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ELEMENTS

OF

GEOMETRY AND TRIGONOMETRY.

TRANSLATED FROM THE FRENCH OF

A. M. LEGENDRE,

BY DAVID BREWSTER, LL. D.

REVISED AND ADAPTED TO THE COURSE OF MATHEMATICAL INSTRUCTION IN THE UNITED STATES,

BY CHARLES DAVIES,

AUTHOR OF ARITHMETIC, ALGEBRA, PRACTICAL GEOMETRY, ELEMENTS OF DESCRIPTIVE AND OF ANALYTICAL GEOMETRY, ELEMENTS OF DIFFERENTIAL AND INTEGRAL CALCULUS, AND SHADES SHADOWS, AND PERSPECTIVE.

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PREFACE

TO THE AMERICAN EDITION.

The Editor, in offering to the public Dr. Brewster's translation of Legendre's Geometry under its present form, is fully impressed with the responsibility he assumes in making alterations in a work of such deserved celebrity.

In the original work, as well as in the translations of Dr. Brewster and Professor Farrar, the propositions are not enunciated in general terms, but with reference to, and by the aid of, the particular diagrams used for the demonstrations. It is believed that this departure from the method of Euclid has been generally regretted. The propositions of Geometry are general truths, and as such, should be stated in general terms, and without reference to particular figures. The method of enunciating them by the aid of particular diagrams seems to have been adopted to avoid the difficulty which beginners experience in comprehending abstract propositions. But in avoiding this difficulty, and thus lessening, at first, the intellectual labour, the faculty of abstraction, which it is one of the primary objects of the study of Geometry to strengthen, remains, to a certain extent, unimproved.

Besides the alterations in the enunciation of the propositions, others of considerable importance have also been made in the present edition. The proposition in Book V., which proves that a polygon and circle may be made to coincide so nearly, as to differ from each other by less than any assignable quantity, has been taken from the Edinburgh Encyclopedia. It is proved in the corollaries that a polygon of an infinite number of sides becomes a circle, and this principle is made the basis of several important demonstrations in Book VIII.

Book II.,on Ratios and Proportions, has been partly adopted from the Encyclopedia Metropolitana, and will, it is believed, supply a deficiency in the or i work.

Very considerable alterations have also been made in the manner of treating the subjects of Plane and Spherical Trigonometry. It has also been thought best to publish with the present edition a table of logarithms and logarithmic sines, and to apply the principles of geometry to the mensuration of surfaces and solids.

Military Academy,
West Point, March, 1834.

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ELEMENTS OF GEOMETRY.

BOOK I.

THE PRINCIPLES.

Definitions.

1. GEOMETRY is the science which has for its object the measurement of extension.

Extension has three dimensions, length, breadth, and height, or thickness.

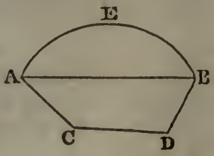
2. A *line* is length without breadth, or thickness.

The extremities of a line are called *points:* a point, therefore, has neither length, breadth, nor thickness, but position only.

3. A *straight line* is the shortest distance from one point to another.

4. Every line which is not straight, or composed of straight lines, is a *curved line.*

Thus, AB is a straight line; ACDB is a *broken* line, or one composed of straight lines; and AEB is a curved line.



The word *line*, when used alone, will designate a straight line; and the word *curve*, a curved line.

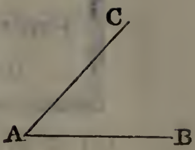
5. A *surface* is that which has length and breadth, without height or thickness.

6. A *plane* is a surface, in which, if two points be assumed at pleasure, and connected by a straight line, that line will lie wholly in the surface.

7. Every surface, which is not a plane surface, or composed of plane surfaces, is a *curved surface.*

8. A *solid* or *body* is that which has length, breadth, and thickness; and therefore combines the three dimensions of extension

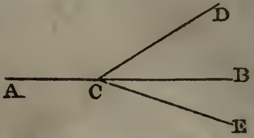
9. When two straight lines, AB, AC, meet each other, their inclination or opening is called an *angle*, which is greater or less as the lines are more or less inclined or opened. The point of intersection A is the vertex of the angle, and the lines AB, AC, are its sides.



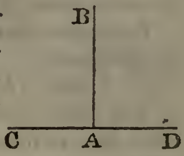
The angle is sometimes designated simply by the letter at the vertex A; sometimes by the three letters BAC, or CAB, the letter at the vertex being always placed in the middle.

Angles, like all other quantities, are susceptible of addition, subtraction, multiplication, and division.

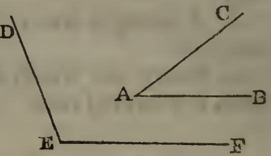
Thus the angle DCE is the sum of the two angles DCB, BCE; and the angle DCB is the difference of the two angles DCE, BCE.



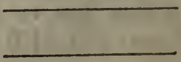
10. When a straight line AB meets another straight line CD, so as to make the adjacent angles BAC, BAD, equal to each other, each of these angles is called a *right angle;* and the line AB is said to be *perpendicular* to CD.



11. Every angle BAC, less than a right angle, is an *acute angle;* and every angle DEF, greater than a right angle, is an *obtuse angle.*



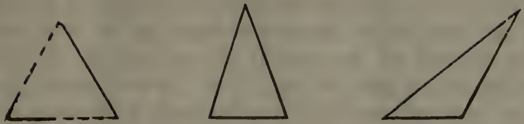
12. Two lines are said to be *parallel,* when being situated in the same plane, they cannot meet, how far soever, either way, both of them be produced.



13. A *plane figure* is a plane terminated on all sides by lines, either straight or curved.

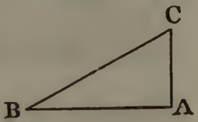
If the lines are straight, the space they enclose is called a *rectilineal figure*, or *polygon,* and the lines themselves, taken together, form the contour, or *perimeter* of the polygon.

14. The polygon of three sides, the simplest of all, is called a *triangle;* that of four sides, a *quadrilateral;* that of five, a *pentagon;* that of six, a *hexagon;* that of seven, a *heptagon:* that of eight, an *octagon;* that of nine, a *nonagon;* that of ten, a *decagon;* and that of twelve, a *dodecagon.*



15. An *equilateral* triangle is one which has its three sides equal; an *isosceles* triangle, one which has two of its sides equal; a *scalene* triangle, one which has its three sides unequal.

16. A right-angled triangle is one which has a right angle. The side opposite the right angle is called the *hypothenuse.* Thus, in the triangle ABC, right-angled at A, the side BC is the hypothenuse.



17. Among the quadrilaterals, we distinguish :

The *square*, which has its sides equal, and its angles right-angles.

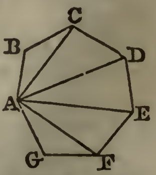
The *rectangle*, which has its angles right angles, without having its sides equal.

The *parallelogram*, or *rhomboid*, which has its opposite sides parallel.

The *rhombus*, or *lozenge*, which has its sides equal, without having its angles right angles.

And lastly, the *trapezoid*, only two of whose sides are parallel.

18. A *diagonal* is a line which joins the vertices of two angles not adjacent to each other. Thus, AF, AE, AD, AC, are diagonals.



19. An *equilateral* polygon is one which has all its sides equal; an *equiangular* polygon, one which has all its angles equal.

20. Two polygons are *mutually equilateral*, when they have their sides equal each to each, and placed in the same order

that is to say, when following their perimeters in the same direction, the first side of the one is equal to the first side of the other, the second of the one to the second of the other, the third to the third, and so on. The phrase, *mutually equiangular*, has a corresponding signification, with respect to the angles.

In both cases, the equal sides, or the equal angles, are named *homologous* sides or angles.

Definitions of terms employed in Geometry.

An *axiom* is a self-evident proposition.

A *theorem* is a truth, which becomes evident by means of a train of reasoning called a *demonstration.*

A *problem* is a question proposed, which requires a *solution.*

A *lemma* is a subsidiary truth, employed for the demonstration of a theorem, or the solution of a problem.

The common name, *proposition*, is applied indifferently, to theorems, problems, and lemmas.

A *corollary* is an obvious consequence, deduced from one or several propositions.

A *scholium* is a remark on one or several preceding propositions, which tends to point out their connexion, their use, their restriction, or their extension.

A *hypothesis* is a supposition, made either in the enunciation of a proposition, or in the course of a demonstration.

Explanation of the symbols to be employed.

The sign $=$ is the sign of equality; thus, the expression $A=B$, signifies that A is equal to B.

To signify that A is smaller than B, the expression $A<B$ is used.

To signify that A is greater than B, the expression $A>B$ is used; the smaller quantity being always at the vertex of the angle.

The sign $+$ is called *plus:* it indicates addition.

The sign — is called *minus:* it indicates subtraction.

Thus, $A+B$, represents the sum of the quantities A and B; $A-B$ represents their difference, or what remains after B is taken from A; and $A-B+C$, or $A+C-B$, signifies that A and C are to be added together, and that B is to be subtracted from their sum.

The sign $\times$ indicates multiplication: thus, $A \times B$ represents the product of A and B. Instead of the sign $\times$, a point is sometimes employed; thus, A.B is the same thing as $A \times B$. The same product is also designated without any intermediate sign, by AB; but this expression should not be employed, when there is any danger of confounding it with that of the line AB, which expresses the distance between the points A and B.

The expression $A \times (B+C-D)$ represents the product of A by the quantity $B+C-D$. If $A+B$ were to be multiplied by $A-B+C$, the product would be indicated thus, $(A+B) \times (A-B+C)$, whatever is enclosed within the curved lines, being considered as a single quantity.

A number placed before a line, or a quantity, serves as a multiplier to that line or quantity; thus, 3AB signifies that the line AB is taken three times; $\frac{1}{2}$ A signifies the half of the angle A.

The square of the line AB is designated by AB^2; its cube by AB^3. What is meant by the square and cube of a line, will be explained in its proper place.

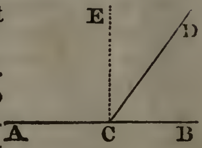
The sign $\surd$ indicates a root to be extracted; thus $\surd 2$ means the square-root of 2; $\sqrt{A \times B}$ means the square-root of the product of A and B.

Axioms.

1. Things which are equal to the same thing, are equal to each other.
2. If equals be added to equals, the wholes will be equal.
3. If equals be taken from equals, the remainders will be equal.
4. If equals be added to unequals, the wholes will be unequal.
5. If equals be taken from unequals, the remainders will be unequal.
6. Things which are double of the same thing, are equal to each other.
7. Things which are halves of the same thing, are equal to each other.
8. The whole is greater than any of its parts.
9. The whole is equal to the sum of all its parts.
10. All right angles are equal to each other.
11. From one point to another only one straight line can be drawn.
12. Through the same point, only one straight line can be drawn which shall be parallel to a given line.
13. Magnitudes, which being applied to each other, coincide throughout their whole extent, are equal.

PROPOSITION I. THEOREM.

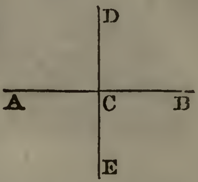
If one straight line meet another straight line, the sum of the two adjacent angles will be equal to two right angles.



Let the straight line DC meet the straight line AB at C, then will the angle ACD + the angle DCB, be equal to two right angles.

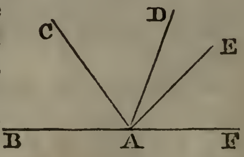
At the point C, erect CE perpendicular to AB. The angle ACD is the sum of the angles ACE, ECD: therefore ACD+DCB is the sum of the three angles ACE, ECD, DCB: but the first of these three angles is a right angle, and the other two make up the right angle ECB; hence, the sum of the two angles ACD and DCB, is equal to two right angles.

Cor. 1. If one of the angles ACD, DCB, is a right angle the other must be a right angle also.



Cor. 2. If the line DE is perpendicular to AB, reciprocally, AB will be perpendicular to DE.

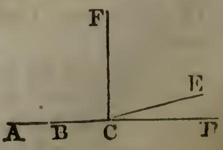
For, since DE is perpendicular to AB, the angle ACD must be equal to its adjacent angle DCB, and both of them must be right angles (Def. 10.). But since ACD is a right angle, its adjacent angle ACE must also be a right angle (Cor. 1.). Hence the angle ACD is equal to the angle ACE, (Ax. 10.) : therefore AB is perpendicular to DE.



Cor. 3. The sum of all the successive angles, BAC, CAD, DAE, EAF, formed on the same side of the straight line BF, is equal to two right angles; for their sum is equal to that of the two adjacent angles, BAC, CAF.

PROPOSITION II. THEOREM.

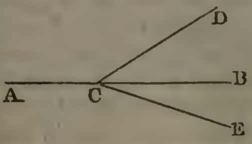
Two straight lines, which have two points common, coincide with each other throughout their whole extent, and form one and the same straight line



Let A and B be the two common points. In the first place it is evident that the two lines must coincide entirely between A and B, for otherwise there would be two straight lines between A and B, which is impossible (Ax. 11). Suppose, however, that on being produced, these lines begin to separate at C, the one becoming CD, the other CE. From the point C draw the line CF, making with AC the right angle ACF. Now, since ACD is a straight line, the angle FCD will be a right angle (Prop. I. Cor. 1.); and since ACE is a straight line, the angle FCE will likewise be a right angle. Hence, the angle FCD is equal to the angel FCE (Ax. 10.); which can only be the case when the lines CD and CE coincide: therefore, the straight lines which have two points A and B common, cannot separate at any point, when produced; hence they form one and the same straight line.

PROPOSITION III. THEOREM.

If a straight line meet two other straight lines at a common point, making the sum of the two adjacent angles equal to two right angles, the two straight lines which are met, will form one and the same straight line.

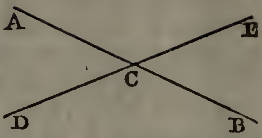


Let the straight line CD meet the two lines AC, CB, at their common point C, making the sum of the two adjacent angles DCA, DCB, equal to two right angles; then will CB be the prolongation of AC, or AC and CB will form one and the same straight line.

For, if CB is not the prolongation of AC, let CE be that prolongation: then the line ACE being straight, the sum of the angles ACD, DCE, will be equal to two right angles (Prop. I.). But by hypothesis, the sum of the angles ACD, DCB, is also equal to two right angles: therefore, ACD+DCE must be equal to ACD+DCB; and taking away the angle ACD from each, there remains the angle DCE equal to the angle DCB, which can only be the case when the lines CE and CB coincide; hence, AC, CB, form one and the same straight line.

PROPOSITION IV. THEOREM.

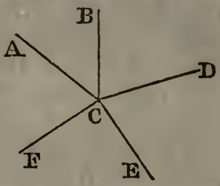
When two straight lines intersect each other, the opposite or vertical angles, which they form, are equal.



Let AB and DE be two straight lines, intersecting each other at C; then will the angle ECB be equal to the angle ACD, and the angle ACE to the angle DCB.

For, since the straight line DE is met by the straight line AC, the sum of the angles ACE, ACD, is equal to two right angles (Prop. I.); and since the straight line AB, is met by the straight line EC, the sum of the angles ACE and ECB, is equal to two right angles: hence the sum ACE+ACD is equal to the sum ACE+ECB (Ax. 1.). Take away from both, the common angle ACE, there remains the angle ACD, equal to its opposite or vertical angle ECB (Ax. 3.).

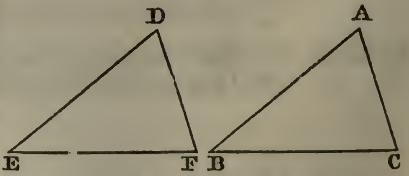
Scholium. The four angles formed about a point by two straight lines, which intersect each other, are together equal to four right angles: for the sum of the two angles ACE, ECB, is equal to two right angles; and the sum of the other two, ACD, DCB, is also equal to two right angles: therefore, the sum of the four is equal to four right angles.



In general, if any number of straight lines CA, CB, CD, &c. meet in a point C, the sum of all the successive angles ACB, BCD, DCE, ECF, FCA, will be equal to four right angles: for, if four right angles were formed about the point C, by two lines perpendicular to each other, the same space would be occupied by the four right angles, as by the successive angles ACB, BCD, DCE, ECF, FCA.

PROPOSITION V. THEOREM.

If two triangles have two sides and the included angle of the one, equal to two sides and the included angle of the other, each to each, the two triangles will be equal



Let the side ED be equal to the side BA, the side DF to the side AC, and the angle D to the angle A; then will the triangle EDF be equal to the triangle BAC.

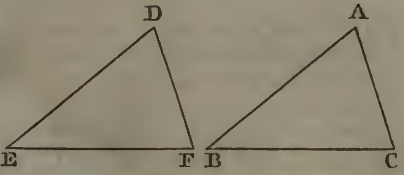
For, these triangles may be so applied to each other, that they shall exactly coincide. Let the triangle EDF, be placed upon the triangle BAC, so that the point E shall fall upon B, and the side ED on the equal side BA; then, since the angle D is equal to the angle A, the side DF will take the direction AC. But

DF is equal to AC; therefore, the point F will fall on C, and the third side EF, will coincide with the third side BC (Ax. 11.): therefore, the triangle EDF is equal to the triangle BAC (Ax. 13.).

Cor. When two triangles have these three things equal, namely, the side ED=BA, the side DF=AC, and the angl D=A, the remaining three are also respectively equal, namely the side EF=BC, the angle E=B, and the angle F=C

PROPOSITION VI. THEOREM.

If two triangles have two angles and the included side of the one, equal to two angles and the included side of the other, each to each, the two triangles will be equal.



Let the angle E be equal to the angle B, the angle F to the angle C, and the included side EF to the included side BC; then will the triangle EDF be equal to the triangle BAC.

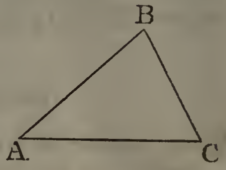
For to apply the one to the other, let the side EF be placed on its equal BC, the point E falling on B, and the point F on C; then, since the angle E is equal to the angle B, the side ED will take the direction BA; and hence the point D will be found somewhere in the line BA. In like manner, since the angle F is equal to the angle C, the line FD will take the direction CA, and the point D will be found somewhere in the line CA. Hence, the point D, falling at the same time in the two straight lines BA and CA, must fall at their intersection A: hence, the two triangles EDF, BAC, coincide with each other, and are therefore equal (Ax. 13.).

Cor. Whenever, in two triangles, these three things are equal, namely, the angle E=B, the angle F=C, and the included side EF equal to the included side BC, it may be inferred that the remaining three are also respectively equal, namely, the angle D=A, the side ED=BA, and the side DF=AC.

Scholium. Two triangles are said to be equal, when being applied to each other, they will exactly coincide (Ax. 13.). Hence, equal triangles have their like parts equal, each to each, since those parts must coincide with each other. The converse of this proposition is also true, namely, that *two triangles which have all the parts of the one equal to the parts of the other, each* *to each, are equal;* for they may be applied to each other, and the equal parts will mutually coincide.

PROPOSITION VII. THEOREM.

The sum of any two sides of a triangle, is greater than the third side.

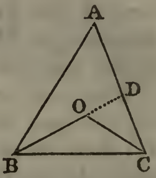


Let ABC be a triangle: then will the sum of two of its sides, as AC, CB, be greater than the third side AB.

For the straight line AB is the shortest distance between the points A and B (Def. 3.); hence $AC+CB$ is greater than AB.

PROPOSITION VIII. THEOREM.

If from any point within a triangle, two straight lines be drawn to the extremities of either side, their sum will be less than the sum of the two other sides of the triangle.



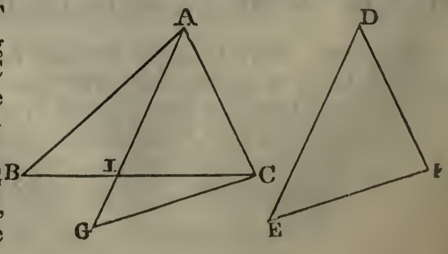
Let any point, as O, be taken within the triangle BAC, and let the lines OB, OC, be drawn to the extremities of either side, as BC; then will $OB+OC<BA+AC$.

Let BO be produced till it meets the side AC in D: then the line OC is shorter than $OD+DC$ (Prop. VII.): add BO to each, and we have $BO+OC<BO+OD+DC$ (Ax. 4.), or $BO+OC<BD+DC$.

Again, $BD<BA+AD$: add DC to each, and we have $BD+DC<BA+AC$. But it has just been found that $BO+OC<BD+DC$; therefore, still more is $BO+OC<BA+AC$.

PROPOSITION IX. THEOREM.

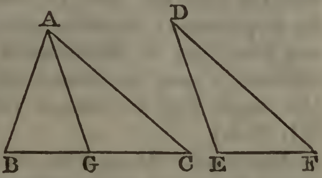
If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angles unequal, the third sides will be unequal; and the greater side will belong to the triangle which has the greater included angle.



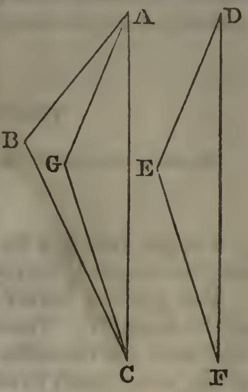
Let BAC and EDF be two triangles, having the side $AB=DE$, $AC=DF$, and the angle $A>D$; then will $BC>EF$.

Make the angle $CAG=D$; take $AG=DE$, and draw CG. The triangle GAC is equal to DEF, since, by construction, they have an equal angle in each, contained by equal sides, (Prop. V.); therefore CG is equal to EF. Now, there may be three cases in the proposition, according as the point G falls without the triangle ABC, or upon its base BC, or within it.

First Case. The straight line $GC < GI + IC$, and the straight line $AB < AI + IB$; therefore, $GC + AB < GI + AI + IC + IB$, or, which is the same thing, $GC + AB < AG + BC$. Take away AB from the one side, and its equal AG from the other; and there remains $GC < BC$ (Ax. 5.); but we have found $GC = EF$, therefore, $BC > EF$.



Second Case. If the point G fall on the side BC, it is evident that GC, or its equal EF, will be shorter than BC (Ax. 8.).



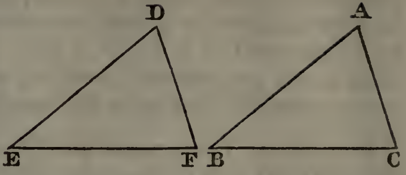
Third Case. Lastly, if the point G fall within the triangle BAC, we shall have, by the preceding theorem, $AG + GC < AB + BC$; and, taking AG from the one, and its equal AB from the other, there will remain $GC < BC$ or $BC > EF$.

Scholium. Conversely, if two sides BA, AC, of the triangle BAC, are equal to the two ED, DF, of the triangle EDF, each to each, while the third side BC of the first triangle is greater than the third side EF of the second; then will the angle BAC of the first triangle, be greater than the angle EDF of the second.

For, if not, the angle BAC must be equal to EDF, or less than it. In the first case, the side BC would be equal to EF, (Prop. V. Cor.); in the second, CB would be less than EF; but either of these results contradicts the hypothesis: therefore, BAC is greater than EDF.

PROPOSITION X. THEOREM.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the three angles will also be equal, each to each, and the triangles themselves will be equal.



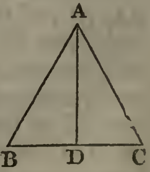
Let the side ED=BA, the side EF=BC, and the side DF=AC; then will the angle D=A, the angle E=B, and the angle F =C.

For, if the angle D were greater than A, while the sides ED, DF, were equal to BA, AC, each to each, it would follow, by the last proposition, that the side EF must be greater than BC; and if the angle D were less than A, it would follow, that the side EF must be less than BC: but EF is equal to BC, by hypothesis; therefore, the angle D can neither be greater nor less than A; therefore it must be equal to it. In the same manner it may be shown that the angle E is equal to B, and the angle F to C: hence the two triangles are equal (Prop. VI. Sch.).

Scholium. It may be observed that the equal angles lie opposite the equal sides: thus, the equal angles D and A, lie opposite the equal sides EF and BC.

PROPOSITION XI. THEOREM

In an isosceles triangle, the angles opposite the equal sides are equal.



Let the side BA be equal to the side AC; then will the angle C be equal to the angle B.

For, join the vertex A, and D the middle point of the base BC. Then, the triangles BAD, DAC, will have all the sides of the one equal to those of the other, each to each; for BA is equal to AC, by hypothesis; AD is common, and BD is equal to DC by construction: therefore, by the last proposition, the angle B is equal to the angle C.

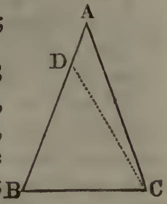
Cor. An equilateral triangle is likewise equiangular, that is to say, has all its angles equal.

Scholium. The equality of the triangles BAD, DAC, proves also that the angle BAD, is equal to DAC, and BDA to ADC, hence the latter two are right angles; *therefore, the line drawn from the vertex of an isosceles triangle to the middle point of its base, is perpendicular to the base, and divides the angle at the vertex into two equal parts.*

In a triangle which is not isosceles, any side may be assumed indifferently as the *base;* and the *vertex* is, in that case, the vertex of the opposite angle. In an isosceles triangle, however, that side is generally assumed as the base, which is not equal to either of the other two.

PROPOSITION XII. THEOREM.

Conversely, if two angles of a triangle are equal, the sides opposite them are also equal, and the triangle is isosceles.

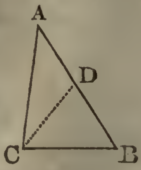


Let the angle ABC be equal to the angle ACB; then will the side AC be equal to the side AB.

For, if these sides are not equal, suppose AB to be the greater. Then, take BD equal to AC, and draw CD. Now, in the two triangles BDC, BAC, we have $BD = AC$, by construction; the angle B equal to the angle ACB, by hypothesis; and the side BC common: therefore, the two triangles, BDC, BAC, have two sides and the included angle in the one, equal to two sides and the included angle in the other, each to each: hence they are equal (Prop. V.). But the part cannot be equal to the whole (Ax. 8.); hence, there is no inequality between the sides BA, AC; therefore, the triangle BAC is isosceles.

PROPOSITION XIII. THEOREM.

The greater side of every triangle is opposite to the greater angle; and conversely, the greater angle is opposite to the greater side.



First, Let the angle C be greater than the angle B; then will the side AB, opposite C, be greater than AC, opposite B.

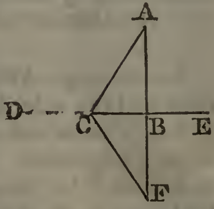
For, make the angle $BCD = B$. Then, in the triangle CDB, we shall have $CD = BD$ (Prop. XII.). Now, the side $AC < AD + CD$; but $AD + CD = AD + DB = AB$: therefore $AC < AB$.

Secondly, Suppose the side $AB > AC$; then will the angle C, opposite to AB, be greater than the angle B, opposite to AC.

For, if the angle $C < B$, it follows, from what has just been proved, that $AB < AC$; which is contrary to the hypothesis. If the angle $C = B$, then the side $AB = AC$ (Prop. XII.); which is also contrary to the supposition. Therefore, when $AB > AC$, the angle C must be greater than B

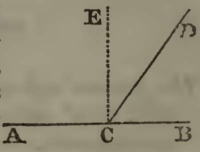
PROPOSITION XIV. THEOREM.

From a given point, without a straight line, only one perpendicular can be drawn to that line.



Let A be the point, and DE the given line.

Let us suppose that we can draw two perpendiculars, AB, AC. Produce either of them, as AB, till BF is equal to AB, and draw FC. Then, the two triangles CAB, CBF, will be equal: for, the angles CBA, and CBF are right angles, the side CB is common, and the side AB equal to BF, by construction; therefore, the triangles are equal, and the angle ACB=BCF (Prop. V. Cor.). But the angle ACB is a right angle, by hypothesis; therefore, BCF must likewise be a right angle. But if the adjacent angles BCA, BCF, are together equal to two right angles, ACF must be a straight line (Prop. III.): from whence it follows, that between the same two points, A and F, two straight lines can be drawn, which is impossible (Ax. 11.): hence, two perpendiculars cannot be drawn from the same point to the same straight line.



Scholium. At a given point C, in the line AB, it is equally impossible to erect two perpendiculars to that line. For, if CD, CE, were those two perpendiculars, the angles BCD, BCE, would both be right angles: hence they would be equal (Ax. 10.); and the line CD would coincide with CE; otherwise, a part would be equal to the whole, which is impossible (Ax. 8.).

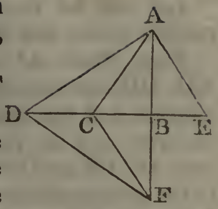
PROPOSITION XV. THEOREM.

If from a point without a straight line, a perpendicular be let fall on the line, and oblique lines be drawn to different points:

1*st, The perpendicular will be shorter than any oblique line.*

2*d, Any two oblique lines, drawn on different sides of the perpendicular, cutting off equal distances on the other line, will be equal.*

3*d, Of two oblique lines, drawn at pleasure, that which is farther from the perpendicular will be the longer.*



Let A be the given point, DE the given line, AB the perpendicular, and AD, AC, AE, the oblique lines.

Produce the perpendicular AB till BF is equal to AB, and draw FC, FD.

First. The triangle BCF, is equal to the triangle BCA, for they have the right angle CBF=CBA, the side CB common, and the side BF=BA; hence the third sides, CF and CA are equal (Prop. V. Cor.). But ABF, being a straight line, is shorter than ACF, which is a broken line (Def. 3.); therefore, AB, the half of ABF, is shorter than AC, the half of ACF; hence, the perpendicular is shorter than any oblique line.

Secondly. Let us suppose BC=BE; then will the triangle CAB be equal to the the triangle BAE; for BC=BE, the side AB is common, and the angle CBA=ABE; hence the sides AC and AE are equal (Prop. V. Cor.): therefore, two oblique, lines, equally distant from the perpendicular, are equal.

Thirdly. In the triangle DFA, the sum of the lines AC, CF, is less than the sum of the sides AD, DF (Prop. VIII.); therefore, AC, the half of the line ACF, is shorter than AD, the half of the line ADF: therefore, the oblique line, which is farther from the perpendicular, is longer than the one which is nearer.

Cor. 1. The perpendicular measures the shortest distance of a point from a line.

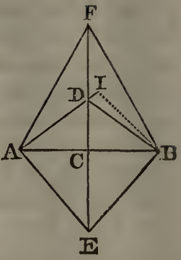
Cor. 2. From the same point to the same straight line, only two equal straight lines can be drawn; for, if there could be more, we should have at least two equal oblique lines on the same side of the perpendicular, which is impossible.

PROPOSITION XVI. THEOREM.

If from the middle point of a straight line, a perpendicular be drawn to this line;

1*st, Every point of the perpendicular will be equally distant from the extremities of the line.*

2*d, Every point, without the perpendicular, will be unequally distant from those extremities.*



Let AB be the given straight line, C the middle point, and ECF the perpendicular.

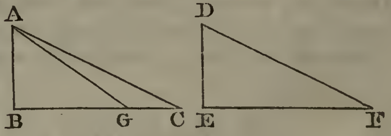
First, Since AC=CB, the two oblique lines AD, DB, are equally distant from the perpendicular, and therefore equal (Prop. XV.). So, likewise, are the two oblique lines AE, EB, the two AF, FB, and so on. Therefore every point in the perpendicular is equally distant from the extremities A and B.

Secondly, Let I be a point out of the perpendicular. If IA and IB be drawn, one of these lines will cut the perpendicular in D; from which, drawing DB, we shall have DB=DA. But the straight line IB is less than ID+DB, and ID+DB=ID+DA=IA; therefore, IB<IA; therefore, every point out of the perpendicular, is unequally distant from the extremities A and B.

Cor. If a straight line have two points D and F, equally distant from the extremities A and B, it will be perpendicular to AB at the middle point C.

PROPOSITION XVII. THEOREM.

If two right angled triangles have the hypothenuse and a side of the one, equal to the hypothenuse and a side of the other, each to each, the remaining parts will also be equal, each to each, and the triangles themselves will be equal.



In the two right angled triangles BAC, EDF, let the hypothenuse AC=DF, and the side BA=ED: then will the side BC=EF, the angle A=D, and the angle C=F.

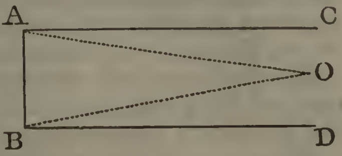
If the side BC is equal to EF, the like angles of the two triangles are equal (Prop. X.). Now, if it be possible, suppose these two sides to be unequal, and that BC is the greater.

On BC take BG=EF, and draw AG. Then, in the two triangles BAG, DEF, the angles B and E are equal, being right angles, the side BA=ED by hypothesis, and the side BG=EF by construction . consequently, AG=DF (Prop. V. Cor.). But by hypothesis AC=DF; and therefore, AC=AG (Ax. 1.) But the oblique line AC cannot be equal to AG, which lies nearer the perpendicular AB (Prop. XV.); therefore, BC and EF cannot be unequal, and hence the angle A=D, and the angle C=F; and therefore, the triangles are equal (Prop. VI Sch.).

PROPOSITION XVIII. THEOREM.

If two straight lines are perpendicular to a third line, they will be parallel to each other: in other words, they will never meet, how far soever either way, both of them be produced.

Let the two lines AC, BD, be perpendicular to AB; then will they be parallel.

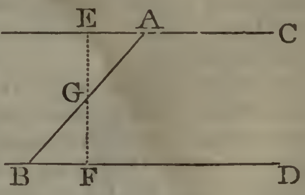


For, if they could meet in a point O, on either side of AB, there would be two perpendiculars OA, OB, let fall from the same point on the same straight line; which is impossible (Prop. XIV.).

PROPOSITION XIX. THEOREM.

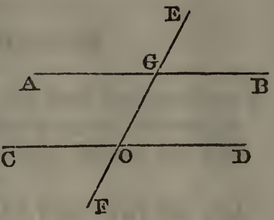
If two straight lines meet a third line, making the sum of the interior angles on the same side of the line met, equal to two right angles, the two lines will be parallel.

Let the two lines EC, BD, meet the third line BA, making the angles BAC, ABD, together equal to two right angles: then the lines EC, BD, will be parallel.



From G, the middle point of BA, draw the straight line EGF, perpendicular to EC. It will also be perpendicular to BD. For, the sum BAC+ABD is equal to two right angles, by hypothesis; the sum BAC+BAE is likewise equal to two right angles (Prop. I.); and taking away BAC from both, there will remain the angle ABD=BAE.

Again, the angles EGA, BGF, are equal (Prop. IV.); therefore, the triangles EGA and BGF, have each a side and two adjacent angles equal; therefore, they are themselves equal, and the angle GEA is equal to the angle GFB (Prop. VI. Cor.). but GEA is a right angle by construction; therefore, GFB is a right angle; hence the two lines EC, BD, are perpendicular to the same straight line, and are therefore parallel (Prop. XVIII.).



Scholium. When two parallel straight lines AB, CD, are met by a third line FE, the angles which are formed take particular names.

Interior angles on the same side, are those which lie within the parallels, and on the same side of the secant line: thus, OGB, GOD, are interior angles on the same side; and so also are the the angles OGA, GOC.

Alternate angles lie within the parallels, and on different sides of the secant line: AGO, DOG, are alternate angles; and so also are the angles COG, BGO.

Alternate exterior angles lie without the parallels, and on different sides of the secant line: EGB, COF, are alternate exterior angles; so also, are the angles AGE, FOD.

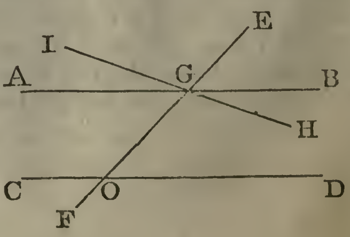
Opposite exterior and interior angles lie on the same side of the secant line, the one without and the other within the parallels, but not adjacent: thus, EGB, GOD, are opposite exterior and interior angles; and so also, are the angles AGE, GOC.

Cor. 1. If a straight line EF, meet two straight lines CD, AB, making the alternate angles AGO, GOD, equal to each other, the two lines will be parallel. For, to each add the angle OGB; we shall then have, AGO+OGB=GOD+OGB; but AGO+OGB is equal to two right angles (Prop. I.); hence GOD+OGB is equal to two right angles: therefore, CD, AB, are parallel.

Cor. 2. If a straight line EF, meet two straight lines CD, AB, making the exterior angle EGB equal to the interior and opposite angle GOD, the two lines will be parallel. For, to each add the angle OGB: we shall then have EGB+OGB=GOD+OGB: but EGB+OGB is equal to two right angles; hence, GOD+OGB is equal to two right angles; therefore, CD, AB, are parallel.

PROPOSITION XX. THEOREM.

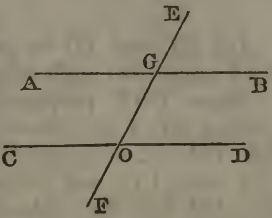
If a straight line meet two parallel straight lines, the sum of the interior angles on the same side will be equal to two right angles.



Let the parallels AB, CD, be met by the secant line FE: then will OGB+GOD, or OGA+GOC, be equal to two right angles.

For, if OGB+GOD be not equal to two right angles, let IGH be drawn, making the sum OGH+GOD equal to two right angles; then IH and CD will be parallel (Prop. XIX.), and hence we shall have two lines GB, GH, drawn through the same point G and parallel to CD, which is impossible (Ax. 12.): hence, GB and GH should coincide, and OGB+GOD is equal to two right angles. In the same manner it may be proved that OGA+GOC is equal to two right angles.

Cor. 1. If OGB is a right angle, GOD will be a right angle also: therefore, *every straight line perpendicular to one of two parallels, is perpendicular to the other.*



Cor. 2. If a straight line meet two parallel lines, the alternate angles will be equal.

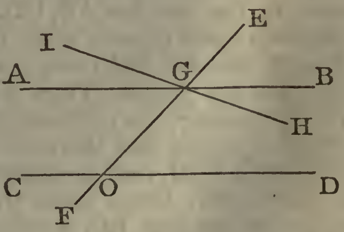
Let AB, CD, be the parallels, and FE the secant line. The sum OGB+GOD is equal to two right angles. But the sum OGB+OGA is also equal to two right angles (Prop. I.). Taking from each, the angle OGB, and there remains OGA=GOD. In the same manner we may prove that GOC=OGB.

Cor. 3. If a straight line meet two parallel lines, the opposite exterior and interior angles will be equal. For, the sum OGB+GOD is equal to two right angles. But the sum OGB +EGB is also equal to two right angles. Taking from each the angle OGB, and there remains GOD=EGB. In the same manner we may prove that AGE=GOC.

Cor. 4. We see that of the eight angles formed by a line cutting two parallel lines obliquely, the four acute angles are equal to each other, and so also are the four obtuse angles.

PROPOSITION XXI. THEOREM.

If a straight line meet two other straight lines, making the sum of the interior angles on the same side less than two right angles, the two lines will meet if sufficiently produced.



Let the line EF meet the two lines CD, IH, making the sum of the interior angles OGH, GOD, less than two right angles: then will IH and CD meet if sufficiently produced.

