D. C. Skinner,

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THE

ELEMENTS OF EUCLID:

CONTAINING

THE FIRST SIX BOOKS,

AND

THE FIRST TWENTY-ONE PROPOSITIONS OF THE ELEVENTH BOOK.

(WITH THE PLANES SHADED,)

CHIEFLY

From the Text of Dr. Simson.

ADAPTED TO

THE USE OF STUDENTS BY MEANS OF SYMBOLS.

BY THE

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OF QUEENS' COLLEGE, CAMBRIDGE; AND CHAPLAIN IN THE HON. EAST INDIA COMPANY'S MADRAS ESTABLISHMENT.

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TO

JOHN PETTY MUSPRATT, ESQ.,

A DIRECTOR OF THE HON. EAST INDIA COMPANY,
ETC. ETC. ETC.

This Work

IS

INSCRIBED AS A TESTIMONY OF THE HIGHEST RESPECT,

AND

AS A GRATEFUL ACKNOWLEDGMENT

OF

MANY AND EXTENSIVE OBLIGATIONS CONFERRED ON

THE AUTHOR.

ADVERTISEMENT.

As the best method of teaching the Elements of Euclid is, undoubtedly, to practise the student in writing out his propositions, it has been one great object of this edition to furnish him with a model, by exhibiting the demonstrations in a form which is at once both brief and compact. With this end in view, the contractions of the first edition have been in some degree restored, and many improvements have been made which will render the subject less heavy to wield, and in consequence less repulsive to the beginner.

A few demonstrations have been added, by way of Appendix, which could not be substituted by the corresponding propositions in the work, without taking too great a liberty with the text.

Such parts of Euclid as are not read in the University of Cambridge, are omitted in this edition.

PREFACE.

Indispensable as a clear perception of the Elements of Euclid is to further progress in the Mathematics, yet the generality of beginners, it is well known, evince considerable reluctance in laying this necessary foundation of scientific knowledge. This may perhaps be mainly attributed to the uninviting form in which they have hitherto met the eye of the student; calculated by its manifest deficiency in perspicuous arrangement to embarrass, and therefore to disgust the mind.

The absence of a systematic arrangement in any kind of argument, cannot fail to occasion an obscurity which ought to be avoided. Each syllogism should be rendered clearly distinct from the other, in order that its force may be fully developed before another, consequent upon it, engage the attention. In the present editions of Euclid's Elements, no stronger mark of distinction exists between the several steps leading to the conclusion, than a colon or period. The student is therefore extremely liable to entertain a confused notion of the whole; the nice discrimination necessary to separate them, requiring more labour and greater abstraction of thought, than the generality of beginners are either capable of, or willing to submit to. This defect, it is hoped, the present work will in a great measure, if not entirely, remove.

The plan is simply this: the appropriation of a single line or paragraph to every individual step throughout the proposition; thus exhibiting the whole train of argument in a perspicuous and systematic arrangement. In order also to facilitate the object in view, by making the sentences shorter and more concise, symbols are substituted for words of frequent occurrence. Considerable attention has been devoted to the selection of these. All that appeared to be mere arbitrary characters have been rejected; while those only are retained whose figure or property makes them appropriate emblems of that which they are intended to indicate.

The text of Dr. R. Simson forms the basis of the work. Wherever that has been deviated from, recourse has been had in every case to the judgment of certain individuals, whose acknowledged scientific learning rendered their advice decisive. By these gentlemen also, the editor has been influenced in the choice of the symbols; and materially assisted in other respects.

It was originally intended to supply algebraical demonstrations to the second and fifth books; this has, however, been relinquished, under the apprehension that the sizo, and consequently the expense of the work, would be so increased, as to hazard the probability of its introduction into schools.*

QUEENS' COLLEGE, *May* 21, 1827.

* "A new Translation of the Elements, &c.," by Mr. George Phillips, embraces all that is requisite on this point.

EXPLANATION OF THE SYMBOLS.

$+$	signifies	*plus*, or *together with.*
$-$	,,	*minus*, or *less by.*
$=$	,,	*equal to.*
$\neq$	,,	*unequal to.*
$>$	,,	*greater than.*
$\ngtr$	,,	*not greater than.*
$<$	,,	*less than.*
$\nless$	,,	*not less than.*
$\perp$	,,	*perpendicular to*, or *at right angles to.*
$\parallel$	,,	*parallel to.*
$\nparallel$	,,	*not parallel to.*
$\because$	,,	*because.*
$\therefore$	,,	*therefore.*
$\angle$	,,	*angle.*
$\angle$s	,,	*angles.*
$\triangle$	,,	*triangle.*
$\square$	,,	*parallelogram.*
$\odot$	,,	*circle.*
$\bigcirc$	,,	*circumference.*
$\frac{1}{2}\odot$	,,	*semicircle.*

$\widehat{AB}$ signifies *arc*, terminated by points A & B.

AB^2 „ *square described on the right line AB.*

$(AB + CD)^2$ *square* described on the whole right line made up of the two AB and CD.

AB · CD is a *rectangle* contained by the right lines AB and CD.

A : B signifies *the ratio of A to B.*

A : B : : C : D *the ratio of A to B is the same as the ratio of C to D*; and is thus read—*as A is to B so is C to D*; or, *A is to B as C to D.*

Dupl. of A : B *the duplicate ratio of A to B.*

Tripl. of A : B *the triplicate ratio of A to B.*

ABBREVIATIONS.

Alt.	for	*altitude.*
Altern.	”	*alternate.*
Circumsc.	”	*circumscribed.*
Coin.	”	*coincide.*
Com.	”	*common.*
Constr.	”	*construction.*
Descr.	”	*described.*
Diag.	”	*diagonal.*
Dist.	”	*distance.*
Ea.	”	*each.*
Ext.	”	*exterior.*
Hex.	”	*hexagon.*
Inscr.	”	*inscribed.*
Int.	”	*interior.*
Mag.	”	*magnitude.*
No.	”	*number.*
Opp.	”	*opposite.*
Pent.	”	*pentagon.*
Pl.	”	*plane.*
Polyg.	”	*polygon.*

Prod.	for	*produced*.
Pt.	„	*point*.
Rem.	„	*remainder*.
Rt.	„	*right*.
Sect.	„	*sector*.
Seg.	„	*segment*.
Sim.	„	*similar to*.
Sq.	„	*square*.

H. C. Skinner.

THE

ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.

I.

A POINT is that which has no parts, or which has no magnitude.

II.

A line is length without breadth.

III.

The extremities of a line are points.

IV.

A right line is that which lies evenly between its extreme points.

V.

A superficies is that which has only length and breadth.

VI.

The extremities of superficies are lines.

VII.

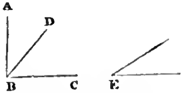
A plane superficies is that in which any two point being taken, the right line between them lies wholly in that superficies.

VIII.

"A plane angle is the inclination of two lines to one "another in a plane, which meet together, but are not in "the same direction."

IX.

A plane rectilineal angle is the inclination of two right lines to one another, which meet together, but are not in the same right line.



'N.B. When several angles are at one point B, any one 'of them is expressed by three letters, of which the letter 'that is at the vertex of the angle, that is, at the point in 'which the right lines that contain the angle meet one 'another, is put between the other two letters, and one of 'these two is somewhere upon one of these right lines, 'and the other upon the other line. Thus the angle which 'is contained by the right lines AB, CB, is named the 'angle ABC, or CBA; that which is contained by AB, 'DB, is named the angle ABD, or DBA; and that which 'is contained by DB, CB, is called the angle DBC, or 'CBD. But, if there be only one angle at a point, it may 'be expressed by the letter at that point; as the angle 'at E.'

X.

When a right line standing on another right line makes the adjacent angles equal to each other, each of these angles is called a right angle ; and the right line which stands on the other is called a perpendicular to it.

XI.

An obtuse angle is that which is greater than a right angle.

XII.

An acute angle is that which is less than a right angle.

XIII.

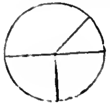
" A term or boundary is the extremity of anything."

XIV.

A figure is that which is enclosed by one or more boundaries.

XV.

A circle is a plain figure contained by one line, which is called the circumference ; and is such, that all right lines, drawn from a certain point within the figure to the circumference, are equal to one another.



XVI.

And this point is called the centre of the circle.

XVII.

A diameter of a circle is a right line drawn through the centre, and terminated both ways by the circumference.

XVIII.

A semicircle is the figure contained by a diameter and that part of the circumference which it cuts off.

XIX.

" A segment of a circle is the figure contained by a " right line and that part of the circumference which it " cuts off."

XX.

Rectilineal figures are those which are contained by right lines.

XXI.

Trilateral figures, or triangles, by three right lines.

XXII.

Quadrilateral, by four right lines.

XXIII.

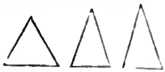
Multilateral figures, or polygons, by more than four right lines.

XXIV.

Of three sided figures, an equilateral triangle is that which has three equal sides.

XXV.

An isosceles triangle is that which has only two sides equal.



XXVI.

A scalene triangle is that which has three unequal sides.

XXVII.

A right angled triangle is that which has a right angle.

XXVIII.

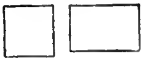
An obtuse angled triangle is that which has an obtuse angle.

XXIX.

An acute angled triangle is that which has three acute angles.

XXX.

Of quadrilateral or four sided figures, a square has all its sides equal and all its angles right angles.



XXXI.

An oblong has all its angles right angles, but has not all its sides equal.

XXXII.

A rhombus has all its sides equal, but its angles are not right angles.

XXXIII.

A rhomboid has its opposite sides equal to each other, but all its sides are not equal, nor its angles right angles.

XXXIV.

All other four sided figures beside these are called trapeziums.

XXXV.

Parallel right lines are such as are in the same plane, and which, however far produced either way, do not meet.

To these may be added :—

1. A problem is a proposition denoting something to be done.

2. A theorem is a proposition which requires to be demonstrated.

3. A corollary is a consequent truth gained from a preceding demonstration.

4. A deduction is a proposition drawn from a preceding demonstration.

POSTULATES.

I.

Let it be granted that a right line may be drawn from any one point to any other point.

II.

That a terminated right line may be produced to any length in a right line.

III.

And that a circle may be described from any centre at any distance from that centre.

AXIOMS.

I.

Things which are equal to the same are equal to each other.

II.

If equals be added to equals the wholes are equal.

III.

If equals be taken from equals the remainders are equal.

IV.

If equals be added to unequals the wholes are unequal.

V.

If equals be taken from unequals the remainders are unequal.

VI.

Things which are double of the same are equal to each other.

VII.

Things which are halves of the same are equal to each other.

VIII.

Magnitudes which coincide with each other, that is, which exactly fill the same space, are equal to each other.

IX.

The whole is greater than its part.

X.

Two right lines cannot inclose a space.

XI.

All right angles are equal to each other.

XII.

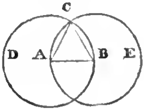
" If a right line meet two right lines, so as to make the " two interior angles on the same side of it taken together " less than two right angles, these right lines being con- " tinually produced shall at length meet on that side on " which are the angles which are less than two right " angles."

ELEMENTS OF EUCLID.

PROP. I.—Problem.

To describe an equilateral triangle upon a given finite right line.

Let AB be given rt. line; it is required to descr. on AB an equilat. △.



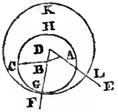
With cr. A, & dist. AB, descr. ⊙ BCD, 3 post.
with cr. B, & dist. BA, descr. ⊙ ACE;
From point C where the ⊙s cut, draw CA, CB. 1 post.
Then ABC is an equilat. △.
For ∵ A is cr. ⊙ BCD.
∴ AC = AB; 15 definition
& ∵ B is cr. ⊙ ACE,
∴ BC = AB,
But AC = AB,
∴ AC = BC, 1 axiom.
∴ AB, BC, CA = each other.

Wherefore △ ABC is equilat. : and it is descr. on AB, Q. E. F.

PROP. II.—Problem.

From a given point to draw a right line equal to a given right line.

Let A be given point, & BC given rt. line ; it is required to draw from A a rt. line = BC.



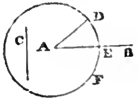
Join BA ;			1 post.
on AB descr. equilat.		$\triangle$ ABD,	1. 1.
prod. DB, DA	to	F & E ;	2 post.
with cr. B, & dist. BC	descr.	$\odot$ CGH,	3 post.
& with cr. D, and dist. DG	descr.	$\odot$ KGL.	
Then AL	=	BC.	
For $\because$ pt. B	is cr.	$\odot$ CGH,	
$\therefore$ BC	=	BG ;	15 def.
& $\because$ D	is cr.	$\odot$ KGL,	
$\therefore$ DL	=	DG,	
but part DA	=	part DB,	constr.
$\therefore$ rem. AL	=	rem. BG :	3 ax.
but BC	=	BG ;	
$\therefore$ AL	=	BC.	1 ax.

Wherefore from given point A has been drawn a rt. line AL = given rt. line BC. Q. E. F.

PROP. III.—Problem.

From the greater of two given right lines to cut off a part equal to the less.

Let C & AB be the given rt. lines, of which AB > C; it is required to cut off from AB a part = C.



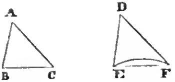
From A draw AD = C; 2. 1.
with cr. A & dist. AD descr. ⊙ DEF 3 post.
cutting AB in E:
then AE = C.
For ∵ A is cr. ⊙ DEF,
∴ AE = AD; 15 def.
But C = AD; constr.
∴ AE = C. 1 ax.

Wherefore from the greater AB is cut off AE = C the less. Q. E. F.

PROP. IV.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each; and have likewise the angles contained by those sides equal to each other; they shall likewise have their bases, or third *sides, equal; and* *the two triangles shall be equal; and their other angles shall be equal, each to each, viz. those to which the equal sides are opposite.*

Let two △s ABC, DEF have AB = DE, AC = DF, & ∠ BAC = ∠ EDF. Then base BC = base EF, △ ABC = △ DEF, ∠ ABC = ∠ DEF, & ∠ BCA = ∠ EFD.



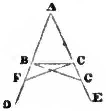
For if △ ABC	be applied	to △ DEF,	
so that pt. A	be on	pt. D,	
& AB	on	DE;	
then ∵ AB	=	DE,	hyp.
∴ B	coincides with	E;	
& ∵ ∠ BAC	=	∠ EDF,	hyp.
∴ AC	coin. with	DF;	
& ∵ AC	=	DF,	hyp.
∴ C	coin. with	F:	
But B	coin. with	E,	
∴ BC	coin. with	EF.	
For if BC do not coin. with		EF,	
then two rt. lines enclose a space,			
which is impossible.			10 ax.
∴ BC	coin. with & =	EF;	8 ax.
Wherefore △ ABC	coin. with & =	△ DEF;	
& ∠ ABC	coin. with & =	∠ DEF;	
& ∠ BCA	coin. with & =	∠ EFD.	

Therefore if two triangles, &c. &c. Q. E. D.

PROP. V.—Theorem.

The angles at the base of an isosceles triangle are equal to each other; and if the equal sides be produced, the angles on the other side of the base shall be equal.

Let ABC be an isosc. △, having side AB = side AC; prod. AB, AC, to D & E; then ∠ ABC = ∠ BCA, & ∠ DBC = ∠ BCE.



In BD take any pt. F;			
From AE cut off AG	=	AF	3. 1.
& join BG, CF.			
∵ AF	=	AG,	constr.
& AC	=	AB,	hyp.
& that ∠ FAG	is com. to	△s AFC, AGB;	
∴ BG	=	CF,	4. 1.
also ∠ ABG	=	∠ ACF,	4. 1.
& ∠ AFC	=	∠ AGB.	4. 1.
Again, ∵ whole AF	=	whole AG,	constr.
& part AB	=	part AC;	hyp.
∴ rem. BF	=	rem. CG;	3 ax.
& ∵ BG	=	CF,	
& BF	=	CG,	
& that ∠ BFC	=	∠ CGB;	
∴ ∠ BCF	=	∠ CBG,	4. 1.
& ∠ BCG	=	∠ CBF;	4. 1.

wh. are ∠s on opp. side base BC.

Again, $\because \angle$ ABG	$=$	$\angle$ ACF,	
& $\angle$ CBG	$=$	$\angle$ BCF,	
$\therefore$ rem. $\angle$ ABC	$=$	rem. $\angle$ BCA.	3 ax.

wh. are $\angle$s at base BC.

Wherefore the angles, &c. &c. Q. E. D.

Cor. Hence every equilateral triangle is also equiangular.

PROP. VI.—Theorem.

If two angles of a triangle be equal to each other, the sides also which subtend, or are opposite to, *the equal angles shall be equal to one another.*

Let $\triangle$ ABC have $\angle$ ABC $=$ $\angle$ BCA ; then AB $=$ AC.

For if AB	$\neq$	AC ;	
One of them is	$>$	the other :	
let AB	$>$	AC :	
From AB cut off DB	$=$	AC.	3. 1

Join DC.

Then $\because$ DB	$=$	AC,	constr.
& BC is come to $\triangle$s DBC, ACB,			
& $\angle$ DBC	$=$	$\angle$ BCA ;	hyp.
$\therefore$ AB	$=$	DC,	4. 1.
& $\triangle$ DBC	$=$	$\triangle$ ACB,	
i. e. less	$=$	greater,	

wh. is absurd.

$\because$ AB not $\neq$ AC,
i. e. AB = AC.

Wherefore if two angles, &c. &c. Q. E. D.

Cor. Hence every equiangular triangle is also equilateral.

PROP. VII.—Theorem.

Upon the same base and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to each other, and likewise those which are terminated in the other extremity.

If possible, on same base AB and on same side, let two $\triangle$s ACB, ADB, have CA of one = DA of other, both wh. are terminated in pt. A of base; and likewise CB = DB, both wh. are terminated in B.

Join CD.

First—Let each of vertices of $\triangle$s fall without the other $\triangle$.

Then $\because$ AC	=	AD,	hyp.
$\therefore \angle$ ACD	=	$\angle$ ADC;	5. 1.
but $\angle$ ACD	>	$\angle$ BCD,	9 ax.
$\therefore \angle$ ADC	>	$\angle$ BCD,	
much more $\therefore \angle$ BDC	>	$\angle$ BCD.	

Again, ∵ BD	=	BC,	hyp.
∴ ∠ BDC	=	∠ BCD,	5. 1.
but also ∠ BDC	>	∠ BCD ;	demon.

wh. is absurd.

SECONDLY—Let vertex D of △ ADB fall within the other △ ACB.

prod. AC, AD, to E, F.

Then ∵ AC	=	AD,	hyp.
∴ ∠ ECD	=	∠ CDF ;	5. 1.
but ∠ ECD	>	BCD,	9 ax.
∴ ∠ CDF	>	BCD,	
much more ∴ ∠ BDC	>	BCD.	
Again, ∵ BD	=	BC,	hyp.
∴ ∠ BDC	=	∠ BCD,	5. 1.
but also ∠ BDC	>	∠ BCD ;	demon.

wh. is absurd.

THIRDLY—The case of the vertex of one △ being on a side of the other, needs no demonstration.

Wherefore upon the same base, &c. &c. Q. E. D.

PROP. VIII.—THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them, of the other.

In △s ABC, DEF, let AB = DE, AC = DF, & base BC = base EF ; then ∠ BAC = ∠ EDF.

For if △ ABC	be appl. to	△ DEF,	
so that pt. B	be on	E,	
& BC	on	EF ;	
then ∵ BC	=	EF,	hyp.
∴ shall pt. C	coin. with	F ;	
& ∵ BC	coin. with	EF,	
∴ BA, AC	coin. with	ED, DF.	

For if BA, AC do not coin. with ED, DF ;
let BA, AC have another direction as EG, GF :
Then upon same base EF are constructed two △s in a manner wh. has been demonstrated to be impossible. 7. 1.

∴ if BC	coin. with	EF,	
BA, AC	must coin. with	ED, DF,	
& ∴ ∠ BAC	coin. with	∠ EDF ;	
∴ ∠ BAC	=	∠ EDF.	8 ax.

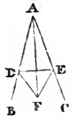
Wherefore if two triangles, &c. &c. Q. E. D.*

PROP. IX.—Problem.

To bisect a given rectilineal angle, that is, to divide it into two equal parts.

Let ∠ BAC be given rectilin. ∠, it is required to bisect it.

* See Appendix.



In AB	take	any pt. D ;	
make AE	=	AD ;	3. 1.
	join DE ;		
on DE	descr.	equilat. △ DEF ;	1. 1.
	join AF ;		

then rectilin. ∠ BAC is bisected by AF.

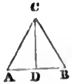
∵ AE	=	AD,	constr.
& AF	is com. to	△s DAF, EAF,	
& base DF	=	base EF ;	constr.
∴ ∠ DAF	=	∠ EAF.	8. 1.

Wherefore rectilin. ∠ BAC is bisected by AF. Q. E. F.

PROP. X.—Problem.

To bisect a given finite right line, that is, to divide it into two equal parts.

Let AB be given rt. line ; it is required to bisect it.



On AB descr. equilat.	△ ABC ;		1. 1.
bisect ∠ ACB	by	CD ;	9. 1.

then AB	is bis. in	D.	
∵ AC	=	CB,	constr.
& CD	is com. to	△s ACD, BCD,	
& ∠ ACD	=	∠ BCD ;	constr.
∴ AD	=	DB.	4. 1.

Wherefore AB is bisected in D. Q. E. F.

PROP. XI.—Problem.

To draw a right line at right angles to a given right line, from a given point in the same.

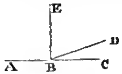
Let AB be given rt. line, & C given point in it; it is required to draw a rt. line from C ⊥ AB.

In AC take any point D ;			
make CE	=	CD ;	3. 1.
on DE	descr.	equilat. △ DEF ;	1. 1.
	join FC ;		
then FC is drawn ⊥ AB.			
For ∵ CD	=	CE,	constr.
& FC	is com. to	△s DFC, EFC ;	
∴ DC, CF	=	EC, CF, ea. to ea. ;	
& base DF	=	base FE,	constr.
∴ ∠ DCF	=	∠ FCE ;	8. 1.

but these are adj. ∠s ;
∴ each of them is a rt. ∠ ; 10 def.
∴ FC is ⊥ AB.

Wherefore from point C in AB, FC has been drawn ⊥ AB. Q. E. F.

Cor. By help of this problem, it may be demonstrated that two right lines cannot have a common segment.



If it be possible,
let seg. AB be com. to two rt. lines ABC, ABD :

draw BE	⊥	AB ;	
& ∵ ABC	is a	rt. line,	
∴ ∠ CBE	=	∠ EBA.	10 def.
Similarly ∵ ABD	is a	rt. line,	
∴ ∠ DBE	=	∠ EBA ;	
∴ ∠ DBE	=	∠ CBE ;	1 ax.
i. e. less	=	greater ;	

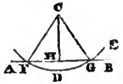
wh. is absurd ;

Therefore two right lines cannot have a common segment.

PROP. XII.—Problem.

To draw a right line perpendicular to a given right line of an unlimited length, from a given point without it.

Let AB be given rt. line, and C given point without it. It is required to draw from C a rt. line ⊥ AB.



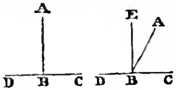
Take any point D on the other side of AB ;
with cr. C & dist. CD desc. ⊙ FDGE ; 3 post.
bisect FG in H ; 10 1.
join CH :
then CH is ⊥ AB.
join CF, CG ;
For ∵ GH = HF, by constr.
& CH is com. to △s FHC, GHC ;
then GH, HC = FH, HC, ea. to ea. ;
& base CF = base CG ; 15 def.
∴ ∠ GHC = ∠ FHC ; 8. 1.
but these are adj. ∠s,
∴ each of them is a rt. ∠,
i. e. CH ⊥ AB. 10 def.

Wherefore from the given point C has been drawn a rt. line CH ⊥ AB. Q. E. F.

PROP. XIII.—Theorem.

The angles which one right line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let rt. line AB make with CD, on same side of it, the ∠s DBA, ABC ; these are either 2 rt. ∠s, or are together = 2 rt. ∠s.



For if $\angle$ DBA $=$ $\angle$ ABC,
each of them is a rt. $\angle$. 10 def.

But if $\angle$ DBA $\neq$ $\angle$ ABC,
draw BE $\perp$ DC ; 11. 1.
$\therefore$ CBE, EBD, are rt. $\angle$s. 10 def.

& $\because$ $\angle$ CBE $=$ $\angle$s CBA+ABE,
add $\angle$ EBD to each,
$\therefore$ $\angle$s CBE+EBD $=$ $\angle$s CBA+ABE+EBD. 2 ax.

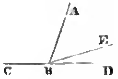
Again, $\because$ $\angle$ DBA $=$ $\angle$s DBE+EBA,
add $\angle$ ABC to each,
$\therefore$ $\angle$s DBA+ABC $=$ $\angle$s DBE+EBA+ABC ; 2 ax.
but $\angle$s CBE+EBD $=$ same 3 $\angle$s ;
$\therefore$ $\angle$s DBA+ABC $=$ $\angle$s CBE+EBD, 1 ax.
$=$ 2 rt. $\angle$s.

Wherefore the angles, &c. &c. Q. E. D.

PROP. XIV.—Theorem.

If, at a point in a right line, two other right lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two right lines shall be in one and the same right line.

At B in AB let BC, BD, on the opp. sides of AB, make the adj. $\angle$s ABC + ABD $=$ 2 rt. $\angle$s ; then shall CB be in the same rt. line with BD.



For if BD be not in the same rt. line with BC,
Let BE be in same rt. line with it.
Then $\because$ AB stands on rt. line CBE,

$\therefore \angle$s ABE + ABC $=$ 2 rt. $\angle$s ; 13. I.
but $\angle$s ABC + ABD $=$ 2 rt. $\angle$s ; by hyp.
$\therefore \angle$s ABE + ABC $=$ $\angle$s ABC + ABD ; 1 ax.
remove com. $\angle$ ABC,
& $\therefore$ rem. $\angle$ ABE $=$ rem. $\angle$ ABD ; 3 ax.
i. e. less $=$ greater ;
which is absurd.
$\therefore$ BE is not in same rt. line with BC :
and similarly none other but BD is in same rt. line with BC.

Wherefore if at a point, &c. &c. Q. E. D.

PROP. XV.—Theorem.

If two right lines cut each other, the vertical or opposite angles shall be equal.

Let AB, CD cut each other in E. Then $\angle$ AEC $= \angle$ BED ; and $\angle$ AED $= \angle$ BEC.

For $\because$ rt. line AE stands on rt. line CD,
$\therefore \angle$s AEC + AED $=$ 2 rt. $\angle$s. 13. I.

Again, ∵ DE stands on AB,

∴ ∠s AED + DEB	=	2 rt. ∠s ;	13. 1.
	=	∠s AEC + AED ;	1 ax.
	remove com. ∠ AED ;		
& rem. ∠ DEB	=	rem. ∠ AEC ;	3 ax.
Similarly ∠ AED	=	∠ BEC.	

Wherefore if two right lines cut each other, &c. &c. Q.E.D.

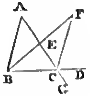
Cor. 1. From this it is manifest, that if two right lines cut each other, the angles they make at the point where they cut, are together equal to four right angles.

Cor. 2. And consequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROP. XVI.—THEOREM.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let side BC of △ ABC be prod. to D.
Then ext. ∠ ACD shall be > ABC or CAB.



Bisect AC in E ;		10. 1.
join BE, and prod. it to F ;		
make EF = EB ;		3 1.
join FC,		

Then $\because$ AE	=	EC,	constr.
& BE	=	EF,	
& that $\angle$ AEB	=	$\angle$ CEF ;	15. 1.
$\therefore$ base AB	=	base FC,	4. 1.
& $\angle$ BAC	=	$\angle$ ACF ;	
but $\angle$ ACD	>	$\angle$ ACF,	9 ax.
$\therefore$ $\angle$ ACD	>	$\angle$ BAC.	

Similarly by bisecting BC, and producing AC to G, it may be demonstrated that $\angle$ BCG,

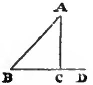
i. e. $\angle$ ACD	>	$\angle$ ABC.	15. 1.

Wherefore if one side, &c. &c. Q. E D.

PROP. XVII.—Theorem.

Any two angles of a triangle are together less than two right angles.

Let ABC be any $\triangle$, any two of its $\angle$s are together < 2 rt. $\angle$s.



Prod. BC	to	D.	
And $\because$ ext. $\angle$ DCA	>	int. & opp. $\angle$ CBA,	16. 1.
add $\angle$ ACB to each,			
$\therefore$ $\angle$s DCA + ACB	>	$\angle$s ACB + CBA.	4 ax.
But $\angle$s DCA + ACB	=	2 rt. $\angle$s ;	13. 1.
$\therefore$ $\angle$s ACB + CBA	<	2 rt. $\angle$s.	

Similarly $\begin{cases} \angle\text{s ABC} + \text{CAB} < 2 \text{ rt. } \angle\text{s}, \\ \& \angle\text{s BAC} + \text{ACB} < 2 \text{ rt. } \angle\text{s}. \end{cases}$

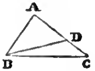
Wherefore any two angles of a triangle, &c. &c. Q. E. D.

Cor. Hence in every triangle having a right or an obtuse angle, the other two angles are acute.

PROP. XVIII.—Theorem.

The greater side of every triangle subtends the greater angle.

In △ ABC let side AC > side AB; then shall ∠ ABC be > ∠ ACB.



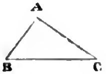
Since AC	>	AB,	
cut from it AD	=	AB;	3. 1.
join BD.			
Then ∵ AD	=	AB,	constr.
∴ ∠ ABD	=	∠ ADB:	5. 1.
but ext. ∠ ADB	>	int. ∠ DCB;	16. 1.
∴ ∠ ABD	>	ACB;	
much more ∴ ∠ ABC	>	∠ ACB.	

Wherefore the greater side of every triangle, &c. &c. Q. E. D.

PROP. XIX.—Theorem.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

In $\triangle$ ABC let $\angle$ ABC be $>$ $\angle$ ACB ; then shall side AC be $>$ side AB.



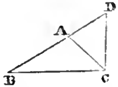
For if AC $\not>$ AB,
it is either $=$ or $<$ AB.
First—assume AC $=$ AB,
then $\angle$ ABC $=$ $\angle$ ACB, 5. 1.
but by hypoth. it is not so ;
$\therefore$ AC $\neq$ AB.
Secondly—assume AC $<$ AB ;
then $\angle$ ABC $<$ $\angle$ ACB, 18. 1.
but by hypoth. it is not so ;
$\therefore$ AC $\not<$ AB ;
& it was also demonstrated $\neq$ AB ;
$\therefore$ AC $>$ AB.

Wherefore the greater angle, &c. &c. Q. E. D.

PROP. XX.—Theorem.

Any two sides of a triangle are together greater than the third side.

In $\triangle$ ABC, any two of its sides together are $>$ third side, viz. BA + AC > BC ; AB + BC > AC ; & BC + CA > AB.



Prod. BA	to	D ;	
make AD	=	AC ;	3. 1.
join DC.			
Then $\because$ AD	=	AC,	
$\therefore \angle$ ADC	=	$\angle$ ACD ;	5. 1.
but $\angle$ BCD	>	$\angle$ ACD,	9 ax.
$\therefore$ it is also	>	$\angle$ ADC :	
& $\because$ in $\triangle$ DCB, $\angle$ BCD	>	$\angle$ BDC,	
$\therefore$ DB	>	BC ;	19. 1.
but DB	=	BA + AC,	by constr.
$\therefore$ sides BA + AC	>	BC.	
Similarly sides { AB + BC	>	AC,	
BC + CA	>	AB.	

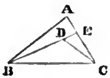
Wherefore any two sides, &c. &c. Q. E. D.

PROP. XXI.—Theorem.

If from the ends of a side of a triangle, there be drawn two right lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

From B, C, ends of side BC of $\triangle$ ABC, let BD, CD be drawn to a point D within $\triangle$ ABC : then shall

BD + DC be < BA + AC, but shall contain an ∠ BDC > ∠ BAC.



Prod. BD to E:

∵ in △ ABE, BA + AE > BE, 20. 1.

add EC to each,

∴ BA + AC > BE + EC. 4 ax.

Again, ∵ DE + EC > CD, 20. 1.

add BD to each.

∴ BE + EC > CD + DB; 4 ax.

but BA + AC > BE + EC,

much more then BA + AC > CD + DB.

Again, ∵ in △ CDE ext. ∠ BDC > int. & opp. ∠ CED, 16. 1.

& that in △ ABE ext. ∠ CEB > int. & opp. ∠ BAC.

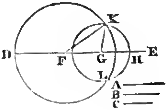
∵ much more ∠ BDC > ∠ BAC.

Wherefore if from, &c. Q. E. D.

PROP. XXII.—Problem.

To make a triangle having its sides equal to three given right lines, of which any two whatever must be greater than the third.

Let A, B, C be three given rt. lines of wh. A + B > C; A + C > B; and B + C > A: it is required to construct a △ having its sides = A, B, C, respectively.



Take DE limited at D, but unlimited towards E.

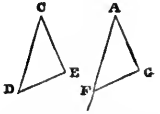
Cut off DF	=	A,	3. 1.
—— FG	=	B,	
—— GH	=	C;	
with cr. F & dist. FD	desc.	⊙ DKL,	
& with cr. G & dist. GH	desc.	⊙ HLK;	
join KF, KG;			
then sides of △ KFG	=	A, B, & C.	
For ∵ F is cr. of ⊙ DKL,			
∴ FK	=	FD;	15 def.
	=	A.	1 ax.
Again, ∵ G is cr. of ⊙ LKH,			
∴ GK	=	GH,	15 def.
	=	C,	1 ax.
and FG	=	B by constr.;	
∴ △ KFG has its sides KF, FG, GK	=	rt. lines A, B, C, respectively.	

Q. E. F.

PROP. XXIII.—Problem.

At a given point in a given right line to construct a rectilineal angle equal to a given rectilineal angle.

Let A be given point in given rt. line AF, & ECD given rectilineal ∠ ; required to make an ∠ at point A in rt. line AF = rectilineal ∠ DCE.



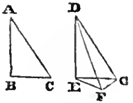
In CD, CE, take any points D, E,
& join ED ;
Constr. a △ AFG,
having its sides AF, FG, GA = CD, DE, EC respectively. 22. 1.
And ∵ DC, CE = FA, AG, ea. to ea.
and base ED = base GF ;
∴ ∠ GAF = ∠ ECD. 8. 1.

Wherefore at given point A, in given rt. line AF, has been constr. a rectilineal ∠ GAF = given rectilineal ∠ ECD. Q. E. F.

PROP. XXIV.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides, equal to them, of the other ; the base of that which has the greater angle, shall be greater than the base of the other.

Let △s ABC, DEF have sides AB, AC = DE, DF, ea. to ea., but $\angle$ BAC > $\angle$ EDF; then base BC > base EF.



Of two sides DE, DF, let DE $\ngtr$ DF;

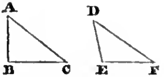
make {	$\angle$ EDG	=	$\angle$ BAC,	23. 1.
	DG	=	AC or DF;	3. 1.
	join EG, GF.			
	$\because$ AB, AC	=	DE, DG, ea. to ea.	hyp.
	& $\angle$ BAC	=	$\angle$ EDG;	constr.
	$\therefore$ base BC	=	base EG.	4. 1.
	& $\because$ DG	=	DF,	constr.
	$\therefore$ $\angle$ DFG	=	$\angle$ DGF:	5. 1.
	but $\angle$ DGF	>	$\angle$ EGF,	9 ax.
	$\therefore$ $\angle$ DFG	>	$\angle$ EGF;	
much more $\therefore$ is	$\angle$ EFG	>	$\angle$ EGF:	
	$\therefore$ EG	>	EF,	19. 1.
	but EG	=	BC:	
	$\therefore$ base BC	>	base EF.	

Wherefore if two triangles, &c. &c. Q. E. D.

PROP. XXV.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; the angle contained by the sides *of the one which has the greater base, shall be greater than the angle contained by the sides, equal to them, of the other.*

Let △s ABC, DEF, have sides AB, AC = sides DE, DF, ea. to ea., but base BC > base EF; then shall ∠ BAC be > ∠ EDF.



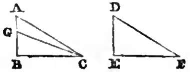
For if ∠ BAC $\ngtr$ ∠ EDF;
it must be either = or < it.
Now ∠ BAC cannot be = ∠ EDF;
for then base BC = base EF; 4. 1.
but it is not by hypoth.;
∴ ∠ BAC $\neq$ ∠ EDF.
Neither can ∠ BAC < ∠ EDF;
for then base BC < base EF; 24. 1.
but it is not by hypoth.;
∴ ∠ BAC $\nless$ ∠ EDF;
and it was also demonstrated that ∠ BAC $\neq$ ∠ EDF;
∴ ∠ BAC must be > ∠ EDF.
Wherefore if two triangles, &c. &c. Q. E. D.

PROP. XXVI.—Theorem.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side; viz. either the sides adjacent to equal angles in each, *or the sides opposite to them ; then shall the other sides be equal each to each, and also the third angle of the one be equal to the third angle of the other.*

Let $\triangle$s ABC, DEF, have $\angle$s ABC, BCA $=$ $\angle$s DEF, EFD, ea. to ea. ; also one side equal to one side.

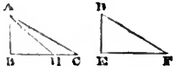
First—let sides adj. to equal $\angle$s in each be equal ; viz. BC $=$ EF : then shall AB $=$ DE, & AC $=$ DF, also $\angle$ BAC $=$ $\angle$ EDF.



	For if AB	$\neq$	DE,	
	then is one	$>$	other ;	
	let AB	$>$	DE ;	
	make BG	$=$	DE ;	3. 1.
	join GC.			
Then $\because$ {	GB, BC	$=$	DE, EF, ea. to ea.,	
	& $\angle$ GBC	$=$	$\angle$ DEF,	hyp.
$\therefore$ {	base GC	$=$	base DF,	
	$\triangle$ GBC	$=$	$\triangle$ DEF,	4. 1.
	& $\angle$ GCB	$=$	$\angle$ DFE ;	
	but $\angle$ ACB	$=$	$\angle$ DFE,	
	$\therefore$ $\angle$ GCB	$=$	$\angle$ ACB,	1 ax.
	i. e. less	$=$	greater ;	
	wh. is absurd.			
	$\therefore$ AB is not	$\neq$	DE,	
	i. e. AB	$=$	DE :	
& $\because$ {	AB, BC	$=$	DE, EF, ea. to ea.,	
	& $\angle$ ABC	$=$	$\angle$ DEF,	hyp.
$\therefore$ {	base AC	$=$	base DF,	4. 1.
	& $\angle$ BAC	$=$	$\angle$ EDF.	

