THE ELEMENTS OF GEOMETRY

BY

BUSH AND CLARKE

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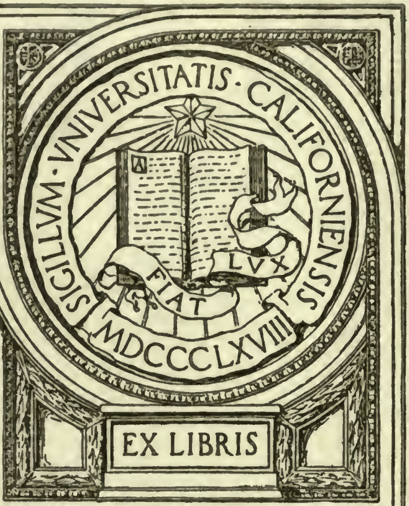

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THE

ELEMENTS OF GEOMETRY

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PREFACE

THE present text, the outgrowth of over twenty years' experience with classes in geometry, carries the work from the simple elements necessary in the beginning class of the high school to the most advanced requirements of university preparation. For many years we have followed in our own classes the plan here presented, with such modifications and improvements as experience has suggested.

It is generally conceded that much of the pupil's difficulty in demonstration arises from his failure to grasp thoroughly and keep vividly in mind as separate and distinct statements, first, the exact data of the proposition, and, second, the precise fact to be established. To remove this stumbling-block, we have stated the hypothesis and conclusion separately for every theorem and corollary demonstrated.

Through the "open" arrangement of the printed matter, we have sought to make each successive step stand out clearly; and by so adjusting diagrams and text that in the course of any single demonstration it is unnecessary to turn the page, we have endeavored to avoid waste of effort on the part of the pupil.

All the original exercises should be mastered with only such help as is given by the book itself. If a proposition has been found difficult for the average pupil, it has been broken up into a series of exercises in such sequence that the difficulties are presented one at a time and in natural order, the truth of the main proposition being established by means of these graded exercises.

It has been our purpose to eliminate discouraging elements, to refresh the memory of the student before he begins inventive work, to arouse his interest and to inspire his confidence in his ability to discover hidden truths.

We desire to express our acknowledgments for valuable counsel and suggestions to Professor F. N. Cole of Columbia University, to Professor Irving Stringham of the University of California, to Professor R. E. Gaines of Richmond College, Virginia, and to J. A. C. Chandler, LL.D., formerly Dean of Richmond Academy, Virginia.

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PLAN AND SCOPE

There are few students who fail to respond to the stimulus of original work in Geometry. The energy expended and enthusiasm displayed in the solution of exercises is in sharp contrast with their apathy toward the study of theorems in the text.

That the student may arrive by the shortest path to the point where the real development of his mental power begins, where his interest in the independent solution of original exercises becomes an active part of his school life, it is of the first importance that the theorems of the text be so classified and demonstrated as to offer his attempts to master them the least resistance. It is of equal importance that these exercises should be so carefully graded as to stimulate and not discourage the pupil, and that some method of systematically attacking and solving problems should be devised and presented for his assistance. Nothing is presented in the pages that follow that has not stood the actual test of class-room experience for many years.

In accordance with these essentials of a text for use in Geometry classes, we call special attention to the following features of this work:

First. The classification of Definitions and Axioms.

Second. The arrangement into groups of Theorems relating to the same topic. For example, theorems concerning isosceles triangles in the "Isosceles Triangle Group"; congruent triangles in the "Congruent Triangle Group"; comparison of areas in the "Areal Ratio Group."

Third. The arrangement of original exercises, second in importance only to the grouping of theorems. The exercises are not only attached to the groups of theorems upon which their solution depends, but are graded according to their degree of difficulty.

Fourth. The elimination of all theorems not essential to a clear understanding of the principles of Geometry; hence, a number of theorems found with their proofs in the usual text are given as exercises.

Fifth. The compactness of each group and the simplicity and clearness of the demonstrations. With few exceptions, not more than three or four recitations are needed to master any one of the groups.

Sixth. The helpful suggestions as to the method of solving original exercises.

Seventh. A simple statement, with illustrations, of the close connection between Algebra and Geometry; classification of principles of the analogy between geometric and algebraic work into Indeterminate, Determinate, and Overdeterminate groups.

In the grouping of Theorems and in the arrangement and grading of Problems, there is constantly employed, as a valuable aid to the student, the principle of the Association of Similars.

In the treatment of the Spherical Geometry, much space and time are saved by utilizing the common properties of the plane and sphere-surface; and thus transferring, where possible, the theorems and proofs of the plane to the sphere-surface.

It is believed that this plan not only gives the pupil a clearer comprehension of the unity of the subject than he will otherwise obtain, but that it is also eminently suggestive of the unlimited possibilities of the extension of geometric truths to other surfaces.

SUGGESTIONS TO TEACHERS

Just as each brick in a building rests upon a brick below it, the whole superstructure standing upon a securely laid foundation, so the proof of each Theorem in Geometry rests upon the proof of the preceding Theorem, which in turn must finally depend upon the Definitions and Axioms. Definitions and Axioms, therefore, must be carefully studied and thoroughly understood.

For example, to prove that the Bisector of the Vertex Angle of an Isosceles Triangle is identical with the Altitude (Group IV, 1 *b*), we must know the definitions of the following words: Bisector, Vertex Angle, Isosceles Triangle, and Altitude. To understand the meaning of Altitude, we must know what a Perpendicular is, which, in turn, requires that we know the meaning of a Right Angle.

The student, therefore, before attempting the proof of a Theorem, should be made to understand the meaning of each and every term found in the Hypothesis and Conclusion of the Theorem.

To cultivate the habit of defining terms before using them, it affords the student valuable if not indispensable exercise to require of him frequent definitions of all the terms found in the Theorems and Corollaries, particularly of the first six groups.

Beginners have great difficulty in keeping in mind the parts given in the Hypothesis distinct from those given in the Conclusion. During the process of demonstration they confuse what was to be proved with what was given. It will relieve this confusion if they form the habit of marking the parts given in the Hypothesis by the usual symbols, *i.e.*:

First. If lines are given parallel, by drawing the symbol ‖ across the lines.

Second. If one line is greater than another, by drawing the symbol >.

Third. If angles are given equal, by making the proper symbol at the vertex, just within the sides of the angle.

Fourth. All parts mentioned in the Conclusion may be marked with an X.

In drawing triangles, unless the triangle is Isosceles, Equilateral, or Right, it is well to adopt the following rule:

First. Draw the base line.

Second. Find approximately its mid-point.

Third. Place the pencil or chalk to the left a convenient distance and erect an imaginary perpendicular.

Fourth. At any suitable point on this perpendicular select the vertex of the vertex angle, from which draw the sides of the triangle.

In this way the beginner may avoid the pitfalls of giving special proofs for the Isosceles, Equilateral, and Right Triangle that will not apply to the general Triangle.

In drawing Parallelograms, unless a Rectangle, Square, or Rhombus be given in the Hypothesis, always construct a Rhomboid.

At the beginning of the course, by written exercises and much blackboard work, familiarize the student with the use of symbols and abbreviations; also with the freehand drawing of the altitudes of obtuse triangles.

After the first month's work require frequent written examinations, insisting, as the course advances, that close attention be paid to the form in which written work is presented. Before the close of the course in Plane Geometry the student should be able to present his examination papers in the compact form found in the text.

CONTENTS

PLANE GEOMETRY

SOLID GEOMETRY

SYMBOLS

1. Letter points with capitals.
2. Letter lines with small letters.
3. Name angles, when there is no ambiguity, with small letters italicized.

4. Points	K	(Greek word *kentron*) center in general.
	K_i	Center of inscribed circle, *i.e.* in-center.
	K_e	Center of escribed circle, *i.e.* ex-center.
	K_c	Center of circumscribed circle, *i.e.* circum-center.
	K_o	Ortho-center (intersection of altitudes).
	K_g	Centroid or Center of Gravity of triangle (intersection of medians).
5. Lines	Sides of a triangle :	Use small letters corresponding to the capitals at the vertices opposite the respective sides ; *a* opposite *A*, etc.
	⊥	Perpendicular, or "is perpendicular to."
	Mid ⊥	Mid-perpendicular.
	⦣	Oblique.
	∥	Parallel.
	⊙	Circumference or circle.
	⌒	Arc.
6. Angle	∠	Angle in general.
	Rt. ∠	Right angle.
7. Triangle	△	Triangle in general.
	Rt. △	Right triangle.
8. Quadrilateral	4-side	Quadrilateral in general.
	▱	Parallelogram or rhomboid.
	▭	Rectangle.
	▱	Rhombus.
	□	Square.
9. Miscellaneous	∴	Therefore.
	∵	Since or because.
	∼	Similar.
	≅	Congruent.
	≡	Identical.
	>	Is greater than.
	<	Is less than.
	≐	Approaches as a limit.

For the plural add the letter *s*.

The cancellation across a symbol means "not," *e.g.* : ≯ means not greater than ; ∦ means not parallel.

ABBREVIATIONS

Alt.	Alternate.
Ax.	Axiom.
Adj.	Adjacent.
Conc.	Conclusion.
Const.	Construction.
Cor.	Corollary.
Corr.	Corresponding.
Def.	Definition.
Dem.	Demonstration.
Ex.	Exercise.
Ext.	Exterior.
Hyp.	Hypothesis.
Hom.	Homologous.
Int.	Interior.
n-gon.	Polygon.
Opp.	Opposite.
Prop.	Proposition.
Prob.	Problem.
Q.E.D.	Quod erat demonstrandum (which was to be proved).
Q.E.F.	Quod erat faciendum (which was to be done).
Supp.	Supplemental.
Th.	Theorem.
Vert.	Vertical.
v.	Vide (see).
q.v.	Quod vide (which see).
cf.	Compare.
r.	Radius.

Groups, Theorems, and Corollaries are read as follows: II. 1. *a* means Group II, Theorem 1, Corollary *a*.

THE ELEMENTS OF GEOMETRY

DEFINITIONS

EXTENSION

The **Definition** of a mathematical term is its explanation in words familiar to the student.

The test of a complete definition is that the subject and predicate may be interchanged without affecting the truth of the statement.

Space extends about us on every side. Every material object occupies a portion of this space. This portion of space is called a geometrical solid or simply a **Solid.** The only properties of the solid with which geometry is concerned are its form and size, and its position with reference to other solids.

A boundary of a solid is called its **Surface.** It is no part of the solid, and therefore has but two dimensions: length and breadth.

A boundary of a surface is called a **Line.** It is no part of the surface, and has therefore but one dimension: length.

A boundary of a line is called a **Point.** A point has no dimension, but position only.

A point, line, or surface that divides any magnitude into two equal parts is called the **Bisector** of that magnitude.

Any definite portion of a line is called a **Line-segment.**

A **Straight Line** is a line that lies evenly between its extreme points; that is, if the ends of one segment may be placed upon the ends of a second segment, the segments *must* coincide throughout their whole extent.

A straight line connecting any two points is called the **Join** of those points.

A **Broken Line** is a series of joins, any two consecutive joins having one point in common.

A **Curved Line** is a line such that no segment of it is straight.

Concurrent Lines are lines passing through the same point.

A **Transversal** is any line intersecting a number of other lines.

NOTE. — In the text the word *line* is used to mean a straight line.

FIGURES

A **Figure** or **Complex** is any collection of points, lines, or points and lines.

Similar figures are figures having the same shape.

Equal figures are figures having the same size.

Congruent figures are figures having the same shape and the same size.

The **Test of Congruency** is that one figure may be placed on the other so that every part of the first will coincide with the corresponding part of the second.

Two figures thus placed are said to be in **Coincident Superposition.**

A **Plane** is a surface such that if any two of its points be joined by a straight line, this line *must* lie wholly within the surface.

A **Plane Figure** is one all the parts of which lie in the same plane.

NOTE. — All figures hereafter defined are assumed to be plane figures.

A **Polygon** is a portion of a plane bounded by a closed broken line called its **Perimeter.**

When only the *form* is considered, the word *polygon* is frequently used to mean the perimeter of the polygon.

A three-sided polygon is called a **Triangle.**

A four-sided polygon is called a **Quadrilateral** or **4-side.**

A five-sided polygon is called a **Pentagon.**

A six-sided polygon is called a **Hexagon.**

A seven-sided polygon is called a **Heptagon.**

An eight-sided polygon is called an **Octagon.**

A nine-sided polygon is called a **Nonagon.**

A ten-sided polygon is called a **Decagon.**

A twelve-sided polygon is called a **Dodecagon.**

A fifteen-sided polygon is called a **Pentadecagon.**

The **Vertices** of a polygon are the points in which its consecutive sides meet.

A **Diagonal** of a polygon is the join of any two non-consecutive vertices.

A **Regular Polygon** is a polygon whose angles are equal and whose sides are equal.

Ex. 1. What is the test of a complete definition ?

Ex. 2. Apply this test to the definition of a line-segment.

Ex. 3. Define (*a*) Bisector. (*b*) Line. (*c*) Join. (*d*) Concurrent lines. (*e*) Transversal.

Ex. 4. Define a straight line.

Can you place two equal portions of a barrel hoop in such a way that the ends of one will coincide with the ends of the other, but the portions (or segments) themselves will not coincide ?

Can you place them so that they will coincide ?

Since, then, equal segments of a curve may or may not be made to coincide, what is the word to emphasize in the definition of a straight line ?

Ex. 5. Draw three concurrent lines.

Ex. 6. Draw three non-concurrent lines.

Ex. 7. Draw a transversal to two lines.

Ex. 8. What are congruent figures ?

Ex. 9. What is the test of congruency ?

Ex. 10. Give illustrations of equal, similar, and congruent figures.

Ex. 11. What are the two dimensions of a surface ?

Ex. 12. Define a plane.

Ex. 13. Using your ruler as a straightedge, show that the top of your desk is a plane.

ANGLES

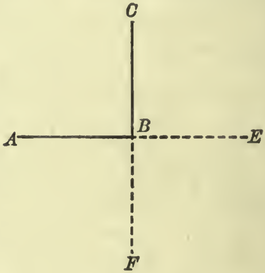
An **Angle** is a figure formed by two lines that meet; the lines being called the **Sides** of the angle.

The point of meeting of the sides of the angle is called the **Vertex.**

The usual **Method of Reading** an angle is to read the three letters on the sides, placing the letter at the vertex between the other two. If there is no ambiguity, the angle may be read by the letter at the vertex.

If, when the sides of an angle are produced, all the angles formed are equal, each angle is called a **Right Angle,** and the lines are said to be **Perpendicular** to each other.

If a line bisects a second line and is also perpendicular to it, the first line is called the **Mid-normal** or **Mid-perpendicular** to the second.



ABC is an angle. Its vertex is B. If the side AB is produced to E and the side CB is produced to F, and if angles ABC, CBE, EBF, and ABF are all equal, then they are right angles, and the line CF is perpendicular to the line AE. If the line CF also bisects the line AE, it is the *Mid-normal* to AE.

By the distance of a point from a line is meant the perpendicular distance; by the distance between two lines, the perpendicular distance.

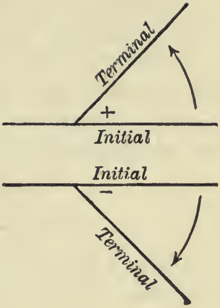
Classes of Angles

(a) *As to their Algebraic Sign*

An angle may be considered as *generated* by the revolution of one line about a fixed point in a second line.

The rotating line, when it comes to rest, is called the **Terminal Line.**

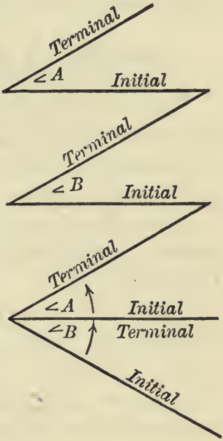
The fixed line is called the **Initial Line.**



If the rotating line moves anti-clockwise, the angle generated we call **Positive**; if clockwise, **Negative**.

If the rotating line, moving from a position coincident with the initial line, complete a revolution, it **generates** four right angles, or a **Perigon**; if half a revolution, it generates two right angles, or a **Straight Angle**.

NOTE. — The size of an angle does not depend upon the length of its sides.



NOTE. — In finding the sum, $\angle a + \angle b$, of 2 $\angle$s, place the initial line and vertex of $\angle b$ on the terminal line and vertex of $\angle a$.

Then generate the $\angle b$ by the rotation indicated by its sign.

Then the angle between the initial line of $\angle a$ and the terminal line of $\angle b$ is $\angle a + \angle b$.

The **Relative Direction** of one line with respect to another is the angle that the first line makes with the second, both the *size* and *sign* of the angle being considered.

The direction of rotation of the line generating the angle will be considered positive unless the contrary is stated.

Relative direction will be the only direction considered in this book; *absolute direction* will not be discussed.

(*b*) *As to Size*

An **Acute** angle is an angle less than a right angle.

An **Obtuse** angle is an angle greater than a right angle.

An obtuse angle equal to two right angles is called a **Straight** angle.

An **Oblique** angle is any angle that is not a right angle.

When two angles are both acute or both obtuse, they are said to be of the **Same Kind.**

If the sum of two angles equals one right angle, they are called **Complemental** angles.

If the sum of two angles equals two right angles, they are called **Supplemental** angles.

(c) *As to Location*

Adjacent angles are angles that have a common side and a common vertex.

Vertical angles are the alternate angles formed by two lines that cross each other.

ANGLES FORMED BY A TRANSVERSAL WITH TWO OTHER LINES

Angles within the two lines crossed by a transversal are called **Interior** angles; angles without, **Exterior** angles.

Non-adjacent angles on the same side of the transversal are called **Corresponding** angles.

There are three classes of corresponding angles: **Corresponding Interior, Corresponding Exterior,** and **Corresponding Exterior-interior** angles.

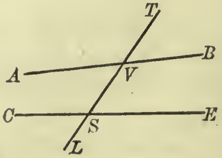
Corresponding exterior-interior angles are **Corresponding Non-adjacent** angles, one of which is interior and the other exterior.

Alternate angles are angles on opposite sides of the transversal that do not have the same vertex.

There are three classes of alternate angles: **Alternate Interior, Alternate Exterior,** and **Alternate Exterior-interior** angles.

Alternate exterior-interior angles are alternate angles, one exterior and the other interior, but not having a common vertex.

Point out in the adjoining figure the following angles: acute, obtuse, oblique, supplemental, adjacent, opposite, of the same kind, corr. ext., corr. int., corr. ext.-int., alt. ext., alt. int., alt. ext.-int.



A **Postulate** is a construction admitted to be possible.

Post. There may always be drawn a pair of lines that make equal corresponding exterior-interior angles with any transversal whatever cutting them.

Two lines are **Parallel** when, if cut by any transversal whatever, the corresponding exterior-interior angles are equal.

Ex. 14. The ruler can be so placed as to lie wholly on the stovepipe. Why, then, is not the stovepipe a plane?

Ex. 15. Show that any two parallels have the same direction with respect to any transversal.

Ex. 16. Define: (*a*) Angle. (*b*) Right angle. (*c*) Perpendiculars. (*d*) Perpendicular bisector (or mid-normal). (*e*) Oblique angles. (*f*) Angles of the same kind.

Ex. 17. What is the complement of two fifths of a right angle?

Ex. 18. What is the supplement of two fifths of a right angle?

Ex. 19. Name the angles formed by the clock hands:

(*a*) 10 minutes after three o'clock. (*c*) 20 minutes after three o'clock.
(*b*) 15 minutes after three o'clock. (*d*) 30 minutes after three o'clock.

Ex. 20. Draw two oblique angles; draw two angles of the same kind.

Ex. 21. Draw two angles that are: (*a*) Adjacent. (*b*) Adjacent and complemental. (*c*) Adjacent and supplemental. (*d*) Non-adjacent and supplemental.

Ex. 22. An angle is one fifth its complement. What is the value of the angle?

Ex. 23. Two angles are equal and at the same time complemental. What is their value?

Ex. 24. Two angles are equal and at the same time supplemental. What is their value?

Ex. 25. Draw two angles that have a common side but not a common vertex.

Ex. 26. Draw two angles that have a common vertex but not a common side.

Ex. 27. Draw two lines so that a transversal crossing them makes:

(*a*) Corresponding exterior-interior angles equal.
(*b*) Corresponding interior angles equal. (*c*) All the angles equal.

Ex. 28. In case (*a*) of the preceding question what other angles are equal?

THE CIRCLE AND THE LOCUS

A **Circle** is a portion of a plane bounded by a curved line, called the **Circumference**, every point of which is equidistant from a point called the **Center**.

Any line from the center to the circumference is called a **Radius**.

Circles having the same center are **Concentric Circles**.

A **Locus** is a line or a complex, all points of which possess a common property that does not belong to any point without this line or complex. *E.g.* it is the common property of every point in the circumference of a circle that is a radial distance from the center; the circumference is the locus of all points that are a radial distance from the center.

Ex. 29. Do the two lines meet?

Ex. 30. In case (*b*) what other angles are equal?

Ex. 31. Do the two lines meet?

Ex. 32. In case (*c*) do the two lines meet? The angles in case (*c*) are all of what kind?

Ex. 33. How may an angle be considered to be generated?

Ex. 34. Define a negative angle.

Ex. 35. The hands of the clock are together at twelve o'clock. If the hour hand is stationary, what is the algebraic sign of the angle formed by the hands at ten minutes after twelve?

Ex. 36. If the hour hand is stationary and the minute hand is moved to the left, what is the algebraic sign of the angle formed at ten minutes before twelve?

Ex. 37. What is the sign of the angle generated by a line from the observer to the moon from moonrise to moonset?

Ex. 38. To mariners what is the fixed or known line? What instrument on every ship serves to determine this line?

Ex. 39. If a ship is sailing southeast, what angle does its course make with this known line?

Ex. 40. What, then, is the relative direction of the ship with respect to this fixed line?

Note. — Answer all questions concerning angles in terms of a right angle until the word *degree* has been defined.

Ex. 41. What angle is one fifth its supplement?

The truths of geometry are expressed by its definitions, axioms, and theorems.

An **Axiom** is a statement, the truth of which is assumed.

AXIOMS

1. Things equal to the same thing or equal things are equal to each other.
2. If equals be added to or subtracted from equals, the results will be equal.
3. If equals be multiplied or divided by equals, the results will be equal.
4. The whole equals the sum of its parts.
   Direct Inference: The whole is greater than any of its parts.
5. The intersection of two lines, straight or curved, fixes the position of a point.
   E.g. the intersection of the latitude and longitude of a ship at sea determines its position.
6. Two points fix the position of a line.
   E.g. if a railroad extending in a straight line passes two stations whose positions are known, the railroad is also determined or fixed in position.
7. A point and the direction of a straight line determine its position.
   E.g. if a railroad extends northwest and passes a known station, the position of the railroad is known.
   - *a.* From a point without a line but one perpendicular can be drawn to the line.
   - *b.* At a point in a line but one perpendicular can be drawn to the line.
8. Through a point one line and only one can be drawn parallel to a given line.
9. Any figure may be transferred from one position in space to any other without change of size or shape.
10. If there be but one x and one y, then, from the fact that x is y, it necessarily follows that y is x.

THREE PRELIMINARY THEOREMS ON INEQUALITY

These propositions are given in many texts as axioms. They are proved, however, in the leading treatises on algebra.

1. *If unequals are added to unequals in the same sense, the results will be unequal in the same sense.*

Dem.: Suppose $a > b$, and $c > e$.

To prove $a + c > b + e$.

$a > b$. $\therefore a = b +$ some quantity, say x.

$c > e$. $\therefore c = e +$ some quantity, say y.

Thus $a = b + x$,

$c = e + y$.

$\therefore a + c = b + e + x + y$. (Ax. 2.)

That is, $a + c > b + e$.

Q.E.D.

2. *If equals are added to unequals, the results are unequal in the same sense.*

Dem.: Suppose $a > b$, and $c = e$

To prove $a + c > b + e$.

$a > b$. $\therefore a = b +$ some quantity, say x.

Then $a = b + x$,

and $c = e$.

$\therefore a + c = b + e + x$. (Ax. 2.)

$\therefore a + c > b + e$.

Q.E.D.

3. *If equals are subtracted from unequals, the results are unequal in the same sense.*

The proof of this proposition is precisely similar to that of Theorem 2.

Ex. 42. What is the supplement of the angle between the hands of a clock at five o'clock?

Ex. 43. What is the complement of the angle in the preceding question? (v. Negative angles.)

GENERAL TERMS

A **Theorem** is a statement to be proved. It consists of two parts:

The **Hypothesis** (Hyp.), or supposition or premise.

The **Conclusion** (Conc.), or what is asserted to follow from the hypothesis.

A **Proof** is a course of reasoning by which the truth of a theorem is established.

The **Converse** of a theorem is obtained by interchanging the hypothesis and conclusion of the original theorem.

Theorem: If A is B, then C is E.
Converse: If C is E, then A is B.

NOTE. — The converse is sometimes called the *indirect* theorem.

The **Contradictory** of a theorem is true if the theorem is false, and *vice versa*.

Theorem: If A is B, then C is E.
Contradictory: If A is B, then C is not E.

The **Opposite** of a theorem is obtained by making both the hypothesis and conclusion negative.

Theorem: If A is B, then C is E.
Opposite: If A is not B, then C is not E.

A **Reciprocal** theorem is formed by replacing, when possible, in the original theorem, the words "point by line," "line by point," "angles of a triangle by the opposite sides of the triangle," "sides of a triangle by the opposite angles of the triangle," "opposite angles of a 4-side" by "the opposite sides of a 4-side," etc.

NOTE. — For every statement in a proof a reason must be given. This reason must be:

1. A hypothesis.
2. A construction.
3. A definition.
4. An axiom.
5. A previously established theorem or corollary.

A **Corollary** is a subordinate statement deduced from, or suggested by, the main statement or its proof.

A **Problem** requires the construction of a geometric figure that will satisfy given conditions.

The **Solution of a Problem** consists of four parts:

First: The **Analysis**, or course of reasoning by which the method of constructing the required figure is discovered or rediscovered.

NOTE. — The analysis of problems is explained in full under the article "Helps to the Solution of Original Problems," Group XII.

Second: The **Construction** of a figure after the method has been discovered.

Third: The **Proof** that the figure satisfies the given conditions.

Fourth: The **Discussion** of the changes in the number of figures that will satisfy the given conditions, made by a change in the size of the given magnitudes, in their relative position, or in both.

A **Proposition** is a general term applying to theorems or problems.

A **Scholium** is a remark upon a particular feature of a proposition.

A **Lemma** is a theorem introduced merely to be used in the proof of one immediately following.

Plane Geometry is that branch of mathematics in which are considered the properties of magnitudes lying on the same plane.

SUGGESTIONS FOR CLASS WORK AT BLACKBOARD

In lettering a figure avoid the use of letters that have the same sound; as B, D, P, and T; M and N, etc.

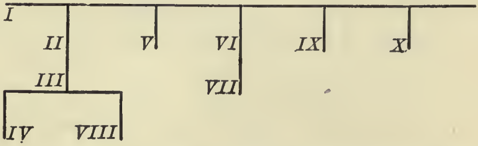
Place tip of pointer within the figure you are to read. To move it from point to point is confusing. If reading a line-segment, place tip about the middle of the segment; if an angle, place it near the vertex; if a polygon of any kind, within the polygon.

In drawing triangles make them, unless otherwise directed, scalene, with the angle at A greater than the angle at B, and C the vertex angle.

TEN EASY EXERCISES IN GEOMETRICAL DRAWING

These problems are introduced at this point to familiarize the student with the use of the ruler and compasses and with geometrical terms. The constructions are simple, and serve to illustrate some of the practical applications of geometry.

The student will observe that the constructions in Probs. II, V, VI, IX, and X are direct applications of the constructions in Prob. I, and that the others are also intimately connected with it. Thus Prob. I may be considered the string from which the other problems are suspended. This dependence is shown by the adjacent diagram.

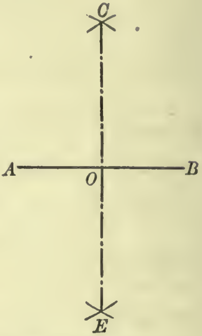


The proofs for these constructions are to be given as soon as the necessary theorems have been established. References to the following problems are made in footnotes and exercises attached to these theorems.

SUMMARY

PROBLEM I.	(*a*) *Bisect a given line-segment.*
	(*b*) *Erect a mid ⊥ to it.*
PROBLEM II.	*Bisect a given arc.*
PROBLEM III.	*Bisect a given angle.*
PROBLEM IV.	*Trisect a given right angle.*
PROBLEM V.	*Erect a ⊥ to a given line at a given point in the line.*
PROBLEM VI.	*Draw a ⊥ to a given line from a given point without the line.*
PROBLEM VII.	*Inscribe a ⊙ in a given triangle.*
PROBLEM VIII.	*Escribe a ⊙ to a given triangle.*
PROBLEM IX.	*Circumscribe a ⊙ to a given triangle.*
PROBLEM X.	*Find the center of a given ⊙.*

PROBLEM I. (*a*) *To bisect a given line-segment.* (*b*) *To erect a mid ⊥ to a given line-segment.*



Given. The line-segment AB.

Required. (*a*) To bisect AB. (*b*) To erect a mid ⊥ to AB.

Const. With A as a center and a $r = AB$, describe arcs above and below AB.

With B as a center and the same r, describe arcs above and below AB.

Let these arcs intersect above the line in C; below in E.

Draw the join CE.

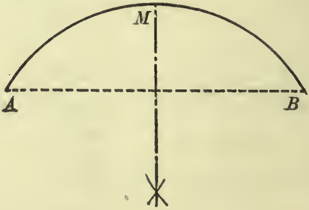
Let it intersect AB in O.

Then (*a*) O bisects AB.

(*b*) CE is the mid ⊥ to AB.

Q.E.F.

PROBLEM II. *To bisect a given arc.*



Given. The arc AB.

Required. To bisect the arc AB.

Const. Draw the chord AB.

Construct the mid ⊥ to this chord by Prob. I.

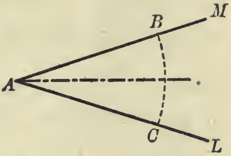
Let this mid ⊥ intersect the arc AB in M.

Then M bisects the arc AB.

Q.E.F.

Ex. 44. In what line do you find all the houses that are one mile from the county courthouse?

Problem III. *To bisect an angle.*



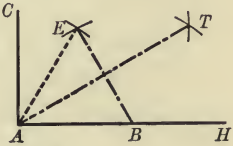
Given. $\angle MAL$.

Required. To bisect $\angle MAL$.

Const. With A as a center and any r, describe an arc BC.
Bisect the arc BC by Prob. II.
The join of A and the mid-point of arc BC bisects $\angle MAL$.

Q.E.F

Problem IV. *To trisect a right angle.*



Given. The rt. $\angle A$.

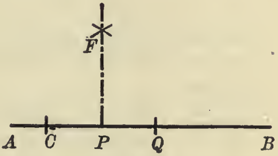
Required. To trisect rt. $\angle A$.

Const. Lay off on AH any line-segment AB.
With A as a center and AB as a r, describe an arc.
With B as a center and the same r, describe an arc.
Let these two arcs intersect at E.
Draw EA and bisect $\angle BAE$ by Prob. III.

$$\angle BAT = \angle TAE = \angle EAC.$$

Q.E.F.

Problem V. *To erect a perpendicular to a line at a point in the line.*



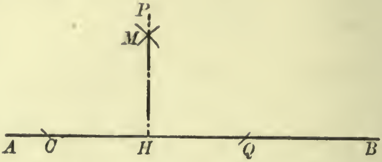
Given. The line AB and P in AB.

Required. To erect a ⊥ to AB at P.

Const. Lay off $PC = PQ$.
Construct the mid ⊥ PF to CQ by Prob. I.
PF is the required ⊥.

Q.E.F.

PROBLEM VI. *To draw a perpendicular to a line from a point without the line.*



Given. The line AB and the point P without AB.

Required. To draw a ⊥ from P to AB.

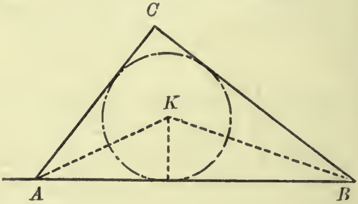
Const. With P as a center and any $r >$ the distance from P to AB, describe an arc intersecting AB in C and Q.

Construct the mid ⊥ PH to the line-segment CQ by Prob. I.

PH is the required ⊥.

Q.E.F.

PROBLEM VII. *To inscribe * a circle in a triangle.*



Given. The △ ABC.

Required. To inscribe a ⊙ in △ ABC.

Const. Bisect ∠ A and ∠ B by Prob. III.

Let the bisectors intersect at some point K.

Construct a ⊥ from K to AB by Prob. VI.

With K as a center and this ⊥ as a r, describe a ⊙.

This ⊙ will be inscribed in △ ABC.

Q.E.F.

Ex. 45. What is the locus of a point one mile from a given point?

Ex. 46. What is the locus of a point that is b distant from a given point F?

Ex. 47. What is a proof or demonstration?

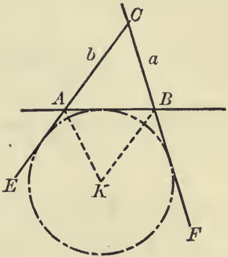
Ex. 48. Into what two parts may a theorem be separated? A problem?

Ex. 49. Name the five classes of reasons, one of which must be given for every statement that is made in the proof.

---

* A ⊙ is inscribed in a △ when the sides of the △ are tangent to the ⊙ (*i.e.* touch the ⊙ in but one point). See p. 78.

PROBLEM VIII. *To escribe*[1] *a circle to a triangle.*



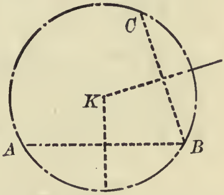
Given. The △ *ABC.*

Required. To escribe a ⊙ to the △ *ABC.*

Const. Produce the sides *b* and *a*.
Bisect ∠ *BAE* and ∠ *FBA* by Prob. III.
Let the bisectors intersect in some point *K.*
Construct a ⊥ from *K* to *a* (or to *b*) produced.
With *K* as a center and this ⊥ as a *r*, describe a ⊙.
This ⊙ will be escribed to △ *ABC.*

Q.E.F.

PROBLEM IX. *To pass a circle through three points.*



Given. The three points, *A, B,* and *C,* not collinear.

Required. To pass a ⊙ through *A, B,* and *C.*

Const. Draw the joins *AB* and *BC.*
Construct the mid ⊥s to *AB* and *BC* by Prob. I.
Let these mid ⊥s intersect at a point *K.*
With *K* as a center and a $r = KA, KB,$ or KC, describe a ⊙.
This ⊙ passes through *A, B,* and *C.*

Q.E.F.

NOTE. — If *A, B,* and *C* are connected, the above ⊙ is said to be circumscribed to △ *ABC.*

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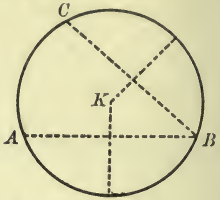
[1] To escribe a circle is to draw it tangent to one side of a triangle and to the other two sides produced.

Problem X. *To find the center of a given circle.*



Given. The ⊙ K.

Required. To find its center.

Const. Take any three points in the ⊙, as A, B, and C.
Draw the joins of any two of these, as AB and BC.
Construct the mid ⊥s to these joins by Prob. I.
Let these mid ⊥s intersect at some point K.
K is the center required.

Q.E.F.

Note. — Prob. X, after the three points have been selected, is evidently identical with Prob. IX.

Note. — The proofs for the solution of these problems will be found as follows: —

I., V., VI., p. 72.
II., p. 79.
III., p. 112.
IV., p. 49, with Def. p. 30.
VII., p. 92, X., 1, a.
VIII., p. 93, Sch.
IX. and X., p. 94.

Ex. 50. What is the opposite of a theorem ?

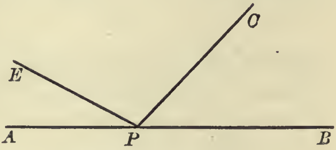
Ex. 51. State the opposite of Group I, Theorem 2.

PLANE GEOMETRY

I. THE GROUP ON ADJACENT AND VERTICAL ANGLES

PROPOSITIONS

I. 1. *If from the same point in a line any number of lines are drawn on the same side of the line, the sum of the successive angles formed equals two right angles.*



Hyp. If from a point in AB, PC and PE are drawn on the same side of AB, forming successively $\angle BPC$, $\angle CPE$, and $\angle EPA$,

Conc.: then $\angle BPC + \angle CPE + \angle EPA = 2$ rt. $\angle$s.

Dem. If PB rotates to PA, it generates a straight $\angle$.
(Def. of a straight $\angle$.)

In this process of rotation, PB generates successively $\angle BPC$, $\angle CPE$, and $\angle EPA$.

$\therefore \angle BPC + \angle CPE + \angle EPA = 2$ rt. $\angle$s. (Ax. 4.)

Q.E.D.

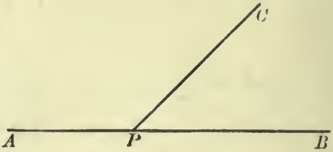
I. 1. Sch. If the angles are on both sides of the line, their sum equals four right angles.

Ex. 1. $\angle A$ is the supplement of $\angle B$. They bear to each other the relation of 4 to 7. What part of a rt. $\angle$ is each ?

Ex. 2. If $\angle AMC$, adjacent to $\angle CMB$, is four thirds of a rt. $\angle$, and $\angle CMB$ is four fifths of a rt. $\angle$, are their exterior sides in the same straight line ?

Ex. 3. Three successive angles about a point on one side of a straight line are in the ratio of the numbers 2, 3, and 5. What is the value of each angle ?

I. 2. *If two angles are adjacent and have their exterior sides in the same straight line, they are supplemental.*



Hyp. If $\angle APC$ and $\angle CPB$ are adjacent, and if PB and PA are in the same straight line,

Conc.: then $\angle CPA + \angle CPB = 2$ rt. $\angle$s.

Dem. If PB rotates to PA, it generates a straight $\angle$.
(Def. of a straight $\angle$.)

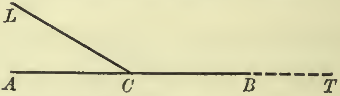
In this process of rotation, PB generates successively $\angle BPC$ and $\angle CPA$.

$\therefore \angle BPC + \angle CPA = 2$ rt. $\angle$s. (Ax. 4.)

Q.E.D.

NOTE. — The above proposition, which is a slight modification of Theorem 1, is introduced to assist the pupil to a clear statement and understanding of its important converse, which directly succeeds.

I. 3. *If two angles are adjacent and supplemental, their exterior sides form the same straight line.*



Hyp. If $\angle ACL$ and $\angle LCB$ are supplemental and adjacent,

Conc.: then AC is in same straight line with CB.

Dem. If CB is not in the same straight line with AC, draw CT that is in the same straight line.

Then $\angle LCT + \angle ACL = 2$ rt. $\angle$s. (I. 1.)

But $\angle LCB + \angle ACL = 2$ rt. $\angle$s. (Hyp.)

$\therefore \angle LCT = \angle LCB$. (Ax. 1.)

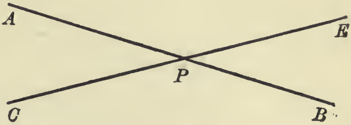
$\therefore CT$ falls on CB. (Ax. 7.)

But CT was drawn in the same straight line with AC.

$\therefore CB$ and CA are in the same straight line.

Q.E.D.

I. 4. *If two straight lines intersect, the vertical angles formed are equal.*



Hyp. If AB intersects CE,

Conc.: then $\angle APC = \angle EPB$.

Dem. $\angle CPA + \angle APE = 2$ rt. $\angle$s. (I. 2.)

$\angle EPB + \angle APE = 2$ rt. $\angle$s. (I. 2.)

$\therefore \angle CPA + \angle APE = \angle EPB + \angle APE$. (Ax. 1.)

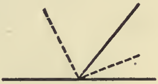
$\therefore \angle CPA = \angle EPB$. (Ax. 2.)

Q.E.D.

The reference number only is given when the reason theorem belongs to the same group as the theorem in course of demonstration.

Ex. 4. The hands of a clock at three o'clock form an angle equal to a rt. $\angle$. This angle will fit the space about the pivot of the hands exactly four times. How many times is two thirds of a rt. $\angle$ contained in a perigon? Four thirds of a rt. $\angle$? Four fifths of a rt. $\angle$?

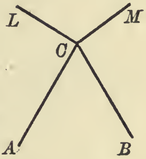
Ex. 5. The bisectors of two supplemental adjacent angles form a rt. $\angle$.



Ex. 6. If the bisectors of two adjacent $\angle$s are $\perp$ to each other, the $\angle$s are supplementary.

Ex. 7. The bisector of an angle is, when produced, the bisector of its vertical angle.

Ex. 8. The bisectors of a pair of vertical angles form the same straight line.



Ex. 9. The bisectors of two pairs of vertical angles are perpendicular to each other.

Ex. 10. If the sides of $\angle LCM$ are perpendicular to the sides of $\angle ACB$, prove that the angles are supplemental.

Ex. 11. If through a point, A, four straight lines, AB, AC, AE, AF, are drawn so that $\angle BAC = \angle EAF$, and $\angle BAF = \angle CAE$, then FAC and BAE are straight lines.

Ex. 12. What relation does Ex. 11 bear to I, 4?

I. SUMMARY OF PROPOSITIONS IN GROUP ON ADJACENT AND VERTICAL ANGLES

1. *If from the same point in a line any number of lines are drawn on the same side of the line, the sum of the successive angles formed equals two right angles.*

SCH. If the angles are on both sides of the line, their sum equals four right angles.

2. *If two angles are adjacent and have their exterior sides in the same straight line, they are supplemental.*

3. *If two angles are adjacent and supplemental, their exterior sides form the same straight line.*

4. *If two straight lines intersect, the vertical angles formed are equal.*

II. THE PARALLEL GROUP

DEFINITIONS

Two lines are said to be **Parallel** when they are so situated that if cut by any transversal the corresponding exterior-interior angles are equal.

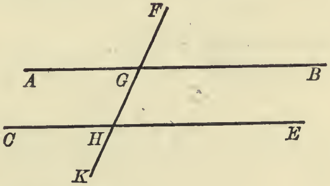
Two Lines Perpendicular to a Third. Direct inferences from the definition of parallels:

(1) If two lines are perpendicular to a third, they are parallel.

(2) A line perpendicular to one of two parallels is perpendicular to the other.

II. 1. *If two parallels are crossed by a third line, the alternate interior angles are equal.*

Hyp. If AB and CE are $\parallel$, and are crossed by the transversal FK,



Conc.: then $\angle AGH = \angle GHE$.

Dem. $\angle FGB = \angle GHE$. (Def. of $\parallel$s.)

$\angle FGB = \angle AGH$.

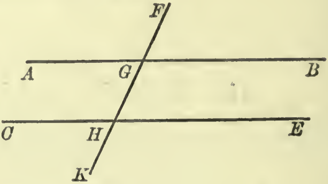
[If two straight lines intersect, the vert. $\angle$s formed are $=$.] (I. 4.)

$\therefore \angle AGH = \angle GHE$. (Ax. 1.)

Q.E.D.

II. 1 *a.* *If two parallels are crossed by a third line, the alternate exterior angles are equal.*

Hyp. If AB and CE are ∥, and are crossed by the transversal KF



Conc.; then $\angle AGF = \angle EHK.$

Dem. $\angle AGF = \angle BGH.$

[If two straight lines intersect, the vert. $\angle$s formed are =.] (I. 4.)

$\angle BGH = \angle GHC.$ (II. 1.)

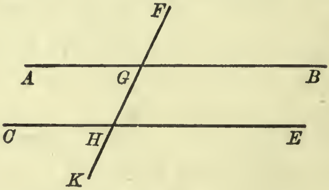
$\angle GHC = \angle EHK.$ (I. 4.)

$\therefore \angle AGF = \angle EHK.$ (Ax. 1.)

Q.E.D.

II. 2. *If two parallels are crossed by a third line, the corresponding interior angles are supplemental.*

Hyp. If AB and CE are ∥, and are crossed by the transversal KF,



Conc.: then $\angle BGH + \angle GHE = 2$ rt. $\angle$s.

Dem. $\angle BGH + \angle BGF = 2$ rt. $\angle$s. (1.)

[If from the same point in a line any number of lines are drawn on the same side of the line, the sum of the successive $\angle$s formed = 2 rt. $\angle$s.] (I. 1.)

$\angle BGF = \angle GHE.$ (Def. of ∥s.)

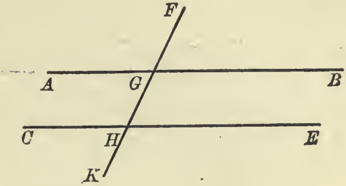
Substituting $\angle GHE$ for its equal $\angle BGF$, we have, from (1),

$\angle BGH + \angle GHE = 2$ rt. $\angle$s.

Q.E.D.

II. 2 *a*. *If two parallels are crossed by a third line, the corresponding exterior angles are supplemental.*

Hyp. If AB and CE are ∥, and are crossed by the transversal KF,



Conc.: then $\angle AGF + \angle CHK = 2$ rt. $\angle$s.

Dem. $\angle AGF + \angle AGH = 2$ rt. $\angle$s.

[If from the same point in a line any number of lines, etc.] (I. 1.)

$\angle AGH = \angle CHK$. (Def. of ∥s.)

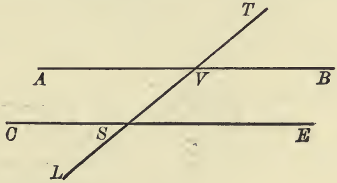
∴ as in II. 2,

$\angle AGF + \angle CHK = 2$ rt. $\angle$s.

Q.E.D.

II. 3. *If two lines are crossed by a third so as to make the alternate interior angles equal, the lines are parallel.*

Hyp. If TL crosses AB and CE so that the alt. int. $\angle$s AVS and VSE are equal,



Conc.: then $AB \parallel CE$.

Dem. $\angle AVS = \angle TVB$.

[If two straight lines intersect, the vert. $\angle$s, etc.] (I. 4.)

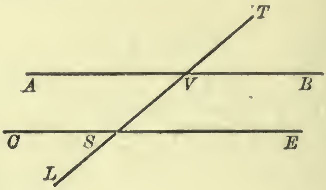
$\angle AVS = \angle VSE$. (Hyp.)

∴ $\angle TVB = \angle VSE$. (Ax. 1.)

∴ $AB \parallel CE$. (Def. of ∥s.)

Q.E.D.

II. 3 *a*. *If two lines are crossed by a third so as to make the alternate exterior angles equal, the lines are parallel.*



Hyp. If AB and CE are crossed by the transversal TL, and if the alt. ext. $\angle$s TVB and CSL are equal,

Conc.: then $AB \parallel CE$.

Dem. $\angle CSL = \angle VSE$.

[If two straight lines intersect, the vert. $\angle$s, etc.] (I. 4.)

$\angle CSL = \angle TVB$. (Hyp.)

$\therefore \angle VSE = \angle TVB$. (Ax. 1.)

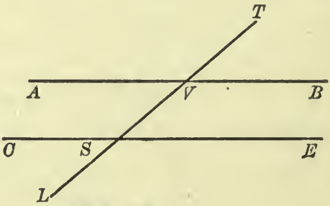
$\therefore AB \parallel CE$. (Def. of $\parallel$s.)

Q.E.D.

II. 4. *If two lines are crossed by a third so as to make*

(1) *the corresponding interior angles supplemental, or*

(2) *the corresponding exterior angles supplemental, the lines are parallel.*



Hyp. (1) If the corr. int. $\angle$s BVS and VSE are supplemental,

Conc.: then $AB \parallel CE$.

Dem. $\angle BVS + \angle VSE = 2$ rt. $\angle$s. (Hyp.)

$\angle BVS + \angle TVB = 2$ rt. $\angle$s.

[If from the same point in a line any number of lines, etc.] (I. 1.)

$\therefore \angle VSE = \angle TVB$. (Ax. 1.)

$\therefore AB \parallel CE$. (Def. of $\parallel$s.)

Q.E.D.

Hyp. (2) If the corr. ext. $\angle$s TVB and LSE are supplemental,

Conc.: then $AB \parallel CE$.

Dem. Similar to that of II. 4 (1).
(Let the pupil supply the proof.)

II. 5. (1) *If two angles have their sides respectively parallel, they are equal, if of the same kind.*

FIG. I.

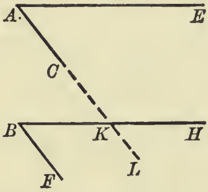
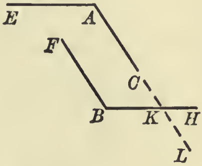


FIG. II.



Hyp. (1) If $\angle A$ and $\angle B$ are of the same kind, and if $AC \parallel BF$ and $AE \parallel BH$,

Conc.: then $\angle A = \angle B$ in Fig. I and Fig. II.

Dem. FIG. I. Extend AC to cross BH at K.

$\angle A = \angle HKL$. (Def. of $\parallel$s.)

$\angle B = \angle HKL$. (Def. of $\parallel$s.)

$\therefore \angle A = \angle B$. (Ax. 1.)

Q.E.D.

NOTE. — In Fig. I the sides of the angles extend in the same direction from the vertices.

Dem. FIG. II. Extend AC to cross BH at K.

$\angle A = \angle AKH$. (II. 1.)

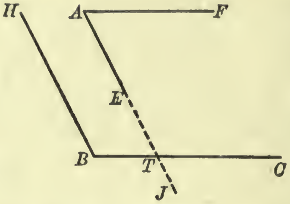
$\angle B = \angle AKH$. (Def. of $\parallel$s.)

$\therefore \angle A = \angle B$. (Ax. 1.)

Q.E.D.

NOTE. — In Fig. II the sides of the angles extend in opposite directions from the vertices.

II. 5. (2) *If two angles have their sides respectively parallel, they are supplemental, if of different kinds.*



Hyp. If $\angle A$ and $\angle B$ are of different kinds, and if $AF \parallel BC$ and $AE \parallel HB$,

Conc.: then $\angle A$ is supplemental to $\angle B$.

Dem. Extend AE to intersect BC at T.

$\angle A$ is supplemental to $\angle ATC$. (II. 2.)

$\angle ATC = \angle B$. (Def. of ‖s.)

$\therefore$ $\angle A$ is supplemental to $\angle B$. (Ax. 1.)

Q.E.D.

Note. — Two sides of $\angle A$ and $\angle B$ extend in the same direction from the vertices A and B, while the other two sides, BH and AE, extend in opposite directions from A and B.

The theorem may therefore be stated thus:

If two angles have their sides respectively parallel, and two extend in the same, two in the opposite directions from their vertices, they are supplemental.

Ex. 1. If two parallels be crossed by a transversal, and any angle is a right angle, what is the value of each of the others?

Ex. 2. Lines ‖ to the same line are ‖ to each other.

Ex. 3. Show that Theorem 3 is the converse of Theorem 1.

Ex. 4. Prove that a line parallel to the base of a triangle cuts off a △ whose angles are respectively equal to the angles of the original △.

Ex. 5. The bisectors of a pair of alt. int. ∠s of parallels are parallel.

Ex. 6. The bisectors of a pair of corresponding interior angles are perpendicular to each other.

Ex. 7. The parallel to the base of an isoangular triangle cuts off another isoangular triangle.

II. SUMMARY OF PROPOSITIONS IN PARALLEL GROUP

1. *If two parallels are crossed by a third line, the alternate interior angles are equal.*

    a *If two parallels are crossed by a third line, the alternate exterior angles are equal.*

2. *If two parallels are crossed by a third line, the corresponding interior angles are supplemental.*

    a *If two parallels are crossed by a third line, the corresponding exterior angles are supplemental.*

3. *If two lines are crossed by a third so as to make the alternate interior angles equal, the lines are parallel.*

    a *If two lines are crossed by a third so as to make the alternate exterior angles equal, the lines are parallel.*

4. *If two lines are crossed by a third so as to make:*

(1) *the corresponding interior angles supplemental, or*

(2) *the corresponding exterior angles supplemental, the lines are parallel.*

5. (1) *If two angles have their sides respectively parallel, they are equal, if of the same kind.*

(2) *If two angles have their sides respectively parallel, they are supplemental, if of different kinds.*

III. THE $(2n-4)$ RIGHT ANGLES GROUP

Briefly: The $(2n-4)$ Group

DEFINITIONS

A triangle is a figure formed by the intersection of three lines not passing through the same point.

If no two sides of a triangle be equal, the triangle is said to be **Scalene** (limping).

If two sides of a triangle be equal, the triangle is said to be **Isosceles.**

If two angles of a triangle be equal, the triangle is said to be **Isoangular.**

If three sides be equal, the triangle is said to be **Equilateral.**

If three angles be equal, the triangle is said to be **Equiangular.**

If three angles be acute, the triangle is called an **Acute Triangle.**

If one angle be obtuse, the triangle is called an **Obtuse Triangle.**

If one angle be right, the triangle is called a **Right Triangle.**

In a right triangle the side opposite the right angle is the **Hypotenuse.**

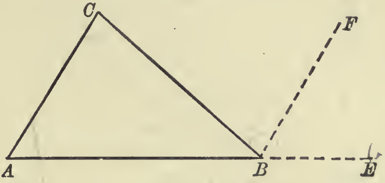
An **Altitude** of a triangle is a perpendicular from a vertex to the opposite side. This side is called the **Base.**

A **Median** of a triangle is a line from a vertex to the middle point of the opposite side.

The **Vertex Angle** of a triangle is the angle opposite the base.

The **Exterior Angle** of a triangle is the angle between one side and a second side produced.

III. 1. *If a figure is a triangle, the sum of the interior angles equals two right angles.*



Hyp. If the figure ABC is a triangle,

Conc.: then $\angle A + \angle C + \angle ABC = 2$ rt. $\angle$s.

Dem. Produce AB to any point E.

Draw $BF \parallel AC$.

Then $\angle A = \angle FBE$. (Def. of $\parallel$s.)

$\angle C = \angle FBC$.

[If two $\parallel$s are crossed by a third line, the alt. int. $\angle$s, etc.] (II. 1.)

But $\angle FBE + \angle FBC + \angle ABC = 2$ rt. $\angle$s.

[If from the same point in a line any number of lines are drawn on the same side of the line, the sum, etc.] (I. 1.)

$\therefore \angle A + \angle C + \angle ABC = 2$ rt. $\angle$s.

Q.E.D.

III. 1 *a*. *One interior angle of a triangle is the supplement of the sum of the second and third angles.*

III. 1 *b*. *In a right triangle, either acute angle is the complement of the other acute angle.*

Ex. 1. In a triangle $\angle a = 2\angle b$, $\angle b = 3\angle c$. What is the value of $\angle a$, $\angle b$, and $\angle c$?

Ex. 2. In a triangle $\angle a + \angle b = \frac{5}{3}$ rt. $\angle$, $\angle a - \angle b = \frac{1}{5}$ rt. $\angle$. What is the value of $\angle a$, $\angle b$, and $\angle c$?

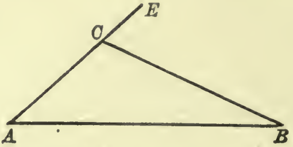
Ex. 3. What is the value of each acute $\angle$ in an isoangular right $\triangle$?

Ex. 4. If 2 $\angle$s of one $\triangle$ are equal respectively to 2 $\angle$s of another, the 3d $\angle$s are equal.

Ex. 5. The vertex $\angle$ of an isoangular triangle is $\frac{5}{9}$ rt. $\angle$. Find the value of the $\angle$ between the base and an altitude on one leg.

III. 2. *Any exterior angle of a triangle equals the sum of the two non-adjacent interior angles.*

Hyp. If the figure *ABC* is a triangle,



Conc.: then $\angle BCE = \angle A + \angle B.$

Dem. $\angle ECB + \angle BCA = 2$ rt. $\angle$s.

[If from the same point in a line any number of lines are drawn on the same side of the line, the sum, etc.] (I. 1.)

$\angle A + \angle B + \angle ACB = 2$ rt. $\angle$s. (III. 1.)

$\therefore \angle BCE + \angle BCA = \angle A + \angle B + \angle ACB.$ (Ax. 1.)

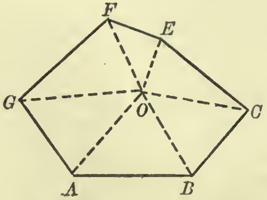
$\therefore \angle BCE = \angle A + \angle B.$ (Ax. 2.)

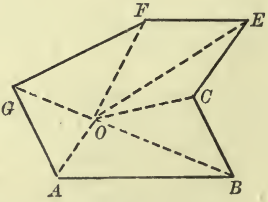
Q.E.D.

III. 2. Sch. The exterior angle of a triangle is greater than either of the non-adjacent interior angles.

III. 2 *a*. *The exterior vertex angle of an isoangular triangle equals twice either interior base angle.*

III. 3. *If a figure is a polygon of* n *sides, the sum of the interior angles equals* $(2n - 4)$ *right angles.*





Hyp. If $ABC \cdots G$ is a polygon of n sides,

Conc.: then $\angle A + \angle B + \angle C$, etc. $= (2n - 4)$ rt. $\angle$s.

Dem. From any point O within the polygon draw the joins, OA, OB, OC, etc.

We thus obtain as many triangles as there are sides, namely, n. The sum of the interior angles of each $\triangle = 2$ rt. $\angle$s. (III. 1.)

$\therefore$ the sum of the int. $\angle$s of n $\triangle$s $= n \times 2$ rt. $\angle$s, or $2n$ rt. $\angle$s.

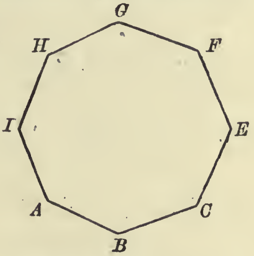
But the base angles only of these triangles compose the angles of the polygon.

$\therefore$ from the $2n$ rt. $\angle$s we must subtract the sum of the angles about O, or 4 rt. $\angle$s. (Sch. to I. 1.)

$\therefore \angle A + \angle B + \angle C$, etc. $= 2n$ rt. $\angle$s $-$ 4 rt. $\angle$s, or $(2n-4)$ rt. $\angle$s.

Q.E.D.

III. 3 *a*. *In a regular polygon each interior angle equals* $\frac{2n-4}{n}$ *right angles.*



Hyp. If a polygon $ABC \cdots HI$ of n sides is regular,

Conc.: then any interior angle, as A, $= \frac{2n-4}{n}$ rt. $\angle$s.

Dem. $\angle A = \angle B = \angle C =$ etc. (Def. of regular polygon.)

$\therefore \angle A + \angle B + \angle C + \cdots = n \angle A = (2n-4)$ rt. $\angle$s. (III. 3.)

$$\therefore \angle A = \frac{1}{n} \text{ of } (2n-4) \text{ rt. } \angle\text{s} = \frac{2n-4}{n} \text{ rt. } \angle\text{s}.$$

Q.E.D.

Ex. 6. The sum of the exterior oblique angles of a right triangle equals 3 rt. $\angle$s.

Ex. 7. What is the sum of the interior angles in a 4-side? a pentagon? a hexagon? an octagon?

Ex. 8. What is the value of each interior angle in the above if each polygon is equiangular?

Ex. 9. How many sides has the regular polygon one of whose interior angles is $\frac{6}{5}$ rt. $\angle$? $\frac{5}{3}$ rt. $\angle$? $\frac{9}{5}$ rt. $\angle$? $\frac{7}{4}$ rt. $\angle$?

III. SUMMARY OF PROPOSITIONS IN THE $(2n-4)$ GROUP

1. *If a figure is a triangle, the sum of the interior angles equals two right angles.*

   a *One interior angle of a triangle is the supplement of the sum of the second and third angles.*

   b *In a right triangle, either acute angle is the complement of the other acute angle.*

2. *Any exterior angle of a triangle equals the sum of the two non-adjacent interior angles.*

SCH. The exterior angle of a triangle is greater than either of the non-adjacent interior angles.

   a *The exterior vertex angle of an isoangular triangle equals twice either interior base angle.*

3. *If a figure is a polygon of n sides, the sum of the interior angles equals $(2n-4)$ right angles.*

   a *In a regular polygon each interior angle equals* $\frac{2n-4}{n}$ *right angles.*

Ex 10. Prove that the sum of the ext. ∠s of a polygon formed by producing the sides in order at all the vertices in succession equals 4 rt. ∠s.

Ex. 11. What polygon has the sum of the interior angles equal to three times the sum of the exterior angles? One half the sum of the ext. ∠s?

Ex. 12. If 2 ∠s have their sides respectively ⊥, they are equal if of the same kind, and supplemental if of different kinds.

Ex. 13. Can the plane space about a point be filled without overlapping by equiangular triangles? By regular pentagons? By regular octagons and squares? By regular dodecagons and equilateral triangles?

Ex. 14. If in a triangle the altitudes from the extremities of the base be drawn to the two sides, prove that the angle formed by them is the supplement of the vertex angle.

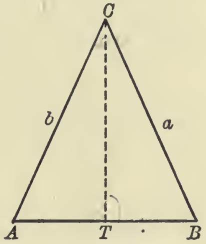
IV. GROUP ON ISOSCELES AND SCALENE TRIANGLES

PROPOSITIONS

The Isosceles Triangle

IV. 1. *If a triangle is isosceles, it is isoangular.*

Hyp. If $\triangle ABC$ is isosceles; that is if $a = b$,



Conc.: then $\angle A = \angle B$.

Dem. Draw CT bisecting $\angle ACB$.
On CT as an axis fold over $\triangle CTB$ to plane of $\triangle CTA$.
Side a will fall on side b, $\because \angle BCT = \angle ACT$ by const. (Ax. 7.)
B will fall on A, $\because$ side $a =$ side b by hyp.

$\therefore$ TB coincides with TA. (Ax. 6.)

$\therefore \angle A = \angle B$.

Q.E.D.

Note.—It will assist the student to remember the sequence of the theorems in Group IV if he observes that in the hypothesis of both IV. 1. and IV. 4. are given the relations between the two sides of the triangle; in the conclusion, the relations between the angles opposite those sides.

IV. 1 *a. If the vertex angle of an isosceles triangle is bisected, the bisector is identical with*

(1) *the altitude to the base,*

(2) *the median to the base.*

Hyp. 1. If $\triangle ABC$ is isosceles, and if CT bisects $\angle ACB$,

Conc.: then CT is identical with the altitude from C to AB.

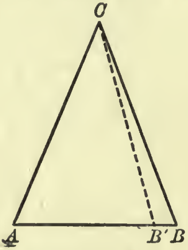
Hyp. 2. If $\triangle ABC$ is isosceles, and if CT bisects $\angle ACB$,

Conc.: then CT is identical with the median from C to AB.

SCH. In an isosceles triangle, the altitude to the base is identical with the median to the base. (Ax. 10.)

IV. 2. *If a triangle is isoangular, it is isosceles.*

Hyp. If $\triangle ABC$ is isoangular; that is, if $\angle A = \angle B$,



Conc.: then $CB = CA$.

Dem. If CB does not equal CA, draw CB' that does.

In other words, suppose $AB'C$ is an isosceles triangle of which CA and CB' are the equal sides.

Then $\angle CB'A = \angle A$. (IV. 1.)

But $\angle B = \angle A$. (Hyp.)

$\therefore \angle CB'A = \angle B$. (Ax. 1.)

$\therefore$ as CB and CB' have C in common, and as both make the same angle with AB, it follows that CB must coincide with CB'. (Ax. 7.)

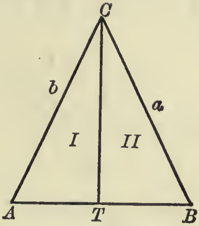
But CB' was drawn equal to CA.

$\therefore CB = CA$.

Q.E.D

IV. 3. *If the altitude of a triangle bisects the vertex angle, the triangle is isosceles.*

Hyp. If, in $\triangle ABC$, $CT \perp AB$, and if CT bisects $\angle ACB$,



Conc.: then $\triangle ACB$ is isosceles; *i.e.* $a = b$.

Dem. On CT as an axis revolve $\triangle$ II to plane of $\triangle$ I.

CB will fall on CA, $\because \angle BCT = \angle ACT$ by hypothesis. (Ax. 7.)

B will fall on b, or its prolongation.

TB will fall on TA, $\because \angle CTB$ and $\angle CTA$ are right angles by hypothesis. (Ax. 7.)

B falls on TA, or its prolongation.

As B lies in b, and also in TA, it must fall on A. (Ax. 5.)

$\therefore a = b$, and $\triangle ACB$ is isosceles.

Q.E.D.

IV. 3 *a*. *If the altitude of a triangle bisects the base, the triangle is isosceles.*

Hyp. If, in $\triangle ABC$, $CM \perp AB$, and if $AM = BM$,

Conc.: then $\triangle ACB$ is isosceles; *i.e.* $a = b$.

Dem. is similar to the preceding. Let the pupil supply it.

Suggestion for Notation. — Letter foot of altitude, H; of median, M; of bisector of vertex angle, T.

General Suggestions. Does not your greatest difficulty in proving a theorem lie in these two points:

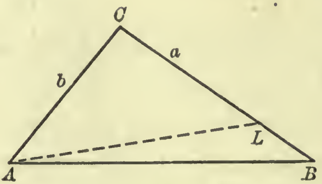
First, that you forget the hypothesis and conclusion?

Second, that you do not clearly remember the definition of the term you are using?

It is most important, therefore, that you should know what an altitude is; what a scalene triangle is; what a median is; in short, the exact meaning of *every term* you use.

THE SCALENE TRIANGLE

IV. 4. *In any triangle the greater angle lies opposite the greater side.*



Hyp. If,
in $\triangle ABC$,
$a > b$,

Conc.: then $\angle A > \angle B$.

Dem. Lay off $CL = b$, and draw AL.
$\triangle LAC$ is isosceles. (Def.)

$\therefore \angle CAL = \angle CLA$. (IV. 1.)

But $\angle CLA > \angle B$.

[The exterior angle of a $\triangle$ is greater, etc.] (III. 1. Sch.)

$\therefore$ its equal, $\angle CAL > \angle B$.

$\therefore$ all the more is $\angle CAB > \angle B$.

Q.E.D.

NOTE. — It does not follow that $\angle A$ is twice $\angle B$, if side a is twice side b. The relative length of the sides of a triangle as compared with the size of the angles opposite which they lie, is treated in trigonometry.

Ex. 1. If a triangle is equilateral, it is equiangular.

Ex. 2. If a triangle is equiangular, it is equilateral.

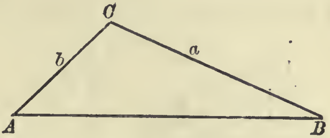
Ex. 3. If the vertex angle of an isosceles triangle is twice either base angle, what is the value of each angle of the triangle?

Ex. 4. Prove that the bisector of the exterior vertex angle of an isosceles triangle is parallel to the base.

Ex. 5. State the converse of Ex. 4.

Ex. 6. If the vertex angles of two isosceles triangles are supplemental, the base angles are complemental.

IV. 5. *In any triangle the greater side lies opposite the greater angle.*



Hyp. If, in $\triangle ABC$, $\angle A > \angle B$,

Conc.: then $a > b$.

Dem. a must be $> b$, $< b$, or $= b$.

If $a < b$, $\angle A < \angle B$. (IV. 4.)

If $a = b$, $\angle A = \angle B$. (IV. 1.)

Both these conclusions are contrary to the hypothesis (*q.v.*).

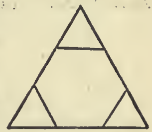
$\therefore a > b$.

Q.E.D.

IV. 5*a*. *In any right triangle the hypotenuse is greater than either side.*

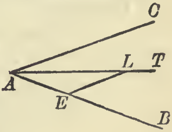
Sch. A perpendicular is the shortest distance from a point to a line.

Note.—Distance from a point to a line is measured on the perpendicular through the point.



Ex. 7. If the sides of a regular hexagon be produced until they meet, prove that an equilateral triangle is formed.

Ex. 8. The angle between the base of an isosceles triangle and the altitude on one of the legs equals one half the vertex angle.

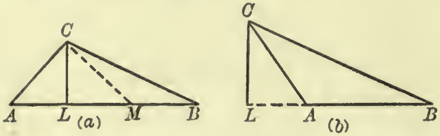


Ex. 9. If AT bisects $\angle CAB$. and we draw $LE = EA$, prove that $EL \parallel CA$.

Ex. 10. If, through the vertex of the vertex angle A of the isosceles $\triangle ABC$, LC is drawn $\perp CA$ and $CM \perp CB$, prove $\angle LCM$ is the double of $\angle A$. (Fig. 1, p. 41.)

Ex. 11. If $\triangle$s ACB and LCM (Fig. 1, p. 41) are isosceles, and the sides of the first are perpendicular to the sides of the second, prove that $\angle L$ is the complement of $\angle A$.

IV. 6. *In any triangle, if the altitude to the base is drawn, the side cutting off the greater distance from the foot of the altitude is the greater.*



Hyp. If, in any scalene $\triangle ABC$, the altitude CL to the base AB is drawn, and $LB > LA$,

Conc.: then $CB > CA$.

Dem. (*a*) If the altitude falls within the triangle, lay off $LM = LA$.

$\triangle ACM$ is isosceles. (IV. 3 *a.*)

$$\therefore \angle A = \angle LMC.$$

$\triangle CLM$ is a right triangle, $\angle CLM$ being a right $\angle$. (Hyp.)

$\therefore \angle LMC$ is acute.

[One interior angle of a triangle is the supplement of the sum of the second and third angles.] (III. 1 *a.*)

$\therefore \angle CMB$ is obtuse, being the supplement of $\angle LMC$.

$\angle B$ is acute, being $< \angle LMC$.

[The exterior angle of a triangle is greater than either opposite interior angle.] (III. 2. Sch.)

$$\therefore CB > CM. \quad \text{(IV. 5.)}$$

$$CM = CA. \quad \text{(IV. 3 } a.\text{)}$$

$$\therefore CB > CA.$$

Or thus: $\angle CMB > \angle CLM = \angle CLA > \angle B$.

$$\therefore CB > CM.$$

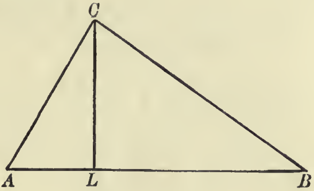
$$CM = CA.$$

$$\therefore CB > CA.$$

Q.E.D.

Dem. (*b*) When the altitude falls without the triangle, the proof is the same as in (*a*), omitting the first step.

IV. 7. *In any triangle, if the altitude to the base is drawn, the greater side cuts off the greater distance from the foot of the altitude.*



Hyp. If, in $\triangle ABC$, $CL \perp AB$, and if $CB > CA$,

Conc.: then $LB > LA$.

Dem. LB must be $< LA$, $= LA$, or $> LA$.

If $LB < LA$, then $CB < CA$. (IV. 6.)

If $LB = LA$, then $CB = CA$. (IV. 3 *a*.)

Each of these conclusions is contrary to the hypothesis, viz., that $CB > CA$.

$\therefore$ LB must be $> LA$.

Q.E.D.

Ex. 12. In Fig. 1, show that $LM \parallel AB$.

Ex. 13. In Fig. 2, show that if CH is the altitude of $\triangle ABC$, CH' the altitude of $\triangle CML$, CH and CH' are in the same straight line.

