGH = \triangle ECD : \triangle LHK$;
$\therefore \triangle ABE : \triangle FGL = \triangle ABE + \triangle EBC + \triangle ECD : \triangle FGL + \triangle LGH + \triangle LHK$, *V.* 12
= polygon $ABCDE$: polygon $FGHKL$.

Lastly, $\triangle ABE : \triangle FGL$ = duplicate of $AB : FG$; *VI.* 19
$\therefore$ $ABCDE : FGHKL$ = duplicate of $AB : FG$. *V.* 11

Cor.—If three straight lines be proportional, as the first is to the third, so is any rectilineal figure described on the first to the similar and similarly described rectilineal figure on the second.

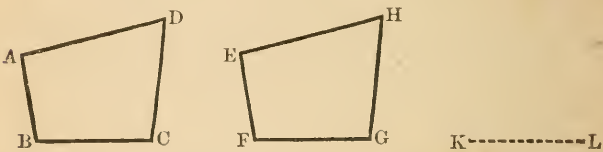
For, take M a third proportional to AB and FG. *VI.* 11
Then since $AB : FG = FG : M$, *Const.*
$\therefore AB : M$ = duplicate of $AB : FG$, *V. Def.* 13, *Cor.*
But $ABCDE : FGHKL$ = duplicate of $AB : FG$, *VI.* 20
$\therefore AB : M = ABCDE : FGHKL$. *V.* 11

1. Squares are to one another in the duplicate ratio of their sides.
2. Similar polygons are to one another as the squares on their homologous sides, or homologous diagonals.
3. The perimeters of similar polygons are to one another as the homologous sides.
4. Polygons are similar which can be divided into the same number of similar and similarly situated triangles.
5. Prove that similar polygons may be divided into the same number of similar triangles having their vertices at points situated within the polygons. (Such points are called *homologous points* with reference to the polygons.)
6. Could homologous points with reference to similar polygons be situated outside the polygons, or on their sides?
7. If two polygons be similar and similarly situated, the straight lines joining their corresponding vertices are concurrent.
   Examine the case when the polygons are similar and oppositely situated.

8. Use the preceding theorem to inscribe a square in a given triangle. How many squares can be inscribed in a triangle?
9. In a given triangle inscribe a rectangle similar to a given rectangle. How many such rectangles can be inscribed?

PROPOSITION 21. Theorem.

*Polygons which are similar and equal have their homologous sides equal.**



Let $ABCD$, $EFGH$ be two similar and equal polygons, having BC and FG homologous sides:
it is required to prove $BC = FG$.

Take KL a third proportional to BC and FG. *VI.* 11

Because $BC : FG = FG : KL$, *Const.*
$\therefore$ $BC : KL = ABCD : EFGH$. *VI.* 20, *Cor.*
But $ABCD = EFGH$; $\therefore$ $BC = KL$. *V.* 14
Again, since $BC : FG = FG : KL$, *Const.*
$\therefore$ $BC \cdot KL = FG^2$. *VI.* 17
But $BC \cdot KL = BC^2$, since $BC = KL$;
$\therefore$ $BC^2 = FG^2$, and $BC = FG$.

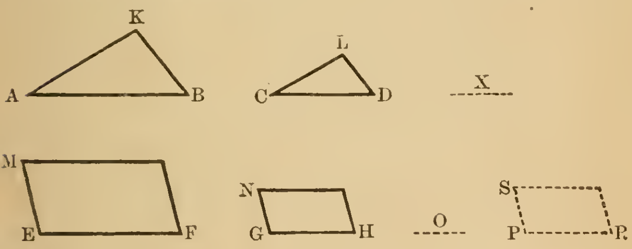
Prove the proposition indirectly.

* Euclid's 21st proposition is 'Rectilineal figures which are similar to the same rectilineal figure are similar to each other,' a theorem which may be regarded as self-evident. In place of it there has been substituted the lemma which occurs after the 22d proposition, and which is assumed in the proof of it. The demonstration of this lemma given in the text is due to Commandine (*Euclidis Elementorum Libri XV.*, 1572).

PROPOSITION 22. Theorems.

If four straight lines be proportional, and there be similarly described on the first and second any two similar polygons, and on the third and fourth any two similar polygons, the polygons shall be proportional.

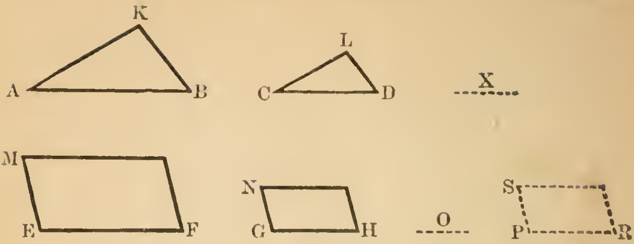
Conversely: If there be similarly described on the first and second of four straight lines two similar polygons, and two similar polygons on the third and fourth, and if the polygons be proportional, the four straight lines shall be proportional.



(1) Let $AB : CD = EF : GH$,
and let there be similarly described on AB and CD the similar polygons KAB, LCD, and on EF and GH the similar polygons MF, NH:
it is required to prove $KAB : LCD = MF : NH$.

Take X a third proportional to AB and CD, *VI.* 11
and O a third proportional to EF and GH. *VI.* 11

Because $AB : CD = EF : GH$, *Hyp.*
and $AB : CD = CD : X$, *Const.*
and $EF : GH = GH : O$; *Const.*
∴ $CD : X = GH : O$. *V.* 11



Now since AB, CD, X are three magnitudes,
and EF, GH, O other three;
and since $AB : CD = EF : GH$,
and $CD : X = GH : O$;
$\therefore$ $AB : X = EF : O$, by direct equality. *V.* 22
But $AB : X = KAB : LCD$, *VI.* 20, *Cor.*
and $EF : O = MF : NH$; *VI.* 20, *Cor.*
$\therefore$ $KAB : LCD = MF : NH$. *V.* 11

(2) Let $KAB : LCD = MF : NH$:
it is required to prove $AB : CD = EF : GH$.

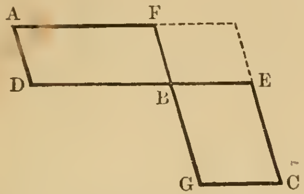
Take PR a fourth proportional to AB, CD, EF, *VI* 12
and on PR let a polygon SR be similar and similarly described to the polygons MF, NH. *VI.* 18

Because $AB : CD = EF : PR$, *Const.*
$\therefore$ $KAB : LCD = MF : SR$. *VI.* 22
But $KAB : LCD = MF : NH$; *Hyp.*
$\therefore$ $MF : SR = MF : NH$; *V.* 11
$\therefore$ $SR = NH$. *V.* 9
Hence $PR = GH$, since SR and NH are similar. *VI.* 21
Now $AB : CD = EF : PR$; *Const.*
$\therefore$ $AB : CD = EF : GH$. *V.* 7

1. If $AB : CD = EF : GH$, then $AB^2 : CD^2 = EF^2 : GH^2$.
2. If two ratios be equal, their duplicates are equal.

PROPOSITION 23. THEOREM.

*Mutually equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.**



Let $\|^m$ AB be equiangular to $\|^m$ BC, having $\angle DBF = \angle GBE$:

it is required to prove $\|^m AB : \|^m BC = \left\{ \begin{matrix} DB : BE \\ FB : BG \end{matrix} \right\}$.

Place the $\|^{ms}$ so that DB and BE may be in one straight line;

then GB and BF are in one straight line. *VI.* 14

Complete the $\|^m$ FE.

Then $\|^m AB : \|^m FE = DB : BE$, *VI.* 1

and $\|^m FE : \|^m BC = FB : BG$; *VI.* 1

$\therefore \left\{ \begin{matrix} \|^m AB : \|^m FE \\ \|^m FE : \|^m BC \end{matrix} \right\} = \left\{ \begin{matrix} DB : BE \\ FB : BG \end{matrix} \right\}$. *V.* 23, *Cor.*

But $\|^m AB : \|^m BC = \left\{ \begin{matrix} \|^m AB : \|^m FE \\ \|^m FE : \|^m BC \end{matrix} \right\}$; *V. Def.* 12

$\therefore \|^m AB : \|^m BC = \left\{ \begin{matrix} DB : BE \\ FB : BG \end{matrix} \right\}$. *V.* 11

1. Triangles which have one angle of the one equal or supplementary to one angle of the other are to one another in the ratio compounded of the ratios of the sides about those angles.

* The proof in the text, due to Franciscus Flussas Candalla, 1566, is somewhat shorter than Euclid's.

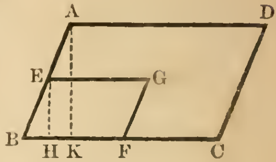
2. Show that VI. 14 is a particular case of the proposition.

3. Show that $\left\{\begin{array}{l} DB : BE \\ FB : BG \end{array}\right\} = DB \cdot BF : EB \cdot BG.$

4. Hence enunciate differently the proposition and the first deduction.

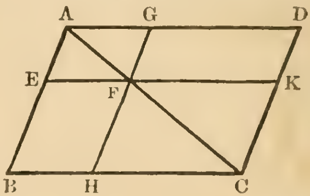
5. Prove the proposition from the accompanying figure.

6. Deduce VI. 19 from the first deduction.



PROPOSITION 24. THEOREM.

Parallelograms about a diagonal of any parallelogram are similar to the whole parallelogram, and to one another.



Let $ABCD$ be a $\|^{m}$, AC one of its diagonals, and let EG, HK be $\|^{ms}$ about AC:

it is required to prove that $\|^{ms}$ *EG, HK are similar to* $\|^{m}$ *$ABCD$, and to one another.*

Because DC is $\|$ GF, $\therefore \angle ADC = \angle AGF$; *I.* 29

and because BC is $\|$ EF, $\therefore \angle ABC = \angle AEF$. *I.* 29

And $\angle$s BCD, EFG are each $= \angle BAD$: *I.* 34

$\therefore \|^{m}$ $ABCD$ is equiangular to $\|^{m}$ $AEFG$.