PROP. XXVI. CONTINUED.

SECONDLY—Let sides *opp.* equal ∠s in each △, be equal to each other; viz. AB = DE; then shall AC = DF, BC = EF, and ∠ BAC = EDF.



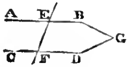
For if BC	≠	EF,	
let BC	>	EF.	
Make BH	=	EF;	3. 1.
join AH.			
& ∵ { AB, BH	=	DE, EF, ea. to ea.,	hyp.
& ∠ ABH	=	DEF; }	hyp.
∴ { base AH	=	base DF,	
∠ ABH	=	△ DEF,	4. 1.
& ∠ BHA	=	∠ EFD; }	
but ∠ EFD	=	∠ BCA,	hyp.
∴ ∠ BHA	=	∠ BCA,	1 ax.
i. e. ext. ∠ BHA of △ AHC	=	int. & opp. ∠ BCA;	
wh. is impossible.			16. 1.
∴ BC is not ≠ EF,			
i. e. BC	=	EF,	
& ∵ { AB, BC	=	DE, EF, ea. to ea.,	
& ∠ ABC	=	∠ DEF, }	hyp.
∴ { base AC	=	base DF,	4. 1.
& ∠ BAC	=	∠ EDF. }	

Wherefore if two triangles, &c. Q. E. D.

PROP. XXVII.—Theorem.

If a right line falling on two other right lines, make the alternate angles equal to each other, these two right lines shall be parallel.

Let rt. line EF, falling on AB, CD, make alt. ∠ AEF = alt. ∠ EFD, then shall AB ∥ CD.



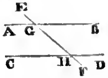
For, if AB ∦ CD,
they will on being prod. meet, either towards A & C, or
B & D; suppose the latter; 35 def.
prod. AB and CD to meet in G,
then EGF is a △,
∴ ext. ∠ AEF > int. & opp. ∠ EFG; 16. 1.
but ∠ AEF = ∠ EFG, hyp.
wh. is impossible.
∴ AB & CD do not meet towards B & D.
Similarly AB & CD do not meet towards A & C;
∴ AB ∥ CD. 35 def.

Wherefore if a right line, &c. &c. Q. E. D.

PROP. XXVIII.—Theorem.

If a right line falling upon two other right lines, make the exterior angle equal to the interior and opposite upon the same side of the line, or make the interior angles upon *the same side together equal to two right angles; the two right lines shall be parallel to each other.*

Let rt. line EF, falling on AB, CD, make ext. $\angle$ EGB $=$ int. & opp. $\angle$ GHD on same side of line; and also $\angle$s BGH $+$ GHD, on same side $=$ 2 rt. $\angle$s: then shall AB $\parallel$ CD.



For $\because \angle$ EGB $= \angle$ GHD, hyp.

& $\angle$ AGH $= \angle$ EGB, 15. 1.

$\therefore \angle$ AGH $= \angle$ GHD: 1 ax.

& these are alt. $\angle$s,

$\therefore$ AB $\parallel$ CD. 27. 1.

Again, $\because \angle$s BGH $+$ GHD $=$ 2 rt. $\angle$s, hyp.

& $\angle$s AGH $+$ BGH $=$ 2 rt. $\angle$s, 13. 1.

$\therefore \angle$s AGH $+$ BGH $= \angle$s BGH $+$ GHD;

take away com. $\angle$ BGH,

$\therefore$ rem. $\angle$ AGH $=$ rem. $\angle$ GHD, 3 ax.

and these are alt. $\angle$s;

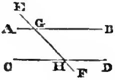
$\therefore$ AB $\parallel$ CD. 27. 1.

Wherefore if a rt. line, &c. &c. Q. E. D.

PROP. XXIX.—Theorem.

If a right line fall on two parallel right lines, it makes the alternate angles equal to each other; and the exterior angle equal to the interior and opposite angle upon the *same side; and likewise the two interior angles on the same side together equal to two right angles.*

Let rt. line EF fall on ||s AB, CD; then shall alt. ∠ AGH = alt. ∠ GHD; & ext. ∠ EGB = int. & opp. ∠ GHD, on same side of line: and also the two int. ∠s BGH + GHD, on same side of line, = 2 rt. ∠s.



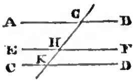
For if ∠ AGH	≠	∠ GHD,	
let ∠ AGH	>	∠ GHD:	
then ∵ ∠ AGH	>	∠ GHD,	
add to each the ∠ BGH,			
∴ ∠s AGH + BGH	>	∠s BGH + GHD;	4 ax.
but ∠s AGH + BGH	=	2 rt. ∠s,	13. 1.
∴ ∠s BGH + GHD	<	2 rt. ∠s,	
∴ AB, CD would meet if prod. far enough;			12 ax.
but they do not meet;			
for AB	||	CD;	hyp.
∴ ∠ AGH is not ≠ ∠ GHD,			
i. e. ∠ AGH	=	∠ GHD;	
but ∠ AGH	=	∠ EGB,	15. 1.
∴ ∠ EGB	=	∠ GHD;	1 ax.
add to each the ∠ BGH,			
∴ ∠s EGB + BGH	=	∠s BGH + GHD;	2 ax.
but ∠s EGB + BGH	=	2 rt. ∠s,	13. 1.
∴ ∠s BGH + GHD	=	2 rt. ∠s.	1 ax.

Wherefore if a rt. line, &c. &c. Q. E. D.

PROP. XXX.—Theorem.

Right lines which are parallel to the same right line are parallel to each other.

Let rt. lines AB, CD be each of them || EF ; then shall AB be || CD.

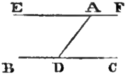


Let a rt. line GK cut AB, EF, CD, in G, H, K.
Then ∵ GK falls on ||s AB, EF,
∴ alt. ∠ AGH = alt. ∠ GHF, 29. 1.
Again, ∵ GK falls on ||s EF, CD,
∴ ext. ∠ GHF = int. & opp. ∠ GKD ; 29. 1.
but ∠ AGH = ∠ GHF,
∴ ∠ AGK = ∠ GKD ; 1 ax.
and they are alt. ∠s,
∴ AB || CD. 27. 1.
Wherefore rt. lines, &c. &c. Q. E. D.

PROP. XXXI.—Problem.

To draw a right line through a given point, parallel to a given right line.

Let A be given point, and BC given rt. line : it is required to draw through A, a rt. line = BC.



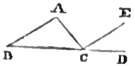
In BC take any point D,
join AD,
make ∠ DAE = ∠ ADC, 23. 1.
& prod. EA to F :
then shall EF || BC.
Because AD falls on rt. lines BC, EF,
& makes alt. ∠ DAE = alt. ∠ ADC,
∴ EF || BC.

Therefore through the given point A has been drawn a rt. line EAF || given rt. line BC. Q. E. F.

PROP. XXXII.—Theorem.

If the side of a triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

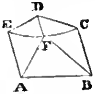
Let side BC of △ ABC be prod. to D. The ext. ∠ ACD = 2 int. & opp. ∠s CAB + ABC; and 3 int. ∠s ABC + BCA + CAB = 2 rt. ∠s.



Draw CE	$\parallel$	BA.	31. 1.
Then $\because$ AC falls on		$\parallel$s BA, CE,	29. 1.
$\therefore$ alt. $\angle$ BAC	$=$	alt. $\angle$ ACE.	29. 1.
Again, $\because$ BD falls on		$\parallel$s BA, CE,	29. 1.
$\therefore$ ext. $\angle$ ECD	$=$	int. & opp. $\angle$ ABC ;	29. 1.
but $\angle$ ACE	$=$	$\angle$ BAC,	
$\therefore$ whole ext. $\angle$ ACD	$=$	2 int. & opp. $\angle$s BAC + ABC.	2 ax.
add to each $\angle$ ACB,			
$\therefore$ $\angle$s ACD + ACB	$=$	$\angle$s BAC + ABC + ACB.	2 ax.
but $\angle$s ACD + ACB	$=$	2 rt. $\angle$s,	13. 1.
$\therefore$ also $\angle$s BAC + ABC + ACB	$=$	2 rt. $\angle$s.	

Wherefore if a side, &c. &c. Q. E. D.

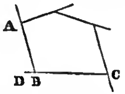
Cor. 1. All the interior angles of any rectilineal figure are, together with four right angles, equal to twice as many right angles as the figure has sides.



For, by drawing rt. lines from any point F within it to each of its angles, any rectilineal Fig. ABCDE may be divided into as many $\triangle$s as there are sides to the figure. Then

$\angle$s of fig. + 4 rt. $\angle$s $=$ $\angle$s of fig. + $\angle$s at point F,
$=$ all $\angle$s of $\triangle$s,
$=$ 2 as many rt. $\angle$s as $\triangle$s,
$=$ 2 as many rt. $\angle$s as sides to fig.

Cor. 2. All the exterior angles of any rectilineal figure are together equal to four right angles.

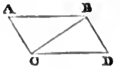


∵ Every int. ∠ABC + its ext. ∠ABD = 2 rt. ∠s, 13. 1.
∴ all int. ∠s + all ext. ∠s of fig. = 2 as many rt. ∠s as fig. has sides ;
= all int. ∠s + 4 rt. ∠s ; Cor. 1.
remove all int. ∠s wh. are com.,
∴ all ext. ∠s = 4 rt. ∠s.

PROP. XXXIII.—Theorem.

The right lines which join the extremities of two equal and parallel right lines towards the same parts, are also themselves equal and parallel.

Let AB, CD be two equal and parallel rt. lines, and let their extrem. be joined towards same parts by rt. lines AC, BD ; AC & BD are also equal and parallel.



Join BC ;
∵ BC falls on ||s AB, CD,
∴ alt. ∠ ABC = alt. ∠ BCD : 29. 1.

& ∵ { AB = CD, hyp.
BC com. to △s ABC, BCD,
& ∠ ABC = ∠ BCD, }

∴ base AC = base BD,
△ ABC = △ BCD, 4. 1.
& ∠ ACB = ∠ CBD;

& ∵ BC falls on AC, BD,
& makes alt. ∠ ACB = alt. ∠ CBD, 27. 1.
∴ AC || BD;
& AC was also proved to be = BD.

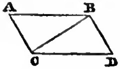
Wherefore rt. lines, &c. &c. Q. E. D.

PROP. XXXIV.—Theorem.

The opposite sides and angles of parallelograms are equal to each other, and the diameter bisects them, that is, divides them into two equal parts.

(N.B. A parallelogram is a four-sided figure, of which the opposite sides are parallel; and the diameter is the straight line joining two of its opposite angles.)

Let AD be a * ▢, and let BC be its diam. Then AB = CD, AC = BD; also ∠ ABD = ∠ DCA & ∠ CAB = ∠ BDC. Also diam. BC bisects ▢ AD.



For ∵ BC falls on ||s AB, CD,
∴ alt. ∠ ABC = alt. ∠ BCD. 29. 1.

* For the sake of brevity, the diagonal letters only of parallelograms are expressed.

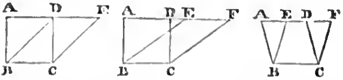
Similarly ∵ AC ∥ BD,
∴ ∠ ACB = ∠ CBD ; 29. 1.
∴ in △s ABC, BCD,
∠s ABC, BCA = ∠s BCD, CBD ea. to ea.
& adj. side BC is com.
∴ AB, AC = CD, DB ea. to ea.
& ∠ CAB = ∠ BDC. } 26. 1.
And ∵ ∠ ABC = ∠ BCD,
& ∠ CBD = ∠ ACB,
∴ whole ∠ ABD = whole ∠ DCA ; 2 ax.
also CAB was demonstr. = ∠ BDC.
Again, ∵ { AB = CD,
BC com. to the △s ABC, BCD,
& ∠ ABC = ∠ BCD, }
∴ △ ABC = △ BCD ; 4. 1.
∴ diam. BC bisects □ AD.

Wherefore opp. &c. &c. Q. E. D.

PROP. XXXV.—Theorem.

Parallelograms on the same base, and between the same parallels, are equal to each other.

Let □s ABCD, EBCF be on same base BC & between same ∥s AF, BC ; □ AC = □ EC.



If AD, DF, sides opp. to BC, be terminated in D,
then each □ AC, DC = 2 △ BDC ; 34. 1.

& ∴ ▱ AC = ▱ DC. 6 ax.

But if they be not terminated in D;

Then ∵ AC is a ▱,

∴ AD = BC; } 34. 1.

Similarly EF = BC; }

∴ AD = EF : 1 ax.

& DE is com. to both;

∴ whole or rem. AE = whole or rem. DF :

& ∵ AE, AB = DF, DC, ea. to ea.

& ext. ∠ FDC = int. ∠ EAB, 29. 1.

∴ EB = FC, } 4. 1.

& △ EAB = △ FDC; }

from trapezium ABCF take △ FDC,

and also from same take △ EAB,

& rem. = rem. 3 ax

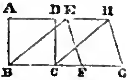
i. e. ▱ AC = ▱ EC.

Therefore ▱s, &c. &c. Q. E. D.

PROP. XXXVI.—Theorem.

Parallelograms on equal bases and between the same parallels are equal to each other.

Let ▱s AC, EG be upon equal bases BC, FG, & between same ||s AH, BG; ▱ AC = EG.



Join BE, CH.

Then ∵ BC = FG, hyp.

& FG = EH ; 34. 1.

∴ BC = EH : 1 ax.

& ∵ BC, EH, which are = and ||, are joined at their extrem. by BE, CH,

∴ BE is || to and = CH ; 33. 1.

∴ EC is a ▭ : 34 def. 1.

& ▭ EC = ▭ AC,

for they are on the same base BC, &c. 35. 1.

& similarly ▭ EC = ▭ EG ;

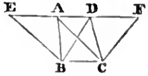
∴ ▭ AC = ▭ EG. 1 ax.

Wherefore ▭s on equal bases, &c. &c. Q. E. D.

PROP. XXXVII.—Theorem.

Triangles on the same base, and between the same parallels, are equal to each other.

Let △s ABC, DBC be on same base BC, and between same ||s AD, BC. Then △ ABC = △ DBC.



Prod. AD both ways to E & F ;

draw { BE || CA, CF || BD, } 31. 1.

∴ each fig. EC, FB is a ▭ : 34 def. 1.

& ∵ they are on the same base BC, &c.

∴ ▭ EC = ▭ FB ; 35. 1.

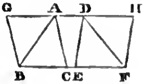
& ∵ diam. AB bisects EC,
∴ △ ABC = ½ ▭ EC ;
similarly △ DBC = ½ ▭ FB ; } 34. 1.
∴ △ ABC = △ DBC. 7 ax.

Wherefore △s, &c. &c. Q. E. D.

PROP. XXXVIII.—Theorem.

Triangles upon equal bases and between the same parallels are equal to each other.

Let △s ABC, DEF be on equal bases BC, EF, and between same ‖s AD, BF ; then △ ABC = △ DEF.



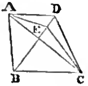
Prod. AD both ways to G & H ;
draw { BG ‖ CA ; FH ‖ ED ; } 31. 1.
then each fig. GC, HE is a ▭ ; 34 def. 1.
& ∵ they are on equal bases, BC, EF, &c.
∴ ▭ GC = ▭ HE : 36. 1.
& ∵ diam. AB bisects ▭ GC,
∴ △ ABC = ½ ▭ GC ;
similarly △ DEF = ½ ▭ HE ; } 34. 1.
∴ △ ABC = △ DEF. 7 ax.

Wherefore △s on equal bases, &c. &c. Q. E. D.

PROP. XXXIX.—Theorem.

Equal triangles on the same base and on the same side of it are between the same parallels.

Let equal $\triangle$s ABC, DBC be on same base BC, and on same side of it ; they are between same $\parallel$s.



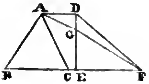
Join AD ;
then AD $\parallel$ BC :
for if not,
draw AE $\parallel$ BC ; 31. 1.
join EC.
Then $\triangle$ ABC $=$ $\triangle$ EBC ; 37. 1.
but $\triangle$ ABC $=$ $\triangle$ DBC, hyp.
$\therefore$ $\triangle$ EBC $=$ $\triangle$ DBC ; 1 ax.
i. e. less $=$ greater ;
wh. is impossible.
$\therefore$ AE $\nparallel$ BC ;
similarly none but AD $\parallel$ BC ;
$\therefore$ AD $\parallel$ BC.

Wherefore equal $\triangle$s, &c. &c. Q. E. D.

PROP. XL.—THEOREM.

Equal triangles on equal bases in the same right line and towards the same parts, are between the same parallels.

Let equal △s ABC, DEF be on equal bases BC, EF in same rt. line BF, & towards same parts; they are between same ||s.



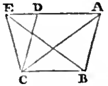
	Join AD :		
then AD	∥	BF :	
	for if not ;		
draw AG	∥	BF ;	31. 1.
	join GF.		
Then △ ABC	=	△ GEF.	38. 1.
but △ ABC	=	△ DEF,	hyp.
∴ △ GEF	=	△ DEF,	1 ax.
i. e. less	=	greater ;	
	wh. is impossible,		
∴ AG	∦	BF.	
Similarly none but AD	∥	BF ;	
∴ AD	∥	BF.	

Wherefore equal △s, &c. &c. Q. E. D.

PROP. XLI.—Theorem.

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.

Let ▢ BD & △ EBC be on same base BC & between same ||s BC, AE ; ▢ BD = 2 △ EBC.



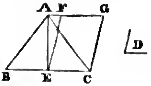
Join AC ;
then △ ABC = △ EBC ;
(for they are on the same base & between same ||s.) 37. 1.
& ∵ diam. AC bisects ▢ BD,
∴ ▢ BD = 2 △ ABC, 34. 1.
= 2 △ EBC. 1 ax.

Therefore if a ▢, &c. &c. Q. E. D.

PROP. XLII.—Problem.

To describe a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be given △, & D given rectil. ∠. It is required to descr. ▢ = △ ABC, and having an ∠ = ∠ D.



Bisect BC in E; — 10. 1.
join AE;
make ∠ CEF = ∠ D; — 23. 1.
draw { AFG ∥ BC; CG ∥ EF; } — 31. 1.
∴ FC is a ▱. — 34 def. 1.
& ∵ BE = EC; and BC is ∥ AG, — constr.
∴ △ ABE = △ ACE; — 38. 1.
& ∴ whole △ ABC = 2 △ ACE:
but ▱ FC = 2 △ ACE, — 41. 1.
∴ ▱ FC = △ ABC; — 6 ax.
& it has ∠ CEF = ∠ D, by constr.

Wherefore a ▱ FECG has been constructed = △ ABC and having an ∠ = ∠ D. Q. E. F.

PROP. XLIII.—Theorem.

The complements of the parallelograms which are about the diameter of any parallelogram, are equal to each other.

Let ABCD be a ▱, of wh. diam. is AC; & EH, GF ▱s about AC; & BK, KD, the comps. The comp. BK = comp. KD.

∵ diam. AC bisects ▱ BD,

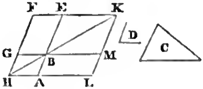
	∴ △ ABC	=	△ ACD ;	
& similarly	△ AEK	=	△ AKH ;	34. 1.
	△ KGC	=	△ KCF :	
	∴ △s AEK + KGC	=	△s AKH + KCF ;	2 ax.
	but whole △ ABC	=	whole △ ACD,	
	∴ the rem. comp. BK	=	rem. comp. KD.	3 ax.

Wherefore the comps., &c. &c. Q. E. D.

PROP. XLIV.—Problem.

To a given right line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be given rt. line, C given △, & D given rectil. ∠. It is required to apply to AB a ▱ = △ C, and having an ∠ = ∠ D.



Make ▱ FB	=	△ C,	42. 1.
having an ∠ at B ;	=	∠ D ;	

& so placed, that AB & BE may be in one rt. line ;

prod. FG to H ;

draw. AH ‖ BG or EF ; 31 1.

join HB.

Then ∵ HF falls on ‖s AH, FE,

∴ ∠s AHF + HFE = 2 rt. ∠s ; 29 1.

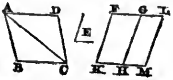
∴ ∠s BHF + HFE < 2 rt. ∠s;
& ∴ HB & FE will meet if prod. far enough; 12 ax.
let them be prod. & meet in K.
draw KL ∥ EA, or FH; 31. 1.
& prod. HA, GB to L, M;
then FL is a ▱:
& HK is its diam.;
also AG, ME are ▱s about HK;
& LB, BF are comps.,
∴ LB = BF 43. 1.
= △ C: 1 ax.
& ∵ ∠ GBE = ∠ ABM, 15. 1.
& also = ∠ D, constr.
∴ ∠ ABM = ∠ D 1 ax.

Therefore to given rt. line AB, ▱ LB is applied = given △ C, having ∠ ABM = given rectil. ∠ D. Q. E. F.

PROP. XLV.—Problem.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be given rectil. fig., & E given rectil. ∠.
Required to descr. a ▱ = fig. BD, & having an ∠ = ∠ E.



Join AC;
make ▱ FH = △ ADC, } 42. 1.
having ∠ FKH = ∠ E; }

to side GH apply □ GM = △ ABC, } 44. 1.
and having ∠ GHM = ∠ E; }
& fig. FM is □ required.
∵ ea. of ∠s, FKH, GHM = ∠ E constr.
∴ ∠ FKH = ∠ GHM; 1 ax.
add to ea. the ∠ KHG,
∴ ∠s FKH + KHG = ∠s KHG + GHM; 2 ax.
but ∠s FKH + KHG = 2 rt. ∠s, 29. 1.
∴ also ∠s KHG + GHM = 2 rt. ∠s; 1 ax.
and ∵ KH, HM are in one rt. line. 14. 1.
& ∵ GH falls on ∥s KM, FG,
∴ alt. ∠ MHG = alt. ∠ HGF; 29. 1.
add ∠ HGL to ea.
∴ ∠s MHG + HGL = ∠s HGF + HGL; 2 ax.
but ∠s MHG + HGL = 2 rt. ∠s, 29. 1.
∴ also ∠s HGF + HGL = 2 rt. ∠s; 1 ax.
& ∴ FG, GL, are in one rt. line: 14. 1.
& ∵ KF ∥ HG,
i. e. ∥ ML; 30. 1.
& also KM ∥ FL, constr.
∴ FM is a □: 34 def. 1.
& ∵ △ ADC = □ FH, } constr.
& △ ABC = □ GM, }
∴ whole fig. BD = whole □ FM. 2 ax.

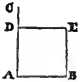
Wherefore a □ FM has been descr. = rectil. fig. BD, & having ∠ FKM = ∠ E. Q. E. F.

Cor. From this it is manifest how to apply to a given right line a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure; viz. by applying to the given right line a parallelogram equal to the first triangle ADC, and having an angle equal to the given angle.

PROP. XLVI.—Problem.

To describe a square on a given right line.

Let AB be given rt. line: required to construct a sq. on AB.



Draw AC	⊥	AB;	11. 1.
make AD	=	AB;	3. 1.
draw { DE	∥	AB;	31. 1.
BE	∥	AD;	31. 1.
∴ fig. AE	is a	▱;	34 def. 1.
& ∴ its opposite sides are equal;			34. 1.
but AB	=	AD by constr.,	
∴ the four sides	=	each other.	1 ax.
Again, ∵ AD falls on ∥s AB, DE,			
∴ ∠s BAD + ADE	=	2 rt. ∠s;	29. 1.
but ∠ BAD	is a	rt. ∠,	constr.
∴ ∠ ADE	is a	rt. ∠;	
∴ opp. ∠s to these are rt. ∠s,			
∴ ▱ AE	is	rectang.	1 ax.
& it was also proved to be equilat.,			
∴ ▱ AE	is a	sq.	30 def.

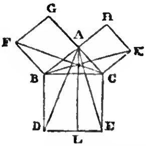
Wherefore a sq. ABED has been descr. on given rt. line AB. Q. E. F.

Cor. Hence every parallelogram which has one right angle has all its angles right angles.

PROP. XLVII.—Theorem *.

In any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the sum of the squares described upon the sides which contain the right angle.

Let rt. $\angle$d $\triangle$ ABC have rt. $\angle$ BAC. Then $BC^2 = BA^2 + AC^2$.



On BC, BA, AC, desc. sqs. BE, BG, AK ; 46. 1.
draw AL || BD, or CE ; 31. 1.
join AD, FC.
Then $\because$ $\angle$s BAC + BAG = 2 rt. $\angle$s, hyp. & 30 def.
$\therefore$ GA is in same rt. line with AC. 14. 1.
Similarly AB is in same rt. line with AH.
& $\because$ $\angle$ DBC = $\angle$ FBA, 11 ax.
add to each $\angle$ ABC,
$\therefore$ whole $\angle$ DBA = whole $\angle$ FBC : 2 ax.
& $\because$ AB, BD = FB, BC ea. to ea., 30 def.
& $\angle$ DBA = $\angle$ FBC,
$\therefore$ $\triangle$ ABD = $\triangle$ CBF. 4. 1.
Now ▭ BL = 2 $\triangle$ ABD, }
also sq. GB = 2 $\triangle$ CBF, } 41. 1.
(for they are respectively on same bases, &c.)

* This proposition has been demonstrated several ways :—vide Clavius, Schouler, Ashby, Leslie, &c. &c. ; but of all these, that given here, which is the original, is most generally admired for its simplicity and elegance.

∴ sq. GB = ▢ BL : 6 ax.

Similarly, by joining AE, BK, it may be demonstr. that sq. AK = ▢ CL ;

∴ sqs. GB + AK = whole sq. BE 2 ax.

but sqs. GB, AK, BE were descr. on AB, AC, BC, respectively,

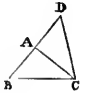
∴ $BC^2 = BA^2 + AC^2$.

Wherefore sq. of side, &c. &c. Q. E. D.

PROP. XLVIII.—Theorem.

If a square described on one of the sides of a triangle, be equal to the squares described on the other two sides of it; the angle contained by these two sides is a right angle.

In △ ABC let $BC^2 = BA^2 + AC^2$; then ∠ BAC is a rt. ∠.



Draw AD ⊥ AC ; 11. 1.

make AD = AB ; 3. 1.

join DC.

Then ∵ DA = AB

∴ $DA^2 = AB^2$;

add AC^2 to each,

∴ $DA^2 + AC^2 = AB^2 + AC^2$; 2 ax.

but $DC^2 = DA^2 + AC^2$; 47. 1.

	(for DAC	is	rt. $\angle$,)	constr.
	also BC^2	$=$	$BA^2 + AC^2$,	hyp.
	$\therefore DC^2$	$=$	BC^2 ;	1 ax.
	& $\therefore$ DC	$=$	BC ;	
& $\therefore$ {	DA	$=$	AB,	constr.
	AC	com. to	$\triangle$s DAC, BAC,	
	& base DC	$=$	base BC,	
	$\therefore \angle$ BAC	$=$	rt. $\angle$ DAC ;	8. 1.
	$\therefore \angle$ BAC	is a	rt. $\angle$.	

Therefore if a sq., &c. &c. Q. E. D.

BOOK II.

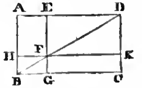
DEFINITIONS.

I.

EVERY right angled parallelogram, *or rectangle*, is said to be contained by any two of the right lines which contain one of the right angles *.

II.

In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon †. "Thus the □ HG + complements AF, FC, "is the gnomon, which is more briefly expressed by the "letters AGK, or EHC, which are at the opposite angles "of the parallelograms which make the gnomon."



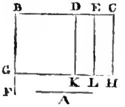
* The opposite sides of parallelograms, and consequently rectangles, being equal; it is evident that the product of any two of the adjacent sides, i. e. of those which contain a right angle, will be the area or content of the whole. And thus, for the sake of brevity, a rectangle is said to be contained as in the definition. And which is expressed by connecting the adjacent sides by sign (·) of multiplication, thus the right angled parallelogram AC is called AB · AD, which is thus read the "rectangle AB, AD."

† From a Greek word, which signifies *a carpenter's square*.

PROP. I.—THEOREM.

If there be two right lines, one of which is divided into any number of parts; the rectangle contained by the two right lines, is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A & BC be 2 rt. lines; & let BC be divided into any number of parts in D & E; then $A \cdot BC = A \cdot BD + A \cdot DE + A \cdot EC$.



Draw BF	$\perp$	BC,	11. 1.
make BG	$=$	A,	3. 1.
draw { DK, EL, CH	$\parallel$	BG, }	31. 1.
GH	$\parallel$	BC, }	

$\therefore EL = DK = BG = A.$ constr.

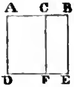
Hence $A \cdot BC = BG \cdot BC$,
$= \text{fig. } BH$,
$= \text{figs. } BK + DL + EH$,
$= BG \cdot BD + DK \cdot DE + EL \cdot EC$,
$= A \cdot BD + A \cdot DE + A \cdot EC$.

Wherefore if two rt. lines, &c. &c. Q. E. D.

PROP. II.—Theorem.

If a right line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the square of the whole line.*

Let AB be divided into any two parts in C ; then $AB^2 = AB \cdot BC + AB \cdot AC$.



On AB descr. sq. AE ; 46. 1.
draw CF $\parallel$ AD or BE. 31. 1.
Then $AB^2 =$ fig. AE,
$=$ figs. AF + CE
$= AD \cdot AC + BE \cdot BC$,
$= AB \cdot AC + AB \cdot BC$. constr.

Wherefore if a rt. line, &c. &c. Q. E. D.

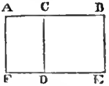
PROP. III.—Theorem.

If a right line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal

* A similar demonstration will apply, should the right line be divided into any number of parts.

to the rectangle contained by the two parts, together with the square of the aforesaid part.

Let AB be divided into any two parts in C; then $AB \cdot BC = AC \cdot CB + CB^2$.



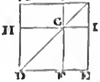
On BC desc. sq. CE;			46. 1.
prod. ED	to	F;	
draw AF	‖	CD or BE.	31. 1.
Then $AF = CD = BE = BC$.			constr.
Now fig. AE	=	figs. AD + CE;	
i. e. $AB \cdot BE$	=	$AC \cdot AF + CB \cdot BE$,	
or $AB \cdot BC$	=	$AC \cdot BC + CB^2$.	

Therefore if a rt. line be divided, &c. &c. Q. E. D.

PROP. IV.—THEOREM *.

If a right line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let AB be divided into any two parts in C; then $AB^2 = AC^2 + CB^2 + 2\,AC \cdot CB$.



* See Appendix.

On AB desc. sq. ADEB & join DB ;

Through C draw CGF $\parallel$ BE or AD ; } 31. 1.
also draw HGI $\parallel$ AB or DE. }

Then $\because$ BD meets $\parallel$s AD, CF,

$\therefore$ ext. $\angle$ CGB = int. & opp. $\angle$ ADB ; 29. 1.

but $\angle$ ADB = $\angle$ ABD ; 5. 1.

(for AD = AB ;) 30. def. 1.

$\therefore$ $\angle$ CGB = $\angle$ CBG, 1 ax.

& BC = CG ; 6. 1.

& CI is a $\square$,

$\therefore$ CB, BI, IG, GC = each other ; 34. 1.

Again, $\because$ $\angle$ ABE is a rt. $\angle$

$\therefore$ the other 3 $\angle$s of $\square$ CI are rt. $\angle$s, cor. 46. 1.

& $\therefore$ $\square$ CI is rectang.

wherefore $\square$ CI is a sq., i. e. CB^2.

Similarly HF is a sq., i. e. AC^2.

(for HG = AC.) 34. 1.

Now fig. AE = figs. HF + CI + AG + GE

= $HG^2 + CB^2 + 2$ fig. AG, 43. 1.

= $HG^2 + CB^2 + 2\, AC \cdot CG$

i. e. $AB^2 = AC^2 + CB^2 + 2\, AC \cdot CB$,

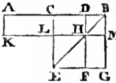
Wherefore if a rt. line, &c. &c. Q. E. D.

Cor. From the demonstration, it is manifest that the parallelograms about the diameter of a square are likewise squares.

PROP. V.—Theorem.

If a right line be divided into two equal parts, and also into two unequal parts; the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

Let AB be bisected in C, & divided into two unequal parts in D. Then shall $BC^2 = AD \cdot BD + CD^2$.



On BC desc. sq CEGB; 46. 1.

join BE;

draw { AK, DHF ∥ CE; MHLK ∥ AB. } 31. 1.

Now BC^2	= fig. CG	constr.
	= figs. CM + MF + LF,	
	= figs. AL + CH + LF	36. 1. & 43. 1.
	= figs. AH + LF,	8 ax.
	= $AD \cdot DH + LH^2$	1 def. 1.
	= $AD \cdot DB + CD^2$.	

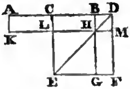
Wherefore if a rt. line, &c. &c. Q. E. D.

From this it is manifest, that the difference of the squares of two unequal lines AC, CD, is equal to the rectangle contained by their sum and difference.

PROP. VI.—Theorem.

If a right line be bisected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the right line which is made up of the half and the part produced.

Let AB be bisected in C, & prod. to D; $AD \cdot DB + BC^2 = CD^2$.



On CD desc. sq. CEFD;
join DE;

draw $\left\{ \begin{array}{l} AK, BHG \parallel CE, \\ MHLK \parallel AD. \end{array} \right\}$ 31. 1.

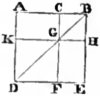
Now CD^2	=	fig. CF.	
	=	figs. CM + MG + LG,	8 ax.
	=	figs. CM + CH + LG,	43. 1.
	=	figs. CM + AL + LG,	36. 1.
	=	figs. AM + LG,	8 ax.
	=	$AD \cdot DM + LH^2$,	1 def. 1.
	=	$AD \cdot DB + CB^2$.	

Wherefore if a rt. line, &c. &c. Q. E. D.

PROP. VII.—Theorem.

If a right line be divided into any two parts, the squares of the whole line and one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let AB be divided into any two parts in C. Then $AB^2 + BC^2 = 2\ AB \cdot BC + AC^2$.



On AB desc. sq. ADEB, 46. 1.
& constr. fig. as in the preceding propositions.

Now AB^2+BC^2	= figs. AE+CH,	constr.
	= figs. AH+CE+KF,	8 ax.
	$= AB \cdot BH+BE \cdot BC+KG^2$,	1 def. 1.
	$= AB \cdot BC+AB \cdot BC+AC^2$.	constr.
	$= 2\ AB \cdot BC+AC^2$.	

Wherefore if a rt. line, &c. &c. Q. E. D.

Cor. It is manifest from this proposition, that AC^2 or $(AB^2-BC)^2$ * $= AB^2 + BC^2 - 2\ AB \cdot BC^2$.

* $(AB - BC)^2$ denotes the square described upon the right line which is the difference of AB & BC.

PROP. VIII.—Theorem.

If a right line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the square of the other part, is equal to the square of the right line, which is made up of the whole and that part.

Let AB be divided into any two parts in C; then $4\ AB \cdot BC + AC^2 = (AB + BC)^2$.*

Prod. AB	to	D ;	
make BD	=	BC ;	
on AD desc. sq. AF ;			
& constr. two figs. as in preced. propositions.			
Then ∵ fig. CK	is a	□	
∴ CB	=	GK,	
but CB	=	BD,	constr.
∴ BD	=	GK ;	
& ∵ figs. BN, GR are sqs.			cor. 4. 2.
& BD	=	GK,	
∴ sq. BN	=	the sq. GR,	
& ∴ △ BKD	=	△ PKR.	
Again, ∵ sq. BN	=	sq. GR,	
∴ BK	=	KR ;	
& ∵ AB	=	MK = XR,	
∴ □ AK	=	□ MR.	

* $(AB + BC)^2$ denotes the square described on the whole line which is made up of the two AB, BC.

Again, $\because \triangle BKD = \triangle PKR$;
add fig. AXPKB to ea.

$\therefore$ whole fig. AXPD $= \begin{cases} \square AR = 2 \square AK = \\ 1\ AB \cdot BC; \end{cases}$

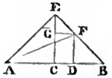
& similarly, whole fig. PHFD $= 2\ AB \cdot BC$;
$\therefore$ gnomon AOH $= 4\ AB \cdot BC$;
$\therefore$ whole fig. AF $= 4\ AB \cdot BC + XH$.
i.e. AD^2 or $(AB + BC)^2 = 4\ AB \cdot BC + AC^2$.

Wherefore if a rt. line, &c. Q. E. D.

PROP. IX.—Theorem.

If a right line be divided into two equal, and also into two unequal parts; the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of section.

Let AB be divided into two unequal parts in D, & two equal parts in C. Then $AD^2 + BD^2 = AC^2 + 2\ CD^2$.



Draw CE $\perp$ AB & $=$ AC or CB;
join EA, EB;
draw DF, FG $\parallel$ CE, AB respectively;
join AF;

Then, $\because$ AC $=$ CE, 5. 1.
& $\angle$ ACE is a rt. $\angle$, constr.

∴ ea. of ∠s EAC, AEC	=	½ rt. ∠.	
Similarly ea. of ∠s CEB, EBC	=	½ rt. ∠ ;	
∴ whole ∠ AEB	=	rt. ∠.	
& ∵ ∠ GEF	is	½ rt. ∠,	
& ∠ EGF	=	int. rt. ∠ ECB ;	29. 1.
∴ rem. ∠ EFG	=	½ rt. ∠ ;	
∴ ∠ GEF	=	∠ EFG ;	1 ax.
& ∴ GE	=	FG.	6. 1.
Again, ∵ ∠ at B	is	½ rt. ∠,	
& ∠ FDB	=	int. rt. ∠ ECB,	29. 1.
∴ rem. ∠ BFD	=	½ rt. ∠ ;	
∴ ∠ B	=	∠ BFD ;	
& ∴ DF	=	DB.	

Hence $AD^2 + DF^2 = AF^2 = AE^2 + EF^2$;

i. e. $AD^2 + DB^2 = AC^2 + CE^2 + EG^2 + GF^2$,

$= 2\ AC^2 + 2\ GF^2$,

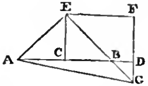
$= 2\ AC^2 + 2\ CD^2$.

