ELEMENTS

OF

GEOMETRY;

WITH

PRACTICAL APPLICATIONS

TO

MENSURATION.

By BENJAMIN GREENLEAF, A. M.,

AUTHOR OF "THE NATIONAL ARITHMETIC," "TREATISE ON ALGEBRA," ETC.

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PREFACE.

The preparation of this treatise has been undertaken at the earnest solicitation of many teachers, who, having used the author's Arithmetics and Algebra with satisfaction, have been desirous of seeing his series rendered more complete by the addition of the Elements of Geometry.

That there are peculiar advantages in a graded series of text-books on the same subject, few, if any, properly qualified to judge, will doubt. The author, therefore, feels justified in introducing this volume to the attention of the public.

In common with most compilers of the present day, he has followed, in the main, the simple and elegant order of arrangement adopted by Legendre; but in the methods of demonstration no particular authority has been closely followed, the aim having been to adapt the work fully to the latest and most approved modes of instruction. In this respect, it is believed, there will be found incorporated a considerable number of important improvements.

More attention than is usual in elementary works of this kind has been given to the *converse* of propositions. In almost all cases where it was possible, the converse of a proposition has been demonstrated.

The demonstration of Proposition XX. of the first book is essentially the one given by M. da Cunha in the *Principes Mathé-*

matiques, which has justly been pronounced by the highest mathematical authorities to be a very important improvement in elementary geometry. It has, however, never before been introduced into a text-book by an American author.

The Application of Geometry to Mensuration, given in the eleventh and twelfth books, are designed to show how the theoretical principles of the science are connected with manifold practical results.

The Miscellaneous Geometrical Exercises, which follow, are calculated to test the thoroughness of the scholar's geometrical knowledge, besides being especially adapted to develop skill and discrimination in the demonstration of theorems and the solution of problems unaided except by principles.

Sufficient Applications of Algebra to Geometry are given to show the relation existing between these two branches of the mathematics. The problems introduced in connection therewith will be found to be, not only of a highly interesting character, but well calculated to secure valuable mental discipline.

In the preparation of this work the author has received valuable suggestions from many eminent teachers, to whom he would here express his sincere thanks. Especially would he acknowledge his great obligations to H. B. Maglathlin, A. M., who for many months has been associated with him in his labors, and to whose experience as a teacher, skill as a mathematician, and ability as a writer, the value of this treatise is largely due.

BENJAMIN GREENLEAF.

BRADFORD, Mass., June 25, 1858.

NOTICE.

A KEY, comprising the Solutions of the Problems contained in the last four Books of this Geometry, has been published, *for Teachers only;* and the same will be mailed, post-paid, to the address of any Teacher who will forward fifty cents in stamps to the Publishers.

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ELEMENTS OF GEOMETRY.

BOOK I.

ELEMENTARY PRINCIPLES.

DEFINITIONS.

1. Geometry is the science of *Position* and *Extension*.

The elements of position are direction and distance.

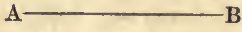
The dimensions of extension are length, breadth, and height or thickness.

2. Magnitude, in general, is that which has one or more of the three dimensions of extension.

3. A Point is that which has position, without magnitude.

4. A Line is that which has length, without either breadth or thickness.

5. A Straight Line, or Right Line, is one which has the same direction in its whole extent; as the line A B.



The word *line* is frequently used alone, to designate a straight line.

6. A Curved Line is one which continually changes its direction; as the line C D.

The word *curve* is frequently used to designate a curved line.

7. A BROKEN LINE is one which is composed of straight lines, not lying in the same direction; as the line E F.

8. A MIXED LINE is one which is composed of straight lines and of curved lines.

9. A SURFACE is that which has length and breadth, without height or thickness.

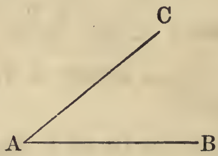
10. A PLANE SURFACE, or simply a PLANE, is one in which any two points being taken, the straight line that joins them will lie wholly in the surface.

11. A CURVED SURFACE is one that is not a plane surface, nor made up of plane surfaces.

12. A SOLID, or VOLUME, is that which has length, breadth, and thickness.

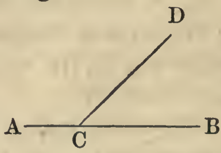
ANGLES AND LINES.

13. A PLANE ANGLE, or simply an ANGLE, is the difference in the direction of two lines, which meet at a point; as the angle A.



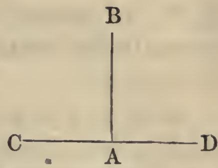
The point of meeting, A, is the *vertex* of the angle, and the lines A B, A C are the *sides* of the angle.

An angle may be designated, not only by the letter at its vertex, as C, but by three letters, particularly when two or more angles have the same vertex; as the angle A C D or D C B, the letter at the vertex always occupying the middle place.



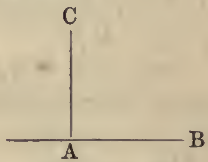
The *quantity* of an angle does not depend upon the length, but entirely upon the position, of the sides; for the angle remains the same, however the lines containing it be increased or diminished.

14. Two straight lines are said to be *perpendicular* to each other, when their meeting forms equal adjacent angles; thus the lines A B and C D are perpendicular to each other.

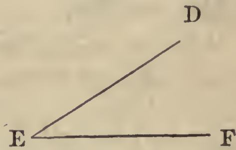


Two adjacent angles, as C A B and B A D, have a common vertex, as A; and a common side, as A B.

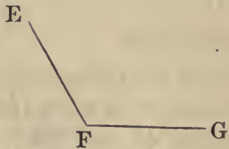
15. A Right Angle is one which is formed by a straight line and a perpendicular to it; as the angle C A B.



16. An Acute Angle is one which is less than a right angle; as the angle D E F.



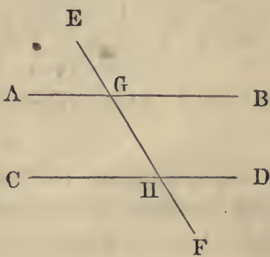
An Obtuse Angle is one which is greater than a right angle; as the angle E F G.



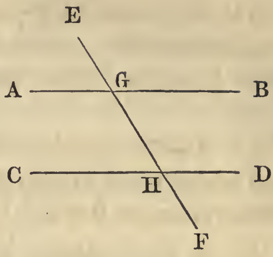
Acute and obtuse angles have their sides oblique to each other, and are sometimes called *oblique angles.*

17. Parallel Lines are such as, being in the same plane, cannot meet, however far either way both of them may be produced; as the lines A B, C D.

18. When a straight line, as E F, intersects two parallel lines, as A B, C D, the angles formed by the intersecting or secant line take particular names, thus: —



Interior Angles on the same Side are those which lie within the parallels, and on the same side of the secant line; as the angles B G H, G H D, and also A G H, G H C.



EXTERIOR ANGLES ON THE SAME SIDE are those which lie without the parallels, and on the same side of the secant line; as the angles B G E, D H F, and also the angles A G E, C H F.

ALTERNATE INTERIOR ANGLES lie within the parallels, and on different sides of the secant line, but are not adjacent to each other; as the angles B G H, G H C, and also A G H, G H D.

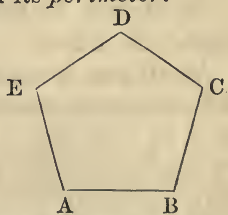
ALTERNATE EXTERIOR ANGLES lie without the parallels, and on different sides of the secant line, but not adjacent to each other; as the angles E G B, C H F, and also the angles A G E, D H F.

OPPOSITE EXTERIOR and INTERIOR ANGLES lie on the same side of the secant line, the one without and the other within the parallels, but not adjacent to each other; as the angles E G B, G H D, and also E G A, G H C, are, respectively, the opposite exterior and interior angles.

PLANE FIGURES.

19. A PLANE FIGURE is a plane terminated on all sides by straight lines or curves.

The boundary of any figure is called its *perimeter*.

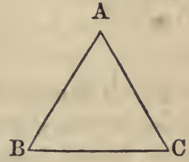


20. When the boundary lines are straight, the space they enclose is called a RECTILINEAL FIGURE, or POLYGON; as the figure A B C D E.

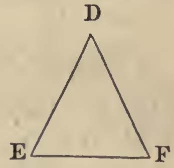
21. A polygon of three sides is called a TRIANGLE; one of four sides, a QUADRILATERAL; one of five, a PENTAGON; one of six, a HEXAGON; one of seven, a HEPTAGON; one

of eight, an OCTAGON; one of nine, a NONAGON; one of ten, a DECAGON; one of eleven, an UNDECAGON; one of twelve, a DODECAGON; and so on.

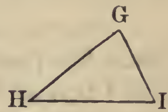
22. An EQUILATERAL TRIANGLE is one which has its three sides equal; as the triangle A B C.



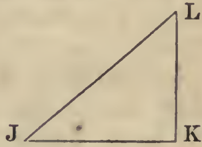
An ISOSCELES TRIANGLE is one which has two of its sides equal; as the triangle D E F.



A SCALENE TRIANGLE is one which has no two of its sides equal; as the triangle G H I.



23. A RIGHT-ANGLED TRIANGLE is one which has a right angle; as the triangle J K L.



The side opposite to the right angle is called the *hypothenuse;* as the side J L.

24. An ACUTE-ANGLED TRIANGLE is one which has three acute angles; as the triangles A B C and D E F, Art. 22.

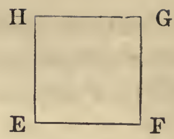
An OBTUSE-ANGLED TRIANGLE is one which has an obtuse angle; as the triangle G H I, Art. 22.

Acute-angled and obtuse-angled triangles are also called *oblique-angled* triangles.

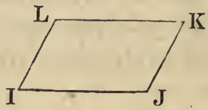
25. A PARALLELOGRAM is a quadrilateral which has its opposite sides parallel.

26. A RECTANGLE is any parallelogram whose angles are right angles; as the parallelogram A B C D.

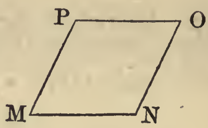
A SQUARE is a rectangle whose sides are equal; as the rectangle EFGH.



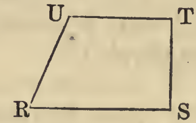
27. A RHOMBOID is any parallelogram whose angles are not right angles; as the parallelogram IJKL.



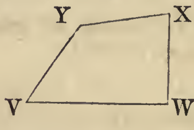
A RHOMBUS is a rhomboid whose sides are equal; as the rhomboid MNOP.



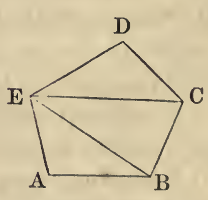
28. A TRAPEZOID is a quadrilateral which has only two of its sides parallel; as the quadrilateral RSTU.



A TRAPEZIUM is a quadrilateral which has no two of its sides parallel; as the quadrilateral VWXY.



29. A DIAGONAL is a line joining the vertices of any two angles which are opposite to each other; as the lines EC and EB in the polygon ABCDE.



30. A BASE of a polygon is the side on which the polygon is supposed to stand. But in the case of the isosceles triangle, it is usual to consider that side the base which is not equal to either of the other sides.

31. An *equilateral* polygon is one which has all its sides equal. An *equiangular* polygon is one which has

all its angles equal. A *regular* polygon is one which is equilateral and equiangular.

32. Two polygons are *mutually equilateral*, when all the sides of the one equal the corresponding sides of the other, each to each, and are placed in the same order.

Two polygons are *mutually equiangular*, when all the angles of the one equal the corresponding angles of the other, each to each, and are placed in the same order.

33. The corresponding equal sides, or equal angles, of polygons mutually equilateral, or mutually equiangular, are called *homologous* sides or angles.

AXIOMS.

34. An AXIOM is a self-evident truth; such as, —

1. Things which are equal to the same thing, are equal to each other.
2. If equals be added to equals, the sums will be equal.
3. If equals be taken from equals, the remainders will be equal.
4. If equals be added to unequals, the sums will be unequal.
5. If equals be taken from unequals, the remainders will be unequal.
6. Things which are double of the same thing, or of equal things, are equal to each other.
7. Things which are halves of the same thing, or of equal things, are equal to each other.
8. The whole is greater than any of its parts.
9. The whole is equal to the sum of all its parts.
10. A straight line is the shortest line that can be drawn from one point to another.
11. From one point to another only one straight line can be drawn.
12. Through the same point only one parallel to a straight line can be drawn.

13. All right angles are equal to one another.

14. Magnitudes which coincide throughout their whole extent, are equal.

POSTULATES.

35. A POSTULATE is a self-evident problem; such as,—

1. That a straight line may be drawn from one point to another.

2. That a straight line may be produced to any length.

3. That a straight line may be drawn through a given point parallel to another straight line.

4. That a perpendicular to a given straight line may be drawn from a point either within or without the line.

5. That an angle may be described equal to any given angle.

PROPOSITIONS.

36. A DEMONSTRATION is a course of reasoning by which a truth becomes evident.

37. A PROPOSITION is something proposed to be demonstrated, or to be performed.

A proposition is said to be the *converse* of another, when the conclusion of the first is used as the supposition in the second.

38. A THEOREM is something to be demonstrated.

39. A PROBLEM is something to be performed.

40. A LEMMA is a proposition preparatory to the demonstration or solution of a succeeding proposition.

41. A COROLLARY is an obvious consequence deduced from one or more propositions.

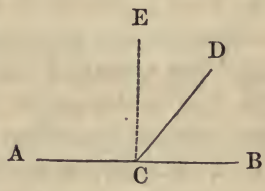
42. A SCHOLIUM is a remark made upon one or more preceding propositions.

43. An HYPOTHESIS is a supposition, made either in the

enunciation of a proposition, or in the course of a demonstration.

Proposition I. — Theorem.

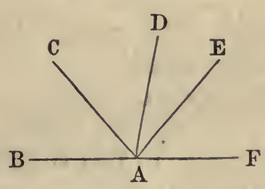
44. *The adjacent angles which one straight line makes by meeting another straight line, are together equal to two right angles.*



Let the straight line D C meet A B, making the adjacent angles A C D, D C B; these angles together will be equal to two right angles.

From the point C suppose C E to be drawn perpendicular to A B; then the angles A C E and E C B will each be a right angle (Art. 15). But the angle A C D is composed of the right angle A C E and the angle E C D (Art. 34, Ax. 9), and the angles E C D and D C B compose the other right angle, E C B; hence the angles A C D, D C B together equal two right angles.

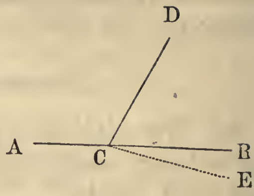
45. *Cor.* 1. If one of the angles A C D, D C B is a right angle, the other must also be a right angle.



46. *Cor.* 2. All the successive angles, B A C, C A D, D A E, E A F, formed on the same side of a straight line, B F, are equal, when taken together, to two right angles; for their sum is equal to that of the two adjacent angles, B A C, C A F.

Proposition II. — Theorem.

47. *If one straight line meets two other straight lines at a common point, making adjacent angles, which together are equal to two right angles, the two lines form one and the same straight line.*

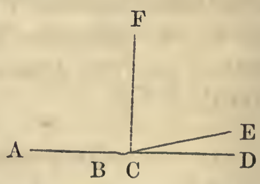


Let the straight line D C meet the two straight lines A C, C B at the common point C, making the adjacent angles A C D, D C B together equal to two right angles; then the lines A C and C B will form one and the same straight line.

If C B is not the straight line A C produced, let C E be that line produced; then the line A C E being straight, the sum of the angles A C D and D C E will be equal to two right angles (Prop. I.). But by hypothesis the angles A C D and D C B are together equal to two right angles; therefore the sum of the angles A C D and D C E must be equal to the sum of the angles A C D and D C B (Art. 34, Ax. 2). Take away the common angle A C D from each, and there will remain the angle D C B, equal to the angle D C E, a part to the whole, which is impossible; therefore C E is not the line A C produced. Hence A C and C B form one and the same straight line.

Proposition III.—Theorem.

48. *Two straight lines, which have two points common, coincide with each other throughout their whole extent, and form one and the same straight line.*



Let the two points which are common to two straight lines be A and B.

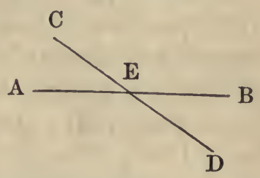
The two lines must coincide between the points A and B, for otherwise there would be two straight lines between A and B, which is impossible (Art. 34, Ax. 11).

Suppose, however, that, on being produced, the lines begin to separate at the point C, the one taking the direction C D, and the other C E. From the point C let the line C F be drawn, making, with C A, the right angle A C F. Now, since A C D is a straight line, the angle F C D will be a right angle (Prop. I. Cor. 1); and since A C E is a straight line, the angle F C E will also be a right angle; therefore the angle F C E is equal to the angle F C D (Art. 34, Ax. 13), a part to the whole, which is impossible; hence two straight lines which have two points common, A and B, cannot separate from each other when produced; hence they must form one and the same straight line.

Proposition IV. — Theorem.

49. *When two straight lines intersect each other, the opposite or vertical angles which they form are equal.*

Let the two straight lines A B, C D intersect each other at the point E; then will the angle A E C be equal to the angle D E B, and the angle C E B to A E D.



For the angles A E C, C E B, which the straight line C E forms by meeting the straight line A B, are together equal to two right angles (Prop. I.); and the angles C E B, B E D, which the straight line B E forms by meeting the straight line C D, are equal to two right angles; hence the sum of the angles A E C, C E B is equal to the sum of the angles C E B, B E D (Art. 34, Ax. 1). Take away from each of these sums the common angle C E B, and there will remain the angle A E C, equal to its opposite angle, B E D (Art. 34, Ax. 3).

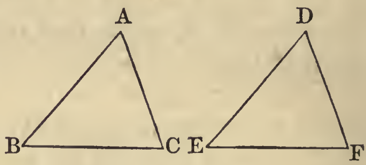
In the same manner it may be shown that the angle C E B is equal to its opposite angle, A E D.

50. *Cor.* 1. The four angles formed by two straight lines intersecting each other, are together equal to four right angles.

51. *Cor.* 2. All the successive angles, around a common point, formed by any number of straight lines meeting at that point, are together equal to four right angles.

Proposition V. — Theorem.

52. *If two triangles have two sides and the included angle in the one equal to two sides and the included angle in the other, each to each, the two triangles will be equal.*



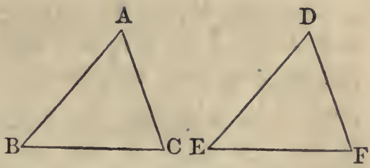
In the two triangles A B C, D E F, let the side A B be equal to the side D E, the side A C to the side D F, and the angle A to the angle D; then the triangles A B C, D E F will be equal.

Conceive the triangle A B C to be applied to the triangle D E F, so that the side A B shall fall upon its equal, D E, the point A upon D, and the point B upon E; then, since the angle A is equal to the angle D, the side A C will take the direction D F. But A C is equal to D F; therefore the point C will fall upon F, and the third side B C will coincide with the third side E F (Art. 34, Ax. 11). Hence the triangle A B C coincides with the triangle D E F, and they are therefore equal (Art. 34, Ax. 14).

53. *Cor.* When, in two triangles, these three parts are equal, namely, the side A B equal to D E, the side A C equal to D F, and the angle A equal to D, the other three corresponding parts are also equal, namely, the side B C equal to E F, the angle B equal to E, and the angle C equal to F.

Proposition VI. — Theorem.

54. *If two triangles have two angles and the included side in the one equal to two angles and the included side in the other, each to each, the two triangles will be equal.*



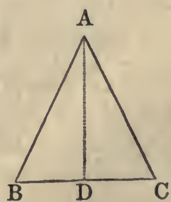
In the two triangles A B C, D E F, let the angle B be equal to the angle E, the angle C to the angle F, and the side B C to the side E F; then the triangles A B C, D E F will be equal.

Conceive the triangle A B C to be applied to the triangle D E F, so that the side B C shall fall upon its equal, E F, the point B upon E, and the point C upon F. Then, since the angle B is equal to the angle E, the side B A will take the direction E D; therefore the point A will be found somewhere in the line E D. In like manner, since the angle C is equal to the angle F, the line C A will take the direction F D, and the point A will be found somewhere in the line F D. Hence the point A, falling at the same time in both of the straight lines E D and F D, must fall at their intersection, D. Hence the two triangles A B C, D E F coincide with each other, and are therefore equal (Art. 34, Ax. 14).

55. *Cor.* When, in two triangles, these three parts are equal, namely, the angle B equal to the angle E, the angle C equal to the angle F, and the side B C equal to the side EF, the other three corresponding parts are also equal; namely, the side B A equal to E D, the side C A equal to F D, and the angle A equal to the angle D.

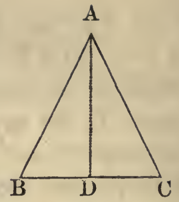
Proposition VII. — Theorem.

56. *In an isosceles triangle, the angles opposite the equal sides are equal.*



Let A B C be an isosceles triangle, in which the side A B is equal to the side A C; then will the angle B be equal to the angle C.

Conceive the angle B A C to be bisected, or divided into two equal parts, by



the straight line A D, making the angle B A D equal to D A C. Then the two triangles B A D, C A D have the two sides A B, A D and the included angle in the one equal to the two sides A C, A D and the included angle in the other, each to each; hence the two triangles are equal, and the angle B is equal to the angle C (Prop. V.).

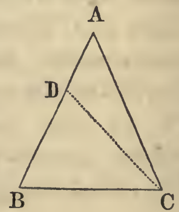
57. *Cor.* 1. The line bisecting the vertical angle of an isosceles triangle bisects the base at right angles.

58. *Cor.* 2. Conversely, the line bisecting the base of an isosceles triangle at right angles, bisects also the vertical angle.

59. *Cor.* 3. Every equilateral triangle is also equiangular.

Proposition VIII. — Theorem.

60. *If two angles of a triangle are equal, the opposite sides are also equal, and the triangle is isosceles.*



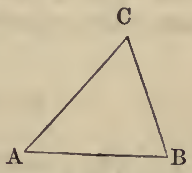
Let A B C be a triangle having the angle B equal to the angle C; then will the side A B be equal to the side A C.

For, if the two sides are not equal, one of them must be greater than the other. Let A B be the greater; then take D B equal to A C the less, and draw C D. Now, in the two triangles D B C, A B C, we have D B equal to A C by construction, the side B C common, and the angle B equal to the angle A C B by hypothesis; therefore, since two sides and the included angle in the one are equal to two sides and the included angle in the other, each to each, the triangle D B C is equal to the triangle A B C (Prop. V.), a part to the whole, which is impossible (Art. 34, Ax. 8). Hence the sides A B and A C cannot be unequal; therefore the triangle A B C is isosceles.

61. *Cor.* Therefore every equiangular triangle is equilateral.

Proposition IX. — Theorem.

62. *Any side of a triangle is less than the sum of the other two.*



In the triangle A B C, any one side, as A B, is less than the sum of the other two sides, A C and C B.

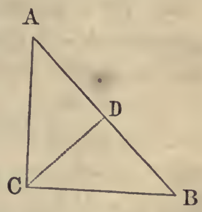
For the straight line A B is the shortest line that can be drawn from the point A to the point B (Art. 34, Ax. 10); hence the side A B is less than the sum of the sides A C and C B.

In like manner it may be proved that the side A C is less than the sum of A B and B C, and the side B C less than the sum of B A and A C.

63. *Cor.* Since the side A B is less than the sum of A C and C B, if we take away from each of these two unequals the side C B, we shall have the difference between A B and C B less than A C; that is, *the difference between any two sides of a triangle is less than the other side.*

Proposition X. — Theorem.

64. *The greater side of any triangle is opposite the greater angle.*



In the triangle C A B, let the angle C be greater than B; then will the side A B, opposite to C, be greater than A C, opposite to B.

Draw the straight line C D, making the angle B C D equal to B. Then, in the triangle B D C, we shall have the side B D equal to D C (Prop. VIII.). But the side A C is less than the sum of A D and D C (Prop. IX.), and the sum of A D and D C is equal to the sum of A D and D B, which is equal to A B; therefore the side A B is greater than A C.

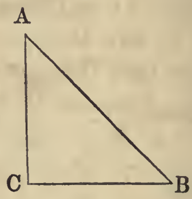
65. *Cor.* 1. Therefore the shorter side is opposite to the less angle.

66. *Cor.* 2. In the right-angled triangle the hypothenuse is the longest side.

Proposition XI. — Theorem.

67. *The greater angle of any triangle is opposite the greater side.*

In the triangle C A B, suppose the side A B to be greater than A C; then will the angle C, opposite to A B, be greater than the angle B, opposite to A C.

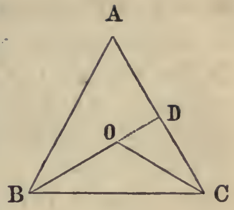


For, if the angle C is not greater than B, it must either be equal to it or less. If the angle C were equal to B, then would the side A B be equal to the side A C (Prop. VIII.), which is contrary to the hypothesis; and if the angle C were less than B, then would the side A B be less than A C (Prop. X. Cor. 1), which is also contrary to the hypothesis. Hence, the angle C must be greater than B.

68. *Cor.* It follows, therefore, that the less angle is opposite to the shorter side.

Proposition XII. — Theorem.

69. *If, from any point within a triangle, two straight lines are drawn to the extremities of either side, their sum will be less than that of the other two sides of the triangle.*

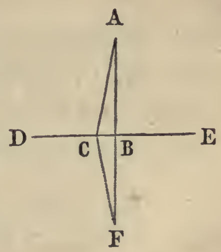


Let the two straight lines B O, C O be drawn from the point O, within the triangle A B C, to the extremities of the side B C; then will the sum of the two lines B O and O C be less than the sum of the sides B A and A C.

Let the straight line B O be produced till it meets the side A C in the point D; and because one side of a triangle is less than the sum of the other two sides (Prop. IX.), the side O C in the triangle C D O is less than the sum of O D and D C. To each of these inequalities add B O, and we have the sum of B O and O C less than the sum of B O, O D, and D C (Art. 34, Ax. 4); or the sum of B O and O C less than the sum of B D and D C. Again, because the side B D is less than the sum of B A and A D, by adding D C to each, we have the sum of B D and D C less than the sum of B A and A C. But it has been just shown that the sum of B O and O C is less than the sum of B D and D C; much more, then, is the sum of B O and O C less than B A and A C.

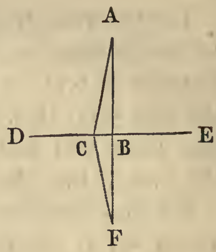
Proposition XIII. — Theorem.

70. *From a point without a straight line, only one perpendicular can be drawn to that line.*

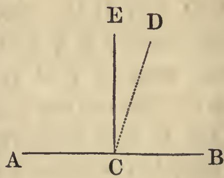


Let A be the point, and D E the given straight line; then from the point A only one perpendicular can be drawn to D E:

Let it be supposed that we can draw two perpendiculars, A B and A C. Produce one of them, as A B, till B F is equal to A B, and join F C. Then, in the triangles A B C and C B F, the angles C B A and C B F are both right angles (Prop. I. Cor. 1), the side C B is common to both, and the side B F is equal to the side A B; hence the two triangles are equal, and the angle B C F is equal to the angle B C A (Prop. V.) But the angle B C A is, by hypothesis, a right angle; therefore B C F must also be a right angle; and if the two adjacent angles, B C A and B C F, are together equal to two right angles, the two lines A C and C F must form one and the same straight line (Prop. II.). Whence it follows, that between the same two points, A and F, two straight lines can be drawn, which is impossible (Art. 34, Ax. 11); hence no more than one perpendicular can be drawn from the same point to the same straight line.



71. *Cor.* At the same point C, in the line A B, it is likewise impossible to erect more than one perpendicular to that line. For, if C D and C E were each perpendicular to A B, the angles B C D, B C E would be right angles; hence the angle B C D would be equal to the angle B C E, a part to the whole, which is impossible.



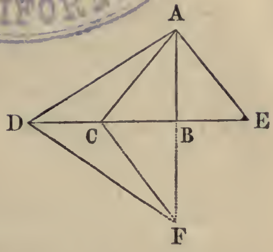
Proposition XIV. — Theorem.

72. *If, from a point without a straight line, a perpendicular be let fall on that line, and oblique lines be drawn to different points in the same line;* —

1st. *The perpendicular will be shorter than any oblique line.*

2d. *Any two oblique lines, which meet the given line at equal distances from the perpendicular, will be equal.*

3d. *Of any two oblique lines, that which meets the given line at the greater distance from the perpendicular will be the longer.*



Let A be the given point, and D E the given straight line. Draw A B perpendicular to D E, and the oblique lines A E, A C, A D. Produce A B till B F is equal to A B, and join C F, D F.

First. The triangle B C F is equal to the triangle B C A, for they have the side C B common, the side A B equal to the side B F, and the angle A B C equal to the angle F B C, both being right angles (Prop. I. Cor. 1); hence the third sides, C F and A C, are equal (Prop. V. Cor.). But A B F, being a straight line, is shorter than A C F, which is a broken line (Art. 34, Ax. 10); therefore A B, the half of A B F, is shorter than A C, the half of A C F; hence the perpendicular is shorter than any oblique line.

Secondly. If B E is equal to B C, then, since A B is common to the triangles, A B E, A B C, and the angles A B E, A B C are right angles, the two triangles are equal (Prop. V.), and the side A E is equal to the side A C (Prop. V. Cor.). Hence the two oblique lines, meeting the given line at equal distances from the perpendicular, are equal.

Thirdly. The point C being in the triangle A D F, the sum of the lines A C, C F is less than the sum of the sides A D, D F (Prop. XII.) But A C has been shown to be equal to C F; and in like manner it may be shown that A D is equal to D F. Therefore A C, the half of the line A C F, is shorter than A D, the half of the line A D F; hence the oblique line which meets the given line the greater distance from the perpendicular, is the longer.

73. *Cor.* 1. The perpendicular measures the shortest distance of any point from a straight line.

74. *Cor.* 2. From the same point to a given straight line only two equal straight lines can be drawn.

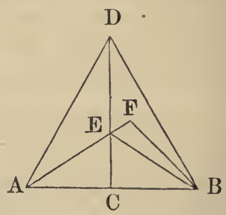
75. *Cor.* 3. Of any two straight lines drawn from a point to a straight line, that which is not shorter than the other will be longer than any straight line that can be drawn between them, from the same point to the same line.

Proposition XV. — Theorem.

76. *If from the middle point of a straight line a perpendicular to this line be drawn,* —

1st. *Any point in the perpendicular will be equally distant from the extremities of the line.*

2d. *Any point out of the perpendicular will be unequally distant from those extremities.*



Let D C be drawn perpendicular to the straight line A B, from its middle point C.

First. Let D and E be points, taken at pleasure, in the perpendicular, and join D A, D B, and also A E, E B. Then, since A C is equal to C B, the two oblique lines D A, D B meet points which are at the same distance from the perpendicular, and are therefore equal (Prop. XIV.). So, likewise, the two oblique lines E A, E B are equal; therefore any point in the perpendicular is equally distant from the extremities A and B.

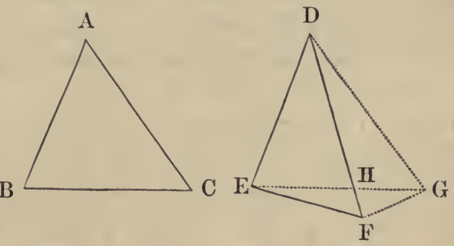
Secondly. Let F be any point out of the perpendicular, and join F A, F B. Then one of those lines must cut the perpendicular, in some point, as E. Join E B; then we have E B equal to E A. But in the triangle F E B, the side F B is less than the sum of the sides E F, E B (Prop. IX.), and since the sum of F E, E B is equal to the sum of F E, E A, which is equal to F A, F B is less than F A. Hence any point out of the perpendicular is at unequal distances from the extremities A and B.

77. *Cor.* If a straight line have two points, of which each is equally distant from the extremities of another

straight line, it will be perpendicular to that line at its middle point.

Proposition XVI. — Theorem.

78. *If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angle of the one greater than the included angle of the other, the third side of that which has the greater angle will be greater than the third side of the other.*



Let ABC, DEF be two triangles, having the side AB equal to DE, and AC equal to DF, and the angle A greater than D; then will the side BC be greater than EF.

Of the two sides DE, DF, let DF be the side which is not shorter than the other; make the angle EDG equal to BAC; and make DG equal to AC or DF, and join EG, GF.

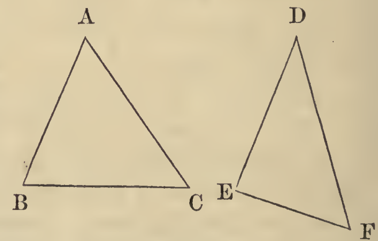
Since DF, or its equal DG, is not shorter than DE, it is longer than DH (Prop. XIV. Cor. 3); therefore its extremity, F, must fall below the line EG. The two triangles, ABC and DEG, have the two sides AB, AC equal to the two sides DE, DG, each to each, and the included angle BAC of the one equal to the included angle EDG of the other; hence the side BC is equal to EG (Prop. V. Cor.).

In the triangle DFG, since DG is equal to DF, the angle DFG is equal to the angle DGF (Prop. VII.); but the angle DGF is greater than the angle EGF; therefore the angle DFG is greater than EGF, and much more is the angle EFG greater than the angle

E G F. Because the angle E F G in the triangle E F G is greater than E G F, and because the greater side is opposite the greater angle (Prop. X.), the side E G is greater than E F; and E G has been shown to be equal to B C; hence B C is greater than E F.

Proposition XVII. — Theorem.

79. *If two triangles have two sides of the one equal to two sides of the other, each to each, but the third side of the one greater than the third side of the other, the angle contained by the sides of that which has the greater third side will be greater than the angle contained by the sides of the other.*

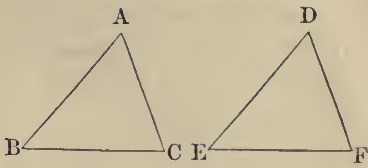


Let A B C, D E F be two triangles, the side A B equal to D E, and A C equal to D F, and the side C B greater than E F, then will the angle A be greater than D.

For, if it be not greater, it must either be equal to it or less. But the angle A cannot be equal to D, for then the side B C would be equal to E F (Prop. V. Cor.), which is contrary to the hypothesis; neither can it be less, for then the side B C would be less than E F (Prop. XVI.), which also is contrary to the hypothesis; therefore the angle A is not less than the angle D, and it has been shown that is not equal to it; hence the angle A must be greater than the angle D.

Proposition XVIII. — Theorem.

80. *If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles themselves will be equal.*



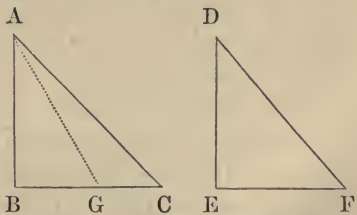
Let the triangles A B C, D E F have the side A B equal to D E, A C to D F, and B C to E F; then will the angle A be equal to D, the angle B to the angle E, and the angle C to the angle F, and the two triangles will also be equal.

For, if the angle A were greater than the angle D, since the sides A B, A C are equal to the sides D E, D F, each to each, the side B C would be greater than E F (Prop. XVI.); and if the angle A were less than D, it would follow that the side B C would be less than E F. But by hypothesis B C is equal to E F; hence the angle A can neither be greater nor less than D; therefore it must be equal to it. In the same manner, it may be shown that the angle B is equal to E, and the angle C to F; hence the two triangles must be equal.

81. *Scholium.* In two triangles equal to each other, the equal angles are opposite the equal sides; thus the equal angles A and D are opposite the equal sides B C and E F.

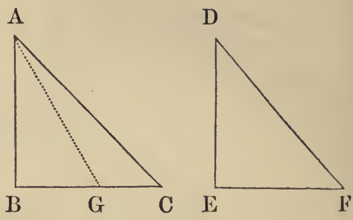
Proposition XIX. — Theorem.

82. *If two right-angled triangles have the hypothenuse and a side of the one equal to the hypothenuse and a side of the other, each to each, the triangles are equal.*



Let the two right-angled triangles ABC, DEF, have the hypothenuse A C equal to D F, and the side A B equal to D E; then will the triangle A B C be equal to the triangle D E F.

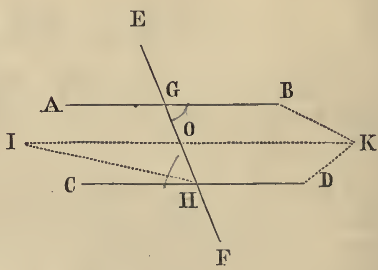
The two triangles are evidently equal, if the sides B C and E F are equal (Prop. XVIII.). If it be possible, let these sides be unequal, and let B C be the greater. Take B G equal to E F, the less side, and join A G. Then, in the two triangles A B G, D E F, the angles B and E are equal, both being right angles, the side A B is equal to D E by hypothesis, and the side B G to E F by construction; hence these triangles are equal (Prop. V.); and therefore A G is equal to D F. But by hypothesis D F is equal to A C, and therefore A G is equal to A C. But the oblique line A C cannot be equal to A G, which meets the same straight line nearer the perpendicular A B (Prop. XIV.); therefore B C and E F cannot be unequal, hence they must be equal; therefore the triangles A B C and D E F are equal.



Proposition XX. — Theorem.

83. *If a straight line, intersecting two other straight lines, makes the alternate angles equal, the two lines are parallel.*

Let the straight line E F intersect the two straight lines A B, C D, making the alternate angles B G H, C H G equal; then the lines A B, C D will be parallel.

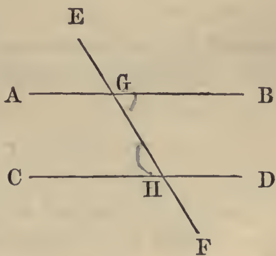


For, if the lines A B, C D are not parallel, let them meet in some point K, and through O, the middle point of G H, draw the straight line I K, making I O equal to O K, and join H I. Then the opposite angles K O G, I O H, formed by the intersection of the two straight lines I K, G H, are equal (Prop. IV.); and the triangles K O G, I O H have the two sides K O, O G and the included angle in the one equal to the two sides I O, O H and the included angle in the other, each to each; hence the angle K G O is equal to the angle I H O (Prop. V. Cor.). But, by hypothesis, the angle K G O is equal to the angle C H O, therefore the angle I H O is equal to C H O, so that H I and H C must coincide; that is, the line C D when produced meets I K in two points, I, K, and yet does not form one and the same straight line, which is impossible (Prop. III.); therefore the lines A B, C D cannot meet, consequently they are parallel (Art. 17).

NOTE. — The demonstration of the proposition is substantially that given by M. da Cunha in the *Principes Mathématiques.* This demonstration Young pronounces "superior to every other that has been given of the same proposition"; and Professor Playfair, in the *Edinburgh Review*, Vol. XX., calls attention to it, as a most important improvement in elementary Geometry.

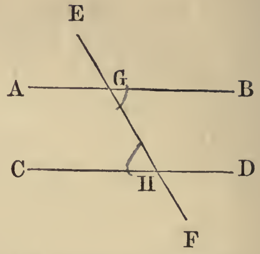
PROPOSITION XXI. — THEOREM.

84. *If a straight line, intersecting two other straight lines, makes any exterior angle equal to the interior and opposite angle, or makes the interior angles on the same side together equal to two right angles, the two lines are parallel.*



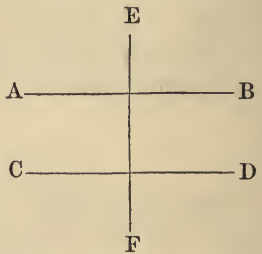
Let the straight line E F intersect the two straight lines A B, C D, making the exterior angle E G B equal to the interior and opposite angle, G H D; then the lines A B, C D are parallel.

For the angle A G H is equal to the angle E G B (Prop. IV.); and E G B is equal to G H D, by hypothesis; therefore the angle A G H is equal to the angle G H D; and they are alternate angles; hence the lines A B, C D are parallel (Prop. XX.).



Again, let the interior angles on the same side, B G H, G H D, be together equal to two right angles; then the lines A B, C D are parallel.

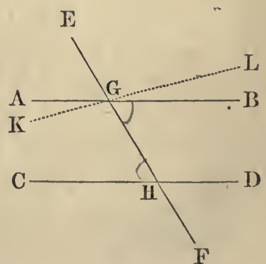
For the sum of the angles B G H, G H D is equal to two right angles, by hypothesis; and the sum of A G H, B G H is also equal to two right angles (Prop. I.); take away B G H, which is common to both, and there remains the angle G H D, equal to the angle A G H; and these are alternate angles; hence the lines A B, C D are parallel.



85. *Cor.* If two straight lines are perpendicular to another, they are parallel; thus A B, C D, perpendicular to E F, are parallel.

PROPOSITION XXII. — THEOREM.

86. *If a straight line intersects two parallel lines, it makes the alternate angles equal; also any exterior angle equal to the interior and opposite angle; and the two interior angles upon the same side together equal to two right angles.*



Let the straight line E F intersect the parallel lines A B, C D; the alternate angles A G H, G H D are equal; the exterior angle E G B is equal to the interior and opposite angle G H D; and the two interior angles B G H, G H D upon the same side are together equal to two right angles.

For if the angle A G H is not equal to G H D, draw the straight line K L through the point G, making the angle K G H equal to G H D; then, since the alternate angles G H D, K G H are equal, K L is parallel to C D (Prop. XX.); but by hypothesis A B is also parallel to C D, so that through the same point, G, two straight lines are drawn parallel to C D, which is impossible (Art. 34, Ax. 12). Hence the angles A G H, G H D are not unequal; that is, they are equal.

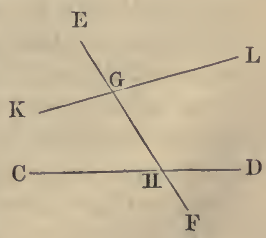
Now, the angle E G B is equal to the angle A G H (Prop. IV.), and A G H has been shown to be equal to G H D; hence E G B is also equal to G H D.

Again, add to each of these equals the angle B G H; then the sum of the angles E G B, B G H is equal to the sum of the angles B G H, G H D. But E G B, B G H are equal to two right angles (Prop. I.); hence B G H, G H D are also equal to two right angles.

87. *Cor.* If a line is perpendicular to one of two parallel lines, it is perpendicular to the other; thus E F (Art. 85), perpendicular to A B, is perpendicular to C D.

Proposition XXIII. — Theorem.

88. *If two straight lines intersect a third line, and make the two interior angles on the same side together less than two right angles, the two lines will meet on being produced.*



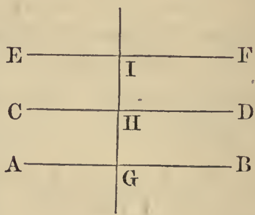
Let the two lines K L, C D make with E F the angles K G H, G H C, together less than two right angles; then K L and C D will meet on being produced.

For if they do not meet, they are parallel (Art. 17). But they are not parallel; for then the sum of the interior angles K G H, G H C would be equal to two right angles (Prop. XXII.); but by hypothesis it is less; therefore the lines K L, C D will meet on being produced.

89. *Scholium.* The two lines K L, C D, on being produced, must meet on the side of E F, on which are the two interior angles whose sum is less than two right angles.

Proposition XXIV. — Theorem.

90. *Straight lines which are parallel to the same line are parallel to each other.*

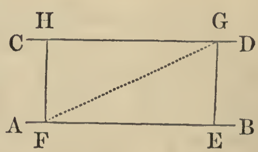


Let the straight lines A B, C D be each parallel to the line E F; then are they parallel to each other.

Draw G H I perpendicular to E F. Then, since A B is parallel to E F, G I will be perpendicular to A B (Prop. XXII. Cor.); and since C D is parallel to E F, G I will for a like reason be perpendicular to C D. Consequently A B and C D are perpendicular to the same straight line; hence they are parallel (Prop. XXI. Cor.).

Proposition XXV. — Theorem.

91. *Two parallel straight lines are everywhere equally distant from each other.*



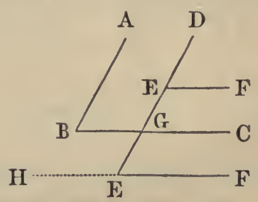
Let A B, C D be two parallel straight lines. Through any two points in A B, as E and F, draw the straight lines E G, F H, perpendicular to A B. These lines will be equal to each other.

For, if G F be joined, the angles G F E, F G H, considered in reference to the parallels A B, C D, will be alternate interior angles, and therefore equal to each other (Prop. XXII.). Also, since the straight lines E G, F H are perpendicular to the same straight line A B, and consequently parallel (Prop. XXI. Cor.), the angles E G F, G F H, considered in reference to the parallels E G, F H, will be alternate interior angles, and therefore equal. Hence, the two triangles E F G, F G H, have a side and the two adjacent angles of the one equal to a side and the two adjacent angles of the other, each to each; therefore these triangles are equal (Prop. VI.); hence the side E G, which measures the distance of the parallels A B, C D, at the point E, is equal to the side F H, which measures the distance of the same parallels at the point F. Hence two parallels are everywhere equally distant.

Proposition XXVI. — Theorem.

92. *If two angles have their sides parallel, each to each, and lying in the same direction, the two angles are equal.*

Let A B C, D E F be two angles, which have the side A B parallel to D E, and B C parallel to E F; then these angles are equal.

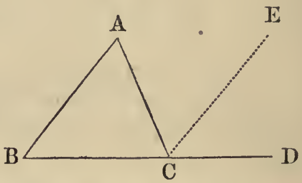


For produce D E, if necessary, till it meets B C in the point G. Then, since E F is parallel to G C, the angle D E F is equal to D G C (Prop. XXII.); and since D G is parallel to A B, the angle D G C is equal to A B C; hence the angle D E F is equal to A B C.

93. *Scholium.* This proposition is restricted to the case where the side E F lies in the same direction with B C, since if F E were produced toward H, the angles D E H, A B C would only be equal when they are right angles.

Proposition XXVII. — Theorem.

94. *If any side of a triangle be produced, the exterior angle is equal to the sum of the two interior and opposite angles.*



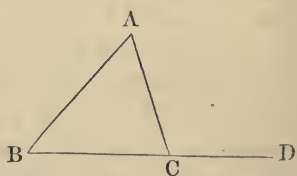
Let A B C be a triangle, and let one of its sides, B C be produced towards D; then the exterior angle A C D is equal to the two interior and opposite angles, C A B, A B C.

For, draw E C parallel to the side A B; then, since A C meets the two parallels A B, E C, the alternate angles B A C, A C E are equal (Prop. XXII.).

Again, since B D meets the two parallels A B, E C, the exterior angle E C D is equal to the interior and opposite angle A B C. But the angle A C E is equal to B A C; therefore, the whole exterior angle A C D is equal to the two interior and opposite angles C A B, A B C (Art. 34, Ax. 2).

Proposition XXVIII. — Theorem.

95. *In every triangle the sum of the three angles is equal to two right angles.*



Let A B C be any triangle; then will the sum of the angles A B C, B C A, C A B be equal to two right angles.

For, let the side B C be produced towards D, making the exterior angle A C D; then the angle A C D is equal to C A B and A B C (Prop. XXVII.). To each of these equals add the angle A C B, and we shall have the sum of

A C B and A C D, equal to the sum of A B C, B C A, and C A B. But the sum of A C B and A C D is equal to two right angles (Prop. I.); hence the sum of the three angles A B C, B C A, and C A B is equal to two right angles (Art. 34, Ax. 2).

96. *Cor.* 1. Two angles of a triangle being given, or merely their sum, the third will be found by subtracting that sum from two right angles.

97. *Cor.* 2. If two angles in one triangle be respectively equal to two angles in another, their third angles will also be equal.

98. *Cor.* 3. A triangle cannot have more than one angle as great as a right angle.

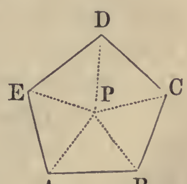
99. *Cor.* 4. And, therefore, every triangle must have at least two acute angles.

100. *Cor.* 5. In a right-angled triangle the right angle is equal to the sum of the other two angles.

101. *Cor.* 6. Since every equilateral triangle is also equiangular (Prop. VII. Cor. 3), each of its angles will be equal to two thirds of one right angle.

Proposition XXIX. — Theorem.

102. *The sum of all the interior angles of any polygon is equal to twice as many right angles, less four, as the figure has sides.*



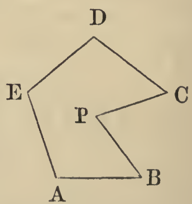
Let A B C D E be any polygon; then the sum of all its interior angles, A, B, C, D, E, is equal to twice as many right angles as the figure has sides, less four right angles.

For, from any point P within the polygon, draw the straight lines P A, P B, P C, P D, P E, to the vertices of all the angles, and the polygon will be divided into as many triangles as it has sides. Now, the sum of the three angles in each of these triangles is equal to two right angles (Prop. XXVIII.); therefore the sum of the angles of all these triangles is equal to twice as many right angles as there are triangles, or sides, to the polygon. But the sum of all the angles about the point P is equal to four right angles (Prop. IV. Cor. 2), which sum forms no part of the interior angles of the polygon; therefore, deducting the sum of the angles about the point, there remain the angles of the polygon equal to twice as many right angles as the figure has sides, less four right angles.

103. *Cor.* 1. The sum of the angles in a *quadrilateral* is equal to four right angles; hence, if all the angles of a quadrilateral are equal, each of them is a right angle; also, if three of the angles are right angles, the fourth is likewise a right angle.

104. *Cor.* 2. The sum of the angles in a *pentagon* is equal to six right angles; in a *hexagon*, the sum is equal to eight right angles, &c.

105. *Cor.* 3. In every *equiangular* figure of more than four sides, each angle is greater than a right angle; thus, in a *regular pentagon*, each angle is equal to one and one fifth right angles; in a *regular hexagon*, to one and one third right angles, &c.

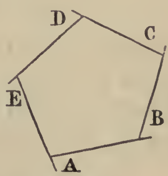


106. *Scholium.* In applying this proposition to polygons which have *re-entrant* angles, or angles whose vertices are directed inward, as B P C, each of these angles must be considered greater than two right angles. But, in order to avoid ambiguity, we shall hereafter limit our reasoning to polygons with *salient* angles, or with angles directed outwards, and which may be called *convex* polygons. Every convex polygon is such that a

straight line, however drawn, cannot meet the perimeter of the polygon in more than two points.

PROPOSITION XXX. — THEOREM.

107. *The sum of all the exterior angles of any polygon, formed by producing each side in the same direction, is equal to four right angles.*

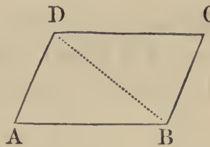


Let each side of the polygon A B C D E be produced in the same direction; then the sum of the exterior angles A, B, C, D, E, will be equal to four right angles.

For each interior angle, together with its adjacent exterior angle, is equal to two right angles (Prop. I.); hence the sum of all the angles, both interior and exterior, is equal to twice as many right angles as there are sides to the polygon. But the sum of the interior angles alone, less four right angles, is equal to the same sum (Prop. XXIX.); therefore the sum of the exterior angles is equal to four right angles.

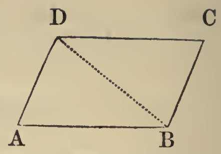
PROPOSITION XXXI. — THEOREM.

108. *The opposite sides and angles of every parallelogram are equal to each other.*



Let A B C D be a parallelogram; then the opposite sides and angles are equal to each other.

Draw the diagonal B D, then, since the opposite sides A B, D C are parallel, and BD meets them, the alternate angles ABD, BDC are equal (Prop. XXII.); and since AD, BC are parallel, and B D meets them, the alternate angles A D B, D B C are likewise equal. Hence, the two triangles A D B, D B C have two angles, A B D, A D B, in the one, equal to two angles, B D C, D B C, in the other, each to each; and since



the side B D included between these equal angles is common to the two triangles, they are equal (Prop. VI.); hence the side A B opposite the angle A D B is equal to the side D C opposite the angle D B C (Prop. VI. Cor.); and, in like manner, the side A D is equal to the side B C; hence the opposite sides of a parallelogram are equal.

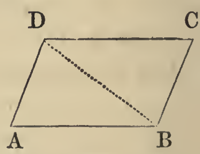
Again, since the triangles are equal, the angle A is equal to the angle C (Prop. VI. Cor.); and since the two angles D B C, A B D are respectively equal to the two angles A D B, B D C, the angle A B C is equal to the angle A D C.

109. *Cor.* 1. The diagonal divides a parallelogram into two equal triangles.

110. *Cor.* 2. The two parallels A D, B C, included between two other parallels, A B, C D, are equal.

Proposition XXXII.—Theorem.

111. *If the opposite sides of a quadrilateral are equal, each to each, the equal sides are parallel, and the figure is a parallelogram.*



Let A B C D be a quadrilateral having its opposite sides equal; then will the equal sides be parallel, and the figure be a parallelogram.

For, having drawn the diagonal B D, the triangles A B D, B D C have all the sides of the one equal to the corresponding sides of the other; therefore they are equal, and the angle A D B opposite the side A B is equal to D B C opposite C D (Prop. XVIII. Sch.); hence the side A D is parallel to B C (Prop. XX.). For a like reason, A B is parallel to C D; therefore the quadrilateral A B C D is a parallelogram.

Proposition XXXIII. — Theorem.

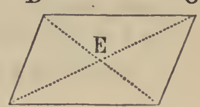
112. *If two opposite sides of a quadrilateral are equal and parallel, the other sides are also equal and parallel, and the figure is a parallelogram.*

Let A B C D be a quadrilateral, having the sides A B, C D equal and parallel; then will the other sides also be equal and parallel.

Draw the diagonal B D; then, since A B is parallel to C D, and B D meets them, the alternate angles A B D, B D C are equal (Prop. XXII.); moreover, in the two triangles A B D, D B C, the side B D is common; therefore, two sides and the included angle in the one are equal to two sides and the included angle in the other, each to each; hence these triangles are equal (Prop. V.), and the side A D is equal to B C. Hence the angle A D B is equal to D B C, and consequently A D is parallel to B C (Prop. XX.); therefore the figure A B C D is a parallelogram.

Proposition XXXIV. — Theorem.

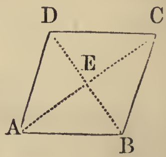
113. *The diagonals of every parallelogram bisect each other.*



Let A B C D be a parallelogram, and A C, D B its diagonals, intersecting at E; then will A E equal E C, and B E equal E D.

For, since A B, C D are parallel, and B D meets them, the alternate angles C D E, A B E are equal (Prop. XXII.); and since A C meets the same parallels, the alternate angles B A E, E C D are also equal; and the sides A B, C D are equal (Prop. XXXI.). Hence the triangles A B E, C D E have two angles and the included side in the one equal to two angles and the included side in the other, each to each; hence the two triangles are equal (Prop. VI.); therefore the side A E opposite the angle A B E is equal to C E opposite C D E; hence, also, the sides B E, D E opposite the other equal angles are equal.

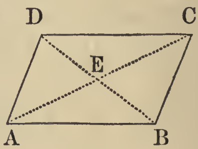
114. *Scholium.* In the case of a rhombus, the sides A B, B C being equal, the triangles A E B, E B C have all the sides of the one equal to the corresponding sides of the other, and are, therefore, equal; whence it follows that the angles A E B, B E C are equal. Therefore the diagonals of a rhombus bisect each other at right angles.



Proposition XXXV. — Theorem.

115. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

Let A B C D be a quadrilateral, and A C, D B its diagonals intersecting at E; then will the figure be a parallelogram.



For, in the two triangles A B E, C D E, the two sides A E, E B and the included angle in the one are equal to the two sides C E, E D and the included angle in the other; hence the triangles are equal, and the side A B is equal to the side C D (Prop. V. Cor.). For a like reason, A D is equal to C B; therefore the quadrilateral is a parallelogram (Prop. XXXII.).

BOOK II.

RATIO AND PROPORTION.

DEFINITIONS.

116. RATIO is the relation, in respect to quantity, which one magnitude bears to another of the same kind; and is the quotient arising from dividing the first by the second.

A ratio may be written in the form of a fraction, or with the sign : .

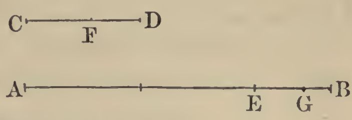
Thus the ratio of A to B may be expressed either by $\frac{A}{B}$, or by A : B.

117. The two magnitudes necessary to form a ratio are called the TERMS of the ratio. The first term is called the ANTECEDENT, and the last, the CONSEQUENT.

118. Ratios of magnitudes may be expressed by numbers, either exactly, or approximately.

This may be illustrated by the operation of finding the numerical ratio of two straight lines, AB, CD.

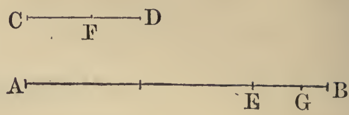
From the greater line AB cut off a part equal to the less CD, as many times as possible; for example, twice, with the remainder BE.



From the line CD cut off a part equal to the remainder BE as many times as possible; once, for example, with the remainder DF.

From the first remainder BE, cut off a part equal to the second DF, as many times as possible; once, for example, with the remainder BG.

From the second remainder D F, cut off a part equal to B G, the third, as many times as possible.



Proceed thus till a remainder arises, which is exactly contained a certain number of times in the preceding one.

Then this last remainder will be the common measure of the proposed lines; and, regarding it as unity, we shall easily find the values of the preceding remainders; and, at last, those of the two proposed lines, and hence their ratio in numbers.

Suppose, for instance, we find G B to be contained exactly twice in F D; B G will be the common measure of the two proposed lines. Let B G equal 1; then will F D equal 2.. But E B contains F D once, plus G B; therefore we have E B equal to 3. C D contains E B once, plus F D; therefore we have C D equal to 5. A B contains C D twice, plus E B; therefore we have A B equal to 13. Hence the ratio of the two lines is that of 13 to 5. If the line C D were taken for unity, the line A B would be $\frac{13}{5}$; if A B were taken for unity, C D would be $\frac{5}{13}$.

It is possible that, however far the operation be continued, no remainder may be found which shall be contained an exact number of times in the preceding one. In that case there can be obtained only an approximate ratio, expressed in numbers, more or less exact, according as the operation is more or less extended.

119. When the greater of two magnitudes contains the less a certain number of times without having a remainder, it is called a MULTIPLE of the less; and the less is then called a SUBMULTIPLE, or measure of the greater.

Thus, 6 is a multiple of 2; 2 and 3 are submultiples, or measures, of 6.

120. EQUIMULTIPLES, or LIKE MULTIPLES, are those which contain their respective submultiples the same number of

times; and EQUISUBMULTIPLES, or LIKE SUBMULTIPLES, are those contained in their respective multiples the same number of times.

Thus 4 and 5 are like submultiples of 8 and 10; 8 and 10 are like multiples of 4 and 5.

121. COMMENSURABLE magnitudes are magnitudes of the same kind, which have a common measure, and whose ratio therefore may be exactly expressed in numbers.

122. INCOMMENSURABLE magnitudes are magnitudes of the same kind, which have no common measure, and whose ratio, therefore, cannot be exactly expressed in numbers.

123. A DIRECT ratio is the quotient of the antecedent by the consequent; an INVERSE ratio, or RECIPROCAL ratio, is the quotient of the consequent by the antecedent, or the reciprocal of the direct ratio.

Thus the direct ratio of a line 6 feet long to a line 2 feet long is $\frac{6}{2}$ or 3; and the inverse ratio of a line 6 feet long to a line 2 feet long is $\frac{2}{6}$ or $\frac{1}{3}$, which is the same as the reciprocal of 3, the direct ratio of 6 to 2.

The word *ratio* when used alone means the direct ratio.