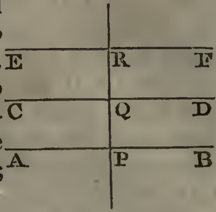
For, if they do not meet they are parallel (Def. 12.). But they are not parallel, for if they were, the sum of the interior angles OGH, GOD, would be equal to two right angles (Prop. XX.), whereas it is less by hypothesis: hence, the lines IH, CD, are not parallel, and will therefore meet if sufficiently produced.

Cor. It is evident that the two lines IH, CD, will meet on that side of EF on which the sum of the two angles OGH, GOD, is less than two right angles

PROPOSITION XXII. THEOREM.

Two straight lines which are parallel to a third line, are parallel to each other.

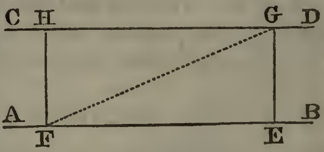
Let CD and AB be parallel to the third line EF; then are they parallel to each other.



Draw PQR perpendicular to EF, and cutting AB, CD. Since AB is parallel to EF, PR will be perpendicular to AB (Prop. XX. Cor. 1.); and since CD is parallel to EF, PR will for a like reason be perpendicular to CD. Hence AB and CD are perpendicular to the same straight line; hence they are parallel (Prop. XVIII.).

PROPOSITION XXIII. THEOREM.

Two parallels are every where equally distant.

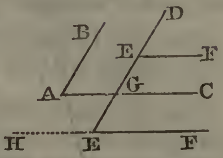


Two parallels AB, CD, being given, if through two points E and F, assumed at pleasure, the straight lines EG, FH, be drawn perpendicular to AB, these straight lines will at the same time be perpendicular to CD (Prop. XX. Cor. 1.): and we are now to show that they will be equal to each other.

If GF be drawn, the angles GFE, FGH, considered in reference to the parallels AB, CD, will be alternate angles, and therefore equal to each other (Prop. XX. Cor. 2.). Also, the straight lines EG, FH, being perpendicular to the same straight line AB, are parallel (Prop. XVIII.); and the angles EGF, GFH, considered in reference to the parallels EG, FH, will be alternate angles, and therefore equal. Hence the two triangles EFG, FGH, have a common side, and two adjacent angles in each equal; hence these triangles are equal (Prop. VI.); therefore, the side EG, which measures the distance of the parallels AB and CD at the point E, is equal to the side FH, which measures the distance of the same parallels at the point F.

PROPOSITION XXIV. THEOREM.

If two angles have their sides parallel and lying in the same direction, the two angles will be equal.



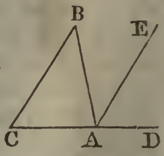
Let BAC and DEF be the two angles, having AB parallel to ED, and AC to EF; then will the angles be equal.

For, produce DE, if necessary, till it meets AC in G. Then, since EF is parallel to GC, the angle DEF is equal to DGC (Prop. XX. Cor. 3.); and since DG is parallel to AB, the angle DGC is equal to BAC; hence the angle DEF is equal to BAC (Ax. 1.).

Scholium. The restriction of this proposition to the case where the side EF lies in the same direction with AC, and ED in the same direction with AB, is necessary, because if FE were produced towards H, the angle DEH would have its sides parallel to those of the angle BAC, but would not be equal to it. In that case, DEH and BAC would be together equal to two right angles. For, DEH+DEF is equal to two right angles (Prop. I.); but DEF is equal to BAC: hence, DEH+BAC is equal to two right angles.

PROPOSITION XXV. THEOREM.

In every triangle the sum of the three angles is equal to two right angles.



Let ABC be any triangle: then will the angle C+A+B be equal to two right angles.

For, produce the side CA towards D, and at the point A, draw AE parallel to BC. Then, since AE, CB, are parallel, and CAD cuts them, the exterior angle DAE will be equal to its interior opposite one ACB (Prop. XX. Cor. 3.); in like manner, since AE, CB, are parallel, and AB cuts them, the alternate angles ABC, BAE, will be equal: hence the three angles of the triangle ABC make up the same sum as the three angles CAB, BAE, EAD; hence, the sum of the three angles is equal to two right angles (Prop. I.).

Cor. 1. Two angles of a triangle being given, or merely their sum, the third will be found by subtracting that sum from two right angles.

Cor. 2. If two angles of one triangle are respectively equal to two angles of another, the third angles will also be equal, and the two triangles will be mutually equiangular.

Cor. 3. In any triangle there can be but one right angle: for if there were two, the third angle must be nothing. Still less, can a triangle have more than one obtuse angle.

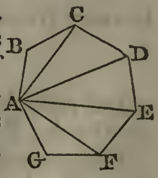
Cor. 4. In every right angled triangle, the sum of the two acute angles is equal to one right angle.

Cor. 5. Since every equilateral triangle is also equiangular (Prop. XI. Cor.), each of its angles will be equal to the third part of two right angles; so that, if the right angle is expressed by unity, the angle of an equilateral triangle will be expressed by $\frac{2}{3}$.

Cor. 6. In every triangle ABC, the exterior angle BAD is equal to the sum of the two interior opposite angles B and C. For, AE being parallel to BC, the part BAE is equal to the angle B, and the other part DAE is equal to the angle C.

PROPOSITION XXVI. THEOREM.

The sum of all the interior angles of a polygon, is equal to two right angles, taken as many times less two, as the figure has sides.



Let ABCDEFG be the proposed polygon. If from the vertex of any one angle A, diagonals AC, AD, AE, AF, be drawn to the vertices of all the opposite angles, it is plain that the polygon will be divided into five triangles, if it has seven sides; into six triangles, if it has eight; and, in general, into as many triangles, less two, as the polygon has sides; for, these triangles may be considered as having the point A for a common vertex, and for bases, the several sides of the polygon, excepting the two sides which form the angle A. It is evident, also, that the sum of all the angles in these triangles does not differ from the sum of all the angles in the polygon: hence the sum of all the angles of the polygon is equal to two right angles, taken as many times as there are triangles in the figure; in other words, as there are units in the number of sides diminished by two.

Cor. 1. The sum of the angles in a quadrilateral is equal to two right angles multiplied by 4—2, which amounts to four right angles: hence, if all the angles of a quadrilateral are equal, each of them will be a right angle; a conclusion which sanctions the seventeenth Definition, where the four angles of a quadrilateral are asserted to be right angles, in the case of the rectangle and the square.

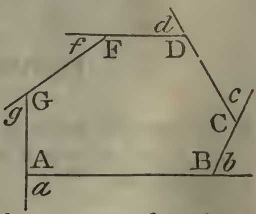
Cor. 2. The sum of the angles of a pentagon is equal to two right angles multiplied by 5—2, which amounts to six right angles: hence, when a pentagon is equiangular, each angle is equal to the fifth part of six right angles, or to $\frac{6}{5}$ of one right angle.

Cor. 3. The sum of the angles of a hexagon is equal to $2\times(6-2,)$ or eight right angles; hence in the equiangular hexagon, each angle is the sixth part of eight right angles, or $\frac{4}{3}$ of one.

Scholium. When this proposition is applied to polygons which have *re-entrant* angles, each re-entrant angle must be regarded as greater than two right angles. But to avoid all ambiguity, we shall henceforth limit our reasoning to polygons with *salient* angles, which might otherwise be named *convex polygons.* Every convex polygon is such that a straight line, drawn at pleasure, cannot meet the contour of the polygon in more than two points.

PROPOSITION XXVII. THEOREM.

If the sides of any polygon be produced out, in the same direction, the sum of the exterior angles will be equal to four right angles.

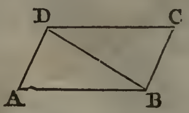


Let the sides of the polygon ABCDFG, be produced, in the same direction; then will the sum of the exterior angles $a+b+c+d+f+g$, be equal to four right angles.

For, each interior angle, plus its exterior angle, as $A+a$, is equal to two right angles (Prop. I.). But there are as many exterior as interior angles, and as many of each as there are sides of the polygon: hence, the sum of all the interior and exterior angles is equal to twice as many right angles as the polygon has sides. Again, the sum of all the interior angles is equal to two right angles, taken as many times, less two, as the polygon has sides (Prop. XXVI.); that is, equal to twice as many right angles as the figure has sides, wanting four right angles. Hence, the interior angles plus four right angles, is equal to twice as many right angles as the polygon has sides, and consequently, equal to the sum of the interior angles plus the exterior angles. Taking from each the sum of the interior angles, and there remains the exterior angles, equal to four right angles.

PROPOSITION XXVIII. THEOREM.

In every parallelogram, the opposite sides and angles are equal.



Let ABCD be a parallelogram: then will AB=DC, AD=BC, A=C, and ADC=ABC.

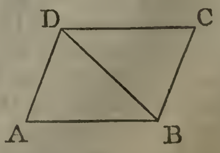
For, draw the diagonal BD. The triangles ABD, DBC, have a common side BD; and since AD, BC, are parallel, they have also the angle ADB=DBC, (Prop. XX. Cor. 2.); and since AB, CD, are parallel, the angle ABD=BDC: hence the two triangles are equal (Prop. VI.); therefore the side AB, opposite the angle ADB, is equal to the side DC, opposite the equal angle DBC; and the third sides AD, BC, are equal: hence the opposite sides of a parallelogram are equal.

Again, since the triangles are equal, it follows that the angle A is equal to the angle C; and also that the angle ADC composed of the two ADB, BDC, is equal to ABC, composed of the two equal angles DBC, ABD: hence the opposite angles of a parallelogram are also equal.

Cor. Two parallels AB, CD, included between two other parallels AD, BC, are equal; and the diagonal DB divides the parallelogram into two equal triangles.

PROPOSITION XXIX. THEOREM.

If the opposite sides of a quadrilateral are equal, each to each, the equal sides will be parallel, and the figure will be a parallelogram.

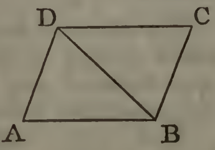


Let ABCD be a quadrilateral, having its opposite sides respectively equal, viz. AB=DC, and AD=BC; then will these sides be parallel, and the figure be a parallelogram.

For, having drawn the diagonal BD, the triangles ABD, BDC, have all the sides of the one equal to the corresponding sides of the other; therefore they are equal, and the angle ADB, opposite the side AB, is equal to DBC, opposite CD (Prop. X.); therefore, the side AD is parallel to BC (Prop. XIX. Cor. 1.). For a like reason AB is parallel to CD: therefore the quadrilateral ABCD is a parallelogram.

PROPOSITION XXX. THEOREM.

If two opposite sides of a quadrilateral are equal and parallel, the remaining sides will also be equal and parallel, and the figure will be a parallelogram.

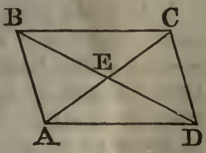


Let ABCD be a quadrilateral, having the sides AB, CD, equal and parallel; then will the figure be a parallelogram.

For, draw the diagonal DB, dividing the quadrilateral into two triangles. Then, since AB is parallel to DC, the alternate angles ABD, BDC, are equal (Prop. XX. Cor. 2.); moreover, the side DB is common, and the side AB=DC; hence the triangle ABD is equal to the triangle DBC (Prop. V.); therefore, the side AD is equal to BC, the angle ADB=DBC, and consequently AD is parallel to BC; hence the figure ABCD is a parallelogram.

PROPOSITION XXXI. THEOREM.

The two diagonals of a parallelogram divide each other into equal parts, or mutually bisect each other.



Let ABCD be a parallelogram, AC and DB its diagonals, intersecting at E, then will AE=EC, and DE=EB.

Comparing the triangles ADE, CEB, we find the side AD=CB (Prop. XXVIII.), the angle ADE=CBE, and the angle DAE=ECB (Prop. XX. Cor. 2.); hence those triangles are equal (Prop. VI.); hence, AE, the side opposite the angle ADE, is equal to EC, opposite EBC; hence also DE is equal to EB.

Scholium. In the case of the rhombus, the sides AB, BC being equal, the triangles AEB, EBC, have all the sides of the one equal to the corresponding sides of the other, and are therefore equal: whence it follows that the angles AEB, BEC, are equal, and therefore, that the two diagonals of a rhombus cut each other at right angles.

BOOK II.

OF RATIOS AND PROPORTIONS.

Definitions.

1. *Ratio* is the quotient arising from dividing one quantity by another quantity of the same kind. Thus, if A and B represent quantities of the same kind, the ratio of A to B is expressed by $\frac{B}{A}$.

The ratios of magnitudes may be expressed by numbers, either exactly or approximatively; and in the latter case, the approximation may be brought nearer to the true ratio than any assignable difference.

Thus, of two magnitudes, one of them may be considered to be divided into some number of equal parts, each of the same kind as the whole, and one of those parts being considered as an unit of measure, the magnitude may be expressed by the number of units it contains. If the other magnitude contain a certain number of those units, it also may be expressed by the number of its units, and the two quantities are then said to be *commensurable.*

If the second magnitude do not contain the measuring unit an exact number of times, there may perhaps be a smaller unit which will be contained an exact number of times in each of the magnitudes. But if there is no unit of an *assignable* value, which shall be contained an exact number of times in each of the magnitudes, the magnitudes are said to be *incommensurable.*

It is plain, however, that the unit of measure, repeated as many times as it is contained in the second magnitude, would always differ from the second magnitude by a quantity less than the unit of measure, since the remainder is always less than the divisor. Now, since the unit of measure may be made as small as we please, it follows, that magnitudes may be represented by numbers to any degree of exactness, or they will differ from their numerical representatives by less than any assignable quantity.

Therefore, of two magnitudes, A and B, we may conceive A to be divided into M number of units, each equal to A': then $A = M \times A'$: let B be divided into N number of equal units, each equal to A'; then $B = N \times A'$; M and N being integral numbers. Now the ratio of A to B, will be the same as the ratio of $M \times A'$ to $N \times A'$; that is the same as the ratio of M to N, since A' is a common unit.

In the same manner, the ratio of any other two magnitudes C and D may be expressed by $P \times C'$ to $Q \times C'$, P and Q being also integral numbers, and their ratio will be the same as that of P to Q.

2. If there be four magnitudes A, B, C, and D, having such values that $\frac{B}{A}$ is equal to $\frac{D}{C}$, then A is said to have the same *ratio* to B, that C has to D, or the ratio of A to B is *equal* to the ratio of C to D. When four quantities have this relation to each other, they are said to be in *proportion.*

To indicate that the ratio of A to B is equal to the ratio of C to D, the quantities are usually written thus, A : B : : C : D, and read, A is to B as C is to D. The quantities which are compared together are called the *terms* of the proportion. The first and last terms are called the *two extremes*, and the second and third terms, the two means.

3. Of four proportional quantities, the first and third are called the *antecedents*, and the second and fourth the *consequents;* and the last is said to be a *fourth proportional* to the other three taken in order.

4. Three quantities are in proportion, when the first has the same ratio to the second, that the second has to the third; and then the middle term is said to be a mean proportional between the other two.

5. Magnitudes are said to be in proportion by *inversion*, or *inversely*, when the consequents are taken as antecedents, and the antecedents as consequents.

6. Magnitudes are in proportion by *alternation*, or alternately when antecedent is compared with antecedent, and consequent with consequent.

7. Magnitudes are in proportion by *composition*, when the sum of the antecedent and consequent is compared either with antecedent or consequent.

8. Magnitudes are said to be in proportion by *division*, when the difference of the antecedent and consequent is compared either with antecedent or consequent.

9. Equimultiples of two quantities are the products which arise from multiplying the quantities by the same number: thus, $m \times A$, $m \times B$, are equimultiples of A and B, the common multiplier being *m*.

10. Two quantities A and B are said to be *reciprocally proportional*, or *inversely proportional*, when one increases in the same ratio as the other diminishes. In such case, either of them is equal to a constant quantity divided by the other, and their product is constant.

PROPOSITION I. THEOREM.

When four quantities are in proportion, the product of the two extremes is equal to the product of the two means

Let A, B, C, D, be four quantities in proportion, and $M : N :: P : Q$ be their numerical representatives; then will $M\times Q = N\times P$; for since the quantities are in proportion $\frac{N}{M}=\frac{Q}{P}$ therefore $N=M\times\frac{Q}{P}$, or $N\times P=M\times Q$.

Cor. If there are three proportional quantities (Def. 4.), the product of the extremes will be equal to the square of the mean.

PROPOSITION II. THEOREM.

If the product of two quantities be equal to the product of two other quantities, two of them will be the extremes and the other two the means of a proportion.

Let $M\times Q=N\times P$; then will $M: N :: P : Q$.

For, if P have not to Q the ratio which M has to N, let P have to Q', a number greater or less than Q, the same ratio that M has to N; that is, let $M : N :: P : Q'$; then $M\times Q' = N\times P$ (Prop. I.) : hence, $Q'=\frac{N\times P}{M}$; but $Q=\frac{N\times P}{M}$; consequently, $Q=Q'$ and the four quantities are proportional; that is, $M : N :: P : Q$.

PROPOSITION III. THEOREM.

If four quantities are in proportion, they will be in proportion when taken alternately.

Let M, N, P, Q, be the numerical representatives of four quanties in proportion ; so that

$M : N :: P : Q$, then will $M : P :: N : Q$.

Since $M : N :: P : Q$, by supposition, $M\times Q=N\times P$; therefore, M and Q may be made the extremes, and N and P the means of a proportion (Prop. II.); hence, $M : P :: N : Q$.

PROPOSITION IV. THEOREM.

If there be four proportional quantities, and four other proportional quantities, having the antecedents the same in both, the consequents will be proportional.

Let $M : N :: P : Q$
and $M : R :: P : S$
then will $N : Q :: R : S$

For, by alternation $M : P :: N : Q$, or $\frac{P}{M} = \frac{Q}{N}$
and $M : P :: R ; S$, or $\frac{P}{M} = \frac{S}{R}$
hence $\frac{Q}{N} = \frac{S}{R}$; or $N : Q :: R : S$.

Cor. If there be two sets of proportionals, having an antecedent and consequent of the first, equal to an antecedent and consequent of the second, the remaining terms will be proportional.

PROPOSITION V. THEOREM.

If four quantities be in proportion, they will be in proportion when taken inversely.

Let $M : N :: P : Q$; then will
$N : M :: Q : P$.

For, from the first proportion we have $M \times Q = N \times P$, or $N \times P = M \times Q$.

But the products $N \times P$ and $M \times Q$ are the products of the extremes and means of the four quantities N, M, Q, P, and these products being equal,

$N : M :: Q : P$ (Prop. II.).

PROPOSITION VI. THEOREM.

If four quantities are in proportion, they will be in proportion by composition, or division.

Let, as before, M, N, P, Q, be the numerical representatives of the four quantities, so that

$$M : N :: P . Q \text{; then will}$$
$$M \pm N : M :: P \pm Q : P.$$

For, from the first proportion, we have

$$M \times Q = N \times P, \text{ or } N \times P = M \times Q \text{;}$$

Add each of the members of the last equation to, or subtract it from M.P, and we shall have,

$$M.P \pm N.P = M.P \pm M.Q \text{; or}$$
$$(M \pm N) \times P = (P \pm Q) \times M.$$

But $M \pm N$ and P, may be considered the two extremes, and $P \pm Q$ and M, the two means of a proportion: hence,

$$\overline{M \pm N} : M :: \overline{P \pm Q} : P.$$

PROPOSITION VII. THEOREM.

Equimultiples of any two quantities, have the same ratio as the quantities themselves.

Let M and N be any two quantities, and m any integral number; then will

$m.\ M : m.\ N :: M : N.$ For

$m.\ M \times N = m.\ N \times M$, since the quantities in each member are the same; therefore, the quantities are proportional (Prop. II.); or

$$m.\ M : m.\ N :: M : N.$$

PROPOSITION VIII. THEOREM.

Of four proportional quantities, if there be taken any equimultiples of the two antecedents, and any equimultiples of the two consequents, the four resulting quantities will be proportional.

Let M, N, P, Q, be the numerical representatives of four quantities in proportion; and let m and n be any numbers whatever, then will

$$m.\ M : n.\ N :: m.\ P : n.\ Q.$$

For, since $M : N :: P : Q$, we have $M \times Q = N \times P$; hence, $m.\ M \times n.\ Q = n.\ N \times m.\ P$, by multiplying both members of the equation by $m \times n$. But $m.\ M$ and $n.\ Q$, may be regarded as the two extremes, and $n.\ N$ and $m.\ P$, as the means of a proportion; hence, $m.\ M : n.\ N :: m.\ P : n.\ Q$.

PROPOSITION IX. THEOREM.

Of four proportional quantities, if the two consequents be either augmented or diminished by quantities which have the same ratio as the antecedents, the resulting quantities and the antecedents will be proportional.

Let	$M : N :: P : Q$, and let also
	$M : P :: m : n$, then will
	$M : P :: N \pm m : Q \pm n$.
For, since	$M : N :: P : Q$, $M \times Q = N \times P$.
And since	$M : P :: m : n$, $M \times n = P \times m$
Therefore,	$M \times Q \pm M \times n = N \times P \pm P \times m$
or,	$M \times (Q \pm n) = P \times (N \pm m)$:
hence	$M : P :: N \pm m : Q \pm n$ (Prop. II.).

PROPOSITION X. THEOREM.

If any number of quantities are proportionals, any one antecedent will be to its consequent, as the sum of all the antecedents to the sum of the consequents.

Let	$M : N :: P : Q :: R : S$, &c. then will
	$M : N :: \overline{M+P+R} : \overline{N+Q+S}$
For, since	$M : N :: P : Q$, we have $M \times Q = N \times P$
And since	$M : N :: R : S$, we have $M \times S = N \times R$
Add	$M \times N = M \times N$
and we have,	$M.N + M.Q + M.S = M.N + N.P + N.R$
or	$M \times (N+Q+S) = N \times (M+P+R)$
therefore,	$M : N :: \overline{M+P+R} : \overline{N+Q+S}$.

PROPOSITION XI. THEOREM.

If two magnitudes be each increased or diminished by like parts of each, the resulting quantities will have the same ratio as the magnitudes themselves.

Let M and N be any two magnitudes, and $\frac{M}{m}$ and $\frac{N}{m}$ be like parts of each: then will

$$M : N :: M \pm \frac{M}{m} : N \pm \frac{N}{m}$$

For, it is obvious that $M \times \left(N \pm \frac{N}{m}\right) = N \times \left(M \pm \frac{M}{m}\right)$ since each is equal to $M.N \pm \frac{N.M}{m}$. Consequently, the four quantities are proportional (Prop. II.).

PROPOSITION XII. THEOREM.

If four quantities are proportional, their squares or cubes will also be proportional.

Let	$M : N : P : Q,$
then will	$M^2 : N^2 :: P^2 : Q^2$
and	$M^3 : N^3 :: P^3 : Q^3$
For,	$M \times Q = N \times P$, since $M : N :: P : Q$
or,	$M^2 \times Q^2 = N^2 \times P^2$, by squaring both members,
and	$M^3 \times Q^3 = N^3 \times P^3$, by cubing both members;
therefore,	$M^2 : N^2 :: P^2 : Q^2$
and	$M^3 : N^3 :: P^3 : Q^3$

Cor. In the same way it may be shown that like powers or roots of proportional quantities are proportionals.

PROPOSITION XIII. THEOREM.

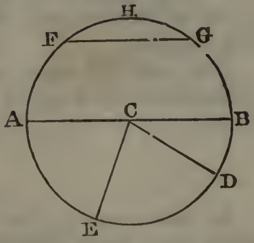
If there be two sets of proportional quantities, the products of the corresponding terms will be proportional.

Let	$M : N :: P : Q$
and	$R : S :: T : V$
then will	$M \times R : N \times S :: P \times T : Q \times V$
For since	$M \times Q = N \times P$
and	$R \times V = S \times T$, we shall have
	$M \times Q \times R \times V = N \times P \times S \times T$
or	$\overline{M \times R} \times \overline{Q \times V} = \overline{N \times S} \times \overline{P \times T}$
therefore,	$\overline{M \times R} : \overline{N \times S} :: \overline{P \times T} : \overline{Q \times V}.$

BOOK III.

THE CIRCLE, AND THE MEASUREMENT OF ANGLES.

Definitions.



1. The *circumference of a circle* is a curved line, all the points of which are equally distant from a point within, called the *centre*.

The *circle* is the space terminated by this curved line.*

2. Every straight line, CA, CE, CD, drawn from the centre to the circumference, is called a *radius* or *semidiameter;* every line which, like AB, passes through the centre, and is terminated on both sides by the circumference, is called a *diameter*.

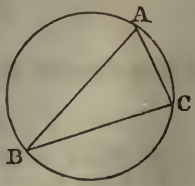
From the definition of a circle, it follows that all the radii are equal; that all the diameters are equal also, and each double of the radius.

3. A portion of the circumference, such as FHG, is called an *arc*.

The *chord*, or *subtense* of an arc, is the straight line FG, which joins its two extremities.†

4. A *segment* is the surface or portion of a circle, included between an arc and its chord.

5. A *sector* is the part of the circle included between an arc DE, and the two radii CD, CE, drawn to the extremities of the arc.



6. A *straight line* is said to be *inscribed in a circle*, when its extremities are in the circumference, as AB.

An *inscribed angle* is one which, like BAC, has its vertex in the circumference, and is formed by two chords.

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* *Note.* In common language, the circle is sometimes confounded with its circumference: but the correct expression may always be easily recurred to if we bear in mind that the circle is a surface which has length and breadth, while the circumference is but a line.

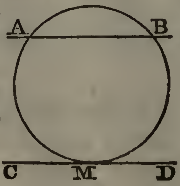
† *Note.* In all cases, the same chord FG belongs to two arcs, FGH, FEG and consequently also to two segments: but the smaller one is always meant, unless the contrary is expressed.

An *inscribed triangle* is one which, like BAC, has its three angular points in the circumference.

And, generally, an *inscribed figure* is one, of which all the angles have their vertices in the circumference. The circle is then said to *circumscribe* such a figure.

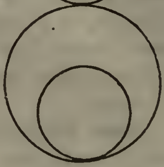
7. A *secant* is a line which meets the circumference in two points, and lies partly within and partly without the circle. AB is a secant.

8. A *tangent* is a line which has but one point in common with the circumference. CD is a tangent.



The point M, where the tangent touches the circumference, is called the *point of contact.*

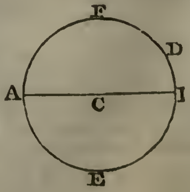
In like manner, two circumferences *touch* each other when they have but one point in common.



9. A polygon is *circumscribed about a circle*, when all its sides are tangents to the circumference: in the same case, the circle is said to be *inscribed* in the polygon.

PROPOSITION I. THEOREM.

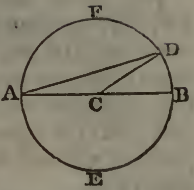
Every diameter divides the circle and its circumference into two equal parts.



Let AEDF be a circle, and AB a diameter. Now, if the figure AEB be applied to AFB, their common base AB retaining its position, the curve line AEB must fall exactly on the curve line AFB, otherwise there would, in the one or the other, be points unequally distant from the centre, which is contrary to the definition of a circle.

PROPOSITION II THEOREM.

Every chord is less than the diameter.



Let AD be any chord. Draw the radii CA, CD, to its extremities. We shall then have $AD < AC + CD$ (Book I. Prop. VII.*); or $AD < AB$.

Cor. Hence the greatest line which can be inscribed in a circle is its diameter.

PROPOSITION III. THEOREM.

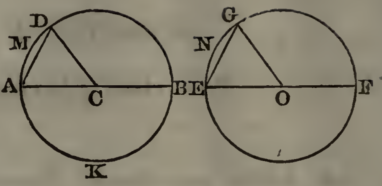
A straight line cannot meet the circumference of a circle in more than two points.

For, if it could meet it in three, those three points would be equally distant from the centre; and hence, there would be three equal straight lines drawn from the same point to the same straight line, which is impossible (Book I. Prop. XV. Cor. 2.).

PROPOSITION IV. THEOREM.

In the same circle, or in equal circles, equal arcs are subtended by equal chords; and, conversely, equal chords subtend equal arcs.

Note. When reference is made from one proposition to another, in the same Book, the number of the proposition referred to is alone given; but when the proposition is found in a different Book, the number of the Book is also given.



If the radii AC, EO, are equal, and also the arcs AMD, ENG; then the chord AD will be equal to the chord EG.

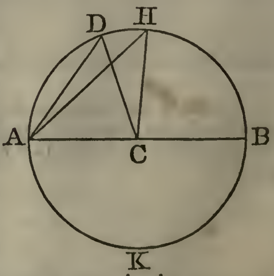
For, since the diameters AB, EF, are equal, the semicircle AMDB may be applied exactly to the semicircle ENGF, and the curve line AMDB will coincide entirely with the curve line ENGF. But the part AMD is equal to the part ENG, by hypothesis; hence, the point D will fall on G; therefore, the chord AD is equal to the chord EG.

Conversely, supposing again the radii AC, EO, to be equal, if the chord AD is equal to the chord EG, the arcs AMD, ENG will also be equal.

For, if the radii CD, OG, be drawn, the triangles ACD, EOG, will have all their sides equal, each to each, namely, AC=EO, CD=OG, and AD=EG; hence the triangles are themselves equal; and, consequently, the angle ACD is equal EOG (Book I. Prop. X.). Now, placing the semicircle ADB on its equal EGF, since the angles ACD, EOG, are equal, it is plain that the radius CD will fall on the radius OG, and the point D on the point G; therefore the arc AMD is equal to the arc ENG

PROPOSITION V. THEOREM.

In the same circle, or in equal circles, a greater arc is subtended by a greater chord, and conversely, the greater chord subtende the greater arc.



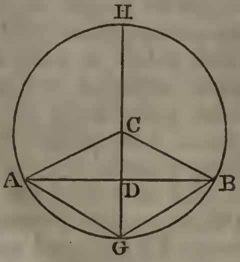
Let the arc AH be greater than the arc AD; then will the chord AH be greater than the chord AD.

For, draw the radii CD, CH. The two sides AC, CH, of the triangle ACH are equal to the two AC, CD, of the triangle ACD, and the angle ACH is greater than ACD; hence, the third side AH is greater than the third side AD (Book I. Prop. IX.); therefore the chord, which subtends the greater arc, is the greater Conversely, if the chord AH is greater than AD, it will follow on comparing the same triangles, that the angle ACH is greater than ACD (Bk. I. Prop. IX. Sch.); and hence that the arc AH is greater than AD; since the whole is greater than its part.

Scholium. The arcs here treated of are each less than the semicircumference. If they were greater, the reverse property would have place; for, as the arcs increase, the chords would diminish, and conversely. Thus, the arc AKBD is greater than AKBH, and the chord AD, of the first, is less than the chord AH of the second.

PROPOSITION VI. THEOREM.

The radius which is perpendicular to a chord, bisects the chord, and bisects also the subtended arc of the chord.



Let AB be a chord, and CG the radius perpendicular to it: then will AD=DB, and the arc AG=GB.

For, draw the radii CA, CB. Then the two right angled triangles ADC, CDB, will have AC=CB, and CD common; hence, AD is equal to DB (Book I. Prop. XVII.).

Again, since AD, DB, are equal, CG is a perpendicular erected from the middle of AB; hence every point of this perpendicular must be equally distant from its two extremities A and B (Book I. Prop. XVI.). Now, G is one of these points; therefore AG, BG, are equal. But if the chord AG is equal to the chord GB, the arc AG will be equal to the arc GB (Prop. IV.); hence, the radius CG, at right angles to the chord AB, divides the arc subtended by that chord into two equal parts at the point G.

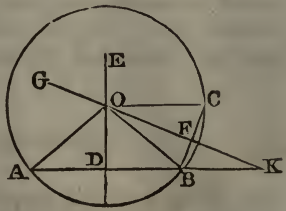
Scholium. The centre C, the middle point D, of the chord AB, and the middle point G, of the arc subtended by this chord, are three points of the same line perpendicular to the chord. But two points are sufficient to determine the position of a straight line; hence every straight line which passes through two of the points just mentioned, will necessarily pass through the third, and be perpendicular to the chord.

It follows, likewise, that *the perpendicular raised from the middle of a chord passes through the centre of the circle, and through the middle of the arc subtended by that chord.*

For, this perpendicular is the same as the one let fall from the centre on the same chord, since both of them pass through the centre and middle of the chord.

PROPOSITION VII. THEOREM.

Through three given points not in the same straight line, one circumference may always be made to pass, and but one.



Let A, B, and C, be the given points.

Draw AB, BC, and bisect these straight lines by the perpendiculars DE, FG: we say first, that DE and FG, will meet in some point O.

For, they must necessarily cut each other, if they are not parallel. Now, if they were parallel, the line AB, which is perpendicular to DE, would also be perpendicular to FG, and the angle K would be a right angle (Book I. Prop. XX. Cor. 1.). But BK, the prolongation of BD, is a different line from BF, because the three points A, B, C, are not in the same straight line; hence there would be two perpendiculars, BF, BK, let fall from the same point B, on the same straight line, which is impossible (Book I. Prop. XIV.); hence DE, FG, will always meet in some point O.

And moreover, this point O, since it lies in the perpendicular DE, is equally distant from the two points, A and B (Book I. Prop. XVI.); and since the same point O lies in the perpendicular FG, it is also equally distant from the two points B and C: hence the three distances OA, OB, OC, are equal; therefore the circumference described from the centre O, with the radius OB, will pass through the three given points A, B, C.

We have now shown that one circumference can always be made to pass through three given points, not in the same straight line: we say farther, that but one can be described through them.

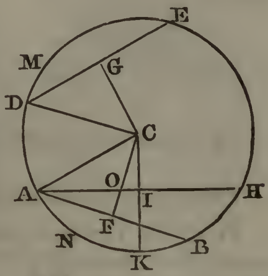
For, if there were a second circumference passing through the three given points A, B, C, its centre could not be out of the line DE, for then it would be unequally distant from A and B (Book I. Prop. XVI.); neither could it be out of the line FG, for a like reason; therefore, it would be in both the lines DE, FG. But two straight lines cannot cut each other in more than one point; hence there is but one circumference which can pass through three given points.

Cor. Two circumferences cannot meet in more than two points; for, if they have three common points, there would be two circumferences passing through the same three points; which has been shown by the proposition to be impossible.

PROPOSITION VIII. THEOREM.

Two equal chords are equally distant from the centre; and of two unequal chords, the less is at the greater distance from the centre.

First. Suppose the chord AB = DE. Bisect these chords by the perpendiculars CF, CG, and draw the radii CA, CD.



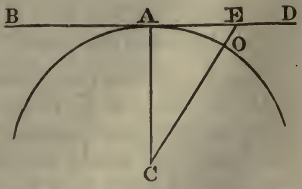
In the right angled triangles CAF, DCG, the hypothenuses CA, CD, are equal; and the side AF, the half of AB, is equal to the side DG, the half of DE: hence the triangles are equal, and CF is equal to CG (Book I. Prop. XVII.); hence, the two equal chords AB, DE, are equally distant from the centre.

Secondly Let the chord AH be greater than DE. The arc AKH will be greater than DME (Prop. V.): cut off from the former, a part ANB, equal to DME; draw the chord AB, and let fall CF perpendicular to this chord, and CI perpendicular to AH. It is evident that CF is greater than CO, and CO than CI (Book I. Prop. XV.); therefore, CF is still greater than CI. But CF is equal to CG, because the chords AB, DE, are equal: hence we have CG>CI; hence of two unequal chords, the less is the farther from the centre.

PROPOSITION IX. THEOREM.

A straight line perpendicular to a radius, at its extremity, is a tangent to the circumference.

Let BD be perpendicular to the radius CA, at its extremity A; then will it be tangent to the circumference.



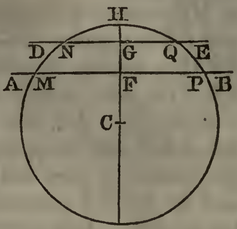
For every oblique line CE, is longer than the perpendicular CA (Book I. Prop. XV.); hence the point E is without the circle; therefore, BD has no point but A common to it and the circumference; consequently BD is a tangent (Def. 8.).

Scholium. At a given point A, only one tangent AD can be drawn to the circumference ; for, if another could be drawn, it would not be perpendicular to the radius CA (Book I. Prop. XIV. Sch.) ; hence in reference to this new tangent, the radius AC would be an oblique line, and the perpendicular let fall from the centre upon this tangent would be shorter than CA ; hence this supposed tangent would enter the circle, and be a secant.

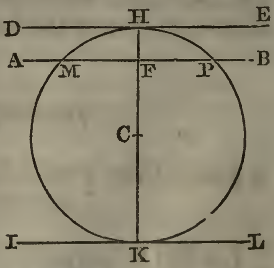
PROPOSITION X. THEOREM.

Two parallels intercept equal arcs on the circumference.

There may be three cases.



First. If the two parallels are secants, draw the radius CH perpendicular to the chord MP. It will, at the same time be perpendicular to NQ (Book I.Prop.XX.Cor.1.); therefore, the point H will be at once the middle of the arc MHP, and of the arc NHQ (Prop. VI.); therefore, we shall have the arc MH=HP, and the arc NH=HQ ; and therefore MH—NH=HP—HQ ; in other words, MN=PQ.



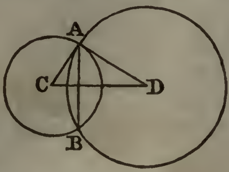
Second. When, of the two parallels AB, DE, one is a secant, the other a tangent, draw the radius CH to the point of contact H ; it will be perpendicular to the tangent DE (Prop. IX.), and also to its parallel MP. But, since CH is perpendicular to the chord MP, the point H must be the middle of the arc MHP (Prop. VI.) ; therefore the arcs MH, HP, included between the parallels AB, DE, are equal.

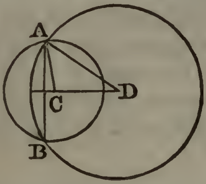
Third. If the two parallels DE, IL, are tangents, the one at H, the other at K, draw the parallel secant AB ; and, from what has just been shown, we shall have MH=HP, MK=KP; and hence the whole arc HMK=HPK. It is farther evident that each of these arcs is a semicircumference

PROPOSITION XI. THEOREM.

If two circles cut each other in two points, the line which passes through their centres, will be perpendicular to the chord which joins the points of intersection, and will divide it into tw equal parts.

For, let the line AB join the points of intersection. It will be a common chord to the two circles. Now if a perpendicular

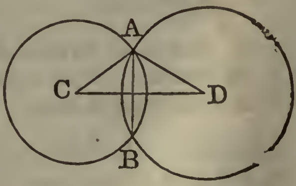




be erected from the middle of this chord, it will pass through each of the two centres C and D (Prop. VI. Sch.). But no more than one straight line can be drawn through two points; hence the straight line, which passes through the centres, will bisect the chord at right angles.

PROPOSITION XII. THEOREM.

If the distance between the centres of two circles is less than the sum of the radii, the greater radius being at the same time less than the sum of the smaller and the distance between the centres, the two circumferences will cut each other.

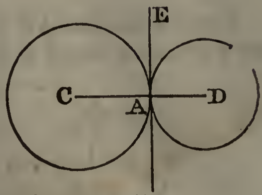


For, to make an intersection possible, the triangle CAD must be possible. Hence, not only must we have $CD<AC+AD$, but also the greater radius $AD< AC+CD$ (Book I. Prop. VII.). And, whenever the triangle CAD can be constructed, it is plain that the circles described from the centres C and D, will cut each other in A and B.