Wherefore if a rt. line, &c. &c. Q. E. D.

PROP. X.—Theorem.

If a right line be bisected, and produced to any point, the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

Let AB be divided into two equal parts in C, & prod. to any point D. Then $AD^2 + DB^2 = 2\ AC^2 + 2\ CD^2$.



Draw CE ⊥ AB & = AC or CB ;
join AE, EB ;
draw EF, FD ∥ AD, EC, respectively.

Then ∵ CF is ▭ rt. ∠d at C, all its ∠s are rt. ∠s. 46. 1. cor.

& ∵ ∠s BEF + EFD < 2 rt. ∠s CEF + EFD,

∴ EB, FD prod. far enough will meet in some point G ;
join AG.

Then ∵ AC = CE & ACE = rt. ∠,

∴ CAE, CEA, ea. = ½ rt. ∠. 6 & 32 1.

Similarly CEB, CBE, ea. = ½ rt. ∠,

∴ rem. ∠ BEF, & vert. ∠ GBD ea. = ½ rt. ∠,

& BGD = alt. ∠ CEB or ½ rt. ∠,

∴ △s ACE, EFG, BDG, are isosc.

∴ AC = CE, EF = FG, BD = DG.

Now $AD^2 + DG^2 = AG^2 = AE^2 + EG^2$, 47. 1.

i. e. $AD^2 + DB^2 = AC^2 + CE^2 + EF^2 + FG^2$,

$= 2\,AC^2 + 2\,EF^2$,

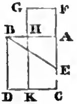
$= 2\,AC^2 + 2\,CD^2$.

Wherefore if a rt. line, &c. &c. Q. E. D.

PROP. XI.—Problem.

To divide a given right line into two such parts, that the rectangle contained by the whole, and one of the parts, shall be equal to the square of the other part.

Let AB be given rt. line ; it is required to divide AB into two such parts, that rectangle contained by whole & one part shall = sq. of other part.

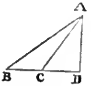


On AB desc. a sq. AD ;
bisect AC in E :
join BE,
prod. CA to F ;
& make EF = EB ; 3. 1.
on AF descr. sq. FH ;
then AB is divided in H so that $AB \cdot BH = AH^2$.
Prod. GH to K ;
then, ∵ AC is bisected in E, and prod. to F,
$\therefore CF \cdot FA + AE^2 = EF^2$ 6. 2.
$= EB^2$; 1 ax.
$= AE^2 + AB^2$; 47. 1.
take away the com. AE^2,
$\therefore CF \cdot FA = AB^2$ 3 ax.
but fig. FK is $CF \cdot FA$,
(for AF = FG & ∠ at F is rt. ∠) 30 def. 1.
also, fig. AD is AB^2, constr.
∴ fig. FK = fig. AD ; 1 ax.
take away com. part AK,
∴ rem. FH = rem. HD 3 ax.
but □ HD is $AB \cdot BH$,
(for AB = BD). 30 def. 1.
Also $FH = AH^2$. constr.
$\because AB \cdot BH = AH^2$.
Wherefore AB is divided as required. Q. E. F.

PROP. XII.—Theorem.

In obtuse angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle, is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the right line intercepted without the triangle between the perpendicular and the obtuse angle.

Let $\triangle$ ABC have obtuse $\angle$ ACB. And from one of acute $\angle$s, as A, let fall AD $\perp$ BC prod. Then $AB^2 < BC^2 + CA^2$ by $2\,BC \cdot CD$.



For $AB^2 = AD^2 + DB^2$, 47. 1.

$= AD^2 + DC^2 + BC^2 + 2\,BC \cdot CD.$ 4. 2.

$= AC^2 + BC^2 + 2\,BC \cdot CD.$

$> AC^2 + BC^2$ by $2\,BC \cdot CD.$

Wherefore in obtuse angled, &c. &c. Q. E. D.

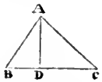
PROP. XIII.—Theorem *.

In every triangle, the square of the side subtending either of the acute angles, is less than the squares of the

* See Appendix.

sides containing that angle, by twice the rectangle contained by either of those sides, and the right line intercepted between the perpendicular let fall upon it from the opposite angle, and the acute angle.

Let △ ABC have acute ∠ ABC, and from opp. ∠ A let fall AD ⊥ BC one of sides containing ∠ B. Then $AC^2 < CB^2 + BA^2$ by 2 CB · BD.



First—Let AD fall within △ ABC.

& ∵ BC is divided into two unequal parts in D,

$\therefore BC^2 + BD^2 = 2\,BC \cdot BD + DC^2$; 7. 2.

add AD^2 to each,

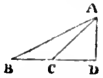
$\therefore BC^2 + (BD^2 + AD^2) = 2\,BC \cdot BD + (AD^2 + DC^2)$; 2 ax.

but $AB^2 = AD^2 + DB^2$, 47. 1.

also $AC^2 = AD^2 + DC^2$, 47. 1.

$\therefore AB^2 + BC^2 = 2\,BC \cdot BD + AC^2$; 1 ax.

i. e. AC^2 alone $<$ $CB^2 + BA^2$ by 2 BC · BD.



Secondly—Let perpendicular fall without △ ABC ;

then, ∵ ∠ D is a rt. ∠, hyp.

* ∴ ∠ ACB > rt. ∠ 16. 1.

& $\therefore AB^2 = AC^2 + CB^2 + 2\,BC \cdot CD$; 12. 2.

add BC^2 to each,

$\therefore AB^2 + BC^2 = AB^2 + (2\,CB^2 + 2\,BC \cdot CD)$; 2 ax.

* For ∠ ACB is ext. ∠ of △ ACD ; and ∴ > int. ∠ ADC.

but DB · BC $=$ BD · CD $+$ BC^2 ; 3. 2.

& ∴ 2 DB · BC $=$ (2 BC · CD $+$ 2 BC^2), 2 ax.

∴ $AB^2 + BC^2$ $=$ AC^2 $+$ (2 DB · BC),

∴ AC^2 alone $<$ $AB^2 + BC^2$ by 2 DB · BC.

LASTLY—Let side AC of △ be ⊥ BC;
then BC is the rt. line between perpendicular & acute ∠ B;

& manifestly $AB^2 + BC^2 = AC^2 + 2\ BC^2$. 47. 1. & 2 ax.

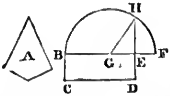
∴ AC^2 alone $<$ $AB^2 + BC^2$ by 2 BC · BC.

Wherefore in every △, &c. &c. Q. E. D.

PROP. XIV.—PROBLEM.

To describe a square that shall be equal to a given rectilineal figure.

Let A be given rectil. fig. It is required to descr. a sq. $=$ fig. A.



Descr. rt. ∠d ▢ BD $=$ fig. A. 45. 1.

Then, if BE $=$ ED

BD is a sq.; 30 def. 1.

& that which was required is done.

But if BE	$\neq$	ED ;	
prod. BE	to	F ;	
make EF	=	ED ;	3. 1.
bisect BF	in	G ;	10. 1.
with cr. G, & dist. GB or GF	desc.	$\frac{1}{2}$ ⊙ BHF ;	
prod. DE	to	H, in ○ :	
then EH^2	=	rectil. fig. A.	

Join GH ;

∵ BF is bisected in G, and divided into two unequal parts in E,

∴ $BE \cdot EF + EG^2$	=	GF^2 ;	5. 2.
	=	GH^2 ;	16 def.
	=	$HE^2 + EG^2$;	47. 1.
take away com. EG^2,			
∴ rem. $BE \cdot EF$	=	rem. EH^2 ;	3 ax.
but $BE \cdot EF$	=	□ BD,	
(for EF	=	ED),	constr.
∴ □ BD	=	EH^2 ;	
but □ BD	=	rectil. fig. A,	constr.
∴ EH^2	=	rectil. fig. A.	

Wherefore a square has been desc. = given rectil. fig. A, viz. the sq. descr. on EH. Q. E. F.

BOOK III.

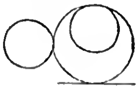
DEFINITIONS.

I.

Equal circles are those of which the diameters are equal, or from the centres of which the right lines to the circumference are equal.

II.

A right line is said to touch a circle, when it meets the circle, and being produced does not cut it.

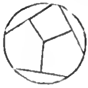


III.

Circles are said to touch each other, which meet, but do not cut each other.

IV.

Right lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.



V.

And the right line on which the greater perpendicular falls, is said to be farther from the centre.

A.

An arc is any part of the circumference of a circle.

VI.

A segment of a circle is a figure contained by a right line, and the circumference which it cuts off.

VII.

The angle of a segment is that which is contained by the right line and the circumference.

VIII.

An angle in a segment is the angle contained by two right lines drawn from any point in the circumference of the segment, to the extremities of the right line which is the base of the segment.

IX.

An angle is said to stand on the circumference intercepted between the right lines that contain the angle.

X.

A sector of a circle is the figure contained by two right lines drawn from the centre, and the circumference between them.

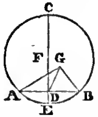
XI.

Similar segments of circles are those in which the angles are equal, or which contain equal angles.

PROP. I.—Problem.

To find the centre of a given circle.

Let ABC be given ⊙ ; it is required to find its cr.



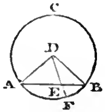
Draw within ⊙ ABC any rt. line AB ;
bisect AB in D ; 10. 1.
draw DC ⊥ AB ; 11. 1.
prod. DC to E in ⊙ ;
bisect EC in F :
then F is cr. of ⊙ ABC.
If not, if possible, let G be cr.
join GA, GD, & GB ;
Then, ∵ in △s ADG, BDG, { AD, DG = BD, DG, ea. to ea. and rad. AG = rad. BG,
∴ ∠ ADG = ∠ BDG ; 8. 1.
∴ they are both rt. ∠s ; 10 def. 1.
but also ∠ FDB is a rt. ∠, constr.
& ∴ = ∠ BDG, 1 ax.
i. e. greater = less,
wh. is impossible.
∴ G is not cr. of ⊙ ABC.
Similarly none other but F is cr. of ⊙ ABC.
Therefore F is cr. of ⊙ ABC. Q. E. F.

Cor. From this it is manifest, that if in a circle a right line bisect another at right angles, the centre of the circle is in the right line which bisects the other.

PROP. II.—Theorem.

If any two points be taken in the circumference of a circle, the right line which joins them shall fall within the circle.

Let ABC be a ⊙, and let any points A & B be taken in ○. The rt. line drawn from A to B shall fall within ⊙.



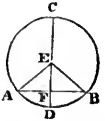
Find D cr. of ⊙.			1. 3.
Join DA, DB,			
& draw from D any rt. line meeting AB in E, & ⊙ in F.			
Then ext. ∠ DEA	>	int. & opp. ∠ DBA ;	16. 1.
but ∵ AD	=	BD,	
∴ ∠ DAB	=	∠ DBA ;	5. 1.
∴ ∠ DEA	>	∠ DAE,	
∴ DE	<	DA,	19. 1.
i. e. DE	<	DF,	
∴ point E is within ⊙ :			
& similarly every other point between A & B is within ⊙,			
& ∴ rt. line which joins A & B is within ⊙.			

Wherefore if any two points, &c. &c. Q. E. D.

PROP. III.—Theorem.

If a right line drawn through the centre of a circle, bisect a right line in it which does not pass through the centre, it shall cut it at right angles; and if it cut it at right angles, it shall bisect it.

First.—Let CD passing through cr. of ⊙ ABC bisect in F any rt. line AB, wh. does not pass through cr.; it shall cut A B at rt. ∠s.



Take E cr. of ⊙ ABC : 1. 3.

join EA, EB.

Then in △s AEF, BEF,

∵ AF, FE, EA = BF, FE, EB, respectively,

∴ ∠ AFE = ∠ BFE ; 8. 1.

& ∴ they are both rt. ∠s ; 10 def. 1.

∴ CD cuts AB at rt. ∠s.

Secondly.—Let CD cut AB at rt. ∠s ; CD shall also bisect AB.

The same constr. being made ;

∵ EA = EB, 15 def. 1.

∴ ∠ EAF = ∠ EBF. 5. 1.

And rt. ∠ AFE = rt. ∠ BFE, 11 ax.

∴ in △s EAF, EBF, 2 ∠s of one = 2 ∠s of other, ea. to ea.

also side EF is com. to △s, and opp. equal ∠s ;

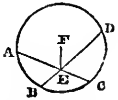
∴ AF = FB. 26. 1.

Wherefore if a rt. line, &c. &c. Q. E. D.

PROP. IV.—Theorem.

If, in a circle, two right lines, not passing through the centre, cut each other, they do not bisect each other.

Let ABCD be a ⊙ ; AEC, BED, two rt. lines in it, not passing through cr. ; they shall not bisect each other.



For, if possible, let AE = EC, & BE = ED ;

If one of lines pass through cr., it is evident that it cannot be bisected by the other wh. does not pass through cr. But if neither of them pass through cr.,

take F, cr. of ⊙ ; 1. 3.

join EF ;

and ∵ EF passing through cr. bisects AC, wh. does not pass through cr.,

∴ EF is ⊥ AC ; 3. 3.

∴ ∠ FEA is a rt. ∠.

Similarly, ∵ EF passing through cr. bisects BD, wh. does not pass through cr.

∴ FE is ⊥ BD ; 3. 3.

∴ ∠ FEB is a rt. ∠ ;

= rt. ∠ FEA ; 11 ax

i. e. greater = less,

wh. is impossible :

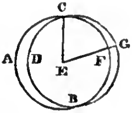
∴ AC, BD do not bisect each other.

Wherefore if in a ⊙, &c. &c. Q. E. D.

PROP. V.—Theorem.

If two circles cut each other, they shall not have the same centre.

Let ⊙s ABC, CDG, cut each other in points C & B; they shall not have same cr.



For, if possible, let E be cr. of both.
Join EC;
& draw any rt. line EFG meeting ○s in F & G;

& ∵ E	is cr. of	⊙ ABC,	
∴ EC	=	EF.	15 def. 1.
Again, ∵ E	is cr. of	⊙ CDG,	
∴ EG	=	EC;	15 def. 1.
	=	EF,	
i. e. greater	=	less,	

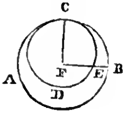
wh. is impossible.
∴ E is not cr. of ⊙s ABC, CDG.

Wherefore if two ⊙s cut each other, &c. &c. Q. E. D.

PROP. VI.—Theorem.

If two circles touch each other internally, they shall not have the same centre.

Let two ⊙s ABC, CDE, touch each other internally in point C; they shall not have same cr.



If possible, let F be centre of both.
Join FC:
& draw any rt. line FEB meeting ⊙s in E & B.

Then ∵ F	is cr. of	⊙ ABC,	
∴ FC	=	FB.	15 def. 1.
Again, ∵ F	is cr. of	⊙ CDE,	
∴ FE	=	FC ;	15 def. 1.
	=	FB ;	1 ax.
i. e. less	=	greater,	

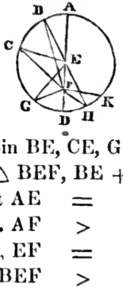
wh. is impossible.
∴ F is not cr. of ⊙s ABC, CDE.

Therefore if two ⊙s touch each other internally, &c. &c.
Q. E. D.

PROP. VII.—Theorem.

If any point be taken in the diameter of a circle which is not the centre, of all the right lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least, and, of any others, that which is nearer to the line which passes through the centre is always greater than one more remote; and from the same point there can only be drawn two right lines that are equal to each other, one upon each side of the shortest line.

In diam. AD of ⨀ ABCD, let any point F, not being cr. be taken; let E be cr. Of all rt. lines FB, FC, FG, &c., that can be drawn from F to ○, the greatest is FA in wh. cr. is; & least is FD, the other part of diam.; and of the others, that wh. is nearer to FA is always > the one more remote; viz. FB > FC, & FC > FG, &c.



Join BE, CE, GE.

Then ∵ in △ BEF, BE + EF > BF,			20. 1.
& that AE	=	BE,	15 def. 1.
∴ AE + EF, i. e. AF	>	BF.	
& ∵ BE, EF	=	CE, EF, ea. to ea.	
but ∠ BEF	>	∠ CEF,	9 ax.
∴ base BF	>	base CF,	24. 1.
Similarly CF	>	GF.	

Again, ∵ GF + FE > EG, 20. 1.
& EG = ED,
∴ GF + FE > ED;
take away com. part FE,
∴ rem. GF > rem. FD; 5 ax.
∴ AF is the greatest & FD is the least } of all rt. lines from F to ○.
& BF > FC.
& FC > FG.

Also only two equal rt. lines can be drawn from F to ○, one upon each side of shortest line.

At E in EF make ∠ FEH = ∠ FEG; 23. 1.
join FH.
Then in △s GEF, HEF, ∵ { GE, EF = HE, EF, ea. to ea.
& ∠ GEF = ∠ HEF; constr.
∴ base FG = base FH. 4. 1.

And besides FH no other rt. line = FG can be drawn from F to ○; for if there can, let it be FK:
& ∵ FK = FG,
& FG = FH,
∴ FK = FH; 1 ax.

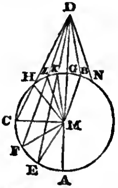
i. e. a line nearer to, = one more remote from, that passing through cr., wh. is impossible.

Therefore if any point be taken, &c. &c. Q. E. D.

PROP. VIII.—Theorem.

If any point be taken without a circle, and right lines be drawn from it to the circumference, whereof one passes through the centre; of those which fall, on the concave circumference, the greatest is that which passes through *the centre; and of the rest, that which is nearer to the one passing through the centre is always greater than one more remote; but, of those which fall on the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote: and only two equal right lines can be drawn from the same point to the circumference, one on each side of the least line.*

Let any point D be taken without ⊙ ABC, & let rt. lines DA, DE, DF, DC be drawn to ○, whereof DA passes through cr. Of those rt. lines wh. fall on concave ○, greatest shall be DA, & of rest, the one nearer to DA shall be > the one more remote; viz. DE > DF, DF > DC. But of those which fall on convex ○ HLKG, least shall be that between D & diam. AG; viz. DG : and of rest, that which is nearer to DG shall be < the one more remote; viz. DK < DL & DL < DH.



Take M cr. of ⊙ ABC;			1. 3.
join ME, MF, MC, MH, ML, MK;			
& ∵ MA	=	EM,	15 def. 1.
add MD to ea.;			
∴ AD	=	EM + MD;	2 ax.
but EM + MD	>	ED;	20. 1.
∴ AD	>	ED.	

Again, ∵ EM, MD = FM, MD, ea. to ea.
but ∠ EMD > ∠ FMD, 9 ax.
∴ base DE > base DF. 24. 1.
Similarly DF > DC,
∵ AD is the greatest ;
& DE > DF, & DF > DC
Again, ∵ MK + KD > MD, 20. 1.
& MK = MG ; 15 def. 1.
∴ rem. KD > rem. GD ; 5 ax.
i. e. GD < KD.
& ∵ MK, DK, are drawn to point within △ MLD ;
∴ MK + KD < ML + LD ; 21. 1.
but MK = ML, 15 def. 1.
∴ rem. DK < rem. DL. 5 ax.
Similarly DL < DH ;
∴ DG is the least ;
& DK < DL, & DL < DH.

Also there can be drawn only two equal rt. lines from same point D to ○, i. e. one on ea. side of least line.

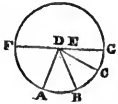
At M, in MD, make ∠ DMB = ∠ DMK ; 23. 1.
& join DB,
& ∵ KM, MD = BM, MD ea. to ea.
& that ∠ KMD = ∠ BMD, constr.
∴ base DK = base DB. 4. 1.
And besides DB, none other = DK can be drawn from D to ○;
for if there can, let it be DN ;
& ∵ DK = DN,
& DB = DK,
∴ BD = DN ;
i. e. a line nearer to the least = one more remote,
wh. is impossible.

Wherefore if any point, &c. &c. Q. E. D.

PROP. IX.—Theorem.

If a point be taken within a circle, from which there fall more than two equal right lines to the circumference, that point is the centre of the circle.

Let point D be taken within ⊙ ABC, from wh. to ○ there fall more than two equal rt. lines, viz. DA, DB, DC ; point D shall be cr. of ⊙.

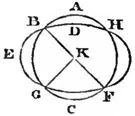


For, if not, let E be cr. of ⊙ ABC.
Join DE ;
prod. DE both ways to the ○ in F, G ;
then FG is diam.
And ∵ point D, not cr. is taken in diam. FG,
∴ DG is greatest of all rt. lines from D to ○ ;
& DC $>$ DB ; 7. 3.
& DB $>$ DA ;
but DA $=$ DB $=$ DC, hyp.
wh. is impossible ;
∴ E is not cr. of ⊙ ABC.
Similarly none other but D is cr. of ⊙ ABC ;
∴ D is cr. of ⊙ ABC.

Wherefore if a point be taken, &c. &c. Q. E. D.

PROP. X.—Theorem.

One circumference of a circle cannot cut another in more than two points.



If possible, let ○ FAB cut ⊙ DEF in more than two points, i. e. B, G, F.

Take K cr. of ⊙ ABC ; 1. 3.

join KB, KG, & KF.

And ∵ from pt. K, in ⊙ DEF, there fall to ○ more than two equal rt. lines KB, KG, KF,

∴ K is cr. of ⊙ DEF ; 9. 3.

but K is also cr. of ⊙ ABC ; constr.

∴ same point is cr. of two ⊙s wh. cut each other ; wh. is impossible. 5. 3.

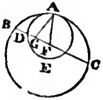
Therefore one ○, &c. &c. Q. E. D.

PROP. XI.—Theorem.

If two circles touch each other internally, the right line which joins their centres, being produced, shall pass through the point of contact.

Let ⊙s ABC, ADE touch each other internally in point A. And let F be cr. of ⊙ ABC, & G of ⊙ ADE ; the rt.

line joining F & G, being prod., shall pass through point of contact A.



If not, if possible let it fall otherwise, as BC,

Join AF, AG ;

& ∵ in △ AGF,

FG + GA > FA, 20. 1.

& FA = FB, 15 def. 1.

∴ FG + GA > FB ;

take away com. part FG,

∴ rem. GA > rem. GB ;

but GA = GD, 15 def. 1.

∴ GD > GB ;

i. e. less > greater ;

wh. is impossible.

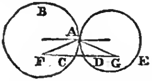
∴ The rt. line joining crs. F & G, being prod., must fall on A, the point of contact.

Wherefore if two ⊙s, &c. &c. Q. E. D.

PROP. XII.—Theorem.

If two circles touch each other externally, the right line which joins their centres shall pass through the point of contact.

Let ⊙ ABC touch ⊙ ADE externally in point A. And let F be cr. of ⊙ ABC, & G of ⊙ ADE; the rt. line joining F & G shall pass through A.



For if not, if possible let it fall otherwise, as FCDG.

Join FA, AG,

& ∵ F is cr. of ⊙ ABC,

∴ FA = FC. 15 def. 1.

also, ∵ G is cr. of ⊙ ADE,

∴ GA = GD; 15 def. 1.

∴ FA + AG = FC + DG; 2 ax.

∴ whole FG > FA + AG;

but also FG < FA + AG, 20. 1.

wh. is impossible.

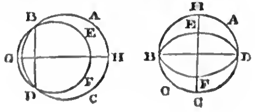
∴ rt. line joining F & G must fall on point of contact A.

Wherefore if two ⊙s, &c. &c. Q. E. D.

PROP. XIII.—Theorem.

One circle cannot touch another in more points than one, whether it touch it internally or externally.

First.—If possible, let ⊙ EBF touch ⊙ ABC *internally* in more than one point, as B & D.



Join BD ;

draw GH, bisecting BD at rt. ∠s. 10. 11. 1.

Then ∵ points B & D are in ○ of ea. ⊙,

∴ BD falls within ea. ⊙ ; 2. 3.

& ∴ GH bisects BD at rt. ∠s.

∴ cr. of ea. ⊙ is in GH ; cor. 1. 3.

∴ GH passes through point of contact ; 11. 3.

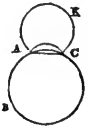
but it does not,

for points B & D are not in rt. line GH ;

wh. is absurd ;

& ∴ one ⊙ cannot touch another *internally* in more than one point.

SECONDLY.—If possible, let ⊙ ACK touch ⊙ ABC *externally* in more than one point, as A & C.



Join AC,

& ∵ points A & C are in ○ of ⊙ ACK,

∴ AC falls within ⊙ ACK, 2. 3.

but ⊙ ACK is without ⊙ ABC, hyp.

∴ AC is without ⊙ ABC ;

but ∵ points A & C are in ○ of ⊙ ABC,

∴ AC is also in ⊙ ABC ; 2. 3.

wh. is absurd.

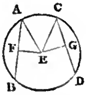
& ∴ one ⊙ cannot touch another *externally* in more points than one.

Wherefore one ⊙, &c. &c. Q. E. D.

PROP. XIV.—Theorem.

Equal right lines in a circle are equally distant from the centre; and those which are equally distant from the centre are equal to each other.

First.—In ⊙ ABDC let AB = CD; they shall be equally dist. from cr.



Take E cr. of ⊙ ABDC;			1. 3.
draw EF	⊥	AB;	12. 1.
& EG	⊥	CD;	12. 1.
join AE, EC,			

Then ∵ EF passing through cr. is ⊥ AB not passing through cr.

∴ AF	=	FB;	3. 3.
& ∴ AB	=	2 AF;	
similarly, CD	=	2 CG;	
but AB	=	CD,	hyp.
∴ AF	=	CG;	7 ax.
& ∵ AE	=	EC,	15 def. 1.
∴ AE^2	=	EC^2;	
but $AF^2 + FE^2$	=	AE^2,	47. 1.
(for ∠ AFE	is a	rt. ∠)	constr.
similarly, $EG^2 + GC^2$	=	EC^2,	
∴ $AF^2 + FE^2$	=	$EG^2 + GC^2$.	1 ax.
Now AF^2	=	CG^2,	

$\therefore$ rem. $FE^2 = $ rem. EG^2, 3 ax.

& $\therefore FE = EG$;

& FE, EG, are drawn from cr. E $\perp$ AB, CD; constr.

$\therefore$ AB & CD are equally dist. from cr. 4 def. 3.

Secondly.—Let AB, CD be equally dist. from cr., i. e. $EF = EG$; then $AB = CD$.

$\because$ as was demonstrated, $AF^2 + FE^2 = EG^2 + GC^2$;

of wh. $FE^2 = EG^2$,

$\therefore$ rem. $AF^2 = $ rem. GC^2; 3 ax.

$\therefore AF = CG$;

but $AB = 2\,AF$,

& $CD = 2\,CG$,

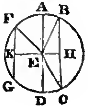
$\therefore AB = CD$. 6 ax.

Wherefore equal rt. lines, &c. &c. Q. E. D.

PROP. XV.—Theorem.

The diameter is the greatest right line in a circle; and of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.

First.—Let ABCD be a ⊙; AD diam. & E cr.; and let a rt. line BC be nearer than another FG is; then shall AD be greater than any rt. line BC, which is not a diam.; and $BC > FG$.



Draw EH, EK $\perp$ BC, FG, respectively; 12. 1.

join EB, EC, EF.

& $\because$ AE $=$ EB, } 15 def. 1.

& ED $=$ EC, }

$\therefore$ AD $=$ BE $+$ EC;

but BE $+$ EC $>$ BC, 20. 1.

$\therefore$ AD $>$ BC.

Again, $BH^2 + EH^2 = EB^2 = EF^2 = EK^2 + KF^2$;

but $EH^2 < EK^2$

(Since BC is nearer cr. than FG) 5 def. 3.

$\therefore BH^2 > KF^2$,

& BH $>$ KF,

but BC $=$ 2 BH; and FG $=$ 2 KF,

$\therefore$ BC $>$ FG.

SECONDLY.—Let BC $>$ FG; then shall BC be nearer to cr. than FG; i. e. EH $<$ EK.

$\because BH^2 + HE^2 = FK^2 + KE^2$;

but $BH^2 > FK^2$ by hypoth.;

$\therefore HE^2 < KE^2$;

$\therefore$ HE $<$ KE;

i. e. BC is nearer cr. than FG.

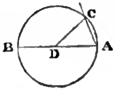
Wherefore the diameter, &c. &c. Q. E. D.

PROP. XVI.—THEOREM.

The right line which is drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no right line can be drawn from the ex-

tremity, between that right line and the circumference which does not cut the circle, or, which is the same thing, no right line can make so great an acute angle with the diameter at its extremity, or so small an angle with the right line which is at right angles to it, as not to cut the circle.

First.—Let ABC be ⊙ ; AB diam. & D its cr. The rt. line drawn from the extremity A ⊥ AB shall fall without ⊙ ABC.



For, if not, let it, if possible, fall within ⊙, as AC.

Draw DC to point C, where AC meets ○ :

Then ∵ DA	=	DC,	15 def. 1.
∴ ∠ DAC	=	∠ ACD ;	5. 1.
but ∠ DAC	is a	rt. ∠,	hyp.
∴ ∠ ACD	is a	rt. ∠ ;	

i. e. two ∠s of △ BCD = two rt. ∠s ;

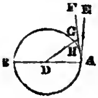
wh. is impossible. 17. 1.

∴ AC does not fall within ⊙ ;

similarly AC does not fall on ○ ;

∴ AC falls without ⊙ ABC, as AE.

Secondly.—Between AE & ○, no rt. line can be drawn from A which does not cut ⊙.



For, if possible, let FA be between them without cutting ⊙;

draw DG	⊥	FA ;	12. 1.
& let DG	meet	○ in H ;	
& ∴ ∠ AGD	is a	rt. ∠,	constr.
& ∠ DAG	<	rt. ∠,	17. 1.
∴ DA	>	DG ;	19. 1.
but DA	=	DH,	15 def. 1.
∴ DH	>	DG ;	
i. e. less	>	greater,	
wh. is impossible.			

Therefore no right line can be drawn from A between AE and the ○ which does not cut the ⊙; or, which amounts to the same thing, however great an acute angle a right line makes with the diameter at A, or however small with AE, the ○ shall pass between that right line and the perpendicular AE. "And this is all that is to be understood, "when in the Greek text, and in translations from it, the "angle of the semicircle is said to be greater than any "acute rectilineal angle, and the remaining angle less "than any rectilineal angle."

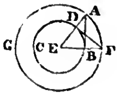
Cor. From this it is manifest that the right line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; and that it touches it only in one point, because if it did meet the circle in two, it would be within it.* "Also it is evident that there can be but one right line which touches the circle in the same point."

* 2. 3.

PROP. XVII.—Problem.

To draw a right line from a given point, either without or in the circumference, which shall touch a given circle.

First.—Let A be given point without given ⊙ BCD; it is required to draw from A, a rt. line wh. shall touch ⊙ BCD.



Find E cr. of ⊙ BCD; 1. 3.
join AE meeting ○ in D;
with cr. E, & dist. EA descr. ⊙ AFG;
draw DF ⊥ EA; 11. 1.
join EF, AB;
then AB touches ⊙ BCD.
For ∵ E is cr. of ⊙s BCD, AFG,
∴ EB = ED, } 15 def. 1.
& EF = EA, }
∴ AE, EB = FE, ED, ea. to ea.
& they contain an ∠ at E com. to △s AEB, FED,
∴ base DF = base AB, 4. 1.
& ∠ EDF = ∠ EBA;
but ∠ EDF is a rt. ∠ by constr.
∴ ∠ EBA is a rt. ∠;
but EB being prod. is a diam.
∴ AB touches ⊙ BCD; 16. 3. cor.
& it is drawn from given point A without ○.

SECONDLY.—Let given point be in ○ of ⊙ as D.

Join DE,

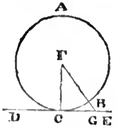
draw DF ⊥ DE,

then DF touches ⊙. cor. 16. 3.

PROP. XVIII.—THEOREM.

If a right line touch a circle, the right line drawn from the centre to the point of contact, shall be perpendicular to the line which touches the circle.

Let DE touch ⊙ ABC in C; & let FC be drawn from cr. F, to C the point of contact; then shall FC ⊥ DE.



For, if FC	be not	⊥ DE;	
draw FG	⊥	DE.	12. 1.
Then ∵ FGC	is a	rt. ∠,	
∴ GCF	$<$	rt. ∠;	17. 1.
∴ ∠ FGC	$>$	∠ GCF;	
∴ also FC	$>$	FG;	19. 1.
but FC	$=$	FB;	15 def. 1.
∴ FC	$>$	FG;	
i. e. less	$>$	greater,	

wh. is impossible,

∴ FG is not ⊥ DE;

similarly, none other but FC is ⊥ DE ;

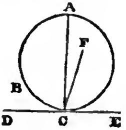
∴ FC ⊥ DE.

Therefore if a rt. line, &c. &c. Q. E. D.

PROP. XIX.—Theorem.

If a right line touch a circle, and from the point of contact a right line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let DE touch ⊙ ABC in C, and let AC be drawn from C ⊥ DE ; cr. of ⊙ shall be in AC.



For, if not, if possible, let F be cr. of ⊙ ABC.

Join CF ;

& ∵ DE touches ⊙ ABC,

& FC is drawn from cr. to point of contact C,

∴ FC ⊥ DE ; 18. 3.

& ∴ ∠ FCE is a rt. ∠ ;

but ∠ ACE is a rt. ∠, by hypoth.

∴ ∠ FCE = ∠ ACE ;

i. e. less = greater,

wh. is impossible.

∴ F is not cr. of ⊙ ABC ;

similarly, none other point without AC is cr. of ⊙ ABC ;

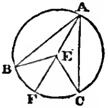
i. e. cr. is in AC.

Wherefore if a rt. line, &c. &c. Q. E. D.

PROP. XX.—Theorem.

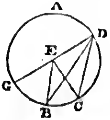
The angle at the centre of a circle is double of the angle at the circumference, upon the same base, that is, upon the same part of the circumference.

In ⊙ ABC let ∠ BEC be at cr. E, & ∠ BAC at ○, having same part BC of ○ for their base. Then shall ∠ BEC = 2 ∠ BAC.



First.—Let cr. E be within ∠ BAC,
Join AE, & prod. it to F in ○ :

& ∵ EA	=	EB,	15 def. 1.
∴ ∠ EAB	=	∠ EBA ;	5. 1.
∴ 2 ∠ EAB	=	∠s EAB + EBA,	
	=	ext. ∠ BEF, by 32. 1. ;	
similarly, 2 EAC	=	∠ FEC ;	
∴ whole ∠ BEC	=	2 whole ∠ BAC.	



Secondly.—Let cr. E be without ∠ BDC.
Join DE, & prod. it to G in ○ :

Then as in first case, 15 def. 1.

$\angle$ GEC = 2 $\angle$ GDC, 5. 1.

2 $\angle$ GEB = 2 $\angle$ GDB,

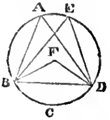
& rem. $\angle$ BEC = 2 rem. $\angle$ BDC.

Therefore the angle, &c. &c. Q. E. D.

PROP. XXI.—Theorem.

The angles in the same segment of a circle are equal to each other.

Let $\angle$s BAD, BED be in same seg. BAED of $\odot$ ABCD. Then shall $\angle$ BAD = $\angle$ BED.



Take F cr. of $\odot$ ABCD. L. 3.

First.—Let seg. be > $\frac{1}{2}$ $\odot$.

Join FB, FD :

& $\because$ $\angle$ BFD is at cr. F,

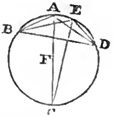
& that $\angle$ BAD is at $\bigcirc$,

& that both have same base $\overparen{BCD}$,

$\therefore$ $\angle$ BFD = 2 $\angle$ BAD, 20. 3.

& similarly, $\angle$ BFD = 2 $\angle$ BED ;

$\therefore$ $\angle$ BAD = $\angle$ BED.

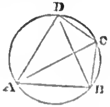


SECONDLY.—Let seg. be < ½ ⊙.
Draw AC through cr. F to ○ at C ;
join CE ;
∴ seg. BADC > ½ ⊙,
& ∠s in it are equal, by 1st case,
i. e. ∠ BAC = ∠ BEC ;
similarly, ∠ CAD = ∠ CED ;
∴ whole ∠ BAD = whole ∠ BED.
Wherefore the angles, &c. &c. Q. E. D.

PROP. XXII.—THEOREM.

The opposite angles of any quadrilateral figure described in a circle, are together equal to two right angles.

Let quadrilat. fig. ABCD be inscr. in ⊙ ABCD ; any two of its opp. ∠s together = 2 rt. ∠s.



Join AC, BD.
Then ∵ ∠s BAC, BDC are in same seg. BADC,
∴ ∠ BDC = ∠ BAC : 21. 3.

similarly, ∠ ADB = ∠ ACB ;

∴ whole ∠ ADC = ∠s BAC + ACB ;

add ∠ CBA to ea.

∴ ∠s ADC + CBA = ∠s CBA+BAC+ACB ;

= 2 rt. ∠s. 32. I.

Similarly, ∠s BAD+DCB = 2 rt. ∠s ;

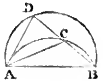
& these are the opp. ∠s of quadrilat. fig. ABCD.

Therefore the opposite angles, &c. &c. Q. E. D.

PROP. XXIII.—Theorem.

Upon the same right line, and upon the same side of it, there cannot be two similar segments of circles, which do not coincide with each other.

If it be possible, let sim. segs. ACB, ADB be on same rt. line AB, on same side of it, & not coincide with each other.



Then ∵ ⊙ ACB cuts ⊙ ADB in points A & B, it cannot cut it in any other point ; 10. 3.

∴ one seg. must fall within other.

Let seg. ACB fall within seg. ADB.

Draw a rt. line BCD, cutting ⊙s in C, D,

join CA, DA ;

& ∵ seg. ACB is sim. to seg. ADB,

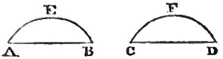
$\therefore \angle$ ACB = $\angle$ADB ; 11 def. 3.
i. e. ext. $\angle$ = int. $\angle$,
wh. is impossible. 16. 1.

Therefore there cannot be on same rt. line, &c. &c. Q. E. D.

PROP. XXIV.—Theorem.

Similar segments of circles upon equal right lines are equal to each other.