124. A COMPOUND ratio is the product of two or more ratios.

Thus the ratio compounded of $A : B$ and $C : D$ is $\frac{A}{B} \times \frac{C}{D}$, or $\frac{A \times C}{B \times D}$.

125. A PROPORTION is an equality of ratios.

Four magnitudes are in proportion, when the ratio of the *first* to the *second* is the same as that of the *third* to the *fourth*.

Thus, the ratios of $A : B$ and $X : Y$, being equal to each other, when written $A : B = X : Y$, or $\frac{A}{B} = \frac{X}{Y}$, form a proportion.

126. Proportion is written not only with the sign =, but, more often, with the sign :: between the ratios.

Thus, A : B :: X : Y, expresses a proportion, and is read, The ratio of A to B is equal to the ratio of X to Y; or, A is to B as X is to Y.

127. The *first* and *third* terms of a proportion are called the ANTECEDENTS; the *second* and *fourth*, the CONSEQUENTS. The *first* and *fourth* are also called the EXTREMES, and the *second* and *third* the MEANS.

Thus, in the proportion A : B :: C : D, A and C are the antecedents; B and D are the consequents; A and D are the extremes; and B and C are the means.

The antecedents are called *homologous* or *like* terms, and so also are the consequents.

128. All the terms of a proportion are called PROPORTIONALS; and the last term is called a FOURTH PROPORTIONAL to the other three taken in their order.

Thus, in the proportion A : B :: C : D, D is the fourth proportional to A, B, and C.

129. When both the means are the same magnitude, either of them is called a MEAN PROPORTIONAL between the extremes; and if, in a series of proportional magnitudes, each consequent is the same as the next antecedent, those magnitudes are said to be in CONTINUED PROPORTION.

Thus, if we have A : B :: B : C :: C : D :: D : E, B is a mean proportional between A and C, C between B and D, D between C and E; and the magnitudes A, B, C, D, E are said to be in continued proportion.

130. When a continued proportion consists of but three terms, the middle term is said to be a MEAN PROPORTIONAL between the other two; and the last term is said to be the THIRD PROPORTIONAL to the first and second.

Thus, when A, B, and C are in proportion, A : B :: B : C; in which case B is called a mean proportional between A and C; and C is called the third proportional to A and B.

131. Magnitudes are in proportion by INVERSION, or INVERSELY, when each antecedent takes the place of its consequent, and each consequent the place of its antecedent.

Thus, let A : B : : C : D ; then, by inversion,

$$B : A :: D : C.$$

132. Magnitudes are in proportion by ALTERNATION, or ALTERNATELY, when antecedent is compared with antecedent, and consequent with consequent.

Thus, let A : B : : D : C ; then, by alternation,

$$A : D :: B : C.$$

133. Magnitudes are in proportion by COMPOSITION, when the sum of the first antecedent and consequent is to the first antecedent, or consequent, as the sum of the second antecedent and consequent is to the second antecedent, or consequent.

Thus, let A : B : : C : D ; then, by composition,

$$A + B : A :: C + D : C, \quad \text{or} \quad A + B : B :: C + D : D.$$

134. Magnitudes are in proportion by DIVISION, when the difference of the first antecedent and consequent is to the first antecedent, or consequent, as the difference of the second antecedent and consequent is to the second antecedent, or consequent.

Thus, let A : B : : C : D ; then, by division,

$$A - B : A :: C - D : C, \quad \text{or} \quad A - B : B :: C - D : D.$$

PROPOSITION I. — THEOREM.

135. *If four magnitudes are in proportion, the product of the two extremes is equal to the product of the two means.*

Let A : B : : C : D ; then will $A \times D = B \times C$.

For, since the magnitudes are in proportion,

$$\frac{A}{B} = \frac{C}{D};$$

and reducing the fractions of this equation to a common denominator, we have

$$\frac{A \times D}{B \times D} = \frac{B \times C}{B \times D},$$

or, the common denominator being omitted,

$$A \times D = B \times C.$$

Proposition II. — Theorem.

136. *If the product of two magnitudes is equal to the product of two others, these four magnitudes form a proportion.*

Let $A \times D = B \times C$; then will $A : B :: C : D$.

For, dividing each member of the given equation by $B \times D$, we have

$$\frac{A \times D}{B \times D} = \frac{B \times C}{B \times D},$$

which, reduced to the lowest terms, gives

$$\frac{A}{B} = \frac{C}{D}.$$

Whence $A : B :: C : D$.

Proposition III. — Theorem.

137. *If three magnitudes are in proportion, the product of the two extremes is equal to the square of the mean.*

Let $A : B :: B : C$; then will $A \times C = B^2$.

For, since the magnitudes are in proportion,

$$\frac{A}{B} = \frac{B}{C},$$

and, by Prop. I.,

$$A \times C = B \times B, \quad \text{or} \quad A \times C = B^2.$$

Proposition IV.—Theorem.

138. *If the product of any two quantities is equal to the square of a third, the third is a mean proportional between the other two.*

Let $A \times C = B^2$; then B is a mean proportional between A and C.

For, dividing each member of the given equation by $B \times C$, we have

$$\frac{A}{B} = \frac{B}{C};$$

whence $A : B :: B : C.$

Proposition V.—Theorem.

139. *If four magnitudes are in proportion, they will be in proportion when taken inversely.*

Let $A : B :: C : D$; then will $B : A :: D : C$.

For, from the given proportion, by Prop. I., we have

$$A \times D = B \times C, \quad \text{or} \quad B \times C = A \times D.$$

Hence, by Prop. II.,

$$B : A :: D : C.$$

Proposition VI.—Theorem.

140. *If four magnitudes are in proportion, they will be in proportion when taken alternately.*

Let $A : B :: C : D$; then will $A : C :: B : D$.

For, since the magnitudes are in proportion,

$$\frac{A}{B} = \frac{C}{D};$$

and multiplying each member of this equation by $\frac{B}{C}$, we have

$$\frac{A \times B}{B \times C} = \frac{C \times B}{D \times C},$$

which, reduced to the lowest terms, gives

$$\frac{A}{C} = \frac{B}{D}.$$

whence $$A : C :: B : D.$$

Proposition VII. — Theorem.

141. *If four magnitudes are in proportion, they will be in proportion by composition.*

Let $A : B :: C : D$; then will $A + B : A :: C + D : C$.

For, from the given proportion, by Prop. I., we have

$$B \times C = A \times D.$$

Adding $A \times C$ to each side of this equation, we have

$$A \times C + B \times C = A \times C + A \times D,$$

and resolving each member into its factors,

$$(A + B) \times C = (C + D) \times A.$$

Hence, by Prop. II.,

$$A + B : A :: C + D : C.$$

Proposition VIII. — Theorem.

142. *If four magnitudes are in proportion, they will be in proportion by division.*

Let $A : B :: C : D$; then will $A - B : A :: C - D : C$.

For, from the given proportion, by Prop. I., we have

$$B \times C = A \times D.$$

Subtracting each side of this equation from $A \times C$, we have

$$A \times C - B \times C = A \times C - A \times D,$$

and resolving each member into its factors,

$$(A - B) \times C = (C - D) \times A.$$

Hence, by Prop. II.,

$$A - B : A :: C - D : C.$$

Proposition IX. — Theorem.

143. *Equimultiples of two magnitudes have the same ratio as the magnitudes themselves.*

Let A and B be two magnitudes, and $m \times A$ and $m \times B$ their equimultiples, then will $m \times A : m \times B :: A : B$.

For $$A \times B = B \times A;$$

Multiplying each side of this equation by any number, m, we have

$$m \times A \times B = m \times B \times A;$$

therefore

$$(m \times A) \times B = (m \times B) \times A.$$

Hence, by Prop. II.,

$$m \times A : m \times B :: A : B.$$

Proposition X. — Theorem.

144. *Magnitudes which are proportional to the same proportionals, will be proportional to each other.*

Let $A : B :: E : F$, and $C : D :: E : F$; then will

$$A : B :: C : D.$$

For, by the given proportions, we have

$$\frac{A}{B} = \frac{E}{F}, \text{ and } \frac{C}{D} = \frac{E}{F}.$$

Therefore, it is evident (Art. 34, Ax. 1),

$$\frac{A}{B} = \frac{C}{D}.$$

Hence $$A : B :: C : D.$$

145. *Cor.* 1. If two proportions have an antecedent and its consequent the same in both, the remaining terms will be in proportion.

146. *Cor.* 2. Therefore, by alternation (Prop. VI.), if two proportions have the two antecedents or the two con-

sequents the same in both, the remaining terms will be in proportion.

Proposition XI. — Theorem.

147. *If any number of magnitudes are proportional, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $A : B :: C : D :: E : F$; then will

$$A : B :: A + C + E : B + D + F.$$

For, from the given proportion, we have

$$A \times D = B \times C, \quad \text{and} \quad A \times F = B \times E.$$

By adding $A \times B$ to the sum of the corresponding sides of these equations, we have

$$A \times B + A \times D + A \times F = A \times B + B \times C + B \times E.$$

Therefore,

$$A \times (B + D + F) = B \times (A + C + E).$$

Hence, by Prop. II.,

$$A : B :: A + C + E : B + D + F.$$

Proposition XII. — Theorem.

148. *If four magnitudes are in proportion, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.*

Let $A : B :: C : D$; then will

$$A + B : A - B :: C + D : C - D.$$

For, from the given proportion, by Prop. VII., we have

$$A + B : A :: C + D : C;$$

and from the given proportion, by Prop. VIII., we have

$$A - B : A :: C - D : C.$$

Hence, from these two proportions, by Prop. X. Cor. 2, we have

$$A + B : A - B :: C + D : C - D.$$

Proposition XIII. — Theorem.

149. *If there be two sets of proportional magnitudes, the products of the corresponding terms will be proportionals.*

Let $A : B :: C : D$, and $E : F :: G : H$; then will

$$A \times E : B \times F :: C \times G : D \times H.$$

For, from the first of the given proportions, by Prop. I., we have

$$A \times D = B \times C;$$

and from the second of the given proportions, by Prop. I., we have

$$E \times H = F \times G.$$

Multiplying together the corresponding members of these equations, we have

$$A \times D \times E \times H = B \times C \times F \times G.$$

Hence, by Prop. II.,

$$A \times E : B \times F :: C \times G : D \times H.$$

Proposition XIV. — Theorem.

150. *If three magnitudes are proportionals, the first will be to the third as the square of the first is to the square of the second.*

Let $A : B :: B : C$; then will $A : C :: A^2 : B^2$.

For, from the given proportion, by Prop. III., we have

$$A \times C = B^2.$$

Multiplying each side of this equation by A gives

$$A^2 \times C = A \times B^2.$$

Hence, by Prop. II.,

$$A : C :: A^2 : B^2.$$

Proposition XV. — Theorem.

151. *If four magnitudes are proportionals, their like powers and roots will also be proportional.*

Let A : B : : C : D ; then will

$$A^n : B^n : : C^n : D^n, \quad \text{and} \quad A^{\frac{1}{n}} : B^{\frac{1}{n}} : : C^{\frac{1}{n}} : D^{\frac{1}{n}}.$$

For, from the given proportion, we have

$$\frac{A}{B} = \frac{C}{D}.$$

Raising both members of this equation to the nth power, we have

$$\frac{A^n}{B^n} = \frac{C^n}{D^n},$$

and extracting the nth root of each member, we have

$$\frac{A^{\frac{1}{n}}}{B^{\frac{1}{n}}} = \frac{C^{\frac{1}{n}}}{D^{\frac{1}{n}}}.$$

Hence, by Prop. II., the last two equations give

$$A^n : B^n : : C^n : D^n,$$

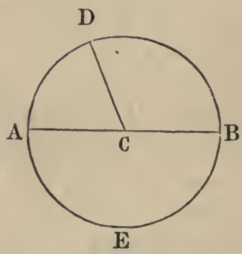
and

$$A^{\frac{1}{n}} : B^{\frac{1}{n}} : : C^{\frac{1}{n}} : D^{\frac{1}{n}}.$$

BOOK III.

THE CIRCLE, AND THE MEASURE OF ANGLES.

DEFINITIONS.



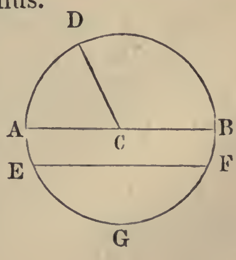
152. A CIRCLE is a plane figure bounded by a curved line, all the points of which are equally distant from a point within called the *centre;* as the figure A D B E.

153. The CIRCUMFERENCE OF PERIPHERY of a circle is its entire bounding line; or it is a curved line, all points of which are equally distant from a point within called the centre.

154. A RADIUS of a circle is any straight line drawn from the centre to the circumference; as the line C A, C D, or C B.

155. A DIAMETER of a circle is any straight line drawn through the centre, and terminating in both directions in the circumference; as the line A B.

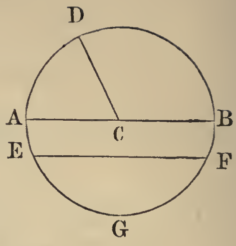
All the radii of a circle are equal; all the diameters are also equal, and each is double the radius.



156. An ARC of a circle is any part of the circumference; as the part A D, A E, or E G F.

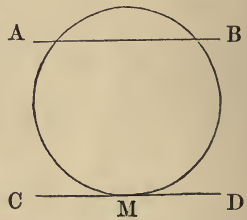
157. The CHORD of an arc is the straight line joining its extremities; thus E F is the chord of the arc E G F.

158. The SEGMENT of a circle is the part of a circle included between an arc and its chord; as the surface included between the arc E G F and the chord E F.



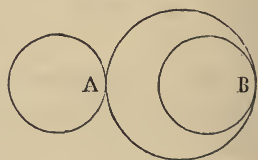
159. The SECTOR of a circle is the part of a circle included between an arc, and the two radii drawn to the extremities of the arc; as the surface included between the arc A D, and the two radii C A, C D.

160. A SECANT to a circle is a straight line which meets the circumference in two points, and lies partly within and partly without the circle; as the line A B.

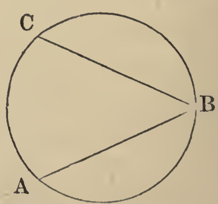


161. A TANGENT to a circle is a straight line which, how far so ever produced, meets the circumference in but one point; as the line C D. The point of meeting is called the POINT OF CONTACT; as the point M.

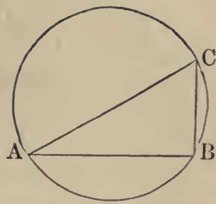
162. Two circumferences TOUCH each other, when they have a point of contact without cutting one another; thus two circumferences touch each other at the point A, and two at the point B.



163. A STRAIGHT LINE is INSCRIBED in a circle when its extremities are in the circumference; as the line A B, or B C.

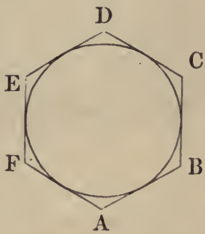


164. An INSCRIBED ANGLE is one which has its vertex in the circumference, and is formed by two chords; as the angle A B C.



165. An INSCRIBED POLYGON is one which has the vertices of all its angles in the circumference of the circle; as the triangle A B C.

166. The circle is then said to be CIRCUMSCRIBED about the polygon.

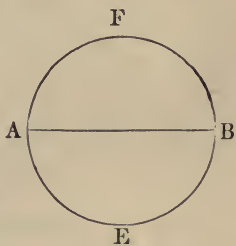


167. A POLYGON is CIRCUMSCRIBED about a circle when all its sides are tangents to the circumference; as the polygon A B C D E F.

168. The circle is then said to be INSCRIBED in the polygon.

PROPOSITION I.—THEOREM.

169. *Every diameter divides the circle and its circumference each into two equal parts.*

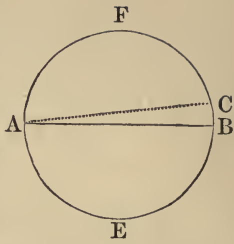


Let A E B F be a circle, and A B a diameter; then the two parts A E B, A F B are equal.

For, if the figure A E B be applied to A F B, their common base A B retaining its position, the curve line A E B must fall exactly on the curve line A F B; otherwise there would be points in the one or the other unequally distant from the centre, which is contrary to the definition of the circle (Art. 152). Hence a diameter divides the circle and its circumference into two equal parts.

170. *Cor.* 1. *Conversely*, a straight line dividing the circle into two equal parts is a diameter.

For, let the line A B divide the circle A E B C F into two equal parts; then, if the centre is not in A B, let A C be drawn through it, which is therefore a diameter, and consequently divides the circle into two equal parts; hence the surface A F C is equal to the surface A F C B, a part to the whole, which is impossible.

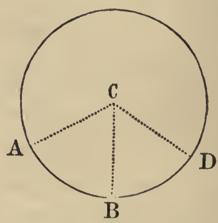


171. *Cor.* 2. The arc of a circle, whose chord is a diameter, is a semi-circumference, and the included segment is a semicircle.

Proposition II. — Theorem.

172. *A straight line cannot meet the circumference of a circle in more than two points.*

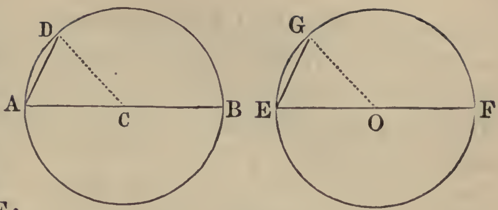
For, if a straight line could meet the circumference A B D, in three points, A, B, D, join each of these points with the centre, C; then, since the straight lines C A, C B, C D are radii, they are equal (Art. 155); hence, three equal straight lines can be drawn from the same point to the same straight line, which is impossible (Prop. XIV. Cor. 2, Bk. I.).



Proposition III. — Theorem.

173. *In the same circle, or in equal circles, equal arcs are subtended by equal chords; and, conversely, equal chords subtend equal arcs.*

Let A D B and E G F be two equal circles, and let the arc A D be equal to E G; then will the chord A D be equal to the chord E G.



For, since the diameters A B, E F are equal, the semicircle A D B may be applied to the semicircle E G F; and the curve line A D B will coincide with the curve line E G F (Prop. I.). But, by hypothesis, the arc A D is equal to the arc E G; hence the point D will fall on G; hence the chord A D is equal to the chord E G (Art. 34, Ax. 11).

Conversely, if the chord A D is equal to the chord E G, the arcs A D, E G will be equal.

For, if the radii C D, O G are drawn, the triangles A C D, E O G, having the three sides of the one equal to the three sides of the other, each to each, are themselves equal (Prop. XVIII. Bk. I.); therefore the angle A C D is equal to the angle E O G (Prop. XVIII. Sch., Bk. I.).

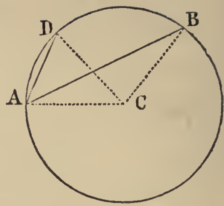
If now the semicircle A D B be applied to its equal E G F, with the radius A C on its equal E O, since the angles A C D, E O G are equal, the radius C D will fall on O G, and the point D on G. Therefore the arcs A D and E G coincide with each other; hence they must be equal (Art. 34, Ax. 14).

Proposition IV. — Theorem.

174. *In the same circle, or in equal circles, a greater arc is subtended by a greater chord; and, conversely, the greater chord subtends the greater arc.*

In the circle of which C is the centre, let the arc A B be greater than the arc A D; then will the chord A B be greater than the chord A D.

Draw the radii C A, C D, and C B. The two sides A C,



C B in the triangle A C B are equal to the two A C, C D in the triangle A C D, and the angle A C B is greater than the angle A C D; therefore the third side A B is greater than the third side A D (Prop. XVI. Bk. I.); hence the chord which subtends the greater arc is the greater.

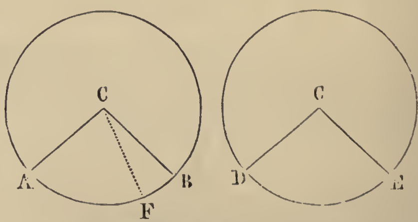
Conversely, if the chord A B be greater than the chord A D, the arc A B will be greater than the arc A D.

For the triangles A C B, A C D have two sides, A C, C B, in the one, equal to two sides, A C, C D, in the other, while the side A B is greater than the side A D; therefore the angle A C B is greater than the angle A C D (Prop. XVII. Bk. I.); hence the arc A B is greater than the arc A D.

175. *Scholium.* The arcs here treated of are each less than the semi-circumference. If they were greater, the contrary would be true; in which case, as the arcs increased, the chords would diminish, and conversely.

Proposition V. — Theorem.

176. *In the same circle, or in equal circles, radii which make equal angles at the centre intercept equal arcs on the circumference; and, conversely, if the intercepted arcs are equal, the angles made by the radii are also equal.*



Let A C B and D C E be equal angles made by radii at the centre of equal circles; then will the intercepted arcs A B and DE be also equal.

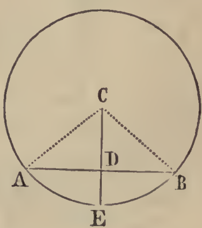
First. Since the angles A C B, D C E are equal, the one may be applied to the other; and since their sides, being radii of equal circles, are equal, the point A will coincide with D, and the point B with E. Therefore the arc A B must also coincide with the arc D E, or there would be points in the one or the other unequally distant from the centre, which is impossible; hence the arc A B is equal to the arc D E.

Second. If the arcs A B and D E are equal, the angles A C B and D C E will be equal.

For, if these angles are not equal, let A C B be the greater, and let A C F be taken equal to D C E. From what has been shown, we shall have the arc A F equal to the arc D E. But, by hypothesis, A B is equal to D E; hence A F must be equal to A B, the part to the whole, which is impossible; hence the angle A C B is equal to the angle D C E.

Proposition VI. — Theorem.

177. *The radius which is perpendicular to a chord bisects the chord, and also the arc subtended by the chord.*



Let the radius C E be perpendicular to the chord A B; then will C E bisect the chord at D, and the arc A B at E.

Draw the radii C A and C B. Then C A and C B, with respect to the perpendicular C E, are equal oblique lines drawn to the chord A B; therefore their extremities are at equal distances from the perpendicular (Prop. XIV. Bk. I.); hence A D and D B are equal.

Again, since the triangle A C B has the sides A C and C B equal, it is isosceles; and the line C E bisects the base A B at right angles; therefore C E bisects also the angle A C B (Prop. VII. Cor. 2, Bk. I.). Since the angles A C D, D C B are equal, the arcs A E, E B are equal

(Prop. V.); hence the radius C E, which is perpendicular to the chord A B, bisects the arc A B subtended by the chord.

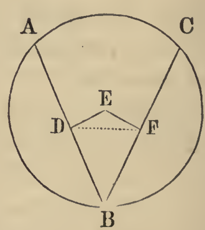
178. *Cor.* 1. Any straight line which joins the centre of the circle and the middle of the chord, or the middle of the arc, must be perpendicular to the chord.

For the perpendicular from the centre C passes through the middle, D, of the chord, and the middle, E, of the arc subtended by the chord. Now, any two of these three points in the straight line C E are sufficient to determine its position.

179. *Cor.* 2. A perpendicular at the middle of a chord passes through the centre of the circle, and through the middle of the arc subtended by the chord, bisecting at the centre the angle which the arc subtends.

Proposition VII. — Theorem.

180. *Through three given points, not in the same straight line, one circumference can be made to pass, and but one.*



Let A, B, and C be any three points not in the same straight line; one circumference can be made to pass through them, and but one.

Join A B and B C; and bisect these straight lines by the perpendiculars D E and F E. Join D F; then, the angles B D E, B F E, being each a right angle, are together equal to two right angles; therefore the angles E D F, E F D are together less than two right angles; hence D E, F E, produced, must meet in some point E (Prop. XXIII. Bk. I.).

Now, since the point E lies in the perpendicular D E, it is equally distant from the two points A and B (Prop. XV. Bk. I.); and since the same point E lies in the perpendicular F E, it is also equally distant from the two points B and C; therefore the three distances, E A, E B, E C, are equal; hence a circumference can be described from the centre E passing through the three points A, B, C.

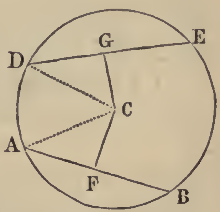
Again, the centre, lying in the perpendicular D E bisecting the chord A B, and at the same time in the perpendicular F E bisecting the chord B C (Prop. VI. Cor. 2), must be at the point of their meeting, E. Therefore, since there can be but one centre, but one circumference can be made to pass through three given points.

181. *Cor.* Two circumferences can intersect in only two points; for, if they have three points in common, they must have the same centre, and must coincide.

Proposition VIII. — Theorem.

182. *Equal chords are equally distant from the centre; and, conversely, chords which are equally distant from the centre are equal.*

Let A B and D E be equal chords, and C the centre of the circle; and draw C F perpendicular to A B, and C G perpendicular to D E; then these perpendiculars, which measure the distance of the chords from the centre, are equal.



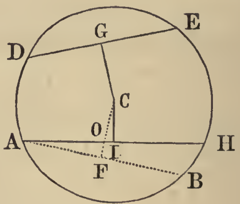
Join C A and C D. Then, in the right-angled triangle C A F, C D G, the hypothenuses C A, C D are equal; and the side A F, the half of A B, is equal to the side D G, the half of D E; therefore the triangles are equal, and C F is equal to C G (Prop. XIX. Bk. I.); hence the two equal chords A B, D E are equally distant from the centre.

Conversely, if the distances C F and C G are equal, the chords A B and D E are equal.

For, in the right-angled triangles A C F, D C G, the hypothenuses C A, C D are equal; and the side C F is equal to the side C G; therefore the triangles are equal, and A F is equal to D G; hence A B, the double of A F, is equal to D E, the double of D G (Art. 34, Ax. 6).

Proposition IX. — Theorem.

183. *Of two unequal chords, the less is the farther from the centre.*

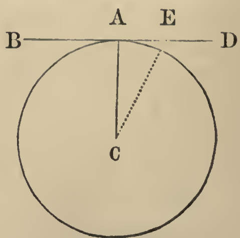


Of the two chords D E and A H, let A H be the greater; then will D E be the farther from the centre C.

Since the chord A H is greater than the chord D E, the arc A H is greater than the arc D E (Prop. IV.). Cut off from the arc A H a part, A B, equal D E; draw C F perpendicular to this chord, C I perpendicular to A H, and C G perpendicular to D E. C F is greater than C O (Art. 34, Ax. 8), and C O than C I (Prop. XIV. Bk. I.); therefore C F is greater than C I. But C F is equal to C G, since the chords A B, D E are equal (Prop. VIII.); therefore, C G is greater than C I; hence, of two unequal chords, the less is the farther from the centre.

Proposition X. — Theorem.

184. *A straight line perpendicular to a radius at its termination in the circumference, is a tangent to the circle.*

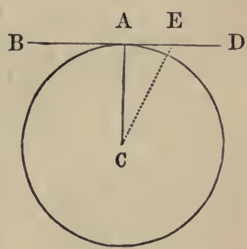


Let the straight line B D be perpendicular to the radius C A at its termination A; then will it be a tangent to the circle.

Draw from the centre C to B D any other straight line, as C E. Then, since C A is perpendicular to B D, it is shorter than the oblique line C E (Prop. XIV. Bk. I.); hence the point E is without the circle. The same may be shown of any other point in the line B D, except the point A; therefore B D meets the circumference at A, and, being produced, does not cut it; hence B D is a tangent (Art. 161).

Proposition XI. — Theorem.

185. *If a line is a tangent to a circumference, the radius drawn to the point of contact with it is perpendicular to the tangent.*



Let B D be a tangent to the circumference, at the point A; then will the radius C A be perpendicular to B D.

For every point in B D, except A, being without the circumference (Prop. X.), any line C E drawn from the centre C to B D, at any point other than A, must terminate at E, without the circumference; therefore the radius C A is the shortest line that can be drawn from the centre to B D; hence C A is perpendicular to the tangent B D (Prop. XIV. Cor. 1, Bk. I.).

186. *Cor.* Only one tangent can be drawn through the same point in a circumference; for two lines cannot both be perpendicular to a radius at the same point.

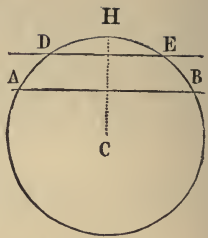
Proposition XII. — Theorem.

187. *Two parallel straight lines intercept equal arcs of the circumference.*

First. When the two parallels are secants, as AB, DE.

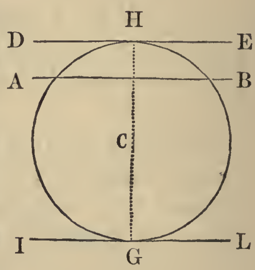
Draw the radius C H perpendicular to A B; and it will also be perpendicular to D E (Prop. XXII. Cor., Bk. I.);

therefore the point H will be at the same time the middle of the arc A H B and of the arc D H E (Prop. VI.); therefore, the arc A H is equal to the arc H B, and the arc D H is equal to the arc H E; hence A H diminished by D H is equal to H B diminished by H E; that is, the intercepted arcs A D, B E are equal.



Second. When of the two parallels, one, as A B, is a secant, and the other, as D E, is a tangent.

Draw the radius C H to the point of contact H. This radius will be perpendicular to the tangent D E (Prop. X.), and also to its parallel A B (Prop. XXII. Cor., Bk. I.). But, since C H is perpendicular to the chord A B, the point H is the middle of the arc A H B; hence the arcs A H, H B, included between the parallels A B, D E, are equal.



Third. When the two parallels are tangents, as D E, I L.

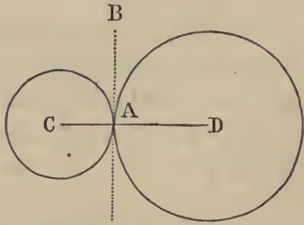
Draw the secant A B parallel to either of the tangents, and it will be parallel to the other (Prop. XXIV. Bk. I.); then, from what has been just shown, the arc A H is equal to the arc H B, and also the arc A G is equal to the arc G B; hence the whole arc H A G is equal to the whole arc H B G.

It is further evident, since the two arcs H A G, H B G are equal, and together make up the whole circumference, that each of them is a semi-circumference.

188. *Cor.* Two parallel tangents meet the circumference at the extremities of the same diameter.

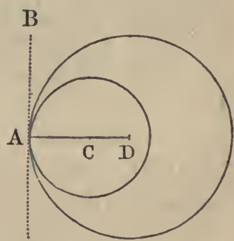
Proposition XIII. — Theorem.

189. *If two circumferences touch each other externally or internally, their centres and the point of contact are in the same straight line.*



Let the two circumferences, whose centres are C and D, touch each other externally in the point A; the points C, D, and A will be all in the same straight line.

Draw from the point of contact A the common tangent A B. Then the radius C A of the one circle, and the radius D A of the other, are each perpendicular to A B (Prop. XI.); but there can be but one straight line drawn through the point A perpendicular to A B (Prop. XIII. Bk. I.); therefore the points C, D, and A are in one perpendicular; hence they are in one and the same straight line.



Also, let the two circumferences touch each other internally in A; then their centres, C and D, and the point of contact, A, will be in the same straight line.

Draw the common tangent A B. Then a straight line perpendicular to A B, at the point A, on being sufficiently produced, must pass through the two centres C and D (Prop. XI.); but from the same point there can be but one perpendicular; therefore the points C, D, and A are in that perpendicular; hence they are in the same straight line.

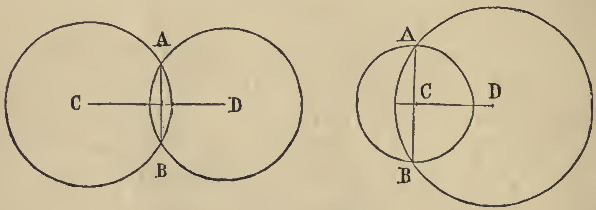
190. *Cor.* 1. When two circumferences touch each other externally, the distance between their centres is equal to the sum of their radii.

191. *Cor.* 2. And when two circumferences touch each other internally, the distance between their centres is equal to the difference of their radii.

Proposition XIV. — Theorem.

192. *If two circumferences cut each other, the straight line passing through their centres will bisect at right angles the chord which joins the points of intersection.*

Let two circumferences cut each other at the points A and B; then the straight line passing through the



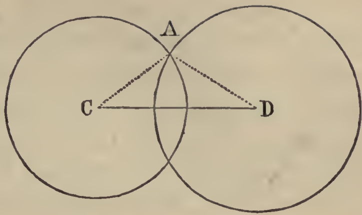
centres C and D will bisect at right angles the chord A B common to the two circles.

For, if a perpendicular be erected at the middle of this chord, it will pass through each of the two centres C and D (Prop. VI. Cor. 1). But no more than one straight line can be drawn through two points; hence the straight line C D, passing through the centres, must bisect at right angles the common chord A B.

193. *Cor.* The straight line joining the points of intersection of two circumferences is perpendicular to the straight line which passes through their centres.

Proposition XV. — Theorem.

194. *If two circumferences cut each other, the distance between their centres will be less than the sum of their radii, and greater than their difference.*

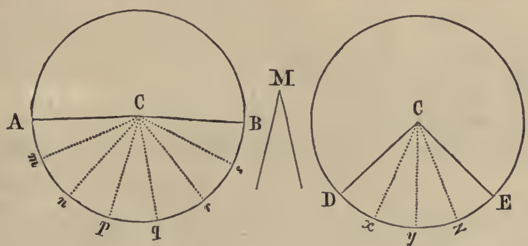


Let two circumferences whose centres are C and D cut each other in the point A, and draw the radii C A and D A. Then, in order that the intersection may take place, the triangle C A D must be possible. And in this triangle the side C D must be less than the sum of A C and A D (Prop. IX. Bk. I.); also C D must be greater than the difference between D A and C A (Prop. IX. Cor., Bk. I.).

Proposition XVI. — Theorem.

195. *In the same circle, or in equal circles, if two angles at the centre are to each other as two whole numbers, the intercepted arcs will be to each other as the same numbers.*

Let us suppose, for example, that the angles A C B, D C E, at the centre of equal circles, are to each other as 7 to 4; or, which amounts to the same thing, that the angle M, which will serve as a common measure, is con-



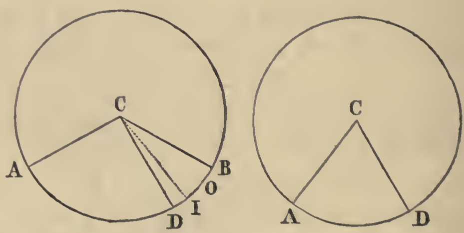
tained seven times in the angle A C B, and four times in the angle D C E. The seven partial angles A C m, m C n, n C p, &c. into which A C B is divided, being each equal to any of the four partial angles into which D C E is divided, each of the partial arcs A m, $m\,n$, $n\,p$, &c. will be also equal to each of the partial arcs D x, $x\,y$, &c. (Prop. V.); therefore the whole arc A B will be to the whole arc D E as 7 to 4. But the same reasoning would apply, if in place of 7 and 4 any numbers whatever were employed; hence, if the ratio of the angles A C B, D C E can be expressed in whole numbers, the arcs A B, D E will be to each other as the angles A C B, D C E.

196. *Cor. Conversely*, if the arcs A B, D E are to each other as two whole numbers, the angles A C B, D C E will be to each other as the same whole numbers, and we shall have A C B : D C E : : A B : D E. For, the partial arcs A m, $m\,n$, &c. and D x, $x\,y$, &c. being equal, the partial angles A C m, m C n, &c. and D C x, x C y, &c. will also be equal.

Proposition XVII. — Theorem.

197. *In the same circle, or in equal circles, any two angles at the centre are to each other as the arcs intercepted between their sides.*

Let A C B be the greater, and A C D the less angle; then will the angle A C B be to the angle A C D as the arc A B is to the arc A D.



Conceive the less angle to be placed on the greater; then, if the proposition be not true, the angle A C B will be to the angle A C D as the arc A B is to an arc greater or less than A D. Suppose this arc to be greater, and let it be represented by A O; we shall have the angle A C B : angle A C D : : arc A B : arc A O. Conceive, now, the arc A B to be divided into equal parts, each of which is less than D O; there will be at least one point of division between D and O; let I be that point; and join C I. The arcs A B, A I will be to each other as two whole numbers, and, by the preceding proposition, we shall have the angle A C B : angle A C I : : arc A B : arc A I. Comparing these two proportions with each other, and observing that the antecedents are the same, we infer that the consequents are proportional (Prop. X. Cor. 2, Bk. II.); hence the angle A C D : angle A C I : : arc A O : arc A I. But the arc A O is greater than the arc A I; therefore, if this proportion is true, the angle A C D must be greater than the angle A C I. But it is less; hence the angle A C B cannot be to the angle A C D as the arc A B is to an arc greater than A D.

By a process of reasoning entirely similar, it may be shown that the fourth term of the proportion cannot be less than A D; therefore it must be A D; hence we have,

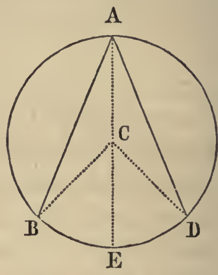
Angle A C B : angle A C D : : arc A B : arc A D.

198. *Scholium* 1. Since the angle at the centre of a circle, and the arc intercepted by its sides, have such a connection, that, if the one be increased or diminished in any ratio, the other will be increased or diminished in the same ratio, we are authorized to take the one of these magnitudes as the measure of the other. Henceforth we shall assume the arc A B as the measure of the angle A C B. It is to be observed, in the comparison of angles with each other, that the arcs which serve to measure them must be described with equal radii.

199. *Scholium* 2. Sectors taken in the same circle, or in equal circles, are to each other as their arcs; for sectors are equal when their angles are so, and therefore are in all respects proportional to their angles.

Proposition XVIII. — Theorem.

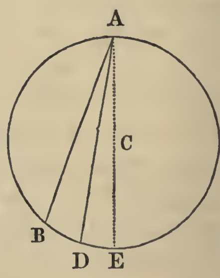
200. *An inscribed angle is measured by half the arc included between its sides.*



Let B A D be an inscribed angle, whose sides include the arc B D; then the angle B A D is measured by half of the arc B D.

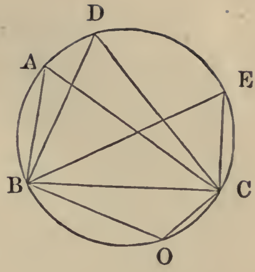
First. Suppose the centre of the circle C to lie within the angle B A D. Draw the diameter A E, and the radii C B, C D.

The angle B C E, being exterior to the triangle A B C, is equal to the sum of the two interior angles C A B, A B C (Prop. XXVII. Bk. I.). But the triangle B A C being isosceles, the angle C A B is equal to A B C; hence, the angle B C E is double B A C. Since B C E lies at the centre, it is measured by the arc B E (Prop. XVII. Sch. 1); hence B A C will be measured by half of B E. For a like reason, the angle C A D will be measured by the half of E D; hence B A C and C A D together, or B A D, will be measured by the half of B E and E D, or half B D.

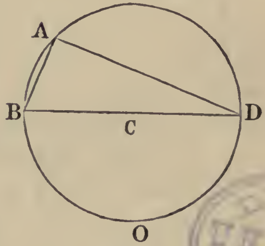


Second. Suppose that the centre C lies without the angle B A D. Then, drawing the diameter A E, the angle B A E will be measured by the half of B E; and the angle D A E is measured by the half of D E; hence, their difference, B A D, will be measured by the half of B E *minus* the half of E D, or by the half of B D.

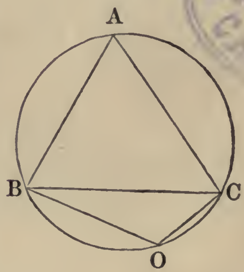
Hence every inscribed angle is measured by the half of the arc included between its sides.



201. *Cor.* 1. All the angles, B A C, B D C, inscribed in the same segment, are equal; because they are all measured by the half of the same arc, B O C.

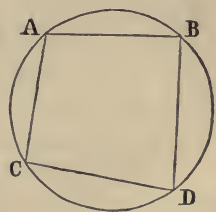


202. *Cor.* 2. Every angle, B A D, inscribed in a semicircle, is a right angle; because it is measured by half the semi-circumference, BOD; that is, by the fourth part of the whole circumference.



203. *Cor.* 3. Every angle, B A C, inscribed in a segment greater than a semicircle, is an acute angle; for it is measured by the half of the arc B O C, less than a semi-circumference.

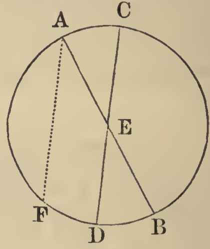
And every angle, B O C, inscribed in a segment less than a semicircle, is an obtuse angle; for it is measured by half of the arc B A C, greater than a semi-circumference.



204. *Cor.* 4. The opposite angles, A and D, of an inscribed quadrilateral, A B D C, are together equal to two right angles; for the angle B A C is measured by half the arc B D C, and the angle B D C is measured by half the arc B A C; hence the two angles B A C, B D C, taken together, are measured by half the circumference; hence their sum is equal to two right angles.

Proposition XIX. — Theorem.

205. *The angle formed by the intersection of two chords is measured by half the sum of the two intercepted arcs.*

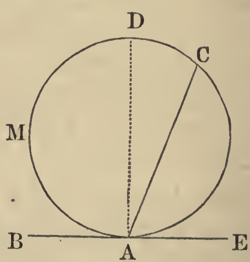


Let the two chords A B, C D intersect each other at the point E; then will the angle D E B, or its equal, A E C, be measured by half the sum of the two arcs D B and A C.

Draw A F parallel to D C; then will the arc F D be equal to the arc A C (Prop. XII.), and the angle F A B equal to the angle D E B (Prop. XXII. Bk. I.). But the angle F A B is measured by half the arc F D B (Prop. XVIII.); that is, by half the arc D B, plus half the arc F D. Hence, since F D is equal to A C, the angle D E B, or its equal angle A E C, is measured by half the sum of the intercepted arcs D B and A C

Proposition XX. — Theorem.

206. *The angle formed by a tangent and a chord is measured by half the intercepted arc.*



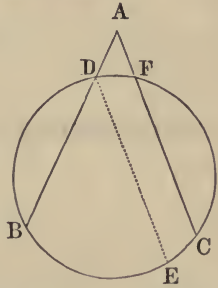
Let the tangent B E form, with the chord A C, the angle B A C; then B A C is measured by half the arc A M C.

From A, the point of contact, draw the diameter A D. The angle B A D is a right angle (Prop. X.), and is measured by half of the semi-circumference A M D (Prop. XVIII.); and the angle D A C is measured by half the arc D C; hence the sum of the angles B A D, D A C, or B A C, is measured by the half of A M D, plus the half of D C; or by half the whole arc A M D C.

In like manner, it may be shown that the angle C A E is measured by half the intercepted arc A C.

Proposition XXI. — Theorem.

207. *The angle formed by two secants is measured by half the difference of the two intercepted arcs.*

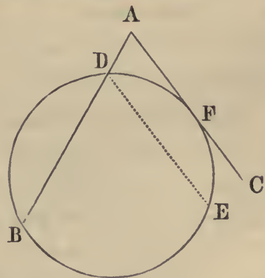


Let A B, A C be two secants forming the angle B A C; then will that angle be measured by half the difference of the two arcs B E C and D F.

Draw D E parallel to A C; then will the arc E C be equal to the arc D F (Prop. XII.); and the angle B D E be equal to the angle B A C (Prop. XXII. Bk. I.). But the angle B D E is measured by half the arc B E (Prop. XVIII.); hence the equal angle B A C is also measured by half the arc B E; that is, by half the difference of the arcs B E C and E C, or, since E C is equal to D F, by half the difference of the intercepted arcs B E C and D F.

Proposition XXII. — Theorem.

208. *The angle formed by a secant and a tangent is measured by half the difference of the two intercepted arcs.*



Let the secant A B form, with the tangent A C, the angle B A C; then B A C is measured by half the difference of the two arcs B E F and F D.

Draw D E parallel to A C; then will the arc E F be equal to the arc D F (Prop. XII.), and the angle B D E be equal to the angle B A C. But the angle B D E is measured by half of the arc B E (Prop. XVIII.); hence the equal angle B A C is also measured by half the arc B E; that is, by half the difference of the arcs B E F and E F, or, since E F is equal to D F, by half the difference of the intercepted arcs B E F and D F.

BOOK IV.

PROPORTIONS, AREAS, AND SIMILARITY OF FIGURES.

DEFINITIONS.

209. The AREA of a figure is its quantity of surface, and is expressed by the number of times which the surface contains some other area assumed as a unit of measure.

Figures have *equal* areas, when they contain the same unit of measure an equal number of times.

210. SIMILAR FIGURES are such as have the angles of the one equal to those of the other, each to each, and the sides containing the equal angles proportional.

211. EQUIVALENT FIGURES are such as have equal areas.

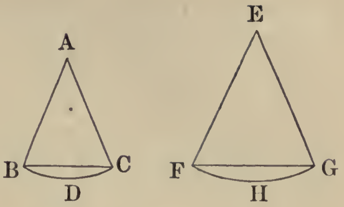
Figures may be equivalent which are not similar. Thus a circle may be equivalent to a square, and a triangle to a rectangle.

212. EQUAL FIGURES are such as, when applied the one to the other, coincide throughout (Art. 34, Ax. 14). Thus circles having equal radii are equal; and triangles having the three sides of the one equal to the three sides of the other, each to each, are also equal.

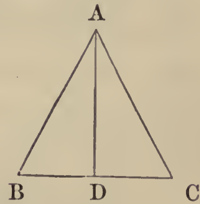
Equal figures are always similar; but similar figures may be very unequal.

213. In different circles, SIMILAR ARCS, SEGMENTS, or SECTORS are such as correspond to equal angles at the centres of the circles.

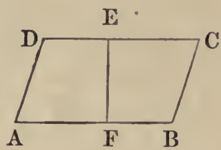
Thus, if the angles A and E are equal, the arc B C will be similar to the arc FG; the segment BDC to the segment F H G, and the sector A B C to the sector E F G.



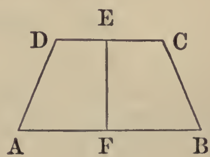
214. The ALTITUDE OF A TRIANGLE is the perpendicular, which measures the distance of any one of its vertices from the opposite side taken as a base; as the perpendicular A D let fall on the base B C in the triangle A B C.



215. The ALTITUDE OF A PARALLELOGRAM is the perpendicular which measures the distance between its opposite sides taken as bases; as the perpendicular E F measuring the distance between the opposite sides, A B, D C, of the parallelogram A B C D.



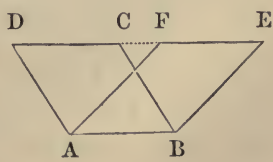
216. The ALTITUDE OF A TRAPEZOID is the perpendicular distance between its parallel sides; as the distance measured by the perpendicular E F between the parallel sides, A B, D C, of the trapezoid A B C D.



PROPOSITION I. — THEOREM.

217. *Parallelograms which have equal bases and equal altitudes are equivalent.*

Let A B C D, A B E F be two parallelograms having equal bases and equal altitudes; then these parallelograms are equivalent.



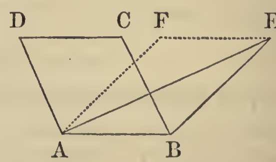
Let the base of the one parallelogram be placed on that of the other, so that A B shall be the common base. Now, since the two parallelograms are of the same altitude, their upper bases, D C, F E, will be in the same straight line, D C E F, parallel to A B. From the nature of parallelograms D C is equal to A B, and F E is equal to A B (Prop. XXXI. Bk. I.); therefore D C is equal to F E (Art. 34, Ax. 1); hence, if D C and F E be taken away from the same line, D E, the remainders C E and D F will be equal (Art. 34, Ax. 3). But A D is equal to B C and A F to B E (Prop. XXXI. Bk. I.); therefore the triangles D A F, C B E, are mutually equilateral, and consequently equal (Prop. XVIII. Bk. I.).

If from the quadrilateral A B E D, we take away the triangle A D F, there will remain the parallelogram A B E F; and if from the same quadrilateral A B E D, we take away the triangle C B E, there will remain the parallelogram A B C D. Hence the parallelograms A B C D, A B E F, which have equal bases and equal altitude, are equivalent.

218. *Cor.* Any parallelogram is equivalent to a rectangle having the same base and altitude.

Proposition II. — Theorem.

219. *If a triangle and a parallelogram have the same base and altitude, the triangle is equivalent to half the parallelogram.*



Let A B E be a triangle, and A B C D a parallelogram having the same base, A B, and the same altitude; then will the triangle be equivalent to half the parallelogram.

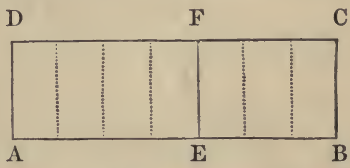
Draw A F, F E so as to form the parallelogram A B E F. Then the parallelograms A B C D, A B E F, having the same base and altitude, are equivalent (Prop. I.). But the triangle A B E is half the parallelogram A B E F (Prop. XXXI. Cor. 1, Bk. I.); hence the triangle A B E is equivalent to half the parallelogram A B C D (Art. 34, Ax. 7).

220. *Cor.* 1. Any triangle is equivalent to half a rectangle having the same base and altitude, or to a rectangle either having the same base and half of the same altitude, or having the same altitude and half of the same base.

221. *Cor.* 2. All triangles which have equal bases and altitudes are equivalent.

Proposition III.—Theorem.

222. *Two rectangles having equal altitudes are to each other as their bases.*



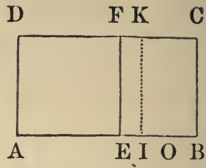
Let A B C D, A E F D be two rectangles having the common altitude A D; they are to each other as their bases A B, A E.

First. Suppose that the bases A B, A E are commensurable, and are to each other, for example, as the numbers 7 and 4. If A B is divided into seven equal parts, A E will contain four of those parts. At each point of division draw lines perpendicular to the base; seven rectangles will thus be formed, all equal to each other, since they have equal bases and the same altitude (Prop. I.). The rectangle A B C D will contain seven partial rectangles, while A E F D will contain four; hence the rectangle A B C D is to A E F D as 7 is to 4, or as A B is to A E. The same reasoning may be applied, whatever be the numbers expressing the ratio of the bases; hence, whatever be that ratio, when its terms are commensurable, we shall have

$$ABCD : AEFD :: AB : AE.$$

Second. Suppose that the bases A B, A E are incommensurable; we shall still have

A B C D : A E F D : : A B : A E.



For, if this proportion be not true, the first three terms remaining the same, the fourth term must be either greater or less than A E. Suppose it to be greater, and that we have

A B C D : A E F D : : A B : A O.

Conceive A B divided into equal parts, each of which is less than E O. There will be at least one point of division, I, between E and O. Through this point, I, draw the perpendicular I K; then the bases A B, A I will be commensurable, and we shall have

A B C D : A I K D : : A B : A I.

But, by hypothesis, we have

A B C D : A E F D : : A B : A O.

In these two proportions the antecedents are equal; hence the consequents are proportional (Prop. X. Cor. 2, Bk. II.), and we have

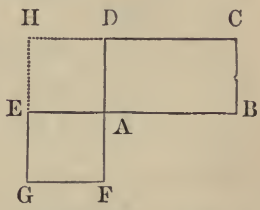
A I K D : A E F D : : A I : A O.

But A O is greater than A I; therefore, if this proportion is correct, the rectangle A E F D must be greater than the rectangle A I K D (Art. 125); on the contrary, however, it is less (Art. 34, Ax. 8); therefore the proportion is impossible. Hence, A B C D cannot be to A E F D as A B is to a line greater than A E.

In the same manner, it may be shown that the fourth term of the proportion cannot be less than A E; therefore it must be equal to A E. Hence, any two rectangles A B C D, A E F D, having equal altitudes, are to each other as their bases A B, A E.

Proposition IV. — Theorem.

223. *Any two rectangles are to eacn other as the products of their bases multiplied by their altitudes.*



Let A B C D, A E G F be two rectangles; then will A B C D be to A E G F as A B multiplied by A D is to A E multiplied by A F. Having placed the two rectangles so that the angles at A are vertical, produce the sides G E, C D till they meet in H. The two rectangles A B C D, A E H D, having the same altitude, A D, are to each other as their bases, A B, A E. In like manner the two rectangles A E H D, A E G F, having the same altitude, A E, are to each other as their bases, AD, A F. Hence we have the two proportions,

$$ABCD : AEHD :: AB : AE,$$
$$AEHD : AEGF :: AD : AF.$$

Multiplying the corresponding terms of these proportions together (Prop. XIII. Bk. II.), and omitting the factor A E H D, which is common to both the antecedent and the consequent (Prop. IX. Bk. II.), we shall have

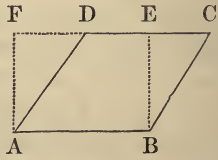
$$ABCD : AEGF :: AB \times AD : AE \times AF.$$

224. *Scholium.* Hence, we may assume as the measure of a rectangle, the product of its base by its altitude, provided we understand by this product the product of two numbers, one of which represents the number of linear units contained in the base, the other the number of linear units contained in the altitude.

The product of two lines is often used to designate their *rectangle*; but the term *square* is used to designate the product of a number multiplied by itself.

Proposition V. — Theorem.

225. *The area of any parallelogram is equal to the product of its base by its altitude.*



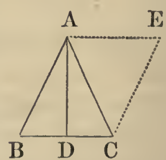
Let A B C D be any parallelogram, A B its base, and B E its altitude; then will its area be equal to the product of A B by B E.

Draw B E and A F perpendicular to A B, and produce C D to F. Then the parallelogram A B C D is equivalent to the rectangle A B E F, which has the same base, A B, and the same altitude, B E (Prop. I. Cor.). But the rectangle A B E F is measured by A B $\times$ B E (Prop. IV. Sch.); therefore A B $\times$ B E is equal to the area of the parallelogram A B C D.

226. *Cor.* Parallelograms having equal bases are to each other as their altitudes, and parallelograms having equal altitudes are to each other as their bases; and, in general, parallelograms are to each other as the products of their bases by their altitudes.

Proposition VI. — Theorem.

227. *The area of any triangle is equal to the product of its base by half its altitude*



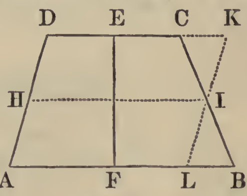
Let A B C be any triangle, B C its base, and A D its altitude; then its area will be equal to the product of B C by half of A D.

Draw A E and C E so as to form the parallelogram A B C E; then the triangle A B C is half the parallelogram A B C E, which has the same base B C, and the same altitude A D (Prop. II.); but the area of the parallelogram is equal to B C $\times$ A D (Prop. V.); hence the area of the triangle must be $\frac{1}{2}$ B C $\times$ A D, or B C $\times$ $\frac{1}{2}$ A D.

228. *Cor.* Triangles of equal altitudes are to each other as their bases, and triangles of equal bases are to each other as their altitudes; and, in general, triangles are to each other as the products of their bases and altitudes.

PROPOSITION VII. — THEOREM.

229. *The area of any trapezoid is equal to the product of its altitude by half the sum of its parallel sides.*



Let A B C D be a trapezoid, E F its altitude, and A B, C D its parallel sides; then its area will be equal to the product of E F by half the sum of A B and C D.

Through I, the middle point of the side B C, draw K L parallel to A D; and produce D C till it meet K L. In the triangles I B L, I C K, we have the sides I B, I C equal, by construction; the vertical angles L I B, C I K are equal (Prop. IV. Bk. I.); and, since C K and B L are parallel, the alternate angles I B L, I C K are also equal (Prop. XXII. Bk. I.); therefore the triangles I B L, I C K are equal (Prop. VI. Bk. I.); hence the trapezoid A B C D is equivalent to the parallelogram A D K L, and is measured by the product of E F by A L (Prop. V.).

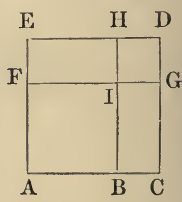
But we have A L equal D K; and since the triangles I B L and K C I are equal, the sides B L and C K are equal; therefore the sum of A B and C D is equal to the sum of A L and D K, or twice A L. Hence A L is half the sum of the bases A B, C D; hence the area of the trapezoid A B, C D is equal to the product of the altitude E F by half the sum of the parallel sides A B, C D.

Cor. If through I, the middle point of B C, the line I H be drawn parallel to the base A B, the point H will also be the middle point of A D. For, since the figure A H I L is a parallelogram, as is likewise D H I K, their opposite sides being parallel, we have A H equal to I L, and D H equal to I K. But since the triangles B I L, C I K are equal, we have I L equal to I K; hence A H is equal to D H.

Now, the line H I is equal to A L, which has been shown to be equal to half the sum of A B and C D; therefore the area of the trapezoid is equal to the product of E F by H I. Hence, the area of a trapezoid is equal to the product of its altitude by the line connecting the middle points of the sides which are not parallel.

Proposition VIII. — Theorem.

230. *If a straight line be divided into two parts, the square described on the whole line is equivalent to the sum of the squares described on the parts, together with twice the rectangle contained by the parts.*



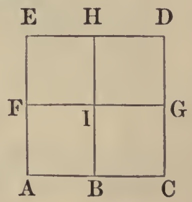
Let A C be a straight line, divided into two parts, A B, B C, at the point B; then the square described on A C is equivalent to the sum of the squares described on the parts A B, B C, together with twice the rectangle contained by A B, B C; that is,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\,AB \times BC.$$

On A C describe the square A C D E; take A F equal to A B; draw F G parallel to A C, and B H parallel to A E.

The square A C D E is divided into four parts; the first, A B I F, is the square described on A B, since A F was taken equal to A B. The second, I G D H, is the square described upon B C; for, since A C is equal to A E, and A B is equal to A F, A C minus A B is equal to A E minus A F, which gives B C equal to E F. But I G is equal to B C, and D G to E F, since the lines are parallels; therefore I G D H is equal to the square described on B C.

These two parts being taken from the whole square, there remain two rectangles B C G I, E F I H, each of which is measured by $AB \times BC$; hence the square on the whole line A C is equivalent to the squares on the parts A B, BC, together with twice the rectangle of the parts.



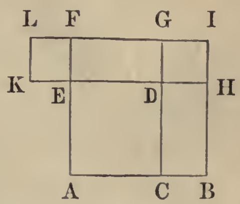
231. *Cor.* The square described on the whole line A C is equivalent to four times the square described on the half A B.

232. *Scholium.* This proposition is equivalent to the algebraical formula,

$$(a + b)^2 = a^2 + 2\,a\,b + b^2.$$

PROPOSITION IX. — THEOREM.

233. *The square described on the difference of two straight lines is equivalent to the sum of the squares described on the two lines, diminished by twice the rectangle contained by the lines.*



Let A B and B C be two lines, and A C their difference; then will the square described on A C be equivalent to the sum of the squares described on A B, B C, diminished by twice the rectangle A B, B C; that is,

$$(AB - BC)^2 \text{ or } \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2\,AB \times BC.$$

On A B describe the square A B I F; take A E equal to A C; draw C G parallel to B I, H K parallel to A B, and complete the square E F L K.

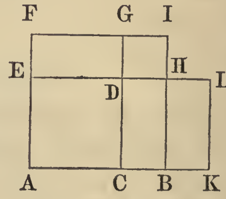
Since A F is equal to A B, and A E to A C, E F is equal to B C, and L F to G I; therefore L G is equal to F I; hence the two rectangles C B I G, G L K D are each measured by $AB \times BC$. Take these rectangles from the whole figure ABILKE, which is equivalent to $AB^2 + BC^2$, and there will evidently remain the square ACDE; hence the square on AC is equivalent to the sum of the squares on AB, BC, diminished by twice the rectangle contained by AB, BC.

234. *Scholium.* This proposition is equivalent to the algebraical formula,

$$(a - b)^2 = a^2 - 2ab + b^2.$$

Proposition X. — Theorem.