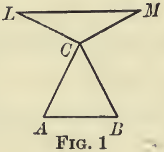


FIG. 1

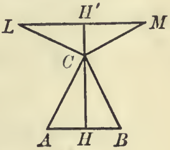


FIG. 2

Ex. 14. If the bisectors of $\angle A$ and $\angle B$ of $\triangle ABC$ (Fig. 3) intersect in M, and through M, EF is drawn $\parallel$ to AB, cutting AC in E and BC in F, prove that the $\triangle$s AEM and MBF are isosceles.

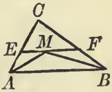


FIG. 3

Ex. 15. Prove in the figure for Ex. 14 that $EF = AE + BF$.

Ex. 16. Prove that if a leg of an isosceles triangle (Fig. 4) is extended its own length from the vertex, the join of its extremity with the extremity of the base is perpendicular to the base.

FIG. 4

IV. SUMMARY OF PROPOSITIONS IN THE GROUP ON THE TRIANGLE

THE ISOSCELES TRIANGLE

1. *If a triangle is isosceles, it is isoangular.*

   a *If the vertex angle of an isosceles triangle is bisected, the bisector is identical with* (1) *the altitude to the base,* (2) *the median to the base.*

SCH. In an isosceles triangle, the altitude to the base is identical with the median to the base.

2. *If a triangle is isoangular, it is isosceles.*

3. *If the altitude of a triangle bisects the vertex angle, the triangle is isosceles.*

   a *If the altitude of a triangle bisects the base, the triangle is isosceles.*

THE SCALENE TRIANGLE

4. *In any triangle the greater angle lies opposite the greater side.*

   a *In any right triangle the hypotenuse is greater than either side.*

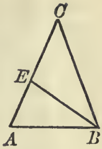
SCH. A perpendicular is the shortest distance from a point to a line.

5. *In any triangle the greater side lies opposite the greater angle.*

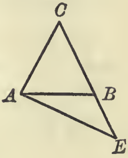
6. *In any triangle, if the altitude to the base is drawn, the side cutting off the greater distance from the foot of the altitude is the greater.*

7. *In any triangle, if the altitude to the base is drawn, the greater side cuts off the greater distance from the foot of the altitude.*

Ex. 17. If one $\angle$ of a $\triangle =$ the sum of the other two, the $\triangle$ can be divided into isosceles $\triangle$s.

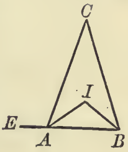


Ex. 18. If, in the isosceles $\triangle ABC$, $\angle C = \frac{2}{5}$ rt. $\angle$, and BE is taken equal to AB, then the angles of $\triangle EAB$ equal the angles of the original triangle, and $\triangle ECB$ is isosceles.



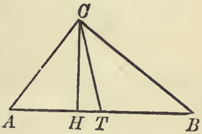
Ex. 19. If, in an isosceles $\triangle ABC$, from any point E in CB produced, EA is drawn, then $\angle BAC = \frac{\angle CAE + \angle E}{2}$.

Ex. 20. The angle formed by the bisectors of the interior base angles of a triangle equals a rt. $\angle + \frac{1}{2}$ the vertex angle.



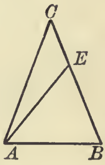
Ex. 21. The exterior base angle of an isosceles triangle equals the angle formed by the bisectors of the two interior base angles; that is, $\angle CAE = \angle AIB$.

Ex. 22. In $\triangle ABC$, if CT bisects $\angle C$, and CH is an altitude, prove that $\angle HCT = \frac{\angle A - \angle B}{2}$.



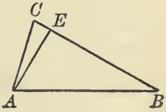
Ex. 23. Prove that in a 4-side the sum of three interior angles minus the exterior angle at the fourth vertex equals 2 rt. $\angle$s.

Ex. 24. In any triangle the three new triangles formed by the bisectors of all the exterior angles of the triangle are mutually equiangular.



Ex. 25. If, in an isosceles $\triangle ABC$, AE is drawn to any point, as E in BC, then $\angle CEA$ is greater than $\angle EAC$.

Ex. 26. If, in the isosceles $\triangle ABC$, E is any point in BC, prove that AE is greater than BE.



Ex. 27. If, in $\triangle ABC$, AE is perpendicular to BC, then $AC + CB$ is greater than $AE + EB$.

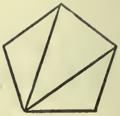
What is the greatest side of the triangle?

Ex. 28. If the vertex angle of an isosceles triangle is twice the base angle, the bisector of the vertex angle divides the triangle into two isosceles triangles.

Ex. 29. If, in an isosceles triangle, either base angle equals twice the vertex angle, the vertex angle is $\frac{2}{5}$ rt. ∠.

Ex. 30. If, in an isosceles triangle, either base angle equals twice the vertex angle, the vertex angle is $\frac{2}{5}$ rt. ∠.

Ex. 31. In a regular pentagon, what is the value of the interior ∠ between two diagonals drawn from the same vertex?



Ex. 32. Prove that these two diagonals are equal.

V. GROUP ON CONGRUENT TRIANGLES

PROPOSITIONS

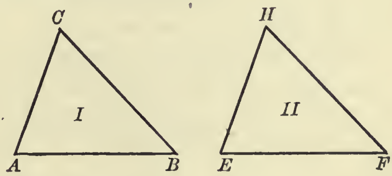
V. 1. *If two triangles have two sides and the included angle of the first equal to two sides and the included angle of the second, they are congruent.*

Hyp. If, in $\triangle$ I and II,

$AB = EF,$

$CA = HE,$

and $\angle A = \angle E,$



Conc.: then $\triangle \text{I} \cong \triangle \text{II}.$

Dem. Place $\triangle$ II on $\triangle$ I with EF in coincident superposition with AB, E falling on A.

Then EH will fall on AC, because, by hyp., $\angle E = \angle A$. (Ax. 7.)

$\therefore$ H will fall on C, because $EH = AC$. (Hyp.)

$\therefore$ HF coincides with CB. (Ax. 6.)

$$\therefore \triangle \text{I} \cong \triangle \text{II}.$$

Q.E.D.

Ex. 1. If two triangles are congruent, the following homologous lines (*i.e.* lines having the same relative position) are equal:

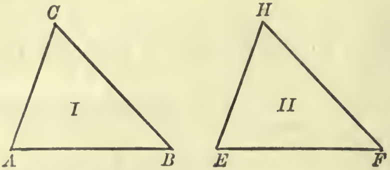
(*a*) the homologous medians;
(*b*) the homologous altitudes;
(*c*) the homologous bisectors of the interior angles.

Ex. 2. If two altitudes of a triangle are equal, the triangle is isosceles.

Ex. 3. Prove that the altitudes of an equilateral triangle are equal.

Ex. 4. Two isosceles triangles are congruent if the vertex angle and its bisector in one are equal to the corresponding parts of the other.

V. 2. *If two triangles have two angles and the included side of the first equal to two angles and the included side of the second, they are congruent.*



Hyp. If, in ⊿ I and II,
$\angle A = \angle E$,
$\angle B = \angle F$,
and $AB = EF$,

Conc.: then $\triangle \text{I} \cong \triangle \text{II}$.

Dem. Place △ II on △ I so that EF is in coincident superposition with AB, E falling on A.

EH will fall on AC, since, by hyp., $\angle A = \angle E$. (Ax. 7.)

$\therefore$ H lies on AC, or its prolongation.

Again, FH will fall on BC, since, by hyp., $\angle F = \angle B$. (Ax. 7.)

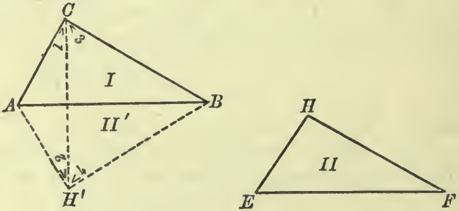
$\therefore$ H lies on BC, or its prolongation.

Since H lies on both AC and BC, it must lie at their intersection. (Ax. 5.)

$$\therefore \triangle \text{I} \cong \triangle \text{II}.$$

Q.E.D.

V. 3. *If two triangles have three sides of the one equal to three sides of the other, they are congruent.*



Hyp. If, in ⊿ I and II,
$AC = EH$,
$AB = EF$,
and $BC = FH$,

Conc.: then $\triangle \text{I} \cong \triangle \text{II}$.

Dem. Place △ II in the position of △ II′ with EF in coincident superposition with AB, E falling on A. Draw CH'.

$\triangle CAH'$ is isosceles; also $\triangle BCH'$. (Hyp.)

$$\therefore \angle 1 = \angle 2 \text{ and } \angle 3 = \angle 4.$$

[If a triangle is isosceles, it is isoangular.] (IV. 1.)

$\angle 1 + \angle 3 = \angle 2 + \angle 4$; that is, $\angle ACB = \angle AH'B$. (Ax. 2.)

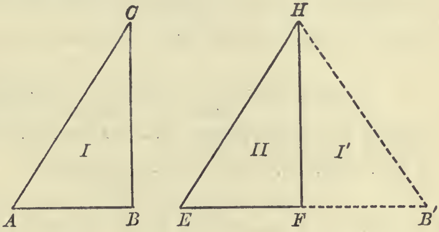
$\therefore \triangle \text{I} \cong \triangle \text{II}'$. (V. 1.)

$\therefore \triangle \text{I} \cong \triangle \text{II}$. (Ax. 1.)

Q.E.D.

V. 4. *If two right triangles have the hypotenuse and a leg of the one equal to the hypotenuse and a leg of the other, they are congruent.*

Hyp. If, in ⊿s I and II, $CB = HF$ and $CA = HE$, and $\angle B$ and $\angle F$ are right angles,



Conc.: then rt. $\triangle \text{I} \cong$ rt. $\triangle \text{II}$.

Dem. Place $\triangle$ I in position of $\triangle$ I′ so that CB is in coincident superposition with HF, C falling on H. EFB is straight.

[If two supp. ⦤s are adj., their ext. sides, etc.] (I. 2.)

$\angle B' = \angle E$.

(An isosceles triangle is isoangular.) (IV. 1.)

$\therefore \angle EHF = \angle FHB'$. (Why ?)

$\triangle \text{I}' \cong \triangle \text{II}$.

[Two ⦤s and the included side of one equal, etc.] (V. 2.)

$\therefore \triangle \text{I} \cong \triangle \text{II}$. (Ax. 1.)

Q.E.D.

Sch. Since congruent triangles may be placed in coincident superposition, it follows that homologous altitudes, medians, angle bisectors, mid-joins, and all other corresponding parts are respectively equal.

V. SUMMARY OF PROPOSITIONS IN THE GROUP ON CONGRUENT TRIANGLES

1. *If two triangles have two sides and the included angle of the first equal to two sides and the included angle of the second, they are congruent.*

2. *If two triangles have two angles and the included side of the first equal to two angles and the included side of the second, they are congruent.*

3. *If two triangles have three sides of the one equal to three sides of the other, they are congruent.*

4. *If two right triangles have the hypotenuse and a leg of the one equal to the hypotenuse and a leg of the other, they are congruent.*

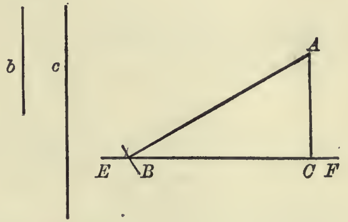
Sch. Since congruent triangles may be placed in coincident superposition, it follows that homologous altitudes, medians, angle bisectors, mid-joins, and all other corresponding parts are respectively equal.

PROBLEMS

Prob. I. *To construct a right triangle having one side and the hypotenuse given.*

Given. The sides b and c.

Required. To construct a triangle.



Const. Take an indefinite line EF.

At C, any point in EF, erect a $\perp$, $CA = b$.

With A as a center and a radius $= c$, describe an arc cutting EF, as in B.

Rt. $\triangle ABC$ is the required right triangle.

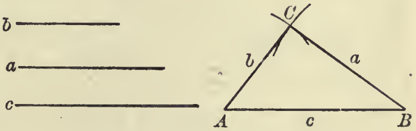
Q.E.F.

Proof. If two right triangles, etc. (V. 4.)

Prob. II. *To construct a triangle, its three sides being given.*

Given. The three sides, a, b, and c.

Required. To construct the triangle.



Const. Draw $AB = c$.

With A as a center and b as a radius, describe an arc.

With B as a center and a as a radius, describe an arc.

Let these two arcs intersect at any point C.

Draw AC and BC.

$\triangle ABC$ is the required triangle.

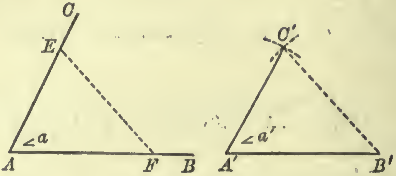
Q.E.F.

Note. — No triangle can be constructed if $a + b < c$.

PROB. III. *To construct an angle equal to a given angle.*

Given. $\angle a$.

Required. To construct an angle equal to $\angle a$.



Const. On AB lay off any distance, AF.
On AC lay off any distance, AE.
Draw the join EF.
Construct a △ whose sides are AF, FE, and AE. (Prob. II.)

$$\triangle A'C'B' \cong \triangle ACB.$$

[Two ⚠ are $\cong$ if three sides of one $=$, etc.] (V. 3.)
Then $\angle a'$ is the required angle.

Q.E.F.

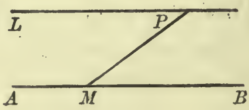
Proof. $\angle a' = \angle a$. (Hom. ∠s of $\cong$ △s.)

Q.E.D.

PROB. IV. *Through a given point to draw a line parallel to a given line.*

Given. The point P and the line AB.

Required. To construct a line through P parallel to AB.



Const. Through P draw any line cutting AB, say at M.
At P draw a line PL, making with PM an $\angle =$ to $\angle PMB$.
PL is the line required.

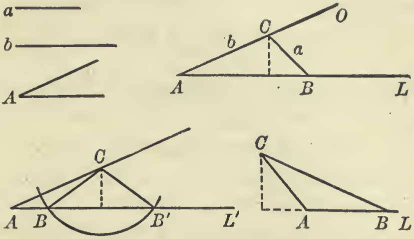
Q.E.F.

Proof. $\angle PMB = \angle LPM$. (Const.)

$\therefore LP \parallel AB$. (II. 3.)

Q.E.D.

Prob. V. *Given two sides and an angle opposite one of them, to construct the triangle.*



Given. The sides a and b, and the angle A.

Required. To construct the triangle.

Const. At one extremity of an indefinite line AL, construct an $\angle LAO = \angle A$.

On AO lay off $AC = b$.

With C as a center and radius equal to a, describe a circle.

The point B, in which the circle cuts AL, will be the third vertex of the triangle.

Q.E.F.

Proof. To be supplied by the student.

From the figures you will observe that several cases arise:

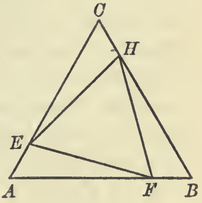
1. $a < b$; two triangles satisfy the conditions.
2. $a > b$; one triangle satisfies the conditions.
3. $a =$ the perpendicular from C to AL; triangle is a right triangle.
4. $a <$ the $\perp$ from C to AL; no $\triangle$ satisfies the conditions.

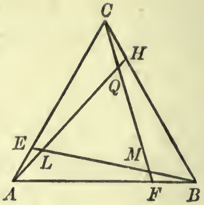
Verify by drawing the figures for each case not shown.

Ex. 5. If $\triangle ABC$ is equilateral and $AE = BF = CH$, show that $\triangle$s ECH, FAE, and FBH are congruent. Hence show that $\triangle EFH$ is equilateral.

Ex. 6. Show that in the $\triangle ABC$ there are three congruent 4-sides.

Ex. 7. If $\triangle ABC$ is equilateral and $AE = CH = BF$, show that ABE, BCF, and ACH are congruent. (See Fig., p. 52.)





Ex. 8. In the adjacent figure show that △ ABH, BCE, and ACF are congruent. (Data as in Ex. 7.)

Ex. 9. Also show that △ AEL, FMB, and CHQ are congruent.

Ex. 10. Hence prove that $\triangle LMQ$ is equilateral.

Ex. 11. Show that △ ALB, BMC, and CQA are congruent.

Ex. 12. Prove that the 4-sides $ALMF$, $BMQH$, and $CQLE$ are congruent.

Ex. 13. Two triangles are congruent if two altitudes and a side to which one of them is drawn in one triangle equal the corresponding parts of the second triangle.

Ex. 14. Two right triangles are congruent, if the altitude to the hypotenuse and the median to the hypotenuse in one triangle are equal to the corresponding parts of the second triangle.

Ex. 15. All the theorems and corollaries concerning congruent triangles in the summary have a condition in common. What is it? What, then, may we infer must be one of the conditions in order to prove two triangles congruent?

Prove that two triangles are congruent if the following parts of one are equal, and are similarly situated, to the corresponding parts of the other:

(Use the following notation: $\angle A$, $\angle B$, $\angle C$; sides opposite these triangles, a, b, c; altitudes to a, b, c, are h_a, h_b, h_c; medians on a, b, c, are m_a, m_b, m_c; angle bisectors are t_a, t_b, t_c; r is the radius of the inscribed circle; r_c is the radius of the circumscribed circle.)

Ex. 16. a, b, and m_b, $\angle A$ being obtuse in both triangles.

Ex. 17. a, b, and h_c.

Ex. 18. c, h_c, m_c.

Ex. 19. b, $\angle A$, h_c, $\angle A$ being acute in both triangles.

Ex. 20. a, $\angle B$, m_c, $\angle B$ being obtuse.

Ex. 21. Two angles and the side opposite one of them being given, to construct the triangle.

Ex. 22. Two rt. △ are $\cong$ if an acute $\angle$ and a leg of one, or an acute $\angle$ and hypotenuse of one, equal the corresponding parts of the other.

Ex. 23. Through a given point without a given line, to draw a line making a given $\angle$ with the given line.

Ex. 24. Construct a triangle, given the two base angles and the bisector of one of them.

VI. GROUP ON PARALLELOGRAMS

DEFINITIONS

THE QUADRILATERAL OR FOUR-SIDE

A **Quadrilateral**, or 4-side, is a figure formed by the intersection of four lines, no three of which pass through the same point. Its alternate angles are called opposite angles.

A **Trapezium** is a 4-side upon which no conditions are imposed.

A **Trapezoid** is a 4-side having one pair of parallel sides, called the bases.

A **Parallelogram** is a 4-side having two pairs of parallel sides.

NOTE.—A parallelogram in the above general form is called a **Rhomboid.**

A **Rectangle** is a parallelogram with two consecutive angles equal.

A **Rhombus** is a parallelogram with two consecutive sides equal.

A **Square** is a rhombus, one of whose angles is a right angle.

The **Mid-join of a Trapezoid** is the line joining the mid-points of the non-parallel sides.

The **Median of a Trapezoid** is the line joining the mid-points of the parallel sides.

NOTE.—If the non-parallel sides of a trapezoid be produced until they meet, the median of the trapezoid becomes a part of the median of the triangle formed by one base of the trapezoid and the non-parallel sides produced. Some authorities give the name *median* to the *mid-join.* The definition above is more consistent with the use of the term *median* in connection with triangles.

An **Isosceles Trapezoid** is a trapezoid having its non-parallel sides equal.

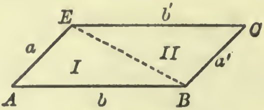
The **Altitude of a Trapezoid or of a Rhomboid** is the perpendicular distance between the bases.

A **Kite** is a 4-side that has 2 pairs of adjacent sides equal.

A **Cyclic Four-side** is one whose vertices lie in a circumference.

PROPOSITIONS

VI. 1. *If a 4-side has two sets of opposite sides equal, it is a parallelogram.*



Hyp. If, in the 4-side A–C, $a = a'$ and $b = b'$,

Conc.: then $a \parallel a'$ and $b \parallel b'$; *i.e.*, the 4-side A–E is a ▱.

Dem. Draw the diagonal EB.

$$\triangle \text{I} \cong \triangle \text{II}.$$

[Two $\triangle$s are $\cong$ if three sides of the first, etc.] (V. 3.)

$\therefore \angle ABE = \angle CEB$. (Hom. $\angle$s of $\cong$ $\triangle$s.)

$$\therefore b \parallel b'.$$

[If two lines be crossed by a transversal, etc.] (II. 3.)

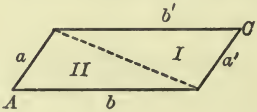
Similarly, $\angle EBC = \angle AEB$.

$$\therefore a \parallel a'.$$

$\therefore$ the 4-side A–C is a parallelogram. (Def. of a ▱.)

Q.E.D.

VI. 1 *a*. *If a 4-side is a parallelogram, its opposite sides are equal.*



Hyp. If the 4-side A–C is a parallelogram,

Conc.: then $a = a'$, $b = b'$.

Dem. Draw a diagonal.

$\triangle \text{I} \cong \triangle \text{II}$. (V. 2.)

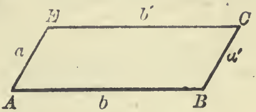
$\therefore a = a'$, and $b = b'$. (Hom. sides of $\cong$ $\triangle$s.)

Q.E.D.

Sch. A diagonal divides a ▱ into two congruent triangles.

Note.—VI. 1. and VI. 1 *a*. are "Direct" theorems; VI. 1'. and VI. 1' *a'*., on page 55, are their "Reciprocals." See definition of reciprocal on p. 11.

VI. 1′. *If a 4-side has two sets of opposite angles equal, it is a parallelogram.*



Hyp. If, in the 4-side A–C, $\angle A = \angle C$ and $\angle B = \angle E$,

Conc.: then $a \parallel a'$ and $b \parallel b'$; *i.e.*, the 4-side A–C is a parallelogram.

Dem. $\angle A = \angle C,\ \angle B = \angle E.$ (Hyp.)

$\therefore \angle A + \angle B = \angle C + \angle E.$ (Ax. 2.)

But $\angle A + \angle B + \angle C + \angle E = 4$ rt. $\angle$s.

[The sum of the interior angles of a 4-side = 4 rt. $\angle$s.] (III. 3.)

$\therefore \angle A + \angle B = 2$ rt. $\angle$s. (Ax. 2.)

$\therefore a \parallel a'.$

[If two lines are crossed by a third, etc.] (II. 4 (1).)

Similarly, $b \parallel b'.$

∴ the 4-side A–C is a parallelogram. (Def. of a ▱.)

Q.E.D.

VI. 1′ a′. *If a 4-side is a parallelogram, its opposite angles are equal.*

Hyp. If the 4-side A–C is a parallelogram,

Conc.: then $\angle A = \angle C$, and $\angle B = \angle E$.

Dem. $a \parallel a'.$ (Def. of a ▱.)

$\therefore \angle A$ is supplementary to $\angle B$.

[If two parallels are crossed by a transversal, etc.] (II. 2.)

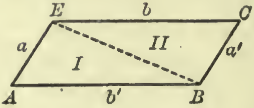
$\angle C$ is supplementary to $\angle B$. (Same reason.)

$\therefore \angle A = \angle C.$ (Ax. 1.)

Similarly. $\angle B = \angle E.$

Q.E.D.

VI. 2. *If a 4-side has one set of sides both equal and parallel, it is a parallelogram.*



Hyp. If $a = a'$ and $a \parallel a'$,

Conc.: then $b \parallel b'$; *i.e.*, the 4-side is a parallelogram.

Dem. Draw the diagonal EB.

$$\triangle \text{I} \cong \triangle \text{II}.$$

[Two $\triangle$ are $\cong$, if two sides and the included $\angle$ of the first = two sides and the included $\angle$ of the second.] (V. 1.)

$\therefore \angle ABE = \angle BEC$. (Hom. $\angle$ of $\cong$ $\triangle$ are =.)

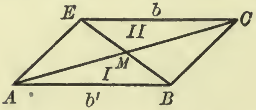
$\therefore b \parallel b'$.

[If two lines are crossed by a third so as to make the alternate interior angles equal, the lines are parallel.] (II. 3.)

$\therefore$ the 4-side A–C is a parallelogram. (Def. of a ▱.)

Q.E.D.

VI. 3. *If a 4-side is a parallelogram, the diagonals bisect each other.*



Hyp. If the 4-side A–C is a parallelogram,

Conc.: then the diagonals AC and BE mutually bisect.

Dem. $b = b'$. (VI. 1 *a*.)

$$\angle ACE = \angle CAB.$$

[If two parallels are crossed by a third line, etc.] (II. 1.)

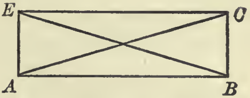
$\angle BEC = \angle ABE$. (Same reason.)

$\therefore \triangle \text{I} \cong \triangle \text{II}$. (Why?)

$\therefore AM = MC$ and $BM = ME$. (Hom. sides of $\cong$ $\triangle$.)

Q.E.D.

VI. 4. *If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.*



Hyp. If, in the ▱ A–C, $AC = BE$,

Conc.: then $\angle EAB = \angle ABC$; *i.e.*, the ▱ A–C is a rectangle.

Dem. $\triangle ABE \cong \triangle ABC$.

[If two △s have three sides of the first equal, etc.] (V. 3.)

$\therefore \angle EAB = \angle CBA$. (Hom. ∠s of $\cong$ △s.)

But $\angle EAB + \angle CBA = 2$ rt. ∠s.

[If two parallels are crossed by a third line, etc.] (II. 2.)

$\therefore \angle EAB = 1$ rt. ∠. (Ax. 2.)

$\angle CBA = 1$ rt. ∠. (Ax. 2.)

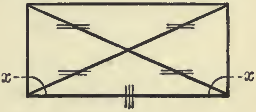
Similarly, $\angle AEC = \angle BCE$.

$\therefore$ ▱ A–C is a rectangle. (Def. of ▭.)

Q.E.D.

VI. 4 *a*. *If two rectangles have the base and altitude of the one equal to the base and altitude of the other, they are congruent.*

Suggestion. — It will be of much assistance to the student in the solution of original exercises, if he marks the parts given in the hypothesis so as to distinguish them from those mentioned in the conclusion. For example, a figure marked thus might be interpreted to mean that if in a parallelogram the diagonals are equal, the parallelogram is a rectangle.



That is, significant symbols may be used to indicate the relations *given in the hypothesis*, while the cross is used to refer to those of the *conclusion*.

Ex. 1. What are the different kinds of quadrilaterals, or 4-sides?

Ex. 2. What is the median and what the mid-join of a trapezoid?

Ex. 3. Produce the non-parallel sides of a trapezoid until they meet. What does the median (produced) of the trapezoid become?

Ex. 4. What is the converse and what the reciprocal of a theorem? Find an illustration of each in Group VI. Find an illustration of each in Group V.

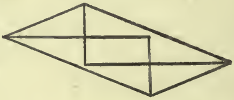
Ex. 5. If through the vertices of any triangle lines are drawn parallel to the opposite sides, point out the three parallelograms formed.

Ex. 6. Prove that the new triangle is four times the size of the original triangle.

Ex. 7. Prove that if through the ends of each diagonal of a 4-side, parallels to the other diagonal are drawn, a parallelogram is formed which is twice as large as the 4-side.

Ex. 8. Prove the converse of VI. 3.

Ex. 9. The bisectors of the corresponding interior angles of two parallels crossed by a transversal meet at right angles. Prove.



Ex. 10. The bisectors of the interior angles of a parallelogram inclose a rectangle. Prove.

Ex. 11. What is a square?

Ex. 12. The bisectors of the interior angles of a rectangle inclose a square. Prove.

Ex. 13. The bisectors of the exterior angles of a rectangle inclose a square. Prove.

Ex. 14. In what parallelograms do these bisectors of the interior angles coincide with the diagonals?

Ex. 15. In such parallelograms, what becomes of the inclosed rectangle? See Ex. 10.

Ex. 16. What is the sum of the interior angles of a triangle?

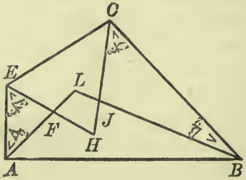
Ex. 17. What is the sum of the interior angles of a 4-side?

Ex. 18. Prove that if HE and LA bisect $\angle E$ and $\angle A$, $\angle HFL$ is supplement of $\frac{\angle A}{2}+\frac{\angle E}{2}$. (Fig. for Ex. 21.)

Ex. 19. Prove that if HC and LB bisect $\angle C$ and $\angle B$, $\angle HJL$ is supplement of $\frac{\angle C}{2}+\frac{\angle B}{2}$. (Fig. for Ex. 21.)

Ex. 20. Why is $\frac{\angle E}{2}+\frac{\angle A}{2}$ the supplement of $\frac{\angle C}{2}+\frac{\angle B}{2}$? (Fig. for Ex. 21.)

Ex. 21. Prove that the four bisectors of the interior angles of a 4-side form a 4-side whose opposite angles are supplemental.



Ex. 22. Prove that the four bisectors of the exterior angles of a 4-side form a 4-side whose opposite angles are supplemental.

Ex. 23. Prove that if two parallelograms have two sides and the included angle of one equal to two sides and the included angle of the other, they are congruent.

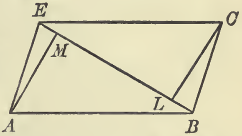
Ex. 24. Prove that if the diagonals of a 4-side not only bisect but are also perpendicular to each other, the 4-side is a rhombus.

Ex. 25. Prove that an isosceles trapezoid is isoangular. (Draw perpendiculars to the longer base from the ends of the shorter.)

Ex. 26. Prove that the diagonals of an isosceles trapezoid are equal.

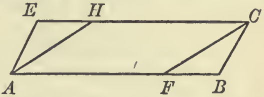
Ex. 27. Why are the opp. ∠s of an isosceles trapezoid supplemental?

Ex. 28. If from the opposite vertices of a parallelogram perpendiculars are let fall on a diagonal, they are equal.

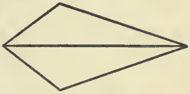


Ex. 29. If a line is drawn parallel to the base of an isosceles triangle, it divides the triangle into an isosceles triangle and an isosceles trapezoid.

Ex. 30. If, in a parallelogram A–C, $BF = EH$, then the 4-side $AFCH$ is a parallelogram.

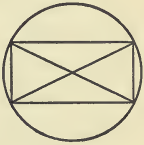


Ex. 31. If one of the diagonals of a 4-side bisects a pair of opposite angles, but the other diagonal does not, the 4-side is a kite.



Ex. 32. The 4-side formed by joining the ends of any two diameters of a circle is a rectangle.

Ex. 33. The 4-side formed by joining the ends of two perpendicular diameters is a square.



Ex. 34. The bisectors of the opposite angles of a rhomboid are parallel.

Ex. 35. Any line through the mid-point of the diagonal of a parallelogram divides the parallelogram into two equal parts.

Ex. 36. Express in terms of n the number of diagonals of an n-gon.

VI. SUMMARY OF PROPOSITIONS IN GROUP ON PARALLELOGRAMS

1. *If a 4-side has two sets of opposite sides equal, it is a parallelogram.*

   a *If a 4-side is a parallelogram, its opposite sides are equal.*

SCH. A diagonal divides a ▱ into two congruent triangles.

1′. *If a 4-side has two sets of opposite angles equal, it is a parallelogram.*

   a′ *If a 4-side is a parallelogram, its opposite angles are equal.*

2. *If a 4-side has one set of sides both equal and parallel, it is a parallelogram.*

3. *If a 4-side is a parallelogram, the diagonals bisect each other.*

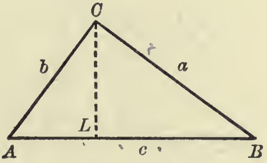
4. *If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.*

   a *If two rectangles have the base and altitude of the one equal to the base and altitude of the other, they are congruent.*

VII. GROUP ON SUM OF LINES AND MID–JOINS

PROPOSITIONS

VII. 1. *The sum of two sides of a triangle is greater than the third side.*



Hyp. If ABC is a triangle,

Conc.: then $b + a > c$.

Dem. If either a or $b > c$, no proof is required.
If each side $< c$, draw $CL \perp c$.
Then AL and BL are each $\perp CL$.
Now $AL < b$, and $BL < a$.
[A perpendicular is shortest distance from a point to a line.]
(IV. 4 a, Sch.)

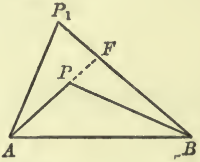
$\therefore \overline{AL + BL}(= c) < b + a$. (Preliminary Th. 1.)

Q.E.D.

VII. 1 a. *The difference between any two sides of a triangle is less than the third side.* (Use Preliminary Th. 3.)

Ex. 1. Prove that a line from the vertex of the vertex angle of an isosceles triangle to any point in the base is less than either of the legs of the triangle.

VII. 2. *The sum of two lines drawn from any point within a triangle to the ends of one side is less than the sum of the two other sides of the triangle.*



Hyp. If from P_1, P_1A and P_1B are drawn, and from P, PA and PB are drawn, but enveloped by P_1A and P_1B,

Conc.: then $P_1A + P_1B > PA + PB$.

Dem. Extend AP to intersect P_1B at F.

$P_1A + P_1F > AF$. (VII. 1.)

$BF + PF > BP$. (VII. 1.)

$\therefore P_1A + P_1F + BF + PF > AP + PF + BP$.

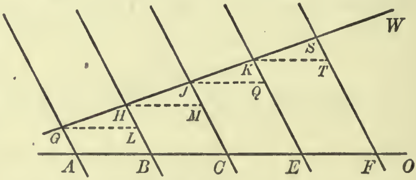
(Preliminary Th. 1.)

Take away PF from both members of the inequality and we have $P_1A + P_1B > PA + PB$. (Preliminary Th. 3.)

Q.E.D.

VII. 3. *If a series of parallels cut off equal segments on one transversal,*

(*a*) *They will cut off equal segments on every transversal.*



Hyp. If GW and AO are any transversals; AG, BH, etc., a set of parallels, and $AB = BC = CE$, etc.,

Conc.: then (*a*) $GH = HJ = JK = KS$, etc.

Dem. Draw GL, HM, JQ, etc., all parallel to AO.

The four-sides $ABGL$, $BCMH$, etc., are ▱s. (Def. of ▱.)

$\therefore GL = AB$, $HM = BC$, $JQ = CE$, etc. (VI. 1 *a*.)

Ex. 2. Show how theorem VII. 3 (*a*) may be practically utilized in solving the problem: To divide a given line-segment into any number of equal parts.

But $AB = BC = CE$, etc. (Hyp.)

$\therefore\ GL = HM = JQ$, etc. (Ax. 1.)

Again, $\angle HGL = \angle JHM = \angle KJQ$, etc. (Def. of $\parallel$s.)

And $\angle HLG = \angle JMH = \angle KQJ$, etc. (II. 5.)

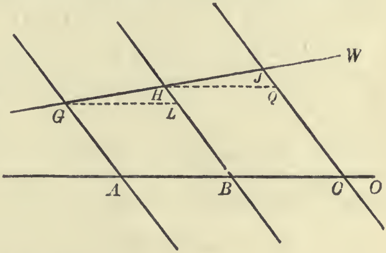
$\therefore\ \triangle GHL \cong \triangle HJM \cong \triangle JKQ$, etc. (V. 2.)

$\therefore\ GH = HJ = JK = KS$, etc.

Q.E.D.

VII. 3 *b*. *If the series consists of three consecutive parallels terminating in the transversals, the mid-parallel equals half the sum of the other two. That is, the mid-join of a trapezoid equals half the sum of the two parallel sides.*

Hyp. If GA, HB, and JC are three successive parallels terminating in the transversals AO and GW,



Conc.: then $HB = \frac{1}{2}(GA + JC)$;

that is, the mid-join of a trapezoid equals one half the sum of the bases.

Dem. $HJ = HG$. (VII. 3 *a*.)

$\therefore$ BH is the mid-join of the trapezoid $ACJG$.

Draw GL and $HQ \parallel AO$.

$GA = LB$, $CQ = LB + LH$. (VI. 1 *a*.)

$CJ = CQ + QJ$, and $LH = QJ$.

(Hom. sides of $\cong$ $\triangle$s are $=$.)

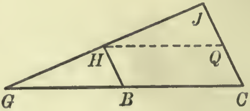
$\therefore\ GA + CQ + CJ = LB + LB + LH + CQ + QJ$, (Ax. 2.)

or $GA + CJ = 2\,LB + 2\,LH$,

or $$\frac{GA + CJ}{2} = LB + LH = HB.$$

Q.E.D.

VII. 3 *c*. *If in a triangle a parallel to the base is drawn through the mid-point of one side, it bisects the other side and equals half the base.*



Hyp. If GCJ is a △, and if B is the mid-point of GC, and $BH \parallel CJ$,

Conc.: then (1) $GH = HJ$

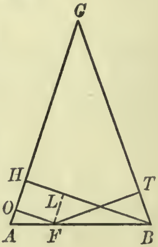
and (2) $HB = \frac{CJ}{2}$.

Dem. (1) The △ GCJ may be considered to be the trapezoid $GACJ$ of VII. 3 (*b*) with the side GA reduced to a point.

∴ the Dem. of VII. 3 (*b*) will apply to VII. 3 (*c*).

Proof. Let the student give the proof in full.

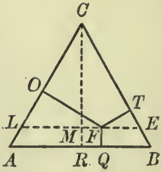
Ex. 3. If, from any point in the base of an isosceles triangle, lines be drawn parallel to the sides, the perimeter of the parallelogram formed equals the sum of the two equal sides of the isosceles triangle.



Ex. 4. If, from any point in the base of an isosceles triangle, perpendiculars to the equal sides are drawn, prove that their sum equals the altitude from either one of the base angles. (Draw $FL \perp BH$.)

Ex. 5. Prove the foregoing proposition for the obtuse isosceles triangle.

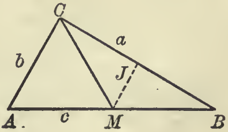
Ex. 6. AB of the isosceles △ is then the locus of what point?



Ex. 7. If, from any point within an equilateral triangle, perpendiculars be drawn to the sides, prove that the sum of these three perpendiculars equals the altitude of the triangle. (Draw LE through $F \parallel AB$.)

Ex. 8. Prove that the mid-join of a triangle is parallel to the third side.

VII. 4. *In a right triangle the median to the hypotenuse equals half the hypotenuse.*



Hyp. If, in the rt. △ ABC, M is the mid-point of the hypotenuse c,

Conc.: then $CM = \frac{c}{2}$.

Dem. Draw $MJ \parallel b$.

$MJ \parallel b$ and $= \frac{b}{2}$. (VII. 3 (*c*).)

$\therefore \angle MJB$ is a rt. ∠, (Why?)

and △ CMB is isosceles.

[If the altitude of a △ bisects the base, etc.] (IV. 3 *a*.)

$\therefore MB = CM$. But $MB = MA$. (Hyp.)

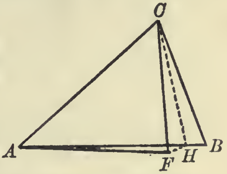
$\therefore CM = MA$. (Ax. 1.)

$$\therefore CM = \frac{MB + MA}{2} = \frac{c}{2}.$$

Q.E.D.

VII. 4 *a*. *Conversely, if a median of a triangle equals half the side that it bisects, the triangle is right.*

Ex. 9. The △ ABC and AFC have AC common, $CF = CB$, and $\angle ACB > \angle ACF$. Show that $AB > AF$.



Note that if $\angle FCB$ be bisected by CH, and HF be drawn, $\triangle FCH \cong \triangle HCB$ (V. 1) and $\therefore FH = HB$. But $AH + FH > AF$, whence $AB > AF$; that is,

If two triangles have two sides of the first equal respectively to two sides of the second and the included angles unequal, the third sides will be unequal; the greater side will belong to the triangle having the greater included angle.

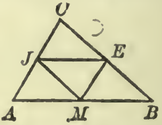
Ex. 10. State and prove the converse of Ex. 9.

Ex. 11. Prove that if in a right triangle one acute angle is two thirds of a right angle, the hypotenuse equals twice the shorter side. (Draw the median to the hypotenuse.)

Ex. 12. Prove that the joins of the mid-points of the adjacent sides of a 4-side form a parallelogram. (Draw the diagonals.)

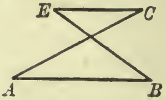
Ex. 13. Prove that the parallelogram formed in Ex. 12 is one half the original 4-side.

Ex. 14. Prove that the mid-join of the diagonals of a trapezoid is parallel to the bases and equals one half their difference. (Draw the join of an end of the upper base and an end of the mid-join. Two congruent triangles are formed with the bases and diagonals.)



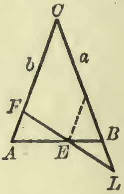
Ex. 15. The mid-joins of the three sides of a triangle divide the triangle into three congruent triangles.

Ex. 16. If we call $ABEC$ a cross trapezoid, prove that $AB + CE$ is less than $AC + BE$.



Ex. 17. In any 4-side the sum of the diagonals is less than the perimeter and greater than the semi-perimeter.

Ex. 18. In the isosceles $\triangle ABC$, E is any point in AB. Through E to draw FL terminating in b and a produced so that $FE = EL$.



Ex. 19. In the isosceles $\triangle ABC$ prove that if FL is bisected in E that $AF = BL$.

Ex. 20. If, in a parallelogram (Fig. 1), M is the mid-point of EC, and L is the mid-point of AB, show that $EL \parallel BM$.

Ex. 21. If, in Fig. 1, EL intersects CA in R, and BM intersects CA in F, show that F is the mid-point of CR.

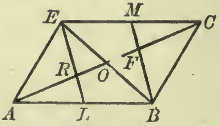
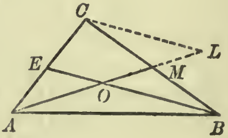


FIG. 1

Ex. 22. In Fig. 1, show that $OF = \frac{CF}{2}$.

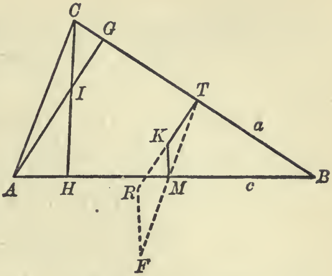
Ex. 23. In Fig. 1, BM and CO are medians of $\triangle BCE$. Prove by means of the three preceding theorems that the medians of a triangle concur.

Ex. 24. If two medians of a triangle intersect in O as in the adjoining figure, then $OM = \frac{AO}{2}$ and $OE = \frac{BO}{2}$; that is, $OM = \frac{AM}{3}$, and $EO = \frac{BE}{3}$.



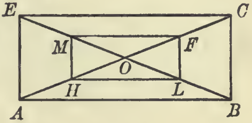
HINT. — Draw $CL \parallel BO$.

Ex. 25. If, in $\triangle ABC$, AG and CH are two altitudes on a and c, respectively, and TK and MK are mid-perpendiculars to a and c, prove that $TK = \frac{AI}{2}$.

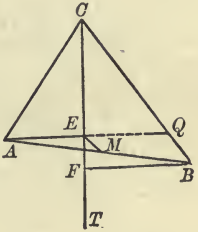


Hint. — Draw $MF = MT$ and $FR \parallel CH$. What triangles are congruent?

Ex. 26. The mid-joins of the four halves of the two diagonals of a rectangle form a second rectangle.



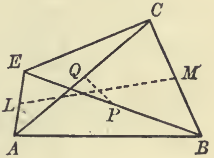
Ex. 27. If, in $\triangle ABC$, CT bisects the vertex angle and AE and BF are drawn perpendicular to CT, then, if M is the mid-point of AB, the join ME (or MF) $= \frac{1}{2}(BC - AC)$.



Ex. 28. If AE and BF are drawn perpendicular to the bisector of the exterior vertex angle, the join $ME = \frac{1}{2}(BC + AC)$.

Ex. 29. The $\angle QAB = \frac{1}{2}(\angle CAB - \angle B)$.

Ex. 30. The join of the mid-points of two opposite sides of a 4-side and the join of the mid-points of its diagonals are the diagonals of a parallelogram.



Ex. 31. State at least three general methods of proving that:

Lines are parallel.
Lines are equal.
Angles are equal.
Angles are supplemental.

VII. SUMMARY OF PROPOSITIONS IN THE GROUP ON SUM OF LINES AND MID-JOINS

1. *The sum of two sides of a triangle is greater than the third side.*

   (a) *The difference between any two sides of a triangle is less than the third side.*

2. *The sum of two lines drawn from any point within a triangle to the ends of one side is less than the sum of the two other sides of the triangle.*

3. *If a series of parallels cut off equal segments on one transversal,*

   (a) *They will cut off equal segments on every transversal.*

   (b) *If the series consists of three consecutive parallels terminating in the transversals, the mid-parallel equals half the sum of the other two. That is, the mid-join of a trapezoid equals half the sum of the two parallel sides.*

   (c) *If in a triangle a parallel to the base is drawn through the mid-point of one side, it bisects the other side and equals half the base.*

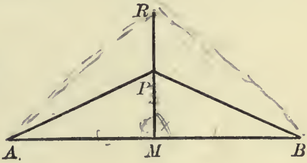
4. *In a right triangle the median to the hypotenuse equals half the hypotenuse.*

   (a) *Conversely, if a median of a triangle equals half the side that it bisects, the triangle is right.*

VIII. GROUP ON POINTS — EQUIDISTANT AND RANDOM

PROPOSITIONS

VIII. 1. *Every point on the mid-perpendicular of a line-segment is equidistant from the ends of the line-segment.*



Hyp. If RM is a mid-perpendicular to AB, and P is any point in RM,

Conc.: then $PA = PB$.

Dem PM is the altitude to base of $\triangle ABP$. (Def. of alt.)

$\therefore \triangle PAB$ is isosceles. (IV. 3 *a*.)

$\therefore PA = PB$. (Def. of isos. $\triangle$.)

Q.E.D.

What is the locus of a point satisfying the following conditions:

Ex. 1. At a distance a from a given point Q?

Ex. 2. At a distance a from a given line AL?

Ex. 3. At a distance a from a given circumference K?

Ex. 4. Equidistant from Q and R?

Ex. 5. Equidistant from two intersecting lines AL and BM?

Ex. 6. Equidistant from two parallels?

Ex. 7. Equidistant from two equal circumferences?

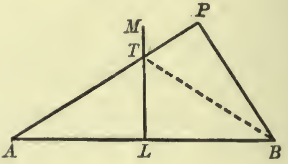
Ex. 8. Equidistant from two concentric circumferences?

Find the points that satisfy the following conditions:

Ex. 9. At a distance a from Q, and a distance b from a line AL.

Ex. 10. At a distance a from Q, and a distance b from a $\odot K$.

VIII. 2. *Every point without the mid-perpendicular of a line-segment is unequally distant from the ends of the line-segment.*



Hyp. If ML is a mid-perpendicular to AB, and if P is any point without ML,

Conc.: then PA is not equal to PB.

Dem. Let AP intersect the mid $\perp ML$ in T.
Draw TB.

$$TB = TA. \qquad \text{(VIII. 1.)}$$

$$TB + TP > PB. \qquad \text{(VII. 1.)}$$

$$TB + TP = TA + TP \text{ (Ax. 2)} = PA.$$

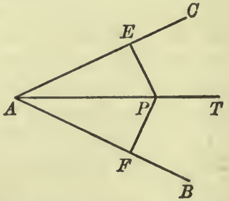
$$\therefore\ PA \text{ does not equal } PB.$$

Q.E.D.

Sch. 1. The mid-perpendicular of a line-segment is the locus of points equidistant from the ends of the line-segment.

Sch. 2. Two points, each of which is equidistant from the ends of a line-segment, determine the mid-perpendicular to the line.

VIII. 3. *Every point in the bisector of an angle is equidistant from the sides of the angle.*



Hyp. If AT is the bisector of $\angle BAC$, P any point in AT, and if PF and PE are perpendiculars to AB and AC, respectively,

Conc.: then P is equidistant from AB and AC. That is, $PF = PE$.

Dem. $\angle EAP = \angle FAP.$ (Hyp.)

$\angle EPA = \angle FPA.$ (Ax. 1.)

$AP \equiv AP.$

$\therefore \triangle EAP \cong \triangle FAP.$

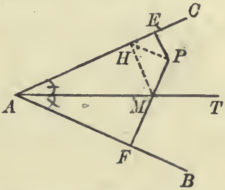
[Two ∡s and the included side of the first, etc.] (V. 2.)

$\therefore PF = PE.$ (Hom. sides of $\cong$ ▲s.)

Q.E.D.

VIII. 4. *Every point without the bisector of an angle is unequally distant from the sides of the angle.*

Hyp. If AT is the bisector of $\angle BAC$, P any point without AT, and if PF and PE are perpendiculars to AB and AC, respectively,



Conc.: then P is unequidistant from AB and AC. That is, PF does not $= PE$.

Dem. Draw $MH \perp AC$; also the join PH.

$PH > PE.$

[A ⊥ is the shortest distance from a point to a line.]

(IV. 4 *a*. Sch.)

$MH + MP > PH.$

[The sum of two sides of a △ is greater, etc.] (VII. 1.)

$MH = FM.$ (VIII. 3.)

$\therefore FM + MP > PH.$

But $FM + MP = PF.$ (Ax. 4.)

$\therefore PF > PH$, which is greater than PE.

$\therefore PF > PE.$

Q.E.D.

SCH. 1. Two points, each of which is equidistant from the sides of an angle, determine the bisector of the angle.

SCH. 2. The bisector of an angle is the locus of all points equidistant from the sides of the angle.

GEN. SCH. Two points in any straight line locus determine the locus.

NOTE. — Sch. 1. of VIII. 2. affords the proof of the solutions of the following problems of the ten easy exercises in geometrical drawing (pp. 14–17):

PROB. I. (*a*) Bisect a given line-segment.
(*b*) Erect a mid-perpendicular to a given line-segment.

PROB. V. Erect a perpendicular to a given line at a given point in the line.

PROB. VI. Draw a perpendicular to a given line from a given point without the line.

Find the points that satisfy the following conditions:

Ex. 11. In a given line AL, and at a distance b from a given line Q.

Ex. 12. In a given line AL, and at a distance b from a $\odot K$.

Ex. 13. In a given circumference, and at a distance b from Q.

Ex. 14. In a given circumference, and at a distance b from a line AL.

Ex. 15. In a given circumference, and at a distance b from a second circumference.

Ex. 16. In a given line, and equidistant from Q and R.

Ex. 17. In a given circumference, and equidistant from Q and R.

Ex. 18. At a distance a from a line AL, and equidistant from Q and R.

Ex. 19. At a distance a from a given circumference, and equidistant from Q and R.

Ex. 20. At a distance a from a given point Q, and equidistant from R and S.

Ex. 21. At a distance a from a given point Q, and equidistant from two intersecting lines AL and AM.

Ex. 22. At a distance a from a point Q, and equidistant from two parallels AL and BM.

Ex. 23. At a distance a from a circumference, and equidistant from two intersecting lines.

Ex. 24. Equidistant from Q and R, and also from two intersecting lines AL and BM.

VIII. SUMMARY OF PROPOSITIONS IN THE GROUP ON POINTS — EQUIDISTANT AND RANDOM

1. *Every point on the mid-perpendicular of a line-segment is equidistant from the ends of the line-segment.*

2. *Every point without the mid-perpendicular of a line-segment is unequally distant from the ends of the line-segment.*

Sch. 1. The mid-perpendicular of a line-segment is the locus of points equidistant from the ends of the line-segment.

Sch. 2. Two points, each of which is equidistant from the ends of a line-segment, determine the mid-perpendicular to the line.

3. *Every point in the bisector of an angle is equidistant from the sides of the angle.*

4. *Every point without the bisector of an angle is unequally distant from the sides of the angle.*

Sch. 1. Two points, each of which is equidistant from the sides of an angle, determine the bisector of the angle.

Sch. 2. The bisector of an angle is the locus of all points equidistant from the sides of the angle.

Gen. Sch. Two points in any straight line locus determine the locus.

NINE ILLUSTRATIONS OF ELEMENTARY PRINCIPLES OF LOCI

1. The locus of a point at a given distance from a given **Point.**
2. The locus of a point at a given distance from a given **Straight Line.**
3. The locus of a point at a given distance from a given **Circle.**
4. The locus of a point equidistant from two given **Points.**
5. The locus of a point equidistant from two given **Straight Lines** which intersect.
6. The locus of a point equidistant from two **Parallels.**
7. The locus of a point equidistant from two **Concentric Circles.**
8. The locus of a point from which perpendiculars may be drawn to a given straight line,
    (*a*) *to* a given point *in* the line ;
    (*b*) *through* a given point *without* the line.
9. The locus of a point from which obliques may be drawn making a given angle with the line,
    (*a*) *to* a given point *in* the line ;
    (*b*) *through* a given point *without* the line.

NINE EXERCISES IN LOCI

Because of the frequent use of the idea of the locus in the subsequent demonstrations, it is of the utmost importance that the pupil become thoroughly familiar with these simple yet fundamental notions of the locus.	The questions on the left are given in familiar, everyday language. The statements below are answers to the questions opposite, and are given in the language of geometry.
1. What line contains all the houses that are 1 mile distant from the city hall ? *Ans.* The circle whose center is the city hall and whose radius is 1 mile.	1. The locus of a point 1 mile distant from a given point is a circle whose center is the given point and whose radius is 1 mile.
What is necessary to determine the size and position of a circle ?	The locus of a point at a given distance from a given point is a circle whose center is the given point and whose radius is the given distance. Ex. Define a circle as a locus.
2. What line or lines contain all the houses 1 mile distant from a main street in your city ?	2. The locus of a point 1 mile distant from a given line consists of two parallels to the given line, one on each side, 1 mile from the line.

NOTE.—By distance is meant, unless otherwise stated, the perpendicular distance. What is necessary to fix or determine the position of a line?	The locus of a point at a given distance from a given line *consists of* two parallels to the given line, one on each side, at a given distance from the line. Ex. By what axiom are the above lines determined in position? Under which of the two ways of stating this axiom does the determination directly fall?
3. What line contains all the flower pots that may be placed 10 feet from a circular path whose diameter is 100 feet? The distance from a circle is always measured on a radius, or radius produced. What is the name of two or more circles that have the same center?	**3.** The locus of a point 10 feet from a given circle whose diameter is 100 feet, *consists of* two concentric circles whose radii, respectively, are 60 feet and 40 feet. The locus of a point at a given distance a from a given circle whose radius is r, *consists of* two concentric circles whose radii, respectively, are $r + a$ and $r - a$. Ex. What determines the position and size of a circle?
4. What line contains all the hydrants that may be placed equidistant from the ends of a straight street? What determines the position of this line?	**4.** The locus of a point equidistant from two given points is the mid-perpendicular to the join of the two points. Ex. What is the direction of this locus with reference to the join of the two points? (v. definition of direction.) NOTE.—The mid-perpendicular to the join of two points is also the locus of the centers of circles, any one of which passes through both points.
5. What line contains all the hydrants that may be placed in a park so as to be equidistant from two intersecting straight paths?	**5.** The locus of a point equidistant from two given intersecting straight lines *consists of* two lines, and bisecting the angles formed by the lines.
6. What line contains all the points that are equidistant from the	**6.** The locus of a point equidistant from two parallels is a line

two rails of a cable street railroad? What determines the position of this line?	parallel to them, midway between them.
7. What line contains all the flower pots that may be placed equidistant from two concentric circular paths with radii of 50 feet and 75 feet, respectively?	7. The locus of a point equidistant from two concentric circles whose radii are, respectively, 50 feet and 75 feet, is a circle concentric with the given circles, whose radius is 62½ feet. The locus of a point equidistant from two concentric circles of radii a and b is a circle concentric with the given circles of radius $\frac{a+b}{2}$.
8. What line contains all the windows in a high building from which a boy may drop apples into a basket standing against the building, on the level sidewalk?	8. (*a*) The locus of a point from which perpendiculars may be drawn to a given line at a given point in the line is the perpendicular to the line at the given point. (*b*) The locus of a point from which perpendiculars may be drawn to a given line through a given point without the line is the perpendicular to the line from the point.
9. (1) Along what line should we find all the telegraph poles on which wires may be strung in northeast and southwest direction, to cross an east and west county road at the schoolhouse (*a*) on the county road, (*b*) 1 mile from the county road? (2) Same question for a northwest and southeast telegraph line.	9. The locus of a point through which obliques may be drawn to a given line, making an angle equal to one half a right angle (*a*) at a given point in the line, (*b*) at a given point without the line, is the line through the given point (1) making half a right angle with the line; (2) making a negative angle equal to half a right angle with the line.

NOTE. — In order to prove that a locus *consists of* a line, or a set of lines, it is necessary to show

First, that every point on the line, or set of lines, fulfills the given conditions.

Second, that no point without the line, or set of lines, does fulfill the given conditions.

CHART PROBLEMS

Note. — In order to answer the questions asked, students are at liberty to change dimensions and must sometimes for theoretical reasons deal with impractical conditions.

Let us assume, for the purpose of illustration, that *The Pirate's Chart* gives the following description of the locations of his buried treasure :

1. The first is a half mile from an oak, and at the same time is three quarters of a mile from a chestnut.

Locate the treasure. (v. Locus, Ex. 1.)

When are there two possible locations ? When none ?

[Draw a diagram for the above and for the following exercises. In the diagram make the given line or lines solid, the loci dotted.]

2. The second is a quarter mile from the shore of a shallow circular pond, whose radius is one mile, and simultaneously is a half mile from a neighboring straight beach.

Locate the treasure. (v. Locus, Exs. 2 and 3.)

When may there be eight such locations ?

3. The third is equidistant from the oak and the chestnut, and simultaneously is one and a half miles distant from the shore of a neighboring circular pond, whose radius is one quarter of a mile.

Locate the treasure. (v. Locus, Exs. 4 and 3.)

When may there be four such locations ?

4. The fourth is equidistant from the turnpike and the valley road, and is simultaneously equidistant from the oak and the chestnut.

Locate the treasure. (v. Locus, Exs. 5 and 4.)

5. The fifth lies on a line making with the turnpike a positive two thirds of a rt. ∠, and passing through the oak ; also on a line making with the turnpike a negative one half of a rt. ∠, and passing through the chestnut.

Locate the treasure. (v. Locus, Ex. 9.)

Suppose the trees were (*a*) upon the turnpike, (*b*) remote from it.

6. The sixth lies on a perpendicular to the turnpike at the schoolhouse ; also on a line passing through the oak and ∥ to the valley road.

Locate the treasure. (v. Locus, Ex. 8.)

7. The seventh lies on a perpendicular through the oak to the join of the oak and the chestnut, and is simultaneously one mile from the oak.

Locate the treasure. (v. Locus, Exs. 8 and 1.)

Note. — Descriptions 5, 6, and 7 are applications of Locus, Exs. 8 and 9. These two exercises should be given careful attention.

IX. GROUP ON THE CIRCLE AND ITS RELATED LINES

DEFINITIONS

A **Secant** is a line cutting a circumference in two points.

A **Chord** is the join of any two points on a circumference. The arcs that have the same extremities as a chord are said to be *subtended* by the chord. The greater of the two arcs is called the major, and the smaller the minor, arc.

A **Diameter** of a circle or circumference is a chord that passes through the center.

A **Tangent** is a line that touches a circle or circumference in but one point.

Two circles are tangent to each other when they are tangent to the same line at the same point.

An **Arc** is any part of a circumference.

A **Segment** of a circle is that portion of the circle contained between an arc and the chord having the same extremities as the arc. This chord is said to *subtend* the arc.

A **Sector** of a circle is that portion of the circle contained between two radii and the arc that they intercept.

COROLLARIES OF THE DEFINITIONS

1. *Circles with equal radii are congruent.* (See Definitions, pp. 2, 8.)

2. *A line that intersects a circumference intersects it in two points and no more.*

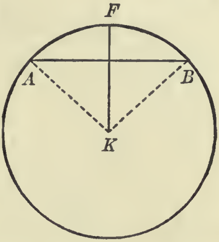
3. *Any diameter of a circle bisects it.*

4. *A tangent may be considered as obtained by revolving a secant about either point of secancy until the two points coincide.*

PROPOSITIONS

IX. 1. *A radius perpendicular to a chord bisects the chord and its subtended arc, and conversely.*

Hyp. If, in ⊙ K, the radius KF is perpendicular to the chord AB,



Conc.: then (*a*) KF bisects the chord AB.
(*b*) KF bisects the arc AFB.

Dem. (*a*) Draw KA and KB.

△ AKB is isosceles. (Sides are radii.)

∴ KF bisects the chord AB.

[In an isosceles △ the altitude to the base, etc.] (IV. 1 *a*, Sch.)

Q.E.D.

Dem. (*b*) On KF as an axis revolve the sector KBF to the plane of sector KFA.

B will fall on A. (Why?)

Moreover, the arcs AF and FB will coincide throughout, as all radii are equal.

∴ arc BF = arc AF.

Q.E.D.

Hyp. Conversely, if, in a circle, a radius KF bisects a chord AB,

Conc.: then $KF \perp$ chord AB.

Dem. $AK = BK$. (Radii of same ⊙.)

∴ △ ABK is isosceles. (Def. of isos. △.)

∴ KF is the altitude to the base of isos. △ ABK.

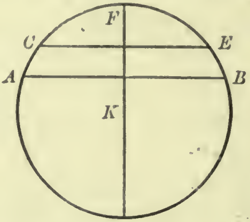
[In an isosceles △ the altitude to the base, etc.] (IV. 1 *a*, Sch.)

∴ $KF \perp$ chord AB.

Q.E.D.

IX. 1 *a.* *A radius perpendicular to a chord is mid-perpendicular to every chord parallel to the first.*

Hyp. If, in ⊙ K, the radius KF is perpendicular to the chord AB, and if CE is any chord parallel to the chord AB,



Conc.: then KF is the mid-⊥ to the chord CE.

Dem. $KF \perp AB$. (Hyp.)

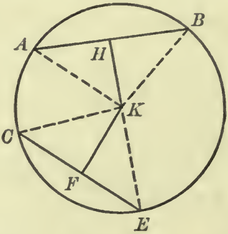
$\therefore KF \perp CE$. (Def. of ∥s, first inference.)

$\therefore KF$ is the mid-⊥ to CE. (IX. 1.)

Q.E.D.

IX. 2. *In the same circle, or in equal circles, equal chords are equidistant from the center, and conversely.*

Hyp. If, in ⊙ K, the chord $AB =$ the chord CE,



Conc.: then the ⊥ $KH =$ the ⊥ KF.

Dem. Draw the radii, KB, KA, KC, and KE.

$$\triangle KAB \cong \triangle KCE.$$

[If two △ have three sides of one equal, etc.] (V. 3.)

$\therefore KH = KF$. (Sch. to Th.'s ≅ △, Group V.)

Q.E.D.

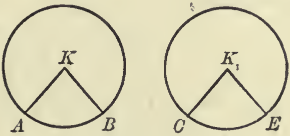
Hyp. Conversely, if the ⊥ $KH =$ the ⊥ KF in the ⊙ K,

Conc.: then the chord $AB =$ the chord CE.

Dem. rt. $\triangle HKA \cong$ rt. $\triangle HKB \cong$ rt. $\triangle FKC \cong$ rt. $\triangle FKE$.
[Right △s are $\cong$ if a leg and hypotenuse of one, etc.] (V. 4.)
$\therefore AH = HB = CF = FE$. (Hom. sides of $\cong$ △s.)
$\therefore$ the chord $AB =$ the chord CE. (Ax. 2.)
Q.E.D.

IX. 3. *In the same circle, or in equal circles, equal angles at the center subtend equal arcs on the circumference, and conversely.*

Hyp. If, in the equal ⓢ K and K_1, $\angle K' = \angle K_1$,



Conc.: then arc $AB =$ arc CE.

Dem. Place $\angle K$ in coincident superposition with its equal $\angle K_1$, A falling on C; then B must fall on E. (Ax. 7.)
$KA = K_1C$ and $KB = K_1E$, being radii of circles equal by hyp.
$\therefore$ arc AB will coincide with arc CE. (The ⓢ are = by hyp.)
Q.E.D.

CONVERSE. Proof of the converse is left to the pupil.

IX. 3 *a*. *In the same circle, or in equal circles, equal chords subtend equal arcs, and conversely.*

Ex. 1. In any circle the greater of two arcs is subtended by the greater chord, and conversely. (Use IX. 3, and Ex. 9, p. 65.)

Ex. 2. In any circle, of two unequal chords the one nearer the center is the greater, and conversely.

Let AB and CE be the chords, KG, the distance of AB from the center K, being greater than KF, the distance of CE from K.

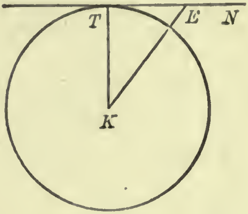
On KG lay off $KM = KF$, and draw through M, $HL \perp KM$. $HL = CE$ (IX. 2). But arc $HABL >$ arc AB by the sum of arcs HA and BL. Hence, chord $HL >$ chord AB by Ex. 1.

Ex. 3. If two circles are concentric, tangents to the first circle that are chords of the second are equal.

Ex. 4. If a radius can be drawn bisecting the angle between two intersecting chords, the chords are equal.

TANGENCY

IX. 4. *A radius to a point of tangency is perpendicular to the tangent.*



Hyp. If TN is a tangent, and KT is a radius to the point of tangency,

Conc.: then $TK \perp TN$.

Dem. Let KE be the join of K and *any point* of TN except T. Then E must be without the circle. (Def. of tangent.)

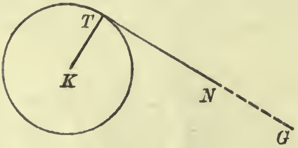
$\therefore KT < KE$. (Def. of ⊙.)

That is, KT is shorter than any other line from K to TN.

$\therefore KT \perp TN$. (IV. 4 *a*, Sch.)

Q.E.D.

IX. 4 *a*. *A line perpendicular to a radius at the outer extremity of the radius is the tangent to the circle at that point.*



Hyp. If TK is a radius, and if $TN \perp TK$ at T,

Conc.: then TN is tangent to ⊙ K at T.

Dem. If TN is not tangent, draw TG, that is.

Then $TK \perp TG$. (IX. 4.)

But $TN \perp TK$. (Hyp.)

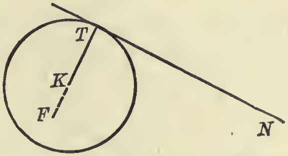
$\therefore TN$ must coincide with TG. (Ax. 7.)

But TG is tangent to ⊙ K by construction.

$\therefore TN$ is tangent.

Q.E.D

IX. 4 *b*. *A perpendicular to a tangent at the point of tangency passes through the center of the circle.*



Hyp. If TN is tangent to ⊙ K, and if $TK \perp TN$ at the point of tangency,

Conc.: then TK passes through the center of the circle.

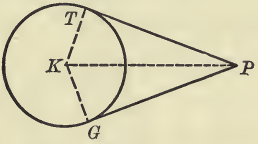
Dem. If TK does not pass through the center, draw TF, that does.

Then	$TF \perp TN$.	(IX. 4.)
But	$TK \perp TN$.	(Hyp.)
	$\therefore$ TF coincides with TK.	(Ax. 7.)

But TF was drawn through the center.

$\therefore$ TK passes through the center. Q.E.D.

IX. 5. *Tangents from the same point to the same circle are equal.*



Hyp. If PG and PT are tangents to ⊙ K,

Conc.: then $PG = PT$.

Dem. Draw radii to the points of tangency T and G; also draw PK.

$PT \perp KT$; $PG \perp KG$. (IX. 4.)

rt. $\triangle PKT \cong$ rt. $\triangle PGK$.

[Two right $\triangle$s are $\cong$ if hypotenuse and leg, etc.] (V. 4.)

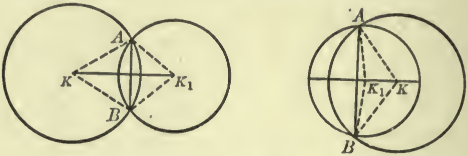
$\therefore$ $PG = PT$. (Hom. sides of $\cong$ $\triangle$s.)

Q.E.D.

IX. 5 *a*. *The join of the center and the intersection of two tangents is the bisector of the angle made by the tangents, and of the angle made by the radii to the points of tangency.*

IX. 6. *If two circles intersect, the line of centers is the mid-perpendicular of the common chord.*

Hyp. If $\odot K$ intersects $\odot K_1$ in the points A and B,



Conc.: then KK_1 is mid-$\perp$ to the common chord AB.

Dem. Draw the radii KA, KB, and K_1A and K_1B.
K is equidistant from A and B. (KA and KB being radii.)
K_1 is equidistant from A and B. (K_1A and K_1B being radii.)

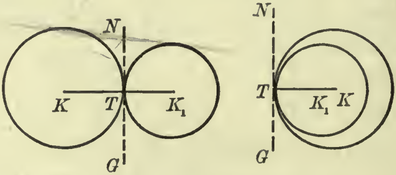
$\therefore$ KK_1 is a mid-perpendicular to AB.

[Two points equidistant from the ends of a line fix the mid-perpendicular to the line.] (VIII. 2, Sch. 2.)

Q.E.D.

IX. 7. *If two circles are tangent, their centers and the point of tangency are in the same straight line.*

Hyp. If $\odot K$ is tangent to $\odot K_1$ at T,



Conc.: then K, T, and K_1 are in the same straight line.

Dem. Draw NG, a common tangent through the common point T. (Def. of tangent Ⓢ.)

$KT \perp NG$.

[A radius to point of tangency is $\perp$ to tangent.] (IX. 4.)

$K_1T \perp NG$. (For the same reason.)

$\therefore$ KT and K_1T are in the same straight line. (Ax. 7.)

Q.E.D

IX. 8. *If two circles are tangent, the distance between their centers is*

(*a*) *the sum of the radii, if the tangency is external;*

(*b*) *the difference of the radii, if the tangency is internal.*

FIG. 1

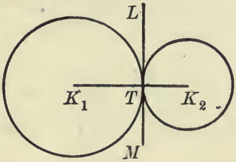
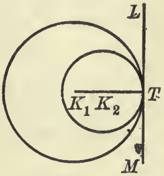


FIG. 2.



Hyp. If K_1 and K_2 are tangent circles at the point T, and their radii are r_1 and r_2,

Conc.: then (Fig. 1) $K_1K_2 = r_1 + r_2$ and (Fig. 2) $K_1K_2 = r_1 - r_2$.

Dem. Draw the common tangent LM through T. (Def. of tangent ⊙.)

Then K_1, K_2, and T are in the same straight line. (IX. 6.)

$\therefore$ (*a*) $K_1T + TK_2 = K_1K_2$, (Ax. 4.)

and (*b*) $K_1T - TK_2 = K_1K_2$; (Ax. 4.)

that is, (*a*) $K_1K_2 = r_1 + r_2$ and (*b*) $K_1K_2 = r_1 - r_2$.

Q.E.D.

IX. 8 *a*. CONVERSELY. *If the distance between the centers of two circles is equal to*

(*a*) *the sum of the radii, the circles are tangent to each other externally;*

(*b*) *the difference of the radii, one circle is tangent to the other internally.*

Ex. 5. Show that if the distance between the centers of two circles is

(*a*) greater than the sum of the radii, each circle is without the other;

(*b*) less than the sum of the radii, but greater than their difference, the circles intersect each other;

(*c*) less than the difference of the radii, one circle is wholly within the other.