Again, because $\angle ABC = \angle AEF$, *I.* 29

and $\angle BAC$ is common;

$\therefore \triangle$s ABC, AEF are mutually equiangular; *I.* 32, *Cor.* 1

$\therefore AB : BC = AE : EF$. *VI.* 4

But since the opposite sides of $\|^{ms}$ are equal, *I.* 34

$\therefore AB : AD = AE : AG,$ *V.* 7

and $CD : BC = FG : EF,$ *V.* 7

and $CD : DA = FG : GA;$ *V.* 7

that is, the sides of the $\|^{ms}$ $ABCD$, $AEFG$ about their equal angles are proportional.

$\therefore \|^{m}$ $ABCD$ is similar to $\|^{m}$ $AEFG$. *VI. Def.* 1

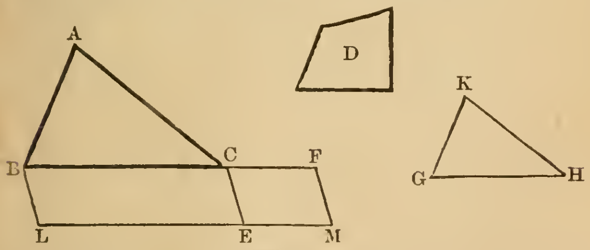
Hence also, $\|^{m}$ $ABCD$ is similar to $\|^{m}$ $FHCK$;

$\therefore \|^{m}$ $AEFG$ is similar to $\|^{m}$ $FHCK$.

1. From this proposition and VI. 14, deduce I. 43.
2. Prove $\|^{m}$ $EG : \|^{m}$ $BF = \|^{m}$ $FD : \|^{m}$ HK.
3. Prove that EG, BD, and HK are parallel.

PROPOSITION 25. Problem.

To describe a rectilineal figure which shall be similar to one and equal to another given rectilineal figure.

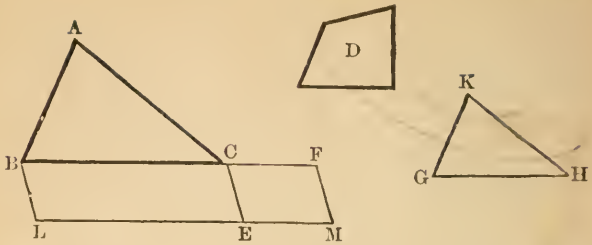


Let ABC be the one, and D the other given rectilineal figure:

it is required to describe a figure similar to ABC, and = D.

On BC describe any $\|^{m}$ BE = the figure ABC, *I.* 45

and on CE describe the $\|^{m}$ CM = the figure D, and having $\angle FCE = \angle CBL$. *I.* 45



Between BC and CF find a mean proportional GH; *VI.* 13 and on GH construct the figure KGH similar and similarly described to the figure ABC. *VI.* 18

KGH is the figure required.

It may be proved as in I. 45 that BC and CF form one straight line, and also LE and EM;

$\therefore\ BC : CF = \|^{m}\ BE : \|^{m}\ CM,$ *VI.* 1

$= ABC : D.$ *V.* 11

But because $BC : GH = GH : CF,$ *Const.*

and because ABC, KGH are similar and similarly described;

$\therefore\ BC : CF = ABC : KGH.$ *VI.* 20, *Cor.*

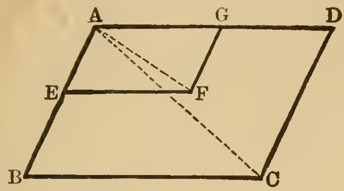
Hence $ABC : D = ABC : KGH;$ *V.* 11

$\therefore\ KGH = D.$ *V.* 9

1. Construct an equilateral triangle = a given square.
2. Construct a square = a given equilateral triangle.
3. Construct a square = a given regular pentagon.
4. Construct a regular pentagon = a given square.
5. Construct an equilateral triangle = a given regular hexagon.
6. Construct a regular hexagon = a given equilateral triangle.
7. Construct a polygon similar to a given polygon, and having a given perimeter.
8. Construct a polygon similar to a given polygon, and having a given ratio to it.
9. Through a given point inside a circle draw a chord so that it shall be divided at the point in a given ratio.

PROPOSITION 26. THEOREM.

If two similar parallelograms have a common angle and be similarly situated, they are about the same diagonal.



Let the $\|^{ms}$ $ABCD$, $AEFG$ be similar and similarly situated, and have the common angle BAD:

it is required to prove that they are about the same diagonal.

Join AC, AF.

Because $\|^{ms}$ $ABCD$, $AEFG$ are similar and similarly situated,

$\therefore$ AC and AF will divide them into similar triangles; *VI.* 20

$\therefore$ $\triangle$ ABC is similar to $\triangle$ AEF;

$\therefore$ $\angle$ $BAC = \angle$ EAF; *VI. Def.* 1

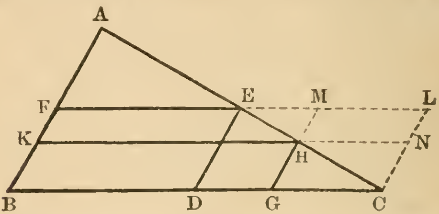
$\therefore$ AF falls along AC.

NOTE.—This proposition is the converse of VI. 24, and should be read immediately after it.

1. Prove the proposition by supposing AC to cut EF at H, and drawing $HK \parallel EA$ to meet AG at K.
2. Extend the proposition to the case of two similar and oppositely situated $\|^{ms}$.
3. From a given $\|^{m}$ cut off a similar $\|^{m}$ having a given ratio to the given $\|^{m}$.

PROPOSITION 27. THEOREM.

*Of all the parallelograms inscribed in a triangle so as to have one of the angles at the base common to them all, the greatest is that which is described on half the base.**



Let ABC be a triangle, having its base BC bisected at D. Let BE and BH be $\|^{ms}$ inscribed in it so as to have $\angle\ B$ of the triangle common to both:

it is required to prove $\|^m\ BE$ *greater than* $\|^m\ BH$.

Complete the $\|^m\ FBCL$, and produce GH, KH to M and N.

Because $BD = DC$, $\therefore\ FE = EL$; *I.* 34

$\therefore$ $\|^m\ KE = \|^m\ EN$. *I.* 36

Again, $\|^m\ MN = \|^m\ DH$. *I.* 43

But $\|^m\ EN$ is greater than $\|^m\ MN$;

$\therefore$ $\|^m\ KE$ is greater than $\|^m\ DH$.

Add to each of these unequals $\|^m\ KD$;

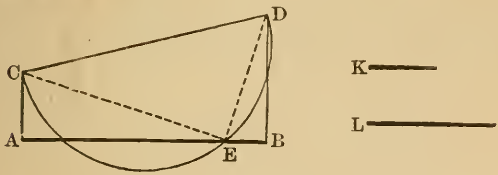
then $\|^m\ BE$ is greater than $\|^m\ BH$.

1. Make the construction and prove the proposition when G lies between B and D.
2. When $AB = BC$ and $\angle\ B$ is right, what does the proposition become?

* The enunciation of this proposition is different from tnat given by Euclid, but the proposition itself is substantially the same. The proof has been somewhat modified.

PROPOSITION 28.* PROBLEM.

To divide a given straight line internally so that the rectangle contained by its segments may be equal to a given rectangle.



Let AB be the given straight line; K and L the sides of the given rectangle:
it is required to divide AB internally so that the rectangle contained by the segments may be $= K \cdot L$.

Draw $AC \perp AB$, and $= K$, *I.* 11, 3
and on the same side of AB draw $BD \perp AB$, and $= L$. *I.* 11, 3
Join CD, and on it as diameter describe the semicircle CED cutting AB at E. $AE \cdot EB$ shall be $= K \cdot L$.

Join CE, ED.
Because $\angle\ CED$ is right, *III.* 31
$\therefore\ \angle\ AEC =$ complement of $\angle\ BED$, *I.* 13
$= \angle\ BDE$; *I.* 32
and $\angle\ CAE = \angle\ EBD$.

* Some editors of Euclid omit this and the following proposition. In the form in which Euclid presents them, they are difficult to understand and apply. The problems in the text are particular cases of Euclid's propositions, and the solutions given are to be found in Willebrord Snell's *Apollonius Batavus*, or Edmund Halley's *Apollonii Pergæi Conica* (1710), Book VIII. Prop. 18, Scholion.

$\therefore$ $\triangle$s AEC, BDE are mutually equiangular; *I.* 32, *Cor.* 1

$\therefore$ $AE : AC = BD : BE$; *VI.* 4

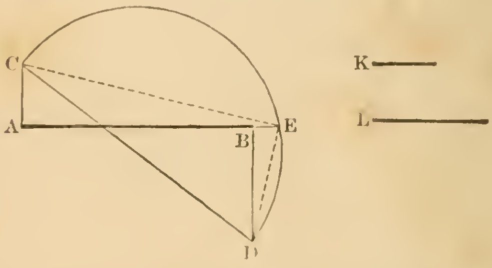
$\therefore$ $AE \cdot EB = AC \cdot BD$, *VI.* 16

$= K \cdot L$.

1. If E' be the other point in which the semicircle cuts AB, prove $AE' \cdot E'B = K \cdot L$.
2. Prove $AE' = BE$ and $E'B = AE$.
3. What limits are there to the size of the rectangle $K \cdot L$?
4. Solve the problem otherwise by converting the rectangle $K \cdot L$ into a square.

PROPOSITION 29. PROBLEM.

To divide a given straight line externally so that the rectangle contained by its segments may be equal to a given rectangle.



Let AB be the given straight line, K and L the sides of the given rectangle:

it is required to divide AB externally so that the rectangle contained by the segments may be $=$ $K \cdot L$.