PROPOSITION XIII. THEOREM.

If the distance between the centres of two circles is equal to the sum of their radii, the two circles will touch each other externally.

Let C and D be the centres at a distance from each other equal to CA+AD.

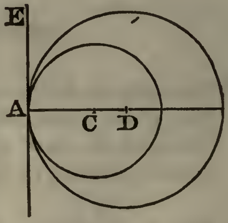


The circles will evidently have the point A common, and they will have no other; because, if they had two points common, the distance between their centres must be less than the sum of their radii.

PROPOSITION XIV. THEOREM.

If the distance between the centres of two circles is equal to the difference of their radii, the two circles will touch each other internally.

Let C and D be the centres at a distance from each other equal to AD—CA.



It is evident, as before, that they will have the point A common: they can have no other; because, if they had, the greater radius AD must be less than the sum of the radius AC and the distance CD between the centres (Prop. XII.); which is contrary to the supposition.

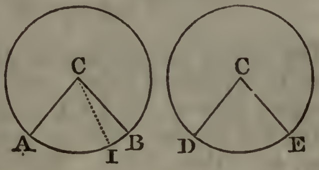
Cor. Hence, if two circles touch each other, either externally or internally, their centres and the point of contact will be in the same right line.

Scholium. All circles which have their centres on the right line AD, and which pass through the point A, are tangent to each other. For, they have only the point A common, and if through the point A, AE be drawn perpendicular to AD, the straight line AE will be a common tangent to all the circles.

PROPOSITION XV THEOREM.

In the same circle, or in equal circles, equal angles having their vertices at the centre, intercept equal arcs on the circumference: and conversely, if the arcs intercepted are equal, the angles contained by the radii will also be equal.

Let C and C be the centres of equal circles, and the angle ACB=DCE.



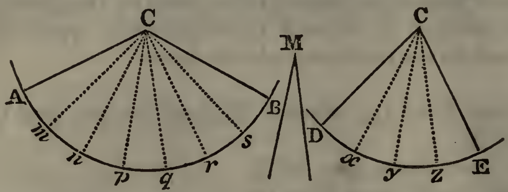
First. Since the angles ACB, DCE, are equal, they may be placed upon each other; and since their sides are equal, the point A will evidently fall on D, and the point B on E. But, in that case, the arc AB must also fall on the arc DE; for if the arcs did not exactly coincide, there would, in the one or the other, be points unequally distant from the centre; which is impossible: hence the arc AB is equal to DE.

Secondly. If we suppose AB=DE, the angle ACB will be equal to DCE. For, if these angles are not equal, suppose ACB to be the greater, and let ACI be taken equal to DCE. From what has just been shown, we shall have AI=DE: but, by hypothesis, AB is equal to DE; hence AI must be equal to AB, or a part to the whole, which is absurd (Ax. 8.): hence, the angle ACB is equal to DCE.

PROPOSITION XVI. THEOREM.

In the same circle, or in equal circles, if two angles at the centre are to each other in the proportion of two whole numbers, the intercepted arcs will be to each other in the proportion of the same numbers, and we shall have the angle to the angle, as the corresponding arc to the corresponding arc.

Suppose, for example, that the angles ACB, DCE, are to each other as 7 is to 4; or, which is the same thing, suppose that the angle M, which may serve as a common measure, is contained 7 times in the angle ACB, and 4 times in DCE

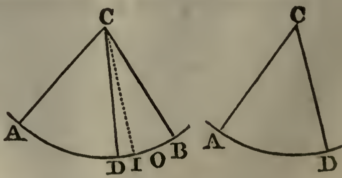


The seven partial angles A*Cm*, *m*C*n*, *n*C*p*, &c., into which ACB is divided, being each equal to any of the four partial angles into which DCE is divided; each of the partial arcs A*m*, *mn*, *np*, &c., will be equal to each of the partial arcs D*x*, *xy*, &c. (Prop. XV.). Therefore the whole arc AB will be to the whole arc DE, as 7 is to 4. But the same reasoning would evidently apply, if in place of 7 and 4 any numbers whatever were employed; hence, if the ratio of the angles ACB, DCE, can be expressed in whole numbers, the arcs AB, DE, will be to each other as the angles ACB, DCE.

Scholium. Conversely, if the arcs, AB, DE, are to each other as two whole numbers, the angles ACB, DCE will be to each other as the same whole numbers, and we shall have ACB : DCE : : AB : DE. For the partial arcs, A*m*, *mn*, &c and D*x*, *xy*, &c., being equal, the partial angles AC*m*, *m*C*n*, &c. and DC*x*, *x*C*y*, &c. will also be equal.

PROPOSITION XVII. THEOREM.

Whatever be the ratio of two angles, they will always be to each other as the arcs intercepted between their sides; the arcs being described from the vertices of the angles as centres with equal radii.



Let ACB be the greater and ACD the less angle.

Let the less angle be placed on the greater. If the proposition is not true, the angle ACB will be to the angle ACD as the arc AB is to an arc greater or less than AD. Suppose this arc to be greater, and let it be represented by AO; we shall thus have, the angle ACB : angle ACD : : arc AB : arc AO. Next conceive the arc

AB to be divided into equal parts, each of which is less than DO; there will be at least one point of division between D and O; let I be that point; and draw CI. The arcs AB, AI, will be to each other as two whole numbers, and by the preceding theorem, we shall have, the angle ACB : angle ACI : : arc AB : arc AI. Comparing these two proportions with each other, we see that the antecedents are the same: hence, the consequents are proportional (Book II. Prop. IV.); and thus we find the angle ACD : angle ACI : : arc AO : arc AI. But the arc AO is greater than the arc AI; hence, if this proportion is true, the angle ACD must be greater than the angle ACI: on the contrary, however, it is less; hence the angle ACB cannot be to the angle ACD as the arc AB is to an arc greater than AD.

By a process of reasoning entirely similar, it may be shown that the fourth term of the proportion cannot be less than AD; hence it is AD itself; therefore we have

Angle ACB : angle ACD : : arc AB : arc AD.

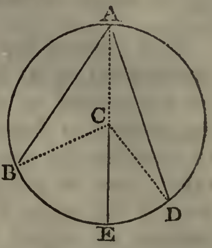
Cor. Since the angle at the centre of a circle, and the arc intercepted by its sides, have such a connexion, that if the one be augmented or diminished in any ratio, the other will be augmented or diminished in the same ratio, we are authorized to establish the one of those magnitudes as the measure of the other; and we shall henceforth assume the arc AB as the measure of the angle ACB. It is only necessary that, in the comparison of angles with each other, the arcs which serve to measure them, be described with equal radii, as is implied in all the foregoing propositions.

Scholium 1. It appears most natural to measure a quantity by a quantity of the same species; and upon this principle it would be convenient to refer all angles to the right angle; which, being made the unit of measure, an acute angle would be expressed by some number between 0 and 1; an obtuse angle by some number between 1 and 2. This mode of expressing angles would not, however, be the most convenient in practice. It has been found more simple to measure them by arcs of a circle, on account of the facility with which arcs can be made equal to given arcs, and for various other reasons. At all events, if the measurement of angles by arcs of a circle is in any degree indirect, it is still equally easy to obtain the direct and absolute measure by this method; since, on comparing the arc which serves as a measure to any angle, with the fourth part of the circumference, we find the ratio of the given angle to a right angle, which is the absolute measure.

Scholium 2. All that has been demonstrated in the last three propositions, concerning the comparison of angles with arcs, holds true equally, if applied to the comparison of sectors with arcs; for sectors are not only equal when their angles are so, but are in all respects proportional to their angles; hence, *two sectors* ACB, ACD, *taken in the same circle, or in equal circles, are to each other as the arcs* AB, AD, *the bases of those sectors.* It is hence evident that the arcs of the circle, which serve as a measure of the different angles, are proportional to the different sectors, in the same circle, or in equal circles.

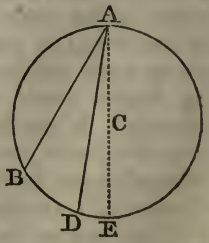
PROPOSITION XVIII. THEOREM.

An inscribed angle is measured by half the arc included between its sides.



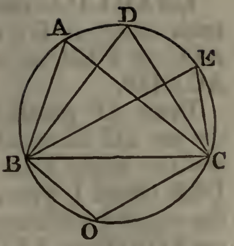
Let BAD be an inscribed angle, and let us first suppose that the centre of the circle lies within the angle BAD. Draw the diameter AE, and the radii CB, CD.

The angle BCE, being exterior to the triangle ABC, is equal to the sum of the two interior angles CAB, ABC (Book I. Prop. XXV. Cor. 6.): but the triangle BAC being isosceles, the angle CAB is equal to ABC; hence the angle BCE is double of BAC. Since BCE lies at the centre, it is measured by the arc BE; hence BAC will be measured by the half of BE. For a like reason, the angle CAD will be measured by the half of ED; hence BAC+CAD, or BAD will be measured by half of BE+ED, or of BED.

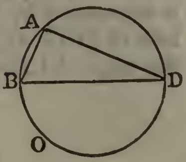


Suppose, in the second place, that the centre C lies without the angle BAD. Then drawing the diameter AE, the angle BAE will be measured by the half of BE; the angle DAE by the half of DE: hence their difference BAD will be measured by the half of BE *minus* the half of ED, or by the half of BD.

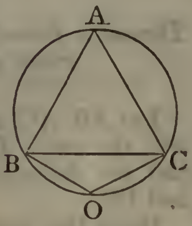
Hence every inscribed angle is measured by half of the arc included between its sides.



Cor. 1. All the angles BAC, BDC, BEC, inscribed in the same segment are equal; because they are all measured by the half of the same arc BOC.

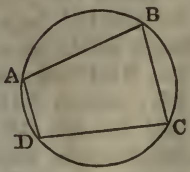


Cor. 2. Every angle BAD, inscribed in a semicircle is a right angle; because it is measured by half the semicircumference BOD, that is, by the fourth part of the whole circumference.



Cor. 3. Every angle BAC, inscribed in a segment greater than a semicircle, is an acute angle; for it is measured by half of the arc BOC, less than a semicircumference.

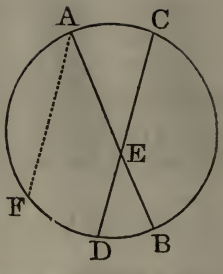
And every angle BOC, inscribed in a segment less than a semicircle, is an obtuse angle; for it is measured by half of the arc BAC, greater than a semicircumference.



Cor. 4. The opposite angles A and C, of an inscribed quadrilateral ABCD, are together equal to two right angles: for the angle BAD is measured by half the arc BCD, the angle BCD is measured by half the arc BAD; hence the two angles BAD, BCD, taken together, are measured by the half of the circumference; hence their sum is equal to two right angles.

PROPOSITION XIX. THEOREM.

The angle formed by two chords, which intersect each other, is measured by half the sum of the arcs included between its sides

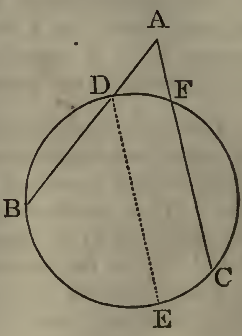


Let AB, CD, be two chords intersecting each other at E: then will the angle AEC, or DEB, be measured by half of AC+DB.

Draw AF parallel to DC: then will the arc DF be equal to AC (Prop. X.); and the angle FAB equal to the angle DEB (Book I. Prop. XX. Cor. 3.). But the angle FAB is measured by half the arc FDB (Prop. XVIII.); therefore, DEB is measured by half of FDB; that is, by half of DB+DF, or half of DB+AC. In the same manner it might be proved tha the angle AED is measured by half of AFD+BC.

PROPOSITION XX. THEOREM.

The angle formed by two secants, is measured by half the difference of the arcs included between its sides.



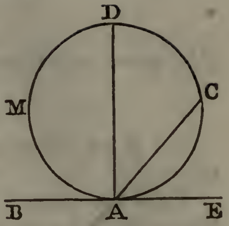
Let AB, AC, be two secants: then will the angle BAC be measured by half the difference of the arcs BEC and DF.

Draw DE parallel to AC: then will the arc EC be equal to DF, and the angle BDE equal to the angle BAC. But BDE is measured by half the arc BE; hence, BAC is also measured by half the arc BE; that is, by half the difference of BEC and EC, or half the difference of BEC and DF.

PROPOSITION XXI. THEOREM.

The angle formed by a tangent and a chord, is measured by half of the arc included between its sides.

Let BE be the tangent, and AC the chord.



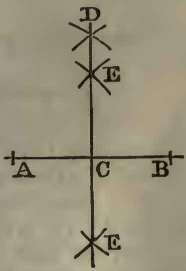
From A, the point of contact, draw the diameter AD. The angle BAD is a right angle (Prop. IX.), and is measured by half the semicircumference AMD; the angle DAC is measured by the half of DC: hence, BAD+DAC, or BAC, is measured by the half of AMD plus the half of DC, or by half the whole arc AMDC.

It might be shown, by taking the difference between the angles DAE, DAC, that the angle CAE is measured by half the arc AC, included between its sides.

PROBLEMS RELATING TO THE FIRST AND THIRD BOOKS.

PROBLEM I.

To divide a given straight line into two equal parts.



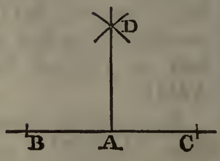
Let AB be the given straight line.

From the points A and B as centres, with a radius greater than the half of AB, describe two arcs cutting each other in D; the point D will be equally distant from A and B. Find, in like manner, above or beneath the line AB, a second point E, equally distant from the points A and B; through the two points D and E, draw the line DE: it will bisect the line AB in C.

For, the two points D and E, being each equally distant from the extremities A and B, must both lie in the perpendicular raised from the middle of AB (Book I. Prop. XVI. Cor.). But only one straight line can pass through two given points; hence the line DE must itself be that perpendicular, which divides AB into two equal parts at the point C.

PROBLEM II.

At a given point, in a given straight line, to erect a perpendicular to this line.



Let A be the given point, and BC the given line.

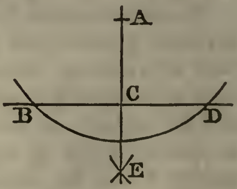
Take the points B and C at equal distances from A; then from the points B and C as centres, with a radius greater than BA, describe two arcs intersecting each other in D; draw AD: it will be the perpendicular required.

For, the point D, being equally distant from B and C, must be in the perpendicular raised from the middle of BC (Book I. Prop. XVI.); and since two points determine a line, AD is that perpendicular.

Scholium. The same construction serves for making a right angle BAD, at a given point A, on a given straight line BC.

PROBLEM III.

From a given point, without a straight line, to let fall a perpendicular on this line.



Let A be the point, and BD the straight line.

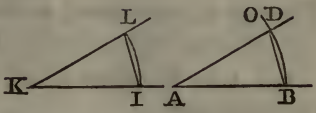
From the point A as a centre, and with a radius sufficiently great, describe an arc cutting the line BD in the two points B and D; then mark a point E, equally distant from the points B and D, and draw AE: it will be the perpendicular required.

For, the two points A and E are each equally distant from the points B and D; hence the line AE is a perpendicular passing through the middle of BD (Book I. Prop. XVI. Cor.).

PROBLEM IV.

At a point in a given line, to make an angle equal to a given angle.

Let A be the given point, AB the given line, and IKL the given angle.

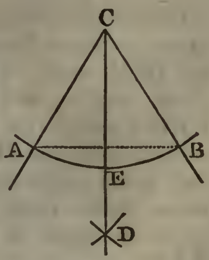


From the vertex K, as a centre, with any radius, describe the arc IL, terminating in the two sides of the angle. From the point A as a centre, with a distance AB, equal to KI, describe the indefinite arc BO; then take a radius equal to the chord LI, with which, from the point B as a centre, describe an arc cutting the indefinite arc BO, in D; draw AD; and the angle DAB will be equal to the given angle K.

For, the two arcs BD, LI, have equal radii, and equal chords; hence they are equal (Prop. IV.); therefore the angles BAD IKL, measured by them, are equal.

PROBLEM V.

To divide a given arc, or a given angle, into two equal parts.



First. Let it be required to divide the arc AEB into two equal parts. From the points A and B, as centres, with the same radius, describe two arcs cutting each other in D; through the point D and the centre C, draw CD: it will bisect the arc AB in the point E.

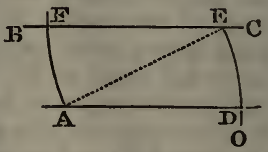
For, the two points C and D are each equally distant from the extremities A and B of the chord AB; hence the line CD bisects the chord at right angles (Book I. Prop. XVI. Cor.); hence, it bisects the arc AB in the point E (Prop. VI.).

Secondly. Let it be required to divide the angle ACB into two equal parts. We begin by describing, from the vertex C as a centre, the arc AEB; which is then bisected as above. It is plain that the line CD will divide the angle ACB into two equal parts.

Scholium. By the same construction, each of the halves AE, EB, may be divided into two equal parts; and thus, by successive subdivisions, a given angle, or a given arc may be divided into four equal parts, into eight, into sixteen, and so on.

PROBLEM VI.

Through a given point, to draw a parallel to a given straight line.



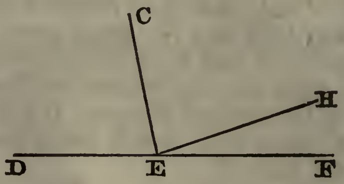
Let A be the given point, and BC the given line.

From the point A as a centre, with a radius greater than the shortest distance from A to BC, describe the indefinite arc EO; from the point E as a centre, with the same radius, describe the arc AF; make ED=AF, and draw AD: this will be the parallel required.

For, drawing AE, the alternate angles AEF, EAD, are evidently equal; therefore, the lines AD, EF, are parallel (Book I. Prop. XIX. Cor. 1.).

PROBLEM VII.

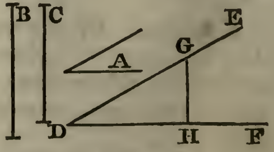
Two angles of a triangle being given, to find the third.



Draw the indefinite line DEF; at any point as E, make the angle DEC equal to one of the given angles, and the angle CEH equal to the other: the remaining angle HEF will be the third angle required; because those three angles are together equal to two right angles (Book I. Prop. I and XXV).

PROBLEM VIII.

Two sides of a triangle, and the angle which they contain, being given, to describe the triangle.

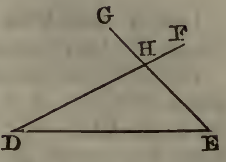


Let the lines B and C be equal to the given sides, and A the given angle.

Having drawn the indefinite line DE, at the point D, make the angle EDF equal to the given angle A; then take DG=B, DH=C, and draw GH: DGH will be the triangle required (Book I. Prop. V.).

PROBLEM IX.

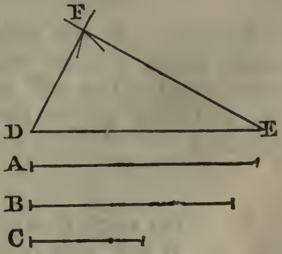
A side and two angles of a triangle being given, to describe the triangle.



The two angles will either be both adjacent to the given side, or the one adjacent, and the other opposite: in the latter case, find the third angle (Prob. VII.); and the two adjacent angles will thus be known: draw the straight line DE equal to the given side: at the point D, make an angle EDF equal to one of the adjacent angles, and at E, an angle DEG equal to the other; the two lines DF, EG, will cut each other in H; and DEH will be the triangle required (Book I. Prop. VI.).

PROBLEM X.

The three sides of a triangle being given, to describe the triangle.



Let A, B, and C, be the sides.

Draw DE equal to the side A; from the point E as a centre, with a radius equal to the second side B, describe an arc; from D as a centre, with a radius equal to the third side C, describe another arc intersecting the former in F; draw DF, EF; and DEF will be the triangle required (Book I. Prop. X.).

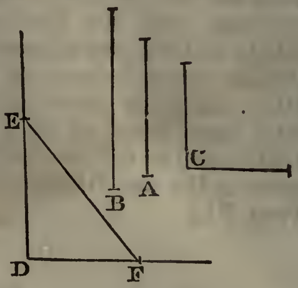
Scholium. If one of the sides were greater than the sum of the other two, the arcs would not intersect each other: but the solution will always be possible, when the sum of two sides, any how taken, is greater than the third.

PROBLEM XI.

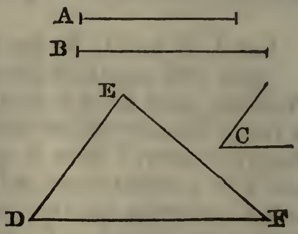
Two sides of a triangle, and the angle opposite one of them, being given, to describe the triangle.

Let A and B be the given sides, and C the given angle. There are two cases.

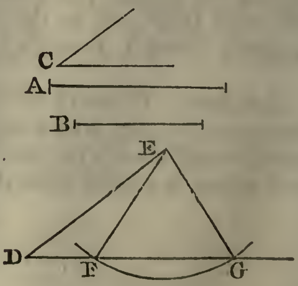
First. When the angle C is a right angle, or when it is obtuse, make the angle EDF=C; take DE=A; from the point E as a centre, with a radius equal to the given side B, describe an arc cutting DF in F; draw EF: then DEF will be the triangle required.



In this first case, the side B must be greater than A; for the angle C, being a right angle, or an obtuse angle, is the greatest angle of the triangle, and the side opposite to it must, therefore, also be the greatest (Book I. Prop. XIII.).



Secondly. If the angle C is acute, and B greater than A, the same construction will again apply, and DEF will be the triangle required.



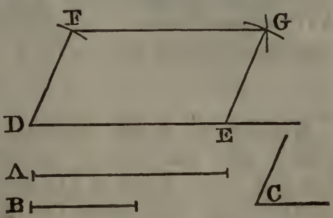
But if the angle C is acute, and the side B less than A, then the arc described from the centre E, with the radius EF=B, will cut the side DF in two points F and G, lying on the same side of D: hence there will be two triangles DEF, DEG, either of which will satisfy the conditions of the problem.

Scholium. If the arc described with E as a centre, should be tangent to the line DG, the triangle would be right angled, and there would be but one solution. The problem would be impossible in all cases, if the side B were less than the perpendicular let fall from E on the line DF.

PROBLEM XII.

The adjacent sides of a parallelogram, with the angle which they contain, being given, to describe the parallelogram

Let A and B be the given sides, and C the given angle.



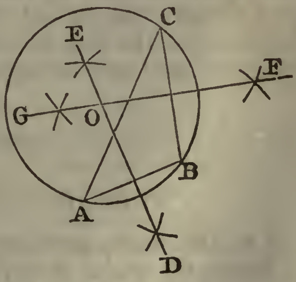
Draw the line DE=A; at the point D, make the angle EDF=C; take DF=B; describe two arcs, the one from F as a centre, with a radius FG=DE, the other from E as a centre, with a radius EG=DF; to the point G, where these arcs intersect each other, draw FG, EG; DEGF will be the parallelogram required.

For, the opposite sides are equal, by construction; hence the figure is a parallelogram (Book I. Prop. XXIX.): and it is formed with the given sides and the given angle.

Cor. If the given angle is a right angle, the figure will be a rectangle; if, in addition to this, the sides are equal, it will be a square.

PROBLEM XIII.

To find the centre of a given circle or arc.

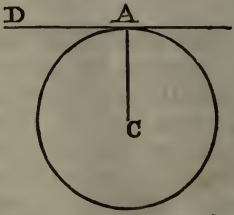


Take three points, A, B, C, any where in the circumference, or the arc; draw AB, BC, or suppose them to be drawn; bisect those two lines by the perpendiculars DE, FG: the point O, where these perpendiculars meet, will be the centre sought (Prop. VI. Sch.).

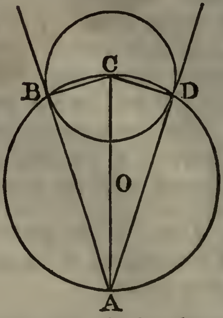
Scholium. The same construction serves for making a circumference pass through three given points A, B, C; and also for describing a circumference, in which, a given triangle ABC shall be inscribed.

PROBLEM XIV.

Through a given point, to draw a tangent to a given circle.



If the given point A lies in the circumference, draw the radius CA, and erect AD perpendicular to it: AD will be the tangent required (Prop. IX.).



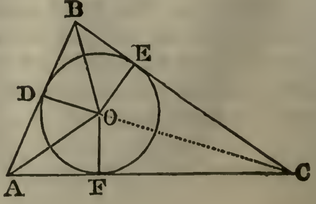
If the point A lies without the circle, join A and the centre, by the straight line CA: bisect CA in O; from O as a centre, with the radius OC, describe a circumference intersecting the given circumference in B; draw AB: this will be the tangent required.

For, drawing CB, the angle CBA being inscribed in a semicircle is a right angle (Prop. XVIII. Cor. 2.); therefore AB is a perpendicular at the extremity of the radius CB; therefore it is a tangent.

Scholium. When the point A lies without the circle, there will evidently be always two equal tangents AB, AD, passing through the point A: they are equal, because the right angled triangles CBA, CDA, have the hypothenuse CA common, and the side CB=CD; hence they are equal (Book I. Prop. XVII.); hence AD is equal to AB, and also the angle CAD to CAB. And as there can be but one line bisecting the angle BAC, it follows, that the line which bisects the angle formed by two tangents, must pass through the centre of the circle.

PROBLEM XV.

To inscribe a circle in a given triangle.



Let ABC be the given triangle.

Bisect the angles A and B, by the lines AO and BO, meeting in the point O; from the point O, let fall the perpendiculars OD, OE, OF, on the three sides of the triangle: these perpendiculars will all be equal. For, by construction, we have the angle DAO=OAF, the right angle ADO=AFO; hence the third angle AOD is equal to the third AOF (Book I. Prop. XXV. Cor. 2.). Moreover, the side AO is common to the two triangles AOD, AOF; and the angles adjacent to the equal side are equal: hence the triangles themselves are equal (Book I. Prop. VI.); and DO is equal to OF. In the same manner it may be shown that the two triangles BOD, BOE, are equal; therefore OD is equal to OE; therefore the three perpendiculars OD, OE, OF, are all equal.

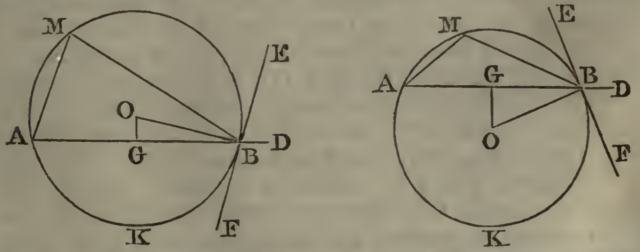
Now, if from the point O as a centre, with the radius OD, a circle be described, this circle will evidently be inscribed in the triangle ABC; for the side AB, being perpendicular to the radius at its extremity, is a tangent; and the same thing is true of the sides BC, AC.

Scholium. The three lines which bisect the angles of a triangle meet in the same point.

PROBLEM XVI.

On a given straight line to describe a segment that shall contain a given angle; that is to say, a segment such, that all the angles inscribed in it, shall be equal to the given angle.

Let AB be the given straight line, and C the given angle.



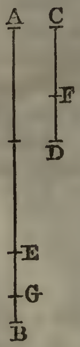
Produce AB towards D; at the point B, make the angle DBE=C; draw BO perpendicular to BE, and GO perpendicular to AB, through the middle point G; and from the point O, where these perpendiculars meet, as a centre, with a distance OB, describe a circle: the required segment will be AMB.

For, since BF is a perpendicular at the extremity of the radius OB, it is a tangent, and the angle ABF is measured by half the arc AKB (Prop. XXI.). Also, the angle AMB, being an inscribed angle, is measured by half the arc AKB: hence we have AMB=ABF=EBD=C: hence all the angles inscribed in the segment AMB are equal to the given angle C.

Scholium. If the given angle were a right angle, the required segment would be a semicircle, described on AB as a diameter.

PROBLEM XVII.

To find the numerical ratio of two given straight lines, these lines being supposed to have a common measure.



Let AB and CD be the given lines.

From the greater AB cut off a part equal to the less CD, as many times as possible; for example, twice, with the remainder BE.

From the line CD, cut off a part equal to the remainder BE, as many times as possible; once, for example, with the remainder DF.

From the first remainder BE, cut off a part equal to the second DF, as many times as possible; once, for example, with the remainder BG.

From the second remainder DF, cut off a part equal to BG the third, as many times as possible.

Continue this process, till a remainder occurs, which is contained exactly a certain number of times in the preceding one.

Then this last remainder will be the common measure of the proposed lines; and regarding it as unity, we shall easily find the values of the preceding remainders; and at last, those of the two proposed lines, and hence their ratio in numbers.

Suppose, for instance, we find GB to be contained exactly twice in FD; BG will be the common measure of the two proposed lines. Put $BG=1$; we shall have $FD=2$: but EB contains FD once, *plus* GB; therefore we have $EB=3$: CD contains EB once, *plus* FD; therefore we have $CD=5$: and lastly, AB contains CD twice, *plus* EB; therefore we have $AB=13$; hence the ratio of the lines is that of 13 to 5. If the line CD were taken for unity, the line AB would be $\frac{13}{5}$; if AB were taken for unity, CD would be $\frac{5}{13}$.

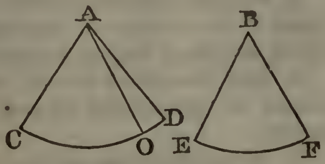
Scholium. The method just explained is the same as that employed in arithmetic to find the common divisor of two numbers: it has no need, therefore, of any other demonstration.

How far soever the operation be continued, it is possible that no remainder may ever be found, which shall be contained an exact number of times in the preceding one. When this happens, the two lines have no common measure, and are said to be *incommensurable.* An instance of this will be seen afterwards, in the ratio of the diagonal to the side of the square. In those cases, therefore, the exact ratio in numbers cannot be found; but, by neglecting the last remainder, an approximate ratio will be obtained, more or less correct, according as the operation has been continued a greater or less number of times.

PROBLEM XVIII.

Two angles being given, to find their common measure, if they have one, and by means of it, their ratio in numbers.

Let A and B be the given angles.



With equal radii describe the arcs CD, EF, to serve as measures for the angles: proceed afterwards in the comparison of the arcs CD, EF, as in the last problem, since an arc may be cut off from an arc of the same radius, as a straight line from a straight line. We shall thus arrive at the common measure of the arcs CD, EF, if they have one, and thereby at their ratio in numbers. This ratio will be the same as that of the given angles (Prop. XVII.); and if DO is the common measure of the arcs, DAO will be that of the angles.

Scholium. According to this method, the absolute value of an angle may be found by comparing the arc which measures it to the whole circumference. If the arc CD, for example, is to the circumference, as 3 is to 25, the angle A will be $\frac{3}{25}$ of four right angles, or $\frac{12}{25}$ of one right angle.

It may also happen, that the arcs compared have no common measure; in which case, the numerical ratios of the angles will only be found approximatively with more or less correctness, according as the operation has been continued a greater or less number of times.

BOOK IV.

OF THE PROPORTIONS OF FIGURES, AND THE MEASUREMENT OF AREAS.

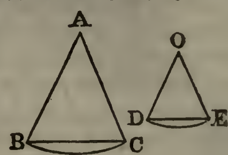
Definitions.

1. Similar figures are those which have the angles of the one equal to the angles of the other, each to each, and the sides about the equal angles proportional.

2. Any two sides, or any two angles, which have like positions in two similar figures, are called *homologous* sides or angles.

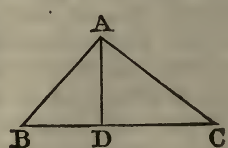
3. In two different circles, *similar arcs*, *sectors*, or *segments*, are those which correspond to equal angles at the centre.

Thus, if the angles A and O are equal, the arc BC will be similar to DE, the sector BAC to the sector DOE, and the segment whose chord is BC, to the segment whose chord is DE.

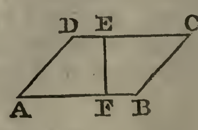


4. The *base* of any rectilineal figure, is the side on which the figure is supposed to stand.

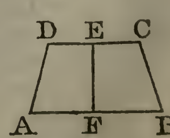
5. The *altitude* of a triangle is the perpendicular let fall from the vertex of an angle on the opposite side, taken as a base. Thus, AD is the altitude of the triangle BAC



6. The *altitude* of a parallelogram is the perpendicular which measures the distance between two opposite sides taken as bases. Thus, EF is the altitude of the parallelogram DB.



7. The altitude of a trapezoid is the perpendicular drawn between its two parallel sides. Thus, EF is the altitude of the trapezoid DB.



8. The *area* and *surface* of a figure, are terms very nearly synonymous. The *area* designates more particularly the superficial content of the figure. The area is expressed numeri-

cally by the number of times which the figure contains some other area, that is assumed for its measuring unit.

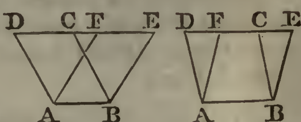
9. Figures have *equal* areas, when they contain the same measuring unit an equal number of times.

10. Figures which have equal areas are called *equivalent.* The term *equal,* when applied to figures, designates those which are equal in every respect, and which being applied to each other will coincide in all their parts (Ax. 13.): the term *equivalent* implies an equality in one respect only: namely, an equality between the measures of figures.

We may here premise, that several of the demonstrations are grounded on some of the simpler operations of algebra, which are themselves dependent on admitted axioms. Thus, if we have $A=B+C$, and if each member is multiplied by the same quantity M, we may infer that $A\times M=B\times M+C\times M$; in like manner, if we have, $A=B+C$, and $D=E-C$, and if the equal quantities are added together, then expunging the $+C$ and $-C$, which destroy each other, we infer that $A+D=B+E$, and so of others. All this is evident enough of itself; but in cases of difficulty, it will be useful to consult some agebraical treatise, and thus to combine the study of the two sciences.

PROPOSITION I. THEOREM.

Parallelograms which have equal bases and equal altitudes, are equivalent.



Let AB be the common base of the two parallelograms ABCD, ABEF: and since they are supposed to have the same altitude, their upper bases DC, FE, will be both situated in one straight line parallel to AB.

Now, from the nature of parallelograms, we have $AD=BC$, and $AF=BE$; for the same reason, we have $DC=AB$, and $FE=AB$; hence $DC=FE$: hence, if DC and FE be taken away from the same line DE, the remainders CE and DF will be equal: hence it follows that the triangles DAF, CBE, are mutually eqilateral, and consequently equal (Book I. Prop. X.).

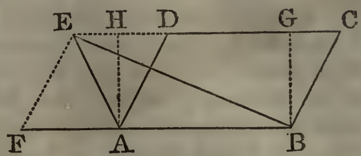
But if from the quadrilateral ABED, we take away the triangle ADF, there will remain the parallelogram ABEF; and if from the same quadrilateral ABED, we take away the equal triangle CBE, there will remain the parallelogram ABCD

Hence these two parallelograms ABCD, ABEF, which have the same base and altitude, are equivalent.

Cor. Every parallelogram is equivalent to the rectangle which has the same base and the same altitude.

PROPOSITION II. THEOREM.

Every triangle is half the parallelogram which has the same base and the same altitude.



Let ABCD be a parallelogram, and ABE a triangle, having the same base AB, and the same altitude: then will the triangle be half the parallelogram.

For, since the triangle and the parallelogram have the same altitude, the vertex E of the triangle, will be in the line EC, parallel to the base AB. Produce BA, and from E draw EF parallel to AD. The triangle FBE is half the parallelogram FC, and the triangle FAE half the parallelogram FD (Book I. Prop. XXVIII. Cor.).

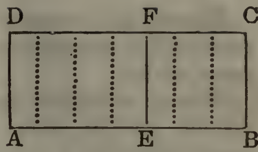
Now, if from the parallelogram FC, there be taken the parallelogram FD, there will remain the parallelogram AC: and if from the triangle FBE, which is half the first parallelogram, there be taken the triangle FAE, half the second, there will remain the triangle ABE, equal to half the parallelogram AC.

Cor 1. Hence a triangle ABE is half of the rectangle ABGH, which has the same base AB, and the same altitude AH: for the rectangle ABGH is equivalent to the parallelogram ABCD (Prop. I. Cor.).

Cor. 2. All triangles, which have equal bases and altitudes, are equivalent, being halves of equivalent parallelograms.

PROPOSITION III. THEOREM.

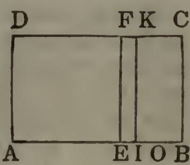
Two rectangles having the same altitude, are to each other as their bases.



Let ABCD, AEFD, be two rectangles having the common altitude AD: they are to each other as their bases AB, AE.

Suppose, first, that the bases are commensurable, and are to each other, for example, as the numbers 7 and 4. If AB be divided into 7 equal parts, AE will contain 4 of those parts: at each point of division erect a perpendicular to the base; seven partial rectangles will thus be formed, all equal to each other, because all have the same base and altitude. The rectangle ABCD will contain seven partial rectangles, while AEFD will contain four: hence the rectangle ABCD is to AEFD as 7 is to 4, or as AB is to AE. The same reasoning may be applied to any other ratio equally with that of 7 to 4: hence, whatever be that ratio, if its terms be commensurable, we shall have

ABCD : AEFD : : AB : AE.



Suppose, in the second place, that the bases AB, AE, are incommensurable: it is to be shown that we shall still have

ABCD : AEFD : : AB : AE.

For if not, the first three terms continuing the same, the fourth must be greater or less than AE. Suppose it to be greater, and that we have

ABCD : AEFD : : AB : AO.

Divide the line AB into equal parts, each less than EO. There will be at least one point I of division between E and O: from this point draw IK perpendicular to AI: the bases AB, AI, will be commensurable, and thus, from what is proved above, we shall have

ABCD : AIKD : : AB : AI.

But by the hypothesis we have

ABCD : AEFD : : AB : AO.

In these two proportions the antecedents are equal; hence the consequents are proportional (Book II. Prop. IV.); and we find

AIKD : AEFD : : AI : AO

But AO is greater than AI; hence, if this proportion is correct, the rectangle AEFD must be greater than AIKD: on the contrary, however, it is less; hence the proportion is impossible; therefore ABCD cannot be to AEFD, as AB is to a line greater than AE

Exactly in the same manner, it may be shown that the fourth term of the proportion cannot be less than AE; therefore it is equal to AE.

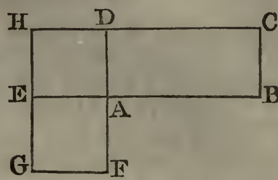
Hence, whatever be the ratio of the bases, two rectangles ABCD, AEFD, of the same altitude, are to each other as their bases AB, AE.

PROPOSITION IV. THEOREM.

Any two rectangles are to each other as the products of their bases multiplied by their altitudes.