IX. SUMMARY OF PROPOSITIONS IN THE GROUP ON THE CIRCLE AND ITS RELATED LINES

1. *A radius perpendicular to a chord bisects the chord and its subtended arc, and conversely.*

   a *A radius perpendicular to a chord is mid-perpendicular to every chord parallel to the first*

2. *In the same circle, or in equal circles, equal chords are equidistant from the center, and conversely.*

3. *In the same circle, or in equal circles, equal angles at the center subtend equal arcs on the circumference, and conversely.*

   a *In the same circle, or in equal circles, equal chords subtend equal arcs, and conversely.*

4. *A radius to a point of tangency is perpendicular to the tangent.*

   a *A line perpendicular to a radius at the outer extremity of the radius is the tangent to the circle at that point.*

   b *A perpendicular to a tangent at the point of tangency passes through the center of the circle.*

5. *Tangents from the same point to the same circle are equal.*

   a *The join of the center and the intersection of two tangents is the bisector of the angle made by the tangents, and of the angle made by the radii to the points of tangency.*

6. *If two circles intersect, the line of centers is the mid-perpendicular of the common chord.*

7. *If two circles are tangent, their centers and the point of tangency are in the same straight line.*

8. *If two circles are tangent, the distance between their centers is*

   (*a*) *the sum of the radii, if the tangency is external;*
   (*b*) *the difference of the radii, if the tangency is internal.*

8 *a*. CONVERSELY. *If the distance between the centers of two circles is equal to*

   (*a*) *the sum of the radii, the circles are tangent to each other externally;*
   (*b*) *the difference of the radii, one circle is tangent to the other internally.*

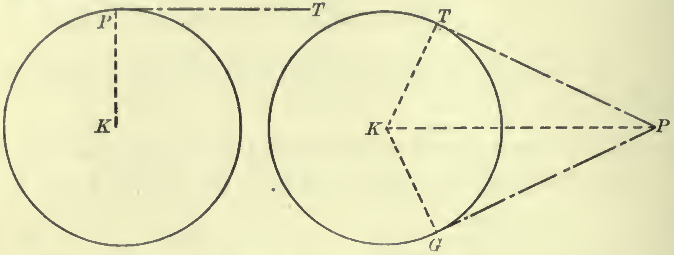
THE ISOSCELES TRIANGLE AS PART OF THE SECTOR OF A CIRCLE

NOTE. — In an isosceles triangle, the altitude to the base is identical with the median to the base. (IV. 1 *a*, Sch.)

NOTE. — Observe that if the vertex of the vertex angle of an isosceles triangle be taken as a center and a circle be described with either leg as a radius, the legs of the triangle are radii of the circle; the base of the triangle is a chord, and the altitude to the base, the median, and the bisector of the vertex angle (which we have seen in (IV. 3) to be the same line) are a part of the radius perpendicular to the chord.

PROBLEMS

Prob. I. *To construct a tangent through a given point to a given circle.*



Given. A circle K, and a point P.

Required. To construct a tangent through P to the circle K.

Case I. When P, the given point, is on the circle.

What is the angle formed by a tangent and a radius drawn to the point of tangency? What is the construction required?

Case II. When P, the given point, is without the circle.

Const. Join P and K, and on PK as a diameter describe a circle intersecting the given circle in T and G.

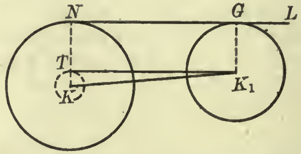
Then PT and PG are the required tangents.

Q.E.F.

Proof. Let the pupil supply the proof.

Why is $\angle PTK$ a right angle? (VII. 4 *a*.)

Prob. II. *To construct a common exterior tangent to two given circles whose radii are* r *and* r′.



Given. Two circles, K and K_1, whose radii $= r$ and r'.

Required. To construct a common exterior tangent.

Const. With K as a center and a radius $= r - r'$, draw the inner concentric circle.

From K_1 draw a tangent to this circle. (Prob. I.)

Draw KT to the point of tangency, T, and produce it to meet the outer circle in N.

Through N draw $NL \parallel TK_1$. (Prob. IV., p. 50.)

Through K_1 draw a line $\parallel$ KN and meeting NL, say at G.

NG is the required common exterior tangent.

Q.E.F.

Proof. The 4-side $T-G$ is a parallelogram. (Def. of a ▱.)

$$\therefore\ K_1G = TN.$$

[Opposite sides of a parallelogram are equal.] (VI. 1 *a*.)

But $TN = r'$ by construction.

$$\therefore\ K_1G = r' \qquad \text{(Ax. 1.)}$$

But $\angle T$ is a right angle. (IX. 4.)

$\therefore$ all the $\angle$s of the 4-side $T-G$ are right angles. (VI. 1′ *a*′.)

$\therefore$ NG is a common exterior tangent to ⓢ K and K_1. (IX. 4 *a*.)

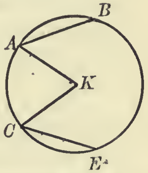
Q.E.D.

Note. — Another common tangent may be found, crossing KK_1 between the circles, and therefore called an *interior* tangent.

In this case the first auxiliary circle has the radius $= r + r'$ instead of $r - r'$. Let the student give the construction in full. Show that four common tangents are possible. When may three only be drawn? When two? When one? When none?

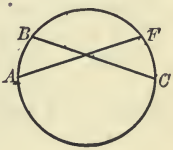
Ex. 6. (*a*) Draw a chord equal and parallel to a given chord.

(*b*) Draw a chord equal and perpendicular to a given chord.



Ex. 7. If, in a circle, two equal chords are drawn, and a radius is drawn to the end of each chord, the angles between the radii and the chords are equal to each other.

Ex. 8. In the same or equal circles the greater of two minor arcs is subtended by the greater chord.



Ex. 9. If chord $AF =$ chord BC, then arc $AB =$ arc CF.

Ex. 10. Show that two chords that are not diameters cannot bisect each other.

Ex. 11. Prove by means of IX. 3 *a* that two triangles are congruent if three sides of the one equal three sides of the other.

Ex. 12. If an inscribed polygon is equiangular, it is not necessarily equilateral.

Ex. 13. If, in Fig. 1, the secant is drawn so that $AB = BK$, show that $\angle CKF = 3\angle A$.

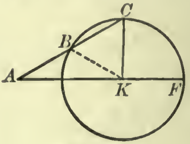


FIG. 1.

Ex. 14. Two parallel chords intercept equal arcs on the circumference. (Fig. 2.)

Ex. 15. A chord and a parallel tangent intercept equal arcs on the circumference.

Ex. 16. Draw:

(1) Two common exterior tangents to two circles.

(2) Two common interior tangents to two circles.

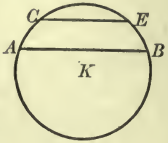


FIG. 2.

Ex. 17. Prove that the above exterior tangents are equal.

Ex. 18. In Fig. 3, $\odot X$ is tangent to $\odot K$ at T, and is also tangent to LF at P.

Why are K, X, and T in the same straight line?

Ex. 19. If $\odot X$ is tangent to LF at P, why is $\angle LPA$ a right angle?

Ex. 20. If $PA = KT$, why is $\triangle KAX$ isosceles?

Ex. 21. How, then, if PA and KA are given in position and length, may the points X and T be determined?

Ex. 22. Problem: Given the line LF, the point P in LF, and the $\odot K$, construct a circle that shall be tangent to LF at P and also tangent to $\odot K$.

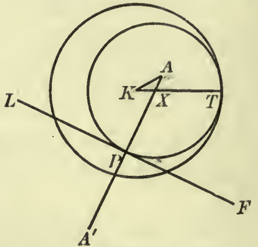
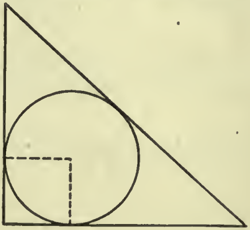


FIG. 3.

Ex. 23. Similarly, by laying off PA' ($= PA$) below LF, find the center X' of a second circle that shall also be tangent to LF at P and likewise tangent to $\odot K$.

Ex. 24. Show that the hypotenuse of a rt. $\triangle$ equals the sum of the two remaining sides, minus twice the radius of the inscribed $\odot$.



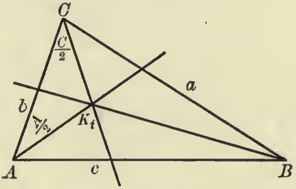
X. GROUP ON CONCURRENT LINES OF A TRIANGLE

DEFINITION

Lines are **Concurrent** when they intersect at the same point.

PROPOSITIONS

X. 1. *The bisectors of the interior angles of a triangle concur.*



Hyp. If ABC is a triangle,

Conc.: then the bisectors of $\angle A$, $\angle B$, and $\angle C$ concur.

Dem. The bisector of $\angle A$ either meets, or is parallel to, the bisector of $\angle C$.

If the bisectors are ∥, the $\frac{\angle A}{2} + \frac{\angle C}{2} = 2$ rt. $\angle$s. (II. 2.)

$\therefore \angle A + \angle C = 4$ rt. $\angle$s. (Ax. 3.)

But this conclusion is impossible, (III. 1.)

$\therefore$ the bisector of $\angle A$ must meet the bisector of $\angle C$.

Let the point of intersection be K_i.

K_i, in the bisector of $\angle C$, is equidistant from a and b, and K_i, in the bisector of $\angle A$, is equidistant from b and c.

[Every point in the bisector of an angle, etc.] (VIII. 3.)

$\therefore$ K_i is equidistant from a and c. (Ax. 1.)

$\therefore$ K_i is in the bisector of $\angle B$. (VIII. 4, Sch.)

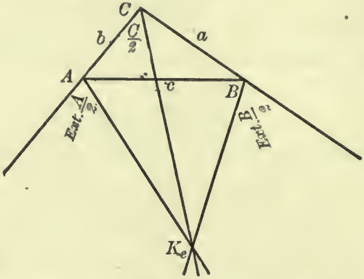
$\therefore$ the bisectors of $\angle A$, $\angle B$, and $\angle C$ concur.

Q.E.D.

This point of concurrence is called the **In-center.**

X. 1 *a. If the in-center be taken as a center, and the distance to any side as a radius, a circle may be drawn tangent to the sides of the triangle.*

X. 2. *The bisectors of one interior angle of a triangle and the two exterior angles non-adjacent to it concur.*



Hyp. If ABC is a triangle,

Conc.: then the bisectors of ext. $\angle A$, ext. $\angle B$, and int. $\angle C$ concur.

Dem. The bisector of $\angle C$ either meets, or is parallel to, the bisector of ext. $\angle A$.

If parallel, then $\frac{\text{ext. } \angle A}{2} = \frac{\angle C}{2}$.

[If two parallels be crossed by a third line, etc.] (II. 1.)

$$\therefore \text{ ext. } \angle A = \angle C. \qquad \text{(Ax. 3.)}$$

But this conclusion is impossible,

[The exterior angle of a △ is greater, etc.] (III. 2, Sch.)

∴ the bisectors of ext. $\angle A$ and of $\angle C$ must meet, say at K_e.

K_e, in the bisector of $\angle C$, is equidistant from a and b (produced).

[Every point in the bisector of an angle, etc.] (VIII. 3.)

K_e, in the bisector of ext. $\angle A$, is equidistant from c and b. (Same reason.)

∴ K_e is equidistant from c and a. (Ax. 1.)

∴ K_e must lie in the bisector of ext. $\angle B$.

[The bisector of an angle is the locus, etc.] (VIII. 4, Sch.)
$\therefore$ the bisectors of ext. $\angle A$, ext. $\angle B$, and int. $\angle C$ concur.

Q.E.D.

The point of concurrence is called an **Ex-center** of the triangle. There are two other ex-centers: viz., $K_{e'}$ on the bisector of $\angle A$, and $K_{e''}$ on the bisector of $\angle B$.

Sch. To three non-concurrent lines, three tangent circles, (besides the inscribed circle already indicated) may be drawn, with K_e, $K_{e'}$, $K_{e''}$ as centers, and the distances from these points to the corresponding lines as respective radii.

Circles tangent to one side of a triangle and to the two other sides produced, are called **Escribed Circles.**

Ex. 1. In $\triangle ABC$, I is the center of the inscribed circle. Show that $\angle I = \text{rt.} \angle + \frac{\angle C}{2}$.

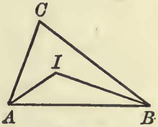


Fig. 1

Ex. 2. In $\triangle ABC$, E is the center of the escribed circle. Show that $\angle E = \text{rt.} \angle - \frac{\angle C}{2}$.

Prove that $\angle E$ of Fig. 2 is the supplement of $\angle I$ of Fig. 1.

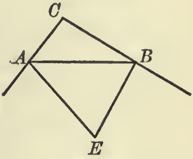


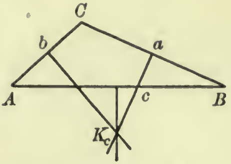
Fig. 2

Find the locus of the center of a circle that satisfies the following conditions:

Ex. 3. That passes through Q, and has a radius a.
Ex. 4. That touches a line AL, and has a radius a.
Ex. 5. That touches a circumference, and has a radius a.
Ex. 6. That passes through two given points.
Ex. 7. That is tangent to two intersecting lines.
Ex. 8. That is tangent to two parallel lines.
Ex. 9. That is tangent to two equal circumferences.
Ex. 10. That is tangent to a line at a given point.
Ex. 11. That is tangent to a circumference at a given point.

X. 3. *The mid-perpendiculars to the three sides of a triangle concur.*

Hyp. If ABC is a triangle,



Conc.: then the mid-perpendiculars to a, b, and c concur.

Dem. The mid-perpendicular to a either intersects, or is parallel to, the mid-perpendicular to b.

If they be parallel, AC and CB would have,

(1) the same direction, ∵ both would be perpendicular to these parallels; and

(2) the point C in common.

∴ AC and CB would lie in the same straight line. (Ax. 7.)

But this conclusion is contrary to the hypothesis that ABC is a triangle.

∴ the mid-perpendicular to a must intersect the mid-perpendicular to b in some point, say K_c.

K_c, in the mid-perpendicular to a, is equidistant from C and B.

[Every point in the mid-perpendicular, etc.] (VIII. 1.)

K_c, in the mid-perpendicular to b, is equidistant from C and A. (Same reason.)

∴ K_c is equidistant from A and B. (Ax. 1.)

∴ K_c is in the mid-perpendicular to c.

[The mid-⊥ to a line-segment, etc.] (VIII. 2, Sch.)

∴ the mid-perpendiculars concur.

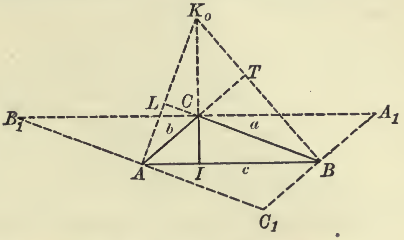
Q.E.D.

The point of concurrence is called the **Circumcenter.**

Sch. Through three non-collinear points an unique circle may be drawn.

X. 4. *The altitudes of a triangle concur.*

Hyp. If ABC is a triangle,



Conc.: then the altitudes AL, BT, and CI concur.

Dem. Through C, A, and B draw parallels to c, a, and b, respectively.

Produce these parallels to intersect in C_1, A_1, and B_1.

The 4-sides $ABCB_1$ and ABA_1C are ▱s. (Def. of ▱.)

$\therefore\ B_1C = AB;\ CA_1 = AB.$ (VI. 1 *a.*)

$\therefore\ B_1C = CA_1.$ (Ax. 1.)

Again, $CI \perp B_1A_1$. (Def. of ∥s, direct inference.)

$\therefore$ CI is the mid-perpendicular to B_1A_1.

Similarly, BT is mid-perpendicular to A_1C_1; AL, mid-perpendicular to B_1C_1.

$\therefore$ the altitudes of the original triangle are mid-perpendiculars to the sides of the larger triangle.

But these mid-perpendiculars concur. (X. 3.)

$\therefore$ the altitudes of the original triangle concur.

Q.E.D.

The point of concurrence is called the **Orthocenter**.

Construct a circle that satisfies the following conditions:

Ex. 12. That has a given radius a, and passes through two given points.

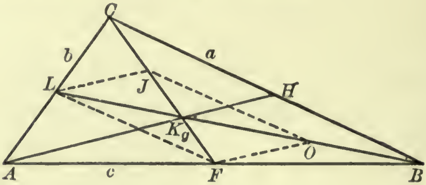
Ex. 13. That has a given radius a, and is tangent to two given intersecting lines AL and BM.

Ex. 14. That has a given radius a, and is tangent to two equal circumferences.

Ex. 15. That has a given radius a, and is tangent to a given line at a given point.

X. 5. *The medians of a triangle concur.*

Hyp. If ABC is a triangle,



Conc.: then the medians AH, BL, and CF concur.

Dem. CF and BL intersect, or are parallel.

If $CF \parallel BL$, all points in CF must lie on the same side of BL.

But, as a consequence of the definition of a median, BL must lie between c and a.

$\therefore$ C and F must lie on opposite sides of BL.

$\therefore$ CF and BL must intersect, say at K_g.

Draw AK_g; draw LJ and $FO \parallel AK_g$; also LF, and OJ.

Then LJ and FO each $\parallel AK_g$ and $= \frac{AK_g}{2}$.

[If in a $\triangle$ a parallel to the base, etc.] (VII. 3 *c*.)

$\therefore$ the 4-side $LJOF$ is a parallelogram.

[A 4-side is a parallelogram if it has one set, etc.] (VI. 2.)

$$\therefore\ K_gF = K_gJ.$$

[The diagonals of a parallelogram mutually bisect.] (VI. 3.)

Now $JK_g = JC$. (Const.)

$\therefore$ $K_gF = \frac{CF}{3}$. Similarly, $K_gL = \frac{BL}{3}$.

$\therefore$ as any two medians cut off the same third of the third median, the three medians must concur.

Q.E.D

This point of concurrence is called the **Center of Gravity**, or **Centroid**, of the triangle.

Ex. 16. Construct a circle that has a given radius a, and is tangent to a given circumference at a given point.

Ex. 17. That has a given radius a, passes through a given point Q, and is tangent to a given line AL.

Ex. 18. That has a given radius a, passes through a given point Q, and is tangent to a given circumference.

Ex. 19. That has a given radius a, is tangent to a given line AL, and is also tangent to a given circumference.

X. SUMMARY OF PROPOSITIONS IN THE GROUP ON CONCURRENT LINES OF A TRIANGLE

1. *The bisectors of the interior angles of a triangle concur.*

   a If the in-center be taken as a center, and the distance to any one side as a radius, a circle may be drawn tangent to the sides of the triangle.

2. *The bisectors of one interior angle of a triangle and the two exterior angles non-adjacent to it concur.*

Sch. To three non-concurrent lines three tangent circles may be drawn, with K_e, $K_{e'}$, $K_{e''}$ as centers and the distances from these points to the corresponding lines as respective radii.

3. *The mid-perpendiculars to the three sides of a triangle concur.*

4. *The altitudes of a triangle concur.*

Sch. Through three non-collinear points an unique circle may be drawn.

5. *The medians of a triangle concur.*

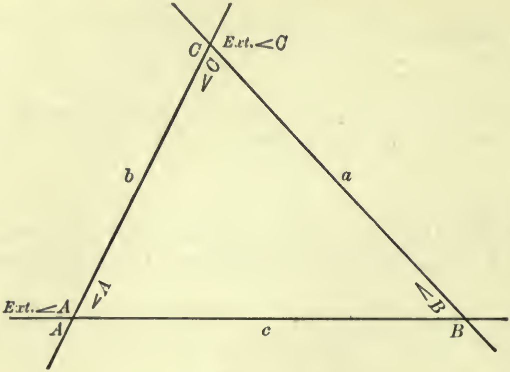


FIG. 1.

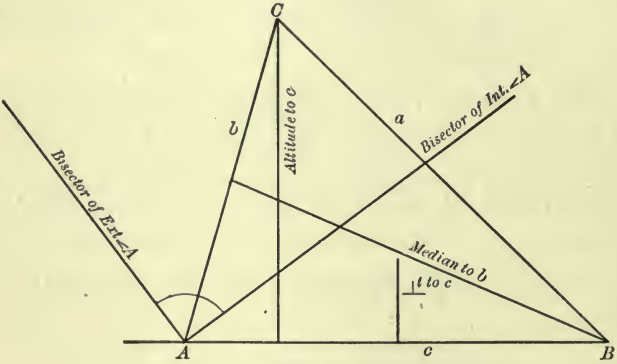


FIG. 2.

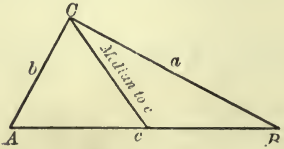


FIG. 3.

SUMMARY OF TRIANGULAR RELATIONS

IMPORTANT PROPERTIES OF THE ANGLES OF A TRIANGLE (Fig. 1)

1. $\angle A + \angle B + \angle C = 2$ rt. $\angle$s. (III. 1.)
2. $\angle A$ is the supplement of $\angle B + \angle C$. (III. 1 *a*.)
3. If $\angle C$ is a right angle, $\angle B$ is the complement of $\angle A$. (III. 1 *b*.)
4. If $\angle A = \angle B$, $a = b$. (IV. 2.)
5. If $\angle A = \angle B = \angle C$, $a = b = c$.
6. If $\angle A > \angle B$, $a > b$. (IV. 5.)
7. Ext. $\angle A = \angle B + \angle C$. (III. 2.)
8. Ext. $\angle A > \angle B$ or $\angle C$. (III. 2, Sch.)
9. If $\angle A = \angle B$, ext. $\angle C = 2\angle A$ or $2\angle B$.
10. The shape (not size) of a triangle is given by any two of its independent angles.

IMPORTANT PROPERTIES OF THE LINES OF A TRIANGLE (Fig. 2)

1. $a + b > c$. (VII. 1.)
2. If $a = b$, $\angle A = \angle B$. (IV. 1.)
3. If $a = b = c$, $\angle A = \angle B = \angle C$.
4. The bisector of $\angle A$ is perpendicular to the bisector of ext. $\angle A$.
5. The bisectors of $\angle A$, $\angle B$, and $\angle C$ concur at the in-center. (X. 1.)
6. The bisectors of ext. $\angle A$, ext. $\angle B$, and of $\angle C$ concur at an ex-center. (X. 2.)
7. The mid-$\perp$s to a, b, and c concur at the circumcenter. (X. 3.)
8. The altitudes to a, b, and c concur at the orthocenter. (X. 4.)
9. The medians to a, b, and c concur at the centroid. (X. 5.)
10. The shape and size of a triangle are determined by any three independent parts.

IMPORTANT PROPERTIES OF THE LINES OF A RIGHT TRIANGLE (Fig. 3)

1. The median to the hypotenuse equals $\frac{1}{2}$ the hypotenuse. (VII. 4.)
2. If $\angle A = \frac{2}{3}$ rt. $\angle$, the median to the hypotenuse equals b.
3. The altitude to a coincides with b, and *vice versa*.
4. The hypotenuse is the diameter of the circumcircle. (VII. 4.)
5. $(a + b) - c$ is the diameter of the inscribed circle. (Ex. 24, p. 90.)
6. The hypotenuse, c, $> a$ or b. (IV. 4 *a*.)

XI. GROUP ON MEASUREMENT

DEFINITIONS

Measurement is (*a*) **Direct**, or (*b*) **Indirect.**[1]

The **Direct Measurement** of a magnitude is the process of finding how many times it contains another magnitude of the same kind, which is called the unit of measure.

E.g. length may be measured in feet, miles, meters, kilometers, etc.; area in acres, square miles, hectares, etc., and weight in kilograms, pounds, tons, etc.

Any line may be assumed as a **Unit of Length.**

The **Indirect Measurement** of a quantity is the process of determining its size by comparing it with some other quantity, the changes in size of which correspond to changes in size of the first magnitude.

E.g. the pressure of steam is measured by the changes in position of a hand on a dial plate.

The height of a mountain is measured by the motion of the index on an aneroid barometer.

The strength of an electric current is measured by the temperature to which it raises a wire of known dimensions.

The pitch of an organ pipe is measured by the length of the pipe.

The amount of acid in a solution is measured by the intensity of the color it produces in a piece of litmus paper.

Angular Measure. Among geometrical magnitudes an angle is often measured by its intercepted arc, or by the quotient of the intercepted arc divided by the radius of the circle

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[1] Whether measurement is direct or indirect, the object attained is the same. The measure of a magnitude is a ratio; therefore, always abstract.

whose center is the vertex of the angle. The latter is called **Radial Measure.**

Ratio and Proportion

The **Geometric Ratio** of one magnitude to another is the quotient obtained by dividing the first by the second.

Thus, the ratio of a to b is $\frac{a}{b} = a \div b = a : b$.

The **Antecedent** of a ratio is the first, or dividend magnitude.

The **Consequent** of a ratio is the second, or divisor magnitude.

The usual **Sign** of ratio is **:**, although any method of indicating division may be used.

Both terms of any ratio may be multiplied or divided by the same quantity, without affecting the value of the ratio.

Commensurable Ratios

When the terms of a ratio can each be expressed as a multiple of a common unit, the terms are said to be commensurable with each other, and the ratio is said to **Commensurable.**

In this case the ratio can always be expressed as a numerical fraction, both of whose terms are whole numbers.

Problem I. *To express the ratio of two line-segments.*

Given. AB and CE.

Required. The ratio $AB : CE$.

Apply CE to AB as often as possible, say twice, with a remainder GB, so that

$$AB = 2\,CE + GB.$$

Apply the remainder GB to CE as often as possible, say four times, with a remainder FE, so that

$$CE = 4\,GB + FE,$$

and

$$\begin{aligned} AB &= 8\,GB + 2\,FE + GB \\ &= 9\,GB + 2\,FE. \end{aligned}$$

Apply the last remainder FE to GB as often as possible, say four times, without remainder, so that

$$GB = 4\,FE,$$

and $$CE = 16\,FE + FE = 17\,FE,$$

and $$AB = 36\,FE + 2\,FE = 38\,FE.$$

Thus the given lines have been expressed in terms of the common unit FE, and their ratio

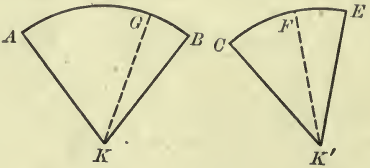
$$\frac{AB}{CE} = \frac{38\,FE}{17\,FE} = \frac{38}{17}.$$

The ratio $$\frac{CE}{AB} = \frac{17}{38}.$$

Apply the same method to the solution of the following problems.

Problem II. *To express the ratio of two arcs of equal circles, arc AB and arc CE.*

Problem III. *To express the ratio of two angles or sectors of equal circles, $\angle AKB$ and $\angle CK'E$.*



Incommensurable Ratios

If, on the other hand, the terms of a ratio cannot be expressed as multiples of the same unit, the terms are incommensurable with each other, and the ratio is said to be **Incommensurable.**

In this case no one of the remainder line-segments, arcs, angles, etc., of the process just indicated will be exactly contained in the preceding remainder line-segment, arc, angle, etc., no matter how long the process be continued.

Hence, an incommensurable ratio cannot be expressed as a fraction the terms of which are whole numbers.

APPROXIMATE EXPRESSION OF INCOMMENSURABLE RATIOS

Such a ratio can usually be expressed, however, in some form that will enable us to state the value of the ratio correctly to any required decimal place, or to any required degree of accuracy.

Hence, all the ratios with which we shall deal may be expressed, either exactly or approximately, as fractions. But the necessity for thus expressing them seldom arises. We shall be concerned chiefly with the *relations* between ratios, and one of the most important of these geometric relations is that of equality. This relation gives us the geometric proportion.

A **Geometric Proportion** is an expression of equality between two or more geometric ratios.

Thus, $a : b = c : m$,
or, as oftener written, $a : b :: c : m$,
and $\square P : \square Q :: h : h'$
are geometric proportions.

If more than two ratios are compared, the proportion is said to be a **Continued Proportion.**

Each ratio of a proportion is called a **Couplet.**

The **Extremes** are the first and fourth terms of a proportion.

The **Means** are the second and third terms of a proportion.

If $a : b :: c : e$, e is said to be a **Fourth Proportional** to a, b, and c. Similarly, b is a fourth proportional to a, c, and e, etc.

If $a : b :: b : c$, c is said to be a **Third Proportional** to a and b. Similarly, a is a third proportional to b and c.

If $a : b :: b : c$, b is said to be a **Mean Proportional** between a and c.

A **Transformation** of a proportion is a change in the proportion, either in the order of the terms or otherwise, that does not destroy the equality of the ratios.

A **Derived Proportion** is one obtained from a given proportion by transformation.

Thus, $a^3 : b^3 :: c^3 : e^3$

is *derived* from $a : b :: c : e$

by cubing the terms of the latter.

PROPOSITIONS

XI. 1. *If four quantities are in proportion, the product of the means equals the product of the extremes, and conversely.*

If $a : b :: c : e$, *i.e.* if $\frac{a}{b} = \frac{c}{e}$,

then $ae = bc$.

[If both members of the given equation be multiplied by be, the results will be equal.] (Ax. 3.)

Q.E.D.

Conversely, if the product of two quantities equals the product of two others, either set of factors may be made the extremes, and the other the means, of a proportion.

Hyp. If $ae = bc$,

Conc.: then $a : b :: c : e$, *i.e.* $\frac{a}{b} = \frac{c}{e}$.

[If both members of the given equation be divided by be, the results will be equal.] (Ax. 3.)

Q.E.D.

SCH. 1. We have seen that in the original proportion the product of the means equals the product of the extremes.

The test of the correctness of every *derived* proportion is that when the product of its extremes is placed equal to the product of its means, the resulting equation is the same as the equation similarly obtained from the *original* proportion, or may be reduced to the same.

Illustration. — If $a:b::c:e$ be the *original* proportion, then, by the test, $a+b:b::c+e:e$ is a correct *derived* proportion; for, by the application of XI, 1 to each, we get in each case $ae=bc$.

(Let the student make the application.)

Sch. 2. The most important transformations give the following derived proportions:

Hyp. If $\mathbf{a:b::c:e},$

Conc.: then (1) $\mathbf{b:a::e:c}.$

Dem. If $\frac{a}{b}=\frac{c}{e}$, then $\frac{1}{\frac{a}{b}}=\frac{1}{\frac{c}{e}}$. (Ax. 3.)

$$\therefore \frac{b}{a}=\frac{e}{c}.$$

Q.E.D.

(This form is said to be derived from the given proportion by *inversion*.)

Conc. (2): $\mathbf{a:c::b:e}.$

Dem. If $\frac{a}{b}=\frac{c}{e}$, then $\frac{a}{b}\times\frac{b}{c}=\frac{c}{e}\times\frac{b}{c}$. (Ax. 3.)

$$\therefore \frac{a}{c}=\frac{b}{e}.$$

Q.E.D.

(This form is said to be derived from the given proportion by *alternation*.)

Conc. (3): $\mathbf{a+b:a::c+e:c}$, and $\mathbf{a+b:b::c+e:e}.$

Dem. If $\frac{a}{b}=\frac{c}{e}$, then $\frac{a}{b}+1=\frac{c}{e}+1$. (Ax. 2.)

$$\therefore \frac{a+b}{b}=\frac{c+e}{e}.$$

Q.E.D.

(This form is said to be derived from the given proportion by *composition*.)

Conc. (4): $\mathbf{a}-\mathbf{b}:\mathbf{a}::\mathbf{c}-\mathbf{e}:\mathbf{c}$, and $\mathbf{a}-\mathbf{b}:\mathbf{b}::\mathbf{c}-\mathbf{e}:\mathbf{e}$.

Dem. If $\frac{a}{b}=\frac{c}{e}$, then, $\frac{a}{b}-1=\frac{c}{e}-1$. (Ax. 2.)

$$\therefore \frac{a-b}{b}=\frac{c-e}{e}.$$

Q.E.D.

(This form is derived from the given proportion by *division.*)

Conc. (5): $\mathbf{a}+\mathbf{b}:\mathbf{a}-\mathbf{b}::\mathbf{c}+\mathbf{e}:\mathbf{c}-\mathbf{e}$.

Dem. Divide the last equation in conclusion (3) by the last equation in conclusion (4), member by member.

$$\therefore \frac{a+b}{a-b}=\frac{c+e}{c-e}. \qquad \text{(Ax. 3.)}$$

Q.E.D.

(This form is said to be derived from the given proportion by *composition and division.*)

XI. 2. *In any number of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Hyp. If $a:b::c:e::f:g::$ etc.,

Conc.: then $a+c+f+\cdots:b+e+g+\cdots::a:b::c:e::$ etc.

Let $\frac{a}{b}=\frac{c}{e}=\frac{f}{g}=r,$

whence $a=br;\ c=er;\ f=gr$, etc.,

and $a+c+f+\cdots=br+er+gr+\cdots.$

Then $a+c+f+\cdots=r(b+e+g+\cdots).$

$$\therefore \frac{a+c+f+\cdots}{b+e+g+\cdots}=r=\frac{a}{b}=\frac{c}{e}=\text{etc.}$$

Q.E.D.

XI. 3. *If two proportions be multiplied together, term by term, the resulting products will be in proportion.*

Let the student supply the proof, by use of Ax. 3.

a *If any number of proportions be multiplied together, term by term, the resulting products will be in proportion.*

b *Like powers of the terms of a proportion are in proportion.*

c *Like roots of the terms of a proportion are in proportion.*

XI. 4. *If the terms of a proportion be divided successively by the terms of a second, the resulting quotients will be in proportion.*

Let the student supply the proof, using Ax. 3.

Ex. 1. If $2f = c$, what is the ratio of c to f?
Ex. 2. If $a = 3e$, what is the ratio of a to e?
Ex. 3. If $a + e : a - e :: 7 : 5$, what is the ratio of a to e?

What is the ratio of x and y in the following:

Ex. 4. $x + y : x :: 13 : 4$?

Ex. 5. $x - y : x :: 5 : 9$?

Ex. 6. $x + y : x - y :: 13 : 5$?

Give the name and value of x in each of the following proportions:

Ex. 7. $5 : x :: 10 : 12.$

Ex. 8. $x : 8 :: 10 : 16.$

Ex. 9. $5 : 11 :: 10 : x.$

What is the name and what the value of x in the following:

Ex. 10. $4 : x :: x : 36$?

Ex. 11. $x : 8 :: 8 : 2$?

Give ten proportions that can be derived from the proportion:

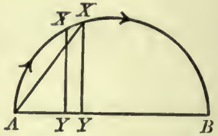
Ex. 12. $3 : 7 :: 9 : x.$

Ex. 13. Test the correctness of your answer by showing that the product of the means and extremes in the derived proportions is identical with the product of the means and extremes in the original proportion.

METHOD OF LIMITS

DEFINITIONS

A **Variable** is a quantity that in the course of a single discussion is always changing its value.

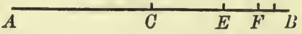


Thus, as the point X moves along the curve AB, its *distance* from the *line* AB, its distance from the *point* A, and the *projection* AY, of this latter distance on the line AB, are all *variables.*

A **Constant** is a quantity that does not change its value in the course of a single discussion.

Thus, if the curve in the above figure be a semicircle on AB as a diameter, AB is a *constant.* The changes in the variables above mentioned produce no changes in AB.

A **Limit** of a variable is a constant which the value of the variable may be made to approach as near as we please, but which the variable cannot be made to reach. That is, the difference between the limit and the variable may be made less than any assignable quantity, but cannot be made zero.



To illustrate: Suppose a point moves from A toward B so as to cover one half the distance in the first second, one half the remaining distance in the second, and so on.

Will the moving point ever coincide with B?

In other words, will the *variable* distance covered by the moving point ever coincide with the *constant* line-segment AB?

What, then, is the Limit of the *variable* distance gone over by the moving point?

If the distance passed over in the first second be called 1, that passed over in the second second will be $\frac{1}{2}$, that passed over in the third second will be $\frac{1}{4}$, and so on.

The whole distance, therefore, say x, will be $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$.

The greater the number of terms we take, the nearer x will approach the value 2.

Thus,

$$\begin{aligned} &1+\tfrac{1}{2}+\tfrac{1}{4} &&= \tfrac{7}{4};\ 2-\tfrac{7}{4}=\tfrac{1}{4};\\ &1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8} &&= \tfrac{15}{8};\ 2-\tfrac{15}{8}=\tfrac{1}{8};\\ &1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16} &&= \tfrac{31}{16};\ 2-\tfrac{31}{16}=\tfrac{1}{16};\end{aligned}$$

and so on.

We can make x as near 2 as we please; *i.e.* differ from 2 by as small a fraction as we choose by taking a sufficiently large number of terms.

But, no matter how great the number of terms we take, their sum will never actually reach 2.

$\therefore$ 2 is said to be the *limit* of the *sum* of the terms.

And AB is the *limit* of the *sum* of the segments AC, CE, EF, etc.; *i.e.* the Limit of the Variable Distance gone over by the moving point is AB.

The symbol $\doteq$ is employed to denote the expression "approaches as a limit," or any equivalent expression.

Postulate of Limits. *If, while approaching their respective limits, two variables are always equal, the limits are equal.*

For, since the two variables are equal at every stage of their progress, we have practically but one variable; and it is impossible that one variable (increasing or decreasing) should be approaching two different limits at the same time.

Direct inferences:

(*a*) The limit of the product of two variables is the product of their limits.

(*b*) If two variables have a constant ratio, and each approaches a limit, these limits, taken in the same order, have the constant limit of the ratios.

Ex. 14. 25 and 49 are perfect squares. By what proposition does it follow that their product must be a perfect square?

Ex. 15. 27 and 125 are perfect cubes. By what proposition does it follow that 3375 is also a perfect cube?

Ex. 16. Verify all the conclusions of XI. 1, Sch. 2 by use of the test given in XI. 1, Sch. 1.

Ex. 17. Show that, if $a : b :: b : c :: c : e$,

then $a : e :: a^3 : b^3$.

XI. SUMMARY OF PROPOSITIONS IN THE GROUP ON MEASUREMENT

1. *If four quantities are in proportion, the product of the means equals the product of the extremes, and conversely.*

2. *In any number of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

3. *If two proportions be multiplied together, term by term, the resulting products will be in proportion.*

   a *If any number of proportions be multiplied together, term by term, the resulting products will be in proportion.*
   b *Like powers of the terms of a proportion are in proportion.*
   c *Like roots of the terms of a proportion are in proportion.*

4. *If the terms of a proportion be divided successively by the terms of a second, the resulting quotients will be in proportion.*

Postulate of Limits. *If, while approaching their respective limits, two variables are always equal, the limits are equal.*

XII. GROUP ON MEASUREMENT OF ANGLES

DEFINITIONS

A **Central Angle** is an angle whose vertex is at the center of a circle.

An **Inscribed Angle** is an angle formed by two chords intersecting on the circumference.

An **Escribed Angle** is an angle formed by one chord with another chord produced.

An angle is **inscribed in a segment** when its vertex is in the arc of the segment and its sides pass through the extremities of this arc.

The **angle between two curves** at any point of intersection is the angle formed by the tangents to the curves at this point. If the angle between two curves is *right*, the curves are said to cut each other **orthogonally**.

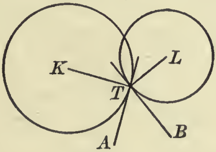


FIG. 1.

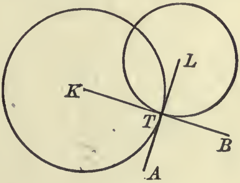


FIG. 2.

Thus, if AT, BT (Fig. 1, Fig. 2) be tangent to the circles K and L, respectively, $\angle ATB$ is the angle between the circles at T. If, as in Fig. 2, $\angle ATB$ is right, the circles are said to cut each other *orthogonally.*

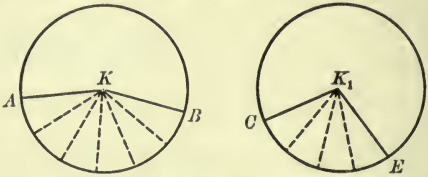
The angle KTL, between the radii to T, is supplemental to $\angle ATB$ (IX, 1 and II, 5). Hence, if $\angle ATB$ is right, $\angle KTL$ is right, and conversely, that is,

Two circles cut each other orthogonally when the radii to either point of intersection are perpendicular to each other.

PROPOSITIONS

XII. 1. *In the same circle, or equal circles, central angles are to each other as their intercepted arcs.*

Hyp. Case I. If $\odot K = \odot K_1$, and arc AB is to arc CE as any two whole numbers, say 6 to 4 (commensurable),



Conc.: then $\angle AKB : \angle CK_1E :: \text{arc } AB : \text{arc } CE$.

Dem. Arcs AB and CE have a common measure. (Hyp.)
Apply it to each of the arcs, and suppose it is contained six times in AB and four times in CE.

Join the points of division with the centers of the circles. All the central angles thus formed will be equal. (IX. 3.)

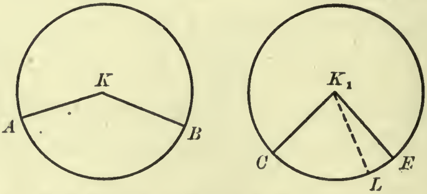
$$\therefore \angle AKB : \angle CK_1E :: 6 : 4.$$

But $$\text{arc } AB : \text{arc } CE :: 6 : 4.$$

$$\therefore \angle AKB : \angle CK_1E :: \text{arc } AB : \text{arc } CE.$$

Q.E.D.

Hyp. Case II. If $\odot K = \odot K_1$, and arc AB and arc CE are incommensurable,



Conc.: then $\angle AKB : \angle CK_1E :: \text{arc } AB : \text{arc } CE$.

Dem. Divide arc AB into any number of equal parts, say 4, each less than arc CE. Let AM be one of these parts, and be contained in CE twice, with a remainder LE.

As the remainder is always less than the divisor, it follows that if we increase the number of equal parts into which AB is divided, we diminish both the divisor and the remainder.

But, as the arcs AB and CE are incommensurable (Hyp.), the remainder can never be 0.

$$\therefore \text{ arc } CL \doteq \text{arc } CE, \text{ and } \angle CK_1L \doteq \angle CK_1E.$$

(XI. Def. of Limit.)

But $$\frac{\angle AKB}{\angle CK_1L} \text{ always} = \frac{\text{arc } AB}{\text{arc } CL}.$$ (XII. 1, Case I.)

$\therefore \angle AKB : \angle CK_1E :: \text{arc } AB : \text{arc } CE.$ (XI. Post. Limits.)

Q.E.D.

Sch. Heretofore, we have measured angles *directly* (see p. 100), using the right angle as the unit of measure. This unit is inconvenient, however, as its use requires us to employ *fractions* too frequently.

The foregoing theorem, due to Thales of Miletus (640 B.C.), introduced a very simple method of *indirect measurement.*

Any change in the magnitude of the central angle produces a proportional change in the intercepted arc. Thus, if the central angle be doubled, the intercepted arc is doubled; if the angle be trebled, the arc is trebled; and so on.

Hence, the intercepted arc may be taken as the *measure* of the central angle.

This very important theorem may be stated thus:

A central angle is measured by its intercepted arc.

The circumference is divided into 360 equal parts, called degrees; each degree into 60 equal parts, called minutes, and each minute into 60 equal parts, called seconds. For brevity, an angle measured by an arc of 1° is called an angle of 1°; etc.

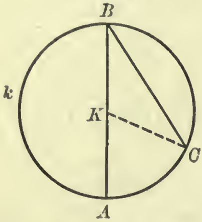
Note. — This division of the circumference, known as the sexagesimal division (or division by sixties), is due to the Babylonians.

The Babylonian year consisted of 12 months of 30 days each, or 360 days. Accordingly the zodiac was divided into 12 signs of 30 degrees each, making 360 degrees for a complete circle.

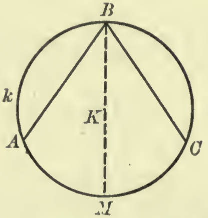
These people were also familiar with the fact that the radius of a circle, used as a chord, divides the circle into 6 equal parts of 60 degrees each. Hence, arose the custom of subdividing the degree by sixties, into minutes, etc., as indicated above.

XII. 2. *An inscribed angle is measured by half the intercepted arc.*

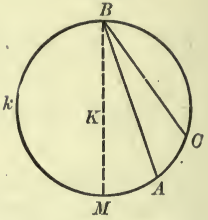
CASE (1)



CASE (2)



CASE (3)



Hyp. If $\angle B$ is inscribed in $\odot K$, and if AC is the intercepted arc,

Conc.: then $\angle B$ is measured by $\frac{\text{arc } AC}{2}$.

Case (1). When one arm of the angle is a diameter through B.

Case (2). When the two arms are on opposite sides of the diameter through B.

Case (3). When both arms are on the same side of the diameter through B.

Dem. (1) Draw the radius CK, forming the isosceles $\triangle CKB$.

Then $\angle B = \angle C$. (IV. 1.)

$\therefore \angle B = \frac{1}{2} \angle AKC$. (III. 2 *a*.)

But $\angle AKC$ is measured by arc AC. (XII. 1. Sch.)

$\therefore \angle B$ is measured by $\frac{1}{2}$ arc AC.

Q.E.D.

Dem. (2) Draw the diameter BM.

Then $\angle MBC$ is measured by $\frac{1}{2}$ arc MC. (XII. 2, Case (1).)

$\angle ABM$ is measured by $\frac{1}{2}$ arc AM. (XII. 2, Case (1).)

$\therefore \angle MBC + \angle ABM$ is measured by $\frac{1}{2}$ arc $MC + \frac{1}{2}$ arc AM.

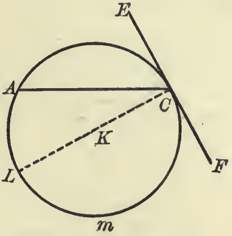
$\therefore \angle ABC$ is measured by $\frac{1}{2}$ arc AC.

Q.E.D.

Dem. (3) Both arms on the same side of the diameter through B.

Proof to be supplied by the pupil.

XII. 2 *a*. *An angle formed by a tangent and a chord is measured by half the intercepted arc.*



Hyp. If AC and EF intersect at C, and if AC is a chord and EF is a tangent,

Conc.: then $\angle ACF$ is measured by $\frac{\text{arc } AmC}{2}$.

Dem. Draw the diameter CL.

$\angle LCF$ is a right angle.

[A radius to point of tangency $\perp$ the tangent.] (IX. 4.)

$$\angle ACF = \text{rt. } \angle LCF + \angle LCA. \qquad \text{(Ax. 4.)}$$

Rt. $\angle LCF$ is measured by $\frac{\text{arc } LmC}{2}$.

[A right angle is measured by one half a semicircumference.] (XII. 1, Sch.)

$\angle LCA$ is measured by $\frac{\text{arc } AL}{2}$. (XII. 2.)

$\therefore$ $\angle ACF$ is measured by $\frac{\text{arc } LmC}{2} + \frac{\text{arc } AL}{2} = \frac{\text{arc } AmC}{2}$.

Q.E.D.

Sch. Angles inscribed in the same segment are equal.

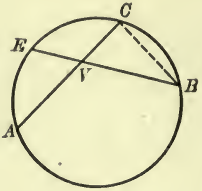
If the segment is greater than a semicircle, the angles are acute.

If the segment is a semicircle, the angles are right.

If the segment is less than a semicircle, the angles are obtuse.

Ex. 1. The bisectors of the vertex angles of all triangles on the same base and inscribed in the same segment are concurrent.

XII. 3. *An angle whose vertex lies between the center and the circumference is measured by half the sum of its intercepted arcs.*



Hyp. If AC and BE intersect at V lying between the center and the circumference,

Conc.: then $\angle AVB$ is measured by $\dfrac{\text{arc } AB + \text{arc } CE}{2}$.

Dem. Draw BC.

$$\angle AVB = \angle C + \angle B.$$

[The ext. ∠ of a △ equals the sum, etc.] (III. 2.)

But $\angle C$ is measured by $\dfrac{\text{arc } AB}{2}$. (XII. 2.)

And $\angle B$ is measured by $\dfrac{\text{arc } CE}{2}$. (XII. 2.)

$\therefore \angle AVB$ is measured by $\dfrac{\text{arc } AB + \text{arc } CE}{2}$.

Q.E.D.

Ex. 2. The opposite angles of an inscribed 4-side are supplemental.

Ex. 3. The bisector of any interior angle of an inscribed 4-side and the bisector of the opposite exterior angle intersect on the circumference.

Ex. 4. How many degrees are there in the arc that subtends an inscribed angle of 25°? Of 25° 40′? Of 25° 40′ 34″?

Ex. 5. What arc measures the supplemental adjacent angles of the preceding inscribed angles?

Ex. 6. What are these angles called?

Ex. 7. An angle between the center and the circumference is 40° 30′. What is the sum of the arcs that measure it?

Ex. 8. An angle of 70° and its supplement are formed by a tangent and a chord. What is the value of each arc that subtends these angles?

Ex. 9. The arc of a segment is 140°. What is the value of each angle inscribed in this segment?

XII. 4. *An angle formed by two secants intersecting without the circle is measured by half the difference of the intercepted arcs.*

Hyp. If AC and BE intersect at V lying without the circumference, and AC and BE are both secants,

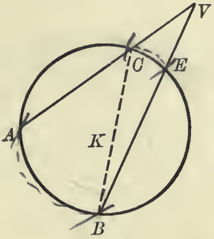


FIG. 1.

Conc.: then $\angle AVB$ is measured by $\frac{\text{arc } AB - \text{arc } CE}{2}$.

Dem. Draw BC.

$$\angle ACB = \angle AVB + \angle B. \qquad (1.)$$

[The ext. ∠ of a △ equals the sum, etc.] (III. 2.)

$$\therefore \angle AVB = \angle ACB - \angle B.$$

((1.) by transposition.)

But $\angle ACB$ is measured by $\frac{\text{arc } AB}{2}$. (XII. 2.)

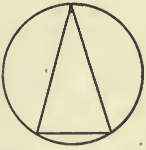
$\angle B$ is measured by $\frac{\text{arc } CE}{2}$. (XII. 2.)

$\therefore \angle AVB$ is measured by

$$\frac{\text{arc } AB}{2} - \frac{\text{arc } CE}{2} = \frac{\text{arc } AB - \text{arc } CE}{2}.$$

Q.E.D.

Prove by means of XII. 2:



Ex. 10. That an isosceles triangle is isoangular.

Ex. 11. That an isoangular triangle is isosceles.

Ex. 12. That the sum of the interior angles of a triangle equals two right triangles.

Ex. 13. That the sum of the exterior angles of a triangle equals four right angles.

Ex. 14. That two mutually equiangular triangles inscribed in the same circle are congruent.

XII. 4 *a*. *An angle formed by a tangent and a secant is measured by half the difference of the intercepted arcs.*

Hyp. If AC is a secant and BE is a tangent (XII. 4 *a*), or if both are tangents (XII. 4 *b*),

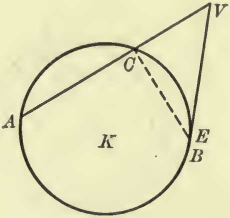


Fig. 2.

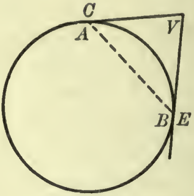


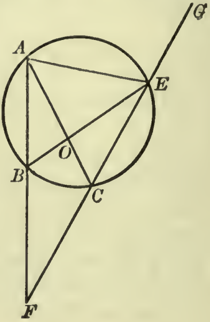
Fig. 3.

Conc.: then $\angle AVB$ is measured by $\dfrac{\text{arc } AB - \text{arc } CE}{2}$.

Dem. Similar to Demonstration of XII. 4.

(Let the pupil give it in full. Note that when the secant VB of Fig. 1 is turned on V as a pivot until it becomes a tangent, the points B and E become coincident, as shown in Fig. 2, and may be denoted by a single letter. Observe also that if the secant VA of Fig. 1 be likewise turned on V as a pivot until it becomes a tangent, the points A and C of Fig. 1 become coincident, as shown in Fig. 3, and may be denoted by a single letter.)

XII. 4 *b*. *An angle formed by two tangents is measured by half the difference of the intercepted arcs.*



Ex. 15. The 4-side $ABCE$ is inscribed in a circle. $\angle F$ is 28°; $\angle BOC$ is 82°. How many degrees in each of the arcs CB and AE?

How many degrees:

Ex. 16. In $\angle OBF$ and $\angle OCF$?

Ex. 17. In $\angle BAE$ and $\angle BAC$?

Ex. 18. In $\angle GEO$?

Ex. 19. What kind of angle with reference to the circle is $\angle GEO$?

Ex. 20. What is the value of all the angles that are inscribed in the major segment that stands on the chord AE?

XII. SUMMARY OF PROPOSITIONS IN THE GROUP ON MEASUREMENT OF ANGLES

1. *In the same, or equal circles, central angles vary (are to each other) as their intercepted arcs.*

SCH. A central angle is measured by its intercepted arc.

2. *An inscribed angle is measured by half the intercepted arc.*

   a *An angle formed by a tangent and a chord is measured by half the intercepted arc.*

3. *An angle whose vertex lies between the center and the circumference is measured by half the sum of its intercepted arcs.*

4. *An angle formed by two secants intersecting without the circle is measured by half the difference of the intercepted arcs.*

   a *An angle formed by a tangent and a secant is measured by half the difference of the intercepted arcs.*

   b *An angle formed by two tangents is measured by half the difference of the intercepted arcs.*

HINTS TO THE SOLUTION OF ORIGINAL EXERCISES

In solving a problem in algebra we proceed as follows:

1. We assume that we have the quantity required and call it x.
2. We form an equation in which x may be surrounded by a number of modifiers — coefficients, exponents, etc.

The value of x is found when by a transformation or by a series of transformations, *x stands alone on one side of the equation*, while the modifiers in some form appear on the other.

In solving a problem in geometry we proceed in a similar way:

1. We assume that we have the figure that satisfies the conditions given in the problem. *This assumed figure* corresponds to *the x of algebra.*
2. We ask ourselves what follows from this assumption; that is, what definitions, axioms, or previously established theorems, corollaries, or problems are suggested by the assumed figure.
3. We ask what one of these theorems or combination of them may be applied in the actual construction of the required figure.

These applied propositions correspond *to the modifiers of x* in algebra.

The drawing of such auxiliary line or lines as will make it possible to apply a suggested theorem, or a combination of suggested theorems, as well as the discovery of these theorems, is the test of the inventional power of the student. No rule can be made that will tell him what theorem to select or what line to draw, but the systematic attack, persistently made, familiarizes him with the principles of geometry.

The solution of a problem consists of:

1. The analysis as outlined above.
2. The construction.
3. The proof.
4. The discussion.

Many problems in the beginning of the course in algebra may be solved without the use of x. That is, they may be considered problems in arithmetic.

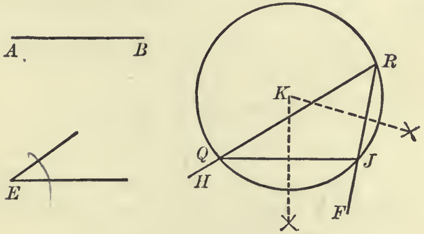
So, in geometry, it is by no means always necessary to give the analysis of the problem; that is, to assume we have the required figure, etc. We pass, however, from the *simple* to the *complex.* We learn best how to use the method in problems where it is indispensable by applying it to the solution of simpler problems first.

PROBLEMS, EXERCISES, AND SPECIAL THEOREMS

Problems

Prob. I. *On a given line to construct a circular segment which shall contain a given angle.*

Given. A line-segment *AB* and an angle *E*.



Required. On *AB* as a chord to construct a circular segment capable of containing an angle equal to $\angle E$.

Const. Construct an $\angle FRH = \angle E$. (V. Prob. III.)

With *AB* as a radius, and any point *J* in *RF* as a center so taken that the arc described will cut *RH*, describe an arc.

Let it intersect *RH* in *Q*.

Draw *QJ*, and draw a circle through *R*, *Q*, and *J*.

(Prob. IX., Ex. in drawing, p. 17.)

The segment *QRJ* is the segment required.

Q.E.D.

Dem. Any angle inscribed in this segment is measured by one half the same arc that subtends $\angle FRH$. (XII. 2.)

$\therefore$ all such angles equal $\angle FRH = \angle E$.

Q.E.D.

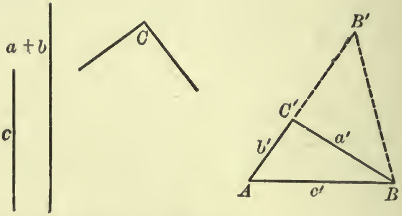
A second construction in common use is the following:

Erect a mid $\perp$ to *AB*. At *B* make $\angle ABC = \angle E$, and erect $BM \perp BC$. The $\odot$ with *K*, the intersection of these $\perp$s, as center and *KB* as radius gives the required segment.

ILLUSTRATION OF THE METHOD OF SOLVING ORIGINAL PROBLEMS

PROB. II. *Given the base, vertex angle, and sum of the legs of a triangle, construct it.*

Given. The base c, the vertex $\angle C$, and the sum $a+b$.



Required. A triangle whose base equals c, vertex angle equals $\angle C$, and the sum of the legs equals $a+b$.

Analysis. Assume $\triangle ABC'$ has $c'=c$, $\angle C'=\angle C$, and $a'+b'=a+b$.

It follows that if AC' be produced, making $C'B'=a'$, and if BB' be drawn, $\triangle C'B'B$ is isosceles.

Suggested theorems are: III. 2 a and IV. 1; also Axioms 7 and 5.

Applicable theorems, etc.: All the above; for the $\angle B'$ and the $\angle C'BB'$ each equal $\frac{\angle C'}{2}$. (IV. 1 and III. Ex. 4.)

$\therefore$ if we start with $a+b$, since $\angle B'=\frac{\angle C}{2}$, the position of $B'B$ is known. (Ax. 7.)

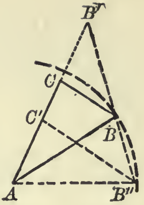
And, since B is c distant from A and also somewhere in BB' or its extension, B is known. (Ax. 5.)

Similarly the position of BC' is known. (Ax. 7.)

Hence, C' is known. (Ax. 5.)

$\therefore$ we have discovered how to *construct* the required triangle.

NOTE. — An analysis when complete shows us clearly the method of construction. In short, the construction then becomes merely an exercise in geometrical drawing.



Const. Let $AB' = a + b$.

At B' construct an angle with $AB' = \frac{\angle C}{2}$.

With A as a center and a radius equal to c, describe arc BB''.

Draw BC making $\angle B'BC = \angle B'$.

Draw AB.

$\triangle ABC$ is the required triangle.

Q.E.F.

Proof. $\angle B'BC = \angle B' = \frac{\angle C}{2}$. (Construction.)

$\therefore \triangle B'BC$ is isoangular.

$\therefore \angle ACB = 2\angle B'$.

[The ext. vert. $\angle$ of an isoangular $\triangle$ equals, etc.] (III. 2 *a.*)

$\therefore \angle ACB =$ given $\angle C$. (1)

Again, $B'C = BC$.

[An isoangular triangle is isosceles.] (IV. 2.)

$\therefore AC + BC = AC + B'C = a + b$. (2)

$AB = c$. (Construction.) (3)

Q.E.D.

Discussion. A second triangle fulfilling the given conditions may be formed; for the circle of line (3) of the above construction cuts $B'B$ in a second point, B''. Therefore, draw $B''C'$ just as BC was drawn and draw the join $B''A$.

$\triangle AB''C'$ is the second triangle meeting the given conditions.

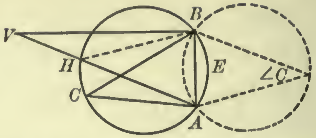
If the circle with A as a center and c as a radius is tangent to BB', the two triangles coincide and become identical.

If this circle does not cut BB', no such triangle can be constructed.

EXERCISES

The following exercises are on the loci of vertices of the equal vertex angles of triangles standing on the same base; of the incenters of such triangles; of their excenters, circumcenters, orthocenters, and centroids.

Ex. *a*. **Hyp.** If any number of angles equal $\angle C$, and their sides pass through the ends of the given line-segment AB,



Conc.: then the locus of their vertices is a circular segment on AB as a chord, and capable of containing an angle equal to $\angle C$.

Dem. On AB, equal to the given line, as a chord, construct a circular segment capable of containing an angle equal to the given $\angle C$.

(Prob. I. Group XII.)

The arc of this segment is the locus required. For,

(1) Any angle inscribed in this segment equals $\angle C$,

$\because$ it is measured by $\frac{\text{arc } AEB}{2}$. (XII. 3.)

(2) No angle whose vertex is without the circle, subtended by AB and to the left of it, can equal $\angle C$. For,

Draw the auxiliary HB. $\angle AHB$ ($= \angle C$. Why ?) is greater than $\angle V$.
[The ext. $\angle$ of a $\triangle >$ either non-adj. int. $\angle$.] (III. 2, Sch.)

Similarly it may be shown that an angle whose vertex lies within the circle cannot equal $\angle C$.

$\therefore$ the arc of the segment to the left of AB, on AB as a chord, is the locus of the vertices of all angles to the left of AB subtended by AB.

The locus of the vertices of such angles on the right of AB and equal to $\angle C$ will be the arc of a segment on AB as a chord on the right of AB, and equal to that on the left.

Q.E.D.

Note 1. — If the given angle be acute, both arcs will be major.

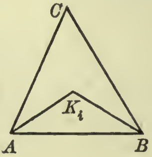
If the given angle be right, both arcs will be semicircles.

If the given angle be obtuse, both arcs will be minor.

Note 2. — The pupil will remember that in order to prove that a line is the locus of a point, two things must be established. What are they ?

Ex. *b*. The locus of the incenters of triangles whose base is AB and whose vertex angles always $= \angle C$, is :

The arc of a segment on AB as a chord that will contain an $\angle =$ rt. $\angle + \frac{\angle C}{2}$.

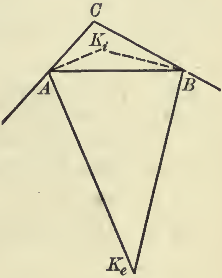


Proof. $\angle AK_iB = 1 \text{ rt. } \angle + \frac{\angle C}{2}$. (IV. Ex. 20.)

$\therefore$ by XII. Ex. (a) the locus of K_i is the arc of a segment on AB as a chord that will contain an angle $= \text{rt. } \angle + \frac{\angle C}{2}$. Q.E.D.

Ex. c. The locus of the excenters of triangles whose base is AB and whose vertex angles always equal $\angle C$, is:

The arc of a segment on AB as a chord that will contain an $\angle = \text{rt. } \angle - \frac{\angle C}{2}$.



Proof. K_eA and K_eB bisect ext. $\angle A$ and ext. $\angle B$, respectively.

Draw K_iA and K_iB bisecting int. $\angle A$ and int. $\angle B$, respectively.

$\angle K_i = \text{rt. } \angle + \frac{\angle C}{2}$. (v. preceding Ex. b.)

$\angle K_iAK_e$ and $\angle K_iBK_e$ are right angles. Why?

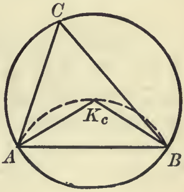
$$\therefore \angle K_i + \angle K_e = 2 \text{ rt. } \measuredangle.$$

$$\therefore \angle K_e = \text{rt. } \angle - \frac{\angle C}{2}.$$

$\therefore$ the locus of K_e is the arc of a segment on AB as a chord that will contain an $\angle = \text{rt. } \angle - \frac{\angle C}{2}$. Q.E.D.

Ex. d. The locus of the circumcenters of triangles whose base is AB and whose vertex angles always equal $\angle C$, is:

The arc of a segment on AB as a chord that will contain an $\angle = 2 \angle C$.



Proof. K_c is by hypothesis the center of the circumcircle of triangle ABC. (v. X. 3.)

$\therefore \angle AK_cB$ is a central angle and is measured by the arc AB. (XII. 1, Sch.)

But $\angle C$ is inscribed in this circle.

$\therefore \angle C$ is measured by half the arc AB.

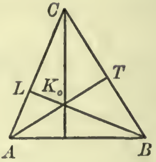
$$\therefore \angle AK_cB = 2 \angle C.$$

$\therefore$ the locus of K_c is the arc of a segment on AB as a chord that will contain an angle equal $2 \angle C$.

Q.E.D.

Ex. *e*. The locus of the orthocenters of triangles whose base is AB and whose vertex angles always equal $\angle C$, is:

The arc of a segment on AB as a chord that will contain an angle equal to the supplement of $\angle C$.



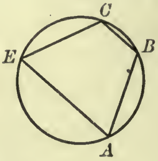
Proof. $\angle AK_oB = \angle LK_oT$.

$\because$ $\angle CLK_o$ is a right angle and $\angle CTK_o$ is a right angle, $\angle C$ is the supplement of $\angle LK_oT$.

$\therefore$ $\angle C$ is the supplement of $\angle AK_oB$.

$\therefore$ the locus of K_o is the arc of a segment on AB as a chord that will contain an angle equal to the supplement of $\angle C$.

Q.E.D.

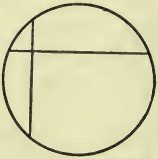


Ex. *f.* If the opposite angles of 4-side $ABCE$ are supplemental, show that a circle passing through A, B, and C, will also pass through E; that is, if the opposite angles of a 4-side are supplemental, it is a cyclic.

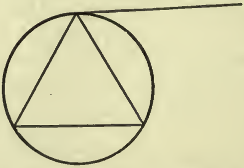
Ex. *g*. Let the student state the locus of each center given.

Ex. 21. Describe a circle. Draw a 4-side so that one of its angles shall be formed by two tangents, one by a tangent and a chord, one by a tangent and a secant, and the fourth by two secants.

Point out the measure of each of the above four angles.



Ex. 22. If two chords are perpendicular to each other in a circle, the sum of either set of opposite intercepted arcs is a semicircle.



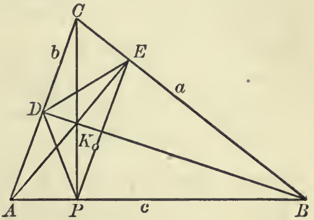
Ex. 23. If at the vertex of an inscribed equilateral triangle a tangent is drawn, find the angle between the tangent and each of the sides meeting at the vertex.

Ex. 24. Tangents are drawn at the vertices of an inscribed triangle. Two angles of the inscribed triangle are respectively 70° and 80°. Find the angles of the circumscribed triangle.

Theorems of Special Interest

Pedal Triangle

Th. 1. If, in $\triangle ABC$, P, E, and D are the feet of the altitudes on c, a, and b, respectively,



Conc.: then CP bisects $\angle DPE$, AE bisects $\angle DEP$, and DB bisects $\angle EDP$.

Dem. CP, BD, and AE concur at K_o. (X. 4.)
The 4-side PK_oDA is cyclic.
[If the opposite $\angle$s of a 4-side be supp., it is cyclic.] (Ex. *f*, p. 126.)

$$\therefore \angle DPK_o = \angle DAK_o.$$

$\left[\text{Both are measured by } \frac{\text{arc } DK_o}{2}.\right]$ (XII. 3.)

The 4-side $APEC$ is cyclic.
[Rt. $\triangle$s CEA and CPA stand on the same hypotenuse AC.]

$$\therefore \angle DAK_o = \angle CPE.$$

$\left[\text{Both are measured by } \frac{\text{arc } CE}{2}.\right]$ (XII. 3.)

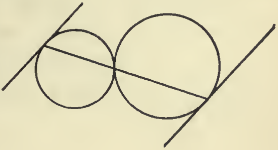
$$\therefore \angle DPK_o = \angle CPE. \quad \text{(Ax. 1.)}$$

$\therefore$ CP bisects $\angle DPE$.
Similarly, AE bisects $\angle DEP$ and DB bisects $\angle PDE$.

Q.E.D.

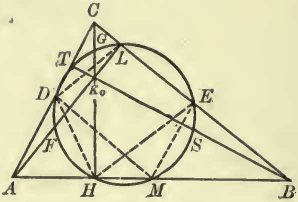
Note. — $\triangle PED$ is known as the pedal triangle of $\triangle ABC$.

Ex. 25. If, through the point of contact of two circles tangent externally, a straight line is drawn terminating in the circles, the tangents at the extremities of this line are parallel.



Nine-Point Circle Theorem

Th. 2. If, in a $\triangle ABC$, H, L, and T be feet of altitudes; M, E, and D feet of medians, and S, G, and F mid-points of BK_o, CK_o, and AK_o, respectively,



Conc.: then H, L, T, M, E, D, S, G, F are cyclic.

Dem. 1. Pass a circle through M, E, and D.

First, prove that this circle passes through H; second, through F.

$$ME \parallel b \text{ and } MD \parallel a.$$

[The mid-join of 2 sides of a $\triangle$ is $\parallel$ to the 3d side.] (VII. Ex. 8.)

$\therefore$ 4-side $MECD$ is a parallelogram. (Def. of ▱.)

$\therefore \angle DME = \angle ACB$. (VI. $1'$ a'.)

$\triangle DHC$ is isosceles. (VII. 4.)

$\therefore \angle DHC = \angle DCH$, and similarly, $\angle CHE = \angle HCE$.

$\therefore \angle DHC + \angle CHE = \angle DCH + \angle HCE (= \angle ACB)$. (Ax. 2.)

$\therefore \angle DHE = \angle DME$. (Ax. 1.)

$\therefore$ a circle through M, E, and D passes through H.

[The arc DME is the locus of vertices of all angles $= \angle DME$, etc.]

Similarly, this identical circle passes through L and T.

Q.E.D.

2. Draw the joins TF and LD.

TF divides rt. $\triangle TAK_o$ into two isosceles triangles. (Why?)

LD divides rt. $\triangle LCA$ into two isosceles triangles.

These two triangles have $\angle CAL$ in common, and are therefore mutually equiangular.

$\therefore \angle TFA = \angle ADL$, whose supplements, ∡ TFL and TDL, are therefore equal.

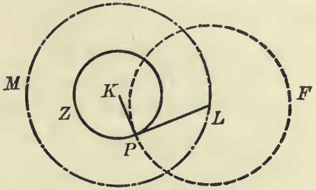
$\therefore$ a circle passing through T, D, and L must pass through F.

Similarly, it may be shown that this circle also passes through S and G.

Q.E.D.

Theorem of Orthogonal Circles

Th. 3. If circles of a given radius are drawn so as to cut a given ⊙ Z orthogonally,



Conc.: then the locus of their centers is a circle concentric with Z, whose radius is the hypotenuse of a right triangle whose legs are the radius of ⊙ Z and the given radius.

Dem. Draw any radius of Z, as KP, and at P erect a perpendicular to KP and equal to the given radius.

With K as a center and KL as a radius, describe the ⊙ M.

This circle is the locus of the centers described in the hypothesis, for

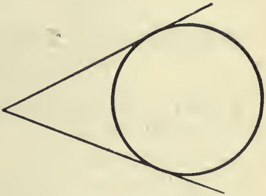
LP and KP are of constant lengths, and the $\angle LPK$ is always a right angle.

∴ LK, the hypotenuse, must always be of the same length.

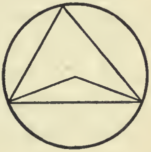
∴ the ⊙ M, with the center K and a radius equal to LK, is the locus required.

Q.E.D.

Ex. 26. If the angle formed by two tangents is 50°, how many degrees in each of the intercepted arcs?



Ex. 27. A circle is circumscribed about a triangle. Prove that the radii drawn to the extremities of the base form an angle equal to twice the angle at the vertex of the triangle.



CLASSIFICATION OF PROBLEMS — INDETERMINATE, DETERMINATE, AND OVERDETERMINATE

Kinds of Equations

In algebra the student has learned that there are three classes of simultaneous equations, to wit:

Indeterminate, having an indeterminate number of roots.

Determinate, having a determinate number of roots.

Overdeterminate, having, in general, no roots.

So in geometry problems may be similarly classified, to wit:

Indeterminate, in which too few conditions are imposed to give a determinate or definite number of figures that will satisfy the given conditions.

E.g. draw a circle tangent to a given line at a given point in the line. An indeterminate number of such circles may be drawn.