235. *The rectangle contained by the sum and difference of two straight lines is equivalent to the difference of the squares of these lines.*



Let AB, BC be two lines; then will the rectangle contained by the sum and difference of AB, BC, be equivalent to the difference of the squares of AB, BC; that is,

$$(AB + BC) \times (AB - BC) = \overline{AB}^2 - \overline{BC}^2.$$

On AB describe the square ABIF, and on AC the square ACDE; produce CD to G; and produce AB until BK is equal to BC, and complete the rectangle AKLE.

The base AK of the rectangle is the sum of the two lines AB, BC; and its altitude AE is the difference of the same lines; therefore the rectangle AKLE is that contained by the sum and the difference of the lines AB, BC. But this rectangle is composed of the two parts ABHE and BHLK; and the part BHLK is equal to the rectangle EDGF, since BH is equal to DE, and BK to EF. Hence the rectangle AKLE is equivalent to ABHE plus EDGF, which is equivalent to the difference between the square A B I F described on A B, and D H I G described on B C; hence

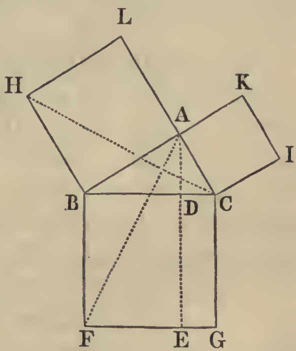
$$(AB + BC) \times (AB - BC) = \overline{AB}^2 - \overline{BC}^2.$$

236. *Scholium.* This proposition is equivalent to the algebraical formula,

$$(a + b) \times (a - b) = a^2 - b^2.$$

Proposition XI. — Theorem.

237. *The square described on the hypothenuse of a right-angled triangle is equivalent to the sum of the squares described on the other two sides.*

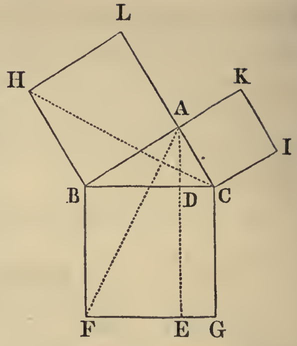


Let A B C be a right-angled triangle, having the right angle at A; then the square described on the hypothenuse B C will be equivalent to the sum of the squares on the sides B A, A C.

On B C describe the square B C G F, and on A B, A C the squares A B H L, A C I K; and through A draw A E parallel to B F or C G, and join A F, H C.

The angle A B F is composed of the angle A B C, together with the right angle C B F; the angle C B H is composed of the same angle A B C together with the right angle A B H; therefore the angle A B F is equal to the angle H B C. But we have A B equal to B H, being sides of the same square; and B F equal to B C, for the same reason; therefore the triangles A B F, H B C have two sides and the included angle of the one equal to two sides and the included angle of the other; hence they are themselves equal (Prop. V. Bk. I.).

But the triangle A B F is equivalent to half the rectangle B D E F, since they have the same base B F, and the



same altitude B D (Prop. II. Cor. 1). The triangle H B C is, in like manner, equivalent to half the square A B H L; for the angles B A C, B A L being both right, A C and A L form one and the same straight line parallel to H B (Prop. II. Bk. I.); and consequently the triangle and the square have the same altitude A B (Prop. XXV. Bk. I.); and they also have the same base B H; hence the triangle is equivalent to half the square (Prop. II.).

The triangle A B F has already been proved equal to the triangle H B C; hence the rectangle B D E F, which is double the triangle A B F, must be equivalent to the square A B H L, which is double the triangle H B C. In the same manner it may be proved that the rectangle C D E G is equivalent to the square A C I K. But the two rectangles B D E F, C D E G, taken together, compose the square B C G F; therefore the square B C G F, described on the hypothenuse, is equivalent to the sum of the squares A B H L, A C I K, described on the two other sides; that is,

$$\overline{BC}^2 \text{ is equivalent to } \overline{AB}^2 + \overline{AC}^2.$$

238. *Cor.* 1. *The square of either of the sides which form the right angle of a right-angled triangle is equivalent to the square of the hypothenuse diminished by the square of the other side;* thus,

$$\overline{AB}^2 \text{ is equivalent to } \overline{BC}^2 - \overline{AC}^2.$$

239. *Cor.* 2. *The square of the hypothenuse is to the square of either of the other sides, as the hypothenuse is to the part of the hypothenuse cut off, adjacent to that side,*

by the perpendicular let fall from the vertex of the right angle. For, on account of the common altitude B F, the square B C G F is to the rectangle B D E F as the base B C is to the base B D (Prop. III.); now, the square A B H L has been proved to be equivalent to the rectangle B D E F; therefore we have,

$$\overline{BC}^2 : \overline{AB}^2 : : BC : BD.$$

In like manner, we have,

$$\overline{BC}^2 : \overline{AC}^2 : : BC : CD.$$

240. *Cor.* 3. *If a perpendicular be drawn from the vertex of the right angle to the hypothenuse, the squares of the sides about the right angle will be to each other as the adjacent segments of the hypothenuse.* For the rectangles B D E F, D C G E, having the same altitude, are to each other as their bases, B D, C D (Prop. III.). But these rectangles are equivalent to the squares A B H L, A C I K; therefore we have,

$$\overline{AB}^2 : \overline{AC}^2 : : BD : DC.$$

241. *Cor.* 4. *The square described on the diagonal of a square is equivalent to double the square described on a side.* For let A B C D be a square, and A C its diagonal; the triangle A B C being right-angled and isosceles, we have,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 = 2\,\overline{AB}^2 = 2 \times ABCD.$$

242. *Cor.* 5. Since $\overline{AC}^2$ is equal to $2\,\overline{AB}^2$, we have

$$\overline{AC}^2 : \overline{AB}^2 : : 2 : 1;$$

and, extracting the square root, we have

$$AC : AB : : \sqrt{2} : 1;$$

hence, *the diagonal of a square is incommensurable with a side.*

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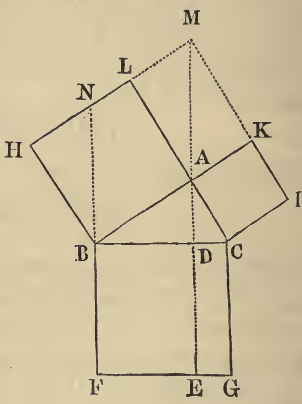
243. NOTE.—The proposition may also be demonstrated as follows:—



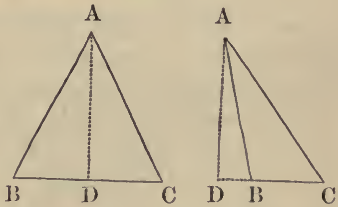
Let A B C be a right-angled triangle, having the right angle at A; then the square described on the hypothenuse B C will be equivalent to the sum of the squares on the sides B A, A C.

On B C describe the square B C G F, and on A B, A C the squares A B H L, A C I K; produce F B to N, H L and I K to M; and through A draw E D A parallel to F B N, and meeting the prolongation of H L in M.

Then, since the angles H B A, N B C are both right angles, if the common angle N B A be taken from each of these equals, there will remain the equal angles H B N, A B C; and, consequently, since the triangles H B N, A B C are both right-angled, and have also the sides B H, B A equal, their hypothenuses B N, B C are equal (Prop. VI. Cor., Bk. I.). But B C is equal to B F; therefore B N is equal to B F; hence the parallelograms B A M N, B D E F, of which the common altitude is B D, have equal bases; therefore the two parallelograms are equivalent (Prop. I.). But the parallelogram B A M N is equivalent to the square A B H L, since they have the same base B A, and the same altitude A L; hence the parallelogram B D E F is also equivalent to the square A B H L. In like manner it may be shown that the rectangle D C G E is equivalent to the square A C I K; hence the two rectangles together, that is, the square B C G F, are equivalent to the sum of the squares A B H L, A C I K.

Proposition XII. — Theorem.

244. *In any triangle, the square of the side opposite an acute angle is less than the sum of the squares of the base and the other side, by twice the rectangle contained by the base and the distance from the vertex of the acute angle to the perpendicular let fall from the vertex of the opposite angle on the base, or on the base produced.*



Let A B C be any triangle, C one of its acute angles, and A D the perpendicular let fall on the base B C, or on B C produced; then, in either case, will the square of A B be less than the sum of the squares of A C, B C, by twice the rectangle B C $\times$ C D.

First. When the perpendicular falls within the triangle A B C, we have B D = B C — C D; and consequently, $\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 - 2\ BC \times CD$ (Prop. IX.). By adding $\overline{AD}^2$ to each of these equals, we have

$$\overline{BD}^2 + \overline{AD}^2 = \overline{BC}^2 + \overline{CD}^2 + \overline{AD}^2 - 2\,BC \times CD.$$

But the two right-angled triangles A D B, A D C give

$$\overline{AB}^2 = \overline{BD}^2 + \overline{AD}^2, \text{ and } \overline{AC}^2 = \overline{CD}^2 + \overline{AD}^2$$

(Prop. XI.); therefore,

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times CD.$$

Secondly. When the perpendicular A D falls without the triangle A B C, we have B D = C D — B C; and consequently, $\overline{BD}^2 = \overline{CD}^2 + \overline{BC}^2 - 2\,CD \times BC$. By adding $\overline{AD}^2$ to each of these equals, we find, as before,

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times CD.$$

PROPOSITION XIII. — THEOREM.

245. *In any obtuse-angled triangle, the square of the side opposite the obtuse angle is equivalent to the sum of the squares of the two other sides plus twice the rectangle contained by the one of those sides into the distance from the vertex of the obtuse angle to the perpendicular let fall from the vertex of the opposite angle to that side produced.*

Let A C B be an obtuse-angled triangle, having the obtuse angle at C, and let A D be perpendicular to the base B C produced; then the square of A B is greater than the sum of the squares of B C, A C, by twice the rectangle $BC \times CD$. Since B D is the sum of the lines $BC + CD$, we have

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 + 2\,BC \times CD$$

(Prop. VIII.). By adding $\overline{AD}^2$ to each of these equals, we have

$$\overline{BD}^2 + \overline{AD}^2 = \overline{BC}^2 + \overline{CD}^2 + \overline{AD}^2 + 2\,BC \times CD.$$

But the two right-angled triangles A D B, A D C give

$$\overline{AB}^2 = \overline{BD}^2 + \overline{AD}^2, \quad \text{and} \quad \overline{AC}^2 = \overline{CD}^2 + \overline{AD}^2$$

(Prop. XI.); therefore,

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\,BC \times CD.$$

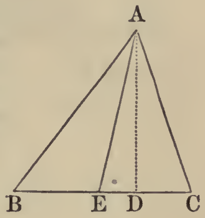
246. *Scholium.* The right-angled triangle is the only one in which the sum of the squares of two sides is equivalent to the square of the third; for if the angle contained by the two sides is acute, the sum of their squares will be greater than the square of the opposite side; if obtuse, it will be less.

PROPOSITION XIV. — THEOREM.

247. *In any triangle, if a straight line be drawn from the vertex to the middle point of the base, the sum of the

squares of the other two sides is equivalent to twice the square of the bisecting line, together with twice the square of half the base.

In any triangle A B C, draw the line A E from the vertex A to the middle of the base B C; then the sum of the squares of the two sides, A B, A C, is equivalent to twice the square of A E together with twice the square of B E.



On B C let fall the perpendicular A D; then, in the triangle A B E,

$$\overline{AB}^2 = \overline{AE}^2 + \overline{EB}^2 + 2\, EB \times ED$$

(Prop. XIII.), and, in triangle A E C,

$$\overline{AC}^2 = \overline{AE}^2 + \overline{EC}^2 - 2\, EC \times ED$$

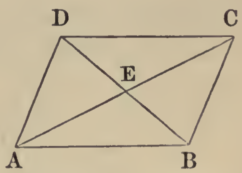
(Prop. XII.). Hence, by adding the corresponding sides together, observing that since E B and E C are equal, $\overline{EB}^2$ is equal to $\overline{EC}^2$, and $EB \times ED$ to $EC \times ED$, we have

$$\overline{AB}^2 + \overline{AC}^2 = 2\, \overline{AE}^2 + 2\, \overline{EB}^2.$$

Proposition XV. — Theorem.

248. *In any parallelogram the sum of the squares of the four sides is equivalent to the sum of the squares of the two diagonals.*

Let A B C D be any parallelogram, the diagonals of which are A C, B D; then the sum of the squares of A B, B C, C D, D A is equivalent to the sum of the squares of A C, B D.



For the diagonals A C, B D bisect each other (Prop. XXXIV. Bk. I.); hence, in the triangle A B C, $\overline{AB}^2 + \overline{BC}^2 = 2\, \overline{AE}^2 + 2\, \overline{BE}^2$ (Prop. XIV.); also, in the triangle A D C,

$$\overline{AD}^2 + \overline{DC}^2 = 2\, AE^2 + 2\, DE^2.$$

Hence, by adding the corresponding sides together, and observing that, since B E and D E are equal, $\overline{BE}^2$ and $\overline{DE}^2$ must also be equal, we shall have,

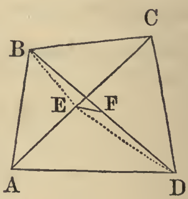
$$\overline{AB}^2 + \overline{BC}^2 + \overline{AD}^2 + \overline{DC}^2 = 4\,\overline{AE}^2 + 4\,\overline{DE}^2.$$

But 4 $\overline{AE}^2$ is the square of 2 A E, or of A C, and 4 $\overline{DE}^2$ is the square of 2 D E, or of B D (Prop. VIII. Cor.); hence,

$$\overline{BA}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{AD}^2 = \overline{AC}^2 + \overline{BD}^2.$$

Proposition XVI. — Theorem.

249. *In any quadrilateral the sum of the squares of the sides is equivalent to the sum of the squares of the diagonals, plus four times the square of the straight line that joins the middle points of the diagonals.*



Let A B C D be any quadrilateral, the diagonals of which are A C, D B, and E F a straight line joining their middle points, E, F; then the sum of the squares of A B, B C, C D, A D is equivalent to $\overline{AC}^2 + \overline{BD}^2 + 4\,\overline{EF}^2$.

Join E B and E D; then in the triangle A B C,

$$\overline{AB}^2 + \overline{BC}^2 = 2\,\overline{AE}^2 + 2\,\overline{BE}^2$$

(Prop. XIV.), and in the triangle A D C,

$$\overline{AD}^2 + \overline{CD}^2 = 2\,\overline{AE}^2 + 2\,\overline{DE}^2.$$

Hence, by adding the corresponding sides, we have

$$\overline{AB}^2 + \overline{BC}^2 + \overline{AD}^2 + \overline{CD}^2 = 4\,\overline{AE}^2 + 2\,\overline{BE}^2 + 2\,\overline{DE}^2.$$

But 4 $\overline{AE}^2$ is equivalent to $\overline{AC}^2$ (Prop. VIII. Cor.), and $2\,\overline{BE}^2 + 2\,\overline{DE}^2$ is equivalent to $4\,\overline{BF}^2 + 4\,\overline{EF}^2$ (Prop. XIV.); hence,

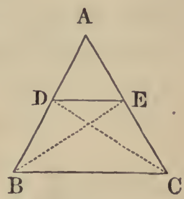
$$\overline{AB}^2 + \overline{BC}^2 + \overline{AD}^2 + \overline{CD}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{EF}^2.$$

250. *Cor.* If the quadrilateral is a parallelogram, the points E and F will coincide; then the proposition will be the same as Prop. XV.

251. *Scholium.* Proposition XV. is only a particular case of this proposition.

Proposition XVII. — Theorem.

252. *If a straight line be drawn in a triangle parallel to one of the sides, it will divide the other two sides proportionally.*



Let A B C be a triangle, and D E a straight line drawn within it parallel to the side B C; then will

A D : D B : : A E : E C.

Join B E and D C; then the two triangles B D E, D E C have the same base, D E; they have also the same altitude, since the vertices B and C lie in a line parallel to the base; therefore the triangles are equivalent (Prop. II. Cor. 2).

The triangles A D E, B D E, having their bases in the same line A B, and having the common vertex E, have the same altitude, and therefore are to each other as their bases (Prop. VI. Cor.); hence

A D E : B D E : : A D : D B.

The triangles A D E, D E C, whose common vertex is D, have also the same altitude, and therefore are to each other as their bases; hence

A D E : D E C : : A E : E C.

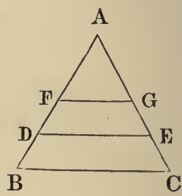
But the triangles B D E, D E C have been shown to be equivalent; therefore, on account of the common ratio in the two proportions (Prop. X. Bk. II.),

A D : D B : : A E : E C.

253. *Cor.* 1. Hence, by composition (Prop. VII. Bk.

II.), we have A D + D B : A D : : A E + E C : A E, or A B : A D : : A C : A E; also, A B : B D : : A C : E C.

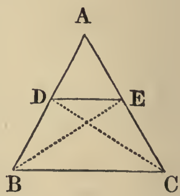
254. *Cor.* 2. If two or more straight lines be drawn in a triangle parallel to one of the sides, they will divide the other two sides proportionally.



For, in the triangle A B C, since D E is parallel to B C, by the theorem, A D : D B : : A E : E C; and, in the triangle A D E, since F G is parallel to D E, by the preceding corollary, A D : F D : : A E : G E. Hence, since the antecedents are the same in the two proportions (Prop. X. Cor. 2, Bk. II.), F D : D B : : G E : E C.

Proposition XVIII. — Theorem.

255. *If a straight line divides two sides of a triangle proportionally, the line is parallel to the other side of the triangle.*



Let A B C be a triangle, and D E a straight line drawn in it dividing the sides A B, A C, so that A D : D B : : A E : E C; then will the line D E be parallel to the side B C.

Join B E and D C; then the triangles A D E, B D E, having their bases in the same straight line A B, and having a common vertex, E, are to each other as their bases A D, D B (Prop. VI. Cor.); that is,

A D E : B D E : : A D : D B.

Also, the triangles A D E, D E C, having the common vertex D, and their bases in the same line, are to each other as these bases, A E, E C; that is,

A D E : D E C : : A E : E C.

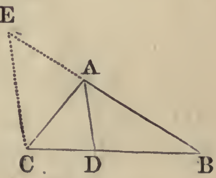
But, by hypothesis, A D : D B : : A E : E C; hence (Prop. X. Bk. II.),

$$ADE : BDE :: ADE : DEC;$$

that is, B D E, D E C have the same ratio to A D E; therefore the triangles B D E, D E C have the same area, and consequently are equivalent (Art. 211). Since these triangles have the same base, D E, their altitudes are equal (Prop. VI. Cor.); hence the line B C, in which their vertices are, must be parallel to D E.

Proposition XIX. — Theorem.

256. *The straight line bisecting any angle of a triangle divides the opposite side into parts, which are proportional to the adjacent sides.*



In any triangle, A B C, let the angle B A C be bisected by the straight line A D; then will

$$BD : DC :: AB : AC.$$

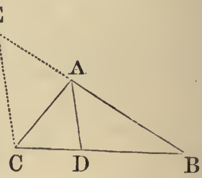
Through the point C draw C E parallel to A D, meeting B A produced in E. Then, since the two parallels A D, E C are met by the straight line A C, the alternate angles D A C, A C E are equal (Prop. XXII. Bk. I.); and the same parallels being met by the straight line B E, the opposite exterior and interior angles B A D, A E C are also equal (Prop. XXII. Bk. I.). But, by hypothesis, the angles D A C, B A D are equal; consequently the angle A C E is equal to the angle A E C; hence the triangle A C E is isosceles, and the side A E is equal to the side A C (Prop. VIII. Bk. I.). Again, since A D, in the triangle E B C, is parallel to E C, we have B D : D C : : A B : A E (Prop. XVII.), and, substituting A C in place of its equal A E,

$$BD : DC :: AB : AC.$$

Proposition XX. — Theorem.

257. *If a straight line drawn from the vertex of any angle of a triangle divides the opposite side into parts which are proportional to the adjacent sides, the line bisects the angle.*

Let the straight line A D, drawn from the vertex of the angle B A C, in the triangle A B C, divide the opposite side B C, so that B D : D C : : A B : A C; then will the line A D bisect the angle B A C.



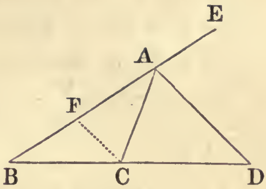
Through the point C draw C E parallel to A D, meeting B A produced in E. Then, by hypothesis, B D : D C : : A B : A C; and since A D is parallel to E C, B D : D C : : A B : A E (Prop. XVII.); then A B : A C : : A B : A E (Prop. X. Bk. II.); consequently A C is equal to A E; hence the angle A E C is equal to the angle A C E (Prop. VII. Bk. I.). But, since C E and A D are parallels, the angle A E C is equal to the opposite exterior angle B A D, and the angle A C E is equal to the alternate angle D A C (Prop. XXII. Bk. I.); hence the angles B A D, D A C are equal, and consequently the straight line A D bisects the angle B A C.

Proposition XXI. — Theorem.

258. *If the exterior angle formed by producing one of the sides of any triangle be bisected by a straight line which meets the base produced, the distances from the extremities of the base to the point where the bisecting line meets the base produced, will be to each other as the other two sides of the triangle.*

Let the exterior angle C A E, formed by producing the side B A of the triangle A B C, be bisected by the straight line A D, which meets the side B C produced in D, then will

B D : D C : : A B : A C.



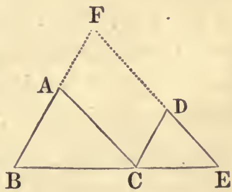
Through C draw C F parallel to A D; then the angle A C F is equal to the alternate angle C A D, and the exterior angle D A E is equal to the interior and opposite angle C F A (Prop. XXII. Bk. I.). But, by hypothesis, the angles C A D, D A E are equal; consequently the angle A C F is equal to the angle C F A; hence the triangle A C F is isosceles, and the side A C is equal to the side A F (Prop. VIII. Bk. I.). Again, since A D is parallel to F C, B D : D C : : B A : A F (Prop. XVII. Cor. 1), and substituting A C in the place of its equal A F, we have

B D : D C : : B A : A C.

Proposition XXII. — Theorem.

259. *Equiangular triangles have their homologous sides proportional, and are similar.*

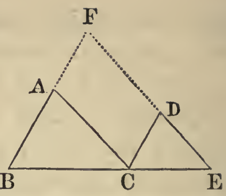
Let the two triangles A B C, D C E be equiangular; the angle B A C being equal to the angle C D E, the angle A B C to the angle D C E, and the angle A C B to the angle D E C, then the homologous sides will be proportional, and we shall have



B C : C E : : A B : C D : : A C : D E.

For, let the two triangles be placed so that two homologous sides, B C, C E, may join each other, and be in the same straight line; and produce the sides B A, E D till they meet in F.

Since B C E is a straight line, and the angle B C A is equal to the angle C E D, A C is parallel to F E (Prop. XXI. Bk. I.); also, since the angle A B C is equal to the



angle D C E, the line B F is parallel to the line C D. Hence the figure A C D F is a parallelogram; and, consequently, A F is equal to C D, and A C to F D (Prop. XXXI. Bk. I.).

In the triangle B E F, since the line A C is parallel to the side F E, we have B C : C E : : B A : A F (Prop. XVII.); or, substituting C D for its equal, A F,

B C : C E : : B A : C D.

Again, C D is parallel to B F; therefore, B C : C E : : F D : D E; or, substituting A C for its equal F D,

B C : C E : : A C : D E.

And, since both these proportions contain the same ratio B C : C E, we have (Prop. X. Bk. II.)

A C : D E : : B A : C D.

Hence, the equiangular triangles B A C, C D E have their homologous sides proportional; and consequently the two triangles are similar (Art. 210).

260. *Cor.* Two triangles having two angles of the one equal to two angles of the other, each to each, are similar; since the third angles will also be equal, and the two triangles be equiangular.

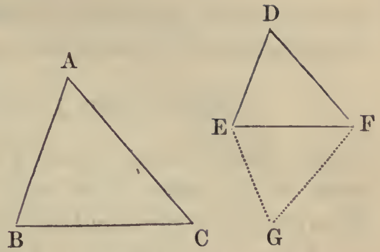
261. *Scholium.* In similar triangles, the homologous sides are opposite to the equal angles; thus the angle A C B being equal to D E C, the side A B is homologous to D C; in like manner, A C and D E are homologous.

Proposition XXIII. — Theorem.

262. *Triangles which have their homologous sides proportional, are equiangular and similar.*

Let the two triangles A B C, D E F have their sides proportional, so that we have B C : E F : : A B : D E : : A C : D F;

then will the triangles have their angles equal; namely, the angle A equal to the angle D, the angle B to the angle E, and the angle C to the angle F.



At the point E, in the straight line EF, make the angle FEG equal to the angle B, and at the point F, the angle EFG equal the angle C; the third angle G will be equal to the third angle A (Prop. XXVIII. Cor. 2, Bk. I.); and the two triangles ABC, EFG will be equiangular. Therefore, by the last theorem, we have

$$BC : EF :: AB : EG;$$

but, by hypothesis, we have

$$BC : EF :: AB : DE;$$

hence, EG is equal to DE.

By the same theorem, we also have

$$BC : EF :: AC : FG;$$

and, by hypothesis,

$$BC : EF :: AC : DF;$$

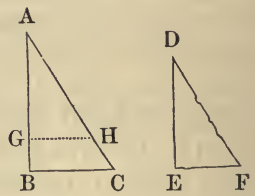
hence FG is equal to DF. Hence, the triangles EGF, DEF, having their three sides equal, each to each, are themselves equal (Prop. XVIII. Bk. I.). But, by construction, the triangle EGF is equiangular with the triangle ABC; hence the triangles DEF, ABC are also equiangular and similar.

263. *Scholium.* The two preceding propositions, together with that relating to the square of the hypothenuse (Art. 237), are the most important and fertile in results of any in Geometry. They are almost sufficient of themselves for all applications to subsequent reasoning, and for the

solution of all problems; since the general properties of triangles include, by implication, those of all figures.

Proposition XXIV. — Theorem.

264. *Two triangles, which have an angle of the one equal to an angle of the other, and the sides containing these angles proportional, are similar.*



Let the two triangles A B C, D E F have the angle A equal to the angle D, and the sides containing these angles proportional, so that A B : D E : : A C : D F; then the triangles are similar.

Take A G equal D E, and draw G H parallel to B C. The angle A G H will be equal to the angle A B C (Prop. XXII. Bk. I.); and the triangles A G H, A B C will be equiangular; hence we shall have

A B : A G : : A C : A H.

But, by hypothesis,

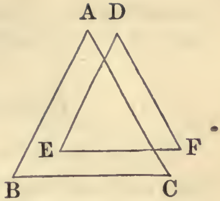
A B : D E : : A C : D F;

and, by construction, A G is equal to D E; hence A H is equal to D F. Therefore the two triangles A G H, D E F, having two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, are themselves equal (Prop. V. Bk. I.). But the triangle A G H is similar to A B C; therefore D E F is also similar to A B C.

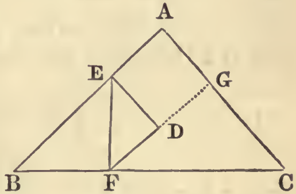
Proposition XXV. — Theorem.

265. *Two triangles, which have their sides, taken two and two, either parallel or perpendicular to each other, are similar.*

Let the two triangles A B C, D E F have the side A B parallel to the side D E, B C parallel to E F, and A C parallel to D F; these triangles will then be similar.



For, since the side A B is parallel to the side D E, and B C to E F, the angle A B C is equal to the angle D E F (Prop. XXVI. Bk. I.). Also, since A C is parallel to D F, the angle A C B is equal to the angle D F E, and the angle B A C to E D F; therefore the triangles A B C, D E F are equiangular; hence they are similar (Prop. XXII.).



Again, let the two triangles A B C, D E F have the side D E perpendicular to the side A B, D F perpendicular to A C, and E F perpendicular to B C; these triangles are similar.

Produce F D till it meets A C at G; then the angles D G A, D E A of the quadrilateral A E D G are two right angles; and since all the four angles are together equal to four right angles (Prop. XXIX. Cor. 1, Bk. I.), the remaining two angles, E D G, E A G, are together equal to two right angles. But the two angles E D G, E D F are also together equal to two right angles (Prop. I. Bk. I.); hence the angle E D F is equal to E A G or B A C.

The two angles, G F C, G C F, in the right-angled triangle F G C, are together equal to a right angle (Prop. XXVIII. Cor. 5, Bk. I.), and the two angles G F C, G F E are together equal to the right angle E F C (Art. 34, Ax. 9); therefore G F E is equal to G C F, or D F E to B C A. Therefore the triangles A B C, D E F have two angles of the one equal to two angles of the other, each to each; hence they are similar (Prop. XXII. Cor.).

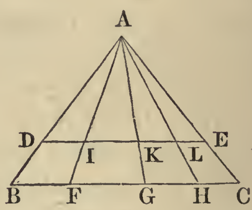
266. *Scholium.* When the two triangles have their sides parallel, the parallel sides are homologous; and when they have them perpendicular, the perpendicular sides are

homologous. Thus, D E is homologous with A B, D F with A C, and E F with B C.

Proposition XXVI. — Theorem.

267. *In any triangle, if a line be drawn parallel to the base, all lines drawn from the vertex will divide the parallel and the base proportionally.*

In the triangle B A C, let D E be drawn parallel to the base B C; then will the lines A F, A G, A H, drawn from the vertex, divide the parallel D E, and the base B C, so that



D I : B F : : I K : F G : : K L : G H.

For, since D I is parallel to B F, the triangles A D I and A B F are equiangular; and we have (Prop. XXII.),

D I : B F : : A I : A F;

and since I K is parallel to F G, we have in like manner,

A I : A F : : I K : F G;

and, since these two propositions contain the same ratio, A I : A F, we shall have (Prop. X. Cor. 1, Bk. II.),

D I : B F : : I K : F G.

In the same manner, it may be shown that

I K : F G : : K L : G H : : L E : H C.

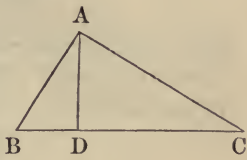
Therefore the line D E is divided at the points I, K, L, as the base B C is, at the points F, G, H.

268. *Cor.* If B C were divided into equal parts at the points F, G, H, the parallel D E would also be divided into equal parts at the points I, K, L.

Proposition XXVII. — Theorem.

269. *In a right-angled triangle, if a perpendicular is drawn from the vertex of the right angle to the hypothe-*

nuse, the triangle will be divided into two triangles similar to the given triangle and to each other.



In the right-angled triangle A B C, from the vertex of the right angle B A C, let A D be drawn perpendicular to the hypothenuse B C; then the triangles B A D, D A C will be similar to the triangle A B C, and to each other.

For the triangles B A D, B A C have the common angle B, the right angle B D A equal to the right angle B A C, and therefore the third angle, B A D, of the one, equal to the third angle, C, of the other (Prop. XXVIII. Cor. 2, Bk. I.); hence these two triangles are equiangular, and consequently are similar (Prop. XXII.). In the same manner it may be shown that the triangles D A C and B A C are equiangular and similar. The triangles B A D and D A C, being each similar to the triangle B A C, are similar to each other.

270. *Cor.* 1. Each of the sides containing the right angle is a mean proportional between the hypothenuse and the part of it which is cut off adjacent to that side by the perpendicular from the vertex of the right angle.

For, the triangles B A D, B A C being similar, their homologous sides are proportional; hence

$$BD : BA :: BA : BC;$$

and, the triangles D A C, B A C being also similar,

$$DC : AC :: AC : BC;$$

hence each of the sides A B, A C is a mean proportional between the hypothenuse and the part cut off adjacent to that side.

271. *Cor.* 2. The perpendicular from the vertex of the right angle to the hypothenuse is a mean proportional between the two parts into which it divides the hypothenuse.

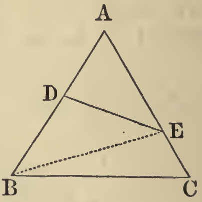
For, since the triangles A B D, A D C are similar, by comparing their homologous sides we have

B D : A D : : A D : D C;

hence, the perpendicular A D is a mean proportional between the parts D B, D C into which it divides the hypothenuse B C.

Proposition XXVIII. — Theorem.

272. *Two triangles, having an angle in each equal, are to each other as the rectangles of the sides which contain the equal angles.*



Let the two triangles A B C, A D E have the angle A in common; then will the triangle A B C be to the triangle A D E as A B × A C to A D × A E.

Join B E; then the triangles A B E, A D E, having the common vertex E, and their bases in the same line, A B, have the same altitude, and are to each other as their bases (Prop. VI. Cor.); hence

A B E : A D E : : A B : A D.

In like manner, since the triangles A B C, A B E have the common vertex B, and their bases in the same line, A C, we have

A B C : A B E : : A C : A E.

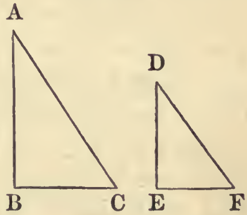
By multiplying together the corresponding terms of these proportions, and omitting the common term A B E, we have (Prop. XIII. Bk. II.),

A B C : A D E : : A B × A C : A D × A E.

273. *Cor.* If the rectangles of the sides containing the equal angles were equivalent, the triangles would be equivalent.

Proposition XXIX. — Theorem.

274. *Similar triangles are to each other as the squares described on their homologous sides.*



Let A B C, D E F be two similar triangles, and let A C, D F be homologous sides; then the triangle A B C will be to the triangle D E F as the square on A C is to the square on D F.

For, the triangles being similar, they have their homologous sides proportional (Art. 210); therefore

$$AB : DE :: AC : DF;$$

and multiplying the terms of this proportion by the corresponding terms of the identical proportion,

$$AC : DF :: AC : DF,$$

we have (Prop. XIII. Bk. II.),

$$AB \times AC : DE \times DF :: \overline{AC}^2 : \overline{DF}^2.$$

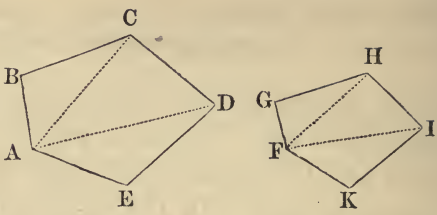
But, by reason of the equal angles A and D, the triangle ABC is to the triangle DEF as $AB \times AC$ is to $DE \times DF$ (Prop. XXVIII.); consequently (Prop. X. Bk. II.),

$$ABC : DEF :: \overline{AC}^2 : \overline{DF}^2.$$

Therefore, the two similar triangles A B C, D E F are to each other as the squares described on the homologous sides A C, D F, or as the squares described on any other two homologous sides.

Proposition XXX. — Theorem.

275. *Similar polygons may be divided into the same number of triangles similar each to each, and similarly situated.*



Let A B C D E, F G H I K be two similar polygons; they may be divided into the same number of triangles similar each to each, and similarly situated. From the homologous angles A and F, draw the diagonals A C, A D and F H, F I.

The two polygons being similar, the angles B and G, which are homologous, must be equal, and the sides A B, B C must also be proportional to F G, G H (Art. 210); that is, A B : F G : : B C : G H. Therefore the triangles A B C, F G H have an angle of the one equal to the angle of the other, and the sides containing these angles proportional; hence they are similar (Prop. XXIV.); consequently the angle B C A is equal to the angle G H F. These equal angles being taken from the equal angles B C D, G H I, the remaining angles A C D, F H I will be equal (Art. 34, Ax. 3). But, since the triangles A B C, F G H are similar, we have

A C : F H : : B C : G H;

and, since the polygons are similar (Art. 210),

B C : G H : : C D : H I;

hence (Prop. X. Cor. 1, Bk. II.),

A C : F H : : C D : H I.

But the terms of the last proportion are the sides about the equal angles A C D, F H I; hence the triangles A C D, F H I are similar (Prop. XXIV.). In the same manner, it may be shown that the corresponding triangles A D E, F I K are similar; hence the similar polygons may be divided into the same number of triangles similar each to each, and similarly situated.

276. *Cor. Conversely, if two polygons are composed*

of the same number of similar triangles, and similarly situated, the two polygons are similar.

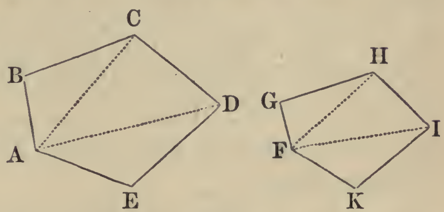
For the similarity of the corresponding triangles give the angles A B C equal to F G H, B C A equal to G H F, and A C D equal to F H I; hence, B C D equal to G H I, likewise C D E equal to H I K, &c. Moreover, we have

$$AB : FG :: BC : GH :: AC : FH :: CD : HI, \&c.;$$

therefore the two polygons have their angles equal and their sides proportional; hence they are similar.

Proposition XXXI. — Theorem.

277. *The perimeters of similar polygons are to each other as their homologous sides; and their areas are to each other as the squares described on these sides.*



Let A B C D E, F G H I K be two similar polygons; then their perimeters are to each other as their homologous sides A B and F G, B C and G H, &c.; and their areas are to each other as $\overline{AB}^2$ is to $\overline{FG}^2$, $\overline{BC}^2$ to $\overline{GH}^2$, &c.

First. Since the two polygons are similar, we have

$$AB : FG :: BC : GH :: CD : HI, \&c.$$

Now the sum of the antecedents A B, B C, C D, &c., which compose the perimeter of the first polygon, is to the sum of the consequents F G, G H, H I, &c., which compose the perimeter of the second polygon, as any one antecedent is to its consequent (Prop. XI. Bk. II.); therefore, as any two homologous sides are to each other, or as A B is to F G.

Secondly. From the homologous angles A and F, draw the diagonals A C, A D and F H, F I. Then, since the triangles A B C, F G H are similar, the triangle

$$ABC : FGH :: \overline{AC}^2 : \overline{FH}^2$$

(Prop. XXIX.); and, since the triangles A C D, F H I are similar, the triangle A C D : F H I :: $\overline{AC}^2 : \overline{FH}^2$. But the ratio $\overline{AC}^2 : \overline{FH}^2$ is common to both of the proportions; therefore (Prop. X. Bk. II.),

$$ABC : FGH :: ACD : FHI.$$

By the same mode of reasoning, it may be proved that

$$ACD : FHI :: ADE : FIK,$$

and so on, if there were more triangles. Therefore the sum of the antecedents A B C, A C D, A D E, which compose the area of the polygon A B C D E, is to the sum of the consequents F G H, F H I, F I K, which compose the area of the polygon F G H I K, as any one antecedent A B C is to its consequent F G H (Prop. XI. Bk. II.), or as $\overline{AB}^2$ is to $\overline{FG}^2$; hence the areas of similar polygons are to each other as the squares described on their homologous sides.

278. *Cor.* 1. The perimeters of similar polygons are also to each other as their corresponding diagonals.

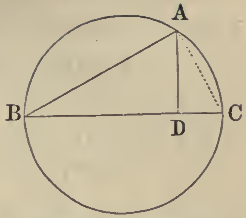
279. *Cor.* 2. The areas of similar polygons are to each other as the squares described on their corresponding diagonals.

Proposition XXXII. — Theorem.

280. *A chord in a circle is a mean proportional between the diameter and the part of the diameter cut off between one extremity of the chord and a perpendicular drawn from the other extremity to the diameter.*

Let A B be a chord in a circle, B C a diameter drawn from one extremity of A B, and A D a perpendicular drawn from the other extremity to B C; then

$$BD : AB :: AB : BC.$$



Join A C; then the triangle A B C, described in a semicircle, is right-angled at A (Prop. XVIII. Cor. 2, Bk. III.); and the triangle B A D is similar to the triangle B A C (Prop. XXVII.); hence, we have (Prop. XXVII. Cor. 1),

$$BD : AB :: AB : BC;$$

therefore the chord A B is a mean proportional between the diameter B C, and the part, B D, cut off between the extremity of the chord and the perpendicular from the other extremity.

281. *Cor.* If from any point, A, in the circumference of a circle, a perpendicular, A D, be drawn to the diameter B C, the perpendicular will be a mean proportional between the parts B D, D C into which it divides the diameter.

For, joining A B and A C, we have the triangle A B C, right-angled at A, and the triangles B A D, D A C similar to it and to each other (Prop. XXVII.); therefore (Prop. XXVII. Cor. 2),

$$BD : AD :: AD : DC,$$

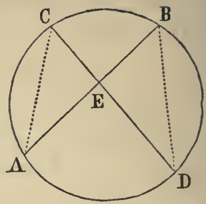
or, what amounts to the same thing (Prop. III. Bk. II.),

$$BD \times DC = \overline{AD}^2.$$

Scholium. A part of a straight line cut off by another is called a *segment* of the line. Thus B D, D C are segments of the diameter B C.

Proposition XXXIII. — Theorem.

282. *If two chords in a circle intersect each other, the segments of the one are reciprocally proportional to the segments of the other.*



Let A B, C D be two chords, which intersect each other at E; then will

A E : D E : : E C : E B.

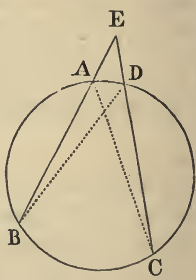
Join A C and B D. In the triangles A E C, B E D, the angles at E are equal being vertical angles (Prop. IV. Bk. I.); the angle A is equal to the angle D, being measured by half the same arc, B C (Prop. XVIII. Cor. 1, Bk. III.); for the same reason, the angle C is equal to the angle B; the triangles are therefore similar (Prop. XXII.), and their homologous sides give the proportion,

A E : D E : : E C : E B.

283. *Cor.* Hence, A E $\times$ E B $=$ D E $\times$ E C; therefore the rectangle of the two segments of the one chord is equal to the rectangle of the two segments of the other.

Proposition XXXIV. — Theorem.

284. *If from the same point without a circle two secants be drawn, terminating in the concave arc, the whole secants will be reciprocally proportional to their external segments.*



Let E B, E C be two secants drawn from the point E without a circle, and terminating in the concave arc at the points B and C; then will

E B : E C : : E D : E A.

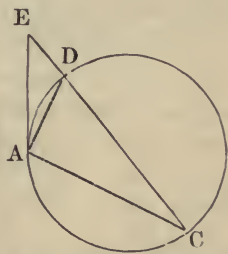
For, joining A C, B D, the triangles A E C, B E D have the angle E common; and the angles B and C, being measured by half the same arc, A D, are equal (Prop. XVIII. Cor. 1, Bk. III.); these triangles are therefore similar (Prop. XXII. Cor.), and their homologous sides give the proportion,

E B : E C : : E D : E A.

285. *Cor.* Hence, $EB \times EA = EC \times ED$; therefore the rectangle contained by the whole of one secant and its external segment is equivalent to the rectangle contained by the whole of the other secant and its external segment.

PROPOSITION XXXV. — THEOREM.

286. *If from a point without a circle there be drawn a tangent terminating in the circumference, and a secant terminating in the concave arc, the tangent will be a mean proportional between the whole secant and its external segment.*



From the point E let the tangent E A, and the secant E C, be drawn; then will $EC : EA :: EA : ED$.

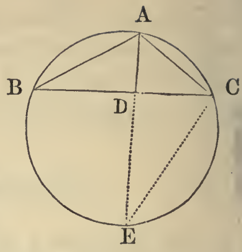
For, joining A D and A C, the triangles E A D, E A C have the angle E common; also, the angle E A D formed by a tangent and a chord has for its measure half the arc A D (Prop. XX. Bk. III.), and the angle C has the same measure; therefore the angle E A D is equal to the angle C; hence the two triangles are similar (Prop. XXII. Cor.), and give the proportion,

$$EC : EA :: EA : ED.$$

287. *Cor.* Hence, $\overline{EA}^2 = EC \times ED$; therefore the square of the tangent is equivalent to the rectangle contained by the whole secant and its external segment.

PROPOSITION XXXVI. — THEOREM.

288. *If any angle of a triangle is bisected by a line terminating in the opposite side, the rectangle of the other two sides is equivalent to the square of the bisecting line plus the rectangle of the segments of the third side.*



Let the triangle A B C have the angle BAC bisected by the straight line A D terminating in the opposite side B C; then the rectangle B A × A C is equivalent to the square of A D plus the rectangle B D × D C. Describe a circle through the three points A, B, C; produce A D till A meets the circumference at E, and join C E.

The triangles B A D, E A C have, by hypothesis, the angle B A D equal to the angle E A C; also the angle B equal to the angle E, being measured by half of the same arc A C (Prop. XVIII. Cor. 1, Bk. III.); these triangles are therefore similar (Prop. XXII. Cor.), and their homologous sides give the proportion,

$$BA : AE :: AD : AC;$$

hence, $$BA \times AC = AE \times AD.$$

But A E is equal to A D + D E, and multiplying each of these equals by A D, we have,

$$AE \times AD = \overline{AD}^2 + AD \times DE;$$

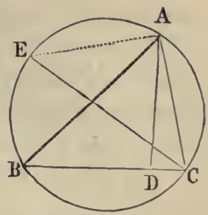
now, A D × D E is equivalent to B D × D C (Prop. XXXIII. Cor.); hence

$$BA \times AC = \overline{AD}^2 + BD \times DC.$$

Proposition XXXVII. — Theorem.

289. *The rectangle contained by any two sides of a triangle is equivalent to the rectangle contained by the diameter of the circumscribed circle and the perpendicular drawn to the third side from the vertex of the opposite angle.*

In any triangle A B C, let A D be drawn perpendicular to B C; and let E C be the diameter of the circle circum-



scribed about the triangle; then will $AB \times AC$ be equivalent to $AD \times CE$.

For, joining AE, the angle EAC is a right angle, being inscribed in a semicircle (Prop. XVIII. Cor. 2, Bk. III.); and the angles B and E are equal, being measured by half of the same arc, AC (Prop. XVIII. Cor. 1, Bk. III.); hence the two right-angled triangles are similar (Prop. XXII. Cor.), and give the proportion $AB : CE :: AD : AC$; hence

$$AB \times AC = CE \times AD.$$

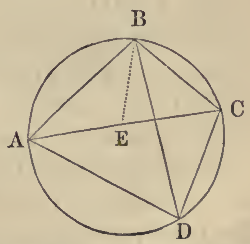
290. *Cor.* If these equals be multiplied by BC, we shall have

$$AB \times AC \times BC = CE \times AD \times BC.$$

But $AD \times BC$ is double the area of the triangle (Prop. VI.); therefore the product of the three sides of a triangle is equal to its area multiplied by twice the diameter of the circumscribed circle.

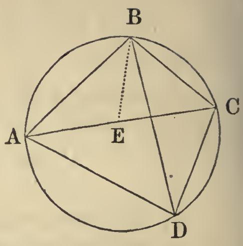
Proposition XXXVIII. — Theorem.

291. *The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equivalent to the sum of the two rectangles of the opposite sides.*



Let $ABCD$ be any quadrilateral inscribed in a circle, and AC, BD its diagonals; then the rectangle $AC \times BD$ is equivalent to the sum of the two rectangles $AB \times CD$, $AD \times BC$.

For, draw BE, making the angle ABE equal to the angle CBD; to each of these equals add the angle EBD, and we shall have the angle ABD equal to the angle EBC; and the angle ADB is equal to the angle BCE, being in the same segment (Prop. XVIII. Cor. 1, Bk. III.); therefore the triangles ABD, BCE are similar; hence the proportion,



$$AD : BD :: CE : BC;$$

and, consequently,

$$AD \times BC = BD \times CE.$$

Again, since the angle ABE is equal to the angle CBD, and the angle BAE is equal to the angle BDC, being in the same segment (Prop. XVIII. Cor. 1, Bk. III.), the triangles ABE, BCD are similar; hence,

$$AB : AE :: BD : CD;$$

and consequently,

$$AB \times CD = AE \times BD.$$

By adding the corresponding terms of the two equations obtained, and observing that

$$BD \times AE + BD \times CE = BD\,(AE + CE) = BD \times AC,$$

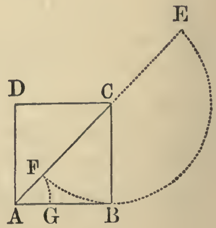
we have

$$BD \times AC = AB \times CD + AD \times BC.$$

Proposition XXXIX.—Theorem.

292. *The diagonal of a square is incommensurable with its side.*

Let ABCD be any square, and AC its diagonal; then AC is incommensurable with the side AB.



To find a common measure, if there be one, we must apply AB, or its equal CB, to CA, as often as it can be done. In order to do this, from the point C as a centre, with a radius CB, describe the semicircle FBE, and produce AC to E. It is evident that CB is contained once in AC, with a remainder A F, which remainder must be compared with B C, or its equal, A B.

The angle A B C being a right angle, A B is a tangent to the circumference, and A E is a secant drawn from the same point, so that (Prop. XXXV.)

$$AF : AB :: AB : AE.$$

Hence, in comparing A F with A B, the equal ratio of A B to A E may be substituted; but A B or its equal C F is contained twice in A E, with a remainder A F; which remainder must again be compared with A B.

Thus, the operation again consists in comparing A F with A B, and may be reduced in the same manner to the comparison of A B, or its equal C F, with A E; which will result, as before, in leaving a remainder A F; hence, it is evident that the process will never terminate; consequently the diagonal of a square is incommensurable with its side.

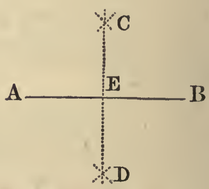
293. *Scholium.* The impossibility of finding numbers to express the exact ratio of the diagonal to the side of a square has now been proved; but, by means of the continued fraction which is equal to that ratio, an approximation may be made to it, sufficiently near for every practical purpose.

BOOK V.

PROBLEMS RELATING TO THE PRECEDING BOOKS.

Problem I.

294. *To bisect a given straight line, or to divide it into two equal parts.*



Let A B be a straight line, which it is required to bisect.

From the point A as a centre, with a radius greater than the half of A B, describe an arc of a circle; and from the point B as a centre, with the same radius, describe another arc, cutting the former in the points C and D. Through C and D draw the straight line C D; it will bisect A B in the point E.

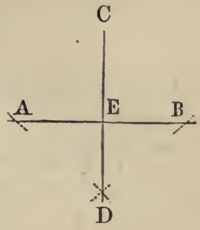
For the two points C and D, being each equally distant from the extremities A and B, must both lie in the perpendicular raised from the middle point of A B (Prop. XV. Cor., Bk. I.). Therefore the line C D must divide the line A B into two equal parts at the point E.

Problem II.

295. *From a given point, without a straight line, to draw a perpendicular to that line.*

Let A B be the straight line, and let C be a given point without the line.

From the point C as a centre, and with a radius sufficiently great, describe an arc cutting the line A B in two points, A and B; then, from the points A and B as centres, with a radius greater than half of A B, describe two arcs cutting each other in D, and draw the straight line C D; it will be the perpendicular required.



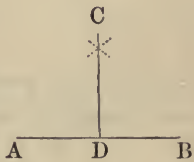
For, the two points C and D are each equally distant from the points A and B; hence, the line C D is a perpendicular passing through the middle of A B (Prop. XV. Cor., Bk. I.).

Problem III.

296. *At a given point in a straight line to erect a perpendicular to that line.*

Let A B be the straight line, and let D be a given point in it.

In the straight line A B, take the points A and B at equal distances from D; then from the points A and B as centres, with a radius greater than A D, describe two arcs cutting each other at C; through C and D draw the straight line C D; it will be the perpendicular required.

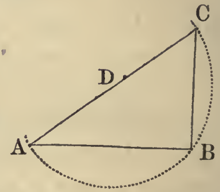


For the point C, being equally distant from A and B, must be in a line perpendicular to the middle of A B (Prop. XV. Cor., Bk. I.); hence C D has been drawn perpendicular to A B at the point D.

297. *Scholium.* The same construction serves for making a right angle, A D C, at a given point, D, on a given straight line, A B.

Problem IV.

298. *To erect a perpendicular at the end of a given straight line.*



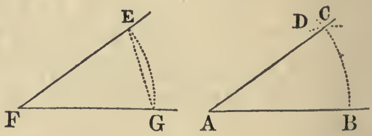
Let A B be the straight line, and B the end of it at which a perpendicular is to be erected.

From any point, D, taken without the line A B, with a radius equal to the distance D B, describe an arc cutting the line A B at the points A and B; through the point A, and the centre D, draw the diameter A C. Then through C, where the diameter meets the arc, draw the straight line C B, and it will be the perpendicular required.

For the angle A B C, being inscribed in a semicircle, is a right angle (Prop. XVIII. Cor. 2, Bk. III.).

Problem V.

299. *At a point in a given straight line to make an angle equal to a given angle.*



Let A be the given point, A B the given line, and E F G the given angle.

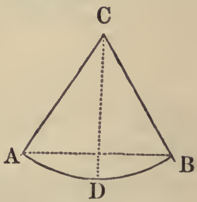
From the point F as a centre, with any radius, describe an arc, G E, terminating in the sides of the angle; from the point A as a centre, with the same radius, describe the indefinite arc B D. Draw the chord G E; then from B as a centre, with a radius equal to G E, describe an arc cutting the arc B D in C. Draw A C, and the angle C A B will be equal to the given angle E F G.

For the two arcs, B C and G E, have equal radii and equal chords; therefore they are equal (Prop. III. Bk.

III.); hence the angles C A B, E F G, measured by these arcs, are also equal (Prop. V. Bk. III.).

Problem VI.

300. *To bisect a given arc, or a given angle.*



First. Let A D B be the given arc which it is required to bisect.

Draw the chord A B; from the centre C draw the line C D perpendicular to A B (Prob. III.); it will bisect the arc A D B in the point D.

For C D being a radius perpendicular to a chord A B, must bisect the arc A D B which is subtended by that chord (Prop. VI. Bk. III.).

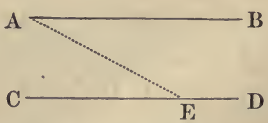
Secondly. Let A C B be the angle which it is required to bisect. From C as a centre, with any radius, describe the arc A D B; bisect this arc, as in the first case, by drawing the line C D; and this line will also bisect the angle A C B.

For the angles A C D, B C D are equal, being measured by the equal arcs A D, D B (Prop. V. Bk. III.).

301. *Scholium.* By the same construction, we may bisect each of the halves A D, D B; and thus, by successive subdivisions, a given angle or a given arc may be divided into four equal parts, into eight, into sixteen, &c.

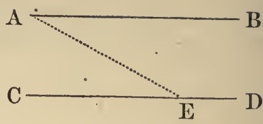
Problem VII.

302. *Through a given point, to draw a straight line parallel to a given straight line.*



Let A be the given point, and C D the given straight line.

From A draw a straight line, A E, to any point, E, in C D. Then draw A B, making the angle E A B equal to the

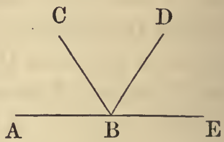


angle A E C (Prob. V.); and A B is parallel to C D.

For the alternate angles E A B, A E C, made by the straight line A E meeting the two straight lines A B, C D, being equal, the lines A B and C D must be parallel (Prop. XX. Bk. I.).

Problem VIII.

303. *Two angles of a triangle being given, to find the third angle.*

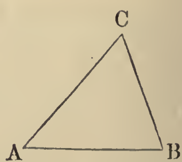


Draw the indefinite straight line A B E. At any point, B, make the angle A B C equal to one of the given angles (Prob. V.), and the angle C B D equal to the other given angle; then the angle D B E will be the third angle required.

For these three angles are together equal to two right angles (Prop. I. Cor. 2, Bk. I.), as are also the three angles of every triangle (Prop. XXVIII. Bk. I.); and two of the angles at B having been made equal to two angles of the triangle, the remaining angle D B E must be equal to the third angle.

Problem IX.

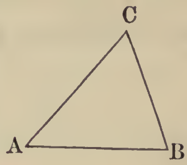
304. *Two sides of a triangle and the included angle being given, to construct the triangle.*



Draw the straight line A B equal to one of the two given sides. At the point A make an angle, C A B, equal to the given angle (Prob. V.); and take A C equal to the other given side. Join B C; and the triangle A B C will be the one required (Prop. V. Bk. I.).

PROBLEM X.

305. *One side and two angles of a triangle being given, to construct the triangle.*

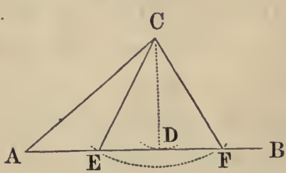


The two given angles will either be both adjacent to the given side, or one adjacent and the other opposite. In the latter case, find the third angle (Prob. VIII.); and the two angles adjacent to the given side will then be known.

In the former case, draw the straight line A B equal to the given side; at the point A, make an angle, B A C, equal to one of the adjacent angles, and at B an angle, A B C, equal to the other. Then the two sides A C, B C will meet, and form with A B the triangle required (Prop. VI. Bk. I.)

PROBLEM XI.

306. *Two sides of a triangle and an angle opposite one of them being given, to construct the triangle.*



Draw the indefinite straight line A B. At the point A make an angle B A C equal to the given angle, and make A C equal to that side which is adjacent to the given angle. Then from C, as a centre, with a radius equal to the other side, describe an arc, which must either touch the line A B in D, or cut it in the points E and F, otherwise a triangle could not be formed.

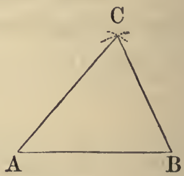
When the arc touches A B, a straight line drawn from C to the point of contact, D, will be perpendicular to A B (Prop. XI. Bk. III.), and the right-angled triangle C A D will be the triangle required.

When the arc cuts A B in two points, E and F, lying on the same side of the point A, draw the straight lines C E, C F; and each of the two triangles C A E, C A F will satisfy the conditions of the problem. If, however, the two points E and F should lie on different sides of the point A, only one of the triangles, as C A F, will satisfy all the conditions; hence that will be the triangle required.

307. *Scholium.* The problem would be impossible, if the side opposite the given angle were less than the perpendicular let fall from the point C on the straight line AB.

Problem XII.

308. *The three sides of a triangle being given, to construct the triangle.*

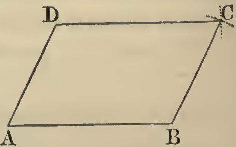


Draw the straight line A B equal to one of the given sides; from the point A as a centre, with a radius equal to either of the other two sides, describe an arc; from the point B, with a radius equal to the third side, describe another arc cutting the former in the point C; draw the straight lines A C, B C; and the triangle A B C will be the one required (Prop. XVIII. Bk. I.).

309. *Scholium.* The problem would be impossible, if one of the given sides were equal to or greater than the sum of the other two.

Problem XIII.

310. *Two adjacent sides of a parallelogram and the included angle being given, to construct the parallelogram.*



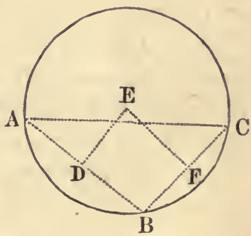
Draw the straight line A B equal to one of the given sides. At the point A make an angle, B A D, equal to the given angle, and take A D equal to the other given side. From the point D, with a radius equal to A B, describe an arc; and from the point B as a centre, with a radius equal to A D, describe another arc cutting the former in the point C. Draw the straight lines C D, C B; and the parallelogram A B C D will be the one required.

For the opposite sides are equal, by construction; hence the figure is a parallelogram (Prop. XXXII. Bk. I.); and it is formed with the given sides and the given angle.

311. *Cor.* If the given angle is a right angle, the figure will be a rectangle; and if the adjacent sides are also equal, the figure will be a square.

PROBLEM XIV.

312. *A circumference, or an arc, being given, to find the centre of the circle.*



Take any three points, A, B, C, on the given circumference, or arc. Draw the chords A B, B C, and bisect them by the perpendiculars D E and F E (Prob. I.); the point E, in which these perpendiculars meet, is the centre required.