Let seg. AEB be sim. to seg. CFD, & let them be on equal rt. lines AB, CD : then shall seg. AEB = seg. CFD.



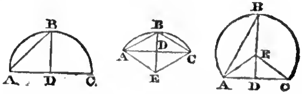
For if seg. AEB be applied to seg. CFD,
so that point A be on point C,
& rt. line AB on rt. line CD ;
then $\because$ AB = CD, hyp.
$\therefore$ also point B shall coincide with point D :
$\therefore$ AB coinciding with CD,
seg. AEB must coincide with seg. CFD ; 23. 3.
& $\therefore$ seg. AEB = seg. CFD.

Wherefore sim. segs. &c. &c. Q. E. D.

PROP. XXV.—Problem.

A segment of a circle being given, to describe the circle of which it is the segment.

Let ABC be given seg.; it is required to descr. ⊙ of wh. it is seg.



Bisect AC in D;
draw DB ⊥ AC;
join AB;

First.—Let ∠ ABD = ∠ BAD,
then BD = DA. 6. 1.
∴ DA, DB, DC = each other,
∴ D is cr. of ⊙; 9. 3.
∴ with cr. D & dist. DA, DB, or DC descr. ⊙;
& this ⊙ shall pass through extrems. of the other two rt. lines;
& ⊙, of wh. ABC is a seg., will be descr.

Secondly.—Let ∠ ABD ≠ ∠ BAD,
make ∠ BAE = ∠ ABD; 23. 1.
if necess. prod. BD to E, & join EC:
& ∵ ∠ ABE = ∠ BAE,
∴ AE = EB; 6. 1.
& ∵ AD = DC, constr.
& DE is com. to △s ADE, CDE,
& that rt. ∠ ADE = rt. ∠ CDE,
∴ base AE = base EC; 4. 1.

but AE = EB,

∴ AE, EB, EC = each other ;

∴ E is cr. of ⊙ ;

∴ With cr. E & dist. AE, EB, or EC descr. a ⊙ ;

& this ⊙ shall pass through extrems. of the other two rt. lines :

& ⊙ of wh. ABC is a seg. will be descr.

And, if ∠ ABD > ∠ BAD, Fig. 2.

it is evident that cr. E shall fall without seg. ABC ;

& ∴ seg. ABC would be < $\frac{1}{2}$ ⊙.

But if ∠ ABD < ∠ BAD, Fig. 3.

then cr. E would fall within seg. ABC ;

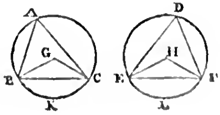
& ∴ seg. ABC would be > $\frac{1}{2}$ ⊙.

Wherefore a seg. of a ⊙ being given, &c. &c. &c. Q. E. F.

PROP. XXVI.—Theorem.

In equal circles, equal angles stand upon equal arcs, whether they be at the centres or circumferences.

Let ABC, DEF be equal ⊙s, and let equal ∠s at crs. be BGC, EHF ; & those at ○s, BAC, EDF. Then shall $\overset{\frown}{BKC} = \overset{\frown}{ELF}$.



Join BC, EF ;

& ∵ ⊙ ABC	=	⊙ DEF.	
∴ BG, GC	=	EH, HF, ea. to ea. ;	1 def. 3.
& ∠ at G	=	∠ at H,	hyp.
∴ base BC	=	base EF ;	4. 1.
& ∵ ∠ at A	=	∠ at D,	

∴ seg. BAC is sim. to seg. EDF ; 11 def. 3.

& they are upon equal rt. lines, BC, EF :

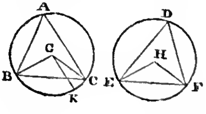
∴ seg. BAC	=	seg. EDF ;	24. 3.
but whole ⊙ ABC	=	whole ⊙ DEF ;	hyp.
∴ rem. seg. BKC	=	rem. seg. ELF ;	3 ax. 1.
& ∴ $\overset{\frown}{BKC}$	=	$\overset{\frown}{ELF}$.	

Wherefore in equal ⊙s, &c. &c. Q. E. D.

PROP. XXVII.—Theorem.

In equal circles, the angles which stand upon equal arcs are equal to each other, whether they be at the centres or circumferences.

In equal ⊙s ABC, DEF, let ∠s BGC, EHF, at crs. & ∠s BAC, EDF at ⊙s, stand upon equal arcs BC, EF. Then ∠ BGC = ∠ EHF, & ∠ BAC = ∠ EDF.



If ∠ BGC	=	∠ EHF,	
it is plain that ∠ BAC	=	∠ EDF ;	20. 3.

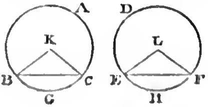
but assume $\angle BGC \neq \angle EHF$,
then one of them is $>$ other ;
let $\angle BGC > \angle EHF$;
make $\angle BGK = \angle EHF$; 23. 1.
$\therefore \overset{\frown}{BK} = \overset{\frown}{EF}$; 26. 3.
but $\overset{\frown}{EF} = \overset{\frown}{BC}$. hyp.
$\therefore \overset{\frown}{BK} = \overset{\frown}{BC}$;
i. e. less $=$ greater ;
which is impossible.
$\therefore \angle BGC$ is not $\neq \angle EHF$,
i. e. $\angle BGC = \angle EHF$;
$\therefore \angle BAC = \angle EDF$. 1 ax.

Wherefore in equal ⊙s, &c. &c. Q. E. D.

PROP. XXVIII.—Theorem.

In equal circles, equal right lines cut off equal arcs, the greater equal to the greater, and the less to the less.

Let ABC, DEF be equal ⊙s, & BC, EF equal rt. lines to them, wh. cut off the two greater arcs BAC, EDF, & the two lesser BGC, EHF. Then the greater $\overset{\frown}{BAC} =$ greater $\overset{\frown}{EDF}$, & the lesser $\overset{\frown}{BGC} =$ lesser $\overset{\frown}{EHF}$.



Take K, L crs. of $\odot$s ; 1. 3.
join BK, KC, EL, LF ;
& $\because$ $\odot$ ABC = $\odot$ EDF,
$\therefore$ BK, KC = FL, LF, ea. to ea., 1 def. 3.
& base BC = base EF, hyp.
$\therefore$ $\angle$ BKC = $\angle$ ELF. 8. 1.

Now $\angle$s at K & L are at crs. of $\odot$s,

$\therefore$ $\overset{\frown}{BGC}$ = $\overset{\frown}{EHF}$; 26. 3.
but whole ○ ABC = whole ○ DEF ; hyp.
$\therefore$ rem. $\overset{\frown}{BAC}$ = rem. $\overset{\frown}{EDF}$. 3 ax.

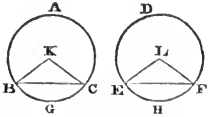
Wherefore in equal $\odot$s, equal rt. lines cut off, &c. &c.

Q. E. D.

PROP. XXIX.—Theorem.

In equal circles, equal arcs are subtended by equal right lines.

Let ABC, DEF be equal $\odot$s, and let arcs BGC, EHF be equal; join BC, EF. Then BC = EF.



Take K, L crs. of $\odot$s ;
join BK, KC, EL, LF ;
& $\because$ $\overset{\frown}{BGC}$ = $\overset{\frown}{EHF}$,

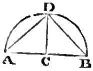
$\therefore \angle$ BKC = $\angle$ ELF; 27. 3.
& $\because$ ⊙ ABC = ⊙ DEF,
$\therefore$ BK, KC = EL, LF, ea. to ea. 1 def. 3.
& they contain equal $\angle$s ;
$\therefore$ base BC = base EF. 4. 1.

Wherefore in equal ⊙s, &c. &c. Q. E. D.

PROP. XXX.—Problem.

To bisect a given arc; that is, to divide it into two equal parts.

Let ADB be given arc ; it is required to bisect it.



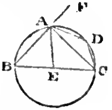
Join AB, and bisect AB in C ;
Draw CD $\perp$ AB ; 11. 1.
Then ADB is bisected in D.
Join AD, DB ;
& $\because$ AC = CB,
& CD is com. to $\triangle$s ACD, BCD,
& that $\angle$ ACD = $\angle$ BCD ; constr.
$\therefore$ base AD = base BD ; 4. 1.
and $\because$ DC passes through the centre, 1 cor. 3.
AD and DB are each < $\frac{1}{2}$ ⊙ ;
$\therefore \widehat{AD} = \widehat{DB}$; 28. 3.

Wherefore $\widehat{ADB}$ is bisected in D. Q. E. F.

PROP. XXXI.—Theorem.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let ABCD be a ⊙, of wh. diam. is BC, & cr. E; & draw CA dividing ⊙ into segs. ABC, ADC, and join BA, AD, DC; then ∠ in $\frac{1}{2}$ ⊙ BAC is a rt. ∠; and ∠ in seg. ABC, wh. is > $\frac{1}{2}$ ⊙, is < a rt. ∠; & ∠ in seg. ADC, wh. is < $\frac{1}{2}$ ⊙, is > a rt. ∠.



First.—Join AE;

prod. BA to F:

& ∵ BE = EA, 15 def. 1.

∴ ∠ BAE = ∠ ABE: 5. 1.

also ∵ AE = EC,

∴ ∠ EAC = ∠ ACE;

∴ whole ∠ BAC = ∠s ABC + ACB:

but ext. ∠ FAC = ∠s ABC + ACB, 32. 1.

∴ ∠ BAC = ∠ FAC;

& ∴ each of ∠s BAC, FAC is a rt. ∠: 10 def. 1.

∴ ∠ BAC in a $\frac{1}{2}$ ⊙ is a rt. ∠.

Secondly.—∵ in △ ABC, ∠s BAC + ABC < 2 rt. ∠s, 17. 1.

& that ∠ BAC = rt. ∠,

$\therefore \angle$ ABC $<$ rt. $\angle$;
i.e. the $\angle$ in a seg. $>$ $\frac{1}{2}$ $\odot$ is $<$ rt. $\angle$.

THIRDLY.—$\because$ ABCD is a quadrilat. fig. in a $\odot$,
any two of its opp. $\angle$s $=$ 2 rt. $\angle$s;
$\therefore \angle$s ABC + ADC $=$ 2 rt. $\angle$s; 22. 3.
but $\angle$ ABC $<$ rt. $\angle$,
$\therefore \angle$ ADC $>$ rt. $\angle$.

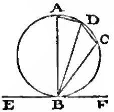
Besides, it is manifest that the arc of the greater segment ABC falls without the right $\angle$ CAB; but the arc of the less segment ADC falls within the right $\angle$ CAF. "And this is all that is meant, when in the Greek text "and the translations from it, the angle of the greater "segment is said to be greater, and the angle of the less "segment is said to be less, than a right $\angle$."

Cor. From this it is manifest, that if one angle of a triangle be equal to the other two, it is a right angle, because the angle adjacent to it is equal to the same two, and when the adjacent angles are equal, they are right angles.

PROP. XXXII.—THEOREM.

If a right line touch a circle, and from the point of contact a right line be drawn cutting the circle, the angles which this makes with the line which touches the circle, shall be equal to the angles which are in the alternate segments of the circle.

Let rt. line EF touch $\odot$ ABCD in B, and let BD be drawn cutting $\odot$; $\angle$s wh. BD makes with EF shall $=$ $\angle$s in altern. segs. of $\odot$: i.e. $\angle$ FBD $=$ $\angle$ in seg. DAB, & $\angle$ DBE $=$ $\angle$ in seg. BCD.



Draw BA $\perp$ EF ; 11. 1.

take any point C in $\overset{\frown}{BD}$;

& join AD, DC, CB.

Then $\because$ EF touches $\odot$ in B,

& BA is $\perp$ EF ;

$\therefore$ cr. of $\odot$ is in AB ; 19. 3.

& $\therefore$ $\angle$ ADB in a $\frac{1}{2}$ $\odot$ is a rt. $\angle$; 31. 3.

& consequently $\angle$s BAD + ABD = rt. $\angle$: 32. 1.

but $\angle$ ABF is a rt. $\angle$,

$\therefore$ $\angle$ ABF = $\angle$s BAD + ABD ;

take away com. $\angle$ ABD,

$\therefore$ rem. $\angle$ DBF = rem. $\angle$ BAD in altern. seg. of $\odot$.

Again, $\because$ ABCD is a quadrilat. fig. in a $\odot$,

$\therefore$ opp. $\angle$s BAD + BCD = 2 rt. $\angle$s ; 22. 3.

but $\angle$s DBF + DBE = 2 rt. $\angle$s ; 13. 1.

$\therefore$ $\angle$s DBF + DBE = $\angle$s BAD + BCD :

but $\angle$ DBF = $\angle$ BAD,

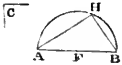
$\therefore$ rem. $\angle$ DBE = rem. $\angle$ BCD in altern. seg. of $\odot$.

Wherefore if a rt. line touch a $\odot$, &c. &c. Q. E. D.

PROP. XXXIII.—Problem.

To describe upon a given right line a segment of a circle which shall contain an angle equal to a given rectilineal angle.

Let AB be given rt. line, & $\angle$ C given rectil. $\angle$; it is required to descr. on AB a seg. of a ⊙, containing an ∠ = ∠ C.

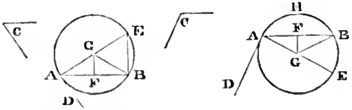


FIRST—Let ∠ C be a rt. ∠.

Bisect AB in F; 10. 1.

with cr. F, & dist. FB descr. $\frac{1}{2}$ ⊙ AHB;

∴ ∠ AHB in $\frac{1}{2}$ ⊙ = rt. ∠ C. 31. 3.



SECONDLY—Let C not be a rt. ∠.

Make ∠ BAD = ∠ C; 23. 1.

draw AE ⊥ AD; 11. 1.

bisect AB in F; 10. 1.

draw FG ⊥ AB;

join GB.

Then ∵ AF, FG = BF, FG ea. to ea.,

& rt. ∠ AFG = rt. ∠ BFG,

∴ base AG = base GB; 4. 1.

& ∴ a ⊙ descr. from G, with dist. GA, shall pass through point B;

let this ⊙ be ABE:

& ∵ from extremity A, of diam. AE, AD is drawn ⊥ AE,

∴ AD touches ⊙; 16 cor. 3.

& ∵ AB drawn from point of contact, cuts ⊙,

∴ ∠ DAB = ∠ in altern. seg. AHB; 32. 3.

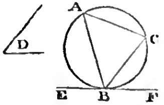
but ∠ DAB = ∠ at C,
∴ also ∠ at C = ∠ in altern. seg. AHB.

Wherefore on given rt. line AB, a seg. of a ⊙ has been descr. wh. contains an ∠ = given rectil. ∠ at C. Q. E. F.

PROP. XXXIV.—Problem.

To cut off a segment from a given circle which shall contain an angle equal to a given rectilineal angle.

Let ABC be given ⊙, and ∠ D given rectil. ∠ ; it is required to cut off from ⊙ ABC a seg. that shall contain an ∠ = given rectil. ∠ D.



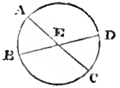
Draw EF, touching ⊙ in B ; 17. 3.
make ∠ FBC = ∠ D. 23. 1.
Then ∵ BC drawn from point of contact B, cuts ⊙,
∴ ∠ FBC = ∠ in altern. seg. BAC ; 32. 3.
but ∠ FBC = ∠ D,
∴ ∠ in altern. seg. BAC = ∠ D.

∴ a seg. BAC is cut from ⊙ ABC containing an ∠ = given rectil. ∠ D. Q. E. F.

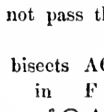
PROP. XXXV.—Theorem.

If two right lines within a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

Let AC, BD cut each other in point E within ⊙ ABCD; then shall AE · EC = BE · ED.



First—Let point E be cr. of ⊙;
then since AE, EC, BE, ED = each other; 15 def. 1.
it is plain that AE · EC = BE · ED.

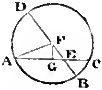


Secondly—Let one of them, BD, pass through cr. & cut other AC, wh. does not pass through cr. at rt. ∠s in point E.

∴ BD bisects AC. 3. 3.
Bisect BD in F;
∴ F is cr. of ⊙ ABCD:
join AF:
& ∵ BD bisects AC,
∴ AE = EC;

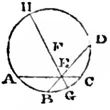
& ∵ BD is bisected in F, & divided into two unequal parts in E,

$\therefore$ BE · ED + EF^2 = FB^2; 5. 2.
= FA^2 15 def. 1.
= $AE^2 + EF^2$; 47. 1.
take away com. EF^2;
$\therefore$ rem. BE · ED = rem. AE^2 = AE · EC. 3 ax.



THIRDLY—Let BD, passing through cr., cut AC, wh. does not pass through cr. in E, but not at rt. ∠F.

Bisect BD in F, wh. is ∴ cr. of ⊙;
join AF;
draw FG ⊥ AC;
$\therefore$ AG = GC. 3. 3.
Now, FA^2 = FB^2
i. e. $AG^2 + GF^2$ = DE · EB + FE^2; } 5. 2.
$\therefore$ AE · EC + GE^2 + GF^2 = DE · EB + FE^2; }
but $GE^2 + GF^2$ = FE^2; 47. 1.
$\therefore$ rem. AE · EC = rem. DE · ED.



LASTLY—Let neither AC nor BD pass through cr. of ⊙.
Take F cr. of ⊙; 1. 3.
through E, where AC, BD cut, draw diam. GEFH:

Then by 3rd case $\begin{cases} AE \cdot EC = GE \cdot EH, \\ \text{and } BE \cdot ED = GE \cdot EH. \end{cases}$

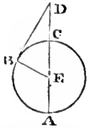
$\therefore$ rect. $AE \cdot EC = BE \cdot ED$.

Wherefore if two rt. lines within a ⊙, &c. &c. Q. E. D.*

PROP. XXXVI.—Theorem.

If from any point without a circle two right lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square of the line which touches it.

Let D be any point without ⊙ ABC; DCA, BD two rt. lines, of wh. DCA cuts ⊙, & DB touches it. Then shall $AD \cdot DC = BD^2$.



Either DCA passes through cr., or it does not.

First—Let DCA pass through cr. E.

Join EB;

$\therefore$ EBD is a rt. ∠; 18. 3.

& ∵ AC is bisected in E, & prod. to D,

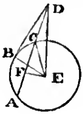
$AD \cdot DC + EC^2 = ED^2$ 6. 2.

$= EB^2 + BD^2$; 47. 1.

but $EC^2 = EB^2$;

$\therefore$ rem. $AD \cdot DC =$ rem. BD^2.

* See Appendix.



SECONDLY—Let DCA not pass through cr. of ⊙.

Take E cr. of ⊙ ; 1. 3.

join ED, EC, EB.

draw EF ⊥ AC ;

∴ AC is bisect. in F, 3. 3.

and is prod. to D,

whence $AD \cdot DC + CF^2 = FD^2$ 6. 2.

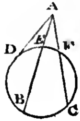
$\therefore AD \cdot DC + CF^2 + FE^2 = FD^2 + FE^2 = DE^2 = BE^2 + BD^2$;

but $CF^2 + FE^2 = CE^2 = EB^2$ 47. 1. & 15 def. 1.

∴ rem. $AD \cdot DC =$ rem. BD^2.

Wherefore, if from a point, &c. &c. Q. E. D.*

Cor. If from a point without a ⊙ two rt. lines as AB, AC be drawn cutting ⊙, then $BA \cdot AE = CA \cdot AF$.



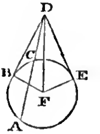
For $BA \cdot AE = AD^2 = CA \cdot AF$.

* See Appendix.

PROP. XXXVII.—Theorem.

If from a point without a circle there be drawn two right lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle and the part of it without the circle be equal to the square of the line which meets it, the line which meets shall touch the circle.

Let any point D be taken without ⊙ ABC, and from it let two rt. lines DCA, DB, be drawn; of which DCA cuts ⊙, and DB meets it; if AD · DC $=$ DB^2, then DB touches the ⊙.



Draw DE touching ⊙ ABC in E; 17. 3.
find F cr. of ⊙; 1. 3.
join FB, FD, FE;
then ∠ FED is a rt. ∠: 18. 3.
& ∵ DE touches ⊙ ABC, & DCA cuts it,
∴ DE^2 $=$ AD · DC; 36. 3.
and AD · DC $=$ DB^2 hyp.
∴ DE $=$ DB:
& ∵ also FE $=$ FB,
∴ DE, EF $=$ DB, BF, ea. to ea.
& base DF is com. to △s DFB, DFE;
∴ ∠ DEF $=$ ∠ DBF; 8. 1.

but ∠ DEF is a rt. ∠,

∴ ∠ DBF is a rt. ∠ ;

but BF prod. is diam.

∴ DB touches ⊙ ABC. 16.

Wherefore, if from a point without a ⊙, &c. &c. Q. E. D.

BOOK IV.

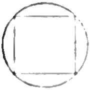
DEFINITIONS.

I.

A RECTILINEAL figure is said to be inscribed in another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.

II.

In like manner, a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

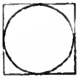


III.

A rectilineal figure is said to be inscribed in a circle, when all the angles of an inscribed figure are upon the circumference of the circle.

IV.

A rectilineal figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.

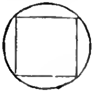


V.

In like manner, a circle is said to be inscribed in a rectilineal figure, when the circumference of the circle touches each side of the figure.

VI.

A circle is said to be described about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.



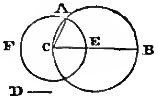
VII.

A right line is said to be placed in a circle, when the extremities of it are in the circumference of the circle.

PROP. I.—Problem.

In a given circle to place a right line, equal to a given right line not greater than the diameter of the circle.

Let ABC be given $\odot$, D given rt. line $\ngtr$ diam. of $\odot$; it is required to place in $\odot$ ABC a rt. line = D.



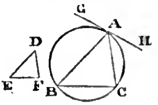
Draw diam. BC of $\odot$ ABC;
& if BC = D,
the thing required is done.
But if BC $\neq$ D,
then BC > D;
from BC, cut off CE = D; 3. 1.
& with cr. C, and dist. CE, desc. $\odot$ AEF,
cutting $\odot$ ABC in A;
join CA: then CA = D,
$\because$ C is cr. of $\odot$ AEF,
$\therefore$ AC = CE;
but CE = D, constr.
$\therefore$ AC = D. 1 ax.

$\therefore$ in given $\odot$ ABC is placed a rt. line AC = D wh. is $\ngtr$ diam. Q. E. F.

PROP. II.—Problem.

In a given circle to inscribe a triangle equiangular to a given triangle.

Let ABC be given $\odot$, DEF given $\triangle$; it is required to inscr. in $\odot$ ABC a $\triangle$ equiang. to $\triangle$ DEF.



Draw GH touching $\odot$ in any point A ; 17. 3.

At A, make $\angle$s $\left\{\begin{array}{lcl} \text{HAC} & = & \text{DEF}, \\ \text{GAB} & = & \text{DFE}; \end{array}\right\}$ 23. 1.

join BC :

ABC is $\triangle$ req.

then, $\because$ GH touches $\odot$ ABC in A,

& AC is drawn from point of contact A,

$\therefore \angle$ ABC = $\angle$ HAC, 32. 3.

but $\angle$ HAC = $\angle$ DEF,

$\therefore \angle$ ABC = $\angle$ DEF ;

similarly $\angle$ ACB = $\angle$ DFE ;

$\therefore$ rem. $\angle$ BAC = rem. $\angle$ EDF ; 32. 1.

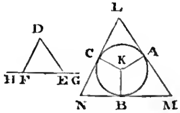
$\therefore \triangle$ ABC is equiang. to $\triangle$ DEF,

And it is inscr. in the $\odot$ ABC. Q. E. F.

PROP. III.—Problem.

About a given circle to describe a triangle equiangular to a given triangle.

Let ABC be given ⊙, DEF given △; it is required to descr. about ⊙ ABC a △ equiang. to △ DEF.



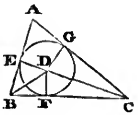
Prod. EF both ways to points G & H;			
find K cr. of ⊙ ABC;			1. 3.
draw KB to ○;			
make ∠s { AKB	=	DEG; }	23. 1.
BKC	=	DFH; }	
through A,B,C, draw LAM, MBN, NCL, touching ⊙ ABC;			
LMN is △ req.			
& ∴ all ∠s at A, B, C,	are	rt. ∠s:	18. 3.
& ∵ 4 ∠s of fig. AMBK	=	4 rt. ∠s,	32. 1. cor. 2.
& that ∠s KAM, MBK	are	2 rt. ∠s,	
∴ ∠s AMB + AKB	=	2 rt. ∠s;	
	=	∠s DEG + DEF;	13. 1.
but by constr. ∠ AKB	=	∠ DEG,	
∴ rem. ∠ LMN	=	rem. ∠ DEF:	
similarly ∠ LNM	=	∠ DFE:	
∴ rem. ∠ MLN	=	rem. ∠ EDF;	32. 1.
& ∴ △ MLN is equiang. to △ DEF.			

And it is descr. about the ⊙ ABC. Q. E. F.

PROP. IV.—Problem.

To inscribe a circle in a given triangle.

Let ABC be given △ ; it is required to inscr. a ⊙ in it.



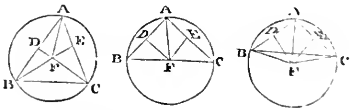
Bisect ∠s ABC, BCA by BD, CD, meeting in D : 9. 1.
draw DE, DF, DG, ⊥ AB, BC, CA ; 12. 1.
& ∵ ∠ EBD = ∠ DBF. constr.
& that rt. ∠ BED = rt. ∠ BFD,
∴ ∠s DBE, BED = ∠s DBF, BFD ea. to ea.
& ∵ side BD is com. to △s DBE, DBF, & opp. to equal ∠s ;
∴ side DE = side DF ; 26. 1.
similarly DG = DF,
∴ DE, DF, DG = each other,
∴ with cr. D & dist. DE, DF, or DG descr. ⊙ EFG ;
& ∵ ∠s at E, F, G, are rt. ∠s,
∴ ⊙ EFG shall touch sides AB, BC, CA ; 16 3.
∴ each of sides AB, BC, CA touches ⊙ EFG ;
& ∴ ⊙ EFG is inscr. in given △ ABC.

Q. E. F.

PROP. V.—Problem.

To describe a circle about a given triangle.

Let ABC be given △; it is required to descr. a ⊙ about △ ABC.



Bisect AB, AC, in D & E ;
draw DF, EF, ⊥ AB, AC ;
then shall DF, EF, meet in F ;
for if they do not meet, they must be || each other,
& ∴ also AB || AC ;
wh. is absurd ;
∴ DF, EF do meet in F ;
join AF,
& if F is not in BC, join BF, FC ;

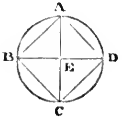
& ∵ { AD = DB,
DF com. to △s ADF, BDF,
& rt. ∠ ADF = rt. ∠ BDF, }
∴ base BF = base AF ; 4. 1.
similarly CF = AF ;
∴ AF, BF, CF = each other.

Therefore a ⊙ descr. with cr. F and dist. any one of them will pass through extrems. of other two, and be descr. about △ ABC. Q. E. F.

PROP. VI.—Problem.

To inscribe a square in a given circle.

Let ABCD be given $\odot$; it is required to inscr. a sq. in it.



Draw two diams. AC, BD, $\perp$ each other ;
join AB, BC, CD, DA ;

& $\because$ { BE $=$ ED,
AE com. to $\triangle$s ABE, AED,
& rt. $\angle$ BEA $=$ rt. $\angle$ AED, }

$\therefore$ base AB $=$ base AD ; 4. 1.
similarly BC, CD, $=$ BA, or AD :
$\therefore$ AB, BC, CD, DA, $=$ each other ;
& $\therefore$ fig. ABCD is equilat.

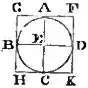
Again, $\because$ $\overset{\frown}{BAD}$ is $\frac{1}{2}$ $\odot$,
$\therefore$ $\angle$ BAD within it is a rt. $\angle$; 31. 3.
similarly $\angle$ ADC, $\angle$DCB, or $\angle$ CBA is a rt. $\angle$;
$\therefore$ fig. ABCD is also equiang. ;
but it was proved to be equilat. ;
& $\therefore$ ABCD is a sq.

Therefore, in given $\odot$ ABCD has been inscr. a sq. Q. E. F.

PROP. VII.—Problem.

To describe a square about a given circle.

Let ABCD be given ⊙. It is required to descr. a sq. about it.



Draw diam. AC, BD, ⊥ each other,
& through points A, B, C, D, draw FG, GH, HK, KF, touching ⊙ : 17. 3.
& ∵ FG touches ⊙ ABCD,
& EA is drawn from cr. E to point of contact A,
∵ ∠s at A are rt. ∠s ;
similarly ∠s at B, C, D, are rt. ∠s ; 18. 3.
& ∵ ∠s AEB, EBG are rt. ∠s,
∴ GH ‖ AC ; 28. 1.
similarly AC ‖ FK ;
& GF, HK ‖ BD ;
∴ figs. GK, GC, AK, FB, BK, are ▱s ;
Hence GH = FK = AC = BD = GF = HK ;
∴ quadrilat. fig. GK is equilat.
Again, ∵ fig. GE is a ▱,
& ∠ BEA is a rt. ∠,
∴ ∠ AGB is a rt. ∠ ;
similarly ∠s GHK, HKF, KFG are rt. ∠s ; 31. 1.
& ∴ fig. GK is equiang. ;
& it has been proved to be equilat.,
∴ GK is a sq. ;
& it is descr. about ⊙ ABCD.

Q. E. F.

PROP. VIII.—Problem.

To inscribe a circle in a given square.

Let ABCD be given sq., it is required to inscr. a ⊙ in it.

Bisect AB, AD in F, E ;

draw { EH ∥ AB or DC ; FK ∥ AD or BC ;

∴ figs. AK, KB, AH, HD, AG, GC, BG, GD, are ▭s ;

& ∴ their opp. sides = each other :

Hence FG = AE = $\frac{1}{2}$ AD = $\frac{1}{2}$ AB = AF = EG :

similarly each of sides GH, GK = FG or GE ;

∴ GE, GF, GH, GK = each other :

& ∴ a ⊙, descr. from cr. G, with dist. any one of them shall pass through extrems. of other three ;

& ∵ ∠s at E, F, H, K, are rt. ∠s, 29. 1.

∴ AB, BC, CD, DA, are ⊥ diams. EH, FK ;

∴ AB, BC, CD, DA, touch ⊙ EFHK ; 16. 3.

& ∴ ⊙ EFHK is inscr. in given sq. ABCD.

Q. E. F.

PROP. IX.—Problem.

To describe a circle about a given square.

Let ABCD be given sq. It is required to descr. a ⊙ about it.

Join AC, BD, cutting each other in E.

Then ∵ { AD = AB,
AC com. to △s ABC, ADC,
& base BC = base DC, }

∴ ∠ DAC = ∠ BAC; 8. 1.

& ∴ ∠ DAB is bisected by AC ;

similarly, ∠s ABC, BCD & CDA are bisected by BD & AC;

& ∵ { ∠ DAB = ∠ ABC,
∠ EAB = $\frac{1}{2}$ ∠ DAB,
∠ EBA = $\frac{1}{2}$ ∠ ABC, }

∴ ∠ EAB = ∠ EBA ;

∴ EA = EB : 6. 1.

similarly, ea. of EC, ED = EA or EB :

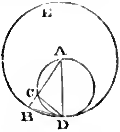
∴ EA, EB, EC, ED = ea. other :

& ∴ a ⊙ descr. from cr. E & dist. any one of them shall pass through extrems. of other three, & be descr. about given sq. ABCD.

Q. E. F.

PROP. X.—Problem.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.



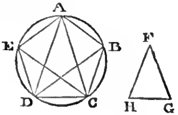
Take any rt. line AB ;
divide AB in C so that AB · BC = AC² ; 11. 2.
with cr. A, & dist. AB, descr. ⊙ BDE ;
in ⊙ BDE place a rt. line BD = AC ⊁ diam. of ⊙ ;
join DA, DC ; and about △ ACD descr. ⊙ ACD :
then △ ABD is such as was required.
For ∵ BD is drawn to meet, and BCA to cut ⊙ ACD,
And BD = AC, & ∴ BD² = AC² = AB · BC by constr.
∴ BD touches ⊙ ACD in D ; 37. 3.
And DC is drawn from D, cutting the ⊙ ACD,
∴ ∠ CAD in altern. seg. = ∠ BDC ; 32. 3.
∴ ∠s CAD + ADC = ∠s BDC + ADC ;
i. e. ext. ∠ DCB = ∠ ADB, 32. 1.
= ∠ ABD, (for AB = AD) 15 def. 1.
∴ DC = DB 6. 1.
= AC by constr.
whence ∠ DAC = ∠ CDA, 5. 1.
∵ 2 ∠ DAC = ∠s DAC + CDA,
= ext. ∠ DCB, 32. 1.
= ∠ DBA, or ∠ ADB, ∠s at base.

Wherefore an isosc. △ is descr. having ea. of ∠s at base double of third ∠. Q. E. F.

PROP. XI.—Problem.

To inscribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be given ⊙ ; it is required to inscr. in it an equilat. & equiang. pent.



Descr. an isosc. △ FGH
having ea. of its ∠s FGH, GHF, at base = 2 ∠ GFH at vertex ; 10. 4.

inscr. in ⊙ ABCDE, a △ ACD equiang. to △ FGH,
so that ∠ CAD = ∠ at F,
& ea. of ∠s ACD, CDA = ∠ at G or ∠ at H ; } 2. 4.

& ∴ ea. of ∠s ACD, CDA = 2 ∠ CAD :
bisect ∠s ACD, CDA by CE, DB : 9. 1.
join AB, BC, DC, DE, EA :
then fig. ABCDE is the required pent.

∵ ea. of ∠s ACD, CDA = 2 ∠ CAD,
& that they are bisected by CE, DB,
∴ 5 ∠s DAC, ACE, ECD, CDB, & BDA = ea. other :
& ∵ equal ∠s stand on equal arcs, 26. 3.
∴ $\frown{AB}, \frown{BC}, \frown{CD}, \frown{DE}, \frown{EA}$ = ea. other ;
& ∴ rt. lines AB, BC, CD, DE, EA = ea. other ; 29. 3.
∴ pent. ABCDE is equilat.

Again, ∵ $\frown{AB}$ = $\frown{DE}$,

add $\overset{\frown}{BCD}$ to ea.,

∴ whole ABD = whole $\overset{\frown}{EDB}$;

& ∵ ∠ AED stands on ABD,

& that ∠ BAE stands on $\overset{\frown}{EDB}$,

∴ ∠ BAE = ∠ AED : 27. 3.

similarly ea. of ∠s ABC, BCD, CDE = ∠ BAE or AED :

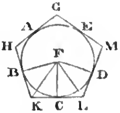
∴ pent. ABCDE is also equiang.

Wherefore in given ⊙ ABCDE, has been inscr. an equilat. & equiang. pent. Q. E. F.

PROP. XII.—Problem.

To describe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be given ⊙ ; it is required to descr. about it an equilat. & equiang. pent.



Let ∠s of a pent. inscr. in ⊙ be in points A, B, C, D, E,

so that $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, $\overset{\frown}{DE}$, $\overset{\frown}{EA}$ = ea. other ; 11. 4.

thro. A,B,C,D,E, draw GH, HK, KL, LM, MG touching ⊙ ;

take F. cr. of ⊙ ;

join FB, FK, FC, FL, FD :

& ∵ HK, KL, LM, touch ⊙,

∴ ∠s at B, C, D are rt. ∠s. 18. 3.

Hence $FB^2 + BK^2 = FK^2 = FC^2 + CK^2$; 47. 1.

but $FB^2 = FC^2$; 15 def. 1.

∴ $BK^2 = CK^2$,

and ∴ BK = CK.

Hence △ FKC is equilat. to △ FKB;

∴ they are also equiang. to ea. other. 8. 1.

Hence ∠ KFC = $\frac{1}{2}$ ∠ BFC;

& similarly ∠ CFL = $\frac{1}{2}$ ∠ CFD.

But ∵ $\overset{\frown}{BC} = \overset{\frown}{CD}$;

∴ ∠ BFC = ∠ CFD; 27. 3.

& ∴ ∠ KFC = ∠ CFL;

also rt. ∠ FCK = rt. ∠ FCL,

& side CF is com. to △s;

∴ CK = CL, 26. 1.

& ∠ FKC = ∠ FLC;

∴ 2 ∠ FKC = 2 ∠ FLC;

i. e. ∠ BKC = ∠ CLD.

Similarly ∠s LMG, MGH, GHK, ea. = HKL or KLM.

∴ pent. GHKLM is equiang.

Again, since CK = CL, & similarly BK = BH;

∴ KL = 2 KC = 2 KB = HK:

& similarly LM, MG, GH, ea. = HK or KL;

∴ pent. is also equilat.

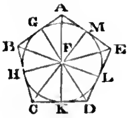
& it is descr. about ⊙ ABCDE.

Wherefore, &c. Q. E. F.

PROP. XIII.—Problem.

To inscribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be given equilat. & equiang. pent.; it is required to inscr. a ⊙ in it.



Bisect ∠s BCD, CDE by CF, DF ; 9. 1.
join FB, FA, FE.

Then ∵ { BC = CD, CF com. to △s BCF, DCF, & ∠ BCF = ∠ DCF,
∴ base BF = base FD, & ∠ CBF = ∠ CDF : } 4. 1.

Hence ∠ CBF = ∠ CDF = $\frac{1}{2}$ ∠ CDE = $\frac{1}{2}$ ∠ CBA,
∴ BF bisects ∠ ABC ;
& similarly ∠s BAE, AED, are bisected by FA, FE.
Draw FG, FH, FK, FL, FM ⊥ AB, BC, CD, DE, EA.

Then ∵ { ∠ HCF = ∠ KCF, rt. ∠ FHC = rt. ∠ FKC, & FC com. to △s ;
∴ FH = FK, 26. 1.
similarly ea. of FL, FM, FG = FH or FK ;
∴ five rt. lines = ea. other :
∴ a ⊙ descr. from F, with dist. any one of them, shall pass through extrems. of other four.

And ∵ ∠s at points G, H, K, L, M are rt. ∠s,
∴ AB, BC, CD, DE, EA touch ⊙ so descr. 16. 3.