Draw $AC \perp AB$, and $= K$, *I.* 11, 3

and on the opposite side of AB draw $BD \perp AB$, and $= L$. *I.* 11, 3

Join CD, and on it as diameter describe the semicircle CED, cutting AB produced at E. $AE \cdot EB$ shall be $= K \cdot L$.

Join CE, ED.

Because $\angle\ CED$ is right, *III.* 31

$\therefore\ \angle\ AEC =$ complement of $\angle\ BED$,

$= \angle\ BDE$; *I.* 32

and $\angle\ CAE = \angle\ EBD$.

$\therefore\ \triangle$s AEC, BDE are mutually equiangular; *I.* 32, *Cor.* 1

$\therefore\ AE : AC = BD : BE$; *VI.* 4

$\therefore\ AE \cdot EB = AC \cdot BD$, *VI.* 16

$= K \cdot L$.

1. If E' be the point in which the semicircle described on the other side of CD cuts AB produced, prove $AE' \cdot E'B = K \cdot L$.
2. Prove $AE' = BE$ and $E'B = AE$.
3. What limits are there to the size of the rectangle $K \cdot L$?
4. Solve the problem otherwise by converting the rectangle $K \cdot L$ into a square.

PROPOSITION 30. Problem.

To divide a given straight line in extreme and mean ratio.

Let AB be the given straight line:
it is required to divide it in extreme and mean ratio.

Divide AB internally at C so that $AB \cdot BC = AC^2$. *II.* 11

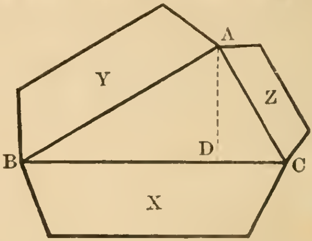
Because $AB \cdot BC = AC^2$; *Const.*

$\therefore\ AB : AC = AC : BC$. *VI.* 17

1. If in the figure to VI. 8, BC be divided in extreme and mean ratio at D, then $AC = BD$; and conversely.
2. AB and DE are two straight lines divided internally at C and F so that $AC : CB = DF : FE$; if $AB \cdot BC = AC^2$, prove $DE \cdot EF = DF^2$.

PROPOSITION 31. Theorem.

Any rectilineal figure described on the hypotenuse of a right-angled triangle is equal to the similar and similarly described figures on the other two sides.



Let $\triangle$ ABC be right-angled at A, and let X, Y, Z be rectilineal figures, similar and similarly described on BC, AB, AC:

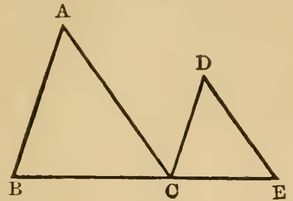
it is required to prove $X = Y + Z$.

Draw $AD \perp BC$.		*I.* 12
Then	$CB : BA = AB : BD$;	*VI.* 8, *Cor.*
$\therefore$	$CB : BD = X : Y$,	*VI.* 20, *Cor.*
and	$BD : CB = Y : X$, by inversion.	*V.* A.
Similarly,	$DC : CB = Z : X$;	
$\therefore$	$BD + DC : CB = Y + Z : X$.	*V.* 24
But $BD + DC = CB$;	$\therefore Y + Z = X$.	

1. From this proposition deduce I. 47.
2. Has I. 47 ever been used in any of the propositions which help to prove VI. 31?
3. Prove VI. 31 from VI. 22 and I. 47.
4. If on AB, AC, BC semicircles are described, those on AB and AC being exterior to the triangle, that on BC not being so, the sum of the areas of the two crescent-shaped figures will $= \triangle ABC$. Assume that semicircles are similar figures. The crescent-shaped figures are often called the *lunules* of Hippocrates of Chios (about 450 B.C.).

PROPOSITION 32. Theorem.

If two triangles, which have two sides of the one proportional to two sides of the other, be joined at one angle so as to have their homologous sides parallel, the remaining sides shall be in the same straight line.



Let ABC, DCE be two triangles, having $BA : AC = CD : DE$, and having $AB \parallel DC$, and $AC \parallel DE$:
it is required to prove BC and CE in the same straight line.

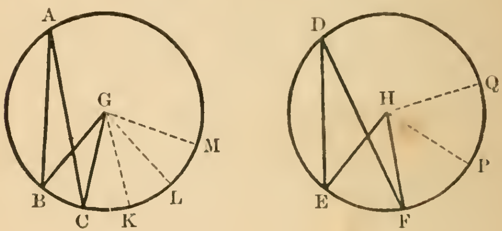
Because AB is $\parallel DC$, and AC is $\parallel DE$, *Hyp.*
$\therefore \angle A = \angle D$. *I.* 34, *Cor.*
And because $BA : AC = CD : DE$, *Hyp.*
$\therefore \triangle$s ABC, DCE are mutually equiangular; *VI.* 6
$\therefore \angle B = \angle DCE$.
To each of these equals add $\angle BCD$;
then $\angle B + \angle BCD = \angle DCE + \angle BCD$.
But $\angle B + \angle BCD = 2$ rt. $\angle$s; *I.* 29
$\therefore \angle DCE + \angle BCD = 2$ rt. $\angle$s;
$\therefore$ BC and CE are in the same straight line. *I.* 14

1. Show, by producing ED its own length to F and joining CF, that the enunciation of the proposition is defective.
2. From the points A and B there are drawn, either in the same or in opposite directions, two parallels AC, BD, and in like manner two other parallels AE, BF; if $AC : BD = AE : BF$, then BA, DC, FE are concurrent. (Simson's *Sectiones Conicæ*, 1735, ii., Lemma 2.)

3. The same things being supposed as in the last deduction, if CH, DL drawn parallel to each other meet AB, or AB produced at H and L, then EH and FL will be parallel. (Lemma 3.)

PROPOSITION 33. Theorem.

*In equal circles, angles, whether at the centres or at the circumferences, have the same ratio as the arcs on which they stand; so also have the sectors.**



Let ABC, DEF be equal circles, and let $\angle$s BGC, EHF be at their centres, and $\angle$s A and D at their ○[ces] :
it is required to prove

$$arc\ BC : arc\ EF = \angle\ BGC : \angle\ EHF,$$
$$arc\ BC : arc\ EF = \angle\ A : \angle\ D,$$
$$arc\ BC : arc\ EF = sector\ BGC : sector\ EHF.$$

Take any number of arcs CK, KL, LM each $= BC$,
and FP, PQ, any number of them, each $= EF$;
and join GK, GL, GM, HP, HQ.

Because arcs BC, CK, KL, LM are all equal, *Const.*
$\therefore$ $\angle$s BGC, CGK, KGL, LGM are all equal. *III.* 27
$\therefore$ whatever multiple arc BM is of arc BC, the same multiple is $\angle$ BGM of $\angle$ BGC.

* The last part of the theorem was added by Theon of Alexandria (about 380 A.D.). The proof in the text is not his.

Similarly, whatever multiple arc EQ is of arc EF, the same multiple is $\angle\ EHQ$ of $\angle\ EHF$.

And if arc BM be equal to, greater, or less than arc EQ,
$\angle\ BGM$ will be equal to, greater, or less than
$\angle\ EHQ$. *III.* 27

Now since there are four magnitudes BC, EF, $\angle\ BGC$, $\angle\ EHF$;
and of BC and $\angle\ BGC$ (the first and third) any equimultiples whatever have been taken, namely, BM and $\angle\ BGM$,
and of EF and $\angle\ EHF$ (the second and fourth) any equimultiples whatever have been taken, namely, EQ and $\angle\ EHQ$;
and since it has been shown that if BM be equal to, greater, or less than EQ,
$\angle\ BGM$ is equal to, greater, or less than $\angle\ EHQ$;
$\therefore$ arc BC : arc $EF = \angle\ BGC : \angle\ EHF$. *V. Def.* 5

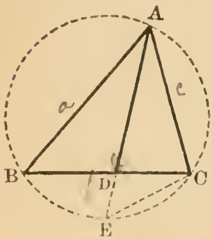
Again, because arc BC : arc $EF = \angle\ BGC : \angle\ EHF$;
$\therefore$ arc BC : arc EF = half $\angle\ BGC$: half $\angle\ EHF$, *V.* 15, 11
$= \angle\ A : \angle\ D$. *III.* 20

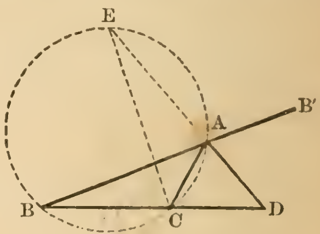
Lastly, because arcs BC, CK, KL, LM are all equal;
$\therefore$ sectors BGC, CGK, KGL, LGM are all equal; *III.* 27, *Cor.*
$\therefore$ whatever multiple arc BM is of arc BC, the same multiple is sector BGM of sector BGC.
Similarly, whatever multiple arc EQ is of arc EF, the same multiple is sector EHQ of sector EHF.
And if arc BM be equal to, greater, or less than arc EQ, sector BGM will be equal to, greater, or less than sector EHQ. *III.* 27, *Cor.*
Hence, as before, arc BC : arc EF = sector BGC : sector EHF. *V. Def.* 5

If arcs of different circles have a common chord, straight lines diverging from one of its extremities will cut the arcs proportionally.

PROPOSITION B.* Theorem.

If the interior or the exterior vertical angle of a triangle be bisected by a straight line which also cuts the base, the square on this bisector shall be equal to the difference between the rectangle contained by the sides of the triangle and the rectangle contained by the segments of the base.





(1) Let ABC be a triangle, having the interior vertical $\angle\ BAC$ bisected by AD:

it is required to prove $AD^2 = AB \cdot AC - BD \cdot DC.$

About the $\triangle\ ABC$ circumscribe a circle; *IV.* 5

produce AD to meet the $\bigcirc^{ce}$ at E, and join EC.