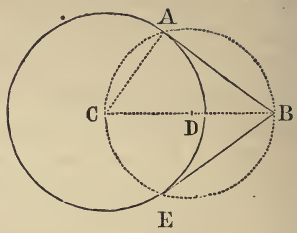
For the perpendiculars D E, F E must both pass through the centre (Prop. VI. Cor. 2, Bk. III.), and E being the only point through which they both pass, E must be the centre.

313. *Scholium.* By the same construction, a circumference may be made to pass through three given points, A, B, C, not in the same straight line; and also a circumference described in which a given triangle, A B C, shall be inscribed.

PROBLEM XV.

314. *Through a given point to draw a tangent to a given circle.*

First. Let the given point A be in the circumference.



Find the centre of the circle, C (Prob. XIV.); draw the radius C A; through the point A draw A B perpendicular to C A (Prob. IV.); and A B will be the tangent required (Prop. X. Bk. III.).

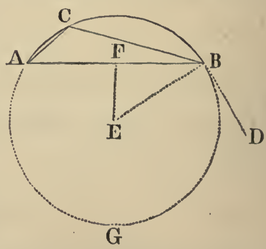
Secondly. Let the given point B be without the circumference.

Join the point B and the centre C by the straight line B C; bisect B C in D; and from D as a centre, with a radius equal to C D or D B, describe a circumference intersecting the given circumference in the points A and E. Draw A B and E B, and each will be a tangent as required.

For, drawing C A, the angle C A B, being inscribed in a semicircle, is a right angle (Prop. XVIII. Cor. 2, Bk. III.); therefore A B is perpendicular to the radius C A at its extremity, A, and consequently is a tangent (Prop. X. Bk. III.). In like manner it may be shown that E B is a tangent.

Problem XVI.

315. *On a given straight line to construct a segment of a circle that shall contain an angle equal to a given angle.*



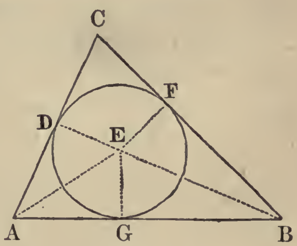
Let A B be the given straight line. Through the point B draw the straight line B D, making the angle A B D equal to the given angle; draw B E perpendicular to B D; bisect A B, and from F erect the perpendicular F E. From the point E, where these perpendiculars meet, as a centre, with the distance E B or E A, describe a circumference, and A C B will be the segment required.

For, since B D is a perpendicular at the extremity of the radius E B, it is a tangent (Prop. X. Bk. III.); and the angle A B D is measured by half the arc A G B (Prop. XX. Bk. III.). Also, the angle A C B, being an inscribed angle, is measured by half the arc A G B; therefore the angle A C B is equal to the angle A B D. But, by construction, the angle A B D is equal to the given angle; hence the segment A C B contains an angle equal to the given angle.

316. *Scholium.* If the given angle were acute, the centre must lie within the segment (Prop. XVIII. Cor. 3, Bk. III.); and if it were right, the centre must be in the middle of the line A B, and the required segment would be a semicircle.

Problem XVII.

317. *To inscribe a circle in any given triangle.*



Bisect any two of the angles, as A and B, by the straight lines A E and B E, meeting in the point E (Prob. VI.). From the point E let fall the perpendiculars E D, E F, E G (Prob. II.) on the three sides of the triangle; these perpendiculars will all be equal.

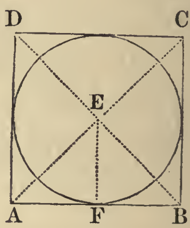
For, by construction, we have the angle D A E equal to the angle E A G, and the right angle A D E equal to the right angle A G E; hence the third angle A E D is equal to the third angle A E G. Moreover, A E is common to the two triangles A E D, A E G; hence the triangles themselves are equal, and E D is equal to E G. In the same manner it may be shown that the two triangles B E F, B E G are equal; therefore E F is equal to E G; hence the three perpendiculars E D, E F, E G are all equal, and if, from the point E as a centre, with the radius E D, a circle be described, it must pass through the points F and G.

318. *Scholium.* The three lines which bisect the angles of a triangle all meet in the centre of the inscribed circle.

Problem XVIII.

319. *To inscribe a circle in a given square.*

Draw the diagonals A C, D B, and from the point E, where the diagonals mutually bisect each other (Prop. XXXIV. Bk. I.), draw the straight line E F perpendicular to a side of the square. From E as a centre, with a radius equal to E F, describe a circle, and it will touch each side of the square.

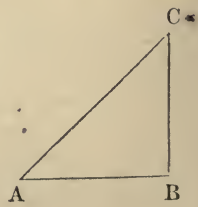


For the square is divided by the diagonals into four equal isosceles triangles; hence, the perpendicular, from the vertex E to the base, is the same in each triangle; therefore the circumference described from the centre E, with the radius E F, passes through the extremities of each perpendicular; consequently, the sides of the square are tangents to the circle (Prop. X. Bk. III.).

Problem XIX.

320. *To find the side of a square which shall be equivalent to the sum of two given squares.*

Draw the two straight lines A B, B C perpendicular to each other, taking A B equal to a side of one of the given squares, and B C equal to a side of the other. Join A C; this will be the side of the square required.

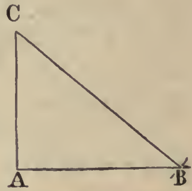


For, the triangle A B C being right-angled, the square that can be described upon the hypothenusé A C is equivalent to the sum of the squares that can be described upon the sides A B and B C (Prop. XI. Bk. IV.).

321. *Scholium.* A square may thus be found equivalent to the sum of any number of squares; for the construction which reduces two of them to one, will reduce three of them to two, and these two to one.

Problem XX.

322. *To find the side of a square which shall be equivalent to the difference of two given squares.*

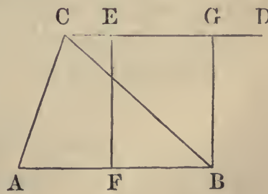


Draw the two straight lines A B, A C perpendicular to each other, making A C equal to the side of the less square. Then from C as a centre, with a radius equal to the side of the other square, describe an arc intersecting A B in the point B, and A B will be the side of the required square.

For, join B C, and the square that can be described upon A B is equivalent to the difference of the squares that can be described on B C and A C (Prop. XI. Cor. 1, Bk. IV.).

Problem XXI.

323. *To construct a rectangle that shall be equivalent to a given triangle.*

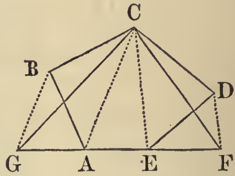


Let A B C be the given triangle. Draw the indefinite straight line C D parallel to the base A B; bisect A B by the perpendicular E F, and make E G equal to F B. Then, by drawing G B, the rectangle E F B G is equal to the triangle A B C.

For the rectangle E F B G has the same altitude, E F, as the triangle A B C, and half its base (Prop. II. Cor. 1, Bk. IV.).

Problem XXII.

324. *To construct a triangle that shall be equivalent to a given polygon.*



Let A B C D E be the given polygon.

Draw the diagonal C E, cutting off the triangle C D E; through the point D draw D F parallel to C E, and meeting A E produced in F. Draw C F; and the polygon A B C D E will be equivalent to the polygon A B C F, which has one side less than the given polygon.

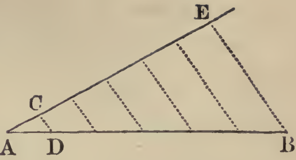
For the triangles C D E, C F E have the base C E common; they have also the same altitude, since their vertices, D, F, are situated in a line, D F, parallel to the base; these triangles are therefore equivalent (Prop. II. Cor. 2, Bk. IV.). Add to each of them the figure A B C E, and the polygon A B C D E will be equivalent to the polygon A B C F.

In like manner, the triangle C G A may be substituted for the equivalent triangle A B C, and thus the pentagon A B C D E will be changed into an equivalent triangle G C F.

The same process may be applied to every other polygon; for, by successively diminishing the number of its sides, one at each step of the process, the equivalent triangle will at last be found.

Problem XXIII.

325. *To divide a given straight line into any number of equal parts.*



Let A B be the given straight line proposed to be divided into any number of equal parts; for example, six.

Through the extremity A draw the indefinite straight line A E, making any angle with A B. Take A C of any convenient length, and apply it six times upon A E. Join the last point of division, E, and the extremity B by the straight line E B; and through the point C draw C D parallel to E B; then A D will be the sixth part of the line A B, and, being applied six times to A B, divides it into six equal parts.

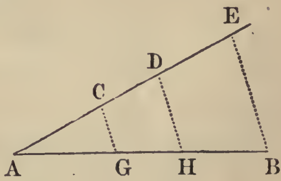
For, since C D is parallel to E B, in the triangle A B E, we have the proportion (Prop. XVII. Bk. IV.),

$$AD : AB :: AC : AE.$$

But A C is the sixth part of A E; hence A D is the sixth part of A B.

Problem XXIV.

326. *To divide a given straight line into parts that shall be proportional to other given lines.*



Let A B be the given straight line proposed to be divided into parts proportional to the given lines A C, C D, D E.

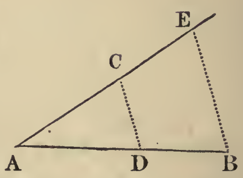
Through the point A draw the indefinite straight line A E, making any angle with A B. On A E lay off A C, C D, and D E. Join the points E and B by the straight line E B, and through the points C and D draw C G and D H parallel to E B; and the line A B will be divided into parts proportional to the given lines.

For, since C G and D H are each parallel to E B, we have the proportion (Prop. XVII. Cor. 2, Bk. IV.),

$$AC : AG :: CD : GH :: DE : HB.$$

PROBLEM XXV.

327. *To find a fourth proportional to three given straight lines.*



Draw the two indefinite straight lines A B, A E, forming any angle with each other.

On A B make A D equal to the first of the proposed lines, and A B equal to the second; and on A E make A E equal to the third. Join B E; and through the point D draw D C parallel to B E, and A C will be the fourth proportional required.

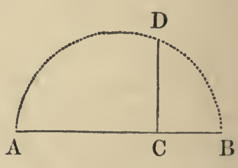
For, since D C is parallel to B E, we have the proportion (Prop. XVII. Cor. 1, Bk. IV.),

$$AB : AD :: AE : AC.$$

328. *Cor.* A third proportional to two given lines, A and B, may be found in the same manner, for it will be the same as a fourth proportional to the three lines, A, B, and B.

PROBLEM XXVI.

329. *To find a mean proportional between two given straight lines.*

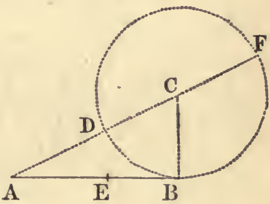


Draw the indefinite straight line A B. On A B take A C equal to the first of the given lines, and C B equal to the second. On A B, as a diameter, describe a semicircle, and at the point C draw the perpendicular C D, meeting the semi-circumference in D; C D will be the mean proportional required.

For the perpendicular C D, drawn from a point in the circumference to a point in the diameter, is a mean proportional between the two segments of the diameter A C, C B (Prop. XXXII. Cor., Bk. IV.); and these segments are equal to the given lines.

PROBLEM XXVII.

330. *To divide a given straight line into two such parts, that the greater part shall be a mean proportional between the whole line and the other part.*



Let A B be the given straight line.

At the extremity, B, of the line A B, erect the perpendicular B C, equal to the half of A B. From the point C as a centre, with the radius C B, describe a circle. Draw A C cutting the circumference in D; and take A E equal to A D. The line A B will be divided at the point E in the manner required; that is,

$$AB : AE :: AE : EB.$$

For A B, being perpendicular to the radius at its extremity, is a tangent (Prop. X. Bk. III.); and if A C be produced till it again meets the circumference, in F, we shall have (Prop. XXXV. Bk. IV.),

$$AF : AB :: AB : AD;$$

hence, by division (Prop. VIII. Bk. II.),

$$AF - AB : AB :: AB - AD : AD.$$

But, since the radius is the half of A B, the diameter D F is equal to A B, and consequently A F — A B is equal to A D, which is equal to A E; also, since A E is equal to A D, we have A B — A D equal to E B; hence,

$$AE : AB :: EB : AD, \text{ or } AE;$$

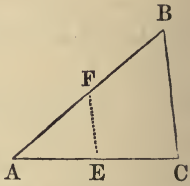
and, by inversion (Prop. V. Bk. II.),

$$AB : AE :: AE : EB.$$

331. *Scholium.* This sort of division of the line A B is called division in *extreme and mean ratio.*

Problem XXVIII.

332. *Through a given point in a given angle, to draw a straight line, which shall have the parts included between that point and the sides of the angle equal to each other.*



Let E be the given point, and A B C the given angle.

Through the point E draw E F parallel to B C, make A F equal to B F. Through the points A and E draw the straight line A E C, and it will be the line required.

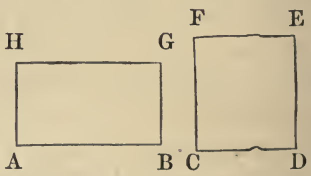
For, E F being parallel to B C, we have (Prop. XVII. Bk. IV.),

$$AF : FB :: AE : EC;$$

but A F is equal to F B; therefore A E is equal to E C.

Problem XXIX.

333. *On a given straight line to construct a rectangle that shall be equivalent to a given rectangle.*



Let A B be the given straight line, and C D E F the given rectangle.

Find a fourth proportional to the three straight lines A B, C D, D E (Prob. XXV.); and let B G be that fourth proportional. The rectangle constructed on A B and B G will be equivalent to the rectangle C D E F.

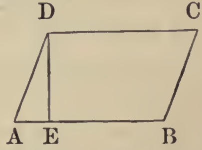
For, since A B : C D : : D E : B G, it follows (Prop. I. Bk. II.) that

$$AB \times BG = CD \times DE;$$

hence, the rectangle A B G H, which is constructed on the line A B, is equivalent to the rectangle C D E F.

PROBLEM XXX.

334. *To construct a square that shall be equivalent to a given parallelogram, or to a given triangle.*



First. Let A B C D be the given parallelogram, A B its base, and D E its altitude.

Find a mean proportional between A B and D E (Prob. XXVI.); and the square constructed on that proportional will be equivalent to the parallelogram A B C D.

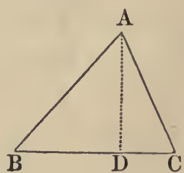
For, denoting the mean proportional by $x y$, we have, by construction,

$$AB : xy :: xy : DE;$$

therefore,

$$\overline{xy}^2 = AB \times DE;$$

but A B $\times$ D E is the measure of the parallelogram, and $\overline{x y}^2$ that of the square; hence they are equivalent.



Secondly. Let A B C be the given triangle, B C its base, and A D its altitude.

Find a mean proportional between B C and the half of A D, and let $x y$ denote that proportional; the square constructed on $x y$ will be equivalent to the triangle A B C.

For since, by construction,

$$BC : xy :: xy : \tfrac{1}{2} AD,$$

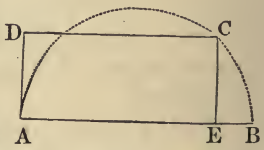
it follows that

$$\overline{x y}^2 = BC \times \tfrac{1}{2} AD;$$

hence the square constructed on $x y$ is equivalent to the triangle A B C.

Problem XXXI.

335. *To construct a rectangle equivalent to a given square, and having the sum of its adjacent sides equal to a given line.*



Let the straight line A B be equal to the sum of the adjacent sides of the required rectangle.

Upon A B as a diameter describe a semicircle; at the point A, draw A D perpendicular to A B, making A D equal to the side of the given square; then draw the line D C parallel to the diameter A B. From the point C, where the parallel meets the circumference, draw C E perpendicular to the diameter; A E and E B will be the sides of the rectangle required.

For their sum is equal to A B; and their rectangle A E × E B is equivalent to the square of C E, or to the square of A D (Prop. XXXII. Cor., Bk. IV.); hence, this rectangle is equivalent to the given square.

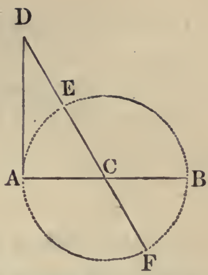
336. *Scholium.* The problem is impossible, when the distance A D is greater than the half the given line A B, for then the line D C will not meet the circumference.

Problem XXXII.

337. *To construct a rectangle that shall be equivalent to a given square, and the difference of whose adjacent sides shall be equal to a given line.*

Let the straight line A B be equal to the difference of the adjacent sides of the required rectangle.

Upon A B as a diameter, describe a circle. At the extremity of the diameter, draw the tangent A D, making it equal to the side of the given square.

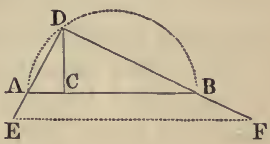


Through the point D and the centre C draw the secant D C F, intersecting the circumference in E; then D E and D F will be the adjacent sides of the rectangle required.

For the difference of these lines is equal to the diameter E F or A B; and the rectangle D E $\times$ D F is equal to $\overline{AD}^2$ (Prop. XXXV. Cor., Bk. IV.); hence it is equivalent to the given square.

PROBLEM XXXIII.

338. *To construct a square that shall be to a given square as one given line is to another given line.*



Draw the indefinite line A B, on which take A C equal to one of the given lines, and C B equal to the other. Upon A B as a diameter, describe a semicircle, and at the point C draw the perpendicular C D, meeting the circumference in D. Through the points A and B draw the straight lines D E, D F, making the former equal to the side of the given square; and through the point E draw E F parallel to A B; D F will be the side of the square required.

For, since E F is parallel to A B,

$$DE : DF :: DA : DB;$$

consequently (Prop. XV. Bk. II.),

$$\overline{DE}^2 : \overline{DF}^2 :: \overline{DA}^2 : \overline{DB}^2.$$

But in the right-angled triangle A D B the square of A D is to the square of D B as the segment A C is to the segment C B (Prop. XI. Cor. 3, Bk. IV.); hence,

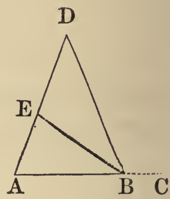
$$\overline{DE}^2 : \overline{DF}^2 :: AC : CB.$$

But, by construction, D E is equal to the side of the given square; also, A C is equal to one of the given lines, and C B to the other; hence, the given square is to that constructed on D F as the one given line is to the other.

Problem XXXIV.

339. *Upon a given base to construct an isosceles triangle, having each of the angles at the base double the vertical angle.*

Let A B be the given base.



Produce A B to some point C till the rectangle A C × B C shall be equivalent to the square of A B (Prob. XXXII.); then, with the base A B and sides each equal to A C, construct the isosceles triangle D A B, and the angle A will double the angle D.

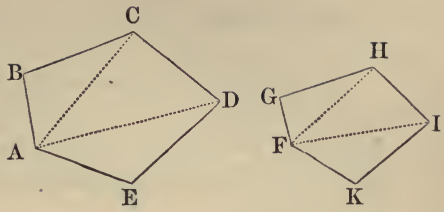
For, make D E equal to A B, or make A E equal to B C, and join E B. Then, by construction,

$$A D : A B : : A B : A E;$$

for A E is equal to B C; consequently the triangles D A B, B A E have a common angle, A, contained by proportional sides; hence they are similar (Prop. XXIV. Bk. IV.); therefore these triangles are both isosceles, for D A B is isosceles by construction, so that A B is equal to E B; but A B is equal to D E; consequently D E is equal to E B, and therefore the angle D is equal to the angle E B D; hence the exterior angle A E B is equal to double the angle D, but the angle A is equal to the angle A E B; therefore the angle A is double the angle D.

Problem XXXV.

340. *Upon a given straight line to construct a polygon similar to a given polygon.*



Let A B C D E be the given polygon, and F G the given straight line.

Draw the diagonals A C, A D. At the point F in the straight line F G, make the angle G F H equal to the angle B A C; and at the point G make the angle F G H equal to the angle A B C. The lines F H, G H will cut each other in H, and F G H will be a triangle similar to A B C. In the same manner, upon F H, homologous to A C, construct the triangle F I H similar to A D C; and upon F I, homologous to A D, construct the triangle F I K similar to A D E. The polygon F G H I K will be similar to A B C D E, as required.

For these two polygons are composed of the same number of triangles, similar each to each, and similarly situated (Prop. XXX. Cor., Bk. IV.).

PROBLEM XXXVI.

341. *Two similar polygons being given, to construct a similar polygon, which shall be equivalent to their sum or their difference.*

Let A and B be two homologous sides of the given polygons.

Find a square equal to the sum or to the difference of the squares described upon A and B; let x be the side of that square; then will x in the polygon required be the side which is homologous to the sides A and B in the given polygons. The polygon itself may then be constructed on x, by the last problem.

For similar figures are to each other as the squares of their homologous sides; but the square of the side x is equal to the sum or the difference of the squares described upon the homologous sides A and B; therefore the figure described upon the side x is equivalent to the sum or to the difference of the similar figures described upon the sides A and B.

Problem XXXVII.

342. *To construct a polygon similar to a given polygon, and which shall have to it a given ratio.*

Let A be a side of the given polygon.

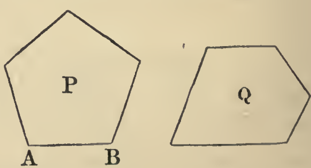
Find the side B of a square, which is to the square on A in the given ratio of the polygons (Prob. XXXIII.).

Upon B construct a polygon similar to the given polygon (Prob. XXXV.), and B will be the polygon required.

For the similar polygons constructed upon A and B have the same ratio to each other as the squares constructed upon A and B (Prop. XXXI. Bk. IV.).

Problem XXXVIII.

343. *To construct a polygon similar to a given polygon,* P, *and which shall be equivalent to another polygon,* Q.



Find M, the side of a square, equivalent to the polygon P, and N, the side of a square equivalent to the polygon Q. Let x be a fourth proportional to the three given lines M, N, A B; upon the side x, homologous to A B, describe a polygon similar to the polygon P (Prob. XXXV.); it will also be equivalent to the polygon Q.

For, representing the polygon described upon the side x by y, we have

$$P : y :: \overline{AB}^2 : x^2 ;$$

but, by construction,

$$AB : x :: M : N, \quad \text{or} \quad \overline{AB}^2 : x^2 :: M^2 : N^2 ;$$

hence,

$$P : y :: M^2 : N^2.$$

But, by construction also, M^2 is equivalent to P, and N^2 is equivalent to Q; therefore,

$$P : y :: P : Q ;$$

consequently y is equal to Q; hence the polygon y is similar to the polygon P, and equivalent to the polygon Q.

BOOK VI.

REGULAR POLYGONS, AND THE AREA OF THE CIRCLE.

DEFINITIONS.

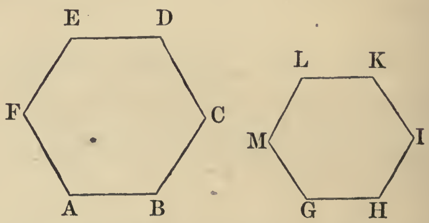
344. A Regular Polygon is one which is both equilateral and equiangular.

345. Regular polygons may have any number of sides: the equilateral triangle is one of three sides; the square is one of four.

Proposition I.—Theorem.

346. *Regular polygons of the same number of sides are similar figures.*

Let ABCDEF, GHIKLM, be two regular polygons of the same number of sides; then these polygons are similar.



For, since the two polygons have the same number of sides, they have the same number of angles; and the sum of all the angles is the same in the one as in the other (Prop. XXIX. Bk. I.). Also, since the polygons are equiangular, each of the angles A, B, C, &c. is equal to each of the angles G, H, I, &c.; hence the two polygons are mutually equiangular.

Again; the polygons being regular, the sides A B, B C, C D, &c. are equal to each other; so likewise are the sides G H, H I, I K, &c. Hence,

A B : G H : : B C : H I : : C D : I K, &c.

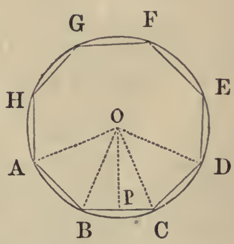
Therefore the two polygons have their angles equal, and their homologous sides proportional; hence they are similar (Art. 210).

347. *Cor.* The perimeters of two regular polygons of the same number of sides, are to each other as their homologous sides, and their areas are to each other as the squares of those sides (Prop. XXXI. Bk. IV.).

348. *Scholium.* The angle of a regular polygon is determined by the number of its sides (Prop. XXIX. Bk. I.).

Proposition II. — Theorem.

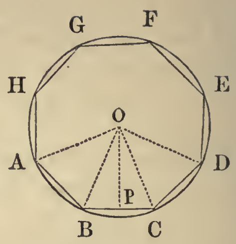
349. *A circle may be circumscribed about, and another inscribed in, any regular polygon.*



Let A B C D E F G H be any regular polygon; then a circle may be circumscribed about, and another inscribed in it.

Describe a circle whose circumference shall pass through the three points A, B, C, the centre being O; let fall the perpendicular O P from O to the middle point of the side B C; and draw the straight lines O A, O B, O C, O D.

Now, if the quadrilateral O P C D be placed upon the quadrilateral O P B A, they will coincide; for the side O P is common, and the angle O P C is equal to the angle O P B, each being a right angle; consequently the side P C will fall upon its equal, P B, and the point C on B. Moreover, from the nature of the polygon, the angle P C D is equal to the angle P B A; therefore C D will take the



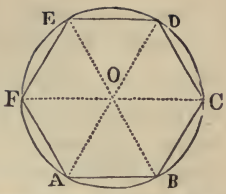
direction B A, and C D being equal to B A, the point D will fall upon A, and the two quadrilaterals will coincide throughout. Therefore O D is equal to A O, and the circumference which passes through the three points A, B, C, will also pass through the point D. By the same mode of reasoning, it may be shown that the circle which passes through the three vertices B, C, D, will also pass through the vertex E, and so on. Hence, the circumference which passes through the three points A, B, C, passes through the vertices of all the angles of the polygon, and is circumscribed about the polygon (Art. 166).

Again, with respect to this circumference, all the sides, A B, B C, C D, &c., of the polygon are equal chords; consequently they are equally distant from the centre (Prop. VIII. Bk. III.). Hence, if from the point O, as a centre, and with the radius O P, a circle be described, the circumference will touch the side B C, and all the other sides of the polygon, each at its middle point, and the circle will be inscribed in the polygon (Art. 168).

350. *Scholium* 1. The point O, the common centre of the circumscribed and inscribed circles, may also be regarded as the centre of the polygon. The angle formed at the centre by two radii drawn to the extremities of the same side is called *the angle at the centre;* and the perpendicular from the centre to a side is called the *apothegm* of the polygon.

Since all the chords A B, B C, C D, &c. are equal, all the angles at the centre must likewise be equal; therefore the value of each may be found by dividing four right angles by the number of sides of the polygon.

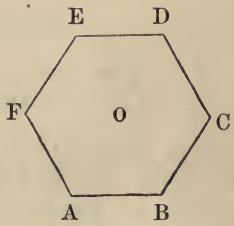
351. *Scholium* 2. To inscribe a regular polygon of any number of sides in a given circle, it is only necessary to



divide the circumference into as many equal parts as the polygon has sides; for the arcs being equal, the chords A B, B C, C D, &c. are also equal (Prop. III. Bk. III.); hence likewise the triangles A O B, B O C, C O D, &c. must be equal, since their sides are equal each to each (Prop. XVIII. Bk. I.); therefore all the angles A B C, B C D, C D E, &c. are equal; hence the figure A B C D E F is a regular polygon.

Proposition III.—Theorem.

352. *If from a common centre a circle can be circumscribed about, and another circle inscribed within, a polygon, that polygon is regular.*



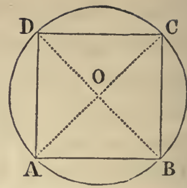
Suppose that from the point O, as a centre, circles can be circumscribed about, and inscribed in, the polygon A B C D E F; then that polygon is regular.

For, supposing it to be described, the inner one will touch all the sides of the polygon; therefore these sides are equally distant from its centre; and consequently, being chords of the outer circle, they are equal; therefore they include equal angles (Prop. XVIII. Cor. 1, Bk. III.). Hence the polygon is at once equilateral and equiangular; consequently it is regular (Art. 344).

Proposition IV.—Problem.

353. *To inscribe a square in a given circle.*

Draw two diameters, A C, B D, intersecting each other at right angles; join their extremities, A, B, C, D, and the figure A B C D will be a square.



For, the angles A O B, B O C, &c. being equal, the chords A B, B C, &c. are also equal (Prop. III. Bk. III.); and the angles A B C, B C D, &c., being inscribed in semicircles, are right angles (Prop. XVIII. Cor. 2, Bk. III.). Hence A B C D is a square, and it is inscribed in the circle A B C D.

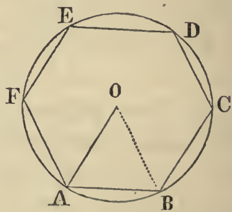
354. *Cor.* Since the triangle A O B is right-angled and isosceles, we have (Prop. XI. Cor. 5, Bk. IV.),

$$AB : AO :: \sqrt{2} : 1;$$

hence, *the side of the inscribed square is to the radius as the square root of 2 is to unity.*

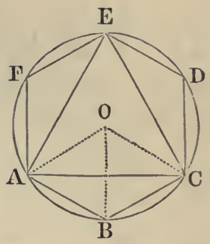
Proposition V. — Theorem.

355. *The side of a regular hexagon inscribed in a circle is equal to the radius of the circle.*



Let A B C D E F be a regular hexagon inscribed in a circle, the centre of which is O; then any side, as B C, will be equal to the radius O A.

Join B O; and the angle at the centre, A O B, is one sixth of four right angles (Prop. II. Sch. 1), or one third of two right angles; therefore the two other angles, O A B, O B A, of the same triangle, are together equal to two thirds of two right angles (Prop. XXVIII. Bk. I.). But A O and B O being equal, the angles O A B, O B A are also equal (Prop. VII. Bk. I.); consequently, each is one third of two right angles. Hence the triangle A O B is equiangular; therefore A B, the side of the regular hexagon, is equal to A O, the radius of the circle (Prop. VIII. Cor. Bk. I.).



356. *Cor.* 1. To inscribe a regular hexagon in a given circle, apply the radius, A O, of the circle six times, as a chord to the circumference. Hence, beginning at any point A, and applying A O six times as a chord to the circumference, we are brought round to the point of beginning, and the inscribed figure A B C D E F, thus formed, is a regular hexagon.

357. *Cor.* 2. By joining the alternate angles of the inscribed regular hexagon by the straight lines A C, C E, E A, the figure A C E, thus inscribed in the circle, will be an equilateral triangle, since its sides subtend equal arcs, A B C, C D E, E F A, on the circumference (Prop. III. Bk. III.).

358. *Cor.* 3. Join O A, O C, and the figure A B C O is a rhombus, for each side is equal to the radius. Hence, the sum of the squares of the diagonals A C, O B is equivalent to the sum of the squares of the sides (Prop. XV. Bk. IV.); or to four times the square of the radius O B; that is, $\overline{AC}^2 + \overline{OB}^2$ is equivalent to $4\ \overline{AB}^2$, or $4\ \overline{OB}^2$; and taking away $\overline{OB}^2$ from both, there remains $\overline{AC}^2$ equivalent to $3\ \overline{OB}^2$; hence

$$\overline{AC}^2 : \overline{OB}^2 :: 3 : 1, \quad \text{or} \quad AC : OB :: \sqrt{3} : 1;$$

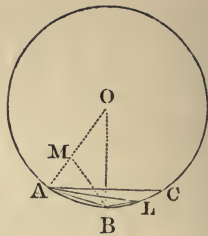
hence, *the side of the inscribed equilateral triangle is to the radius as the square root of* 3 *is to unity.*

Proposition VI. — Problem.

359. *To inscribe a regular decagon in a given circle.*

Divide the radius, O A, of the given circle, in extreme and mean ratio, at the point M (Prob. XXVII. Bk. V.).

Take the chord A B equal to O M, and A B will be the side of a regular decagon inscribed in the circle. For we have by construction,



$$AO : OM :: OM : AM;$$

or, since A B is equal to O M,

$$AO : AB :: AB : AM.$$

Draw M B and B O; and the triangles A B O, A M B have a common angle, A, included between proportional sides; hence the two triangles are similar (Prop. XXIV. Bk. IV.). Now, the triangle O A B being isosceles, A M B must also be isosceles, and A B is equal to B M; but A B is equal to O M, consequently M B is equal to M O; hence the triangle M B O is isosceles.

Again, the angle A M B, being exterior to the isosceles triangle B M O, is double the interior angle O (Prop. XXVII. Bk. I.). But the angle A M B is equal to the angle M A B; hence the triangle O A B is such, that each of the angles at the base, O A B, O B A, is double the angle O, at its vertex. Hence the three angles of the triangle are together equal to five times the angle O, which consequently is a fifth part of two right angles, or the tenth part of four right angles; therefore the arc A B is the tenth part of the circumference, and the chord A B is the side of an inscribed regular decagon.

360. *Cor.* 1. By joining the vertices of the alternate angles A, C, &c. of the regular decagon, a regular pentagon may be inscribed. Hence, the chord A C is the side of an inscribed regular pentagon.

361. *Cor.* 2. A B being the side of the inscribed regular decagon, let A L be the side of an inscribed regular hexagon (Prop. V. Cor. 1). Join B L; then B L will be the side of an inscribed regular pentedecagon, or regular polygon of fifteen sides. For A B cuts off an arc equal to a tenth part of the circumference; and A L subtends an

arc equal to a sixth of the circumference; therefore B L, the difference of these arcs, is a fifteenth part of the circumference; and since equal arcs are subtended by equal chords, it follows that the chord B L may be applied exactly fifteen times around the circumference, thus forming a regular pentedecagon.

362. *Scholium.* If the arcs subtended by the sides of any inscribed regular polygon be severally bisected, the chords of those semi-arcs will form another inscribed polygon of double the number of sides. Thus, from having an inscribed square, there may be inscribed in succession polygons of 8, 16, 32, 64, &c. sides; from the hexagon may be formed polygons of 12, 24, 48, 96, &c. sides; from the decagon, polygons of 20, 40, 80, &c. sides; and from the pentedecagon, polygons of 30, 60, 120, &c. sides.

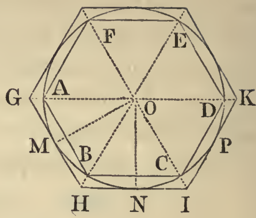
NOTE. — For a long time the polygons above noticed were supposed to include all that could be inscribed in a circle. In the year 1801, M. Gauss, of Göttingen, made known the curious discovery that the circumference of a circle could be divided into any number of equal parts capable of being expressed by the formula $2^n + 1$, provided it be a prime number. Now, the number 3 is the simplest of this kind, it being the value of the above formula when the exponent n is 1; the next prime number is 5, and this is contained in the formula. But the polygons of 3 and of 5 sides have already been inscribed. The next prime number expressed by the formula is 17, so that it is possible to inscribe a regular polygon of 17 sides in a circle. The investigations, however, which establish this geometrical fact involve considerations of a nature that do not enter into the elements of Geometry.

PROPOSITION VII. — PROBLEM.

363. *A regular inscribed polygon being given, to circumscribe a similar polygon about the same circle.*

Let A B C D E F be a regular polygon inscribed in a circle whose centre is O.

Through M, the middle point of the arc A B, draw the tangent, G H; also draw tangents at the middle points of the arcs B C, C D, &c.; these tangents are parallel to the chords A B, B C, C D, &c. (Prop. XI. Bk. III., and Prop. VI. Cor. 1, Bk. III.), and by their intersections form the regular circumscribed polygon G H I, &c. similar to the one inscribed.



Since M is the middle point of the arc A B, and N the middle point of the equal arc B C, the arcs B M, B N are halves of equal arcs, and therefore are equal; that is, the vertex, B, of the inscribed polygon is at the middle point of the arc M N. Draw O H; the line O H will pass through the point B. For, the right-angled triangles O M H, O N H, having the common hypothenuse O H, and the side O M equal to O N, must be equal (Prop. XIX. Bk. I.), and consequently the angle M O H is equal to H O N, wherefore the line O H passes through the middle point, B, of the arc M N. In like manner, it may be shown that the line O I passes through the middle point, C, of the arc N P; and so with the other vertices.

Since G H is parallel to A B, and H I to B C, the angle G H I is equal to the angle A B C (Prop. XXVI. Bk. I.); in like manner, H I K is equal to B C D; and so with the other angles; hence, the angles of the circumscribed polygon are equal to those of the inscribed polygon. And, further, by reason of these same parallels, we have

$$GH : AB :: OH : OB, \text{ and } HI : BC :: OH : OB;$$

therefore (Prop. X. Bk. II.),

$$GH : AB :: HI : BC.$$

But A B is equal to B C, therefore G H is equal to H I. For the same reason, H I is equal to I K, &c.; consequently, the sides of the circumscribed polygon are all equal; hence this polygon is regular, and similar to the inscribed one.

364. *Cor.* 1. Conversely, if the circumscribed polygon G H I K, &c. is given, and it is required, by means of it, to construct a similar inscribed polygon, draw the straight lines O G, O H, &c. from the vertices of the angles G, H, I, &c. of the given polygon to the centre; the lines will meet the circumference in the points A, B, C, &c. Join these points by the chords A B, B C, &c., which will form the inscribed polygon. Or simply join the points of contact, M, N, P, &c., by chords, M N, N P, &c., which likewise would form an inscribed polygon similar to the circumscribed one.

365. *Cor.* 2. Hence, we may circumscribe about a circle any regular polygon similar to an inscribed one, and conversely.

366. *Cor.* 3. It has been shown that N H and H M are equal; therefore the sum of N H and H M, which is equal to the sum of H M and M G, is equal to H G, one of the equal sides of the polygon.

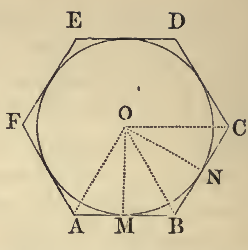
367. *Scholium.* From having a circumscribed regular polygon, another having double the number of sides may be readily constructed, by drawing tangents to the points of bisection of the arcs, intercepted by the sides of the proposed polygon, limiting these tangents by those sides. In like manner other circumscribed polygons may be formed; but it is plain that each of the polygons so formed will be less than the preceding polygon, being entirely comprehended in it.

PROPOSITION VIII. — THEOREM.

368. *The area of a regular polygon is equivalent to the product of its perimeter by half of the radius of the inscribed circle.*

Let A B C D E F be a regular polygon, and O the centre of the inscribed circle.

From O let the straight lines O A, O B, &c. be drawn to

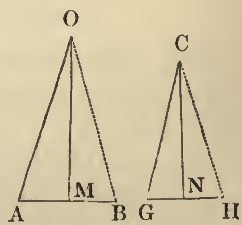


the vertices of all the angles of the polygon, and the polygon will be divided into as many equal triangles as it has sides; and let the radii O M, O N, &c. of the inscribed circle be drawn to the centres of the sides of the polygon, or to the points of tangency M, N, &c., and these radii are perpendicular to the sides respectively (Prop. XI. Bk. III.); therefore the radius of the circle is equal to the altitude of the several triangles.

Now, the triangle A O B is measured by the product of A B by half of O M (Prop. VI. Bk. IV.); the triangle O B C by the product of B C by half of O N. But O M is equal to O N; hence the two triangles taken together are measured by the sum of A B and B C by half of O M. In like manner the measure of the other triangles may be found; hence, the sum of all the triangles, or the whole polygon, is equal to the sum of the bases A B, B C, &c., or the perimeter of the polygon, multiplied by half of O M, or half the radius of the inscribed circle.

Proposition IX. — Theorem.

369. *The perimeters of two regular polygons, having the same number of sides, are to each other as the radii of the circumscribed circles, and, also, as the radii of the inscribed circles; and their areas are to each other as the squares of those radii.*



Let A B be a side of one polygon, O the centre, and consequently O A the radius of the circumscribed circle, and O M, perpendicular to A B, the radius of the inscribed circle. Let G H be a side of the other polygon, C the centre, C G and C N the radii of the circumscribed and the inscribed circles.

The perimeters of the two polygons are to each other as the sides A B and G H (Prop. XXXI. Bk. IV.); but the angles A and G are equal, being each half of the angle of the polygon; so also are the angles B and H; hence, drawing O B and C H, the isosceles triangles A B O, G H C are similar, as are likewise the right-angled triangles A M O, G N C; hence

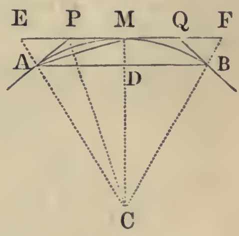
$$AB : GH :: AO : GC :: MO : NC.$$

Hence, the perimeters of the polygons are to each other as the radii A O, G C of the circumscribed circles, and, also, as the radii M O, N C of the inscribed circles.

The surfaces of these polygons are to each other as the squares of the homologous sides A B, G H (Prop. XXXI. Bk. IV.); they are therefore to each other as the squares of A O, G C, the radii of the circumscribed circles, or as the squares of O M, C N, the radii of the inscribed circles.

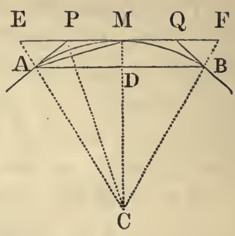
Proposition X. — Problem.

370. *The surface of a regular inscribed polygon, and that of a similar circumscribed polygon, being given; to find the surfaces of regular inscribed and circumscribed polygons having double the number of sides.*



Let A B be a side of the given inscribed polygon; E F, parallel to A B, a side of the circumscribed polygon, and C the centre of the circle. Draw the chord A M, and the tangents A P, B Q; then A M will be a side of the inscribed polygon, having twice the number of sides; and P Q, the double of P M, will be a side of the similar circumscribed polygon.

Let A, then, be the surface of the inscribed polygon whose side is A B, B that of the similar circumscribed polygon; A′ the surface of the polygon whose side is A M,



B′ that of the similar circumscribed polygon: A and B are given; we have to find A′ and B′.

First. The triangles A C D, A C M, whose common vertex is A, are to each other as their bases C D, C M (Prop. VI. Cor., Bk. IV.); they are likewise as the polygons A and A′; hence

$$A : A' :: CD : CM.$$

Again, the triangles C A M, C M E, whose common vertex is M, are to each other as their bases C A, C E; they are likewise to each other as the polygons A′ and B; hence

$$A' : B :: CA : CE.$$

But, since A D and M E are parallel, we have,

$$CD : CM :: CA : CE;$$

hence

$$A : A' :: A' : B;$$

hence, *the polygon A′ is a mean proportional between the two given polygons.*

Secondly. The altitude C M being common, the triangle C P M is to the triangle C P E as P M is to P E; but since C P bisects the angle M C E, we have (Prop. XIX. Bk. IV.),

$$PM : PE :: CM : CE :: CD : CA :: A : A';$$

hence

$$CPM : CPE :: A : A';$$

and, consequently,

$$CPM : CPM + CPE \text{ or } CME :: A : A + A'.$$

But C M P A or 2 C M P and C M E are to each other as the polygons B′ and B; hence

$$B' : B :: 2\,A : A + A';$$

which gives

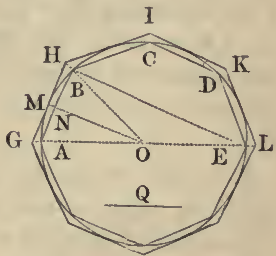
$$B' = \frac{2\,A \times B}{A + A'};$$

or, *the polygon* B′ *is equal to the quotient of twice the product of the given polygons divided by the sum of the inscribed polygons.*

Thus, by means of the polygons A and B, it is easy to find the polygons A′ and B′, which have double the number of sides.

PROPOSITION XI. — THEOREM.

371. *A circle being given, two similar polygons can always be formed, the one circumscribed about the circle, the other inscribed in it, which shall differ from each other by less than any assignable surface.*



Let Q be the side of a square less than the given surface.

Bisect A C, a fourth part of the circumference, and then bisect the half of this fourth, and so proceed until an arc is found whose chord A B is less than Q. As this arc must be an exact part of the circumference, if we apply the chords A B, B C, &c., each equal to A B, the last will terminate at A, and there will be inscribed in the circle a regular polygon, A B C D E, &c. Next describe about the circle a similar polygon, G H I K L, &c. (Prop. VII.); and the difference of these two polygons will be less than the square of Q.

Find the centre, O; from the points G and H draw the straight lines G O, H O, and they will pass through the points A and B (Prop. VII.). Draw also O M to the point of tangency, M; and it will bisect A B in N, and be perpendicular to it (Prop. VI. Cor. 1, Bk. III.). Produce A O to E, and draw B E.

Let P represent the circumscribed polygon, and p the inscribed polygon. Then, since these polygons are similar, they are as the squares of the homologous sides G H, A B (Prop. XXXI. Bk. IV.); but the triangles G O H, A O B are similar (Prop. XXIV. Bk. IV.); hence they are to each other as the squares of the homologous sides O G and O A (Prop. XXIX. Bk. IV.); therefore

$$P : p :: \overline{OG}^2 : \overline{OA}^2 \text{ or } \overline{OM}^2.$$

Again, the triangles O G M, E A B, having their sides respectively parallel, are similar; therefore

$$P : p :: \overline{OG}^2 : \overline{OM}^2 :: \overline{AE}^2 : \overline{BE}^2;$$

and, by division,

$$P : P - p :: \overline{AE}^2 : \overline{AE}^2 - \overline{EB}^2 \text{ or } \overline{AB}^2.$$

But P is less than the square described on the diameter A E; therefore $P - p$ is less than the square described on A B, that is, less than the given square Q. Hence, the difference between the circumscribed and inscribed polygons may always be made less than any given surface.

372. *Cor.* Since the circle is obviously greater than any inscribed polygon, and less than any circumscribed one, it follows that *a polygon may be inscribed or circumscribed, which will differ from the circle by less than any assignable magnitude.*

Proposition XII. — Problem.

373. *To find the approximate area of a circle whose radius is unity.*

Let the radius of the circle be 1, and let the first inscribed and circumscribed polygons be squares; the side of the inscribed square will be $\sqrt{2}$ (Prop. IV. Cor.), and that of the circumscribed square will be equal to the diameter 2. Hence the surface of the inscribed square is 2, and that of the circumscribed square is 4. Let, therefore $A = 2$, and $B = 4$. Now it has been proved, in Proposition X., that the surface of the inscribed octagon, or, as it has been represented, A', is a mean proportional between the two squares A and B, so that $A' = \sqrt{8} = 2.8284271$; and it has also been proved, in the same proposition, that the circumscribed octagon, represented by B', $= \frac{2 A \times B}{A + A'}$; so that $B' = \frac{16}{2 + \sqrt{8}} = 3.3137085$. The inscribed and the circumscribed octagons being thus determined, we can easily, by means of them, determine the polygons having twice the number of sides. We have only in this case to put $A = 2.8284271$, $B = 3.3137085$; and we shall find $A' = \sqrt{A \times B} = 3.0614674$, and $B' = \frac{2 A \times B}{A + A'} = 3.1825979$.

In like manner may be determined the area of polygons of sixteen sides, and thence the area of polygons of thirty-two sides, and so on till we arrive at an inscribed and a circumscribed polygon differing so little from each other, and consequently from the circle, that the difference shall be less than any assignable magnitude (Prop. XI. Cor.).

The subjoined table exhibits the area, or numerical expression for the surface, of these polygons, carried on till they agree as far as the seventh place of decimals.

Number of sides.	Inscribed Polygons.	Circumscribed Polygons.
4	2.0000000	4.0000000
8	2.8284271	3.3137085
16	3.0614674	3.1825979
32	3.1214451	3.1517249
64	3.1365485	3.1441148
128	3.1403311	3.1422236
256	3.1412772	3.1417504
512	3.1415138	3.1416321
1024	3.1415729	3.1416025
2048	3.1415877	3.1415951
4096	3.1415914	3.1415933
8192	3.1415923	3.1415928
16384	3.1415925	3.1415927
32768	3.1415926	3.1415926

It appears, therefore, that the inscribed and circumscribed polygons of 32768 sides differ so little from each other that the numerical value of each, as far as seven places of decimals, is absolutely the same; as the circle is between the two, it cannot, strictly speaking, differ from either so much as they do from each other; so that the number 3.1415926 expresses the area of a circle whose radius is 1, correctly, as far as seven places of decimals.

Some doubt may exist, perhaps, about the last decimal figure, owing to errors proceeding from the parts omitted; but the calculation has been carried on with an additional figure, that the final result here given might be absolutely correct even to the last decimal place.

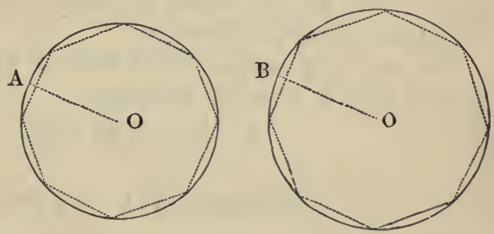
374. *Cor.* Since the inscribed and circumscribed polygons are regular, and have the same number of sides, they are similar (Prop. I.); therefore, by increasing the number of the sides, the corresponding polygons formed will approach to an equality with the circle. Now if, by continual bisections, the polygons formed shall have their number of sides indefinitely great, each side will become indefinitely small, and the inscribed and circumscribed polygons will ultimately coincide with each other. But when they coincide with each other, they must each coincide with the circle, since no part of an inscribed polygon can be without the circle, nor can any part of a circumscribed one be within it; hence, *the perimeters of the polygons must coincide with the circumference of the circle, and be equal to it.*

375. *Scholium.* Every circle, therefore, may be regarded as a polygon of an infinite number of sides.

NOTE. — This new definition of the circle, if it does not appear at first view to be very strict, has at least the advantage of introducing more simplicity and precision into demonstrations. (*Cours de Géométrie Élémentaire*, par Vincent et Bourdon.)

Proposition XIII. — Theorem.

376. *The circumferences of circles are to each other as their radii, and their areas are to each other as the squares of their radii.*



Let C denote the circumference of one of the circles, R its radius O A, A its area; and let C′ denote the circumference of the other circle, r its radius O B, A′ its area; then will

$$C : C' : : R : r,$$

and

$$A : A' : : R^2 : r^2.$$

Inscribe within the given circles two regular polygons of the same number of sides; and, whatever be the number of sides, the perimeters of the polygons will be to each other as the radii O A and O B (Prop. IX.). Now, conceive the arcs subtending the sides of the polygon to be continually bisected, forming other inscribed polygons, until polygons are formed of an indefinite number of sides, and therefore having perimeters coinciding with the circumference of the circumscribed circles (Prop. XII. Cor.); and we shall have

$$C : C' : : R : r.$$

Again, the areas of the inscribed polygons are to each other as $\overline{OA}^2$ to $\overline{OB}^2$ (Prop. IX.). But when the number of sides of the polygons is indefinitely increased, the areas of the polygons become equal to the areas of the circles; hence we shall have

$$A : A' : : R^2 : r^2.$$

377. *Cor.* 1. The circumferences of circles are to each other as twice their radii, or as their diameters.

For, multiplying the terms of the second ratio in the first proportion by 2, we have

$$C : C' : : 2\,R : 2\,r.$$

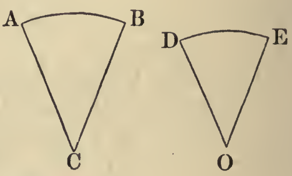
378. *Cor.* 2. The areas of circles are to each other as the squares of their diameters.

For, multiplying the second ratio of the second proportion by 4, or 2 squared, we have

$$A : A' : : 4\,R^2 : 4\,r^2.$$

Proposition XIV. — Theorem.

379. *Similar arcs are to each other as their radii; and similar sectors are to each other as the squares of their radii.*



Let A B, D E be similar arcs; A C B, D O E, similar sectors; and denote the radii C A and O D by R and r; then will

$$A\,B : D\,E : : R : r,$$

and

$$A\,C\,B : D\,O\,E : : R^2 : r^2.$$

For, since the arcs are similar, the angle C is equal to the angle O (Art. 213). But the angle C is to four right angles as the arc A B is to the whole circumference described with the radius C A (Prop. XVII. Sch. 2, Bk. III.); and the angle O is to four right angles as the arc D E is to the circumference described with the radius O D. Hence, the arcs A B, D E are to each other as the circumferences of which they form a part. But these circumferences are to each other as their radii, C A, O D (Prop. XIII.); therefore

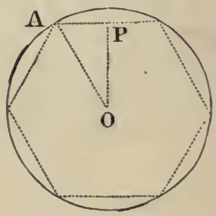
$$Arc\ A\,B : Arc\ D\,E : : R : r.$$

By like reasoning, the sectors A C B, D O E are to each other as the whole circles of which they are a part; and these are as the squares of their radii (Prop. XIII.); therefore

$$\textit{Sector}\ ACB : \textit{Sector}\ DOE :: R^2 : r^2.$$

Proposition XV. — Theorem.

380. *The area of a circle is equal to the product of the circumference by half the radius.*

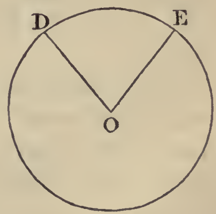


Let C denote the circumference of the circle, whose centre is O, R its radius O A, and A its area; then will

$$A = C \times \tfrac{1}{2} R.$$

For, inscribe in the circle any regular polygon, and from the centre draw O P perpendicular to one of the sides. The area of the polygon, whatever be the number of sides, will be equal to its perimeter multiplied by half of O P (Prop. VIII.). Conceive the arcs subtending the sides of the polygon to be continually bisected, until a polygon is formed having an indefinite number of sides; its perimeter will be equal to the circumference of the circle (Prop. XII. Cor.), and O P be equal to the radius O A; therefore the area of the polygon is equal to that of the circle; hence

$$A = C \times \tfrac{1}{2} R.$$



381. *Cor.* 1. The area of a sector is equal to the product of its arc by half of its radius.

For, let C denote the circumference of the circle of which the sector D O E is a part, R its radius O D, and A its area; then we shall have (Prop. XVII. Sch. 2, Bk. III.),

$$\textit{Sector}\ DOE : A :: \textit{Arc}\ DE : C;$$

hence, since equimultiples of two magnitudes have the same ratio as the magnitudes themselves (Prop. IX. Bk. II.),

$$\textit{Sector}\ \mathrm{D\,O\,E} : \mathrm{A} :: \textit{Arc}\ \mathrm{D\,E} \times \tfrac{1}{2}\,\mathrm{R} : \mathrm{C} \times \tfrac{1}{2}\,\mathrm{R}.$$

But A, or the area of the whole circle, is equal to $\mathrm{C} \times \frac{1}{2}\,\mathrm{R}$; hence, the area of the sector D O E is equal to the arc $\mathrm{D\,E} \times \frac{1}{2}\,\mathrm{R}$.

382. *Cor.* 2. Let the circumference of the circle whose diameter is unity be denoted by π (which is called *pi*), the radius by R, and the diameter by D; and the circumference of any other circle by C, and its area by A. Then, since circumferences are to each other as their diameters (Prop. XIII. Cor. 1), we shall have,

$$\mathrm{C} : \mathrm{D} :: \pi : 1;$$

therefore

$$\mathrm{C} = \mathrm{D} \times \pi = 2\,\mathrm{R} \times \pi.$$

Multiplying both numbers of this equation by $\frac{1}{2}\,\mathrm{R}$, we have

$$\mathrm{C} \times \tfrac{1}{2}\,\mathrm{R} = \mathrm{R}^2 \times \pi, \quad \text{or} \quad \mathrm{A} = \mathrm{R}^2 \times \pi;$$

that is, *the area of a circle is equal to the product of the square of its radius by the constant number* π.

383. *Cor.* 3. The circumference of every circle is equal to the product of its diameter, or twice its radius, by the constant number π.

384. *Cor.* 4. The constant number π denotes the ratio of the circumference of any circle to its diameter; for

$$\frac{\mathrm{C}}{\mathrm{D}} = \pi.$$

385. *Scholium* 1. The exact numerical value of the ratio denoted by π can be only approximately expressed. The approximate value found by Proposition XII. is 3.1415926; but, for most practical purposes, it is sufficiently accurate to take $\pi = 3.1416$. The symbol π is the first letter of the Greek word *περίμετρον*, *perimetron*, which signifies *circumference*.

386. *Scholium* 2. The QUADRATURE OF THE CIRCLE is the problem which requires the finding of a square equivalent in area to a circle having a given radius. Now, it has just been proved that a circle is equivalent to the rectangle contained by its circumference and half its radius; and this rectangle may be changed into a square, by finding a mean proportional between its length and its breadth (Prob. XXVI. Bk. V.). To square the circle, therefore, is to find the circumference when the radius is given; and for effecting this, it is enough to know the ratio of the circumference to its radius, or its diameter.

But this ratio has never been determined except approximately; but the approximation has been carried so far, that a knowledge of the exact ratio would afford no real advantage whatever beyond that of the approximate ratio. Professor Rutherford extended the approximation to 208 places of decimals, and Dr. Clausen to 250 places. The value of π, as developed to 208 places of decimals, is
3.14159265358979323846264338327950288419716939937
51058209749445923078164062862089986280348253427170
6798214808651328230664709384460955058223172535940
81284847378139203863383021574739960082593125912940
183280651744.

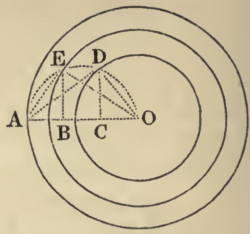
Such an approximation is evidently equivalent to perfect correctness; the root of an imperfect power is in no case more accurately known.

PROPOSITION XVI. — PROBLEM.

387. *To divide a circle into any number of equal parts by means of concentric circles.*

Let it be proposed to divide the circle, whose centre is O, into a certain number of equal parts, — three for instance, — by means of concentric circles.

Draw the radius A O; divide A O into three equal parts, A B, B C, C O. Upon A O describe a semi-circumference,



and draw the perpendiculars, B E, C D, meeting that semi-circumference in the points E, D. Join O E, O D, and with these lines as radii from the centre, O, describe circles; these circles will divide the given circle into the required number of equal parts.

For join A E, A D; then the angle A D O, being in a semicircle, is a right angle (Prop. XVIII. Cor. 2, Bk. III.); hence the triangles D A O, D C O are similar, and consequently are to each other as the squares of their homologous sides; that is,

$$DAO : DCO :: \overline{OA}^2 : \overline{OD}^2;$$

but

$$DAO : DCO :: OA : OC;$$

hence

$$\overline{OA}^2 : \overline{OD}^2 :: OA : OC;$$

consequently, since circles are to each other as the squares of their radii (Prop. XIII.), it follows that the circle whose radius is O A, is to that whose radius is O D, as O A to O C; that is to say, the latter is one third of the former.

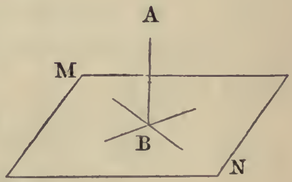
In the same manner, by means of the right-angled triangles E A O, E B O, it may be proved that the circle whose radius is O E, is two thirds that whose radius is O A. Hence, the smaller circle and the two surrounding *annular* spaces are all equal.

NOTE. — This useful problem was first solved by Dr. Hutton, the justly distinguished English mathematician.

BOOK VII.

PLANES. — DIEDRAL AND POLYEDRAL ANGLES.

DEFINITIONS.



388. A STRAIGHT line is *perpendicular to a plane*, when it is perpendicular to every straight line which it meets in that plane.

Conversely, the plane, in the same case, is *perpendicular to the line.*

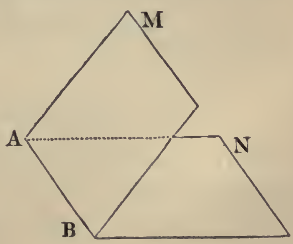
The *foot* of the perpendicular is the point in which it meets the plane.

Thus the straight line A B is perpendicular to the plane M N; the plane M N is perpendicular to the straight line A B; and B is the foot of the perpendicular A B.