Determinate, in which enough conditions are imposed to give a determinate number of figures that will satisfy the given conditions.

E.g. draw a circle, with a given radius, that shall be tangent to a given line at a given point in the line. Two such circles may be drawn.

Overdeterminate, in which too many conditions are imposed.

E.g. draw a circle, with a given radius, that shall be tangent to a given line at a given point in the line, and at the same time (simultaneously) pass through a given point. In general, no such circle can be drawn.

EXERCISES IN INDETERMINATE, DETERMINATE, AND OVERDETERMINATE PROBLEMS

1. Add to the following indeterminate problems a condition that will make each determinate:

(*a*) Draw a line through a given point.
(*b*) Draw a perpendicular to a given line.
(*c*) Draw a circle tangent to two intersecting lines.
(*d*) Construct a △, having given one side and an angle adjacent to it.
(*e*) Construct a triangle, having given 2 ∡s.

2. By what axiom is the following problem determinate?

Draw the bisector of the vertex angle of a given triangle.

3. Why is the following problem overdeterminate? Draw a bisector of the vertex angle of a triangle that shall be perpendicular to the base.

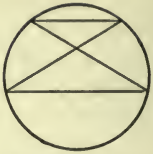
4. Arrange a summary of quadrilateral relations, similar to that of triangular relations on page 99, giving ten properties of the angles and lines of a 4-side.

PROBLEMS — THEIR CLASSIFICATION ILLUSTRATED

	INDETERMINATE	DETERMINATE	OVERDETERMINATE
	Construct:	Construct:	Construct:
1.	A point a distant from P.	A point a distant from P, and b distant from P_1.	A point a distant from P, b distant from P_1, and c distant from P.
2.	A point a distant from a given line l.	A point a distant from l, and b distant from P.	A point a distant from l, b distant from P, and c distant from P_1.
3.	A point a distant from a given $\odot K$.	A point a distant from $\odot K$, and b distant from a given line l.	A point a distant from $\odot K$, b distant from l, and c distant from l_1.
4.	A point equidistant from two parallels.	A point equidistant from two parallels, and a distant from $\odot K$.	A point equidistant from two parallels, a distant from $\odot K$, and b distant from P.
5.	A point equidistant from two concentric circles whose radii are a and b, respectively.	A point equidistant from two concentric circles whose radii are a and b, and at the same time c distant from $\odot K$.	A point equidistant from two concentric circles whose radii are a and b, and at the same time c distant from $\odot K$ and e distant from l.

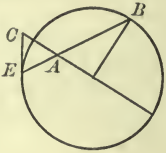
Ex. 28 (*a*). Parallel chords intercept equal arcs.

Ex. 28 (*b*). If the opposite ends of two parallel chords are joined, two isosceles triangles are formed.

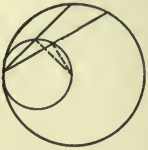


Ex. 29. If a 4-side is circumscribed about a circle, prove that the sum of two opposite sides equals the sum of the other two sides.

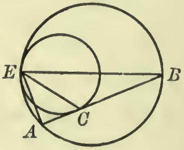
Ex. 30. If A is any point in a diameter, B the extremity of a radius perpendicular to the diameter, E the point in which AB meets the circumference, C the point in which the tangent through E meets the diameter produced, then $AC = EC$.



Ex. 31. If two circles are internally tangent, and the diameter of the less equals the radius of the larger, the circumference of the less bisects every chord of the larger which can be drawn through the point of contact.



Ex. 32. Two circles are internally tangent in the point E, and AB is a chord of the larger circle tangent to the less in the point C. Prove that EC bisects $\angle AEB$.



Ex. 33. Show that in a circumscribed hexagon the sum of one set of alternate sides (first, third, fifth) equals the sum of the other set (second, fourth, sixth).

Show also that the sum of one set of alternate sides of a circumscribed octagon equals the sum of the other set.

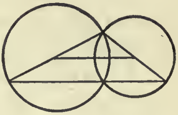
Ex. 34. Show that any circumscribed polygon with an even number of sides has the sum of one set of alternate sides equal to the sum of the other set.

Ex. 35. Inscribe a square in a given circle. Show how to obtain from this square, by bisection of sides, etc., a regular inscribed octagon and a regular inscribed hexa-decagon.

Ex. 36. An equilateral inscribed polygon is equiangular.

Ex. 37. What is the converse of Ex. 36? Is it true? Illustrate your answer by a figure.

Ex. 38. If through one of the points of intersection of two circles the diameters of the circles be drawn, the join of the other extremities of these diameters passes through the other point of intersection of the circles.



Ex. 39. The join spoken of in Ex. 38 is parallel to the line of centers of the circles.

Ex. 40. This join is longer than any other line through a point of intersection of the circumferences and terminated by them.

Ex. 41. What is a cyclic 4-side?

Ex. 42. What kind of angle with reference to the circle is $\angle ECB$? $\angle ECF$?

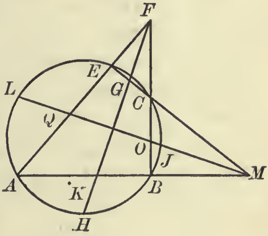
Ex. 43. What is the measure of $\angle ECF$? Why?

Ex. 44. What is the measure of $\angle EFC$?

Ex. 45. What is the measure of $\angle COQ$? Of $\angle EQO$?

Ex. 46. If FH bisects $\angle F$ and ML bisects $\angle M$, show that:

(*a*) Arc LE $-$ arc $CJ =$ arc $LA -$ arc JB.
(*b*) Arc LE $+$ arc $JB =$ arc $LA +$ arc CJ.
(*c*) Arc $LEC +$ arc $JB =$ arc $LA +$ arc ECJ.



Ex. 47. The first member of (*c*) is the measure of what angle?

Ex. 48. The second member of (*c*) is the measure of what angle?

Ex. 49. Therefore, what kind of triangle is FOQ?

Ex. 50. Why, then, is $FH \perp LM$?

Ex. 51. Similarly, prove that $\triangle MGK$ is isosceles.

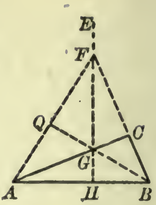
Ex. 52. Combine 41, 46, and 50 into one theorem.

Ex. 53. If two straight lines are drawn through the point of contact of two tangent circles, the chords of the arcs intercepted by these lines are parallel.

Ex. 54. What parallelograms may be inscribed in a circle?

Ex. 55. The apparent size of a circular object is determined by the angle between two tangents drawn from the eye to the object.

What is the locus of the point from which a given circle always appears to have the same size?



Ex. 56. Given the rt. $\triangle ABC$.

At any point H of the hypotenuse AB erect a $\perp HE$.

Let HE intersect BC (produced) in F.

Draw AF and GB.

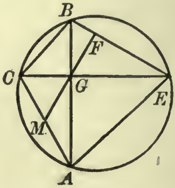
Let GB (produced) meet AF in Q.

As the $\perp HE$ moves along AB, what is the locus of Q?

Ex. 57. What part of the hypotenuse of a right triangle is the median to the hypotenuse?

Ex. 58. Into what kind of triangles does the median to the hypotenuse divide the right triangle?

Ex. 59. Draw a right triangle and its altitude to the hypotenuse. Name the three sets of complemental angles in your figure.



Ex. 60. What is the measure of an inscribed angle?

If, in the adjacent figure, $CE \perp BA$, $GF \perp BE$, and the 4-side is inscribed, point out three angles equal to $\angle BGF$, and give reasons.

Ex. 61. Give two reasons why $\angle FEG = \angle CAG$.

Ex. 62. Prove that if the diagonals of a cyclic 4-side be perpendicular to each other, and from their intersection a perpendicular be let fall on one side of the 4-side, this perpendicular will bisect the opposite side.

In the following exercises the angles, sides, and principal lines of a triangle are represented thus:

Angles: A, B, C.

Sides: a, b, c opposite $\angle A$, $\angle B$, and $\angle C$, respectively.

Altitudes: h_a, h_b, and h_c, altitudes to sides a, b, and c, respectively.

Medians: m_a, m_b, and m_c, medians to sides a, b, and c, respectively.

Bisectors of $\measuredangle$s: t_A, t_B, and t_C, bisectors of A, B, and C, respectively.

(Note. — When one angle of an isosceles triangle is given, all the angles are given.)

Construct an isosceles triangle, given:

Ex. 63. c and $\angle A$.

Ex. 64. c and $\angle C$.

Ex. 65. c and the radius of the inscribed circle.

Ex. 66. $c + a$ and $\angle B$.

(Anal.: Extend c to C', making $BC' = c + a$. Draw CC'. Use III. 2 *a*.)

Ex. 67. a and h_c.

Ex. 68. c and h_c.

Ex. 69. $\angle B$ and h_c.

Ex. 70. $\angle C$ and $a + b$.

Ex. 71. c and h_b. **Ex. 73.** $\angle C$ and $a + c$.

Ex. 72. h_c and $\angle C$. **Ex. 74.** $\angle C$ and m_b.

Ex. 75. $b + h_c$ and $\angle C$.

(Anal. : Let h_c with its extension to $H' = h_c + b$. Draw $H'A$. Use III. 2 a.)

Ex. 76. $b + h_c$ and c.

(Anal. : Let h_c with its extension to $H = h_c + b$. Take $\frac{c}{2}$. Use III. 2.)

Ex. 77. $a + b + c$ and $\angle A$.

(Note.—If one acute angle of a right triangle is given, the other is also given.)

Construct a right triangle (right angle at C), given:

Ex. 78. c and $\angle A$. **Ex. 79.** c and h_c.

Ex. 80. c and the radius of the inscribed circle.

Ex. 81. $\angle A$ and the radius of the inscribed circle.

Ex. 82. a and the radius of the inscribed circle.

Ex. 83. m_c and h_c. (Anal.: Use VII. 4.)

Ex. 84. $\angle A$ and h_c.

Ex. 85. The two segments of c made by h_c.

Ex. 86. The two segments of c made by t_c.

(Anal. : Extend t_c to meet ⊙ on c. Use XII. 2.)

Ex. 87. c and the distance from the vertex of $\angle C$ to a given line.

Ex. 88. $a + b$ and $\angle A$.

Ex. 89. The radius of the inscribed and the radius of the circumscribed circle.

Ex. 90. $a - b$ and c.

Ex. 91. $c - a$ and $\angle A$.

(Anal. : $\angle B$ is known. Const. a $\triangle$, given $c - a$ and its $\angle$s.)

Ex. 92. $a + b + c$ and $\angle A$.

Construct an equilateral triangle, given:

Ex. 93. The perimeter. **Ex. 94.** The altitude. **Ex. 95.** $a + h$.

Ex. 96. The radius of the inscribed circle.

Ex. 97. The radius of the circumscribed circle.

Construct a triangle, given:

Ex. 98. The perimeter and $\angle A$ and $\angle B$.

Ex. 99. c, h_c, and $\angle C$. **Ex. 100.** c, m_c, and $\angle C$.

Ex. 101. c, h_c, and h_a. (Use IX. 4.)

Ex. 102. h_c, $\angle A$, and $\angle B$.

Ex. 103. a, h_c, and c.

Ex. 104. a, c, and m_c.

Ex. 105. c, h_c, and m_c.

Ex. 106. a, b, and m_c. (Double the median. Use Conv. of VI. 3.)

Ex. 107. m_c, h_a, and h_b. (Double the median. Use IX. 4.)

Construct a square, given:

Ex. 108. Its apothem.

Ex. 109. The difference between its diagonal and its side.

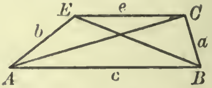
Construct a rhombus, given:

Ex. 110. The two diagonals.

Ex. 111. One diagonal and a side.

Construct a trapezoid, given:

Notation:



Ex. 112. a, c, $\angle A$, and $\angle B$.

Ex. 113. a, c, h, and $\angle A$.

Ex. 114. c, e, AC, and BE.

Ex. 115. a, b, c, and BE.

Ex. 116. c, a, b, e, and AC.

Construct a rhomboid, given:

Ex. 117. c and AC and BE.

Ex. 118. c, AC, and h.

Ex. 119. AC, h, and $\angle A$.

Ex. 120. AC, h, and b.

Ex. 121. If a 4-side is circumscribed about a circle, the central angles subtended by the opposite sides are supplemental.

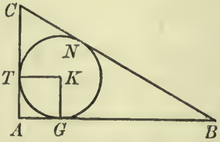
Ex. 122. In rt. $\triangle ABC$, inscribe $\odot K$.

Ex. 123. If the points of tangency of this $\odot$ be N, G, and T, why does $BG = BN$?

Ex. 124. Show that 4-side $KGAT$ is a square.

Ex. 125. Show $CB = CA + BA$ minus the diameter of the inscribed circle.

Ex. 126. Show that the diameter of the circumcircle plus the diameter of the incircle equals the sum of the legs of the right triangle.



Ex. 127. If AB is produced to the left, making $AC' = AC$, what is the value of $\angle AC'C$?

Ex. 128. If BC' is the sum of the legs in the rt. $\triangle ABC$, and BC is the diameter of the circumcircle, what are the two loci of C?

Ex. 129. Construct a right triangle, given the sum of the legs and the radius of the incircle.

Ex. 130. If the radius of the circum-⊙ is given, and also that of the in-⊙, how from these do you obtain the hypotenuse and the sum of the legs?

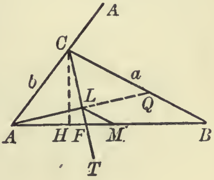
Ex. 131. Construct a right triangle, given the radius of the incircle and radius of the circumcircle.

Ex. 132. If, in $\triangle ABC$, CT bisects $\angle ACB$ and AL is drawn perpendicular to CT, by what theorem is $\triangle ACQ$ isosceles?

Ex. 133. If M is the mid-point of AB, why is LM parallel to QB?

Ex. 134. Prove that $LM = \frac{1}{2}(a - b)$.

Ex. 135. If BL' is drawn perpendicular to CT, prove that the join $L'M$ is also $= \frac{1}{2}(a - b)$.



Ex. 136. If CT' bisect ext. $\angle BCA'$, and from B a $\perp$ to CT' is drawn, then the join of M and the foot of this perpendicular $= \frac{1}{2}(a + b)$. Prove.

Ex. 137. $\angle LAM = \angle CAB - \angle CAQ$. Prove, then, that $\angle LAM = \frac{1}{2}(\angle CAB - \angle B)$.

Ex. 138. If $CH \perp AB$, prove that

$$\angle HCT = \tfrac{1}{2}(\angle CAB - \angle B).$$

Ex. 139. State the preceding theorem in general terms.

Ex. 140. Draw FQ, and prove that as $\triangle AFQ$ is isosceles (Why?),

$$\angle QFB = \angle CAB - \angle B.$$

Ex. 141. Show that $\angle AQB = \angle ACB + \frac{1}{2}$ the supplement of $\angle ACB$.

Ex. 142. Construct a triangle, having given the base, the difference of the two sides, and the vertex angle.

Ex. 143. If from M, the middle point of the arc AMB, any two chords MF, MG are drawn, cutting the chord AB in E and C, then $CEFG$ is cyclic.

XIII. GROUP ON AREAS OF RECTANGLES AND OTHER POLYGONS

(Briefly, the Areal Group)

DEFINITIONS

The **Area** of a plane figure is the ratio of its surface to some other surface taken as a unit of area.

This **Unit of Area** is usually a square whose base and altitude are each a unit of length.

A **Polygon is circumscribed** to a circle when the sides of the polygon are tangents to the circle.

A **Polygon is inscribed** in a circle when the sides of the polygon are chords of the circle, that is, when its vertices are in the circumference.

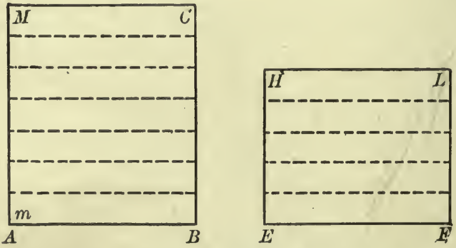
PROPOSITIONS

XIII. 1. *Two rectangles*

(*a*) *having equal bases, vary as their altitudes;*

(*b*) *having equal altitudes, vary as their bases.*

Hyp. 1. (*a*) If, in the ▭'s A–C and E–L, the bases AB and EF are equal,



Conc.: then ▭ A–C : ▭ E–L : : altitude AM : altitude EH.

CASE I. Commensurable case.

Dem. If AM and EH have common measure (v. XI. Prob. I.), say m, apply it as many times as possible to AM and to EH.

Suppose it is contained in AM seven times, in EH five.

$$\therefore AM : EH :: 7 : 5.$$

Through the points of division draw parallels to the bases.

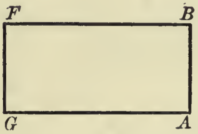
The ▭ $A\text{–}C$ will be divided into seven ≅ ▭'s, and the ▭ $E\text{–}L$ will be divided into five ≅ ▭'s. (VI. 4 *a.*)

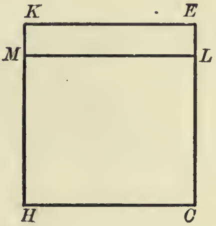
Furthermore, all of these smaller ▭'s are ≅. (VI. 4 *a.*)

$$\therefore \text{▭}\, A\text{–}C : \text{▭}\, E\text{–}L :: 7 : 5.$$

$\therefore$ ▭ $A\text{–}C$: ▭ $E\text{–}L$ :: altitude AM : altitude EH. (Ax. 1.)

Q.E.D.





CASE II. Incommensurable case; that is, h and h_1 have no common measure.

Dem. Divide AB into any number of equal parts, say n.

Apply one of these as a divisor to CE until there is a remainder LE less than the divisor.

Draw $LM \parallel HC$.

▭ $G\text{–}B$: ▭ $C\text{–}M$ :: $AB : CL$. (XIII. 1. (*a*) Case I.)

Now, if we decrease the divisor, we decrease the remainder without affecting the equality of the quotients.

That is, as $LE \doteq 0$.

and $CL \doteq CE$,

so ▭ $CM \doteq$ ▭ CK. (XI. Def. of Limit.)

The ratios ▭ G–B : ▭ C–M and $AB : CL$, however, remain equal as they approach their limits, viz.:

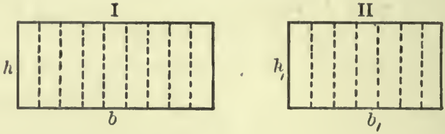
▭ G–B : ▭ C–K and $AB : CE$. (Case I.)

$\therefore$ ▭ G–B : ▭ C–K : : $AB : CE$.

[If, while approaching their respective limits, etc.]

(XI. Post. Limits.) **Q.E.D.**

Hyp. 1. (*b*) If, in ▭'s I and II, the altitudes h, h_1 are equal, and the bases are b and b_1,



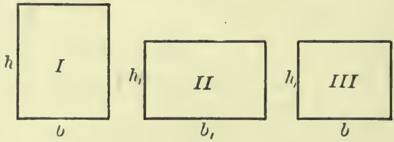
Conc.: then ▭ I : ▭ II : : $b : b_1$.

Dem. The proof of 1. (*b*) is exactly similar to that of 1. (*a*).

Q.E.D.

XIII. 1 *a*. *Any two rectangles are to each other as the products of their bases and altitudes.*

Hyp. If the two ▭'s I and II have bases b and b_1, and altitudes h and h_1,



Conc.: then ▭ I : ▭ II : : $b \cdot h : b_1 \cdot h_1$.

Dem. Construct a ▭ III with base equal to b and altitude equal to h_1.

▭ I : ▭ III : : $h : h_1$. (1) (XIII. 1 (*a*).)

▭ III : ▭ II : : $b : b_1$. (2) (XIII. 1 (*b*).)

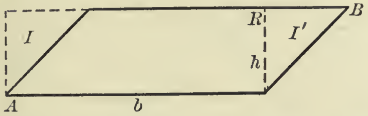
Multiply proportion (1) by (2), and we have

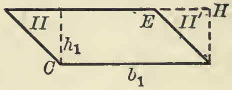
▭ I : ▭ II : : $b \cdot h : b_1 \cdot h_1$.

[If two proportions be multiplied together, etc.] (XI. 3.)

Q.E.D.

XIII. 1 *b.* *Any two parallelograms are to each other as the products of their bases and altitudes.*





Hyp. If two ▱s A–B and C–E have b and b_1, h and h_1 as bases and altitudes, respectively,

Conc.: then ▱ $A\text{–}B$: ▱ $C\text{–}E :: b \cdot h : b_1 \cdot h_1$.

Dem. Draw the perpendiculars to meet sides (produced if necessary) as in figure.

Rt. △ I $\cong$ rt. △ I′ and rt. △ II $\cong$ rt. △ II′.

[If two rt. △s have the leg and hypotenuse of one, etc.] (V. 4.)

∴ ▱ $A\text{–}B$ = ▭ $A\text{–}R$; ▱ $C\text{–}E$ = ▭ $C\text{–}H$.

b and h are identical in ▱ $A\text{–}B$ and ▭ $A\text{–}R$.
b_1 and h_1 are identical in ▱ $C\text{–}E$ and ▭ $C\text{–}H$.

Now ▭ $A\text{–}R$: ▭ $C\text{–}H :: b \cdot h : b_1 \cdot h_1$. (XIII. 1 *a*.)

∴ ▱ $A\text{–}B$: ▱ $C\text{–}E :: b \cdot h : b_1 \cdot h_1$.

Q.E.D.

SCH. A parallelogram is equal to a rectangle of the same base and altitude.

Ex. 1. Two rectangles are equal. The bases are 27 and 18, respectively, and the altitude of the first is 8. What is the altitude of the second?

Ex. 2. Construct three equal triangles on the same base, the first of which shall be acute, the second right, and the third obtuse.

Ex. 3. A number of equal triangles stand on the same base.

How do their altitudes compare?

Show, then, that the locus of their vertices consists of two lines parallel to the base.

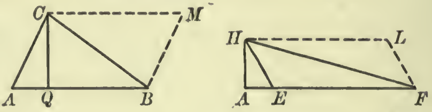
Ex. 4. The base of a triangle is divided into five equal parts, and the points of division are joined to the vertex.

How do the altitudes of the resulting triangles compare?

How, then, do the areas of these triangles compare?

XIII. 1 *c.* *Any two triangles are to each other as the products of their bases and altitudes.*

Hyp. If the △s ABC and EFH have AB and EF for bases, and CQ and HA for altitudes,



Conc.: then $\triangle ABC : \triangle EHF ::$ ▭ of $AB \cdot CQ :$ ▭ of $EF \cdot HA$.

Dem. Complete the parallelograms as in the figures.

▱ A–M : ▱ E–L :: ▭ of $AB \cdot CQ$: ▭ of $EF : HA$. (XIII. 1 *b.*)

But $\triangle ABC$ is $\frac{1}{2}$ ▱ A–M and $\triangle EFH$ is $\frac{1}{2}$ ▱ E–L.
[The diagonal of a ▱ divides it into two $\cong$ △s.] (VI. 1 *a*, Sch.)

$\therefore \triangle ABC : \triangle EHF ::$ ▭ of $AB \cdot CQ :$ ▭ of $EF \cdot HA$.

Q.E.D.

Sch. 1. Parallelograms (or triangles) with equal bases are to each other as their altitudes.

Sch. 2. Parallelograms (or triangles) with equal altitudes are to each other as their bases.

Sch. 3. If parallelograms (or triangles) have equal altitudes and equal bases, they are equal.

Ex. 5. How would you divide a triangle into n equal parts by lines passing through the vertex?

Ex. 6. If the altitude, 26 ft., of a rectangle is to be reduced to 20 ft., how much must be added to the base, 30 ft., to keep the area unchanged?

Ex. 7. If one angle of a right triangle be 30°, how does the hypotenuse compare with the shortest side?

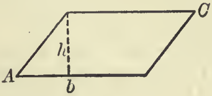
Find the area of a right triangle, one of whose sides is 12 ft. and one of whose angles is 30°.

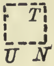
Ex. 8. Show, by drawing a figure, that the square on one half a line is one fourth the square on the line.

Ex. 9. Show also that the square on one third a line is one ninth the square on the line.

XIII. 2. *The area of a parallelogram equals the product of its base and altitude.*

Hyp. If ▱ $A\text{–}C$ has b as a base and h as an altitude,





Conc.: then the area of ▱ $A\text{–}C = b \cdot h$.

Dem. On any line as UN assumed as a unit of length (v. Def. in XI.) construct a □ $U\text{–}T$.

This square may be taken as the unit of area (v. Def. in XIII.).

$$\therefore \text{ the } \square\, U\text{–}T = 1.$$

$$▱\, A\text{–}C : \square\, U\text{–}T :: b \cdot h : 1 \times 1.$$

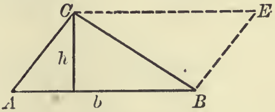
[Any two ▱s are to each other, etc.] (XIII. 1 *b*.)

∴ ▱ $A\text{–}C : 1 :: b \cdot h : 1$; that is, the area of ▱ $A\text{–}C = b \cdot h$.
(Def. of area (XIII.).)

Q.E.D.

XIII. 2 *a*. *The area of a triangle equals one half the product of its base and altitude.*

Hyp. If b and h are the base and altitude, respectively, of △ ABC,



Conc.: then the area of △ $ABC = \frac{1}{2} b \cdot h$.

Dem. Complete ▱ $A\text{–}E$ as in figure.

$$\triangle\, ABC = \tfrac{1}{2}\, ▱\, A\text{–}E.$$

[The diagonal of a ▱ divides it into 2 ≅ △s.] (VI. 1, Sch.)

The area of ▱ $A\text{–}E = b \cdot h$. (XIII. 2.)

$$\therefore \text{ the area of } \triangle\, ABC = \tfrac{1}{2}\, b \cdot h.$$

Q.E.D.

Ex. 10. The legs of one right triangle are 8 ft. and 6 ft.; of another 5 ft. and 12 ft. What is the ratio of their areas?

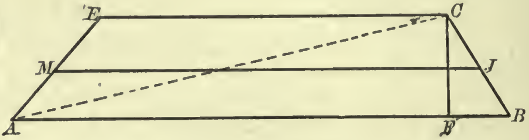
Ex. 11. A parallelogram has a base of 9 ft. and an altitude of 16 ft. What is the side of a square equivalent to the parallelogram?

AREAS OF IRREGULAR FIGURES

SCH. To obtain the area of an irregular polygon, join any point within the polygon with the vertices of the figure.

Find the area of each triangle thus formed. The sum of these areas equals the area of the polygon.

XIII. 2 *b*. *The area of a trapezoid equals the product of the altitude and the mid-join of the non-parallel sides.*



Hyp. If the 4-side A–C is a trapezoid, MJ its mid-join, and CF its altitude,

Conc.: then the area of trapezoid A–C = ▭ of $CF \cdot MJ$.

Dem. Draw AC.

The area of $$\triangle ACB = CF \cdot \frac{AB}{2}. \qquad \text{(XIII. 2 } a.\text{)}$$

The area of $$\triangle AEC = CF \cdot \frac{CE}{2}. \qquad \text{(XIII. 2 } a.\text{)}$$

$$\therefore \triangle ACB + \triangle ACE = CF \cdot \frac{AB + CE}{2}. \qquad \text{(Ax. 2.)}$$

That is, the area of trapezoid A–$C = CF \cdot \dfrac{AB + CE}{2}$.

But $$\frac{AB + CE}{2} = MJ.$$

[The mid-join of a trapezoid $= \frac{1}{2}$ the sum of ∥ sides.] (VII. 3 *b*.)

∴ the area of the trapezoid A–C equals the rectangle of $CF \cdot MJ$.

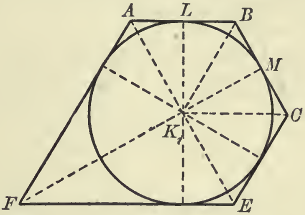
Q.E.D.

Ex. 12. In the last problem substitute "triangle" for "parallelogram," and solve.

Ex. 13. If the area of a trapezoid be 65 sq. ft., and the parallel sides respectively 10 ft. and 16 ft., what is the altitude?

Ex. 14. A square and a rectangle have the same perimeter. Which has the greater area? Why?

XIII. 3. *The area of a circumscribed polygon equals one half the product of its perimeter and the radius of the inscribed circle.*



Hyp. If an n-gon $ABC\text{–}F$ is circumscribed to a circle of center K_i and of radius r_i,

Conc.: then the area of n-gon

$$ABC\text{–}F = \tfrac{1}{2}(AB + BC + CE + \cdots)r_i.$$

Dem. Draw K_iA, K_iB, etc.; also K_iL, K_iM, etc.

K_iL, K_iM, etc., are altitudes of $\triangle\!\!\triangle$ AK_iB, BK_iC, etc. (IX. 4.)

That is, the radius of the inscribed circle equals the altitude of the triangles that make up the n-gon.

But the area of $\triangle AK_iB = \dfrac{AB}{2} \cdot r_i$.

Similarly, for the remaining triangles.

∴ the sum of the areas of the triangles, or the area of the n-gon $ABC\text{–}F = \frac{1}{2}(AB + BC + CE + \cdots) \cdot r_i$.

Q.E.D.

Ex. 15. What is the area of a triangle if the base is 1384 ft., and the altitude is 256 ft.?

Ex. 16. What is the area of a rt. isosceles △ one leg of which is 1414 ft.?

Ex. 17. What is the area of a right triangle whose perimeter is 840 ft., and whose sides are to each other as 3 : 4 : 5?

Ex. 18. What is the area of a right triangle whose perimeter is 300 ft., if its sides are as 5 : 12 : 13?

Ex. 19. The base of a triangle is to its altitude as 11 : : 60; the area of the triangle is 1320 sq. ft. What is the length of the base?

Ex. 20. The altitude of a trapezoid is 16 ft.; the mid-join is 32 ft. What is the area of the trapezoid?

Ex. 21. A △ whose altitude is 10 ft. and base 24 ft. is transformed into a rhombus. Its longer diagonal is 16 ft. What is the length of the shorter?

What are the dimensions of a rectangle if the

Ex. 22. Area is 3822 sq. ft. and the sides are as 6 : 13?

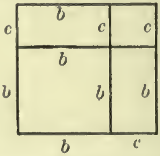
Ex. 23. Area is 59100 sq. ft. and perimeter 994 ft.?

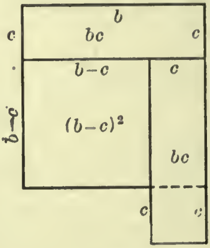
Ex. 24. In a parallelogram a line is drawn that cuts off one fourth of one side and three fifths of the opposite side. What part of the parallelogram is each trapezoid thus formed?

Ex. 25. Show geometrically that if b and c represent lines,

$$(b+c)^2 = b^2 + 2\,bc + c^2.$$

State the proposition in words, *i.e.* without the use of symbols.

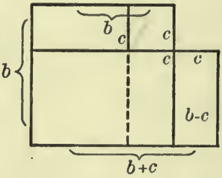




Ex. 26. Show geometrically that if b and c represent lines,

$$\begin{aligned}(b-c)^2 &= b^2 + c^2 - 2\,bc.\\ &= b^2 - 2\,bc + c^2.\end{aligned}$$

State the proposition in words.



Ex. 27. Show, from the figure, that if b and c be any two lines,

$$(b+c)(b-c) = b^2 - c^2.$$

State the proposition in words.

Ex. 28. If, in a trapezoid $ABCE$, AC and BE, the two diagonals, are drawn, prove $\triangle ABC = \triangle ABE$.

Ex. 29. If, in the same trapezoid, the diagonals intersect in M, prove that $\triangle AME = \triangle BMC$.

Ex. 30. If two equal triangles have the same base, and lie on opposite sides of this base, the join of the vertices of the triangles is bisected by the common base.

Ex. 31. If, in a ▱ $ABCE$, perpendiculars from B and E are let fall on the diagonal AC, prove that they are equal.

Ex. 32. In the above parallelogram, if P is any point in the diagonal AC, prove that $\triangle APB = \triangle APE$.

Ex. 33. Prove that if from a vertex of a parallelogram a line is drawn to the mid-point of one of the opposite sides, it cuts off one third of the diagonal it intersects.

Ex. 34. Prove that the line referred to in the preceding theorem cuts off a triangle equal to one fourth of the parallelogram.

XIII. SUMMARY OF PROPOSITIONS IN THE AREAL GROUP

1. *Two rectangles*

  (*a*) *having equal bases, vary as their altitudes,*

  (*b*) *having equal altitudes, vary as their bases.*

    a. Any two rectangles are to each other as the products of their bases and altitudes.

    b. Any two parallelograms are to each other as the products of their bases and altitudes.

SCH. A parallelogram is equal to a rectangle of the same base and altitude.

  c. Any two triangles are to each other as the products of their bases and altitudes.

SCH. 1. Parallelograms (or triangles) with equal bases are to each other as their altitudes.

2. Parallelograms (or triangles) with equal altitudes are to each other as their bases.

3. Parallelograms (or triangles) with equal bases and equal altitudes are equal.

2. *The area of a parallelogram equals the product of its base and altitude.*

  a The area of triangle equals one half the product of its base and altitude.

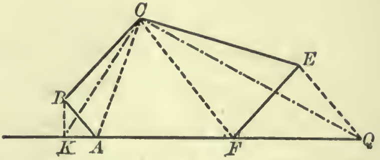
SCH. Areas of irregular figures.

  b The area of a trapezoid equals the product of the altitude and the mid-join of the non-parallel sides.

3. *The area of a circumscribed polygon equals one half the product of its perimeter and the radius of the inscribed circle.*

PROBLEM

Prob. I. *To reduce a polygon to an equivalent triangle.*



Given. The polygon $ABCEF$.

Required. To reduce it to an equivalent triangle.

Const. Extend AF indefinitely.

Draw FC, a diagonal.

Draw $EQ \parallel CF$, meeting AF produced in Q.

Draw CQ.

$$\triangle CFE = \triangle CFQ.$$

[$\triangle$s having = bases and = altitudes are =.] (XIII. 1. *c*, Sch. 3.)

$$\therefore ABCF + \triangle CEF = ABCF + \triangle CFQ. \qquad \text{(Ax. 2.)}$$

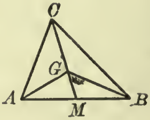
$$\therefore n\text{-gon } ABCQ = n\text{-gon } ABCEF.$$

Similarly, draw CA, BK, and CK.

The number of vertices of n-gon is thus reduced to three.

$$\therefore \triangle KCQ = n\text{-gon } ABCEF.$$

Q.E.F.



Ex. 35. What is the centroid of a triangle? (X., 5.) If G be the centroid of ABC, what is the ratio of the areas of ACM and AGM? Of the areas of BCM and BGM? Of the areas of AGB and ACB?

Ex. 36. How, then, do the $\triangle$s AGB, AGC, and BGC compare in area?

State as a theorem the property of the centroid thus established.

Ex. 37. The centroid of a material triangle uniform in thickness and of the same material throughout is called the "center of gravity" of the triangle. How might this name be suggested by the property just established?

Ex. 38. On a side of a given triangle as a base, to construct a triangle equal to the first and having its vertex on a given line.

Ex. 39. Show that the greatest number of solutions possible in Ex. 38, is two. Under what conditions is the solution impossible? Under what conditions is the problem indeterminate? Draw figures to illustrate your answers.

XIV. PYTHAGOREAN GROUP

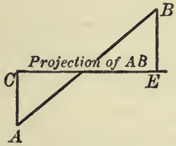
DEFINITIONS

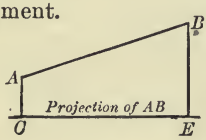
PROJECTION ON A LINE

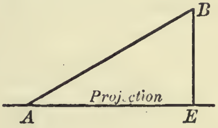
The **Projection of a Point** on a straight line is the foot of the perpendicular from the point to the line.

The line on which the perpendicular is dropped is called the **Base of Projection.**

The **Projection of a Line-Segment** is that portion of the base of projection which lies between the projections of the extremities of the given line-segment.

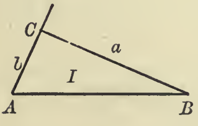


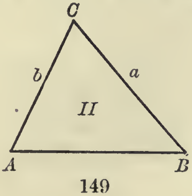


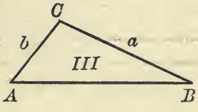


NOTE. — This group is named after Pythagoras, an eminent Greek mathematician of the sixth century B.C., who was the first to publish a proof of the first theorem of the group. The truth of this proposition was known before his time, but a proof had long been sought in vain. The theorem, the 47th of Euclid, is often called the *pons asinorum* of geometry.

Ex. 1. In the right, acute, and obtuse $\triangle$ I, II, III, draw the projections of a on b, and of b on a; also of a on c, and of c on a.



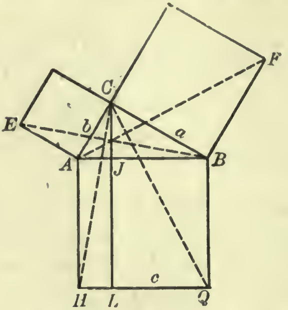




PROPOSITIONS

XIV. 1. *If a triangle is right, the square on the hypotenuse equals the sum of the squares on the legs.*

Hyp. If, in a rt. $\triangle ABC$, $\angle C$ is the right angle,



Conc.: then the □ on $c =$ the □ on $a +$ the □ on b.

Dem. Draw the altitude CJ and extend it to L on the □ on c. Draw CH, CQ, BE, and AF.

$AB = AH$ and $AE = AC$. (Def. of □.)

$\angle EAB = \angle CAH$. (Each = a rt. $\angle + \angle CAB$.)

$\therefore \triangle ABE \cong \triangle CAH$. (V. 1.)

$\triangle ABE = \frac{1}{2}$ the □ on b.[1]

(Having the same base EA and = altitudes.)

[The area of a $\triangle = \frac{1}{2}$ the product of $b \cdot h$.] (XIII. 2 a.)

$\triangle CAH = \frac{1}{2}$ the ▭ A–L.[1]

(Having the same base AH and = altitudes.)

$\therefore \frac{1}{2}$ □ on $b = \frac{1}{2}$ the ▭ A–L. (Ax. 1.)

$\therefore$ the □ on $b =$ the ▭ A–L. (Ax. 2.)

Similarly, we may show that the □ on $a =$ ▭ L–B.

$\therefore$ the □ on $a +$ the □ on $b =$ ▭ A–L + ▭ L–B = □ on c.

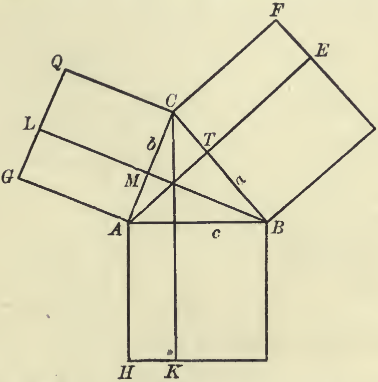
Q.E.D.

XIV. 1 *a*. *The square on a leg of a right triangle equals the difference between the square on the hypotenuse and the square on the other leg.*

---

[1] Draw altitude in $\triangle ABE$ from B to EA; in $\triangle CAH$ from C to AH.

XIV. 2. *In any triangle, the square on a side opposite an acute angle equals the sum of the squares on the other two sides, diminished by twice the rectangle of either of these sides and the projection of the other upon it.*



Hyp. If, in the △ ABC, ∠ C is acute; CT, the projection of b on a, and CM, the projection of a on b,

Conc.: then □ on $c =$ □ on $a +$ □ on $b - 2 \cdot$ ▭ of $a \cdot CT$, or $2 \cdot$ ▭ of $b \cdot CM$.

Dem. Draw the altitudes and extend them to meet sides of the □'s as shown in the figure. These altitudes must pass through M and T. (Def. of projection.)

If, as in figure for XIV. 1, CH and BG were drawn, △ CAH would be ≅ △ BAG. ∴ ▭ $A\text{–}K =$ ▭ $G\text{–}M$. (Ax. 3.)

Similarly, ▭ $B\text{–}K =$ ▭ $E\text{–}B$.

Again, if BQ and AF were drawn,

△ BCQ ≅ △ ACF.

∴ ▭ $M\text{–}Q =$ ▭ $T\text{–}F$. (Ax. 3.)

▭ $A\text{–}K +$ ▭ $B\text{–}K =$ ▭ $G\text{–}M +$ ▭ $E\text{–}B$, (Ax. 2.)

$=$ □ on $b -$ ▭ $M\text{–}Q +$ □ on $a -$ ▭ $T\text{–}F$,

or (since ▭ $T\text{–}F =$ ▭ $M\text{–}Q$)

$=$ □ on $b +$ □ on $a - 2$ ▭ $M\text{–}Q$,

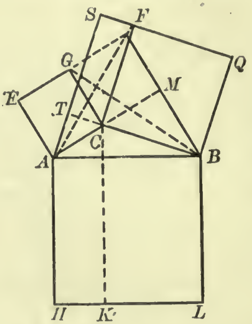
$=$ □ on $b +$ □ on $a - 2$ ▭ of b and CM,

$=$ □ on $b +$ □ on $a - 2$ ▭ of a and CT.

Q.E.D.

XIV. 3. *In any obtuse triangle, the square on the side opposite the obtuse angle equals the sum of the squares on the other two sides, increased by twice the rectangle of either of these sides and the projection of the other upon it.*

Hyp. If, in the $\triangle ABC$, $\angle C$ is obtuse; CT the projection of b on a, and CM the projection of a on b,



Conc.: then □ on c = □ on a + □ on b + 2 · ▭ of b and CM,
= □ on a + □ on b + 2 · ▭ of a and CT.

Dem. Draw the altitudes of $\triangle ABC$ and produce them as in the figure; also draw AF and BG.

If EB and CH be drawn, ▭ E–M may be proved = ▭ A–K.
[Each ▭ being twice one of the $\cong$ △s ABE and ACH (why $\cong$?).]

Similarly, ▭ B–S = ▭ B–K.

(Draw CL and AQ and give remainder of proof.)

∴ ▭ A–K + ▭ B–K = ▭ E–M + ▭ B–S.

That is, the □ on c = ▭ E–M + ▭ B–S.

But ▭ E–M = the □ on b + ▭ G–M,

and ▭ B–S = the □ on a + ▭ C–S.

∴ the □ on c = the □ on b + the □ on a + ▭ G–M + ▭ C–S.

But ▭ G–M = ▭ C–S, ∵ $\triangle ACF \cong \triangle GCB$. (Why?)

But ▭ C–S = $CT \cdot CF$ or $CT \cdot a$, and ▭ G–M = $b \cdot CM$.

∴ □ on c = □ on b + □ on a + 2 · ▭ of a and CT,
= □ on b + □ on a + 2 · ▭ of b and CM.

Q.E.D.

Ex. 2. The radius of a circle is r. A tangent of length 1 is drawn to this circle.

Draw the hypotenuse and find an expression for its length. What is the locus of the extremity of this tangent?

Ex. 3. A ladder 50 ft. long just reaches the top of a wall 40 ft. high. If the ground be level, how far is the foot of the ladder from the wall?

Ex. 4. (*a*) The length of a chord is 12 ft. What is its distance from the center of a circle whose radius is 20 ft.?

(*b*) If the length of a chord be $2a$, how far is this chord from the center of a circle of radius r?

Ex. 5. What is the length of a chord in a circle of radius r, and at a distance d from the center?

Ex. 6. Find the area of the cross section of a ditch, the section being an isosceles trapezoid 20 ft. wide at the bottom, 30 ft. wide at the top, and each slope 25 ft. from top to bottom.

Ex. 7. If a public square is 200 yds. on a side, how much is gained by crossing the square from corner to corner on a diagonal walk instead of using the sidewalk around the square?

Ex. 8. Find the side and area of the □ inscribed in a ⊙ of radius a.

Ex. 9. The sides of a triangle are 6, 8, and 10, respectively. What kind of triangle is it? Why?

Ex. 10. The sides of a △ are 5, 7, and 10, respectively. What kind of △ is it?

Ex. 11. If the three sides of a triangle are given, we may find:

(1) The lengths of the projections of two of the sides on the third side;
(2) The length of the altitude to the third side;
(3) The area of the triangle,

in the following manner:

(1) Let the sides of the triangle be 8, 11, and 14.

Let x and y be the projections on 14 of 11 and 8, respectively.

$x + y = 14.$

$x^2 - y^2 = 57$ (XIV. 1 and ∵ the altitude is common to both ▲.)

$14(x - y) = 57$, or $x - y = \frac{57}{14}$.

Knowing $x + y$ and $x - y$, we may find x and y.

(2) Knowing either x or y, we may find the altitude by XIV. 1 *a*.

(3) Knowing the base and altitude, we have the area.

XIV. SUMMARY OF PROPOSITIONS IN PYTHAGOREAN GROUP

1. *If a triangle is right, the square on the hypotenuse equals the sum of the squares on the other two sides.*

   a *The square on the side of a right triangle equals the difference between the square on the hypotenuse and the square on the other side.*

2. *In any triangle, the square on a side opposite an acute angle equals the sum of the squares on the other two sides, diminished by twice the rectangle of either of these sides and the projection of the other upon it.*

3. *In any obtuse triangle, the square on the side opposite the obtuse angle equals the sum of the squares on the other two sides, increased by twice the rectangle of either of these sides and the projection of the other upon it.*

Ex. 12. Having the length of two sides of a △ and of the altitude to the third side, how do you find the length of the third side?

Ex. 13. If each side of an equilateral triangle equals 10, show that the altitude is $\sqrt{75}=5\sqrt{3}$, and the area $25\sqrt{3}$.

Ex. 14. In an equilateral triangle, if one of the sides is a and the altitude is h, show that $h=\frac{a}{2}\sqrt{3}$.

Ex. 15. What is the area of an equilateral triangle whose side is 20 ft.?

Ex. 16. Prove that in any triangle, ABC, if p and q are the segments of c made by an altitude on c, and a and b are the remaining sides, then

$$a+b:p+q::p-q:a-b.$$

Ex. 17. Prove that if a line-segment makes an ∠ of 60° with the base of projection, the length of the projection is one half the original line-segment.

Ex. 18. If a line-segment of length a makes an angle of 30° with the base of projection, how long is the projection of a?

Ex. 19. The sides of a trapezoid are 12, 32, 12, and 40, respectively. Find its area.

Ex. 20. An extension ladder, 75 ft. long, just reaches a window 60 ft. from the ground. How far is the foot of the ladder from the side of the building?

Ex. 21. The sides of a right triangle are three consecutive integral numbers. What is the length of the hypotenuse?

Ex. 22. What is the length of the diagonal of a rectangular floor 24 ft. by 22 ft.?

Ex. 23. The diagonal of a rectangle is 2.9 ft.; the perimeter is 8.2 ft. What is the length of the rectangle?

Ex. 24. The hypotenuse of a right triangle is 58 ft.; one leg is 42 ft. What is the length of the altitude to the hypotenuse?

Ex. 25. The sides of a triangle are 5, 6, and 7.
Find the segments of each side made by the altitude upon it.

Ex. 26. Draw the projection of the line-segment on the base of projection in each of the following cases:

1. The line-segment above the base of projection.
2. The line-segment meeting the base of projection.
3. The line-segment intersecting the base of projection.

Ex. 27. If, in a scalene triangle, a median is drawn, prove that one of the angles it forms with the side is acute and the other is obtuse.

Ex. 28. Hence, prove that in any triangle the sum of the squares on two sides equals twice the square on half the third side plus twice the square on the median to that side. (XIV. 2 and 3, and combine by addition.)

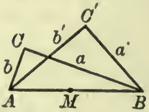
Ex. 29. Prove that in any triangle the difference of the squares on any two sides equals twice the rectangle of the third side and the projection of the median on that side. (Use XIV. 2 and 3; combine by subtraction.)

Ex. 30. Verify by means of Ex. 28 the theorem that the square on the hypotenuse equals the sum of the squares on the two sides.

Ex. 31. Prove that the sum of the squares on the four sides of a parallelogram equals the sum of the squares on its diagonals.

Ex. 32. Prove that in a right triangle twice the sum of the squares of the medians equals three times the square of the hypotenuse.

Ex. 33. If, in rt. $\triangle s$ ABC and ABC', M is the mid-point of hypotenuse AB, what does $AC^2 + BC^2$ equal? What does $AC'^2 + BC'^2$ equal?



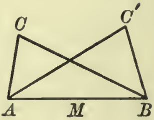
Ex. 34. Deduce from Ex. 28 that $\overline{CM}^2$ (or $\overline{C'M}^2$) $= \overline{AM}^2$.

Ex. 35. If the point C moves so that it is always the vertex of a right angle whose sides pass through A and B, then $\overline{CA}^2 + \overline{CB}^2$ always equals what constant quantity?

Ex. 36. What is the center and what the radius of the circle that is the locus of all points, the sum of the squares of the distances from which points to A and B is a square on the join of A and B?

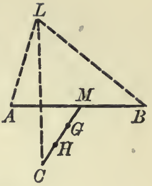
Ex. 37. In an oblique $\triangle ABC$, M is the mid-point of AB.
What does $a^2 + b^2$ equal? (v. Ex. 28.)
What does $BC'^2 + AC'^2$ equal?

Ex. 38. If point C in the annexed diagram moves so that $a^2 + b^2$ always equals a given square, say Q^2, show that $\overline{CM}^2 = \frac{Q^2}{2} - \overline{AM}^2$.



Ex. 39. Construct a square $= \frac{Q^2}{2} - \overline{AM}^2$.

Ex. 40. What is the center and what the radius of the circle that is the locus of all points, the sum of the squares of the distances from which points to A and B equals a given square Q^2?



Ex. 41. If, in the annexed diagram, G is the intersection of the medians of the $\triangle ABC$, in what ratio does G divide CM?

Ex. 42. If L is any point in the plane, prove that

$$\overline{LA}^2 + \overline{LB}^2 + \overline{LC}^2 = \overline{AG}^2 + \overline{BG}^2 + \overline{CG}^2 + 3\,\overline{LG}^2.$$

Proof. Let H be mid-point of CG.

$$\overline{LA}^2 + \overline{LB}^2 = 2\,\overline{AM}^2 + 2\,\overline{LM}^2. \quad \text{(Why ?)}$$

$$\overline{LC}^2 + \overline{LG}^2 = 2\,\overline{LH}^2 + 2\,\overline{GM}^2. \quad \text{(Why ?)}$$

$$\overline{LH}^2 + \overline{LM}^2 = 2\,\overline{LG}^2 + 2\,\overline{GM}^2.$$

(Add, and combine terms.)

Ex. 43. If L moves so that the sum of the squares of its distances from A, B, and C = a given square; that is, so that $\overline{LA}^2 + \overline{LB}^2 + \overline{LC}^2$ equals, say, Q^2, what is the center and what the radius of the locus circle?

Ex. 44. The sum of the squares of the medians of a triangle equals three fourths of the sum of the squares of the three sides. Prove.

If the area of a triangle is 112 sq. ft. and its altitude is 4 ft.:

Ex. 45. What is the length of its base?

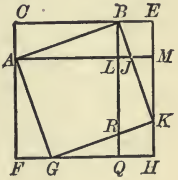
Ex. 46. What is the length of the median to the base if its projection on the base is 3 ft.?

Ex. 47. What does the sum of the squares on the two sides equal?

Ex. 48. What does the difference of the squares on the two sides equal?

Ex. 49. What, then, are the lengths of the two sides?

Ex. 50. Draw a trapezium, its diagonals, and the mid-join of the diagonals. Prove that the sum of the squares on the four sides equals the sum of the squares on the diagonals plus four times the square on the mid-join. (Euler's Theorem.) (Use Ex. 28.)



In the accompanying figure, $\triangle ABL$ is right-angled at L. A–K is a square on AB, L–F on AL, L–E on BL. Prove the following relations:

Ex. 51. (1) $\triangle AFG \cong \triangle ABL$.

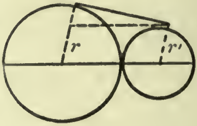
Ex. 52. (2) $\triangle GRQ \cong \triangle JMK$.

Ex. 53. (3) 4-side $JKRL = \triangle BEK$.

Ex. 54. (4) Hence, by means of this figure, prove XIV. 1.

Ex. 55. (5) Also prove XIV. 1 by showing that the remainder obtained by subtracting from the large square, the sum of the medium and small squares, is equal to that obtained by subtracting from the large square, the square on the hypotenuse AB.

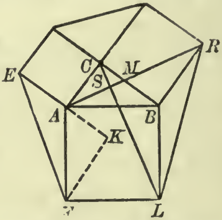
Ex. 56. If, in the figure for Ex. 40, A and B are the centers of circles of radii r and r', respectively, what is the locus of a point that moves so that the sum of the squares of the tangents from the moving point to the two given circles equals a given square? (Use Ex. 40 and XIV. 1 *a*.)



Ex. 57. Prove that the external common tangent to two tangent circles is a mean proportional to the diameters of the two circles.

(Draw line as in the annexed diagram. Hypotenuse of rt. $\triangle = r + r'$; short leg $= r - r'$. Use XIV. 1 *a*.)

Ex. 58. Find the locus of the extremities of lines that all pass through the same point M on a given line AB, and have the same projection on AB.

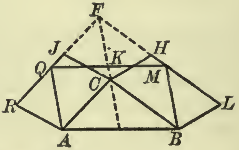


Ex. 59. If squares are drawn on the three sides of rt. $\triangle ABC$, prove that $RA \perp CL$.

Ex. 60. Prove that if $KF \perp EA$ in this figure, $\triangle AKF \cong \triangle ABC$.

Ex. 61. In the same figure prove that the area of $\triangle KAF =$ the area of $\triangle RBL$.

Ex. 62. If the diagonals of a 4-side are perpendicular to each other, show that the sum of the squares of one set of opposite sides equals the sum of the squares of the other set.



Ex. 63. The 4-side A–J is any parallelogram on AC, and 4-side C–L is any parallelogram on CB. If LH and RJ are produced to intersect at F, and then if a ▱ A–M is constructed on AB as a base, whose sides are equal and parallel to FC, prove:

(1) By Ax. 7, that join FM must fall on FL.
(2) By XIII. 1 *c*, Sch. 3, that ▱ L–C = ▱ B–F.
(3) That ▱ B–K = ▱ B–F.
(4) Why, then, is ▱ B–K = ▱ L–C?

Ex. 64. Show in a similar manner that ▱ A–K = ▱ A–J.

Ex. 65. Prove therefore that ▱ A–M = ▱ L–C + ▱ R–C.

Ex. 66. By means of the foregoing, prove that if $\angle ACB$ is a right angle, then the square on $AB =$ the square on $AC +$ the square on BC.

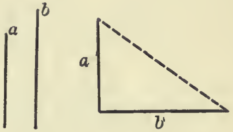
PROBLEMS

Prob. I. *To construct a square equal to the sum of two given squares.*

Given. a and b, the sides of 2 □'s.

Required. To construct a □ equal to $a^2 + b^2$.

[Construction is left to the student.]



Prob. II. *To construct a square equal to twice a given square.*

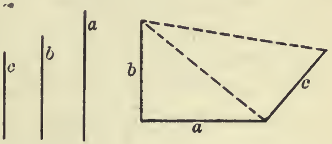
[Construction is left to the student.]

Prob. III. *To construct a square equal to the sum of three given squares.*

Given. a, b, and c, the sides of 3 □'s.

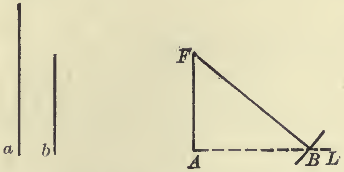
Required. To construct a □ equal to $a^2 + b^2 + c^2$.

[Construction is left to the student.]



Prob. IV. *To construct a square equal to the difference of two given squares.*

Given. The sides a and b of two squares.



Required. To construct a square equal to their difference.

Const. Const. is left to the student.

XV. GROUP ON SIMILAR FIGURES

DEFINITIONS

Two Polygons are Similar when the angles of the first are respectively equal to the angles of the second, and the homologous sides are proportional.

Homologous Sides of Similar Polygons are the sides connecting the vertices of corresponding angles.

Homologous Lines are lines similarly drawn in the two polygons.

ILLUSTRATIONS: Homologous lines in similar triangles are the corresponding medians, altitudes, angle bisectors, etc.

Homologous lines in similar polygons are the corresponding diagonals, radii of circumscribed and of inscribed circles, etc.

The **Ratio of Similitude** of any two similar figures is the ratio of any two homologous lines of the figures.

Two or more figures are in **Perspective**, when they are so placed that the joins of **Corresponding Points** concur.

In polygons the corresponding points are usually vertices.

A **Center of Similitude** is a point of concurrence of corresponding joins of similar figures in perspective.

Draw

Ex. 1. Two similar triangles.

Ex. 2. Two triangles that are not similar.

Ex. 3. Two quadrilaterals that are mutually equiangular but not similar.

Ex. 4. Two quadrilaterals that have the sides of the one proportional to the sides of the other, but are not similar.

Two triangles are similar. Show that

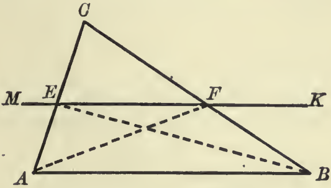
Ex. 5. If the first be isosceles, the second will also be isosceles.

Ex. 6. If the first be equilateral, the second will also be equilateral.

Ex. 7. If the first be right, the second will also be right.

XV. 1. *If a line is parallel to one side of a triangle, it divides the other two sides proportionally.*

Hyp. If, in $\triangle ABC$, $MK \parallel AB$, and cuts CA in E and CB in F,



Conc.: then $CB : CF :: CA : CE$.

Dem. Draw AF and BE.

The ◬ EBF and EFC have the same altitude; for a perpendicular dropped from E to the line CFB would be the altitude of each. (Def. of altitude of $\triangle$.)

Similarly, the ◬ AFE and EFC have the same altitude.

$$\therefore \triangle EFC : \triangle EBF :: CF : FB. \quad (1)$$

[◬ with equal altitudes are to each other, etc.] (XIII. 1 *c*, Sch. 2.)

$$\triangle EFC : \triangle AFE :: CE : EA. \quad (2) \quad \text{(Same reason.)}$$

The ◬ EBF and AFE have the same base EF.

Their altitudes are equal; for perpendiculars dropped from A and B on MK would be opposite sides of a rectangle, and equal. (VI. 1 *a*.)

$$\therefore \triangle EBF = \triangle AFE. \quad \text{(XIII. 1 } c\text{, Sch. 3.)}$$

$\therefore$ from proportions (1) and (2),

$$CF : FB = CE : EA. \quad \text{(Ax. 1.)}$$

Taking this proportion by composition,

$$CF + FB : CF :: CE + EA : CE, \quad \text{(XI. 1 } a\text{.)}$$

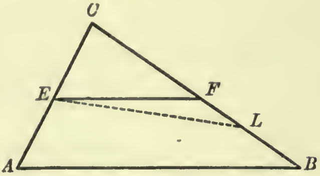
or

$$CB : CF :: CA : CE.$$

Q.E.D.

Ex. 8. The sides of a triangle are 3, 5, 7. Find the sides of three ◬ each of which shall be similar to the first and shall have one side equal to 10.5.

XV. 1 *a*. CONVERSELY. *If a line divides two sides of a triangle proportionally, this line is parallel to the third side.*



Hyp. If, in the $\triangle ABC$, EF is drawn so that $CB : CF :: CA : CE$,

Conc.: then $EF \parallel AB$.

Dem. If $EF \nparallel AB$, draw $EL \parallel AB$.

Then $CB : CL :: CA : CE$, (XV. 1.)

and $CB : CF :: CA : CE$. (Hyp.)

$\therefore\ CB : CF :: CB : CL$. (Ax. 1.)

$\therefore\ CF = CL$.

$\therefore$ F must be the same point as L; *i.e.* EF must coincide with EL. (Ax. 6.)

$\therefore$ EF must be $\parallel AB$.

Q.E.D.

SCH. Proportional division of the sides of a triangle may take place in three different ways.

Thus, in the proof of XV. 1, we have established two proportions involving CB and CA and their segments, viz.:

$$CF : FB :: CE : EA, \quad (1)$$

$$CB : CF :: CA : CE. \quad (2)$$

We may also obtain from XV. 1 by composition (XI. 8),

$$CF + FB : FB :: CE + EA : EA,$$

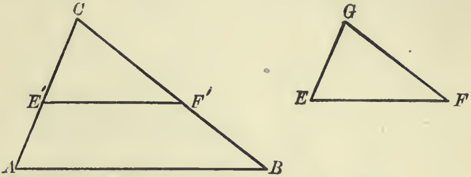
or

$$CB : FB :: CA : EA. \quad (3)$$

Ex. 9. The sides of a right triangle are 3, 4, 5. The hypotenuse of a similar right triangle is 60. Find the other sides of the second right triangle.

Ex. 10. The sides of a triangle are 7, 10, 12, and the longest side of a similar triangle is 18. Find the remaining sides of the second triangle.

XV. 2. *If in two triangles the angles of the one are equal to the angles of the other, the sides of the triangles are proportional, and the triangles are similar.*



Hyp. If, in the $\triangle$s ABC and EFG,

$\angle A = \angle E$,

$\angle B = \angle F$,

and $\angle C = \angle G$,

Conc.: then $CB : GF :: CA : GE :: AB : EF$,

and $\triangle ABC \sim \triangle EFG$

Dem. We may place the $\triangle EFG$ on the $\triangle ABC$ so that it takes the position $CE'F'$. (Why?)

$\angle CE'F' = \angle A$. (Hyp.)

$\therefore E'F' \parallel AB$. (Def. of ∥s.)

$\therefore \dfrac{CB}{CF'} = \dfrac{CA}{CE'}$; (XV. 1.)

i.e. $\dfrac{CB}{GF} = \dfrac{CA}{GE}$. (Const.)

Similarly, by making E coincide with A, we may show that

$$\frac{CA}{GE} = \frac{AB}{EF}.$$

$\therefore \dfrac{CB}{GF} = \dfrac{CA}{GE} = \dfrac{AB}{EF}$. (Ax. 1.)

$\therefore \triangle ABC \sim \triangle EFG$. (Def. of $\sim$ $\triangle$s.)

Q.E.D.

XV. 2 *a*. *Two right triangles are similar if an acute angle of one equals an acute angle of the other.*

Two 4-sides are similar. Show that

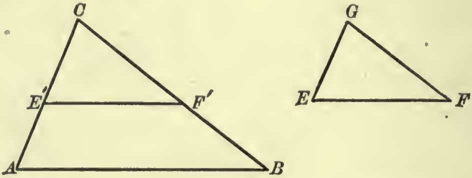
Ex. 11. If the first be a trapezoid, the second will also be a trapezoid.

Ex. 12. If the first be a parallelogram, the second will also be a parallelogram.

XV. 3. *If in two triangles the sides of the one are respectively proportional to the sides of the other, the angles of the first are equal to the angles of the second, and the triangles are similar.*

Hyp. If, in the $\triangle$s ABC and EFG,

$$\frac{CB}{GF} = \frac{CA}{GE} = \frac{AB}{EF},$$



Conc.: then $\angle A = \angle E$; $\angle B = \angle F$; $\angle C = \angle G$, and $\triangle ABC \sim \triangle EFG$.

Dem. Lay off $CF' = GF$, and $CE' = GE$; draw $E'F'$.

Substituting CE' and CF' for their equals GE and GF in the given proportion, we have

$$\frac{CB}{CF'} = \frac{CA}{CE'}.$$

$$\therefore E'F' \parallel AB. \qquad \text{(XV. 1 } a.\text{)}$$

$$\therefore \angle CE'F' = \angle A;\ \angle CF'E' = \angle B. \qquad \text{(Def. of } \parallel\text{s.)}$$

$$\therefore \triangle CE'F' \sim \triangle ABC. \qquad \text{(XV. 2.)}$$

$$\therefore \frac{CA}{CE'} = \frac{AB}{E'F'}; \qquad \text{(Def. of } \sim \text{ polygons.)}$$

i.e.

$$\frac{CA}{GE} = \frac{AB}{E'F'}. \qquad (CE' = GE;\ \text{Const.})$$

But

$$\frac{CA}{GE} = \frac{AB}{EF}. \qquad \text{(Hyp.)}$$

$$\therefore \frac{AB}{EF} = \frac{AB}{E'F'}. \qquad \text{(Ax. 1.)}$$

$$\therefore E'F' = EF.$$

$$\therefore \triangle CE'F' \cong \triangle EFG. \qquad \text{(V. 3.)}$$

But

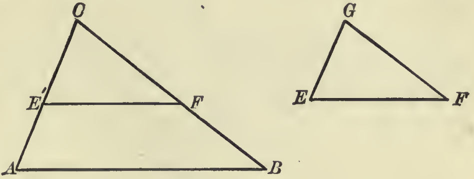
$$\triangle CE'F' \sim \triangle ABC. \qquad \text{(v. line 9 of Dem.)}$$

$$\therefore \triangle EFG \sim \triangle ABC.$$

Q.E.D.

XV. 4. *If two sides of one triangle are proportional to two sides of a second, and the included angles are equal, the triangles are similar.*

Hyp. If, in the $\triangle$s ABC and EFG, $\angle C = \angle G$, and $CA : GE :: CB : GF$,



Conc.: then $\triangle ABC \sim \triangle EFG$.

Dem. Place $\triangle EFG$ on $\triangle ABC$ so that it will take the position $E'F'C$.

$$CB : CF' :: CA : CE'. \qquad \text{(Hyp.)}$$

$$\therefore E'F' \parallel AB. \qquad \text{(XV. 1 } a.)$$

$$\therefore \angle CE'F' = \angle A \text{ and } \angle CF'E' = \angle B. \qquad \text{(Def. of } \parallel\text{s.)}$$

$$\angle C \equiv \angle G.$$

$$\therefore \triangle ABC \sim \triangle EFG. \qquad \text{(XV. 2.)}$$

Q.E.D.

Sch. If two right triangles have the legs of the first proportional to the legs of the second, the two right triangles are similar.

Two 4-sides are similar. Show that

Ex. 13. If the first be a rectangle, the second will also be a rectangle.

Ex. 14. If the diagonals of the first be equal, the diagonals of the second will also be equal.

Ex. 15. If the first be cyclic, the second will also be cyclic.

Ex. 16. If the first be circumscriptible, the second will also be circumscriptible.

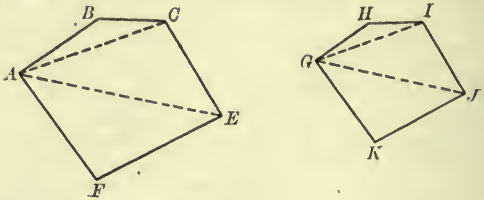
Ex. 17. If the diagonals of the first be at right angles, the diagonals of the second will be at right angles.

Ex. 18. If two isosceles triangles have their vertex angles equal, the triangles are similar.

Ex. 19. If in two isosceles triangles a leg and the base of one be proportional to a leg and the base of the other, the triangles are similar.

XV. 5. *If two polygons are similar, they may be divided into the same number of triangles, similar in corresponding pairs.*

Hyp. If the two polygons *A–F* and *G–K* are similar,



Conc.: then *A–F* and *G–K* may be divided into the same number of triangles, similar in corresponding pairs.

Dem. Draw the homologous diagonals *AC*, *AE*, *GI*, *GJ*.

The polygons are thus divided into the same number of triangles.

To prove these triangles similar in corresponding pairs.

$\triangle ABC \sim \triangle GHI.$ (XV. 4.)

$\triangle ACE \sim \triangle GIJ.$

For $\angle BCE = \angle HIJ.$ (Def. $\sim$ polygons.)

$\angle BCA = \angle HIG.$ (Homologous $\angle$s of $\sim$ $\triangle$s.)

$\therefore \angle ACE = \angle GIJ.$ (By subtraction; Ax. 2.)

Again, $\frac{AC}{GI} = \left(\frac{BC}{HI}\right) = \frac{CE}{IJ}.$ (Hom. sides of $\sim$ $\triangle$s.)

$\therefore \triangle ACE \sim \triangle GIJ.$ (XV. 4.)

Similarly for the remaining pairs of triangles.

$\therefore$ the triangles are similar in corresponding pairs.

Q.E.D.

Ex. 20. Show, from Ex. 19, by drawing the altitudes of the isosceles triangles, that

If in two right $\triangle$s the hypotenuse and a leg of the first are proportional to the hypotenuse and a leg of the second, the right $\triangle$s are similar.

Ex. 21. All equilateral triangles are similar.

XV. 5 *a*. CONVERSELY. *If two polygons may be divided into the same number of triangles, similar in corresponding pairs and similarly placed, the polygons are similar.*

Let the student supply the proof, showing that:

(*a*) The homologous angles of the polygons are equal, using addition where in XV. 5 we have used subtraction.

(*b*) The homologous sides are proportional, *e.g.*

$$BC : HI :: AC : GI,$$

and $$CE : IJ :: AC : GI;$$

whence, $$BC : HI :: CE : IJ, \text{ etc.} \qquad \text{(Ax. 1.)}$$

Ex. 22. If two ▱ have an angle of the first equal to an angle of the second and the including sides proportional, the ▱ are similar.

Ex. 23. From a triangle of altitude 20 ft. and base 50 ft. a small triangle is cut off by a line parallel to the base at a distance of 4 ft. from the vertex. What is the area of the trapezoid remaining?

Find the values that will satisfy the following given conditions:

Ex. 24. Find b, given $a = 8$, $e = 3$, $s = 12$.

Ex. 25. Find e, given $b + e = 20$, $a + s = 35$, $a = 25$.

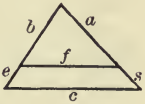
Ex. 26. Find a, given $a = b + e$, $b = 16$, $a + s = 25$.

Ex. 27. Find $b + e$, given $b = 4$, $a = 5$, $s = 1$.

Ex. 28. Find b, given $f = 2b$, $c = 24$, $e = 3$.

Ex. 29. Given $b = 10$ ft., $e = 4$ ft., $c = 35$ ft. Find f.

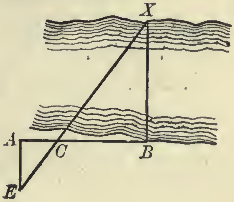
Ex. 30. Given $b + e = 25$ ft., $c = 40$ ft., $b - e = 7$ ft. Find f.



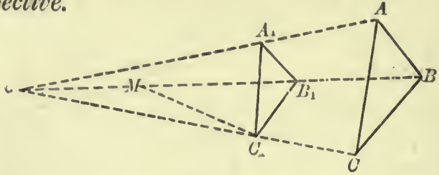
Ex. 31. Given $b + e = 25$ ft., $c = 35$ ft., $f = 24$. Find b and e.

Ex. 32. Given $f = 2b$, $c = 30$ ft., $p - e = 3$. Find b and e.

Ex. 33. Wishing to find the breadth (BX) of a river, I measure off the line AB, and at A draw AE parallel to BX, and EX, crossing AB at C. Show how to find the distance BX.



XV. 6. *If two triangles are similar, they may be placed in perspective.*



Hyp. If $\triangle ABC$ and $\triangle A_1B_1C_1$ are similar,

Conc.: then they may be so placed that A_1A, B_1B, and C_1C will concur.

Dem. Place the triangles so that any two pairs of sides, as AB and A_1B_1, BC and B_1C_1, are parallel.

Draw AA_1 and BB_1, and let them intersect at O.

$$\triangle ABO \sim \triangle A_1B_1O. \qquad \text{(XV. 2.)}$$

$$\therefore OA : OA_1 :: OB : OB_1 :: AB : A_1B_1.$$

Now $AB : A_1B_1$ is the ratio of similitude of the $\triangle$s ABC and $A_1B_1C_1$.