Let ABCD, AEGF, be two rectangles; then will the rectangle,

$$ABCD : AEGF :: AB.AD : AF.AE.$$



Having placed the two rectangles, so that the angles at A are vertical, produce the sides GE, CD, till they meet in H. The two rectangles ABCD, AEHD, having the same altitude AD, are to each other as their bases AB, AE: in like manner the two rectangles AEHD, AEGF, having the same altitude AE, are to each other as their bases AD, AF: thus we have the two proportions,

$$ABCD : AEHD :: AB : AE,$$
$$AEHD : AEGF :: AD : AF.$$

Multiplying the corresponding terms of these proportions together, and observing that the term AEHD may be omitted, since it is a multiplier of both the antecedent and the consequent, we shall have

$$ABCD : AEGF :: AB \times AD : AE \times AF.$$

Scholium. Hence the product of the base by the altitude may be assumed as the measure of a rectangle, provided we understand by this product, the product of two numbers, one of which is the number of linear units contained in the base, the other the number of linear units contained in the altitude. This product will give the number of superficial units in the surface; because, for one unit in height, there are as many superficial units as there are linear units in the base; for two units in height twice as many; for three units in height, three times as many, &c.

Still this measure is not absolute, but relative: it supposes that the area of any other rectangle is computed in a similar manner, by measuring its sides with the same linear unit; a second product is thus obtained, and the ratio of the two products is the same as that of the rectangles, agreeably to the proposition just demonstrated.

For example, if the base of the rectangle A contains three units, and its altitude ten, that rectangle will be represented by the number 3×10, or 30, a number which signifies nothing while thus isolated; but if there is a second rectangle B, the base of which contains twelve units, and the altitude seven, this second rectangle will be represented by the number $12 \times 7 = 84$; and we shall hence be entitled to conclude that the two rectangles are to each other as 30 is to 84; and therefore, if the rectangle A were to be assumed as the unit of measurement in surfaces, the rectangle B would then have $\frac{84}{30}$ for its absolute measure, in other words, it would be equal to $\frac{84}{30}$ of a superficial unit.

A

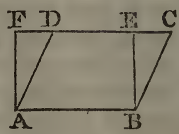
It is more common and more simple, to assume the square as the unit of surface; and to select that square, whose side is the unit of length. In this case the measurement which we have regarded merely as relative, becomes absolute: the number 30, for instance, by which the rectangle A was measured, now represents 30 superficial units, or 30 of those squares, which have each of their sides equal to unity, as the diagram exhibits.

In geometry the product of two lines frequently means the same thing as their *rectangle*, and this expression has passed into arithmetic, where it serves to designate the product of two unequal numbers, the expression *square* being employed to designate the product of a number multiplied by itself.

The arithmetical squares of 1, 2, 3, &c. are 1, 4, 9, &c. So likewise, the geometrical square constructed on a double line is evidently four times greater than the square on a single one; on a triple line it is nine times greater, &c.

PROPOSITION V. THEOREM.

The area of any parallelogram is equal to the product of its base by its altitude.



For, the parallelogram ABCD is equivalent to the rectangle ABEF, which has the same base AB, and the same altitude BE (Prop. I. Cor.): but this rectangle is measured by $AB \times BE$ (Prop. IV. Sch.); therefore, $AB \times BE$ is equal to the area of the parallelogram ABCD.

Cor. Parallelograms of the same base are to each other as their altitudes; and parallelograms of the same altitude are to each other as their bases: for, let B be the common base, and C and D the altitudes of two parallelograms:

then, $B \times C : B \times D :: C : D$, (Book II. Prop. VII.)

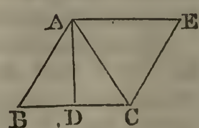
And if A and B be the bases, and C the common altitude, we shall have

$$A \times C : B \times C :: A : B.$$

And parallelograms, generally, are to each other as the products of their bases and altitudes.

PROPOSITION VI. THEOREM.

The area of a triangle is equal to the product of its base by half its altitude.

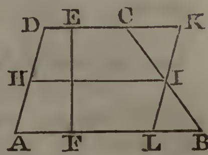


For, the triangle ABC is half of the parallelogram ABCE, which has the same base BC, and the same altitude AD (Prop. II.); but the area of the parallelogram is equal to $BC \times AD$ (Prop. V.); hence that of the triangle must be $\frac{1}{2}BC \times AD$, or $BC \times \frac{1}{2}AD$.

Cor. Two triangles of the same altitude are to each other as their bases, and two triangles of the same base are to each other as their altitudes. And triangles generally, are to each other, as the products of their bases and altitudes.

PROPOSITION VII. THEOREM.

The area of a trapezoid is equal to its altitude multiplied by the half sum of its parallel bases.



Let ABCD be a trapezoid, EF its altitude, AB and CD its parallel bases; then will its area be equal to $EF \times \frac{1}{2}(AB+CD)$.

Through I, the middle point of the side BC, draw KL parallel to the opposite side AD; and produce DC till it meets KL.

In the triangles IBL, ICK, we have the side $IB=IC$, by construction; the angle $LIB=CIK$; and since CK and BL are parallel, the angle $IBL=ICK$ (Book I. Prop. XX. Cor. 2.); hence the triangles are equal (Book I. Prop. VI.); therefore, the trapezoid ABCD is equivalent to the parallelogram ADKL, and is measured by $EF \times AL$.

But we have $AL=DK$; and since the triangles IBL and KCI are equal, the side $BL=CK$: hence, $AB+CD=AL+DK=2AL$; hence AL is the half sum of the bases AB, CD; hence the area of the trapezoid ABCD, is equal to the altitude EF multiplied by the half sum of the bases AB, CD, a result which is expressed thus: $ABCD=EF \times \dfrac{AB+CD}{2}$.

Scholium. If through I, the middle point of BC, the line IH be drawn parallel to the base AB, the point H will also be the middle of AD. For, since the figure AHIL is a parallelogram, as also DHIK, their opposite sides being parallel, we have $AH=IL$, and $DH=IK$; but since the triangles BIL, CIK, are equal, we already have $IL=IK$; therefore, $AH=DH$.

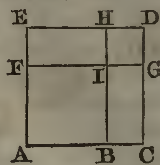
It may be observed, that the line $HI=AL$ is equal to $\dfrac{AB+CD}{2}$; hence the area of the trapezoid may also be expressed by $EF \times HI$: it is therefore equal to the altitude of the trapezoid multiplied by the line which connects the middle points of its inclined sides.

PROPOSITION VIII. THEOREM.

If a line is divided into two parts, the square described on the whole line is equivalent to the sum of the squares described on the parts, together with twice the rectangle contained by the parts.

Let AC be the line, and B the point of division; then, is

$$AC^2 \text{ or } (AB+BC)^2=AB^2+BC^2+2AB\times BC.$$



Construct the square ACDE; take AF=AB; draw FG parallel to AC, and BH parallel to AE.

The square ACDE is made up of four parts; the first ABIF is the square described on AB, since we made AF=AB: the second IDGH is the square described on IG, or BC; for since we have AC=AE and AB=AF, the difference, AC—AB must be equal to the difference AE—AF, which gives BC=EF; but IG is equal to BC, and DG to EF, since the lines are parallel; therefore IGDH is equal to a square described on BC. And those two squares being taken away from the whole square, there remains the two rectangles BCGI, EFIH, each of which is measured by AB×BC: hence the large square is equivalent to the two small squares, together with the two rectangles.

Cor. If the line AC were divided into two equal parts, the two rectangles EI, IC, would become squares, and the square described on the whole line would be equivalent to four times the square described on half the line.

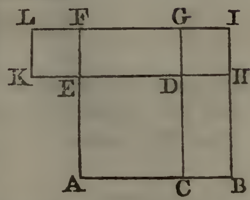
Scholium. This property is equivalent to the property demonstrated in algebra, in obtaining the square of a binominal; which is expressed thus:

$$(a+b)^2=a^2+2ab+b^2.$$

PROPOSITION IX. THEOREM.

The square described on the difference of two lines, is equivalent to the sum of the squares described on the lines, minus twice the rectangle contained by the lines.

Let AB and BC be two lines, AC their difference; then is AC^2, or $(AB—BC)^2=AB^2+BC^2—2AB\times BC$.



Describe the square ABIF; take AE $=$AC; draw CG parallel to to BI, HK parallel to AB, and complete the square EFLK.

The two rectangles CBIG, GLKD, are each measured by $AB\times BC$; take them away from the whole figure ABILKEA, which is equivalent to AB^2+BC^2, and there will evidently remain the square ACDE; hence the theorem is true.

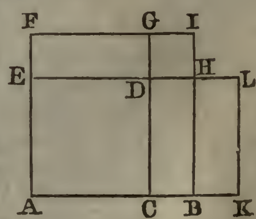
Scholium. This proposition is equivalent to the algebraical formula, $(a—b)^2=a^2—2ab+b^2$.

PROPOSITION X. THEOREM.

The rectangle contained by the sum and the difference of two lines, is equivalent to the difference of the squares of those lines.

Let AB, BC, be two lines; then, will

$$(AB+BC)\times(AB—BC)=AB^2—BC^2.$$



On AB and AC, describe the squares ABIF, ACDE; produce AB till the produced part BK is equal to BC; and complete the rectangle AKLE.

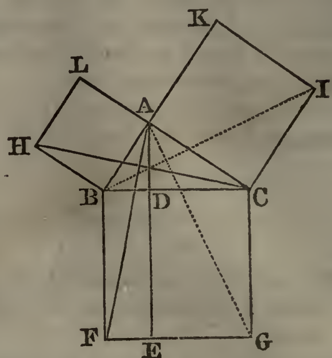
The base AK of the rectangle EK, is the sum of the two lines AB, BC; its altitude AE is the difference of the same lines; therefore the rectangle AKLE is equal to $(AB+BC)\times(AB—BC)$. But this rectangle is composed of the two parts ABHE $+$BHLK; and the part BHLK is equal to the rectangle EDGF, because BH is equal to DE, and BK to EF; hence AKLE is equal to ABHE$+$EDGF. These two parts make up the square ABIF *minus* the square DHIG, which latter is equal to a square described on BC: hence we have

$$(AB+BC)\times(AB—BC)=AB^2—BC^2.$$

Scholium. This proposition is equivalent to the algebraical formula, $(a+b)\times(a—b)=a^2—b^2$.

PROPOSITION XI. THEOREM.

The square described on the hypothenuse of a right angled triangle is equivalent to the sum of the squares described on the other two sides.



Let the triangle ABC be right angled at A. Having described squares on the three sides, let fall from A, on the hypothenuse, the perpendicular AD, which produce to E; and draw the diagonals AF, CH.

The angle ABF is made up of the angle ABC, together with the right angle CBF; the angle CBH is made up of the same angle ABC, together with the right angle ABH; hence the angle ABF is equal to HBC. But we have AB=BH, being sides of the same square; and BF=BC, for the same reason: therefore the triangles ABF, HBC, have two sides and the included angle in each equal; therefore they are themselves equal (Book I. Prop. V.).

The triangle ABF is half of the rectangle BE, because they have the same base BF, and the same altitude BD (Prop. II. Cor. 1.). The triangle HBC is in like manner half of the square AH: for the angles BAC, BAL, being both right angles, AC and AL form one and the same straight line parallel to HB (Book I. Prop. III.); and consequently the triangle HBC, and the square AH, which have the common base BH, have also the common altitude AB; hence the triangle is half of the square.

The triangle ABF has already been proved equal to the triangle HBC; hence the rectangle BDEF, which is double of the triangle ABF, must be equivalent to the square AH, which is double of the triangle HBC. In the same manner it may be proved, that the rectangle CDEG is equivalent to the square AI. But the two rectangles BDEF, CDEG, taken together, make up the square BCGF: therefore the square BCGF, described on the hypothenuse, is equivalent to the sum of the squares ABHL, ACIK, described on the two other sides; in other words, $BC^2=AB^2+AC^2$.

Cor. 1. Hence the square of one of the sides of a right angled triangle is equivalent to the square of the hypothenuse diminished by the square of the other side; which is thus expressed: $AB^2=BC^2-AC^2$.

Cor. 2. It has just been shown that the square AH is equivalent to the rectangle BDEF; but by reason of the common altitude BF, the square BCGF is to the rectangle BDEF as the base BC is to the base BD; therefore we have

$$BC^2 : AB^2 :: BC : BD.$$

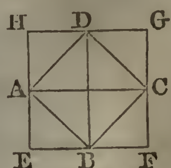
Hence *the square of the hypothenuse is to the square of one of the sides about the right angle, as the hypothenuse is to the segment adjacent to that side.* The word *segment* here denotes that part of the hypothenuse, which is cut off by the perpendicular let fall from the right angle: thus BD is the segment adjacent to the side AB; and DC is the segment adjacent to the side AC. We might have, in like manner,

$$BC^2 : AC^2 :: BC : CD.$$

Cor. 3. The rectangles BDEF, DCGE, having likewise the same altitude, are to each other as their bases BD, CD. But these rectangles are equivalent to the squares AH, AI; therefore we have $AB^2 : AC^2 :: BD : DC$.

Hence *the squares of the two sides containing the right angle, are to each other as the segments of the hypothenuse which lie adjacent to those sides.*

Cor. 4. Let ABCD be a square, and AC its diagonal: the triangle ABC being right angled and isosceles, we shall have $AC^2=AB^2+BC^2=2AB^2$: hence *the square described on the diagonal* AC, *is double of the square described on the side* AB.



This property may be exhibited more plainly, by drawing parallels to BD, through the points A and C, and parallels to AC, through the points B and D. A new square EFGH will thus be formed, equal to the square of AC. Now EFGH evidently contains eight triangles each equal to ABE; and ABCD contains four such triangles: hence EFGH is double of ABCD.

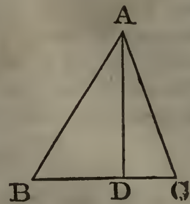
Since we have $AC^2 : AB^2 :: 2 : 1$; by extracting the square roots, we shall have $AC : AB :: \sqrt{2} : 1$; hence, *the diagonal of a square is incommensurable with its side;* a property which will be explained more fully in another place.

PROPOSITION XII. THEOREM.

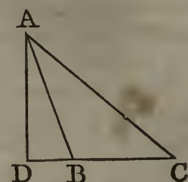
In every triangle, the square of a side opposite an acute angle is less than the sum of the squares of the other two sides, by twice the rectangle contained by the base and the distance from the acute angle to the foot of the perpendicular let fall from the opposite angle on the base, or on the base produced.

Let ABC be a triangle, and AD perpendicular to the base CB; then will $AB^2=AC^2+BC^2-2BC\times CD$.

There are two cases.



First. When the perpendicular falls within the triangle ABC, we have $BD=BC-CD$, and consequently $BD^2=BC^2+CD^2-2BC\times CD$ (Prop. IX.). Adding AD^2 to each, and observing that the right angled triangles ABD, ADC, give $AD^2+BD^2=AB^2$, and $AD^2+CD^2=AC^2$, we have $AB^2=BC^2+AC^2-2BC\times CD$.

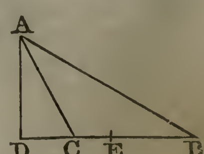


Secondly. When the perpendicular AD falls without the triangle ABC, we have $BD=CD-BC$; and consequently $BD^2=CD^2+BC^2-2CD\times BC$ (Prop. IX.). Adding AD^2 to both, we find, as before, $AB^2=BC^2+AC^2-2BC\times CD$.

PROPOSITION XIII. THEOREM.

In every obtuse angled triangle, the square of the side opposite the obtuse angle is greater than the sum of the squares of the other two sides by twice the rectangle contained by the base and the distance from the obtuse angle to the foot of the perpendicular let fall from the opposite angle on the base produced.

Let ACB be a triangle, C the obtuse angle, and AD perpendicular to BC produced; then will $AB^2=AC^2+BC^2+2BC\times CD$.



The perpendicular cannot fall within the triangle; for, if it fell at any point such as E, there would be in the triangle ACE, the right angle E, and the obtuse angle C, which is impossible (Book I. Prop. XXV. Cor. 3.):

hence the perpendicular falls without; and we have $BD=BC+CD$. From this there results $BD^2=BC^2+CD^2+2BC\times CD$ (Prop. VIII.). Adding AD^2 to both, and reducing the sums as in the last theorem, we find $AB^2=BC^2+AC^2+2BC\times CD$.

Scholium. The right angled triangle is the only one in which the squares described on the two sides are together equivalent to the square described on the third; for if the angle contained by the two sides is acute, the sum of their squares will be greater than the square of the opposite side; if obtuse, it will be less.

PROPOSITION XIV. THEOREM.

In any triangle, if a straight line be drawn from the vertex to the middle of the base, twice the square of this line, together with twice the square of half the base, is equivalent to the sum of the squares of the other two sides of the triangle.

Let ABC be any triangle, and AE a line drawn to the middle of the base BC; then will

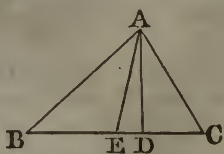
$$2AE^2+2BE^2=AB^2+AC^2.$$

On BC, let fall the perpendicular AD. Then, by Prop. XII.

$$AC^2=AE^2+EC^2-2EC\times ED.$$

And by Prop. XIII.

$$AB^2=AE^2+EB^2+2EB\times ED.$$

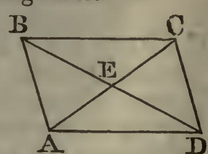


Hence, by adding, and observing that EB and EC are equal, we have

$$AB^2+AC^2=2AE^2+2EB^2.$$

Cor. Hence, *in every parallelogram the squares of the sides are together equivalent to the squares of the diagonals.*

For the diagonals AC, BD, bisect each other (Book I. Prop. XXXI.); consequently the triangle ABC gives



$$AB^2+BC^2=2AE^2+2BE^2.$$

The triangle ADC gives, in like manner.

$$AD^2+DC^2=2AE^2+2DE^2.$$

Adding the corresponding members together, and observing that BE and DE are equal, we shall have

$$AB^2+AD^2+DC^2+BC^2=4AE^2+4DE^2.$$

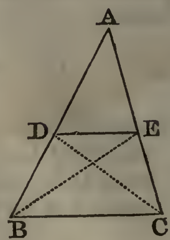
But $4AE^2$ is the square of 2AE, or of AC; $4DE^2$ is the square of BD (Prop. VIII. Cor.): hence the squares of the sides are together equivalent to the squares of the diagonals.

PROPOSITION XV. THEOREM.

If a line be drawn parallel to the base of a triangle, it will divide the other sides proportionally.

Let ABC be a triangle, and DE a straight line drawn parallel to the base BC; then will

AD : DB : : AE : EC.



Draw BE and DC. The two triangles BDE, DEC having the same base DE, and the same altitude, since both their vertices lie in a line parallel to the base, are equivalent (Prop. II. Cor. 2.).

The triangles ADE, BDE, whose common vertex is E, have the same altitude, and are to each other as their bases (Prop. VI. Cor.); hence we have

ADE : BDE : : AD : DB.

The triangles ADE, DEC, whose common vertex is D, have also the same altitude, and are to each other as their bases; hence

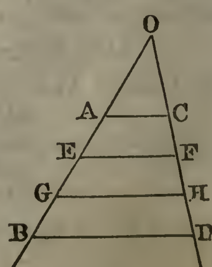
ADE : DEC : : AE : EC.

But the triangles BDE, DEC, are equivalent; and therefore, we have (Book II. Prop. IV. Cor.)

AD : DB : : AE : EC.

Cor. 1. Hence, by composition, we have AD + DB : AD : : AE + EC : AE, or AB : AD : : AC : AE; and also AB : BD : : AC : CE.

Cor. 2. If between two straight lines AB, CD, any number of parallels AC, EF, GH, BD, &c. be drawn, those straight lines are cut proportionally, and we shall have AE : CF : : EG : FH : GB : HD.

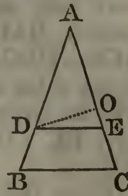


For, let O be the point where AB and CD meet. In the triangle OEF, the line AC being drawn parallel to the base EF, we shall have OE : AE : : OF : CF, or OE : OF : : AE : CF. In the triangle OGH, we shall likewise have OE : EG : : OF : FH, or OE : OF : : EG : FH. And by reason of the common ratio OE : OF, those two proportions give AE : CF : : EG : FH. It may be proved in the same manner, that EG : FH : : GB : HD, and so on; hence the lines AB, CD, are cut proportionally by the parallels AC, EF, GH, &c.

PROPOSITION XVI. THEOREM.

Conversely, if two sides of a triangle are cut proportionally by a straight line, this straight line will be parallel to the third side.

In the triangle ABC, let the line DE be drawn, making AD : DB : : AE : EC : then will DE be parallel to BC.



For, if DE is not parallel to BC, draw DO parallel to it. Then, by the preceding theorem, we shall have AD : DB : : AO : OC. But by hypothesis, we have AD : DB : : AE : EC : hence we must have AO : OC : : AE : EC, or AO : AE : : OC : EC; an impossible result, since AO, the one antecedent, is less than its consequent AE, and OC, the other antecedent, is greater than its consequent EC. Hence the parallel to BC, drawn from the point D, cannot differ from DE; hence DE is that parallel.

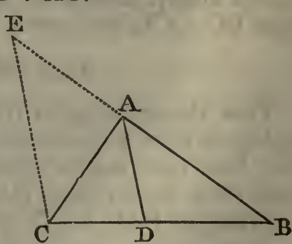
Scholium. The same conclusion would be true, if the proportion AB : AD : : AC : AE were the proposed one. For this proportion would give AB—AD : AD : : AC—AE : AE, or BD : AD : : CE : AE.

PROPOSITION XVII. THEOREM.

The line which bisects the vertical angle of a triangle, divides the base into two segments, which are proportional to the adjacent sides.

In the triangle ACB, let AD be drawn, bisecting the angle CAB; then will

BD : CD : : AB : AC.



Through the point C, draw CE parallel to AD till it meets BA produced.

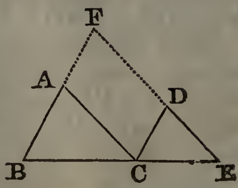
In the triangle BCE, the line AD is parallel to the base CE; hence we have the proportion (Prop. XV.),

BD : DC : : AB : AE.

But the triangle ACE is isosceles: for, since AD, CE are parallel, we have the angle ACE =DAC, and the angle AEC=BAD (Book I. Prop. XX. Cor. 2 & 3.); but, by hypothesis, DAC=BAD; hence the angle ACE=AEC, and consequently AE=AC (Book I. Prop. XII.). In place of AE in the above proportion, substitute AC, nd we shall have BD : DC : : AB : AC.

PROPOSITION XVIII. THEOREM.

Two equiangular triangles have their homologous sides proportional, and are similar.



Let ABC, CDE be two triangles which have their angles equal each to each, namely, BAC=CDE, ABC=DCE and ACB=DEC; then the homologous sides, or the sides adjacent to the equal angles, will be proportional, so that we shall have BC : CE : : AB : CD : : AC : DE.

Place the homologous sides BC, CE in the same straight line; and produce the sides BA, ED, till they meet in F.

Since BCE is a straight line, and the angle BCA is equal to CED, it follows that AC is parallel to DE (Book I. Prop. XIX. Cor. 2.). In like manner, since the angle ABC is equal to DCE, the line AB is parallel to DC. Hence the figure ACDF is a parallelogram.

In the triangle BFE, the line AC is parallel to the base FE; hence we have BC : CE : : BA : AF (Prop. XV.); or putting CD in the place of its equal AF,

BC : CE : : BA : CD.

In the same triangle BEF, CD is parallel to BF which may be considered as the base; and we have the proportion BC : CE : : FD : DE; or putting AC in the place of its equal FD,

BC : CE : : AC : DE.

And finally, since both these proportions contain the same ratio BC : CE, we have

AC : DE : : BA : CD.

Thus the equiangular triangles BAC, CED, have their homologous sides proportional. But two figures are similar when they have their angles equal, each to each, and their homologous sides proportional (Def. 1.); consequently the equiangular triangles BAC, CED, are two similar figures.

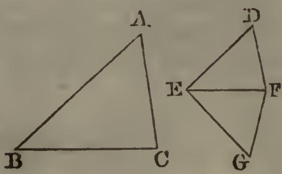
Cor. For the similarity of two triangles, it is enough that they have two angles equal, each to each: since then, the third will also be equal in both, and the two triangles will be equiangular.

Scholium. Observe, that in similar triangles, the homologous sides are opposite to the equal angles; thus the angle ACB being equal to DEC, the side AB is homologous to DC; in like manner, AC and DE are homologous, because opposite to the equal angles ABC, DCE. When the homologous sides are determined, it is easy to form the proportions:

AB : DC : : AC : DE : : BC : CE.

PROPOSITION XIX. THEOREM.

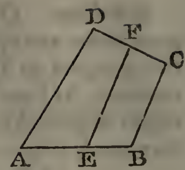
Two triangles, which have their homologous sides proportional, are equiangular and similar.



In the two triangles BAC, DEF, suppose we have BC : EF : : AB : DE : : AC : DF; then will the triangles ABC, DEF have their angles equal, namely, A=D, B=E, C=F.

At the point E, make the angle FEG=B, and at F, the angle EFG=C; the third G will be equal to the third A, and the two triangles ABC, EFG will be equiangular (Book I. Prop. XXV. Cor. 2.). Therefore, by the last theorem, we shall have BC : EF : : AB : EG; but, by hypothesis, we have BC : EF : : AB : DE; hence EG=DE. By the same theorem, we shall also have BC : EF : : AC : FG; and by hypothesis, we have BC : EF : : AC : DF; hence FG=DF. Hence the triangles EGF, DEF, having their three sides equal, each to each, are themselves equal (Book I. Prop. X.). But by construction, the triangles EGF and ABC are equiangular: hence DEF and ABC are also equiangular and similar.

Scholium 1. By the last two propositions, it appears that in triangles, equality among the angles is a consequence of proportionality among the sides, and conversely; so that either of those conditions sufficiently determines the similarity of two triangles. The case is different with regard to figures of more than three sides: even in quadrilaterals, the proportion between the sides may be altered without altering the angles, or the angles may be altered without altering the proportion between the sides; and thus proportionality among the sides cannot be a consequence of equality among the angles of two quadrilaterals, or *vice versa.* It is evident, for example, that

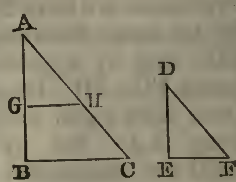


by drawing EF parallel to BC, the angles of the quadrilateral AEFD, are made equal to those of ABCD, though the proportion between the sides is different; and, in like manner, without changing the four sides AB, BC, CD, AD, we can make the point B approach D or recede from it, which will change the angles.

Scholium 2. The two preceding propositions, which in strictness form but one, together with that relating to the square of the hypothenuse, are the most important and fertile in results of any in geometry: they are almost sufficient of themselves for every application to subsequent reasoning, and for solving every problem. The reason is, that all figures may be divided into triangles, and any triangle into two right angled triangles. Thus the general properties of triangles include, by implication, those of all figures.

PROPOSITION XX. THEOREM.

Two triangles, which have an angle of the one equal to an angle of the other, and the sides containing those angles proportional, are similar.



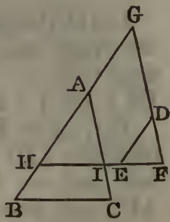
In the two triangles ABC, DEF, let the angles A and D be equal; then, if AB : DE : : AC : DF, the two triangles will be similar.

Take AG=DE, and draw GH parallel to BC. The angle AGH will be equal to the angle ABC (Book I. Prop. XX. Cor 3.); and the triangles AGH, ABC, will be equiangular: hence we shall have AB : AG : : AC : AH. But by hypothesis, we have AB : DE : : AC : DF; and by construction, AG=DE: hence AH=DF. The two triangles AGH, DEF, have an equal angle included between equal sides; therefore they are equal; but the triangle AGH is similar to ABC: therefore DEF is also similar to ABC.

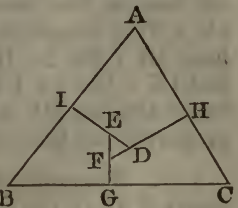
PROPOSITION XXI. THEOREM.

Two triangles, which have their homologous sides parallel, or perpendicular to each other, are similar

Let BAC, EDF, be two triangles.



First. If the side AB is parallel to DE, and BC to EF, the angle ABC will be equal to DEF (Book I. Prop. XXIV.); if AC is parallel to DF, the angle ACB will be equal to DFE, and also BAC to EDF; hence the triangles ABC, DEF, are equiangular; consequently they are similar (Prop. XVIII.).



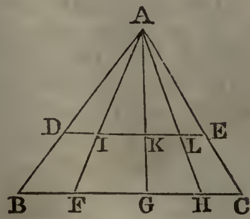
Secondly. If the side DE is perpendicular to AB, and the side DF to AC, the two angles I and H of the quadrilateral AIDH will be right angles; and since all the four angles are together equal to four right angles (Book I. Prop. XXVI. Cor. 1.), the remaining two IAH, IDH, will be together equal to two right angles. But the two angles EDF, IDH, are also equal to two right angles: hence the angle EDF is equal to IAH or BAC. In like manner, if the third side EF is perpendicular to the third side BC, it may be shown that the angle DFE is equal to C, and DEF to B: hence the triangles ABC, DEF, which have the sides of the one perpendicular to the corresponding sides of the other, are equiangular and similar.

Scholium. In the case of the sides being parallel, the homologous sides are the parallel ones: in the case of their being perpendicular, the homologous sides are the perpendicular ones. Thus in the latter case DE is homologous with AB, DF with AC, and EF with BC.

The case of the perpendicular sides might present a relative position of the two triangles different from that exhibited in the diagram. But we might always conceive a triangle DEF to be constructed within the triangle ABC, and such that its sides should be parallel to those of the triangle compared with ABC; and then the demonstration given in the text would apply.

PROPOSITION XXII. THEOREM.

In any triangle, if a line be drawn parallel to the base, then, all lines drawn from the vertex will divide the base and the parallel into proportional parts.



Let DE be parallel to the base BC, and the other lines drawn as in the figure ; then will
DI : BF : : IK : FG : : KL : GH.

For, since DI is parallel to BF, the triangles ADI and ABF are equiangular ; and we have DI : BF : : AI : AF ; and since IK is parallel to FG, we have in like manner AI : AF : : IK : FG ; hence, the ratio AI : AF being common, we shall have DI : BF : : IK : FG. In the same manner we shall find IK : FG : : KL : GH ; and so with the other segments : hence the line DE is divided at the points I, K, L, in the same proportion, as the base BC, at the points F, G, H.

Cor. Therefore if BC were divided into equal parts at the points F, G, H, the parallel DE would also be divided into equal parts at the points I, K, L.

PROPOSITION XXIII. THEOREM.

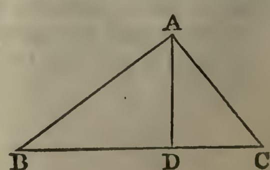
If from the right angle of a right angled triangle, a perpendicular be let fall on the hypothenuse ; then,

1*st. The two partial triangles thus formed, will be similar to each other, and to the whole triangle.*

2*d. Either side including the right angle will be a mean proportional between the hypothenuse and the adjacent segment.*

3*d. The perpendicular will be a mean proportional between the two segments of the hypothenuse.*

Let BAC be a right angled triangle, and AD perpendicular to the hypothenuse BC.



First. The triangles BAD and BAC have the common angle B, the right angle BDA=BAC, and therefore the third angle BAD of the one, equal to the third angle C, of the other (Book I. Prop. XXV. Cor 2.) : hence those two triangles are equiangular and similar. In the same manner it may be shown that the triangles DAC and BAC are similar; hence all the triangles are equiangular and similar.

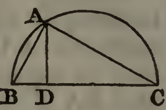
Secondly. The triangles BAD, BAC, being similar, their homologous sides are proportional. But BD in the small triangle, and BA in the large one, are homologous sides, because they lie opposite the equal angles BAD, BCA; the hypothenuse BA of the small triangle is homologous with the hypothenuse BC of the large triangle: hence the proportion BD : BA : : BA : BC. By the same reasoning, we should find DC : AC : : AC : BC; hence, each of the sides AB, AC, is a mean proportional between the hypothenuse and the segment adjacent to that side.

Thirdly. Since the triangles ABD, ADC, are similar, by comparing their homologous sides, we have BD : AD : : AD : DC; hence, the perpendicular AD is a mean proportional between the segments BD, DC, of the hypothenuse.

Scholium. Since BD : AB : : AB : BC, the product of the extremes will be equal to that of the means, or $AB^2=BD.BC$. For the same reason we have $AC^2=DC.BC$; therefore $AB^2+AC^2=BD.BC+DC.BC=(BD+DC).BC=BC.BC=BC^2$; or the square described on the hypothenuse BC is equivalent to the squares described on the two sides AB, AC. Thus we again arrive at the property of the square of the hypothenuse, by a path very different from that which formerly conducted us to it: and thus it appears that, strictly speaking, the property of the square of the hypothenuse, is a consequence of the more general property, that the sides of equiangular triangles are proportional. Thus the fundamental propositions of geometry are reduced, as it were, to this single one, that equiangular triangles have their homologous sides proportional.

It happens frequently, as in this instance, that by deducing consequences from one or more propositions, we are led back to some proposition already proved. In fact, the chief characteristic of geometrical theorems, and one indubitable proof of their certainty is, that, however we combine them together, provided only our reasoning be correct, the results we obtain are always perfectly accurate. The case would be different, if any proposition were false or only approximately true: it would frequently happen that on combining the propositions together, the error would increase and become perceptible. Examples of this are to be seen in all the demonstrations, in which the *reductio ad absurdum* is employed. In such demonstrations, where the object is to show that two quantities are equal, we proceed by showing that if there existed the smallest inequality between the quantities, a train of accurate reasoning would lead us to a manifest and palpable absurdity; from which we are forced to conclude that the two quantities are equal.

Cor. If from a point A, in the circumference of a circle, two chords AB, AC, be drawn to the extremities of a diameter BC, the triangle BAC will be right angled at A (Book III. Prop. XVIII. Cor. 2.); hence, first, *the perpendicular* AD *is a mean proportional between the two segments* BD, DC, *of the diameter*, or what is the same, $AD^2 = BD.DC$.



Hence also, in the second place, *the chord* AB *is a mean proportional between the diameter* BC *and the adjacent segment* BD, or, what is the same, $AB^2 = BD.BC$. In like manner, we have $AC^2 = CD.BC$; hence $AB^2 : AC^2 :: BD : DC$: and comparing AB^2 and AC^2, to BC^2, we have $AB^2 : BC^2 :: BD : BC$, and $AC^2 : BC^2 :: DC : BC$. Those proportions between the squares of the sides compared with each other, or with the square of the hypothenuse, have already been given in the third and fourth corollaries of Prop. XI.

PROPOSITION XXIV. THEOREM.

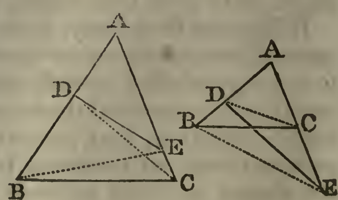
Two triangles having an angle in each equal, are to each other as the rectangles of the sides which contain the equal angles.

In the two triangles ABC, ADE, let the angle A be equal to the angle A; then will the triangle

$$ABC : ADE :: AB.AC : AD.AE.$$

Draw BE. The triangles ABE, ADE, having the common vertex E, have the same altitude, and consequently are to each other as their bases (Prop. VI. Cor.): that is,

$$ABE : ADE :: AB : AD.$$



In like manner,

$$ABC : ABE :: AC : AE.$$

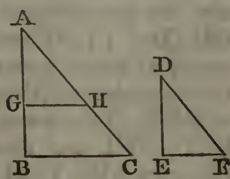
Multiply together the corresponding terms of these proportions, omitting the common term ABE; we have

$$ABC : ADE : AB.AC : AD.AE.$$

Cor. Hence the two triangles would be equivalent, if the rectangle AB.AC were equal to the rectangle AD.AE, or if we had AB : AD : : AE : AC ; which would happen if DC were parallel to BE.

PROPOSITION XXV. THEOREM.

Two similar triangles are to each other as the squares describea on their homologous sides.



Let ABC, DEF, be two similar triangles, having the angle A equal to D, and the angle B=E.

Then, first, by reason of the equal angles A and D, according to the last proposition, we shall have

ABC : DEF : : AB.AC : DE.DF.

Also, because the triangles are similar,

AB : DE : : AC : DF,

And multiplying the terms of this proposition by the corresponding terms of the identical proportion,

AC : DF : : AC : DF,

there will result

$$AB.AC : DE.DF : : AC^2 : DF^2.$$

Consequently,

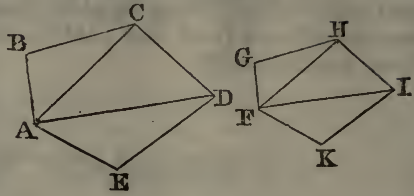
$$ABC : DEF : : AC^2 : DF^2.$$

Therefore, two similar triangles ABC, DEF, are to each other as the squares described on their homologous sides AC, DF, or as the squares of any other two homologous sides.

PROPOSITION XXVI. THEOREM.

Two similar polygons are composed of the same number of triangles, similar each to each, and similarly situated.

Let ABCDE, FGHIK, be two similar polygons.



From any angle A, in the polygon ABCDE, draw diagonals AC, AD to the other angles. From the homologous angle F, in the other polygon FGHIK, draw diagonals FH, FI to the other angles.

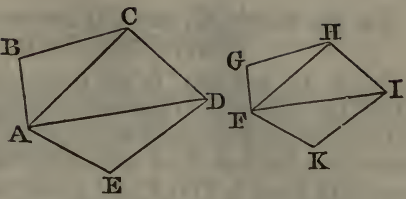
These polygons being similar, the angles ABC, FGH, which are homologous, must be equal, and the sides AB, BC, must also be proportional to FG, GH, that is, AB : FG : : BC : GH (Def. 1.). Wherefore the triangles ABC, FGH, have each an equal angle, contained between proportional sides; hence they are similar (Prop. XX.); therefore the angle BCA is equal to GHF. Take away these equal angles from the equal angles BCD, GHI, and there remains ACD=FHI. But since the triangles ABC, FGH, are similar, we have AC : FH : : BC : GH; and, since the polygons are similar, BC : GH : : CD : HI; hence AC : FH : : CD : HI. But the angle ACD, we already know, is equal to FHI; hence the triangles ACD, FHI, have an equal angle in each, included between proportional sides, and are consequently similar (Prop. XX.). In the same manner it might be shown that all the remaining triangles are similar, whatever be the number of sides in the polygons proposed: therefore two similar polygons are composed of the same number of triangles, similar, and similarly situated.

Scholium. The converse of the proposition is equally true: *If two polygons are composed of the same number of triangles similar and similarly situated, those two polygons will be similar.*

For, the similarity of the respective triangles will give the angles, ABC=FGH, BCA=GHF, ACD=FHI: hence BCD=GHI, likewise CDE=HIK, &c. Moreover we shall have AB : FG : : BC : GH : : AC : FH : : CD : HI, &c.; hence the two polygons have their angles equal and their sides proportional; consequently they are similar.

PROPOSITION XXVII. THEOREM.

The contours or perimeters of similar polygons are to each other as the homologous sides: and the areas are to each other as the squares described on those sides.