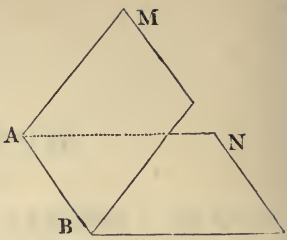
389. A line is *parallel to a plane* when it cannot meet the plane, however far both of them may be produced.

Conversely, the plane, in the same case, is parallel to the line.

390. Two *planes are parallel to each other*, when they cannot meet, however far both of them may be produced.



391. A DIEDRAL ANGLE is an angle formed by the intersection of two planes, and is measured by the inclination of two straight lines drawn from any point in the line of intersection, perpendicular to that line, one being drawn in each plane.

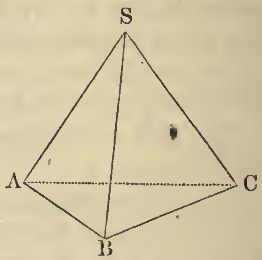


The line of common section is called the *edge*, and the two planes are called the faces, of the diedral angle.

Thus the two planes A B M, A B N, whose line of intersection is A B, form a diedral angle, of which the line A B is the edge, and the planes A B M, A B N are the faces.

392. A diedral angle may be acute, right, or obtuse.

If the two faces are perpendicular to each other, the angle is right.



393. A POLYEDRAL ANGLE is an angle formed by the meeting at one point of more than two plane angles, which are not in the same plane.

The common point of meeting of the planes is called the *vertex*, each of the plane angles a *face*, and the line of common section of any two of the planes an *edge* of the polyedral angle.

Thus the three plane angles A S B, B S C, C S A form a polyedral angle, whose vertex is S, whose faces are the plane angles, and whose edges are the sides, A S, B S, C S, of the same angles.

394. A polyedral angle formed by three faces is called a *triedral* angle; by four faces, a *tetraedral;* by five faces, a *pentaedral*, &c.

PROPOSITION I. — THEOREM.

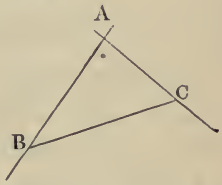
395. *A straight line cannot be partly in a plane, and partly out of it.*

For, by the definition of a plane (Art. 10), a straight line which has two points in common with a plane lies wholly in that plane.

396. *Scholium.* To determine whether a surface is a plane, apply a straight line in different directions to that surface, and ascertain whether the line throughout its whole extent touches the surface.

Proposition II. — Theorem.

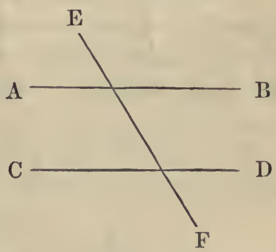
397. *Two straight lines which intersect each other lie in the same plane and determine its position.*



Let A B, A C be two straight lines which intersect each other in A; then these lines will be in the same plane.

Conceive a plane to pass through A B, and to be turned about A B, until it pass through the point C; then, the two points A and C being in this plane, the line A C lies wholly in it (Art. 10). Hence, the position of the plane is determined by the condition of its containing the two straight lines A B, A C.

398. *Cor.* 1. A triangle, A B C, or three points, A, B, C, not in a straight line, determine the position of a plane.

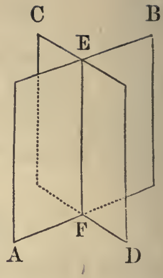


399. *Cor.* 2. Hence, also, two parallels, A B, C D, determine the position of a plane; for, drawing the secant E F, the plane of the two straight lines A B, E F is that of the parallels A B, C D.

Proposition III. — Theorem.

400. *If two planes cut each other, their common section is a straight line.*

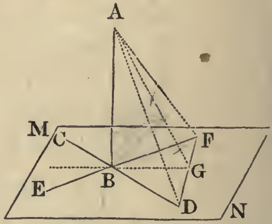
Let the two planes A B, C D cut each other, and let E, F be two points in their common section. Draw the straight line E F. Now, since the points E and F are in the plane A B, and also in the plane C D, the straight line E F, joining E and F, must be wholly in each plane, or is common to both of them. Therefore, the common section of the two planes A B, C D is a straight line.



PROPOSITION IV. — THEOREM.

401. *If a straight line is perpendicular to each of two straight lines, at their point of intersection, it is perpendicular to the plane in which the two lines lie.*

Let the straight line A B be perpendicular to each of the straight lines C D, E F, at B, the point of their intersection, and M N the plane in which the lines C D, E F lie; then will A B be perpendicular to the plane M N.



Through the point B draw any straight line, B G, in the plane M N; and through any point G draw D G F, meeting the lines C D, E F in such a manner that D G shall be equal to G F (Prob. XXVIII. Bk. V.). Join A D, A G, A F.

The line D F being divided into two equal parts at the point G, the triangle D B F gives (Prop. XIV. Bk. IV.)

$$\overline{BF}^2 + \overline{BD}^2 = 2\,\overline{BG}^2 + 2\,\overline{GF}^2.$$

The triangle D A F, in like manner, gives

$$\overline{AF}^2 + \overline{AD}^2 = 2\,\overline{AG}^2 + 2\,\overline{GF}^2.$$

Subtracting the first equation from the second, and observing that the triangles A B F, A B D, each being right-angled at B, give

$$\overline{AF}^2 - \overline{BF}^2 = \overline{AB}^2, \quad \text{and} \quad \overline{AD}^2 - \overline{BD}^2 = \overline{AB}^2,$$

we shall have

$$\overline{AB}^2 + \overline{AB}^2 = 2\ \overline{AG}^2 - 2\ \overline{BG}^2.$$

Therefore, by taking the halves of both members, we have

$$\overline{AB}^2 = \overline{AG}^2 - \overline{BG}^2, \quad \text{or} \quad \overline{AG}^2 = \overline{AB}^2 + \overline{BG}^2;$$

hence, the triangle A B G is right-angled at B, and the side A B is perpendicular to B G.

In the same manner, it may be shown that A B is perpendicular to any other straight line in the plane M N, which it may meet at B; therefore A B is perpendicular to the plane M N (Art. 388).

402. *Scholium.* Thus it is evident, not only that a straight line may be perpendicular to all the straight lines which pass through its foot, in a plane, but it always must be so whenever it is perpendicular to two straight lines drawn in the plane; which shows the accuracy of the first definition (Art. 388).

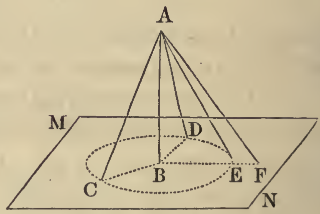
403. *Cor.* 1. The perpendicular A B is shorter than any oblique line A G; therefore it measures the shortest distance from the point A to the plane M N.

404. *Cor.* 2. From any given point, B, in a plane, only one perpendicular to that plane can be drawn. For if there could be two, conceive a plane to pass through them, intersecting the plane M N in B G; the two perpendiculars would then be perpendicular to the straight line B G at the same point, and in the same plane, which is impossible (Prop. XIII. Cor., Bk. I.).

It is also impossible to let fall from a given point out of a plane two perpendiculars to that plane. For, suppose A B, A G to be two such perpendiculars, then the triangle A B G will have two right angles, A B G, A G B, which is impossible (Prop. XXVIII. Cor. 3, Bk. I.).

Proposition V. — Theorem.

405. *Oblique lines drawn from a point to a plane at equal distances from a perpendicular drawn from the same point to it, are equal; and of two oblique lines unequally distant from the perpendicular, the more remote is the longer.*



Let A B be perpendicular to the plane M N; and A C, A D, A E be oblique lines, from the point A, meeting the plane at equal distances, B C, B D, B E, from the perpendicular; and A F a line meeting the plane more remote from the perpendicular; then will A C, A D, A E be equal to each other, and A F be longer than A C.

For, the angles A B C, A B D, A B E being right angles, and the distances B C, B D, B E being equal to each other, the triangles A B C, A B D, A B E have in each an equal angle contained by equal sides; consequently they are equal (Prop. V. Bk. I.); therefore, the hypothenuses, or the oblique lines A C, A D, A E, are equal to each other.

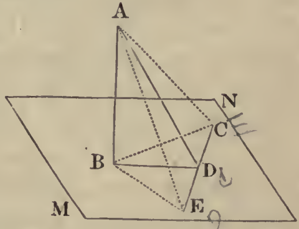
In like manner, since the distance B F is greater than B C, or its equal B E, the oblique line A F must be greater than A E, or its equal A C (Prop. XIV. Bk. I.).

406. *Cor.* All the equal oblique lines A C, A D, A E, &c. terminate in the circumference of a circle, C D E, described from B, the foot of the perpendicular, as a centre; therefore, a point, A, being given out of a plane, the point B, at which the perpendicular let fall from it would meet that plane, may be found by taking upon the plane three points, C, D, E, equally distant from the point A, and then finding the centre of the circle which passes through these points; this centre will be the point B required.

407. *Scholium.* The angle A C B is called *the inclination of the oblique line* A C *to the plane* M N; which inclination is evidently equal with respect to all such lines, A C, A D, A E, as are equally distant from the perpendicular; for all the triangles A C B, A D B, A E B, &c. are equal to each other.

Proposition VI. — Theorem.

408. *If from the foot of a perpendicular a straight line be drawn at right angles to any straight line of the plane, and a straight line be drawn from the point of intersection to any point of the perpendicular, this last line will be perpendicular to the line of the plane.*



Let A B be perpendicular to the plane M N, and B D a straight line drawn through B, cutting at right angles the straight line C E in the plane; draw the straight line A D from the point of intersection, D, to any point, A, in the perpendicular A B; and A D will be perpendicular to C E.

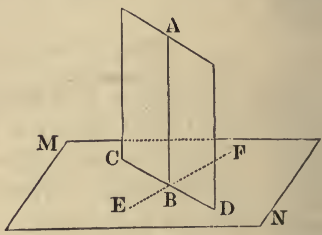
For, take D E equal to D C, and join B E, B C, A E, A C. Since D E is equal to D C, the two right-angled triangles B D E, B D C are equal, and the oblique line B E is equal to B C (Prop. V. Bk. I.); and since B E is equal to B C, the oblique line A E is equal to A C (Prop. V. Bk. I.); therefore the line A D has two of its points, A and D, equally distant from the extremities E and C; hence, A D is a perpendicular to E C, at its middle point, D (Prop. XV. Cor., Bk. I.).

409. *Cor.* It is also evident that C E is perpendicular to the plane of the triangle A B D, since C E is perpendicular at the same time to the two straight lines A D and B D (Prop. IV.).

Proposition VII. — Theorem.

410. *If a straight line is perpendicular to a plane, every plane which passes through that line is also perpendicular to the plane.*

Let A B be a straight line perpendicular to the plane M N; then will any plane, A C, passing through A B, be perpendicular to M N.



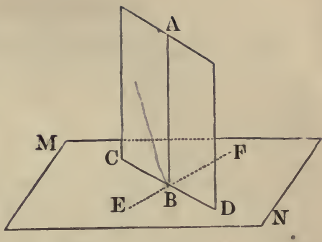
For, let C D be the intersection of the planes A C, M N; in the plane M N draw E F, through the point B, perpendicular to C D; then the line A B, being perpendicular to the plane M N, is perpendicular to each of the two straight lines C D, E F (Art. 388). But the angle A B E, formed by the two perpendiculars A B, E F to their common section, C D, measures the angle of the two planes A C, M N (Art. 391); hence, since that angle is a right angle, the two planes are perpendicular to each other.

411. *Cor.* When three straight lines, as A B, C D, E F, are perpendicular to each other, each of those lines is perpendicular to the plane of the other two, and the three planes are perpendicular to each other.

Proposition VIII. — Theorem.

412. *If two planes are perpendicular to each other, a straight line drawn in one of them, perpendicular to their common section, will be perpendicular to the other plane.*

Let A C, M N be two planes perpendicular to each other, and let the straight line A B be drawn in the plane A C perpendicular to the common section C D; then will A B be perpendicular to the plane M N.

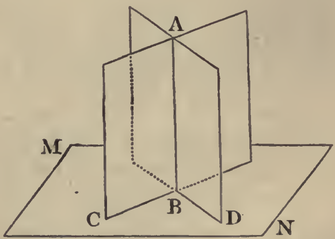


For, in the plane M N, draw E F, through the point B, perpendicular to C D; then, since the planes A C, M N are perpendicular, the angle A B E is a right angle (Art. 391); therefore the line A B is perpendicular to the two straight lines C D, E F, at the point of their intersection; hence it is perpendicular to their plane, M N (Prop. IV.).

413. *Cor.* If the plane A C is perpendicular to the plane M N, and if at a point B of the common section we erect a perpendicular to the plane M N, that perpendicular will be in the plane A C. For, if not, there may be drawn in the plane A C a line, A B, perpendicular to the common section C D, which would be at the same time perpendicular to the plane M N. Hence, at the same point B there would be two perpendiculars to the plane M N, which is impossible (Prop. IV. Cor. 2).

Proposition IX. — Theorem.

414. *If two planes which cut each other are perpendicular to a third plane, their common section is perpendicular to the same plane.*

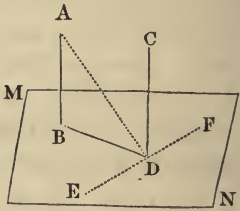


Let the two planes C A, D A, which cut each other in the straight line A B, be each perpendicular to the plane M N; then will their common section A B be perpendicular to M N.

For, at the point B, erect a perpendicular to the plane M N; that perpendicular must be at once in the plane C A and in the plane D A (Prop. VIII. Cor.); hence, it is their common section, A B.

Proposition X.—Theorem.

415. *If one of two parallel straight lines is perpendicular to a plane, the other is also perpendicular to the same plane.*



Let AB, CD be two parallel straight lines, of which AB is perpendicular to the plane MN; then will CD also be perpendicular to it.

For, pass a plane through the parallels AB, CD, cutting the plane MN in the straight line BD. In the plane MN draw the straight line EF, at right angles with BD; and join AD.

Now, EF is perpendicular to the plane ABDC (Prop. VI. Cor.); therefore the angle CDE is a right angle; but the angle CDB is also a right angle, since AB is perpendicular to BD, and CD parallel to AB (Prop. XXII. Cor., Bk. I.); therefore the line CD is perpendicular to the two straight lines EF, BD; hence it is perpendicular to their plane, MN (Prop. IV.).

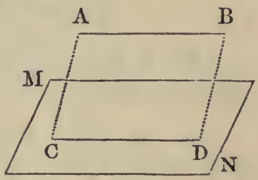
416. *Cor.* 1. Conversely, if the straight lines AB, CD are perpendicular to the same plane, MN, they must be parallel. For, if they be not so, draw, through the point D, a line parallel to AB; this parallel will be perpendicular to the plane MN; hence, through the same point D more than one perpendicular may be erected to the same plane, which is impossible (Prop. IV. Cor. 2).

417. *Cor.* 2. Two lines, A and B, parallel to a third, C, are parallel to each other; for, conceive a plane perpendicular to the line C; the lines A and B, being parallel to C, will be perpendicular to the same plane; hence, by the preceding corollary, they will be parallel to each other.

The three lines are supposed to be not in the same plane; otherwise the proposition would be already demonstrated (Prop. XXIV. Bk. I.).

Proposition XI. — Theorem.

418. *If a straight line without a plane is parallel to a line within the plane, it is parallel to the plane.*

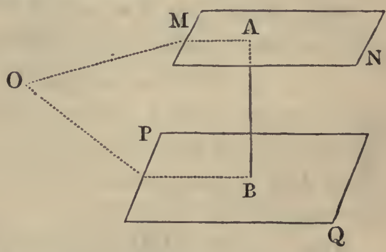


Let the straight line A B, without the plane M N, be parallel to the line C D in that plane; then will A B be parallel to the plane M N.

Conceive a plane A B C D to pass through the parallels A B, C D. Now, if the line A B, which lies in the plane A B C D, could meet the plane M N, it could only be in some point of the line C D, the common section of the two planes; but the line A B cannot meet C D, since they are parallel (Art. 17); therefore it will not meet the plane M N; hence it is parallel to that plane (Art. 389).

Proposition XII. — Theorem.

419. *If two planes are perpendicular to the same straight line, they are parallel to each other.*



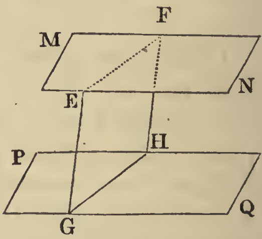
Let the planes M N, P Q, be each perpendicular to the straight line A B; then will they be parallel to each other.

For, if they can meet, on being produced, let O be one of their common points; and join O A, O B. The line A B, which is perpendicular to the plane M N, is perpendicular to the straight line O A, drawn through its foot in that plane (Art. 388). For the same reason, A B is perpendicular to B O. Therefore O A and O B are two perpendiculars let fall from the same point, O, upon the same straight line, A B, which is impossible (Prop. XIII. Bk. I.).

Therefore, the planes M N, P Q cannot meet on being produced; hence they are parallel to each other.

Proposition XIII. — Theorem.

420. *If two parallel planes are cut by a third plane, the two intersections are parallel.*

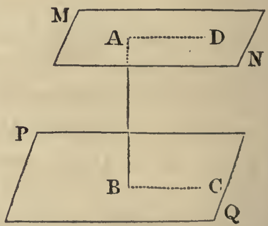


Let the two parallel planes MN and P Q be cut by the plane E F G H, and let their intersections with it be E F, G H; then E F is parallel to G H.

For, if the lines E F, G H, lying in the same plane, were not parallel, they would meet each other on being produced; therefore the planes M N, P Q, in which those lines are situated, would also meet, which is impossible, since these planes are parallel.

Proposition XIV. — Theorem.

421. *A straight line which is perpendicular to one of two parallel planes, is also perpendicular to the other plane.*



Let M N, P Q be two parallel planes, and A B a straight line perpendicular to the plane M N; then A B is also perpendicular to the plane P Q.

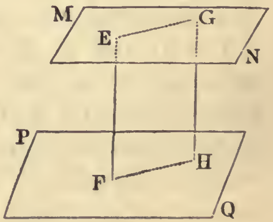
Draw any line, B C, in the plane P Q; and through the lines A B, B C, conceive a plane, A B C, to pass, intersecting the plane M N in A D; the intersection A D will be parallel to B C (Prop. XIII.). But the line A B, being perpendicular to the plane M N, is perpendicular to the straight line A D; consequently it will be perpendicular to its parallel B C (Prop. XXII. Cor., Bk. I.).

Hence the line A B, being perpendicular to any line, B C, drawn through its foot in the plane P Q, is consequently perpendicular to the plane P Q (Art. 388).

PROPOSITION XV. — THEOREM.

422. *Parallel straight lines included between two parallel planes are equal.*

Let E F, G H be two parallel straight planes, included between two parallel planes, M N, P Q; then E F and G H are equal.



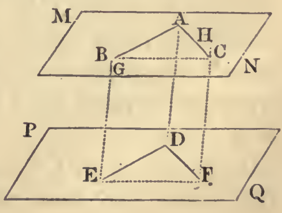
For, through the parallels E F, G H conceive the plane E F G H to pass, intersecting the parallel planes in E G, F H. The intersections E G, F H are parallel to each other (Prop. XIII.); and E F, G H are also parallel; consequently the figure E F G H is a parallelogram; hence E F is equal to G H (Prop. XXXI. Bk. I.).

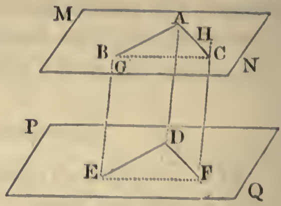
423. *Cor. Two parallel planes are everywhere equidistant.* For, if E F, G H are perpendicular to the two planes M N, P Q, they will be parallel to each other (Prop. X. Cor. 1); and consequently equal.

PROPOSITION XVI. — THEOREM.

424. *If two angles not in the same plane have their sides parallel and lying in the same direction, these angles will be equal, and their planes will be parallel.*

Let B A C, E D F be two triangles, lying in different planes, M N and P Q, having their sides parallel and lying in the same direction; then the angles B A C, E D F will be equal, and their planes, M N, P Q, be parallel.





For, take A B equal to E D, and A C equal to D F; and join B C, E F, B E, A D, C F. Since A B is equal and parallel to E D, the figure A B E D is a parallelogram (Prop. XXXIII. Bk. I.); therefore A D is equal and parallel to B E. For a similar reason, C F is equal and parallel to A D; hence, also, B E is equal and parallel to C F; hence the figure B C F E is a parallelogram, and the side B C is equal and parallel to E F; therefore the triangles B A C, E D F have their sides equal, each to each; hence the angle B A C is equal to the angle E D F.

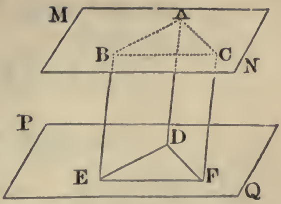
Again, the plane B A C is parallel to the plane E D F. For, if not, suppose a plane to pass through the point A, parallel to E D F, meeting the lines B E, C F, in points different from B and C, for instance G and H. Then the three lines G E, A D, H F will be equal (Prop. XV.). But the three lines B E, A D, C F are already known to be equal; hence B E is equal to G E, and H F is equal to C F, which is absurd; hence the plane B A C is parallel to the plane E D F.

425. *Cor.* If two parallel planes M N, P Q, are met by two other planes, A B E D, A C F D, the angles B A C, E D F, formed by the intersections of the parallel planes, are equal; for the intersection A B is parallel to E D, and A C to D F (Prop. XIII.); therefore the angle B A C is equal to the angle E D F.

Proposition XVII. — Theorem.

426. *If three straight lines not in the same plane are equal and parallel, the triangles formed by joining the extremities of these lines will be equal, and their planes will be parallel.*

Let B E, A D, C F be three equal and parallel straight lines, not in the same plane, and let B A C, E D F be two

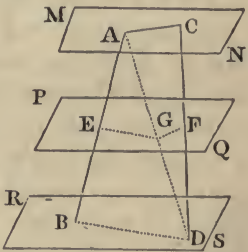


triangles formed by joining the extremities of these lines; then will these triangles be equal, and their planes parallel.

For, since B E is equal and parallel to A D, the figure A B E D is a parallelogram; hence, the side A B is equal and parallel to D E (Prop. XXXIII. Bk. I.). For a like reason, the sides B C, E F are equal and parallel; so also are A C, D F; hence, the two triangles B A C, E D F, having their sides equal, are themselves equal (Prop. XVIII. Bk. I.); consequently, as shown in the last proposition, their planes are parallel.

Proposition XVIII. — Theorem.

427. *If two straight lines are cut by three parallel planes, they will be divided proportionally.*



Let the straight line A B meet the parallel planes, M N, P Q, R S, at the points A, E, B; and the straight line C D meet the same planes at the points C, F, D; then will

A E : E B : : C F : F D.

Draw the line A D, meeting the plane P Q in G, and draw A C, E G, B D. Then the two parallel planes P Q, R S, being cut by the plane A B D, the intersections E G, B D are parallel (Prop. XIII.); and, in the triangle A B D, we have (Prop. XVII. Bk. IV.),

A E : E B : : A G : G D.

In like manner, the intersections A C, G F being parallel, in the triangle A D C, we have

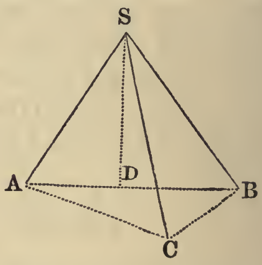
A G : G D : : C F : F D;

hence, since the ratio A G : G D is common to both proportions, we have

A E : E B : : C F : F D.

Proposition XIX. — Theorem.

428. *The sum of any two of the plane angles which form a triedral angle is greater than the third.*



The proposition requires demonstration only when the plane angle, which is compared to the sum of the other two, is greater than either of them.

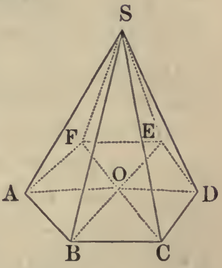
Let the triedral angle whose vertex is S be formed by the three plane angles A S B, A S C, B S C; and suppose the angle A S B to be greater than either of the other two; then the angle A S B is less than the sum of the angles A S C, B S C.

In the plane A S B make the angle B S D equal to B S C; draw the straight line A D B at pleasure; make S C equal S D, and draw A C, B C.

The two sides B S, S D are equal to the two sides B S, S C, and the angle B S D is equal to the angle B S C; therefore the triangles B S D, B S C are equal (Prop. V. Bk. I.); hence the side B D is equal to the side B C. But A B is less than the sum of A C and B C; taking B D from the one side, and from the other its equal, B C, there remains A D less than A C. The two sides A S, S D of the triangle A S D, are equal to the two sides A S, S C, of the triangle A S C, and the third side A D is less than the third side A C; hence the angle A S D is less than the angle A S C (Prop. XVII. Bk. I.). Adding B S D to one, and its equal, B S C, to the other, we shall have the sum of A S D, B S D, or A S B, less than the sum of A S C, B S C.

Proposition XX. — Theorem.

429. *The sum of the plane angles which form any polyedral angle is less than four right angles.*



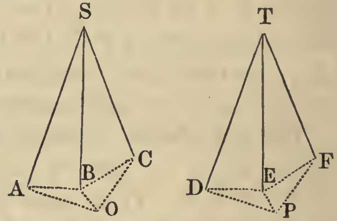
Let the polyedral angles whose vertex is S be formed by any number of plane angles, A S B, B S C, C S D, &c.; the sum of all these plane angles is less than four right angles.

Let the planes forming the polyedral angle be cut by any plane, A B C D E F. From any point, O, in this plane, draw the straight lines A O, B O, C O, D O, E O, F O. The sum of the angles of the triangles A S B, B S C, &c. formed about the vertex S, is equal to the sum of the angles of an equal number of triangles A O B, B O C, &c. formed about the point O. But at the point B the sum of the angles A B O, O B C, equal to A B C, is less than the sum of the angles A B S, S B C (Prop. XIX.); in the same manner, at the point C we have the sum of B C O, O C D less than the sum of B C S, S C D; and so with all the angles at the points D, E, &c. Hence, the sum of all the angles at the bases of the triangles whose vertex is O, is less than the sum of all the angles at the bases of the triangles whose vertex is S; therefore, to make up the deficiency, the sum of the angles formed about the point O is greater than the sum of the angles formed about the point S. But the sum of the angles about the point O is equal to four right angles (Prop. IV. Cor. 2, Bk. I.); therefore the sum of the angles about S must be less than four right angles.

430. *Scholium.* This demonstration supposes that the polyedral angle is convex; that is, that no one of the faces would, on being produced, cut the polyedral angle; if it were otherwise, the sum of the plane angles would no longer be limited, and might be of any magnitude.

Proposition XXI. — Theorem.

431. *If two triedral angles are formed by plane angles which are equal each to each, the planes of the equal angles will be equally inclined to each other.*



Let the two triedral angles whose vertexes are S and T, have the angle ASC equal to DTF, the angle ASB equal to DTE, and the angle BSC equal to ETF; then will the inclination of the planes ASC, ASB be equal to that of the planes DTF, DTE.

For, take SB at pleasure; draw BO perpendicular to the plane ASC; from the point O, at which the perpendicular meets the plane, draw OA, OC, perpendicular to SA, SC; and join AB, BC. Next, take TE equal SB; draw EP perpendicular to the plane DTE; from the point P draw PD, PF, perpendicular respectively to TD, TF; and join DE, EF.

The triangle SAB is right-angled at A, and the triangle TDE at D; and since the angle ASB is equal to DTE, we have SBA equal to TED. Also, SB is equal to TE; therefore the triangle SAB is equal to TDE; hence SA is equal to TD, and AB is equal to DE.

In like manner it may be shown that SC is equal to TF, and BC is equal to EF. We can now show that the quadrilateral ASCO is equal to the quadrilateral DTFP; for, place the angle ASC upon its equal DTF; since SA is equal to TD, and SC is equal to TF, the point A will fall on D, and the point C on F; and, at the same time, AO, which is perpendicular to SA, will fall on DP, which is perpendicular to TD, and, in like manner, CO on FP; wherefore the point O will fall on the point P, and AO will be equal to DP.

But the triangles A O B, D P E are right-angled at O and P; the hypotenuse A B is equal to D E, and the side A O is equal to D P; hence the two triangles are equal (Prop. XIX. Bk. I.); and, consequently, the angle O A B is equal to the angle P D E. The angle O A B is the inclination of the two planes A S B, A S C; and the angle P D E is that of the two planes D T E, D T F; hence, those two inclinations are equal to each other.

432. *Scholium* 1. It must, however, be observed, that the angle A of the right-angled triangle O A B is properly the inclination of the two planes A S B, A S C only when the perpendicular B O falls on the same side of S A with S C; for if it fell on the other side, the angle of the two planes would be obtuse, and joined to the angle A of the triangle O A B it would make two right angles. But, in the same case, the angle of the two planes D T E, D T F would also be obtuse, and joined to the angle D of the triangle D P E it would make two right angles; and the angle A being thus always equal to the angle D, it would follow in the same manner that the inclination of the two planes A S B, A S C must be equal to that of the two planes D T E, D T F.

433. *Scholium* 2. If two triedral angles are formed by three plane angles respectively equal to each other, and if at the same time the equal or homologous angles are *similarly situated*, the two angles are equal. For, by the proposition, the planes which contain the equal angles of the triedral angles are equally inclined to each other.

434. *Scholium* 3. When the equal plane angles forming the two triedral angles are *not similarly situated*, these angles are equal in all their constituent parts, but, not admitting of superposition, are said to be *equal by symmetry*, and are called *symmetrical angles*.

BOOK VIII.

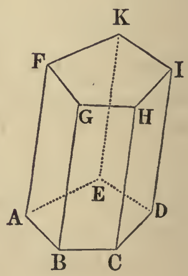
POLYEDRONS.

DEFINITIONS.

435. A Polyedron is a solid, or volume, bounded by planes.

The bounding planes are called the *faces* of the polyedron; and the lines of intersection of the faces are called the *edges* of the polyedron.

436. A Prism is a polyedron having two of its faces equal and parallel polygons, and the other faces parallelograms.



The equal and parallel polygons are called the *bases* of the prism, and the parallelograms its *lateral* faces. The lateral faces taken together constitute the *lateral* or *convex surface* of the prism.

Thus the polyedron A B C D E – K is a prism, having for its bases the equal and parallel polygons A B C D E, F G H I K, and for its lateral faces the parallelograms A B G F, B C H G, &c.

The *principal edges* of a prism are those which join the corresponding angles of the bases; as A F, B G, &c.

437. The altitude of a prism is a perpendicular drawn from any point in one base to the plane of the other.

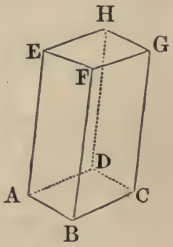
438. A Right Prism is one whose principal edges are perpendicular to the planes of its bases. Each of the

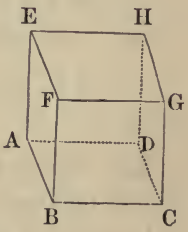
edges is then equal to the altitude of the prism. Every other prism is *oblique*, and has each edge greater than the altitude.

439. A prism is *triangular*, *quadrangular*, *pentangular*, *hexangular*, &c., according as its base is a triangle, a quadrilateral, a pentagon, a hexagon, &c.

440. A PARALLELOPIPEDON is a prism whose bases are parallelograms; as the prism A B C D – H.

The parallelopipedon is *rectangular* when all its faces are rectangles; as the parallelopipedon A B C D – H.

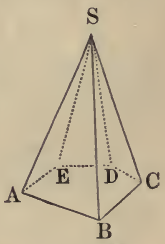




441. A CUBE, or REGULAR HEXAEDRON, is a rectangular parallelopipedon having all its faces equal squares; as the parallelopipedon A B C D – H.

442. A PYRAMID is a polyedron of which one of the faces is any polygon, and all the others are triangles meeting at a common point.

The polygon is called the *base* of the pyramid, the triangles its *lateral faces*, and the point at which the triangles meet its *vertex*. The lateral faces taken together constitute the *lateral* or *convex surface* of the pyramid.



Thus the polyedron A B C D E – S is a pyramid, having for its base the polygon A B C D E, for its lateral faces the triangles A S B, B S C, C S D, &c., and for its vertex the point S.

443. The ALTITUDE of a pyramid is a perpendicular drawn from the vertex to the plane of the base.

444. A pyramid is *triangular*, *quadrangular*, &c., according as its base is a triangle, a quadrilateral, &c.

445. A RIGHT PYRAMID is one whose base is a regular polygon, and the perpendicular drawn from the vertex to the base passes through the centre of the base. In this case the perpendicular is called the *axis* of the pyramid.

446. The SLANT HEIGHT of a right pyramid is a line drawn from the vertex to the middle of one of the sides of the base.

447. A FRUSTUM of a pyramid is the part of the pyramid included between the base and a plane cutting the pyramid parallel to the base.

448. The ALTITUDE of the frustum of a pyramid is the perpendicular distance between its parallel bases.

449. The SLANT HEIGHT of a frustum of a right pyramid is that part of the slant height of the pyramid which is intercepted between the bases of the frustum.

450. The AXIS of the frustum of a pyramid is that part of the axis of the pyramid which is intercepted between the bases of the frustum.

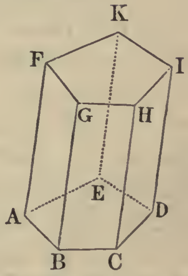
451. The DIAGONAL of a polyedron is a line joining the vertices of any two of its angles which are not in the same face.

452. SIMILAR POLYEDRONS are those which are bounded by the same number of similar faces, and have their polyedral angles respectively equal.

453. A REGULAR POLYEDRON is one whose faces are all equal and regular polygons, and whose polyedral angles are all equal to each other.

Proposition I. — Theorem.

454. *The convex surface of a right prism is equal to the perimeter of its base multiplied by its altitude.*



Let ABCDE–K be a right prism; then will its convex surface be equal to the perimeter of its base,

$$AB + BC + CD + DE + EA,$$

multiplied by its altitude AF.

For, the convex surface of the prism is equal to the sum of the parallelograms AG, BH, CI, DK, EF (Art. 436). Now, the area of each of those parallelograms is equal to its base, AB, BC, CD, &c., multiplied by its altitude, AF, BG, CH, &c. (Prop. V. Bk. IV.). But the altitudes AF, BG, CH, &c. are each equal to AF, the altitude of the prism. Hence, the area of these parallelograms, or the convex surface of the prism, is equal to

$$(AB + BC + CD + DE + EA) \times AF;$$

or the product of the perimeter of the prism by its altitude.

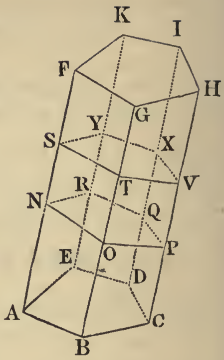
455. *Cor.* If two right prisms have the same altitude, their convex surfaces are to each other as the perimeters of their bases.

Proposition II. — Theorem.

456. *In every prism, the sections formed by parallel planes are equal polygons.*

Let the prism ABCDE–K be intersected by the parallel planes NP, SV; then are the sections NOPQR, STVXY equal polygons.

For the sides ST, NO are parallel, being the intersections of two parallel planes with a third plane ABGF

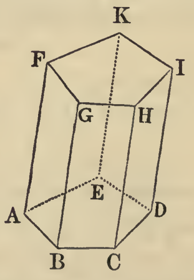


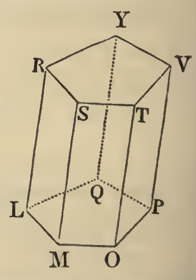
(Prop. XIII. Bk. VII.); these same sides S T, N O, are included between the parallels N S, O T, which are sides of the prism; hence N O is equal to S T. For like reasons, the sides O P, P Q, Q R, &c. of the section N O P Q R, are respectively equal to the sides T V, V X, X Y, &c. of the section S T V X Y; and since the equal sides are at the same time parallel, it follows that the angles N O P, O P Q, &c. of the first section are respectively equal to the angles S T V, T V X of the second (Prop. XVI. Bk. VII.). Hence, the two sections N O P Q R, S T V X Y, are equal polygons.

457. *Cor.* Every section made in a prism parallel to its base, is equal to that base.

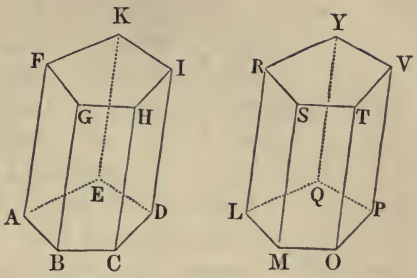
Proposition III. — Theorem.

458. *Two prisms are equal, when the three faces which form a triedral angle in the one are equal to those which form a triedral angle in the other, each to each, and are similarly situated.*





Let the two prisms A B C D E – K and L M O P Q – Y have the faces which form the triedral angle B equal to the faces which form the triedral angle M; that is, the base A B C D E equal to the base L M N O P Q, the parallelogram A B G F equal to the parallelogram L M S R, and the parallelogram B C H G equal to M O T S; then the two prisms are equal.



For, apply the base A B C D E to the equal base LMOPQ; then, the triedral angles B and M, being equal, will coincide, since the plane angles which form these triedral angles are equal each to each, and similarly situated (Prop. XXI. Sch. 2, Bk. VII.); hence the edge B G will fall on its equal M S, and the face B H will coincide with its equal M T, and the face B F with its equal M R. But the upper bases are equal to their corresponding lower bases (Art. 436); therefore the bases F G H I K, R S T V Y are equal; hence they coincide with each other. Therefore H I coincides with T V, I K with V Y, and K F with Y R; and consequently the lateral faces coïncide. Hence the two prisms coincide throughout, and are equal.

459. *Cor.* Two right prisms, which have equal bases and equal altitudes, are equal.

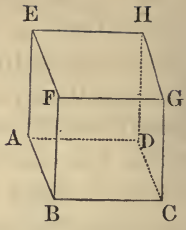
For, since the side A B is equal to L M, and the altitude B G to M S, the rectangle A B G F is equal to the rectangle L M S R; so, also, the rectangle B G H C is equal to M S T O; and thus the three faces which form the triedral angle B, are equal to the three faces which form the triedral angle M. Hence the two prisms are equal.

Proposition IV. — Theorem.

460. *In every parallelopipedon the opposite faces are equal and parallel.*

Let A B C D – H be a parallelopipedon; then its opposite faces are equal and parallel.

The bases A B C D, E F G H are equal and parallel (Art. 436), and it remains only to be shown that the same is

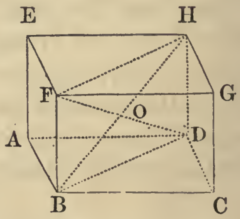


true of any two opposite lateral faces, as B C G F, A D H E. Now, since the base A B C D is a parallelogram, the side A D is equal and parallel to B C. For a similar reason, A E is equal and parallel to B F; hence the angle D A E is equal to the angle C B F (Prop. XVI. Bk. VII.), and the planes D A E, C B F are parallel; hence, also, the parallelogram B C G F is equal to the parallelogram A D H E. In the same way, it may be shown that the opposite faces A B F E, D C G H are equal and parallel.

461. *Cor.* Any two opposite faces of a parallelopipedon may be assumed as its bases, since any face and the one opposite to it are equal and parallel.

Proposition V. — Theorem.

462. *The diagonals of every parallelopipedon bisect each other.*



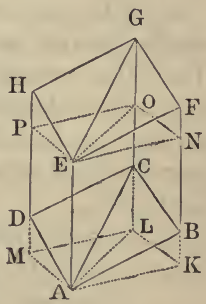
Let A B C D – H be a parallelopipedon; then its diagonals, as B H, D F, will bisect each other.

For, since B F is equal and parallel to D H, the figure B F H D is a parallelogram; hence the diagonals B H, D F bisect each other at the point O (Prop. XXXIV. Bk. I.). In the same manner it may be shown that the two diagonals A G and C E bisect each other at the point O; hence the several diagonals bisect each other.

463. *Scholium.* The point at which the diagonals mutually bisect each other may be regarded as the centre of the parallelopipedon.

Proposition VI. — Theorem.

464. *Any parallelopipedon may be divided into two equivalent triangular prisms by a plane passing through its opposite diagonal edges.*

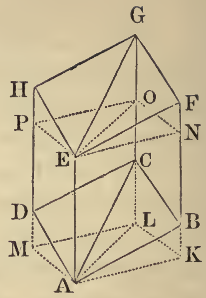


Let any parallelopipedon, A B C D–H, be divided into two prisms, A B C – G, A D C – G, by a plane, A C G E, passing through opposite diagonal edges; then will the two prisms be equivalent.

Through the vertices A and E, draw the planes A K L M, E N O P, perpendicular to the edge A E, and meeting B F, C G, D H, the three other edges of the parallelopipedon, in the points K, L, M, and in N, O, P. The sections A K L M, E N O P are equal, since they are formed by planes perpendicular to the same straight lines, and hence parallel (Prop. II.). They are parallelograms, since the two opposite sides of the same section, A K, L M, are the intersections of two parallel planes, A B F E, D C G H, by the same plane, A K L M (Prop. XIII. Bk. VII.).

For a like reason, the figure A M P E is a parallelogram; so, also, are A K N E, K L O N, L M P O, the other lateral faces of the solid A K L M – P; consequently, this solid is a prism (Art. 436); and this prism is right, since the edge A E is perpendicular to the plane of its base. This right prism is divided by the plane A L O E into the two right prisms A K L – O, A M L – O, which, having equal bases, A K L, A M L, and the same altitude, A E, are equal (Prop. III. Cor.).

Now, since A E H D, A E P M are parallelograms, the sides D H, M P, being each equal to A E, are equal to each other; and taking away the common part, D P, there remains D M equal to H P. In the same manner it may be shown that C L is equal to G O.



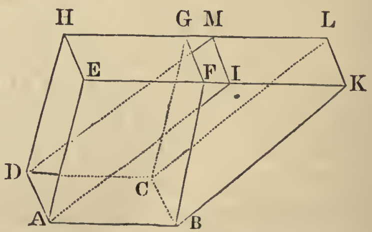
Conceive now E P O, the base of the solid E P O – G, to be applied to its equal A M L, the point P falling upon M, and the point O upon L; the edges G O, H P will coincide with their equals C L, D M, since they are all perpendicular to the same plane, A K L M. Hence the two solids coincide throughout, and are therefore equal. To each of these equals add the solid A D C – P, and the right prism A M L – O is equivalent to the prism A D C – G.

In the same manner, it may be proved that the right prism A K L – O is equivalent to the prism A B C – G. The two right prisms A K L – O, A M L – O being equal, it follows that two triangular prisms, A B C – G, A D C – G, are equivalent to each other.

465. *Cor.* Every triangular prism is half of a parallelopipedon having the same triedral angle, with the same edges.

Proposition VII. — Theorem.

466. ***Two parallelopipedons, having a common lower base, and their upper bases in the same plane and between the same parallels, are equivalent to each other.***



Let the two parallelopipedons A G, A L have the common base ABCD, and their upper bases, E F G H, I K L M, in the same plane, and between the same parallels, E K, H L; then the parallelopipedons will be equivalent.

There may be three cases, according as E I is greater or less than, or equal to, E F; but the demonstration is the same for each.

Since A E is parallel to B F, and H E to G F, the plane angle A E I is equal to B F K, H E I to G F K, and H E A to G F B. Of these six plane angles, the three first form the polyedral angle E, the three last the polyedral angle F; consequently, since these plane angles are equal each to each, and similarly situated, the polyedral angles, E, F, must be equal. Now conceive the prism A E I – M to be applied to the prism B F K – L; the base A E I, being placed upon the base B F K, will coincide with it, since they are equal; and, since the polyedral angle E is equal to the polyedral angle F, the side E H will fall upon its equal, F G. But the base A E I and its edge E H determine the prism A E I – M, as the base B F K and its edge F G determine the prism B F K – L (Prop. III.); hence the two prisms coincide throughout, and therefore are equal to each other.

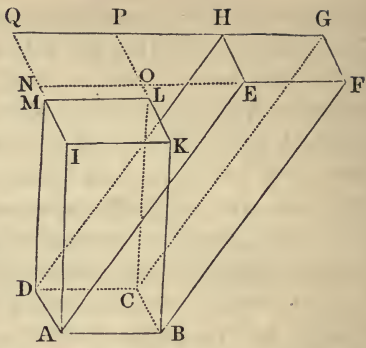
Take away, now, from the whole solid A E L C, the prism A E I – M, and there will remain the parallelopipedon A L; and take away from the same solid A L the prism B F K – L, and there will remain the parallelopipedon A G; hence the two parallelopipedons A L, A G are equivalent.

Proposition VIII. — Theorem.

467. *Two parallelopipedons having the same base and the same altitude are equivalent.*

Let the two parallelopipedons A G, A L have the common base A B C D, and the same altitude; then will the two parallelopipedons be equivalent.

For, the upper bases E F G H, I K L M being in the same plane, produce the edges E F, H G, L K, I M, till by their intersections they form the parallelogram N O P Q; this parallelogram is equal to either of the bases I L, E G, and is between the same parallels; hence N O P Q is equal to the common base A B C D, and is parallel to it.



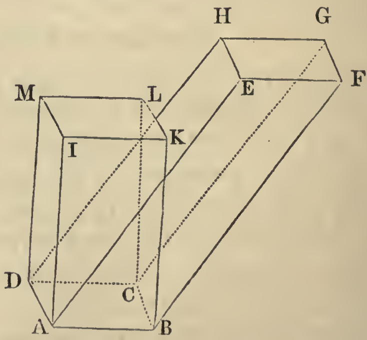
Now, if a third parallelopipedon be conceived, which, with the same lower base A B C D, has for its upper base NOPQ, this third parallelopipedon will be equivalent to the parallelopipedon A G, since the lower base is the same, and the upper bases lie in the same plane and between the same parallels, G Q, F N (Prop. VII.).

For the same reason, this third parallelopipedon will also be equivalent to the parallelopipedon A L; hence the two parallelopipedons A G, A L, which have the same base and the same altitude, are equivalent.

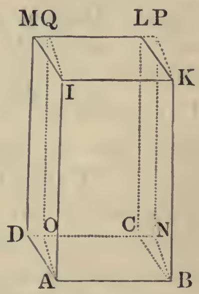
Proposition IX. — Theorem.

468. *Any oblique parallelopipedon is equivalent to a rectangular parallelopipedon having the same altitude and an equivalent base.*

Let A G be any parallelopipedon; then A G will be equivalent to a rectangular parallelopipedon having the same altitude and an equivalent base.



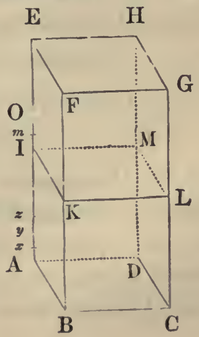
From the points A, B, C, D, draw A I, B K, C L, D M, perpendicular to the lower base, and equal in altitude to A G; there will thus be formed the parallelopipedon A L, equivalent to A G (Prop. VIII.), and having its lateral faces, A K, B L, &c., rectangular. Now, if the base A B C D is a rectangle, A L will be a rectangular parallelopipedon equivalent to A G.



But if A B C D is not a rectangle, draw A O, B N, each perpendicular to C D; also O Q, N P, each perpendicular to the base; then we shall have a rectangular parallelopipedon A B N O – Q. For, by construction, the bases A B N O, I K P Q are rectangles; so, also, are the lateral faces, the edges A I, O Q, &c. being perpendicular to the plane of the base; therefore the solid A P is a rectangular parallelopipedon. But the two parallelopipedons A P, A L may be considered as having the same base, A B K I, and the same altitude, A O; hence they are equivalent. Hence the parallelopipedon A G, which was shown to be equivalent to the parallelopipedon A L, is also equivalent to the rectangular parallelopipedon A P, having the same altitude, A I, and a base, A B N O, equivalent to the base A B C D.

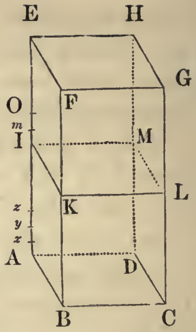
Proposition X.—Theorem.

469. *Two rectangular parallelopipedons, which have the same base, are to each other as their altitudes.*



Let the two parallelopipedons A G, A L have the same base, A B C D; then they are to each other as their altitudes, A E, A I.

First. Suppose the altitudes A E, A I are to each other as two whole numbers; for example, as 15 is to 8. Divide A E into 15 equal parts, of which A I will contain 8. Through x, y, z, &c., the points of division, conceive planes to



pass parallel to the common base. These planes will divide the solid A G into 15 small parallelopipedons, all equal to each other, having equal bases and equal altitudes; equal bases, since every section, as I K L M, parallel to the base A B C D, is equal to that base (Prop. II.), and equal altitudes, since the altitudes are the equal divisions A x, x y, y z, &c. But of those 15 equal parallelopipedons, 8 are contained in A L; hence the parallelopipedon A G is to the parallelopipedon A L as 15 is to 8, or, in general, as the altitude A E is to the altitude A I.

Secondly. If the ratio of A E to A I cannot be exactly expressed by numbers, we shall still have the proportion,

Solid A G : *Solid* A L : : A E : A I.

For, if this proportion is not correct, suppose we have

Solid A G : *Solid* A L : : A E : A O greater than A I.

Divide A E into equal parts, each of which shall be less than I O; there will be at least one point of division, m, between I and O. Let P represent the parallelopipedon, whose base is A B C D, and altitude A m; since the altitudes A E, A m are to each other as two whole numbers, we shall have

Solid A G : P : : A E : A m.

But, by hypothesis, we have

Solid A G : *Solid* A L : : A E : A O;

hence (Prop. X. Cor. 2, Bk. II.),

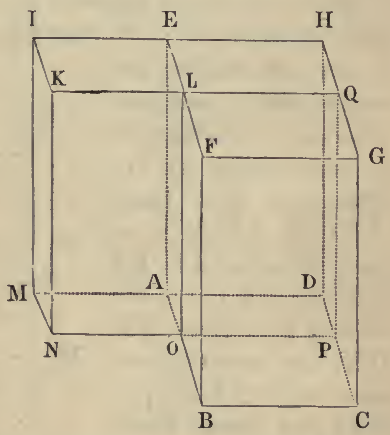
Solid A L : P : : A O : A m.

But A O is greater than A m; hence, if the proportion is correct, the parallelopipedon A L must be greater than P. On the contrary, however, it is less; consequently the solid A G cannot be to the solid A L as the line A E is to a line greater than A I.

By the same mode of reasoning, it may be shown that the fourth term of the proportion cannot be less than A I; therefore it must be equal to A I. Hence rectangular parallelopipedons, having the same base, are to each other as their altitudes.

Proposition XI. — Theorem.

470. *Two rectangular parallelopipedons, having the same altitude, are to each other as their bases.*



Let the two rectangular parallelopipedons A G, A K have the same altitude, A E; then they are to each other as their bases.

Place the two solids so that their faces, B E, O E, may have the common angle B A E; produce the plane O N K L till it meets the plane D C G H in P Q; we shall thus have a third parallelopipedon, A Q, which may be compared with each of the parallelopipedons A G, A K. The two solids, A G, A Q, having the same base, A E H D, are to each other as their altitudes A B, A O (Prop. X.); in like manner, the two solids A Q, A K, having the same base, A O L E, are to each other as their altitudes A D, A M. Hence we have the two proportions,

$$\textit{Solid } AG : \textit{Solid } AQ :: AB : AO,$$

$$\textit{Solid } AQ : \textit{Solid } AK :: AD : AM.$$

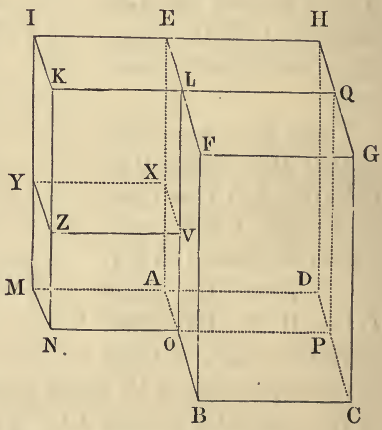
Multiplying together the corresponding terms of these proportions, and omitting, in the result, the common factor *Solid* A Q, we shall have,

Solid A G : *Solid* A K : : A B × A D : A O × A M.

But A B × A D measures the base A B C D (Prop. IV. Sch., Bk. IV.); and A O × A M measures the base A M N O; hence two rectangular parallelopipedons of the same altitude are to each other as their bases.

Proposition XII. — Theorem.

471. *Any two rectangular parallelopipedons are to each other as the product of their bases by their altitudes.*



Let A G, A Z be two rectangular parallelopipedons; then they are to each other as the product of their bases, A B C D, A M N O, by their altitudes, A E, A X.

Place the two solids so that their faces, B E, O X, may have the common angle B A E; produce the planes necessary for completing the third parallelopipedon, A K, having the same altitude with the parallelopipedon A G. By the last proposition, we shall have

Solid A G : *Solid* A K : : A B C D : A M N O.

But the two parallelopipedons A K, A Z, having the same base, A M N O, are to each other as their altitudes, A E, A X (Prop. X.); hence we have

Solid A K : *Solid* A Z : : A E : A X.

Multiplying together the corresponding terms of these proportions, and omitting, in the result, the common factor *Solid* A K, we shall have

Solid A G : *Solid* A Z : : A B C D $\times$ A E : A M N O $\times$ A X.

Hence, any two rectangular parallelopipedons are to each other as the products of their bases by their altitudes.

472. *Scholium* 1. We are consequently authorized to assume, as the measure of a rectangular parallelopipedon, the product of its base by its altitude; in other words, *the product of its three dimensions.* But by the product of two or more lines is always meant the product of the numbers which represent them; those numbers themselves being determined by the particular linear unit, which may be assumed as the standard. It is necessary, therefore, in comparing magnitudes, that the measuring unit be the same for each of the magnitudes compared.

473. *Scholium* 2. The measured magnitude of a solid, or volume, is called its *volume*, *solidity*, or *solid contents.* We assume as the *unit of volume*, or *solidity*, the cube, each of whose edges is the linear unit, and each of whose faces is the unit of surface.

Proposition XIII. — Theorem.

474. *The solid contents of a parallelopipedon, and of any other prism, are equal to the product of its base by its altitude.*

First. Any parallelopipedon is equivalent to a rectangular parallelopipedon having the same altitude and an equivalent base (Prop. IX.). But the solid contents of a rectangular parallelopipedon are equal to the product of its base by its altitude; therefore the solid contents of any parallelopipedon are equal to the product of its base by its altitude.

Second. Any triangular prism is half of a parallelopipedon, so constructed as to have the same altitude, and a base twice as great (Prop. VI.). But the solid contents of the parallelopipedon are equal to the product of its base by its altitude; hence, that of the triangular prism is also equal to the product of its base, or half that of the parallelopipedon, by its altitude.

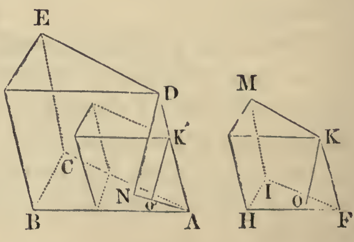
Third. Any prism may be divided into as many triangular prisms of the same altitude, as there are triangles in the polygon taken for a base. But the solid contents of each triangular prism are equal to the product of its base by its altitude; and, since the altitude is the same in each, it follows that the sum of all these prisms is equal to the sum of all the triangles taken as bases multiplied by the common altitude.

Hence the solid contents of any prism are equal to the product of its base by its altitude.

475. *Cor.* When any two prisms have the same altitude, the products of the bases by the altitudes will be as the bases (Prop. IX. Bk. II.); hence, *prisms of the same altitude are to each other as their bases.* For a like reason, *prisms of the same base are to each other as their altitudes.*

Proposition XIV. — Theorem.

476. *Similar prisms are to each other as the cubes of their homologous edges.*



Let A B C–E, F H I–M be two similar prisms; these prisms are to each other as the cubes of their homologous edges, A B and F H.

For, from D and K, homologous angles of the two prisms, draw the perpendiculars D N, K O, to the bases A B C, F H I. Take A K′ equal to F K, and join A N.

Draw $K'O'$ perpendicular to AN in the plane AND, and $K'O$ will be perpendicular to the plane ABC, and equal to KO, the altitude of the prism $FHI-M$. For, conceive the triedral angles A and F to be applied the one to the other; the planes containing them, and therefore the perpendiculars $K'O'$, KO, will coincide.

Now, since the bases ABC, FHI are similar, we have (Prop. XXIX. Bk. IV.),

$$\textit{Base}\ ABC : \textit{Base}\ FHI :: \overline{AB}^2 : \overline{FH}^2;$$

and, because of the similar triangles DAN, KFO, and of the similar parallelograms DB, KH, we have

$$DN : KO :: DA : KF :: AB : FH.$$

Hence, multiplying together the corresponding terms of these proportions, we have

$$\textit{Base}\ ABC \times DN : \textit{Base}\ FHI \times KO : \overline{AB}^3 : \overline{FH}^3.$$

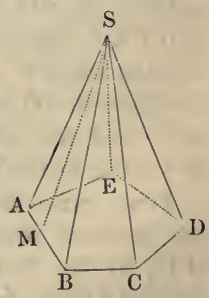
But the product of the base by the altitude is equal to the solidity of a prism (Prop. XIII.); hence

$$\textit{Prism}\ ABC-E : \textit{Prism}\ FHI-M :: \overline{AB}^3 : \overline{FH}^3.$$

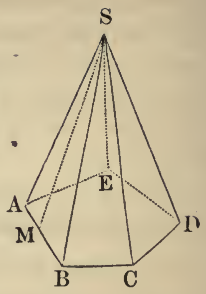
Proposition XV. — Theorem.

477. *The convex surface of a right pyramid is equal to the perimeter of its base, multiplied by half the slant height.*

Let $ABCDE-S$ be a right pyramid, and SM its slant height; then the convex surface is equal to the perimeter $AB + BC + CD + DE + EA$ multiplied by $\frac{1}{2}SM$.



The triangles SAB, SBC, SCD, &c. are all equal; for the sides AB, BC, CD, &c. are equal (Art. 445), and the sides SA, SB, SC, &c., being oblique lines meeting the base at equal



distances from a perpendicular let fall from the vertex S to the centre of the base, are also equal (Prop. V. Bk. VII.). Hence, these triangles are all equal (Prop. XVIII. Bk. I.); and the altitude of each is equal to the slant height S M. But the area of a triangle is equal to the product of its base multiplied by half its altitude (Prop. VI. Bk. IV.). Hence, the areas of the triangles S A B, S B C, S C D, &c. are equal to the sum of the bases A B, B C, C D, &c. multiplied by half the common altitude, S M; that is, the convex surface of the pyramid is equal to the perimeter of the base multiplied by half the slant height.

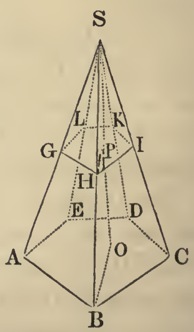
478. *Cor.* The lateral faces of a right pyramid are equal isosceles triangles, having for their bases the sides of the base of the pyramid.

Proposition XVI.—Theorem.

479. *If a pyramid be cut by a plane parallel to its base,—*

1*st. The edges and the altitude will be divided proportionally.*

2*d. The section will be a polygon similar to the base.*



Let the pyramid A B C D E–S, whose altitude is S O, be cut by a plane, G H I K L, parallel to its base; then will the edges S A, S B, S C, &c., with the altitude S O, be divided proportionally; and the section G H I K L will be similar to the base A B C D E.

First. Since the planes A B C, G H I are parallel, their intersections A B, G H, by the third plane S A B, are parallel (Prop. XIII. Bk. VII.); hence the triangles S A B, S G H are similar (Prop. XXV. Bk. IV.), and we have

S A : S G : : S B : S H.