In $\triangle$s $ABD, AEC, \begin{cases} \angle\ BAD = \angle\ EAC & \textit{Hyp.} \\ \angle\ ABD = \angle\ AEC; & \textit{III. } 21 \end{cases}$

$\therefore$ these triangles are mutually equiangular. *I.* 32, *Cor.* 1

$\therefore\ AB : AD = AE : AC;$ *VI.* 4

$\therefore\ AB \cdot AC = AE \cdot AD,$ *VI.* 16

$= ED \cdot AD + AD^2,$ *II.* 3

$= BD \cdot DC + AD^2;$ *III.* 35

$\therefore\quad AD^2 = AB \cdot AC - BD \cdot DC.$

* The first part of the theorem is given in Schooten's *Exercitationes Mathematicæ* (1657), p. 65.

(2) Let ABC be a triangle, having the exterior vertical $\angle\ B'AC$ bisected by AD:
it is required to prove $AD^2 = BD \cdot DC - AB \cdot AC$.

About the $\triangle\ ABC$ circumscribe a circle; *IV.* 5
produce DA to meet the $\bigcirc^{ce}$ at E, and join EC.

Because $\angle\ B'AD = \angle\ CAD$; *Hyp.*

$\therefore$ supplement of $\angle\ B'AD$ = supplement of $\angle\ CAD$;

$\therefore$ $\angle\ BAD = \angle\ EAC$. *I.* 13

In $\triangle$s ABD, AEC, $\left\{ \begin{array}{l} \angle\ BAD = \angle\ EAC \\ \angle\ ABD = \angle\ AEC; \end{array} \right.$ *III.* 21

$\therefore$ these triangles are mutually equiangular. *I.* 32, *Cor.* 1

$\therefore$ $AB : AD = AE : AC$; *VI.* 4

$\therefore$ $AB \cdot AC = AE \cdot AD$, *VI.* 16

$= ED \cdot AD - AD^2$, *II.* 3

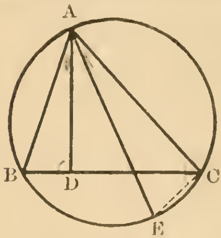
$= BD \cdot DC - AD^2$; *III.* 35, *Cor.*

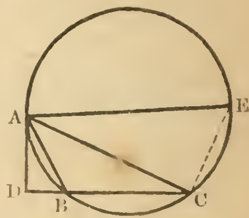
$\therefore$ $AD^2 = BD \cdot DC - AB \cdot AC$.

1. If, in fig. 1, AE be a diameter of the circle, of what shape will $\triangle\ ABC$ be?
2. In that case prove $AD^2 = AB \cdot AC - BD \cdot DC$, if AD be any straight line drawn to the base BC.
3. Could the bisector of the exterior vertical angle of a triangle be a diameter of the circle circumscribed about the triangle?
4. Prove $AE \cdot ED = BE^2$ or CE^2.
5. If a straight line be cut internally and externally in the same ratio, the square on the segment between the points of section = the difference between the rectangle contained by the external segments, and the rectangle contained by the internal segments.
6. Prove that the converse of the proposition is true except when $AB = AC$.
7. Express in terms of a, b, c, the sides of a triangle, the bisectors of the interior and the exterior vertical angles.
8. Construct a triangle having given two sides and
   (1) the bisector of the angle included by them,
   (2) the bisector of the angle adjacent to that included by them.

PROPOSITION C.* THEOREM.

If from the vertical angle of a triangle a perpendicular be drawn to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle circumscribed about the triangle.





Let ABC be a triangle, AD the perpendicular from A on the base BC, and AE a diameter of the circle circumscribed about ABC:

it is required to prove $AB \cdot AC = AD \cdot AE$.

Join EC.

In $\triangle$s ABD, AEC, $\begin{cases} \angle ADB = \angle ACE & III.\ 31 \\ \angle ABD = \angle AEC; & III.\ 21, \text{or } 22, Cor. \end{cases}$

$\therefore$ these triangles are mutually equiangular. *I.* 32, *Cor.* 1

$\therefore AB : AD = AE : AC$; *VI.* 4

$\therefore AB \cdot AC = AD \cdot AE$. *VI.* 16

1. Conversely, if ABC be a triangle, AE the diameter of the circumscribed circle, and if AD be drawn to BC so that $AD \cdot AE = AB \cdot AC$, then AD is $\perp$ BC.
2. Construct a triangle, having given the base, the vertical angle, and the rectangle contained by the sides.
3. If a circle be circumscribed about a triangle, and two straight lines be drawn from the vertex making equal angles with the sides, one of the straight lines meeting the base, or the base

* Given by Brahmegupta, an Indian mathematician (born 598 A.D.).

produced, and the other the ○ce, the rectangle contained by these straight lines = the rectangle contained by the sides of the triangle.

4. From the preceding deduction deduce VI. B and C.

5. Express the circumscribed radius of a triangle in terms of any two sides and the perpendicular on the third side; and the area of the triangle in terms of the three sides and the circumscribed radius.

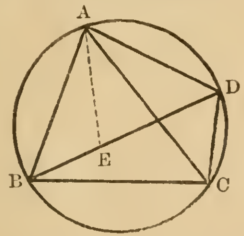
6. The rectangles contained by any two sides of triangles inscribed in the same or equal circles are proportional to the perpendiculars on the third sides.

7. If in the figure to VI. D the diagonals intersect at F, prove $BA \cdot BC : CB \cdot CD = BF : CF$, and conversely.

8. In the same figure prove
$$AB \cdot AD + CB \cdot CD : BA \cdot BC + DA \cdot DC = AC : BD.$$

PROPOSITION D.* Theorem.

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two rectangles contained by its opposite sides.

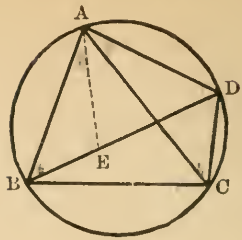


Let $ABCD$ be a quadrilateral inscribed in a circle, and AC, BD its two diagonals:

it is required to prove $AC \cdot BD = AB \cdot CD + AD \cdot BC$.

Make $\angle\ BAE = \angle\ DAC$. *I.* 23

* This theorem is often called Ptolemy's (about 140 A.D.) because it occurs in his *Almagest*, I. 9.



To each of these equals add $\angle\ EAC$;

$\therefore\ \angle\ BAC = \angle\ EAD$.

In $\triangle$s ABC, AED, $\begin{cases} \angle\ BAC = \angle\ EAD \\ \angle\ ACB = \angle\ ADE; \end{cases}$ *III.* 21

$\therefore$ these triangles are mutually equiangular. *I.* 32, *Cor.* 1

$\therefore\ BC : CA = ED : DA$; *VI.* 4

$\therefore\ AD \cdot BC = AC \cdot ED$. *VI.* 16

In $\triangle$s ABE, ACD, $\begin{cases} \angle\ BAE = \angle\ CAD \\ \angle\ ABE = \angle\ ACD; \end{cases}$ *Const.* *III.* 21

$\therefore$ these triangles are mutually equiangular. *I.* 32, *Cor.* 1

$\therefore\ AB : BE = AC : CD$; *VI.* 4

$\therefore\ AB \cdot CD = AC \cdot BE$. *VI.* 16

Hence, $AB \cdot CD + AD \cdot BC = AC \cdot BE + AC \cdot ED$,

$= AC \cdot BD$. *II.* 1

1. An equilateral triangle is inscribed in a circle, and from any point on the $\bigcirc^{ce}$ straight lines are drawn to the vertices; prove that one of these is equal to the sum of the other two.
2. In all quadrilaterals that cannot be inscribed in a circle, the rectangle contained by the diagonals is less than the sum of the two rectangles contained by the opposite sides.
3. Prove the converse of the proposition.
4. ABC is a triangle inscribed in a circle; D, E are taken on AB, AC so that B, D, E, C are concyclic; the circle ADE cuts the former in F. Prove that $FE + FB : FC + FD = AB : AC$. (R. Tucker.)

APPENDIX VI.

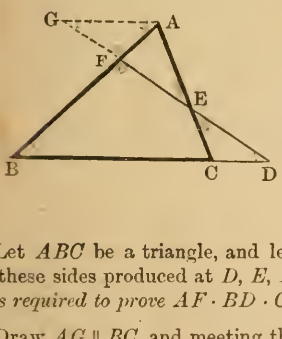
TRANSVERSALS.

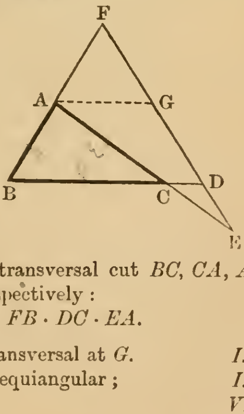
Def. 1.—When a straight line intersects a system of straight lines, it is called a **transversal**.

This definition of a transversal is not the most general (that is, comprehensive) one, but it will suffice for our present purpose.

Proposition 1.

*If a transversal cut the sides, or the sides produced, of a triangle, the product of three alternate segments taken cyclically is equal to the product of the other three, and conversely.**

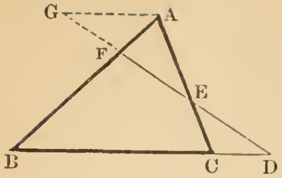


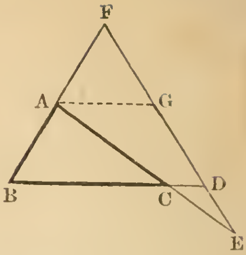


Let ABC be a triangle, and let a transversal cut BC, CA, AB, or these sides produced at D, E, F respectively :
it is required to prove $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$.

Draw $AG \parallel BC$, and meeting the transversal at G.	*I.* 31
Then $\triangle$s AFG, BFD are mutually equiangular ;	*I.* 29
$\therefore AF : AG = BF : BD$;	*VI.* 4
$\therefore AF \cdot BD = AG \cdot BF$. (1)	*VI.* 16

* Given in the third book of the *Spherics* of Menelaus, who lived at Alexandria towards the close of the first century A.D. For a full account of the theorem, see Chasles' *Aperçu Historique sur l'origine et le développement des Méthodes en Géométrie*, p. 291.