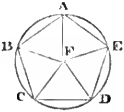
Therefore a ⊙ has been inscr. in given pent. ABCDE.

Q. E. F.

PROP. XIV.—Problem.

To describe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be equilat. & equiang. pent.; it is required to descr. a ⊙ about it.



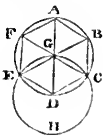
Bisect ∠s BCD, CDE by CF, DF meeting in F;
join FB, FA, FE.
Then it may be shown as in preced. prop.
that FA, FB, FE bisect ∠s CBA, BAE, AED:
Then ∠ FCD = ½ ∠ BCD = ½ ∠ CDE = ∠ FDC;
∴ FC = FD. 6. 1.
Similarly FB, FA, or FE = FC, or FD:
∴ five rt. lines = ea. other.

Therefore a ⊙ descr. from cr. F, with dist. any one of them, shall pass through points A, B, C, D, E, and be descr. about given pent. ABCDE. Q. E. F.

PROP. XV.—Problem.

To inscribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be given ⊙ ; it is required to inscr. an equilat. and equiang. hex. in it.



Find G cr. of ⊙ ;
draw diam. AGD ;
with cr. D, & dist. DG, descr. ⊙ EGCH ;
join EG, GC, and prod. them to B & F ;
join AB, BC, CD, DE, EF, FA.
Then hex. ABCDEF is equilat. & equiang.
For ∵ G is cr. of ⊙ ABCDEF,
∴ GE = GD ;
& ∵ D is cr. of ⊙ EGCH,
∴ DE = DG ;
∴ △ EGD is equilat.,
& ∴ it is equiang. 5 cor. 1.
& ∵ 3 ∠s of a △ = 2 rt. ∠s, 32. 1.
∴ ∠ EGD = $\frac{1}{3}$ of 2 rt. ∠s :
similarly ∠ DGC = $\frac{1}{3}$ of 2 rt. ∠s :
& ∵ rt. line GC stands on rt. line EB,
& ∴ makes adj. ∠s EGC + CGB = 2 rt. ∠s ; 13. 1.

∴ rem. ∠ CGB = $\frac{1}{3}$ of 2 rt. ∠s;

∴ ∠s EGD, DGC, CGB = ea. other;

but also vert. ∠s BGA, AGF, FGE = ∠s EGD, DGC, CGB, 15. 1.

∴ six ∠s = ea. other;

& ∴ $\frown{AB}$, $\frown{BC}$, $\frown{CD}$, $\frown{DE}$, $\frown{EF}$, $\frown{FA}$ = ea. other; 26. 3.

∴ AB, BC, CD, DE, EF, FA = ea. other: 29. 3.

∴ hex. is equilat.

Again, ∵ $\frown{AF}$ = $\frown{ED}$,

add $\frown{ACD}$ to each,

∴ whole $\frown{FBD}$ = the whole $\frown{ECA}$;

& ∴ ∠s standing upon these equal arcs are equal,

i. e. ∠ AFE = ∠ FED; 27. 3.

similarly ea. of other four ∠s = ∠ AFE, or ∠ FED:

& ∴ six ∠s = ea. other:

∴ hex. ABCDEF is also equiang.

Therefore an equilat. & equiang. hex. has been inscr. in given ⊙. Q. E. F.

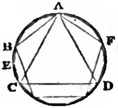
Cor. From this it is manifest, that the side of the hexagon is equal to the right line from the centre, that is to the semidiameter of the circle.

And if through the points A, B, C, D, E, F, there be drawn right lines touching the circle, an equilateral and equiangular hexagon shall be described about it, which may be demonstrated by what has been said of the pentagon; and likewise a circle may be inscribed in a given equilateral and equiangular hexagon, and circumscribed about it, by a method like that used for the pentagon.

PROP. XVI.—Problem.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Let ABCD be given ⊙ ; it is required to inscr. an equilat. & equiang. quindec. in it.



Let AC be the side of an equil. ▽ inscr. in ⊙, 2. 4.
And AB the side of an equil. and equiang. pentagon inscr. in the same, 11. 4.

then $\overset{\frown}{ABC}$ = $\frac{1}{3}$ of whole ○ :
& $\overset{\frown}{AB}$ = $\frac{1}{5}$ of whole ○ :
& ∴ if whole ○ contain 15 equal parts,
then $\overset{\frown}{ABC}$ contains 5 such parts ;
& $\overset{\frown}{AB}$ contains 3 such parts ;
& ∴ their difference $\overset{\frown}{BC}$ contains 2 such parts :
now bisect $\overset{\frown}{BC}$ in E, 30. 3.
& ∴ $\overset{\frown}{BE}$, or $\overset{\frown}{EC}$ will contain 1 such part.

And consequently if rt. lines BE, or EC be drawn, & their equals extended round whole of ⊙ ; an equilat. & equiang. quindec. shall be inscr. in it. Q. E. F.

And, in the same manner as was done in the pentagon, if, through the point of division made by inscribing the quindecagon, right lines be drawn touching the circle, an equilateral and equiangular quindecagon shall be described about it; and likewise, as in the pentagon, a circle may be inscribed in a given equilateral and equiangular quindecagon, and circumscribed about it.

BOOK V.

DEFINITIONS.

I.

A LESS magnitude is said to be a part of a greater magnitude when the less measures the greater; that is, 'when the less is contained a certain number of times 'exactly in the greater.'

II.

A greater magnitude is said to be a multiple of a less, when the greater is measured by the less, that is, 'when 'the greater contains the less a certain number of times 'exactly.'

III.

"Ratio is a mutual relation of two magnitudes of the "same kind to one another, in respect of quantity."

IV.

Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

V.

The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth ; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth : or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth : or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

VI.

Magnitudes which have the same ratio are called proportionals. 'N.B. When four magnitudes are proportionals, 'it is usually expressed by saying, the first is to the second, 'as the third to the fourth.'

VII.

When of the equimultiples of four magnitudes, (taken as in the fifth definition,) the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth ; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth : and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

VIII.

"Analogy or proportion, is the similitude of ratios."

IX.

Proportion consists in three terms at least.

X.

When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

XI.

When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on, quadruplicate, &c., increasing the denomination still by unity, in any number of proportionals.

Definition A, *to wit, of compound ratio.*

Where there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D ; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the same ratio which E has to F ; and B to C the same ratio that G has to H ; and C to D the same that K has to L ; then, by this definition, A is said to have to D the ratio compounded of ratios which are the same with the ratios of E to F, G to H, and K to L. And the same thing is to be understood when it is more briefly expressed by saying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D; then, for shortness' sake, M is said to have to N the ratio compounded of the ratios of E to F, G to H, and K to L.

XII.

In proportionals, the antecedent terms are called homologous to one another, as also the consequents to one another.

'Geometers make use of the following technical words 'to signify certain ways of changing either the order or 'magnitude of proportionals, so that they continue still to 'be proportionals.'

XIII.

Permutando, or alternando, by permutation or alternately. This word is used when there are four proportionals, and it is inferred that the first has the same ratio to the third which the second has to the fourth; or that the first is to the third as the second is to the fourth: as is shown in the 16th Prop. of this Fifth Book.

XIV.

Invertendo, by inversion; when there are four proportionals, and it is inferred that the second is to the first as the fourth to the third. Prop. B. Book 5.

XV.

Componendo, by composition; when there are four proportionals, and it is inferred that the first together with the second, is to the second, as the third together with the fourth, is to the fourth. 18th Prop. Book 5.

XVI.

Dividendo, by division ; when there are four proportionals, and it is inferred that the excess of the first above the second, is to the second, as the excess of the third above the fourth, is to the fourth. 17th Prop. Book 5.

XVII.

Convertendo, by conversion ; when there are four proportionals, and it is inferred that the first is to its excess above the second, as the third to its excess above the fourth. Prop. E. Book 5

XVIII.

Ex æquali, (sc. distantiâ,) or ex æquo, from equality of distance : when there is any number of magnitudes more than two, and as many others, such that they are proportionals when taken two and two of each rank, and it is inferred that the first is to the last of the first rank of magnitudes, as the first is to the last of the others : 'Of this 'there are the two following kinds, which arise from the 'different order in which the magnitudes are taken, two 'and two.'

XIX.

Ex æquali, from equality. This term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank ; and as the second is to the third of the first rank, so is the second to the third of the other ; and so on in order : and the inference is as mentioned in the preceding definition ; whence this is called ordinate proportion. It is demonstrated in the 22nd Prop. Book 5.

XX.

Ex æquali in proportione perturbatâ seu inordinatâ, from equality in perturbate or disorderly proportion.* This term is used when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank ; and as the second is to the third of the first rank, so is the last but two, to the last but one of the second rank ; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank ; and so on in a cross order : and the inference is in the 18th definition. It is demonstrated in 23 Prop. Book 5.

AXIOMS.

I.

Equimultiples of the same, or of equal magnitudes, are equal to one another.

II.

Those magnitudes, of which the same or equal magnitudes are equimultiples, are equal to one another.

III.

A multiple of a greater magnitude is greater than the same multiple of a less.

IV.

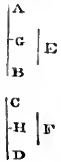
That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

* Prop. lib. 2, Archimedis de sphærâ et cylindro.

PROP. I.—Theorem.

If any number of magnitudes be equimultiples of as many, each of each; what multiple soever any one of them is of its part, the same multiple shall all the first magnitudes be of all the other.

Let any number of mags. AB, CD be equimults. of as many others, E, F, ea. of ea.; then shall AB + CD be same mult. of E + F, that AB is of E.



∵ AB is same mult. of E, that CD is of F,
∴ (No. mags. in AB wh. = E) = (No. mags. in CD wh. = F.)
Divide AB into mags. AG, GB ea. = E;
& CD into mags. CH, HD ea. = F;

then No. mags. CH, HD	=	No. mags. AG, GB;	
& ∵ AG	=	E,	
& CH	=	F,	
∴ AG + CH	=	E + F;	2 ax. 1.
similarly GB + HD	=	E + F:	
∴ (No. mags. in AB wh. = E)	=	(No. mags. in AB + CD wh. = E + F);	

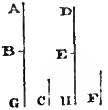
∴ whatever mult. AB is of E, same is AB + CD of E + F.
∴ if any number of mags., &c. &c.

"For same demonstr. holds in any number of mags. wh. is here applied to two." Q. E. D.

PROP. II.—Theorem.

If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.

Let AB, 1st, be same mult. of C, 2d, that DE, 3d, is of F, 4th; also BG, 5th, same mult. of C, 2d, that EH, 6th, is of F, 4th. Then AG, (1st + 5th,) is same mult. of C that DH, (3d + 6th,) is of F.



∵ AB is same mult. of C that DE is of F,

∴ (No. mags. in AB wh. = C) = (No. mags. in DE wh. = F):

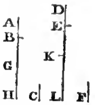
similarly, (No. mags. in BG wh. = C) = (No. mags. in EH wh. = F:

∴ (No. mags. in whole AG wh. = C) = (No. mags. in whole DH wh. = F):

∴ AG is same mult. of C that DH is of F;

i. e. AG, 1st + 5th, is same mult. of C, 2d, that DH, 3d + 6th, is of F, 4th.

If ∴ first be same mult., &c. &c. Q. E. D.

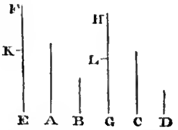


Cor. "From this it is plain, that if any number of mag-"nitudes AB, BG, GH, be multiples of another C; and "as many DE, EK, KL, be the same multiples of F, each "of each; the whole of the first, viz. AH is the same "multiple of C that the whole of the last, viz. DL, is "of F."

PROP. III.—Theorem.

If the first be the same multiple of the second, which the third is of the fourth; and if of the first and third there be taken equimultiples, these shall be equimultiples, the one of the second, and the other of the fourth.

Let A, 1st, be same mult. of B, 2d, that C, 3d, is of D, 4th; & of A, C, let equimults. EF, GH be taken: then EF is same mult. of B that GH is of D.



∵ EF is same mult. of A, that GH is of C,

∴ (No. mags. in EF wh. = A) = (No. mags. in GH wh. = C)

Divide EF into mags. EK, KF, ea. = A;
& GH into mags. GL, LH, ea. = C:
∴ No. mags. EK, KF = No. mags. GL, LH.
& ∵ A is same mult. of B, that C is of D,
& that EK = A,
& GL = C,
∴ EK is same mult. of B, that GL is of D:
similarly, KF is same mult. of B, that LH is of D.

And so on, if there are more parts in EF, GH wh. = A, C.
Now ∵ EK, 1st, is same mult. of B, 2d, that GL, 3d, is of D, 4th, and that KF, 5th, is same mult. of B, 2d, that LH, 6th, is of D, 4th,
∴ EF, 1st + 5th, is same mult. of B, 2d, that GH, 3d + 6th, is of D, 4th. 2. 5.

If ∴ first be same mult., &c. &c. Q. E. D.

PROP. IV.—Theorem.

If the first of four magnitudes have the same ratio to the second which the third has to the fourth; then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth, viz. 'the equimultiple of the first shall have the 'same ratio to that of the second, which the equimultiple 'of the third has to that of the fourth.'

Let A, 1st, : B, 2d, : : C, 3d, : D, 4th; & of A, C let there be taken any equimults. E, F; & of B, D any equimults. G, H; then E : G : : F : H.

Of E, F take any equimults. K, L ;
& of G, H take any equimults. M, N :
then ∵ E is same mult. of A, that F is of C,
& that K is same mult. of A, that L is of F,
∴ K is same mult. of A, that L is of C. 3. 5.
Similarly, M is same mult. of B, that N is of D.
And, ∵ A : B : : C : D, hyp.
& that K is same mult. of A, that L is of C,
& that M is same mult. of B, that N is of D,
if K > M,
then L > N.
if equal, equal ; if less, less. 5 def. 5.
But K is same mult. of E, that L is of F, constr.
also M is same mult. of G, that N is of H,
∴ E : G : : F : H, 5 def. 5.

Therefore, &c. &c. Q. E. D.

Cor. Likewise, if the first have the same ratio to the second, which the third has to the fourth, then also any equimultiples of the first and third have the same ratio to the second and fourth ; and in like manner, the first and the third have the same ratio to any equimultiples whatever of the second and fourth.

Let A, 1st, : B, 2d, : : C, 3d, : D, 4th ; & of A & C let E & F be any equimults. whatever ; then E : B : : F : D.

Of E & F take any equimults. K, L,
& of B & D take any equimults. G, H :
then it may be demonstrated as before,
that K is same mult. of A, that L is of C :
& ∵ A : B : : C : D, hyp.
& that of A, C, are taken equimults. K & L,
& of B, D, are taken equimults. G & H,
if K > G,
then L > H,
if equal, equal ; if less, less. 5 def. 5.
Now K, L, are any equimults. of E, F, constr.
& G, H, are any equimults. of B, D.
∴ E : B : : F : D. 5 def. 5.

And in the same way the other case may be demonstrated.

PROP. V.—Theorem.

If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other ; the remainder shall be the same multiple of the remainder, that the whole is of the whole.

Let AB be same mult. of CD, that AE a part taken from 1st is of CF a part taken from 2d ; then rem. EB is same mult. of rem. FD, that whole AB is of whole CD.

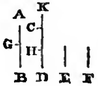
Take AG same mult. of FD, that AE is of CF,
∴ AE is same mult. of CF, that EG is of CD ; 1. 5.
but, AE is same mult. of CF, that AB is of CD, hyp.
∴ EG is same mult. of CD, that AB is of CD ;
∴ EG = AB ; 1 ax. 5.
take away com. mag. AE,
& rem. AG = rem. EB ;
& since AE is the same mult. of CF, that AG is of FD,
& that AG = EB,
∴ AE is same mult. of CF, that EB is of FD :
but AE is same mult. of CF, that AB is of CD, hyp.
∴ EB is same mult. of FD, that AB is of CD.

Therefore, if any mags., &c. &c. Q. E. D.

PROP. VI.—THEOREM.

If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two ; the remainders are either equal to these others, or equimultiples of them.

Let two mags. AB, CD be equimults. of two others E, F ; & AG, CH, taken from first two, be equimults. of same E, F. Then rems. GB, HD, are either = E, F, or equimults. of them.



FIRST.—Let GB = E. Then HD = F.

Make CK = F.

& ∵ AG is same mult. of E, that CH is of F,

& that GB = E.

& CK = F,

∴ AB is same mult. of E, that KH is of F ;

but AB is same mult. of E, that CD is of F,

∴ KH is same mult. of F, that CD is of F :

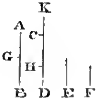
∴ KH = CD ; 1 ax. 5.

take away com. mag. CH,

& rem. KC = rem. HD :

but KC = F, constr.

∴ HD = F.



SECONDLY.—Let GB be a mult. of E : then HD is same mult. of F, that GB is of E.

Make CK same mult. of F, that GB is of E ;

& ∵ AG is same mult. of E, that CH is of F, hyp.

& GB is same mult. of E, that CK is of F,

∴ AB is same mult. of E, that KH is of F ; 2. 5.

but AB is same mult. of E, that CD is of F, hyp.

∴ KH is same mult. of F, that CD is of F ;

∴ KH = CD ; 1 ax. 5.

take from both CH,

& rem. KC = rem. HD ;

∴ HD is same mult. of F, that GB is of E.

Therefore, if two mags., &c. &c. Q. E. D.

PROP. A.—Theorem.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less.

Take any equimults. of ea. of them, such as the doubles of ea.

Then, if 2 first	$>$	2 second,	
$\therefore$ 2 third	$>$	2 fourth:	5 def. 5.
but, if first	$>$	second,	
then, 2 first	$>$	2 second;	
$\therefore$ also 2 third	$>$	2 fourth.	
& $\therefore$ third	$>$	fourth.	

Similarly, if first $=$ or $<$ second,
then third $=$ or $<$ fourth.

Therefore, if the first, &c. &c. Q. E. D.

PROP. B.—Theorem.

If four magnitudes are proportionals, they are proportionals also when taken inversely.

If A : B : : C : D then also inversely B : A : : D : C.

Of B & D take any equimults. E & F ;
& of A & C any equimults. G & H.

Let E $>$ G,
then G $<$ E.
& ∵ A : B : : C : D, hyp.
& that G is same mult. of A, 1st, that H is of C, 3d,
& that E is same mult. of B, 2d, that F is of D, 4th,
& that G $<$ E,
∴ H $<$ F ; 5 def. 5.
i. e. F $>$ H :
if, then E $>$ G,
∴ F $>$ H.
Similarly, if E $=$ G,
then F $=$ H,
and if less, less.

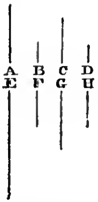
Now E is same mult. of B, that F is of D, } constr.
& G is same mult. of A, that H is of C, }
∴ B : A : : D : C. 5 def. 5.

Therefore, if four mags., &c. &c. Q. E. D.

PROP. C.—Theorem.

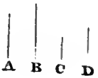
If the first be the same multiple of the second, or the same part of it, that the third is of the fourth ; the first is to the second, as the third is to the fourth.

First.—Let A, 1st, be same mult. of B, 2d, that C, 3d, is of D, 4th ; then A : B : : C : D.



Of A & C, take any equimults. E & G ;
& of B & D, take any equimults. F & H.
Then ∵ A is same mult. of B, that C is of D, hyp.
& that E is same mult. of A, that G is of C, constr.
∴ E is same mult. of B, that G is of D ; 3. 5.
∴ E & G are same mults. of B & D :
but F & H are equimults. of B & D : constr.
∴ if E be a mult. of B > F is of B,
then G is a mult. of D > H is of D ;
i. e. if E > F,
then G > H.
Similarly, if E = F,
then G = H,
and if less, less.
But, E & G are any equimults. of A & C, }
& F & H are any equimults. of B & D, } constr.
∴ A : C : : C : D, 5 def. 5.

SECONDLY.—Let A, 1st, be same part of B, 2d, that C, 3d, is of D, 4th ; then also A : B : : C : D.



For, B is same mult. of A, that D is of C,
∴ by preced. case, B : A : : D : C,
& inversely A : B : : C : D. B. 5.

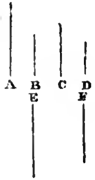
Therefore, if the first, &c. &c. Q. E. D.

PROP. D.—Theorem.

If the first be to the second as the third to the fourth, and if the first be a multiple, or a part of the second; the third is the same multiple, or the same part of the fourth.

Let A : B : : C : D.

First.—Let A be a mult. of B; then C is same mult. of D.



Take E = A;
& make F same mult. of D, that A or E is of B.
Then ∵ A : B : : C : D, hyp.
& that E, F are any equimults. of B, 2d, and D, 4th,
∴ A : E : : C : F; cor. 4. 5.
but A = E, constr.
∴ C = F; A. 5.

& F is same mult. of D, that A is of B, constr.

∴ C is same mult. of D, that A is of B.

SECONDLY.—Let A be a part of B ; then C is same part of D.*

For, ∵ A : B : : C : D,

then, inversely, B : A : : D : C.

But A is a part of B,

∴ B is a mult. of A :

& by preced. case, D is same mult. of C, that B is of A,

i. e. C is same part of D, that A is of B.

Therefore, if first, &c. &c. Q. E. D.

PROP. VII.—THEOREM.

Equal magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.

Let A & B be equal mags., and C any other ; then A : C : : B : C; also C : A : : C : B.

FIRST.—Of A, B take any equimults. D, E,

& of C take any mult. F.

* See fig. at foot of p. 161.

Then, ∵ D is same mult. of A, that E is of B, constr.
& that A $=$ B, hyp.
∴ D $=$ E ; 1 ax. 5.
& if D $>$ F,
then E $>$ F,
if equal, equal ; and if less, less.
Now D, E, are any equimults. of A & B, constr.
& F is any mult. of C,
∴ A : C : : B : C. 5 def. 5.

SECONDLY.—Also C : A : : C : B,
For with same constr. it may be demonstr.
that D $=$ E,
& ∴ if F $>$ D,
then F $>$ E,
if equal, equal ; if less, less.
Now F is any mult. of C,
& D, E any equimults. of A, B,
∴ C : A : : C : B. 5 def. 5.

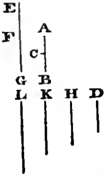
Therefore, equal mags., &c. &c. Q. E. D.

PROP. VIII.—THEOREM.

Of unequal magnitudes the greater has the greater ratio to the same than the less has: and the same magnitude has a greater ratio to the less than it has to the greater.

Let AB, BC be unequal mags. of which AB is the greater : and let D be any mag. whatever ; then AB : D $>$ BC : D, also D : BC $>$ D : AB.

Fig. 1.



FIRST.—If that mag. wh. is $\not>$ other, of AC, CB, be $>$ D,
take EF, & FG $=$ 2 AC, & 2 CB : (fig. 1st ;)
but, if that wh. is $\not>$ other, of AC, CB be $<$ D,
(as in figs. 2d & 3d,)
then this mag. AC or CB can be multiplied
so as to become $>$ D ;
let it be multiplied until it become $>$ D ;
& let other be multiplied as often.
And let EF be mult. thus taken, of AC ;
& FG same mult. of CB :
$\therefore$ EF or FG $>$ D.
Now in every one of the cases
take H $=$ 2 D,
K $=$ 3 D,
& so on until mult. of D be first wh. becomes $>$ FG :
let L be that mult. of D wh. is first $>$ FG ;
& K the mult. of D wh. is next $<$ L.

Fig. 2.

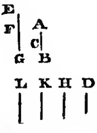
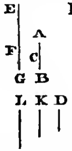


Fig. 3.



Then, $\because$ L is that mult. of D wh. first becomes $>$ FG,

$\therefore$ K, the next preced. mult. of D, is $\not>$ FG ;

i. e. FG $\not<$ K.

And since EF is same mult. of AC, that FG is of CB,

$\therefore$ FG is same mult. of BC that EG is of AB ; 1. 5.

$\therefore$ EG & FG are equimults. of AB & BC.

Now FG $\not<$ K, demon.

& EF $>$ D, constr.

$\therefore$ whole EG $>$ K + D ;

but K + D $=$ L,

$\therefore$ FG $>$ L ;

but FG $\not>$ L,

& EG, FG are equimults. of AB & BC,

& L is a mult. of D, constr.

$\therefore$ AB : D $>$ BC : AB. 7 def. 5.

SECONDLY.—D : BC $>$ D : AB.

For with same constr. it may be demonstr.

that L $>$ FG ;

but that L $\not>$ EG ;

now L is a mult. of D, constr.

& FG, EG are equimults. of CD, AB,

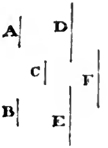
$\therefore$ D : BC $>$ D : AB. 7 def. 5.

Therefore, if unequal mags., &c. &c. Q. E. D.

PROP. IX.—THEOREM.

Magnitudes which have the same ratio to the same magnitudes are equal to one another ; and those to which the same magnitude has the same ratio are equal to one another.

FIRST.—Let A : C : : B : C; then A = B.



For, if A ≠ B,
one is > other;
let A > B.
Then there are some equimults. of A & B, 8. 5.
& some mult. of C,
such, that mult. of A > mult. of C;
but mult. of B ≯ mult. of C.
Let such mults. be taken:
& let D, E be equimults. of A, B;
& F a mult. of C;
so that D > F,
& E ≯ F,
But, ∵ A : C : : B : C, hyp.
& that D, E are equimults. of A, B,
& F is a mult. of C,
& that D > F;
then also E > F; 5 def. 5.
but E ≯ F, constr.
wh. is impossible.
∴ A is not ≠ B,
i. e. A = B.

SECONDLY.—Let C : B : : C : A; then also A = B.
For if A ≠ B,
then one > other:

let A > B.

Then of C, there is some mult. F,

& of A, B, there are some equimults. D, E, 8. 5.

such that F > E,

but ⯒ D.

But ∵ C : B : : C : A, hyp.

& that F, a mult. of first > E, a mult. of second,

∴ F, a mult. of third, > D, a mult. of fourth ; 5 def. 5.

but F ⯒ D : constr.

wh. is impossible.

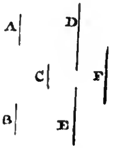
∴ A = B.

Wherefore mags. wh., &c. &c. Q. E. D.

PROP. X.—Theorem.

That magnitude which has a greater ratio than another has unto the same magnitude, is the greater of the two ; and that magnitude to which the same has a greater ratio than it has unto another magnitude, is the less of the two.

First.—Let A : C > B : C ; then A > B.



For, ∵ A : C > B : C,

∴ of A, B, there are some equimults. D, E,

& of C some mult. F, 7 def. 5.

such that $D > F$,

but $E \not> F$;

& $\therefore D > E$:

& $\because$ D, E are equimults. of A, B,

& that $D > E$,

$\therefore A > B$. 4 ax. 5.

SECONDLY.—Let $C : B > C : A$; then $B < A$.

For of C there is some mult. F,

& of B, A, some equimults. E, D, 7 def. 5.

such that $F > E$, but $\not> D$,

$\therefore B < D$:

& $\because$ E, D, are equimults. of B, A,

$\therefore B < A$. 4 ax. 5.

Therefore that mag., &c. &c. Q. E. D.

PROP. XI.—THEOREM.

Ratios that are the same to the same ratio are the same to each other.

Let $A : B :: C : D$, & also $C : D :: E : F$; then shall $A : B :: E : F$.

Of A, C, E take any equimults. G, H, K,

& of B, D, F take any equimults. L, M, N.

Then, ∵ A : B : : C : D,
& that G, H are any equimults. of A, C,
& L, M are any equimults. of B, D,
if G > L,
then H > M,
if equal, equal ; if less, less. 5 def. 5.
Again, ∵ C : D : : E : F,
& that H, K are any equimults. of C, E,
& M, N are any equimults. of D, F,
if H > M,
then K > N,
& if equal, equal ; if less, less. 5 def. 5.
But it has been shown
that if G > L,
then H > M,
if equal, equal ; if less, less.
∴ if G > L,
K > N,
if equal, equal ; if less, less.
Now G, K are any equimults. of A, E,
& L, N are any equimults. of B, F,
∴ A : B : : E : F. 5 def. 5.

Therefore ratios, &c. &c. Q. E. D.

PROP. XII.—Theorem.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

Let any number of mags. A, B, C, D, E, F, be proportionals; i. e. A : B : : C : D : : E : F; then A : B : : A + C + E : C + D + F.

Of A, C, E take any equimults. G, H, K,
& of B, D, F take any equimults. L, M, N.
Then, ∵ A : B : : C : D : : E : F,
& that G, H, K are equimults. of A, C, E,
& L, M, N are equimults. of B, D, F,
if G $>$ L,
then H $>$ M,
& K $>$ N,
if equal, equal; if less, less. 5 def. 5.
∴ if G $>$ L,
then G + H + K $>$ D + M + N,
& if equal, equal; if less, less.
Now G & G + H + K are any equimults. of A & A + C + E, [1. 5.
also L & L + M + N are any equimults. of B & B + D + F,
∴ A : B : : A + C + E : B + D + F. 5 def. 5.

Wherefore if any number, &c. &c. Q. E. D.

PROP. XIII.—Theorem.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater *ratio than the fifth has to the sixth ; the first shall also have to the second a greater ratio than the fifth has to the sixth.*

Let A, 1st, : B, 2d, : : C, 3d, : D, 4th ; but C, 3d, : D, 4th, > E, 5th, : F, 6th ; then shall A : B > E : F,

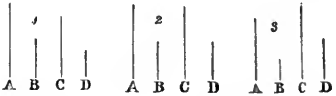
$\because$ C : D > E : F,
there are some equimults. of C & E, as G & H,
& some equimults. of D & F, as K & L,
such that G > K,
but H $\ngtr$ L. 7 def. 5.
Take M, same mult. of A that G is of C ;
& N, same mult. of B that K is of D.
Then, $\because$ A : B : : C : D, constr.
& that M, G are equimults. of A, C,
& N, K are equimults. of B, D,
if M > N,
then G > K,
& if equal, equal ; if less, less. 5 def. 5.
but G > K, constr.
$\therefore$ M > N ;
but H $\ngtr$ L. constr.
Now, M, H are equimults. of A, E,
& N, L are equimults. of B, F,
$\therefore$ A : B > E : F. 7 def. 5.

Wherefore if the first, &c. &c. Q. E. D.

PROP. XIV.—Theorem.*

If the first has the same ratio to the second which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

Let A, 1st, : B, 2d, : : C, 3d, : D, 4th,



First.—Let $A > C$; then $B > D$.

$\because A > C$,

& B is another mag.;

$\therefore A : B > C : B$: 8. 5.

but $A : B :: C : D$, hyp.

$\therefore C : D > C : B$; 13. 5.

$\therefore D < B$; 10. 5.

i. e. $B > D$.

Secondly.—Let $A = C$; then $B = D$.

For $A : B :: C$, i. e. $A : D$.

$\therefore B = D$. 9. 5.

Thirdly.—Let $A < C$; then $B < D$.

For, $C > A$;

& $\because C : D :: A : B$,

$\therefore D > B$, by first case;

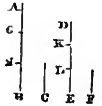
i. e. $B < D$.

Therefore, if first, &c. &c. Q. E. D.

PROP. XV.—Theorem.

Magnitudes have the same ratio to each other which their equimultiples have.

Let AB be same mult. of C, that DE is of F; then C : F : : AB : DE.



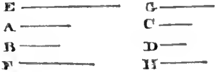
∵ AB is same mult. of C that DE is of F,
∴ (No. mags. in AB wh. = C) = (No. mags. in DE wh. = F).
Divide AB into mags. AG, GH, HB, ea. = C;
& DE into mags. DK, KL, LE, ea. = F;
∴ No. mags. AG, GH, HB = No. of mags. DK, KL, LE.
And ∵ AG, GH, HB = ea. other,
& that DK, KL, LE = ea. other,
∴ AG : DK : : GH : KL : : HB : LE; 7. 5.
& ∴ AG : DK : : AB : DE. 12. 5.
But AG = C,
& DK = F,
∴ C : F : : AB : DE.

Therefore mags., &c. &c. Q. E. D.

PROP. XVI.—Theorem.

If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.

Let A, B, C, D be four proportionals; viz. A : B : : C : D, they are proportionals when taken alternately, i. e. A : C : : B : D.



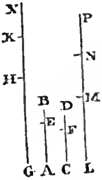
Of A, B, take any equimults. E, F,
& of C, D, take any equimults. G, H :
& ∵ E is same mult. of A, that F is of B,
∴ A : B : : E : F; 15. 5.
but A : B : : C : D, hyp.
∴ C : D : : E : F. 11. 5.
Again, ∵ G is same mult. of C, that H is of D,
∴ C : D : : G : H; 15. 5.
but C : D : : E : F,
∴ E : F : : G : H; 11. 5.
∴ if E > G
then F > H,
if equal, equal; if less, less. 14. 5.
Now E, F are any equimults. of A, B, }
& G, H are any equimults. of C, D, } constr.
∴ A : C : : B : D. 5 def. 5.

If ∴ four mags., &c. &c. Q. E. D.

PROP. XVII.—Theorem.

If magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately: that is, if two magnitudes together have to one of them the same ratio which two others have to one of these, the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.

Let AB, BE, CD, DF, be mags. taken jointly, wh. are proportionals, i.e. AB : BE : : CD : DF ; they shall also be proportionals when taken separately, viz. AE : EB : : CF : FD.



Of AE, EB, CF, FD take any equimults. GH, HK, LM, MN ;

& again of EB, FD take any equimults. KX, NP.

And ∵ GH is same mult. of AE, that HK is of EB,

∴ GH is same mult. of AE, that GK is of AB ; [1. 5.

but GH is same mult. of AE, that LM is of CF,

∴ GK is same mult. of AB, that LM is of CF.

Again, ∵ LM is same mult. of CF, that MN is of FD,

∴ LM is same mult. of CF, that LN is of CD ; [1. 5.

but LM is same mult. of CF, that GK is of AB, [demon.

∴ GK is same mult. of AB, that LN is of CD ;
i. e. GK, LN are equimults. of AB, CD.

Next, ∵ HK is same mult. of EB, that MN is of FD,
& that KX is same mult. of EB, that NP is of FD,
∴ HX is same mult. of EB, that MP is of FD ; [2. 5.

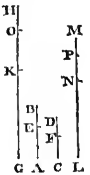
& ∵ AB : BE : : CD : DF, hyp.
& that GK, LN are equimults. of AB, CD,
& HX, MP are equimults. of EB, FD,
if GK > HX,
then LN > MP,
if equal, equal ; if less, less. 5 def. 5.
But, if GH > KX ;
add to both HK,
then GK > HX ;
∴ also LN > MP ;
take from both MN,
then LM > NP ;
∴ if GH > KX,
LM > NP,
if equal, equal ; if less, less. 5 def. 5.
Now GH, LM are any equimults. of AE, CF,
& KX, NP are any equimults. of EB, FD,
∴ AE : EB : : CF : FD.

Therefore, if mags., &c. &c. Q. E. D.

PROP. XVIII.—Theorem.

If magnitudes, taken separately, be proportionals, they shall also be proportionals when taken jointly: that is, if the first be to the second, as the third to the fourth, the first and second together shall be to the second, as the third and fourth together to the fourth.

Let AE, EB, CF, FD be proportionals; that is, AE : EB : : CF : FD; they shall also be proportionals when taken jointly, viz. AB : BE : : CD : DF.



Of AB, BE, CD, DF take any equimults. GH, HK, LM, MN;

& again of BE, DF take any equimults. KO, NP,

& ∵ KO, NP are equimults. of BE, DF,

& that KH, NM are also equimults. of BE, DF;

if KO, a mult. of BE, > KH, a mult. of BE,

then NP, a mult. of DF, > NM, a mult. of DF;

& if KO = KH,

then NP = NM;

and if less, less. 5 def. 5.

First.—Let KO $\not>$ KH;

∴ NP $\not>$ NM:

& ∵ GH, HK are equimults. of AB, BE,

& that AB $>$ BE,

∴ GH $>$ HK ; 3 ax. 5.

but KO $\ngtr$ KH,

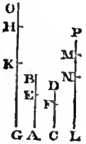
∴ GH $>$ KO.

Similarly LM $>$ NP :

∴ if KO $\ngtr$ KH,

then GH, a mult. of AB, $>$ KO, a mult. of BE.

Similarly LM, a mult. of CD, $>$ NP, a mult. of DF.



Secondly.—Let KO $>$ KH ;

∴ also NP $>$ NM : demon.

& ∵ whole GH is same mult. of whole AB, that HK is of BE,

∴ rem. GK is same mult. of rem. AE, that GH is of AB ; [5. 5.

wh. is same that LM is of CD ;

similarly, ∵ LM is same mult. of CD, that MN is of DF,

∵ rem. LN is same mult. of rem. CF, that whole LM is of whole CD. 5. 5.

But LM is same mult. of CD, that GK is of AE, demon.

∴ GK is same mult. of AE, that LN is of CF ;

i. e. GK, LN are equimults. of AE, CF ;

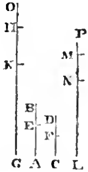
& ∵ KO, NP are equimults. of BE, DF,

& that KH, NM are also equimults. of BE, DF,

if KH, NM be taken from KO, NP,

∴ rems. HO, MP are either = or equimults. of, BE, DF. [6.5.

FIRST.—Let HO, MP = BE, DF ;
& ∵ AE : EB :: CF : FD,
& that GK, LN are equimults. of AE, CF,
∴ GK : EB :: LN : FD : cor. 4. 5.
but HO = EB,
& MP = FD,
∴ GK : HO :: LN : MP :
if, ∴ GK > HO,
then LN > MP ;
if equal, equal ; if less, less. A. 5.



SECONDLY.—Let HO, MP be equimults. of EB, FD :
& ∴ AE : EB :: CF : FD, hyp.
& that GK, LN are any equimults. of AE, CF,
& HO, MP are any equimults. of EB, FD ;
if GK > HO,
then LN > MP ;
if equal, equal ; if less, less ; 5 def. 5.
wh. was also shown in preced. case :
if ∴ GH > KO,
take from both KH,
then GK > HO ;
∴ also LN > MP ;
& consequently, adding NM to both,
LM > NP ;
if ∴ GH > KO,

then LM $>$ NP :

similarly, if equal, equal ; if less, less.

Now in FIRST case where KO was assumed $\not>$ KH,

it was shown that GH $>$ KO always ;

and also LM $>$ NP ;

but GH, LM are any equimults. of AB, CD, constr.