For the same reason, we have

S B : S H : : S C : S I;

and so on. Hence all the edges, S A, S B, S C, &c., are cut proportionally in G, H, I, &c. The altitude S O is likewise cut in the same proportion, at the point P; for B O and H P are parallel; therefore we have

S O : S P : : S B : S H.

Secondly. Since G H is parallel to A B, H I to B C, I K to C D, &c. the angle G H I is equal to A B C, the angle H I K to B C D, and so on (Prop. XVI. Bk. VII.). Also, by reason of the similar triangles S A B, S G H, we have

A B : G H : : S B : S H;

and by reason of the similar triangles S B C, S H I, we have

S B : S H : : B C : H I;

hence, on account of the common ratio S B : S H,

A B : G H : : B C : H I.

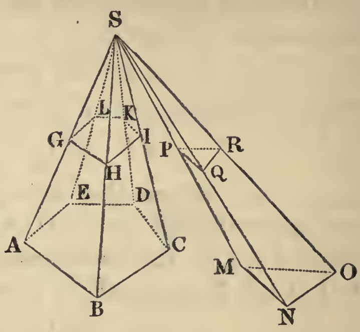
For a like reason, we have

B C : H I : : C D : I K,

and so on. Hence the polygons A B C D E, G H I K L have their angles equal, each to each, and their homologous sides proportional; hence they are similar.

480. *Cor.* 1. *If two pyramids have the same altitude, and their bases in the same plane, their sections made by a plane parallel to the plane of their bases are to each other as their bases.*

Let A B C D E – S, M N O – S be two pyramids, having the same altitude, and their bases in the same plane; and let G H I K L, P Q R be sections made by a plane parallel to the plane of their bases; then these sections are to each other as the bases A B C D E, M N O.



For, the two polygons A B C D E, G H I K L being similar, their surfaces are as the squares of the homologous sides A B, G H (Prop. XXXI. Bk. IV.). But

$$AB : GH :: SA : SG.$$

Hence,

$$ABCDE : GHIKL :: \overline{SA}^2 : \overline{SG}^2.$$

For the same reason,

$$MNO : PQR :: \overline{SM}^2 : \overline{SP}^2.$$

But since G H I K L and P Q R are in the same plane, we have also (Prop. XVIII. Bk. VII.),

$$SA : SG :: SM : SP;$$

hence,

$$ABCDE : GHIKL :: MNO : PQR;$$

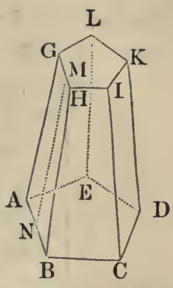
therefore the sections G H I K L, P Q R are to each other as the bases A B C D E, M N O.

481. *Cor.* 2. If the bases A B C D E, M N O are equivalent, any sections, G H I K L, P Q R, made at equal distances from those bases, are likewise equivalent.

Proposition XVII. — Theorem.

482. *The convex surface of a frustum of a right pyramid is equal to half the sum of the perimeters of its two bases, multiplied by its slant height.*

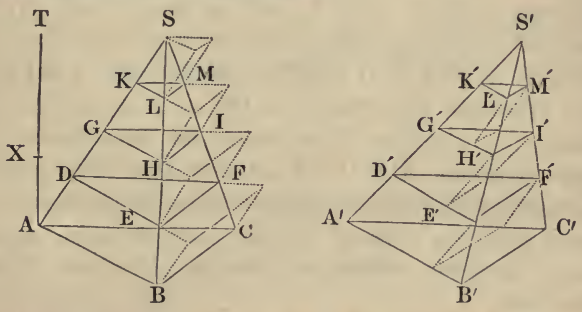
Let A B C D E – L be the frustum of a right pyramid, and M N its slant height; then the convex surface is equal to the sum of the perimeters of the two bases A B C D E, G H I K L, multiplied by half of M N.



For the upper base G H I K L is similar to the base A B C D E (Prop. XVI.), and A B C D E is a regular polygon (Art. 445); hence the sides G H, H I, I K, K L, and L G are all equal to each other. The angles G A B, A B H, H B C, &c. are equal (Prop. XV. Cor.), and the edges A G, B H, C I, &c. are also equal (Prop. XVI.); therefore the faces A H, B I, C K, &c. are all equal trapezoids (Art. 28), having a common altitude, M N, the slant height of the frustum. But the area of either trapezoid, as A H, is equal to $\frac{1}{2}$ (A B + G H) $\times$ M N (Prop. VII. Bk. IV.); hence the areas of all the trapezoids, or the convex surface of frustum, are equal to half the sum of the perimeters of the two bases multiplied by the slant height.

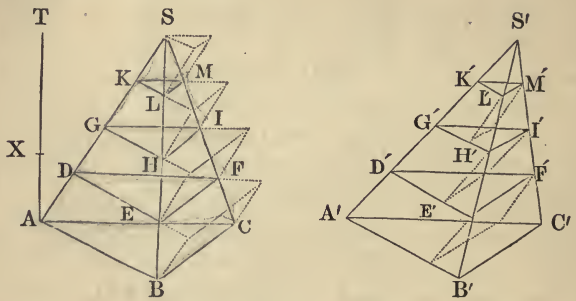
Proposition XVIII. — Theorem.

483. *Triangular pyramids, having equivalent bases and the same altitude, are equivalent.*



Let A B C – S, A′ B′ C′ – S′ be two triangular pyramids, having equivalent bases, A B C, A′ B′ C′, situated in the same plane; and let them have the same altitude, A T; then these pyramids are equivalent.

For, if the two pyramids are not equivalent, let A′ B′ C′ – S′ be the smaller, and suppose A X to be the altitude of a prism, which, having A B C for its base, is equal to their difference.



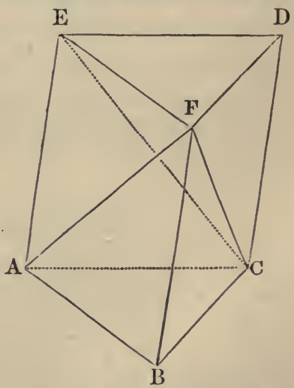
Divide the altitude A T into equal parts, each less than A X; through each point of division pass a plane parallel to the plane of the base, thus forming corresponding sections in the two pyramids, equivalent each to each, namely, D E F to D′ E′ F′, G H I to G′ H′ I′, &c.

Upon the triangles A B C, D E F, G H I, &c., taken as bases, construct exterior prisms, having for edges the parts A D, D G, G K, &c. of the edge S A; in like manner, on the bases D′ E′ F′, G′ H′ I′, &c. in the second pyramid, construct interior prisms, having for edges the corresponding parts of S′ A′. It is plain that the sum of all the exterior prisms of the pyramid A B C – S is greater than this pyramid; and also that the sum of all the interior prisms of the pyramid A′ B′ C′ – S′ is less than this pyramid. Hence, the difference between the sum of all the exterior prisms and the sum of all the interior ones, must be greater than the difference between the two pyramids themselves.

Now, beginning with the bases A B C, A′ B′ C′, the second exterior prism, D E F – G, is equivalent to the first interior prism, D′ E′ F′ – A′, since they have equal altitudes, and their bases, D E F, D′ E′ F′, are equivalent. For a like reason, the third exterior prism, G H I – K, and the second interior prism, G′ H′ I′ – D′, are equivalent; and so on to the last in each series. Hence, all the exterior prisms of the pyramid A B C – S, excepting the first prism, A B C – D, have equivalent corresponding ones in the interior prisms of the pyramid A B C′ – S′. Therefore the prism A B C – D is the difference between the sum of all the exterior prisms of the pyramid A B C – S, and the sum of the interior prisms of the pyramid A′ B′ C′ – S′. But the difference between these two sets of prisms has been proved to be greater than that of the two pyramids, which latter difference we supposed to be equal to the prism A B C – X. Hence, the prism A B C – D must be greater than the prism A B C – X, which is impossible, since they have the same base, A B C, and the altitude of the first is less than A X, the altitude of the second. Hence, the supposed inequality between the two pyramids cannot exist; therefore the two pyramids A B C – S, A′ B′ C′ – S′, having the same altitude and equivalent bases, are themselves equivalent.

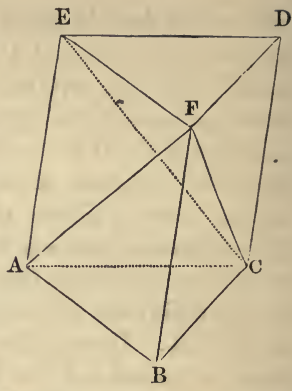
Proposition XIX.—Theorem.

484. *Every triangular pyramid is a third part of a triangular prism having the same base and the same altitude.*



Let A B C – F be a triangular pyramid, and A B C – D E F a triangular prism of the same base and the same altitude; then the pyramid is one third of the prism.

Cut off the pyramid A B C – F from the prism, by the plane F A C; there will remain the solid A C D E – F, which may be considered as a quadrangular pyramid, whose vertex is F, and whose base is the parallelogram A C D E. Draw the



diagonal C E, and pass the plane F C E, which will cut the quadrangular pyramid into two triangular ones, ACE–F, EDC–F. These two triangular pyramids have for their common altitude the perpendicular let fall from F on the plane A C D E; they have equal bases, since the triangles A C E, C D E are halves of the same parallelogram; hence the two pyramids A C E – F, C D E – F are equivalent (Prop. XVIII.). But the pyramid C D E – F and the pyramid A B C – F have equal bases, A B C, D E F; they have also the same altitude, namely, the distance between the parallel planes A B C, D E F; hence the two pyramids are equivalent. Now, the pyramid C D E – F has been proved equivalent to A C E – F; hence the three pyramids A B C – F, C D E – F, A C E – F, which compose the whole prism A B C – D E F, are all equivalent; therefore, either pyramid, as A B C – F, is the third part of the prism, which has the same base and the same altitude.

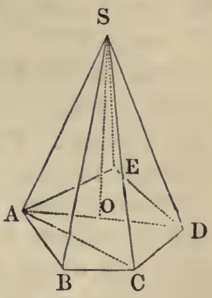
485. *Cor.* 1. Every triangular prism may be divided into three equivalent triangular pyramids.

486. *Cor.* 2. The solidity of a triangular pyramid is equal to a third part of the product of its base by its altitude.

Proposition XX. — Theorem.

487. *The solidity of every pyramid is equal to the product of its base by one third of its altitude.*

Let A B C D E – S be any pyramid, whose base is A B C D E, and altitude S O; then its solidity is equal to A B C D E $\times \frac{1}{3}$ S O.



Draw the diagonals A C, A D, and pass the planes S A C, S A D through these diagonals and the vertex S; the polygonal pyramid A B C D E – S will be divided into several triangular pyramids, all having the same altitude, S O. But each of these pyramids is measured by the product of its base, B A C, C A D, D A E, by a third part of its altitude, S O (Prop. XIX. Cor. 2); hence, the sum of these triangular pyramids, or the polygonal pyramid A B C D E – S, will be measured by the sum of the triangles B A C, C A D, D A E, or the polygon A B C D E, multiplied by one third of S O; hence, every pyramid is measured by the product of its base by one third of its altitude.

488. *Cor.* 1. Every pyramid is the third part of the prism which has the same base and the same altitude.

489. *Cor.* 2. Pyramids having the same altitude are to each other as their bases.

490. *Cor.* 3. Pyramids having the same base, or equivalent bases, are to each other as their altitudes.

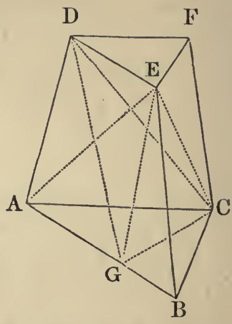
491. *Cor.* 4. Pyramids are to each other as the products of their bases by their altitudes.

492. *Scholium.* The solidity of any polyedron may be found by dividing it into pyramids, by passing planes through its vertices.

Proposition XXI. — Theorem.

493. *A frustum of a pyramid is equivalent to the sum of three pyramids, having for their common altitude the altitude of the frustum, and whose bases are the two bases of the frustum and a mean proportional between them.*

First. Let A B C – D E F be the frustum of a pyramid, whose base is a triangle. Pass a plane through the points



A, E, C; it cuts off the triangular pyramid A B C – E, whose altitude is that of the frustum, and whose base, A B C, is the lower base of the frustum. Pass another plane through the points D, E, C; it cuts off the triangular pyramid D E F – C, whose altitude is that of the frustum, and whose base, D E F, is the upper base of the frustum.

There now remains of the frustum the pyramid A C D – E. Draw E G parallel to A D; join C G and D G. Then, since E G is parallel to A D, it is parallel to the plane A C D (Prop. XI. Bk. VII.); and the pyramid A C D – E is equivalent to the pyramid A C D – G, since they have the same base, A C D, and their vertices, E and G, lie in the same straight line parallel to the common base. But the pyramid A C D – G is the same as the pyramid A G C – D, whose altitude is that of the frustum, and whose base, A G C, as will be proved, is a mean proportional between the bases A B C and D E F.

The two triangles A G C, D E F have the angles A and D equal to each other (Prop. XVI. Bk. VII.); hence we have (Prop. XXVIII. Bk. IV.),

$$AGC : DEF :: AG \times AC : DE \times DF;$$

but since A G is equal to D E,

$$AGC : DEF :: AC : DF.$$

We have, also (Prop. VI. Cor., Bk. IV.),

$$ABC : AGC :: AB : AG \text{ or } DE.$$

But the similar triangles A B C, D E F give

$$AB : DE :: AC : DF;$$

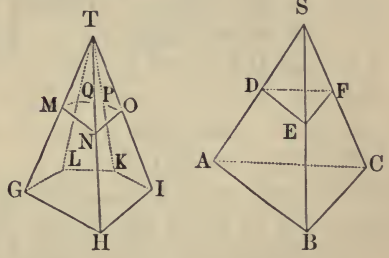
hence (Prop. X. Bk. II.),

$$ABC : AGC :: AGC : DEF;$$

that is, the base A G C is a mean proportional between the bases A B C, D E F of the frustum.

Secondly. Let G H I K L – M N O P Q be the frustum of a pyramid, whose base is any polygon.

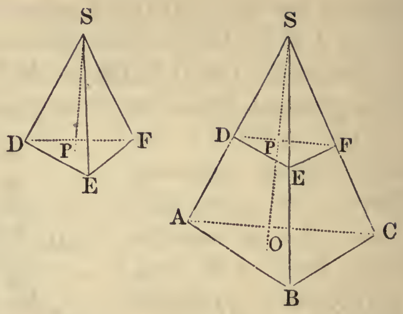
Let A B C – S be a triangular pyramid having the same altitude, and an equivalent base, with any polygonal pyramid, G H I K L – T; these pyramids are equivalent (Prop. XX. Cor. 3.)



The bases of the two pyramids may be regarded as situated in the same plane, in which case the plane M N O P Q produced will form in the triangular pyramid a section, D E F, at the same distance above the common plane of the bases; and therefore the section D E F will be to the section M N O P Q as the base A B C is to the base G H I K L (Prop. XVI. Cor. 1); and since the bases are equivalent, the sections will be so likewise. Hence, the pyramids M N O P Q – T, D E F – S, having the same altitude and equivalent bases, are equivalent. For the same reason, the entire pyramids G H I K L – T, A B C – S are equivalent; consequently, the frustums G H I K L – M N O P Q, A B C – D E F, are equivalent. But the frustum A B C – D E F has been shown to be equivalent to the sum of three pyramids having for their common altitude the altitude of the frustum, and whose bases are the two bases of the frustum, and a mean proportional between them. Hence the proposition is true of the frustum of any pyramid.

Proposition XXII. — Theorem.

494. *Similar pyramids are to each other as the cubes of their homologous edges.*



Let $ABC-S$ and $DEF-S$ be two similar pyramids; these pyramids are to each other as the cubes of their homologous edges AB and DE, or BC and EF, &c.

For, the two pyramids being similar, the homologous polyedral angles at the vertices are equal (Art. 452); hence the smaller pyramid may be so applied to the larger, that the polyedral angle S shall be common to both.

In that case, the bases ABC, DEF will be parallel; for, since the homologous faces are similar, the angle SDE is equal to SAB, and SEF to SBC; hence the plane ABC is parallel to the plane DEF (Prop. XVI. Bk. VII.). Then let SO be drawn from the vertex S perpendicular to the plane ABC, and let P be the point where this perpendicular meets the plane DEF. From what has already been shown (Prop. XVI.), we shall have

$$SO : SP :: SA : SD :: AB : DE;$$

and consequently,

$$\tfrac{1}{3}SO : \tfrac{1}{3}SP :: AB : DE.$$

But the bases ABC, DEF are similar; hence (Prop. XXIX. Bk. IV.),

$$ABC : DEF :: \overline{AB}^2 : \overline{DE}^2.$$

Multiplying together the corresponding terms of these two proportions, we have

$$ABC \times \tfrac{1}{3}SO : DEF \times \tfrac{1}{3}SP :: \overline{AB}^3 : \overline{DE}^3.$$

Now, $ABC \times \frac{1}{3}SO$ represents the solidity of the pyramid $ABC-S$, and $DEF \times \frac{1}{3}SP$ that of the pyramid $DEF-S$ (Prop. XX.); hence two similar pyramids are to each other as the cubes of their homologous edges.

Proposition XXIII. — Theorem.

495. *There can be no more than five regular polyedrons.*

For, since regular polyedrons have equal regular polygons for their faces, and all their polyedral angles equal, there can be but few regular polyedrons.

First. If the faces are equilateral triangles, polyedrons may be formed of them, having each polyedral angle contained by *three* of these triangles, forming a solid bounded by four equal equilateral triangles; or by *four*, forming a solid bounded by eight equal equilateral triangles; or by *five*, forming a solid bounded by twenty equal equilateral triangles. No others can be formed with equilateral triangles. For six of these angles are equal to four right angles, and cannot form a polyedral angle (Prop. XX. Bk. VII.).

Secondly. If the faces are squares, their angles may be arranged by threes, forming a solid bounded by six equal squares. Four angles of a square are equal to four right angles, and cannot form a polyedral angle.

Thirdly. If the faces are regular pentagons, their angles may be arranged by threes, forming a solid bounded by twelve equal and regular pentagons.

We can proceed no farther. Three angles of a regular hexagon are equal to four right angles; three of a heptagon are greater. Hence, there can be formed no more than five regular polyedrons, — three with equilateral triangles, one with squares, and one with pentagons.

496. *Scholium.* The regular polyedron bounded by four equilateral triangles is called a TETRAEDRON; the one bounded by eight is called an OCTAEDRON; the one bounded by twenty is called an ICOSAEDRON. The regular polyedron bounded by six equal squares is called a HEXAEDRON, or CUBE; and the one bounded by twelve equal and regular pentagons is called a DODECAEDRON.

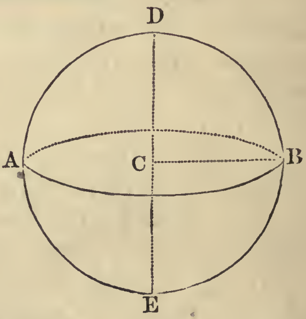
BOOK IX.

THE SPHERE, AND ITS PROPERTIES.

DEFINITIONS.

497. A SPHERE is a solid, or volume, bounded by a curved surface, all points of which are equally distant from a point within, called the centre.

The sphere may be conceived to be formed by the revolution of a semicircle, D A E, about its diameter, D E, which remains fixed.



498. The RADIUS of a sphere is a straight line drawn from the centre to any point in surface, as the line C B.

The DIAMETER, or AXIS, of a sphere is a line passing through the centre, and terminated both ways by the surface, as the line D E.

Hence, all the radii of a sphere are equal; and all the diameters are equal, and each is double the radius.

499. A CIRCLE, it will be shown, is a section of a sphere.

A GREAT CIRCLE of the sphere is a section made by a plane passing through the centre, and having the centre of the sphere for its centre; as the section A B, whose centre is C.

500. A SMALL CIRCLE of the sphere is any section made by a plane not passing through the centre.

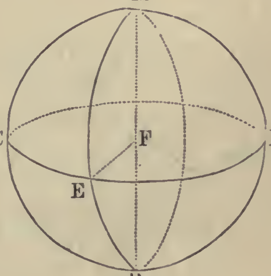
501. The POLE of a circle of the sphere is a point in the

surface equally distant from every point in the circumference of the circle.

502. It will be shown (Prop. V.) that every circle, great or small, has two poles.

503. A PLANE is TANGENT to a sphere, when it meets the sphere in but one point, however far it may be produced.

504. A SPHERICAL ANGLE is the difference in the direction of two arcs of great circles of the sphere; as A E D, formed by the arcs E A, D E.



It is the same as the angle resulting from passing two planes through those arcs; as the angle formed on the edge EF, by the planes EAF, EDF.

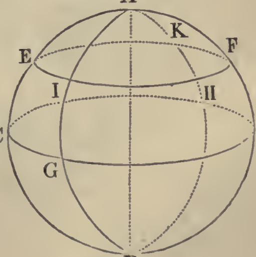
505. A SPHERICAL TRIANGLE is a portion of the surface of a sphere bounded by three arcs of great circles, each arc being less than a semi-circumference; as A E D.

These arcs are named the *sides* of the triangle; and the angles which their planes form with each other are the *angles* of the triangle.

506. A spherical triangle takes the name of *right-angled*, *isosceles*, *equilateral*, in the same cases as a plane triangle.

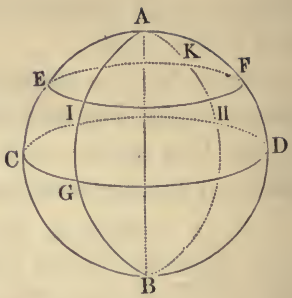
507. A SPHERICAL POLYGON is a portion of the surface of a sphere bounded by several arcs of great circles.

508. A LUNE is a portion of the surface of a sphere comprehended between semi-circumferences of two great circles; as A I G B D F.



509. A SPHERICAL WEDGE, or UNGULA, is that portion of a sphere comprehended between

two great semicircles having a common diameter.



510. A Zone is a portion of the surface of a sphere cut off by a plane, or comprehended between two parallel planes; as E I F K – A, or C G D H – E I F K.

511. A Spherical Segment is a portion of the sphere cut off by a plane, or comprehended between two parallel planes.

512. The Altitude of a Zone or of a Spherical Segment is the perpendicular distance between the two parallel planes which comprehend the zone or segment.

In case the zone or segment is a portion of the sphere cut off, one of the planes is a tangent to the sphere.

513. A Spherical Sector is a solid described by the revolution of a circular sector, in the same manner as the semicircle of which it is a part, by revolving round its diameter, describes a sphere.

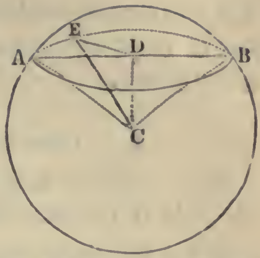
514. A Spherical Pyramid is a portion of the sphere comprehended between the planes of a polyedral angle whose vertex is the centre.

The *base* of the pyramid is the spherical polygon intercepted by the same planes.

Proposition I. — Theorem.

515. *Every section of a sphere made by a plane is a circle.*

Let A B E be a section made by a plane in the sphere whose centre is C. From the centre, C, draw C D perpendicular to the plane A B E; and draw the lines C A, C B, C E, to different points of the curve A B E, which bounds the section.



The oblique lines C A, C B, C E are equal, being radii of the sphere; therefore they are equally distant from the perpendicular, C D (Prop. V. Cor., Bk. VII.). Hence, the lines D A, D B, D E, and, in like manner, all the lines drawn from D to the boundary of the section, are equal; and therefore the section A B E is a circle whose centre is D.

516. *Cor.* 1. If the section passes through the centre of the sphere, its radius will be the radius of the sphere; hence all great circles are equal.

517. *Cor.* 2. Two great circles always bisect each other. For, since the two circles have the same centre, their common intersection, passing through the centre, must be a common diameter bisecting both circles.

518. *Cor.* 3. Every great circle divides the sphere and its surface into two equal parts. For if the two hemispheres were separated, and afterwards placed on the common base, with their convexities turned the same way, the two surfaces would exactly coincide.

519. *Cor.* 4. The centre of a small circle, and that of the sphere, are in a straight line perpendicular to the plane of the small circle.

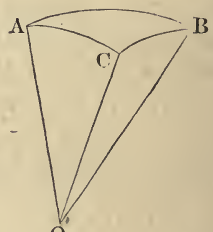
520. *Cor.* 5. Small circles are less according to their distance from the centre; for, the greater the distance C D, the smaller the chord A B, the diameter of the small circle A B E.

521. *Cor.* 6. The arc of a great circle may be made to pass through any two points on the surface of a sphere; for the two given points and the centre of the sphere determine the position of a plane. If, however, the two given points be the extremities of a diameter, these two points

and the centre would be in a straight line, and any number of great circles may be made to pass through the two given points.

Proposition II. — Theorem.

522. *Any one side of a spherical triangle is less than the sum of the other two.*

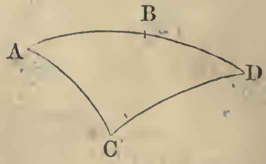


Let A B C be any spherical triangle; then any side, as A B, is less than the sum of the other two sides, A C, B C.

For, draw the radii O A, O B, O C, and the plane angles A O B, A O C, C O B will form a triedral angle, O. The angles A O B, A O C, C O B will be measured by A B, A C, B C, the side of the spherical triangle. But each of the three plane angles forming a triedral angle is less than the sum of the other two (Prop. XIX. Bk. VII.). Hence, any side of a spherical triangle is less than the sum of the other two.

Proposition III. — Theorem.

523. *The shortest path from one point to another, on the surface of a sphere, is the arc of the great circle which joins the two given points.*



Let A B D be the arc of the great circle which joins the points A and D; then the line A B D is the shortest path from A to D on the surface of the sphere.

For, if possible, let the shortest path on the surface from A to D pass through the point C, out of the arc of the great circle A B D. Draw A C, D C, arcs of great circles, and take D B equal to D C. Then in the spherical triangle A B D C the side A B D is less than the sum of the sides A C, D C (Prop. II.); and subtracting the equal D B and D C, there will remain A B less than A C.

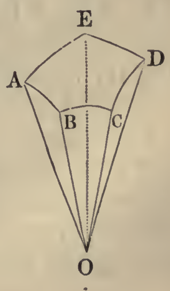
Now, the shortest path, on the surface, from D to C, whether it is the arc D C, or any other line, is equal to the shortest path from D to B; for, revolving D C about the diameter which passes through D, the point C may be brought into the position of the point B, and the shortest path from D to C be made to coincide with the shortest path from D to B. But, by hypothesis, the shortest path from A to D passes through C; consequently, the shortest path on the surface from A to C cannot be greater than that from A to B.

Now, since A B has been proved to be less than A C, the shortest path from A to C must be greater than that from A to B; but this has just been shown to be impossible. Hence, no point of the shortest path from A to D can lie out of the arc A B D; consequently, this arc of a great circle is itself the shortest path between its extremities.

524. *Cor.* The distance between any two points of surface, on the surface of a sphere, is measured by the arc of a great circle joining the two points.

PROPOSITION IV. — THEOREM.

525. *The sum of all the sides of any spherical polygon is less than the circumference of a great circle.*



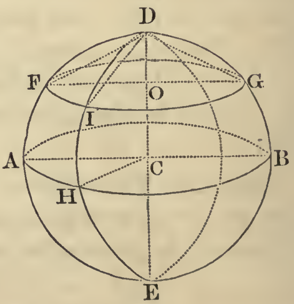
Let A B C D E be a spherical polygon; then the sum of the sides A B, B C, C D, &c. is less than the circumference of a great circle.

For, from O, the centre of the sphere, draw the radii O A, O B, O C, &c., and the plane angles A O B, B O C, C O B, &c. will form a polyedral angle at O. Now, the sum of the plane angles which form a polyedral angle is less than four right angles (Prop. XX. Bk. VII.). Hence, the sum of the arcs A B, B C, C D, &c., which measure these angles, and bound the spherical polygon, is less than the circumference of a great circle.

526. *Cor.* The sum of the three sides of a spherical triangle is less than the circumference of a great circle, since a triangle is a polygon of three sides.

PROPOSITION V. — THEOREM.

527. *The extremities of a diameter of a sphere are the poles of all circles of the sphere whose planes are perpendicular to that diameter.*



Let D E be a diameter perpendicular to A H B, a great circle of a sphere, and also to the small circle F I G; then D and E, the extremities of this diameter, are the poles of these two circles.

For, since D E is perpendicular to the plane A H B, it is perpendicular to all the straight lines, A C, H C, B C, &c., drawn through its foot in this plane; hence, all the arcs D A, D H, D B, &c. are quarters of the circumference. So, likewise, are all the arcs E A, E H, E B, &c.; hence the points D and E are each equally distant from all the points of the circumference, A H B; consequently D and E are poles of that circumference (Art. 501).

Again, since the radius D C is perpendicular to the plane A H B, it is perpendicular to the parallel plane F I G; hence it passes through O, the centre of the circle F I G (Prop. I. Cor. 4). Hence, if the oblique lines D F, D I, D G, &c. be drawn, these lines will be equally distant from the perpendicular D O, and will themselves be equal (Prop. V. Bk. VII.). But the chords being equal, the arcs are equal; hence the point D is a pole of the small circle F I G; and, for like reasons, the point E is the other pole.

528. *Cor.* 1. Every arc of a great circle, D H, drawn from a point in the arc of a great circle, A H B, to its pole, is a quarter of the circumference, and is called a quadrant. This quadrant makes a right angle with the arc A H. For, the line D C being perpendicular to the plane A H C, every plane D H C passing through the line D C is perpendicular to the plane A H C (Prop. VII. Bk. VII.); hence the angle of those planes, or the angle A H D, is a right angle (Art. 506).

529. *Cor.* 2. To find the pole of a given arc, A H, draw the indefinite arc H D perpendicular to A H, and take H D equal to a quadrant; the point D will be one of the poles of the arc A H D; or at each of the two points A and H, draw the arcs A D and H D perpendicular to A H; the point of their intersection, D, will be the pole required.

530. *Cor.* 3. Conversely, if the distance of the point D from each of the points A and H is equal to a quadrant, the point D will be the pole of the arc A H; and the angles D A H, A H D will be right.

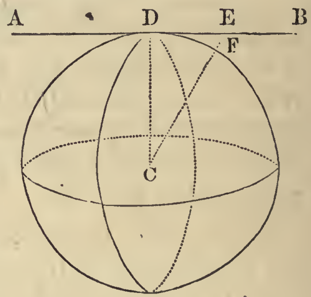
For, let C be the centre of the sphere, and draw the radii C A, C D, C H. Since the angles A C D, H C D are right, the line C D is perpendicular to the two straight lines C A, C H; hence it is perpendicular to their plane (Prop. IV. Bk. VII.). Hence the point D is the pole of the arc A H; and consequently the angles D A H, A H D are right angles.

531. *Scholium.* A circle may be described on the surface of a sphere with the same facility as on a plane surface. For instance, by turning the arc D F, or any other line extending to the same distance, round the point D the

extremity, F, will describe the small circle F I G; and by turning the quadrant D F A round the point D, its extremity, A, will describe the great circle A H B.

PROPOSITION VI. — THEOREM.

532. *A plane perpendicular to a radius, at its termination in the surface, is tangent to the sphere.*



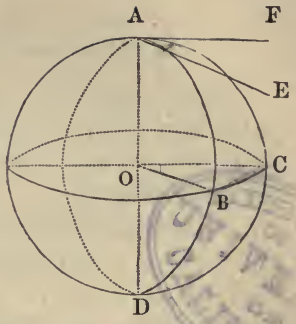
Let A D B be a plane perpendicular to a radius, C D, at its termination, D; then the plane A D B is a tangent to the sphere.

For, draw from the centre, C, any other straight line, C E, to the plane, A D B. Then, since C D is perpendicular to the plane, it is shorter than the oblique line C E; hence the radius C F is shorter than C E; consequently the point E is without the sphere. The same may be shown of any other point in the plane A D B, except the point D; hence the plane can meet the sphere in but one point, and therefore is a tangent to the sphere (Art. 503).

533. *Scholium.* In the same manner, it may be proved that two spheres are tangent to each other, when the distance between their centres is equal to the sum or the difference of their radii; in which case the centres and the point of contact lie in the same straight line.

PROPOSITION VII. — THEOREM.

534. *The angle formed by two arcs of great circles is equal to the angle formed by the tangents of those arcs at the point of their intersection, and is measured by the arc of a great circle described from its vertex as a pole, and intercepted between its sides, produced if necessary.*

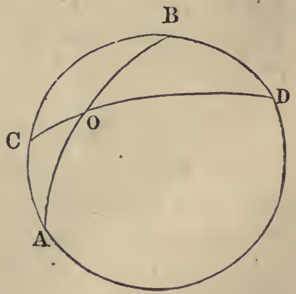


Let B A C be an angle formed by the two arcs A B, A C; then will it be equal to the angle E A F, formed by the tangents A E, A F, and it is measured by B C, the arc of a great circle described from the vertex A as a pole.

For the tangent A E, drawn in the plane of the arc A B, is perpendicular to the radius A O (Prop. X. Bk. III.); and the tangent A F, drawn in the plane of the arc A C, is perpendicular to the same radius A O. Hence the angle E A F is equal to the angle of the planes A O B, A O C (Art. 391); which is that of the arcs A B, A C.

Also, if the arcs A B, A C are both quadrants, the lines O B, O C will be perpendicular to A O, and the angle B O C will be equal to the angle of the planes A O B, A O C; hence the arc B C is the measure of the angle of these planes, or the measure of the angle C A B.

535. *Cor.* 1. The angles of spherical triangles may be compared together, by means of the arcs of great circles described from their vertices as poles, and included between their sides; hence it is easy to make an angle of this kind equal to a given angle.

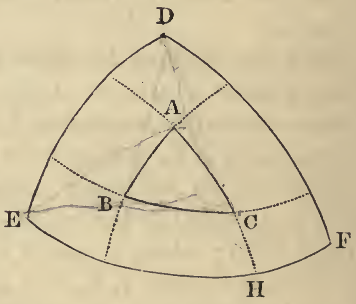


536. *Cor.* 2. Vertical angles, such as A O C and B O D, are equal; for each of them is equal to the angle formed by the two planes A O B, C O D.

It is also evident that the two adjacent angles, A O C, C O B, taken together, are equal to two right angles.

Proposition VIII. — Theorem.

537. *If from the vertices of any spherical triangle, as poles, arcs of great circles are described, a second triangle is formed, whose vertices will be poles to the sides of the first triangle.*



Let A B C be any spherical triangle; and from the vertices, A, B, C, as poles, let the arcs E F, F D, D E be described, and a second triangle, D E F, is formed, whose vertices, D, E, F, will be poles to the sides of the triangle A B C.

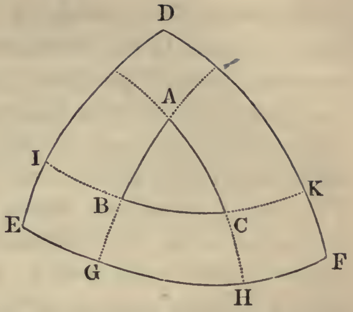
For, the point A being the pole of the arc E F, the distance A E is a quadrant; the point C being the pole of the arc D E, the distance C E is also a quadrant; hence the point E is at the distance of a quadrant from each of the points A and C; hence it is the pole of the arc A C (Prop. V. Cor. 3). In like manner, it may be shown that D is the pole of the arc B C, and F that of the arc A B.

538. *Scholium.* Hence the triangle A B C may be described by means of D E F, as D E F may be by means of A B C. Spherical triangles thus described are said to be *polar to each other*, and are called *polar* or *supplemental* triangles.

Proposition IX. — Theorem.

539. *Each of the angles of a spherical triangle is measured by a semi-circumference minus the side lying opposite to it in the polar triangle.*

Let A B C be a spherical triangle, and D E F a triangle polar to it; then each of the angles of A B C is measured



by a semi-circumference minus the side lying opposite to it in D E F.

For, produce the sides A B, A C, if necessary, till they meet E F in G and H. The point A being the pole of the arc G H, the angle A will be measured by that arc (Prop. VII.). But, E being the pole of A H, the arc E H is a quadrant; and F being the pole of A G, F G is a quadrant. Hence, E H and G F together are equal to a semi-circumference. Now, the sum of E H and G F is equal to the sum of E F and G H; hence the arc G H, which measures the angle A, is equal to a semi-circumference minus the side E F. In like manner, the angle B will be measured by a semi-circumference minus D F; and the angle C by a semi-circumference minus D E.

540. *Cor.* This property must be reciprocal in the two triangles, since they are polar to each other. The angle D, for example, of the triangle D E F, is measured by the arc I K; but the sum of I K and B C is equal to the sum of I C and B K, which is equal to a semi-circumference; hence the arc I K, the measure of D, is equal to a semi-circumference minus B C. In like manner, it may be shown that E is measured by a semi-circumference minus A C, and F by a semi-circumference minus A B.

Proposition X.—Theorem.

541. *The sum of the angles in any spherical triangles is less than six right angles, and greater than two.*

First. Every angle of a spherical triangle is less than two right angles; hence, the sum of the three is less than six right angles.

Secondly. The measure of each angle of a spherical triangle is equal to the semi-circumference minus the corresponding side of the polar triangle (Prop. IX.); hence, the sum of the three is measured by three semi-circumferences minus the sum of the sides of the polar triangle. Now, this latter sum is less than a circumference (Prop. IV. Cor.); therefore, taking it away from three semi-circumferences, the remainder will be greater than one semi-circumference, which is the measure of two right angles; hence, the sum of the three angles of a spherical triangle is greater than two right angles.

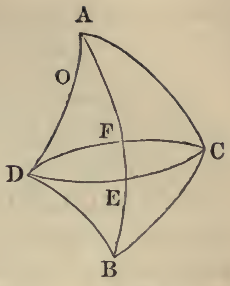
542. *Cor*. 1. The sum of the angles of a spherical triangle is not constant, like that of the angles of a rectilineal triangle. It varies between two right angles and six, without ever arriving at either of these limits. Two given angles, therefore, do not serve to determine the third.

543. *Cor*. 2. A spherical triangle may have two, or even three right angles, or obtuse angles.

544. *Scholium*. If a spherical triangle has two right angles, it is said to be *bi-rectangular;* and if it has three right angles, it is said to be *tri-rectangular*, or *quadrantal*. The quadrantal triangle is evidently contained eight times in the surface of the sphere.

Proposition XI. — Theorem.

545. *If around the vertices of any two angles of a given spherical triangle, as poles, the circumferences of two circles be described, which shall pass through the third angle of the triangle, and then if through the other point in which these circumferences intersect, and the vertices of the first two angles of the triangles, arcs of two great circles be drawn, the triangle thus formed will have all its parts equal to those of the given triangle, each to each.*



Let A B C be the given spherical triangle, and C E D, D F C arcs described about the vertices of any two of its angles, A and B, as poles; then will the triangle A D B have all its parts equal to those of A B C.

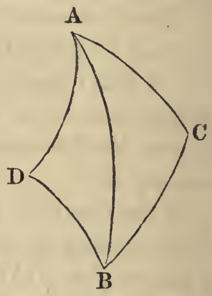
For, by construction, the side A D is equal to A C, D B is equal to B C, and A B is common; hence the two triangles have their sides equal, each to each. We are now to show that the angles opposite these equal sides are also equal.

If the centre of the sphere is supposed to be at O, a triedral angle may be conceived as formed at O by the three plane angles A O B, A O C, B O C; also, another triedral angle may be conceived as formed by the three plane angles A O B, A O D, B O D. Now, since the sides of the triangle A B C are equal to those of the triangle A D B, the plane angles forming the one of these triedral angles are equal to the plane angles forming the other, each to each. Therefore the planes, in which the equal angles lie, are equally inclined to each other (Prop. XXI. Bk. VII.); hence, all the angles of the spherical triangle D A B are respectively equal to those of the triangle C A B; namely, D A B is equal to B A C, D B A to A B C, and A D B to A C B; hence, the sides and angles of the triangle A D B are equal to the sides and the angles of the triangle A C B, each to each.

546. *Scholium.* The equality of these triangles is not, however, an absolute equality, or one of superposition; for it would be impossible to apply them to each other exactly, unless they were isosceles. The equality here meant is that by *symmetry;* therefore the triangles A C B, A D B are termed *symmetrical triangles.*

Proposition XII.—Theorem.

547. *If two triangles on the same sphere, or on equal spheres, are mutually equilateral, they are mutually equiangular; and their equal angles are opposite to equal sides.*



Let A B C, A B D be two triangles on the same sphere, or on equal spheres, having the sides of the one respectively equal to those of the other; then the angles opposite to the equal sides, in the two triangles, are equal.

For, with three given sides, A B, A C, B C, there can be constructed only two triangles, A C B, A B D, and these triangles will be equal, each to each, in the magnitude of all their parts (Prop. XI.). Hence, these two triangles, which are mutually equilateral, must be either absolutely equal, or equal by *symmetry;* in either case they are mutually equiangular, and the equal angles lie opposite to equal sides.

Proposition XIII.—Theorem.

548. *If two triangles on the same sphere, or on equal spheres, are mutually equiangular, they are mutually equilateral.*

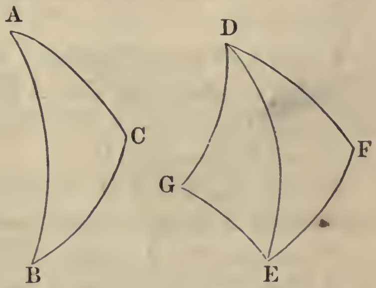
Let A and B be the two given triangles; P and Q, their polar triangles.

Since the angles are equal in the triangles A and B, the sides will be equal in the polar triangles P and Q (Prop. IX.). But since the triangles P and Q are mutually equilateral, they must also be mutually equiangular (Prop. XII.); and, the angles being equal in the triangles P and Q, it follows that the sides are equal in their polar triangles A and B. Hence, the triangles A and B, which are mutually equiangular, are at the same time mutually equilateral.

Proposition XIV. — Theorem.

549. *If two triangles on the same sphere, or on equal spheres, have two sides and the included angle in the one equal to two sides and the included angle in the other, each to each, the two triangles are equal in all their parts.*

In the two triangles A B C, D E F, let the side A B be equal to the side D E, the side A C to the side D F; and the angle B A C to the angle E D F; then the triangles will be equal in all their parts.



Let the triangle D E G be symmetrical with the triangle D E F (Prop. XI. Sch.), having the side E G equal to E F, the side G D equal to F D, and the side E D common, and consequently the angles of the one equal to those of the other (Prop. XII.).

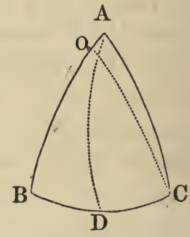
Now, the triangle A B C may be applied to the triangle D E F, or to D E G symmetrical with D E F, just as two rectilineal triangles are applied to each other, when they have an equal angle included between equal sides. Hence, all the parts of the triangle A B C will be equal to all the parts of the triangle D E F, each to each; that is, besides the three parts equal by hypothesis, we shall have the side B C equal to E F, the angle A B C equal to D E F, and the angle A C B equal to D F E.

550. *Cor.* If two triangles, A B C, D E F, on the same sphere, or on equal spheres, have two angles and the included side in the one equal to two angles and the included side in the other, each to each, the two triangles are equal in all their parts.

For one of these triangles, or the triangle symmetrical with it, may be applied to the other, as is done in the corresponding case of rectilineal triangles.

Proposition XV. — Theorem.

551. *In every isosceles spherical triangle, the angles opposite the equal sides are equal; and, conversely, if two angles of a spherical triangle are equal, the triangle is isosceles.*



Let A B C be an isosceles spherical triangle, in which the side A B is equal to the side A C; then will the angle B be equal to the angle C.

For, if the arc A D be drawn from the vertex A to the middle point, D, of the base, the two triangles A B D, A C D will have all the sides of the one respectively equal to the corresponding sides of the other, namely, A D common, B D equal to D C, and A B equal to A C; hence their angles must be equal; consequently, the angles B and C are equal.

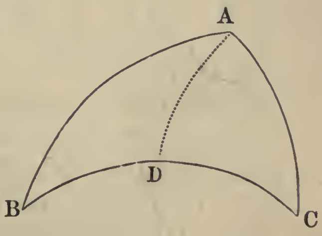
Conversely. Let the angles B and C be equal; then will the side A C be equal to A B.

For, if A C and A B are not equal, let A B be the greater of the two; take B O equal to A C, and draw O C. The two sides B O, B C in the triangle B O C are equal to the two sides A B, B C in the triangle B A C; the angle O B C, contained by the first two, is equal to A C B, contained by the second two. Hence, the two triangles B O C, B A C have all their other parts equal (Prop. XIV. Cor.); hence the angle O C B is equal to A B C. But, by hypothesis, the angle A B C is equal to A C B; hence we have O C B equal to A C B, which is impossible; therefore A B cannot be unequal to A C; consequently the sides A B, A C, opposite the equal angles B and C, are equal.

552. *Cor.* The angle B A D is equal to D A C, and the angle B D A is equal to A D C; the last two are therefore right angles; hence the arc drawn from the vertex of an isosceles spherical triangle to the middle of the base, is perpendicular to the base, and bisects the vertical angle.

Proposition XVI. — Theorem.

553. *In a spherical triangle, the greater side is opposite the greater angle; and, conversely, the greater angle is opposite the greater side.*



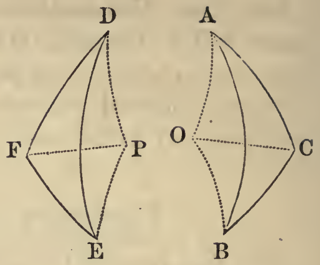
In the triangle A B C, let the angle A be greater than B; then will the side B C, opposite to A, be greater than A C, opposite to B.

Take the angle B A D equal to the angle B; then, in the triangle A B D, we shall have the side A D equal to D B (Prop. XV.). But the sum of A D plus D C is greater than A C; hence, putting D B in the place of A D, we shall have the sum of D B plus D C, or B C, greater than A C.

Conversely. Let the side B C be greater than A C; then the angle B A C will be greater than A B C. For, if B A C were equal to A B C, we should have B C equal to A C; and if B A C were less than A B C, we should then have, as has just been shown, B C less than A C. Both of these results are contrary to the hypothesis; hence the angle B A C is greater than A B C.

Proposition XVII. — Theorem.

554. *If two triangles on the same sphere, or on equal spheres, are mutually equilateral, they are equivalent.*



Let A B C, D E F be two triangles, having the three sides of the one equal to the three sides of the other, each to each, namely, A B to D E, A C to D F, and C B to E F; then their triangles will be equivalent.

Let O be the pole of the small circle passing through the three points A, B, C; draw the arcs O A, O B, O C, and they will all be equal (Prop. V. Sch.). At the point F make the angle D F P equal to A C O; make the arc F P equal to C O; and draw D P, E P.

The sides D F, F P are equal to the sides A C, C O, and the angle D F P is equal to the angle A C O; hence the two triangles D F P, A C O are equal in all their parts (Prop. XIV.); hence the side D P is equal to A O, and the angle D P F is equal to A O C.

In the triangles D F E, A B C, the angles D F E, A C B, opposite to the equal sides D E, A B, are equal (Prop. XII.). Taking away the equal angles D F P, A C O, there will remain the angle P F E, equal to O C B. The sides P F, F E are equal to the sides O C, C B; hence the two triangles F P E, C O B are equal in all their parts (Prop. XIV.); hence the side P E is equal to O B, and the angle F P E is equal to C O B.

Now, the triangles D F P, A C O, which have the sides equal, each to each, are at the same time isosceles, and may be applied the one to the other. For, having placed O A upon its equal P D, the side O C will fall on its equal P F, and thus the two triangles will coincide; consequently they are equal, and the surface D P F is equal to A O C. For a like reason, the surface F P E is equal to C O B, and the surface D P E is equal to A O B; hence we have

$$AOC + COB - AOB = DPF + FPE - DPE,$$

or,
$$ABC = DEF.$$

Hence the two triangles A B C, D E F are equivalent.

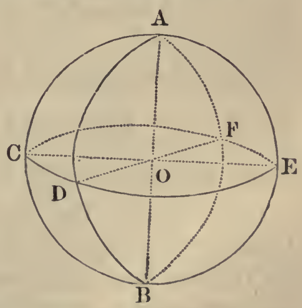
555. *Cor.* 1. If two triangles on the same sphere, or on equal spheres, are mutually equiangular, they are equivalent. For in that case the triangles will be mutually equilateral.

556. *Cor.* 2. Hence, also, if two triangles on the same sphere, or on equal spheres, have two sides and the included angle, or have two angles and the included side, in the one equal to those in the other, the two triangles are equivalent.

557. *Scholium.* The poles O and P might lie within the triangles A B C, D E F; in which case it would be requisite to add the three triangles D P F, F P E, D P E together, to form the triangle D E F; and in like manner to add the three triangles A O C, C O B, A O B together, to form the triangle A B C; in all other respects the demonstration would be the same.

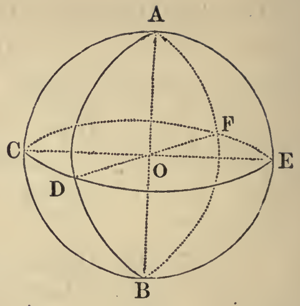
PROPOSITION XVIII. — THEOREM.

558. *The area of a lune is to the surface of the sphere as the angle of the lune is to four right angles, or as the arc which measures that angle is to the circumference.*



Let A C B D be a lune upon a sphere whose diameter is A B; then will the area of the lune be to the surface of the sphere as the angle D O C to four right angles, or as the arc D C to the circumference of a great circle.

For, suppose the arc C D to be to the circumference C D E F in the ratio of two whole numbers, as 5 to 48, for example.



Then, if the circumference C D E F be divided into 48 equal parts, C D will contain 5 of them; and if the pole A be joined with the several points of division by as many quadrants, we shall have 48 triangles on the surface of the hemisphere ACDEF, all equal, since all their parts are equal. Hence, the whole sphere must contain 96 of these triangles, and the lune A C B D 10 of them; consequently, the lune is to the sphere as 10 is to 96, or as 5 to 48; that is, as the arc C D is to the circumference.

If the arc C D is not commensurable with the circumference, it may still be shown, by a mode of reasoning exemplified in Prop. XVI. Bk. III., that the lune is to the sphere as C D is to the circumference.

559. *Cor.* 1. Two lunes on the same sphere, or on equal spheres, are to each other as the angles included between their planes.

560. *Cor.* 2. It has been shown that the whole surface of the sphere is equal to eight quadrantal triangles (Prop. X. Sch.). Hence, if the area of a quadrantal triangle be represented by T, the surface of the sphere will be represented by 8 T. Now, if the right angle be assumed as unity, and the angle of the lune be represented by A, we have,

$$\text{Area of the lune} : 8\,T : : A : 4,$$

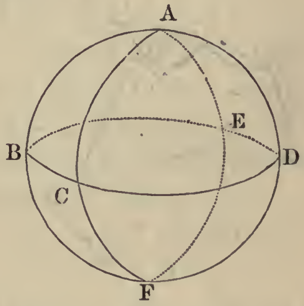
which gives the area of lune equal to $2\,A \times T$.

561. *Cor.* 3. The spherical ungula included by the planes A C B, A D B, is to the whole sphere as the angle D O C is to four right angles. For, the lunes being equal, the spherical ungulas will also be equal; hence, two spherical ungulas on the same sphere, or on equal spheres,

are to each other as the angles included between their planes.

Proposition XIX. — Theorem.

562. *If two great circles intersect each other on the surface of a hemisphere, the sum of the opposite triangles thus formed is equivalent to a lune, whose angle is equal to the angle formed by the circles.*



Let the great circles B A D, C A E intersect on the surface of a hemisphere, A B C D E; then will the sum of the opposite triangles, B A C, D A E, be equal to a lune whose angle is D A E.

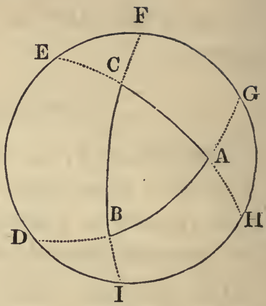
For, produce the arcs A D, A E till they meet in F; and the arcs B A D, A D F will each be a semi-circumference. Now, if we take away A D from both, we shall have D F equal to B A. For a like reason, we have E F equal to C A. D E is equal to B C. Hence, the two triangles B A C, D E F are mutually equilateral; therefore they are equivalent (Prop. XVII.). But the sum of the triangles D E F, D A E is equivalent to the lune A D F E, whose angle is D A E.

Proposition XX. — Theorem.

563. *The area of a spherical triangle is equal to the excess of the sum of its three angles above two right angles, multiplied by the quadrantal triangle.*

Let A B C be a spherical triangle; its area is equal to the excess of the sum of its angles, A, B, C, above two right angles multiplied by the quadrantal triangle.

For produce the sides of the triangle A B C till they meet the great circle DEFGHI, drawn without the triangle. The two triangles ADE, AGH are together equivalent to the lune whose angle is A (Prop. XIX.), and whose area is expressed by $2A \times T$ (Prop. XVIII. Cor. 2). Hence we have



$$ADE + AGH = 2A \times T;$$

and, for a like reason,

$$BGF + BID = 2B \times T, \text{ and } CIH + CFE = 2C \times T.$$

But the sum of these six triangles exceeds the hemisphere by twice the triangle ABC; and the hemisphere is represented by $4T$; consequently, twice the triangle ABC is equivalent to

$$2A \times T + 2B \times T + 2C \times T - 4T;$$

therefore, once the triangle ABC is equivalent to

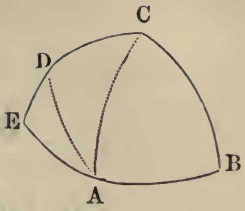
$$(A + B + C - 2) \times T.$$

Hence the area of a spherical triangle is equal to the excess of the sum of its three angles above two right angles multiplied by the quadrantal triangle.

564. *Cor.* If the sum of the three angles of a spherical triangle is equal to three right angles, its area is equal to the quadrantal triangle, or to an eighth part of the surface of the sphere; if the sum is equal to four right angles, the area of the triangle is equal to two quadrantal triangles, or to a fourth part of the surface of the sphere, &c.

Proposition XXI. — Theorem.

565. *The area of a spherical polygon is equal to the excess of the sum of all its angles above two right angles taken as many times as the polygon has sides, less two, multiplied by the quadrantal triangle.*



Let A B C D E be any spherical polygon. From one of the vertices, A, draw the arcs A C, A D to the opposite vertices; the polygon will be divided into as many spherical triangles as it has sides less two. But the area of each of these triangles is equal to the excess of the sum of its three angles above two right angles multiplied by the quadrantal triangle (Prop. XX.); and the sum of the angles in all the triangles is evidently the same as that of all the angles in the polygon; hence the area of the polygon A B C D E is equal to the excess of the sum of all its angles above two right angles taken as many times as the polygon has sides, less two, multiplied by the quadrantal triangle.

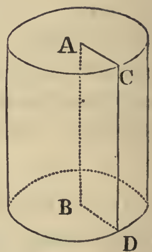
566. *Cor.* If the sum of all the angles of a spherical polygon be denoted by S, the number of sides by n, the quadrantal triangle by T, and the right angle be regarded as *unity*, the area of the polygon will be expressed by

$$\overline{S - 2\,(n - 2)} \times T = (S - 2\,n + 4) \times T.$$

BOOK X.

THE THREE ROUND BODIES.

DEFINITIONS.

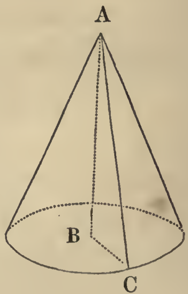


567. A CYLINDER is a solid, which may be described by the revolution of a rectangle turning about one of its sides, which remains immovable; as the solid described by the rectangle A B C D revolving about its side A B.

The BASES of the cylinder are the circles described by the sides, A C, B D, of the revolving rectangle, which are adjacent to the immovable side, A B.

The AXIS of the cylinder is the straight line joining the centres of its two bases; as the immovable line A B.

The CONVEX SURFACE of the cylinder is described by the side C D of the rectangle, opposite to the axis A B.



568. A CONE is a solid which may be described by the revolution of a right-angled triangle turning about one of its perpendicular sides, which remains immovable; as the solid described by the right-angled triangle A B C revolving about its perpendicular side A B.

The BASE of the cone is the circle described by the revolution of the side B C, which is perpendicular to the immovable side.

The CONVEX SURFACE of a cone is described by the hypothenuse, A C, of the revolving triangle.

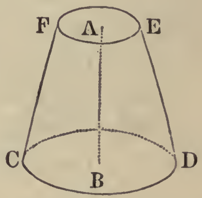
The VERTEX of the cone is the point A, where the hypothenuse meets the immovable side.

The AXIS of the cone is the straight line joining the vertex to the centre of the base; as the line A B.

The ALTITUDE of a cone is a line drawn from the vertex perpendicular to the base; and is the same as the axis, A B.

The SLANT HEIGHT, or SIDE, of a cone, is a straight line drawn from the vertex to the circumference of the base; as the line A C.

569. The FRUSTUM of a cone is the part of a cone included between the base and a plane parallel to the base; as the solid C D – F.



The AXIS, or ALTITUDE, of the frustum, is the perpendicular line A B included between the two bases; and the SLANT HEIGHT, or SIDE, is that portion of the slant height of the cone which lies between the bases; as F C.

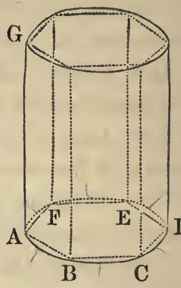
570. SIMILAR CYLINDERS, or CONES, are those whose axes are to each other as the radii, or diameters, of their bases.

571. The sphere, cylinder, and cone are termed the THREE ROUND BODIES of elementary Geometry.

PROPOSITION I. — THEOREM.

572. *The convex surface of a cylinder is equal to the circumference of its base multiplied by its altitude.*

Let A B C D E F – G be a cylinder, whose circumference is the circle A B C D E F, and whose altitude is the line A G; then its convex surface is equal to A B C D E F multiplied by A G.



In the base of the cylinder inscribe any regular polygon, A B C D E F, and on this polygon construct a right prism of the same altitude with the cylinder. The prism will be inscribed in the convex surface of the cylinder. The convex surface of this prism is equal to the perimeter of its base multiplied by its altitude, A G (Prop. I. Bk. VIII.).

Conceive now the arcs subtending the sides of the polygon to be continually bisected, until a polygon is formed having an indefinite number of sides; its perimeter will then be equal to the circumference of the circle ABCDEF (Prop. XII. Cor., Bk. VI.); and thus the convex surface of the prism will coincide with the convex surface of the cylinder. But the convex surface of the prism is always equal to the perimeter of its base multiplied by its altitude; hence, the convex surface of the cylinder is equal to the circumference of its base multiplied by its altitude.

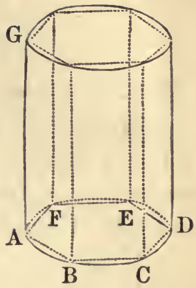
573. *Cor.* 1. If two cylinders have the same altitude, their convex surfaces are to each other as the circumferences of their bases.

574. *Cor.* 2. If H represent the altitude of a cylinder, and R the radius of its base, then we shall have the circumference of the base represented by $2\,R \times \pi$ (Prop. XV. Cor. 3, Bk. VI.), and the convex surface of the cylinder by $2\,R \times \pi \times H$.

Proposition II. — Theorem.

575. *The solid contents of a cylinder are equal to the product of its base by its altitude.*

Let A B C D E F – G be a cylinder whose base is the circle A B C D E F, and whose altitude is the line A G; then its solid contents are equal to the product of A B C D E F by A G.