Again, $\triangle$s AEG, CED are mutually equiangular ; *I.* 29

$\therefore AG : AE = CD : CE ;$ *VI.* 4

$\therefore AG \cdot CE = CD \cdot AE.$ (2) *VI.* 16

Multiply equations (1) and (2) together, and strike out the common factor AG; then $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$.

Cor. 1.—The equation $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$ may be put in any of the following four useful forms :

$$AF : FB = DC \cdot EA : BD \cdot CE,$$
$$BD : DC = EA \cdot FB : CE \cdot AF,$$
$$CE : EA = FB \cdot DC : AF \cdot BD,$$
$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

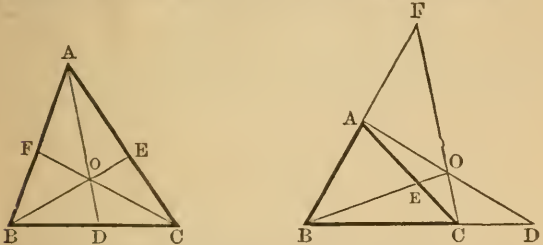
Cor. 2.—Consider ABC as the triangle, DEF as the transversal ;
then $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$. (1)
Consider AFE as the triangle, BCD as the transversal ;
then $AB \cdot FD \cdot EC = BF \cdot DE \cdot CA$. (2)
Consider BDF as the triangle, AEC as the transversal ;
then $BC \cdot DE \cdot FA = CD \cdot EF \cdot AB$. (3)
Consider CED as the triangle, AFB as the transversal ;
then $CB \cdot DF \cdot EA = BD \cdot FE \cdot AC$. (4)

Any one of these four equations may be deduced from the other three by multiplying them together and striking out the factors common to both sides.

The converse of the theorem (which may be proved indirectly) is, If two points be taken in the sides of a triangle, and a third point in the third side produced, or if three points be taken in the three sides produced of a triangle, such that the product of three alternate segments taken cyclically is equal to the product of the other three, the three points are collinear.

Proposition 2.

*If three concurrent straight lines be drawn from the vertices of a triangle to meet the opposite sides, or two of those sides produced, the product of three alternate segments of the sides taken cyclically is equal to the product of the other three; and conversely.**



Let ABC be a triangle, and let AD, BE, CF, which pass through any point O, meet the opposite sides in D, E, F:
it is required to prove $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$.

Consider ABD as a triangle cut by the transversal COF;
then $AF \cdot BC \cdot DO = FB \cdot CD \cdot OA$. (1) *App. VI.* 1
Consider ADC as a triangle cut by the transversal BOE;
then $AO \cdot DB \cdot CE = OD \cdot BC \cdot EA$. (2) *App. VI.* 1
Multiply equations (1) and (2) together, and strike out the common factors AO, DO, BC;
then $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$.

Cor.—Repeat Cor. 1 to the preceding theorem.

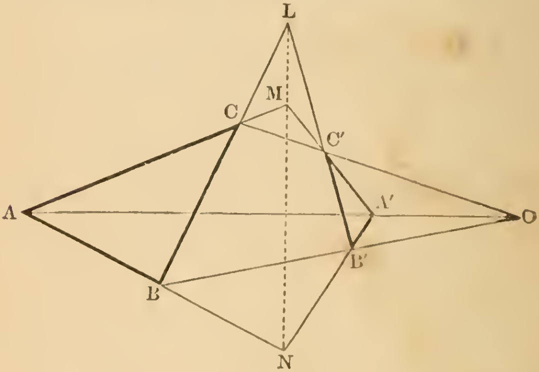
The converse of the theorem (which may be proved indirectly) is, If three straight lines be drawn from the vertices of a triangle to meet the opposite sides, or two of those sides produced, so that the product of three alternate segments of the sides taken cyclically is equal to the product of the other three, the three straight lines are concurrent.

* This theorem is first found in a work of the Marquis Giovanni Ceva, *De lineis rectis se invicem secantibus, statica constructio* (1678), Book I., Prop. 10. The proof given in the text is due to Carnot, the founder of the Theory of Transversals. See his *Essai sur la Théorie des Transversales* (1806), p. 74.

NOTE.—To distinguish readily between the converse of Menelaus's theorem and that of Ceva's, it should be observed that in the first case an even number of the points D, E, F are situated on the sides, and an odd number on the sides produced; in the second case matters are reversed.

PROPOSITION 3.

*If two triangles be situated so that the straight lines joining corresponding vertices are concurrent, the points of intersection of corresponding sides are collinear; and conversely.**



Let ABC, $A'B'C'$ be two triangles such that AA', BB', CC' are concurrent at O; and let the corresponding sides BC, $B'C'$ meet in L, AC, $A'C'$ in M, AB, $A'B'$ in N:
it is required to prove L, M, N collinear.

Consider AOB as a triangle cut by the transversal $A'B'N$;
then $AN \cdot BB' \cdot OA' = NB \cdot B'O \cdot A'A$. (1) *App. VI.* 1
Consider AOC as a triangle cut by the transversal $A'C'M$;
then $AA' \cdot OC' \cdot CM = A'O \cdot C'C \cdot MA$. (2) *App. VI.* 1
Consider BOC as a triangle cut by the transversal $B'C'L$;
then $B'O \cdot C'C \cdot LB = BB' \cdot OC' \cdot CL$. (3) *App. VI.* 1

* Due to Girard Desargues, an architect of Lyon, who was born 1593, and died 1662. See Poudra's *Œuvres de Desargues*, tome i. pp. 413, 430.

Multiply the equations (1), (2), (3) together, and strike out the common factors;
then $AN \cdot BL \cdot CM = NB \cdot LC \cdot MA$;
$\therefore$ L, M, N are collinear. *App. VI.* 1

The converse of the theorem (which may be proved indirectly) is, If two triangles be situated so that the points of intersection of corresponding sides are collinear, the straight lines joining corresponding vertices are concurrent.

HARMONICAL PROGRESSION.

Def. 2.—If a straight line be cut internally and externally in the same ratio it is said to be cut **harmonically**; and the two points of section are said to form with the ends of the straight line a **harmonic range**.

Thus, if AB be cut internally at C, and externally at D, in the same ratio, AB is said to be cut harmonically; and the points A, C, B, D are said to form a harmonic range.

Def. 3.—The points C and D are said to be **harmonically conjugate** to each other (harmonic conjugates) with respect to the points A and B. The segments AB, CD are sometimes (Chasles' *Géométrie Supérieure*, § 58) called **harmonic conjugates.**

Since a straight line can be cut internally, and therefore externally in any ratio, it may be cut harmonically in an infinite number of ways.

The ancient Greek mathematicians* defined three magnitudes to be in harmonical progression when the first is to the third as the difference between the first and second is to the difference between the second and third. Now, if AB be cut internally at C and externally at D in the same ratio,

$$AD : DB = AC : CB;$$

$$\therefore \quad AD : AC = DB : CB \quad \text{by alternation,} \qquad V.\ 16$$

$$= AD - AB : AB - AC.$$

Hence, if AD, AB, AC be regarded as the three magnitudes, it will be seen that they are in harmonical progression, since they conform to the definition.

* Pythagoras probably first. On the different progressions, see Pappus, III., section 12.

Proposition 4.

If C and D are harmonic conjugates with respect to A and B, then A and B are harmonic conjugates with respect to C and D.

Since C and D are harmonic conjugates with respect to A and B,
$\therefore AB$ is cut internally at C and externally at D in the same ratio;
$\therefore AD : DB = AC : CB$; *App. VI. Def.* 3
$\therefore AD : AC = DB : CB$, by alternation, *V.* 16
that is, CD is cut externally at A and internally at B in the same ratio;
$\therefore A$ and B are harmonic conjugates with respect to C and D.

Cor. 1.—Hence, if A, C, B, D form a harmonic range, not only are AD, AB, AC in harmonic progression, but also AD, CD, BD.

Cor. 2.—The points which are harmonic conjugates to two given points are always situated on the same side of the middle of the line joining the two given points.

(1)

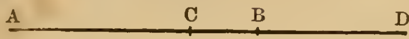
(2)

(3)

(4)

Suppose A and B the given points, O the middle of AB.
Since C and D are harmonic conjugates with respect to A and B,
$\therefore AD : DB = AC : CB$. *App. VI. Def.* 3
Now if D be situated (as in figs. 1 and 2) to the right of O,
then AD must be greater than DB;
$\therefore AC$ must be greater than CB,
that is, C also is situated to the right of O.
If D be situated (as in figs. 3 and 4) to the left of O,
then AD must be less than DB;
$\therefore AC$ must be less than CB,
that is, C also is situated to the left of O.

Cor. 3.—If any three of the points forming a harmonic range be given, the fourth may be determined.



Four cases are all that can arise, namely, when A, C, B, or D is to be found.

(1) If A, C, B are given, D can be found by dividing AB externally in the ratio $AC : CB$.

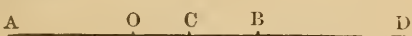
(2) If C, B, D are given, A can be found by dividing DC externally in the ratio $DB : BC$.

(3) If A, B, D are given, C can be found by dividing AB internally in the ratio $AD : DB$.

(4) If A, C, D are given, B can be found by dividing DC internally in the ratio $DA : AC$.

Proposition 5.

*If AD, AB, AC are in harmonical progression, and the mean AB is bisected at O, then OD, OB, OC are in geometrical progression; and conversely.**



Since AD, AB, AC are in harmonical progression,

$\therefore \quad AD : DB = AC : CB;$ *App. VI. Def.* 2

$\therefore OD + OB : OD - OB = OB + OC : OB - OC;$

$\therefore \quad OD : OB = OB : OC.$ *Converse of* V. D

Cor. 1.—Since $OD : OB = OB : OC$, $\therefore OB^2 = OC \cdot OD$. *VI.* 17
Now if A and B are fixed points, OB^2 is constant;
$\therefore OC \cdot OD$ is constant.