First. Since, by the nature of similar figures, we have AB : FG : : BC : GH : : CD : HI, &c. we conclude from this series of equal ratios that the sum of the antecedents AB+BC+CD, &c., which makes up the perimeter of the first polygon, is to the sum of the consequents FG+GH+HI, &c., which makes up the perimeter of the second polygon, as any one antecedent is to its consequent; and therefore, as the side AB is to its corresponding side FG (Book II. Prop. X.).

Secondly. Since the triangles ABC, FGH are similar, we shall have the triangle ABC : FGH : : AC^2 : FH^2 (Prop. XXV.); and in like manner, from the similar triangles ACD, FHI, we shall have ACD : FHI : : AC^2 : FH^2; therefore, by reason of the common ratio, AC^2 : FH^2, we have

ABC : FGH : : ACD : FHI.

By the same mode of reasoning, we should find

ACD : FHI : : ADE : FIK;

and so on, if there were more triangles. And from this series of equal ratios, we conclude that the sum of the antecedents ABC+ACD+ADE, or the polygon ABCDE, is to the sum of the consequents FGH+FHI+FIK, or to the polygon FGHIK, as one antecedent ABC, is to its consequent FGH, or as AB^2 is to FG^2 (Prop. XXV.); hence the areas of similar polygons are to each other as the squares described on the homologous sides.

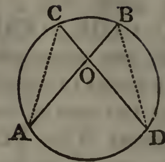
Cor. If three similar figures were constructed, on the three sides of a right angled triangle, the figure on the hypothenuse would be equivalent to the sum of the other two: for the three figures are proportional to the squares of their homologous sides; but the square of the hypothenuse is equivalent to the sum of the squares of the two other sides; hence, &c.

PROPOSITION XXVIII. THEOREM.

The segments of two chords, which intersect each other in a circle, are reciprocally proportional.

Let the chords AB and CD intersect at O: then will

AO : DO : : OC : OB.



Draw AC and BD. In the triangles ACO, BOD, the angles at O are equal, being vertical; the angle A is equal to the angle D, because both are inscribed in the same segment (Book III. Prop. XVIII. Cor. 1.); for the same reason the angle C=B; the triangles are therefore similar, and the homologous sides give the proportion

AO : DO : : CO : OB.

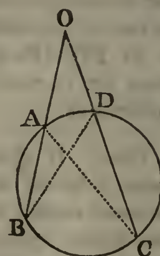
Cor. Therefore AO.OB=DO.CO: hence the rectangle under the two segments of the one chord is equal to the rect angle under the two segments of the other.

PROPOSITION XXIX. THEOREM.

If from the same point without a circle, two secants be drawn terminating in the concave arc, the whole secants will be reciprocally proportional to their external segments.

Let the secants OB, OC, be drawn from the point O: then will

OB : OC : : OD : OA.



For, drawing AC, BD, the triangles OAC, OBD have the angle O common; likewise the angle B=C (Book III. Prop. XVIII. Cor. 1.); these triangles are therefore similar; and their homologous sides give the proportion,

OB : OC : : OD : OA.

Cor. Hence the rectangle OA.OB is equal to the rectangle OC.OD.

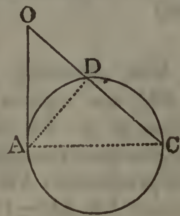
Scholium. This proposition, it may be observed, bears a great analogy to the preceding, and differs from it only as the two chords AB, CD, instead of intersecting each other within, cut each other without the circle. The following proposition may also be regarded as a particular case of the proposition just demonstrated.

PROPOSITION XXX. THEOREM.

If from the same point without a circle, a tangent and a secant be drawn, the tangent will be a mean proportional between the secant and its external segment.

From the point O, let the tangent OA, and the secant OC be be drawn ; then will

$$OC : OA :: OA : OD, \text{ or } OA^2 = OC.OD.$$



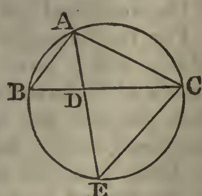
For, drawing AD and AC, the triangles OAD, OAC, have the angle O common; also the angle OAD, formed by a tangent and a chord, has for its measure half of the arc AD (Book III. Prop. XXI.) ; and the angle C has the same measure : hence the angle OAD= C; therefore the two triangles are similar, and we have the proportion OC : OA : : AO · OD, which gives $OA^2 = OC.OD$.

PROPOSITION XXXI. THEOREM.

If either angle of a triangle be bisected by a line terminating in the opposite side, the rectangle of the sides including the bisected angle, is equivalent to the square of the bisecting line together with the rectangle contained by the segments of the third side.

In the triangle BAC, let AD bisect the angle A ; then will

$$AB.AC = AD^2 + BD.DC.$$



Describe a circle through the three points A, B, C ; produce AD till it meets the circumference, and draw CE.

The triangle BAD is similar to the triangle EAC ; for, by hypothesis, the angle BAD=EAC ; also the angle B=E, since they are both measured by half of the arc AC ; hence these triangles are similar, and the homologous sides give the proportion BA : AE : : AD . AC ; hence BA.AC=AE.AD ; but AE=AD+DE, and multiplying each of these equals by AD, we have $AE.AD = AD^2 + AD.DE$; now AD.DE=BD.DC (Prop. XXVIII.) ; hence, finally,

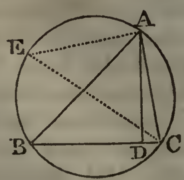
$$BA.AC = AD^2 + BD.DC.$$

PROPOSITION XXXII. THEOREM.

In every triangle, the rectangle contained by two sides is equivalent to the rectangle contained by the diameter of the circumscribed circle, and the perpendicular let fall upon the third side.

In the triangle ABC, let AD be drawn perpendicular to BC; and let EC be the diameter of the circumscribed circle: then will

$$AB.AC=AD.CE.$$

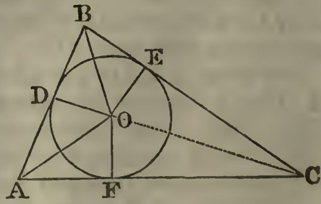


For, drawing AE, the triangles ABD, AEC, are right angled, the one at D, the other at A: also the angle B=E; these triangles are therefore similar, and they give the proportion AB : CE : : AD : AC; and hence AB.AC=CE.AD.

Cor. If these equal quantities be multiplied by the same quantity BC, there will result AB.AC.BC=CE.AD.BC; now AD.BC is double of the area of the triangle (Prop. VI.); therefore *the product of three sides of a triangle is equal to its area multiplied by twice the diameter of the circumscribed circle.*

The product of three lines is sometimes called a *solid*, for a reason that shall be seen afterwards. Its value is easily conceived, by imagining that the lines are reduced into numbers, and multiplying these numbers together.

Scholium. It may also be demonstrated, that the area of a triangle is equal to its perimeter multiplied by half the radius of the inscribed circle.



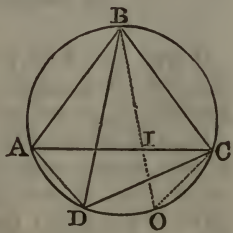
For, the triangles AOB, BOC, AOC, which have a common vertex at O, have for their common altitude the radius of the inscribed circle; hence the sum of these triangles will be equal to the sum of the bases AB, BC, AC, multiplied by half the radius OD; hence the area of the triangle ABC is equal to the perimeter multiplied by half the radius of the inscribed circle

PROPOSITION XXXIII. THEOREM.

In every quadrilateral inscribed in a circle, the rectangle of the two diagonals is equivalent to the sum of the rectangles of the opposite sides.

In the quadrilateral ABCD, we shall have

$$AC.BD = AB.CD + AD.BC.$$



Take the arc CO=AD, and draw BO meeting the diagonal AC in I.

The angle ABD=CBI, since the one has for its measure half of the arc AD, and the other, half of CO, equal to AD; the angle ADB=BCI, because they are both inscribed in the same segment AOB; hence the triangle ABD is similar to the triangle IBC, and we have the proportion AD : CI : : BD : BC; hence AD.BC=CI.BD. Again, the triangle ABI is similar to the triangle BDC; for the arc AD being equal to CO, if OD be added to each of them, we shall have the arc AO=DC; hence the angle ABI is equal to DBC; also the angle BAI to BDC, because they are inscribed in the same segment; hence the triangles ABI, DBC, are similar, and the homologous sides give the proportion AB : BD : : AI : CD; hence AB.CD=AI.BD.

Adding the two results obtained, and observing that

$$AI.BD + CI.BD = (AI + CI).BD = AC.BD,$$

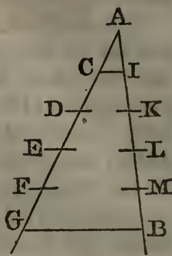
we shall have

$$AD.BC + AB.CD = AC.BD.$$

PROBLEMS RELATING TO THE FOURTH BOOK.

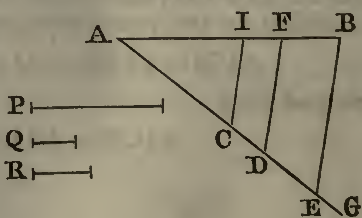
PROBLEM I.

To divide a given straight line into any number of equal parts, or into parts proportional to given lines.



First. Let it be proposed to divide the line AB into five equal parts. Through the extremity A, draw the indefinite straight line AG; and taking AC of any magnitude, apply it five times upon AG; join the last point of division G, and the extremity B, by the straight line GB; then draw CI parallel to GB: AI will be the fifth part of the line AB; and thus, by applying AI five times upon AB, the line AB will be divided into five equal parts.

For, since CI is parallel to GB, the sides AG, AB, are cut proportionally in C and I (Prop. XV.). But AC is the fifth part of AG, hence AI is the fifth part of AB,

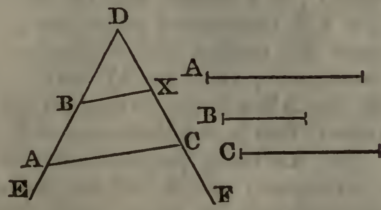


Secondly. Let it be proposed to divide the line AB into parts proportional to the given lines P, Q, R. Through A, draw the indefinite line AG; make AC = P, CD = Q, DE = R; join the extremities E and B; and through the points C, D, draw CI, DF, parallel to EB; the line AB will be divided into parts AI, IF, FB, proportional to the given lines P, Q, R.

For, by reason of the parallels CI, DF, EB, the parts AI, IF, FB, are proportional to the parts AC, CD, DE; and by construction, these are equal to the given lines P, Q, R.

PROBLEM II.

To find a fourth proportional to three given lines, A, B, C.

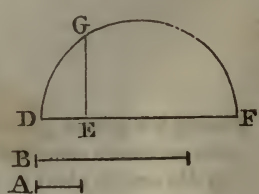


Draw the two indefinite lines DE, DF, forming any angle with each other. Upon DE take DA=A, and DB=B; upon DF take DC=C; draw AC; and through the point B, draw BX parallel to AC; DX will be the fourth proportional required; for, since BX is parallel to AC, we have the proportion DA : DB : : DC : DX; now the first three terms of this proportion are equal to the three given lines: consequently DX is the fourth proportional required.

Cor. A third proportional to two given lines A, B, may be found in the same manner, for it will be the same as a fourth proportional to the three lines A, B, B.

PROBLEM III.

To find a mean proportional between two given lines A *and* B.



Upon the indefinite line DF, take DE=A, and EF=B; upon the whole line DF, as a diameter, describe the semicircle DGF; at the point E, erect upon the diameter the perpendicular EG meeting the circumference in G; EG will be the mean proportional required.

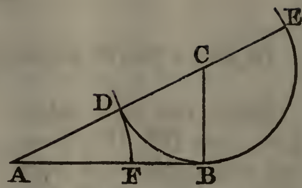
For, the perpendicular EG, let fall from a point in the circumference upon the diameter, is a mean proportional between DE, EF, the two segments of the diameter (Prop. XXIII. Cor.); and these segments are equal to the given lines A and B.

PROBLEM IV.

To divide a given line into two parts, such that the greater part shall be a mean proportional between the whole line and the other part.

Let AB be the given line.

At the extremity B of the line AB, erect the perpendicular BC equal to the half of AB; from the point C, as a centre, with the radius CB, describe a semicircle; draw AC cutting the circumference in D; and take AF=AD: the line AB will be divided at the point F in the manner required; that is, we shall have AB : AF : : AF : FB.



For, AB being perpendicular to the radius at its extremity, is a tangent; and if AC be produced till it again meets the circumference in E, we shall have AE : AB : : AB : AD (Prop. XXX.); hence, by division, AE—AB : AB : : AB—AD : AD. But since the radius is the half of AB, the diameter DE is equal to AB, and consequently AE—AB=AD=AF; also, because AF=AD, we have AB—AD=FB; hence AF : AB : : FB : AD or AF; whence, by exchanging the extremes for the means, AB : AF : : AF : FB.

Scholium. This sort of division of the line AB is called division in extreme and mean ratio: the use of it will be perceived in a future part of the work. It may further be observed, that the secant AE is divided in extreme and mean ratio at the point D; for, since AB=DE, we have AE : DE : : DE : AD.

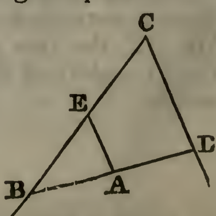
PROBLEM V.

Through a given point, in a given angle, to draw a line so that the segments comprehended between the point and the two sides of the angle, shall be equal.

Let BCD be the given angle, and A the given point.

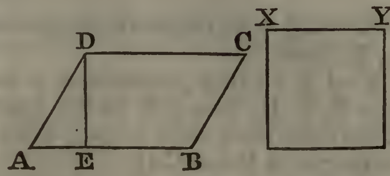
Through the point A, draw AE parallel to CD, make BE=CE, and through the points B and A draw BAD; this will be the line required.

For, AE being parallel to CD, we have BE : EC : : BA : AD; but BE=EC; therefore BA=AD.



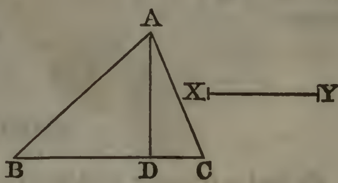
PROBLEM VI.

To describe a square that shall be equivalent to a given parallelogram, or to a given triangle.



First. Let ABCD be the given parallelogram, AB its base, DE its altitude: between AB and DE find a mean proportional XY; then will the square described upon XY be equivalent to the parallelogram ABCD.

For, by construction, AB : XY : : XY : DE; therefore, $XY^2 = AB.DE$; but AB.DE is the measure of the parallelogram, and XY^2 that of the square; consequently, they are equivalent.



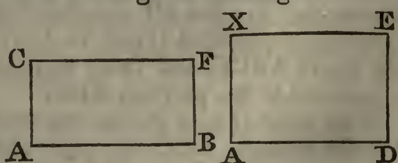
Secondly. Let ABC be the given triangle, BC its base, AD its altitude: find a mean proportional between BC and the half of AD, and let XY be that mean; the square described upon XY will be equivalent to the triangle ABC.

For, since BC : XY : : XY : $\frac{1}{2}$AD, it follows that $XY^2 = BC.\frac{1}{2}AD$; hence the square described upon XY is equivalent to the triangle ABC.

PROBLEM VII.

Upon a given line, to describe a rectangle that shall be equivalent to a given rectangle.

Let AD be the line, and ABFC the given rectangle.



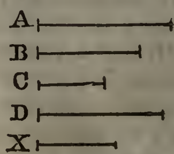
Find a fourth proportional to the three lines AD, AB, AC, and let AX be that fourth proportional; a rectangle constructed with the lines AD and AX will be equivalent to the rectangle ABFC.

For, since AD : AB : : AC : AX, it follows that AD.AX= AB.AC; hence the rectangie ADEX is equivalent to the rectangle ABFC.

PROBLEM VIII.

To find two lines whose ratio shall be the same as the ratio of two rectangles contained by given lines.

Let A.B, C.D, be the rectangles contained by the given lines A, B, C, and D.



Find X, a fourth proportional to the three lines B, C, D; then will the two lines A and X have the same ratio to each other as the rectangles A.B and C.D.

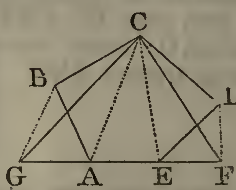
For, since B : C : : D : X, it follows that C.D=B.X; hence A.B : C.D : : A.B : B.X : : A : X.

Cor. Hence to obtain the ratio of the squares described upon the given lines A and C, find a third proportional X to the lines A and C, so that A : C : : C : X; you will then have

$$A.X = C^2, \text{ or } A^2.X = A.C^2; \text{ hence}$$
$$A^2 : C^2 : : A : X.$$

PROBLEM IX.

To find a triangle that shall be equivalent to a given polygon.



Let ABCDE be the given polygon. Draw first the diagonal CE cutting off the triangle CDE; through the point D, draw DF parallel to CE, and meeting AE produced; draw CF: the polygon ABCDE will be equivalent to the polygon ABCF, which has one side less than the original polygon.

For, the triangles CDE, CFE, have the base CE common, they have also the same altitude, since their vertices D and F, are situated in a line DF parallel to the base: these triangles are therefore equivalent (Prop. II. Cor. 2.). Add to each of them the figure ABCE, and there will result the polygon ABCDE, equivalent to the polygon ABCF.

The angle B may in like manner be cut off, by substituting for the triangle ABC the equivalent triangle AGC, and thus the pentagon ABCDE will be changed into an equivalent triangle GCF.

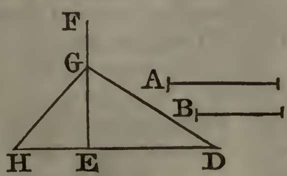
The same process may be applied to every other figure; for, by successively diminishing the number of its sides, one being retrenched at each step of the process, the equivalent triangle will at last be found.

Scholium. We have already seen that every triangle may be changed into an equivalent square (Prob. VI.); and thus a square may always be found equivalent to a given rectilineal figure, which operation is called *squaring* the rectilineal figure, or finding the *quadrature* of it.

The problem of *the quadrature of the circle,* consists in finding a square equivalent to a circle whose diameter is given.

PROBLEM X.

To find the side of a square which shall be equivalent to the sum or the difference of two given squares.



Let A and B be the sides of the given squares.

First. If it is required to find a square equivalent to the sum of these squares, draw the two indefinite lines ED, EF, at right angles to each other; take ED=A, and EG=B; draw DG: this will be the side of the square required.

For the triangle DEG being right angled, the square described upon DG is equivalent to the sum of the squares upon ED and EG.

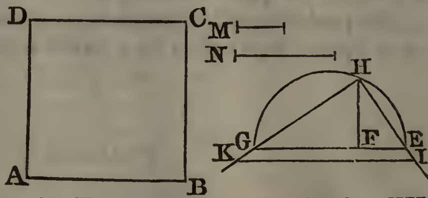
Secondly. If it is required to find a square equivalent to the difference of the given squares, form in the same manner the right angle FEH; take GE equal to the shorter of the sides A and B; from the point G as a centre, with a radius GH, equal to the other side, describe an arc cutting EH in H: the square described upon EH will be equivalent to the difference of the squares described upon the lines A and B.

For the triangle GEH is right angled, the hypothenuse GH=A, and the side GE=B; hence the square described upon EH, is equivalent to the difference of the squares A and B.

Scholium. A square may thus be found, equivalent to the sum of any number of squares; for a similar construction which reduces two of them to one, will reduce three of them to two, and these two to one, and so of others. It would be the same, if any of the squares were to be subtracted from the sum of the others.

PROBLEM XI.

To find a square which shall be to a given square as a given line to a given line.



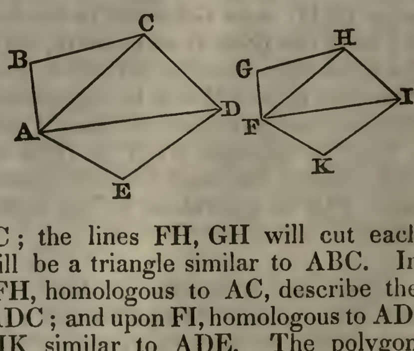
Let AC be the given square, and M and N the given lines.

Upon the indefinite line EG, take EF=M, and FG=N; upon EG as a diameter describe a semicircle, and at the point F erect the perpendicular FH. From the point H, draw the chords HG, HE, which produce indefinitely: upon the first, take HK equal to the side AB of the given square, and through the point K draw KI parallel to EG; HI will be the side of the square required.

For, by reason of the parallels KI, GE, we have HI : HK : : HE : HG; hence, $HI^2 : HK^2 :: HE^2 : HG^2$: but in the right angled triangle EHG, the square of HE is to the square of HG as the segment EF is to the segment FG (Prop. XI. Cor. 3.), or as M is to N; hence $HI^2 : HK^2 :: M : N$. But HK=AB; therefore the square described upon HI is to the square described upon AB as M is to N.

PROBLEM XII.

Upon a given line, to describe a polygon similar to a given polygon.



Let FG be the given line, and AEDCB the given polygon.

In the given polygon, draw the diagonals AC, AD; at the point F make the angle GFH=BAC, and at the point G the angle FGH=ABC; the lines FH, GH will cut each other in H, and FGH will be a triangle similar to ABC. In the same manner upon FH, homologous to AC, describe the triangle FIH similar to ADC; and upon FI, homologous to AD, describe the triangle FIK similar to ADE. The polygon FGHIK will be similar to ABCDE, as required.

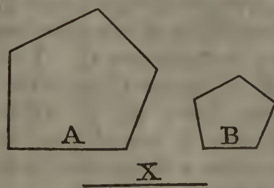
For, these two polygons are composed of the same number of triangles, which are similar and similarly situated (Prop. XXVI. Sch.).

PROBLEM XIII.

Two similar figures being given, to describe a similar figure which shall be equivalent to their sum or their difference.

Let A and B be two homologous sides of the given figures.

Find a square equivalent to the sum or to the difference of the squares described upon A and B; let X be the side of that square; then will X in the figure required, be the side which is homologous to the sides A and B in the given figures. The figure itself may then be constructed on X, by the last problem.

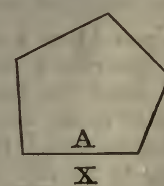


For, the similar figures are as the squares of their homologous sides; now the square of the side X is equivalent to the sum, or to the difference of the squares described upon the homologous sides A and B; therefore the figure described upon the side X is equivalent to the sum, or to the difference of the similar figures described upon the sides A and B.

PROBLEM XIV.

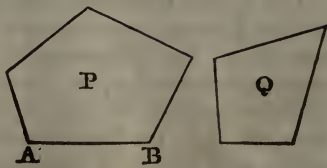
To describe a figure similar to a given figure, and bearing to it the given ratio of M to N.

Let A be a side of the given figure, X the homologous side of the figure required. The square of X must be to the square of A, as M is to N: hence X will be found by (Prob. XI.), and knowing X, the rest will be accomplished by (Prob. XII.).



PROBLEM XV.

To construct a figure similar to the figure P, and equivalent to the figure Q.



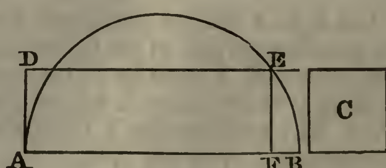
Find M, the side of a square equivalent to the figure P, and N, the side of a square equivalent to the figure Q. Let X be a fourth proportional to the three given lines, M, N, AB; upon the side X, homologous to AB, describe a figure similar to the figure P; it will also be equivalent to the figure Q.

For, calling Y the figure described upon the side X, we have $P : Y :: AB^2 : X^2$; but by construction, $AB : X :: M : N$, or $AB^2 : X^2 :: M^2 : N^2$; hence $P : Y :: M^2 : N^2$. But by construction also, $M^2 = P$ and $N^2 = Q$; therefore $P : Y :: P : Q$; consequently $Y = Q$; hence the figure Y is similar to the figure P, and equivalent to the figure Q.

PROBLEM XVI.

To construct a rectangle equivalent to a given square, and having the sum of its adjacent sides equal to a given line.

Let C be the square, and AB equal to the sum of the sides of the required rectangle.



Upon AB as a diameter, describe a semicircle; draw the line DE parallel to the diameter, at a distance AD from it, equal to the side of the given square C; from the point E, where the parallel cuts the circumference, draw EF perpendicular to the diameter; AF and FB will be the sides of the rectangle required.

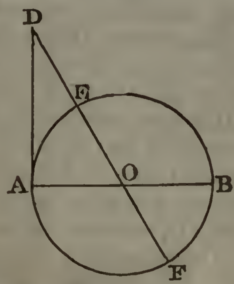
For their sum is equal to AB; and their rectangle AF.FB is equivalent to the square of EF, or to the square of AD; hence that rectangle is equivalent to the given square C.

Scholium. To render the problem possible, the distance AD must not exceed the radius; that is, the side of the square C must not exceed the half of the line AB.

PROBLEM XVII.

To construct a rectangle that shall be equivalent to a given square, and the difference of whose adjacent sides shall be equal to a given line.

Suppose C equal to the given square, and AB the difference of the sides.

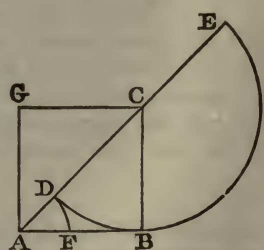


Upon the given line AB as a diameter, describe a semicircle: at the extremity of the diameter draw the tangent AD, equal to the side of the square C; through the point D and the centre O draw the secant DF; then will DE and DF be the adjacent sides of the rectangle required.

For, first, the difference of these sides is equal to the diameter EF or AB; secondly, the rectangle DE, DF, is equal to AD^2 (Prop. XXX.): hence that rectangle is equivalent to the given square C.

PROBLEM XVIII.

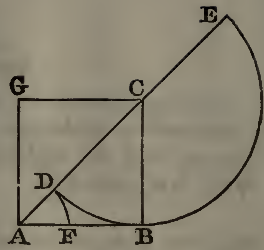
To find the common measure, if there is one, between the diagonal and the side of a square.



Let ABCG be any square whatever, and AC its diagonal.

We must first apply CB upon CA, as often as it may be contained there. For this purpose, let the semicircle DBE be described, from the centre C, with the radius CB. It is evident that CB is contained once in AC, with the remainder AD; the result of the first operation is therefore the quotient 1, with the remainder AD, which latter must now be compared with BC, or its equal AB.

We might here take AF=AD, and actually apply it upon AB; we should find it to be contained twice with a remainder: but as that remainder, and those which succeed it, con-



tinue diminishing, and would soon elude our comparisons by their minuteness, this would be but an imperfect mechanical method, from which no conclusion could be obtained to determine whether the lines AC, CB, have or have not a common measure. There is a very simple way, however, of avoiding these decreasing lines, and obtaining the result, by operating only upon lines which remain always of the same magnitude.

The angle ABC being a right angle, AB is a tangent, and AE a secant drawn from the same point; so that AD : AB : : AB : AE (Prop. XXX.). Hence in the second operation, when AD is compared with AB, the ratio of AB to AE may be taken instead of that of AD to AB; now AB, or its equal CD, is contained twice in AE, with the remainder AD; the result of the second operation is therefore the quotient 2 with the remainder AD, which must be compared with AB.

Thus the third operation again consists in comparing AD with AB, and may be reduced in the same manner to the comparison of AB or its equal CD with AE; from which there will again be obtained 2 for the quotient, and AD for the remainder.

Hence, it is evident that the process will never terminate; and therefore there is no common measure between the diagonal and the side of a square: a truth which was already known by arithmetic, since these two lines are to each other : : $\sqrt{2} : 1$ (Prop. XI. Cor. 4.), but which acquires a greater degree of clearness by the geometrical investigation.

BOOK V.

REGULAR POLYGONS, AND THE MEASUREMENT OF THE CIRCLE.

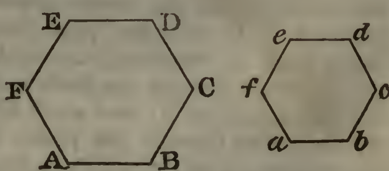
Definition.

A Polygon, which is at once equilateral and equiangular, is called a *regular polygon.*

Regular polygons may have any number of sides: the equilateral triangle is one of three sides; the square is one of four.

PROPOSITION I. THEOREM.

Two regular polygons of the same number of sides are similar figures.



Suppose, for example, that ABCDEF, *abcdef,* are two regular hexagons. The sum of all the angles is the same in both figures, being in each equal to eight right angles (Book I. Prop. XXVI. Cor. 3.). The angle A is the sixth part of that sum; so is the angle a: hence the angles A and a are equal; and for the same reason, the angles B and b, the angles C and c, &c. are equal.

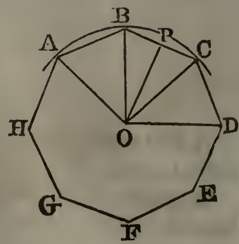
Again, since the polygons are regular, the sides AB, BC, CD, &c. are equal, and likewise the sides ab, bc, cd, &c. (Def.); it is plain that AB : ab : : BC : bc : : CD : cd, &c.; hence the two figures in question have their angles equal, and their homologous sides proportional; consequently they are similar (Book IV. Def. 1.).

Cor. The perimeters of two regular polygons of the same number of sides, are to each other as their homologous sides, and their surfaces are to each other as the squares of those sides (Book IV. Prop. XXVII.).

Scholium. The angle of a regular polygon, like the angle of an equiangular polygon, is determined by the number of its sides (Book I. Prop. XXVI.).

PROPOSITION II. THEOREM.

Any regular polygon may be inscribed in a circle, and circumscribed about one.



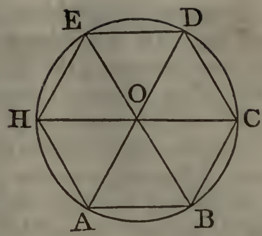
Let ABCDE &c. be a regular polygon: describe a circle through the three points A, B, C, the centre being O, and OP the perpendicular let fall from it, to the middle point of BC: draw AO and OD.

If the quadrilateral OPCD be placed upon the quadrilateral OPBA, they will coincide; for the side OP is common; the angle OPC=OPB, each being a right angle; hence the side PC will apply to its equal PB, and the point C will fall on B: besides, from the nature of the polygon, the angle PCD=PBA; hence CD will take the direction BA; and since CD=BA, the point D will fall on A, and the two quadrilaterals will entirely coincide. The distance OD is therefore equal to AO; and consequently the circle which passes through the three points A, B, C, will also pass through the point D. By the same mode of reasoning, it might be shown, that the circle which passes through the three points B, C, D, will also pass through the point E; and so of all the rest: hence the circle which passes through the points A, B, C, passes also through the vertices of all the angles in the polygon, which is therefore inscribed in this circle.

Again, in reference to this circle, all the sides AB, BC, CD, &c. are equal chords; they are therefore equally distant from the centre (Book III. Prop. VIII.): hence, if from the point O with the distance OP, a circle be described, it will touch the side BC, and all the other sides of the polygon, each in its middle point, and the circle will be inscribed in the polygon, or the polygon described about the circle.

Scholium 1. The point O, the common centre of the inscribed and circumscribed circles, may also be regarded as the centre of the polygon; and upon this principle the angle AOB is called *the angle at the centre*, being formed by two radii drawn to the extremities of the same side AB.

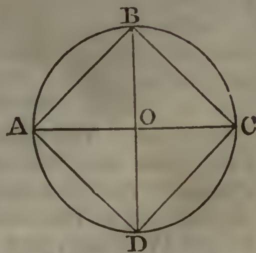
Since all the chords AB, BC, CD, &c. are equal, all the angles at the centre must evidently be equal likewise; and therefore the value of each will be found by dividing four right angles by the number of sides of the polygon.



Scholium 2. To inscribe a regular polygon of a certain number of sides in a given circle, we have only to divide the circumference into as many equal parts as the polygon has sides: for the arcs being equal, the chords AB, BC, CD, &c. will also be equal; hence likewise the triangles AOB, BOC, COD, must be equal, because the sides are equal each to each; hence all the angles ABC, BCD, CDE, &c. will be equal; hence the figure ABCDEH, will be a regular polygon.

PROPOSITION III. PROBLEM.

To inscribe a square in a given circle.

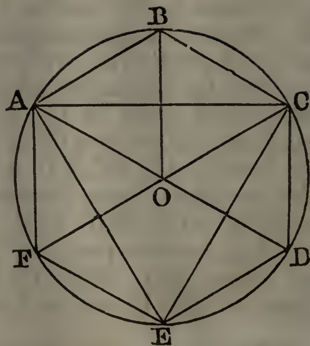


Draw two diameters AC, BD, cutting each other at right angles; join their extremities A, B, C, D: the figure ABCD will be a square. For the angles AOB, BOC, &c. being equal, the chords AB, BC, &c. are also equal: and the angles ABC, BCD, &c. being in semicircles, are right angles.

Scholium. Since the triangle BCO is right angled and isosceles, we have BC : BO : : $\sqrt{2}$: 1 (Book IV. Prop. XI. Cor. 4.); hence *the side of the inscribed square is to the radius, as the square root of* 2, *is to unity.*

PROPOSITION IV. PROBLEM.

In a given circle, to inscribe a regular hexagon ana an equilateral triangle.



Suppose the problem solved, and that AB is a side of the inscribed hexagon; the radii AO, OB being drawn, the triangle AOB will be equilateral.

For, the angle AOB is the sixth part of four right angles; therefore, taking the right angle for unity, we shall have $AOB=\frac{4}{6}=\frac{2}{3}$: and the two other angles ABO, BAO, of the same triangle, are together equal to $2-\frac{2}{3}=\frac{4}{3}$; and being mutually equal, each of them must be equal to $\frac{2}{3}$; hence the triangle ABO is equilateral; therefore the side of the inscribed hexagon is equal to the radius.

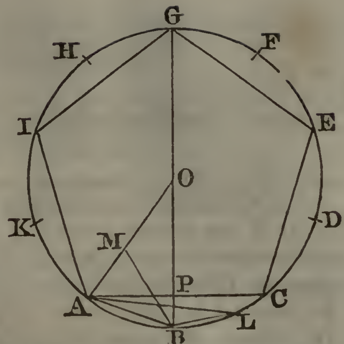
Hence to inscribe a regular hexagon in a given circle, the radius must be applied six times to the circumference; which will bring us round to the point of beginning.

And the hexagon ABCDEF being inscribed, the equilateral triangle ACE may be formed by joining the vertices of the alternate angles.

Scholium. The figure ABCO is a parallelogram and even a rhombus, since $AB=BC=CO=AO$; hence the sum of the squares of the diagonals AC^2+BO^2 is equivalent to the sum of the squares of the sides, that is, to $4AB^2$, or $4BO^2$ (Book IV. Prop XIV. Cor.): and taking away BO^2 from both, there will remain $AC^2=3BO^2$; hence $AC^2 : BO^2 :: 3 : 1$, or $AC : BO :: \sqrt{3} : 1$; hence *the side of the inscribed equilateral triangle is to the radius as the square root of three is to unity.*

PROPOSITION V. PROBLEM.

In a given circle, to inscribe a regular decagon; then a pentagon, and also a regular polygon of fifteen sides.



Divide the radius AO in extreme and mean ratio at the point M (Book IV. Prob. IV.); take the chord AB equal to OM the greater segment; AB will be the side of the regular decagon, and will require to be applied ten times to the circumference.

For, drawing MB, we have by construction, AO : OM : : OM : AM; or, since AB =OM, AO : AB : : AB : AM; since the triangles ABO, AMB, have a common angle A, included between proportional sides, they are similar (Book IV. Prop. XX.). Now the triangle OAB being isosceles, AMB must be isosceles also, and AB=BM; but AB=OM; hence also MB=OM; hence the triangle BMO is isosceles.

Again, the angle AMB being exterior to the isosceles triangle BMO, is double of the interior angle O (Book I. Prop. XXV. Cor. 6.): but the angle AMB=MAB; hence the triangle OAB is such, that each of the angles OAB or OBA, at its base, is double of O, the angle at its vertex; hence the three angles of the triangle are together equal to five times the angle O, which consequently is the fifth part of the two right angles, or the tenth part of four; hence the arc AB is the tenth part of the circumference, and the chord AB is the side of the regular decagon.

2d. By joining the alternate corners of the regular decagon, the pentagon ACEGI will be formed, also regular.

3d. AB being still the side of the decagon, let AL be the side of a hexagon; the arc BL will then, with reference to the whole circumference, be $\frac{1}{6}-\frac{1}{10}$, or $\frac{1}{15}$; hence the chord BL will be the side of the regular polygon of fifteen sides, or pentedecagon. It is evident also, that the arc CL is the third of CB.

Scholium. Any regular polygon being inscribed, if the arcs subtended by its sides be severally bisected, the chords of those semi-arcs will form a new regular polygon of double the number of sides: thus it is plain, that the square will enable us to inscribe successively regular polygons of 8, 16, 32, &c. sides. And in like manner, by means of the hexagon, regular polygons of 12, 24, 48, &c. sides may be inscribed; by means of the decagon, polygons of 20, 40, 80, &c. sides; by means of the pentedecagon, polygons of 30, 60, 120, &c. sides.

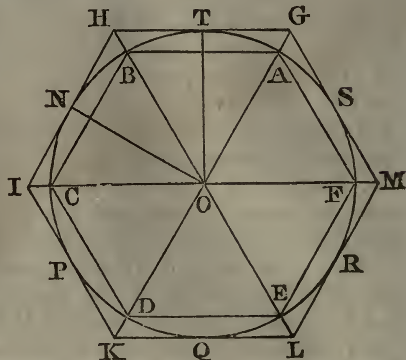
It is further evident, that any of the inscribed polygons will be less than the inscribed polygon of double the number of sides, since a part is less than the whole.

PROPOSITION VI. PROBLEM.

A regular inscribed polygon being given, to circumscribe a similar polygon about the same circle.

Let CBAFED be a regular polygon.

At T, the middle point of the arc AB, apply the tangent GH, which will be parallel to AB (Book III. Prop. X.); do the same at the middle point of each of the arcs BC, CD, &c.; these tangents, by their intersections, will form the regular circumscribed polygon GHIK &c. similar to the one inscribed.



Since T is the middle point of the arc BTA, and N the middle point of the equal arc BNC, it follows, that BT=BN; or that the vertex B of the inscribed polygon, is at the middle point of the arc NBT. Draw OH. The line OH will pass through the point B.

For, the right angled triangles OTH, OHN, having the common hypothenuse OH, and the side OT=ON, must be equal (Book I. Prop. XVII.), and consequently the angle TOH=HON, wherefore the line OH passes through the middle point B of the arc TN. For a like reason, the point I is in the prolongation of OC; and so with the rest.

But, since GH is parallel to AB, and HI to BC, the angle GHI=ABC (Book I. Prop. XXIV.); in like manner HIK=BCD; and so with all the rest: hence the angles of the circumscribed polygon are equal to those of the inscribed one. And further, by reason of these same parallels, we have GH : AB : : OH : OB, and HI : BC : : OH : OB; therefore GH : AB : : HI : BC. But AB=BC, therefore GH=HI. For the same reason, HI=IK, &c.; hence the sides of the circumscribed polygon are all equal; hence this polygon is regular and similar to the inscribed one.