Draw CC_1 and, if possible, let it intersect BB_1 in M.

$$\angle ABC = \angle A_1B_1C_1 \text{ and } \angle OBA = \angle OB_1A_1. \qquad (\text{Def. of} \sim \triangle\text{s.})$$

$$\therefore \angle OBC = \angle OB_1C_1. \qquad \text{(Ax. 2.)}$$

$$\therefore BC \parallel B_1C_1. \qquad (\text{Def. of} \parallel\text{s.})$$

$$\therefore \triangle MBC \sim \triangle MB_1C_1. \qquad \text{(XV. 2.)}$$

$$\therefore MB : MB_1 :: BC : B_1C_1. \qquad (\text{Hom. sides of} \sim \triangle\text{s.})$$

But $$BC : B_1C_1 :: AB : A_1B_1. \qquad \text{(Hyp.)}$$

$$\therefore MB : MB_1 :: AB : A_1B_1 :: OB : OB_1. \qquad \text{(Ax. 1.)}$$

That is, $OB : OB_1 :: MB : MB_1$, (or by division)

$$\frac{OB - OB_1\ (= BB_1)}{OB} = \frac{MB - MB_1\ (= BB_1)}{MB}.$$

As $BB_1 \equiv BB_1$, OB must be $\equiv MB$; that is, M must fall on O.

$\therefore A_1A$, B_1B, and C_1C must concur.

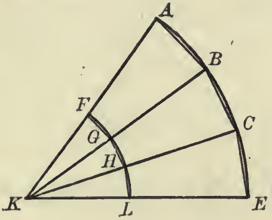
Q.E.D.

XV. 6 *a*. *If two figures in perspective have their sides parallel, they are similar.*

XV. 6 *b*. *If two figures are in perspective, and the joins of homologous points are divided proportionally by the figures, they are similar.*

XV. 7. *If two sectors have equal central angles, their arcs are proportional to the radii.*

Hyp If the sectors *AKE* and *FKL* have the common $\angle K$,



Conc.: then arc $ABCE$: arc $FGHL$ :: KA : KF.

Dem. Divide AE into any number of equal arcs AB, BC, etc., and draw the chords AB, BC, etc. Join A, B, etc., to K, and connect the points of intersection with arc FL by FG, GH, etc.

$$\triangle KAB \sim \triangle KFG. \qquad \text{(XV. 4.)}$$

Similarly, $\triangle KBC \sim \triangle KGH$, etc.

$$\therefore n\text{-gon } K\text{–}ABCE \sim n\text{-gon } K\text{–}FGHL. \qquad \text{(XV. 5 } a.)$$

$$\therefore \frac{AB+BC+CE}{FG+GH+HL}=\frac{KA}{KF}. \qquad (1)$$

(Hom. lines of ~ figures.)

If, now, the arcs AB, BC, etc., be bisected, and new polygonal sectors be formed having double the number of angles, these sectors will still be similar, and proportion (1) will continue to be true, no matter how far the process be carried.

But by continuing the process of bisection, we may make the polygonal sector approach as near as we please to the circular sectors, which are therefore the limits of the similar polygons.

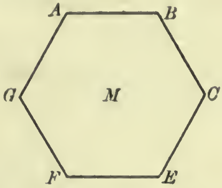
∴ at the limits we have

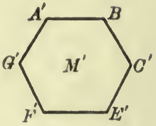
$$\frac{\text{arc } ABCE}{\text{arc } FGHL}=\frac{KA}{KF}. \qquad \text{(XI. } B.)$$

Q.E.D.

XV. 8. *Regular polygons of the same number of sides are similar.*

Hyp. If two regular polygons have the same number of sides,





Conc.: then the polygons are similar.

Dem. Let M and M' be the polygons, n being the number of sides in each.

Any angle of M equals any angle of $M' = \frac{2n-4}{n}$ rt. $\angle$s.

[In a regular polygon each int. $\angle = \frac{2n-4}{n}$ rt. $\angle$s.] (III. 3 *a*.)

$\therefore$ M and M' are mutually equiangular.

Again, $AB = BC$; $A'B' = B'C'$, etc. (Def. of reg. polygons.)

$\therefore$ $AB : A'B' :: BC : B'C'$, etc.; (Ax. 3.)

i.e. the sides of M are proportional to the sides of M'.

$\therefore$ $M \sim M'$. (Def. of $\sim$ polygons.)

Q.E.D.

Ex. 34. In $\triangle$s CLF and CAB, $\angle CLF = \angle B$, and $\angle CFL$ is supplemental to $\angle CAB$. CL and BA are produced to intersect at R, and LF is produced to meet CB at Q.

Prove that $\triangle CRB$ is isosceles.

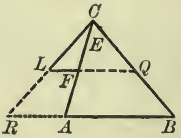
Ex. 35. Prove that $CL : CB :: CF : CA$.

Ex. 36. Prove that $LF : FQ :: RA : AB$.

Ex. 37. Prove, then, that $LQ : RB :: CF : CA$.

Ex. 38. If $LF : RA :: FQ : AB$ and if RL and AF meet at C while $LQ \parallel RB$, prove that

$$CF : CA :: LF : RA.$$



Ex. 39. Similarly, if QB and AF meet at E, show that $EF : EA :: FQ : AB$.

Ex. 40. Show, then, that $CF : CA :: EF : EA$.

Ex. 41. Take the last proportion by inversion and then by division and show that $EF = CF$.

XV. SUMMARY OF PROPOSITIONS IN THE GROUP ON SIMILAR FIGURES

1. *If a line is parallel to one side of a triangle, it divides the other two sides proportionally.*

> *a* CONVERSELY. *If a line divides two sides of a triangle proportionally, this line is parallel to the third side.*

SCH. Proportional division of the sides of a triangle may take place in three different ways.

2. *If in two triangles the angles of the one are equal to the angles of the other, the sides of the triangles are proportional, and the triangles are similar.*

> *a* *Two right triangles are similar if an acute angle of one equals an acute angle of the other.*

3. *If in two triangles the sides of the one are respectively proportional to the sides of the other, the angles of the first are equal to the angles of the second, and the triangles are similar.*

4. *If two sides of one triangle are proportional to two sides of a second, and the included angles are equal, the triangles are similar.*

SCH. If two right triangles have the legs of the first proportional to the legs of the second, the two right triangles are similar.

5. *If two polygons are similar, they may be divided into the same number of triangles, similar in corresponding pairs.*

a Conversely. *If two polygons may be divided into the same number of triangles, similar in corresponding pairs, the given polygons are similar.*

6. *If two triangles are similar, they may be placed in perspective.*

a *If two figures are in perspective, and have their sides parallel, they are similar.*

b *If two figures are in perspective, and the joins of homologous points are divided proportionally by the figures, they are similar.*

7. *If two sectors have equal central angles, their arcs are proportional to the radii.*

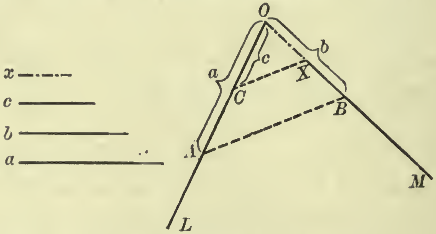
8. *Regular polygons of the same number of sides are similar.*

PROBLEM

Prob. I. *To find a fourth proportional to three given lines.*

Given. The lines a, b, and c,

Required. To find a fourth proportional to a, b, and c.



Anal. Suppose x to be the required fourth proportional.

Then $a : b :: c : x$. (See XV. 1.)

Const. On the indefinite line OL, take $OA = a$ and $OC = c$.

On any intersecting line OM, take $OB = b$.

Draw CX parallel to the join AB.

Then OX is the required fourth proportional. (XV. 1.)

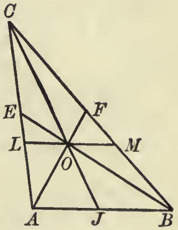
Q.E.F.

Proof. To be supplied by the pupil.

Ex. 42. Prove, then, that if three or more transversals intercept proportional segments on two parallels, the transversals are concurrent.

Ex. 43. Prove that in a triangle the median to the base bisects all lines parallel to the base.

Ex. 44. Prove, then, that the non-parallel sides of a trapezoid meet the extended median of the trapezoid in the same point.



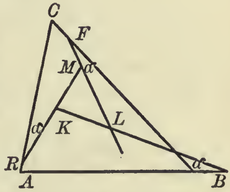
Ex. 45. The two points E and F divide the two sides of $\triangle ABC$ proportionally. Prove the following relations:

(1) $\triangle ABE = \triangle ABF$.

(2) If, through the point of intersection O, LM is drawn parallel to AB, prove $OL = OM$.

(3) Prove that if CO is produced to meet AB in J, $AJ = BJ$.

Ex. 46. In the figure for the preceding theorem, state of what points the median CJ is the locus. (Two loci.)



Ex. 47. The sides of $\triangle MKL$ are equally inclined to those of $\triangle ABC$; that is, each when produced makes an angle alpha with the respective sides of $\triangle ABC$. Prove the following relations:

(1) That the 4-side $CFMR$ is cyclic.

(2) That the two triangles are similar.

Ex. 48. If two angles of one triangle equal two angles of another triangle, and the third pair of angles are supplemental, what kind of triangles are they?

Ex. 49. Why can we not have two angles of one triangle supplemental to two angles of another triangle?

Ex. 50. Prove, then, that two triangles are similar, if the sides of one triangle are respectively perpendicular to the sides of the other.

Ex. 51. Prove that if the sides of one triangle are parallel to the sides of a second, the triangles are similar.

Ex. 52. From any point O, lines are drawn to the vertices of any figure and these lines are divided internally in the ratio of two given lines. Prove that the figure formed by the joins of these points of division is similar to the original figure.

Ex. 53. If the lines in the preceding theorem are extended beyond the point O and divided externally in the ratio of the two given lines, prove that the figure formed by the joins of these points is also similar to the original figure.

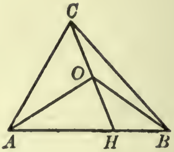
Ex. 54. To divide a given line into three segments, x, y, and z, so that

$$x : y :: a : b$$

and

$$y : z :: c : e;$$

a, b, c, and e being any four given lines.



Ex. 55. If, in the $\triangle ABC$, a line CH is drawn to any point H in AB, and from any point O in CH, OA and OB are drawn, then

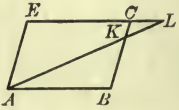
$$AOB : ABC :: OH : CH.$$

Prove.

Ex. 56. If, in the above $\triangle ABC$, lines AF, BG, CH are drawn through a point O to the sides a, b, and c, respectively, then

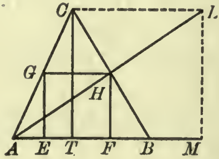
$$\frac{OF}{AF} + \frac{OG}{BG} + \frac{OH}{CH} = 1.$$

Ex. 57. Prove that if, in the preceding theorem, O is taken at a vertex, the equation reduces to the identity $1 \equiv 1$.



Ex. 58. $ABCE$ is any parallelogram. AL is any line through A. Prove that the $\triangle$s AEL and ABK are similar.

Ex. 59. From the preceding exercise, show that in the figure of that exercise the ▭ $EL \cdot BK$ has the same value (that is, is constant), no matter how AL is drawn.



Ex. 60. In the $\triangle ABC$ a square is described on the altitude CT. Let AL intersect CB in H. If HF is parallel to CT and HG is perpendicular to CT, prove that the 4-side $EGHF$ is a square. (This square is said to be inscribed in the triangle.)

Ex. 61. How, then, would you inscribe a square in a given triangle?

Ex. 62. If, in the preceding theorem, we had drawn BL in place of AL, then BL produced would in general meet AC as at G'. Draw lines as before, and prove that the new 4-side is also a square. (Escribed square.)

Ex. 63. How do you construct an escribed square to a triangle?

Ex. 64. If a circle is circumscribed about a triangle whose two sides are a and b, and if the altitude to side c is h, and d the diameter of the circle, prove that $a : h :: d : b$.

(Join end of diameter CD with A, and prove triangles similar.)

Ex 65. By multiplying means and extremes of the above proportion, what rectangles do you find equal?

Ex. 66. If a series of parallels cut any two transversals, the segments determined by the parallels will be proportional. By using this theorem, show how to divide a given line-segment AB into parts proportional to any number of given line-segments, m, k, l, etc.

Of what proposition in Group VII is the foregoing theorem a generalization?

Hint. — If AT, CG, EH, and BI be the parallels, and FI, AB the transversals, draw $AL \parallel FI$. XV. 1 a, Sch. then gives

$$\frac{AE}{CE}=\frac{AJ}{JK}=\frac{FH}{GH}, \qquad \text{(VI. 1 } a.)$$

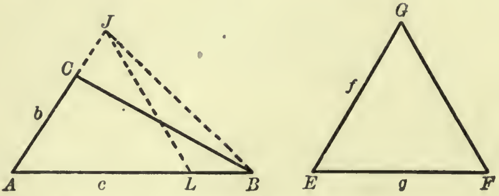
whence,

$$\frac{AE}{FH}=\frac{CE}{GH}=\frac{EB}{HI}. \qquad \text{(XV. 1.)}$$

XVI. GROUP ON AREAL RATIOS

PROPOSITIONS

XVI. 1. *If two triangles have an angle of one equal to an angle of the other, they are to each other as the rectangles of the sides respectively including the equal angles.*



Hyp. If, in the $\triangle$s ABC and EFG, $\angle A = \angle E$,

Conc.: then $\triangle ABC : \triangle EFG :: \square b \cdot c : \square f \cdot g$.

Dem. Place $\triangle EFG$ on $\triangle ABC$ so that it takes the position AJL. Draw JB.

$$\triangle AJL : \triangle AJB :: AL : AB. \qquad (1)$$

[$\triangle$s with equal altitudes are to each other, etc.] (XIII. 1 *c.*, Sch. 2.)

$$\triangle AJB : \triangle ABC :: AJ : AC. \quad \text{(Same reason.)} \qquad (2)$$

$$\therefore \triangle AJL : \triangle ABC :: AL \cdot AJ : AB \cdot AC.$$

(Multiplying (1) by (2).)

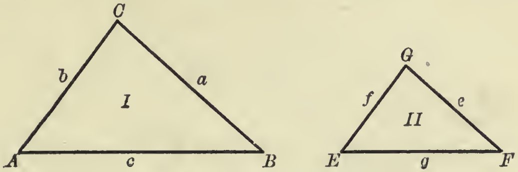
$$\therefore \triangle EFG : \triangle ABC :: \square f \cdot g : \square b \cdot c.$$

Or, by inversion,

$$\triangle ABC : \triangle EFG :: \square b \cdot c : \square f \cdot g.$$

Q.E.D

XVI. 2. *If two triangles are similar, they are to each other as the squares on any two homologous sides.*



Hyp. If the triangles I and II are similar,

Conc. : then $\triangle \text{I} : \triangle \text{II} :: a^2 : e^2 :: b^2 : f^2 :: c^2 : g^2$.

Dem. $$\frac{\triangle \text{I}}{\triangle \text{II}} = \frac{\square\, a \cdot b}{\square\, e \cdot f}. \qquad \text{(XVI. 1.) (1)}$$

Now $$\frac{b}{f} = \frac{a}{e}. \qquad \text{(Hom. sides of} \sim \triangle\text{.)}$$

Substituting in (1) for $\frac{b}{f}$, its equal, $\frac{a}{e}$, we have

$$\triangle \text{I} : \triangle \text{II} :: a^2 : e^2 \text{ and similarly as } b^2 : f^2.$$

Q.E.D.

Ex. 1. Two equilateral △ are as 5 : 4. Compare their altitudes.

Ex. 2. The base of one equilateral triangle equals the altitude of another. What is the ratio of their areas ?

Ex. 3. The homologous sides of two similar polygons are in the ratio of 3 : 7. What is the ratio of the areas of the polygons ?

Ex. 4. What are homologous lines of similar figures ?

Ex. 5. Why are the perimeters of similar polygons homologous lines ?

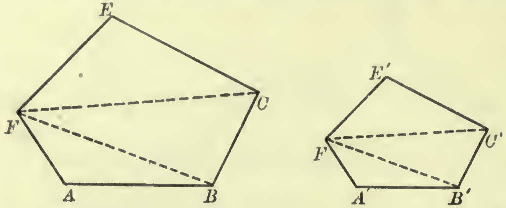
Ex. 6. If two similar polygons have equal perimeters, what do you know of their areas ?

Ex. 7. Under what conditions are two dissimilar triangles equal ?

Ex. 8. What is the relation between the areas of two triangles that have an angle of one equal to an angle of the other ?

Ex. 9. Draw a triangle. Draw a second triangle whose vertex angle is the supplemental adjacent angle of the other. Prove that the areas of these triangles vary (or are to each other) as the rectangles of the sides including the supplemental angles.

XVI. 3. *If two polygons are similar, they are to each other as the squares of any two homologous sides.*



Hyp. If polygon $ABC\cdots$ is similar to polygon $A'B'C'\cdots$, and if their areas be Q and Q', respectively,

Conc.: then $Q : Q' :: \overline{AB}^2 : \overline{A'B'}^2$.

Dem. Draw FC, FB, and $F'C'$, $F'B'$.

[Then the triangles of the first polygon are similar to the similarly placed triangles of the second polygon.] (XV. 5.)

$$\therefore\ \frac{\overline{AB}^2}{\overline{A'B'}^2}=\frac{\triangle FAB}{\triangle F'A'B'}=\frac{\overline{FB}^2}{\overline{F'B'}^2}=\frac{\triangle BFC}{\triangle B'F'C'}=\frac{\overline{FC}^2}{\overline{F'C'}^2}=\frac{\triangle FEC}{\triangle F'E'C'}. \qquad \text{(XVI. 2.)}$$

$$\therefore\ \frac{\triangle FAB}{\triangle F'A'B'}=\frac{\triangle BFC}{\triangle B'F'C'}=\frac{\triangle FEC}{\triangle F'E'C'}\left(=\frac{\overline{AB}^2}{\overline{A'B'}^2}\right). \qquad \text{(Ax. 1.)}$$

$$\therefore\ \frac{\triangle FAB+BFC+FEC}{\triangle F'A'B'+B'F'C'+F'E'C'}=\frac{\triangle FAB}{\triangle F'A'B'}=\frac{\overline{AB}^2}{\overline{A'B'}^2}. \qquad \text{(XI. 2.)}$$

That is, $Q : Q' :: \overline{AB}^2 : \overline{A'B'}^2$.

Q.E.D.

XVI. 3 *a*. *The areas of two similar polygons are to each other as the squares of any two homologous lines.*

Dem. $AB : A'B' :: FB : F'B'$. (Hom. sides $\sim \triangle$.)

$\therefore\ \overline{AB}^2 : \overline{A'B'}^2 :: \overline{FB}^2 : \overline{F'B'}^2$. (XI. 3, *b*.)

But $Q : Q' :: \overline{AB}^2 : \overline{A'B'}^2$. (XVI. 3.)

$\therefore\ Q : Q' :: \overline{FB}^2 : \overline{F'B'}^2$.

Q.E.D.

XVI. SUMMARY OF PROPOSITIONS IN THE GROUP ON AREAL RATIOS

1. *If two triangles have an angle of one equal to an angle of the other, they are to each other as the rectangles of the sides respectively including the equal angles.*

2. *If two triangles are similar, they are to each other as the squares on any two homologous sides.*

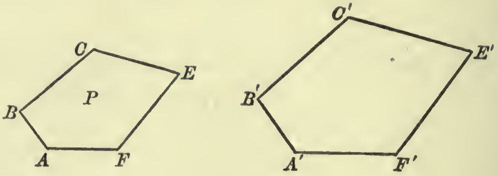
3. *If two polygons are similar, they are to each other as the squares of any two homologous sides.*

   a *The areas of any two similar polygons are to each other as the squares of any two homologous lines.*

PROBLEMS

Prob. I. *On a given line that is to be homologous to a given side of a given polygon, to construct a polygon similar to the given polygon.*

Given. Any line l, and any polygon P.



Required. A polygon on l, $\sim P$ and of which the side equal to l shall be homologous to AF.

Const. Lay off $A'F' = l$, at A' construct $\angle A' = \angle A$; at F' construct $\angle F' = \angle F$.

Find a fourth proportional to AF, $A'F'$, and AB.

Lay off this fourth proportional on the terminal line of $\angle A'$, as $A'B'$.

[Completion of construction and proof left to student as an exercise.]

Prob. II. *To construct a polygon similar to each of two given similar polygons and equal to their difference.*

Hint. — Construct a right triangle as in Prob. I, and use XVI. 3.

Prob. III. *To construct a polygon similar to each of two given polygons and equal to their sum.*

Hint. — Construct a right triangle as in XIV. Prob. I, and use XVI. 3.

Ex. 10. Draw a triangle. Draw a second triangle within it, two of whose sides are perpendicular to two sides of the first. Prove that the areas of these triangles vary as the rectangles of the sides that are perpendicular to each other.

The areas of two similar triangles are respectively 196 sq. ft. and 256 sq. ft.

Ex. 11. What is the ratio of any pair of their homologous sides?

Ex. 12. What is the ratio of the rectangles of the sides including a pair of homologous angles?

To construct a triangle similar to *ABC* and satisfying the following conditions:

Ex. 13. Having a perimeter three times as long as the perimeter of *ABC*.

Ex. 14. Having an area equal to four ninths the area of *ABC*.

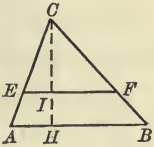
Ex. 15. Having an area twice as great as that of *ABC*.

Ex. 16. The areas of two triangles, I and II, having one angle in common, are 20 sq. ft. and 60 sq. ft., respectively. The sides in triangle I about the common angle are 5 ft. and 6 ft. One of the corresponding sides of triangle II is 12 ft. What is the length of the other side?

Ex. 17. The sides of a triangle are 4, 9, 10.

Divide the triangle into two equal parts in three ways by drawing, in succession, parallels to the three sides.

Ex. 18. Prove Theorem 2 by using the adjoining figure, *CH* being the altitude of the $\triangle ABC$.



Ex. 19. Each side of a regular pentagon is 3. Construct a similar pentagon twice as large as the first.

Ex. 20. To construct a hexagon similar to a given hexagon and having one third the area of the given hexagon.

Ex. 21. The areas of two similar polygons are 324 sq. ft. and 576 sq. ft. They are divided into three sets of similar triangles by diagonals drawn from each of two homologous vertices.

What is the ratio of the areas of corresponding pairs of the similar triangles?

What is the ratio of the homologous diagonals?

Ex. 22. Given two similar hexagons *Q* and *R*. To construct a hexagon similar to *Q* and *R* and equal to their sum.

Ex. 23. Given two similar pentagons *Q* and *R*. To construct a pentagon similar to *Q* and *R* and equal to their difference.

Ex. 24. Given any number of similar polygons. Construct a polygon similar to each and equal to their sum.

(Use XVI. 3 and XIV. Prob. III.)

Ex. 25. To divide a triangle into two equal parts by a line parallel to a given line.

Ex. 26. To divide a triangle into two equal parts by a line through a given point on one side of the triangle.

Ex. 27. To divide a triangle into three equal parts by parallels to one side.

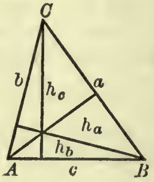
Ex. 28. What does the area of a triangle equal in terms of the base and altitude?

Ex. 29. The sides of a triangle are a, b, and c. The altitudes to these sides are h_a, h_b, and h_c.

Show that $a \cdot h_a = b \cdot h_b = c \cdot h_c$.

Ex. 30. Divide each member of the preceding equation by $h_a \cdot h_b$ and show that

$$\frac{a}{h_b} = \frac{b}{h_a} = \frac{c}{\frac{h_a h_b}{h_c}}.$$



Ex. 31. Place $\frac{h_a \cdot h_b}{h_c} = x$ and construct x (a fourth proportional to h_a, h_b, and h_c).

Ex. 32. If x equals a line m, then $\frac{a}{h_b} = \frac{b}{h_a} = \frac{c}{m}$.

Ex. 33. Why, then, would a triangle whose sides are h_b, h_a, and m be similar to a triangle whose sides are a, b, and c?

Ex. 34. Construct such a triangle, $A'B'C'$, and draw the altitude corresponding to h_c of $\triangle ABC$. Produce this altitude, if necessary, to equal h_c. Through its foot, draw the parallel to c'. Produce the sides, if necessary, to meet this parallel. Prove this new triangle congruent with $\triangle ABC$.

Ex. 35. Construct a triangle having given the three altitudes.

XVII. GROUP ON LINEAR APPLICATION OF PROPORTION

PROPOSITIONS

XVII. 1. *If the bisectors of the interior and exterior angles at the vertex of a triangle are drawn, these bisectors will divide the base into segments proportional to the other two sides.*

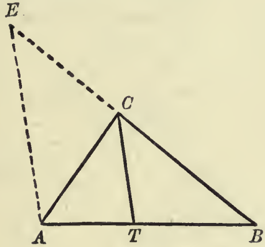


FIG. 1.

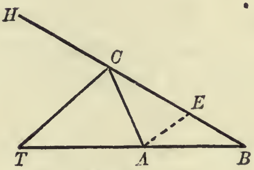


FIG. 2.

Hyp. If CT bisects $\angle ACB$ (Fig. 1), or $\angle ACH$ (Fig. 2),

Conc.: then, in either figure, $AT : TB :: AC : BC$.

Dem. CASE I. Extend BC, making $EC = AC$.

Draw EA. $\triangle CAE$ is isoangular. (IV. 1.)

$\therefore \angle ACB = 2\angle E.$ (III. 2 *a*.)

$\therefore \angle BCT\left(=\frac{\angle ACB}{2}\right) = \angle E.$ (Ax. 2.)

$\therefore CT \parallel AE.$ (Def. of ∥s.)

$\therefore AT : TB :: EC : BC$ (XV. 1.)

Substituting AC for EC, we have

$$AT : TB :: AC : BC.$$

Q.E.D.

Dem. CASE II. Make $EC = AC$.

Draw EA. $\triangle CAE$ is isoangular. (IV. 1.)

$$\therefore \angle ACH = 2 \angle CEA. \qquad \text{(III. 2 } a.\text{)}$$

$$\therefore \angle TCH \left(= \frac{\angle ACH}{2} \right) = \angle CEA. \qquad \text{(Ax. 2.)}$$

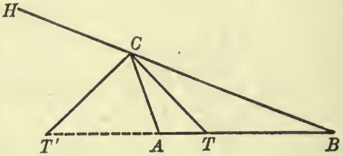
$$\therefore CT \parallel AE. \qquad \text{(Def. of } \parallel\text{s.)}$$

$$\therefore AT : TB :: AC : BC.$$

Q.E.D.

Def. When a line-segment is divided internally and externally in the same absolute geometric ratio, the line-segment is said to be divided harmonically.

XVII. 1 *a*. *The bisectors of the interior and exterior vertex angles of a triangle divide the base harmonically in the ratio of the sides.*



Hyp. If CT and CT' bisect int $\angle ACB$ and ext. $\angle ACH$, respectively, and cut AB in T and T',

Conc.: then $AT : TB :: AT' : T'B$.

Dem.

$$\frac{AT}{TB} = \frac{AC}{CB}; \text{ also } \frac{AT'}{T'B} = \frac{AC}{CB}. \qquad \text{(XVII. 1.)}$$

$$\therefore AT : TB :: AT' : T'B. \qquad \text{(Ax. 1.)}$$

Q.E.D.

XVII. 1 *b*. *If T and T' divide AB harmonically, then the points A and B divide the line TT' harmonically.*

For, if $AT : TB :: AT' : T'B$,

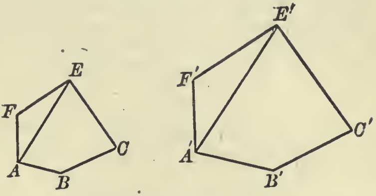
then, by alternation, $AT : AT' :: TB : T'B$.

That is, the ratio of the distances of A from T and T' equals the ratio of the distances of B from T and T'.

A, B and T, T' are called two pairs of *conjugate harmonic points.*

XVII. 2. *The perimeters of similar polygons are to each other as any two homologous lines*

Hyp. If p and p' be the perimeters of two similar polygons, AB and $A'B'$ two homologous sides, and AE and $A'E'$ any two homologous lines (*e.g.* homologous diagonals),



Conc.: then $p : p' :: AB : A'B' :: AE : A'E'$.

Dem. $AB : A'B' :: BC : B'C' :: CE : C'E'$, etc.

(Hom. sides of ∼ polygons.)

$$\therefore \frac{AB + BC + CE + \cdots}{A'B' + B'C' + C'E' + \cdots} = \frac{AB}{A'B'} = \frac{BC}{B'C'}, \text{ etc.}$$

[In a series of equal ratios the sum, etc. ;] (XI. 2.)

i.e. $p : p' :: AB : A'B'$, etc.

But $AE : A'E' :: AB : A'B'$ (XV. 5.)

$\therefore p : p' :: AB : A'B' :: AE : A'E'$. (Ax. 1.)

Q.E.D.

Ex. 1. The sides of a triangle are 7, 9, and 12. Find the segments of the side 12 made by the interior and exterior bisectors of the opposite angle.

Ex. 2. To divide a line-segment harmonically in a given ratio; *i.e.* so that the ratio of the segments of internal and external division shall equal the ratio of two given lines a and b.

Ex. 3. If, in the last exercise, $a = b$, what becomes of the external point of division of the given line-segment?

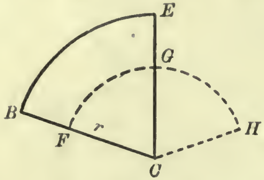
Ex. 4. If the diagonal of one pentagon is twice as long as the corresponding diagonal of a similar pentagon,

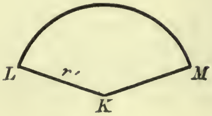
How does the perimeter of the first figure compare with that of the second?

Ex. 5. If the diagonal of a pentagon is equal to the sum of the corresponding diagonals of two pentagons similar to the first,

What relation exists between the perimeter of the first figure and the perimeters of the other two?

XVII. 3. *If two angles have their vertices at the centers of two unequal circles, the angles are to each other as the arc of the first divided by its radius is to the arc of the second divided by its radius.*





Hyp. If $\angle BCE$ and $\angle K$ have their vertices C and K at the centers of circles of radii r and r', respectively,

Conc.: then $\angle BCE : \angle K :: \frac{\text{arc } BE}{r} : \frac{\text{arc } LM}{r'}$.

Dem. With C as a center and a radius $= r'$, describe arc FG.
Produce arc FG, making arc $FH =$ arc LM.
Draw CH.
Then $$\angle FCH = \angle K.$$
[In equal ⓢ equal $\angle$s at the centers intercept, etc.] (IX. 3.)
$$\therefore \angle FCG : \angle FCH :: \text{arc } FG : \text{arc } FH.$$
[In equal ⓢ central $\angle$s vary as their intercepted arcs.] (XII. 1.)

But $$\angle FCG \equiv \angle BCE.$$
$$\therefore \angle BCE : \angle FCH :: \frac{\text{arc } FG}{r'} : \frac{\text{arc } FH}{r'}.$$
But the sectors BCE and FCG have $=$ central $\angle$s.
$$\therefore \frac{\text{arc } FG}{r'} = \frac{\text{arc } BE}{r}. \qquad \text{(XV. 7.)}$$
$$\therefore \angle BCE : \angle FCH :: \frac{\text{arc } BE}{r} : \frac{\text{arc } FH}{r'}.$$
$$\therefore \angle BCE : \angle K :: \frac{\text{arc } BE}{r} : \frac{\text{arc } LM}{r'}.$$

Q.E.D.

SCH. Upon this theorem is based what is called the **Radial** method of measuring angles.

The unit angle of this system is of course the angle whose measure is 1; *i.e.* the angle for which $\frac{\text{arc}}{r} = 1$; or arc = radius. This unit is called a **Radian**, and will hereafter be shown to denote an angle of about 57°.3.

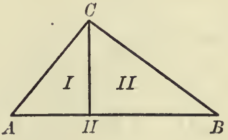
The advantage of radial measure over the ordinary measurement in degrees is that the former does not depend at all upon the size of the circle.

The radian is practically the only unit employed in advanced work.

APPLICATION TO THE SIDES OF A RIGHT TRIANGLE AND THE LINE-SEGMENTS DEPENDENT UPON THE ALTITUDE

XVII. 4. LEMMA. *If the altitude of a right triangle is drawn to the hypotenuse, the three triangles of the resulting figure are similar.*

Hyp. If, in rt. △ *ABC*, the altitude *CH* is drawn to the hypotenuse,



Conc.: then rt. △ I ∼ rt. △ *ABC* ∼ rt. △ II.

Dem. Rt. △ *ABC* ∼ rt. △ I.

∵ ∠ *A* is common to both of them.

[Two rt. ⚠ are ∼ if an acute ∠ of one, etc.] (XV. 2 *a*.)

Similarly, rt. △ *ABC* ∼ rt. △ II.

∴ the ∠s of rt. △ I = respectively the ∠s of rt. △ II. (Ax. 1.)

∴ rt. △ I ∼ rt. △ II. (XV. 2.)

∴ rt. △ I ∼ rt. △ *ABC* ∼ rt. △ II.

Q.E.D

XVII. 5. *If the altitude of a right triangle to the hypotenuse is drawn:*

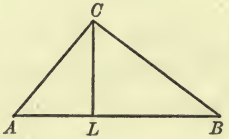
(*a*) *The altitude is a mean proportional between the segments of the hypotenuse.*

(*b*) *Either leg is a mean proportional* (*or geometric mean*) *between the hypotenuse and the segment adjacent to that leg.*

(*c*) *The segments of the hypotenuse are to each other as the squares on the legs respectively adjacent to them.*

(*d*) *The square of the length of the hypotenuse equals the sum of the squares of the lengths of the legs.*

Hyp. If CL is the altitude to AB, the hypotenuse of the rt. $\triangle ABC$,



Conc.: then

(*a*) $AL : CL :: CL : LB$, or $\overline{CL}^2 = AL \cdot LB$.

(*b*) $AL : CA :: CA : AB$, or $\overline{CA}^2 = AL \cdot AB$.

(*c*) $AC^2 : \overline{BC}^2 :: AL : LB$.

(*d*) $\overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2$.

Dem. (*a*) Rt. $\triangle ALC \sim$ rt. $\triangle CLB$. (XVII. 4.)

$\therefore AL : CL :: CL : LB$. (Hom. sides of $\sim$ ⚠, etc.)

$\therefore \overline{CL}^2 = AL \cdot LB$. (XI. 1.)

Q.E.D.

Dem. (*b*) Rt. $\triangle ALC \sim$ rt. $\triangle ACB$. (XVII. 4.)

$\therefore AL : CA :: CA : AB$, or $\overline{CA}^2 = AL \cdot AB$; (1)

also, $LB : BC :: BC : AB$, or $\overline{BC}^2 = LB \cdot AB$. (2)

Q.E.D.

Dem. (*c*) Divide (1) by (2), member by member, and we have

$$\overline{AC}^2 : \overline{BC}^2 :: AL : LB.$$

Q.E.D.

Dem. (*d*) Add (1) to (2), member to member, and we have

$$\begin{aligned}\overline{AC}^2 + \overline{BC}^2 &= (AL + LB) \cdot AB \\ &= AB \cdot AB \\ &= \overline{AB}^2.\end{aligned}$$

Q.E.D.

Def. If O be any point on the straight line AB (whether between the points A and B, or on AB produced), the distances OA and OB are called the **segments** of AB made by the point O.

Ex. 6. What are homologous sides of similar triangles?

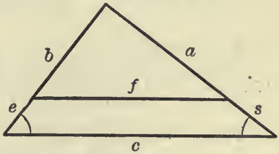
Name the sets of similar triangles in the figure of XVII, 4.

Read the homologous angles of each set; also the homologous sides.

Ex. 7. Find a mean proportional to a and b by using XVII. 6 a.

Ex. 8. Find a third proportional to a and b by using XVII. 6 a.

Ex. 9. Given $a = 8$ ft., $c = 13$ ft., and $s = 1\frac{1}{2}$ ft. Find f.



Ex. 10. Given: perimeter of small triangle is 46 ft.; of large, 138 ft.

What is the ratio of any two homologous sides?

Ex. 11. In two similar triangles the sides of the first are 4 ft., 9 ft., and 11 ft.

The shortest side of the second is 12 ft.

What is the length of the other two sides?

What is the ratio of their perimeters?

The sides of a triangle are 7 ft., 10 ft., and 12 ft. Find:

Ex. 12. The segments determined on each side by the bisector of the interior angle at the opposite vertex.

Ex. 13. The segments determined on each side by the bisector of the exterior angle at the opposite vertex.

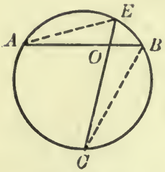
Ex. 14. The projections of each side on the other two.

Ex. 15. The projection of the bisector of each interior angle on the opposite side.

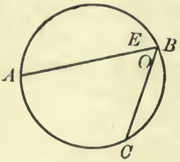
XVII. 6. *If through a fixed point any line is drawn cutting a circle in two points, the rectangle of the segments of this line is constant, in whatever direction the line is drawn.*

THREE CASES.

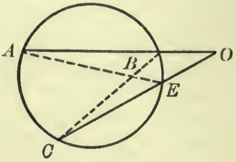
(1) *The fixed point within the circle.*
(2) *The fixed point on the circumference.*
(3) *The fixed point without the circle.*



CASE (1).



CASE (2).



CASE (3).

Hyp. If O is a fixed point, and AB is any line through O, and EC is any other line through O, each cutting the circle,

Conc.: then the ▭ of $BO \cdot OA =$ the ▭ of $EO \cdot OC$.

Dem. Draw AE and BC in each of the three cases.

$$\triangle AOE \sim \triangle BOC.$$

$$\because \angle A = \angle C. \quad \left(\text{Each is measured by } \frac{\text{arc } EB}{2}.\right)$$

$$\angle AOE = \angle BOC. \quad \text{(Why?)}$$

$\therefore \triangle AOE \sim \triangle BOC$, and $OA : OE :: OC : OB$. (XV. 2.)

$\therefore$ the rectangle of $BO \cdot OA =$ the rectangle of $EO \cdot OC$.

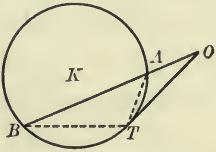
Q.E.D.

SCH. The proportion $OA : OE :: OC : OB$ may be written

$$\frac{OA}{OE} = \frac{OC}{OB}, \text{ or } \frac{OA}{OE} = 1 \div \frac{OB}{OC}.$$

That is: The ratio of two segments equals the *reciprocal* of the ratio of the two corresponding segments; in other words, the segments are *reciprocally proportional.*

XVII. 6 *a*. *If from the same point to the same circle a tangent and a secant be drawn, the tangent will be a mean proportional between the secant and the part without the circle.*



Hyp. If from O, OT a tangent, and OB a secant are drawn to $\odot K$,

Conc.: then $OB : OT :: OT : OA$, or $\overline{OT}^2 = OB \cdot OA$.

Dem. Draw AT and BT.

$\triangle OAT \sim \triangle OTB$. (XV. 2.)

$\therefore OB : OT :: OT : OA$, or $\overline{OT}^2 = OB \cdot OA$. (XI. 1.)

Q.E.D.

Sch. OA is a third proportional to OB and OT.
OB is a third proportional to OA and OT.

The sides of a triangle are 7 ft., 10 ft., and 12 ft. Find:

Ex. 16. The projection of the bisector of one of the exterior angles on the opposite side.

Ex. 17. The lengths of any of the angle bisectors.

Ex. 18. The area of the triangle.

Ex. 19. The acute angles of a right triangle are 60° and 30°. What is the ratio of the segments into which the shortest side is divided by the bisector of the opposite angle?

Ex. 20. Given a circle of radius a; through a point at a distance $2a$ from the center, tangents are drawn. Find:

(*a*) The length of each tangent. (*b*) The length of the chord of contact.
(*c*) The angle between the tangents.
(*d*) The angle at the center subtended by the chord of contact.
(*e*) The distance of the chord of contact from the center.

XVII. SUMMARY OF PROPOSITIONS IN THE GROUP ON THE LINEAR APPLICATION OF PROPORTION

APPLICATION TO ANGLE BISECTORS, PERIMETERS, AND TO ANGULAR MEASUREMENT

1. *If the bisector of the interior and exterior angles at the vertex of a triangle are drawn, these bisectors will divide the base into segments proportional to the other two sides.*

   a *The bisectors of the interior and exterior vertex angle of a triangle divide the base harmonically in the ratio of the sides.*

2. *The perimeters of similar polygons are to each other as any two homologous lines.*

3. *If two angles have their vertices at the centers of two unequal circles, the angles are to each other as the arc of the first divided by its radius is to the arc of the second divided by its radius.*

   SCH. Unit of angular magnitude: radian.

APPLICATION TO THE SIDES OF A RIGHT TRIANGLE AND THE LINE-SEGMENTS DEPENDENT UPON THE ALTITUDE

4. LEMMA. *If an altitude of a right triangle is drawn to the hypotenuse, the three triangles of the resulting figure will be similar.*

5. *If the altitude of a right triangle to the hypotenuse is drawn,*

(*a*) *The altitude is a mean proportional between the segments of the hypotenuse.*

(*b*) *Either leg is a mean proportional (or geometric mean) between the hypotenuse and the segment adjacent to that leg.*

(*c*) *The segments of the hypotenuse are to each other as the squares on the legs respectively adjacent to them.*

(*d*) *The square of the length of the hypotenuse equals the sum of the squares of the lengths of the legs.*

APPLICATION TO CHORDS, TANGENTS, AND SECANTS

6. *If through a fixed point any line is drawn cutting a circle in two points, the rectangle of the segments of this line is constant, in whatever direction the line is drawn.*

THREE CASES.

(1) *The fixed point within the circle.*

(2) *The fixed point on the circumference.*

(3) *The fixed point without the circle.*

SCH. The ratio of two corresponding segments equals the reciprocal of the ratio of the other two corresponding segments. That is, the segments are reciprocally proportional.

a. If from the same point to the same circle a tangent and a secant are drawn, the tangent will be a mean proportional between the secant and the part without the circle.

SCH. On third proportionals.

PROBLEMS

PROB. I. *To find a fourth proportional to three given lines.*

Given. The three lines, a, b, and c.

Required. A fourth proportional to a, b, and c.

Analysis. Suppose x to be the required fourth proportional.

Then $$a : b :: c : x.$$

Suggested Theorem, XVII. 6, Cases (1) and (3).

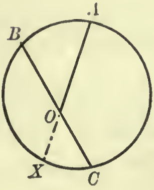


FIG. 1.

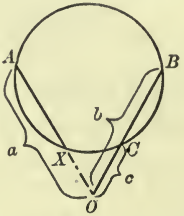


FIG. 2.

The "dash and dot" line in figures is the line required.

Const. CASE (1). Draw a circle whose diameter $> b + c$. (Fig. 1.)

In this circle draw a chord $BC = b + c$.

Take $OA = a$ and produce it to intersect circle at X.

Then OX is the required fourth proportional. (XVII. 6 (1).)

Q.E.F.

Const. CASE (2). Draw a circle whose diameter $> b - c$. (Fig. 2.)

In this circle draw a chord $BC = b - c$.

Produce it so that $CO = c$.

Take $OA = a$.

Then OX is the required fourth proportional. (XVII. 6 (3).)

Q.E.F.

Note. — The student should suggest still other ways of laying off the given segments in Problem I. x is a fourth proportional to a, b, and c in any one of the following arrangements:

(1) $a : b :: c : x$. (3) $a : x :: b : c$.
(2) $a : b :: x : c$. (4) $x : b :: a : c$.

Also in their transformations.

Sch. If two of the given line-segments are equal, *e.g.* if $b = c$, each of the constructions given above will determine a *third* proportional to a and b, a and c, or b and c, provided that the line-segments that become equal are not in the same couplet of the proportion.

Prob. II. *To construct a mean proportional to two given lines.*

Given. The lines a and b.

Required. To construct a mean proportional to a and b.

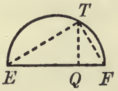


Fig. 1.

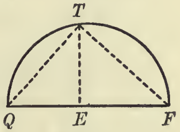


Fig. 2.

Const. Case (1). Draw $EF = b$ and lay off $EQ = a$.
Describe a semicircle on EF as a diameter.
At Q erect a perpendicular to EF intersecting semicircle in T.
ET is the required mean proportional. Q.E.F.

Proof. $EF : ET :: ET : EQ$ or $b : ET :: ET : a$.

[Either leg of a rt. △ is a mean proportional, etc.] (XVII. 5 *b*.)

∴ ET of Fig. 1 is a mean proportional to a and b. Q.E.D.

Const. Case (2). Draw $EQ = a$ and $EF = b$.

Then $QF = a + b$.

On QF as a diameter describe a semicircle.
At E erect a $\perp$ to QF intersecting the semicircle at T.
ET is the required mean proportional. Q.E.F.

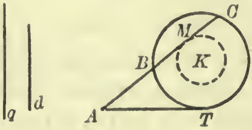
Proof. $QE : ET :: ET : EF$ or $a : ET :: ET : b$.

For QTF is a right △. (Why?)

∴ ET of Fig. 2 is a mean proportional to a and b. Q.E.D

Prob. III. *To draw a tangent of given length to a circle so that the chord of the secant drawn from the outer extremity of the tangent to the same circle shall equal a given line.*

Given. The length q, the line d, and the circle K.



Required. To draw a tangent $= q$, so that the chord of the secant from the outer extremity of the tangent $= d$.

Analysis. Suppose tangent $AT = q$ and that $AC - AB = d$. It follows that if KM be drawn $\perp$ BC, M is the mid-point of BC. (Why?)

Propositions, etc., suggested are:

Find the locus of the mid-points of all chords in the given circle equal to d, and

Draw to the given circle a tangent equal to q, and

From the outer extremity of this tangent draw a tangent to the locus circle.

We have now discovered a method of constructing the line required.

Discussion. As two tangents may be drawn from A to the locus circle, there are evidently two solutions.

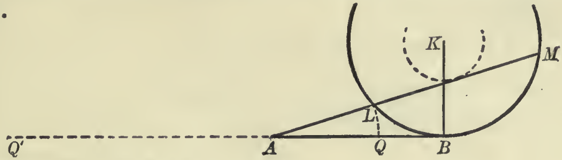
These solutions become coincident when d equals the diameter of the given circle.

There is no solution when d is greater than the diameter of the given circle.

Def. A line is divided **in extreme and mean ratio** (or **in golden section**) when one segment of the line is a mean proportional between the whole line and the other segment.

Ex. 21. The line of centers of two circles is divided externally in the ratio of their radii by its point of intersection with an external common tangent.

PROB. IV. *To divide a given line in extreme and mean ratio.*



Given. The line AB.

Required. To divide AB in extreme and mean ratio.

Const. Draw $KB \perp AB$. Let it equal or be greater than $\frac{AB}{2}$.

From A draw a secant AM to this circle so that the chord

$$LM = AB. \qquad \text{(Prob. III.)}$$

With A as a center, and a radius equal to AL, describe an arc cutting AB, as at Q.

Then AB is divided at Q in extreme and mean ratio.

That is $$AB : AQ :: AQ : QB.$$

Q.E.F.

Proof. AB is tangent to large circle K.

[If a line is ⊥ to a radius at its extremity, etc.] (IX. 4 *a*.)

$$\therefore AM : AB :: AB : AL. \qquad (1)$$

$$\therefore AM - AB : AB :: AB - AL : AL.$$

(From (1) by division.)

But $$AB = LM,$$

and $$AL = AQ. \qquad \text{(Const.)}$$

$$\therefore AM - AB = AM - LM = AL,$$

and $$AB - AL = AB - AQ = QB.$$

$$\therefore AQ : AB :: QB : AQ. \qquad (2)$$

$$\therefore AB : AQ :: AQ : QB.$$

(From (2) by inversion.)

Q.E.D.

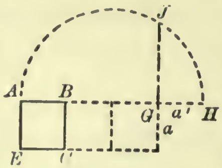
Discussion. With A as a center, and a radius equal to AM, describe an arc, cutting AB produced to the left, as at Q'.

Then $$AB : AQ' :: AQ' : Q'B.$$

Prove by taking (1) by composition.

$\therefore$ Q' is a second point of golden section.

Prob. V. *To construct a square that shall be three times a given square.*



Given. The square A–C.

Required. To construct a square equal to three times the square A–C.

Const. Produce AB, making $AG = 3\,AB$.

Produce AG, making $a' = a$.

On AH as a diameter, describe a semicircle.

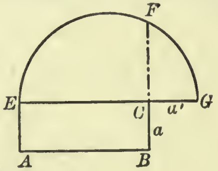
At G erect a perpendicular to AH, meeting the semicircle in J.

JG is the side of the required square.

Proof is left to the pupil.

Def. To transform a figure means to change it to another figure that is equal to the first.

Prob. VI. *To transform a rectangle into a square.*



Given. The ▭ E–B.

Required. To transform it into an equal square.

Const. Produce EC, making $a' = a$.

On EG as a diameter describe a semicircle.

At C erect a perpendicular to EG, meeting the semicircle in F.

CF is the side of the required square.

Proof is left to the pupil.

Ex. 22. The line of centers of two circles is divided internally in the ratio of their radii by its point of intersection with an internal common tangent.

Ex. 23. The points of intersection of the pairs of internal and external common tangents to two circles divide the line of centers harmonically.

PROB. VII. *To transform a triangle into a square.*

Construct a mean proportional between the base and half the altitude.

Ex. 24. Two sides of a right triangle are 20 and 21.

Determine: (a) The projections of these sides on the hypotenuse. (b) The altitude on the hypotenuse.

Ex. 25. If the segments of a diameter be 3 and 7, what is the length of the perpendicular to the diameter at the point of division and extending to the circumference?

Ex. 26. Show how to construct $\sqrt{14}$ geometrically.

Ex. 27. What is the expression for the rectangle of the segments of a diameter determined by any point in the diameter?

From this expression show that the rectangle of the segments of a diameter is greatest when the segments are equal.

Ex. 28. If two chords be drawn from any point of a semicircle to the extremities of a diameter, the squares of the chords will be to each other as the projections of the chords on the diameter.

Ex. 29. The area of a triangle equals the rectangle of its base by half the altitude or $b \cdot \frac{h}{2}$.

Ex. 30. Construct a square equal to twice a given triangle.

Ex. 31. Construct a square equal to the sum of two given triangles.

Ex. 32. Construct a square equal to the sum of a given triangle and a given parallelogram.

Ex. 33. Construct a square equal to the sum of a given pentagon and a given rectangle.

Ex. 34. What is the length of the side of a square equal to an equilateral triangle each side of which is 10 ft.?

What is the geometric meaning of the following expressions?

Ex. 35. $x = \frac{ab}{c}$; $x = \frac{a^2}{b}$.

Ex. 36. $x = \sqrt{\overline{AB}^2}$; $x = \sqrt{\frac{2}{3}\,\overline{AB}^2}$.

Ex. 37. $x = \sqrt{2\,\overline{AB}^2 - \square \text{ of } CE \cdot HJ}$.

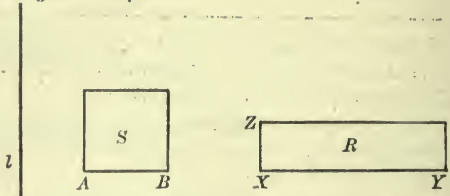
Ex. 38. $x = \sqrt{2 \square \text{ of } a \cdot b}$.

Ex. 39. $x = \sqrt{\frac{\overline{AQ}^2 - 2\,\overline{AM}^2}{2}}$.

Prob. VIII. *To construct a rectangle whose perimeter is equal to twice the length of a given line and whose area equals that of a given square.*

Given. The line l, and the square S.

Required. To construct a rectangle whose perimeter equals $2l$ and whose area equals S.

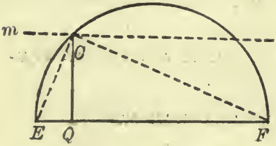


Analysis. Suppose ▭ R has perimeter $= 2l$ and area $=$ □ S.
That is, that rectangle of $XY \cdot XZ = \overline{AB}^2$, (1)

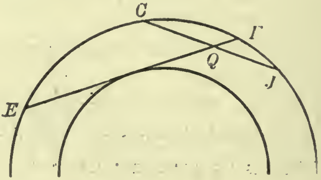
and
$$XY + XZ = l. \qquad (2)$$

It follows from (1) and (2) that AB is a mean proportional between two line-segments whose sum $= l$.

Several previously established theorems are suggested by the last statement, viz.: XVII. 5 (a) and 5 (b); XVII. 6 and 6 a; Problems I, II, III. Of these, let us try XVII. 5 a and XVII. 6.



Case 1.



Case 2.

Const. Case 1. Describe a semicircle on $EF = l$ as diameter. Let m be the locus of a point that is AB distance from EF. From C, the intersection of this locus with the circle, draw $CQ \perp EF$.

The rectangle of $EQ \cdot QF$ is the rectangle required.

Q.E.F

Proof. $CQ = AB$, $\because$ m is the locus of all points AB distant from EF.

$EF = l$ (Const.)

$EQ : CQ :: CQ : QF$ or $\overline{CQ}^2 =$ rectangle of $EQ \cdot QF$. (XVII. 5 *a*.)

Q.E.D.

Const. Case 2 Draw any circle whose diameter $> l$.

Draw a chord $CJ = 2\,AB$.

Draw the circle that is the locus of mid-points of chords $= l$.

From Q, the mid-point of CJ, draw a chord of the larger circle that is tangent to the locus circle.

The rectangle of $EQ \cdot QF$ is the rectangle required

Q.E.F.

Proof to be given by the pupil from XVII. 6.

Ex. 40. If a tangent to a circle be terminated by two parallel tangents, the rectangle of the segments of this tangent determined by the point of tangency equals the square on the radius.

Ex. 41. Given a circle of radius 20 ft.; through a point 16 ft. from the center a chord is drawn.

Find the rectangle of the segments into which the chord is divided at the point.

Ex. 42. Given a circle of radius 24 ft.; through a point 40 ft. from the center a tangent is drawn, and also a secant 58 ft. long. Find:

(*a*) The length of the tangent.
(*b*) The length of the chord cut from the secant.
(*c*) The distance of the secant from the center.

Ex. 43. The radius of a circle is 12 ft.; tangents are drawn to this circle through a point 20 ft. from the center. What is the length of the chord joining the points of tangency?

Ex. 44. Given a line m and a parallelogram of base b and altitude h.

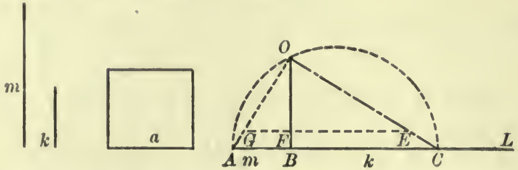
Use Theorem 6 to construct on m as a base a rectangle equal in area to the given parallelogram.

Ex. 45. To pass a circle through two given points and tangent to a given line. (Use XVII. 6 *a*.)

Ex. 46. To transform a parallelogram into an equal square.

Ex. 47. To transform a scalene triangle into an equilateral triangle of the same area.

PROB. IX. *To construct a square that shall be to a given square in a given ratio.*



Given. The square on a and the ratio $k : m$.

Required. To construct a square that shall be to a^2 as k is to m.

Const. On indefinite line AL, lay off $AB = m$ and $BC = k$.
On AC as a diameter describe a semicircle.
At B erect a perpendicular to AC, cutting semicircle in O.
Draw OA and OC and lay off $OG = a$.
Draw $GE \parallel AC$.
OE is the side of the required square.

Q.E.F.

Proof. $\triangle FOG \sim \triangle BOA$ and $\triangle FOE \sim \triangle BOC$. (XV. 2.)

$\therefore\ m : GF :: BO : FO$ (Hom. sides of $\sim\triangle$s.)

and $k : FE :: BO : FO$. (Hom. sides of $\sim\triangle$s.)

$\therefore\ m : GF :: k : FE$ (1). (Ax. 1.)

$\therefore\ m : k :: GF : FE$. (Taking (1) by alt.)

But $GF : FE :: \overline{OG}^2 : \overline{OE}^2$, (XVII. 5 (c).)

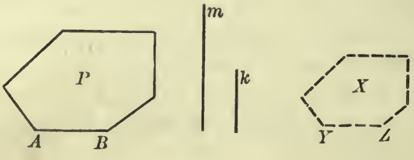
or $m : k :: a^2 : \overline{OE}^2$.

Q.E.D.

PROB. X. *To construct a polygon similar to a given polygon and having a given ratio to it.*

Given. The polygon P, and the ratio $k : m$.

Required. To construct a polygon similar to P and having with P the ratio $k : m$.



Analysis. Suppose $X \sim P$ and $X : P :: k : m$.

If $X \sim P$,
then $X : P :: \overline{YZ}^2 : \overline{AB}^2$; (XVI. 3.)
and, if $X : P :: k : m$,
then $k : m :: \overline{YZ}^2 : \overline{AB}^2$. (Ax. 1.)

This proportion suggests XVII. 5 (*c*) and the preceding problem.

Hence, the construction.

Const. Construct a line $A'B'$ such that the square upon it shall be to the square on AB as $k : m$. (Prob. IX.)

On this line $\overline{A'B'}$, homologous to AB, construct a polygon Q similar to P.

Q is the polygon required.

Q.E.F.

Proof. (Let pupil give proof.)

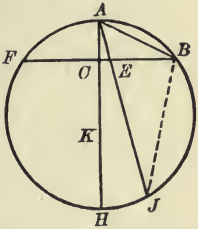
Ex. 48. To transform a scalene triangle into an equal isosceles triangle with a given base angle.

Ex. 49. To construct a square that shall be to a given pentagon as 4 is to 9.

Ex. 50. How does an angle inscribed in a segment compare with the angle between the base of the segment and a tangent at either extremity of the base? State the converse of the proposition.

Ex. 51. A circle is described on a given line AH as a diameter. AB is a fixed chord. BF is drawn perpendicular to AH. AJ is any chord cutting BF in E.

Why is $\angle AJB = \angle ABF$?



Ex. 52. Why, then, is AB tangent to a circle through E, B, and J?

Prove, then, that $\overline{AB}^2 =$ rectangle of $AE \cdot AJ$.

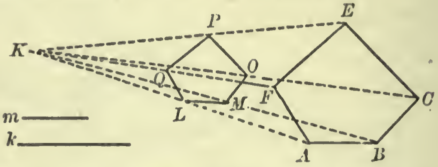
Ex. 53. (*a*) Show that the 4-side $CEHJ$ is cyclic.

(*b*) Noting that $\angle ABH$ is a right angle, and that BC is the altitude on the hypotenuse, give another proof that $\overline{AB}^2 =$ rectangle of $AE \cdot AJ$.

Ex. 54. What is the locus of a point E that divides all chords through A so that EA is a third proportional to AB and the whole chord, say AJ?

PROB. XI. *To construct a polygon whose sides shall be in a given ratio to the sides of a given polygon.*

Given. Any n-gon $ABC \cdots$ and two lines k and m, representing any given ratio.



Required. An n-gon $\sim LMO \cdots$ whose sides are to the sides of n-gon $ABC \cdots$ in the ratio of k to m.

That is, the ratio of similitude of the required to the given n-gon is $k : m$.

Const. Find a fourth proportional to k, m, and any side AB of n-gon $ABC \cdots$. (v. Prob. I.)

Let this line be LM, and draw it parallel to AB at any convenient distance from AB.

Draw AL and BM, and let them intersect at K.

Draw KC, KE, and KF.

Through L draw $LQ \parallel AF$ and terminating on KF.

Similarly, draw QP and PO. Draw OM.

The n-gon $LMO \cdots$ is the n-gon required.

Q.E.F.

Proof. $LM : AB :: k : m$ (Const.)

$:: KM : KB :: KL : KA :: LQ : AF$, etc.

$:: KO : KC$.

$\therefore OM \parallel BC$.

[If a line divides two sides of a $\triangle$ proportionally, etc.] (XV. 1 *a*.)

$\therefore$ n-gon $LMO \cdots \sim n$-gon $ABC \cdots$ (XV. 6 *a*.)

and its ratio of similitude to n-gon $ABC \cdots$ is $k : m$.

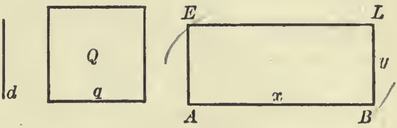
Q.E.D.

Ex. 55. What is locus of J (Fig. for Ex. 53) so taken that the rectangle of $AE \cdot AJ = \overline{AB}^2$, where A is a fixed point in the mid-perpendicular to the given line-segment FB?

Prob. XII. *To construct a rectangle whose area shall equal a given square, and the difference of whose base and altitude shall equal a given line.*

Given. The square Q and the line d.

Required. To construct a rectangle whose area shall equal Q, and the difference of whose base and altitude shall equal d.



Analysis. Suppose ▭ A–L = the rectangle required.

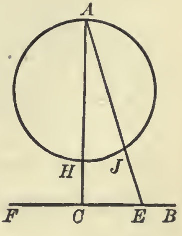
That is, that ▭ A–L = □ Q and that $x - y = d$.

If ▭ A–L = □ Q, then $x \cdot y = q^2$. (1) (XIII. 2.)

Equation (1) suggests any one of the theorems concerning mean proportionals, the most convenient of which for practical purposes is that on the tangent and secant.

Again, equation (1) is indeterminate; but, if we examine it and remember that $x - y = d$, we see that the problem may now be stated: Draw a tangent of a given length (q) to a circle so that the chord of the secant drawn from the extremity of the tangent to the same circle shall equal a given line (d). (XVII. Prob. III.)

We have, therefore, discovered or rediscovered the method of constructing the required rectangle. For the tangent equals the side of the square, the whole secant equals the base, and its external segment equals the altitude, of the required rectangle.



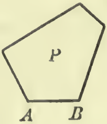
Ex. 56. If AH is the diameter of a circle, and CE is perpendicular to AH produced, prove that the locus of a point J which divides any line from A to BF so that the rectangle of $AE \cdot AJ$ = rectangle of $AH \cdot AC$, is the circle on AH as a diameter.

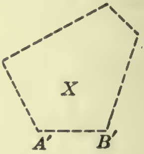
(Note that 4-side $HCJE$ is cyclic.)

Prob. XIII. *To construct a polygon similar to one given polygon and equal in area to another given polygon.*

Given. The polygons P and Q.

Required. To construct a polygon similar to P and equal in area to Q.





Analysis. Suppose polygon $X \sim P$ and $= Q$.

If $X \sim P$, then $P : X :: \overline{AB}^2 : \overline{A'B'}^2$. (XVI. 3.)

In this proportion substitute for X its equal Q, and we have

$$P : Q :: \overline{AB}^2 : \overline{A'B'}^2.$$

This proportion contains but one unknown quantity, namely, $\overline{A'B'}^2$.

The question arises, How shall we construct, geometrically, $\overline{A'B'}^2$?

Now P and Q are dissimilar polygons by hypothesis; but, if we change each to an equal square whose sides are respectively m and k, proportion (1) becomes

$$m^2 : k^2 :: \overline{AB}^2 : \overline{A'B'}^2.$$

$$\therefore\ m : k :: AB : A'B'.$$

That is, $A'B'$ is a fourth proportional to m, k, and AB.

Hence, the construction.

Const. Change P to an equal triangle; also Q. (XIII. Prob. I.)

Find a square equal to each of these triangles. (XVII. Prob. VII.)

Find a fourth proportional to the sides of these squares and AB. (XVII. Prob. I.)

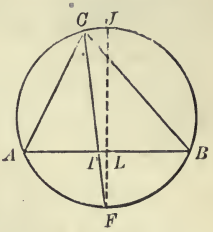
On this fourth proportional construct a polygon similar to P. (XVII. Prob. XII.)

It will be the polygon required.

Q.E.F.

Proof is left to the pupil.

Ex. 57. $\triangle ACB$ is inscribed. CT bisects $\angle ACB$. FJ is a mid-perpendicular to AB.



Why does CT pass through F?

Why is FJ a diameter of the circumcircle?

Why is F the mid-point of arc AB?

Why is TF a fourth proportional to JF, LF, and CF?

Ex. 58. Let $x = TF$, $t = CT$, $s = LF$, and $d = JF$.

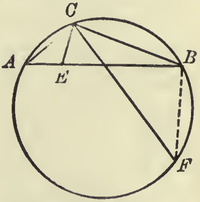
Then $x : d :: s : x + t$, or $x^2 + tx = ds$. Solve for x.

$$\therefore x = -\tfrac{1}{2}(t \pm \sqrt{4 \text{ ▭ of } d \cdot s + \square \text{ on } t}).$$

Construct x.

Ex. 59. By the aid of the foregoing, construct a triangle, having given the base, the vertex angle, and the bisector of the vertex angle.

Ex. 60. Given, $\triangle ABC$ inscribed in a circle. CE and CF are so drawn that



$$\angle ACE = \angle FCB.$$

Prove that $\triangle ACE \sim \triangle FCB$.

Hence, show that

$$CA \cdot CB = CE \cdot CF.$$

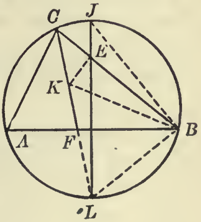
Ex. 61. In the figure, CF bisects $\angle ACB$. Circumscribe a circle to $\triangle ABC$, and produce CF to L.

Show that $\triangle ACF \sim LCB$.

Why, then, is $AC : CF :: CL : CB$?

Ex. 62. Why does the rectangle of $CF \cdot FL$ = rectangle of $AF \cdot FB$? Prove, then, that the rectangle of $AC \cdot CB = \overline{CF}^2 +$ rectangle of $AF \cdot FB$.

Ex. 63. State the above equation in general terms.



Ex. 64. If LJ is a diameter of the circumcircle of $\triangle ABC$, K the center of the incircle, and if KB is drawn, and $KE \perp CB$, show that $\angle ABL = \angle LCB$.

Show that $\angle KBA = \angle KBC$, and therefore that $\angle LBK = \angle LKB$, and therefore $LK = BL$.

Ex. 65. Why is $\triangle LJB$ a right triangle? Why is rt. $\triangle LJB \sim KEC$?

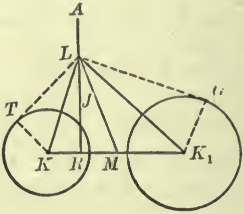
Why, then, does the ▭ of $LJ \cdot KE =$ ▭ of $BL \cdot KC =$ ▭ of $LK \cdot KC$?

Ex. 66. Prove from Exercise 65 that the rectangle of the diameter of the circumcircle of a triangle, and the radius of the incircle, equals the rectangle of the segments of any chord of the circumcircle that passes through the center of the incircle.

Ex. 67. If M is the mid-point of KK_1, and RA is a perpendicular to KK_1 at any point R, then $\overline{LK_1}^2 - \overline{LK}^2 = 2\,KK_1 \cdot RM$. (Why?)

Prove that if L be taken anywhere in the $\perp RA$, say at J,

$$\overline{JK_1}^2 - \overline{JK}^2 = 2\,KK_1 \cdot RM.$$



Ex. 68. Of what point, then, is the $\perp RA$ the locus?

Ex. 69. In the expression

$$\overline{LK_1}^2 - \overline{LK}^2 = 2\square\, KK_1 \cdot RM,$$

construct RM.

(Factor the first member. XVII. Prob. 1.)

Ex. 70. Find, then, the locus of a point the difference of the squares of whose distances from two given points equals a given square.

Ex. 71. If two circles are drawn with any radii, and K and K_1 as centers, and from any point L in a perpendicular to KK_1, as AR, tangents are drawn to these circles, show that

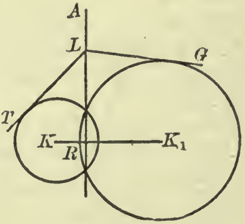
$$\overline{LG}^2 - \overline{LT}^2 = 2\,KK_1 \cdot RM - (r'^2 - r^2).$$

Ex. 72. In the second member of the last equation, is there any term whose value changes as L moves up or down the $\perp AR$?

If, then, the value of $\overline{LG}^2 - \overline{LT}^2$ is the same, or constant, no matter where, in the $\perp AR$, L may lie, construct the locus of a point the difference of the squares of the tangents from which to two given circles equals a given square.

Def. The **Radical Axis** of two circles is the locus of a point from which the tangents to the two circles are equal.

In such case RA is called the *radical axis* of the two circles.



Ex. 73. If $\odot K$ intersect $\odot K_1$, show that the radical axis of the two circles is their common chord produced.

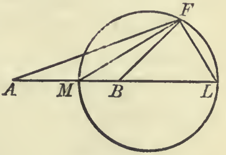
NOTE. — The radical axis of two circles is the locus of the centers of circles which cut the two circles orthogonally.

Ex. 74. In a system of three circles the radical axes concur. For, if tangents be drawn to the three circles from the point where two of the radical axes intersect, these tangents will be equal. Therefore, the third radical axis must pass through this point.

Def. This point of concurrence is called the **Radical Center** of the three circles.

Thus, if RA is so taken that $\overline{LG}^2 - \overline{LT}^2 = 0$, then $\overline{LG}^2 = \overline{LT}^2$, or $LG = LT$.

Ex. 75. If AB is divided internally in the ratio of $a : b$ at the point M, and also divided externally at L in the same ratio, and if on ML as a diameter a circle is described, and the point F is any point in the circle, prove the following relations:



(1) MFL is a right angle.

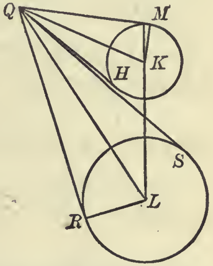
(2) The rectangle of $AM \cdot LB =$ the rectangle of $AL \cdot MB$.

(3) Prove that the circle on ML as a diameter is the locus of a point the ratio of whose distances from A and B is that of $a : b$.

(4) If $AM = MB$, that is, if the ratio $a : b = 1$, what becomes of the point L?

(5) If the ratio $a : b$ is less than one, where does the point L reappear?

Ex. 76. The apparent size of an object is determined by the angle that it subtends at the eye of the observer.



Thus, if at any point Q, the tangents to two unequal ⓢ K and L make equal angles, *i.e.* if $\angle RQS = \angle HQM$, the circles will appear equal, as seen from Q.

Draw the radii LR and KM, and show that, if Q be such a point, $\triangle LQR$ is similar to $\triangle KQM$.

Hence, show that if a and b be the radii of the circles, $QL : QK :: a : b$.

Hence, find the locus of the point from which two unequal circles seem to be equal. (Ex. 68.)

Def. This locus is called the **Circle of Similitude** of the given circles.

XVIII. GROUP ON CIRCUMSCRIBED AND INSCRIBED REGULAR POLYGONS

DEFINITIONS

A **Regular Polygon** is a polygon that is both equiangular and equilateral.

The **Apothem** of a regular polygon is the radius of its inscribed circle.

The **Radius** of a regular polygon is the radius of the circumscribed circle.

The **Center** of a regular polygon is the common center of the inscribed and circumscribed circles.

PROPOSITIONS

XVIII. 1. *If a polygon is regular,*

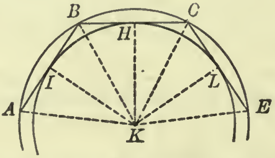
(1) *A circle may be circumscribed about the polygon.*

(2) *A concentric circle may be inscribed in the polygon.*

Hyp. If a polygon

$ABC\text{–}E$

is regular,



Conc.: then (1) a circle may be passed through A, B, C, etc.,
(2) a concentric circle may be described tangent to AB, BC, etc.