Hence if OC diminishes, OD increases, that is, if C moves nearer to O, D moves farther away; and if OC increases, OD diminishes, that is, if C moves away from O, D moves nearer to O. In other words, if C and D move in such a manner as always to remain harmonic conjugates with respect to the fixed points A and B, they must move in opposite directions. Also, the nearer C approaches to O, the farther does D recede from it; and when C coincides with O, D must be infinitely distant from it, or as it is often expressed, at infinity.

* Pappus, VII. 160.

Cor. 2. $OD : CD = OB^2 : AC \cdot CB$.

Cor. 3. $OC : OD = AC^2 : AD^2$.

[Corr. 2, 3 are given in De La Hire's *Sectiones Conicæ*, 1685, p. 3.]

Proposition 6.

*If AD, AB, AC are in harmonical progression, and the mean AB is bisected at O, then AD, OD, CD, BD are proportionals; and conversely.**

For $AD \cdot DB = (OD + OB) \cdot (OD - OB)$,

$= OD^2 - OB^2$, *II.* 5, *Cor.*

$= OD^2 - OD \cdot OC$, *App. VI.* 5

$= OD \cdot CD$; *II.* 3

$\therefore \quad AD : OD = CD : BD$. *VI.* 16

Cor. 1.—Since $OD \cdot CD = AD \cdot DB$;

$\therefore \quad 2\,OD \cdot CD = 2\,AD \cdot DB$;

$\therefore \quad (AD + DB) \cdot CD = 2\,AD \cdot DB$,

a result which, considering AD, CD, BD as the terms in harmonical progression, may be stated thus:

The rectangle under the harmonic mean and the sum of the extremes is equal to twice the rectangle under the extremes.

Cor. 2.—The geometric mean between two straight lines is a geometric mean between the arithmetic and the harmonic means of the same straight lines. [The arithmetic mean between two magnitudes is half their sum.]

Denote the arithmetic, geometric, and harmonic means between AD and DB by a, g, h respectively;

then $a = \frac{1}{2}(AD + DB) = OD$, $g^2 = AD \cdot DB$, $h = CD$.

Now since $AD \cdot DB = OD \cdot CD$, $\therefore g^2 = a \cdot h$;

$\therefore a : g = g : h$.

* Pappus, VII. 160.

Proposition 7.

*If AD, AB, AC are in harmonical progression, and the mean AB is bisected at O, then $CB : CD = CO : CA$; and conversely.**

For $AC \cdot CB = (OB + OC) \cdot (OB - OC)$,

$= OB^2 - OC^2$, *II.* 5, *Cor.*

$= OC \cdot OD - OC^2$, *App. VI.* 5

$= OC \cdot CD$; *II.* 3

$\therefore \quad CB : CD = CO : CA.$ *VI.* 16

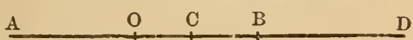
Cor. 1. $DB : DC = AO : AC$. (De La Hire's *Sectiones Conicæ*, p. 3.)

Cor. 2. $AB \cdot CD = 2\,AC \cdot BD = 2\,AD \cdot BC$.

Cor. 3. $AB^2 + CD^2 = (AC + BD)^2$.

Proposition 8.

If AD, AB, AC are in harmonical progression, then $AD \cdot DB - AC \cdot CB = CD^2$; and conversely.



Bisect AB at O.

Then $AD \cdot DB = (OD + OB) \cdot (OD - OB) = OD^2 - OB^2$, *II.* 5, *Cor.*

and $AC \cdot CB = (OB + OC) \cdot (OB - OC) = OB^2 - OC^2$; *II.* 5, *Cor.*

$\therefore AD \cdot DB - AC \cdot CB = OD^2 - 2\,OB^2 + OC^2$,

$= OD^2 - 2\,OD \cdot OC + OC^2$, *App. VI.* 5

$= (OD - OC)^2 = CD^2$. *II.* 7

The theorem may also be proved without bisecting AB.

The following definitions are necessary for some of the deductions :

Def. 4.—If four points A, C, B, D forming a harmonic range be joined to another point O, the straight lines OA, OC, OB, OD are said to form a **harmonic pencil**. OA, OC, OB, OD are called the **rays** of the pencil, and the pencil is denoted by $O \cdot ACBD$.

* Pappus, VII. 160.

Def. 5.—If the straight line joining the centres of two circles be divided internally and externally in the ratio of the radii, the points of section are called the **internal** and **external centres of similitude** of the two circles. (The phrase 'centre of similitude' is due to Euler, 1777. See *Nov. Act. Petrop.*, ix. 154.)

Def. 6.—The figure which results from producing all the sides of any ordinary quadrilateral till they intersect is called a **complete quadrilateral**; and the straight line joining the intersections of pairs of opposite sides is called the **third diagonal**. (Carnot, *Essai sur la Théorie des Transversales*, p. 69.)

To the notation adopted for points and lines connected with the triangle ABC on pp. 98–100, 252, 253, should be added the following:

N, P, Q denote the points where the bisectors of the interior $\angle$ s A, B, C meet the opposite sides.

N', P', Q' denote the points where the bisectors of the exterior $\angle$ s A, B, C meet the opposite sides.

$\triangle$ by itself denotes the area of $\triangle\ ABC$.

ρ denotes the radius of the circle inscribed in the orthocentric $\triangle\ XYZ$.

DEDUCTIONS.

1. C and D are two points both in AB, or both in AB produced: show that $AC : CB$ is not $= AD : DB$.
2. Find the geometric mean between the greatest and the least straight lines that can be drawn to the $\bigcirc^{ce}$ of a circle from a point (1) within, (2) without the circle.
3. In the figure to IV. 10, $\triangle$ s ABD, ACD, DCB are in geometrical progression.
4. Construct a right-angled triangle whose sides shall be in geometrical progression.
5. If a straight line be a common tangent to two circles which touch each other externally, that part of the tangent between the points of contact is a geometric mean between the diameters of the circles.
6. Any regular polygon inscribed in a circle is a geometric mean between the inscribed and circumscribed regular polygons of half the number of sides.

7. To find a mean proportional between AB and BC, C being situated between A and B. Produce AB to E, making $BE = AC$; with A and E as centres and AB as radius, describe arcs cutting in D; join BD. BD is the mean proportional. (See Wallis's *Algebra*, Additions and Emendations, 1685, p. 164.)

Of three straight lines in geometrical progression:

8. Given the mean and the sum of the extremes, to find the extremes.
9. Given the mean and the difference of the extremes, to find the extremes.
10. Given one extreme and the sum of the mean and the other extreme, to find the mean and the other extreme.
11. Given one extreme and the difference of the mean and the other extreme, to find the mean and the other extreme.
12. Find two straight lines from any two of the six following data: their sum, their difference, the sum of their squares, the difference of their squares, their rectangle, their ratio.
13. If two triangles have two angles supplementary and other two angles equal, the sides about their third angles are proportional.
14. Divide a straight line into two parts, the squares on which shall have a given ratio.
15. Describe a square which shall have a given ratio to a given polygon.
16. Cut off from a given triangle another similar to it, and in a given ratio to it.
17. Cut off from a given angle a triangle = a given space, and such that the sides about that angle shall have a given ratio.
18. ACB is a semicircle whose diameter is AB, and on AB is described a rectangle $ADEB$, whose altitude = the chord of half the semicircle; from C, any point in the $\bigcirc^{ce}$, CD, CE are drawn cutting AB at F and G. Prove $AG^2 + BF^2 = AB^2$. (Due to Fermat, 1658. See Wallis's *Opera Mathematica*, 1695, vol. i. p. 858.)
19. If two chords AB, CD intersect each other at a point E inside a circle, the straight lines AD, BC cut off equal segments from the chord which passes through E and is there bisected.
20. Enunciate and prove the preceding theorem when the chords AB, CD intersect each other outside the circle.

Prove the following properties of $\triangle\ ABC$:

21. $s(s-a) : \Delta = \Delta : (s-b)(s-c)$.
22. $s : s-a = (s-b)(s-c) : r^2 = r_1^2 : (s-b)(s-c)$.
23. $rr_1r_2r_3 = \Delta^2$.
24. $s^2 = r_1r_2 + r_2r_3 + r_3r_1$.
25. $sa = r_1(r_2 + r_3)$, $sb = r_2(r_3 + r_1)$, $sc = r_3(r_1 + r_2)$.
26. $r(r_1 + r_2 + r_3) = AF \cdot FB + BD \cdot DC + CE \cdot EA$.
27. $\Delta = \frac{1}{2}R\,(XY + YZ + ZX)$.
28. $2s : XY + YZ + ZX = R : r$.
29. $\triangle\ ABC : \triangle\ XYZ = R : \rho$.
30. $2R\rho = AO \cdot OX = BO \cdot OY = CO \cdot OZ$.
31. $a^2 + b^2 + c^2 = 8R^2 + 4R\rho$.
32. $SI^2 = R(R - 2r)$.
33. $SI_1^2 = R(R + 2r_1)$, $SI_2^2 = R(R + 2r_2)$, $SI_3^2 = R(R + 2r_3)$.
34. $SI^2 + SI_1^2 + SI_2^2 + SI_3^2 = 12R^2$.
35. $a^2 + b^2 + c^2 + r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2$.
36. $II_1^2 + II_2^2 + II_3^2 + I_1I_2^2 + I_2I_3^2 + I_3I_1^2 = 48R^2$.
37. $HD^2 = HD_1^2 = HX \cdot HN$; $HD_2^2 = HD_3^2 = HX \cdot HN'$.
38. $HX \cdot ND = HD \cdot DX$; $HX \cdot ND_1 = HD_1 \cdot D_1X$.
39. $HX \cdot N'D_2 = HD_2 \cdot D_2X$; $HX \cdot N'D_3 = HD_3 \cdot D_3X$.
40. $HN \cdot NX = DN \cdot ND_1$; $HN' \cdot N'X = D_2N' \cdot N'D_3$.