Cor. 1. Reciprocally, if the circumscribed polygon GHIK &c. were given, and the inscribed one ABC &c. were required to be deduced from it, it would only be necessary to draw from the angles G, H, I, &c. of the given polygon, straight lines OG, OH, &c. meeting the circumference in the points A, B, C, &c.; then to join those points by the chords AB, BC, &c.; this would form the inscribed polygon. An easier solution of this problem would be simply to join the points of contact T, N, P, &c. by the chords TN, NP, &c. which likewise would form an inscribed polygon similar to the circumscribed one.

Cor. 2. Hence we may circumscribe about a circle any regular polygon, which can be inscribed within it, and conversely.

Cor. 3. It is plain that $NH+HT=HT+TG=HG$, one of the equal sides of the polygon.

PROPOSITION VII. PROBLEM.

A circle and regular circumscribed polygon being given, it is required to circumscribe the circle by another regular polygon having double the number of sides.

Let the circle whose centre is P, be circumscribed by the square CDEG: it is required to find a regular circumscribed octagon.

Bisect the arcs AH, HB, BF, FA, and through the middle points *c*, *d*, *a*, *b*, draw tangents to the circle, and produce them till they meet the sides of the square: then will the figure A*p*H*d*B &c. be a regular octagon.

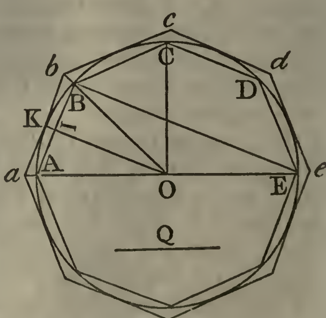
For, having drawn P*d*, P*a*, let the quadrilateral P*dg*B, be applied to the quadrilateral PB*fa*, so that PB shall fall on PB. Then, since the angle *d*PB is equal to the angle BP*a*, each being half a right angle, the line P*d* will fall on its equal P*a*, and the point *d* on the point *a*. But the angles P*dg*, P*af*, are right angles (Book III. Prop. IX.); hence the line *dg* will take the direction *af*. The angles PB*g*, PB*f*, are also right angles; hence B*g* will take the direction B*f*; therefore, the two quadrilaterals will coincide, and the point *g* will fall at *f*; hence, $Bg=Bf$, $dg=af$, and the angle $dgB=Bfa$. By applying in a similar manner, the quadrilaterals PB*fa*, PF*ha*, it may be shown, that $af=ah$, $fB=Fh$, and the angle $Bfa=ahF$. But since the two tangents *fa*, *f*B, are equal (Book III. Prob. XIV. Sch.), it follows that fh, which is twice fa, is equal to fg, which is twice fB.

In a similar manner it may be shown that $hf=hi$, and the angle Fit=Fha, or that any two sides or any two angles of the octagon are equal: hence the octagon is a regular polygon (Def.). The construction which has been made in the case of the square and the octagon, is equally applicable to other polygons.

Cor It is evident that the circumscribed square is greater than the circumscribed octagon by the four triangles, Cnp, kDg, hEf, Git; and if a regular polygon of sixteen sides be circumscribed about the circle, we may prove in a similar way, that the figure having the greatest number of sides will be the least; and the same may be shown, whatever be the number of sides of the polygons: hence, in general, *any circumscribed regular polygon, will be greater than a circumscribed regular polygon having double the number of sides.*

PROPOSITION VIII. THEOREM.

Two regular polygons, of the same number of sides, can always be formed, the one circumscribed about a circle, the other inscribed in it, which shall differ from each other by less than any assignable surface.



Let Q be the side of a square less than the given surface. Bisect AC, a fourth part of the circumference, and then bisect the half of this fourth, and proceed in this manner, always bisecting one of the arcs formed by the last bisection, until an arc is found whose chord AB is less than Q. As this arc will be an exact part of the circumference, if we apply chords AB, BC, CD, &c. each equal to AB, the last will terminate at A, and there will be formed a regular polygon ABCDE &c. in the circle.

Next, describe about the circle a similar polygon $abcde$ &c. (Prop. VI.): the difference of these two polygons will be less than the square of Q.

For, from the points a and b, draw the lines aO, bO, to the centre O: they will pass through the points A and B, as was shown in Prop. VI. Draw also OK to the point of contact K: it will bisect AB in I, and be perpendicular to it (Book III. Prop. VI. Sch.). Produce AO to E, and draw BE.

Let P represent the circumscribed polygon, and p the inscribed polygon: then, since the triangles aOb, AOB, are like parts of P and p, we shall have

$$aOb : AOB :: P : p \text{ (Book II. Prop. XI.)}:$$

But the triangles being similar,

$$aOb : AOB :: Oa^2 : OA^2, \text{ or } OK^2.$$

Hence, $\quad P : p :: Oa^2 : OK^2.$

Again, since the triangles OaK, EAB are similar, having their sides respectively parallel,

$$Oa^2 : OK^2 :: AE^2 : EB^2, \text{ hence,}$$
$$P : p :: AE^2 : EB^2, \quad \text{or by division,}$$
$$P : P-p :: AE^2 : AE^2-EB^2, \text{ or } AB^2.$$

But P is less than the square described on the diameter AE (Prop. VII. Cor.); therefore $P-p$ is less than the square described on AB; that is, less than the given square on Q: hence the difference between the circumscribed and inscribed polygons may always be made less than a given surface.

Cor. 1. A circumscribed regular polygon, having a given number of sides, is greater than the circle, because the circle makes up but a part of the polygon: and for a like reason, the inscribed polygon is less than the circle. But by increasing the number of sides of the circumscribed polygon, the polygon is diminished (Prop. VII. Cor.), and therefore approaches to an equality with the circle; and as the number of sides of the inscribed polygon is increased, the polygon is increased (Prop. V. Sch.), and therefore approaches to an equality with the circle.

Now, *if the number of sides of the polygons be indefinitely increased, the length of each side will be indefinitely small, and the polygons will ultimately become equal to each other, and equal also to the circle.*

For, if they are not ultimately equal, let D represent their smallest difference.

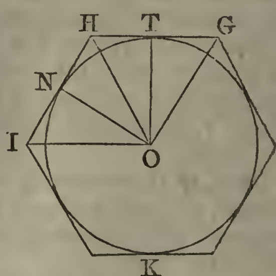
Now, it has been proved in the proposition, that the difference between the circumscribed and inscribed polygons, can be made less than any assignable quantity: that is, less than D: hence the difference between the polygons is equal to D, and less than D at the same time, which is absurd: therefore, the polygons are ultimately equal. But when they are equal to each other, each must also be equal to the circle, since the circumscribed polygon cannot fall within the circle, nor the inscribed polygon without it.

Cor. 2. Since the circumscribed polygon has the same number of sides as the corresponding inscribed polygon, and since the two polygons are regular, they will be similar (Prop. I.); and therefore when they become equal, they will exactly coincide, and have a common perimeter. But as the sides of the circumscribed polygon cannot fall within the circle, nor the sides of the inscribed polygon without it, it follows *that the perimeters of the polygons will unite on the circumference of the circle, and become equal to it.*

Cor. 3. When the number of sides of the inscribed polygon is indefinitely increased, and the polygon coincides with the circle, the line OI, drawn from the centre O, perpendicular to the side of the polygon, will become a radius of the circle, and any portion of the polygon, as ABCO, will become the sector OAKBC, and the part of the perimeter AB+BC, will become the arc AKBC.

PROPOSITION IX. THEOREM.

The area of a regular polygon is equal to its perimeter, multiplied by half the radius of the inscribed circle.

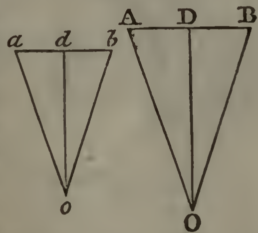


Let there be the regular polygon GHIK, and ON, OT, radii of the inscribed circle. The triangle GOH will be measured by $GH \times \frac{1}{2}OT$; the triangle OHI, by $HI \times \frac{1}{2}ON$: but $ON = OT$; hence the two triangles taken together will be measured by $(GH+HI) \times \frac{1}{2}OT$. And, by continuing the same operation for the other triangles, it will appear that the sum of them all, or the whole polygon, is measured by the sum of the bases GH, HI, &c. or the perimeter of the polygon, multiplied into $\frac{1}{2}OT$, or half the radius of the inscribed circle.

Scholium. The radius OT of the inscribed circle is nothing else than the perpendicular let fall from the centre on one of the sides: it is sometimes named the *apothem* of the polygon.

PROPOSITION X. THEOREM.

The perimeters of two regular polygons, having the same number of sides, are to each other as the radii of the circumscribed circles, and also, as the radii of the inscribed circles; and their areas are to each other as the squares of those radii.



Let AB be the side of the one polygon, O the centre, and consequently OA the radius of the circumscribed circle, and OD, perpendicular to AB, the radius of the inscribed circle; let *ab*, in like manner, be a side of the other polygon, *o* its centre, *oa* and *od* the radii of the circumscribed and the inscribed circles. The perimeters of the two polygons are to each other as the sides AB and *ab* (Book IV. Prop. XXVII.): but the angles A and *a* are equal, being each half of the angle of the polygon; so also are the angles B and *b*; hence the triangles ABO, *abo* are similar, as are likewise the right angled triangles ADO, *ado*; hence AB : *ab* : : AO : *ao* : : DO : *do*; hence the perimeters of the polygons are to each other as the radii AO, *ao* of the circumscribed circles, and also, as the radii DO, *do* of the inscribed circles.

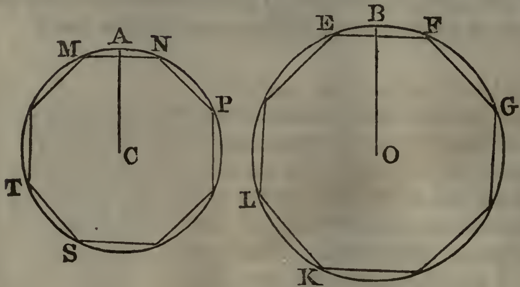
The surfaces of these polygons are to each other as the squares of the homologous sides AB, *ab*; they are therefore likewise to each other as the squares of AO, *ao*, the radii of the circumscribed circles, or as the squares of OD, *od*, the radii of the inscribed circles.

PROPOSITION XI. THEOREM.

The circumferences of circles are to each other as their radii, and the areas are to each other as the squares of their radii.

Let us designate the circumference of the circle whose radius is CA by *circ.* CA; and its area, by *area* CA: it is then to be shown that

$$circ.\ CA : circ.\ OB :: CA : OB, \text{ and that}$$
$$area\ CA : area\ OB :: CA^2 : OB^2$$



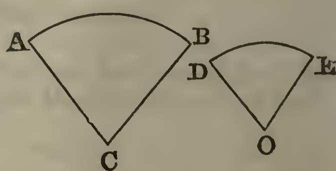
Inscribe within the circles two regular polygons of the same number of sides. Then, whatever be the number of sides, their perimeters will be to each other as the radii CA and OB (Prop. X.). Now, if the arcs subtending the sides of the polygons be continually bisected, until the number of sides of the polygons shall be indefinitely increased, the perimeters of the polygons will become equal to the circumferences of the circumscribed circles (Prop. VIII. Cor. 2.), and we shall have

$$circ.\ CA : circ.\ OB :: CA : OB.$$

Again, the areas of the inscribed polygons are to each other as CA^2 to OB^2 (Prop. X.). But when the number of sides of the polygons is indefinitely increased, the areas of the polygons become equal to the areas of the circles, each to each, (Prop. VIII. Cor. 1.); hence we shall have

$$area\ CA : area\ OB :: CA^2 : OB^2.$$

Cor. The similar arcs AB, DE are to each other as their radii AC, DO; and the similar sectors ACB, DOE, are to each other as the squares of their radii.



For, since the arcs are similar, the angle C is equal to the angle O (Book IV. Def. 3.); but C is to four right angles, as the arc AB is to the whole circumference described with the radius AC (Book III. Prop. XVII.); and O is to the four right angles, as the arc DE is to the circumference described with the radius OD: hence the arcs AB, DE, are to each other as the circumferences of which they form part: but these circumferences are to each other as their radii AC, DO; hence

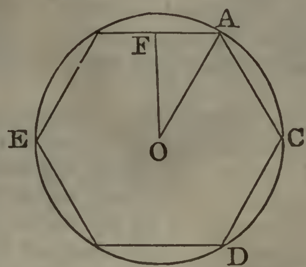
$$arc\ AB : arc\ DE :: AC : DO.$$

For a like reason, the sectors ACB, DOE are to each other as the whole circles; which again are as the squares of their radii; therefore

$$sect.\ ACB : sect.\ DOE :: AC^2 : DO^2.$$

PROPOSITION XII. THEOREM.

The area of a circle is equal to the product of its circumference by half the radius.



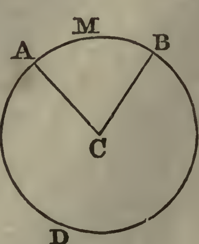
Let ACDE be a circle whose centre is O and radius OA: then will

$area\ OA = \frac{1}{2}OA \times circ.\ OA.$

For, inscribe in the circle any regular polygon, and draw OF perpendicular to one of its sides. Then the area of the polygon will be equal to $\frac{1}{2}$OF, multiplied by the perimeter (Prop. IX.). Now, let the number of sides of the polygon be indefinitely increased by continually bisecting the arcs which subtend the sides: the perimeter will then become equal to the circumference of the circle, the perpendicular OF will become equal to OA, and the area of the polygon to the area of the circle (Prop. VIII. Cor. 1. & 3.). But the expression for the area will then become

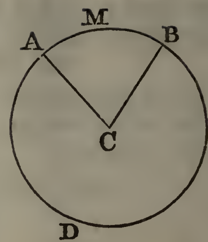
$$area\ OA = \tfrac{1}{2}OA \times circ.\ OA:$$

consequently, the area of a circle is equal to the product of half the radius into the circumference.



Cor. 1. The area of a sector is equal to the arc of that sector multiplied by half its radius.

For, the sector ACB is to the whole circle as the arc AMB is to the whole circumference ABD (Book III. Prop. XVII. Sch. 2.), or as AMB $\times \frac{1}{2}$AC is to ABD $\times \frac{1}{2}$AC. But the whole circle is equal to ABD $\times \frac{1}{2}$AC; hence the sector ACB is measured by AMB $\times \frac{1}{2}$AC



Cor. 2. Let the circumference of the circle whose diameter is unity, be denoted by π: then, because circumferences are to each other as their radii or diameters, we shall have the diameter 1 to its circumference π, as the diameter 2CA is to the circumference whose radius is CA, that is, $1 : \pi :: 2CA :$ *circ.* CA, therefore *circ.* $CA = \pi \times 2CA$. Multiply both terms by $\frac{1}{2}CA$; we have $\frac{1}{2}CA \times$ *circ.* $CA = \pi \times CA^2$, or *area* $CA = \pi \times CA^2$: hence the area of a circle is equal to the product of the square of its radius by the constant number π, which represents the circumference whose diameter is 1, or the ratio of the circumference to the diameter.

In like manner, the area of the circle, whose radius is OB, will be equal to $\pi \times OB^2$; but $\pi \times CA^2 : \pi \times OB^2 :: CA^2 : OB^2$; hence *the areas of circles are to each other as the squares of their radii*, which agrees with the preceding theorem.

Scholium. We have already observed, that the problem of the quadrature of the circle consists in finding a square equal in surface to a circle, the radius of which is known. Now it has just been proved, that a circle is equivalent to the rectangle contained by its circumference and half its radius; and this rectangle may be changed into a square, by finding a mean proportional between its length and its breadth (Book IV. Prob. III.). To square the circle, therefore, is to find the circumference when the radius is given; and for effecting this, it is enough to know the ratio of the circumference to its radius, or its diameter.

Hitherto the ratio in question has never been determined except approximatively; but the approximation has been carried so far, that a knowledge of the exact ratio would afford no real advantage whatever beyond that of the approximate ratio. Accordingly, this problem, which engaged geometers so deeply, when their methods of approximation were less perfect, is now degraded to the rank of those idle questions, with which no one possessing the slightest tincture of geometrical science will occupy any portion of his time.

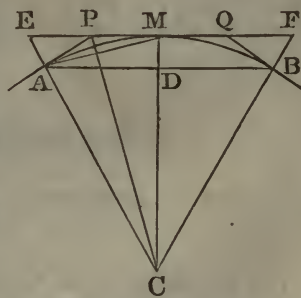
Archimedes showed that the ratio of the circumference to the diameter is included between $3\frac{10}{70}$ and $3\frac{10}{71}$; hence $3\frac{1}{7}$ or $\frac{22}{7}$ affords at once a pretty accurate approximation to the number above designated by π; and the simplicity of this first approximation has brought it into very general use. *Metius*, for the same number, found the much more accurate value $\frac{355}{113}$. At last the value of π, developed to a certain order of decimals, was found by other calculators to be 3.1415926535897932, &c.:

and some have had patience enough to continue these decimals to the hundred and twenty-seventh, or even to the hundred and fortieth place. Such an approximation is evidently equivalent to perfect correctness: the root of an imperfect power is in no case more accurately known.

The following problem will exhibit one of the simplest elementary methods of obtaining those approximations.

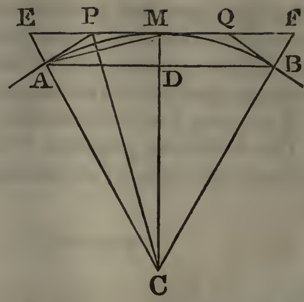
PROPOSITION XIII. PROBLEM.

The surface of a regular inscribed polygon, and that of a similar polygon circumscribed, being given; to find the surfaces of the regular inscribed and circumscribed polygons having double the number of sides.



Let AB be a side of the given inscribed polygon; EF, parallel to AB, a side of the circumscribed polygon; C the centre of the circle. If the chord AM and the tangents AP, BQ, be drawn, AM will be a side of the inscribed polygon, having twice the number of sides; and $AP+PM=2PM$ or PQ, will be a side of the similar circumscribed polygon (Prop. VI. Cor. 3.). Now, as the same construction will take place at each of the angles equal to ACM, it will be sufficient to consider ACM by itself, the triangles connected with it being evidently to each other as the whole polygons of which they form part. Let A, then, be the surface of the inscribed polygon whose side is AB, B that of the similar circumscribed polygon; A' the surface of the polygon whose side is AM, B' that of the similar circumscribed polygon: A and B are given; we have to find A' and B'.

First. The triangles ACD, ACM, having the common vertex A, are to each other as their bases CD, CM; they are likewise to each other as the polygons A and A', of which they form part: hence $A : A' :: CD : CM$. Again, the triangles CAM, CME, having the common vertex M, are to each other as their bases CA, CE; they are likewise to each other as the polygons A' and B of which they form part; hence $A' : B :: CA : CE$. But since AD and ME are parallel, we have $CD : CM :: CA : CE$; hence $A : A' :: A' : B$; hence the polygon A', one of those required, is a mean proportional between the two given polygons A and B and consequently $A' = \sqrt{A \times B}$.



Secondly. The altitude CM being common, the triangle CPM is to the triangle CPE as PM is to PE; but since CP bisects the angle MCE, we have PM : PE : : CM : CE (Book IV. Prop. XVII.) : : CD : CA : : A : A′ : hence CPM : CPE : : A : A′; and consequently CPM : CPM + CPE or CME : : A : A + A′. But CMPA, or 2CMP, and CME are to each other as the polygons B′ and B, of which they form part: hence B′ : B : : 2A : A + A′. Now A′ has been already determined; this new proportion will serve for determining B′, and give us $B' = \frac{2A.B}{A+A'}$; and thus by means of the polygons A and B it is easy to find the polygons A′ and B′, which shall have double the number of sides.

PROPOSITION XIV. PROBLEM.

To find the approximate ratio of the circumference to the diameter.

Let the radius of the circle be 1; the side of the inscribed square will be $\sqrt{2}$ (Prop. III. Sch.), that of the circumscribed square will be equal to the diameter 2; hence the surface of the inscribed square is 2, and that of the circumscribed square is 4. Let us therefore put A=2, and B=4; by the last proposition we shall find the inscribed octagon $A' = \sqrt{8} = 2.8284271$, and the circumscribed octagon $B' = \frac{16}{2+\sqrt{8}} = 3.3137085$. The inscribed and the circumscribed octagons being thus determined, we shall easily, by means of them, determine the polygons having twice the number of sides. We have only in this case to put A=2.8284271, B=3.3137085; we shall find $A' = \sqrt{A.B} = 3.0614674$, and $B' = \frac{2\,A.B}{A+A'} = 3.1825979$. These polygons of 16 sides will in their turn enable us to find the polygons of 32; and the process may be continued, till there remains no longer any difference between the inscribed and the circumscribed polygon, at least so far as that place of decimals where the computation stops, and so far as the seventh place, in this example. Being arrived at this point, we shall infer that the last result expresses the area of the circle, which, since it must always lie between the inscribed and the circumscribed polygon, and since those polygons agree as far as a certain place of decimals, must also agree with both as far as the same place.

We have subjoined the computation of those polygons, carried on till they agree as far as the seventh place of decimals.

Number of sides	Inscribed polygon.	Circumscribed polygon.
4	2.0000000	4.0000000
8	2.8284271	3.3137085
16	3.0614674	3.1825979
32	3.1214451	3.1517249
64	3.1365485	3.1441184
128	3.1403311	3.1422236
256	3.1412772	3.1417504
512	3.1415138	3.1416321
1024	3.1415729	3.1416025
2048	3.1415877	3.1415951
4096	3.1415914	3.1415933
8192	3.1415923	3.1415928
16384	3.1415925	3.1415927
32768	3.1415926	3.1415926

The area of the circle, we infer therefore, is equal to 3.1415926. Some doubt may exist perhaps about the last decimal figure, owing to errors proceeding from the parts omitted; but the calculation has been carried on with an additional figure, that the final result here given might be absolutely correct even to the last decimal place.

Since the area of the circle is equal to half the circumference multiplied by the radius, the half circumference must be 3.1415926, when the radius is 1; or the whole circumference must be 3.1415926, when the diameter is 1: hence the ratio of the circumference to the diameter, formerly expressed by π, is equal to 3.1415926. The number 3.1416 is the one generally used

BOOK VI.

PLANES AND SOLID ANGLES.

Definitions.

1. A straight line is *perpendicular to a plane*, when it is perpendicular to all the straight lines which pass through its *foot* in the plane. Conversely, the plane is perpendicular to the line.

The *foot* of the perpendicular is the point in which the perpendicular line meets the plane.

2. A line is *parallel to a plane*, when it cannot meet that plane, to whatever distance both be produced. Conversely, the plane is parallel to the line.

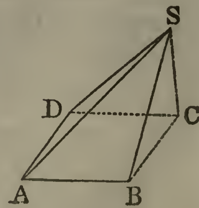
3. Two *planes* are *parallel* to each other, when they cannot meet, to whatever distance both be produced.

4. The *angle* or mutual *inclination of two planes* is the quantity, greater or less, by which they separate from each other; this angle is measured by the angle contained between two lines, one in each plane, and both perpendicular to the common intersection at the same point.

This angle may be acute, obtuse, or a right angle.

If it is a right angle, the two *planes* are perpendicular to each other.

5. A *solid angle* is the angular space included between several planes which meet at the same point.



Thus, the solid angle S, is formed by the union of the planes ASB, BSC, CSD, DSA.

Three planes at least, are requisite to form a solid angle.

PROPOSITION I. THEOREM.

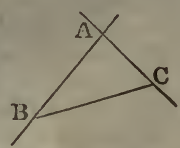
A straight line cannot be partly in a plane, and partly out of it.

For, by the definition of a plane, when a straight line has two points common with a plane, it lies wholly in that plane.

Scholium. To discover whether a surface is plane, it is necessary to apply a straight line in different ways to that surface, and ascertain if it touches the surface throughout its whole extent.

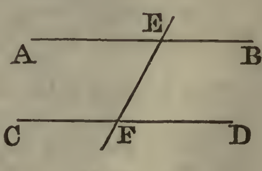
PROPOSITION II. THEOREM.

Two straight lines, which intersect each other, lie in the same plane, and determine its position.



Let AB, AC, be two straight lines which intersect each other in A; a plane may be conceived in which the straight line AB is found; if this plane be turned round AB, until it pass through the point C, then the line AC, which has two of its points A and C, in this plane, lies wholly in it; hence the position of the plane is determined by the single condition of containing the two straight lines AB, AC.

Cor. 1. A triangle ABC, or three points A, B, C, not in a straight line, determine the position of a plane.

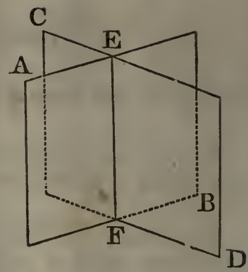


Cor. 2. Hence also two parallels AB, CD, determine the position of a plane; for, drawing the secant EF, the plane of the two straight lines AE, EF, is that of the parallels AB, CD.

PROPOSITION III. THEOREM.

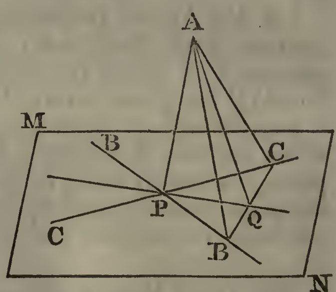
If two planes cut each other, their common intersection will be a straight line.

Let the two planes AB, CD, cut each other. Draw the straight line EF, joining any two points E and F in the common section of the two planes. This line will lie wholly in the plane AB, and also wholly in the plane CD (Book I. Def. 6.): therefore it will be n both planes at once, and consequently is their common intersection.



PROPOSITION IV. THEOREM.

If a straight line be perpendicular to two straight lines at their point of intersection, it will be perpendicular to the plane of those lines.



Let MN be the plane of the two lines BB, CC, and let AP be perpendicular to them at their point of intersection P; then will AP be perpendicular to every line of the plane passing through P, and consequently to the plane itself (Def. 1.).

Through P, draw in the plane MN, any straight line as PQ, and through any point of this line, as Q, draw BQC, so that BQ shall be equal to QC (Book IV. Prob. V.); draw AB, AQ, AC.

The base BC being divided into two equal parts at the point Q, the triangle BPC will give (Book IV. Prop. XIV.),

$$PC^2+PB^2=2PQ^2+2QC^2.$$

The triangle BAC will in like manner give,

$$AC^2+AB^2=2AQ^2+2QC^2.$$

Taking the first equation from the second, and observing that the triangles APC, APB, which are both right angled at P, give

$$AC^2-PC^2=AP^2, \text{ and } AB^2-PB^2=AP^2;$$

we shall have

$$AP^2+AP^2=2AQ^2-2PQ^2.$$

Therefore, by taking the halves of both, we have

$$AP^2=AQ^2-PQ^2, \text{ or } AQ^2=AP^2+PQ^2;$$

hence the triangle APQ is right angled at P; hence AP is perpendicular to PQ.

Scholium. Thus it is evident, not only that a straight line may be perpendicular to all the straight lines which pass through its foot in a plane, but that it always must be so, whenever it is perpendicular to two straight lines drawn in the plane; which proves the first Definition to be accurate.

Cor. 1. The perpendicular AP is shorter than any oblique line AQ; therefore it measures the true distance from the point A to the plane MN.

Cor. 2. At a given point P on a plane, it is impossible to erect more than one perpendicular to that plane; for if there could be two perpendiculars at the same point P, draw through these two perpendiculars a plane, whose intersection with the plane MN is PQ; then these two perpendiculars would be perpendicular to the line PQ, at the same point, and in the same plane, which is impossible (Book I. Prop. XIV. Sch.).

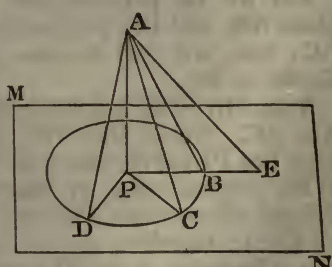
It is also impossible to let fall from a given point out of a plane two perpendiculars to that plane; for let AP, AQ, be these two perpendiculars, then the triangle APQ would have two right angles APQ, AQP, which is impossible.

PROPOSITION V. THEOREM.

If from a point without a plane, a perpendicular be drawn to the plane, and oblique lines be drawn to different points,

1*st. Any two oblique lines equally distant from the perpendicular will be equal.*

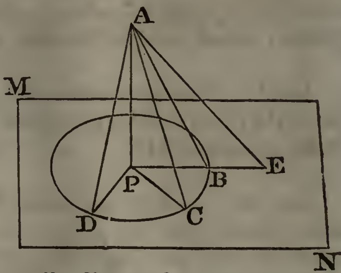
2*d. Of any two oblique lines unequally distant from the perpendicular, the more distant will be the longer.*



Let AP be perpendicular to the plane MN; AB, AC, AD, oblique lines equally distant from the perpendicular, and AE a line more remote: then will AB=AC=AD; and AE will be greater than AD.

For, the angles APB, APC, APD, being right angles, if we suppose the distances PB, PC, PD, to be equal to each other, the triangles APB, APC, APD, will have in each an equal angle contained by two equal sides; therefore they will be equal; hence the hypothenuses, or the oblique lines AB, AC, AD, will be equal to each other. In like manner, if the distance PE is greater than PD or its equal PB, the oblique line AE will evidently be greater than AB, or its equal AD.

Cor. All the equal oblique lines, AB, AC, AD, &c. terminate in the circumference BCD, described from P the foot of the perpendicular as a centre; therefore a point A being given out of a plane, the point P at which the perpendicular let fall from A would meet that plane, may be found by marking upon that plane three points B, C, D, equally distant from the point A, and then finding the centre of the circle which passes through these points; this centre will be P, the point sought.

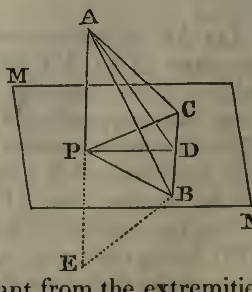


Scholium. The angle ABP is called the *inclination of the oblique line* AB *to the plane* MN; which inclination is evidently equal with respect to all such lines AB, AC, AD, as are equally distant from the perpendicular; for all the triangles ABP, ACP, ADP, &c. are equal to each other.

PROPOSITION VI. THEOREM.

If from a point without a plane, a perpendicular be let fall on the plane, and from the foot of the perpendicular a perpendicular be drawn to any line of the plane, and from the point of intersection a line be drawn to the first point, this latter line will be perpendicular to the line of the plane.

Let AP be perpendicular to the plane NM, and PD perpendicular to BC; then will AD be also perpendicular to BC.



Take DB=DC, and draw PB, PC, AB, AC. Since DB=DC, the oblique line PB=PC: and with regard to the perpendicular AP, since PB=PC, the oblique line AB=AC (Prop. V. Cor.); therefore the line AD has two of its points A and D equally distant from the extremities B and C; therefore AD is a perpendicular to BC, at its middle point D (Book I. Prop. XVI. Cor.).

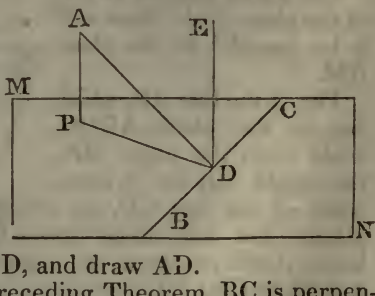
Cor. It is evident likewise, that BC is perpendicular to the plane APD, since BC is at once perpendicular to the two straight lines AD, PD.

Scholium. The two lines AE, BC, afford an instance of two lines which do not meet, because they are not situated in the same plane. The shortest distance between these lines is the straight line PD, which is at once perpendicular to the line AP and to the line BC. The distance PD is the shortest distance between them, because if we join any other two points, such as A and B, we shall have AB>AD, AD>PD; therefore AB>PD.

The two lines AE, CB, though not situated in the same plane, are conceived as forming a right angle with each other, because AE and the line drawn through one of its points parallel to BC would make with each other a right angle. In the same manner, the line AB and the line PD, which represent any two straight lines not situated in the same plane, are supposed to form with each other the same angle, which would be formed by AB and a straight line parallel to PD drawn through one of the points of AB.

PROPOSITION VII. THEOREM.

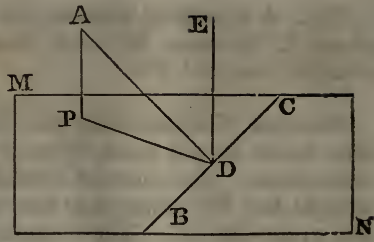
If one of two parallel lines be perpendicular to a plane, the other will also be perpendicular to the same plane.



Let the lines ED, AP, be parallel; if AP is perpendicular to the plane NM, then will ED be also perpendicular to it.

Through the parallels AP, DE, pass a plane; its intersection with the plane MN will be PD; in the plane MN draw BC perpendicular to PD, and draw AD.

By the Corollary of the preceding Theorem, BC is perpendicular to the plane APDE; therefore the angle BDE is a right angle; but the angle EDP is also a right angle, since AP is perpendicular to PD, and DE parallel to AP (Book I. Prop. XX. Cor. 1.); therefore the line DE is perpendicular to the two straight lines DP, DB; consequently it is perpendicular to their plane MN (Prop. IV.)



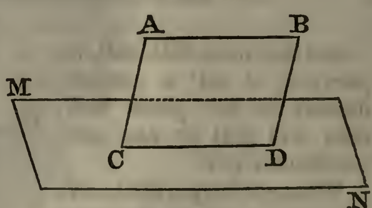
Cor. 1. Conversely, if the straight lines AP, DE, are perpendicular to the same plane MN, they will be parallel; for if they be not so, draw through the point D, a line parallel to AP, this parallel will be perpendicular to the plane MN; therefore through the same point D more than one perpendicular might be erected to the same plane, which is impossible (Prop. IV. Cor. 2.).

Cor. 2. Two lines A and B, parallel to a third C, are parallel to each other; for, conceive a plane perpendicular to the line C; the lines A and B, being parallel to C, will be perpendicular to the same plane; therefore, by the preceding Corollary, they will be parallel to each other.

The three lines are supposed not to be in the same plane; otherwise the proposition would be already known (Book I. Prop. XXII.).

PROPOSITION VIII. THEOREM.

If a straight line is parallel to a straight line drawn in a plane, it will be parallel to that plane.

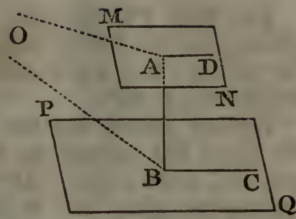


Let AB be parallel to CD of the plane NM; then will it be parallel to the plane NM.

For, if the line AB, which lies in the plane ABDC, could meet the plane MN, this could only be in some point of the line CD, the common intersection of the two planes: but AB cannot meet CD, since they are parallel; hence it will not meet the plane MN; hence it is parallel to that plane (Def. 2.).

PROPOSITION IX. THEOREM.

Two planes which are perpendicular to the same straight line are parallel to each other.

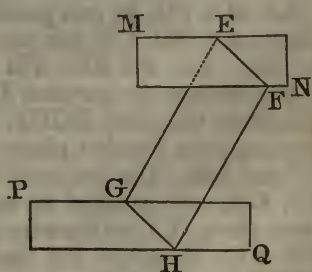


Let the planes NM, QP, be perpendicular to the line AB, then will they be parallel.

For, if they can meet any where, let O be one of their common points, and draw OA, OB ; the line AB which is perpendicular to the plane MN, is perpendicular to the straight line OA drawn through its foot in that plane ; for the same reason AB is perpendicular to BO ; therefore OA and OB are two perpendiculars let fall from the same point O, upon the same straight line; which is impossible (Book I. Prop. XIV.); therefore the planes MN, PQ, cannot meet each other; consequently they are parallel.

PROPOSITION X. THEOREM.

If a plane cut two parallel planes, the lines of intersection will be parallel.

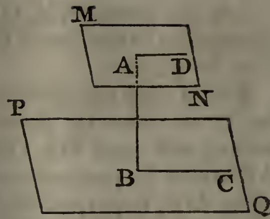


Let the parallel planes NM, QP, be intersected by the plane EH; then will the lines of intersection EF, GH, be parallel.

For, if the lines EF, GH, lying in the same plane, were not parallel, they would meet each other when produced; therefore, the planes MN, PQ, in which those lines lie, would also meet; and hence the planes would not be parallel.

PROPOSITION XI. THEOREM.

If two planes are parallel, a straight line which is perpendicular to one, is also perpendicular to the other.

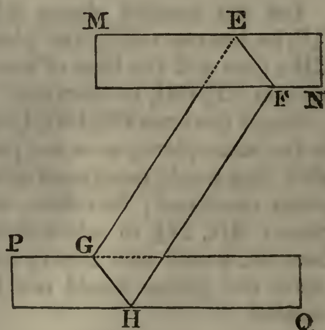


Let MN, PQ, be two parallel planes, and let AB be perpendicular to NM; then will it also be perpendicular to QP.

Having drawn any line BC in the plane PQ, through the lines AB and BC, draw a plane ABC, intersecting the plane MN in AD; the intersection AD will be parallel to BC (Prop. X.); but the line AB, being perpendicular to the plane MN, is perpendicular to the straight line AD; therefore also, to its parallel BC (Book I. Prop. XX. Cor. 1.): hence the line AB being perpendicular to any line BC, drawn through its foot in the plane PQ, is consequently perpendicular to that plane (Def. 1.).

PROPOSITION XII. THEOREM.

The parallels comprehended between two parallel planes are equal.



Let MN, PQ, be two parallel planes, and FH, GE, two parallel lines: then will EG=FH

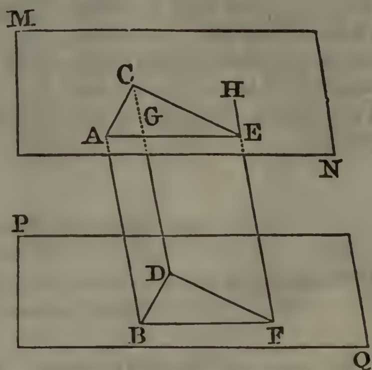
For, through the parallels EG, FH, draw the plane EGHF, intersecting the parallel planes in EF and GH. The intersections EF, GH, are parallel to each other (Prop. X.); so likewise are EG, FH; therefore the figure EGHF is a parallelogram; consequently, EG=FH.

Cor. Hence it follows, that *two parallel planes are every where equidistant:* for, suppose EG were perpendicular to the plane PQ; the parallel FH would also be perpendicular to it (Prop. VII.), and the two parallels would likewise be perpendicular to the plane MN (Prop. XI.); and being parallel, they will be equal, as shown by the Proposition.

PROPOSITION XIII. THEOREM.

If two angles, not situated in the same plane, have their sides parallel and lying in the same direction, those angles will be equal and their planes will be parallel.

Let the angles be CAE and DBF.



Make AC=BD, AE=BF; and draw CE, DF, AB, CD, EF. Since AC is equal and parallel to BD, the figure ABDC is a parallelogram; therefore CD is equal and parallel to AB. For a similar reason, EF is equal and parallel to AB; hence also CD is equal and parallel to EF; hence the figure CEFD is a parallelogram, and the side CE is equal and parallel to DF; therefore the triangles CAE, DBF, have their corresponding sides equal; therefore the angle CAE=DBF.