& KO, NP are any equimults. of BE, DF ;

∴ AB : BE : : CD : DF. 5 def. 5.

Therefore if mags. &c. &c. Q. E. D.

PROP. XIX.—THEOREM.

If a whole magnitude be to a whole, as a magnitude taken from the first, is to a magnitude taken from the other ; the remainder shall be to the remainder, as the whole to the whole.

Let whole AB : whole CD : : AE (a mag. taken from AB) : CF (a mag. taken from CD); then shall rem. EB : rem. FD : : AB : CD.

For ∵ AB : CD : : AE : CF,

∴ alternando, AB : AE : : CD : CF ; 16. 5.

& dividendo EB : AE : : FD : CF ; 17. 5.

again, alternando, EB : FD : : AE : CF ;

but AE : CF	: :	AB : CD ;	hyp.
∴ EB : FD	: :	AB : CD.	11. 5.

Therefore if the whole, &c. &c. Q. E. D.

Cor. If the whole be to the whole, as a magnitude taken from the first, is to a magnitude taken from the other ; the remainder likewise is to the remainder, as the magnitude taken from the first to that taken from the other. The demonstration is contained in the preceding.

PROP. E.—THEOREM.

If four magnitudes be proportionals, they are also proportionals by conversion, that is, the first is to its excess above the second, as the third to its excess above the fourth.

Let AB : BE : : CD : DF ; then BA : AE : : DC : CF.

For ∵ AB : BE	: :	CD : DF ;	
dividendo AE : EB	: :	CF : FD ;	17. 5.
invertendo BE : EA	: :	DF : FC ;	B. 5.
∴ componendo BA : AE	: :	DC : CF.	18. 5.

Therefore, if four mags., &c. &c. Q. E. D.

PROP. XX.—Theorem.

If there be three magnitudes, and other three, which, taken two and two, have the same ratio ; then, if the first be greater than the third, the fourth shall be greater than the sixth ; and if equal, equal ; and if less, less.

Let A, B, C, be three mags. ; and D, E, F, three others, wh. taken two & two, have same ratios, viz. A : B : : D : E ; & B : C : : E : F ; &

First.—Let A > C ; then shall D > F.

∵ A	>	C,	
& B is any other mag.			
∴ A : B	>	C : B ;	8. 5.
but D : E	: :	A : B,	
∴ D : E	>	C : B ;	13. 5.
& ∵ B : C	: :	E : F,	
invertendo C : B	: :	F : E,	B. 5.
∴ D : E	>	F : E ;	cor. 13. 5.
∴ D	>	F.	10. 5.

SECONDLY.—Let A = C; then shall D = F.

$\because$ A = C,
$\therefore$ A : B :: C : B; 7. 5.
but A : B :: D : E,
& C : B :: F : E,
$\therefore$ D : E = F : E; 11. 5.
$\therefore$ D = F. 9. 5.

THIRDLY.—Let A > C; then shall D > F.

For C > A;
& as by 1st case C : B :: F : E;
similarly B : A :: E : D;
$\therefore$ by 1st case, F > D:
& $\therefore$ D < F.

Therefore, if there be three mags., &c. &c. Q. E. D.

PROP. XXI.—THEOREM.

If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Let A, B, C, be three mags. & D, E, F, three others, wh. have same ratio, taken two & two, but in a cross order, viz. A : B :: E : F, & B : C :: D : E; &

First.—Let A $>$ C; then shall D $>$ F.

$\because$ A $>$ C,

& B is any other mag.;

$\therefore$ A : B $>$ C : B; 8. 5.

but E : F :: A : B,

$\therefore$ E : F $>$ C : B; 13. 5.

& $\because$ B : C :: D : E,

$\therefore$ invertendo C : B :: E : D;

& E : F $>$ C : B, demon.

$\therefore$ E : F $>$ E : D; cor. 13. 5.

& $\therefore$ F $<$ D; 10. 5.

i. e. D $>$ F.

Secondly.—Let A $=$ C; then shall D $=$ F.

$\because$ A $=$ C,

$\therefore$ A : B :: C : B; 7. 5.

but A : B :: E : F,

& C : B :: E : D,

$\therefore$ E : F :: E : D; 11. 5.

$\therefore$ D $=$ F. 9. 5.

THIRDLY.—Let A $<$ C ; then shall D $<$ F.

For C $>$ A,

&, as was shown, C : B : : E : D ;

similarly B : A : : F : E ;

$\therefore$ by 1st case, F $>$ D ;

$\therefore$ D $<$ F.

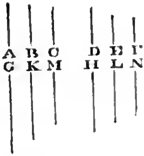
Therefore, if there be three mags., &c. &c. Q. E. D.

PROP. XXII.—THEOREM.

If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last of the same.

N.B. This is usually cited by the words "ex æquali," or "ex æquo."

FIRST.—Let there be three mags. A, B, C, & three others D, E, F, wh. taken two & two, have same ratio; i.e. A : B : : D : E, & B : C : : E : F. Then shall A : C : : D : F.



Of A, D, of B, E, & of C, F,	take any equimults.	G, H ; K, L ; M, N.

Then ∵ A : B :: D : E,
& that G, H are equimults. of A, D,
& K, L are equimults of B, E ;
∴ G : K :: H : L. 4. 5.
Similarly K : M :: L : N.

Now, ∵ there are three mags. G, K, M, & also three others H, L, N, wh., taken two & two, have same ratio ;
if G > M,
then H > N ;
if equal, equal ; if less, less. 20. 5.
Now, G, H are any equimults. of A, D, & M, N are any equimults. of C, F, constr.
∴ A : C :: D : F. 5 def. 5.

Secondly.—Let A, B, C, D, be four mags., & four others E, F, G, H, wh., taken two and two, have same ratio ; viz. A : B : : E : F ; B : C : : F : G ; & C : D : : G : H. Then shall A : D : : E : H.

For, ∵ A, B, C, are three mags. & E, F, G, three others, wh. taken two & two have same ratio ;

∴ by 1st case, A : C : : E : G ;
but C : D : : G : H ;
∴ again, by 1st case A : D : : E : H.

A. B. C. D.
E. F. G. H.

& so on, whatever be the number of mags.

Wherefore, if there be any number, &c. &c. Q. E. D.

PROP. XXIII.—Theorem.

If there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last of the same.

N.B. This is usually cited by the words "ex æquali in proportione perturbatâ;" or "ex æquo perturbato."

First.—Let there be three mags. A, B, C, & three others D, E, F, wh. taken two & two in cross order, have same ratio, i. e. A : B : : E : F, & B : C : : D : E. Then A : C : : D : F.

Of A, B, D, take any equimults. G, H, K ;
& of C, E, F, take any equimults. L, M, N ;
& ∵ G, H, are equimults. of A, B,

$\therefore$ A : B :: G : H : 15. 5.
similarly E : F :: M : N :
but A : B :: E : F,
$\therefore$ G : H :: M : N ; 11. 5.
& $\because$ B : C :: D : E,
& that H, K, are equimults. of B, D,
& L, M, are equimults. of C, E,
$\therefore$ H : L :: K : M : 4. 5.
& it was shown,
that G : H :: M : N.

Now $\because$ there are three mags. G, H, L, & three others, K, M, N, wh. taken two & two in cross order, have same ratio ;

if G > L,
then K > N ;
if equal, equal ; if less, less. 21. 5.
Now G, K, are equimults. of A, D,
& L, N, are equimults. of C, F ;
$\therefore$ A : C :: D : F. 5 def. 5.

SECONDLY.—Let there be four mags. A, B, C, D, & four others E, F, G, H, wh. taken two & two in cross order, have same ratio, viz. A : B : : G : H ; B : C : : F : G & C : D : : E : F. Then shall A : D : : E : H.

For $\because$ A, B, C are three mags. & F, G, H are three others, wh. taken two & two in cross order have same ratio ;

$\therefore$ by 1st case, A : C : : F : H ;
but C : D : : E : F,
$\therefore$ by 1st case, A : D : : E : H.

A. B. C. D.
E. F. G. H.

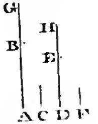
And so on, whatever be number of mags.

Therefore, if there be any number, &c. &c. Q. E. D.

PROP. XXIV.—Theorem.

If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second the same which the sixth has to the fourth; the first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.

Let AB, 1st, : C, 2d, : : DE, 3d, : F, 4th ; & let BG, 5th, : C, 2d, : : EH, 6th, : F, 4th ; then AG, 1st + 5th, : C, 2d, : : DH, 3d + 6th, : F, 4th.



∵ BG : C	: :	EH : F,	
∴ invertendo C : BG	: :	F : EH :	B. 5.
& ∵ AB : C	: :	DE : F,	
& that C : BG	: :	F : EH,	
∴ ex æquali, AB : BG	: :	DE : EH ;	22. 5.
∴ componendo AG : GB	: :	DH : HE :	18. 5.
but GB : C	: :	HE : F,	hyp.
∴ ex æquali AG : C	: :	DH : F.	22. 5.

Therefore, if the first, &c. &c. Q. E. D.

Cor. 1. If the same hypothesis be made as in the proposition, the excess of the first and fifth shall be to the second, as the excess of the third and sixth to the fourth. The demonstration of this is the same with that of the proposition, if division be used instead of composition.

Cor. 2. The proposition holds true of two ranks of magnitudes, whatever be their number, of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to a fourth magnitude; as is manifest.

PROP. XXV.—Theorem.

If four magnitudes of the same kind are proportionals, the greatest and least of them together are greater than the other two together.

Let four mags. AB, CD, E, F, be proportionals, viz. AB : CD : : E : F; & let AB be greatest of them, & consequently F least.* Then shall AB + F > CD + E. * A. 14 & 15. 5.

Take AG = E,
& CH = F.
Then ∵ AB : CD : : E : F,
& that AG = E,
& CH = F;
∴ AB : CD : : AG : CH;
& ∵ whole AB : whole CD : : AG : CH;
∴ rem. GB : rem. HD : : whole AB : whole CD; [19. 5.

but AB > CD,
∴ GB > HD; A. 5.
& ∵ AG = E,
& CH = F,
& AG + F = CH + E.

If ∴ $\left\{\begin{array}{l}\text{AG + F,}\\ \text{CH + E,}\end{array}\right\}$ be added to unequal mags. $\left\{\begin{array}{l}\text{GB,}\\ \text{HD,}\end{array}\right.$

then ∵ GB > HD,
∴ AB + F > CD + E.

Therefore, if four mags., &c. &c. Q. E. D.

BOOK VI.

DEFINITIONS.

I.

SIMILAR rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.

II.

" Reciprocal figures, viz. triangles and parallelograms, " are such as have their sides about two of their angles " proportionals in such a manner, that a side of the first " figure is to a side of the other, as the remaining side " of this other is to the remaining side of the first."

III.

A right line is said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment is to the less.

IV.

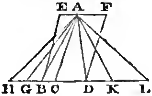
The altitude of any figure is the right line drawn from its vertex perpendicular to the base.

K

PROP. I.—Theorem.

Triangles and parallelograms of the same altitude are to each other as their bases.

Let △s ABC, ACD, & ▱s EC, CF have same alt.; viz. the perpendicular drawn from A to BD; then base BC : base CD : : △ ABC : △ ACD : : ▱ EC : ▱ CF.



Prod. BD both ways to points H, L;
take any number of rt. lines, 3. 1.
viz. { BG, GH, ea. = base BC,
DK, KL, ea. = base CD;
join AG, AH, AK, AL.
Then ∵ △s AHG, AGB, ABC are on equal bases CB, BG, GH, & also between same ∥s EF, HC,
∴ △s AHG, AGB, ABC = ea. other; 38. 1.
∴ △ AHC is same mult. of △ ABC that base HC is of base BC; similarly, △ ALC is same mult. of △ ADC that base LC is of base DC.
& if HC = CL,
then △ AHC = △ ALC; 38. 1.
if greater, greater; if less, less.
Now ∵ of base BC & △ ABC, 1st & 3d, are taken any equimults., i. e. base HC, & △ AHC,
& of base CD & △ ACD, 2d & 4th, are taken any equimults., i. e. base CL & △ ALC;
& that, if HC > CL,

then △ AHC	>	△ ALC ;	
if equal, equal ; if less, less ;			
∴ base BC : base CD	: :	△ ABC : ACD.	5 def. 5.
And ∵ □ CE	=	2 △ ABC,	41. 1.
& that □ CF	=	2 △ ACD,	
∴ △ ABC : △ ACD	: :	□ CE : □ CF :	15. 5.
& ∵ also, BC : CD	: :	△ ABC : △ ACD,	
∴ base BC : base CD	: :	□ CE : □ CF.	11. 5.

∴ if △s, &c. &c. Q. E. D.

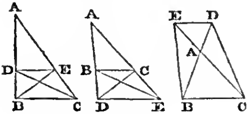
Cor. From this it is plain, that triangles and parallelograms which have equal altitudes, are to each other as their bases.

Let the figures be placed so as to have their bases in the same right line ; and having drawn perpendiculars from the vertices of the triangles to the bases, the right line which joins the vertices is parallel to that in which the bases are, because the perpendiculars are both equal and parallel to each other : then if the same construction be made as in the proposition, the demonstration will be the same. 28. 1. 33. 1.

PROP. II.—Theorem.

If a right line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or these produced, proportionally : and if the sides, or the sides produced, be cut proportionally, the right line which joins the points of section shall be parallel to the remaining side of the triangle.

First.—Let DE be drawn ∥ BC, a side of △ ABC ; then BD : DA : : CE : EA.



Join BE, CD.

Then △ BDE = △ CDE, 37. 1.

(for they are on same base DE & between same ||s DE, BC;)

& ∵ △ ADE is another mag.

∴ △ BDE : △ ADE : : △CDE : △ADE ; 7. 5.

but △ BDE : △ ADE : : BD : DA, 1. 6.

(for they have same alt. DE).

Similarly △ CDE : △ ADE : : CE : EA.

∴ BD : DA : : CE : EA. 11. 5.

Secondly.—Let AB, AC sides of △ ABC, or these prod. be cut in points D, E, so that BD : DA : : CE : EA ; then DE || BC.

Same constr. being made,

∵ BD : DA : : CE : EA,

& BD : DA : : △BDE : △ADE, } 1. 6.

& that CE : EA : : △CDE : △ADE, }

∴ △ BDE : △ ADE : : △CDE : △ADE ; 11. 5.

i. e. △s BDE, CDE have each same ratio to △ ADE ;

& ∴ △ BDE = △ CDE ; 9. 5.

& they are on same side of base DE ;

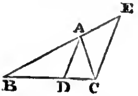
∴ DE || BC. 39. 1.

Wherefore if a rt. line, &c. &c. Q. E. D.

PROP. III.—Theorem.

If the angle of a triangle be divided into two equal angles by a right line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangle have to each other: and if the segments of the base have the same ratio which the other sides of the triangle have to each other, the right line drawn from the vertex to the point of section, divides the vertical angle into two equal angles.

First.—Let $\angle$ BAC, of any $\triangle$ ABC, be bisected by AD, cutting base in D; then BD : DC : : BA : AC.



Draw CE	$\parallel$	DA;	31. 1.
& let BA prod. meet CE in E;			
Then $\because$ AD & EC are $\parallel$s			
int. $\angle$ AEC	$=$	ext. $\angle$ BAD,	29. 1.
	$=$	$\angle$ DAC by hypoth.	
	$=$	altern. $\angle$ ACE,	29. 1.
$\therefore$ AE	$=$	AC.	6. 1.
& $\because$ AD	$\parallel$	EC a side of $\triangle$ BCE,	
$\therefore$ BD : DC	: :	BA : AE;	2. 6.
but AE	$=$	AC,	
$\therefore$ BD : DC	: :	BA : AC.	7. 5.

Secondly.—Let BD : DC : : BA : AC; join AD; then $\angle$ BAC is bisected by AD, i. e. $\angle$ BAD $=$ $\angle$ CAD.

Same constr. being made ;

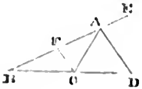
∵ BD : DC	: :	BA : AC,	
& that BD : DC	: :	BA : AE,	2. 6.
(for AD	‖	EC,)	
∴ BA : AC	: :	BA : AE ;	11. 5.
∴ AC	=	AE ;	9. 5.
& ∴ ∠ AEC	=	∠ ACE :	5. 1.
but ∠ AEC	=	ext. ∠ BAD,	29. 1.
also ∠ ACE	=	altern. ∠ CAD,	
∴ ∠ BAD	=	∠ CAD.	

Wherefore, if the angle, &c. &c. Q. E. D.

PROP. A.—Theorem.

If the outward angle of a triangle made by producing one of its sides, be divided into two equal angles, by a right line which also cuts the base produced ; the segments between the dividing line and the extremities of the base have the same ratio which the other sides of the triangle have to each other : and if the segments of the base produced have the same ratio which the other sides of the triangle have, the right line drawn from the vertex to the point of section divides the outward angle of the triangle into two equal angles.

First.—Let ext. ∠ CAE of any △ ABC be bisected by AD wh. meets base prod. in D ; then BD : DC : : BA : AC.



Draw CF	$\parallel$	AD.	31. 1.
Then $\because$ CF & AD are $\parallel$s,			
int. $\angle$ AFC	$=$	ext. $\angle$ EAD,	29. 1.
	$=$	$\angle$ CAD by hypoth.	
	$=$	altern. $\angle$ ACF,	29. 1.
$\therefore$ AF	$=$	AC.	6. 1.
& $\because$ AD	$\parallel$	FC a side of $\triangle$ BCF,	
$\therefore$ BD : DC	: :	BA : AF ;	2. 6.
now AF	$=$	AC,	
$\therefore$ BD : DC	: :	BA : AC.	7. 5.

Secondly.—Let BD : DC : : BA : AC ; then $\angle$ EAD $= \angle$ CAD.

Same constr. being made,

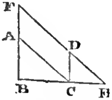
$\because$ BD : DC	: :	BA : AC,	
& that BD : DC	: :	BA : AF,	11. 5.
$\therefore$ BA : AC	: :	BA : AF ;	
$\therefore$ AC	$=$	AF ;	9. 5.
$\therefore$ $\angle$ AFC	$=$	$\angle$ ACF :	5. 1.
but $\angle$ AFC	$=$	ext. $\angle$ EAD,	29. 1.
also $\angle$ ACF	$=$	$\angle$ CAD,	(29. 1.)
$\therefore$ $\angle$ EAD	$=$	$\angle$ CAD.	

Wherefore the outward angle, &c. &c. Q. E. D.

PROP. IV.—Theorem.

The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents, or consequents of the ratios.

Let ABC, DCE be equiang. $\triangle$s, having $\angle$ ABC $=$ $\angle$ DCE & $\angle$ ACB $=$ $\angle$ DEC & consequently $\angle$ BAC* $=$ $\angle$ CDE. Then sides about equal $\angle$s of $\triangle$s ABC, DCE are proportionals; and those are homolog. sides wh. are opp. to equal $\angle$s. * 32. 1.



Let $\triangle$ DCE be so placed, that its side CE may be contiguous to & in same rt. line with BC.

$\because$ $\angle$s ABC + ACB	$<$	2 rt. $\angle$s,	
& that $\angle$ ACB	$=$	$\angle$ DEC,	
$\therefore$ $\angle$s ABC + DEC	$<$	2 rt. $\angle$s ;	
& $\therefore$ BA, ED, if prod. far enough, will meet ;			12 ax. 1.
let them be prod. to meet in F :			
& $\because$ $\angle$ ABC	$=$	$\angle$ DCE,	hyp.
$\therefore$ BF	$\parallel$	CD.	28. 1.
Again, $\because$ $\angle$ ACB	$=$	$\angle$ DEC,	
$\therefore$ AC	$\parallel$	FE ;	28. 1.
$\therefore$ fig. FC	is a	$\square$;	
& $\therefore$ AF	$=$	CD ;	31. 1.
& AC	$=$	FD :	31. 1.
& $\because$ AC	$\parallel$	FE a side of $\triangle$ FBE,	
$\therefore$ BA : AF	: :	BC : CE ;	2. 6.
but AF	$=$	CD,	
$\therefore$ BA : CD	: :	BC : CE ;	7. 5.
& alternando AB : BC	: :	DC : CE.	16. 5.
Again, $\because$ CD	$\parallel$	BF,	
$\therefore$ BC : CE	: :	FD : DE ;	2. 6.
but FD	$=$	AC,	

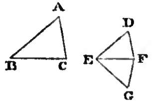
∴ BC : CE	: :	AC : DE ;	7. 5.
& alternando BC : CA	: :	CE : ED.	16. 5.
Now ∵ AB : BC	: :	DC : CE,	demon.
& that BC : CA	: :	CE : ED,	
∴ also ex æquali BA : AC	: :	CD : DE.	22. 5.

Therefore the sides, &c. &c. Q. E. D.

PROP. V.—Theorem.

If the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular, and have their equal angles opposite to the homologous sides.

Let △s ABC, DEF have their sides proportionals, so that AB : BC : : DE : EF ; & BC : CA : : EF : FD ; & consequently ex æquali BA : AC : : ED : DF. Then △ ABC is equiang. to △ DEF, & their equal ∠s are opp. to homolog. sides, viz. ∠ ABC = ∠ DEF, ∠ BCA = ∠ EFD, & ∠ BAC = ∠ EDF.



Make { ∠ FEG	=	∠ ABC, }	23. 1.
{ & ∠ EFG	=	∠ BCA, }	
∴ rem. ∠ EGF	=	rem. ∠ BAC ;	32. 1.
& ∴ △ ABC is equiang. to △ GEF ;			
& ∴ AB : BC	: :	GE : EF :	4. 6.
but AB : BC	: :	DE : EF,	hyp.

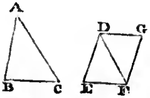
∴ DE : EF : : GE : EF ; 11. 5.
∴ DE = GE ; 9. 5.
similarly DF = FG :
& ∵ DE = EG,
& EF is com. to △s DEF, GEF,
∴ DE, EF = EG, EF ea. to ea. ;
& ∵ base DF = base FG,
∵ ∠ DEF = ∠ GEF ; 8. 1.
& consequently ∠ DFE = ∠ GFE ; } 4. 1.
& ∠ EDF = ∠ EGF : }
& ∵ ∠ DEF = ∠ GEF,
& that ∠ GEF = ∠ ABC,
∴ ∠ ABC = ∠ DEF.
Similarly { ∠ ACB = ∠ DFE,
& ∠ BAC = ∠ EDF.
∴ △ ABC is equiang. to △ DEF.

Wherefore, if the sides, &c. &c. Q. E. D.

PROP. VI.—Theorem.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

Let △s ABC, DEF have ∠ BAC in one = ∠ EDF in other, & sides about those ∠s proportionals ; i. e. BA : AC : : ED : DF. Then △s ABC, DEF are equiang., & have ∠ ABC = ∠ DEF and ∠ ACB = ∠ DFE.



Make $\angle$s $\left\{\begin{array}{lll} \text{FDG} & = & \text{BAC or EDF,} \\ \text{DFG} & = & \text{ACB;} \end{array}\right\}$ 23. 1.

$\therefore$ rem. $\angle$ at B = rem. $\angle$ at G ; 32. 1.

& $\therefore$ $\triangle$ ABC is equiang. to $\triangle$ DGF ;

& $\therefore$ BA : AC :: GD : DF ; 4. 6.

but BA : AC :: ED : DF,

$\therefore$ ED : DF :: GD : DF ; 11. 5.

$\therefore$ ED = GD : 9. 5.

& $\because$ DF is com. to $\triangle$s EDF, GDF,

then ED, DF = GD, DF ea. to ea. ;

& $\angle$ EDF = $\angle$ GDF, constr.

$\therefore$ base EF = base FG,

$\left.\begin{array}{rll} \text{\& } \triangle \text{ EDF} & = & \triangle \text{ GDF,} \\ \text{\& } \therefore \text{ also } \angle \text{ DFG} & = & \angle \text{ DFE,} \\ \text{\& } \angle \text{ DGF} & = & \angle \text{ DEF,} \end{array}\right\}$ 4. 1.

but $\angle$ DFG = $\angle$ ACB,

$\therefore$ $\angle$ ACB = $\angle$ DFE ;

also $\angle$ BAC = $\angle$ EDF. hyp.

$\therefore$ rem. $\angle$ ABC = rem. $\angle$ DEF.

& $\therefore$ $\triangle$ ABC is equiang. to $\triangle$ DEF.

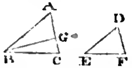
Wherefore, if two triangles, &c. &c. Q. E. D.

PROP. VII.—Theorem.*

If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals; then, if each of the remaining angles be either less or not less than a right angle, or if one of them be a right angle; the triangles shall be equiangular, and shall have those angles equal about which the sides are proportionals.

Let △s ABC, DEF have ∠ BAC = ∠ EDF, & sides about two other ∠s ABC, DEF proportionals, i. e. AB : BC : : DE : EF ; &,

First.—Let each of rem. ∠s at C, F be < rt. ∠. Then △ ABC is equiang. to △ DEF, viz. ∠ ABC = ∠ DEF & rem. ∠ at C = rem. ∠ at F.



For if ∠ ABC	≠	∠ DEF,	
the one	>	other ;	
let ∠ ABC	>	∠ DEF.	
Make ∠ ABG	=	∠ DEF ;	23. 1.
& ∵ ∠ BAC	=	∠ EDF,	
& that ∠ ABG	=	∠ DEF,	
∴ rem. ∠ AGB	=	rem. ∠ DFE ;	32. 1.
∴ △ ABG is equiang. to △ DEF ;			
∴ AB : BG	: :	DE : EF ;	4. 6.
but AB : BC	: :	DE : EF,	

* See Appendix.

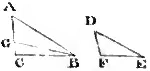
$\therefore$ AB : BC	: :	AB : BG ;	11. 5.
$\therefore$ BC	$=$	BG ;	9. 5.
& $\therefore \angle$ BGC	$=$	$\angle$ BCG ;	5. 1.
but $\angle$ BCG	$<$	rt. $\angle$,	hyp.
$\therefore$ also $\angle$ BGC	$<$	rt. $\angle$;	
& $\therefore$ adj. $\angle$ AGB	$>$	rt. $\angle$;	13. 1.
but $\angle$ AGB	$=$	$\angle$ DFE ;	demon.
$\therefore \angle$ DFE	$>$	rt. $\angle$;	
but $\angle$ DFE	$<$	rt. $\angle$;	hyp.

wh. is absurd.

$\therefore \angle$ ABC is not	$\neq$	$\angle$ DEF ;	
i. e. $\angle$ ABC	$=$	$\angle$ DEF ;	
& $\angle$ at A	$=$	$\angle$ at D, by hypoth.	
$\therefore$ rem. $\angle$ at C	$=$	rem. $\angle$ at F ;	

$\therefore \triangle$ ABC is equiang. to $\triangle$ DEF.

Secondly.—Let each of $\angle$s at C, F be $\not<$ rt. $\angle$; then $\triangle$ ABC is equiang. to $\triangle$ DEF.



Same constr. being made, it may be proved as before,

that BC	$=$	BG,	
& $\therefore \angle$ BCG	$=$	$\angle$ BGC ;	5. 1.
but $\angle$ BCG	$\not<$	rt. $\angle$,	
$\therefore \angle$ BGC	$\not<$	rt. $\angle$;	

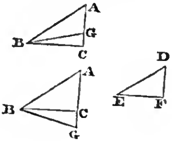
$\therefore$ in $\triangle$ BGC are two $\angle$s which together are $\not<$ two rt. $\angle$s ;

wh. is impossible ; 17. 1.

& $\therefore$ it may be proved as in first case,

that $\triangle$ ABC is equiang. to $\triangle$ DEF.

THIRDLY.—Let one of ∠s C, F, viz. ∠ at C, be a rt. ∠ : then likewise △ ABC is equiang. to △ DEF.



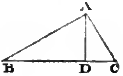
For if △ ABC be not equiang. to △ DEF ;
make ∠ ABG = ∠ DEF :
then it may be proved as in first ease,
that BG = BC ;
& ∴ ∠ BCG = ∠ BGC ; 5. 1.
but ∠ BCG is a rt. ∠ ;
∴ ∠ BGC is a rt. ∠ ;
∴ in △ BGC are two ∠s together = two rt. ∠s,
wh. is impossible. 17. 1.
∴ △ ABC is equiang. to △ DEF.

Wherefore, if two △s, &c. &c. Q. E. D.

PROP. VIII.—THEOREM.

In a right angled triangle, if a perpendicular be drawn from the right angle to the base ; the triangles on each side of it are similar to the whole triangle, and to each other.

Let ABC be a rt. ∠d △, having rt. ∠ BAC ; let AD be drawn ⊥ base BC ; then △s ABD, ADC, are sim. to whole △ ABC, & to each other.



∴ ∠ BAC = ADB, 11 ax. 1.
& that ∠ ABC is com. to △s ABC, ABD,
∴ rem. ∠ ACB = rem. ∠ BAD ; 32. 1.
∴ △ ABC is equiang. to △ ABD;
& their sides about equal ∠s are proportional, 4. 6.
∴ △ ABC is sim. to △ ADB : 1 def. 6.
simly. △ ADC is equiang. & sim. to △ ABC ;
∴ △ ABD is sim. to △ ADC.

Therefore, in a rt. ∠d △, &c. &c. Q. E. D.

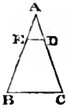
Cor. From this it is manifest that the perpendicular, drawn from the rt. ∠ of a rt. ∠d △ to the base, is a mean proportional between the segments of the base ; and also that each of the sides is a mean proportional between the base, and the segment adjacent to that side ;

for in △s BDA, ADC.—BD : DA : : DA : DC ;
& in △s ABC, DBA.—BC : BA : : BA : BD ; } 4. 6.
& in △s ABC, ACD.—BC : CA : : CA : CD.

PROP. IX.—Problem.

From a given right line to cut off any part required.

Let AB be given rt. line ; it is required to cut off any part from it.



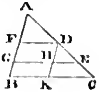
Draw AC, making any ∠ with AB ;
in AC take any point D ;
& take AC, same mult. of AD that AB is of part to be cut off;
join BC ;
draw DE || BC ; 31. 1.
then AE is part required to be cut off.
∵ ED || BC a side of △ ABC,
∴ CD : DA :: BE : EA ; 2. 6.
but componendo CA : AD :: BA : AE, 18. 5.
∴ BA is same mult. of AE that CA is of AD ; D. 5.
& ∴ AE is same part of BA that AD is of CA.

Therefore, from AB, the part required is cut off. Q. E. F.

PROP. X.—Problem.

To divide a given right line similarly to a given divided right line, that is, into parts that shall have the same ratios to each other which the parts of the divided right line have.

Let AB be rt. line given to be divided, & AC divided rt. line ; it is required to divide AB simly. to AC.



Let AC be divided in points D, E ;
& let AB, AC be placed so as to contain any ∠ ;
join BC ;

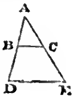
draw { DF, EG	‖	BC ; }	31. 1.
DHK	‖	AB ; }	
∴ each fig. FH, HB	is a	▱ ;	
∴ DH	=	FG, }	34. 1.
& HK	=	GB : }	
& ∵ HE	‖	KC a side of △ DKC,	
∴ CE : ED	: :	KH : HD ;	2. 6.
but KH	=	BG,	
& HD	=	GF,	
∴ CE : ED	: :	BG : GF.	7. 5.
Again, ∵ FD	‖	EG a side of △ AGE,	
∴ ED : DA	: :	GF : FA ;	2. 6.
also CE : ED	: :	BG : GF.	demon.

Therefore, AB is divided simly. to AC. Q. E. F.

PROP. XI.—PROBLEM.

To find a third proportional to two given right lines.

Let AB, AC be the two given rt. lines, and let them be placed so as to contain any ∠ ; it is required to find a third proportional to AB, AC.



Prod. AB, AC to D, E ;

make BD = AC ; 3. 1.

join BC ;

draw DE ∥ BC. 31. 1.

Then, ∵ BC ∥ DE a side of △ ADE,

∴ AB : BD :: AC : CE ; 2. 6.

but BD = AC,

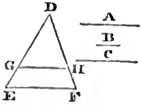
∴ AB : AC :: AC : CE. 7. 5.

Therefore, to the two given rt. lines AB, AC, a third proportional CE is found. Q. E. F.

PROP. XII.—Problem.

To find a fourth proportional to three given right lines.

Let A, B, C be the three given rt. lines ; it is required to find a fourth proportional to them.



Take two rt. lines DE, DF, containing any ∠ EDF ;

in these make { DG = A, GE = B, DH = C :

join GH ;

draw EF ∥ GH.

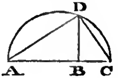
Then ∵ GH ∥ EF a side of △ DEF,
∴ DG : GE :: DH : HF ; 2. 6.
but DG = A,
GE = B,
& DH = C,
∴ A : B :: C : HF. 7. 5.

Therefore, to A, B, C a fourth proportional HF has been found. Q. E. F.

PROP. XIII.—Problem.

To find a mean proportional between two given right lines.

Let AB, BC be the two given rt. lines ; it is required to find a mean proportional to them.



Place AB, BC in a rt. line ;
On AC descr. ½ ⊙ ADC ;
draw BD ⊥ AC ; 11. 1.
join AD, DC.

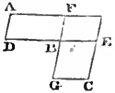
And ∵ ∠ ADC, in a ½ ⊙, is a rt. ∠, 31. 3.
& that in rt. ∠d △ ADC, DB is drawn from rt. ∠ ⊥ base,
∴ DB is a mean proportional between AB, BC the segs. of base. cor. 8. 6.

Therefore between the given rt. lines AB, BC a mean proportional DB is found. Q. E. F.

PROP. XIV.—Theorem.

Equal parallelograms, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and parallelograms that have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to each other.

First.—Let AB, BC be equal □s, wh. have their ∠s at B equal; and let sides DB, BE be placed in same rt. line, ∴ also FB, BG will be in same rt. line;* then sides of □s AB, BC, about equal ∠s, are reciprocally proportional, viz. DB : BE : : GB : BF.



Complete □ FE.

And ∵ □ AB	=	□ BC,	hyp.
& that EF is another mag.;			
∴ AB : FE	: :	BC : FE;	7. 5.
but AB : FE	: :	DB : BE,	1. 6.
also BC : FE	: :	GB : BF,	
∴ DB : BE	: :	GB : BF,	11. 5.

∴ sides of □s AB, BC, about equal ∠s, are reciprocally proportional.

Secondly.—Let sides about equal ∠s be reciprocally

* By 14. 1. For ∠s GBE + EBF = ∠s DBF + EBF = 2 rt. ∠s.

proportional, viz. DB : BE : : GB : BF, then □ AB = □ BC.

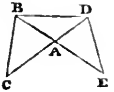
∵ DB : BE	: :	GB : BF,	
& DB : BE	: :	□ AB : □ FE,	1. 6.
& GB : BF	: :	□ BC : □ FE,	
∴ AB : FE	: :	BC : FE;	11. 5.
∴ □ AB	=	□ BC.	9. 5.

Wherefore equal □s, &c. &c. Q. E. D.

PROP. XV.—Theorem.

Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angle reciprocally proportional; and triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to each other.

First.—Let ABC, ADE be equal △s, wh. have ∠ BAC = ∠ DAE; then sides about equal ∠s are reciprocally proportional,—viz. CA : AD : : EA : AB.



Let △s be placed, so that CA, AD be in same rt. line;
& ∴ EA, AB are in same rt. line: 14. 1.
join BD.

And ∵ △ ABC = △ ADE,
& that △ ABD is another mag.
∴ △ CAB : △ BAD :: △ EAD : △ DAB ; 7. 5.
but △ CAB : △ BAD :: base CA : base AD, } 1. 6.
& EAD : DAB :: base EA : base AB ; }
∴ CA : AD :: EA : AB. 11. 5.
∴ sides of △s, about equal ∠s, are reciprocally proportional.

Secondly.—Let sides of △s ABC, ADE, about equal ∠s, be reciprocally proportional, viz. CA : AD : : EA : AB ; then △ ABC = △ ADE.

Join BD as before.

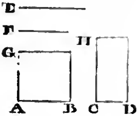
And ∴ CA : AD :: EA : AB.
& that CA : AD :: △ ABC : △ BAD, } 1. 6.
& EA : AB :: △ EAD : △ BAD, }
∴ ABC : BAD :: EAD : BAD ; 11. 5.
∴ △ ABC = △ AED. 9. 5.

Therefore equal triangles, &c. &c. Q. E. D.

PROP. XVI.—Theorem.

If four right lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four right lines are proportionals.

First.—Let four rt. lines AB, CD, E, F be proportionals, viz. AB : CD : : E : F. Then AB · F = CD · E.



Draw AG, CH ⊥ AB, CD respectively;

make { AG = F;
CH = E;

& complete □s BG, DH.

And ∵ AB : CD :: E : F,

& that E = CH,

& F = AG;

∴ AB : CD :: CH : AG; 7. 5.

∴ sides of □s BG, DH, about equal ∠s, are reciprocally proportional;

∴ □ BG = □ DH; 14. 6.

but □ BG = AB · F,

also □ DH = CD · E,

∴ AB · F = CD · E.

SECONDLY.—Let AB · F = CD · E; then AB : CD :: E : F.

Same constr. being made,

∵ AB · F = CD · E,

& that □ BG = AB · F,

& □ DH = CD · E,

∴ □ BG = DH;

& they are equiang.;

∴ AB : CD :: CH : AG : 14. 6.

but CH = E,

& AG = F;

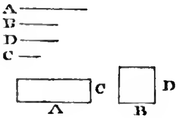
∴ AB : CD :: E : F. 7. 5.

Wherefore, if four rt. lines, &c. &c. Q. E. D.

PROP. XVII.—Theorem.

If three right lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean: and if the rectangle contained by the extremes be equal to the square of the mean, the three right lines are proportionals.

First.—Let three rt. lines A, B, C be proportionals, i. e. $A : B :: B : C$; then $A \cdot C = B^2$.



Take D	$=$	B ;	
then $A : B$	$::$	$D : C$;	7. 5.
$\therefore A \cdot C$	$=$	$B \cdot D$:	16. 6.
but $B \cdot D$	$=$	B^2,	
$\therefore A \cdot C$	$=$	B^2.	