In the base of the cylinder inscribe any regular polygon, A B C D E F, and on this polygon construct a right prism of the same altitude with the cylinder. The prism will be inscribed in the convex surface of the cylinder. The solid contents of this prism are equal to the product of its base by its altitude (Prop. XIII. Bk. VIII.).

Conceive now the number of the sides of the polygon to be indefinitely increased, until its perimeter coincides with the circumference of the circle A B C D E F (Prop. XII. Cor., Bk. VI.), and the solid contents of the prism will equal those of the cylinder. But the solid contents of the prism will still be equal to the product of its base by its altitude; hence the solid contents of the cylinder are equal to the product of its base by its altitude.

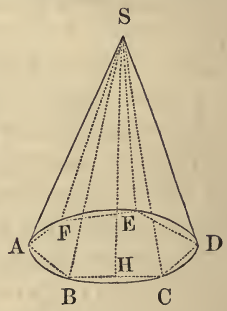
576. *Cor.* 1. Cylinders of the same altitude are to each other as their bases; and cylinders of equal bases are to each other as their altitudes.

577. *Cor.* 2. Similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the diameters of their bases. For the bases are as the squares of their radii (Prop. XIII. Bk. VI.), and the cylinders being similar, the radii of their bases are to each other as their altitudes (Art. 570); therefore the bases are as the squares of the altitudes; hence, the products of the bases by the altitudes, or the cylinders themselves, are as the cubes of the altitudes.

578. *Cor.* 3. If the altitude of a cylinder be represented by H, and the area of its base by $R^2 \times \pi$ (Prop. XV. Cor. 2, Bk. VI.), the solid contents of the cylinder will be represented by $R^2 \times \pi \times H$.

PROPOSITION III. — THEOREM.

579. *The convex surface of a cone is equal to the circumference of the base multiplied by half the slant height.*



Let A B C D E F – S be a cone whose base is the circle A B C D E F, and whose slant height is the line S A; then its convex surface is equal to A B C D E F multiplied by $\frac{1}{2}$ S A.

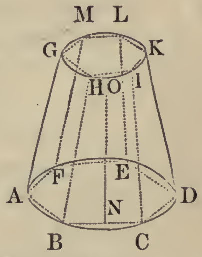
In the base of the cone inscribe any regular polygon, A B C D E F, and on this polygon construct a regular pyramid having the same vertex, S, with the cone. Then a right pyramid will be inscribed in the cone.

From S draw S H perpendicular to B C, a side of the polygon. The convex surface of the pyramid is equal to the perimeter of its base, multiplied by half its slant height, S H (Prop. XV. Bk. VIII.). Conceive now the arcs subtending the sides of the polygon to be continually bisected, until a polygon is formed having an indefinite number of sides; its perimeter will equal the circumference of the circle A B C D E F; its slant height, S H, will equal that of the cone, and its convex surface coincide with the convex surface of the cone. But the convex surface of every right pyramid is equal to the perimeter of its base, multiplied by half the slant height; hence the convex surface of the cone is equal to the circumference of its base multiplied by half its slant height.

580. *Cor.* If S A represent the slant height of a cone, and R the radius of the base, then, since the circumference of the base is represented by $2\,R \times \pi$ (Prop. XV. Cor. 3, Bk. VI.), the convex surface of the cone will be represented by $2\,R \times \pi \times \frac{1}{2}\,S\,A$, equal to $\pi \times R \times S\,A$.

Proposition IV. — Theorem.

581. *The convex surface of a frustum of a cone is equal to half the sum of the circumference of the two bases multiplied by its slant height.*



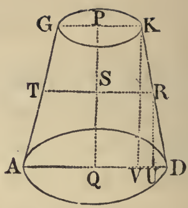
Let A B C D E F – M be the frustum of a cone, and A G its slant height; then the convex surface is equal to half the sum of the circumferences of the two bases A B C D E F, G H I K L M, multiplied by A G.

For, inscribe in the bases of the frustum two regular polygons of the same number of sides, having their sides parallel, each to each. Draw the straight lines A G, B H, C I, &c., joining the vertices of the corresponding angles, and these lines will be the edges of the frustum of a pyramid inscribed in the frustum of the cone. The convex surface of the frustum of the pyramid is equal to half the sum of the perimeters of the two bases multiplied by its slant height, O N (Prop. XVII. Bk. VIII.).

Conceive now the number of sides of the inscribed polygons to be indefinitely increased; the perimeters of the polygons will then coincide with the circumferences of the circles A B C D E F, G H I K L M; and the slant height, O N, of the frustum of the pyramid, will equal the slant height, A G, of the frustum of the cone; and the surfaces of the two frustums will coincide.

But the convex surface of every frustum of a right pyramid is equal to half the sum of the perimeters of its two bases, multiplied by its slant height; hence, the convex surface of the frustum of the cone is equal to half the sum of the circumference of its two bases multiplied by half its slant height.

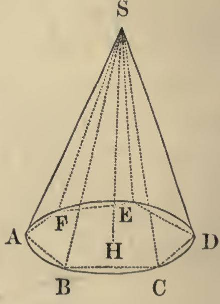
582. *Cor.* Through R, the middle point of the side K D,



draw the diameter R S T, parallel to the diameter A Q D, and the straight lines R U, K V, parallel to the axis P Q. Then, since D R is equal to R K, D U is equal to U V (Prop. XVII. Cor. 2, Bk. IV.); hence, the radius S R is equal to half the sum of the radii Q D, P K. But the circumferences of circles being to each other as their radii (Prop. XIII. Bk. VI.), the circumference of the section of which S R is the radius is equal to half the sum of the circumferences of which Q D, P K are the radii; hence, the convex surface of a frustum of a cone is equal to the slant height multiplied by the circumference of a section at equal distances between the two bases.

Proposition V. — Theorem.

583. *The solidity of a cone is equal to the product of its base by one third of its altitude.*



Let A B C D E F – S be a cone, whose base is A B C D E F, and altitude S H; then its solidity is equal to A B C D E F $\times \frac{1}{3}$ S H.

In the base of the cone inscribe any regular polygon, A B C D E F, and on this polygon construct a regular pyramid, having the same vertex, S, with the cone. Then a right pyramid will be inscribed in the cone; and its solidity will be equal to the product of its base by one third of its altitude (Prop. XX. Bk. VIII.).

Conceive, now, the number of sides of the polygon to be indefinitely increased, and its perimeter will become equal to the circumference of the cone, and the pyramid will exactly coincide with the cone. But the solidity of every right pyramid is equal to the product of the base by one third of its altitude; hence, the solidity of a cone is equal to the product of its base by one third of its altitude.

584. *Cor.* 1. A cone is the third of a cylinder having the same base and the same altitude; hence it follows,—

1. That cones of equal altitudes are to each other as their bases;

2. That cones of equal bases are to each other as their altitudes;

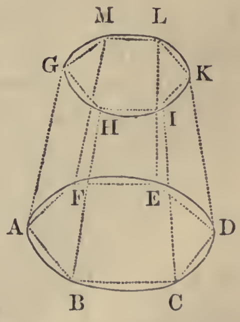
3. That similar cones are as the cubes of the diameters of their bases, or as the cubes of their altitudes.

585. *Cor.* 2. If the altitude of a cone be represented by H, and the radius of its base by R, the solidity of the cone will be represented by

$$R^2 \times \pi \times \tfrac{1}{3} H, \quad \text{or} \quad \tfrac{1}{3} \pi \times R^2 \times H.$$

Proposition VI.—Theorem.

586. *The solidity of the frustum of a cone is equivalent to the sum of three cones, having for their common altitude the altitude of the frustum, and whose bases are the two bases of the frustum, and a mean proportional between them.*



Let ABCDEF–M be the frustum of a cone; then will its solidity be equivalent to the sum of three cones having the same altitude as the frustum, and whose bases are the two bases of the frustum, and a mean proportional between them.

For, inscribe in the two bases of the frustum two regular polygons having the same number of sides, and having their sides parallel, each to each. Let the vertices of the corresponding angles be joined by the straight lines BH, CI, &c., and there is inscribed in the frustum of the cone the frustum of a regular pyramid. The solidity of the frustum of this pyramid is equivalent to the sum of three pyramids, having for their common altitude the altitude of the frustum, and whose bases are the two bases of the frustum, and a mean proportional between them.

Conceive now the number of the sides of the polygons to be indefinitely increased; and the bases of the frustum of the pyramid will equal the bases of the frustum of the cone; and the two frustums will coincide. Hence the frustum of a cone is equivalent to the sum of three cones, having for their common altitude the altitude of the frustum, and whose bases are the two bases of the frustum, and a mean proportional between them.

Proposition VII. — Theorem.

587. *If any regular semi-polygon be revolved about a line passing through the centre and the vertices of opposite angles, the surface described will be equal to the product of its axis by the circumference of its inscribed circle.*

Let the regular semi-polygon A B C D E F be revolved about A F as an axis; then the surface described by the sides A B, B C, C D, &c. will equal the product of A F by the inscribed circle.

For, from the vertices B, C, D, E of the semi-polygon, draw B G, C H, D M, E N, perpendicular to the axis A F; and from the centre, O, draw O I perpendicular to one of the sides; also draw I K perpendicular to A F, and B L perpendicular to C H.

Now O I is the radius of the inscribed circle (Prop. II. Bk. VI.); and the surface described by the revolution of a side, B C, of a regular polygon, is equal to B C multiplied by the circumference, I K (Prop. IV. Cor.).

The two triangles O I K, B C L, having their sides perpendicular to each other, are similar (Prop. XXV. Bk. IV.); therefore,

B C : B L or G H : : O I : I K : : *Circ.* O I : *Circ.* I K.

Hence (Prop. I. Bk. II.),

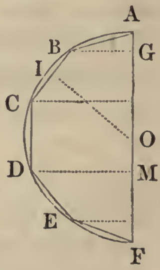
$$BC \times \textit{Circ.}\ IK = GH \times \textit{Circ.}\ OI;$$

that is, the surface described by B C is equal to the product of the altitude G H by the circumference of the inscribed circle. The same may be shown of each of the other sides; hence, the surface described by all the sides taken together is equal to the product of the sum of the altitudes A G, G H, H M, M N, N F, by the circ. O I, or to the product of the axis A F by the circ. O I.

Proposition VIII. — Theorem.

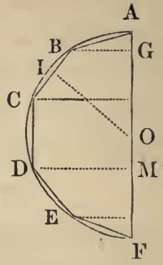
588. *The surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*

Let A B C D E F be a semicircle in which is inscribed any regular semi-polygon; from the centre, O, draw O I perpendicular to one of the sides.



If now the semicircle and the semi-polygon be revolved about the axis A F, the surface described by the semicircle will be the surface of a sphere (Art. 497), and that described by the semi-polygon will be equal to the product of its axis, A F, by the circumference, O I (Prop. VII.); and the same is true, whatever be the number of sides of the polygon.

Conceive the number of sides of the semi-polygon to be made, by continual bisections, indefinitely great; then its perimeter will coincide with the semi-circumference A B C D E F, and the perpendicular O I will be equal to the radius O A; hence, the surface of the sphere is equal to the product of the diameter by the circumference of a great circle.



589. *Cor.* 1. *The surface of a sphere is equal to the area of four of its great circles.*

For the area of a circle is equal to the product of the circumference by half the radius, or one fourth of the diameter (Prop. XV. Bk. VI.).

590. *Cor.* 2. *The surface of a zone or segment is equal to the product of its altitude by the circumference of a great circle.*

For the surface described by the sides B C, C D of the inscribed polygon is equal to the product of the altitude G M by the circumference of the inscribed circle O I. If, now, the number of the sides of an inscribed polygon be indefinitely increased, its perimeter will equal the circle, and B C, C D will coincide with the arc B C D; consequently, the surface of the zone described by the revolution of B C D is equal to the product of its altitude by the circumference of a great circle. In like manner, the same may be proved true of a segment, or a zone having but one base.

591. *Cor.* 3. The surfaces of two zones, or segments upon the same sphere, are to each other as their altitudes; and any zone or segment is to the surface of the sphere as the altitude of that zone or segment is to the diameter.

592. *Cor.* 4. If the radius of a sphere is represented by R, and its diameter by D, its surface will be represented by

$$4\pi \times R^2, \quad \text{or} \quad \pi \times D^2.$$

593. *Cor.* 5. Hence, the surfaces of spheres are to each other as the squares of their radii or diameters.

594. *Cor.* 6. If the altitude of a zone or segment is

represented by H, the surface of a zone or segment will be represented by

$$2\pi \times R \times H, \quad \text{or} \quad \pi \times D \times H.$$

PROPOSITION IX. — THEOREM.

595. *The solidity of a sphere is equal to the product of its surface by one third of its radius.*

For a sphere may be regarded as composed of an indefinite number of pyramids, each having for its base a part of the surface of the sphere, and for its vertex the centre of the sphere; consequently, all these pyramids have the radius of the sphere as their common altitude.

Now, the solidity of every pyramid is equal to the product of its base by one third of its altitude (Prop. XX. Bk. VIII.); hence, the sum of the solidities of these pyramids is equal to the product of the sum of their bases by one third of their common altitude. But the sum of their bases is the surface of the sphere, and their common altitude its radius; consequently, the solidity of the sphere is equal to the product of its surface by one third of its radius.

596. *Cor.* 1. *The solidity of a spherical pyramid or sector is equal to the product of the polygon or zone which forms its base, by one third of the radius.*

For the polygon or zone forming the base of the spherical pyramid or sector may be regarded as composed of an indefinite number of planes, each serving as a base to a pyramid, having for its vertex the centre of the sphere.

597. *Cor.* 2. Spherical pyramids, or sectors of the same sphere or of equal spheres, are to each other as their bases.

598. *Cor.* 3. A spherical pyramid or sector is to the sphere of which it is a part, as its base is to the surface of the sphere.

599. *Cor.* 4. Hence, spherical sectors upon the same

sphere are to each other as the altitudes of the zones forming their bases (Prop. VIII. Cor. 3); and any spherical sector is to the sphere as the altitude of the zone forming its base is to the diameter of the sphere.

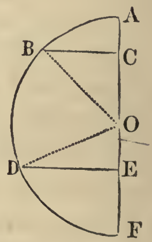
600. *Cor.* 5. If the radius of a sphere is represented by R, its diameter by D, and its surface by S, its solidity will be represented by

$$S \times \tfrac{1}{3} R = 4 \pi \times R^2 \times \tfrac{1}{3} R = \tfrac{4}{3} \pi \times R^3 \text{ or } \tfrac{1}{6} \pi \times D^3.$$

601. *Cor* 6. Hence, the solidities of spheres are to each other as the cubes of their radii.

602. *Cor.* 7. If the altitude of the zone which forms the base of a sector be represented by H, the solidity of the sector will be represented by

$$2 \pi \times R \times H \times \tfrac{1}{3} R = \tfrac{2}{3} \pi \times R^2 \times H.$$



603. *Scholium.* The solidity of the spherical segment less than a hemisphere, and of one base, formed by the revolution of a portion, A B C, of a semicircle about the radius O A, is equivalent to the solidity of the spherical sector formed by A O B, less the solidity of the cone formed by O B C.

The solidity of the spherical segment greater than a hemisphere, and of one base, formed by the revolution of A D E, is equivalent to the solidity of the spherical sector formed by A O D, plus the solidity of the cone formed by O D E.

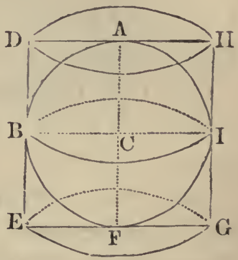
The solidity of the spherical segment of two bases formed by the revolution of C B D E about the axis A F, is equivalent to the solidity of the segment formed by A D E, less the solidity of the segment formed by A B C.

Proposition X.—Theorem.

604. *The surface of a sphere is equivalent to the convex surface of the circumscribed cylinder, and is two thirds*

of the whole surface of the cylinder; also, the solidity of the sphere is two thirds of that of the circumscribed cylinder.

Let A B F I be a great circle of the sphere; D E G H the circumscribed square; then, if the semicircle A B F and the semi-square A D E F be revolved about the diameter A F, the semicircle will describe a sphere, and the semisquare a cylinder circumscribing the sphere.



The convex surface of the cylinder is equal to the circumference of its base multiplied by its altitude (Prop. I.). But the base of the cylinder is equal to the great circle of the sphere, its diameter E G being equal to the diameter B I, and the altitude D E is equal to the diameter A F; hence, the convex surface of the cylinder is equal to the circumference of the great circle multiplied by its diameter. This measure is the same as that of the surface of the sphere (Prop. VIII.); hence, the surface of the sphere is equal to the convex surface of the circumscribed cylinder.

But the surface of the sphere is equal to four great circles of the sphere (Prop. VIII. Cor. 1); hence, the convex surface of the cylinder is also equal to four great circles; and adding the two bases, each equal to a great circle, the whole surface of the circumscribed cylinder is equal to six great circles of the sphere; hence, the surface of the sphere is $\frac{4}{6}$ or $\frac{2}{3}$ of the whole surface of the circumscribed sphere.

In the next place, since the base of the circumscribed cylinder is equal to a great circle of the sphere, and its altitude to the diameter, the solidity of the cylinder is equal to a great circle multiplied by its diameter (Prop. II.). But the solidity of the sphere is equal to its surface, or four great circles, multiplied by one third of its radius (Prop. IX.), which is the same as one great circle multiplied by $\frac{4}{3}$ of the radius, or by $\frac{2}{3}$ of the diameter; hence, the solidity of the sphere is equal to $\frac{2}{3}$ of that of the circumscribed cylinder.

605. *Cor.* 1. Hence the sphere is to the circumscribed cylinder as 2 to 3; and their solidities are to each other as their surfaces.

606. *Cor.* 2. Since a cone is one third of a cylinder of the same base and altitude (Prop. V. Cor. 1), if a cone has the diameter of its base and its altitude each equal to the diameter of a given sphere, the solidities of the cone and sphere are to each other as 1 to 2; and the solidities of the cone, sphere, and circumscribing cylinder are to each other, respectively, as 1, 2, and 3.

BOOK XI.

APPLICATIONS OF GEOMETRY TO THE MENSURATION OF PLANE FIGURES.

DEFINITIONS.

607. MENSURATION OF PLANE FIGURES is the process of determining the areas of plane surfaces.

608. The AREA of a figure, or its quantity of surface, is determined by the number of times the given surface contains some other area, assumed as the unit of measure.

609. The MEASURING UNIT assumed for a given surface is called the *superficial* unit, and is usually a *square*, taking its name from the *linear* unit forming its side; as a square whose side is 1 inch, 1 foot, 1 yard, &c.

Some superficial units, however, have no corresponding linear unit; as the rood, acre, &c.

610. TABLE OF LINEAR MEASURES.

12	Inches	make	1 Foot.
3	Feet	"	1 Yard.
$5\frac{1}{2}$	Yards	"	1 Rod or Pole.
40	Rods	"	1 Furlong.
8	Furlongs	"	1 Mile.

Also,

$7\frac{92}{100}$	Inches	"	1 Link.
25	Links	"	1 Rod or Pole.
100	Links	"	1 Chain.
10	Chains	"	1 Furlong.
8	Furlongs	"	1 Mile.

NOTE.—For other linear measures, see National Arithmetic, Art. 133, 134, 136.

611. Table of Surface Measures.

144	Square Inches	make	1 Square Foot.
9	Square Feet	"	1 Square Yard.
$30\frac{1}{4}$	Square Yards	"	1 Square Rod or Pole.
40	Square Rods	"	1 Rood.
4	Roods	"	1 Acre.
640	Acres	"	1 Square Mile.

Also,

625	Square Links	"	1 Square Rod.
16	Square Rods	"	1 Square Chain.
10	Square Chains	"	1 Acre.

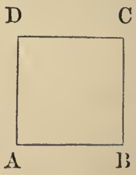
612. Since an acre is equal to 10 chains, or 100,000 links, square chains may be readily reduced to acres by pointing off one decimal place from the right, and square links by pointing off five decimal places from the right.

Problem I.

613. To find the area of a PARALLELOGRAM.

Multiply the base by the altitude, and the product will be the area (Prop. V. Bk. IV.).

Examples.



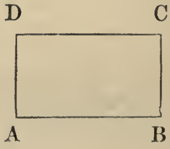
1. What is the area of a square, A B C D, whose side is 25 feet?

$$25 \times 25 = 625 \text{ feet, Ans.}$$

2. What is the area of a square field whose side is 35.25 chains? Ans. 124 A. 1 R. 1 P.

3. How many square feet of boards are required to lay a floor 21 ft. 6 in. square?

4. Required the area of a square farm, whose side is 3,525 links.



5. What is the area of the rectangle A B C D, whose length, A B, is 56 feet, and whose width, A D, is 37 feet?

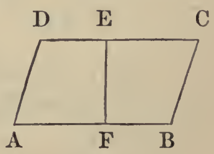
$$56 \times 37 = 2{,}072 \text{ feet, Ans.}$$

6. How many square feet in a plank, of a rectangular form, which is 18 feet long and 1 foot 6 inches wide?

7. How many acres in a rectangular garden, whose sides are 326 and 153 feet? Ans. 1 A. 23 P. $6\frac{1}{4}$ yd.

8. A rectangular court 68 ft. 3 in. long, by 56 ft. 8 in. broad, is to be paved with stones of a rectangular form, each 2 ft. 3 in. by 10 in.; how many stones will be required? Ans. $2,062\frac{2}{3}$ stones.

9. Required the area of the rhomboid A B C D, of which the side A B is 354 feet, and the perpendicular distance, E F, between A B and the opposite side C D, is 192 feet.



$$354 \times 192 = 67,968 \text{ feet, Ans.}$$

10. How many square feet in a flower-plat, in the form of a rhombus, whose side is 12 feet, and the perpendicular distance between two opposite sides of which is 8 feet?

11. How many acres in a rhomboidal field, of which the sides are 1,234 and 762 links, and the perpendicular distance between the longer sides of which is 658 links?

Ans. 8 A. 19 P. 4 yd. $6\frac{1}{4}$ ft.

Problem II.

614. The area of a SQUARE being given, to find the side.

Extract the square root of the area.

Scholium. This and the two following problems are the converse of Prob. I.

Examples.

1. What is the side of a square containing 625 square feet?

$$\sqrt{625} = 25 \text{ feet, the side required.}$$

2. The area of a square farm is 124 A. 1 R. 1 P.; how many links in length is its side?

3. A certain corn-field in the form of a square contains

15 A. 2 R. 20 P. If the corn is planted on the margin, 4 hills to a rod in length, how many hills are there on the margin of the field ? Ans. 800 hills.

Problem III.

615. The area of a RECTANGLE and either of its sides being given, to find the other side.

Divide the area by the given side, and the quotient will be the other side.

Examples.

1. The area of a rectangle is 2,072 feet, and the length of one of the sides is 56 feet; what is the length of the other side ?

$$2072 \div 56 = 37 \text{ feet, the side required.}$$

2. How long must a rectangular board be, which is 15 inches in width, to contain 11 square feet ?

3. A rectangular piece of land containing 6 acres is 120 rods long; what is its width ? Ans. 8 rods.

4. The area of a rectangular farm is 266 A. 3 R. 8 P., and the breadth 46 chains; what is the length ?

Ans. 58 chains.

Problem IV.

616. The area of a RHOMBOID or RHOMBUS and the length of the base being given, to find the altitude; or the area and the altitude being given, to find the base.

Divide the area by the length of the base, and the quotient will be the altitude; or divide the area by the altitude, and the quotient will be the length of the base.

Examples.

1. The area of a rhomboid is 67,968 square feet, and the length of the side taken as its base 354 feet; what is the altitude ?

$$67{,}968 \div 354 = 192 \text{ feet, the altitude required.}$$

2. The area of a piece of land in the form of a rhombus

is 69,452 square feet, and the perpendicular distance between two of its opposite sides is 194 feet; required the length of one of the equal sides. Ans. 358 ft.

3. On a base 12 feet in length it is required to find the altitude of a rhomboid containing 968 square feet.

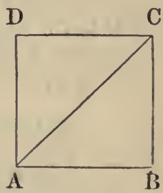
4. The area of a rhomboidal-shaped park is 1 A. 3 R. 34 P. $5\frac{1}{2}$ yd.; and the perpendicular distance between the two shorter sides is 96 yards; required the length of each of these sides? Ans. 18 rods.

Problem V.

617. The diagonal of a SQUARE being given, to find the area.

Divide the square of the diagonal by 2, *and the quotient will be the area.* (Prop. XI. Cor. 4, Bk. IV.)

Examples.



1. The diagonal, A C, of the square A B C D, is 30 feet; what is the area?

$30^2 = 900$; $900 \div 2 = 450$ square feet, [the area required.

2. The diagonal of a square field is 45 chains; how many acres does it contain?

3. The distance across a public square diagonally is 27 rods; what is the area of the square?

Problem VI.

618. The area of a SQUARE being given, to find the diagonal.

Extract the square root of double the area.

Scholium. This problem is the converse of the last.

Examples.

1. The area of a square is 450 square feet; what is its diagonal?

$450 \times 2 = 900$; $\sqrt{900} = 30$ feet, the diagonal required.

2. The area of a public square is 4 A. 2 R. 9 P.; what is the distance across it diagonally?

3. The area of a square farm is 57.8 acres; what is the diagonal in chains? Ans. 34 chains.

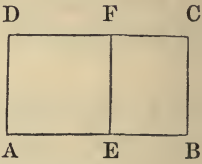
Problem VII.

619. The sides of a RECTANGLE being given, to cut off a given area by a line parallel to either side.

Divide the given area by the side which is to retain its length or width, and the quotient will be the length or width of the part to be cut off. (Prop. IV. Sch., Bk. IV.)

Examples.

1. If the sides of a rectangle, ABCD, are 25 and 14 feet, how wide an area, EBCF, to contain 154 square feet, can be cut off by a line parallel to the side AD?



$$154 \div 14 = 11 \text{ feet, the width required.}$$

2. A farmer has a field 16 rods square, and wishes to cut off from one side a rectangular lot containing exactly one acre; what must be the width of the lot?

3. A carpenter sawed off, from the end of a rectangular plank, in a line parallel to its width, 5 square feet. From the remainder he then sawed off, in a line parallel to the length, 8 square feet. Required the dimensions of the part still remaining, provided the original dimensions of the plank were 20 feet by 15 inches.

Ans. 16 feet by 9 inches.

4. The length of a certain rectangular lot is 64 rods, and its width 50 rods; how far from the longer side must a parallel line be drawn to cut off an area of 4 acres, and how far from the shorter side of the remaining portion to cut off 5 acres and 2 roods? How many acres will remain after the two portions are cut off?

Problem VIII.

620. To find the area of a TRIANGLE, the base and altitude being given.

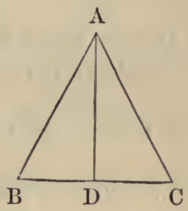
Multiply the base by half the altitude (Prop. VI. Bk. IV.).

621. *Scholium.* The same result can be obtained by multiplying the altitude by half the base, or by multiplying together the base and altitude and taking half the product.

Examples.

1. Required the area of the triangle A B C, whose base, B C, is 210, and altitude, A D, is 190 feet.

$$210 \times \frac{190}{2} = 19{,}950$$ square feet, the [area required.



2. A piece of land is in the form of a right-angled triangle, having the sides about the right angle, the one 254 and the other 136 yards; required the area in acres.

Ans. 3 A. 2 R. 10 P. 29½ yd.

3. Required the number of square feet in a triangular board whose base is 27 inches and altitude 27 feet.

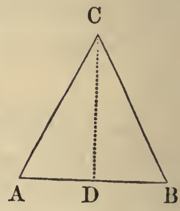
4. What is the area of a triangle whose base is 15.75 chains, and the altitude 10.22 chains?

5. What is the area of a triangular field whose base is 97 rods, and the perpendicular distance from the base to the opposite angle 40 rods? Ans. 12 A. 20 P.

Problem IX.

622. To find the area of a TRIANGLE, the three sides being given.

From half the sum of the three sides subtract each *side; multiply the half sum and the three remainders together, and the square root of the product will be the area required.*



For, let A B C be a triangle whose three sides, A B, B C, A C, are given, but not the altitude C D, and let the side B C be represented by a, A C by b, and A B by c.

Now, since A is an acute angle of the triangle A B C, we have (Prop. XII. Bk. IV.),

$$a^2 = b^2 + c^2 - 2c \times AD, \quad \text{or} \quad AD = \frac{b^2 + c^2 - a^2}{2c}.$$

Hence, in the right-angled triangle A D C, we have (Prop. XI. Cor. 1, Bk. IV.),

$$CD^2 = b^2 - \frac{(b^2 + c^2 - a^2)^2}{4c^2} = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4c^2};$$

and, by extracting the square root,

$$CD = \frac{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}{2c}.$$

But the area of the triangle A B C is equivalent to the product of c by half of C D (Prob. VIII.); hence

$$ABC = \tfrac{1}{4}\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}.$$

The expression $4b^2c^2 - (b^2 + c^2 - a^2)^2$, being the difference of two squares, can be decomposed into

$$(2bc + b^2 + c^2 - a^2) \times (2bc - b^2 - c^2 + a^2).$$

Now, the first of these factors may be transformed to $(b + c)^2 - a^2$, and consequently may be resolved into $(b + c + a) \times (b + c - a)$; and the second is the same thing as $a^2 - (b - c)^2$, which is equal to $(a + b - c) \times (a - b + c)$. We have then

$$4b^2c^2 - (b^2 + c^2 - a^2)^2 = (a + b + c) \times (b + c - a) \times (a + c - b) \times (a + b - c).$$

Let S represent half the sum of the three sides of the triangle; then

$$a+b+c=2\,S;\quad b+c-a=2\,(S-a);$$
$$a+c-b=2\,(S-b);\quad a+b-c=2\,(S-c);$$

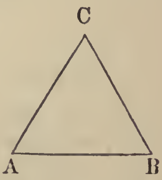
hence

$$ABC=\tfrac{1}{4}\sqrt{16\,S\,(S-a)\times(S-b)\times(S-c)},$$

which, being reduced, gives as the area of the triangle, as given above,

$$\sqrt{S\,(S-a)\times(S-b)\times(S-c)}.$$

Examples.



1. What is the area of a triangle, ABC, whose sides, AB, BC, CA, are 40, 30, and 50 feet?

$\overline{30+40+50}\div 2 = 60$, half the sum of the three sides.
$60-30=30$, first remainder.
$60-40=20$, second remainder.
$60-50=10$, third remainder.
$60\times 30\times 20\times 10=180{,}000$; $\sqrt{180{,}000}=424.26$ square feet, the area required.

2. How many square feet in a triangular floor, whose sides are 15, 16, and 21 feet?

3. Required the area of a triangular field whose sides are 834, 658, and 423 links.

Ans. 1 A. 1 R. 20 P. 4 yd. 1.6 ft.

4. Required the area of an equilateral triangle, of which each side is 15 yards.

5. What is the area of a garden in the form of a parallelogram, whose sides are 432 and 263 feet, and a diagonal 342 feet? Ans. 2 A. 10 P. 11.46 yd.

6. Required the area of an isosceles triangle, whose base is 25 and each of its equal sides 40 rods.

7. What is the area of a rhomboidal field, whose sides are 57 and 83 rods, and the diagonal 127 rods?

Ans. 22 A. 3 R. 21 P. 26 yd. 5 ft.

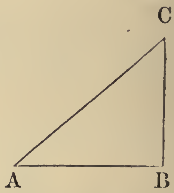
Problem X.

623. Any two sides of a RIGHT-ANGLED TRIANGLE being given, to find the third side.

To the square of the base add the square of the perpendicular; and the square root of the sum will give the hypothenuse (Prop. XI. Bk. IV.).

From the square of the hypothenuse subtract the square of the given side, and the square root of the difference will be the side required (Prop. XI. Cor. 1, Bk. IV.).

Examples.



1. The base, AB, of the triangle ABC is 48 feet, and the perpendicular, BC, 36 feet; what is the hypothenuse?

$48^2 + 36^2 = 3600$; $\sqrt{3600} = 60$ feet,
[the hypothenuse required.

2. The hypothenuse of a triangle is 53 feet, and the perpendicular 28 feet; what is the base?

3. Two ships sail from the same port, one due west 50 miles, and the other due south 120 miles; how far are they apart? Ans. 130 miles.

4. A rectangular common is 25 rods long and 20 rods wide; what is the distance across it diagonally?

5. If a house is 40 feet long and 25 feet wide, with a pyramidal-shaped roof 10 feet in height, how long is a rafter which reaches from the vertex of the roof to a corner of the building?

6. There is a park in the form of a square containing 10 acres; how many rods less is the distance from the centre to each corner, than the length of the side of the square? Ans. 11.716 rods.

Problem XI.

624. The sum of the hypothenuse and perpendicular

and the base of a RIGHT-ANGLED TRIANGLE being given, to find the hypothenuse and the perpendicular.

To the square of the sum add the square of the base, and divide the amount by twice the sum of the hypothenuse and perpendicular, and the quotient will be the hypothenuse.

From the sum of the hypothenuse and perpendicular subtract the hypothenuse, and the remainder will be the perpendicular.

625. *Scholium.* This problem may be regarded as equivalent to the sum of two numbers and the difference of their squares being given, to find the numbers (National Arithmetic, Art. 553).

NOTE. — The learner should be required to give a geometrical demonstration of the problem, as an exercise in the application of principles.

EXAMPLES.

1. The sum of the hypothenuse and the perpendicular of a right-angled triangle is 160 feet, and the base 80 feet; required the hypothenuse and the perpendicular.

Ans. Hypothenuse, 100 ft.; perpendicular, 60 ft.

$$160^2 + 80^2 = 32{,}000;\ 32{,}000 \div (160 \times 2) = 100;$$
$$160 - 100 = 60.$$

2. Two ships leave the same anchorage; the one, sailing due north, enters a port 50 miles from the place of departure, and the other, sailing due east, also enters a port, but by sailing thence in a direct course enters the port of the first; now, allowing that the second passed over, in all, 90 miles, how far apart are the two ports?

3. A tree 100 feet high, standing perpendicularly on a horizontal plane, was broken by the wind, so that, as it fell, while the part broken off remained in contact with the upright portion, the top reached the ground 40 feet from the foot of the tree; what is the length of each part?

Ans. The part broken off, 58 ft.; the upright, 42 ft.

PROBLEM XII.

626. The area and the base of a TRIANGLE being given, to find the altitude; or the area and altitude being given, to find the base.

Divide double the area by the base, and the quotient will be the altitude; or divide double the area by the altitude, and the quotient will be the base.

627. *Scholium.* This problem is the converse of Prob. VIII.

EXAMPLES.

1. The area of a triangle is 1300 square feet, and the base 65 feet; what is the altitude?

$1300 \times 2 = 2600$; $2600 \div 65 = 40$ ft., altitude required.

2. The area of a right-angled triangle is 17,272 yards, of which one of the sides about the right angle is 136 yards; required the other perpendicular side.

3. The area of a triangle is 46.25 chains, and the altitude 5.2 chains; what is the base?

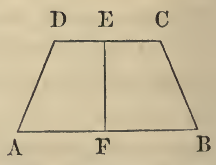
4. A triangular field contains 30 A. 3 R. 27 P.; one of its sides is 97 rods; required the perpendicular distance from the opposite angle to that side. Ans. 102 rods.

PROBLEM XIII.

628. To find the area of a TRAPEZOID.

Multiply half the sum of its parallel sides by its altitude (Prop. VII. Bk. IV.).

EXAMPLES.



1. What is the area of the trapezoid A B C D, whose parallel sides, A B, D C, are 32 and 24 feet, and the altitude, E F, 20 feet?

$32 + 24 = 56$; $56 \div 2 = 28$; $28 \times 20 = 560$ sq. ft., [the area required.

2. How many square feet in a board in the form of a trapezoid, whose width at one end is 2 feet 3 inches, and at the other 1 foot 6 inches, the length being 16 feet?

3. Required the area of a garden in the form of a trapezoid, whose parallel sides are 786 and 473 links, and the perpendicular distance between them 986 links.

Ans. 6 A. 33 P. 3 yd.

4. How many acres in a quadrilateral field, having two parallel sides 83 and 101 rods in length, and which are distant from each other 60 rods?

Problem XIV.

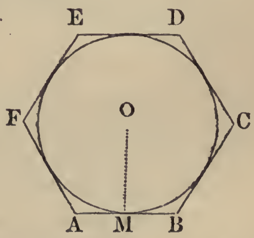
629. To find the area of a REGULAR POLYGON, the perimeter and apothegm being given.

Multiply the perimeter by half the apothegm, and the product will be the area (Prop. VIII. Bk. VI.).

630. *Scholium.* This is in effect resolving the polygon into as many equal triangles as it has sides, by drawing lines from the centre to all the angles, then finding their areas, and taking their sum.

Examples.

1. Required the area of a regular hexagon, A B C D E F, whose sides, A B, B C, &c. are each 15 yards, and the apothegm, O M, 13 yards.



$15 \times 6 = 90$; $90 \div \frac{13}{2} = 585$ yd.,
[the area required.

2. What is the area of a regular pentagon, whose sides are each 25 feet, and the perpendicular from the centre to a side 17.205 feet?

3. A park is laid out in the form of a regular heptagon, whose sides are each 19.263 chains; and the perpendicular

distance from the centre to each of the sides is 20 chains. How many acres does it contain?

Ans. 134 A. 3 R. 14 P.

Problem XV.

631. To find the area of a REGULAR POLYGON, its side or perimeter being given.

Multiply the square of the side of the polygon by the area of a similar polygon whose side is unity or 1 (Prop. XXXI. Bk. IV.).

632. A Table of Regular Polygons whose Side is 1.

NAMES.	AREAS.	NAMES.	AREAS.
Triangle,	0.4330127	Octagon,	4.8284271
Square,	1.0000000	Nonagon,	6.1818242
Pentagon,	1.7204774	Decagon,	7.6942088
Hexagon,	2.5980762	Undecagon,	9.3656399
Heptagon,	3.6339124	Dodecagon,	11.1961524

The apothegm of any regular polygon whose side is 1 being ascertained, its area is computed readily, by Prob. XIV.

Examples.

1. Required the area of an equilateral triangle, whose side is 100 feet.

$100^2 = 10{,}000$; $10{,}000 \times 0.4330127 = 4330.127$ square [feet, the area required.

2. What is the area of a regular pentagon, whose side is 37 yards?

3. How many acres in a field in the form of a regular undecagon, whose side is 27 yards?

Ans. 1 A. 1 R. 25 P. 21 yd. 2.7 ft.

4. What is the area of an octagonal floor, whose side is 15 ft. 6 in.?

5. How many acres in a regular nonagon, whose perimeter is 2286 feet? Ans. 9 A. 24 P. 28 yd.

PROBLEM XVI.

633. To find the side of any REGULAR POLYGON, its area being given.

Divide the given area by the area of a similar polygon whose side is 1, *and the square root of the quotient will be the side required.*

634. *Scholium.* This problem is the converse of Prob. XV.

EXAMPLES.

1. The area of an equilateral triangle is 4330.127 square feet; what is its side?

$4330.127 \div .4330127 = 10{,}000$; $\sqrt{10{,}000} = 100$ feet, [the side required.

2. The area of a regular hexagon is 1039.23 feet; what is its side?

3. The area of a regular decagon is 7 P. 18 yd. 5 ft. 128.55 in.; what is its side? Ans. 16 ft. 5 in.

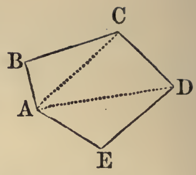
PROBLEM XVII.

635. To find the area of an IRREGULAR POLYGON.

Divide the polygon into triangles, or triangles and trapezoids, and find the areas of each of them separately; the sum of these areas will be the area required.

636. *Scholium.* When the irregular polygon is a quadrilateral, the area may be found by multiplying together the diagonal and half the sum of the perpendiculars drawn from it to the opposite angles.

Examples.



1. Required the area of the irregular pentagon A B C D E, of which the diagonal A C is 20 feet, and A D 36 feet; and the perpendicular distance from the angle B to A C is 8 feet, from C to A D 12 feet, and from E to A D 6 feet.

$20 \times \frac{8}{2} = 80$; $36 \times \frac{12}{2} = 216$; $36 \times \frac{6}{2} = 108$;
$80 + 216 + 108 = 504$ sq. ft., the area required.

2. What is the area of a trapezium, whose diagonal is 42 feet, and the two perpendiculars from the diagonal to the opposite angles are 16 and 18 feet?

3. In an irregular hexagon, A B C D E F, are given the sides A B 536, B C 498, C D 620, D E 580, E F 398, and A F 492 links, and the diagonals A C 918, C E 1048, and A E 652 links; required the area.

Ans. 6 A. 2 R. 9 P. 23 yd. 8.4 ft.

4. In measuring along one side, A B, of a quadrangular field, A B C D, that side and the perpendiculars let fall on it from two opposite corners measured as follows: A B 1110, A E 110, A F 745, D E 352, C F 595 links. What is the area of the field? Ans. 4 A. 1 R. 5 P. 24 yd.

5. In a four-sided rectilineal field, A B C D, on account of obstructions, there could be taken only the following measures: the two sides B C 265 and A D 220 yards, the diagonal A C 378, and the two distances of the perpendiculars from the ends of the diagonal, namely, A E 100, and C F 70 yards. Required the area in acres.

Problem XVIII.

637. To find the circumference of a circle, when the diameter is given, or the diameter when the circumference is given.

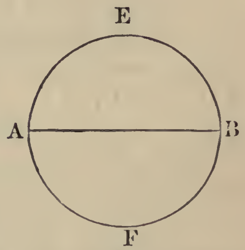
Multiply the diameter by 3.1416, *and the product will be the circumference; or, divide the circumference by*

3.1416, *and the quotient will be the diameter* (Prop. XV. Cor. 3, Bk. VI.).

638. *Scholium.* The diameter may also be found by multiplying the circumference by .31831, the reciprocal of 3.1416.

EXAMPLES.

1. The diameter, A B, of the circle A E B F is 100 feet; what is its circumference?



$100 \times 3.1416 = 314.16$ feet, the [circumference required.

2. Required the circumference of a circle whose diameter is 628 links. Ans. 1 fur. 38 rd. 5 yd. 1.56 in.

3. If the diameter of the earth is 7912 miles, what is its circumference?

4. Required the diameter of a circular pond whose circumference is 928 rods.

Ans. 7 fur. 15 rd. 2 yd. 5.55 in.

5. The circumference of a circular garden is 1043 feet; what is its radius? Ans. 10 rd. 1 ft.

PROBLEM XIX.

639. To find the length of an arc of a circle containing any number of degrees, the radius or diameter being given.

Multiply the number of degrees in the given arc by 0.01745, *and the product by the radius of the circle.*

For, when the diameter of a circle is 1, the circumference is 3.1416 (Prop. XV. Sch. 1, Bk. VI.); hence, when the radius is 1, the circumference is 6.2832; which, divided by 360, the number of degrees into which every circle is supposed to be divided, gives 0.01745, the length of the arc of 1 degree, when the radius is 1.

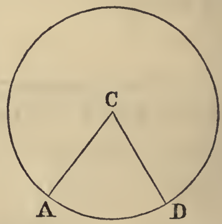
640. *Scholium.* Each of the 360 degrees of a circle,

marked thus, 360°, is divided into 60 minutes, marked thus, 60′, and each minute into 60 seconds, marked thus, 60″ (National Arithmetic, Art. 143).

EXAMPLES.

1. What is the length of an arc, A D, containing 60° 30′ on the circumference of a circle whose radius, A C, is 100 feet?

60° 30′ = 60.5°; 60.5 × 0.01745 = 1.055725; 1.055725 × 100 = 105.5725 ft., arc required.



2. Required the length of an arc of 31° 15′, the radius being 12 yards.

3. Required the length of an arc of 12° 10′, the diameter being 20 feet. Ans. 2.1231 feet.

4. What is the length of an arc of 57° 17′ 44½″, the radius being 25 feet? Ans. 25 feet.

PROBLEM XX.

641. To find the area of a circle.

Multiply the circumference by half the radius (Prop. XV. Bk. VI.); *or, multiply the square of the radius by* 3.1416 (Prop. XV. Cor. 2, Bk VI.).

642. *Scholium.* Multiplying the circumference by half the radius is the same as multiplying the circumference and diameter together, and taking one fourth of the product. Now, denoting the circumference by c, and the diameter by d, since $c = 3.1416 \times d$ (Prob. XVIII.), we have $(d \times 3.1416 \times d) \div 4 = d^2 \times 0.7854 =$ the area of a circle. Again, since $d = c \div 3.1416$ (Prob. XVIII.), we have $c \div 3.1416 \times c \div 4 = c^2 \div 12.5664$, which is, by taking the reciprocal of 12.5664, equal to $c^2 \times 0.07958$ = the area of the circle. Hence the area of the circle may also be found by *multiplying the square of the diam-*

eter by 0.7854; *or by multiplying the square of the circumference by* 0.07958.

EXAMPLES.

1. The circumference of a circle is 314.16 feet, and its radius 50 feet; what is its area?

$$314.16 \times \frac{50}{2} = 7854 \text{ feet, the area required.}$$

2. If the circumference of a circle is 355 feet, and its diameter 113 feet, what is the area?

3. What is the area of a circular garden, whose radius is $281\frac{1}{2}$ links? Ans. 2 A. 1 R. 38 rd. 9 yd. 5 ft.

4. A horse is tethered in a meadow by a cord 39.25075 yards long; over how much ground can he graze?

5. Required the area of a semicircle, the diameter of the whole circle being 751 feet.

Ans. 5 A. 13 P. 16 yd.

PROBLEM XXI.

643. To find the DIAMETER or CIRCUMFERENCE, the area being given.

Divide the area by 0.7854, *and the square root of the quotient will be the diameter; or, divide the area by* 0.07958, *and the square root of the quotient will be the circumference.*

644. *Scholium.* This problem is the converse of Prob. XX.

EXAMPLES.

1. The area of a circle is 314.16 feet; what is the diameter?

$314.16 \div 0.7854 = 400$; $\sqrt{400} = 20$ feet, the diameter [required.

2. What must be the length of a cord to be used as a radius in describing a circle which shall contain exactly 1 acre?

3. The area of a circular pond is 6 A. 1 R. 27 P. 18.2 yd.; what is the circumference? Ans. 625 yd.

4. The area of a circle is 7856 feet; what is the circumference?

5. The length of a rectangular garden is 32, and its width 18 rods; required the diameter of a circular garden having the same area. Ans. 27 rd. 1 ft. 4 in.

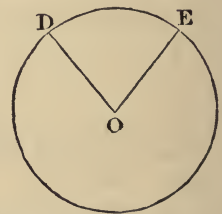
Problem XXII.

645. To find the area of a sector of a circle.

Multiply the arc of the sector by half of its radius (Prop. XV. Cor. 1, Bk. VI.); *or,*

As 360° *are to the degrees in the arc of the sector, so is the area of the circle to the area of the sector.*

Examples.



1. Required the area of a sector, D E, whose arc is 80 feet, and its radius, O E, 70 feet.

$80 \times \frac{70}{2} = 2800$ square feet, the area [required.

2. Required the area of a sector, of which the arc is 90 and the radius 112 yards.

3. Required the area of a sector, of which the angle is 137° 20′, and the radius 456 links.

Ans. 2 A. 1 R. 38 P. 21.92 yd.

Problem XXIII.

646. To find the area of a segment of a circle.

Find the area of the sector having the same arc with the segment, and also the area of the triangle formed by the chord of the segment and the radii of the sector. Then, if the segment is less than a semicircle, take the difference of these areas; but if greater, take their sum.

647. *Scholium.* When the height of the segment and

the diameter of the circle are given, the area may be readily found by means of a table of segments, *by dividing the height by the diameter, and looking in the table for the quotient in the column of heights, and taking out, in the next column on the right hand, the corresponding area; which, multiplied by the square of the diameter, will give the area required.*

When the quotient cannot be exactly found in the table, proportions may be instituted so as to find the area between the next higher and the next lower, in the same ratio that the given height varies from the next higher and lower heights.

648. TABLE OF SEGMENTS.

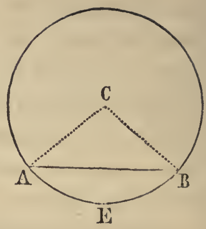
Height.	Seg. Area.	Height.	Seg. Area.	Height.	Seg. Area.	Height.	Seg. Area.	Height.	Seg. Area.
.01	.00133	.11	.04701	.21	.11990	.31	.20738	.41	.30319
.02	.00375	.12	.05339	.22	.12811	.32	.21667	.42	.31304
.03	.00687	.13	.06000	.23	.13646	.33	.22603	.43	.32293
.04	.01054	.14	.06683	.24	.14494	.34	.23547	.44	.33284
.05	.01468	.15	.07387	.25	.15354	.35	.24498	.45	.34278
.06	.01924	.16	.08111	.26	.16226	.36	.25455	.46	.35274
.07	.02417	.17	.08853	.27	.17109	.37	.26418	.47	.36272
.08	.02944	.18	.09613	.28	.18002	.38	.27386	.48	.37270
.09	.03502	.19	.10390	.29	.18905	.39	.28359	.49	.38270
.10	.04088	.20	.11182	.30	.19817	.40	.29337	.50	.39270

The segments in the table are those of a circle whose diameter is 1, and the first column contains the corresponding heights divided by the diameter. The method of calculating the areas of segments from the elements in the table depends upon the principle that similar plane figures are to each other as the squares of their like linear dimensions.

EXAMPLES.

1. What is the area of the segment A B E, its arc A E B being 73.74°, its chord A B being 12 feet, and

the radius, C B, of the circle 10 feet?



$0.7854 \times 20^2 = 314.16$, area of circle; then $360° : 73.74° :: 314.16 : 64.3504$, area of sector A E B C; and, by Problem IX., 48 is the area of the triangle A B C; $64.3504 - 48 = 16.3504$ feet, the area required.

2. Required the area of a segment whose height is 18, and the diameter of the circle 50 feet.

$18 \div 50 = .36$; to which the corresponding area in the table is .25455; $.25455 \times 50^2 = 636.375$, area required.

3. Required the area of a segment whose arc is 100°, chord 153.208 feet, and the diameter of the circle 200 feet.

4. What is the area of a segment whose height is 4 feet, and the radius 51 feet? Ans. 106 feet.

5. Required the area of a segment, the arc being 160°, chord 196.9616 feet, and the radius of the circle 100 feet.

Problem XXIV.

649. To find the area of a CIRCULAR ZONE, or the space included between two parallel chords and their intercepted arcs.

From the area of the whole circle subtract the areas of the segments on the sides of the zone.

Examples.

1. What is the area of a zone whose chords are each 12 feet, subtending each an arc of 73.74°, when the radius of the circle is 10 feet?

Area of the whole circle by Prob. XX. $= 314.16$; area of each segment by Prob. XXIII. $= 16.3504$; $16.3504 \times 2 = 32.7008 =$ area of both segments; $314.16 - 32.7008 = 281.4592$, the area required.

2. What is the area of a circular zone whose longer chord is 20 yards, subtending an arc of 60°, and the shorter chord 14.66 yards, subtending an arc of 43°, the diameter of the circle being 40 yards?

3. A circle whose diameter is 20 feet is divided into three parts by two parallel chords; one of the segments cut off is 8 feet in height, and the other 6 feet; what is the area of the circular zone? Ans. 117.544 ft.

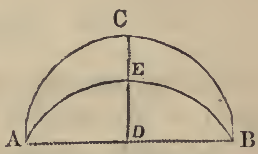
Problem XXV.

650. To find the area of a crescent.

Find the difference of the areas of the two segments formed by the arcs of the crescent and its chord.

Examples.

1. The arcs A C B, A E B, of circles having the same radius, 50 rods, intersecting, form the crescent A C B E; the height, D C, of the segment A C B is 60 rods, and the height, D E, of the segment A B E is 40 rods; what is the area of the crescent?



The area of the segment A C B, by Prob. XXIII., is 4920.3 rods, and that of the segment A B E is 2933.7 rods; $4920.3 - 2933.7 = 1986.6$ rods, the area of the crescent.

2. If the arc of a circle whose diameter is 24 yards intersects a circle whose diameter is 20 yards, forming a crescent, so that the height of the segment of the first circle is 5.072 yards, and that of the segment of the second circle is 8 yards, what is the area of the crescent?

Problem XXVI.

651. To find the area of a circular ring, or the space included between two concentric circles.

Find the areas of the two circles separately (Prob. XX.), *and take the difference of these areas; or sub-*

tract the square of the less diameter from the square of the greater, and multiply their difference by 0.7854 (Prob. XX. Sch.).

EXAMPLES.

1. Required the area of the ring formed by two circles whose diameters are 30 and 50 feet.

$50^2 - 30^2 = 1400$; $1400 \times 0.7854 = 1099.56$ sq. feet, [the area of the ring.

2. What is the area of a ring formed by two circles whose radii are 36 and 24 feet?

3. A circular park, 256 yards in diameter, has a carriage-way running around it 29 feet wide; what is the area of the carriage-way?

Ans. 1 A. 2 R. 26 P. 21.5 yd.

PROBLEM XXVII.

652. The diameter or circumference of a CIRCLE being given, to find the side of an EQUIVALENT SQUARE.

Multiply the diameter by 0.8862, *or the circumference by* 0.2821; *the product in either case will be the side of an equivalent square.*

For, since 0.7854 is the area of a circle whose diameter is 1 (Prob. XX. Sch.), the square root of 0.7854, which is 0.8862, is the side of a square which is equivalent to a circle whose diameter is 1. Now when the circumference is 1, the side of an equivalent square must have the same ratio to 0.8862 as the diameter 1 has to its circumference 3.1416 (Prop. XV. Cor. 4, Bk. VI.); and $0.8862 \div 3.1416$ gives 0.2821 as the side of the equivalent square when the circumference is 1.

EXAMPLES.

1. The diameter of a circle is 120 feet; what is the side of an equivalent square?

$120 \times 0.8862 = 106.344$ feet, the side required.

2. The circumference of a circle is 100 yards; what is the side of an equivalent square? Ans. 28.21 yd.

3. There is a circular floor 30 feet in diameter; what is the side of a square floor containing the same area?

4. If 500 feet is the circumference of a circular island, what is the side of a square of equal area?

Ans. 141.05 ft.

PROBLEM XXVIII.

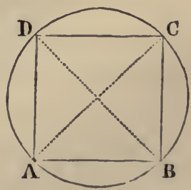
653. The diameter or circumference of a CIRCLE being given, to find the side of the INSCRIBED SQUARE.

Multiply the diameter by 0.7071, *or the circumference by* 0.2251; *the product in either case will be the side of the inscribed square.*

For 0.7071 is the side of the inscribed square when the diameter of the circumscribed circle is 1, since the side of the inscribed square is to the radius of the circle as the square root of 2 to 1 (Prop. IV. Cor., Bk. VI.); consequently, the side is to the diameter, or twice the radius, as half the square root of 2 is to 1, and half the square root of 2 is 0.7071, approximately. Now, the ratio of the diameter of a circle to the side of its inscribed square being as 1 to 0.7071, and the ratio of the circumference of a circle to its diameter as 3.1416 to 1, the ratio of the inscribed square is to the circumference of the circle as 0.7071 to 3.1416; and 0.7071 ÷ 3.1416 gives 0.2251 as the side of the inscribed square when the circumference is 1.

EXAMPLES.

1. The diameter, A C, of a circle is 110 feet; what is the side, A B, of the inscribed square?



110 × 0.7071 = 77.781 feet, the side [required.

2. The circumference of a circle is 300 feet; what is the side of the inscribed square? Ans. 67.53 ft.

3. A log is 36 inches in diameter; of how many inches square can a stick be hewn from it?

4. There is a circular field 1000 rods in circuit; what is the side of the largest square that can be described in it? Ans. 225.10 rods.

PROBLEM XXIX.

654. The diameter or circumference of a CIRCLE being given, to find the side of an INSCRIBED EQUILATERAL TRIANGLE.

Multiply the diameter by 0.8660, *or the circumference by* 0.2757; *the product in either case will be the side of the inscribed equilateral triangle.*

For 0.8660 is the side of the inscribed equilateral triangle when the diameter of the circumscribed circle is 1, since the side of the inscribed equilateral triangle is to the radius of the circle as the square root of 3 is to 1 (Prop. V. Cor. 3, Bk. VI.); consequently, the side is to the diameter, or twice the radius, as half the square root of 3 is to 1, and half the square root of 3 is 0.8660, approximately. Also, since the ratio of the circumference of a circle to its diameter is as 3.1416 to 1, the side of the inscribed equilateral triangle, when the circumference is 1, equals $0.8660 \div 3.1416$, or 0.2757.

EXAMPLES.

1. Required the side of an equilateral triangle that may be inscribed in a circle 101 feet in diameter.

$101 \times 0.8660 = 87.4660$ feet, the side required.

2. Required the side of an equilateral triangle that may be inscribed in a circle 80 rods in circumference. Ans. 22.05 rods.

3. Required the side of the largest equilateral triangular beam that can be hewn from a piece of round timber 36 inches in diameter.

4. Required the side of an equilateral triangle that can be inscribed in a circle 251.33 feet in circumference.

5. How much less is the area of an equilateral triangle that can be inscribed in a circle 100 feet in diameter, than the area of the circle itself? Ans. 4606.4 sq. ft.

THE ELLIPSE.

655. An ELLIPSE is a plane figure bounded by a curve, from any point of which the sum of the distances to two fixed points is equal to a straight line drawn through those two points, and terminated both ways by the curve.

Thus A D B C is an ellipse. The two fixed points G and H are called the *foci*. The longest diameter, A B, of the ellipse is called its *major* or *transverse axis*, and its shortest diameter, C D, is called its *minor* or *conjugate axis*.

656. The AREA of an ellipse is a mean proportional between the areas of two circles whose diameters are the two axes of the ellipse.

This, however, can only be well demonstrated by means of Analytical Geometry, a branch of the mathematics with which the learner here is not supposed to be acquainted.

PROBLEM XXX.

657. To find the area of an ELLIPSE, the major and minor axes being given.

Multiply the axes together, and their product by 0.7854, *and the result will be the area.*

For $AB^2 \times 0.7854$ expresses the area of a circle whose diameter is A B, and $CD^2 \times 0.7854$ expresses the area of a circle whose diameter is C D; and the product of these two areas is equal to $AB^2 \times CD^2 \times 0.7854^2$, which is equal to the square of $AB \times CD \times 0.7854$; hence, $AB \times CD \times 0.7854$ is a mean proportional between the areas of the two circles whose diameters are AB and CD (Prop. IV. Bk. II.); consequently it measures the area of an ellipse whose axes are AB and CD (Art. 656).

Examples.

1. Required the area of an ellipse, of which the major axis is 60 feet, and the minor axis 40 feet.

$60 \times 40 \times 0.7854 = 1884.96$ sq. ft., the area required.

2. What is the area of an ellipse whose axes are 75 and 35 feet?

3. Required the area of an ellipse whose axes are 526 and 354 inches. Ans. 112 yd. 7 ft. 84.62 in.

4. How many acres in an elliptical pond whose semi-axes are 436 and 254 feet?

Ans. 7 A. 3 R. 37 P. 27 yd. 7 ft.

BOOK XII.

APPLICATIONS OF GEOMETRY TO THE MENSURATION OF SOLIDS.

DEFINITIONS.

658. MENSURATION OF SOLIDS, or VOLUMES, is the process of determining their contents.

The SUPERFICIAL CONTENTS of a body is its quantity of surface.

The SOLID CONTENTS of a body is its measured magnitude, volume, or solidity.

659. The UNIT OF VOLUME, or SOLIDITY, is a cube, whose faces are each a *superficial* unit of the surface of the body, and whose edges are each a *linear* unit of its linear dimensions.