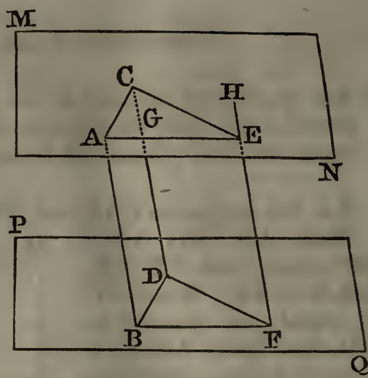
Again, the plane ACE is parallel to the plane BDF. For suppose the plane drawn through the point A, parallel to BDF, were to meet the lines CD, EF, in points different from C and E, for instance in G and H; then, the three lines AB, GD, FH, would be equal (Prop. XII.): but the lines AB, CD, EF, are already known to be equal; hence CD=GD, and FH=EF, which is absurd; hence the plane ACE is parallel to BDF.

Cor. If two parallel planes MN, PQ are met by two other planes CABD, EABF, the angles CAE, DBF, formed by the intersections of the parallel planes will be equal; for, the intersection AC is parallel to BD, and AE to BF (Prop. X.); therefore the angle CAE=DBF.

PROPOSITION XIV. THEOREM.

If three straight lines, not situated in the same plane, are equal and parallel, the opposite triangles formed by joining the extremities of these lines will be equal, and their planes will be parallel

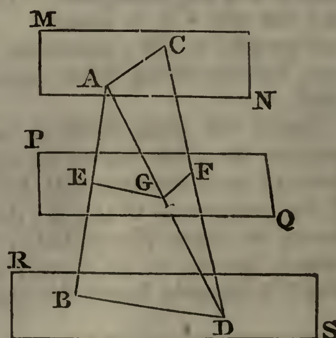
Let AB, CD, EF, be the lines.



Since AB is equal and parallel to CD, the figure ABDC is a parallelogram; hence the side AC is equal and parallel to BD. For a like reason the sides AE, BF, are equal and parallel, as also CE, DF; therefore the two triangles ACE, BDF, are equal; hence, by the last Proposition, their planes are parallel.

PROPOSITION XV. THEOREM.

If two straight lines be cut by three parallel planes, they will be divided proportionally.



Suppose the line AB to meet the parallel planes MN, PQ, RS, at the points A, E, B; and the line CD to meet the same planes at the points C, F, D: we are now to show that

AE : EB : : CF : FD.

Draw AD meeting the plane PQ in G, and draw AC, EG, GF, BD; the intersections EG, BD, of the parallel planes PQ, RS, by the plane ABD, are parallel (Prop. X.); therefore

AE : EB : : AG : GD;

in like manner, the intersections AC, GF, being parallel,

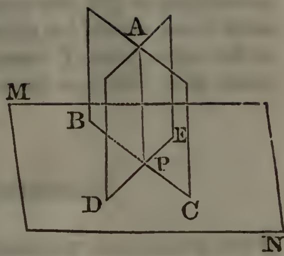
AG : GD : : CF : FD;

the ratio AG : GD is the same in both; hence

AE : EB : : CF : FD.

PROPOSITION XVI. THEOREM.

If a line is perpendicular to a plane, every plane passed through the perpendicular, will also be perpendicular to the plane.



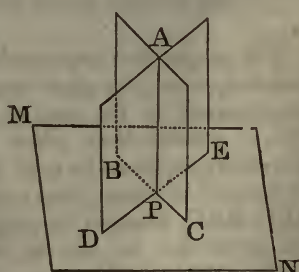
Let AP be perpendicular to the plane NM; then will every plane passing through AP be perpendicular to NM.

Let BC be the intersection of the planes AB, MN; in the plane MN, draw DE perpendicular to BP: then the line AP, being perpendicular to the plane MN, will be perpendicular to each of the two straight lines BC, DE; but the angle APD, formed by the two perpendiculars PA, PD, to the common intersection BP, measures the angle of the two planes AB, MN (Def. 4.); therefore, since that angle is a right angle, the two planes are perpendicular to each other.

Scholium. When three straight lines, such as AP, BP, DP, are perpendicular to each other, each of those lines is perpendicular to the plane of the other two, and the three planes are perpendicular to each other.

PROPOSITION XVII. THEOREM.

If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their common intersection, will be perpendicular to the other plane.



Let the plane AB be perpendicular to NM; then if the line AP be perpendicular to the intersection BC, it will also be perpendicular to the plane NM.

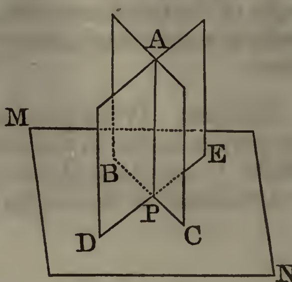
For, in the plane MN draw PD perpendicular to PB; then, because the planes are perpendicular, the angle APD is a right angle; therefore, the line AP is perpendicular to the two straight lines PB, PD; therefore it is perpendicular to their plane MN (Prop. IV.).

Cor. If the plane AB is perpendicular to the plane MN, and if at a point P of the common intersection we erect a perpendicular to the plane MN, that perpendicular will be in the plane AB: for, if not, then, in the plane AB we might draw AP perpendicular to PB the common intersection, and this AP, at the same time, would be perpendicular to the plane MN; therefore at the same point P there would be two perpendiculars to the plane MN, which is impossible (Prop. IV. Cor. 2.).

PROPOSITION XVIII. THEOREM.

If two planes are perpendicular to a third plane, their common intersection will also be perpendicular to the third plane.

Let the planes AB, AD, be perpendicular to NM; then will their intersection AP be perpendicular to NM.

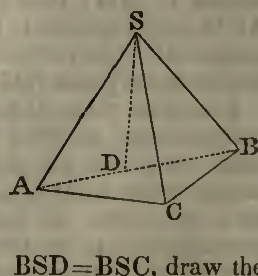


For, at the point P, erect a perpendicular to the plane MN; that perpendicular must be at once in the plane AB and in the plane AD (Prop. XVII. Cor.); therefore it is their common intersection AP.

PROPOSITION XIX. THEOREM.

If a solid angle is formed by three plane angles, the sum of any two of these angles will be greater than the third.

The proposition requires demonstration only when the plane angle, which is compared to the sum of the other two, is greater than either of them. Therefore suppose the solid angle S to be formed by three plane angles ASB, ASC, BSC, whereof the angle ASB is the greatest; we are to show that $ASB < ASC + BSC$.

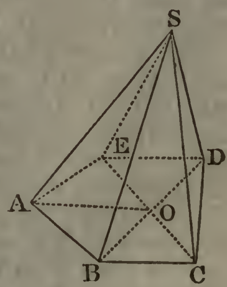


In the plane ASB make the angle $BSD = BSC$, draw the straight line ADB at pleasure; and having taken $SC = SD$ draw AC, BC.

The two sides BS, SD, are equal to the two BS, SC; the angle $BSD = BSC$; therefore the triangles BSD, BSC, are equal; therefore $BD = BC$. But $AB < AC + BC$; taking BD from the one side, and from the other its equal BC, there re mains $AD<AC$. The two sides AS, SD, are equal to the two AS, SC; the third side AD is less than the third side AC; therefore the angle $ASD<ASC$ (Book I. Prop. IX. Sch.). Adding $BSD=BSC$, we shall have $ASD+BSD$ or $ASB<ASC+BSC$.

PROPOSITION XX. THEOREM.

The sum of the plane angles which form a solid angle is always less than four right angles.



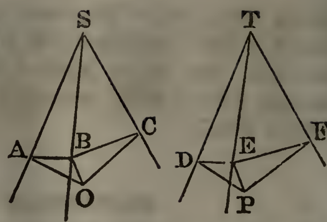
Cut the solid angle S by any plane ABCDE; from O, a point in that plane, draw to the several angles the straight lines AO, OB, OC, OD, OE.

The sum of the angles of the triangles ASB, BSC, &c. formed about the vertex S, is equal to the sum of the angles of an equal number of triangles AOB, BOC, &c. formed about the point O. But at the point B the sum of the angles ABO, OBC, equal to ABC, is less than the sum of the angles ABS, SBC (Prop. XIX.); in the same manner at the point C we have $BCO+OCD<BCS+SCD$; and so with all the angles of the polygon ABCDE: whence it follows, that the sum of all the angles at the bases of the triangles whose vertex is in O, is less than the sum of the angles at the bases of the triangles whose vertex is in S; hence to make up the deficiency, the sum of the angles formed about the point O, is greater than the sum of the angles formed about the point S. But the sum of the angles about the point O is equal to four right angles (Book I. Prop. IV. Sch.); therefore the sum of the plane angles, which form the solid angle S, is less than four right angles.

Scholium. This demonstration is founded on the supposition that the solid angle is convex, or that the plane of no one surface produced can ever meet the solid angle; if it were other wise, the sum of the plane angles would no longer be limited, and might be of any magnitude.

PROPOSITION XXI. THEOREM.

If two solid angles are contained by three plane angles which are equal to each other, each to each, the planes of the equal angles will be equally inclined to each other



Let the angle ASC=DTF, the angle ASB=DTE, and the angle BSC=ETF; then will the inclination of the planes ASC, ASB, be equal to that of the planes DTF, DTE.

Having taken SB at pleasure, draw BO perpendicular to the plane ASC; from the point O, at which the perpendicular meets the plane, draw OA, OC perpendicular to SA, SC; draw AB, BC; next take TE=SB; draw EP perpendicular to the plane DTF; from the point P draw PD, PF, perpendicular respectively to TD, TF; lastly, draw DE, EF.

The triangle SAB is right angled at A, and the triangle TDE at D (Prop. VI.): and since the angle ASB=DTE we have SBA=TED. Likewise SB=TE; therefore the triangle SAB is equal to the triangle TDE; therefore SA=TD, and AB=DE. In like manner, it may be shown, that SC=TF, and BC=EF. That granted, the quadrilateral SAOC is equal to the quadrilateral TDPF: for, place the angle ASC upon its equal DTF; because SA=TD, and SC=TF, the point A will fall on D, and the point C on F; and at the same time, AO, which is perpendicular to SA, will fall on PD which is perpendicular to TD, and in like manner OC on PF; wherefore the point O will fall on the point P, and AO will be equal to DP. But the triangles AOB, DPE, are right angled at O and P; the hypothenuse AB=DE, and the side AO=DP: hence those triangles are equal (Book I. Prop. XVII.); and consequently, the angle OAB=PDE. The angle OAB is the inclination of the two planes ASB. ASC; and the angle PDE is that of the two planes DTE, DTF; hence those two inclinations are equal to each other.

It must, however, be observed, that the angle A of the right angled triangle AOB is properly the inclination of the two planes ASB, ASC, only when the perpendicular BO falls on the same side of SA, with SC; for if it fell on the other side, the angle of the two planes would be obtuse, and the obtuse angle together with the angle A of the triangle OAB would make two right angles. But in the same case, the angle of the two planes TDE, TDF, would also be obtuse, and the obtuse angle together with the angle D of the triangle DPE, would make two right angles; and the angle A being thus always equal to the angle at D, it would follow in the same manner that the inclination of the two planes ASB, ASC, must be equal to that of the two planes TDE, TDF.

Scholium. If two solid angles are contained by three plane angles, respectively equal to each other, and if at the same time the equal or homologous angles are *disposed in the same manner* in the two solid angles, these angles will be equal, and they will coincide when applied the one to the other. We have already seen that the quadrilateral SAOC may be placed upon its equal TDPF; thus placing SA upon TD, SC falls upon TF, and the point O upon the point P. But because the triangles AOB, DPE, are equal, OB, perpendicular to the plane ASC, is equal to PE, perpendicular to the plane TDF; besides, those perdendiculars lie in the same direction; therefore, the point B will fall upon the point E, the line SB upon TE, and the two solid angles will wholly coincide.

This coincidence, however, takes place only when we suppose that the equal plane angles are *arranged in the same manner* in the two solid angles; for if they were *arranged in an inverse order*, or, what is the same, if the perpendiculars OB, PE, instead of lying in the same direction with regard to the planes ASC, DTF, lay in opposite directions, then it would be impossible to make these solid angles coincide with one another. It would not, however, on this account, be less true, as our Theorem states, that the planes containing the equal angles must still be equally inclined to each other; so that the two solid angles would be equal in all their constituent parts, without, however, admitting of superposition. This sort of equality, which is not absolute, or such as admits of superposition, deserves to be distinguished by a particular name: we shall call it *equality by symmetry*.

Thus those two solid angles, which are formed by three plane angles respectively equal to each other, but disposed in an inverse order, will be called *angles equal by symmetry*, or simply *symmetrical angles*.

The same remark is applicable to solid angles, which are formed by more than three plane angles: thus a solid angle, formed by the plane angles A, B, C, D, E, and another solid angle, formed by the same angles in an inverse order A, E, D, C, B, may be such that the planes which contain the equal angles are equally inclined to each other. Those two solid angles, are likewise equal, without being capable of superposition, and are called *solid angles equal by symmetry*, or *symmetrical solid angles*.

Among plane figures, equality by symmetry does not properly exist, all figures which might take this name being absolutely equal, or equal by superposition; the reason of which is, that a plane figure may be inverted, and the upper part taken indiscriminately for the under. This is not the case with solids; in which the third dimension may be taken in two different directions.

BOOK VII.

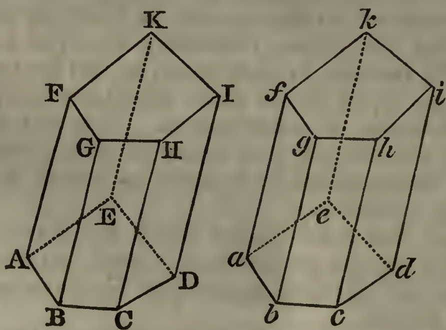
POLYEDRONS.

Definitions.

1. The name *solid polyedron*, or simple *polyedron*, is given to every solid terminated by planes or plane faces; which planes, it is evident, will themselves be terminated by straight lines.

2. The common intersection of two adjacent faces of a polyedron is called the *side*, or *edge* of the polyedron.

3. The *prism* is a solid bounded by several parallelograms, which are terminated at both ends by equal and parallel polygons.



To construct this solid, let ABCDE be any polygon; then if in a plane parallel to ABCDE, the lines FG, GH, HI, &c. be drawn equal and parallel to the sides AB, BC, CD, &c. thus forming the polygon FGHIK equal to ABCDE; if in the next place, the vertices of the angles in the one plane be joined with the homologous vertices in the other, by straight lines, AF, BG, CH, &c. the faces ABGF, BCHG, &c. will be parallelograms, and ABCDE-K, the solid so formed, will be a prism.

4. The equal and parallel polygons ABCDE, FGHIK, are called the *bases of the prism;* the parallelograms taken together constitute the *lateral* or *convex surface of the prism;* the equal straight lines AF, BG, CH, &c. are called the *sides*, or *edges of the prism.*

5. The *altitude of a prism* is the distance between its two bases, or the perpendicular drawn from a point in the upper base to the plane of the lower base.

6. A *prism* is *right*, when the sides AF, BG, CH, &c. are perpendicular to the planes of the bases; and then each of them is equal to the altitude of the prism. In every other case the prism is *oblique*, and the altitude less than the side.

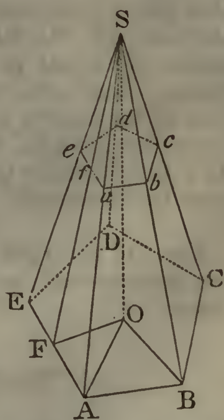
7. A *prism* is *triangular*, *quadrangular*, *pentagonal*, *hexagonal*, &c. when the base is a triangle, a quadrilateral, a pentagon, a hexagon, &c.

8. A prism whose base is a parallelogram, and which has all its faces parallelograms, is named a *parallelopipedon*.

The *parallelopipedon* is *rectangular* when all its faces are rectangles.

9. Among rectangular parallelopipedons, we distinguish the *cube*, or regular hexaedron, bounded by six equal squares.

10. A *pyramid* is a solid formed by several triangular planes proceeding from the same point S, and terminating in the different sides of the same polygon ABCDE.



The polygon ABCDE is called the base of the pyramid, the point S the vertex; and the triangles ASB, BSC, CSD, &c. form its *convex* or *lateral* surface.

11. If from the pyramid S-ABCDE, the pyramid S-*abcde* be cut off by a plane parallel to the base, the remaining solid ABCDE-*d*, is called a *truncated pyramid*, or the frustum of a pyramid.

12. The *altitude* of a pyramid is the perpendicular let fall from the vertex upon the plane of the base, produced if necessary.

13. A pyramid is *triangular*, *quadrangular*, &c. according as its base is a triangle, a quadrilateral, &c.

14. A pyramid is *regular*, when its base is a regular polygon, and when, at the same time, the perpendicular let fall from the vertex on the plane of the base passes through the centre of the base. That perpendicular is then called the *axis* of the pyramid.

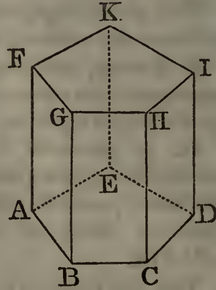
15. Any line, as SF, drawn from the vertex S of a regular pyramid, perpendicular to either side of the polygon which forms its base, is called the *slant height* of the pyramid.

16. The *diagonal* of a polyedron is a straight line joining the vertices of two solid angles which are not adjacent to each other.

17. Two polyedrons are *similar* when they are contained by the same number of similar planes, similarly situated, and having like inclinations with each other.

PROPOSITION I. THEOREM.

The convex surface of a right prism is equal to the perimeter of its base multiplied by its altitude.



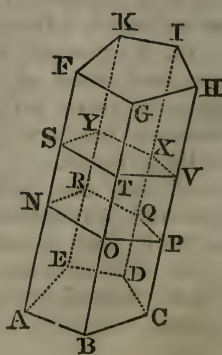
Let ABCDE-K be a right prism: then will its convex surface be equal to $(AB+BC+CD+DE+EA)\times AF$.

For, the convex surface is equal to the sum of all the rectangles AG, BH, CI, DK, EF, which compose it. Now, the altitudes AF, BG, CH, &c. of the rectangles, are equal to the altitude of the prism. Hence, the sum of these rectangles, or the convex surface of the prism, is equal to $(AB+BC+CD+DE+EA)\times AF$; that is, to the perimeter of the base of the prism multiplied by its altitude.

Cor. If two right prisms have the same altitude, their convex surfaces will be to each other as the perimeters of their bases.

PROPOSITION II. THEOREM.

In every prism, the sections formed by parallel planes, are equal polygons.



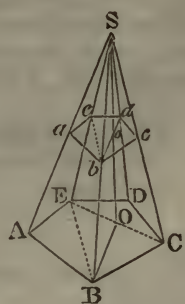
Let the prism AH be intersected by the parallel planes NP, SV; then are the polygons NOPQR, STVXY equal.

For, the sides ST, NO, are parallel, being the intersections of two parallel planes with a third plane ABGF; these same sides, ST, NO, are included between the parallels NS, OT, which are sides of the prism: hence NO is equal to ST. For like reasons, the sides OP, PQ, QR, &c. of the section NOPQR, are equal to the sides TV, VX, XY, &c. of the section STVXY, each to each. And since the equal sides are at the same time parallel, it follows that the angles NOP, OPQ, &c. of the first section, are equal to the angles STV, TVX, &c. of the second, each to each (Book VI. Prop. XIII.). Hence the two sections NOPQR, STVXY, are equal polygons.

Cor. Every section in a prism, if drawn parallel to the base is also equal to the base.

PROPOSITION III. THEOREM.

If a pyramid be cut by a plane parallel to its base,
1*st. The edges and the altitude will be divided proportionally.*
2*d. The section will be a polygon similar to the base.*

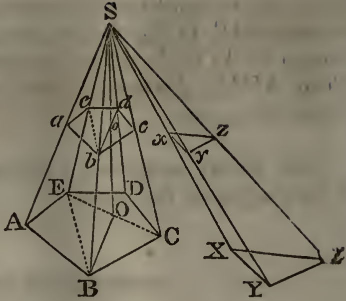


Let the pyramid S-ABCDE, of which SO is the altitude, be cut by the plane *abcde*; then will Sa : SA : : So : SO, and the same for the other edges: and the polygon *abcde*, will be similar to the base ABCDE.

First. Since the planes ABC, *abc*, are parallel, their intersections AB, *ab*, by a third plane SAB will also be parallel (Book VI. Prop. X.); hence the triangles SAB, Sab are similar, and we have SA : Sa : : SB : Sb; for a similar reason, we have SB : Sb : : SC : Sc; and so on. Hence the edges SA, SB, SC, &c. are cut proportionally in a, b, c, &c. The altitude SO is likewise cut in the same proportion, at the point o; for BO and bo are parallel, therefore we have

SO : So : : SB : Sb.

Secondly. Since *ab* is parallel to AB, *bc* to BC, *cd* to CD, &c. the angle *abc* is equal to ABC, the angle *bcd* to BCD, and so on (Book VI. Prop. XIII.). Also, by reason of the similar triangles SAB, Sab, we have AB : *ab* : : SB : Sb; and by reason of the similar triangles SBC, Sbc, we have SB : Sb : : BC : *bc*; hence AB : *ab* : : BC : *bc*; we might likewise have BC : *bc* : : CD : *cd*, and so on. Hence the polygons ABCDE. *abcde* have their angles respectively equal and their homologous sides proportional; hence they are similar.



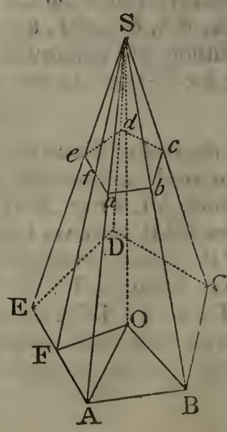
Cor. 1. Let S-ABCDE, S-XYZ be two pyramids, having a common vertex and the same altitude, or having their bases situated in the same plane; if these pyramids are cut by a plane parallel to the plane of their bases, giving the sections *abcde*, *xyz*, *then will the sections abcde, xyz, be to each other as the bases* ABCDE, XYZ.

For, the polygons ABCDE, *abcde*, being similar, their surfaces are as the squares of the homologous sides AB, *ab*; but AB : *ab* : : SA : S*a*; hence ABCDE : *abcde* : : $SA^2 : Sa^2$. For the same reason, XYZ : *xyz* : : $SX^2 : Sx^2$. But since *abc* and *xyz* are in one plane, we have likewise SA : S*a* : : SX : S*x* (Book VI. Prop. XV.); hence ABCDE : *abcde* : : XYZ : *xyz*; hence the sections *abcde*, *xyz*, are to each other as the bases ABCDE, XYZ.

Cor. 2. If the bases ABCDE, XYZ, are equivalent, any sections *abcde*, *xyz*, made at equal distances from the bases, will be equivalent likewise.

PROPOSITION IV. THEOREM.

The convex surface of a regular pyramid is equal to the perimeter of its base multiplied by half the slant height.



For, since the pyramid is regular, the point O, in which the axis meets the base, is the centre of the polygon ABCDE (Def. 14.); hence the lines OA, OB, OC, &c. drawn to the vertices of the base, are equal.
In the right angled triangles SAO, SBO, the bases and perpendiculars are equal: since the hypothenuses are equal: and it may be proved in the same way that all the sides of the right pyramid are equal. The triangles, therefore, which form the convex surface of the prism are all equal to each other. But the area of either of these triangles, as ESA, is equal

to its base EA multiplied by half the perpendicular SF, which is the slant height of the pyramid : hence the area of all the triangles, or the convex surface of the pyramid, is equal to the perimeter of the base multiplied by half the slant height.

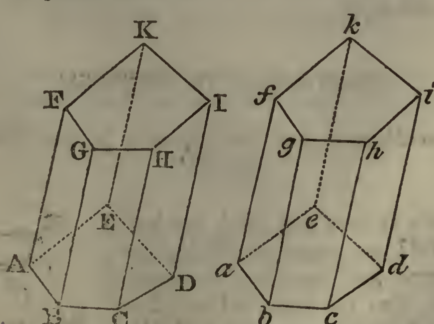
Cor. The convex surface of the frustum of a regular pyramid is equal to half the perimeters of its upper and lower bases multiplied by its slant height.

For, since the section *abcde* is similar to the base (Prop. III.), and since the base ABCDE is a regular polygon (Def. 14.), it follows that the sides *ea*, *ab*, *bc*, *cd* and *de* are all equal to each other. Hence the convex surface of the frustum ABCDE-*d* is formed by the equal trapezoids EA*ae*, AB*ba*, &c. and the perpendicular distance between the parallel sides of either of these trapezoids is equal to F*f*, the slant height of the frustum. But the area of either of the trapezoids, as AE*ea*, is equal to $\frac{1}{2}(EA+ea)\times Ff$ (Book IV. Prop. VII.) : hence the area of all of them, or the convex surface of the frustum, is equal to half the perimeters of the upper and lower bases multiplied by the slant height.

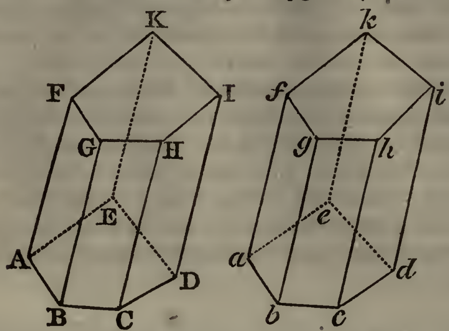
PROPOSITION V. THEOREM.

If the three planes which form a solid angle of a prism, are equal to the three planes which form the solid angle of another prism, each to each, and are like situated, the two prisms will be equal to each other.

Let the base ABCDE be equal to the base *abcde*, the parallelogram ABGF equal to the parallelogram *abgf*, and the parallelogram BCHG equal to *bchg*; then will the prism ABCDE-K be equal to the prism *abcde-k*.



For, lay the base ABCDE upon its equal *abcde* ; these two bases will coincide. But the three plane angles which form the solid angle B, are respectively equal to the three plane angles, which form the solid angle b, namely, ABC$=abc$, ABG$=abg$, and GBC$=gbc$; they are also similarly situated. hence the solid angles B and b are equal (Book VI. Prop. XXI. Sch.); and therefore the side BG will fall on its equal bg. It is likewise evident, that by reason of the equal parallelograms ABGF, $abgf$, the side GF will fall on its equal gf, and in the same manner GH on gh; hence, the plane of the upper base, FGHIK will coincide with the plane $fghik$ (Book VI. Prop. II.).



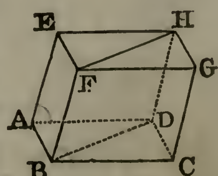
But the two upper bases being equal to their corresponding lower bases, are equal to each other. hence HI will coincide with hi, IK with ik, and KF with kf; and therefore the lateral faces of the prisms will coincide: therefore, the two prisms coinciding throughout are equal (Ax. 13.).

Cor. Two right prisms, which have equal bases and equal altitudes, are equal. For, since the side AB is equal to ab, and the altitude BG to bg, the rectangle ABGF will be equal to $abgf$; so also will the rectangle BGHC be equal to $bghc$; and thus the three planes, which form the solid angle B, will be equal to the three which form the solid angle b. Hence the two prisms are equal.

PROPOSITION VI. THEOREM.

faces

In every parallelopipedon the opposite planes are equal and parallel.



By the definition of this solid, the bases ABCD, EFGH, are equal parallelograms, and their sides are parallel: it remains only to show, that the same is true of any two opposite lateral faces, such as AEHD, BFGC. Now AD is equal and parallel to BC, because the figure ABCD is a parallelogram; for a like reason, AE is parallel to BF: hence the angle DAE is equal to the angle CBF, and the planes DAE, CBF, are parallel (Book VI. Prop. XIII.); hence also the parallelogram DAEH is equal to the parallelogram CBFG. In the same way, it might be shown that the opposite parallelograms ABFE, DCGH, are equal and parallel.

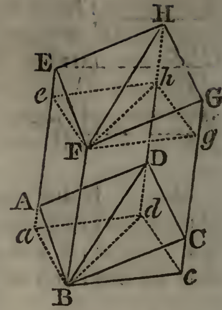
Cor. 1. Since the parallelopipedon is a solid bounded by six planes, whereof those lying opposite to each other are equal and parallel, it follows that any face and the one opposite to it, may be assumed as the bases of the parallelopipedon.

Cor. 2. *The diagonals of a parallelopipedon bisect each other.* For, suppose two diagonals EC, AG, to be drawn both through opposite vertices: since AE is equal and parallel to CG, the figure AEGC is a parallelogram; hence the diagonals EC, AG will mutually bisect each other. In the same manner, we could show that the diagonal EC and another DF bisect each other; hence the four diagonals will mutually bisect each other, in a point which may be regarded as the centre of the parallelopipedon.

Scholium. If three straight lines AB, AE, AD, passing through the same point A, and making given angles with each other, are known, a parallelopipedon may be formed on those lines. For this purpose, a plane must be passed through the extremity of each line, and parallel to the plane of the other two; that is, through the point B a plane parallel to DAE, through D a plane parallel to BAE, and through E a plane parallel to BAD. The mutual intersections of these planes will form the parallelopipedon required.

PROPOSITION VII. THEOREM.

The two triangular prisms into which a parallelopipedon is divided by a plane passing through its opposite diagonal edges, are equivalent.



Let the parallelopipedon ABCD-H be divided by the plane BDHF passing through its diagonal edges : then will the triangular prism ABD-H be equivalent to the triangular prism BCD-H.

Through the vertices B and F, draw the planes B*adc*, F*ehg*, at right angles to the side BF, the former meeting AE, DH, CG, the three other sides of the parallelopipedon, in the points *a*, *d*, *c*, the latter in *e*, *h*, *g*: the sections B*adc*, F*ehg*, will be equal parallelograms. They are equal, because they are formed by planes perpendicular to the same straight line, and consequently parallel (Prop. II.); they are parallelograms, because *a*B, *dc*, two opposite sides of the same section, are formed by the meeting of one plane with two parallel planes ABFE, DCGH.

For a like reason, the figure B*ae*F is a parallelogram; so also are BF*gc*, *cdhg*, *adhe*, the other lateral faces of the solid B*adc-g*; hence that solid is a prism (Def. 6.); and that prism is right, because the side BF is perpendicular to its base.

But the right prism B*adc-g* is divided by the plane BH into two equal right prisms B*ad-h*, B*cd-h*; for, the bases B*ad*, B*cd*, of these prisms are equal, being halves of the same parallelogram, and they have the common altitude BF, hence they are equal (Prop. V. Cor.).

It is now to be proved that the oblique triangular prism ABD-H will be equivalent to the right triangular prism B*ad-h*; and since those prisms have a common part ABD-*h*, it will only be necessary to prove that the remaining parts, namely, the solids B*a*AD*d*, F*e*EH*h*, are equivalent.

Now, by reason of the parallelograms ABFE, *a*BF*e*, the sides AE, *ae*, being equal to their parallel BF, are equal to each other; and taking away the common part A*e*, there remains A*a*=E*e*. In the same manner we could prove D*d*=H*h*.

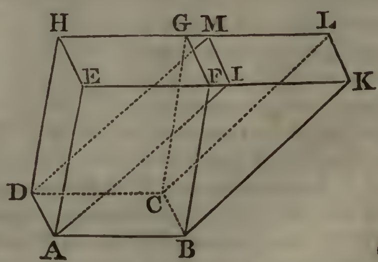
Next, to bring about the superposition of the two solids B*a*AD*d*, F*e*EH*h*, let us place the base F*eh* on its equal B*ad*: the point *e* falling on *a*, and the point *h* on *d*, the sides *e*E, *h*H, will fall on their equals *a*A, *d*D, because they are perpendicular to the same plane B*ad*. Hence the two solids in question will coincide exactly with each other; hence the oblique prism BAD-H, is equivalent to the right one B*ad-h*.

In the same manner might the oblique prism BCD-H, be proved equivalent to the right prism B*cd-h*. But the two right prisms B*ad-h*, B*cd-h*, are equal, since they have the same altitude BF, and since their bases B*ad*, B*dc*, are halves of the same parallelogram (Prop. V. Cor.). Hence the two triangular prisms BAD-H, BDC-G, being equivalent to the equal right prisms, are equivalent to each other.

Cor. Every triangular prism ABD-HEF is half of the parallelopipedon AG described with the same solid angle A, and the same edges AB, AD, AE.

PROPOSITION VIII. THEOREM.

If two parallelopipedons have a common base, and their upper bases in the same plane and between the same parallels, they will be equivalent.



Let the parallelopipedons AG, AL, have the common base AC, and their upper bases EG, MK, in the same plane, and between the same parallels HL, EK; then will they be equivalent.

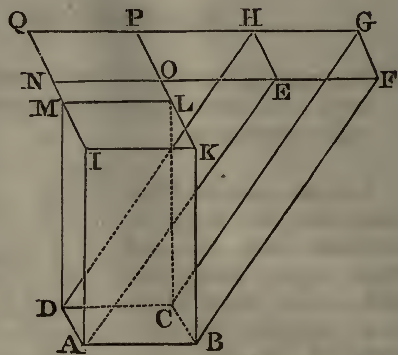
There may be three cases, according as EI is greater, less than, or equal to, EF; but the demonstration is the same for all. In the first place, then we shall show that the triangular prism AEI-MDH, is equal to the triangular prism BFK-LCG.

Since AE is parallel to BF, and HE to GF, the angle AEI =BFK, HEI=GFK, and HEA=GFB. Also, since EF and IK are each equal to AB, they are equal to each other. To each add FI, and there will result EI equal to FK: hence the triangle AEI is equal to the triangle BFK (Bk. I. Prop. V), and the paralellogram EM to the parallelogram FL. But the parallelogram AH is equal to the parallelogram CF (Prop. VI): hence, the three planes which form the solid angle at E are respectively equal to the three which form the solid angle at F, and being like placed, the triangular prism AEI-M is equal to the triangular prism BFK-L.

But if the prism AEI-M is taken away from the solid AL, there will remain the parallelopipedon BADC-L; and if the prism BFK-L is taken away from the same solid, there will remain the parallelopipedon BADC-G; hence those two parallelopipedons BADC-L, BADC-G, are equivalent.

PROPOSITION IX. THEOREM.

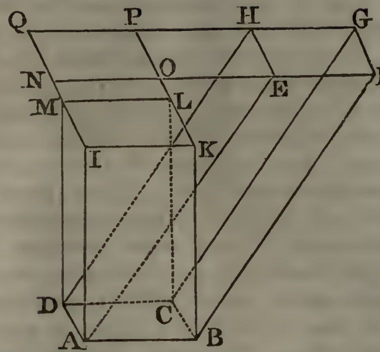
Two parallelopipedons, having the same base and the same altitude, are equivalent.



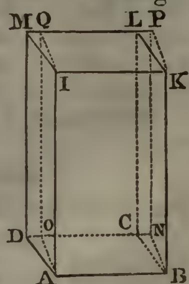
Let ABCD be the common base of the two parallelopipedons AG, AL; since they have the same altitude, their upper bases EFGH, IKLM, will be in the same plane. Also the sides EF and AB will be equal and parallel, as well as IK and AB; hence EF is equal and parallel to IK; for a like reason, GF is equal and parallel to LK. Let the sides EF, GH, be produced, and likewise KL, IM, till by their intersections they form the parallelogram NOPQ; this parallelogram will evidently be equal to either of the bases EFGH, IKLM. Now if a third parallelopipedon be conceived, having for its lower base the parallelogram ABCD, and NOPQ for its upper, the third parallelopipedon will be equivalent to the parallelopipedon AG, since with the same lower base, their upper bases lie in the same plane and between the same parallels, GQ, FN (Prop. VIII.). For the same reason, this third parallelopipedon will also be equivalent to the parallelopipedon AL; hence the two parallelopipedons AG, AL, which have the same base and the same altitude, are equivalent.

PROPOSITION X. THEOREM.

Any parallelopipedon may be changed into an equivalent rectangular parallelopipedon having the same altitude and an equivalent base.



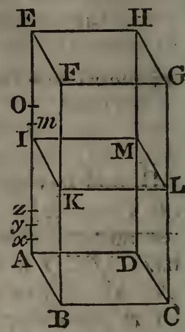
Let AG be the parallelopipedon proposed. From the points A, B, C, D, draw AI, BK, CL, DM, perpendicular to the plane of the base; you will thus form the parallelopipedon AL equivalent to AG, and having its lateral faces AK, BL, &c. rectangles. Hence if the base ABCD is a rectangle, AL will be a rectangular parallelopipedon equivalent to AG, and consequently the parallelopipedon required. But if ABCD is not a rectangle, draw AO and BN perpendicular to CD, and OQ and NP perpendicular to the base; you will then have the solid ABNO-IKPQ, which will be a rectangular parallelopipedon: for by construction, the bases ABNO, and IKPQ are rectangles; so also are the lateral faces, the edges AI, OQ, &c. being perpendicular to the plane of the base; hence the solid AP is a rectangular parallelopipedon. But the two parallelopipedons AP, AL may be conceived as having the same base ABKI and the same altitude AO: hence the parallelopipedon AG, which was at first changed into an equivalent parallelopipedon AL, is again changed into an equivalent rectangular parallelopipedon AP, having the same altitude AI, and a base ABNO equivalent to the base ABCD.



PROPOSITION XI. THEOREM.

Two rectangular parallelopipedons, which have the same base, are to each other as their altitudes.

Let the parallelopipedons AG, AL, have the same base BD; then will they be to each other as their altitudes AE, AI.



First, suppose the altitudes AE, AI, to be to each other as two whole numbers, as 15 is to 8, for example. Divide AE into 15 equal parts; whereof AI will contain 8; and through x, y, z, &c. the points of division, draw planes parallel to the base. These planes will cut the solid AG into 15 partial parallelopipedons, all equal to each other, because they have equal bases and equal altitudes—equal bases, since every section MIKL, made parallel to the base ABCD of a prism, is equal to that base (Prop. II.), equal altitudes, because the altitudes are the equal divisions Ax, xy, yz, &c. But of those 15 equal parallelopipedons, 8 are contained in AL; hence the solid AG is to the solid AL as 15 is to 8, or generally, as the altitude AE is to the altitude AI.

Again, if the ratio of AE to AI cannot be exactly expressed in numbers, it is to be shown, that notwithstanding, we shall have

solid AG : *solid* AL : : AE : AI.

For, if this proportion is not correct, suppose we have

sol. AG : *sol.* AL : : AE : AO greater than AI.

Divide AE into equal parts, such that each shall be less than OI; there will be at least one point of division m, between O and I. Let P be the parallelopipedon, whose base is ABCD, and altitude Am; since the altitudes AE, Am, are to each other as the two whole numbers, we shall have

sol. AG : P : : AE : Am.

But by hypothesis, we have

sol. AG : *sol.* AL : : AE : AO;

therefore,

sol. AL : P : : AO : Am.