Dem. (1) Pass a circle through A, B, and C; let its center be K. Join K to the vertices and to the mid-points of the sides of the polygon (by KA, KB, etc.; KH, KL, etc.).

$$KH \perp BC.$$

[A radius $\perp$ to a chord bisects the chord, etc.] (IX. 1.)

Revolve $KHCE$ on KH as an axis, until it falls on $KHBA$.

The angles at H are right angles, as just shown.

$\therefore$ HC takes the direction of HB. (Ax. 7.)

$HC = HB$ (Const.). $\therefore$ C falls on B.

$\angle HCE = \angle HBA$ and $CE = BA$.

(Def. of regular polygon.)

$\therefore$ first, CE takes the direction BA, and second, E falls on A;

i.e. $KA = KE$;

and the circle through A, B, and C passes through E.

Similarly, this circle may be shown to pass through any other vertex of the polygon.

(2) The sides AB, BC, etc., are equal chords of the same circle.

$\therefore$ these sides are equidistant from the center.

[In the same ⊙ equal chords are equidistant, etc.] (IX. 2.)

$\therefore$ A circle with K as center and a radius $= KL = KH$, etc., is tangent to every side of the polygon.

Q.E.D.

Ex. 1. In a regular n-gon, the central angle is the supplement of any one of the interior angles.

Ex. 2. Divide a regular dodecagon into triangles by drawing radii. Join any two alternate vertices.

Prove, by finding the area of the triangles crossed by the join, that:

The area of a regular dodecagon equals three times the square on the radius.

Ex. 3. Draw a figure showing a circumscribed equilateral polygon that is not regular.

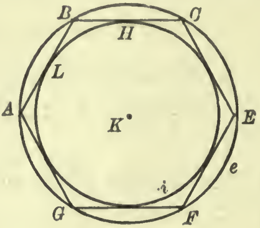
Ex. 4. If a circumscribed polygon is equilateral, the polygon will be regular, provided the number of sides be odd.

(Use IX. 5 and V. 1.)

Ex. 5. Explain why the polygon of Ex. 4 may not be regular if the number of sides be even. That is, show that while there will be two sets of equal angles when the number of sides is even, the angles of one set will not necessarily be equal to those of the other.

XVIII. 2. Conversely. *If a polygon is inscribed in a circle and circumscribed to a concentric circle, the polygon is regular.*

Hyp. If a polygon *ABC–G* is inscribed in a ⊙ *e* and also circumscribed to a concentric ⊙ *i*,



Conc.: then the polygon is regular.

Dem. *AB, BC*, etc., tangents to the ⊙ *i*, are perpendicular to the radii *KL, KH*, etc., of the ⊙ *i*.

[A tangent is ⊥ to the radius through the point, etc.] (IX. 4.)

But these tangents to the ⊙ *i* are chords of the ⊙ *e*. (Hyp.)

Again the distances of these chords from the common center of the ⓢ *i* and *e* are equal. ∴ the chords themselves are equal.

[Equal chords are equidistant from center, etc.] (IX. 2.)

∴ the polygon is equilateral.

Again the arcs *AB, BC*, etc., are all equal.

[In the same ⊙, or in equal ⓢ, equal chords, etc.] (IX. 3 *a*.)

Each angle of the polygon intercepts $(n-2)$ of these equal arcs, if n be the number of sides in the polygon.

∴ each angle of the polygon is measured by $\frac{1}{2}(n-2)$ of these equal arcs.

[An inscribed ∠ is measured by one half, etc.] (XII. 3.)

∴ each angle has the same measure as any other, n being the same for all.

∴ the polygon is equiangular.

∵ the polygon is both equilateral and equiangular, it is regular.

Q.E.D.

XVIII. 2 *a*. *The area of a regular polygon equals one half of the product of its perimeter and apothem.*

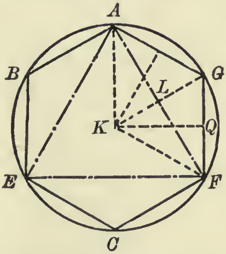
(See XIII. 3.)

XVIII. 3. *If an inscribed polygon is equilateral, the polygon is regular.*

Prove as above that the polygon is equiangular.

XVIII. 4. *If a regular hexagon is inscribed in a circle, the side of the polygon equals the radius of the circle.*

Hyp. If a hexagon AB–G is regular and inscribed in a $\odot K$,



Conc.: then the side of the hexagon equals the radius of the circle.

Dem. Join K to the vertices of the hexagon, and draw the apothem, KQ.

The $\triangle GKF$ is isosceles. ($KG = KF =$ radius of given $\odot$.)

$\angle GKF = \frac{2}{3}$ rt. $\angle$. (Central $\angle$ of a hexagon.)

$\therefore$ each of the two other angles $= \frac{2}{3}$ rt. $\angle$, and GKF is an equilateral triangle.

$\therefore$ $GF = GK =$ the radius.

Q.E.D.

Ex. 6. Draw figures showing inscribed equiangular polygons that are not regular.

Ex. 7. If an inscribed polygon is equiangular, the polygon is regular, provided the number of sides be odd.

(Use IX. 3 *a* and XII. 3.)

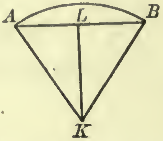
Ex. 8. Explain why the polygon may not be regular if the number of sides be even.

Ex. 9. Construct an angle of 4° 30′.

Ex. 10. Construct an angle of 72°.

Ex. 11. Construct an angle of 24°.

XVIII. 5. *The radius is the limit to which the apothem of the inscribed regular polygon approaches, as the number of sides is increased indefinitely.*



Hyp. If AB be a side of a regular n-gon of apothem KL,

Conc.: then $KL \doteq KA$ as n is indefinitely increased.

Dem. $AK - KL < AL$. (VII. 2 a.)

$AL = \frac{1}{2} AB$.

As n increases, AB diminishes, and may be made as small as we please.

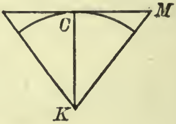
$\therefore AL \doteq 0$. (Def. of limit.)

$\therefore AK - KL \doteq 0$; (Same reason.)

i.e. $KL \doteq AK$. (Same reason.)

Q.E.D.

XVIII. 5 *a*. *The radius of a regular circumscribed* n-*gon approaches as a limit the radius of the inscribed circle as* n *is indefinitely increased.*



Let the student supply the proof, which is exactly similar to the above, using the adjoining figure.

XVIII. 5 *b*. *The square on the apothem approaches as a limit the square on the radius.* (Fig. of Theorem.)

XVIII. 6. *The circumference is the common limit to which the perimeters of similar inscribed and circumscribed regular polygons approach as the number of sides is increased indefinitely.*

Hyp. If C is the circumference of a circle, of radius r, and P and P' are the perimeters of the regular circumscribed and similar inscribed n-gons,

Conc.: then, as n is indefinitely increased,

$$P \doteq C \text{ and } P' \doteq C.$$

Dem. Let AB and EH be the sides of the n-gons; KM and KL, the apothems.

Then $\quad P : P' :: KM : KL.$ (XVII. 2.)

$\therefore P - P' : P :: KM - KL : KM.$ (XI. 1. Sch. Conc. (4).)

$\therefore (P - P') \cdot KM = P \cdot (KM - KL) = P \cdot (r - KL).$ (XI. 1.)

$\therefore P - P' = P \cdot \dfrac{r - KL}{KM}.$ (Ax. 3.)

But $\quad r - KL \doteq 0$ (XVIII. 5.),

and P decreases as n increases. (Why ?)

$\therefore P \cdot \dfrac{r - KL}{KM} \doteq 0$; *i.e.* $P - P' \doteq 0.$ (Def. Limit.)

Now $\quad C > P'$, and $C < P.$ (Why ?)

$\therefore P - C < P - P'.$

$\therefore$ since $\quad P - P' \doteq 0$, $P - C \doteq 0$, and $P \doteq C.$

Similarly, $C - P' < P - P'$, $C - P' \doteq 0$ and $P' \doteq C.$

Q.E.D

XVIII. 6 *a*. *The circle is the common limit to which the areas of similar inscribed and circumscribed regular polygons approach as the number of sides is increased indefinitely.*

XVIII. 7. *The area of a circle equals one half the product of its circumference and radius.*

Let the pupil supply the proof by using 2 *a* and 6 *a*.

Ex. 12. Construct an angle of 6°.

Ex. 13. The perimeter of an inscribed square is 40 ft. What is the radius of the circle ?

Ex. 14. The perimeter of a square is 40 ft. What is the length of the apothem of the square ?

What is the length of the radius of the square ?

XVIII. SUMMARY OF PROPOSITIONS IN THE GROUP ON CIRCUMSCRIBED AND INSCRIBED REGULAR POLYGONS

1. *If a polygon is regular,*

   (1) *A circle may be circumscribed about the polygon;*
   (2) *A concentric circle may be inscribed in the polygon.*

2. CONVERSELY. *If a polygon is inscribed in a circle and circumscribed to a concentric circle, the polygon is regular.*

   a *The area of a regular polygon equals one half of the product of its perimeter and apothem.*

3. *If an inscribed polygon is equilateral, the polygon is regular.*

4. *If a regular hexagon is inscribed in a circle, the side of the polygon equals the radius of the circle.*

5. *The radius is the limit to which the apothem of the inscribed regular polygon approaches as the number of sides is increased indefinitely.*

   a *The radius of a regular circumscribed* n*-gon approaches as a limit the radius of the inscribed circle as* n *is indefinitely increased.*
   b *The square on the apothem approaches as a limit the square on the radius.*

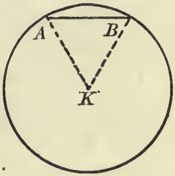
6. *The circumference is the common limit to which the perimeters of similar inscribed and circumscribed regular polygons approach as the number of sides is increased indefinitely.*

a *The circle is the common limit to which the areas of similar inscribed and circumscribed regular polygons approach as the number of sides is increased indefinitely.*

7. *The area of a circle equals half the product of its circumference and radius.*

PROBLEMS

Prob. I. *To inscribe a regular hexagon in a given circle.*



Given. The $\odot K$.

Required. To inscribe a regular hexagon in $\odot K$.

Analysis. Suppose AB is a side of a regular hexagon inscribed in $\odot K$.

Then $AB = AK$, the radius of the circle. (XVIII. 4.)
Hence, the construction.

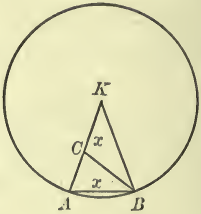
Const. With any point in the circumference as a center, apply the radius of the circle from this starting point six times as a chord; a regular hexagon is thus formed.

Q.E.D.

Note 1.—The joins of the alternate points will form an inscribed equilateral triangle.

Note 2.—The joins of the vertices of the regular hexagon, with the mid-points of the arcs subtended by the sides of the hexagon, will form a regular inscribed dodecagon.

Prob. II. *To inscribe a regular decagon in a given circle.*



Given. The $\odot K$.

Required. To inscribe a regular decagon in $\odot K$.

Analysis. Suppose x is a side of a regular inscribed decagon.

If x be a side of a regular inscribed decagon, then if radii KA and KB be drawn, $\angle K = \frac{1}{10}$ of 4 rt. $\angle$s, or $\frac{2}{5}$ rt. $\angle$, and each base angle $= \frac{4}{5}$ rt. $\angle$. Again, if BC bisect $\angle ABK$, then $\triangle ABC$ and $\triangle BCK$ are isosceles.

Now $\triangle ACB$ is similar to $\triangle AKB$. (XV. 4.)

$\therefore$ KB (or r) : AB (or x) : : BC (or its equal, x) : AC. (Def. of $\sim$ n-gons.)

That is, $r : x :: x : AC$,

or x is the greater segment of the radius divided in extreme and mean ratio.

Suggested theorems, etc., are IV. Ex. 18, and XV. 2.

Applicable theorems: both the above.

Const. Divide the radius KA in extreme and mean ratio.

The greater segment is the side of the required regular decagon.

Proof. $r : x :: x : CA$, or $KA : AB :: AB : CA$. (Const.)

$\therefore$ KAB is similar to ACB. (XV. 4.)

$\therefore$ ACB is isosceles, and BCK is isosceles.

$\therefore \angle K = \angle CBK = \frac{1}{2} ACB$; (III. 2 *a*.)

that is, $\angle K = \frac{1}{2} KAB$.

$\therefore \angle K = \frac{2}{5}$ rt. $\angle$.

Q.E.D.

Ex. 15. Find the cost, at $2.30 a yard, of building a stone wall around a lot, in the shape of a regular hexagon, containing 260 sq. yds.

Ex. 16. The perimeters of a regular hexagon and a regular octagon are each 240 ft. What is the difference in their areas?

Ex. 17. What is the area of a garden in the shape of a regular decagon, one side of which is 18 ft. long?

Ex. 18. The perimeter of a regular hexagon is 42 ft. What is its area?

Ex. 19. The perimeter of a regular hexagon is 30 ft. What is the length of the radius of the hexagon?

Ex. 20. Two regular octagons contain 108 sq. ft. and 96 sq. ft., respectively. What is the length of the side of a third regular octagon equal in area to the sum of the first two?

Ex. 21. A regular decagon is inscribed in a circle whose radius is 10 ft. Find its area. (Use Prob. II, p. 218.)

Ex. 22. A lawn in the shape of a regular octagon measures 186 ft. on each side. What is its area?

Ex. 23. If the sides of three regular decagons are 3 ft., 4 ft., and 12 ft., respectively, what is the side of a regular octagon whose area is equal to the sum of the three given regular decagons?

Let r be the radius of any regular polygon, s the side of the polygon, p the apothem, and A the area. Show that:

Ex. 24. In a square $p = \frac{r}{2}\sqrt{2}$; $A = 2\,r^2$.

Ex. 25. In an equilateral triangle $p = \frac{r}{2}$; $A = \frac{3\,r^2}{4}\sqrt{3}$.

Ex. 26. In a regular hexagon $p = \frac{r}{2}\sqrt{3}$; $A = \frac{3\,r^2}{2}\sqrt{3}$.

Ex. 27. In an equilateral triangle $r = 2\,p$; $s = 2\,p\sqrt{3}$; $A = \frac{1}{4}\,s^2\sqrt{3}$.

Ex. 28. In a regular octagon $s = r\sqrt{2 - \sqrt{2}}$.

Ex. 29. In a regular decagon $s = \frac{r}{2}(\sqrt{5} - 1)$.

Ex. 30. In a regular hexagon $A = \frac{3\,s^2}{2}\sqrt{3}$.

Ex. 31. In a regular dodecagon $A = 3\,r^2$.

Ex. 32. In a regular octagon $A = 2\,r^2\sqrt{2}$.

Ex. 33. The square on the side of the inscribed equilateral triangle equals three times the square on the side of the inscribed regular hexagon.

NOTE. — Let $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$.

Ex. 34. In a square $r = 14$ ft. Find p and A.

Ex. 35. Find the area of an equilateral triangle inscribed in a circle whose radius is 10 ft.

Ex. 36. Find the apothem of a regular hexagon whose side is 12 meters.

Ex. 37. Find the side of an equilateral triangle whose apothem is 6 ft.

Ex. 38. Find the radius of a regular octagon whose side is 4 ft.

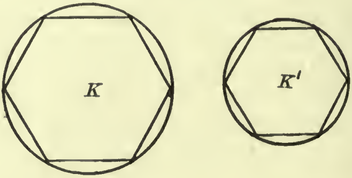
Ex. 39. What is the side of a regular decagon whose radius is 10 ft.?

Ex. 40. The area of a regular hexagon is 1732 sq. ft. What is the length of each side?

XIX. GROUP ON THE AREA OF THE CIRCLE

PROPOSITIONS

XIX. 1. *The circumferences of two circles are to each other as their radii and as their diameters.*



Hyp. If the circumferences of any two circles be denoted by C and C', and their radii by r and r',

Conc.: then $\frac{C}{r}=\frac{C'}{r'}$ and $\frac{C}{2r}=\frac{C'}{2r'}$.

Dem. Inscribe in the circles regular polygons of the same number of sides.

These two polygons will be similar. (XV. 8.)

Denoting the perimeters of the polygons by P and P' we have

$$\frac{P}{r}=\frac{P'}{r'}. \qquad \text{(XVII. 2.)}$$

Now, if the number of sides be increased indefinitely, P and P' will approach their limits C and C'. That is, $\frac{P}{r}$ and $\frac{P'}{r'}$, while remaining equal, will approach as their limits $\frac{C}{r}$ and $\frac{C'}{r'}$.

$$\therefore \frac{C}{r}=\frac{C'}{r'}. \qquad \text{(Postulate of Limits.)}$$

Dividing by 2 we have

$$\frac{C}{2r}=\frac{C'}{2r'}.$$

Q.E.D.

XIX. 1 *a*. *The ratio of the circumference to the diameter is constant.*

Dem. If C be the first circumference and C' the second, and r and r' the respective radii,

then $$\frac{C}{2r}=\frac{C'}{2r'}. \qquad \text{(XIX. 1.)}$$

That is, the ratio $\frac{C}{2r}$ is the same whatever the size of the circle. In other words,

$$C \div 2r \text{ is a constant.}$$

Q.E.D.

Sch. This constant is called π.

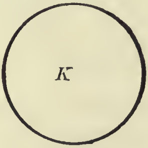
[π is the initial letter of the Greek word for perimeter.]

Hence, $$\frac{C}{2r}=\pi, \text{ or } C=2\pi r.$$

If $$r=1, \quad \pi=\frac{C}{2}.$$

XIX. 2. *The area of a circle is* πr^2.

Hyp. If K be a circle of radius r,



Conc.: then the area of this circle $=\pi r^2$.

Dem. The area of the circle $=\odot\times\frac{1}{2}r$.

[The circumference is the common limit to which the perimeters, etc.] (XVIII. 6.)

But the circumference $=2\pi r$. (XIX. 1. Sch.)

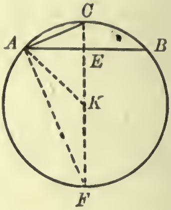
$\therefore$ the area of the circle $=2\pi r\times\frac{r}{2}$

$=\pi r^2$.

Q.E.D.

XIX. 2 *a*. *The areas of two circles are to each other as the squares of the radii and as the squares of the diameters.*

XIX. 3. PROB. *Given the length of a side of a regular polygon, inscribed in a circle of radius* r, *to find the side of a regular polygon of double the number of sides, inscribed in the same circle.*



Given. The value a of a side AB of a regular n-gon inscribed in a circle of radius r.

Required. To find the side of the regular $(2n)$-gon inscribed in the same circle.

Const. Draw FKE the mid-perpendicular of AB, intersecting the circumference at C; draw AC. Draw AK and AF.

AC will be the side of the regular $(2n)$-gon. (Why?)

It is required to compute the value of AC in terms of a.

Computation. In the rt. $\triangle AEK$, $\overline{EK}^2 = \overline{AK}^2 - \overline{AE}^2$ (XIV. 1 *a*.)

that is,
$$\overline{EK}^2 = r^2 - \left(\frac{a}{2}\right)^2 = r^2 - \frac{a^2}{4} = \frac{4r^2 - a^2}{4}.$$

$$\therefore\ EK = \sqrt{\frac{4r^2 - a^2}{4}} = \tfrac{1}{2}\sqrt{4r^2 - a^2}.$$

Again, $\angle CAF$ is a right angle. (XII. 2 *a*. Sch.)

$$\therefore\ \overline{AC}^2 = CF \cdot CE. \quad \text{(XVII. 5 } b.)$$

But $CF = 2r;\ CE = CK - EK = r - EK.$

$$\begin{aligned}\therefore\ \overline{AC}^2 = CF \cdot CE &= 2r(r - EK)\\ &= 2r(r - \tfrac{1}{2}\sqrt{4r^2 - a^2})\\ &= 2r^2 - r\sqrt{4r^2 - a^2}.\end{aligned}$$

$$\therefore\ AC = \sqrt{2r^2 - r\sqrt{4r^2 - a^2}}.$$

Q.E.F.

SCH. The perimeter of the regular $2n$-gon $= 2nAC = 2n\sqrt{2r^2 - r\sqrt{4r^2 - a^2}}$. It is most convenient to take $r = 1$, when the expression becomes $2n\sqrt{2 - \sqrt{4 - a^2}}$.

XIX. 4. PROB. *To compute the value of π approximately.*

Sol. $\pi = \frac{\odot}{2r}$ (XIX. 1. Sch.). Hence, if $r = 1$, $\pi = \frac{1}{2}\odot$.

Let P_n denote the perimeter of a polygon of n-sides.

Beginning with the value $n = 6$, we find by XIX. 3. Sch.:

$$\begin{aligned} P_6 &= 6. \\ P_{12} &= 6.2116571 \\ P_{24} &= 6.2652572 \\ P_{48} &= 6.2787004 \\ P_{96} &= 6.2820640 \\ P_{192} &= 6.2829051 \\ P_{384} &= 6.2831154 \\ P_{768} &= 6.2831694 \end{aligned}$$

We may continue this operation as far as we please, but for all practical purposes the perimeter of the polygon of 768 sides coincides with the circle.

$$\therefore\ \pi = 6.283169 \div 2 = 3.14159 \text{ (nearly).}$$

Q.E.F.

NOTE. — Lambert (1750) proved that π is incommensurable with 1.

Lindemann (1882) went further and showed that π cannot be expressed algebraically.

Ex. 1. The minute hand of a tower clock is 6 ft. long. What is the length of the circumference of the clock ?

Ex. 2. What is the area of the face of the clock ?

Ex. 3. What is the circumference of a circle whose diameter is 10 in. ?

Ex. 4. What is the diameter of a circle whose circumference is 27 ft. 8 in. ?

Ex. 5. The minute hands of two tower clocks are respectively 6 ft. and 8 ft. long. What is the ratio between the lengths of the circumferences ?

Ex. 6. What is the ratio between their areas ?

Ex. 7. What is the length of an arc of 36° in a circle whose diameter is 24 in. ?

Ex. 8. What is the diameter of a circle whose area is 40 A. ?

XIX. SUMMARY OF PROPOSITIONS IN THE GROUP ON THE AREA OF THE CIRCLE

1. *The circumferences of two circles are to each other as their radii and as their diameters.*

   a *The ratio of a circumference to its diameter is constant.*

SCH. The circumference of a circle is $2\pi r$.

   b *The areas of any two circles are to each other as the squares of the radii, and as the squares of the diameters.*

2. *The area of a circle is* πr^2.

3. PROB. *Given the length of a side of a regular polygon, inscribed in a circle of radius* r, *to find the side of a regular polygon of double the number of sides, inscribed in the same circle.*

SCH. Application of the foregoing problem to the calculation of perimeters.

4. PROB. *To compute the value of* π *approximately.*

Ex. 9. The radius of a circle is 8 ft. What is the radius of a circle 100 times as large?

Ex. 10. A wheel makes 420 revolutions in traveling half a mile. What is its diameter?

Ex. 11. To find a circle whose circumference is two thirds a given circumference.

Ex. 12. A ten-inch water pipe discharges 200 gallons a minute. What is the diameter of a pipe that discharges 800 gallons a minute under the same pressure?

Ex. 13. A circular pipe 10 in. in diameter delivers 100 gallons per minute. If the capacity is to be increased fourfold, what will be the diameter of the new pipe?

Ex. 14. The radius of a circle is 12 in., and the length of an arc is the same. What is the angle subtended at the center by the arc?

Ex. 15. The minute hand of a clock is 6 ft. long. How long is the arc described by the hand in 10 min.?

Ex. 16. What is the ratio between the central angles of two circles of unequal radii?

Ex. 17. In a clock whose minute hand is 6 ft. long, how many degrees in the central angle that subtends an arc 10 ft. long?

Ex. 18. Having found the value of the central angle in the preceding question, by what proportion may you determine the central angle subtending an arc of 10 ft. in a clock whose minute hand is 8 ft. long?

Ex. 19. In a square closet whose side measures 28 ft. is to be made a circular shelf 1 ft. wide, with its circumference touching the walls of the room. Find the area of the shelf.

Ex. 20. The area of a circular mirror is 314 sq. in. The frame is 5 in. wide. If $\pi = \frac{22}{7}$, what is the area of the frame?

Ex. 21. To construct a circle equal to the difference between two given circles of radii a and b, respectively.

Ex. 22. To construct a circle equal to the sum of two given circles, of radii a and b, respectively.

Ex. 23. To construct a circle equal to the sum of several circles of radii a, b, c, e, etc.

Ex. 24. The perimeters of an equilateral triangle, a square, and a circle are each equal to 264 ft. Compare the areas of the three figures.

Ex. 25. The area of a square is 256 sq. ft. What is the area of the circle inscribed in the square?

Ex. 26. What is the ratio of the area of a circle to that of the inscribed regular hexagon?

Ex. 27. What is the area of a circle if the area of the inscribed regular hexagon is 17.32 sq. ft.?

Ex. 28. The span (chord) of an arch in a doorway in the form of a circular arc is 26 ft., and its height above the stone piers is 5 ft. What is the radius of the circle?

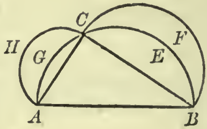
Ex. 29. The altitude of a segment is 3 ft.; the radius of the circle is 6 ft. 2 in. Find the base (chord) of the segment.

Ex. 30. The length of an arc is 42 ft., and the radius of the circle is 29 ft. What is the area of the sector?

Ex. 31. In a given equilateral triangle, describe three circles tangent to the sides of the triangle and to each other.

Ex. 32. Semicircles are described on the sides of a right triangle as shown in the figure.

What is the expression for the area of each?



Ex. 33. How does the sum of the areas of the smaller semicircles compare with the area of the largest?

Ex. 34. If a segment AGC be subtracted from the semicircle on AC, what area remains?

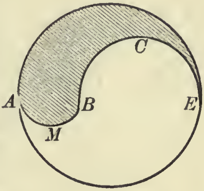
If segment CEB be subtracted from the semicircle on BC, what area remains?

Ex. 35. Hence, show that area rt. $\triangle\, ACB =$ area $AHCG +$ area $BFCE$.

NOTE. — These curvilinear figures are called the *Lunulæ of Hippocrates* (470 B.C.).

This is the first equation established between rectilinear and curvilinear areas.

Ex. 36. In the figure, the diameter $AE = 6$, $AB = 2$, and $BE = 4$. What is the ratio of the circumference on AE to that on BE?



Ex. 37. What is the ratio of the semicircumference on AE to that on BE?

Ex. 38. What is the length of the diameter of a circumference equal to the circumference on BE plus the circumference on AE?

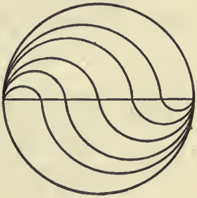
Ex. 39. Prove that the semicircumference on AE equals the sum of the semicircumferences on AB and BE; that is, that the curve $AMBCE$ equals the semicircumference on AE.

Ex. 40. What is the area of AMB? Of BCE?

Ex. 41. How much, then, is added to the upper semicircle by AMB? How much is subtracted from the same semicircle by BCE?

Ex. 42. What, then, is the area of the shaded horn?

Ex. 43. What is the area of the unshaded horn?



Ex. 44. In the figure the diameter is divided into six equal parts.

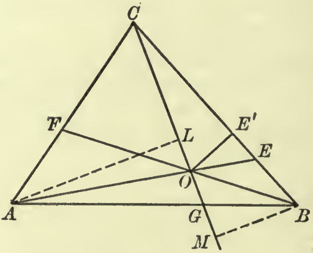
Prove that the contour or boundary line of any one of the figures equals that of any other — the figures to lie between two consecutive lines.

Ex. 45. Similarly with the areas.

XX. GROUP ON CONCURRENT TRANSVERSALS AND NORMALS

PROPOSITIONS

XX. 1. *If three transversals through the vertices of a triangle are concurrent, the product of one set of three alternate segments determined by the transversals on the sides of the triangle equals the product of the other set, and conversely.*



Hyp. If, in the $\triangle ABC$, AE, BF, and CG concur,

Conc.: then $GB \cdot EC \cdot FA = AG \cdot BE \cdot CF$.

Dem. Draw AL and BM, the altitudes of $\triangle$s AOC and BOC, respectively.

The base CO is common to the triangles.

Then $\triangle AOC : \triangle BOC :: AL : BM$. (1)

[$\triangle$s with equal bases are to each other, etc.] (XIII. 1 *c*. Sch. 1.)

But $\triangle ALG \sim \triangle BMG$.

[They are right triangles, and $\angle LGA = \angle BGM$.] (XV. 2 *a*.)

$$\therefore\ AL : BM :: AG : GB. \tag{2}$$

$$\therefore\ \triangle AOC : \triangle BOC :: AG : GB,$$

or

$$\frac{\triangle AOC}{\triangle BOC} = \frac{AG}{GB}. \tag{3}$$

Similarly,

$$\frac{\triangle BOA}{\triangle COA} = \frac{BE}{EC}, \tag{4}$$

and

$$\frac{\triangle COB}{\triangle AOB} = \frac{CF}{FA}. \tag{5}$$

Multiplying the equations (3), (4), and (5) together, member by member, and canceling in the first member, we obtain

$$1 = \frac{AG}{GB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA},$$

or

$$GB \cdot EC \cdot FA = AG \cdot BE \cdot CF.$$

Q.E.D.

Conversely. If

$$GB \cdot EC \cdot FA = AG \cdot BE \cdot CF,$$

Conc.: then AE, BF, and CG concur.

Dem. If the transversals do not concur, at least two of them, say BF and CG, will meet, say at O.

Join A and O, and suppose that this join, instead of meeting BC in E, meets it in E'.

Then $\dfrac{AG}{GB} \cdot \dfrac{BE'}{E'C} \cdot \dfrac{CF}{FA} = 1$, by the theorem;

and $\dfrac{AG}{GB} \cdot \dfrac{BE}{EC} \cdot \dfrac{CF}{FA} = 1$, by hypothesis.

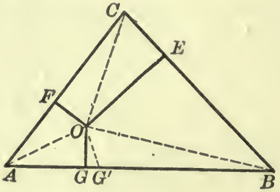
$\therefore\ \dfrac{BE'}{E'C} = \dfrac{BE}{EC}$, which is impossible, for if $BE' > BE$, then $E'C < EC$ (see figure), and the first fraction is greater than the second. If $BE' < BE$, then $E'C > EC$, and the first fraction is less than the second.

$\therefore\ E'$ must coincide with E, and AE, BF, CG concur.

Q.E.D.

XX. 2. *Three concurrent perpendiculars divide the sides of a triangle so that the sum of the squares of one set of alternate segments equals the sum of the squares of the other set, and conversely.*

Hyp. If OE, OF, OG are concurrent ⊥s on the sides of the $\triangle ABC$,



Conc.: then $\overline{AG}^2 + \overline{BE}^2 + \overline{CF}^2 = \overline{GB}^2 + \overline{EC}^2 + \overline{FA}^2$.

Dem. Join O with A, B, and C.

$\overline{AO}^2 - \overline{AG}^2 = \overline{GO}^2 = \overline{BO}^2 - \overline{GB}^2$. (XIV. 1 *a*, and Ax. 1.)

Similarly on the other sides.

$$\therefore\ \overline{AG}^2 - \overline{BG}^2 + \overline{BE}^2 - \overline{EC}^2 + \overline{CF}^2 - \overline{FA}^2 =$$
$$\overline{AO}^2 - \overline{BO}^2 + \overline{BO}^2 - \overline{CO}^2 + \overline{CO}^2 - \overline{AO}^2 = 0.$$

Transposing,

$$\overline{AG}^2 + \overline{BE}^2 + \overline{CF}^2 = \overline{GB}^2 + \overline{EC}^2 + \overline{FA}^2.$$

Q.E.D.

Conversely. If

$$\overline{AG}^2 + \overline{BE}^2 + \overline{CF}^2 = \overline{GB}^2 + \overline{EC}^2 + \overline{FA}^2,$$

Conc.: then perpendiculars to the sides of the triangle at E, F, and G concur.

Dem. Suppose the perpendiculars do not concur; that those at F and E meet at O, while the perpendicular from O to AB meets AB at G'.

Then $\overline{AG'}^2 + \overline{BE}^2 + \overline{CF}^2 = \overline{G'B}^2 + \overline{EC}^2 + \overline{FA}^2$. (1)

(By the direct Th.)

$\overline{AG}^2 + \overline{BE}^2 + \overline{CF}^2 = \overline{GB}^2 + \overline{EC}^2 + \overline{FA}^2$. (2)

(By Hyp.)

Subtracting (2) from (1), member from member,

$$\overline{AG'}^2 - \overline{AG}^2 = \overline{G'B}^2 - \overline{GB}^2,$$

which is impossible unless $G \equiv G'$; for if $G \not\equiv G'$, the first member is + and the second is −, or *vice versa*, and a + quantity cannot equal a − quantity.

∴ the perpendicular from O on AB must fall at G; *i.e.* the three perpendiculars to the sides at E, F, and G must concur.

Q.E.D.

Ex. 1. Show, by using XX. 1, that the following sets of angle-transversals of a triangle are concurrent:

(*a*) The medians.

(*b*) The altitudes.

(*c*) The joins of the vertices to the points of contact of the inscribed circle.

(*d*) The bisectors of the interior angles.

(*e*) The bisectors of two exterior angles and an interior angle not adjacent to either of these exterior angles.

Ex. 2. Show that the perpendiculars erected to the sides of a triangle at the points of contact of the escribed circles are concurrent.

Ex. 3. Show from XX. 2 that if each of three circles intersect both the others, the three common chords are concurrent.

XX. SUMMARY OF PROPOSITIONS IN THE GROUP ON CONCURRENT TRANSVERSALS AND NORMALS

1. *If three transversals through the vertices of a triangle are concurrent, the product of one set of three alternate segments determined by the transversals on the sides of the triangle equals the product of the other set, and conversely.*

2. *Three concurrent perpendiculars divide the sides of a triangle so that the sum of the squares of one set of alternate segments equals the sum of the squares of the other set, and conversely.*

SOLID GEOMETRY

XXI. GROUP ON THE PLANE AND ITS RELATED LINES

DEFINITIONS

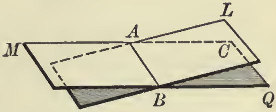
A **Plane** has already been defined to be a surface such that if any two of its points be joined by a straight line, this line will lie wholly within the surface.

A plane is said to be *determined* when it fulfills such conditions that its position is fixed.

No two planes can fulfill the same set of determining conditions without coinciding throughout their whole extent.

COROLLARIES OF THE DEFINITION

(*a*) *A straight line and a point without the line determine a plane.*



Dem. Let AB be the given line and C the given point.

Let MQ be *any* plane through AB.

Revolve MQ on AB as an axis, until it contains C.

Let this position of the plane be BL.

The plane BL is fixed.

For if it be revolved in either direction about AB, it will no longer contain C.

Q.E.D.

(b) *Three points not in the same straight line determine a plane.*

Dem. Join any two of the points; apply Cor. (*a*).

(c) *Two intersecting lines determine a plane.*

(d) *Two parallels determine a plane.*

(e) *The plane determined by a line and a point is identical with the plane determined by this line and the parallel to it that contains the given point.*

(*f*) *A straight line cannot intersect a plane in more than one point.*

(g) *If four points, A, B, C, and E, are not in the same plane* (i.e. *are not coplanar*), *no three of the points can be collinear.*

Dem. If, for example, A, B, and C lie in the same straight line, then this line determines with E a plane, *i.e.* all four points are coplanar, which contradicts the hypothesis.

∴ no three of the points can be collinear.

Q.E.D.

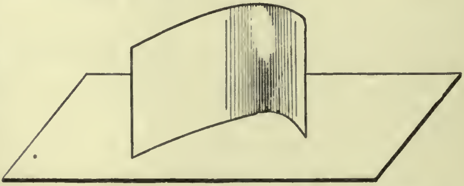
The point in which a line meets a plane is called the **Foot of the Line.**

A **Perpendicular to a Plane** is a line perpendicular to every line of the plane that passes through its foot.

The **Projection of a Point** on a plane is the foot of the perpendicular from the point to the plane.

The **Projection of a Line** (straight or curved) on a plane is the locus of the projections of the points of the line.

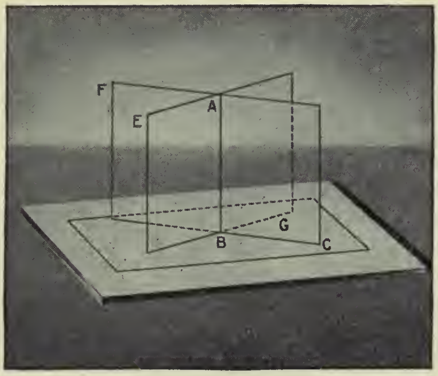
The **Angle** that a straight line makes with a plane is the angle formed by the line and its projection on the plane.



A line is **parallel to a plane** when any plane through the line intersects the given plane in a line parallel to the given line.

PROPOSITIONS

XXI. 1. *The line of intersection of two planes is straight.*



Hyp. If plane CF intersects plane EG in the line AB,

Conc.: then AB is a straight line.

Dem. If AB is not a straight line, it must contain at least a third point that is not in the same straight line with A and B.

But three points not in the same straight line determine the position of a plane. (Def. of Plane, Cor. (*b*).)

That is, if A, B, and the third point are not in the same straight line, planes FC and EG coincide.

But this conclusion is contrary to the hypothesis.

$\therefore$ AB is a straight line.

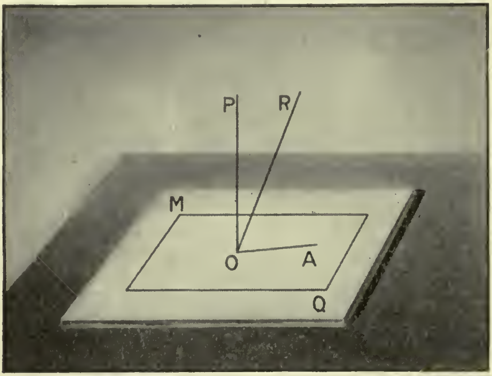
Q.E.D.

Ex. 1. To make sure that a surface is perfectly "flat," a mechanic applies his "straightedge" to the surface in various directions and sees that the "straightedge" touches the surface along its whole length in every position. On what definition is his action based?

Ex. 2. How may three points be so situated that more than one plane may be passed through them?

Ex. 3. Show that four different planes may be passed so as to contain three out of four given points, if no three of the points be collinear. In how many planes would each of the four given points lie?

XXI. 2. *If through a given point a perpendicular is drawn to a plane, it is the only perpendicular that can be drawn through the point to the plane.*



Hyp. Case I. If O is a given point in plane MQ, and PO is perpendicular to plane MQ,

Conc.: then PO is the only perpendicular that can be drawn to plane MQ at O.

Dem. If possible, let RO be a second $\perp$ to MQ at O.

Let the plane of PO and RO intersect the plane MQ in OA.

Then RO must be $\perp$ to OA. (Def. of $\perp$ to a plane.)

But PO is perpendicular to OA. (Def. of $\perp$ to a plane.)

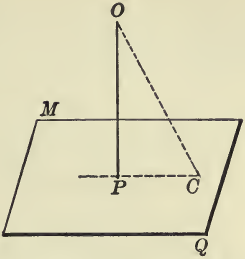
$\therefore$ we have OA in plane of PO and RO perpendicular to PO and RO.

But this conclusion is impossible. (Ax. 7. Direct Inf.)

$\therefore$ PO is the only perpendicular that can be drawn to plane MQ at O.

Q.E.D.

Ex. 4. Hold two pencils so as to show that if two lines do not intersect, and are not parallel, a plane cannot contain both of them.



Hyp. CASE II. If O is a given point without the plane MQ, and PO is perpendicular to plane MQ,

Conc.: then PO is the only perpendicular that can be drawn to plane MQ through O.

Dem. If possible, let OL be a second perpendicular to MQ through O.

Draw LP.

Then, in the plane OPL, we have OP and OL each perpendicular to PC.

But this conclusion is impossible. (Ax. 7. Direct Inf.)

$\therefore$ PO is the only perpendicular that can be drawn to MQ through O.

Q.E.D.

Ex. 5. Why are the projections of straight lines on a plane always straight?

Show that if the projection of a line on a plane is straight, the line need not be straight.

Ex. 6. Show how a circle may be so situated with respect to a plane that its projection on the plane will be a straight line.

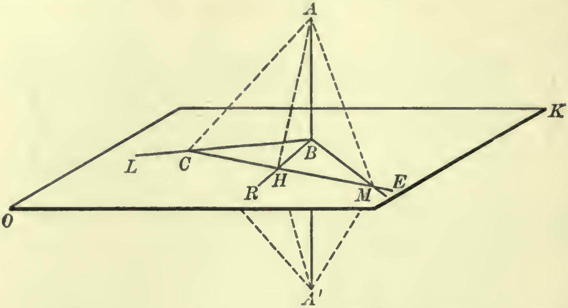
Ex. 7. Show that if the projection of the line AB on each of two intersecting planes be straight, the line itself must be straight.

Ex. 8. Why does a three-legged stool always stand firmly on a level floor, while a table or chair with four legs may be unsteady?

Ex. 9. Show that if two lines lie in the same plane, they must either intersect or be parallel.

Ex. 10. Show that if four lines concur, the greatest number of planes that can be determined by the lines two and two is six.

XXI. 3. *A line perpendicular to two lines at their intersection is perpendicular to the plane of these lines.*



Hyp. If AB is perpendicular to BM and BL at B, and if OK is the plane determined by BM and BL, and if BR is any other line of OK through B,

Conc.: then $AB \perp BR$, or AB is perpendicular to plane OK.

Dem. Draw EC, any line of OK.
Suppose EC cuts BL in C, BR in H, and BM in E.
Produce AB to A', making $BA' = AB$.
Draw AC, AH, AE, and $A'C$, $A'H$, $A'E$.
EB and CB are mid-perpendiculars to AA'. (Const.)

$$\therefore\ EA = EA',\ CA = CA'.$$

EC is common to the $\triangle$s AEC, $A'EC$.

$\therefore\ \triangle AEC \cong \triangle A'EC.$ (V. 3.)

$\therefore\ \angle ACH = \angle HCA'.$ (Hom. $\angle$s of $\cong$ $\triangle$s.)

$\therefore\ \triangle ACH \cong \triangle HCA'.$ (V. 1.)

$\therefore\ HA = HA'.$ (Hom. sides of $\cong$ $\triangle$s.)

$\therefore$ two points (B and H) of BR are equidistant from the ends of AA'.

$\therefore$ RB is a mid-perpendicular to AA'.

$\therefore$ AB is perpendicular to any line of OK passing through B.

$$\therefore\ AB \perp \text{plane } OK.$$

Q.E.D.

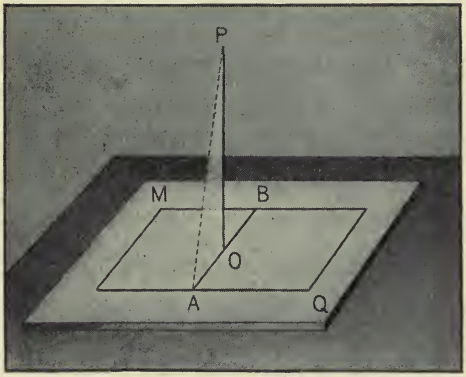
XXI. 3 *a.* *If three lines are perpendicular to a given line at a given point, these perpendiculars lie in a plane perpendicular to the given line at the given point.*

XXI. 3 *b.* *The plane mid-normal to the join of two points is the locus of all points equidistant from the given points.*

XXI. 3 *c.* *The plane through a given point perpendicular to a given line, is unique, whether the given point be on the given line or without the given line.*

XXI. 4. *Of all straight lines drawn from a given point to a given plane,*

(1) *The perpendicular is the shortest line, and conversely.*



Hyp. If PO is perpendicular to plane MQ, and PA is any other line from P to plane MQ,

Conc.: then $PO < PA$.

Dem. Draw AO.

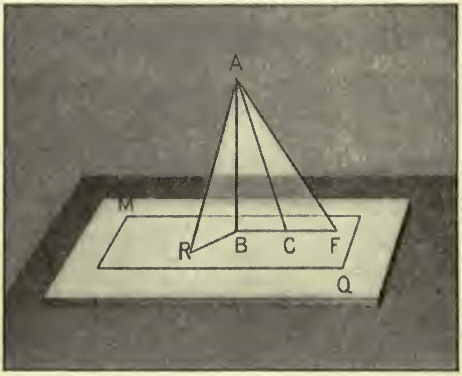
In rt. $\triangle POA$, $PO < PA$. (IV. 4 *a.* Sch.)

$\therefore PO < PA$.

Q.E.D.

Proof of converse is left as an exercise for the pupil.

(2) *Obliques with equal projections are equal, and conversely.*



Hyp. If AR and AC are obliques drawn from the point A to the plane MQ, with the equal projections, BR and BC, respectively,

Conc.: then $AR = AC$.

Dem. Rt. $\triangle ABR \cong$ rt. $\triangle ABC$. (V. 1.)

$\therefore AR = AC$. (Hom. sides of $\cong$ $\triangle$s.)

Q.E.D.

Proof of converse is left as an exercise for the pupil.

(3) *Of two obliques with unequal projections, that one is the greater which has the greater projection, and conversely.*

Hyp. If projection BF is greater than projection BR,

Conc.: then $AF > AR$.

Dem. Lay off $BC = BR$, and draw AC

$AC = AR$. (Why ?)

But $AF > AC$.

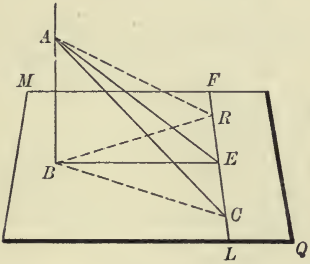
$\therefore AF > AR$.

Q.E.D.

Proof of converse is left as an exercise for the pupil.

Theorem of the Three Perpendiculars

XXI. 5. *If from the foot of a perpendicular to a plane a line is drawn at right angles to a second line of the plane, and if the point of intersection of these two lines is joined to any point in the perpendicular, this third line is perpendicular to the second line of the plane.*



Hyp. If AB is perpendicular to plane MQ, and if BE in MQ is drawn from the foot of AB perpendicular to LF, any line of MQ, and if from E, the point of intersection of BE and LF, AE is drawn to A, any point in AB,

Conc.: then $AE \perp FL$.

Dem. Take $ER = EC$. Draw BR, BC, AR, and AC.

$\triangle BEC \cong \triangle BER$. (V. 1.)

$\therefore BC = BR$. (Hom. sides of $\cong$ ⩟.)

$\therefore AC = AR$. (XXI. 4. (2).)

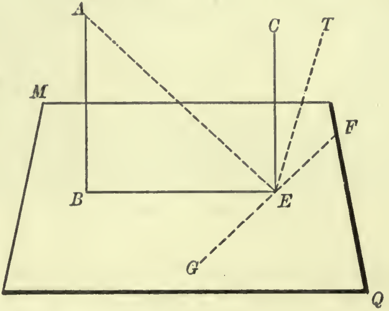
$\therefore AE \perp FL$. (VIII. 2. Sch. 2.)

Q.E.D.

XXI. 5 *a*. *The second line FL in the above figure is perpendicular to the plane of the first and third lines, namely, plane AEB.*

XXI. 5 *b*. *If two lines intersect at right angles, and through their point of intersection a perpendicular to the second is drawn without the plane of the lines, the first is the projection of the third perpendicular on the plane of the given lines.*

XXI. 6. *If one of two parallels is perpendicular to a plane, the other is also perpendicular to the plane, and conversely.*



Hyp. If $AB \parallel CE$, and AB is perpendicular to plane MQ,

Conc.: then CE is perpendicular to plane MQ.

Dem. Draw BE; also, in plane MQ, $GF \perp BE$ at E; draw AE.

GF is perpendicular to plane of AE and EB. (XXI. 5 *a*.)
But CE lies in plane EAB. (XXI. Def. of plane, (*e*).)

$\therefore GE \perp CE$. (Def. ⊥ to a plane.)

$\therefore CE \perp GE$.

But $CE \perp BE$. (Def. ∥s, 2d Dir. Inf.)

$\therefore CE \perp$ plane MQ. (XXI. 3.)

Q.E.D.

Conversely. If AB and CE are perpendicular to plane MQ,

Conc.: then $AB \parallel CE$.

Dem. If EC is not parallel to BA, draw ET that is.

Then ET is perpendicular to plane MQ. (XXI. 6.)

But EC is perpendicular to plane MQ. (Hyp.)

And EC is the only perpendicular that can be drawn to plane MQ at E. (XXI. 2.)

$\therefore ET$ and EC must coincide.

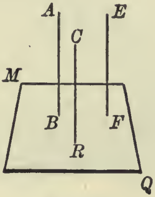
$$\therefore AB \parallel CE.$$

Q.E.D.

XXI. 6 *a*. *If two lines are parallel to a third, they are parallel to each other.*

Hyp. If $AB \parallel CR$

and $EF \parallel CR$,



Conc.: then $AB \parallel EF$.

Dem. Pass a plane MQ perpendicular to CR.

Then AB and EF are each perpendicular to MQ. (XXI. 6.)

$\therefore AB \parallel EF$. (XXI. 6. Conv.)

Q.E.D.

Ex. 11. A ruled surface is one that may be generated by the motion of a straight line. Show that a plane is a ruled surface.

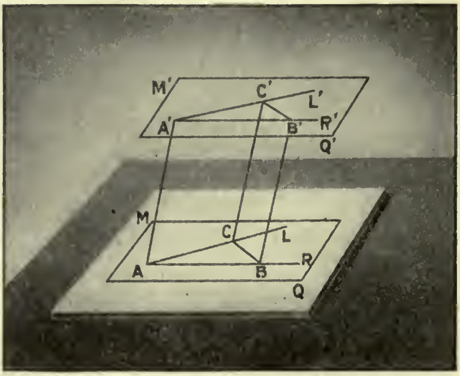
Ex. 12. To get a "straightedge" we sometimes fold a sheet of paper and use the edge of the fold. What proposition are we illustrating?

Ex. 13. In how many positions can the pendulum of a clock be perpendicular to the floor on which the clock stands? Why?

Ex. 14. To find whether or not a square post is perpendicular to a floor, a carpenter applies a square to the floor and post on two sides of the post. On what theorem is his action based? Does it make any difference what sides of the post he selects? Why?

Ex. 15. To keep a vertical sign in an upright position, it is fastened at the foot to two horizontal crosspieces nailed together in the shape of an X. What proposition is illustrated by this device?

XXI. 7. *If two angles not in the same plane have their sides respectively parallel and lying on the same side of the join of their vertices, the angles are equal.*



Hyp. If $A'L' \parallel AL$ and $A'R' \parallel AR$, and these lines lie on same side of the join AA' and also lie in different planes,

Conc.: then $\angle L'A'R' = \angle LAR$.

Dem. Lay off $AB = A'B'$, and $AC = A'C'$; and join CC', BB', BC, and $B'C'$.

The 4-sides $A\text{–}B'$ and $A\text{–}C'$ are parallelograms. (VI. 2.)

$\therefore$ $BB' = AA'$ and $CC' = AA'$. (VI. 1 *a.*)

$\therefore$ $BB' = CC'$. (Ax. 1.)

$BB' \parallel CC'$. (XXI. 6 *a.*)

$\therefore$ the 4-side $B\text{–}C'$ is a parallelogram. (VI. 2.)

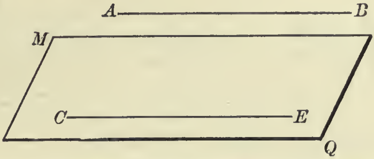
$\therefore$ $BC = B'C'$.

$\therefore$ $\triangle BCA \cong \triangle B'C'A'$. (V. 3.)

$\therefore$ $\angle L'A'R' = \angle LAR$. (Hom. $\angle$s of $\cong$ $\triangle$s.)

Q.E.D.

XXI. 8. *If a line is parallel to a plane, any parallel to the given line through a point of the plane lies wholly in the plane.*



Hyp. If AB is parallel to plane MQ, and $CE \parallel AB$ through C, a point of MQ,

Conc.: then CE lies wholly in the plane MQ.

Dem. But one parallel to AB can pass through C. (Ax. 7.) The line of intersection of the planes ABC and MQ is parallel to AB and passes through C. (Def. of ∥ to a plane.) ∴ this line of intersection is the parallel CE, that is, CE lies wholly in the plane MQ.

Q.E.D.

Ex. 16. How many lines may be drawn perpendicular to a given line at a given point? How are all these lines situated?

Ex. 17. How many lines can be drawn perpendicular to a given line through a point without this line? Prove.

Ex. 18. Prove that one plane, and only one, may be passed perpendicular to a given line at a given point.

Ex. 19. Show how to pass a plane through a given point perpendicular to a given line passing through the point.

Ex. 20. Show how to pass a plane through a given point perpendicular to a line that does not pass through the point.

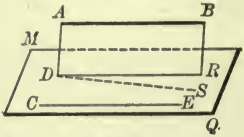
Ex. 21. What surface is generated by the hand of a clock as it passes around the dial? Why?

Ex. 22. A vertical flagstaff 75 ft. high stands in the center of a grassplot 40 ft. in diameter. How far is the top of the pole from any point in the circumference of the grassplot?

Ex. 23. The blades of a windmill are 15 ft. long, and are fastened to the axle at an angle of 60°. How far does the tip of a blade travel in 10 minutes, when the wheel is making 80 revolutions a minute?

XXI. 8 *a*. CONVERSELY. *If a line is parallel to a line of a plane, it is parallel to the plane.*

Hyp. If $AB \parallel CE$, a line of the plane MQ,



Conc.: then AB is parallel to plane MQ.

Dem. Through AB pass any plane AR.
Let this plane intersect MQ in DR.

If $DR \nparallel AB$, $DR \nparallel CE$. (XXI. 6 *a*.)

If $DR \nparallel CE$, draw $DS \parallel CE$.

DS lies in MQ. (Def. of plane, (*d*).)

$DS \parallel AB$. (XXI. 6 *a*.)

∴ DS is coplanar with AB. (Def. of plane, (*d*).)
But DR is coplanar with AB. (Const.)
∴ the two planes $ABDR$ and $ABDS$ have three points (A, B, D) common.
But this is absurd. (Def. of plane, (*b*).)

∴ to suppose $DR \nparallel AB$ is absurd.

∴ $DR \parallel AB$.

That is, AB is parallel to plane MQ. (Def. of $\parallel$ to plane.)

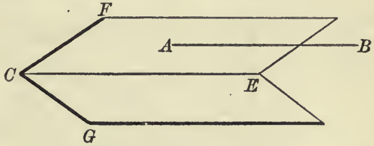
Q.E.D.

Ex. 24. The arm of a derrick is 50 ft. long; it is so fastened to the mast as to revolve at a constant angle of 30° with the vertical upright. How far does the end of the arm travel in a quarter revolution?

Ex. 25. The ceiling of a room is 10 ft. high. How would you determine, by means of a 12-ft. pole, a point in the floor directly under a gas drop in the ceiling?

Ex. 26. If two columns are perpendicular to the same floor, how are they situated with respect to each other?

XXI. 8 *b*. *If a line is parallel to each of two planes, it is parallel to their line of intersection.*



Hyp. If AB is ∥ to plane EF, and also to plane EG,

Conc.: then $AB \parallel EC$.

Dem. A parallel to AB through C must lie in the plane EF. (XXI. 8.)

This parallel must also lie in the plane EG. (XXI. 8.)

∴ the parallel must be CE.

Q.E.D.

Ex. 27. A column is perpendicular to a level floor. The capital and base of a second column in the same wall are respectively 20 ft. from the capital and base of the first. By what proposition do you know the second column to be vertical?

Ex. 28. How would you make use of XXI. 5 *b* to let fall a perpendicular to a plane from a point without the plane?

Ex. 29. Show how to draw, through a given point in a plane, a perpendicular to the plane by using XXI. 5 *b*.

Ex. 30. Show how to pass a plane through a given point parallel to a given line.

Ex. 31. How many planes may be passed through a given point parallel to a given line?

Ex. 32. Show how to draw a line through a given point parallel to a given plane.

Ex. 33. How many lines may be drawn through a given point parallel to a given plane?

Ex. 34. If a number of lines be drawn through a given point parallel to a given plane, how will these lines be situated? Why?

Ex. 35. Show that if a number of lines be parallel to the same plane, and one of the lines intersect all the others, then all the lines must lie in the same plane.

Ex. 36. If a number of planes be passed through a given point parallel to a given line, how will these planes be situated with respect to each other? Why?

XXI. SUMMARY OF PROPOSITIONS IN THE GROUP ON THE PLANE AND ITS RELATED LINES

1. *The line of intersection of two planes is straight.*

2. *If through a given point a perpendicular is drawn to a given plane, it is the only one that can be drawn through the point to the plane.*

Case I. *Point in plane.*

Case II. *Point without plane.*

3. *A line perpendicular to each of two other lines at their intersection is perpendicular to the plane of these lines.*

a. *If three lines are perpendicular to a given line at a given point, these perpendiculars lie in a plane perpendicular to the given line at the given point.*

b. *The plane mid-normal to the join of two points is the locus of all points equidistant from the given points.*

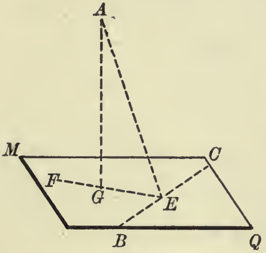
c. *The plane through a given point perpendicular to a given line, is unique, whether the given point be on the given line or without the given line.*

4. *Of all straight lines drawn from a given point to a given plane,*

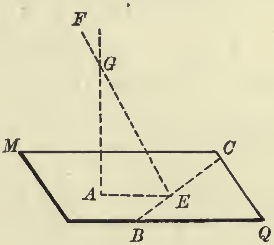
(1) *The perpendicular is the shortest line, and conversely.*

PROBLEMS

XXI. Prob. 1. *Through a given point to draw a perpendicular to a given plane.*



Case I.



Case II.

Given. The plane MQ and the point A.

Required. A perpendicular to MQ through A.

Case I. A is without MQ.

Const. Draw any line, BC, in MQ.
Draw $AE \perp BC$.
On MQ draw $EF \perp BC$.
Draw $AG \perp EF$.
AG is the perpendicular required.

Q.E.F.

Dem. The demonstration is supplied by XXI. 5 *b*.

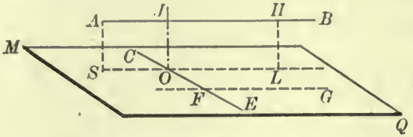
Case II. A is in MQ.

Const. Draw any line, AE, in MQ.
In MQ draw $BC \perp AE$.
Draw EF, without MQ, and perpendicular to BC.
In the plane of FE and AE draw $AG \perp AE$.
AG is the perpendicular required.

Q.E.F.

Dem. The demonstration is supplied by XXI. 5 *a*.

XXI. Prob. 2. *To draw a common perpendicular to two lines not in the same plane.*



Given. The lines AB and CE, not in the same plane.

Required. A common perpendicular to AB and CE.

Const. Through any point of CE, as F, draw $FG \parallel AB$.
CE and FG determine a plane MQ. (Def. of plane, (*a*).)
Project BA on this plane, by the perpendiculars HL, AS.
Let CE intersect SL in O.
Through O draw $OJ \perp$ to the plane MQ. (XXI. Prob. 1.)
OJ is the perpendicular required.

Q.E.F.

Dem. $AB \parallel MQ$. (XXI. 8 *a*.)

$\therefore$ $AB \parallel SL$. (Def. of $\parallel$ to a plane.)

$OJ \perp SL$. (Def. of $\perp$ to plane.)

$\therefore$ $OJ \parallel AS$. (XXI. 6, converse.)

$\therefore$ OJ lies in the plane through O, S, and A;
i.e. in the plane of the parallels AB and SL.
$\therefore$ OJ intersects AB, and $OJAS$ is a rectangle. (Def. of ▭.)

i.e. $OJ \perp AB$.

But $OJ \perp CE$. (Def. of $\perp$ to a plane.)

$\therefore$ OJ is a common perpendicular to AB and CE.

Q.E.D.

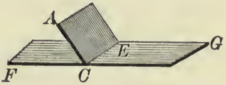
Sch. But one common perpendicular can be drawn to AB and CE; for a line perpendicular to AB must be perpendicular to SL, which is parallel to AB. A common perpendicular to AB and CE must therefore be perpendicular to CE and SL, and hence perpendicular to MQ at O. But the line perpendicular to MQ at O is unique; that is, the common perpendicular is unique.

XXII. GROUP ON PLANAL ANGLES

DEFINITIONS

(*a*) Dihedrals

A **Dihedral** angle, or simply a dihedral, is the figure formed by two planes that intersect.



The **Faces** of a dihedral are the planes by which it is formed (AE, FG).

The **Edge** of a dihedral is the line of intersection of the faces (CE).

When two planes intersect so that the four dihedrals formed are equal, each of the dihedrals is called a **Right Dihedral**, and the planes are said to be perpendicular to each other.

The terms *acute*, *obtuse*, *oblique*, *angles of the same kind*, *complemental*, *supplemental*, *adjacent*, *opposite*, *exterior*, *interior*, *corresponding*, *alternate*, etc., have the same meaning in solid geometry as in the plane, the *face* of the dihedral replacing the *side* of the plane angle.

Method of reading Dihedrals. A dihedral may be read, when there is no ambiguity, by merely reading the edge; as,

dihedral CE.

Otherwise, the angle is indicated by reading one letter from each face, with the edge between these letters; thus,

$A\text{–}CE\text{–}G.$

The dihedral may be assumed to be **generated** by the revolution of a plane, upon the edge as an axis, from an initial plane (as FG) to a terminal plane (as AE).

As in plane geometry, so in solid geometry, rotation is **Positive** when anti-clockwise, and **Negative** when clockwise.

The **Rectilineal Angle** of a dihedral is the plane angle formed by two lines, one drawn in each face, and perpendicular to the edge at the same point.

NOTE. — It is easily shown that

(*a*) If two dihedrals are equal, their rectilinear angles are equal, and conversely.

(*b*) Any two dihedrals are to each other as their rectilinear angles.

Accordingly, the rectilineal angle is usually called the **Measure** of the dihedral.

A **Transversal Plane** is a plane intersecting a number of other planes.

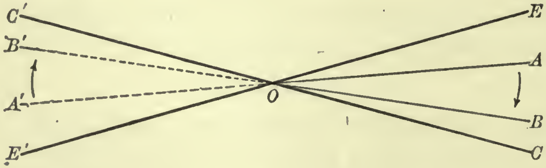
Parallel Planes. Two planes are said to be parallel when they are so situated that if any transversal plane cuts them, the corresponding exterior-interior dihedrals are equal and have their edges parallel.

The plane through a given point and parallel to a given plane is **unique**.

(*b*) POLYHEDRALS

A **Polyhedral** (*n*-dral) is the figure formed by the intersection of three or more planes at a single point.

Dihedral and polyhedral angles are called, collectively, **Planal Angles.**



The **Vertex** of the polyhedral is the point in which the planes intersect (*O*).

The **Faces** of a polyhedral are the intersecting planes (*AOB*, *BOC*, etc.).

The **Edges** of a polyhedral are the lines of intersection of the faces (*OA*, *OB*, etc.).

The **Face Angles** of a polyhedral are the plane angles formed at the vertex by consecutive edges ($\angle AOC$, $\angle BOC$, etc.).

Symmetric Polyhedrals. As the faces and edges of a polyhedral are infinite (unlimited) in extent, the polyhedral will have two parts, lying on opposite sides of the vertex, the angles of which, dihedral and plane, are equal in pairs.

Thus, $B\text{–}AO\text{–}C \equiv B'\text{–}A'O\text{–}C'$.

(Their faces are the same planes.)

Similarly, $A\text{–}BO\text{–}C \equiv A'\text{–}B'O\text{–}C'$, etc.

Again, $\angle AOC \equiv \angle A'OC'$ (vertical angles), etc.

But the equal angles of the two parts of the polyhedral occur in reverse order, as indicated by the arrowheads.

For this reason the two parts of the polyhedral are called *Symmetrical Polyhedrals.* This name is due to Legendre.

Unless the contrary is stated, or evidently implied, we shall, in using the word *polyhedral,* refer to the part on *one side of the vertex only.* The other part will be called the *symmetrical.*

Polyhedrals are classified according to the number of faces as **tri**(3)hedrals, **tetra**(4)hedrals, **penta**(5)hedrals, etc.

The **Trihedral** is analogous to the triangle, the faces of the trihedral corresponding to the sides of the triangle, and the dihedrals of the trihedral to the angles of the triangle.

Hence, the following terms will be self-explanatory:

Scalene,
Isosceles,
Equilateral,
Isoangular,
Equiangular.

A **Rectangular** trihedral is one that has a single right dihedral.

A **Bi-rectangular** trihedral is one that has two right dihedrals, and no more.

A **Tri-rectangular** trihedral is one that has three right dihedrals.

Method of reading Polyhedrals. A polyhedral is read by taking first the letter at the vertex, and then, in succession, the letters at the other extremities of the edges, as $O\text{–}ABCE$.

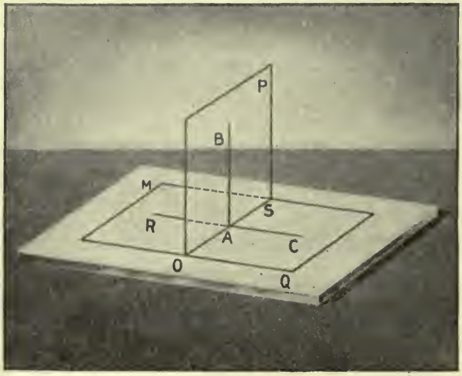
If no misunderstanding is likely to result, the letter at the vertex may be used alone, as in the case of plane angles.

A **Convex Polyhedral** is one whose intersection with any plane not passing through its vertex is a convex polygon.

PROPOSITIONS

(*a*) Dihedrals

XXII. 1. *In a right dihedral a line drawn in one face, perpendicular to the edge, is perpendicular to the other face.*



Hyp. If $P\text{–}SO\text{–}Q$ is a right dihedral, and if AB of plane PO is perpendicular to SO, the edge of the dihedral,

Conc.: then AB is perpendicular to plane MQ.

Dem. Through A draw CR in plane MQ, perpendicular to SO.

$\angle CAB$ is the measure of the dihedral $P\text{–}SO\text{–}Q$.

(Def. of measure of a dihedral.)

$\therefore \angle CAB$ is a right angle.

But $AB \perp SO$. (Hyp.)

$\therefore AB$ is perpendicular to plane MQ. (XXI. 3.)

Q.E.D.

XXII. 1 *a.* *A perpendicular to either face of a right dihedral, at any point of the edge, lies in the other face.*

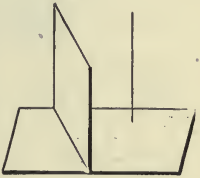
Hyp. If *AB* is perpendicular to plane *MQ*, one face of the dihedral *P–SO–Q*,

Conc.: then *AB* lies in plane *OP*.

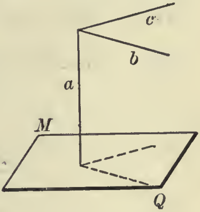
Dem. A line in plane *OP* perpendicular to *SO* is perpendicular to plane *MQ*. (XXII. 1.)
Only one ⊥ to plane *MQ* can be drawn at *A*. (XXI. 2.)
But plane *OP* is perpendicular to plane *MQ*. (Hyp.)

∴ *AB* must lie in plane *OP*.

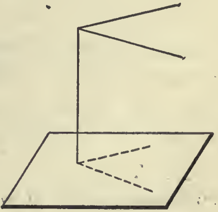
Q.E.D.



Ex. 1. If a line and a plane are each perpendicular to a second plane, the line and plane are parallel.



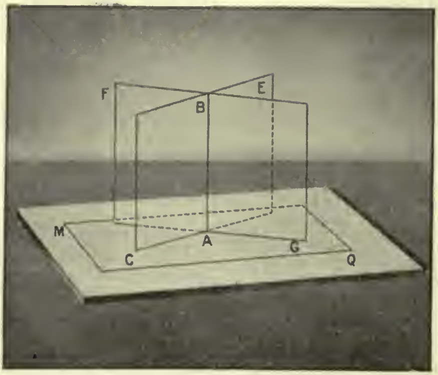
Ex. 2. If a line *a* be perpendicular to a plane *MQ*, and the lines *b* and *c* be both perpendicular to *a*, then *b* and *c* will both be parallel to plane *MQ*.



Ex. 3. If two intersecting lines be parallel to a given plane, the plane of the lines will be parallel to the given plane.

Ex. 4. Hence, show how to pass, through a given point, a plane parallel to a given plane.

XXII. 2. *If each of two intersecting planes is perpendicular to a third, their line of intersection is perpendicular to the third.*



Hyp. If plane CE is perpendicular to plane MQ, and plane FG is perpendicular to plane MQ, and AB is their line of intersection,

Conc.: then AB is perpendicular to plane MQ.

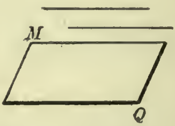
Dem. Dihedral AG is a right dihedral. (Hyp.)

$\therefore$ a $\perp$ to plane MQ at A lies in plane FG. (XXII. 1 *a*.)

Also a $\perp$ to plane MQ at A lies in plane CE. (XXII. 1 *a*.)

$\therefore$ AB is perpendicular to plane MQ.

Q.E.D.

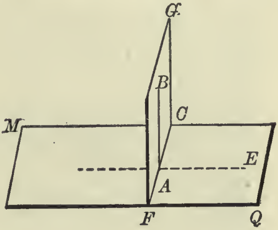


Ex. 5. If one of two parallels is parallel to a given plane MQ, the other is also parallel to MQ.

Ex. 6. If each of two planes is parallel to a third, these two planes are parallel to each other.

Ex. 7. Through a given point but one plane can be passed parallel to two lines that are not parallel to each other.

XXII. 2 *a*. *If a line is perpendicular to a plane, every plane through the line is also perpendicular to the plane.*



Hyp. If AB is perpendicular to plane MQ, and plane GF is drawn through AB,

Conc.: then plane GF is perpendicular to plane MQ.

Dem. Draw $AE \perp CF$ and in plane MQ.

$AB \perp AE$. (Def. of $\perp$ to a plane.)

$AE \perp CF$. (Const.)

$\therefore$ AE is perpendicular to plane GF. (XXI. 3.)

But rt. $\angle EAB$ is the measure of dihedral $Q\text{–}FC\text{–}G$.

$\therefore$ plane GF is perpendicular to plane MQ.

Q.E.D.

Ex. 8. What is the smallest number of plane angles that can be brought together at a point to form a convex polyhedral? What must be true of the sum of these angles if they form a polyhedral?

Ex. 9. If three equilateral triangles be brought together so as to have a common vertex, what will be the sum of the plane angles at this vertex? Why, then, must a polyhedral be formed?

Ex. 10. Show that a polyhedral may be formed with four equilateral triangles; also with five.

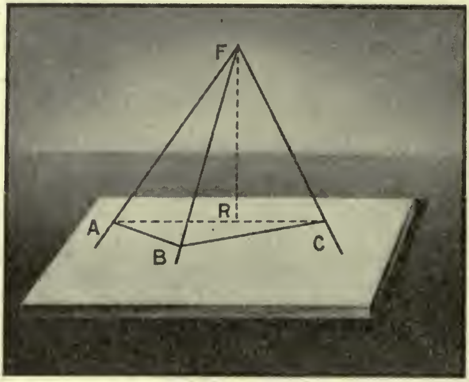
Ex. 11. Show that no polyhedral can be formed by using any larger number of equilateral triangles than five.