[Regarding theorem 21, see p. 145. It has, however, been conjectured, and with probability, that the treatise in which it occurs is a work of Heron the younger, and therefore long subsequent to the date of the elder Heron. The theorem was known to Brahmegupta, 628 A.D. For theorems 22, 36, 25, 26, see Davies in *Ladies' Diary*, 1835, pp. 56, 59; 1836, p. 50; and *Philosophical Magazine* for June 1827, p. 28. For 23 and 24, see Lhuilier, *Élémens d'Analyse*, p. 224. For 27, 28, 29, 30, 31, 34, 35, see Feuerbach, *Eigenschaften*, &c., section vi., theorems 3, 4, 5, 6, 7; section iv., § 50; section ii., § 29. Theorem 32 is usually attributed to Euler, who gave it in 1765. It occurs, however, in vol. i. page 123, by William Chapple, of the *Miscellanea Curiosa Mathematica*, and probably appeared about 1746. Theorem 33 is given in John Landen's *Mathematical Lucubrations*, 1755, p. 8. Some of the properties 37–40 are well known; but I cannot trace them to their sources. Hundreds of other beautiful properties of the triangle may be found in Thomas Weddle's papers in the *Lady's and Gentleman's Diary* for 1843, 1845, 1848.]

Construct a triangle, having given :

41. The vertical angle, the ratio of the sides containing it, and the base. (Pappus, VII. 155.)
42. The vertical angle, the ratio of the sides containing it, and the diameter of the circumscribed circle.
43. The vertical angle, the median from it, and the angle which the median makes with the base.
44. The vertical angle, the perpendicular from it to the base, and the ratio of the segments of the base made by the perpendicular.
45. The vertical angle, the perpendicular from it to the base, and the sum or difference of the other two sides.
46. The base, the perpendicular from the vertex to the base, and the ratio of the other two sides.
47. The base, the perpendicular from the vertex to the base, and the rectangle contained by the other two sides.
48. The segments into which the perpendicular from the vertex divides the base, and the ratio of the other two sides.
49. The perpendiculars from the vertices to the opposite sides.
50. The sides containing the vertical angle, and the distance of the vertex from the centre of the inscribed circle.

TRANSVERSALS.

The following five triads of straight lines are concurrent :

1. The medians of a triangle.
2. The bisectors of the angles of a triangle.
3. The bisector of any angle of a triangle and the bisectors of the two exterior opposite angles.
4. The perpendiculars from the vertices of a triangle on the opposite sides.
5. AL, BK, CF in the figure to I. 47.
6. If two sides of a triangle be cut proportionally (as in VI. 2), the straight lines drawn from the points of section to the opposite vertices will intersect on the median from the third vertex ; and conversely.
7. The points in which the bisectors of any two angles of a triangle and the bisector of the exterior third angle cut the opposite sides are collinear.
8. The points in which the bisectors of the three exterior angles of a triangle meet the opposite sides are collinear.

9. If a circle be circumscribed about a triangle, the points in which tangents at the vertices meet the opposite sides are collinear.
10. The perpendiculars to the bisectors of the angles of a triangle at their middle points meet the sides opposite those angles in three points which are collinear. (G. de Longchamps.)
11. OA, $O'A'$, $O''A''$ are three parallel straight lines; OO', AA' meet at B''; $O'O''$, $A'A''$ at B; $O''O$, $A''A$ at B'. Prove B, B', B'' collinear.
12. If a transversal cut the sides, or the sides produced, of any polygon, the product of one set of alternate segments taken cyclically is equal to the product of the other set. (Carnot's *Essai sur la Théorie des Transversales*, p. 70.)
13. If a hexagon be inscribed in a circle, and the opposite sides be produced to meet, the three points of intersection are collinear. (Particular case of Pascal's theorem.)
14. Prove with reference to fig. on p. 345.

$$AO \cdot BO \cdot CO : DO \cdot EO \cdot FO = AB \cdot BC \cdot CA : AF \cdot BD \cdot CE.$$

(Davies's edition of *Hutton's Mathematics*, 1843, vol. ii. p. 219.)

15. If a point A be joined with three collinear points B, C, D, then will

$$AC^2 \cdot BD \pm AB^2 \cdot CD = AD^2 \cdot BC \pm BD \cdot DC \cdot BC,$$

the upper sign being taken when D lies between B and C, and the lower when it does not. (Matthew Stewart's *Some General Theorems of considerable use in the higher parts of Mathematics*, 1746, Prop. II.) Deduce from the preceding theorem, App. II. 1; deduction 1 on p. 151; VI. B; and App. VI. 8.

16. If the $\bigcirc^{ce}$ of a circle cut the sides BC, CA, AB, or those sides produced, of $\triangle\ ABC$ at the points D,D', E,E', F,F', then will $AF \cdot AF' \cdot BD \cdot BD' \cdot CE \cdot CE' = FB \cdot F'B \cdot DC \cdot D'C \cdot EA \cdot E'A$. (Carnot's *Essai*, &c., p. 72.)
17. Prove with reference to fig. on p. 251.

$$AI \cdot BI \cdot CI : AB \cdot BC \cdot CA = AB \cdot BC \cdot CA : AI_1 \cdot BI_2 \cdot CI_3.$$

(C. Adams's *Die merkwürdigsten Eigenschaften des geradlinigen Dreiecks*, 1846, p. 20.)

18. Prove the following triads of straight lines connected with $\triangle\ ABC$ concurrent:

(1) $AD,\ BE,\ CF$	(7) $AD_3,\ BE,\ CF_1$	(13) $BC,\ E_2F_3,\ E_1F_1$
(2) $AD_1,\ BE_1,\ CF_1$	(8) $AD_2,\ BE_1,\ CF$	(14) $CA,\ F_3D_1,\ F_2D_2$
(3) $AD_2,\ BE_2,\ CF_2$	(9) $AB,\ DE,\ D_2E_1$	(15) $AB,\ NP,\ I_1I_2$
(4) $AD_3,\ BE_3,\ CF_3$	(10) $BC,\ EF,\ E_3F_2$	(16) $BC,\ PQ,\ I_2I_3$
(5) $AD_1,\ BE_2,\ CF_3$	(11) $CA,\ FD,\ F_1D_3$	(17) $CA,\ QN,\ I_3I_1$
(6) $AD,\ BE_3,\ CF_2$	(12) $AB,\ D_1E_2,\ D_3E_3$	

19. If the triads (5), (6), (7), (8) meet at the points I', I_1', I_2', I_3' respectively, prove that these four points are the inscribed and escribed centres of the triangle formed by drawing through A, B, C parallels to the opposite sides.

20. If the triads (9), (10), (11) meet at the points Q_1, N_1, P_1,
 " (12), (13), (14) " " Q_2, N_2, P_2,
 " (15), (16), (17) " " Q_3, N_3, P_3;
then Q_1, N_1, P_1 will lie on one straight line n,
 Q_2, N_2, P_2 " " p,
 Q_3, N_3, P_3 " " q;
and the three straight lines n, p, q will be concurrent. (Stephen Watson in the *Lady's and Gentleman's Diary* for 1867, p. 72.)

HARMONICAL PROGRESSION.

1. When a straight line is cut in extreme and mean ratio, the difference of the segments equals half the harmonic mean between them.

Of three straight lines in harmonical progression, having given

2. The mean and the greater extreme, find the less extreme.
3. The mean and the less extreme, find the greater extreme.
4. The two extremes, find the mean. (Pappus, III. 9, 10, 11.)

5. If from any point in the ○ᶜᵉ of a circle straight lines be drawn to the extremities of a chord, and meeting the diameter ⊥ the chord, they will divide the diameter harmonically. (Pappus, VII. 156.)

6. If two tangents be drawn to a circle, any third tangent is cut harmonically by the two former, by their chord of contact, and by the circle.

7. In the figures to VI. 2, if BE, CD intersect at F, then AF is cut harmonically by DE and BC; and AF bisects BC.

8. If from a point outside a circle two tangents and a secant be drawn, that part of the secant between the external point

and the chord of contact of the tangents is cut harmonically by the $\bigcirc^{ce}$. (Pappus, VII. 154.)

9. A, C, B, D form a harmonic range; on AB as diameter a circle is described, and from D there is drawn a perpendicular to AD. If E be any point in this perpendicular, EC is cut harmonically by the $\bigcirc^{ce}$ of the circle. (Pappus, VII. 161.)

10. AFB is a circle, of which AB is a diameter; D is any point in AB, and DF is $\perp$ AB; EDC is a chord drawn through D such that DE equals the radius. Show that DE, DF, DC are the arithmetic, geometric, and harmonic means between AD and DB; and prove App. VI., 6, Cor. 2.

11. AFB is a circle of which AB is a diameter; D is any point in AB produced, and DF is a tangent to the circle; FC is drawn $\perp$ AB, and E is the middle point of AB. Show that DE, DF, DC are the arithmetic, geometric, and harmonic means between AD and DB; and prove App. VI., 6, Cor. 2.

12. If one of the four rays of a pencil be $\parallel$ a transversal, and the alternate ray bisect the segment of the transversal between the remaining rays, the pencil is harmonic.

13. If a transversal be $\parallel$ one ray of a harmonic pencil, the conjugate ray will bisect that segment of the transversal intercepted by the other pair of rays.

14. The base of a triangle is cut harmonically by the bisectors of the interior and exterior vertical angles.

15. If two alternate rays of a harmonic pencil be at right angles, one of them bisects the angle included by the remaining pair of rays, and the other bisects the supplementary angle.