Secondly.—Let $A \cdot C = B^2$; then $A : B :: B : C$.

Make same constr.

& $\because A \cdot C$	$=$	B^2,	
& that B^2	$=$	$B \cdot D$,	
$\therefore A \cdot C$	$=$	$B \cdot D$;	
$\therefore A : B$	$::$	$D : C$;	16. 6.
but B	$=$	D,	
$\therefore A : B$	$::$	$B : C$.	

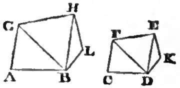
Wherefore, if three rt. lines, &c. &c. Q. E. D.

PROP. XVIII.—Problem.

On a given right line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure.

Let AB be given rt. line, and CDEF given rectil. fig. of four sides;

First.—It is required to descr. on AB a rectil. fig. sim. & simly. situated to CDEF.



Join DF:

& make { ∠ BAG = ∠ FCD; ∠ ABG = ∠ CDF; } 23. 1.

& ∴ rem. ∠ AGB = rem. ∠ CFD; 32. 1.

& ∴ △ FCD is equiang. to △ ABG.

Again, make { ∠ BGH = ∠ DFE, ∠ GBH = ∠ FDE;

∴ rem. ∠ FED = rem. ∠ GHB;

& ∴ △ FDE is equiang. to △ GBH.

Then, ∵ ∠ AGB = ∠ CFD,

& that also ∠ BGH = ∠ DFE,

∴ whole ∠ AGH = whole ∠ CFE:

similarly ∠ ABH = ∠ CDE;

& by constr. ∠ GAB = ∠ FCD;

& ∠ GHB = ∠ FED;

∴ rectil. fig. ABHG is equiang. to rectil. fig. CDEF.

And also these figs. have their sides about equal ∠s, proportionals.

For ∵ △ GAB is equiang. to △ FCD,

∴ BA : AG :: DC : CF : 4. 6.

& ∵ AG : GB :: CF : FD,

& that GB : GH :: FD : FE,

(for △ BGH is equiang. to △ DFE,)

∴ ex æquali AG : GH :: CF : FE ; 22. 5.

similarly AB : BH :: CD : DE,

and GH : HB :: FE : ED.

Now ∵ fig. ABHG is equiang. to fig. CDEF,

and that both have sides about equal ∠s proportionals ;

∴ rectil. fig. ABHG is sim. to rectil. fig. CDEF.

Secondly.—It is required to descr. on AB a rectil. fig. sim. to a given rectil. fig. CDKEF of *five* sides.

Join DE ;

On AB descr. a rectil. fig. ABHG, sim. and simly. situated to rectil. fig. CDEF : 1st case.

make { ∠ HBL = ∠ EDK ;
∠ BHL = ∠ DEK ;

∴ rem. ∠ DKE = rem. ∠ BLH. 32. 1.

And ∵ fig. ABHG is sim. to fig. CDEF,

∴ ∠ GHB = ∠ FED ;

but also ∠ BHL = ∠ DEK ; constr.

∴ whole ∠ GHL = whole ∠ FEK :

similarly ∠ ABL = ∠ CDK,

∴ rectil. fig. AGHLB is equiang. to rectil. fig. CFEKD.

And ∵ fig. ABHG is sim. to fig. CDEF,

∴ GH : HB :: FE : ED ;

& HB : HL :: ED : EK ; 4. 6.

∴ ex æquali GH : HL :: FE : EK : 22. 5.

similarly AB : BL :: CD : DK ;

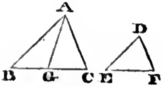
& BL : LH : : DK : KE;
(for △ BLH is equiang. to △ DKE).
Now, ∵ rectil. fig. AGHLB is equiang. to rectil. fig. CFEKD, and that they have sides about equal ∠s proportionals;
∴ fig. AGHLB is sim. to fig. CFEKD.

And in same manner a rectil. fig. may be, descr. sim. and simly. situated to a given rectil. fig. of six or more sides. Q. E. F.

PROP. XIX.—Theorem.

Similar triangles are to each other in the duplicate ratio of their homologous sides.

Let ABC, DEF be sim. △s, having ∠ B = ∠ E; & let AB : BC : : DE : EF, so that side BC be hom. to side EF: *then △ ABC : △ DEF : : dupl. of BC : EF.
* 12 def. 5.



Take BG a third proportional to BC, EF, 11. 6.
so that BC : EF : : EF : BG;
join GA.
Then ∵ AB : BC : : DE : EF,
∴ alternando AB : DE : : BC : EF; 16. 5.

but BC : EF :: EF : BG ;
∴ AB : DE :: BF : BG ; 11. 5.
∴ sides of △s ABG, DEF about equal ∠s are reciprocally proportionals :
∴ △ ABG = △ DEF ; 15. 6.
& ∵ BC : EF :: EF : BG,
∴ BC : BG :: dupl. of BC : EF ; 10 def. 5.
but BC : BG :: △ ABC : △ ABG, 1. 6.
∴ △ ABC : △ ABG :: dupl. of BC : EF ;
but △ ABG = △ DEF,
∴ △ ABC : △ DEF :: dupl. of BC : EF.

Therefore sim. △s, &c. &c. Q. E. D.

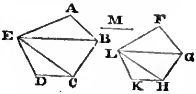
Cor. From this it is manifest, that if three right lines be proportionals, as the first is to the third, so is any triangle upon the first to a similar and similarly described triangle upon the second.

PROP. XX.—Theorem.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to each other that the polygons have; and the polygons have to each other the duplicate ratio of that which their homologous sides have.

Let ABCDE, FGHKL be sim. polygons, & let AB, FG be hom. sides. Then,

First.—The polygons ABCDE, FGHKL may be divided into same number of sim. △s.



Join BE, EC ; GL, LH ;

& ∵ fig. ABCDE is sim. to fig. FGHKL,

∴ ∠ BAE = ∠ GFL ; 1 def. 6.

& ∴ BA : AE : : GF : FL, 1 def. 6.

& ∴ ABE is equiang. to △ FGL ; 6. 6.

and the sides about the equal ∠s are proportionals ;

& ∴ also △ ABE is sim. to △ FGL ; 4. 6.

∴ ∠ ABE = ∠ FGL.

Again, ∵ fig. ABCDE is sim to fig. FGHKL,

∴ whole ∠ ABC = whole ∠ FGH ; 1 def. 6.

& ∴ rem. ∠ EBC = rem. ∠ LGH :

& ∵ △ ABE is sim. to △ FGL,

∴ EB : BA : : LG : GF : 1 def. 6.

also ∵ fig. ABCDE is sim. to fig. FGHKL,

∴ AB : BC : : FG : GH ; 1 def. 6.

∴ ex æquali EB : BC : : LG : GH ; 22 5.

i. e. sides about equal ∠s are proportionals ;

∴ △ EBC is equiang. to △ LGH ; 6 6.

i. e. △ EBC is sim. to △ LGH : 4. 6.

similarly △ ECD is sim. to △ LHK.

∴ sim. polygons ABCDE, FGHKL are divided into same number of sim. △s.

Secondly.—These △s have ea. to ea. the same ratio wh. polygons have, antecedents being △s ABE, EBC, ECD, and consequents △s FGL, LGH, LHK ; also ABCDE : FGHKL : : dupl. of AB : FG.

∵ △ ABE is sim. to △ FGL,

∴ △ ABE : △ FGL : : dupl. of BE : GF : } 19. 6.
simly. △ EBC : △ LGH : : dupl. of BE : GL ; }

$\therefore \triangle ABE : \triangle FGL :: \triangle EBC : \triangle LGH.$ 11. 5.

Again, $\because \triangle$ EBC is sim. to $\triangle$ LGH ;

$\therefore \triangle EBC : \triangle LGH ::$ dupl. of EC : LH ;

Simly. $\triangle ECD : \triangle LKH ::$ dupl. of EC : LH ;

& $\therefore \triangle EBC : \triangle LGH :: \triangle ECD : \triangle LHK$; 11. 5.

but $\triangle EBC : \triangle LGH :: \triangle ABE : \triangle FGL$; demon.

$\therefore$ ABE : FGL : : EBC : LGH : : ECD : LHK ;

$\therefore$ ABE : FGL : : fig. ABCDE : fig. FGHKL,

(for one antec. : its conseq. : : all antecs. : all conseqs.) ; 12. 5.

but $\triangle ABE : \triangle FGL ::$ dupl. of AB : FG,

& $\therefore$ ABCDE : FGHKL : : dupl. of AB : FG.

Wherefore sim. polygons, &c. &c. Q. E. D.

Cor. 1. In like manner it may be proved that similar four-sided figures, or of any number of sides, are to each other in the duplicate ratio of their homologous sides; and it has already been proved in triangles : therefore universally, similar rectilineal figures are to each other in the duplicate ratio of their homologous sides.

Cor. 2. And if to AB, FG, two of the homologous sides, a third proportional M be taken, AB has to M the duplicate ratio of that which AB has to FG ; but the four-sided figure or polygon upon AB has to the four-sided figure or polygon upon FC, likewise, the duplicate ratio of that which AB has to FG ; therefore, as AB is to M, so is the figure upon AB to the figure upon FG ; which was also proved in triangles : therefore, universally, it is manifest, that if three right lines be proportionals, as the first is to the third, so is any rectilineal figure upon the first, to a similar and similarly described rectilineal figure upon the second.

PROP. XXI.—Theorem.

Rectilineal figures which are similar to the same rectilineal figure, are also similar to each other.

Let each of the rectil. figs. A, B, be sim. to rectil. fig. C ; then fig. A is sim. to fig. B.

∵ A is sim. to C,

∴ A is equiang. to C,

& they have their sides about equal ∠s proportionals. 1 def. 6.

Again, ∵ B is sim. to C,

∴ B is equiang. to C,

& they have their sides about equal ∠s proportionals ;

∴ each of figs. A, B is equiang. to fig. C,

&, of each of them, & of C, sides about equal ∠s are proportionals ;

∴ fig. A is equiang. to fig. B, 1 ax. 1.

& they have sides about equal ∠s proportionals ; 11. 5.

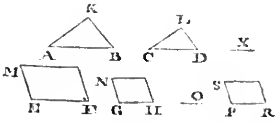
& ∴ rectil. fig. A is sim. to rectil. fig. B. 1 def. 6.

Therefore, rectil. figs., &c. &c. Q. E. D.

PROP. XXII.—Theorem.

If four right lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals; and if the similar rectilineal figures similarly described upon four right lines be proportionals, those right lines shall be proportionals.

First.—Let four rt. lines AB, CD, EF, GH be proportionals, i. e. AB : CD : : EF : GH, & on AB, CD let sim. rectil. figs. KAB, LCD be simly. descr.; & on EF, GH, sim. rectil. figs. MF, NH be descr. in like manner. Then rectil. fig. KAB : LCD : : MF : NH.



To AB, CD take a third proportional X : & to EF, GH take a third proportional O ;			11. 6.
& ∵ AB : CD	: :	EF : GH,	
& that CD : X	: :	GH : O ;	11. 5.
∴ ex æquali AB : X	: :	EF : O ;	22. 5.
but AB : X	: :	KAB : LCD,	2 cor. 20. 6.
& EF : O	: :	MF : NH,	
∴ KAB : LCD	: :	MF : NH.	11. 5.

Secondly.—Let rectil. fig. KAB : LCD : : MF : NH ; then shall AB : CD : : EF : GH.

Make AB : CD	: :	EF : PR ;	12. 6.

& on PR descr. a rectil. fig. SR,
sim. & simly. situated to MF, or NK. 18. 6.

Then ∵ AB : CD	: :	EF : PR,	
∴ by 1st case KAB : LCD	: :	MF : SR ;	
but KAB : LCD	: :	MF : NH,	hyp.
∴ NH	=	SR :	9. 5.

& these are also sim. and simly. situated :

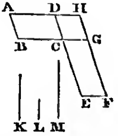
∴ GH	=	PR.
And ∵ AB : CD	: :	EF : PR,
& that PR	=	GH,
∴ AB : CD	: :	EF : GH.

Therefore, if four rt. lines, &c. &c. Q. E. D.

PROP. XXIII.—Theorem.

Equiangular parallelograms have to each other the ratio which is compounded of the ratios of their sides.

Let AC, CF be equiang. ▭s, having ∠ BCD = ∠ ECG. Then ▭ AC : ▭ CF, is same with ratio wh. is comp. of ratio of their sides. Def. A. 5.



Let BC, CG be placed in same rt. line ;
∴ DC, CE are also in same rt. line. 14. 1.

Complete ▱ DG,
take any rt. line K;

& make { BC : CG	: :	K : L; }	
DC : CE	: :	L : M; }	12. 6.

∴ K : L & L : M are same as BC : CG & DC : CE:
now K : M is comp. of K : L & L : M, A. def. 5.
∴ also K : M is comp. of BC : CG & DC : CE:

& ∵ BC : CG	: :	▱ AC : ▱ CH,	1. 6.
& that BC : CG	: :	K : L,	
∴ K : L	: :	▱ AC : ▱ CH.	11. 5.
Again, ∵ DC : CE	: :	▱ CH : ▱ CF,	
& that DC : CE	: :	L : M,	
∴ L : M	: :	▱ CH : ▱ CF;	11. 5.
& since also K : L	: :	▱ AC : ▱ CH,	
∴ ex æquali, K : M	: :	▱ AC : ▱ CF:	22. 5.

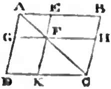
but K : M is comp. of BC : CG & DC : CE,
∴ ▱ AC : ▱ CF : is comp. of BC : CG & DC : CE.

Wherefore, equiang. ▱s, &c. &c. Q. E. D.

PROP. XXIV.—Theorem.

Parallelograms about the diameter of any parallelograms, are similar to the whole, and to each other.

Let ABCD be a ▱, of wh. diam. is AC; & EG, HK ▱s about diam. Then ▱s EG, HK are sim. to ▱ ABCD, & to each other.



∵ DC ∥ GF,

∴ ∠ ADC = ∠ AGF : 29. 1.

simly. ∠ ABC = ∠ AEF :

& ea. of ∠s BCD, EFG = opp. ∠ BAD ; 34. 1.

∴ ∠s BCD, EFG = each other ;

& ∴ □s BD, EG are equiang. :

& ∵ ∠ ABC = ∠ AEF,

& that ∠ BAC is com. to △s BAC, EAF,

∴ △s BAC, EAF are equiang. ;

∴ AB : BC :: AE : EF ; 4. 6.

& ∵ opp. sides of □s = ea. other, 34. 1.

∴ AB : AD :: AE : AG ; 7. 5.

& CD : DA :: FG : GA :

∴ sides of □s BD, EG about equal ∠s are proportionals ;

& ∴ □s BD, EG are sim. to each other : 1 def. 6.

similarly □ BD is sim. to □ KH ;

∴ ea. of □s EG, KH is sim. to □ BD,

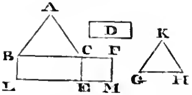
& ∴ □ EG is sim. □ KH. 21. 6.

Wherefore the □s, &c. &c. Q. E. D.

PROP. XXV.—Problem.

To describe a rectilineal figure which shall be similar to one and equal to another given rectilineal figure.

Let ABC be given rectil. fig. to wh. fig. to be descr. is required to be sim. & D that to wh. it must be equal ; required to descr. a rectil. fig. sim. to ABC & = D.



On BC descr. ▱ BE,
so that ▱ BE = fig. ABC ; cor. 45. 1.
& on CE descr. ▱ CM = fig. D,
& having ∠ FCE = ∠ CBL ;
∴ BC, CF are in one rt. line, & also LE, EM. 29. and 14. 1.
Between BC, CF find a mean proportional GH ; 13. 6.
& on GH descr. the rectil. fig. HKG, sim. & simly. situated to rectil. fig. ABC. 18. 6.

Now ∵ BC : GH	: :	GH : CF,	
∴ BC : CF	: :	fig. ABC : KCH ;	2 cor. 20. 6.
but BC : CF	: :	▱ BE : ▱ EF ;	1. 6.
∴ ABC : KGH	: :	BE : EF ;	11. 5.
but ABC	=	BE,	constr.
∴ KGH	=	EF ;	14. 5.
but EF	=	D ;	
∴ KGH	=	D ;	

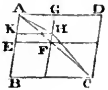
& also KGH is sim. to ABC.

Therefore a rectil. fig. KGH is drawn sim. to a given rectil. fig. ABC, & = given rectil. fig. D. Q. E. F.

PROP. XXVI.—Theorem.

If two similar parallelograms have a common angle, and be similarly situated, they are about the same diameter.

Let ▱s BD, EG be sim. & simly. situated, and have ∠ DAB com.; then ▱s BD, EG are about the same diam.



For, if not, if possible, let ▱ BD have diam. AHC,
but in a diff. direc. from AF, diam. of ▱ EG.
Let GF meet AHC in H;
draw HK ∥ AD or BC;
∴ ▱s BD, GK are about same diam. AHC;
& ∵ ▱s BD, GK are sim. to each other; 24. 6.
∴ DA : AB :: GA : AK: 1 def. 6.
& ∵ ▱s BD, EG are sim. to each other, hyp.
∴ DA : AB :: GA : AE;
& ∴ GA : AE :: GA : AK; 11. 5.
∴ AK = AE;
i. e. less = greater,
wh. is impossible.
∴ ▱s BD, GK are not about same diam.
& ∴ ▱s BD, EG must be about same diam.

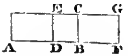
Therefore, if two sim. ▱s, &c. &c. Q. E. D.

'To understand the three following propositions more 'easily, it is to be observed,

1. 'That a parallelogram is said to be applied to a right 'line, when it is described upon it as one of its sides. Ex.

'gr. the parallelogram AC is said to be applied to the 'right line AB.

2. 'But a parallelogram AE is said to be applied to 'a right line AB, deficient by a parallelogram, when AD 'the base of AE is less than AB, and therefore AE is 'less than the parallelogram AC described upon AB in 'the same angle, and between the same parallels, by the 'parallelogram DC; and DC is therefore called the defect 'of AE.

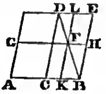


3. 'And a parallelogram AG is said to be applied to a 'right line AB, exceeding by a parallelogram, when AF 'the base of AG is greater than AC, and therefore AG 'exceeds AC, the parallelogram described upon AB in 'the same angle, and between the same parallels, by the 'parallelogram BG.'

PROP. XXVII.—Theorem.

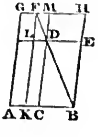
Of all parallelograms applied to the same right line and deficient by parallelograms, similar and similarly situated to that which is described upon the half of the line; that which is applied to the half, and is similar to its defect, is the greatest.

Let AB be a rt. line bisected in C; and let □ AD be applied to the half, AC; wh. is ∴ deficient from □ upon whole lime AB by □ CE upon other half CB. Of all □s applied to any other parts of AB, & deficient by □s that are sim. & simly. situated to CE, AD is greatest.



Let AF be any □ applied to AK, any other part of AB but its half, & so as to be deficient from □ AE by □ KH sim. & simly. situated to □ CE; then AD > AF.

First.—Let AK, base of AF > AC, ½ of AB.
And ∵ □ CE is sim. to □ KH,
∴ they are about same diam.; 26. 6.
draw diam. DB, & complete the diagram.
And ∵ □ CF = □FE, 43. 1.
add to each □ KH;
∴ whole □ CH = whole □ KE;
but □ CH = □ CG, 36. 1.
(for base AC = base CB,)
∴ □ CG = □ KE;
add to each □ CF,
∴ whole □ AF = gnomon CHL;
∴ □ CE or □ AD > □ AF.



Secondly.—Let AK < AC;
& ∵ BC = CA,

∴ HM = MG ; 34. 1.

& ∴ □ DH = □ DG ; 36. 1.

& ∴ □ DH > □ LG ;

now □ DH = □ DK ; 43. 1.

∴ □ DK > □ LG ;

add to ea. □ AL ;

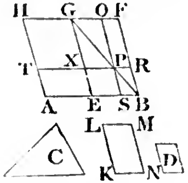
∴ whole □ AD > whole □ AF.

Therefore, of all □s, &c. &c. Q. E. D.

PROP. XXVIII.—Problem.

To a given right line to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram similar to a given parallelogram ; but the given rectilineal figure, to which the parallelogram to be applied, is to be equal, must not be greater than the parallelogram applied to half of the given line, having its defect similar to the defect of that which is to be applied : that is, to the given parallelogram.

Let AB be given rt. line, and C given rectil. fig. wh. must not be > □ applied to ½ of given line, having its defect from that upon whole line sim. to defect of that wh. is to be applied ; & let D be □ to wh. this defect is required to be sim. It is required to apply a □ to AB wh. shall = fig. C, & be deficient from □ upon whole line by a □ sim. to □ D.



Bisect AB in E; 10. 1.

on EB descr. □ EF,

so that EF be sim. & simly. situated to □ D; 18. 6.

complete □ AG.

Now AG must be either = or > C;

& if AG = C,

then that is done wh. was required.

But if □ AG ≠ C,

then □ AG > C:

& □ EF = □ AG; 36. 1.

∴ □ EF > C:

make □ KM = □ EF−C, 25. 6.

so that KM be sim. & simly. situated to □ D;

but □ D is sim. to □ EF,

∴ □ KM is sim. to □ EF: 21. 6.

Let side KL be hom. to EG,

& let LM be hom. to GF:

& ∵ □ EF = C + KM,

∴ □ EF > □ KM;

∴ EG > KL;

& ∴ GF > LM;

make GX = KL,

& GO = LM;

& complete □ XO;

∴ XO is = & sim. to KM;
but ▭ KM is sim. to ▭ EF;
∴ ▭ XO is sim. to ▭ EF;
∴ ▭s XO, EF are about same diam. 26. 6.

Let GPB be their diam., & complete the diagram.

Then ∵ ▭ EF = C + KM,
& part XO = part KM;
∴ rem. gnomon ERO = rem. fig. C:
& ∵ ▭ OR = ▭ XS, 34. 1.
add to each ▭ SR;
∴ whole ▭ OB = whole ▭ XB;
but ▭ XB = ▭ TE, 36. 1.
(for base AE = base EB;)
∴ ▭ TE = ▭ OB;
add to each ▭ XS;
∴ whole ▭ TS = whole gnomon ERO;
but ERO = C;
∴ ▭ TS = C.

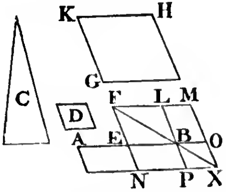
Therefore to rt. line AB a ▭ TS is applied = given rectil. fig. C, & deficient by ▭ SR, sim. to given ▭ D, ∵ SR is sim. to EF.* Q. E. F. * 24. 6.

PROP. XXIX.—Problem.

To a given right line to apply a parallelogram equal to a given rectilineal figure, exceeding by a parallelogram similar to another given.

Let AB be given rt. line, & C given rectil. fig. to wh. ▭ to be applied is required to be equal, & D ▭ to wh.

excess of the one to be applied above that upon AB, is required to be sim. It is required to apply to a given rt. line a ☐ = C, exceeding by a ☐ sim. to D.



Bisect AB in E ;

on EB descr. ☐ EL, sim. & simly. situated to D ;

make ☐ GH = ☐ EL + fig. C, 25. 6.

& also sim. & simly. situated to D ;

∴ ☐ GH is sim. to ☐ EL. 21. 6.

Let side KH be hom. to FL ;

& KG be hom. to FE.

& ∵ ☐ GH > ☐ EL ;

∴ KH > FL,

& KG > FE ;

prod. FL & FE ;

& make FLM $\stackrel{\angle}{=}$ KH ;

& FEN = KG ;

& complete ☐ MN ;

∴ ☐ MN = & sim. to ☐ GH ;

but ☐ GH is sim. to ☐ EL ;

∴ ☐ MN is sim. to ☐ EL :

& ∴ EL & MN are about same diam. 26. 6.

draw their diam. FX, & complete the diagram.

& since ☐ GH = ☐ EL + C ;

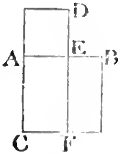
& that GH = MN ;
∴ MN = EL + C ;
take away com. □ EL ;
∴ rem. gnomon NOL = rem. fig. C :
& ∵ AE = EB ;
∴ □ AN = □ NB, i.e. BM ; 36. & 43. 1.
add to each □ NO ;
∴ whole □ AX = gnomon NOL ;
but NOL = fig. C,
∴ □ AX = fig. C.

Therefore to rt. line AB is applied a □ AX = rectil. fig. C, & exeeeding by □ PO sim. to □ D, for PO is sim. to EL.* Q. E. F. * 24. 6.

PROP. XX.—Problem.

To cut a given right line in extreme and mean ratio.

Let AB be given rt. line ; it is required to cut it in extreme and mean ratio.



On AB descr. sq. BC ; 46. 1.
to AC apply a □ CD = sq. BC,
& exceeding by a fig. AD sim. to fig. BC. } 29. 6.

But BC is a sq.;
∴ AD is a sq.;
& ∵ sq. BC = □ CD, constr.
take from each com. □ CE,
∴ rem. □ BF = rem. □ AD;
& □s BF, AD are equiang.;
∴ sides about equal ∠s are recip. proport.
i. e. FE : ED :: AE : EB.
Now FE = AC, i. e. AB, 34. 1.
& ED = AE;
∴ BA : AE :: AE : EB;
but AB > AE,
∴ AE > EB; 14. 5.
∴ AB is cut in extreme & mean ratio in C. 3 def. 6.

Q. E. F.

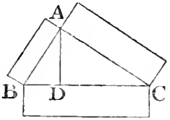
Otherwise;
divide AB in C,
so that AB · BC = AC^2. 11. 2.
Then ∵ AB · BC = AC^2,
∴ BA : AC :: AC : CB : 17. 6.
∴ AB is cut in extreme & mean ratio in C. 3 def. 6.

Q. E. F.

PROP. XXXI.—Theorem.

In right angled triangles, the rectilineal figure described upon the side opposite to the right angle, is equal to the *similar and similarly described figures upon the sides containing the right angle.*

Let ABC be a rt. ∠d △, having rt. ∠ BAC ; the rectil. fig. descr. upon BC is = sim. & simly. descr. figs. upon BA, AC.



Draw AD ⊥ BC.

Then, ∵ in △ ABC, AD is drawn from rt. ∠ A ⊥ base BC,
∴ △s ABD, ADC are sim. to △ ABC, & to ea. other : [8. 6.

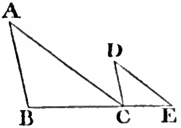
& ∵ △ ABC is sim. to △ ADB,
∴ CB : BA : : BA : BD ; 4. 6.
& ∴ CB : BD : : fig. descr. on CB : sim. & simly. descr. fig. on BA ; 2 cor. 20. 6.
& ∴ invertendo DB : BC : : fig. on BA : fig. on BC ; B.5.
similarly DC : CB : : fig. on CA : fig. on CB :
∴ BD + DC : BC : : figs. on BA & AC : fig. on [BC ; 24. 1.
but BD + DC = BC ;
∴ fig. descr. on BC = sim. & simly. descr. figs. on BA, AC.

Wherefore, in rt. ∠d △s, &c. &c. Q. E. D.

PROP. XXXII.—Theorem.

If two triangles which have two sides of the one proportional to two sides of the other, be joined at one angle so as to have their homologous sides parallel to each other; the remaining sides shall be in a right line.

Let ABC, DCE be two $\triangle$s wh. have two sides BA, AC proport. to two CD, DE, i. e. BA : AC : : CD : DE; & let AB $\parallel$ CD, & AC $\parallel$ DE. Then BC, CE are in a rt. line.



$\because$ AC falls on $\parallel$s AB, DC;

$\therefore \angle$ BAC $=$ $\angle$ ACD: 29. 1.

similarly $\angle$ CDE $=$ $\angle$ ACD;

& $\therefore \angle$ BAC $=$ $\angle$ CDE;

& $\because$ in $\triangle$ ABC, $\angle$ at A $=$ $\angle$ D in $\triangle$ DCE,

and that sides about these equal $\angle$s are proportionals,

i. e. BA : AC : : CD : DE;

$\therefore \triangle$ ABC is equiang. to $\triangle$ DCE; 6. 6.

& $\therefore \angle$ ABC $=$ $\angle$ DCE:

now $\angle$ BAC $=$ $\angle$ ACD; demon.

$\therefore$ whole $\angle$ ACE $=$ $\angle$s ABC + BAC;

add com. $\angle$ ACB to each,

$\therefore \angle$s ACE+ACB $=$ $\angle$s AB+BAC+ACB;

but $\angle$s ABC+BAC +ACB $=$ 2 rt. $\angle$s; 32. 1.

∵ ∠s ACE + ACB = 2 rt. ∠s :
∴ BC, CE are in one rt. line. 14. 1.

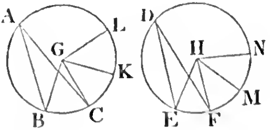
Therefore, if two △s, &c. &c. Q. E. D.

PROP. XXXIII.—Theorem.

In equal circles, angles, whether at the centres or circumferences, have the same ratios which the arcs on which they stand have to each other: so also have the sectors.

Let ABC, DEF be equal ⊙s; and let ∠s BGC, EHF be angles at their crs., & ∠s BAC, EDF, be ∠s at their ○s ; then

First.—$\overset{\frown}{BC} : \overset{\frown}{EF}$: : ∠BGC : ∠EHF : : ∠BAC : ∠EDF.



Take any number of arcs.

viz. { $\overset{\frown}{CK}$, $\overset{\frown}{KL}$, ea. = $\overset{\frown}{BC}$,
& $\overset{\frown}{FM}$, $\overset{\frown}{MN}$, ea. = $\overset{\frown}{EF}$;

join GK, GL ; HM, HN.

And ∵ $\overset{\frown}{BC}$, $\overset{\frown}{CK}$, $\overset{\frown}{KL}$ = ea. other,
∴ ∠s BGC, CGK, KGL = ea. other ; 27. 3.
& ∴ ∠ BGL is same mult. of ∠ BGC that $\overset{\frown}{BL}$ is of $\overset{\frown}{BC}$:

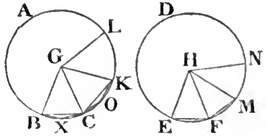
simly. ∠ EHN is same mult. of ∠ EHF that $\widehat{EN}$ is of $\widehat{EF}$:

& if $\widehat{BL}$ = $\widehat{EN}$,
then ∠ BGL = ∠ EHN ;
& if greater, greater ; if less, less.

Now ∵ there are four mags. $\widehat{BC}$, $\widehat{EF}$, ∠ BGC & ∠ EHF, & that of $\widehat{BC}$ & ∠ BGC are taken any equimults. $\widehat{BL}$ & ∠ BGL,
& also, of $\widehat{EF}$ & ∠ EHF are taken any equimults. $\widehat{EN}$ & ∠ EHN ;

& that if $\widehat{BL}$ > $\widehat{EN}$,
then ∠ BGL > ∠ EHN ;
and if equal, equal ; if less, less.

∴ $\widehat{BC}$: $\widehat{EF}$: : ∠ BGC : ∠ EHF ; 5 def.5.
but ∠ BGC : ∠ EHF : : ∠ BAC : ∠ EDF, 15. 6.
(for ea. is double of ea.) 20. 3.
∴ $\widehat{BC}$: $\widehat{EF}$: : ∠ BAC : ∠ EDF.



SECONDLY.—Also $\widehat{BC}$: $\widehat{EF}$: : sec. BGC : sec. EHF.

Join BC, CK ;
in $\widehat{BC}$, $\widehat{CK}$ take any points X, O ;
join BX, XC, CO, OK.

Then ∵ in △ GBC, BG, GC = CG, GK, in △ GCK,
& that ∠ BGC = ∠ CGK ;

∴ base BC = base CK, } 4. 1.
& △ GBC = △ GCK ; }
& ∵ $\overset{\frown}{BC}$ = $\overset{\frown}{CK}$,
rem. ○ BALC = rem. ○ CBLK ;
∴ ∠ BXC = ∠ COK ; 27. 3.
& ∴ seg. BXC is sim. to seg. COK : 11 def. 3.
& ∵ they are on equal rt. lines,
∴ seg. BXC = seg. COK ; 24. 3.
& △ BGC = △ CGK,
∴ whole sec. BGC = whole sec. CGK ;
& simly. sec. KGL = ea. of sec. BGC, CGK :
& simly. it may be proved, that sec. EHF, FHM, MHN = ea. other.

∴ sec. BGL is same mult. of sec. BGC that $\overset{\frown}{BL}$ is of $\overset{\frown}{BC}$;
& sec. EHN is same mult. of sec. EHF that $\overset{\frown}{EN}$ is of $\overset{\frown}{EF}$;
& if $\overset{\frown}{BL}$ = $\overset{\frown}{EN}$,
then sec. BGL = sec. EHN :
if greater, greater ; if less, less.

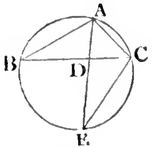
Now, ∵ there are four mags. $\overset{\frown}{BC}$, $\overset{\frown}{EF}$, sec. BGC, & EHF ;
& that of $\overset{\frown}{BC}$ & BGC are taken equimults. $\overset{\frown}{BL}$, BGL,
also of $\overset{\frown}{EF}$ & EHF are taken any equimults. $\overset{\frown}{EN}$, EHN ;
& that if $\overset{\frown}{BL}$ > $\overset{\frown}{EN}$,
then sec. BGL > sec. EHN :
if equal, equal ; & if less, less ;
∴ $\overset{\frown}{BC}$: $\overset{\frown}{EF}$: : sec. BGC : sec. EHF.

Wherefore, in equal ⊙s, &c. &c. Q. E. D.

PROP. B.—Theorem.

If an angle of a triangle be bisected by a right line, which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the right line bisecting the angle.

Let ABC be a △, & let ∠ BAC be bisected by rt. line AD : then BA · AC = BD · DC + AD².



About △ ABC descr. ⊙ ACB ; 5. 4.
prod. AD to E in ○ :
join EC.

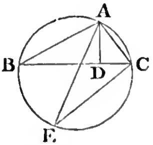
Then, ∵ ∠ BAD	=	∠ CAE,	
& that ∠ ABD	=	∠ AEC,	21. 3.
(for they are in same seg. ;)			
∴ △s ABD, AEC are equiang. to ea. other :			
∴ BA : AD	: :	EA : AC ;	4. 6.
& ∴ BA · AC	=	EA · AD ;	16. 6.
	=	ED · DA + AD² ;	3. 2.
	=	BD · DC + AD².	

Wherefore, if an angle, &c. &c. Q. E. D.

PROP. C.—Theorem.

If from any angle of a triangle a right line be drawn perpendicular to the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

Let ABC be a △, & AD ⊥ BC; then BA · AC = AD · diam. of ⊙ descr. about △.



About △ ABC descr. ⊙ ACB; 5. 4.
draw diam. AE:
join EC.

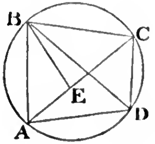
Then, ∵ rt. ∠ BDA = rt. ∠ ECA in a $\frac{1}{2}$ ⊙, 31. 3.
& ∠ ABD = ∠ AEC in same seg. 21. 3.
∴ △s ABD, AEC are equiang.
∴ BA : AD :: EA : AC; 4. 6.
& ∴ BA · AC = EA · AD.

Therefore, if from any angle, &c. &c. Q. E. D.

PROP. D.—Theorem.

The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle, is equal to both the rectangles together, contained by its opposite sides.

Let ABCD be any quadrilat. inscr. in a $\odot$, & draw AC, BD its diags. ; then $AC \cdot BD = AB \cdot CD + AD \cdot BC$.



Make $\angle$ ABE $=$ $\angle$ DBC ;

add to ea. com. $\angle$ FBD ;

$\therefore$ $\angle$ ABD $=$ $\angle$ EBC :

& $\angle$ BDA $=$ $\angle$ BCE in same seg. 21. 3.

$\therefore$ $\triangle$s ABD, BCE are equiang.

$\therefore$ BC : CE : : BD : DA ; 4. 6.

& $\therefore$ $BC \cdot AD$: : $BD \cdot CE$. 16. 6.

Again, $\because$ $\angle$ ABE $=$ $\angle$ DBC.

& $\angle$ BAE $=$ $\angle$ BDC ; 21. 3.

$\therefore$ $\angle$s ABE, BCD are equiang.

$\therefore$ BA : AE : : BD : DC ;

& $\therefore$ $BA \cdot DC = BD \cdot AE$;

but $BC \cdot AD = BC \cdot CE$;

$\therefore$ whole $AC \cdot BD = AB \cdot CD + AD \cdot BC$.

Wherefore the rectangle, &c. &c. Q. E. D.

BOOK XI.

DEFINITIONS.

I.

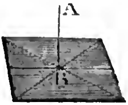
A SOLID is that which hath length, breadth, and thickness.

II.

That which bounds a solid is a superficies.

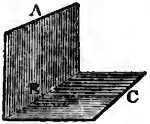
III.

A right line is perpendicular, or at right angles, to a plane, when it makes right angles with every right line which meets it in that plane.



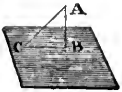
IV.

A plane is perpendicular to a plane, when the right lines drawn in one of the planes perpendicular to the common section of the two planes, are perpendicular to the other planes.



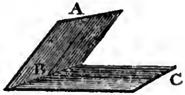
V.

The inclination of a right line to a plane, is the acute angle contained by that right line, and another drawn from the point in which the first line meets the plane, to the point in which a perpendicular to the plane drawn from any point of the first line above the plane, meets the same plane.



VI.

The inclination of a plane to a plane is the acute angle contained by two right lines drawn from any the same point of their common section at right angles to it, one upon one plane, and the other upon the other plane.



VII.

Two planes are said to have the same or a like inclination to each other which two other planes have, when the said angles of inclination are equal to each other.

VIII.

Parallel planes are such as do not meet each other though produced.

IX.

A solid angle is that which is made by the meeting of more than two plain angles, which are not in the same plane, in one point.

X.

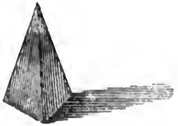
Equal and similar solid figures are such as are contained under an equal number of equal and similar planes.*

XI.

Similar solid figures are such as have all their solid angles equal, each to each, and are contained by the same number of similar planes.

XII.

A pyramid is a solid figure contained by planes that are constituted betwixt one plane and one point above it in which they meet.