Ex. 12. Show that if all the faces of a polyhedral be regular n-gons, the only polyhedrals possible are those in which $n = 3$, 4, or 5.

(b) POLYHEDRALS

XXII. 3. *The sum of the two face angles of a trihedral is greater than the third angle.*

NOTE. — No proof is necessary unless the third angle is greater than each of the others.



Hyp. If $F\text{–}ABC$ is a trihedral angle, and $\angle AFC$ is greater than either $\angle AFB$ or $\angle BFC$,

Conc.: then $\angle AFB + \angle BFC > \angle AFC$.

Dem. Draw FR in plane AFC so that $\angle AFR = \angle AFB$.

Lay off $FR = FB$ and pass a plane through B and R, cutting the edges in A, B, and C.

Draw AB, BC, and AC.

$$\triangle AFB \cong \triangle AFR. \qquad \text{(V. 1.)}$$

$$\therefore AB = AR.$$

But $$AB + BC > AC. \qquad \text{(VII. 1.)}$$

$$\therefore BC > RC. \qquad \text{(Preliminary Th. 3.)}$$

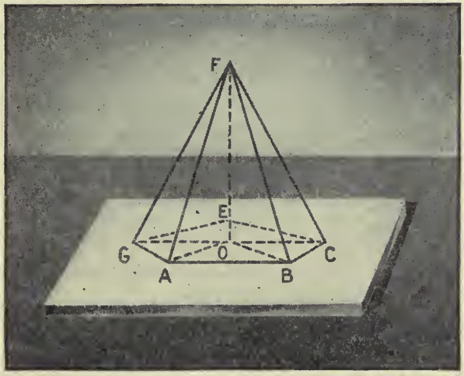
$$\therefore \angle BFC > \angle RFC.$$

[If two $\triangle$ have two sides of one equal, etc.] (VII. **Ex.** 9.)

$$\therefore \angle AFB + \angle BFC > \angle AFR + \angle RFC (= \angle AFC).$$

Q.E.D.

XXII. 4. *The sum of the face angles of any convex polyhedral is less than four right angles.*



Hyp. If F is a convex polyhedral,

Conc.: then $\angle AFB + \angle BFC + \angle CFE$, etc., $<$ 4 rt. $\angle$s.

Dem. Pass a plane cutting the edges of polyhedral F in $A, B, C, \cdots$.

From O, any point in this plane, draw $OA, OB, OC, \cdots$.

In the base there are n triangles; there are also n triangle faces in solid angle F.

∴ the sum of the int. $\angle$s of the base $\triangle$s $= 2n$ rt. $\angle$s, (III. 1.)
and the sum of the int. $\angle$s of the face $\triangle$s $= 2n$ rt. $\angle$s. (III. 1.)

Now $\angle GAB$ of the base $< \angle FAG + \angle FAB$. (XXII. 3.)
Similarly, $\angle ABC < \angle FBA + \angle FBC$. (XXII. 3.)

∴ the sum of the angles of the base n-gon is less than the sum of the base angles of the face triangles.

Now the sum of the interior angles of the

n-gon $= (2n - 4)$ rt. $\angle$s; (III. 3.)

that is, the sum of the angles about $O = 4$ rt. $\angle$s.

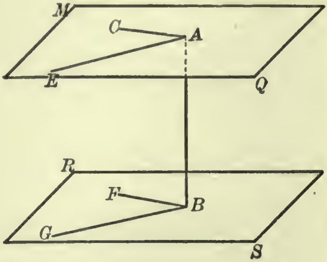
∴ as the sum of the base angles of the face triangles is greater than the sum of the interior angles of the n-gon, it follows that

$\angle AFB + \angle BFC + \angle CFE \cdots < 4$ rt. $\angle$s. **Q.E.D.**

XXII. 4. Sch. If planal angle F were concave in any of its dihedrals, the sum of the angles about angle F might exceed four right angles.

XXII. 5. *A line perpendicular to one of two parallel planes is perpendicular to the other, and conversely.*

Hyp. If plane MQ is parallel to plane RS and $AB \perp RS$,



Conc.: then $AB \perp MQ$.

Dem. Through B draw any two lines, BF and BG in RS.
Let planes FBA and GBA intersect MQ in AC and AE, respectively.

Then, $AC \parallel BF$ and $AE \parallel BG$. (Def. ∥ pls.)
But, $AB \perp BF$, (Def. ⊥ pl.)
$\therefore AB \perp AC$. (Def. ∥'s. Direct Inf. (2).)
Similarly, $AB \perp AE$.
$\therefore AB \perp RS$. (XXI. 3.)

Q.E.D.

Conversely;

Hyp. If planes MQ and RS are $\perp AB$,

Conc.: then $MQ \parallel RS$.

Dem. The plane through A, $\parallel RS$, is unique. (Def. ∥ pls. *a*.)
The plane through $A \perp AB$ is unique. (XXI. 3 *c*.)
The ∥ plane is $\perp AB$. (Direct. Th.)
$\therefore$ the plane $\perp AB$ is the ∥ plane. (Ax. 10.)
That is, $MQ \parallel RS$.

Q.E.D.

XXII. SUMMARY OF PROPOSITIONS IN THE GROUP ON PLANAL ANGLES

(*a*) Dihedrals

1. *In a right dihedral a line drawn in one face, perpendicular to the edge, is perpendicular to the other face.*

   a. A perpendicular to either face of a right dihedral, at any point of the edge, lies in the other face.

2. *If each of two intersecting planes is perpendicular to a third, their line of intersection is perpendicular to the third.*

   a. If a line is perpendicular to a plane, every plane through the line is perpendicular to the given plane.

(*b*) Polyhedrals

3. *The sum of any two face angles of a trihedral is greater than the third angle.*

4. *The sum of the face angles of any convex polyhedral is less than four right angles.*

Sch. If planal angle F were concave in any of its dihedrals, the sum of the angles about angle F might exceed four right angles.

5. *A line perpendicular to one of two parallel planes is perpendicular to the other, and conversely.*

Show that if the faces of a polyhedral are to be regular polygons of different kinds, it is always possible to form a polyhedral with the following combinations of figures:

Ex. 13. Two equilateral triangles, with any regular polygon whatever.

Ex. 14. Three equilateral triangles, with any regular polygon whatever.

Ex. 15. Two squares, with any regular polygon whatever.

In how many ways can polyhedrals be formed by using, at a common vertex:

Ex. 16. Squares? **Ex. 17.** Regular pentagons?

Ex. 18. Squares and equilateral triangles?

Ex. 19. Squares and regular pentagons?

Ex. 20. Squares and regular hexagons?

Ex. 21. Squares and regular heptagons?

Ex. 22. Regular pentagons?

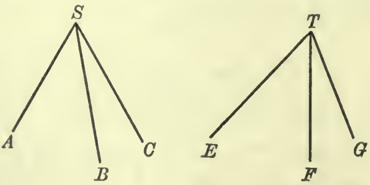
Ex. 23. Regular pentagons and equilateral triangles?

Ex. 24. Regular pentagons and regular hexagons?

Ex. 25. Regular hexagons and equilateral triangles?

Ex. 26. Regular heptagons and equilateral triangles?

Ex. 27. Show that if two trihedrals have two face angles and the included dihedral of the first equal to two face angles and the included dihedral of the second, the trihedrals will be congruent.



Proof. If $\angle ASB = \angle ETF$, $\angle ASC = \angle ETG$, and $B\text{-}AS\text{-}C = F\text{-}ET\text{-}G$, place trihedral T on trihedral S so that $\angle ETG$ shall coincide with $\angle ASC$, TE falling along SA.

Plane ETF falls in plane ASB. (Why?)
TF falls along SB. (Why?)
(See proof of V. 1.)

Ex. 28. Show by superposition that:

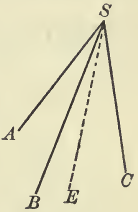
If two trihedrals have two dihedrals and the included face angle of the first equal to two dihedrals and the included face angle of the second, the trihedrals will be congruent.

(See proof of V. 2.)

Ex. 29. An isosceles trihedral is isoangular.

Proof. In the figure, if $\angle ASB = \angle BSC$, then $B\text{–}SA\text{–}C = B\text{–}SC\text{–}A$.

Bisect $A\text{–}SB\text{–}C$ by the plane BSE. Then use Ex. 27 to show the trihedrals $S\text{–}ABE$ and $S\text{–}CBE$ congruent.



XXIII. GROUP ON THE PRISM AND THE CYLINDER

DEFINITIONS

(*a*) The Prism

A **Polyhedron** is a solid bounded by polygons called **Faces.**

The **Edges** of a polyhedron are the sides of its faces.

The **Vertices** of a polyhedron are the vertices of its faces.

A **Section** of a polyhedron is a polygon obtained by passing a plane through the polyhedron.

A **Convex** polyhedron is one of which the sections are all convex.

In all subsequent definitions polyhedra will be assumed to be convex.

A **Regular** polyhedron is one whose faces are congruent regular polygons and whose polyhedrals are congruent.

Polyhedra are classified according to the *number of faces.*

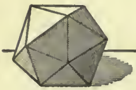
A **Tetra**hedron is a polyhedron of four faces.

A **Hexa**hedron is a polyhedron of six faces.

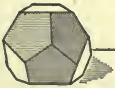
An **Octa**hedron is a polyhedron of eight faces.

A **Dodeca**hedron is a polyhedron of twelve faces.

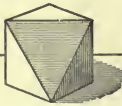
An **Icosa**hedron is a polyhedron of twenty faces.



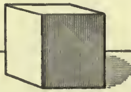
Icosahedron



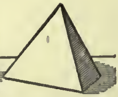
Dodecahedron



Octahedron



Hexahedron



Tetrahedron

A **Prism** is a polyhedron, two of whose faces (called bases) are parallel polygons, and whose lateral faces are parallelograms whose vertices are all vertices of the respective bases.

The **Lateral Area** of a prism is the sum of the areas of the lateral faces.

The **Lateral Edges** of the prism are the edges in which the lateral faces intersect.

The **Altitude** of a prism is the perpendicular distance between the bases.

Prisms are classified according to the *number of sides* of the bases.

A **Triangular** prism is a prism whose base is a triangle.

A **Quadrangular** prism is a prism whose base is a quadrilateral.

A **Pentagonal** prism is a prism whose base is a pentagon.

A **Right Section** of a prism is a section made by a plane perpendicular to a lateral edge.

A **Right** prism is a prism in which the lateral edges are perpendicular to the bases.

An **Oblique** prism is a prism whose lateral edges are oblique to the bases.

A **Regular** prism is a right prism in which the bases are regular polygons.

A **Parallelepiped** is a prism in which the bases are parallelograms; that is, a prism all of whose faces are parallelograms.

A **Rectangular** parallelepiped is a right parallelepiped in which the bases are rectangles; that is, one in which all the faces are rectangles.

A **Cube** is a regular parallelepiped in which the lateral faces are squares.

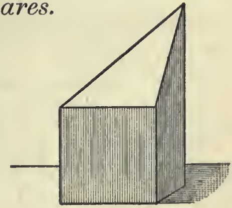
Corollaries of the Definitions

(*a*) *The lateral edges of a prism are equal and parallel.*

(*b*) *The faces of a cube are equal squares.*

A **Truncated** prism is that portion of a prism which is comprised between either base and a section not parallel to it.

A **Right Truncated** prism is a portion of a right prism comprised between either base and a section not parallel to it.



The **Volume** of a polyhedron is its ratio to some other polyhedron, called the unit of volume.

The **Unit of Volume** usually taken is the cube, each edge of which equals the unit of length.

Two polyhedra are said to be equal when their volumes are equal.

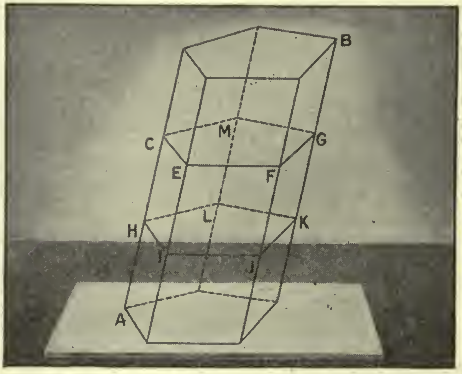
Two polyhedra are said to be congruent when they may be placed in coincident superposition.

Similar polyhedra are polyhedra that have the same number of faces, which are similar, each to each, and similarly placed.

PROPOSITIONS

(a) THE PRISM

XXIII. 1. *Parallel sections of a prism are congruent.*



Hyp. If, in the prism AB, section $CEFGM$ is parallel to section $HIJKL$,

Conc.: then $CEF \cdots \cong HIJ \cdots$.

Dem. $HI \parallel CE,\ IJ \parallel EF$, etc. (Def. of ∥ pls.)

$\therefore\ \angle HIJ = \angle CEF,\ \angle IJK = EFG$, etc. (XXI. 7.)

Now $HI = CE,\ IJ = EF$, etc. (VI. 1 a.)

$\therefore$ as the two sections are mutually equiangular and equilateral, they may be placed in coincident superposition and are therefore congruent. (Def. of $\cong$ figs.)

Q.E.D.

XXIII. 1 *a.* *The bases of a prism are congruent.*

Ex. 1. Find the volume of a rectangular parallelepiped 20 ft. long, 3 ft. wide, and 5 ft. high.

Ex. 2. A bushel contains 2150.4 cu. in. Find the height of a bin the bottom of which is 25 ft. by 15 ft., and the capacity of which is 2500 bu.

Ex. 3. A "lumber foot" is 12 in. square and 1 in. thick. At $18 per M. how much will it cost to build a cubical bin of three-inch lumber to contain 500 bushels, allowing $\frac{1}{8}$ extra material for studding and $10 for labor?

Ex. 4. The number of cubic feet in the volume of a cubical block is equal to the number of square feet in its entire surface. Find the length of the edge of the block.

Ex. 5. The edges of a rectangular parallelepiped are 6 ft., 10 ft., and 15 ft.; the edges of a second are 12 ft., 14 ft., and 18 ft. Find the edge of a cube whose volume equals the sum of the volumes of the parallelepipeds.

Ex. 6. A rectangular parallelepiped is often called an "oblong block." The dimensions of an oblong block are in the ratio of $2:3:5$. The number of cubic feet in its volume is 10 times the number of square feet in its entire surface. Find its dimensions.

Ex. 7. The diagonal of one cube equals the edge of a second. Find the ratio of the volumes of the cubes.

Ex. 8. The diagonal of one cube is 3 times as long as the edge of a second. What is the ratio of the surfaces of the two cubes?

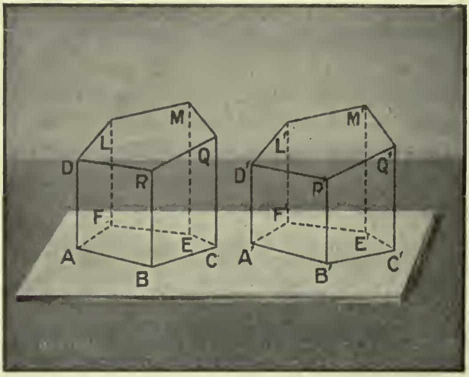
Ex. 9. The diagonal of one cube is a times as long as the edge of a second. What is the ratio of the volumes of the two figures?

Ex. 10. The volume of a prism is 324 cu. ft. Its altitude is 36 ft. What is the area of the base?

Ex. 11. The altitude of a prism is 20 yd. Its base is an equilateral triangle each side of which is 15 ft. Find the volume of the prism.

Ex. 12. The altitude of a regular prism is 10. Its base is a hexagon each side of which is 6. Find the total surface and the volume of the prism.

XXIII. 2. *Two right truncated prisms are congruent, if three faces including a trihedral of one are congruent respectively to three faces including a trihedral of the other and are similarly placed.*



Hyp. If the right truncated prisms *AM* and *A'M'* have the three faces of trihedral *B* congruent with the three faces of trihedral *B'* and the faces are similarly placed,

Conc.: then right truncated prism *AM* is congruent to right truncated prism *A'M'*.

Dem. Place the base of *AM* in coincident superposition with the base of *A'M'*; *AB* falling on *A'B'*.

Then *AD* must fall on *A'D'*, *BR* on *B'R'*, etc. (XXI. 2.)

Then *D* must fall on *D'*, *R* on *R'*, and *Q* on *Q'*.

(These faces are ≅ by hyp.)

∴ the plane of *D*, *R*, and *Q* must fall on the plane of *D'*, *R'*, and *Q'*. (Def. of plane (*b*).)

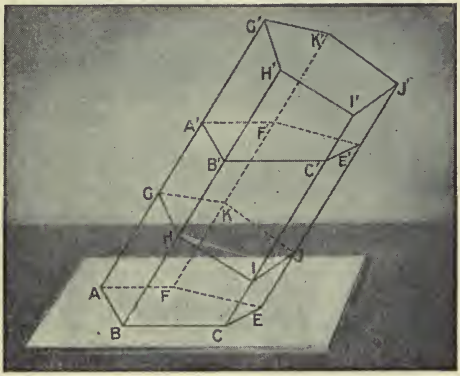
And as the truncated prisms are right, *M* must fall on *M'* and *L* on *L'*. (XXI. 2.)

∴ the right truncated prism *AM* is congruent with right truncated prism *A'M'*.

Q.E.D.

XXIII. 2 *a.* *Two right prisms having congruent bases and equal altitudes are congruent.*

XXIII. 3. *Any oblique prism is equal to a right prism of which the altitude equals a lateral edge of the oblique prism and the bases are right sections of the oblique prism.*



Hyp. If the right prism GJ' has its bases right sections of the oblique prism AE' and its altitude JJ' equal to the edge EE',

Conc.: then right prism GJ' equals oblique prism AE'.

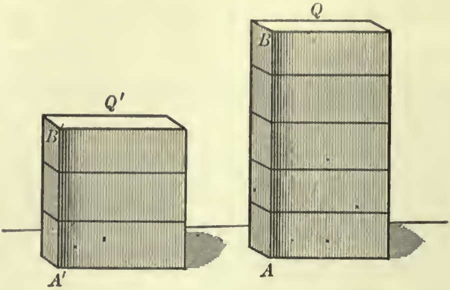
Dem. AJ and $A'J'$ are right truncated prisms. (Const.)
Their bases A–E and A'–E' are congruent. (XXIII. 1 *a.*)
The lateral faces BG and $B'G'$ are congruent.
Likewise, the lateral faces BI and $B'I'$ are congruent.

$\therefore$ Rt. Tr. Prism $AJ \cong$ Rt. Tr. Prism $A'J'$. (XXIII. 2.)

To each of the right truncated prisms add the right truncated prism GE', and we have

right prism GJ' equals oblique prism AE'. (Ax. 2.)
Q.E.D.

XXIII. 4. *Two rectangular parallelepipeds that have equal bases are to each other as their altitudes.*



Hyp. If the two rectangular parallelepipeds Q and Q' have equal bases and their altitudes are AB and $A'B'$,

Conc.: then rectangular parallelepiped Q : rectangular parallelepiped $Q' :: AB : A'B'$.

Case I. AB and $A'B'$ commensurable.

Case II. AB and $A'B'$ incommensurable.

Dem. Case I. Find a common measure of AB and $A'B'$. Let it be contained in AB five, and in $A'B'$ three times.

Then $$AB : A'B' :: 5 : 3.$$

Through the points of division draw planes parallel to the bases.

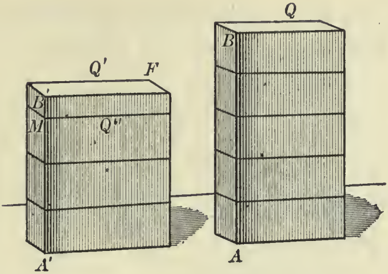
The small rectangular parallelepipeds thus obtained are all congruent. (XXIII. 2 *a.*)

In Q there are five, in Q' three of these equal parallelepipeds.

$$\therefore\ Q : Q' :: AB : A'B'. \qquad \text{(Ax. 1.)}$$

Q.E.D.

Ex. 13. The volume of a regular octagonal prism of altitude 8 is equal to the volume of a regular hexagonal prism of altitude 12. The radius of the base of the octagonal prism is 6. Find the lateral surface of each prism.



Dem. Case II. Divide AB into any number of equal parts. Suppose one of these equal parts is contained in $A'B'$ three times, with a remainder MB'.

Through M pass a plane parallel to the base.

Then $\quad Q'' : Q :: A'M : AB. \quad$ (Case I.)

If the number of equal parts in AB be indefinitely increased, the remainder MB' will be indefinitely decreased, but can never equal zero, because AB and $A'B'$ are incommensurable.

$\therefore$ $A'M$ approaches $A'B'$ as a limit, and Q'' approaches Q' as a limit.

$\therefore$ $\dfrac{A'M}{AB}$ approaches $\dfrac{A'B'}{AB}$ as a limit, and $\dfrac{Q''}{Q}$ approaches $\dfrac{Q'}{Q}$ as a limit.

But $\dfrac{Q''}{Q}$ is always equal to $A'B' : AB$. (Case I.)

$\therefore$ the limits of the variables being equal,

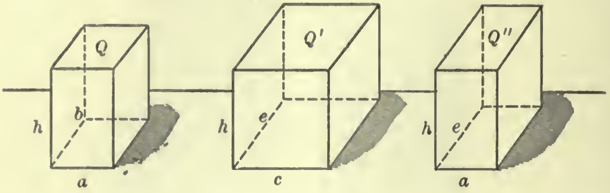
Rt. parallelepiped Q' : rt. parallelepiped $Q :: A'M : AB$.

Q.E.D.

Sch. Two rectangular parallelepipeds which have two dimensions in common are to each other as their third dimensions.

Ex. 14. The volume of each of two prisms is 1386 cu. ft. The base of the first is an equilateral triangle whose altitude is 20 ft. The base of the second is a square, each side of which is 20 ft. Find the ratio of the altitudes of the prisms.

XXIII. 4 *a*. *Two rectangular parallelepipeds which have equal altitudes are to each other as their bases.*



Hyp. If the rectangular parallelepipeds Q and Q' have their altitudes equal, and the base of Q, $a \cdot b$ and of Q', $c \cdot e$,

Conc.: then $\quad Q : Q' :: a \cdot b : c \cdot e.$

Dem. Construct a rectangular parallelepiped Q'' whose altitude is h, and whose base is $a \cdot e$.

Then $\quad Q : Q'' :: b : e.$ (XXIII. 4. Sch.)

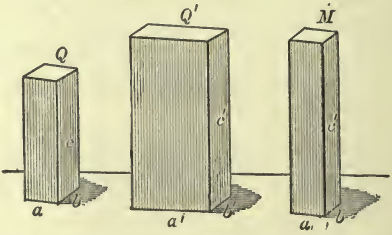
But $\quad Q'' : Q' :: a : c.$ (XXIII. 4. Sch.)

$\therefore \; Q : Q' :: a \cdot b : c \cdot e.$ (By mult.)

Q.E.D.

XXIII. 5. *Two rectangular parallelepipeds are to each other as the products of their three dimensions.*

Hyp. If Q and Q' are two rectangular parallelepipeds whose bases are $a \cdot b$ and $a' \cdot b'$, respectively, and whose altitudes are c and c', respectively,



Conc.: then $\quad Q : Q' :: a \cdot b \cdot c : a' \cdot b' \cdot c'.$

Dem. Construct a rectangular parallelepiped M, whose base is $a \cdot b$ and whose altitude is c'.

Then $Q : M :: c : c'$. (XXIII. 4.)

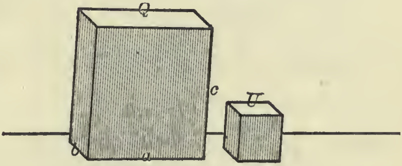
And $M : Q' :: a \cdot b : a' \cdot b'$. (XXIII. 4 *a*.)

$\therefore\ Q : Q' :: a \cdot b \cdot c : a' \cdot b' \cdot c'$. (By mult.)

Q.E.D.

XXIII. 5 *a*. *The volume of a rectangular parallelepiped equals the product of its three dimensions.*

Hyp. If Q is a rectangular parallelepiped whose dimensions are a, b, and c,



Conc.: then volume of $Q = a \cdot b \cdot c$.

Dem. Construct a cube U, whose edge is the linear unit.

Then $Q : U :: a \cdot b \cdot c : 1 \cdot 1 \cdot 1$. (XXIII. 5.)

But $Q : U$ is the volume of Q. (Def. of vol.)

And
$$\frac{a \cdot b \cdot c}{1 \cdot 1 \cdot 1} = a \cdot b \cdot c.$$

$\therefore$ the volume of $Q = a \cdot b \cdot c$.

Q.E.D.

Sch. The volume of a rectangular parallelepiped equals the product of its base by its altitude.

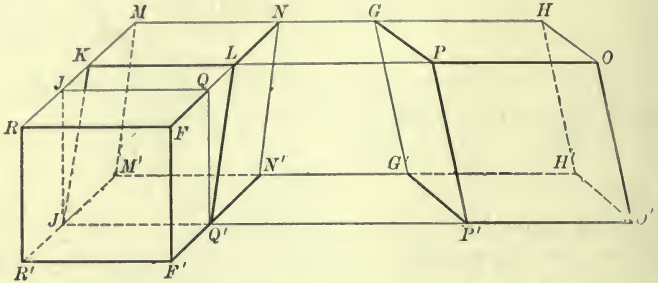
Ex. 15. The interior dimensions of a water tank are 8 ft., 4 ft., and 5 ft., respectively. How many gallons will the tank hold?

Ex. 16. How much will it cost to line the tank with zinc at 85¢ a square yard, allowing a waste of $\frac{1}{12}$ of the material for seams?

Ex. 17. The convex surface of a right circular cylinder is equal to the total surface of a cube; the diameter of the cylinder and its altitude each equals 10. Find the edge of the cube.

Ex. 18. The altitude of a right circular cylinder is a, the radius of its base is b. To find the radius of a circle equal in area to the convex surface of the cylinder.

XXIII. 6. *The volume of any parallelepiped equals the product of its base and its altitude.*



Hyp. If $P'H$ is any parallelepiped whose base is the parallelogram $P'–H'$ and whose altitude is the perpendicular between the bases,

Conc.: then the volume of $P'H$ equals the area of its base times its altitude.

Dem. Produce PO, making $KL = PO$, and through K and L pass planes $J'M$ and $Q'N \perp KL$.

Extend the faces HP, $H'P'$, PO', and GH' to intersect the planes $J'M$ and $Q'N$, forming the right parallelepiped MQ'.

Oblique parallelepiped $P'H =$ right parallelepiped MQ'.
(XXIII. 3.)

Again, produce $N'Q'$ making $Q'F' = N'Q'$ and through Q' and F' pass planes $Q'J$ and $F'R \perp Q'F'$.

Extend the faces $Q'M'$ and LM, $Q'N$ and $M'K$ to intersect the planes $Q'J$ and $F'R$, forming the rectangular parallelepiped $Q'R$. (Def. of rt. parallelepiped.)

Rt. parallelepiped $MQ' =$ rect. parallelepiped $Q'R$. (XXIII. 3.)

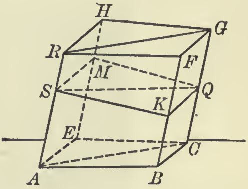
$\therefore$ rect. parallelepiped $Q'R =$ obl. parallelepiped $P'H$. (Ax. 1.)

Now the volume of a rectangular parallelepiped equals the product of the base and an altitude. (XXIII. 5 *a*.)

And as the base and an altitude of $Q'R$ equal, respectively, the base and altitude of $P'H$, the volume of $P'H$ equals the area of its base times the altitude.

Q.E.D.

XXIII. 7. *The plane through two diagonally opposite edges of a parallelepiped divides the figure into two equal triangular prisms.*



Hyp. If AG is a parallelepiped and a plane $ACGR$ is passed through AR and CG,

Conc.: then triangular prism $ACB–F$ equals triangular prism $ACE–H$.

Dem. Through S, any point of AR, pass a plane $SMQK$ perpendicular to RA and intersecting $ACGR$ in SQ.

Plane BR is parallel to plane CH. (Def. of parallelepiped.)

$\therefore SK \parallel MQ$. (Def. of ∥ planes.)

Similarly, $SM \parallel KQ$.

∴ the 4-side $S–Q$ is a parallelogram. (Def. of ▱.)

$\therefore \triangle SKQ \cong \triangle SQM$. (VI. 1 *a*. Sch.)

Prism $ABC–F$ equals a right prism whose base is $\triangle SKQ$ and whose altitude is FB. (XXIII. 3.)

Prism $ACE–H$ equals a right prism whose base is $\triangle SRQ$ and whose altitude is FB (or EH). (XXIII. 3.)

But these right prisms are equal. (XXIII. 2 *a*.)

∴ triangular prism $ACB–F$ equals triangular prism $ACE–H$. (Ax. 1.)

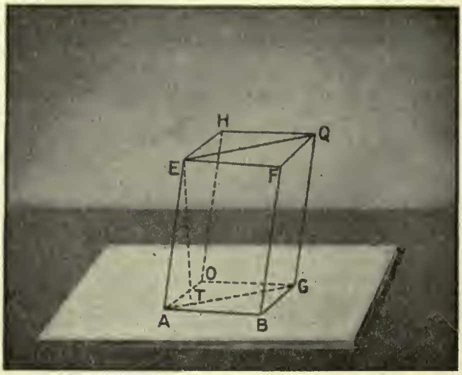
Q.E.D.

Ex. 19. The total surface of a right circular cylinder whose height is twice the radius of its base is equal to the surface of a cube. If the edge of the cube is 12 in., what is the height of the cylinder?

Ex. 20. The altitude of a cylinder is 12 ft. Its base is a circle of radius 8 in. Find the volume and the total surface of the cylinder.

Ex. 21. Find the diameter of a cylindrical tank 10 ft. deep, whose capacity is 8000 gal.

XXIII. 8. *The volume of a triangular prism equals the product of its base by its altitude.*



Hyp. If $ABG–F$ is a triangular prism whose base is ABG and whose altitude is ET,

Conc.: then the volume of $AGB–F = \triangle ABG \times ET$.

Dem. Complete the parallelograms $ABGO$ and $EFQH$.

Draw OH completing the parallelepiped $ABGO–F$.

The volume of parallelepiped $ABGO–F = \square ABGO \times ET$. (XXIII. 6.)

But the volume of prism $AGB–F$ equals one half that of parallelepiped $AGBO–F$. (XXIII. 7.)

And $\triangle ABG = \frac{1}{2} \square ABGO$. (VI. 1 *a*. Sch.)

$\therefore$ volume of prism $AGB–F = \triangle ABG \times ET$. (Ax. 3.)

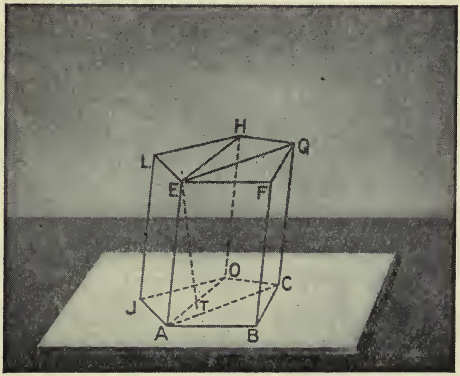
Q.E.D.

Ex. 22. How much sheet iron would it take to make such a tank (Ex. 21), allowing $\frac{1}{10}$ for waste and seams?

Ex. 23. A cylindrical pipe of diameter 20 in. discharges 800 gal. a second. What is the velocity of the water in the pipe?

Ex. 24. The total surface of a right circular cylinder whose height is three times the diameter of its base is 2513.28 sq. ft. Find the volume of the cylinder.

XXIII. 8 *a*. *The volume of any prism equals the product of its base by its altitude.*



Hyp. If *ABCO–F* is any prism, and *ABCOJ* is its base, and *ET* is its altitude,

Conc.: then its volume equals the product of its base times its altitude.

Dem. Through *EA* and *OH*, *EA* and *CQ*, pass planes.

These planes divide the prism into triangular prisms.

The volume of each triangular prism equals its base times its altitude. (XXIII. 8.)

But the sum of the bases of the triangular prisms equals the base of the given prism, and the altitude of the triangular prisms is the altitude of the given prism.

∴ the volume of prism *ABCO–F* equals the product of its base times its altitude.

Q.E.D.

Ex. 25. The diameter and the altitude of a cylinder are each equal to the edge of a cube. What is the ratio of the volumes of the two figures?

Ex. 26. A cubic foot of cast iron weighs 445 lb. What is the weight of a cast-iron pipe 16 ft. long, $1\frac{1}{2}$ in. thick, and 10 in. in diameter, internal measurement?

XXIII. 8 *b.* *Any two prisms are to each other as the products of the bases by the altitudes.*

If the bases are equal, the prisms are to each other as the altitudes.

If the altitudes are equal, the prisms are to each other as the bases.

If the bases are equal and also the altitudes, the prisms are equal.

XXIII. SUMMARY OF PROPOSITIONS IN THE GROUP ON (*a*) THE PRISM

1. *Parallel sections of a prism are congruent.*

   a. *The bases of a prism are congruent.*

2. *Two right truncated prisms are congruent, if three faces including a trihedral of the one are equal respectively to three faces including a trihedral of the other and are similarly placed.*

   a. *Two right prisms having equal bases and equal altitudes are congruent.*

3. *Any oblique prism is equal to a right prism of which the altitude equals a lateral edge of the oblique prism, and the bases are right sections of the oblique prism.*

4. *Two rectangular parallelepipeds that have equal bases are to each other as their altitudes.*

   a. *Two rectangular parallelepipeds that have equal altitudes are to each other as their bases.*

5. *Two rectangular parallelepipeds are to each other as the products of their three dimensions.*

   a. *The volume of a rectangular parallelepiped equals the product of its three dimensions.*

6. *The volume of any parallelepiped equals the product of its base by its altitude.*

7. *The plane through two diagonally opposite edges of a parallelepiped divides the figure into two equal triangular prisms.*

8. *The volume of a triangular prism equals the product of its base by its altitude.*

   a. *The volume of any prism equals the product of its base by its altitude.*

   b. *Any two prisms are to each other as the products of the bases by the altitudes.*

   If the bases are equal, the prisms are to each other as the altitudes.

   If the altitudes are equal, the prisms are to each other as the bases.

   If the bases are equal, and also the altitudes, the prisms are equal.

DEFINITIONS

(*b*) THE CYLINDER

A **Cylindrical Surface** is a surface generated by a straight line that moves parallel to its first position along a curve not coplanar with the moving line.

The moving line is called the **Generatrix**.

The curve that directs the motion is called the **Directrix**.

The successive positions of the generatrix are called the **Elements of the Surface**.

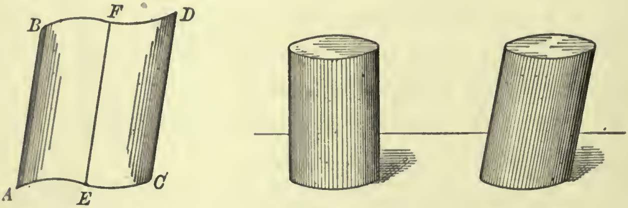
A **Cylinder** is a solid inclosed by a cylindrical surface and two parallel planes.

The **Bases** of a cylinder are the parallel plane sections.

The **Elements of the Cylinder** are the portions of the elements of the cylindrical surface determined by the bases.

The **Altitude** of a cylinder is the perpendicular distance between the bases.

A **Right Section** of a cylinder is a section made by a plane perpendicular to an element.



CYLINDERS

A **Right** cylinder is a cylinder the elements of which are perpendicular to the bases.

A **Circular** cylinder is a cylinder the bases of which are circles.

A point is said to revolve around a fixed line when it generates a circle whose plane is perpendicular to the fixed line and having its center on the fixed line.

The fixed line is called the **Axis of Revolution,** or simply the **axis.**

A line or surface is said to revolve about the axis, when every point in the moving line or surface revolves about the axis.

The surface generated by the revolution of a line (straight or curved) about an axis is called a **Surface of Revolution.**

The volume (or solid) generated by the revolution of a surface about an axis is called a **Volume** (or **Solid**) **of Revolution.**

The axis of revolution is often called the **Axis of the Surface** or **Volume** generated by the revolution.

COROLLARIES OF THE DEFINITIONS

(*a*) *Any section of a cylinder through an element is a parallelogram.*

(*b*) *A right circular cylinder is a cylinder of revolution.*

As the circle has been shown (XVIII) to be the limit of the regular polygon as the number of sides is increased indefinitely, so the circular cylinder is the limit in surface and volume of the prism with regular bases, as the number of sides of the bases is increased beyond any assignable number. (It will be a good exercise for the student to give the detailed proof of the statement.)

Accordingly, every proposition that is true of every prism with a regular base, *whatever may be the number of lateral faces,* is true of the circular cylinder. We therefore obtain from the corresponding propositions of XXIII (*a*), the summary on the following page.

Ex. 27. A horse power is the force necessary to raise 33,000 lb. 1 ft. in 1 min. The cylinder of an engine is 4 ft. in diameter and 6 ft. high; the piston is 6 in. thick and the piston-rod 8 in. in diameter.

Find the horse power of the engine when it is making 200 revolutions a minute with a steam pressure of 60 lb. to the square inch.

Ex. 28. Show that no polyhedron can have less than four faces nor less than six edges.

XXIII. SUMMARY OF PROPOSITIONS IN THE GROUP ON (*b*) THE CIRCULAR CYLINDER

1. *Parallel sections of a circular cylinder are congruent.*

   a. The bases of a circular cylinder are congruent.

2. *The volume of a circular cylinder equals the product of the area of its base by its altitude.*

   a. If **H** *be the altitude of any circular cylinder, and* **R** *the radius of either base, the volume of the cylinder equals* $\pi R\,H$.

   b. If **H** *be the altitude of any right circular cylinder, and* **R** *the radius of either base, the area of the convex surface of the cylinder equals* $2\pi RH$.

   c. The volumes of any two circular cylinders are to each other as the products of the areas of the bases by the altitudes.

   If the bases are equal, the cylinders are to each other as the altitudes.

   If the altitudes are equal, the cylinders are to each other as the areas of the bases.

   If the bases are equal and also the altitudes, the cylinders are equal.

Ex. 29. Every plane section of a parallelepiped is a parallelogram if the plane of the section intersects 4 parallel edges.

Ex. 30. What kind of quadrilateral is cut from a parallelepiped by a diagonal plane? How do the diagonals of the section cut each other?

Ex. 31. Show that the diagonals of a parallelepiped concur in a point at which each is bisected.

Def. The point in which the diagonals of a parallelepiped concur is called the **center** of the parallelepiped.

Ex. 32. Show that the converse of this proposition is true.

Ex. 33. Show that any line that passes through the center (K) of a parallelepiped and terminates in opposite faces of the parallelepiped is bisected at K.

Ex. 34. Show that the sum of the squares of the diagonals of a parallelepiped equals the sum of the squares of the edges.

Ex. 35. Show that the diagonals of a rectangular parallelepiped are equal.

Ex. 36. Prove that the converse of the proposition is also true.

Ex. 37. The edges of a rectangular parallelepiped are 12, 15, and 20 ft., respectively. What is the length of the diagonal?

Ex. 38. Show how to construct a parallelepiped that shall have its edges on three given straight lines.

Ex. 39. A plane through any edge of the upper base and the diagonally opposite edge of the lower base of a prism cuts from the figure a rectangle. What kind of prism is the original figure?

Ex. 40. Every plane, through an edge of the upper base, that cuts the lower base of a prism, cuts from the prism a parallelogram. What kind of prism is the original figure?

Ex. 41. The volume of any regular prism is equal to the product of the lateral area by the apothem of either base.

Ex. 42. Show how to cut from a cube a regular hexagon.

Ex. 43. Show that if a prism be cut by two $\nparallel$ planes and the corresponding sides of the sections be produced, the points of intersection of these sides will be collinear.

Ex. 44. Show that the volume of a prism equals the product of the area of a right section by the length of a lateral edge.

Ex. 45. The total surface of a circular cylinder is equal to the convex surface of a cylinder having the same base as the given cylinder, and having an altitude equal to the altitude of the given cylinder plus the radius of the base.

Ex. 46. The volume of a cylinder equals the product of the area of a right section by an element of the cylinder.

Ex. 47. Show that the volume of a cylinder is equal to its convex surface multiplied by $\frac{1}{2}$ the radius of its base.

Ex. 48. What is the locus of a point whose distance from a given straight line is equal to 10 in.?

Ex. 49. What is the locus of a point whose distance from a given plane is a and whose distance from a given line parallel to the plane is b?

Ex. 50. What is the locus of a point in a plane at a distance d from a line that intersects the plane?

Ex. 51. What surface is generated by the axis of a circular cylinder of radius a that rolls on the inner surface of a circular cylinder of radius b?

Ex. 52. Find a point equidistant from two given points, A and B, and also at a distance d from a given straight line.

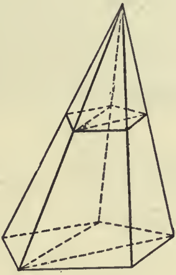
Ex. 53. Find a point equidistant from three given points, A, B, and C, and also at a given distance from a given straight line.

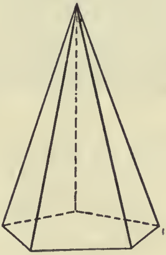
XXIV. GROUP ON THE PYRAMID AND THE CONE

DEFINITIONS

(*a*) The Pyramid

A **Pyramid** is a polyhedron, one face of which is a polygon, while the other faces are triangles that have a common vertex. This common vertex is called the **Vertex** of the pyramid.





The **Base** of a pyramid is the polygon on which the pyramid is supposed to rest.

If all the faces of a pyramid are triangles, any one may be taken as the base.

The faces of a pyramid other than the base are called the **Lateral Faces.**

The sum of the areas of the lateral faces is called the **Lateral** (or **Convex**) **Surface** of the pyramid.

The **Lateral Edges** are the edges that meet in the vertex.

The **Altitude** of a pyramid is the perpendicular dropped from the vertex to the base.

Pyramids are said to be **Triangular, Quadrangular, Pentagonal,** etc., according to the number of sides of the bases.

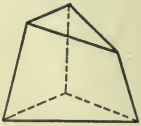
A **Regular** pyramid is one whose base is a regular polygon that has for its center the foot of the altitude of the pyramid.

Corollary of the Definition

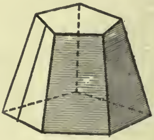
(*a*) *The lateral faces of a regular pyramid are congruent isosceles triangles.*

The **Slant Height** of a regular pyramid is the altitude of any of its lateral faces.

A **Truncated** pyramid is that portion of a pyramid which is comprised between the base and a plane section not parallel to the base.



A **Frustum** of a pyramid is that portion of a pyramid which is comprised between the base and a plane section parallel to the base.



A **Prismoid** is a polyhedron, two of whose faces, called **Bases,** are parallel, while the other faces (**lateral faces**) are triangles or trapezoids that have their vertices at the vertices of the bases.

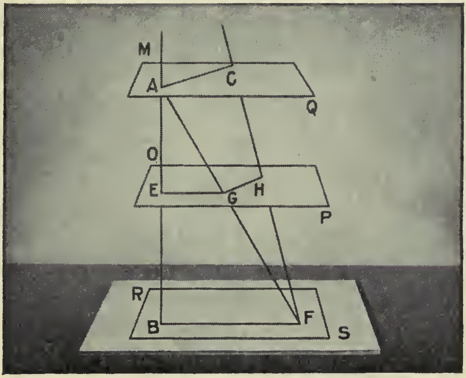
The **Altitude** of a frustum or of a prismoid is the perpendicular distance between the bases.

The **Slant Height** of a regular frustum (*i.e.* a frustum cut from a regular pyramid) is the altitude of any of its lateral faces.

PROPOSITIONS

(*a*) PYRAMIDS

XXIV. 1. *If a set of lines be cut by three parallel planes, the lines are cut proportionally.*



Hyp. If AB and CF are cut by the parallel planes SR, PO, and QM in B, E, A, and F, H, C,

Conc.: then $AE : EB :: CH : HF$.

Dem. Draw the join AF cutting plane PO in G.
Draw the joins BF, EG, GH, and AC.

Then $AC \parallel GH$ and $EG \parallel BF$. (Def. of ∥ planes.)

$\therefore AE : EB :: AG : GF$. (XV. 1.)

But $AG : GF :: CH : HF$. (XV. 1.)

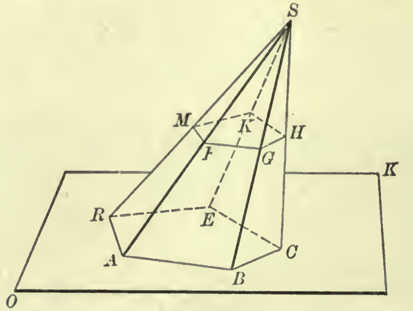
$\therefore AE : EB :: CH : HF$. (Ax. 1.)

Q.E.D.

NOTE. — So also, $AB : AE :: CF : CH$,
and $AB : BE :: CF : HF$.

SCH. The same course of reasoning may be extended to any number of lines and any number of planes.

XXIV. 2. *Any section of a pyramid parallel to the base is similar to the base.*



Hyp. If, in the pyramid $S\text{–}ABCER$, the section $FGHKM$ is parallel to $ABCER$,

Conc.: then n-gon $ABC \cdots \sim n$-gon $FGH \cdots$.

Dem. $AB \parallel FG$ and $RA \parallel MF$. (Def. of $\parallel$ planes.)

$\therefore \angle RAB = \angle MFG$. (XXI. 7.)

Similarly, $\angle ABC = \angle FGH$; $\angle BCE = \angle GHK$, etc.;

i.e. $ABCE$ and $FGHK$ are mutually equiangular.

Again, $\triangle SHG \sim \triangle SCB$;

$\triangle SGF \sim \triangle SBA$, etc. (XV. 2.)

$\therefore BC : GH :: BS : GS$. (Hom. sides of $\sim \triangle$s.)

And $AB : FG :: BS : GS$. (Same reason.)

$\therefore AB : FG :: BC : GH$. (Ax. 1.)

Similarly, the other pairs of homologous sides are proportional.

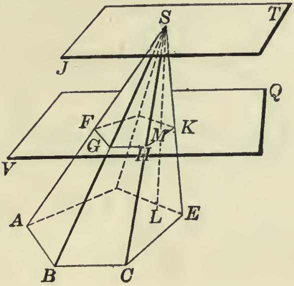
$\therefore n$-gon $ABCE \cdots \sim n$-gon $FGHK \cdots$.

(Def. of $\sim$ figs.)

Q.E.D.

Ex. 1. From a point A without a plane obliques are drawn terminating on the plane. On each oblique a point P is so taken as to divide the line in the ratio of $2 : 3$. Show that the locus of P is a plane. (XXIV. 1.)

XXIV. 2 *a*. *The perimeters of parallel sections of a pyramid are to each other as the distances of the sections from the vertex.*



Hyp. If SM is perpendicular to plane VQ in M and perpendicular to plane AE in L, and if section $FGHK \cdots$ is parallel to section $ABCE \cdots$,

Conc.: then perim. $FGHK$: perim. $ABCE \cdots :: SM : SL$.

Dem. Through S pass a plane JT parallel to plane VQ parallel to plane AE.

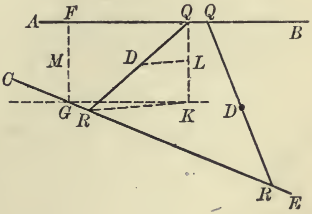
Perim. $FGHK$: perim. $ABCE :: FG : AB$. (XXIV. 2.)

But $FG : AB :: SG : SB$. (Hom. sides of $\sim \triangle$.)

And $SG : SB :: SM : SL :: SF : SA$, etc. (XXIV. 1.)

$\therefore$ perim. $FGHK$: perim. $ABCE :: SM : SL$. (Ax. 1.)

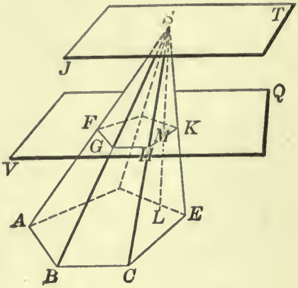
Q.E.D.



Ex. 2. AB and CE are two lines not in the same plane. Any number of lines QR terminating in AB and CE are bisected at D. Show that the locus of D is a plane through the midpoint M of the common $\perp$ FG.

Pass a plane through M parallel to AB and CE. Through AB and CE pass planes parallel to the first plane. Use XXIV. 1.

XXIV. 2 *b*. *The areas of two parallel sections of a pyramid are to each other as the squares of their distances from the vertex.*



Hyp. If SM is perpendicular to plane VQ in M and perpendicular to plane AE in L, and if section $FGHK \cdots$ is parallel to section $ABC \cdots$,

Conc: then area of

$$n\text{-gon } FGHK : \text{area of } n\text{-gon } ABCE :: \overline{SM}^2 : \overline{SL}^2.$$

Dem. n-gon $FGHK : n$-gon $ABCE :: \overline{FG}^2 : \overline{AB}^2$. (XVI. 3.)

But $FG : AB :: SG : SB :: SM : SL$. (XXIV. 1.)

$\therefore \overline{FG}^2 : \overline{AB}^2 :: \overline{SG}^2 : \overline{SB}^2 :: \overline{SM}^2 : \overline{SL}^2$. (XI. ($C$).)

$\therefore$ area of n-gon $FGHK$: area of n-gon $ABCE :: \overline{SM}^2 : \overline{SL}^2$.

Q.E.D.

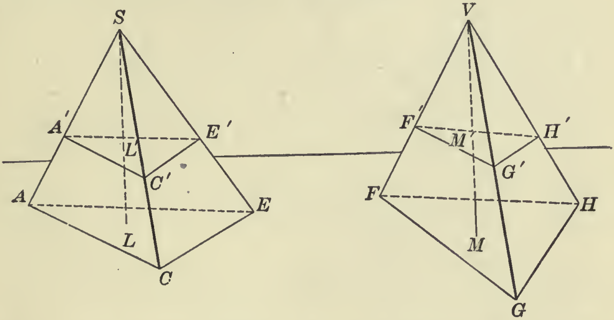
Ex. 3. The base of a pyramid is a regular octagon whose radius is 10 ft. The altitude of the pyramid is 24 ft. Find the perimeter and the area of a section parallel to the base and 6 ft. from the vertex.

Ex. 4. Two pyramids have the same volume. The area of the base of the first is 120 sq. yd. and its altitude 60 ft. The altitude of the second is 35 ft. What is the area of its base?

Ex. 5. The altitude of a regular pyramid is 15 ft. Its base is a square, each side of which is 4 ft. Find the volume of the pyramid.

Find also the lateral surface and the total surface of the pyramid.

XXIV. 2 c. *If two pyramids have equal altitudes and equal bases, sections parallel to their bases and equally distant from their vertices are equal.*



Hyp. If $\triangle ACE = \triangle FGH$, altitude $SL =$ altitude VM, and $SL' = VM'$,

Conc.: then n-gon $A'C'E' = n$-gon $F'G'H'$.

Dem. Area ACE : area $A'C'E' :: \overline{SL}^2 : \overline{SL'}^2$. (XXIV. 2 *b*.)

Area FGH : area $F'G'H' :: \overline{VM}^2 : \overline{VM'}^2$.

But $\overline{SL}^2 : \overline{SL'}^2 :: \overline{VM}^2 : \overline{VM'}^2$. (Hyp.)

$\therefore$ area ACE : area FGH :: area $A'C'E'$: area $F'G'H'$.

But area $ACE =$ area FGH.

$\therefore$ area $A'C'E' =$ area $F'G'H'$.

Q.E.D.

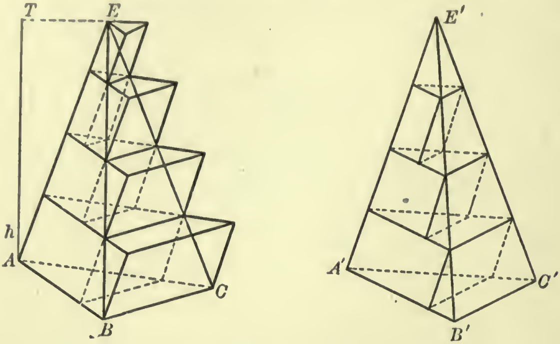
Note.—From the fourth proportion of the demonstration, it follows that if two pyramids have equal altitudes, the areas of sections parallel to the bases and equidistant from them have the same ratio as the bases.

Ex. 6. The base of a regular pyramid is a dodecagon whose radius is 20 in. The volume of the pyramid is 4000 cu. in. Find the altitude.

Ex. 7. Find the total surface and the volume of a regular triangular pyramid, each edge of which is *a*.

Ex. 8. Each lateral edge of a regular triangular pyramid is 29 ft.; the altitude is 21 ft. Find the total surface and the volume of the pyramid.

XXIV. 3. *Two triangular pyramids that have equal bases and equal altitudes are equal.*



Hyp. If, in the triangular pyramids E and E', base $ABC =$ base $A'B'C'$, and if the altitudes of the two pyramids are equal,

Conc.: then Pyr. $E =$ Pyr. E'.

Dem. If $E \neq E'$, let $E - E' = r$.

Divide altitude AT into any number, say four, equal parts.

Through each point of division pass parallel sections.

On the base ABC construct a prism whose lateral edges are parallel to AE and whose altitude equals $\frac{1}{4}\,AT$.

Similarly, construct a prism upon each section as a base.

This set of prisms is circumscribed about Pyr. E.

With the topmost section of Pyr. E' as an upper base, construct a prism whose lateral edges are parallel to $A'E'$ and whose altitude equals $\frac{1}{4}\,AT$.

Similarly, construct prisms with the remaining sections as upper bases.

This set of prisms is inscribed in Pyr. E'.

Each prism in Pyr. E' equals the prism next above it in Pyr. E. (XXIII. 8 *b*.)

$\therefore$ the difference between the sums of prisms of Pyr. E and Pyr. E' is the lowest prism of Pyr. E.

Let sum of prisms in Pyr. $E = S$, and sum of prisms in Pyr. $E' = S'$, and let volume of lowest prism in $E = v$.

Then $$S - S' = v.$$

But $$E < S \text{ and } S' < E'.$$

$$\therefore E + S' < S + E'. \quad \text{(Preliminary Th. 1.)}$$

$$\therefore E - E' < S - S'. \quad \text{(Preliminary Th. 3.)}$$

That is, $$E - E' < v, \text{ or } r < v. \qquad (1)$$

Now, if the number of equal parts into which altitude AT is divided is increased, the difference, v, is correspondingly decreased.

Evidently, by increasing the number of equal parts of AT indefinitely, this difference, or v, can be made smaller than any assignable value except zero.

In other words, v may be less than r, a constant. (2)

But we have just proved that $r < v$. (1)

$\therefore$ unless r equals zero, (2) contradicts (1).

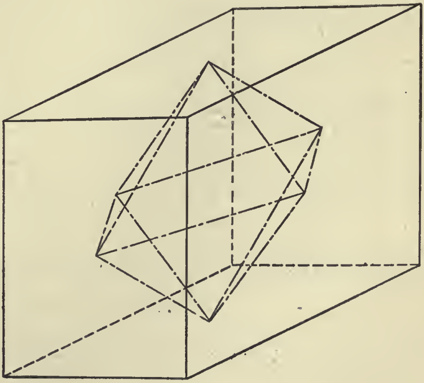
$\therefore$ r must equal zero.

That is, $$\text{Pyr. } E = \text{Pyr. } E'.$$

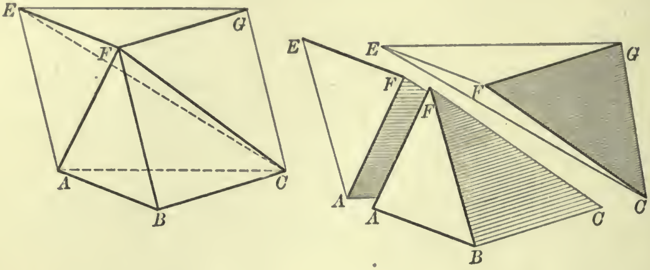
Q.E.D.

Ex. 9. The edges of an oblong block are a, b, and c. The centers of the faces are joined as shown in the figure. Find the volume of the octahedron thus formed.

Ex. 10. Join the center of a cube to its vertices. By considering the pyramids thus formed, show that the volume of a cube equals one sixth the product of its total surface by one edge.



XXIV. 4. *The volume of a triangular pyramid is one third the product of its base and altitude.*



Hyp. If $F\text{–}ABC$ is a triangular pyramid of base b and altitude h,

Conc.: then volume of pyramid $F\text{–}ABC = \frac{1}{3} b \cdot h$.

Dem. Complete the triangular prism $ABC\text{–}GFE$.
Draw EC.
This prism minus pyramid $F\text{–}ABC$ equals pyramid $F\text{–}ACGE$.

$$\text{Pyr. } F\text{–}ACGE = \text{Pyr. } F\text{–}AEC + \text{Pyr. } F\text{–}GEC.$$

$$\triangle AEC \cong \triangle GEC,$$

and the altitudes of these two pyramids are equal.

$$\therefore \text{ Pyr. } F\text{–}AEC = \text{Pyr. } F\text{–}GEC. \qquad \text{(XXIV. 3.)}$$

But pyramid $F\text{–}ABC$ may be read $C\text{–}ABF$, and pyramid $F\text{–}AEC$ may be read $C\text{–}AEF$.

But $\qquad \text{Pyr. } C\text{–}ABF = \text{Pyr. } C\text{–}AEF. \qquad$ (XXIV. 3.)

$$\therefore \text{ Pyr. } C\text{–}ABF = \text{Pyr. } C\text{–}AEF = \text{Pyr. } F\text{–}GEC. \qquad \text{(Ax. 1.)}$$

But the sum of these three pyramids equals the prism.

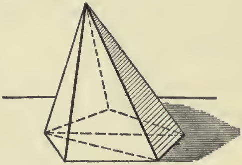
$$\therefore \text{ volume of prism} = b \cdot h.$$

$$\therefore \text{ volume of Pyr. } F\text{–}ABC = \tfrac{1}{3} b \cdot h.$$

Q.E.D.

XXIV. 4 *a*. *The volume of any pyramid equals one third the product of its base and altitude.*

NOTE. — Let the student prove this corollary by dividing the given pyramid into triangular pyramids with a common altitude and taking their sum.



XXIV. 4 *b*. *The volumes of any two pyramids are to each other as the products of their bases and altitudes.*

(Let the student give the proof, using XXIV. 4 *a*.)

SCH. 1. If the bases are equal, the pyramids are to each other as their altitudes.

If the altitudes are equal, the pyramids are to each other as their bases.

SCH. 2. The volume of any polyhedron may be found by dividing the figure into pyramids and adding the volumes of these pyramids.

Ex. 11. The great pyramid of Cheops is 486 ft. high and its base is a square 768 ft. on a side. Find the lateral surface in square yards.

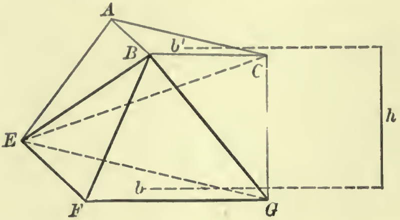
Ex. 12. Find the volume and the total surface of a regular quadrangular pyramid each of whose base edges is 10 ft. and each of whose lateral edges is 20 ft.

Ex. 13. Each edge of the base of a regular octagonal pyramid is 12 ft. Each lateral edge is 40 ft. Find the convex surface and the volume of the pyramid.

Ex. 14. The lateral edges of a regular pentagonal pyramid are each 5 a. Each side of the base is a. Find the volume and the total surface of the pyramid.

Ex. 15. The altitude of a square pyramid, each edge of whose base is a, is equal to the diagonal of the base. Find the volume and the total surface of the pyramid.

XXIV. 5. *The volume of any triangular frustum equals one third the product of the altitude into the sum of the two bases and a mean proportional between them.*



Hyp. If $ABC\text{–}G$ is a triangular frustum of bases b and b', and of altitude h,

Conc.: then volume of $ABC\text{–}G = \frac{1}{3}hb + \frac{1}{3}hb' + \frac{1}{3}h\sqrt{bb'}$

$$= \tfrac{1}{3}h(b + b' + \sqrt{bb'}).$$

Dem. Draw BE, BG, and EC.

Fr. $ABC\text{–}G =$ Pyr. $B\text{–}EFG +$ Pyr. $E\text{–}ABC +$ Pyr. $B\text{–}GCE$.

Vol. Pyr. $B\text{–}EFG = \frac{1}{3}h \cdot b$. (1) (XXIV. 4.)

Vol. Pyr. $E\text{–}ABC = \frac{1}{3}h \cdot b'$. (2)

Pyramids $B\text{–}GCE$ and $B\text{–}ACE$ have the same vertex B, and their bases lie in the same plane $ACGE$.

$\therefore$ Pyr. $B\text{–}GCE :$ Pyr. $B\text{–}ACE :: \triangle GCE : \triangle ACE$.

(XXIV. 4 *b*. Sch.)

But $\triangle GCE : \triangle ACE :: EG : AC$. (XIII. 1 *c*. Sch. 2.)

And $EG : AC :: \sqrt{b} : \sqrt{b'}$. (XVI. 2.)

$\therefore$ Pyr. $B\text{–}GCE :$ Pyr. $B\text{–}ACE :: \sqrt{b} : \sqrt{b'}$.

But Pyr. $B\text{–}ACE \equiv$ Pyr. $E\text{–}ABC = \frac{1}{3}h \cdot b'$.

$\therefore$ Pyr. $B\text{–}GCE : \frac{1}{3}h \cdot b' :: \sqrt{b} : \sqrt{b'}$.

$\therefore$ Pyr. $B\text{–}GCE : \frac{1}{3}h \cdot \sqrt{b'} :: \sqrt{b} : 1$.

[Dividing the consequents by the common factor $\sqrt{b'}$.]

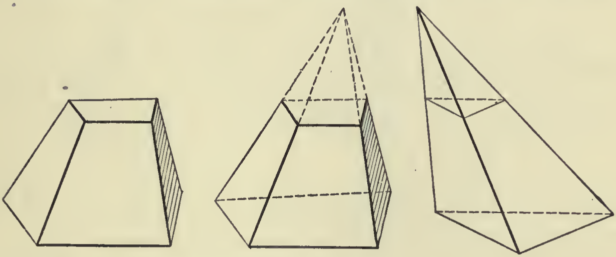
$$\therefore \text{ Pyr. } B\text{–}GCE = \tfrac{1}{3} h \cdot \sqrt{bb'}. \qquad (3)$$

$$\therefore \text{ volume of } ABC\text{–}G = \tfrac{1}{3} hb + \tfrac{1}{3} hb' + \tfrac{1}{3} h\sqrt{bb'} \text{ (adding (1), (2), (3))}$$

$$= \tfrac{1}{3} h\,(b + b' + \sqrt{bb'}).$$

Q.E.D.

XXIV. 6. *The volume of any frustum equals one third the product of the altitude into the sum of the two bases and a mean proportional between them.*



(Let the student supply the proof.)

Ex. 16. If a, b, c, etc., be the sides of any section of a pyramid, and a', b', c', etc., the sides of any other section of the pyramid, then the points of intersection (a, a'), (b, b'), etc., lie in one straight line.

Ex. 17. The slant height of a regular frustum is 20 in.; its bases are squares; each side of the upper base is 12 in. and each side of the lower base is 8 in. Find the lateral surface and the whole surface of the frustum.

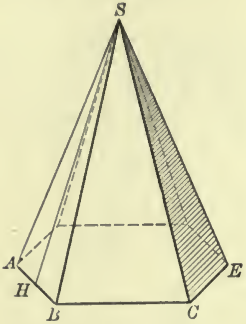
Ex. 18. The altitude of a regular frustum is 8 ft.; its lower base is a square, each side of which is 6 ft., and its upper base has an area of 16 sq. ft. What is the volume of the frustum?

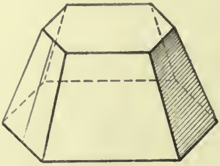
Ex. 19. The area of the upper base of a frustum is 125 sq. yd.; the area of the lower base is 500 sq. yd.; the altitude is 60 yd. Find the volume.

Ex. 20. The altitude of a regular hexagonal frustum is 10 ft.; the radius of the upper base is 6 ft., and the radius of the lower base 10 ft. Find the volume of the frustum.

Ex. 21. Find the volume of a regular hexagonal frustum the upper base of which has a radius of 6 ft.; the lower base a radius of 10 ft.; and of which each lateral edge is 5 ft.

XXIV. 7. *The lateral surface of a regular pyramid equals one half the rectangle of the slant height and the perimeter of the base.*





Hyp. If the slant height SH of a regular pyramid equals H', and the perimeter of the base equals P,

Conc.: then the lateral surface $= \frac{1}{2} P \cdot H'$.

Dem. SAB, SBC, etc., are congruent isosceles triangles. (Def. regular pyramid. Cor. (*a*).)

∴ their altitudes are each equal to SH.

$$\text{Area } SCE = \tfrac{1}{2} CE \cdot SH. \qquad \text{(XIII. 2 } a.\text{)}$$

$$\text{Area } SCB = \tfrac{1}{2} BC \cdot SH.$$

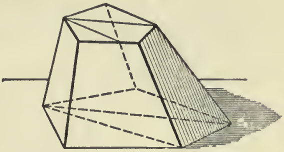
Similarly, for the other faces.

∴ by addition,

$$\text{lateral surface} = \tfrac{1}{2}(BC + CE + \text{etc.})\, SH = \tfrac{1}{2} P \cdot H'.$$

Q.E.D.

XXIV. 7 *a*. *The lateral surface of a regular pyramid equals the rectangle of the slant height and the perimeter of a section parallel to the base and midway between the vertex and the base (called the mid-section).*



XXIV. 7 *b*. *The lateral surface of a regular frustum equals the rectangle of the slant height and the perimeter of the mid-section.*

XXIV. 7 *c*. *The lateral surface of a regular frustum equals the rectangle of the slant height and half the sum of the perimeters of the bases.*

Ex. 22. The lower base of a regular pentagonal frustum has a radius of a. The area of the upper base is one third the area of the lower. The altitude of the pyramid from which the frustum is cut is b. Find the convex surface of the frustum.

Ex. 23. The lower base of a regular octagonal frustum is 9 times as large as the upper. The radius of the upper base is a and the slant height of the pyramid from which the frustum is cut is b. Find the volume of the frustum.

Ex. 24. The radius of the upper base of a regular octagonal frustum is 12 ft., the altitude of the frustum is 24 ft. The area of the lower base is to the area of the upper as 16 : 9. Find the convex surface, the total surface, and the volume of the frustum.

Ex. 25. The volume of a cube is equal to the volume of a regular triangular frustum whose altitude is c and the sides of whose bases are a and b respectively. Find the length of the edge of the cube.

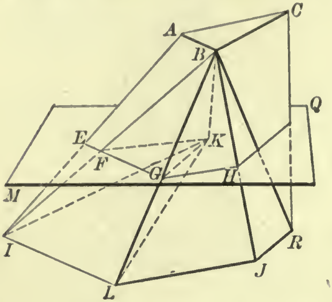
Ex. 26. The total surface of a right circular cone, the radius of whose base is 12 in., is $1178\frac{4}{7}$ sq. in. Find the slant height of the cone.

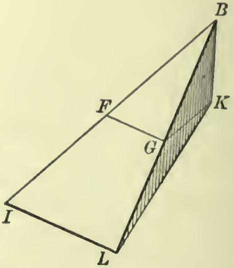
Ex. 27. The convex surface of a right circular cone is two thirds of the total surface. Show that the slant height of the cone equals the diameter of the base.

Ex. 28. The total surface of a right circular cone 8 ft. in diameter is equal to the total surface of a right circular cylinder 5 ft. in diameter; the altitude of the cone is 10 ft. Find the altitude of the cylinder.

Ex. 29. Show that the total surface of a right circular cone equals the convex surface of a cone of the same base whose slant height is the sum of the radius of the base and the slant height of the given cone.

XXIV. 8. *The volume of a prismoid equals one sixth the product of its altitude by the sum of the areas of the bases and four times the area of the mid-section.*





Hyp. If $\mathbf{b}_1$ denotes the area of ABC, $\mathbf{b}_2$ the area of $ILJ \cdots$, $\mathbf{m}$ the area of $EGH \cdots$, $\mathbf{h}$ the altitude, and $\mathbf{V}$ the volume, of the prismoid $L\text{–}C$,

Conc.: then $$\mathbf{V} = \frac{\mathbf{h}}{6}(\mathbf{b}_1 + \mathbf{b}_2 + 4\,\mathbf{m}).$$

Dem. Join any point K, of $EGH \cdots$ to the vertices of the prismoid. Draw diagonals BI, BR, etc. in each trapezoid face.

Pyramid $K\text{–}ABC$ has b_1 for base and $\frac{h}{2}$ for altitude.

(Def. of mid-section.)

$$\therefore \text{ vol. } K\text{–}ABC = \frac{h}{6} \cdot b_1. \qquad \text{(XXIV. 4.)}$$

Similarly, $$\text{vol. } K\text{–}ILJ \cdots = \frac{h}{6} \cdot b_2.$$

The bases of the other set of pyramids are $\triangle$s, cut in mid-joins by the plane MQ of the mid-section. (XXIV. 1.)

Let $K\text{–}IBL$ be any one of these pyramids. Draw KG, KF.

$$\frac{\text{Pyramid } K\text{–}IBL}{\text{Pyramid } K\text{–}FBG} = \frac{\text{Area } IBL}{\text{Area } FBG} \qquad \text{(XXIV. 4 } b\text{. Sch.)}$$

$$= \frac{BL^2}{BG^2} \qquad \text{(XVI. 2.)}$$

$$= 4. \quad (\because BL = 2\,BG.)$$

But $$K\text{–}FBG \equiv B\text{–}KFG = \frac{h}{6} \cdot \text{area } KFG. \quad \text{(Why?)}$$

$$\therefore\ K\text{–}IBL = 4\,(K\text{–}FBG) = \frac{4\,h}{6} \cdot \text{area } KFG.$$

The sum of the areas of the $\triangle$, KFG, etc. $= m$. (Ax. 4.)
$\therefore$ the sum of the pyramids whose bases are the lateral faces of the prismoid $= \frac{4\,h}{6} m$.

$$\therefore\ V = \frac{h}{6}(b_1 + b_2 + 4\,m).$$

Q.E.D.

Sch. The formula just established is often called the **Prismoidal Formula.**

Ex. 30. The elements of a right circular cone make an angle of 45° with the plane of the base. Find the ratio of the area of the base to the convex surface of the cone.

Ex. 31. The elements of a right circular cone make an angle of 60° with the plane of the base. Find the ratio of the area of the base to the total surface of the cone.

Ex. 32. Find the volume of a right circular cone of which the altitude is 6 ft. and the base has a radius of 30 in.

Ex. 33. Find the volume of an oblique cone of which the altitude is 12 yd. and the base has a diameter of 3 yd.

Ex. 34. Find the volume of a cone whose base has a radius of 10 ft. and whose slant height is 27 ft.

Ex. 35. The convex surface of a right circular cone is 523.6 sq. ft.; the slant height is 20 ft. Find the total surface and the volume of the cone.

Ex. 36. The slant height of a right circular cone is 60 ft.; each element makes an angle of 60° with the plane of the base. Find the total surface and the volume of the cone.