16. If a pencil divide one transversal harmonically, it will divide all transversals harmonically.

17. A, C, B, D and A, C', B', D' are harmonic ranges. Prove that CC', BB', DD' are concurrent.

18. AB is a straight line, and C any point in it; on AB any $\triangle$ ABE is described, and CE is joined; in CE any point O is taken, and AO, BO are joined and produced to meet BE and AE in F and G. Prove that FG produced will cut AB at D, the point harmonically conjugate to C with respect to A and B.

[The last seven theorems are given in De La Hire's *Sectiones Conicæ*, pp. 5–9; theorem 16, however, is a particular case of Pappus, VII. 129.]

19. ABC is a triangle; AD and AE, the bisectors of the interior and exterior vertical angles, meet the base BC at D and E. Prove that the rectangles $BD \cdot DC$, $BA \cdot AC$, $BE \cdot EC$ are in arithmetical progression when the difference of the base angles is equal to a right angle, in geometrical progression when one of the base angles is right, and in harmonical progression when the vertical angle is right. (Lardner's *Elements of Euclid*, 1843, p. 206.)

20. If K and L represent two regular polygons of the same number of sides, the one inscribed in, and the other circumscribed about, the same circle, and if M and N represent the inscribed and circumscribed polygons of twice the number of sides; M shall be a geometric mean between K and L, and N shall be a harmonic mean between L and M. (Library of Useful Knowledge, *Geometry*, 1847, p. 96.)

CENTRES OF SIMILITUDE.

1. When is the internal centre of similitude situated on both circles? How, in that case, is the external centre situated?
2. When is the external centre of similitude situated on both circles? How, in that case, is the internal centre situated?
3. When are both centres of similitude outside both circles, and when inside both circles?
4. When is the internal centre of similitude inside both circles, and the external centre outside both?
5. When two circles intersect, the straight line joining either point of intersection to the internal centre of similitude bisects the angle between the radii drawn to this point, and the straight line joining it to the external centre of similitude bisects the external angle between the radii.
6. The direct common tangents to two circles pass through the external, and the transverse common tangents through the internal, centre of similitude.
7. If from either centre of similitude of two circles a tangent be drawn to one of the circles, it will be a tangent also to the other. (Pappus, VII. 118.)
8. The vertices of a triangle are the external centres of similitude of the inscribed circle and each of the escribed circles, and

the internal centres of similitude of every pair of the escribed circles.

9. The points in which the bisectors of the interior angles of a triangle meet the opposite sides are the internal centres of similitude of the inscribed circle and each of the escribed circles.

10. The points in which the bisectors of the exterior angles of a triangle meet the opposite sides are the external centres of similitude of every pair of the escribed circles.

11. The secants drawn through the ends of parallel radii of two circles pass through the two centres of similitude. (Compare Pappus, VII. 110.)

12. If through either centre of similitude of two circles a common secant be drawn, and the points of intersection on each circle joined with the centre of that circle, the resulting radii will be parallel in pairs.

13. Any common secant drawn through either centre of similitude divides the circles into pairs of similar segments.

14. The straight line joining the vertex of a triangle to the escribed point of contact on the base, intersects the inscribed radius perpendicular to the base on the inscribed circle.

15. Enunciate and prove the corresponding property for the inscribed point of contact on the base.

16. The middle point of the base of a triangle, the inscribed centre, and the middle of the line drawn from the vertex to the point of inscribed contact on the base, are collinear.

17. Enunciate and prove the corresponding property for the escribed centre.

18. If a variable circle have with two fixed circles, contacts of the same species (that is, either both external, or both internal), the chord of contact will pass through the external centre of similitude of the two fixed circles; if contacts of different species, through the internal centre of similitude. (Poncelet, *Propriétés Projectives*, § 261. Compare Pappus, IV. 13.)

19. If each of two circles have contacts with another pair of circles either both of the same species, or both of different species, a centre of similitude of either pair lies on the radical axis of the other pair. (Poncelet, *Propriétés Projectives*, § 268.)

20. The six centres of similitude of three circles lie three and three

on four straight lines, called *axes of similitude.* (This theorem is attributed sometimes to D'Alembert, 1716–1783, sometimes to Monge.)

Loci.

1. Straight lines are drawn parallel to the base of a triangle and terminated by the other sides or the other sides produced; find the locus of their middle points.
2. Straight lines are drawn from a given point to a given straight line, and are cut internally or externally in a given ratio; find the locus of the points of section.
3. Straight lines are drawn from a given point to the $\bigcirc^{ce}$ of a given circle, and are cut internally or externally in a given ratio; find the locus of the points of section.
4. Hence find the locus of the centroid of a triangle whose base and vertical angle are given.
5. Find the locus of the points the ratio of whose distances from two given straight lines is equal to a given ratio.
6. If A, B, C be three points in a straight line, and D a point at which AB and BC subtend equal angles, the locus of D is the $\bigcirc^{ce}$ of a circle.
7. Given the base of a triangle and the ratio of the other two sides; find the locus of the vertex.
8. Find the locus of the intersection of the diagonals of all the rectangles that can be inscribed in a triangle.
9. ABC and ADE are similar triangles; ABC remains fixed, but ADE is rotated round A. Find the locus of the intersection of the straight lines which join the corresponding vertices B and D, C and E.
10. $ABCD$ is a rhombus whose diagonal AC is equal to each of its sides; through D a straight line PQ is drawn to meet BA and BC produced at P and Q, and AQ, CP are joined, intersecting at M. Find the locus described by M when PQ turns round D.
11. A series of triangles have the same base BC, and the sides which terminate at B are equal to a given length; find the locus of the point at which the bisector of the angle B intersects the opposite side.

    Examine the case of the bisector of the exterior angle at B.

12. AB is a diameter of a circle. A right angle, whose vertex is at A, revolves round A, and its sides intersect the tangent at B in the points C and D; find the locus of the intersection of the tangents drawn to the circle from the points C and D.

13. XY and $X'Y'$ are two parallel straight lines, and A, B, C three fixed collinear points. A straight line revolves round A and meets XY and $X'Y'$ at D and E; find the loci of the intersections of BE and CD, and of BD and CE.

14. XY and $X'Y'$ are two parallel straight lines, and O is a point midway between them. Through O straight lines are drawn terminated by XY and $X'Y'$, and equilateral triangles are described on these straight lines; find the locus of the third vertices of the triangles.

15. XY and $X'Y'$ are two parallel straight lines, and O is a fixed point. Through O straight lines are drawn to XY and $X'Y'$, and on the segments intercepted between XY and $X'Y'$ similar triangles are described; find the locus of the third vertices of the triangles. (The last three examples are taken from Vuibert's *Journal de Mathématiques Élémentaires*, 1re Année, pp. 13, 20; 3e Année, p. 5.)

MISCELLANEOUS.

1. Show that the perpendiculars of a triangle are concurrent, by a method which will prove at the same time that the circumscribed centre, the centroid, and the orthocentre are collinear, and that their distances from each other are in a constant ratio.

2. The circumscribed centre, the centroid, the medioscribed centre, and the orthocentre form a harmonic range; and the centroid and the orthocentre are the internal and external centres of similitude of the circumscribed and medioscribed circles.

3. All straight lines drawn from the orthocentre to the $\bigcirc^{ce}$ of the circumscribed circle are bisected by the $\bigcirc^{ce}$ of the medioscribed circle.

4. What is the analogous property for the straight lines drawn from the centroid to the $\bigcirc^{ce}$ of the circumscribed circle?

5. The inscribed centre, the centroid, and the point I' (see the 19th deduction on p. 357) are collinear, and their distances from each other are in a constant ratio.

6. The middle point J of $I'I$ is the centre of the circle inscribed in the centroidal $\triangle$ HKL.
7. The points H, J, and the middle point of AI' are collinear.
8. The points I', J, G, I form a harmonic range; and G and I' are the internal and external centres of similitude of the circles inscribed in $\triangle$s ABC, HKL.
9. The inscribed circle of $\triangle$ HKL is also the inscribed circle of the triangle formed by joining the middle points of AI', BI', CI'.
10. Deduce the properties corresponding to those in the last five deductions for the escribed centres, the centroid, and the points I_1', I_2', I_3'. (See the 19th deduction on p. 357.)
11. To find the centre of the circle ABC. With any point P on the $\bigcirc^{ce}$ as centre, and any radius PB, describe the circle ABD cutting the given circle at A and B. In this circle place the chord $BD = BP$, and join AD, meeting the given circle at E; EB or ED will be the radius of the given circle. (J. H. Swale. See *Philosophical Magazine*, 1851, p. 541.)
12. ABC is a triangle, right-angled at C. Angle B is bisected by BD, which meets AC at D; prove $2\ BC^2 : BC^2 - CD^2 = CA : CD$. (John Pell, 1644. This theorem is susceptible of a good many proofs.)
13. Of the four triangles formed by I, I_1, I_2, I_3 (see fig. on p. 251), the centroid of any one is the orthocentre of the triangle formed by the centroids of the other three.
14. The middle points of the three diagonals of a complete quadrilateral are in one straight line. The circles described on these three diagonals as diameters have the same radical axis; this radical axis is perpendicular to the straight line through the middle points of the diagonals, and it contains the orthocentres of the four triangles formed by taking the sides of the quadrilateral three and three. (The first part of this theorem is ascribed to Gauss, 1810; the last part is due to Steiner. See his *Gesammelte Werke*, vol. i. p. 128.)
15. In a given circle to inscribe a triangle

    (*a*) whose three sides shall be parallel to three given straight lines.

    (*b*) two of whose sides shall be parallel to two given straight lines, and the third shall pass through a given point.

(c) two of whose sides shall pass through two given points, and the third shall be parallel to a given straight line.

(d) whose three sides shall pass through three given points.

[The last of these problems is often called Castillon's, whose solution was published in 1776. A very full history, by T. S. Davies, both of it and of the more general problems to which it gave rise, will be found in *The Mathematician*, vol. iii. (1856), pp. 75–87, 140–154, 225–233, 311–322. It may be interesting to compare also Pappus, VII., 105, 107, 108, 109, 117.]

THE END