660. TABLE OF SOLID MEASURES.

1728	Cubic	Inches	make	1	Cubic	Foot
27	"	Feet	"	1	"	Yard.
$4492\frac{1}{8}$	"	Feet	"	1	"	Rod.
32,768,000	"	Rods	"	1	"	Mile.

Also,

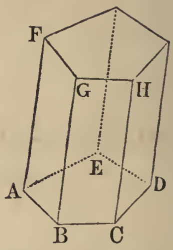
231	"	Inches	"	1 Liquid Gallon.
$268\frac{4}{5}$	"	Inches	"	1 Dry Gallon.
$2150\frac{42}{100}$	"	Inches	"	1 Bushel.
128	"	Feet	"	1 Cord.

PROBLEM I.

661. To find the surface of a RIGHT PRISM.

Multiply the perimeter of the base by the altitude, and the product will be the CONVEX *surface* (Prop. I. Bk.

VIII.). *To this add the areas of the two bases, and the result will be the* ENTIRE *surface.*



EXAMPLES.

1. Required the entire surface of a pentangular prism, having each side of its base, A B C D E, equal to 2 feet, and its altitude, A F, equal to 5 feet.

$2 \times 5 = 10$; $10 \times 5 = 50$ square feet, [the surface required.

2. The altitude of a hexangular prism is 12 feet, two of its faces are each 2 feet wide, three are each $2\frac{1}{2}$ feet wide, and the remaining face is 9 inches wide; what is the convex surface of the prism?

3. Required the entire surface of a cube, the length of each edge being 25 feet.

4. Required, in square yards, the wall surface of a rectangular room, whose height is 20 feet, width 30 feet, and length 50 feet. Ans. $355\frac{5}{9}$ sq. yd.

PROBLEM II.

662. To find the solidity of a PRISM.

Multiply the area of its base by its altitude, and the product will be its solidity (Prop. XIII. Bk. VIII.).

EXAMPLES.

1. Required the solidity of a pentangular prism, having each side of its base equal to 2 feet, and its altitude equal to 5 feet.

$2^2 \times 1.72048 = 6.88192$; $6.88192 \times 5 = 34.40960$ cubic [feet, the solidity required.

2. Required the solidity of a triangular prism, whose length is 10 feet, and the three sides of whose base are 3, 4, and 5 feet. Ans. 60.

3. A slab of marble is 8 feet long, 3 feet wide, and 6 inches thick; required its solidity.

4. There is a cistern in the form of a cube, whose edge is 10 feet; what is its capacity in liquid gallons?

Ans. 7480.519 gallons.

5. Required the solid contents of a quadrilateral prism, the length being 19 feet, the sides of the base 43, 54, 62, and 38, and the diagonal between the first and second sides, 70 inches. Ans. 306.047 cu. ft.

6. How many cords in a range of wood cut 4 feet long, the range being 4 feet 6 inches high and 160 feet long?

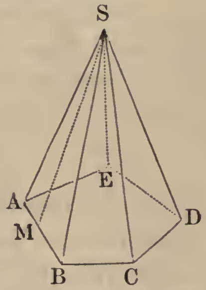
PROBLEM III.

663. To find the surface of a RIGHT PYRAMID.

Multiply the perimeter of the base by half its slant height, and the product will be the CONVEX *surface* (Prop. XV. Bk. VIII.). *To this add the area of the base, and the result will be the* ENTIRE *surface.*

664. *Scholium.* The surface of an oblique pyramid is found by taking the sum of the areas of its several faces.

EXAMPLES.



1. Required the convex surface of a pentangular pyramid, A B C D E – S, each side of whose base, A B C D E, is 5 feet, and whose slant height, S M, is 20 feet.

$5 \times 5 = 25$; $25 \times \frac{20}{2} = 250$ square [feet, the surface required.

2. What is the entire surface of a triangular pyramid, of which the slant height is 18 feet, and each side of the base 42 inches? Ans. 99.804 sq. ft.

3. Required the convex surface of a triangular pyramid, the slant height being 20 feet, and each side of the base 3 feet.

4. What is the entire surface of a quadrangular pyramid, the sides of the base being 40 and 30 inches, and the slant height upon the greater side 20.04, and upon the less side 20.07 feet? Ans. 125.308 ft.

Problem IV.

665. To find the surface of a FRUSTUM OF A RIGHT PYRAMID.

Multiply half the sum of the perimeters of its two bases by its slant height, and the product will be the CONVEX *surface* (Prop. XVII. Bk. VIII.); *to this add the areas of the two bases, and the result will be the* ENTIRE *surface.*

Examples.

1. What is the entire surface of a rectangular frustum whose slant height is 12 feet, and the sides of whose bases are 5 and 2 feet?

$5 \times 4 = 20$; $2 \times 4 = 8$; $20 + 8 = 28$; $\frac{28}{2} \times 12 = 168$; $5^2 + 2^2 = 29$; $168 + 29 = 197$ sq. ft., area required.

2. Required the convex surface of a regular hexangular frustum, whose slant height is 16 feet, and the sides of whose bases are 2 feet 8 inches and 3 feet 4 inches.

3. What is the entire surface of a regular pentangular frustum, whose slant height is 11 feet, and the sides of whose bases are 18 and 34 inches?

Ans. 136.849 sq. ft.

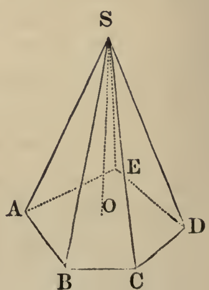
Problem V.

666. To find the solidity of a PYRAMID.

Multiply the area of its base by one third of its altitude (Prop. XX. Bk. VIII.).

Examples.

1. Required the solidity of a pentangular pyramid, ABCDE–S, each side of whose base, ABCDE, is 5 feet, and whose altitude, SO, is 15 feet.



$5^2 \times 1.7205 = 43.0125$; $43.0125 \times \frac{15}{3} = 215.0575$ cu. ft., the solidity required.

2. What is the solidity of a hexangular pyramid, the altitude of which is 9 feet, and each side of the base 29 inches?

3. What is the solidity of a square pyramid, each side of whose base is 30 feet, and whose perpendicular height is 25 feet? Ans. 7500.

4. Required the solid contents of a triangular pyramid, the perpendicular height of which is 24 feet, and the sides of the base 34, 42, and 50 inches. Ans. 39.2354 cu. ft.

Problem VI.

667. To find the solidity of a FRUSTUM OF A PYRAMID.

Add together the areas of the two bases and a mean proportional between them, and multiply that sum by one third of the altitude of the frustum (Prop. XXI. Bk. VIII.).

Examples.

1. Required the solidity of the frustum of a quadrangular pyramid, the sides of whose bases are 3 feet and 2 feet, and whose altitude is 15 feet.

$3 \times 3 = 9$; $2 \times 2 = 4$; $\sqrt{9 \times 4} = 6$ (Prop. IV. Bk. II.);
$(9 + 4 + 6) \times \frac{15}{3} = 95$ cu. ft., solidity required.

2. How many cubic feet in a stick of timber in the form of a quadrangular frustum, the sides of whose bases are 15 inches and 6 inches, and whose altitude is 20 feet?

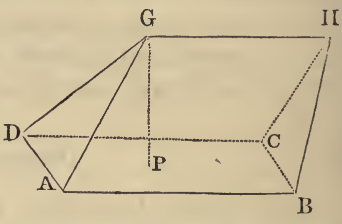
3. Required the solid contents of a pentangular frustum, whose altitude is 5 feet, each side of whose lower base is 18 inches, and each side of whose upper base is 6 inches. Ans. 9.319 cu. ft.

4. Required the solidity of the frustum of a triangular pyramid, the altitude of which is 14 feet, the sides of the lower base 21, 15, and 12, and those of the upper base 14, 10, and 8 feet. Ans. 868.752 cu. ft.

THE WEDGE.

668. A WEDGE is a polyedron bounded by a rectangle, called the base of the wedge; by two trapezoids, called the sides, which meet in an edge parallel to the base; and by two triangles, called the ends of the wedge.

Thus A B C D - G H is a wedge, of which A B C D is the rectangular base; A B H G, D C H G, the trapezoidal sides, which meet in the edge GH; and ADG, B C H, the triangular ends.



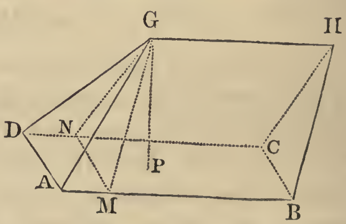
The *altitude* of a wedge is the perpendicular distance from its edge to the plane of its base; as G P.

PROBLEM VII.

669. To find the solidity of a WEDGE.

Add the length of the edge to twice the length of the base; multiply the sum by one sixth of the product of the altitude of the wedge and the breadth of the base.

For, let L equal A B, the length of the base; l equal GH, the length of the edge; b equal B C, the breadth of the base; and h equal P G, the height of the wedge. Then $L - l = AB - GH = AM$.



Now, if the length of the base and the edge be *equal*, the polyedron is equal to half a parallelopipedon having the same base and altitude (Prop. VI. Bk. VIII.), and its solidity will be equal to $\frac{1}{2}\, b\, l\, h$ (Prop. XIII. Bk. VIII.).

If the length of the base is *greater* than that of the edge, let a section, M N G, be made parallel to B C H.

This section will divide the whole wedge into the quadrangular pyramid A M N D – G, and the triangular prism B C H – G.

The solidity of A M N D – G is equal to $\frac{1}{3} b h \times (L - l)$ (Prob. V.); and the solidity of B C H – G is equal to $\frac{1}{2} b l h$; hence the solidity of the whole wedge is equal to

$$\frac{1}{2} b h l + \frac{1}{3} b h \times (L - l) = \frac{1}{6} b h\, 3 l + \frac{1}{6} b h\, 2 L - \frac{1}{6} b h\, 2 l = \frac{1}{6} b h \times (2 L + l).$$

But, if the length of the base is *less* than that of the edge, the solidity of the wedge will be equal to the prism less the pyramid; or to

$$\frac{1}{2} b h l - \frac{1}{3} b h \times (l - L) = \frac{1}{6} b h\, 3 l - \frac{1}{6} b h\, 2 l + \frac{1}{6} b h\, 2 L = \frac{1}{6} b h \times (2 L + l).$$

Examples.

1. Required the solidity of a wedge, the edge of which is 10 inches, the sides of the base 12 inches and 6 inches, and the altitude 14 inches.

$10 + (12 \times 2) = 34$; $34 \times \frac{14 \times 6}{6} = 476$ cu. in., the [solidity required.

2. What is the solidity of a wedge, of which the edge is 24 inches, the sides of the base 36 inches and 9 inches, and the altitude 22 inches?

3. How many solid feet in a wedge, of which the sides of the base are 35 inches and 15 inches, the length of the edge 55 inches, and the altitude $17\frac{3}{20}$ inches?

Ans. 3 cu. ft. $175\frac{3}{8}$ cu. in.

Rectangular Prismoid.

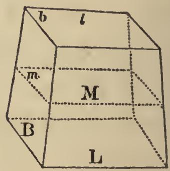
670. A RECTANGULAR PRISMOID is a polyedron bounded by two rectangles, called the bases of the prismoid, and by four trapezoids called the lateral faces of the prismoid.

The *altitude* of a prismoid is the perpendicular distance between its bases.

Problem VIII.

671. To find the solidity of a RECTANGULAR PRISMOID.

Add the area of the two bases to four times the area of a parallel section at equal distances from the bases; multiply the sum by one sixth of the altitude.



Let L and B be the length and breadth of the lower base, l and b the length and breadth of the upper base, M and m the length and breadth of the parallel section equidistant from the bases, and h the altitude of the prismoid.

If a plane be passed through the opposite edges L and l, the prismoid will be divided into two wedges, having for bases the bases of the prismoid, and for edges L and l.

The solidity of these wedges, which compose the prismoid, is (Prob. VII.),

$$\tfrac{1}{6}\,\mathrm{B}\,h \times (2\,\mathrm{L} + l) + \tfrac{1}{6}\,b\,h \times (2\,l + \mathrm{L}) = \tfrac{1}{6}\,h\,(2\,\mathrm{B}\,\mathrm{L} + \mathrm{B}\,l + 2\,b\,l + b\,\mathrm{L}).$$

But M being equally distant from L and l, $2\,\mathrm{M} = \mathrm{L} + l$, and $2\,m = \mathrm{B} + b$ (Prop. VII. Cor., Bk. IV.); consequently,

$$4\,\mathrm{M}\,m = (\mathrm{L} + l) \times (\mathrm{B} + b) = \mathrm{B}\,\mathrm{L} + \mathrm{B}\,l + b\,\mathrm{L} + b\,l.$$

Substituting $4\,\mathrm{M}\,m$ for its base, in the preceding equation, we have, as the expression of the solidity of a prismoid,

$$\tfrac{1}{6}\,h\,(\mathrm{B}\,\mathrm{L} + b\,l + 4\,\mathrm{M}\,m).$$

672. *Scholium.* This demonstration applies to prismoids of other forms. For, whatever be the form of the two bases, there may be inscribed in each such a number of small rectangles that the sum of them in each base shall differ less from that base than any assignable quantity; so that the sum of the rectangular prismoids that may be

constructed on these rectangles will differ from the given prismoid by less than any assignable quantity.

Examples.

1. Required the solidity of a prismoid, the larger base of which is 30 inches by 27 inches, the smaller base 24 inches by 18 inches, and the altitude 48 inches.

$30 \times 27 = 810$; $24 \times 18 = 432$; $\frac{30+24}{2} \times \frac{27+18}{2} \times 4$
$= 2430$; $(810 + 432 + 2430) \times \frac{48}{6} = 29{,}376$ cu. in.
$= 17$ cu. ft., the solidity required.

2. What is the solidity of a stick of timber, whose larger end is 24 inches by 20 inches, the smaller end 16 inches by 12 inches, and the length 18 feet?

3. What is the solidity of a block, whose ends are respectively 30 by 27 inches and 24 by 18 inches, and whose length is 36 inches?

4. What is the capacity in gallons of a cistern $47\frac{1}{4}$ inches deep, whose inside dimensions are, at the top $81\frac{1}{2}$ and 55 inches, and at the bottom 41 and $29\frac{1}{2}$ inches?

Ans. 546.929 gall.

Problem IX.

673. To find the surface of a REGULAR POLYEDRON.

Multiply the area of one of the faces by the number of faces; or multiply the square of one of the edges of the polyedron by the surface of a similar polyedron whose edges are 1.

For, since the faces of a regular polyedron are all equal, it is evident that the area of one face multiplied by the number of faces will give the area of the whole surface. Also, since the surfaces of regular polyedrons of the same name are bounded by the same number of similar polygons (Prop. I. Bk. VI.), their surfaces are to each other as the squares of the edges of the polyedrons (Prop. I. Cor., Bk. VI.).

674. TABLE OF SURFACES AND SOLIDITIES OF POLYEDRONS WHOSE EDGE IS 1.

NAMES.	NO. OF FACES.	SURFACES.	SOLIDITIES.
Tetraedron,	4	1.7320508	0.1178511
Hexaedron,	6	6.0000000	1.0000000
Octaedron,	8	3.4641016	0.4714045
Dodecaedron,	12	20.6457288	7.6631189
Icosaedron,	20	8.6602540	2.1816950

The surfaces in the table are obtained by multiplying the area of one of the faces of the polyedron, as given in Art. 632, by the number of faces.

EXAMPLES.

1. What is the surface of an octaedron whose edge is 16 inches?

$16^2 \times 3.4641016 = 886.81$ sq. in., the area required.

2. Required the surface of an icosaedron whose edge is 20 inches.

3. Required the surface of a dodecaedron whose edge is 12 feet. Ans. 2972.985 sq. ft.

PROBLEM X.

675. To find the solidity of a REGULAR POLYEDRON.

Multiply the surface by one third of the perpendicular distance from the centre to one of the faces; or multiply the cube of one of the edges by the solidity of a similar polyedron whose edge is 1.

For any regular polyedron may be divided into as many equal pyramids as it has faces, the common vertex of the pyramids being the centre of the polyedron; hence, the solidity of the polyedron must equal the product of the areas of all its faces by one third the perpendicular distance from the centre to each face of the polyedron.

Also, since similar pyramids are to each other as the cubes of their homologous edges (Prop. XXII. Bk. VIII.), two polyedrons containing the same number of similar pyramids are to each other as the cubes of their edges; hence, the solidity of a polyedron whose edge is 1 (Art. 673), may be used to measure other similar polyedrons.

EXAMPLES.

1. Required the solidity of an octaedron whose edge is 16 inches.

$16^3 \times 0.4714045 = 1930.8728$ cu. in., solidity required.

2. What is the solidity of a tetraedron whose edge is 2 feet?

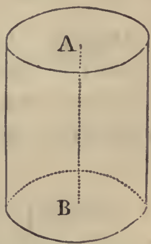
3. Required the solidity of an icosaedron whose edge is 15 inches. Ans. 7363.2206 cu. in.

PROBLEM XI.

676. To find the surface of a CYLINDER.

Multiply the circumference of its base by its altitude, and the product will be the CONVEX *surface* (Prop. I. Bk. X.). *To this add the areas of its two bases, and the result will be the* ENTIRE *surface.*

EXAMPLES.



1. What is the entire surface of a cylinder, the altitude of which, A B, is 10 feet, and the circumference of the base 20 feet?

$10 \times 20 = 200$; $20^2 \times 0.07958 \times 2 = 63.264$; $200 + 63.264 = 263.264$ sq. ft., the surface required.

2. Required the convex surface of a cylinder whose altitude is 16 feet, and the circumference of whose base is 21 feet.

3. What is the entire surface of a cylinder whose altitude is 10 inches, and whose circumference is 4 feet?

4. How many times must a cylinder 5 feet 3 inches long, and 21 inches in diameter, revolve, to roll an acre?

Ans. 1509.18 times.

Problem XII.

677. To find the solidity of a CYLINDER.

Multiply the area of the base by the altitude, and the product will be the solidity (Prop. II. Bk. X.).

Examples.

1. What is the solidity of a cylinder, whose altitude is 10 feet, and the circumference of whose base is 20 feet?

$20^2 \times 0.07958 \times 10 = 318.32$ cu. ft., solidity required.

2. Required the solidity of a cylindrical log, whose length is 9 feet, and the circumference of whose base is 6 feet.

Ans. 25.7831 cu. ft.

3. The Winchester bushel is a hollow cylinder $18\frac{1}{2}$ inches in diameter, and 8 inches deep; what is its capacity in cubic inches?

Problem XIII.

678. To find the surface of a CONE.

Multiply the circumference of the base by half the slant height (Prob. III. Bk. X.), *and the product will be the convex surface. To this add the area of the base, and the result will be the entire surface.*

Examples.

1. What is the convex surface of a cone, whose slant height is 28 feet, and the circumference of whose base is 40 feet?

$40 \times \frac{28}{2} = 560$ sq. ft., the surface required.

2. Required the entire surface of a cone, whose slant height is 14 feet, and the circumference of whose base is 92 inches.

3. What is the surface of a cone, whose slant height is 9 feet, and the diameter of whose base is 36 inches?

4. How many yards of canvas are required for the covering of a conical tent, the slant height of which is 30 feet, and the circumference of the base 900 feet?

Ans. 1500 sq. yd.

Problem XIV.

679. To find the surface of a FRUSTUM OF A CONE.

Multiply half the sum of the circumferences of its two bases by its slant height, and the product will be the convex surface (Prop. IV. Bk. X.). *To this add the area of its bases, and the result will be the entire surface.*

680. *Scholium.* The convex surface of a frustum of a cone may also be found *by multiplying the slant height* by the circumference of a section at equal distances between the two bases (Prop. IV. Cor., Bk. X.).

Examples.

1. Required the convex surface of a frustum of a cone, whose slant height is 20 feet, and the circumferences of whose bases are 30 feet and 40 feet.

$$\frac{30+40}{2} \times 20 = 700 \text{ sq. ft., the surface required.}$$

2. Required the surface of a frustum of a cone, the diameters of the bases being 43 inches and 23 inches, and the slant height 9 feet.

3. What is the convex surface of a frustum of a cone, of which a section equidistant from its two bases is 24 feet in circumference, the slant height of the frustum being 19 feet?

4. From a cone the circumference of whose base is 10 feet, and whose slant height is 30 feet, a cone has been cut off, whose slant height is 8 feet. What is the convex surface of the frustum? Ans. $139\frac{1}{3}$ sq. ft.

PROBLEM XV.

681. To find the solidity of a CONE.

Multiply the area of its base by one third of its altitude, and the product will be the solidity (Prop. V. Bk. X.).

EXAMPLES.

1. What is the solidity of a cone whose altitude is 42 feet, and the diameter of whose base is 10 feet?

$10^2 \times 0.7854 \times \frac{42}{3} = 1099.56$ cu. ft., solidity required.

2. Required the solidity of a cone whose altitude is 63 feet, and the radius of whose base is 12 feet 6 inches.

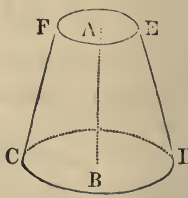
3. How many cubic feet in a conical stick of timber, whose length is 18 feet, the diameter at the larger end being 42 inches? Ans. 57.7269 cu. ft.

PROBLEM XVI.

682. To find the solidity of the FRUSTUM OF A CONE.

Add together the areas of the two bases and a mean proportional between them, and multiply that sum by one third of the altitude of the frustum; and the result will be the solidity required (Prop. VI. Bk. X.).

EXAMPLES.



1. What is the solidity of a frustum of a cone, C D E F, whose altitude, A B, is 21 feet, and the area of whose bases, F E, C D, are 80 square feet and 300 square feet?

$(80 + 300 + \sqrt{80 \times 300}) \times \frac{21}{3} =$ 3732.96 cu. ft., solidity required.

2. Required the solidity of a frustum of a cone, the diameters of the bases being 38 and 27 inches, and the altitude 11 feet.

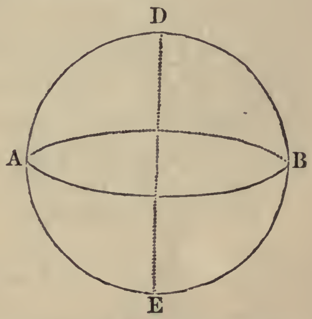
3. If a cask, which is two equal frustums of cones joined together at the larger bases, have its bung diameter 28

inches, the head diameter 20 inches, and length 40 inches, how many gallons of wine will it hold? Ans. 79.06.

Problem XVII.

683. To find the surface of a SPHERE.

Multiply the diameter by the circumference of a great circle of the sphere (Prop. VIII. Bk. X.); *or multiply the area of one great circle of the sphere by* 4 (Prop. VIII. Cor 1, Bk. X.); *or multiply* 3.1416 *by the square of the diameter* (Prop. VIII. Cor. 4, Bk. X.).



Examples.

1. What is the surface of a sphere, whose diameter, ED, is 40 feet, and whose circumference, A E B D, is 125.664?

$125.664 \times 40 = 5026.56$ sq. [ft., the surface required.

2. Required the surface of a sphere whose diameter is 30 inches.

3. What is the surface of a globe whose diameter is 7 feet and circumference 21.99 feet? Ans. 153.93.

4. How many square miles of surface has the earth, its diameter being 7912 miles?

Problem XVIII.

684. To find the surface of a ZONE OR SEGMENT OF A SPHERE.

Multiply the altitude of the zone or segment by the circumference of a great circle of the sphere (Prop. VIII. Cor. 2, Bk. X.); *or multiply the product of the diameter and altitude by* 3.1416 (Prop. VIII. Cor. 6, Bk. X.).

Examples.

1. What is the surface of a segment of a sphere, the altitude of the segment being 10 feet, and the diameter of the sphere 50 feet?

$50 \times 10 \times 3.1416 = 1570.80$ sq. ft., surface required.

2. The altitude of a segment of a sphere is 38 inches, and the circumference of the sphere is 25 feet; what is the surface of the segment?

3. Required the surface of a zone or segment, the diameter of the sphere being 72 feet, and the altitude of the zone 24 feet. Ans. 5428.6848 sq. ft.

4. If the earth be regarded as a perfect sphere whose axis is 7912 miles, and the part of the axis corresponding to each of the frigid zones is 327.192848, to each of the temperate zones 2053.468612, and to the torrid zone 3150.67708 miles; what is the surface of each zone?

Ans. Each frigid zone 8132797.39568; each temperate zone 51041592.99898; torrid zone 78314115.07768 miles.

Problem XIX.

685. To find the solidity of a sphere.

Multiply the surface of the sphere by one third of its radius (Prop. IX. Bk. X.); *or multiply the cube of the diameter of the sphere by* 0.5236 (Prop. IX. Cor. 5, Bk. X.).

Examples.

1. What is the solidity of a sphere whose diameter is 40 inches?

$40^3 \times 0.5236 = 33510.4$ cu. in., the solidity required.

2. Required the solidity of a globe whose circumference is 60 inches.

3. What is the solidity of the moon in cubic miles, supposing it a perfect sphere with a diameter of 2160 miles?

4. Required the solidity of the earth, supposing it to be a perfect sphere, whose diameter is 7912 miles.

Ans. 259332805349.80493 cu. miles.

PROBLEM XX.

686. To find the surface of a SPHERICAL POLYGON.

From the sum of all the angles subtract the product of two right angles by the number of sides less two; divide the remainder by 90°, and multiply the quotient by one eighth of the surface of the sphere; and the result will be the surface of the spherical polygon (Prop. XX. Bk. IX.).

EXAMPLES.

1. Required the surface of a spherical polygon having five sides, described on a sphere whose diameter is 100 feet, the sum of the angles being 720 degrees.

$2 \times 90° \times (5 - 2) = 540°$; $(720° - 540°) \div 90° = 2$; $100^2 \times 3.1416 = 31416$; $2 \times \frac{31416}{8} = 7854$ sq. ft., the surface required.

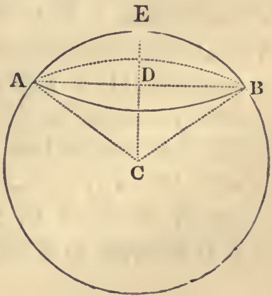
2. What is the surface of a triangle on a sphere whose diameter is 20 feet, the angles being 150°, 90°, and 54°?

PROBLEM XXI.

687. To find the solidity of a SPHERICAL PYRAMID or SECTOR.

Multiply the area of the polygon or zone which forms the base of the pyramid or sector by one third of the radius (Prop. IX. Cor. 1, Bk. X.); *or multiply the altitude of the base by the square of the radius, and that product by* 2.0944 (Prop. IX. Cor. 7, Bk. X.).

EXAMPLES.



1. Required the solidity of a spherical sector, A C B E, the altitude, E D, of the zone forming its base being 5 feet, and the radius, C B, of the sphere being 12 feet.

$5 \times 24 \times 3.1416 = 376.992$; $376.992 \times \frac{12}{3} = 1507.968$ cu. ft., the solidity required.

2. What is the solidity of a spherical pyramid, the area of its base being 364 square feet, and the diameter of the sphere 60 feet?

3. Required the solidity of a spherical sector, whose base is a zone 16 inches in altitude, in a sphere 3 feet in diameter.

4. What is the solidity of a spherical sector, whose base is a zone 6 feet in altitude, in a sphere 18 feet in diameter? Ans. 1017.88 cu. ft.

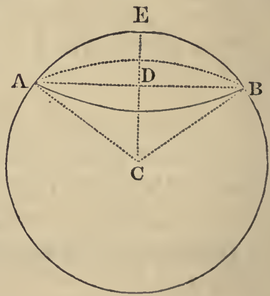
Problem XXII.

688. To find the solidity of a SPHERICAL SEGMENT.

When the segment is LESS *than a hemisphere, from the solidity of the spherical* SECTOR *whose base is the zone of the segment, take the solidity of the cone whose vertex is the centre of the sphere, and whose base is the circular base of the segment; but when the segment is* GREATER *than a hemisphere, take the sum of these solidities* (Prop. IX. Sch., Bk. X.).

689. *Scholium.* If the segment has two plane bases, its solidity may be found *by taking the difference of the two segments which lie on the same side of its two bases* (Prop. IX. Sch., Bk. X.).

Examples.



1. What is the solidity of a segment, A B E, whose altitude, E D, is 5 feet, cut from a sphere whose radius, C E, is 20 feet?

The altitude of the cone A B C is equal to C E — E D, or 20 — 5, which is equal to 15 feet; and the radius of its base is equal to $\sqrt{CA^2 - CD^2}$, or $\sqrt{20^2 - 15^2}$, which is equal to 13.23; consequently the diameter A B is equal to 26.46 feet; $5 \times 20^2 \times 2.0944 = 4188.8$

cubic feet, the solidity of the sector A C D E (Prob. XXI.); $26.46^2 \times 0.7854 \times \frac{15}{3} = 2946.99$ cubic feet, the solidity of the cone A B – C (Prob. XV.); $4188.8 - 2946.99 = 1241.81$ cubic feet, the solidity of the segment A B E required.

2. Required the solidity of a segment, whose altitude is 57 inches, the diameter of the sphere being 153 inches.

3. What is the solidity of a spherical segment, whose altitude is 13 feet, and the diameter of the sphere 33 feet 6 inches?

4. Required the solidity of the segments of the earth which are bounded severally by its five zones, the earth's diameter being 7912 miles, and the part of the diameter corresponding to each of the frigid zones being 327.19, to each temperate zone 2053.47, and to the torrid zone 3150.68.

Ans. Each frigid zone 1293793463.32, each temperate zone 55013912318.45, and the torrid zone 146717393786.26 cubic miles.

The Spheroid.

690. A SPHEROID is a solid which may be described by the revolution of an ellipse about one of its axes, which remains immovable.

An *oblate* spheroid is one described by the revolution of the ellipse about its *minor* or *conjugate* axis.

A *prolate* spheroid is one described by the revolution of the ellipse about its major or transverse axis.

Problem XXIII.

691. To find the solidity of a SPHEROID.

Multiply the square of the axis of revolution by the fixed axis, and that product by 0.5236.

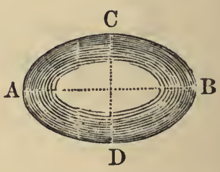
A full demonstration of this, which is based upon the principle that a spheroid is two thirds of its circumscribing cylinder, would require a knowledge of Conic Sections, or of the Differential and Integral Calculi, with neither of which is the learner here supposed to be acquainted.

The relation, however, of the spheroid to its circumscribing cylinder, is that which the sphere sustains to its circumscribing cylinder (Prop. X. Bk. X.).

Now the area of the base of the cylinder is found by multiplying the square of the axis of revolution by 0.7854, and the solidity of the cylinder by multiplying that product by the fixed axis (Prop. II. Bk. X.). But the solidity of the spheroid is only two thirds of that of the cylinder; hence, to obtain the solidity of the former, instead of multiplying by 0.7854, we must use a factor only two thirds as large, which will be 0.5236.

Examples.

1. What is the solidity of the oblate spheroid A C B D, whose fixed axis, C D, is 30 inches, and the axis of revolution, A B, 40 inches.



$40^2 \times 30 \times 0.5236 = 25132.8$ cubic inches, the solidity required.

2. Required the solidity of a prolate spheroid, whose fixed axis is 50 feet, and the axis of revolution 36 feet.

3. What is the solidity of a prolate, and also of an oblate spheroid, the axes of each being 25 and 15 inches?

Ans. Prolate, 2945.25 cu. in.; oblate, 4908.75 cu. in.

4. What is the solidity of a prolate, and also of an oblate spheroid, the axes of each being 3 feet 6 inches and 2 feet 10 inches?

5. Required the solidity of the earth, its figure being that of an oblate spheroid whose axes are 7925.3 and 7898.9 miles. Ans. 259774584886.834 cubic miles.

BOOK XIII.

MISCELLANEOUS GEOMETRICAL EXERCISES.

1. If the opposite angles formed by four lines meeting at a point are equal, these lines form but two straight lines.

2. If the equal sides of an isosceles triangle are produced, the two exterior angles formed with the base will be equal.

3. The sum of any two sides of a triangle is greater than the third side.

4. If from any point within a triangle two straight lines are drawn to the extremities of either side, they will include a greater angle than that contained by the other two sides.

5. If two quadrilaterals have the four sides of the one equal to the four sides of the other, each to each, and the angle included by any two sides of the one equal to the angle contained by the corresponding sides of the other, the quadrilaterals are themselves equal.

6. The sum of the diagonals of a trapezium is less than the sum of any four lines which can be drawn to the four angles from any point within the figure, except from the intersection of the diagonals.

7. Lines joining the corresponding extremities of two equal and parallel straight lines, are themselves equal and parallel, and the figure formed is a parallelogram.

8. If, in the sides of a square, at equal distances from the four angles, points be taken, one in each side, the straight lines joining these points will form a square.

9. If one angle of a parallelogram is a right angle, all its angles are right angles.

10. Any straight line drawn through the middle point of a diagonal of a parallelogram to meet the sides, is bisected in that point, and likewise bisects the parallelogram.

11. If four magnitudes are proportionals, the first and second may be multiplied or divided by the same magnitude, and also the third and fourth by the same magnitude, and the resulting magnitudes will be proportionals.

12. If four magnitudes are proportionals, the first and third may be multiplied or divided by the same magnitude, and also the second and fourth by the same magnitude, and the resulting magnitudes will be proportionals.

13. If there be two sets of proportional magnitudes, the quotients of the corresponding terms will be proportionals.

14. If any two points be taken in the circumference of a circle, the straight line joining them will lie wholly within the circle.

15. The diameter is the longest straight line that can be inscribed in a circle.

16. If two straight lines intercept equal arcs of a circle, and do not cut each other within the circle, the lines will be parallel.

17. If a straight line be drawn to touch a circle, and be parallel to a chord, the point of contact will be the middle point of the arc cut off by that chord.

18. If two circles cut each other, and from either point of intersection diameters be drawn, the extremities of these diameters and the other point of intersection will be in the same straight line.

19. If one of the equal sides of an isosceles triangle be the diameter of a circle, the circumference of the circle will bisect the base of the triangle.

20. If the opposite angles of a quadrilateral be together equal to two right angles, a circle may be circumscribed about the quadrilateral.

21. Parallelograms which have two sides and the included angle equal in each, are themselves equal.

22. Equivalent triangles upon the same base, and upon the same side of it, are between the same parallels.

23. If the middle points of the sides of a trapezoid, which are not parallel, be joined by a straight line, that line will be parallel to each of the two parallel sides, and be equal to half their sum.

24. If, in opposite sides of a parallelogram, at equal distances from opposite angles, points be taken, one in each side, the straight line joining these points will bisect the parallelogram.

25. The perimeter of an isosceles triangle is greater than the perimeter of a rectangle, which is of the same altitude with, and equivalent to, the given triangle.

26. If the sides of the square described upon the hypothenuse of a right-angled triangle be produced to meet the sides (produced if necessary) of the squares described upon the other two sides of the triangle, the triangles thus formed will be similar to the given triangle, and two of them will be equal to it.

27. A square circumscribed about a given circle is double a square inscribed in the same circle.

28. If the sum of the squares of the four sides of a quadrilateral be equivalent to the sum of the squares of the two diagonals, the figure is a parallelogram.

29. Straight lines drawn from the vertices of a triangle, so as to bisect the opposite sides, bisect also the triangle.

30. The straight lines which bisect the three angles of a triangle meet in the same point.

31. The area of a triangle is equal to its perimeter multiplied by half the radius of the inscribed circle.

32. If the points of bisection of the sides of a given triangle be joined, the triangle so formed will be one fourth of the given triangle.

33. To describe a square upon a given straight line.

34. To find in a given straight line a point equally distant from two given points.

35. To construct a triangle, the base, one of the angles at the base, and the sum of the other two sides being given.

36. To trisect a right angle.

37. To divide a triangle into two parts by a line drawn parallel to a side, so that these parts shall be to each other as two given straight lines.

38. To divide a triangle into two parts by a line drawn perpendicular to the base, so that these parts shall be to each other as two given lines.

39. To divide a triangle into two parts by a line drawn from a given point in one of the sides, so that the parts shall be to each other as two given lines.

40. To divide a triangle into a square number of equal triangles, similar to each other and to the original triangle.

41. To trisect a given straight line.

42. To inscribe a square in a given right-angled isosceles triangle.

43. To inscribe a square in a given quadrant.

44. To describe a circle that shall pass through a given point, have a given radius, and touch a given straight line.

45. To describe a circle, the centre of which shall be in the perpendicular of a given right-angled triangle, and the circumference of which shall pass through the right angle and touch the hypothenuse.

46. To describe three circles of equal diameters which shall touch each other, and to describe another circle which shall touch the three circles.

47. If, on the diameter of a semicircle, two equal circles be described, and in the curvilinear space included by the three circumferences a circle be inscribed, its diameter will be to that of the equal circles in the ratio of two to three.

48. If two points be taken in the diameter of a circle,

equidistant from the centre, the sum of the squares of two lines drawn from these points to any point in the circumference will always be the same.

49. Given the vertical angle, and the radii of the inscribed and circumscribed circles, to construct the triangle.

50. If a diagonal cuts off three, five, or any odd number of sides from a regular polygon, the diagonal is parallel to one of the sides.

51. The area of a regular hexagon inscribed in a circle is double that of an equilateral triangle inscribed in the same circle.

52. The side of a square circumscribed about a circle is equal to the diagonal of a square inscribed in the same circle.

53. To describe a circle equal to half a given circle.

54. A regular duodecagon is equivalent to three fourths of the square constructed on the diameter of its circumscribed circle; or is equal to the square constructed on the side of the equilateral triangle inscribed in the same circle.

55. If semicircles be described on the sides of a right-angled triangle as diameters, the one described on the hypothenuse will be equal to the sum of the other two.

56. If on the sides of a triangle inscribed in a semicircle, semicircles be described, the two crescents thus formed will together equal the area of the triangle.

57. If the diameter of a semicircle be divided into any number of parts, and on them semicircles be described, their circumferences will together be equal to the circumference of the given semicircle.

58. To divide a circle into any number of parts, which shall all be equal in area and equal in perimeter, and not have the parts in the form of sectors.

59. To draw a straight line perpendicular to a plane, from a given point above the plane.

60. Two straight lines not in the same plane being

given in position, to draw a straight line which shall be perpendicular to them both.

61. The solidity of a triangular prism is equal to the product of the area of either of its rectangular sides as a base multiplied by half its altitude on that base.

62. All prisms of equal bases and altitudes are equal in solidity, whatever be the figure of their bases.

63. The convex surface of a regular pyramid exceeds the area of its base in the ratio that the slant height of the pyramid exceeds the radius of the circle inscribed in its base.

64. If from any point in the circumference of the base of a cylinder, a straight line be drawn perpendicular to the plane of the base, it will be wholly in the surface of the cylinder.

65. A cylinder and a parallelopipedon of equal bases and altitudes are equivalent to each other.

66. If two solids have the same height, and if their sections made at equal altitudes, by planes parallel to the bases, have always the same ratio which the bases have to one another, the solids have to one another the same ratio which their bases have.

67. The side of the largest cube that can be inscribed in a sphere, is equal to the square root of one third of the square of the diameter of the sphere.

68. To cut off just a square yard from a plank 14 feet 3 inches long, and of a uniform width, at what distance from the edge must a line be struck? Ans. $7\frac{1}{19}$ in.

69. How much carpeting a yard wide will be required to cover the floor of an octagonal hall, whose sides are 10 feet each?

70. The perambulator, or surveying-wheel, is so constructed as to turn just twice in the length of a rod; what is its diameter? Ans. 2.626 ft.

71. What is the excess of a floor 50 feet long by 30 broad, above two others, each of half its dimensions?

72. The four sides of a trapezium are 13, 13.4, 24, and 18 feet, and the first two contain a right angle. Required the area. Ans. 253.38 sq. ft.

73. If an equilateral triangle, whose area is equal to 10,000 square feet, be surrounded with a walk of uniform width, and equal to the area of the inscribed circle, what is the width of the walk? Ans. 11.701 ft.

74. A right-angled triangle has its base 16 rods, and its perpendicular 12 rods, and a triangle is cut off from it by a line parallel to its base, of which the area is 24 rods. Required the sides of that triangle. Ans. 8, 6, and 10 rods.

75. There is a circular pond whose area is $5028\frac{4}{7}$ square feet, in the middle of which stood a pole 100 feet high; now, the pole having been broken off, it was observed that the top portion resting on the stump just reached the brink of the pond. What is the height of the piece left standing? Ans. 41.9968 ft.

76. The area of a square inscribed in a circle is 400 square feet; required the diagonal of a square circumscribed about the same circle.

77. The four sides of a field, whose diagonals are equal, are known to be 25, 35, 31, and 19 rods, in a successive order; what is the area of the field?

Ans. 4 A. 1 R. $38\frac{1}{4}$ p.

78. The wheels of a chaise, each 4 feet high, in turning within a ring, moved so that the outer wheel made two turns while the inner made one, and their distance from one another was 5 feet; what were the circumferences of the tracks described by them?

Ans. Outer, 62.8318 ft.; inner, 31.4159 ft.

79. The girt of a vessel round the outside of the hoop is 22 inches, and the hoop is 1 inch thick; required the true girt of the vessel.

80. If one of the Egyptian pyramids is 490 feet high, having each slant side an equilateral triangle and the base a square, what is the area of the base?

Ans. 11 A. 3 rd. $223\frac{1}{4}$ ft.

81. An ellipse is surrounded by a wall 14 inches thick; its axes are 840 links and 612 links; required the quantity of ground enclosed, and the quantity occupied by the wall.

Ans. 4 A. 6 rd. enclosed, and 1760.49 sq. ft., area occupied by the wall.

82. There is a meadow of 1 acre in the form of a square; what must be the length of the rope by which a horse, tied equidistant from each angle, can be permitted to graze over the entire meadow?

83. A gentleman has a rectangular garden, whose length is 100 feet and breadth 80 feet; what must be the uniform width of a walk half-way round the same, to take up just half the garden? Ans. 25.9688 ft.

84. Two trees, 100 feet asunder, are placed, the one at the distance of 100 feet, and the other 50 feet from a wall; what is the distance that a person must pass over in running from one tree to touch the wall, and then to the other tree, the lines of distance making equal angles with the wall? Ans. 173.205 ft.

85. There is a rectangular park 400 feet long and 300 feet broad, all round which, and close by the wall, is a border 10 feet broad; close by the border there is a walk, and also two others, crossing each other and the park at right angles, in the middle of the garden. The walks are all of one breadth, and their area takes up one tenth of the whole park; required the breadth of the walks.

Ans. 6.2375 ft.

86. A farmer borrowed a cubical pile of wood, which measured 6 feet every way, and repaid it by two cubical piles, of which the sides were 3 feet each; what part of the quantity borrowed has he returned?

87. A board is 10 feet long, 8 inches in breadth at the greater end, and 6 inches at the less; how much must be cut off from the less end to make a square foot?

Ans. 23.2493 in.

88. A piece of timber is 10 feet long, each side of the

greater base 9 inches, and each side of the less 6 inches; how much must be cut off from the less end to contain a solid foot? Ans. 3.39214 ft.

89. What must be the inside dimensions of a cubical box to hold 200 balls, each $2\frac{1}{2}$ inches in diameter?

90. Near my house I intend making a hexagonal or six-sided seat around a tree, for which I have procured a pine plank $16\frac{1}{2}$ feet long and 11 inches broad; what must be the inner and outer lengths of each side of the seat, that there may be the least loss in cutting up the plank?

Ans. 26.64915 in. inner, and 39.35085 in. outer length.

91. Required the capacity of a tub in the form of a frustum of a cone, of which the greatest diameter is 48 inches, the inside length of the staves 30 inches, and the diagonal between the farthest extremities of the diameters 50 inches. Ans. 165.34 gals.

92. The front of a house is of such a height, that, if the foot of a ladder of a certain length be placed at the distance of 12 feet from it, the top of the ladder will just reach to the top of the house; but if the foot of the ladder be placed 20 feet from the front, its top will fall 4 feet below the top of the house. Required the height of the house, and the length of the ladder.

Ans. 34 feet, the height of the building; 36.0555 feet, the length of the ladder.

93. A sugar-loaf in form of a cone is 20 inches high; it is required to divide it equally among three persons, by sections parallel to the base; what is the height of each part?

Ans. Upper 13.8672, next 3.6044, lowest 2.5284 in.

94. Within a rectangular court, whose length is four chains, and breadth three chains, there is a piece of water in the form of a trapezium, whose opposite angles are in a direct line with those of the court, and the respective distances of the angles of the one from those of the other are 20, 25, 40, and 45 yards, in a successive order; required the area of the water. Ans. 960 sq. yd.

95. What will the diameter of a sphere be, when its solidity and the area of its surface are expressed by the same numbers? Ans. 6.

96. There is a circular fortification, which occupies a quarter of an acre of ground, surrounded by a ditch coinciding with the circumference, 24 feet wide at bottom, 26 at top, and 12 deep; how much water will fill the ditch, if it slope equally on both sides? Ans. 135483.25 cu. ft.

97. A father, dying, left a square field containing 30 acres to be divided among his five sons, in such a manner that the oldest son may have 8 acres, the second 7, the third 6, the fourth 5, and the fifth 4 acres. Now, the division fences are to be so made that the oldest son's share shall be a narrow piece of equal breadth all around the field, leaving the remaining four shares in the form of a square; and in like manner for each of the other shares, leaving always the remainders in form of squares, one within another, till the share of the youngest be the innermost square of all, equal to 4 acres. Required a side of each of the enclosures.

Ans. 17.3205, 14.8324, 12.2474, 9.4868, and 6.3246 chains.

98. Required the dimensions of a cone, its solidity being 282 inches, and its slant height being to its base diameter as 5 to 4.

Ans. 9.796 in. the base diameter; 12.246 in. the slant height; and 11.223 in. the altitude.

99. A gentleman has a piece of ground in form of a square, the difference between whose side and diagonal is 10 rods. He would convert two thirds of the area into a garden of an octagonal form, but would have a fish-pond at the centre of the garden, in the form of an equilateral triangle, whose area must equal five square rods. Required the length of each side of the garden, and of each side of the pond.

Ans. 8.9707 rods, each side of the garden, and 3.398 rods, each side of the pond.

BOOK XIV.

APPLICATIONS OF ALGEBRA TO GEOMETRY.

692. When it is proposed to solve a geometrical problem by aid of Algebra, draw a figure which shall represent the several parts or conditions of the problem, both known and required.

Represent the known parts by the first letters of the alphabet, and the required parts by the last letters.

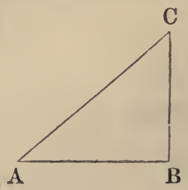
Then, observing the geometrical relations that the parts of the figure have to each other, make as many independent equations as there are unknown quantities introduced, and the solution of these equations will determine the unknown quantities or required parts.

To form these equations, however, no definite rules can be given; but the best aids may be derived from experience, and a thorough knowledge of geometrical principles.

It should be the aim of the learner to effect the simplest solution possible of each problem.

Problem I.

693. *In a right-angled triangle, having given the hypothenuse, and the sum of the other two sides, to determine these sides.*



Let A B C be the triangle, right-angled at B. Put $AC = a$, the sum $AB + BC = s$, $AB = x$, and $BC = y$.

Then, $$x + y = s.$$

and (Prop. XI. Bk. IV.),

$$x^2 + y^2 = a^2.$$

From the first equation, $$x = s - y.$$

Substitute in second equation this value of x,

$$s^2 - 2\,s\,y + 2\,y^2 = a^2.$$

Or, $$2\,y^2 - 2\,s\,y = a^2 - s^2,$$

Or, $$y^2 - s\,y = \tfrac{1}{2}\,a^2 - \tfrac{1}{2}\,s^2.$$

By completing the square,

$$y^2 - s\,y + \tfrac{1}{4}\,s^2 = \tfrac{1}{2}\,a^2 - \tfrac{1}{4}\,s^2,$$

Extracting sq. root, $$y - \tfrac{1}{2}\,s = \pm\sqrt{\tfrac{1}{2}\,a^2 - \tfrac{1}{4}\,s^2},$$

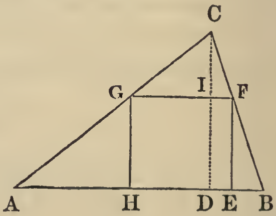
Or, $$y = \tfrac{1}{2}\,s \pm \sqrt{\tfrac{1}{2}\,a^2 - \tfrac{1}{4}\,s^2}.$$

If $AC = 5$, and the sum $AB + BC = 7$, $y = 4$ or 3, and $x = 3$ or 4.

Problem II.

694. *Having given the base and perpendicular of a triangle, to find the side of an inscribed square.*

Let ABC be the triangle, and $HEFG$ the inscribed square. Put $AB = b$, $CD = a$, and GF or $GH = DI = x$; then will $CI = CD - DI = a - x$.



Since the triangles ABC, GFC are similar,

$$AB : CD :: GF : CI,$$

or $$b : a :: x : a - x.$$

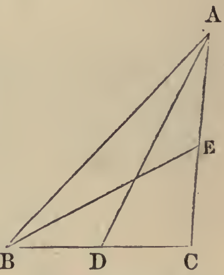
Hence, $$a\,b - b\,x = a\,x,$$

or, $$x = \frac{a\,b}{a + b}.$$

that is, *the side of the inscribed square is equal to the product of the base by the altitude, divided by their sum.*

Problem III.

695. *Having given the lengths of two straight lines drawn from the acute angles of a right-angled triangle to the middle of the opposite sides, to determine those sides.*



Let A B C be the given triangle, and A D, B E the given lines.

Put $AD = a$, $BE = b$, CD or $\frac{1}{2}$ $CB = x$, and CE or $\frac{1}{2}$ $CA = y$; then, since $CD^2 + CA^2 = AD^2$, and $CE^2 + CB^2 = BE^2$,

we have $$x^2 + 4\,y^2 = a^2,$$

and $$y^2 + 4\,x^2 = b^2.$$

By subtracting the second equation from four times the first,

$$15\,y^2 = 4\,a^2 - b^2,$$

or, $$y = \sqrt{\frac{4\,a^2 - b^2}{15}};$$

by subtracting the first equation from four times the second,

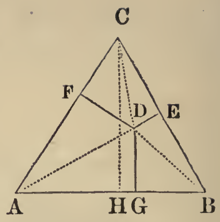
$$15\,x^2 = 4\,b^2 - a^2,$$

or, $$x = \sqrt{\frac{4\,b^2 - a^2}{15}};$$

which values of x and y are half the base and perpendiculars of the triangle.

Problem IV.

696. *In an equilateral triangle, having given the lengths of the three perpendiculars drawn from a point within to the three sides, to determine these sides.*



Let A B C be the equilateral triangle, and D E, D F, D G the given perpendiculars from the point D. Draw D A, D B, D C to the vertices of the three angles, and let fall the perpendicular, C H, on the base, A B.

Put $DE = a$, $DF = b$, $DG = c$, and A H or B H, half the side of the equilateral triangle, $= x$. Then AC or $BC = 2x$, and $CH = \sqrt{AC^2 - AH^2} = \sqrt{4x^2 - x^2} = \sqrt{3x^2} = x\sqrt{3}$. Now, since the area of a triangle is equal to the product of half its base by its altitude (Prop. VI. Bk. IV.),

The triangle $ACB = \frac{1}{2}AB \times CH = x \times x\sqrt{3} = x^2\sqrt{3}.$

$$ABD = \tfrac{1}{2}AB \times DG = x \times c \quad = cx.$$
$$BCD = \tfrac{1}{2}BC \times DE = x \times a \quad = ax.$$
$$ACD = \tfrac{1}{2}AC \times DF = x \times b \quad = bx.$$

But the three triangles A B D, B C D, A C D are together equal to the triangle A C B.

Hence, $x^2\sqrt{3} = ax + bx + cx = x(a + b + c)$,

or, $x\sqrt{3} = a + b + c$;

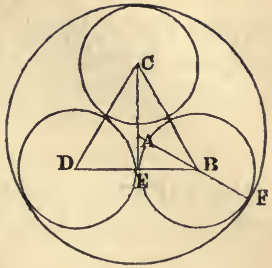
or, $x = \dfrac{a + b + c}{\sqrt{3}}$.

Hence each side, or $2x = \dfrac{2(a + b + c)}{\sqrt{3}}$.

697. *Cor.* Since the perpendicular, C H, is equal to $x\sqrt{3}$, it is equal to $a + b + c$; that is, *the whole perpendicular of an equilateral triangle is equal to the sum of all the perpendiculars let fall from any point in the triangle to each of its sides.*

Problem V.

698. *To determine the radii of three equal circles described within and tangent to a given circle, and also tangent to each other.*



Let AF be the radius of the given circle, and BE the radius of one of the equal circles described within it. Put $AF = a$, and $BE = x$; then each side of the equilateral triangle, BCD, formed by joining the centres of the required circles, will be represented by $2x$, and its altitude, CE, by $\sqrt{4x^2 - x^2}$, or $x\sqrt{3}$.

The triangles BCE, ABE aré similar, since the angles BCE and ABE are equal, each being half as great as one of the angles of the equilateral triangle, and the angle BEC is common.

Hence, $$CE : BE :: BC : AB,$$

or $$x\sqrt{3} : x :: 2x : AB,$$

and $$AB = \frac{2x}{\sqrt{3}}.$$

But $$AB + BF = AF;$$

hence, $$\frac{2x}{\sqrt{3}} + x = a,$$

or $$2x + x\sqrt{3} = a\sqrt{3},$$

or $$(2 + \sqrt{3})\,x = a\sqrt{3}.$$

Hence, $$x = \frac{a\sqrt{3}}{2 + \sqrt{3}} = \frac{a}{2.1547} = a \times 0.4641.$$

Problem VI.

699. In a right-angled triangle, having given the base, and the sum of the perpendicular and hypothenuse, to find these two sides.

Problem VII.

700. In a rectangle, having given the diagonal and perimeter, to find the sides.

Problem VIII.

701. In a right-angled triangle, having given the base, and the difference between the hypothenuse and perpendicular, to find both these two sides.

Problem IX.

702. Having given the area of a rectangle inscribed in a given triangle, to determine the sides of the rectangle.

Problem X.

703. In a triangle, having given the ratio of the two sides, together with both the segments of the base, made by a perpendicular from the vertical angle, to determine the sides of the triangle.

Problem XI.

704. In a triangle, having given the base, the sum of the other two sides, and the length of a line drawn from the vertical angle to the middle of the base, to find the sides of the triangle.

Problem XII.

705. In a triangle, having given the two sides about the vertical angle together with the line bisecting that angle, and terminating in the base, to find the base.

Problem XIII.

706. To determine a right-angled triangle, having given the perimeter and the radius of its inscribed circle.

Problem XIV.

707. To determine a triangle, having given the base, the perpendicular, and the ratio of the two sides.

Problem XV.

708. To determine a right-angled triangle, having given the hypothenuse, and the side of the inscribed square.

PROBLEM XVI.

709. In a right-angled triangle, having given the perimeter, or sum of all the sides, and the perpendicular let fall from the right angle on the hypothenuse, to determine the triangle, that is, its sides.

PROBLEM XVII.

710. To determine a right-angled triangle, having given the hypothenuse, and the difference of two lines drawn from the two acute angles to the centre of the inscribed circle.

PROBLEM XVIII.

711. To determine a triangle, having given the base, the perpendicular, and the difference of the two other sides.

PROBLEM XIX.

712. To determine a triangle, having given the lengths of three lines drawn from the three angles to the middle of the opposite sides.

PROBLEM XX.

713. In a triangle, having given all the three sides, to find the radius of the inscribed circle.

PROBLEM XXI.

714. To determine a right-angled triangle, having given the side of the inscribed square, and the radius of the inscribed circle.

PROBLEM XXII.

715. To determine a triangle, having given the base, the perpendicular, and the rectangle of the two other sides.

Problem XXIII.

716. To determine a right-angled triangle, having given the hypothenuse, and the radius of the inscribed circle.

Problem XXIV.

717. To determine a right-angled triangle, having given the hypothenuse and the difference between a side and the radius of the inscribed circle.

Problem XXV.

718. To determine a triangle, having given the base, the line bisecting the vertical angle, and the diameter of the circumscribing circle.

Problem XXVI.

719. There are two stone pillars in a garden, whose perpendicular heights are 20 and 30 feet, and the distance between them 60 feet. A ladder is to be placed at a certain point in the line of distance, of such a length, that it may just reach the top of both the pillars. What is the length of the ladder, and how far from each pillar must it be placed?

Ans. 39.5899 feet, length of the ladder; $34\frac{1}{6}$ feet, distance of the foot of the ladder from the bottom of the lower pillar; and $25\frac{5}{6}$ feet, distance of the foot of the ladder from the bottom of the higher pillar.

Problem XXVII.

720. There is a cistern, the sum of the length and breadth of which is 84 inches, the diagonal of the top 60 inches, and the ratio of the breadth to the depth as 25 to 7. What are its dimensions, provided it has the form of a rectangular parallelopipedon?

Ans. Length 48 inches; width 36 inches; depth 10.08 inches.

Problem XXVIII.

721. The three distances from an oak, growing in an open plain, to the three visible corners of a square field, lying at some distance, are known to be 78, 59.161, and 78 poles, in successive order. What are the dimensions of the field, and its area?

Ans. Side of the square 24 rd.; area 3 A. 2 R. 16 rd.

Problem XXIX.

722. There is a house of three equal stories in height. Now a ladder being raised against it, at 20 feet distance from the foot of the building, reaches the top; whilst another ladder, 12 feet shorter, raised from the same point, reaches only to the top of the second story. What is the height of the building? Ans. 41.696 ft.

Problem XXX.

723. The solidity of a cone is 2513.28 cubic inches, and the slant side of a frustum of it, whose solidity is 2474.01, is 19.5 inches. Required the dimensions of the cone.

Ans. Altitude 24 inches; base diameter 20 inches.

Problem XXXI.

724. Within a rectangular garden containing just an acre of ground, I have a circular fountain, whose circumference is 40, 28, 52, and 60 yards distant from the four angles of the garden. From these dimensions, the length and breadth of the garden, and likewise the diameter of the fountain, are required.

Ans. Length 94.996 yds.; width 50.949 yds.; diameter of the fountain 20 yds.

Problem XXXII.

725. There is a vessel in the form of a frustum of a cone, standing on its lesser base, whose solidity is 8.67 feet, the depth 21 inches, its greater base diameter to that

of the lesser as 7 to 5, into which a globe had accidentally been put, whose solidity was $2\frac{1}{2}$ times the measure of its surface. Required the diameters of the vessel and of the globe, and how many gallons of water would be requisite just to cover the latter within the former.

Ans. 35 and 25 inches, top and bottom diameters of the frustum; 15 inches, diameter of the globe; and 34.2 gallons, the water required.

Problem XXXIII.

726. Three trees, A, B, C, whose respective heights are 114, 110, and 98 feet, are standing on a horizontal plane, and the distance from A to B is 112, from B to C is 104, and from A to C is 120 feet. What is the distance from the top of each tree to a point in the plane which shall be equally distant from each? Ans. 126.634 ft.

Problem XXXIV.

727. A person possessed a rectangular meadow, the fences of which had been destroyed, and the only mark left was an oak-tree in the east corner; he however recollected the following particulars of the dimensions. It had once been resolved to divide the meadow into two parts by a hedge running diagonally; and he recollected that a segment of the diagonal intercepted by a perpendicular from one of the corners was 16 chains, and the same perpendicular, produced 2 chains, met the other side of the meadow. Now the owner has bequeathed it to four grandchildren, whose shares are to be bounded by the diagonal and perpendicular produced. What is the area of the meadow, and what are the several shares?

Ans. Area of the whole meadow, 16 acres; shares, 1 R. 24 rd.; 1 A. 2 R. 16 rd.; 6 A. 1 R. 24 rd.; 7 A. 2 R. 16 rd.

THE END