Ex. 37. The volume of a right circular cone the radius of whose base is 12 in. is a cubic foot. Find the altitude and the convex surface of the cone.

Ex. 38. One angle of a right triangle is 30°. Find the ratio between the volumes of the cones generated by revolving the triangle first about the shorter leg as an axis and then about the longer.

XXIV. SUMMARY OF PROPOSITIONS IN THE GROUP ON (*a*) THE PYRAMID

1. *If a set of lines be cut by three parallel planes, the lines are cut proportionately.*

2. *Any section of a pyramid parallel to the base is similar to the base.*

   a. *The perimeters of parallel sections of a pyramid are to each other (vary) as the distances of the sections from the vertex.*
   b. *The areas of two parallel sections of a pyramid vary as the squares of the distances of the sections from the vertex.*
   c. *If two pyramids have equal bases and equal altitudes, any sections of the two pyramids equidistant from the vertices are equal.*

3. *Two triangular pyramids that have equal bases and equal altitudes are equal.*

4. *The volume of a triangular pyramid is one third the product of its base and altitude.*

   a. *The volume of any pyramid equals one third the product of its base and altitude.*
   b. *The volumes of any two pyramids are to each other as the products of their bases and altitudes.*

Sch. 1. If the bases are equal, the pyramids are to each other as their altitudes.

If the altitudes are equal, the pyramids are to each other as their bases.

SCH. 2. The volume of any polyhedron may be found by dividing the figure into pyramids and adding the volumes of these pyramids.

5. *The volume of a triangular frustum equals one third the product of the altitude into the sum of the two bases and a mean proportional between them.*

6. *The volume of any frustum equals one third the product of the altitude into the sum of the two bases and a mean proportional between them.*

7. *The lateral surface of a regular pyramid equals one half the rectangle of the slant height and the perimeter of the base.*

   a. *The lateral surface of a regular pyramid equals the rectangle of the slant height and the perimeter of a section parallel to the base and midway between the vertex and the base (called the mid-section).*
   b. *The lateral surface of a regular frustum equals the rectangle of the slant height and the perimeter of the mid-section.*
   c. *The lateral surface of a regular frustum equals the rectangle of the slant height and half the sum of the bases.*

8. *The volume of a prismoid equals one sixth the product of its altitude by the sum of the areas of the bases and four times the area of the mid-section.*

SCH. The formula just established is often called the **Prismoidal Formula.**

DEFINITIONS

(*b*) THE CONE

A **Conical Surface** is a surface generated by a straight line that moves along a fixed curve and always passes through a fixed point not coplanar with the curve.

The moving line is called the **Generatrix.**

The fixed curve is called the **Directrix.**

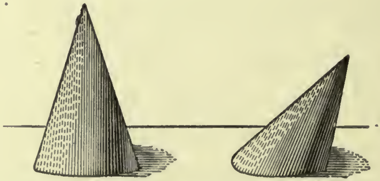
It is not necessary that the directrix be a plane curve.

The successive positions of the generatrix are called the **Elements of the Surface.**

The fixed point through which the generatrix passes is called the **Vertex.**

The **Nappes** of the surface are the two parts into which it is divided at the vertex.

A **Cone** is a portion of space bounded by one nappe of a conical surface and a plane not passing through the vertex.



The **Base** of the cone is the plane section that forms part of its bounding surface.

The **Elements of the Cone** are the segments of the elements of the conical surface determined by the vertex and the base.

The **Altitude** of a cone is the perpendicular from the vertex to the base.

A **Circular** cone is one that has a circle for its base.

A **Right** cone is one in which the altitude falls at the center of the base.

The terms **Truncated** and **Frustum** have the same meaning in the case of the cone as in that of the pyramid.

Note. — A frustum of a right circular cone is called a right circular frustum.

Corollaries from the Definitions

(*a*) *Every section of a cone that passes through the vertex is a triangle.*

(*b*) *The right circular cone is a cone of revolution, of which the axis is the altitude.*

General Sch. As the **Circle** is the limit of the **Regular Polygon** when the number of sides of the latter is increased beyond any assignable value, so the **Circular Cone** is the limit of the **Pyramid** *with a Regular Base* when the number of lateral faces is increased beyond any assignable value.

Hence, whatever propositions are true of such a **Pyramid**, irrespective of the number of its lateral faces, are true of the **Circular Cone**.

Accordingly, from the corresponding propositions on the pyramid, we have the following summary:

XXIV. SUMMARY OF PROPOSITIONS IN THE GROUP ON (*b*) THE CIRCULAR CONE

9. *Sections of a cone parallel to the base are similar to the base.*

10. *If sections of a cone are parallel to the base,*
    - (*a*) *their circumferences are to each other as their distances from the vertex, and*
    - (*b*) *their areas are to each other as the squares of their distances from the vertex.*

11. *The convex surface of a right circular cone equals one half the product of the circumference of its base by its slant height.*

a. If $\mathbf{R}$ *is the radius of the base of a right circular cone, and* $\mathbf{H}'$ *the slant height of the cone, the convex surface of the cone equals* $\pi\mathbf{RH}'$, *and the total surface equals* $\pi\mathbf{R}(\mathbf{H}'+\mathbf{R})$.

b. The convex surface of a right circular frustum equals the product of the slant height by the circumference of the mid-section.

c. If $\mathbf{R}_1$ *is the radius of the upper base of a right circular frustum,* $\mathbf{R}_2$ *the radius of the lower base, and* $\mathbf{h}'$ *the slant height, the convex surface of the frustum equals* $\pi\mathbf{h}'(\mathbf{R}_1+\mathbf{R}_2)$.

12. *The volume of a circular cone equals one third the product of the area of its base by its altitude.*

a. If $\mathbf{R}$ *is the radius of the base of a circular cone and* $\mathbf{H}$ *the altitude, the volume of the cone equals* $\frac{1}{3}\pi\mathbf{R}^2\mathbf{H}$.

b. Any two cones are to each other as the products of the areas of the bases by the altitudes.

If the altitudes are equal, the cones are to each other as the areas of the bases.

If the bases are equal, the cones are to each other as the altitudes.

If the bases are equal and also the altitudes, the cones are equal.

13. *The volume of the frustum of a cone equals one third the product of the altitude by the sum of the areas of the bases and a mean proportional between them.*

Ex. 39. The radius of the upper base of a right circular frustum is 25 in.; the radius of the lower base is 36 in.; the slant height is 15 in. Find the convex surface and also the total surface of the frustum.

Ex. 40. Show that the convex surface of a right circular frustum equals the convex surface of a cylinder whose altitude is half the sum of the diameters of the bases of the frustum and whose diameter is the slant height of the frustum.

Ex. 41. The radius of the upper base of a right circular frustum equals 12 ft.; the slant height is 14 ft.; the convex surface is 4928 sq. ft. What is the radius of the lower base? (Take $\pi = \frac{22}{7}$.)

Ex. 42. The radii of the bases of a right circular frustum are 16 in. and 24 in. respectively. Its convex surface is one half its total surface. What is the slant height of the frustum? What is the altitude of the frustum?

Ex. 43. The convex surface of a right circular frustum is three fourths of the convex surface of the cone from which it has been cut. Find the ratio of the altitude of the frustum to the altitude of the cone.

Ex. 44. The radius of the upper base of a frustum is 5 ft.; the radius of the lower base is 8 ft.; the altitude is 12 ft. What is the volume of the frustum?

Ex. 45. The radii of the bases of a right circular frustum are 20 ft. and 24 ft. respectively. Its volume is 4928 cu. ft. What is its altitude?

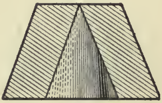
Ex. 46. The volume of a right circular frustum is 2200 cu. ft.; the slant height is 13 ft. and the altitude is 12 ft. Find the radii of the bases.

Ex. 47. How much sheet tin will be required to construct a water pail 18 in. in diameter at the top, 14 in. in diameter at the bottom, and having a capacity of 8 gal., allowing a waste of 10 per cent in the material in seams and cuttings?

Ex. 48. The volume of a right circular frustum is three fourths of the volume of the cone from which it has been cut. What is the ratio of the altitude of the frustum to the altitude of the cone?

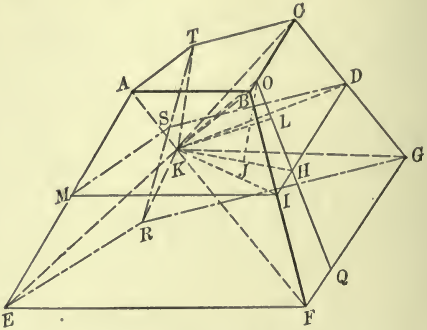
Ex. 49. The area of the lower base of a right circular frustum is four times the area of the upper base. The volume of the frustum equals the volume of a right circular cylinder whose base is the upper base of the frustum. Find the ratio of the altitude of the cylinder to the altitude of the frustum.

Ex. 50. From a right circular frustum whose upper diameter is $2a$ and lower diameter $4a$ is bored out, as shown in the figure, a right circular cone whose base has a diameter $2a$. Find the ratio of the volume of the original frustum to the volume of the hollow frustum.



$A \cdots G$ is a **prismoid**; $MIDS$ is a mid-section parallel to the bases. K, any point whatever of $MIDS$, is joined to the vertices of the prismoid.

Denote the altitude by h, the upper base by b, the lower by b', and the mid-section by m.



Ex. 51. Show that the volume of pyramid

$$K\text{-}ABCT = \frac{h}{6} b,$$

and the volume of pyramid

$$K\text{-}EFGR = \frac{h}{6} b'.$$

Draw $KH \perp DI$ and $OJ \perp$ both CB and KH.

Draw OQ through H and $KL \perp OQ$.

(XXI. Prob. 2.)

$OQ \perp DI$ and $KL \perp BCGF$. (XXI. 5, 5 *a*, 5 *b*, 3, 3 *a*.)

Ex. 52. Show that $$OJ \cdot KH = KL \cdot OH,$$

whence, $$2\, OJ \cdot KH \cdot DI = KL \cdot 2\, OH \cdot DI,$$

and the volume of pyramid $K\text{-}CBFG = \frac{1}{3} KL \cdot OQ \cdot DI = \frac{h}{3} \cdot 2 \triangle DKI$

$$= \frac{h}{6} \cdot 4 \triangle DKI.$$

Ex. 53. Hence show, by considering all the pyramids, that the

$$\text{volume of the prismoid} = \frac{h}{6}(b + b' + 4\,m).$$

Ex. 54. Show that if the prismoid be the frustum of a pyramid, the above expression may be reduced to that given in XXIV. 6.

Ex. 55. Show that any plane through the center of a parallelepiped divides the figure into two equal parts.

Ex. 56. Divide a parallelepiped into two equal parts by a right section.

Ex. 57. Divide a parallelepiped into two equal parts by a plane parallel to a given plane.

Ex. 58. A parallelepiped is given and also two lines in space. Show how to pass a plane parallel to the given lines that shall divide the parallelepiped into two equal parts.

XXV. GROUP ON THE SPHERE

DEFINITIONS

A **Spherical Surface** is a surface generated by the revolution of a semicircumference (the generatrix) about its diameter as an axis.

A **Sphere** is a portion of space inclosed by a spherical surface.

NOTE. — As in the case of *circle* and *circumference*, the terms *sphere* and *spherical surface* are used interchangeably where no confusion is likely to result.

The **Radius** of the generatrix is the radius of the sphere.

A **Great Circle** of a sphere is a circle on the sphere whose plane passes through the center of the sphere.

A **Small Circle** of a sphere is a circle on the sphere whose plane does not pass through the center of the sphere.

The **Axis of a Circle** of a sphere is the perpendicular to the plane of the circle at its center.

The **Poles** of a circle are the points in which the axis of the circle intersects the surface of the sphere.

The **Polar Distance** of a point on a circumference on the sphere is the length of the great circle arc joining the point to the nearer pole of that circumference.

A **Plane** is **Tangent** to a sphere when it has one point, and only one, in common with the surface of the sphere.

A sphere is **Circumscribed** to a polyhedron when all the vertices of the polyhedron lie on the surface of the sphere.

A sphere is **Inscribed** in a polyhedron when the faces of the polyhedron are all tangent to the sphere.

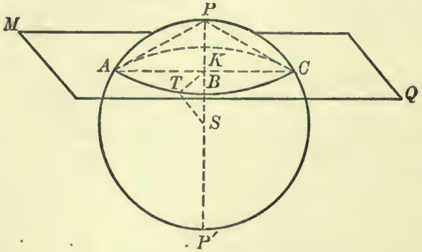
A **Zone** is a portion of the surface of the sphere comprised between two parallel planes.

A **Spherical Segment** is a portion of the volume of the sphere comprised between two parallel planes.

A **Spherical Sector** is the portion of the sphere generated by the revolution of a sector of the generatrix.

PROPOSITIONS

XXV. 1. *Every plane section of a sphere is a circle.*



Hyp. If the plane MQ cuts the sphere S in the line $ABC \cdots$,

Conc.: then $ABC \cdots$ is a circle.

Dem. Through S draw PP' perpendicular to plane MQ, and intersecting the plane MQ in K.

Suppose T to be any point whatever in $ABC \cdots$.

Draw ST and KT.

Let $ST = R$, $KT = r$, and $SK = d$.

Then, in the right triangle SKT,

$$r = \sqrt{R^2 - d^2}. \qquad \text{(XIV. 1 } a.\text{)}$$

But R is constant for the given sphere, and d is constant for the given plane.

$\therefore$ r is constant for all points in their line of intersection.

That is, the distance of T from K is the same for all positions of T.

$\therefore$ $ABC \cdots$ is a circle whose center is K.

Q.E.D

XXV. 1 *a*. *The join of the center of the sphere and the center of any circle on the sphere is the axis of the circle, and conversely.*

XXV. 1 *b*. *The locus of the centers of all spheres that pass through three given points is the axis of the circle that passes through the points.*

XXV. 1 *c*. *Circles cut out by planes equidistant from the center of the sphere are equal, and conversely.*

Dem. $r = \sqrt{R^2 - d^2}$. (XIV. 1 *a*.)

$\therefore$ if d remains the same, r must remain the same, and if r remains the same, d must remain the same.

Q.E.D.

XXV. 1 *d*. *Of two circles cut by planes unequally distant from the center, the one nearer the center is the greater, and conversely.*

Dem. $r = \sqrt{R^2 - d^2}$. (XIV. 1 *a*.)

When $d = R$, $r = \sqrt{R^2 - R^2} = 0$.

As d diminishes, $R^2 - d^2$ increases.

$\therefore$ r, which $= \sqrt{R^2 - d^2}$, increases.

$\therefore$ the nearer a circle is to the sphere center, the greater is the radius of the circle; *i.e.* the greater is the circle.

Q.E.D.

XXV. 1 *e*. *The polar distances of all points in the circumference of a circle of the sphere are equal.*

Sch. From this property the polar distance of the points of a circle are called arc-radii of the circle.

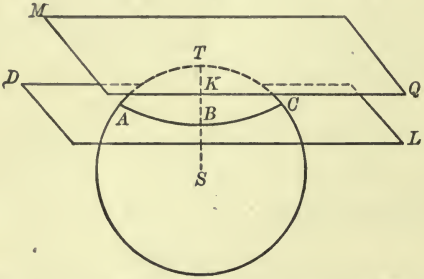
The arc-radius of a great circle is a great-circle quadrant, called simply a quadrant.

The arc-radius of a small circle is less than a quadrant.

XXV. 1 *f.* *Three points on the sphere surface (not on the same great circle, no two of which are the extremities of a diameter) are necessary and sufficient to determine a small circle of the sphere.*

XXV. 1 *g.* *Two points on the sphere surface (not the extremities of a diameter) are necessary and sufficient to determine a great circle of the sphere.*

XXV. 2. *A plane perpendicular to a radius at its extremity is tangent to the sphere, and conversely.*



Hyp. If the plane MQ is perpendicular to ST at T,

Conc.: then plane MQ is tangent to the sphere; and conversely.

Dem. Let DL be any plane perpendicular to ST, and cutting the sphere.

All points common to DL and the sphere lie in the circle ABC, of radius $r = \sqrt{R^2 - d^2}$. (XIV. 1 *a.*)

Move the plane DL away from S, keeping it $\perp$ to ST.

As d increases, r decreases; until, when

$$d = R,\ r = \sqrt{R^2 - R^2} = 0;$$

i.e. the circle ABC which contains all points common to the plane and the sphere becomes itself a point, T.

$\therefore$ this point is a unique point common to the sphere and the plane DL, *i.e.* DL is tangent to the sphere.

(Def. of tangent plane.)

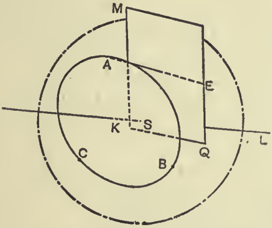
But when $d = R$, the planes MQ and DL are both perpendicular to ST at T.

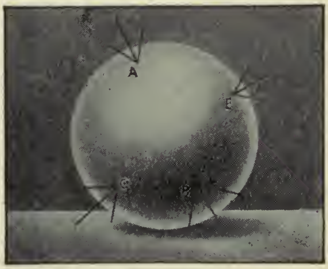
∴ plane MQ is identical with plane DL. (XXI. 3 *a*.)

∴ plane MQ is tangent to the sphere at T, and is unique.

Q.E.D.

XXV. 3. *Through any four points, not in the same plane, a unique sphere may be passed.*





Hyp. If A, B, C, E, are not in the same plane,

Conc.: then a unique sphere may be passed through A, B, C, and E.

Dem. No three of the points can be collinear.

(Def. of plane, *g*, XXI.)

The locus of the centers of spheres through A, B, and C is KL, perpendicular to plane ABC at K, the center of the circle through A, B, and C. (XXV. 1 *b*.)

All points equidistant from A and E lie on the plane MQ perpendicular to AE. (XXI. 3 *b*.)

This plane must cut KL, as at S. (Why ?)

∴ S is equidistant from A, B, C, and E. (Why ?)

∴ a sphere with S as center and SA as radius passes through A, B, C, and E.

Again, the point S is unique. (Why ?)

∴ the sphere through A, B, C, and E is unique.

Q.E.D.

XXV. 3 *a.* *The perpendiculars to the faces of a tetrahedron at their circumcenters are concurrent.*

Outline Dem. Draw *AB, BC,* etc., forming the tetrahedron *ABCE.*

Then *K* is the circumcenter of triangle *ABC,* and *KL* the axis of the circle through *A, B,* and *C.*

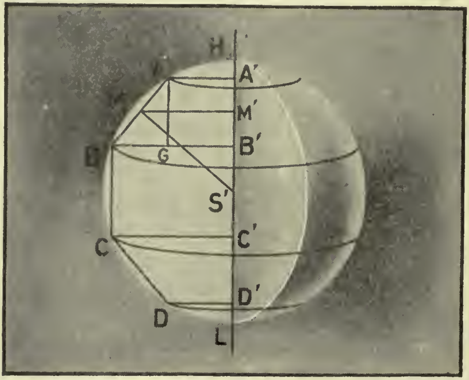
Similarly with each of the other faces of *ABCE.*

∴ the perpendiculars concur at the center of the circumsphere. (XXV. 1 *a.*)

Q.E.D.

XXV. 3 *b.* *The six planes mid-normal to the edges of a tetrahedron have a unique point in common.*

XXV. 4. *If R is the radius of a sphere, the area of the surface of the sphere equals* $4\pi R^2$.



Hyp. If R is the radius of the sphere generated by the semicircle *HBL,*

Conc.: then the area of the surface is $4\pi R^2$.

Dem. Let *AB* be a side of a regular n-gon inscribed in the circle, $A'B'$ its projection on the axis *HL,* and M' the projection of its middle point *M.*

Draw AG parallel to HL, and draw MS'.

The surface generated by AB when HBL generates the sphere, or

$$\text{Surf. } AB = 2\pi MM' \cdot AB.$$

[The convex surface of the frustum, etc. (XXIV. 11 *b*.).]

But $$\text{rt.} \triangle MM'S' \sim \text{rt.} \triangle ABG. \quad \text{(XV. Exs. 50, 51.)}$$

$$\therefore AB : S'M = AG (= A'B') : MM'.$$

$$\therefore AB \cdot MM' = S'M \cdot A'B'.$$

$$\therefore \text{Surf. } AB = 2\pi S'M \cdot A'B'.$$

Similarly for the surface generated by HA, BC, etc.

$\therefore$ the surface generated by the semi-polygon

$$HAB \cdots L = 2\pi S'M(HA' + A'B' + B'C' + \cdots) = 2\pi S'M \cdot HL$$
$$= 2\pi S'M \cdot 2R \qquad\qquad = 4\pi R \cdot S'M.$$

But if n be indefinitely increased, the semi-polygon $\doteq$ the generatrix, and the surface generated by the semi-polygon $\doteq$ the sphere surface and

$$S'M \doteq R.$$

$\therefore$ the sphere surface $= 4\pi R \cdot R = 4\pi R^2$.

Q.E.D.

XXV. 4 *a*. *If h is the altitude of a zone on a sphere of radius R, the area of the zone equals $2\pi Rh$.*

Outline Dem. The surface generated by any portion of the semi-polygon, *e.g.* ABC, is $2\pi R \cdot A'C'$. At the limit, this surface becomes a zone of altitude $A'C' = h$.

Ex. 1. Find the surface and the volume of a sphere whose radius is 10 ft.

Ex. 2. Find the total surface of a hemisphere whose radius is 12 ft.

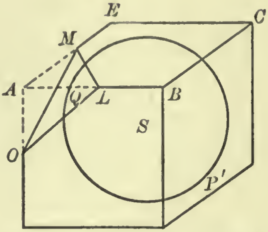
Ex. 3. The surface of a sphere is 1256.64 sq. ft. What is the radius?

Ex. 4. Show that the area of the surface of a sphere is the same as that of a circle whose radius is the diameter of the sphere.

Ex. 5. Find the area of a zone whose altitude is 10 ft. and which is situated on a sphere of 20 ft. radius.

XXV. 5. *If* R *is the radius of a sphere, the volume of the sphere is* $\frac{4}{3}\pi R^3$.

Hyp. If R is the length of the radius of the sphere S,



Conc.: then the volume of the sphere S is $\frac{4}{3}\pi R^3$.

Dem. Circumscribe the cube P_1 about S.

Draw SA, SB, etc., thus dividing P_1 into pyramids.

Volume of pyramid $S\text{–}ABCE = ABCE \cdot \frac{1}{3}R$. (XXIV. 4 *a.*)

Similarly, for the other pyramids of P_1.

$$\therefore \text{ volume of } P_1 = (\text{surface of } P_1) \cdot \tfrac{1}{3}R.$$

At Q, $\cdots$, where SA, etc., cut the surface of S, pass planes tangent to S and truncating (cutting off the corners) symmetrically the cube P_1.

Denote the polyhedron thus obtained by P_2.

Draw SL, SO, etc., dividing P_2 into pyramids.

Volume of pyramid $S\text{–}LMO = LMO \cdot \frac{1}{3}R$. (XXIV. 4.)

Similarly, for the other pyramids of P_2.

$$\therefore \text{ volume of } P_2 = (\text{surface of } P_2) \cdot \tfrac{1}{3}R.$$

By passing a new set of tangent planes truncating P_2, we obtain a new circumscribed polyhedron, P_3, such that

$$\text{volume of } P_3 = (\text{surface of } P_3) \cdot \tfrac{1}{3}R.$$

This process may be continued indefinitely.

But each successive polyhedron is nearer in volume to the sphere than the preceding.

Again, the volume of any polyhedron thus obtained will be greater than the volume of S, since its vertices are without S, and its faces are tangent planes.

But the excess of volume may be made as small as we please by continuing the truncation far enough.

∴ if P be any of the polyhedra,

$$\text{volume of } P \doteq \text{volume of } S.$$

But $$\text{volume of } P = (\text{surface of } P) \cdot \tfrac{1}{3} R.$$

$$\therefore \text{ volume of } S = (\text{surface of } S) \cdot \tfrac{1}{3} R$$

$$= 4\pi R^2 \cdot \tfrac{1}{3} R \qquad \text{(XXV. 4.)}$$

$$= \tfrac{4}{3}\pi R^3.$$

Q.E.D.

XXV. 5 *a*. *The volume of a spherical sector equals one third the product of the zone that forms its base and the radius of the sphere.*

Ex. 6. The surface of a sphere is equal to the surface of a cube. Find the ratio of the diameter of the sphere to the edge of the cube.

Assuming the radius of the earth to be 3957.2 miles, and the chord of an arc of $23\frac{1}{2}°$ to be .407 of the radius, find:

Ex. 7. The area of each Frigid Zone.

Ex. 8. The area of each Temperate Zone.

Ex. 9. The area of the Torrid Zone.

Ex. 10. The total surface of a cylinder of revolution is equal to the total surface of a hemisphere. Find the ratio of the two volumes, if the diameter of the cylinder equals its altitude.

Ex. 11. The total surface of a cone of which the altitude equals the radius of the base is equal to the surface of a sphere. Find the ratio of the altitude of the cone to the diameter of the sphere.

Ex. 12. Show that if spheres be described on the sides of a right triangle as diameters the surface of the sphere on the hypotenuse will be equal to the sum of the surfaces of the other two spheres.

Ex. 13. The radii of two spheres are a and b. Show how to construct the sphere whose area is equal to the sum of the areas of the given spheres.

Ex. 14. Show that if spheres be described on three concurrent edges of a rectangular parallelepiped as diameters the sum of the surfaces of the spheres will equal the surface of the sphere described on the diagonal of the parallelepiped as diameter.

XXV. SUMMARY OF PROPOSITIONS IN THE GROUP ON THE SPHERE

1. *Every plane section of a sphere is a circle.*

   a. *The join of the center of a sphere and the center of any circle on the sphere is the axis of this circle, and conversely.*

   b. *The locus of the centers of all spheres that pass through three given points is the axis of the circle that passes through the points.*

   c. *Circles cut off by planes equidistant from the center of the sphere are equal, and conversely.*

   d. *Of two circles cut by planes unequally distant from the center of the sphere, the one nearer the center is the greater, and conversely.*

   e. *The polar distances of all points in the circumference of a circle of the sphere are equal.*

SCH. From this property the polar distances of the points of a circle are called arc-radii of the circle.

The arc-radius of a great circle is a great-circle quadrant, called simply a quadrant.

The arc-radius of a small circle is less than a quadrant.

   f. *Three points on the sphere surface (not on the same great circle, and no two of which are the extremities of a diameter) are necessary and sufficient to determine a small circle of the sphere.*

   g. *Two points on the sphere surface (not the extremities of a diameter) are necessary and sufficient to determine a great circle of the sphere.*

2. *A plane perpendicular to a radius at its extremity is tangent to the sphere, and conversely.*

3. *Through any four points not in the same plane a unique sphere may be passed.*

   a. *The perpendiculars to the faces of a tetrahedron erected at their circumcenters are concurrent.*
   b. *The six planes mid-normal to the edges of a tetrahedron have a unique point in common.*

4. *If* R *is the radius of a sphere, the area of the surface of the sphere is* $4\pi R^2$.

   a. *If* h *is the altitude of a zone on a sphere of radius* R, *the area of the zone equals* $2\pi Rh$.

5. *If* R *is the radius of a sphere, the volume of the sphere equals* $\frac{4}{3}\pi R^3$.

   a. *The volume of a spherical sector equals one third the product of the zone that forms its base and the radius of the sphere.*

Ex. 15. Find the volume of a sphere that will just fit into a cubical box each edge of the inside measurement of which is 10 in.

Ex. 16. Find, to within .01 ft., the edge of a cube whose volume equals the volume of a sphere of radius 12 ft.

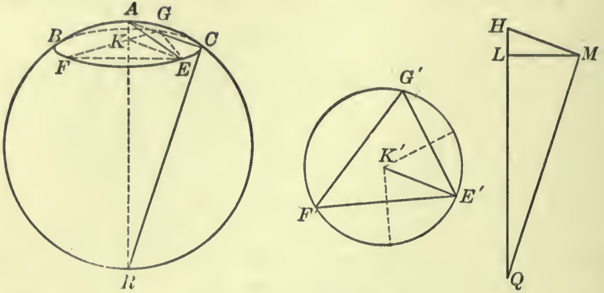
Ex. 17. The volume of a sphere is $7241\frac{1}{7}$ cu. ft. What is the radius?

Ex. 18. The number that expresses the volume of a certain sphere is the same as the number that expresses the surface of the sphere. Find the radius of the sphere.

Ex. 19. A cube and a sphere have the same surface. Which has the greater volume? Prove.

PROBLEMS

XXV. Prob. 1. *To find the diameter of a material sphere.*



Given. The material sphere $AB \cdots R$.

Required. The diameter of this sphere.

Const. With any point, as A, as a pole, and any convenient opening of the dividers, describe a circle, $B \cdots C$.

In this circle select any 3 points, F, E, G, and with the dividers set off $E'F' = EF$, $F'G' = FG$, and $G'E' = GE$ so as to form the triangle $E'F'G'$ congruent to triangle EFG.

Find the circum-radius of this triangle, $K'E'$, which will also be the circum-radius of EFG, *i.e.* the radius of circle $B \cdots C$.

Construct rt. $\triangle HLM$ with hypotenuse $HM =$ chord AE and leg $LM = K'E'$.

Erect a perpendicular to HM at M, and extend this perpendicular to meet HL, say at Q.

HQ is the diameter required.

Proof. Suppose the diameter AR drawn; also AC, CR.

rt. $\triangle HLM \cong$ rt. $\triangle AKE$. (Const.)

$\therefore$ In rt. $\triangle$s ACR, HMQ,

$\angle A = \angle H$. (Hom. $\angle$s $\cong$ $\triangle$s.)

$\angle ACR = \angle HMQ$. (Both rt. $\angle$s.)

$AC = HM.$ (Const.)

$\therefore$ rt. $\triangle ACR \cong$ rt. $\triangle HMQ.$ (V. 2.)

$\therefore AR = HQ.$ (Hom. s's. $\sim$ ⊿s.)

Q.E.D.

Ex. 20. An iron cannon ball 12 in. in diameter weighs 225 lbs. What is the diameter of a ball of the same material weighing 1800 lbs.?

Ex. 21. The volume of a sphere equals the volume of a cube. Find the ratio of the diameter of the sphere to the edge of the cube.

Find also the ratio of the diameter of the sphere to the diagonal of the cube.

Ex. 22. The volume of a sphere is equal to the volume of a cone whose slant height is double the radius of its base. Find the ratio of the total surfaces of the two figures.

Ex. 23. The sum of the surfaces of three spheres is equal to a circle of which the radius is twice the diagonal of an oblong block whose edges are a, b, and c. The volumes of the three spheres are in the ratio of $a^3 : b^3 : c^3$. Find the radii of the spheres.

Ex. 24. A sphere just fits into a regular triangular prism each base edge of which is a. Find the volume of the sphere.

Ex. 25. The surface of a cube equals the surface of a sphere. Find the ratio of the volume of the cube to the volume of the sphere.

Ex. 26. A sphere and a regular tetrahedron have the same surface. Find the ratio of their volumes.

Find the ratio:

Ex. 27. Of the surface of a sphere to the surface of the circumscribed cube.

Ex. 28. Of the surface of a sphere to the surface of the inscribed cube.

Ex. 29. Of the volume of a sphere to the volume of the circumscribed cube.

Ex. 30. Of the volume of a sphere to the volume of the inscribed cube.

Def. A **Principal Section** of a Surface of Revolution is a section that passes through the axis.

The principal sections of a certain right circular cylinder are squares each side of which is $2a$.

Find the ratio:

Ex. 31. Of the convex surface of the cylinder to the surface of the inscribed sphere.

Ex. 32. Of the total surface of the cylinder to the surface of the inscribed sphere.

Ex. 33. Of the volume of the cylinder to the volume of the inscribed sphere.

NOTE. — The three preceding problems were first solved by Archimedes.

Ex. 34. Of the convex surface of the cylinder to the surface of the circumscribed sphere.

Ex. 35. Of the total surface of the cylinder to the surface of the circumscribed sphere.

Ex. 36. Of the volume of the cylinder to the volume of the circumscribed sphere.

Ex. 37. Of the volume of the circumscribed sphere to the volume of the inscribed sphere.

The vertex angle of a right circular cone is 60°; its slant height is $2a$.

Find the ratio:

Ex. 38. Of the convex surface of the cone to the surface of a sphere of radius a.

Ex. 39. Of the total surface of the cone to the surface of the sphere of radius $2a$.

Ex. 40. Of the total surface of the cone to the surface of a sphere whose radius is the altitude of the cone.

Ex. 41. Of the volume of the cone to the volume of the sphere of radius a.

Ex. 42. Of the volume of the cone to the volume of the sphere whose radius is the altitude of the cone.

Ex. 43. Of the volume of the cone to the volume of the sphere whose surface equals the convex surface of the cone.

Ex. 44. Of the volume of the cone to the volume of the sphere whose surface equals the total surface of the cone.

The surface of a sphere equals the total surface of a right circular cone whose principal sections are equilateral triangles; and also equals the total surface of a right circular cylinder whose principal sections are squares.

Find the ratio:

Ex. 45. Of the volume of the sphere to the volume of the cylinder.

Ex. 46. Of the volume of the sphere to the volume of the cone.

Ex. 47. Of the volume of the cylinder to the volume of the cone.

Ex. 48. Of the convex surface of the cylinder to the convex surface of the cone.

Ex. 49. An iron sphere whose radius is 8 in. is melted and recast in the form of a hollow right circular cylinder whose altitude and interior diameter are each 6 in. Find the thickness of the cylinder.

Ex. 50. Find the weight of a hollow spherical cast-iron shell whose exterior diameter is 20 in. and whose interior diameter is 17 in.

Ex. 51. A hollow spherical shell has a capacity of 2 gal.; its exterior diameter is 12 in. Find the thickness of the shell.

Ex. 52. Find the locus of the center of a sphere 10 in. in diameter, the surface of which passes through a given point A.

What is the locus of a point at a constant distance (d) from a given point?

What is the locus:

Ex. 53. Of the center of a sphere passing through two given points, A and B?

Ex. 54. Of the center of a sphere of radius r whose surface passes through two given points, A and B?

Ex. 55. Of the center of a sphere whose surface passes through three given points, A, B, and C?

Ex. 56. Of the centers of spheres whose surfaces all contain a given circle?

What is the locus of the center of a sphere:

Ex. 57. Tangent to three given planes?

Ex. 58. Tangent to three intersecting lines?

Ex. 59. Tangent to a given plane at a given point?

Ex. 60. Tangent to a given cylindrical surface at a given point?

Ex. 61. Tangent to a given sphere at a given point?

Ex. 62. Tangent to a cone of revolution at a given point?

Ex. 63. Tangent to two concentric spheres?

Find the locus of the center of a sphere of given radius r that satisfies the conditions that follow:

Ex. 64. Tangent to a given plane.

Ex. 65. Tangent to two intersecting planes.

Ex. 66. Tangent to a cylindrical surface of revolution.

Ex. 67. Tangent to a conical surface of revolution.

Ex. 68. Tangent to a sphere of radius m.

Ex. 69. Tangent to two cylindrical surfaces of revolution, of diameters a and b.

Ex. 70. Tangent to two spheres of equal radius a.

Ex. 71. To find a point equidistant from three given points and also equidistant from two other points not in the plane of the first three.

To construct a sphere of radius r:

Ex. 72. That shall pass through a given point and be tangent to two given planes.

Ex. 73. That shall pass through two given points and shall also be tangent to a given sphere.

Ex. 74. That shall pass through a given point and shall also be tangent to a cylinder of revolution along a given element.

Ex. 75. That shall pass through two given points and shall also be tangent to a cylinder of revolution along a given element.

Ex. 76. That shall pass through a given point and be tangent to a given sphere and a given plane.

Ex. 77. Tangent to three given planes.

Ex. 78. Tangent to three given concurrent lines.

Ex. 79. That shall pass through three given points.

Ex. 80. Tangent to two intersecting given lines and passing through a given point.

Ex. 81. Tangent to two non-intersecting lines and passing through a given point.

Ex. 82. Tangent to two given spheres of radii b and c, respectively. and passing through a given point.

Show that, given a point and a sphere,

Ex. 83. In general, an infinite number of tangent lines can be drawn through the given point tangent to the given sphere.

Ex. 84. These tangents either lie in the same plane or form the elements of a cone.

Ex. 85. The cone of tangents is a cone of revolution.

Ex. 86. In general, an infinite number of planes may be passed through the point tangent to the sphere.

Ex. 87. These planes will each be tangent to the cone of tangent lines through the point.

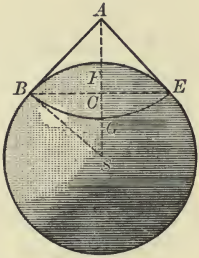
Ex. 88. When will it be impossible to pass more than one tangent plane through the point?

Ex. 89. When will it be impossible to pass either a tangent line or a tangent plane through the point?

To pass a plane tangent to a given sphere,

Ex. 90. At a point on the sphere.

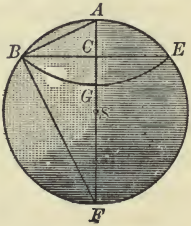
Ex. 91. Through a point without the sphere.



Ex. 92. If AB and AE are tangent to the sphere S, show that ABS is a right triangle. Show further that $AB^2 = AC \cdot AS$. Hence, show how to find FC when AF and the radius SB are given.

Ex. 93. An electric light is placed at A, 3 ft. from the nearest point of a sphere 20 ft. in diameter. Find the area of that portion of the sphere which is illuminated by the light at A.

Ex. 94. How far is the light A from the surface when the illuminated portion of the sphere is $\frac{1}{16}$ of the whole surface? When the illuminated portion is $\frac{1}{3}$ of the whole?

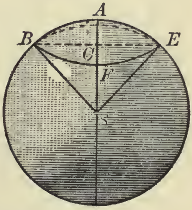


Ex. 95. Why is the illuminated portion of the surface always less than $\frac{1}{2}$ the whole surface?

Ex. 96. A–BGE is a zone of altitude AC. If AB, BF be drawn, what is the value of $\angle B$?

In the $\triangle ABF$, what relation connects AB, AC, AF? (XVII. 5 (*a*).)

Prove that the area of the zone A–BGE = the area of a circle of which AB is the radius.



Ex. 97. The altitude CA of the zone A–BFE equals H; the radius of the sphere equals R.

What is the area of the zone? What is the volume of the spherical sector S–$ABFE$? Show that, from the rt. $\triangle BCS$, $BC^2 = 2R \cdot H - H^2$.

Find the expression for volume of cone S–BFE.

Ex. 98. Show that the volume of the spherical segment

$$A\text{–}BFE = \frac{2\pi R^2 H}{3} - \frac{\pi(R-H)(2R \cdot H - H^2)}{3}$$

$$= \frac{\pi H}{3}\{2R^2 - (R-H)(2R-H)\}$$

$$= \frac{\pi H}{3}\{3HR - H^2\}$$

$$= \frac{\pi H^2}{3}(3R - H).$$

XXVI. GROUP ON GEOMETRY OF THE SPHERE SURFACE; BRIEFLY, SPHERICAL GEOMETRY

DEFINITIONS

The **Distance** between two points on the surface is the shorter arc of a great circle that passes through the points.

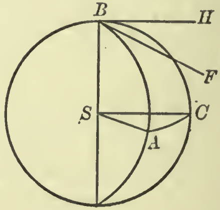
The **Arc-Radius** of a small circle on the sphere is the shorter polar distance of any one of its points.

A **Spherical Angle** is the figure formed by two great-circle arcs that intersect in one point.

The **Angle between any two circles** that have a common point is the plane angle formed by tangents to the circles at this point.

A spherical angle, therefore, is the same as the angle between the tangents to its sides drawn at the point of intersection.

Thus, the spherical angle ABC is the same as the plane angle FBH formed by the tangents FB and HB, to the sides AB and BC, respectively, at B.



Corollaries to the Definition

(*a*) *Any spherical angle is the measure of the dihedral formed by the planes of its sides.*

(*b*) *The measure of a spherical angle* (ABC), *plane angle FBH, is the arc AC intercepted by its sides on the great circle of which its vertex* (B) *is the pole.*

Dem. Draw AS, BS, CS to the center S.

$\angle ASB = \angle CSB =$ rt. $\angle$. (Def. of pole.)

$\angle FBS = \angle HBS =$ rt. $\angle$. (IX. 4.)

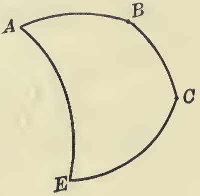
$\therefore FB \parallel AS$ and $HB \parallel CS$. (Def. of ‖s.)

$\therefore \angle FBH = \angle ASC$. (XXI. 7.)

But angle ASC is measured by arc AC. (XII. 1.)

$\therefore \angle FBH$ is measured by arc AC.

Q.E.D.



A **Spherical Polygon** is a portion of the surface of the sphere bounded by arcs of great circles.

A **Convex Polygon** is one the perimeter of which cannot be cut by a great circle in more than two points. Only convex polygons will be discussed in this group.

Spherical polygons are classified in the some manner as plane polygons.

A right spherical triangle is said to be **Rectangular**, when it contains one right angle; **Birectangular**, when it has two right angles, and **Trirectangular**, when it contains three right angles.

The term *right triangle* will hereafter be used to denote a spherical triangle with *but one* right angle, unless the context evidently implies the existence of more than one such angle in the triangle.

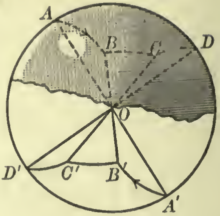
COROLLARIES TO THE DEFINITION

(*a*) *A polyhedral that has its vertex at the center of the sphere cuts from the surface a spherical polygon whose sides are the measures of the face angles of the polyhedral and whose angles are the measures of the dihedrals of the polyhedrals.*

(*b*) *The sum of any two sides of a spherical triangle is greater than the third.* (XXII. (*b*) 3.)

(*c*) *The sum of the sides of a spherical polygon (i.e. the perimeter) is less than the circumference of a great circle.* (XXII. (*b*) 4.)

(*d*) *An isosceles spherical triangle is isoangular.* (XXII. Ex. 29.)



Two spherical polygons are **Symmetric**, when the parts of the one are respectively equal to the parts of the other, but are arranged in opposite order.

If two symmetric polyhedrals have their vertex at the center of the sphere, they cut from the surface two symmetric spherical polygons.

Thus $\angle AOB = \angle A'OB'$. (Vertical angles.)

$\therefore$ arc $AB =$ arc $A'B'$. (Measures of equal angles.)

Again, dihedral $A\text{–}BO\text{–}C =$ dihedral $A'\text{–}B'O\text{–}C'$. (Vertical dihedrals.)

$\therefore \angle B = \angle B'$. (Measures of = dihedrals.)

Similarly, for the other parts of the two figures.

The opposite order of parts in the two figures is shown by the arrow heads.

Symmetric polygons so situated that the joins of corresponding vertices concur at the center of the sphere are said to be in **Perspective** with regard to this center.

Two symmetric polygons cannot, in general, be placed in coincident superposition.

For, on account of the curvature of the sphere surface, it will be impossible to make the parts of the one coincide with the corresponding parts of the other without breaking or tearing the surface.

For the same reason, it is impossible to *revolve* one of two symmetric polygons having homologous sides in common about this common side, so as to make the polygons coincide. That is, *revolution*, as a step in a proof, cannot occur in the geometry of the sphere-surface; an *arc* cannot be taken as an *axis*.

It will be shown, however (XXVI. 3 *a*), that if polygons are symmetric, they are equal in area.

A spherical polygon is the **Polar** of a second, when the vertices of the first are the poles of the sides of the second.

A spherical polygon is **Supplemental** to a second, when the angles of the first are the supplements of the corresponding sides of the second.

The **Spherical Excess** of a polygon of n sides is the excess of the sum of its angles over $(2n-4)$ right angles.

A **Lune** is a portion of a sphere-surface bounded by two great semicircles.

An **Ungula** or **Spherical Wedge** is the portion of the volume of the sphere comprised between two great semicircles.

COROLLARIES TO THE DEFINITIONS.

(*a*) *A lune is the same fraction of the total surface of a sphere that the angle of the lune is of four right angles.*

(*b*) *An ungula is the same fraction of the volume of the sphere that the angle of the ungula is of four right angles.*

PRELIMINARY SCHOLIUM

Correspondence between Plane and Spherical Geometry

The Line. Since any two points (not the extremities of a diameter) determine a great circle of the sphere (XXV. 1 *g*), the great-circle of the sphere corresponds to the straight line of the plane.

A *great-circle arc* may therefore be called simply a *line* of the sphere-surface.

But two great-circle arcs perpendicular to a third meet in the poles of this third, and from either of these poles an infinite number of perpendiculars can be drawn to the great-circle arc.

Parallels. Hence, there are *no* lines on the sphere corresponding to the *parallels* of the plane.

Axioms. It follows that:

(1) There is *no parallel axiom* on the sphere surface.

(2) All the axioms of the plane, except *Axioms* 7 and 8, are true on the surface of the sphere.

The Circle. Three non-collinear points in a plane determine a circle.

Three non-collinear points on a sphere-surface determine a small circle of this surface.

The **small circle of the sphere-surface** therefore corresponds to the **circle of the plane**, the word *Pole* (meaning the nearer pole) replacing the word *Center* of the plane geometry.

Propositions. The propositions of spherical geometry are therefore derived from those of plane geometry by replacing the "straight line" of the plane by the "great-circle arc," or simply the "line" of the sphere-surface, and dropping all propositions that *cannot be proved without using* Axiom 7 or Axiom 8.

The proofs for the sphere-surface are identical with those already given for the plane where Axioms 7 and 8 are not involved.

Summary of Propositions Common to Plane and Spherical Geometry

The following summary presents the geometry of the sphere as it corresponds to the geometry of the plane.

PLANE	SPHERE
Group I.	All propositions true on the sphere.
Group II.	No propositions true on the sphere.
Group III.	No propositions true on the sphere.
Group IV.	All propositions true on the sphere.
Group V.	All propositions true on the sphere, if "congruent" be replaced by "congruent or symmetric."
Group VI.	No propositions true on the sphere.
Group VII.	Only Theorems 1 and 2 true on the sphere.
Groups VIII., IX., X., XI.	All propositions true on the sphere.
Group XII.	Only Theorem 1 true on the sphere, but not used as a basis of measurement.

Areal Measurement in the Plane. The plane is boundless (*i.e.* without boundaries of any sort), and is infinite in extent. The ratio of the surface of a plane figure to the total surface of the plane on which it lies cannot be expressed in numbers. Hence, a purely arbitrary unit, the square on the linear unit, is selected, and the areal unit changes when the linear unit changes. All the propositions of plane geometry that deal with area involve the use of the square.

Neither the square nor any analogous figure can be drawn on the sphere-surface.

Areal Measurement on the Sphere. The sphere-surface, though boundless, is not infinite (XXV. 4). The ratio of the surface of a spherical polygon to the whole sphere-surface on which it lies can be expressed in numbers. Hence, the sphere-surface itself (or some definite fraction of it) is the natural unit of area for the figures that lie upon it, a unit that never changes for a given surface.

Accordingly, the treatment of surface measurement on the sphere is essentially different from the treatment of this subject in the plane, and no propositions of plane geometry that deal with areas or areal relations can have any correspondents on the sphere-surface.

Again, the **Doctrine of Similarity**, with all its varied applications, is based on Axioms 7 and 8.

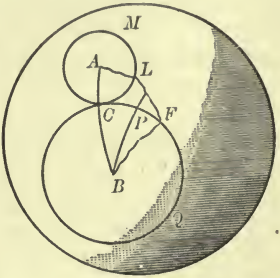
There is, therefore, *no Theory of Similarity* in spherical geometry.

For these reasons, no use can be made of the plane geometry beyond Group XII.

Detailed proofs will be required only for those propositions which are peculiar to the sphere-surface, or which have been proved in plane geometry by use of Axioms 7 and 8, when a proof not involving these axioms might have been used.

PROPOSITIONS

XXVI. 1. *The shortest line that can be drawn on the surface of the sphere, between any two points on the surface, is the great-circle arc (not greater than a semicircumference) that joins them.*



Hyp. If AB is a great-circle arc less than a semicircle, and $ALFB$ any other line whatever from A to B,

Conc.: then $AB < ALFB$.

Dem. With A as a pole and arc-radius less than AB, describe $\odot M$, cutting AB in C and $ALFB$ in L.

With B as a pole and an arc-radius BC, describe $\odot Q$, cutting $ALFB$ in F. This circle will be tangent to $\odot M$ at C.

(IX. 8 *a*, converse (*a*).)

Whatever the form of the line AL may be, a line equal to AL may be drawn from A to C.

For, if the zone whose base is $\odot M$ be revolved on the axis of $\odot M$, L, moving along the circumference, will reach the position C, and the line AL will itself extend from A to C.

Similarly, a line equal to BF can be drawn from B to C.

But L is without $\odot Q$ ($\odot Q$ is externally tangent to $\odot M$ at C).

$\therefore BF + FL > BC$. (Preliminary Th. 1.)

$\therefore BF + FL + LA > BC + CA$; (Preliminary Th. 1.)

i.e. $BFLA > BA$.

Q.E.D.

Sch. If A and B be extremities of a diameter, an infinite number of equal great-circle arcs may be drawn from A to B, but each will be less than any line not a great-circle arc that can be drawn from A to B.

Note. — In speaking of *two points on the sphere*, we shall hereafter assume, unless the contrary be stated or evidently implied, that these points are not the extremities of a diameter of the sphere.

The proof of the theorem emphasizes the analogy between the great circle on the sphere and the straight line of the plane. Each is determined by two points of the surface; each measures the shortest distance on the surface between any two of its points.

Ex. 1. Show that a spherical triangle may have 3 obtuse angles.

Ex. 2. Two spherical polygons of the same number of sides are equal, if the sum of the angles of the one equals the sum of the angles of the other.

Ex. 3. Two spherical right triangles are congruent or symmetric, if the oblique angles of the one equal respectively the oblique angles of the other.

Ex. 4. Two birectangular triangles are congruent or symmetric, if the oblique angle of the one equals the oblique angle of the other.

Ex. 5. If the diagonals of a spherical polygon bisect each other, the opposite sides are equal.

Ex. 6. If two spheres intersect, the tangent lines drawn to the spheres from any point in the plane of the circle of intersection are equal.

Def. If two spheres be generated by the revolution of two circles about their line of centers as an axis, the radical axis of the circles (XVII. Ex. 72, Def.) will generate a plane perpendicular to the line of centers of the spheres. (XXI. 3.)

This plane is called the **Radical Plane** of the Spheres.

Ex. 7. The radical plane of two spheres is the locus of the point from which pairs of equal tangent lines may be drawn to the spheres.

Ex. 8. The radical planes of three spheres taken two and two intersect in a line perpendicular to the plane of the centers of the spheres.

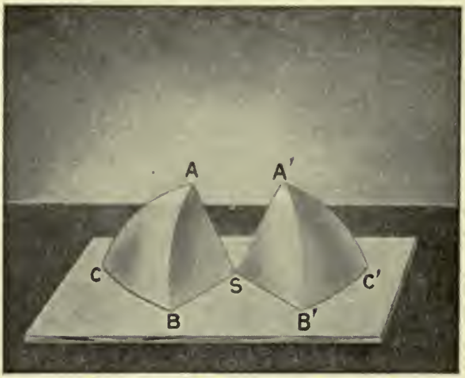
The radius of a sphere is 20 ft. Find the area:

Ex. 9. Of a triangle on the sphere whose angles are 82°, 104°, and 84°, respectively.

Ex. 10. Of a birectangular triangle whose vertex angle is 72°.

Ex. 11. Of a trirectangular triangle.

XXVI. 2. *Isosceles symmetric triangles are congruent.*



Hyp. If the $\triangle$ ABC, $A'B'C'$ are symmetric, and if $AB = AC$, $A'B' = A'C'$,

Conc.: then $\triangle ABC \cong \triangle A'B'C'$.

Dem. $AB = AC = A'B' = A'C'$. (Hyp.)

Place $\triangle ABC$ on $\triangle A'B'C'$, so that A shall fall at A' and B shall fall at C'.

The two triangles fall on the same side of $A'C'$, since the parts occur in reverse order.

$\angle A = \angle A'$. (Def. Sph. n-gon. d.)

$\therefore$ AC takes the direction of $A'B'$.

$AC = A'B'$. (Hyp.)

$\therefore$ C falls at B'.

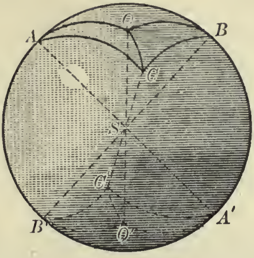
$\therefore$ BC coincides with $B'C'$.

[Two points, etc., determine a great circle.] (XXV. 1 g.)

$\therefore$ $\triangle ABC \cong \triangle A'B'C'$.

Q.E.D.

XXVI. 3. *Symmetric triangles are equal.*



Hyp. If $\triangle ABC$ is symmetric to $\triangle A'B'C'$,

Conc.: then $\triangle ABC = \triangle A'B'C'$.

Dem. Place $\triangle ABC$ in perspective with $\triangle A'B'C'$, with respect to S. Let Q be the pole of the circle through A, B, and C.

[Three points, etc., determine a small circle.] (XXV. 1 *f.*)

Draw QSQ' and the joins $Q'A'$, $Q'B'$, and $Q'C'$.

Trihedral $S\text{–}ABQ$ is symmetric with trihedral $S\text{–}A'B'Q'$.

(Def. of symmetric trihedrals.)

$\therefore \triangle ABQ$ is symmetric with $\triangle A'B'Q'$.

(Def. of symmetric *n*-gons, etc.)

$\therefore QA = Q'A'$ and $QB = Q'B'$.

But $QA = QB = Q'A' = Q'B'$; (XXV. 1 *e.*)

i.e. $\triangle$s AQB and $A'Q'B'$ are isosceles.

$\therefore \triangle AQB \cong \triangle A'Q'B'$. (XXVI. 2.)

Similarly, $\triangle BQC \cong \triangle B'Q'C'$,

and $\triangle AQC \cong \triangle A'Q'C'$.

$\therefore \triangle AQB + \triangle BQC + \triangle AQC = \triangle A'Q'B' + \triangle B'Q'C' + \triangle A'Q'C'$;

i.e. $\triangle ABC = \triangle A'B'C'$.

Q.E.D.

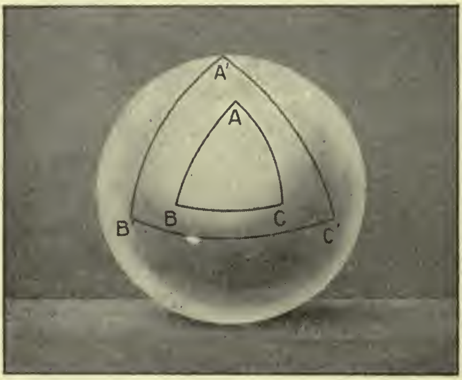
XXVI. 3 *a.* *Symmetric polygons are equal.*

Outline Dem. Divide the polygons into triangles by diagonals from homologous vertices.

These triangles will be symmetric in pairs and therefore equal in pairs, therefore the polygons are equal.

Q.E.D.

XXVI. 4. *If one triangle is polar to another, the second is polar to the first.*



Hyp. If $A, B, C,$ are the poles of $B'C'$, $A'C'$, $A'B'$, respectively,

Conc. : then A', B', C', are the poles of BC, AC, AB, respectively.

Dem. A is the pole of $B'C'$. (Hyp.)

$$\therefore\ AC' = 90°.$$

[The arc-radius of a great ⊙ is a quadrant (XXV. 1 *e.*) Sch.]

Similarly, $BC' = 90°$.

$\therefore$ A and B lie in a great circle of which C' is the pole.

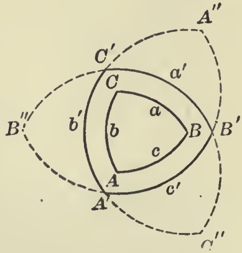
But only one great ⊙ can pass through A and B, (XXV. 1 *g.*) and the side AB is a great-circle arc.

$$\therefore\ C' \text{ is the pole of the side } AB.$$

Similarly, B' is the pole of the side AC and A' is the pole of the side BC.

Q.E.D.

Sch. If the sides of $\triangle A'B'C'$ be extended to meet in A'', B'', C'', three new triangles are formed each of which is polar to A, B, C. A triangle therefore has four polars.



But in speaking of *the polar* of a triangle, we always mean the central figure, for which *each* of the distances Aa', Bb', Cc', is less than a quadrant.

XXVI. 4 *a*. *If one polygon is polar to another, the second is polar to the first.*

XXVI. 4 *b*. *A trirectangular triangle is its own polar* (i.e. *is self-polar*).

The radius of a sphere is 20 feet. Find the area:

Ex. 12. Of a quadrilateral whose angles are 112°, 85°, 92°, and 126°, respectively.

Ex. 13. Of a pentagon, whose angles expressed in radians, are 2.25, 2.83, 2.7, 3.72, and 1.7, respectively.

Ex. 14. Of a hexagon each of whose angles is 130°.

The radius of a sphere is 20 ft. A plane passes through the sphere at a distance 12 ft. from the center. Find:

Ex. 15. The total surface of the minor spherical segment cut off by the plane.

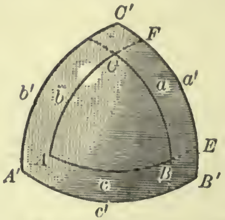
Ex. 16. The angles of an equiangular triangle on the sphere whose area is equal to the zone surface of the segment.

Ex. 17. The angles of an equiangular triangle equal in area to the total surface of the segment.

Ex. 18. The third angle of a triangle, two of whose angles are 100° and 75°, and which equals in area one face of the inscribed cube.

XXVI. 5. *Polar triangles are supplemental.*

Hyp. If $\triangle ABC$ is the polar of $\triangle A'B'C'$,



Conc.: then $\angle A = 180° - a'$; $\angle B = 180° - b'$; $\angle C = 180° - c'$,

and $\angle A' = 180° - a$; $\angle B' = 180° - b$; $\angle C' = 180° - c$.

Dem. Extend AB, AC to meet $B'C'$ as at E and F. $\angle A$ is measured by FE. (Def. of measure of spherical $\angle$.)

C' is the pole of ABE. (Def. of polar n-gons.)

$$\therefore\ C'E = 90°;$$

i.e. $$C'F + FE = 90°.$$

Similarly, $$FB' = 90°.$$

$$\therefore\ C'F + FB' + FE = 180°;$$

i.e. $$C'B' + FE = 180°.$$

$$\therefore\ FE = 180° - C'B',$$

or $$\angle A = 180° - a'.$$

Similarly, for $\angle B$ and $\angle C$.

In like manner, we may show that

$\angle A' = 180° - a$ (by extending BC as shown), etc.

Q.E.D.

XXVI. 5 *a*. *Polar polygons are supplemental.*

XXVI. 5 *b*. *Two triangles are congruent or symmetric, if the angles of the one are equal respectively to the angles of the other.*

Dem. Let P and P' be the polars of the given triangles.

The sides of P equal the sides of P'. (XXVI. 5.)

$\therefore$ P is congruent (or symmetric) to P'. (V. 3.)

$\therefore$ the angles of P equal the angles of P'.

(Def. of symmetric n-gons.)

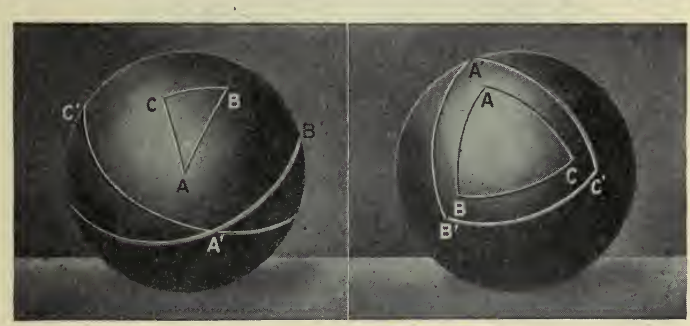
$\therefore$ the sides of the given triangles are equal, respectively.

(XXVI. 5.)

$\therefore$ the given triangles are congruent or symmetric. (V. 3.)

Q.E.D.

XXVI. 6. *The sum of the interior angles of a triangle lies between two right angles and six right angles.*



Hyp. If ABC is a spherical triangle,

Conc.: then $\angle A + \angle B + \angle C > 2$ rt. $\angle$s and < 6 rt. $\angle$s.

Dem. Let $\triangle A'B'C'$ be the polar of $\triangle ABC$.

Then $\angle A = 2$ rt $\angle\text{s} - a'$. (XXVI. 5.)

$\angle B = 2$ rt. $\angle\text{s} - b'$.

$\angle C = 2$ rt. $\angle\text{s} - c'$.

$\therefore \angle A + \angle B + \angle C = 6$ rt. $\angle\text{s} - (a' + b' + c')$.

But $a' + b' + c' > 0$ and < 4 rt. $\angle$s. (Cor. to the def. XXVI. *c*.)

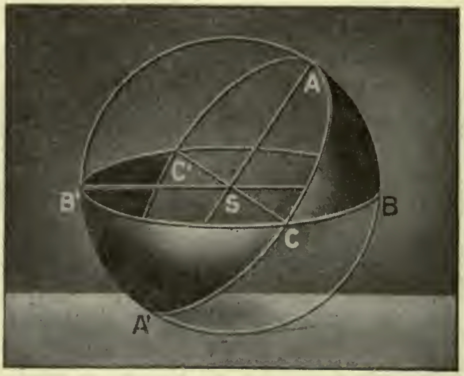
$\therefore \angle A + \angle B + \angle C > 6$ rt. $\angle\text{s} - 4$ rt. $\angle$s, or 2 rt. $\angle$s,

and < 6 rt. $\angle\text{s} - 0$, or 6 rt. $\angle$s.

Q.E.D.

XXVI. 6 *a*. *The sum of the interior angles of a polygon lies between* $(2\text{n} - 4)$ *right angles and* 2n *right angles* (n *being the number of sides of the polygon*).

XXVI. 7. *If two great semicircles intersect on the surface of a hemisphere, the sum of either set of opposite triangles equals the lune whose angle is the corresponding angle of the semicircles.*



Hyp. If the lines ACA', BCB' meet the base of the hemisphere S in A, A', B, B',

Conc.: then $\triangle ABC + \triangle A'B'C =$ lune $CA'C'B'$.

Dem. $CA' + A'C' = CA'C' = 180°$. (Def. of great circle.)

$CA' + AC = ACA' = 180°$. (Same reason.)

$\therefore CA' + AC = CA' + A'C'$. (Ax. 1.)

$\therefore AC = A'C'$. (Ax. 2.)

Similarly, $BC = B'C'$,

and $AB = A'B'$.

$\therefore \triangle ABC$ is symmetric and equal to $\triangle A'B'C'$. (XXVI. 3.)

$\therefore \triangle ABC + \triangle A'B'C = \triangle A'B'C' + \triangle A'B'C$ (Ax. 2.)

$=$ lune $CA'C'B'$ (or lune C).

Q.E.D.

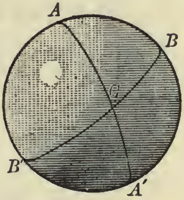
XXVI. 7 *a*. *A trirectangular triangle is one eighth of the surface of the sphere on which it lies.*

SCH. UNIT OF AREA. ANGULAR UNIT. The trirectangular triangle, or one eighth of the sphere-surface, will be taken as the unit of areal measure on the sphere, and the right angle as the unit of angle measure. Angles must be expressed in terms of the right angle in all formulæ for area.

Thus, a lune whose angle is a right angle is twice the trirectangular triangle; *i.e.* its area is 2; a lune whose angle is 50°, or $\frac{5}{9}$ rt. $\angle$, is $\frac{5}{9}$ of 2, or $\frac{10}{9}$; a lune whose angle is A has an area of $2A$, etc. The area of the sphere is 8; that of the hemisphere is 4.

XXVI. 8. *The area of a spherical triangle equals its spherical excess.*

Hyp. If the angles of the $\triangle ABC$ be expressed in rt. $\angle$s, and the trirectangular $\triangle$ be taken as the unit of area,



Conc.: then the area of $\triangle ABC = A + B + C - 2$.

Dem. On the hemisphere $ABB'A'$,

$\triangle ABC + \triangle AB'C = \text{lune } B = 2B$, (XXVI. 7. Sch.)

$\triangle ABC + \triangle A'BC = \text{lune } A = 2A$, (Same Reason.)

$\triangle ABC + \triangle A'B'C = \text{lune } C = 2C$. (Same Reason.)

But

$\triangle ABC + \triangle AB'C + \triangle A'BC + \triangle A'B'C =$ the hemisphere

$= 4$. (XXVI. 7. Sch.)

$\therefore$ (by addition) $2\triangle ABC + 4 = 2(A + B + C)$,

or $\triangle ABC + 2 = A + B + C$.

$\therefore \triangle ABC = A + B + C - 2$.

Q.E.D.

XXVI. 8 *a*. *The area of a spherical polygon equals its spherical excess.*

Let the student supply the proof by dividing the polygon into triangles and using the theorem.

Ex. 19. The angles of an equiangular triangle equal in area to the convex surface of the cone of tangents extending to the sphere from a point whose distance from the center is 35 ft.

Note. — The few exercises appended to Group XXVI are intended merely as illustrations of the method of utilizing the theorems and exercises of the plane geometry as exercises in spherical geometry. As stated in the General Scholium, every proposition of the plane geometry that does not involve Ax. 7 or Ax. 8 as a necessary element in its proof, is true on the sphere-surface as in the plane.

In the following problems, the word *line* (unqualified) means *arc of a great circle;* the word *circle* and the symbol ⊙ (unqualified) mean *small circle.* All data are supposed to be given on the sphere-surface.

What is the locus of a point:

Ex. 20. At a given distance from a given point?

Ex. 21. At a given distance from a given line?

Ex. 22. At a given distance from a given circle of radius r?

Ex. 23. Equidistant from two given points?

Ex. 24. Equidistant from two given lines?

What is the locus of the centers of circles:

Ex. 25. Of given radius, passing through a given point?

Ex. 26. Of given radius tangent to a given line?

Ex. 27. Of given radius tangent to a given circle?

Ex. 28. Passing through two given points?

Ex. 29. Tangent to two given lines?

Ex. 30. Tangent to two concentric circles?

Ex. 31. Tangent to two equal circles?

Ex. 32. Bisect a given line-segment.

Ex. 33. Circumscribe a ⊙ about a △.

Ex. 34. Bisect a given ∠.

Ex. 35. Inscribe a ⊙ in a given △.

Ex. 36. Escribe a ⊙ to a given △.

XXVI. SUMMARY OF PROPOSITIONS IN THE GROUP ON SPHERICAL GEOMETRY

1. *The shortest line that can be drawn on the sphere, between any two points on the surface, is the great-circle arc (not greater than a semicircumference) that joins them.*

SCH. There is no "shortest line" on the sphere-surface between the extremities of a diameter of the sphere.

2. *Isosceles symmetric triangles are congruent.*

3. *Symmetric triangles are equal.*

   a. Symmetric polygons are equal.

4. *If one triangle is polar to another, the second is polar to the first.*

SCH. A triangle has four polars.

   a. If one polygon is polar to another, the second is polar to the first.
   b. A trirectangular triangle is its own polar (i.e. *is self-polar*).

5. *Polar triangles are supplemental.*

   a. Polar polygons are supplemental.
   b. Two triangles are congruent or symmetric, if the angles of the one are equal respectively to the angles of the other.

6. *The sum of the interior angles of a triangle lies between two right angles and six right angles.*

a. The sum of the interior angles of a polygon lies between (**2 n – 4**) *right angles and* **2 n** *right angles,* **n** *being the number of sides of the polygon.*

7. *If two great semicircles intersect on the surface of a hemisphere, the sum of the opposite triangles formed equals the lune whose angle is the angle of the two semicircles.*

a. A trirectangular triangle is one eighth of the surface of the sphere on which it lies.

Sch. Unit of Area and Angular Unit.

8. *The area of a spherical triangle equals its spherical excess.*

a. The area of any spherical polygon equals its spherical excess.

NOTES AND BIOGRAPHICAL SKETCHES

Alexandrian School (The First). From earliest times to about 100 B.C. To this school belonged Euclid. The Second Alexandrian School began with the Christian Era. Menelaus, Theon, and Hypatia belonged to this school.

Apollonius, Greek. Lived in third century B.C. He ranked with Euclid and Archimedes. Known as the "Great Geometer."

Archimedes, Greek. Third century B.C. Born in Syracuse. Greatest mathematician of antiquity.

Ceulen, Ludolp van, Netherlands. Carried the value of π to 35 places, known as Ludolp's Numbers.

Ceva, G. (1648–1737), Italian. Discovered the theorem that bears his name.

Chasles, M. (1793–1880), French. He did much to develop modern projective geometry.

Cissoid. A curve by means of which may be found two mean proportionals between two given straight lines.

Cube, Duplication of. One of the three ancient, unsolved problems of geometry, the other being the quadrature of a circle and the trisection of an angle, to which the ancients devoted much time. It is now generally admitted that they cannot be solved by the geometry of the straight line and circle.

Descartes, René (1596–1650), French. He introduced into geometry the use of the algebraic equation for the purpose of analysis.

Euclid (about 300 B.C.), Greek. Lived in Alexandria. Probably went there from Athens. Not much is reliably known of him. His "Elements," containing 13 books, has been the basis of all elementary geometrical instruction for over 2000 years. His geometry does not touch upon the subject of mensuration.

Eudoxus (408 B.C.). Pupil of Plato. Astronomer and legislator, as well as mathematician.

Euler, L. (1707–1783), Swiss. He wrote a great number of works on mathematics. He introduced much of current notation into trigonometry; also use of π.

Golden Section. It cuts a line in extreme and mean ratio. It was much studied by Eudoxus.

Harmonics. Its fundamental theorem was discovered by Serenus of the Second Alexandrian School.

Hippocrates of Chios (430 B.C.), Greek. Developed the subject of similar figures; also discovered important relations between areas of circles. He was the first author of an elementary text-book on geometry.

Legendre, A. M. (1752–1833), French. Wrote an "Elements of Geometry" in 1794, which was largely adopted in continental Europe and the United States as a substitute for the more difficult Euclid.

Menelaus (98 A.D.), Greek. Of the Second Alexandrian School. The theorem of Menelaus is the foundation of the modern theory of transversals. He contributed largely to our knowledge of trigonometry.

π. For its history and values see Cajori's "History of Mathematics."

Pascal (1632–1662), French. A great mathematician and metaphysician.

Plato (429–348 B.C.), Greek. A pupil of Socrates and at head of a school in Athens. Many of Euclid's definitions are ascribed to him. He was the first to use the method of analysis to discover proof of theorem. He added much to geometrical knowledge.

Poncelet, J. V. (1788–1867), French. He investigated and developed modern projective geometry.

Ptolemæus, Claudius (about 139 A.D.), Egyptian. A celebrated astronomer. The fundamental idea of his system as opposed to that of Copernicus was that the earth is the center of the universe, and that the sun and planets revolve about it.

Pythagoras (580 B.C.), Greek, but lived many years in Egypt. In lower Italy he founded a school for the teaching of mathematics, philosophy and the natural sciences, chiefly the first.

Steiner (1796–1863), Swiss-German. "The greatest geometrician since Euclid." He laid the foundation for modern synthetic geometry.

Thales (640–546 B.C.), Greek. Lived long in Egypt. He measured the height of the Pyramids from their shadows. He was one of the Seven Wise Men of Greece.

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