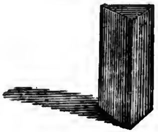


* Dr. Simson has omitted this definition altogether. He says that it is properly a theorem, and requires demonstration; and therefore accuses Theon of the interpolation.

That figures are similar, he observes, ought to be proved from the

XIII.

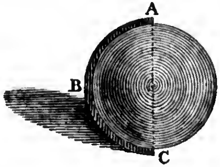
A prism is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to each other ; and the others are parallelograms.



XIV.

A sphere is a solid figure described by the revolution of a semicircle about its diameter, which remains unmoved.

Thus the inner side of the semicircle ABC revolving round the diameter AC, which remains fixed, generates a sphere.



XV.

The axis of a sphere is the fixed right line about which the semicircle revolves.

Thus AC, in the figure above, is the axis of the sphere.

definitions of similar figures ; and that they are equal, ought to be demonstrated from the axiom, " Magnitudes that wholly coincide, are equal ;" or from Props. A or 9th or 14th of 5th Book, from one of which the equality of all kinds of figures must be ultimately deduced.

XVI.

The centre of a sphere is the same with that of the semicircle.

XVII.

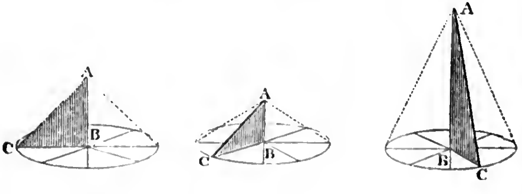
The diameter of a sphere is any right line which passes through the centre, and is terminated both ways by the superficies of the sphere.

XVIII.

A cone is a solid figure described by the revolution of a right angled triangle about one of the sides containing the right angle, which side remains fixed.

If the fixed side be equal to the other side containing the right angle, the cone is called a right angled cone; if it be less than the other side, an obtuse angled; and if greater, an acute angled cone.

Thus the side AC, revolving round AB fixed, one of the sides which contains the right angle, generates a cone.



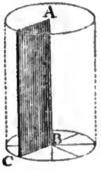
XIX.

The axis of a cone is the fixed right line about which the triangle revolves.

In fig. above, AB is the axis.

XX.

The base of a cone is the circle described by that side containing the right angle which revolves.



XXI.

A cylinder is a solid figure described by the revolution of a right angled parallelogram about one of its sides which remains fixed.

Thus the revolution of the parallelogram AC about its side AB, which remains fixed, generates a cylinder.

XXII.

The axis of a cylinder is the fixed right line about which the parallelogram revolves.

XXIII.

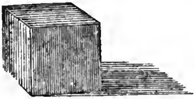
The bases of a cylinder are the circles described by the two revolving opposite sides of the parallelogram.

XXIV.

Similar cones and cylinders are those which have their axes and the diameters of their bases proportionals.

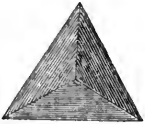
XXV.

A cube is a solid figure contained by six equal squares.



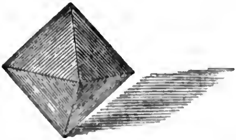
XXVI.

A tetrahedron is a solid figure contained by four equal and equilateral triangles.



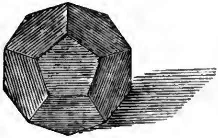
XXVII.

An octahedron is a solid figure contained by eight equal and equilateral triangles.



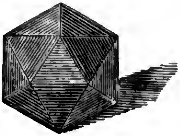
XXVIII.

A dodecahedron is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.



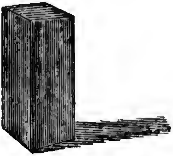
XXIX.

An icosahedron is a solid figure contained by twenty equal and equilateral triangles.



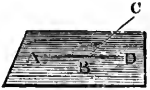
Def. A.

A parallelopiped is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel.



PROP. I.—Theorem.

One part of a right line cannot be in a plane and another part above it.



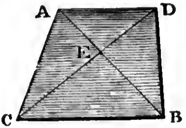
If possible,
let AB, part of rt. line ABC, be in a pl.,
& part BC above pl.
And ∵ AB is in a pl.
it can be prod. in that pl.
Let AB be prod. to D.
And let any pl. pass through AD, and be turned about it
till it pass through point C.
Then ∵ pts. B, C, are both in same pl. ;
∴ rt. line BC is in it. 7 def. 1.
∴ there are two rt. lines ABC, ABD, in same pl.
wh. have com. seg. AB ;
wh. is impossible. cor. 11. 1.

Therefore one part, &c. &c. Q. E. D.

PROP. II.—Theorem.

Two right lines which cut each other are in one plane, and three right lines which meet each other are in one plane.

Let two rt. lines AB, CD, cut ea. other in E; AB, CD are in one pl.: & the three rt. lines EC, CB, BE, wh. meet ea. other, are in one pl.



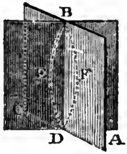
Let any pl. pass through EB;
& let it be turned about EB;
& prod. if necessary, until it pass through C.
Then ∴ E, C, are in same pl.
∴ rt. line EC is in the pl. 7 def. 11.
Similarly BC is in same pl.
but by hypoth. EB is in same pl.
∴ EC, CB, BE are in one pl.
Now CD, AB, are in same pl. with EC, EB. 1. 11.
∴ AB, CD, are in one pl.

Wherefore, rt. lines, &c. &c. Q. E. D.

PROP. III.—Theorem.

If two planes cut each other, their common section is a right line.

Let pl. AB cut pl. BC; and let DB be their com. sec. then DB is a rt. line.



If not;

draw rt. lines { DEB in pl. AB ; DFB in pl. BC ;

consequently DEB, DFB have same extrems.

& ∴ rt. lines DEB, DFB inclose a space ;

wh. is impossible. 10 ax. 1.

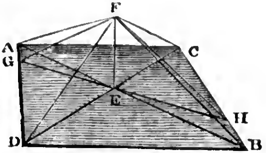
∴ BD com. sec. of pls. AB, BC is a rt. line.

Wherefore, if two pls., &c. &c. Q. E. D.

PROP. IV.—Theorem.

If a right line stand at right angles to each of two right lines in the point of their intersection, it shall also be at right angles to the plane which passes through them, that is, to the plane in which they are.

Let rt. line EF be ⊥ each of rt. lines AB, CD, in E the point of their intersec. ; EF is also ⊥ pl. passing through AB, CD.



Take rt. lines AE, EB, CE, ED, = ea. other :
join AD, CB ;
draw GEH in pl. in wh. are AB, DC ;
take any point F in EF ; join FA, FG, FD, FB, FH, FC.

And ∵ AE, ED	=	BE, EC, ea. to ea.	
& that ∠ AED	=	∠ BEC ;	15 1.
∴ base AD	=	base BC,	4. 1.
& ∠ DAE	=	∠ EBC :	4. 1.
& ∠ AEG	=	∠ BEH,	15. 1.
∴ in △ AEG, ∠s GAE, AEG	=	∠s EBH, HEB in △ BEH ;	

also sides adj. to equal ∠s are = ea. other.

i. e. AE	=	EB ;	
& ∴ also GE	=	EH,	26. 1.
& AG	=	BH,	26. 1.
& ∵ AE	=	EB,	

& that EF is com. & ⊥ them,

∴ base AF	=	base FB :	4. 1.
similarly CF	=	FD ;	
& ∵ AD	=	BC,	
& AF	=	FB,	
& base DF	=	base FC ;	
∴ ∠ FAD	=	∠ FBC.	8. 1.

Again, ∵ { GA = BH, demon.
AF = FB,
& ∠ FAG = ∠ FBH,

∴ base FG = base FH. 4. 1.

Again, ∵ { GE = EH, demon.
EF com. to both,
& base GF = base FH,

∴ ∠ GEF = ∠ HEF :

& these are adj. ∠s ;

∴ ea. of ∠s GEF, HEF, is a rt. ∠ : 10 def. 1.

∴ FE makes rt. ∠s with GH ;

i. e. FE makes rt. ∠s with any rt. line drawn through E, in pl. passing through AB, CD.

In same manner it may be proved, that FE makes rt. ∠s with every rt. line wh. meets it in that pl. Now a rt. line is ⊥ a pl. when it makes rt. ∠s with every rt. line which meets it in that pl.*

* 3 def. 11.

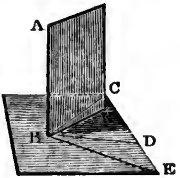
∴ EF is ⊥ pl. passing through AB, CD.

Wherefore if a rt. line, &c. &c. Q. E. D.

PROP. V.—Theorem.

If three right lines meet all in one point, and a right line stand at right angles to each of them in that point: these three right lines are in one and the same plane.

Let rt. line AB be ⊥ ea. of rt. lines BC, BD, BE, in B the point where they meet. BC, DB, BE, are in same pl.



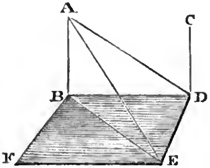
If not, if possible,
let BD, BE be in one pl.
& BC be above it ;
& let a pl. pass through AB, BC :
then the sec. of this pl. with pl. passing through BD, DE,
is a rt. line. 3 11.
let this rt. line be BF ;
∴ AB, BC, BF, are in one pl.
viz. in that wh. passes through AB, BC.
Now ∵ AB is ⊥ BD & BE,
∴ AB is ⊥ pl. passing through BD, BE ; 4. 11.
& ∴ AB is ⊥ every rt. line meeting it in that pl. ; 3 def. 11.
now BF, wh. is in that pl. meets AB,
∴ ∠ ABF is a rt. ∠ ;
but ∠ ABC is a rt. ∠, hyp.
∴ ∠ ABF = ∠ ABC ;
& they are both in same pl.
wh. is impossible ;
∴ BC is not above pl. in wh. are BD, BE ;
i. e. BC, BD, BE, are in same pl.

Wherefore, if three rt. lines, &c. &c. Q. E. D.

PROP. VI.—Theorem.

If two right lines be at right angles to the same plane, they shall be parallel to each other.

Let rt. lines AB, CD, be ⊥ same pl. FD; then is AB ∥ CD.



Let AB, CD, meet pl. in B, D;
join BD;
draw DE ⊥ BD in pl. FD;
make DE = AB;
join BE, AE, AD.
Then ∵ AB ⊥ pl. FD,
∴ AB is ⊥ every rt. line wh. meets it in FD; 3 def. 11.
now BD, BE, wh. are in FD, meet AB;
∴ ea. of ∠s ABD, ABE is a rt. ∠:
& simly. ea. of ∠s CDB, CDE is a rt. ∠.
And ∵ AB = DE,
& BD is com. to △s ABD, BDE,
& that rt. ∠ ABD = rt. ∠ BDE;
∴ base AD = base BE. 4. 1.
Again, ∵ AB = DE,
& that BE = AD,

& base AE is com. to $\triangle$s ABE, EDA ;

$\therefore \angle$ ABE $=$ $\angle$ EDA ; 8. 1.

but $\angle$ ABE is a rt. $\angle$,

$\therefore \angle$ EDA is a rt. $\angle$;

& $\therefore$ ED $\perp$ DA

but also ED $\perp$ BD & DC ;

$\therefore$ ED is $\perp$ ea. of BD, DA, DC, in pt. where they meet ;

$\therefore$ BD, DA, DC, are in one pl. BC : 5. 11.

now AB is in same pl. with BD, DA ;

(for three rt. lines meeting ea. other are in one pl.) 2. 11.

$\therefore$ AB, BD, DC are in one pl.

& ea. of $\angle$s ABD, BDC, is a rt. $\angle$;

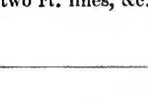
$\therefore$ AB $\parallel$ CD.

Wherefore, if two rt. lines, &c. &c. Q. E. D.

PROP. VII.—Theorem.

If two right lines be parallel, the right line drawn from any point in the one to any point in the other, is in the same plane with the parallels.

Let AB, CD, be parallel rt. lines, & take any points, E in AB and F in CD. The rt. line wh. joins E & F is in same pl. AD with parallels.



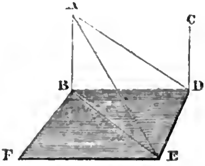
If not, if possible,
let it be above pl. AD, as EGF :
& in pl. AD draw EHF from E to F :
& ∵ EGF is also a rt. line ;
∴ EGF, EHF, include a space ;
wh. is impossible : 10 ax. 1.
∴ rt. line joining points E, F, is not above pl. AD ;
i. e. it is in same pl. with AB, CD.

Wherefore, if two rt. lines, &c. &c. Q. E. D.

PROP. VIII.—Theorem.

If two right lines be parallel, and one of them is at right angles to a plane ; the other also shall be at right angles to the same plane.

Let AB, CD, be parallel rt. lines, & let AB be ⊥ pl. FD ; then CD is ⊥ same pl.



Let AB, CD, meet pl. FD in B, D ;
join BD ;
∴ AB, CD, BD, are in one pl. BC : 7. 11.
in pl. FD, draw DE ⊥ BD ;

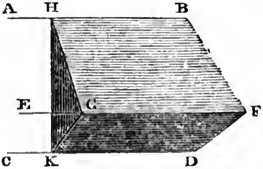
make DE $=$ AB ;
join BE, AE, AD :
then $\because$ AB $\perp$ pl. FD,
$\therefore$ AB $\perp$ BD, BE ; 3 def. 11.
$\therefore$ ea. of $\angle$s ABD, ABE is a rt. $\angle$;
& $\because$ BD meets $\parallel$s AB, CD ;
$\therefore$ $\angle$s ABD + CDB $=$ 2 rt. $\angle$s ; 29. 1.
but $\angle$ ABD is a rt. $\angle$,
$\therefore$ $\angle$ CDB is a rt. $\angle$;
& $\therefore$ CD $\perp$ BD :
& $\because$ AB $=$ DE,
& BD com. to both $\triangle$s ABD, EDB,
& that rt. $\angle$ ABD $=$ rt. $\angle$ EDB ;
$\therefore$ base AD $=$ base BE. 4. 1.
Again, $\because$ AB $=$ DE,
& BE $=$ AD, 8. 1.
& that base AE is com. to $\triangle$s ABE, EDA ;
$\therefore$ $\angle$ ABE $=$ $\angle$ EDA ;
but $\angle$ ABE is a rt. $\angle$,
$\therefore$ $\angle$ EDA is a rt. $\angle$;
& $\therefore$ ED $\perp$ DA ;
but also ED $\perp$ BD ;
$\therefore$ ED is $\perp$ pl. BC passing through BD, DA : 4. 11.
now DC is also in pl. BC,
(for all these are in pl. passing through $\parallel$s AB, CD,)
$\therefore$ ED is $\perp$ DC, 3 def. 11.
but also CD is $\perp$ DB,
$\therefore$ CD is $\perp$ DE & DB, in pt. of intersec. D ;
& $\therefore$ CD is $\perp$ pl. passing through DE, DB ;
i. e. CD is $\perp$ pl. FD, to wh. AB is $\perp$.

Wherefore, if two rt. lines, &c. &c. Q. E. D.

PROP. IX.—Theorem.

Two right lines which are each of them parallel to the same right line, and nat in the same plane with it, are parallel to each ather.

Let AB, CD, be ea. || EF, & not in same pl. with it; AB shall be || CD.



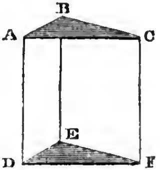
In EF take any point G;
In pl. EB, passing through AB, EF, draw GH ⊥ EF;
& in pl. ED passing through EF, CD,
draw GK ⊥ EF:
& ∵ EF ⊥ GH, GK:
∴ EF ⊥ pl. HGK through GH, GK: 11.
Now EF || AB,
∴ AB ⊥ pl. HGK: 8. 11.
& simly. CD ⊥ pl. HGK;
∴ AB & CD are ea. ⊥ pl. HGK;
∴ AB || CD. 6. 11.

Wherefore, two rt. lines, &c. &c. Q. E. D.

PROP. X.—Theorem.

If two right lines meeting each other be parallel to two others which meet each other, and are not in the same plane with the first two; the first two and the other two shall contain equal angles.

Let two rt. lines AB, BC, wh. meet ea. other, be || two DE, EF, wh. meet ea. other, & are not in same pl. with AB, BC; then ∠ ABC = ∠ DEF.



Take AB, BC, DE, EF = ea. other;
join AD, BE, CF, AC, DF.

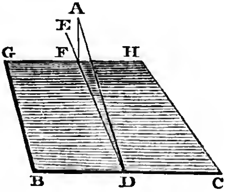
Then ∵ AB	=	& || DE,	
∴ AD	=	& || BE;	33. 1.
Simly. CF	=	& || BE;	
& ∴ AD	=	& || CF:	9. 11. & 1 ax. 1.
∴ AC	=	& || DF:	33. 1.
& ∵ AB, BC	=	DE, EF ea. to ea.	
& base AC	=	base DF;	
∴ ∠ ABC	=	∠ DEF.	8. 1.

Therefore if two rt. lines, &c. &c. Q. E. D.

PROP. XI.—Problem.

To draw a right line perpendicular to a plane, from a given point above it.

Let A be given point above pl. BH ; it is required to draw from A a rt. line ⊥ pl. BH.



In pl. BH draw any rt. line BC ;
draw AD ⊥ BC ;
then, if AD ⊥ pl. BH,
the thing required is done.
But if not ;
in pl. BH, draw DE ⊥ BC ;
draw AF ⊥ DE ;
& through F draw GH ∥ BC.
& ∵ BC is ⊥ ED, & DA ;
∴ BC is ⊥ pl. passing through ED, DA : 4. 11.
& ∴ GH ∥ BC,
∴ GH is ⊥ pl. passing through ED, DA : 8. 11.
& ∵ AF, in pl. with ED, DA, meets GH,
∴ GH ⊥ AF ; 3 def. 11.
but AF ⊥ DE ;

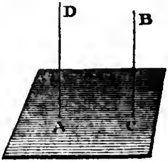
∴ AF ⊥ GH, & DE, in pt. of inters. F ;
∴ AF is ⊥ pl. passing through GH, DE : 4. 11.
now BH is that pl.
∴ AF ⊥ pl. BH.

Therefore, from point A, a rt. line AF is drawn ⊥ pl. BH.
Q. E. F.

PROP. XII.—Problem.

To erect a right line at right angles to a given plane, from a point given in the plane.

Let A be given point in pl.; it is required to erect a rt. line from A ⊥ same pl.



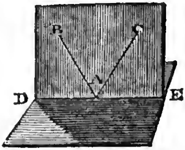
From any point B above pl. }
draw BC ⊥ pl.; } 11. 11.
Draw AD || BC.
Then ∵ AD, CB are two parallel rt. lines
& that one BC ⊥ given pl.;
∴ AD is ⊥ same pl. 8. 11.

Therefore, a rt. line AD has been erected from given point A, in given pl. ⊥ that pl. Q. E. F.

PROP. XIII.—Theorem.

From the same point in a given plane, there cannot be two right lines at right angles to the plane, upon the same side of it: and there can be but one perpendicular to a plane from a point above the plane.

For, if possible, let AC, AB, be ea. ⊥ given pl. from one point A in same pl. & on same side of it.



Let a pl. pass through BA, AC;
then com. sec. of two pls. is a rt. line. 3. 11.
Let DAE be their com. sec.;
∴ AB, AC, DAE are in one pl.:
& ∵ AC is ⊥ given pl.,
& that rt. line DAE meets AC in that pl.;
∴ ∠ CAE is a rt. ∠: 3 def. 11.
simly. ∠ BAE is a rt. ∠;
∴ ∠ CAE = ∠ BAE;
& they are in one pl.;
wh. is impossible.

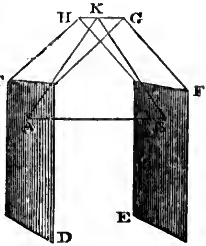
Also from a point above a pl. there can be but one perpend. to that pl.; for if there could be two, they would be || ea. other:* * 6. 11.
wh. is absurd.

Therefore, from same point, &c. &c. Q. E. D.

PROP. XIV.—Theorem.

Planes to which the same right line is perpendicular, are parallel to each other.

Let rt. line AB be ⊥ ea. of pls. CD, EF; then pls. are || ea. other.



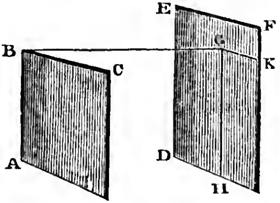
If not;
they shall meet when prod.,
& their sec. shall be a rt. line GH;
in GH take any point K;
join AK, BK.
Then ∵ AB ⊥ pl. EF;
∴ AB ⊥ rt. line BK in that pl.; 3 def. 11.
& ∴ ∠ ABK is a rt. ∠:
simly. ∠ BAK is a rt. ∠,
∴ two ∠s ABK, BAK of △ ABK = 2 rt. ∠s;
wh. is impossible: 17. 1.
∴ pls. CD, EF, being prod. do not meet;
i.e. pls. CD, EF, || ea. other.

Wherefore pls., &c. &c. Q. E. D.

PROP. XV.—THEOREM.

If two right lines meeting each other, be parallel to the other lines which meet, but are not in the same plane with the first two; the plane which passes through these is parallel to the plane passing through the others.

Let AB, BC, two rt. lines meeting ea. other, be || DE, EF wh. meet ea. other, but are not in same pl. with AB BC. Then pls. wh. pass through AB, BC, & DE, EF, shall not meet, though prod.



Draw BG ⊥ pl. DF passing thro' DE, EF;
& let BG meet DF in G;

draw { GH || ED,
GK || EF:

& ∵ BG ⊥ pl. DF,
& that GH, GK, meet BG in that pl.
∴ BG is ⊥ GH, GK; 3 def. 11.
∴ ea. of ∠s BGH, BGK is a rt. ∠.
& ∵ BA || GH, 9. 11.
(for ea. of them is || DE & not in same pl. with it),

$\therefore \angle$s GBA + BGH = 2 rt. $\angle$s: 29. 1.

now $\angle$ BGH is a rt. $\angle$,

$\therefore \angle$ GBA is a rt. $\angle$;

& $\therefore$ GB $\perp$ BA:

simly. GB $\perp$ BC:

& $\therefore$ GB is $\perp$ rt. lines BA, BC in point of intersec. B;

$\therefore$ GB $\perp$ pl. AC; 4. 11.

but also GB $\perp$ pl. DF;

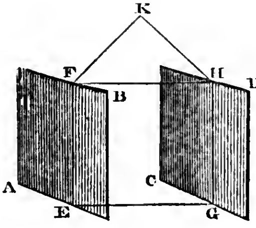
$\therefore$ pl. passing thro' AB, BC $\parallel$ pl. passing through DE, EF. 14. 11.

Wherefore if two rt. lines, &c. &c. Q. E. D.

PROP. XVI.—Theorem.

If two parallel planes be cut by another plane, their common sections with it are parallels.

Let two parallel pls. AB, CD be cut by pl. EH; & let their secs. with it be rt. lines EF, GH: then EF $\parallel$ GH.



For if EF be not || GH,
then they will meet if prod. either on side of FH or EG.

First.—Let EF, GH meet on side of FH, in K.
And ∵ rt. line EFK is in pl. AB,
∴ every point in EFK is in that pl.;
but K is a point in EFK,
∴ K is in pl. AB;
simly. K is in pl. CD;
∴ pls. AB, CD if prod. would meet ea. other;
but pl. AB || pl. CD, hyp.
∴ AB, CD do not meet ea. other;
∴ EF, GH do not meet if prod. on side of FH.

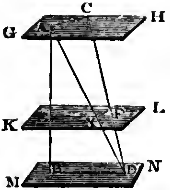
Secondly.—In same manner it may be demonstr.
that EF, GH do not meet if prod. on side of EG;
∴ EF || GH, 35 def. 1.

Wherefore, if two parallel pls., &c. &c. Q. E. D.

PROP. XVII.—Theorem.

If two right lines be cut by parallel planes, they shall be cut in the same ratio.

Let rt. lines AB, CD be cut by parallel pls. GH, KL, MN, in points A, E, B; C, F, D: then AE : EB : : CF : FD.



Join AC, BD, AD ;
& let AD meet pl. KL in X ;
join EX, XF :

∴ parallel pls. KL, MN are cut by pl. BX,
∴ com. secs. BD, EX are || ea. other ; 16. 11.
Again ∵ parallel pls. KL, GH, are cut by pl. CX,
∴ com. secs. AC, XF are || ea. other.

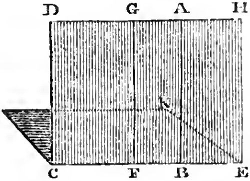
Now ∵ EX	||	BD a side of △ ABD,	
∴ AE : ED	: :	AX : XD.	2. 6.
Again ∵ XF	||	AC a side of △ ADC,	
∴ AX : XD	: :	CF : FD :	
but AX : XD	: :	AE : EB,	demon.
∴ AE : EB	: :	CF : FD.	11. 5.

Wherefore, if two rt. lines, &c. &c. Q. E. D.

PROP. XVIII.—Theorem.

If a right line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane.

Let rt. line AB be ⊥ pl. CK ; then every pl. wh. passes through AB shall be ⊥ pl. CK.



Let any pl. DE pass through AB ;
& let rt. line CE be sec. of pls. CK, DE ;
take any point F, in CE ;
draw FG, in pl. DE, ⊥ CE.
And ∵ AB ⊥ pl. CK ;
∴ AB ⊥ CE ; 3 def. 11.
& ∴ ∠ ABF is a rt. ∠ ;
but ∠ GFB is a rt. ∠,
∴ AB ∥ FG ; 28. 1.
but AB is ⊥ pl. CK ;
∴ FG is ⊥ pl. CK : 8. 11.
Now ∵ in pl. DE, FG ⊥ pl. CK,
& that it is also ⊥ CE com. sec. ; constr.
∴ pl. DE is ⊥ pl. CK ; 4 def. 11.

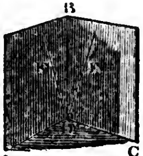
simly. it may be demonstr. that all pls. passing through AB are ⊥ pl. CK.

Wherefore, if a rt. line, &c. &c. Q. E. D.

PROP. XIX.—Theorem.

If two planes which cut each other be each of them perpendicular to a third plane; their common section shall be perpendicular to that third plane.

Let two pls. AB, BC be ea. ⊥ a third pl. ADC, & let BD be sec. of AB, BC. Then BD is ⊥ pl. ADC.



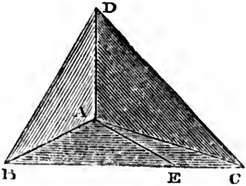
If BD be not ⊥ pl. ADC,
then in pl. AB, draw DE ⊥ AD sec. of pls. AB, ADC;
& in pl. BC draw DF ⊥ DC sec. of pls. BC, ADC;
Now ∵ pl. AB ⊥ pl. ADC,
& that in AB, BE is drawn ⊥ AD their com. sec.
∴ DE ⊥ pl. ADC : 4 def. 11.
simly. DF ⊥ pl. ADC;
∴ from one point D, two rt. lines are ⊥ same pl. ADC, on same side of it;
wh. is impossible. 13. 11.
∴ from D, no rt. line can be drawn ⊥ pl. ADC, except BD, com. sec. of two pls. AB, BC;
∴ BD ⊥ pl. ADC.

Therefore, if two pls., &c. &c. Q. E. D.

PROP. XX.—Theorem.

If a solid angle be contained by three plane angles, any two of them are greater than the third.

Let solid ∠ at A be contained by three pl. ∠s BAC, CAD, DAB ; any two of them shall be > third.



If ∠s BAC, CAD, DAB = ea. other,
it is evident that any two together are > third ;
but if they are ≠ ea. other,
let ∠ BAC be that wh. is ≮ either of others,
but > DAB.
Then in pl. passing through BA, AC,
make an ∠ BAE = ∠ DAB ; 23. 1.
and make AE = AD ;
draw BEC cutting AB, AC in B, C ;
join BD, DC.
Then ∵ DA = AE,
& AB is com. to both △s,
& that ∠ EAB = ∠ DAB ;
∴ base DB = base BE : 4. 1.
& ∵ BD + DC > BC, 20. 1.

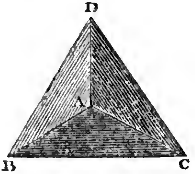
& that BD	=	BE a part of BC,	
∴ DC	>	rem. part EC.	
Again ∵ DA	=	AE,	
& AC is com. to		both △s,	
& that base DC	>	base EC ;	
∴ ∠ DAC	>	∠ EAC :	25. 1.
now ∠ DAB	=	BAE,	constr.
∴ ∠s DAB + DAC	>	∠s BAE + EAC ;	
i. e. ∠s DAB + DAC	>	∠ BAC ;	
but ∠ BAC	≮	either of ∠s DAB, DAC,	
∴ ∠BAC + either of them	>	the other.	

Wherefore, if a solid angle, &c. &c. Q. E. D.

PROP. XXI.—Theorem.

Every solid angle is contained by plane angles which together are less than four right angles.

First.—Let solid $\angle$ at A be cont. by 3 pl. $\angle$s BAC, CAD, DAB. Then these 3 together are $<$ 4 rt. $\angle$s.



In AB, AC, AD, take any points B, C, D ;
join BC, CD, DB.

Then $\because$ sol. $\angle$ at B is cont. by 3 pl. $\angle$s CBA, ABD, DBC,

$\therefore$ any two of them $>$ third ; 20. 11.

$\therefore$ $\angle$s CBA + ABD $>$ $\angle$ DBC :

simly. $\begin{cases} \angle\text{s BCA} + \text{ACD} > \angle\text{ DCB}; \\ \& \angle\text{s CDA} + \text{ADB} > \angle\text{ BDC}; \end{cases}$

$\therefore 6 \angle\text{s} \begin{Bmatrix} \text{CBA, ABD, BCA,} \\ \text{ACD, CDA, ADB,} \end{Bmatrix} > 3 \angle\text{s} \begin{Bmatrix} \text{DBC, BCD,} \\ \text{CDB}; \end{Bmatrix}$

but $\angle$s DBC + BCD + CDB = 2 rt. $\angle$s, 32. 1.

$\therefore 6 \angle\text{s} \begin{Bmatrix} \text{CBA, ABD, BCA,} \\ \text{ACD, CDA, ADB,} \end{Bmatrix} > 2 \text{ rt. } \angle\text{s}:$

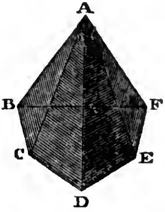
now $\because$ the 3 $\angle$s of ea. $\triangle$ ABC, ACD, ADB, = 2 rt. $\angle$s, [32. 1.

$$\therefore \text{ whl. } 9 \angle\text{s} \left\{ \begin{array}{l} \text{CBA, BAC, ACB,} \\ \text{ACD, CDA, DAC,} \\ \text{ADB, DBA, BAD,} \end{array} \right\} = 6 \text{ rt. } \angle\text{s};$$

but it was demonstr. that 6 of these 9 $\angle$s are $>$ 2 rt. $\angle$s ; [demon.

$\therefore$ rem. 3 $\angle$s BAC, DAC, BAD $<$ 4 rt. $\angle$s.

SECONDLY.—Let solid $\angle$ at A be cont. by any number of pl. $\angle$s BAC, CAD, DAE, EAF, FAB ; these together shall be $<$ 4 rt. $\angle$s.



Let pls. in wh. the $\angle$s are, be cut by a pl.
& let secs. of it with these pls. be BC, CD, DE, EF, FB.
Then $\because$ sol. $\angle$ at B is cont. by 3 pl. $\angle$s CBA, ABF, FBC, of wh. any two are $>$ third ;

$$\therefore \angle\text{s ABC} + \text{ABF} > \angle \text{ CBF}:$$

$$\text{simly. } \angle\text{s} \left\{ \begin{array}{l} \text{ACD} + \text{ACB} > \angle \text{ BCD,} \\ \text{ADE} + \text{ADC} > \angle \text{ CDE,} \\ \text{AED} + \text{AEF} > \angle \text{ DEF,} \\ \text{\& AFE} + \text{AFB} > \angle \text{ EFB}: \end{array} \right.$$

$$\text{but } \angle\text{s} \left\{ \begin{array}{c} \text{FBC, BCD,} \\ \text{CDE, DEF,} \\ \text{\& EFB,} \end{array} \right\} \text{ are } \angle\text{s of fig. BCDEF};$$

$\therefore$ all $\angle$s at bases of the $\triangle$s $>$ all $\angle$s of polygon :

& ∵ all ∠s of △s together = { 2 number of rt. ∠s as there are △s. 32. 1.

i. e. = 2 number of rt. ∠s as sides in fig.

& that all ∠s of fig. + 4 rt. ∠s } = { 2 number of rt. ∠s as sides in fig.

1 cor. 32. 1.

∴ all ∠s of △s together = all ∠s of fig. + 4 rt. ∠s;

but all ∠s at bases of △s > all ∠s of fig. demon.

∴ rem. ∠s of △s, wh. cont. sol. ∠ A < 4 rt. ∠s.

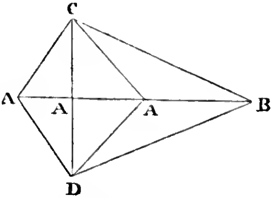
Therefore every solid angle, &c. &c. Q. E. D.

APPENDIX.

APPENDIX.

PROP. VIII.—THEOREM.—BOOK I.

PROCLUS gives a *direct* demonstration of this proposition, nearly as follows.



Let the two △s be so applied, that their equal base may coincide in the same rt. line AB, but that their vertices C, D, may be on opposite sides of it ; join CD if necessary.

Then ∵ BC	=	BD,	hypoth.
∴ ∠ BCD	=	∠ BDC ;	5. 1.
& ∵ AC	=	AD,	hypoth.
∴ ∠ ACD	=	∠ ADC :	5. 1.
∴ ∠ ACB	=	∠ ADB.	ax. 2. & 3.

Q. E. D.

PROP. VI.—Theorem.—BOOK II.

This proposition may be very briefly deduced from Prop. I. Book II. as follows:

A C B

$$\begin{aligned} AB^2 &= AB \cdot AB, \\ &= AB \cdot AC, \\ &+ AB \cdot CB, \\ &= AC \cdot AC + CB \cdot AC \\ &+ AC \cdot CB + CB \cdot CB \\ &= AC^2 + 2\, AC \cdot CB + CB^2. \end{aligned} \qquad \text{1. 2.}$$

Q. E. D.

PROP. XIII.—Theorem.—BOOK II.

The demonstration of this proposition would be shorter thus:—

Fig. 1.

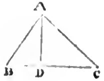
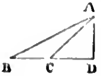


Fig. 2.

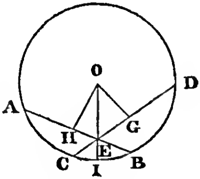


$$
\begin{aligned}
AB^2+BC^2 &= BD^2+DA^2+(BD\pm DC)^2, \begin{cases} +\text{in } (1) \\ -\text{in } (2) \end{cases} \\
&= BD^2+DA^2 \\
&\quad + BD^2+DC^2\pm 2\ BD\cdot DC, \qquad \text{4. 2. \& cor. to 7. 2.} \\
&= AD^2+DC^2+2\ BD^2\pm BD\cdot DC, \\
&= AC^2+2\ BD\cdot(BD\pm DC) \qquad \text{1. 2.} \\
&= AC^2+2\ BD\cdot BC.
\end{aligned}
$$

Q. E. D.

PROP. XXXV.—Theorem.—BOOK III.

The last case of this proposition, which is *general*, may be demonstrated as follows :—



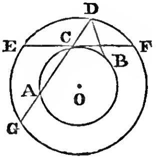
ABCD the ⊙, O its cr., AB, CD, two rt. lines intersecting in E ; draw OH, OG ⊥ AB, CD ; join OE & prod. it to I.

$$
\begin{aligned}
\text{Then } OI^2 &= OH^2 + HB^2, \qquad \text{47. 1.} \\
&= OH^2 + HE^2 + AE\cdot EB, \qquad \text{5. 2.} \\
&= OE^2 + AE\cdot EB\,; \\
\text{Simly. } OI^2 &= OE^2 + DE\cdot EC\,; \\
\therefore\ AE\cdot EB &= DE\cdot EC. \qquad \text{ax. 3.}
\end{aligned}
$$

Q. E. D.

PROP. XXXVI.—Theorem.—BOOK III.

This proposition may be made to depend upon the last.



ABC the giv. ⊙, O its cr., DCA the rt. line cutting ⊙ and DB touching it.

With cr. O & dist. OD desc. ⊙ DEG ; prod. DA to G, & draw ECF, touching ⊙ ACB.

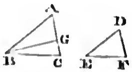
Then it is easily proved that EC = CF = BD, and that GC = AD.

Hence DC · DA = DC · CG,
= EC · CF,
= DB^2.

Q. E. D.

PROP. VII.—Theorem.—BOOK VI.

The demonstration of this proposition may be shortened thus :—



Let $\triangle$s ABC, DEF have $\angle$ A = $\angle$ D, & AB : BC : : DE : EF ; then, if each of $\angle$s C & F, be either > or < a rt. $\angle$, or if one of them, as $\angle$ C, be a rt. $\angle$, the $\triangle$s shall be equiang.

For if $\angle$ ABC ≠ $\angle$ E,
make $\angle$ ABG = $\angle$ E.
Then ∵ $\angle$ A = $\angle$ D, hypoth.
& $\angle$ ABG = $\angle$ E; constr.
& rem. $\angle$ AGB = rem. $\angle$ F ;
∴ $\triangle$s ABG, DEF, are equiang. :
Hence, AB : BG : : DE : EF : : AB : BC ; hypoth.
∴ BG = BC, 9. 5.
∴ $\angle$ BGC = $\angle$ C :

First.—If now $\angle$ C be a rt. $\angle$,
∴ $\angle$ BGC is also a rt. $\angle$,
wh. is impossible. 17. 1.

Secondly.—If $\angle$s C & F be either each > or each < a rt. $\angle$, their equals, $\angle$s BGC, BGA, are either each > or each < a rt. $\angle$, which is impossible. 13. 1.

Hence $\angle$ ABG ≠ $\angle$ E,
∴ $\angle$ ABC = $\angle$ E,
& rem. $\angle$ C = rem. $\angle$ E ;
∴ $\triangle$s ABC, DEF, are equiang.

Q. E. D.

THE END.

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