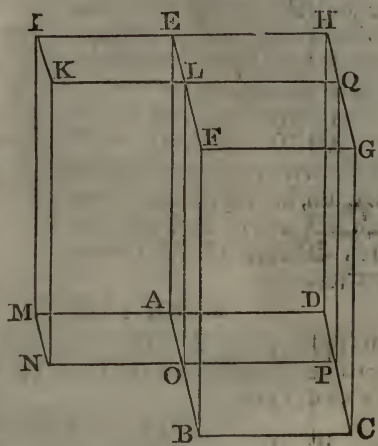
But AO is greater than Am; hence if the proportion is correct, the solid AL must be greater than P. On the contrary, however, it is less: hence the fourth term of this proportion

sol. AG : *sol.* AL : : AE : x,

cannot possibly be a line greater than AI. By the same mode of reasoning, it might be shown that the fourth term cannot be less than AI; therefore it is equal to AI; hence rectangular parallelopipedons having the same base are to each other as their altitudes.

PROPOSITION XII. THEOREM.

Two rectangular parallelopipedons, having the same altitude are to each other as their bases.



Let the parallelopipedons AG, AK, have the same altitude AE; then will they be to each other as their bases AC, AN.

Having placed the two solids by the side of each other, as the figure represents, produce the plane ONKL till it meets the plane DCGH in PQ; you will thus have a third parallelopipedon AQ, which may be compared with each of the parallelopipedons AG, AK. The two solids AG, AQ, having the same base AEHD are to each other as their altitudes AB, AO; in like manner, the two solids AQ, AK, having the same base AOLE, are to each other as their altitudes AD, AM. Hence we have the two proportions,

$$sol.\ AG : sol.\ AQ :: AB : AO,$$
$$sol.\ AQ : sol.\ AK :: AD : AM.$$

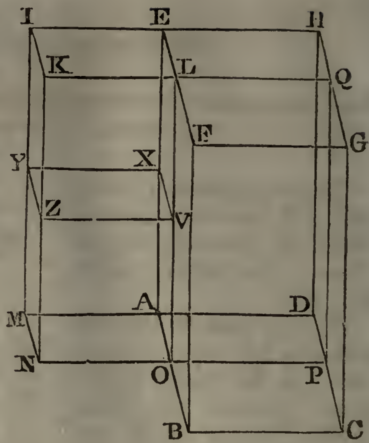
Multiplying together the corresponding terms of these proportions, and omitting in the result the common multiplier *sol.* AQ; we shall have

$$sol.\ AG : sol.\ AK :: AB \times AD : AO \times AM.$$

But AB × AD represents the base ABCD; and AO × AM represents the base AMNO; hence two rectangular parallelopipedons of the same altitude are to each other as their bases.

PROPOSITION XIII. THEOREM.

Any two rectangular parallelopipedons are to each other as the products of their bases by their altitudes, that is to say, as the products of their three dimensions.



For, having placed the two solids AG, AZ, so that their surfaces have the common angle BAE, produce the planes necessary for completing the third parallelopipedon AK having the same altitude with the parallelopipedon AG. By the last proposition, we shall have

sol. AG : *sol.* AK : : ABCD : AMNO.

But the two parallelopipedons AK, AZ, having the same base AMNO, are to each other as their altitudes AE, AX; hence we have

sol. AK : *sol.* AZ : : AE : AX.

Multiplying together the corresponding terms of these proportions, and omitting in the result the common multiplier *sol.* AK; we shall have

sol. AG : *sol.* AZ : : ABCD $\times$ AE : AMNO $\times$ AX.

Instead of the bases ABCD and AMNO, put AB $\times$ AD and AO $\times$ AM it will give

sol. AG : *sol.* AZ : : AB $\times$ AD $\times$ AE : AO $\times$ AM $\times$ AX.

Hence any two rectangular parallelopipedons are to each other, &c.

Scholium. We are consequently authorized to assume, as the measure of a rectangular parallelopipedon, the product of its base by its altitude, in other words, the product of its three dimensions.

In order to comprehend the nature of this measurement, it is necessary to reflect, that the number of linear units in one dimension of the base multiplied by the number of linear units in the other dimension of the base, will give the number of superficial units in the base of the parallelopipedon (Book IV. Prop. IV. Sch.). For each unit in height there are evidently as many solid units as there are superficial units in the base. Therefore, the number of superficial units in the base multiplied by the number of linear units in the altitude, gives the number of solid units in the parallelopipedon.

If the three dimensions of another parallelopipedon are valued according to the same linear unit, and multiplied together in the same manner, the two products will be to each other as the solids, and will serve to express their relative magnitude.

The magnitude of a solid, its volume or extent, forms what is called its *solidity;* and this word is exclusively employed to designate the measure of a solid: thus we say the solidity of a rectangular parallelopipedon is equal to the product of its base by its altitude, or to the product of its three dimensions.

As the cube has all its three dimensions equal, if the side is 1, the solidity will be $1\times1\times1=1$: if the side is 2, the solidity will be $2\times2\times2=8$; if the side is 3, the solidity will be $3\times3\times3=27$; and so on: hence, if the sides of a series of cubes are to each other as the numbers 1, 2, 3, &c. the cubes themselves or their solidities will be as the numbers 1, 8, 27, &c. Hence it is, that in arithmetic, the *cube* of a number is the name given to a product which results from three factors, each equal to this number.

If it were proposed to find a cube double of a given cube, the side of the required cube would have to be to that of the given one, as the cube-root of 2 is to unity. Now, by a geometrical construction, it is easy to find the square root of 2; but the cube-root of it cannot be so found, at least not by the simple operations of elementary geometry, which consist in employing nothing but straight lines, two points of which are known, and circles whose centres and radii are determined.

Owing to this difficulty the problem of the *duplication of the cube* became celebrated among the ancient geometers, as well as that of the *trisection of an angle,* which is nearly of the same species. The solutions of which such problems are susceptible, have however long since been discovered; and though less simple than the constructions of elementary geometry, they are not, on that account, less rigorous or less satisfactory.

PROPOSITION XIV. THEOREM.

The solidity of a parallelopipedon, and generally of any prism, is equal to the product of its base by its altitude.

For, in the first place, any parallelopipedon is equivalent to a rectangular parallelopipedon, having the same altitude and an equivalent base (Prop. X.). Now the solidity of the latter is equal to its base multiplied by its height; hence the solidity of the former is, in like manner, equal to the product of its base by its altitude.

In the second place, any triangular prism is half of the parallelopipedon so constructed as to have the same altitude and a double base (Prop. VII.). But the solidity of the latter is equal to its base multiplied by its altitude; hence that of a triangular prism is also equal to the product of its base, which is half that of the parallelopipedon, multiplied into its altitude.

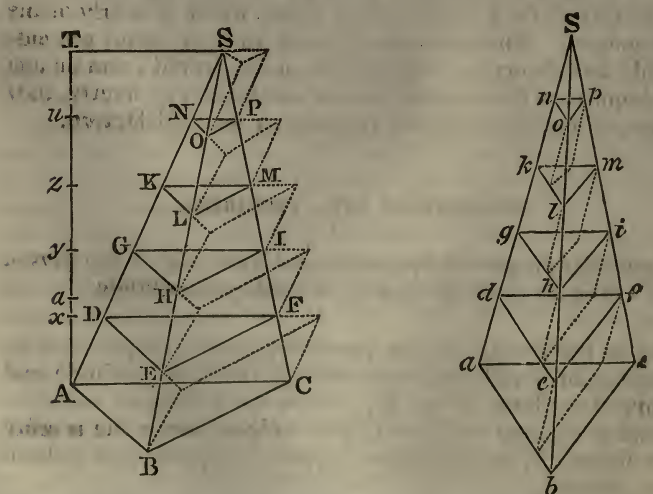
In the third place, any prism may be divided into as many triangular prisms of the same altitude, as there are triangles capable of being formed in the polygon which constitutes its base. But the solidity of each triangular prism is equal to its base multiplied by its altitude; and since the altitude is the same for all, it follows that the sum of all the partial prisms must be equal to the sum of all the partial triangles, which constitute their bases, multiplied by the common altitude.

Hence the solidity of any polygonal prism is equal to the product of its base by its altitude.

Cor. Comparing two prisms, which have the same altitude, the products of their bases by their altitudes will be as the bases simply; hence *two prisms of the same altitude are to each other as their bases.* For a like reason, *two prisms of the same base are to each other as their altitudes.* And when neither their bases nor their altitudes are equal, their solidities will be to each other as the products of their bases and altitudes.

PROPOSITION XV. THEOREM.

Two triangular pyramids, having equivalent bases and equal altitudes, are equivalent, or equal in solidity.



Let S-ABC, S-*abc*, be those two pyramids; let their equivalent bases ABC, *abc*, be situated in the same plane, and let AT be their common altitude. If they are not equivalent, let S-*abc* be the smaller: and suppose Aa to be the altitude of a prism, which having ABC for its base, is equal to their difference.

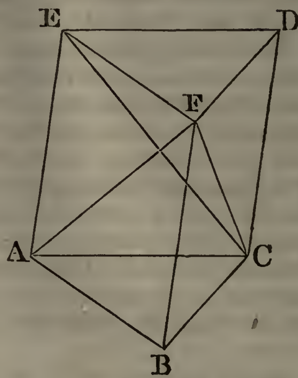
Divide the altitude AT into equal parts Ax, xy, yz, &c. each less than Aa, and let k be one of those parts; through the points of division pass planes parallel to the plane of the bases; the corresponding sections formed by these planes in the two pyramids will be respectively equivalent, namely DEF to *def*, GHI to *ghi*, &c. (Prop. III. Cor. 2.).

This being granted, upon the triangles ABC, DEF, GHI, &c. taken as bases, construct exterior prisms having for edges the parts AD, DG, GK, &c. of the edge SA; in like manner, on bases *def*, *ghi*, *klm*, &c. in the second pyramid, construct interior prisms, having for edges the corresponding parts of Sa. It is plain that the sum of all the exterior prisms of the pyramid S-ABC will be greater than this pyramid; and also that the sum of all the interior prisms of the pyramid S-*abc* will be less than this pyramid. Hence the difference, between the sum of all the exterior prisms and the sum of all the interior ones, must be greater than the difference between the two pyramids themselves.

Now, beginning with the bases ABC, *abc*, the second exterior prism DEF-G is equivalent to the first interior prism *def-a*, because they have the same altitude k, and their bases DEF, *def*, are equivalent; for like reasons, the third exterior prism GHI-K and the second interior prism *ghi-d* are equivalent; the fourth exterior and the third interior; and so on, to the last in each series. Hence all the exterior prisms of the pyramid S-ABC, excepting the first prism ABC-D, have equivalent corresponding ones in the interior prisms of the pyramid S-*abc*: hence the prism ABC-D, is the difference between the sum of all the exterior prisms of the pyramid S-ABC, and the sum of the interior prisms of the pyramid S-*abc*. But the difference between these two sets of prisms has already been proved to be greater than that of the two pyramids; which latter difference we supposed to be equal to the prism a-ABC: hence the prism ABC-D, must be greater than the prism a-ABC. But in reality it is less; for they have the same base ABC, and the altitude Ax of the first is less than Aa the altitude of the second. Hence the supposed inequality between the two pyramids cannot exist; hence the two pyramids S-ABC, S-*abc*, having equal altitudes and equivalent bases, are themselves equivalent.

PROPOSITION XVI. THEOREM.

Every triangular pyramid is a third part of the triangular prism having the same base and the same altitude.



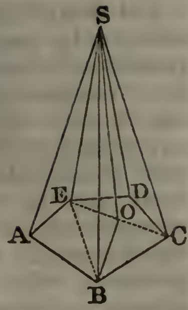
Let F-ABC be a triangular pyramid, ABC-DEF a triangular prism of the same base and the same altitude; the pyramid will be equal to a third of the prism.

Cut off the pyramid F-ABC from the prism, by the plane FAC; there will remain the solid F-ACDE, which may be considered as a quadrangular pyramid, whose vertex is F, and whose base is the parallelogram ACDE. Draw the diagonal CE; and pass the plane FCE, which will cut the quadrangular pyramid into two triangular ones F-ACE, F-CDE. These two triangular pyramids have for their common altitude the perpendicular let fall from F on the plane ACDE; they have equal bases, the triangles ACE, CDE being halves of the same parallelogram; hence the two pyramids F-ACE, F-CDE, are equivalent (Prop. XV.). But the pyramid F-CDE and the pyramid F-ABC have equal bases ABC, DEF; they have also the same altitude, namely, the distance between the parallel planes ABC, DEF; hence the two pyramids are equivalent. Now the pyramid F-CDE has already been proved equivalent to F-ACE; hence the three pyramids F-ABC, F-CDE, F-ACE, which compose the prism ABC-DEF are all equivalent. Hence the pyramid F-ABC is the third part of the prism ABC-DEF, which has the same base and the same altitude.

Cor. The solidity of a triangular pyramid is equal to a third part of the product of its base by its altitude.

PROPOSITION XVII. THEOREM.

The solidity of every pyramid is equal to the base multiplied by a third of the altitude.



Let S-ABCDE be a pyramid.

Pass the planes SEB, SEC, through the diagonals EB, EC; the polygonal pyramid S-ABCDE will be divided into several triangular pyramids all having the same altitude SO. But each of these pyramids is measured by multiplying its base ABE, BCE, or CDE, by the third part of its altitude SO (Prop. XVI. Cor.); hence the sum of these triangular pyramids, or the polygonal pyramid S-ABCDE will be measured by the sum of the triangles ABE, BCE, CDE, or the polygon ABCDE, multiplied by one third of SO; hence every pyramid is measured by a third part of the product of its base by its altitude.

Cor. 1. Every pyramid is the third part of the prism which has the same base and the same altitude.

Cor. 2. Two pyramids having the same altitude are to each other as their bases.

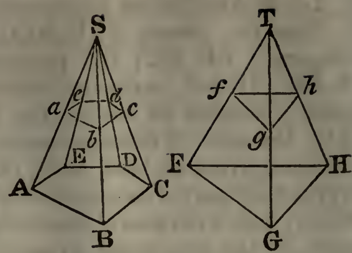
Cor. 3. Two pyramids having equivalent bases are to each other as their altitudes.

Cor. 4. Pyramids are to each other as the products of their bases by their altitudes.

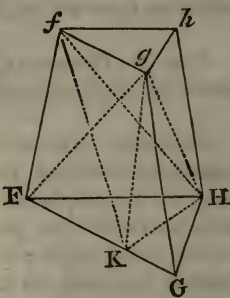
Scholium. The solidity of any polyedral body may be computed, by dividing the body into pyramids; and this division may be accomplished in various ways. One of the simplest is to make all the planes of division pass through the vertex of one solid angle; in that case, there will be formed as many partial pyramids as the polyedron has faces, *minus* those faces which form the solid angle whence the planes of division proceed.

PROPOSITION XVIII. THEOREM.

If a pyramid be cut by a plane parallel to its base, the frustum that remains when the small pyramid is taken away, is equivalent to the sum of three pyramids having for their common altitude the altitude of the frustum, and for bases the lower base of the frustum, the upper base, and a mean proportional between the two bases.



Let S-ABCDE be a pyramid cut by the plane *abcde*, parallel to its base; let T-FGH be a triangular pyramid having the same altitude and an equivalent base with the pyramid S-ABCDE. The two bases may be regarded as situated in the same plane; in which case, the plane *abcd*, if produced, will form in the triangular pyramid a section *fgh* situated at the same distance above the common plane of the bases; and therefore the section *fgh* will be to the section *abcde* as the base FGH is to the base ABD (Prop. III.), and since the bases are equivalent, the sections will be so likewise. Hence the pyramids S-*abcde*, T-*fgh* are equivalent, for their altitude is the same and their bases are equivalent. The whole pyramids S-ABCDE, T-FGH are equivalent for the same reason; hence the frustums ABD-*dab*, FGH-*hfg* are equivalent; hence if the proposition can be proved in the single case of the frustum of a triangular pyramid, it will be true of every other.



Let FGH-*hfg* be the frustum of a triangular pyramid, having parallel bases: through the three points F, *g*, H, pass the plane F*g*H; it will cut off from the frustum the triangular pyramid *g*-FGH. This pyramid has for its base the lower base FGH of the frustum; its altitude likewise is that of the frustum, because the vertex *g* lies in the plane of the upper base *fgh*.

This pyramid being cut off, there will remain the quadrangular pyramid *g*-*fh*HF, whose vertex is *g*, and base *fh*HF. Pass the plane *fg*H through the three points *f*, *g*, H; it will divide the quadrangular pyramid into two triangular pyramids *g*-F*f*H, *g*-*fh*H. The latter has for its base the upper base *gfh* of the frustum; and for its altitude, the altitude of the frustum, because its vertex H lies in the lower base. Thus we already know two of the three pyramids which compose the frustum.

It remains to examine the third *g*-F*f*H. Now, if *g*K be drawn parallel to *f*F, and if we conceive a new pyramid K-F*f*H, having K for its vertex and F*f*H for its base, these two pyramids will have the same base F*f*H; they will also have the same altitude, because their vertices *g* and K lie in the line *g*K, parallel to F*f*, and consequently parallel to the plane of the base: hence these pyramids are equivalent. But the pyramid K-FfH may be regarded as having its vertex in f, and thus its altitude will be the same as that of the frustum: as to its base FKH, we are now to show that this is a mean proportional between the bases FGH and fgh. Now, the triangles FHK, fgh, have each an equal angle $F=f$; hence

$$FHK : fgh :: FK \times FH : fg \times fh \text{ (Book IV. Prop. XXIV.)}$$

but because of the parallels, $FK=fg$, hence

$$FHK : fgh :: FH : fh.$$

We have also,

$$FHG : FHK :: FG : FK \text{ or } fg.$$

But the similar triangles FGH, fgh give

$$FG : fg :: FH : fh;$$

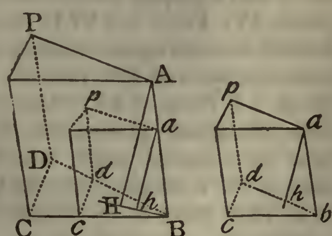
hence,

$$FGH : FHK :: FHK : fgh;$$

or the base FHK is a mean proportional between the two bases FGH, fgh. Hence the frustum of a triangular pyramid is equivalent to three pyramids whose common altitude is that of the frustum and whose bases are the lower base of the frustum, the upper base, and a mean proportional between the two bases.

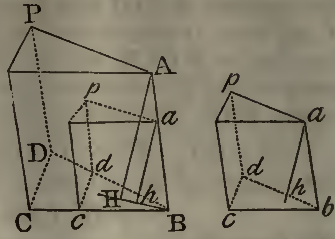
PROPOSITION XIX. THEOREM.

Similar triangular prisms are to each other as the cubes of their homologous sides.



Let CBD-P, *cbd-p*, be two similar triangular prisms, of which BC, *bc*, are homologous sides: then will the prism CBD-P be to the prism *cbd-p*, as BC^3 to bc^3.

For, since the prisms are similar, the planes which contain the homologous solid angles B and b, are similar, like placed, and equally inclined to each other (Def. 17.): hence the solid angles B and b, are equal (Book VI. Prop. XXI. Sch.). If these solid angles be applied to each other, the angle cbd will coincide with CBD, the side ba with BA, and the prism *cbd-p* will take the position B*cd-p*. From A draw AH perpendicular to the common base of the prisms: then will the plane BAH be perpendicular to the plane of the common base (Book VI. Prop. XVI.). Through a, in the plane BAH, draw ah perpendicular to BH: then will ah also be perpendicular to the base BDC (Book VI. Prop. XVII.); and AH, ah will be the altitudes of the two prisms.



Now, because of the similar triangles ABH, aBh, and of the similar parallelograms AC, ac, we have

$$AH : ah :: AB : ab :: BC : bc.$$

But since the bases are similar, we have

base BCD : *base bcd* :: $BC^2 : bc^2$ (Book IV. Prop. XXV.); hence,

$$base\ BCD : base\ bcd :: AH^2 : ah^2.$$

Multiplying the antecedents by AH, and the consequents by ah, and we have

$$base\ BCD \times AH : base\ bcd \times ah :: AH^3 \quad ah^3.$$

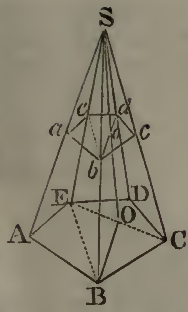
But the solidity of a prism is equal to the base multiplied by the altitude (Prop. XIV.); hence, the

prism BCD-P : *prism bcd-p* :: $AH^3 : ah^3 :: BC^3 : bc^3$, or as the cubes of any other of their homologous sides.

Cor. Whatever be the bases of similar prisms, the prisms will be to each other as the cubes of their homologous sides. For, since the prisms are similar, their bases will be similar polygons (Def. 17.); and these similar polygons may be divided into an equal number of similar triangles, similarly placed (Book IV. Prop. XXVI.): therefore the two prisms may be divided into an equal number of triangular prisms, having their faces similar and like placed; and therefore, equally inclined (Book VI. Prop. XXI.); hence the prisms will be similar. But these triangular prisms will be to each other as the cubes of their homologous sides, which sides being proportional, the sums of the triangular prisms, that is, the polygonal prisms, will be to each other as the cubes of their homologous sides.

PROPOSITION XX. THEOREM.

Two similar pyramids are to each other as the cubes of their homologous sides.



For, since the pyramids are similar. the solid angles at the vertices will be contained by the same number of similar planes, like placed, and equally inclined to each other (Def. 17.). Hence, the solid angles at the vertices may be made to coincide, or the two pyramids may be so placed as to have the solid angle S common.

In that position, the bases ABCDE, *abcde*, will be parallel; because, since the homologous faces are similar, the angle S*ab* is equal to SAB, and S*bc* to SBC; hence the plane ABC is parallel to the plane *abc* (Book VI. Prop. XIII.). This being proved, let SO be the perpendicular drawn from the vertex S to the plane ABC, and *o* the point where this perpendicular meets the plane *abc*: from what has already been shown, we shall have

$$\mathrm{SO} : \mathrm{S}o :: \mathrm{SA} : \mathrm{S}a :: \mathrm{AB} : ab \text{ (Prop. III.)};$$

and consequently,

$$\tfrac{1}{3}\mathrm{SO} : \tfrac{1}{3}\mathrm{S}o :: \mathrm{AB} : ab.$$

But the bases ABCDE, *abcde*, being similar figures, we have

$$\mathrm{ABCDE} : abcde :: \mathrm{AB}^2 : ab^2 \text{ (Book IV. Prop. XXVII.)}.$$

Multiply the corresponding terms of these two proportions; there results the proportion,

$$\mathrm{ABCDE} \times \tfrac{1}{3}\mathrm{SO} : abcde \times \tfrac{1}{3}\mathrm{S}o :: \mathrm{AB}^3 : ab^3.$$

Now $\mathrm{ABCDE} \times \frac{1}{3}\mathrm{SO}$ is the solidity of the pyramid S-ABCDE, and $abcde \times \frac{1}{3}\mathrm{S}o$ is that of the pyramid S-*abcde* (Prop. XVII.); hence two similar pyramids are to each other as the cubes of their homologous sides.

General Scholium.

The chief propositions of this Book relating to the solidity of polyedrons, may be exhibited in algebraical terms, and so recapitulated in the briefest manner possible.

Let B represent the base of a *prism;* H its altitude: the solidity of the prism will be $\mathrm{B} \times \mathrm{H}$, or BH.

Let B represent the base of a *pyramid;* H its altitude: the solidity of the pyramid will be $\mathrm{B} \times \frac{1}{3}\mathrm{H}$, or $\mathrm{H} \times \frac{1}{3}\mathrm{B}$, or $\frac{1}{3}\mathrm{BH}$.

Let H represent the altitude of the *frustum of a pyramid*, having parallel bases A and B; $\sqrt{\mathrm{AB}}$ will be the mean proportional between those bases; and the solidity of the frustum will be $\frac{1}{3}\mathrm{H} \times (\mathrm{A} + \mathrm{B} + \sqrt{\mathrm{AB}})$.

In fine, let P and p represent the *solidities of two similar prisms or pyramids;* A and a, two homologous edges: then we shall have

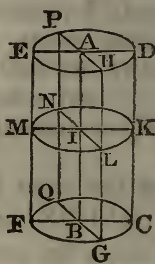
$$\mathrm{P} : p :: \mathrm{A}^3 : a^3$$

BOOK VIII.

THE THREE ROUND BODIES.

Definitions.

1. A *cylinder* is the solid generated by the revolution of a rectangle ABCD, conceived to turn about the immoveable side AB.

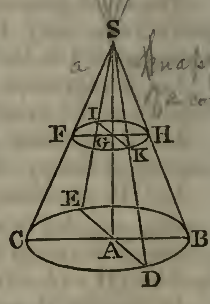
In this movement, the sides AD, BC, continuing always perpendicular to AB, describe equal circles DHP, CGQ, which are called the *bases of the cylinder*, the side CD at the same time describing the *convex surface.*

The immoveable line AB is called the *axis of the cylinder.*

Every section KLM, made in the cylinder, at right angles to the axis, is a circle equal to either of the bases; for, whilst the rectangle ABCD turns about AB, the line KI, perpendicular to AB, describes a circle, equal to the base, and this circle is nothing else than the section made perpendicular to the axis at the point I.

Every section PQG, made through the axis, is a rectangle double of the generating rectangle ABCD.

2. A *cone* is the solid generated by the revolution of a right-angled triangle SAB, conceived to turn about the immoveable side SA.

In this movement, the side AB describes a circle BDCE, named the *base of the cone;* the hypothenuse SB describes the *convex surface of the cone.*



The point S is named the *vertex of the cone,* SA the *axis* or the *altitude,* and SB the *side* or the *apothem.*

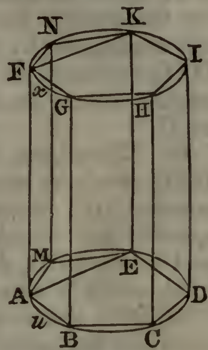
Every section HKFI, at right angles to the axis, is a circle; every section SDE, through the axis, is an isosceles triangle double of the generating triangle SAB.

3. If from the cone S-CDB, the cone S-FKH be cut off by a plane parallel to the base, the remaining solid CBHF is called a *truncated cone,* or the *frustum of a cone*

We may conceive it to be generated by the revolution of a trapezoid ABHG, whose angles A and G are right angles, about the side AG. The immoveable line AG is called the *axis* or *altitude of the frustum*, the circles BDC, HEK, are its *bases*, and BH is its *side*.

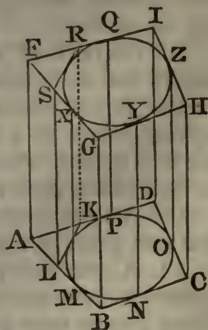
4. Two cylinders, or two cones, are *similar*, when their axes are to each other as the diameters of their bases.

5. If in the circle ACD, which forms the base of a cylinder, a polygon ABCDE be inscribed, a right prism, constructed on this base ABCDE, and equal in altitude to the cylinder, is said to be *inscribed in the cylinder*, or the cylinder to be *circumscribed about the prism*.



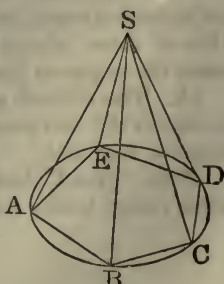
The edges AF, BG, CH, &c. of the prism, being perpendicular to the plane of the base, are evidently included in the convex surface of the cylinder; hence the prism and the cylinder touch one another along these edges.

6. In like manner, if ABCD is a polygon, circumscribed about the base of a cylinder, a right prism, constructed on this base ABCD, and equal in altitude to the cylinder, is said to be *circumscribed about the cylinder*, or the cylinder to be *inscribed in the prism*.

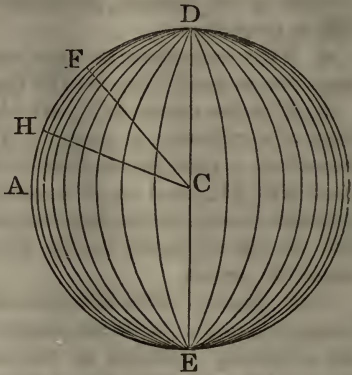


Let M, N, &c. be the points of contact in the sides AB, BC, &c.; and through the points M, N, &c. let MX, NY, &c. be drawn perpendicular to the plane of the base: these perpendiculars will evidently lie both in the surface of the cylinder, and in that of the circumscribed prism; hence they will be their lines of contact.

7. If in the circle ABCDE, which forms the base of a cone, any polygon ABCDE be inscribed, and from the vertices A, B, C, D, E, lines be drawn to S, the vertex of the cone, these lines may be regarded as the sides of a pyramid whose base is the polygon ABCDE and vertex S. The sides of this pyramid are in the convex surface of the cone, and the pyramid is said to be *inscribed* in the cone.



8. The *sphere* is a solid terminated by a curved surface, all the points of which are equally distant from a point within, called the *centre*.



The sphere may be conceived to be generated by the revolution of a semicircle DAE about its diameter DE: or the surface described in this movement, by the curve DAE, will have all its points equally distant from its centre C.

9. Whilst the semicircle DAE revolving round its diameter DE, describes the sphere; any circular sector, as DCF or FCH, describes a solid, which is named a *spherical sector*.

10. The *radius of a sphere* is a straight line drawn from the centre to any point of the surface; the *diameter* or *axis* is a line passing through this centre, and terminated on both sides by the surface.

All the radii of a sphere are equal; all the diameters are equal, and each double of the radius.

11. It will be shown (Prop. VII.) that every section of the sphere, made by a plane, is a circle: this granted, a *great circle* is a section which passes through the centre; a *small circle*, is one which does not pass through the centre.

12. A *plane* is *tangent* to a sphere, when their surfaces have but one point in common.

13. A *zone* is a portion of the surface of the sphere included between two parallel planes, which form its *bases*. One of these planes may be tangent to the sphere; in which case, the zone has only a single base.

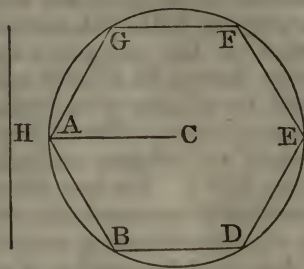
14. A *spherical segment* is the portion of the solid sphere, included between two parallel planes which form its bases. One of these planes may be tangent to the sphere; in which case, the segment has only a single base.

15. The *altitude of a zone* or *of a segment* is the distance between the two parallel planes, which form the bases of the zone or segment.

Note. The Cylinder, the Cone, and the Sphere, are the *three round bodies* treated of in the Elements of Geometry.

PROPOSITION I. THEOREM.

The convex surface of a cylinder is equal to the circumference of its base multiplied by its altitude.

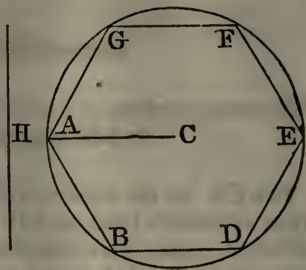


Let CA be the radius of the given cylinder's base, and H its altitude: the circumference whose radius is CA being represented by *circ.* CA, we are to show that the convex surface of the cylinder is equal to *circ.* CA $\times$ H.

Inscribe in the circle any regular polygon, BDEFGA, and construct on this polygon a right prism having its altitude equal to H, the altitude of the cylinder: this prism will be inscribed in the cylinder. The convex surface of the prism is equal to the perimeter of the polygon, multiplied by the altitude H (Book VII. Prop. I.). Let now the arcs which subtend the sides of the polygon be continually bisected, and the number of sides of the polygon indefinitely increased: the perimeter of the polygon will then become equal to *circ.* CA (Book V. Prop. VIII. Cor. 2.), and the convex surface of the prism will coincide with the convex surface of the cylinder. But the convex surface of the prism is equal to the perimeter of its base multiplied by H, whatever be the number of sides: hence, the convex surface of the cylinder is equal to the circumference of its base multiplied by its altitude.

PROPOSITION II. THEOREM.

The solidity of a cylinder is equal to the product of its base by its altitude.



Let CA be the radius of the base of the cylinder, and H the altitude. Let the circle whose radius is CA be represented by *area* CA, it is to be proved that the solidity of the cylinder is equal to *area* CA × H.

Inscribe in the circle any regular polygon BDEFGA, and construct on this polygon a right prism having its altitude equal to H, the altitude of the cylinder: this prism will be inscribed in the cylinder. The solidity of the prism will be equal to the area of the polygon multiplied by the altitude H (Book VII. Prop. XIV.). Let now the number of sides of the polygon be indefinitely increased: the solidity of the new prism will still be equal to its base multiplied by its altitude.

But when the number of sides of the polygon is indefinitely increased, its area becomes equal to the *area* CA, and its perimeter coincides with *circ.* CA (Book V. Prop. VIII. Cor. 1. & 2.); the inscribed prism then coincides with the cylinder, since their altitudes are equal, and their convex surfaces perpendicular to the common base: hence the two solids will be equal; therefore the solidity of a cylinder is equal to the product of its base by its altitude.

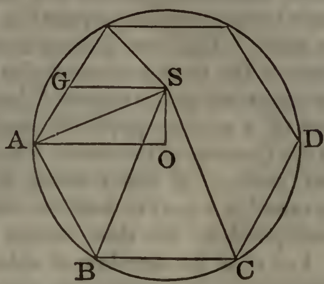
Cor. 1. Cylinders of the same altitude are to each other as their bases; and cylinders of the same base are to each other as their altitudes.

Cor. 2. Similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the diameters of their bases. For the bases are as the squares of their diameters; and the cylinders being similar, the diameters of their bases are to each other as the altitudes (Def. 4.); hence the bases are as the squares of the altitudes; hence the bases, multiplied by the altitudes, or the cylinders themselves, are as the cubes of the altitudes.

Scholium. Let R be the radius of a cylinder's base; H the altitude: the surface of the base will be $\pi.R^2$ (Book V. Prop. XII. Cor. 2.); and the solidity of the cylinder will be $\pi R^2 \times H$ or $\pi.R^2.H$.

PROPOSITION III. THEOREM.

The convex surface of a cone is equal to the circumference of its base, multiplied by half its side.



Let the circle ABCD be the base of a cone, S the vertex, SO the altitude, and SA the side: then will its convex surface be equal to *circ.* OA $\times \frac{1}{2}$SA.

For, inscribe in the base of the cone any regular polygon ABCD, and on this polygon as a base conceive a pyramid to be constructed having S for its vertex: this pyramid will be a regular pyramid, and will be inscribed in the cone.

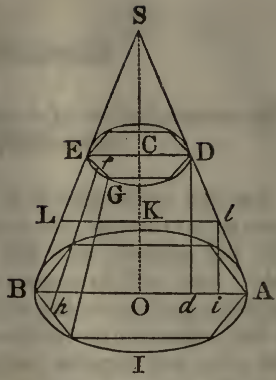
From S, draw SG perpendicular to one of the sides of the polygon. The convex surface of the inscribed pyramid is equal to the perimeter of the polygon which forms its base, multiplied by half the slant height SG (Book VII. Prop. IV.). Let now the number of sides of the inscribed polygon be indefinitely increased; the perimeter of the inscribed polygon will then become equal to *circ.* OA, the slant height SG will become equal to the side SA of the cone, and the convex surface of the pyramid to the convex surface of the cone. But whatever be the number of sides of the polygon which forms the base, the convex surface of the pyramid is equal to the perimeter of the base multiplied by half the slant height: hence the convex surface of a cone is equal to the circumference of the base multiplied by half the side.

Scholium. Let L be the side of a cone, R the radius of its base; the circumference of this base will be $2\pi.R$, and the surface of the cone will be $2\pi R \times \frac{1}{2}L$, or πRL.

PROPOSITION IV. THEOREM.

The convex surface of the frustum of a cone is equal to its side multiplied by half the sum of the circumferences of its two bases

Let BIA-DE be a frustum of a cone: then will its convex surface be equal to $AD \times \left(\frac{circ.OA + circ.CD}{2}\right)$.



For, inscribe in the bases of the frustums two regular polygons of the same number of sides, and having their homologous sides parallel, each to each. The lines joining the vertices of the homologous angles may be regarded as the edges of the frustum of a regular pyramid inscribed in the frustum of the cone. The convex surface of the frustum of the pyramid is equal to half the sum of the perimeters of its bases multiplied by the slant height fh (Book VII. Prop. IV. Cor.).

Let now the number of sides of the inscribed polygons be indefinitely increased: the perimeters of the polygons will become equal to the circumferences BIA, EGD; the slant height fh will become equal to the side AD or BE, and the surfaces of the two frustums will coincide and become the same surface.

But the convex surface of the frustum of the pyramid will still be equal to half the sum of the perimeters of the upper and lower bases multiplied by the slant height: hence the surface of the frustum of a cone is equal to its side multiplied by half the sum of the circumferences of its two bases.

Cor. Through l, the middle point of AD, draw lKL parallel to AB, and li, Dd, parallel to CO. Then, since Al, lD, are equal, Ai, id, will also be equal (Book IV. Prop. XV. Cor. 2.): hence, Kl is equal to $\frac{1}{2}(OA + CD)$. But since the circumferences of circles are to each other as their radii (Book V. Prop. XI.), the $circ.\ Kl = \frac{1}{2}(circ.\ OA + circ.\ CD)$; therefore, *the convex surface of a frustum of a cone is equal to its side multiplied by the circumference of a section at equal distances from the two bases.*

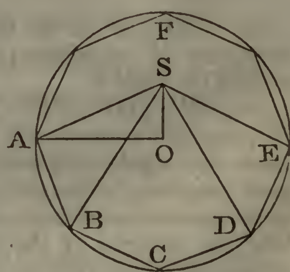
Scholium. If a line AD, lying wholly on one side of the line OC, and in the same plane, make a revolution around OC, the surface described by AD will have for its measure $AD \times \left(\frac{circ.AO + circ.DC}{2}\right)$, or $AD \times circ.\ lK$; the lines AO, DC, lK, being perpendiculars, let fall from the extremities and from the middle point of AD, on the axis OC.

For, if AD and OC are produced till they meet in S, the surface described by AD is evidently the frustum of a cone having AO and DC for the radii of its bases, the vertex of the whole cone being S. Hence this surface will be measured as we have said.

This measure will always hold good, even when the point D falls on S, and thus forms a whole cone; and also when the line AD is parallel to the axis, and thus forms a cylinder. In the first case DC would be nothing; in the second, DC would be equal to AO and to *l*K.

PROPOSITION V. THEOREM.

The solidity of a cone is equal to its base multiplied by a third of its altitude.



Let SO be the altitude of a cone, OA the radius of its base, and let the area of the base be designated by *area* OA: it is to be proved that the solidity of the cone is equal to *area* $OA \times \frac{1}{3}SO$.

Inscribe in the base of the cone any regular polygon ABDEF, and join the vertices A, B, C, &c. with the vertex S of the cone: then will there be inscribed in the cone a regular pyramid having the same vertex as the cone, and having for its base the polygon ABDEF. The solidity of this pyramid is equal to its base multiplied by one third of its altitude (Book VII. Prop. XVII.). Let now the number of sides of the polygon be indefinitely increased: the polygon will then become equal to the circle, and the pyramid and cone will coincide and become equal. But the solidity of the pyramid is equal to its base multiplied by one third of its altitude, whatever be the number of sides of the polygon which forms its base: hence the solidity of the cone is equal to its base multiplied by a third of its altitude.

Cor. A cone is the third of a cylinder having the same base and the same altitude; whence it follows,

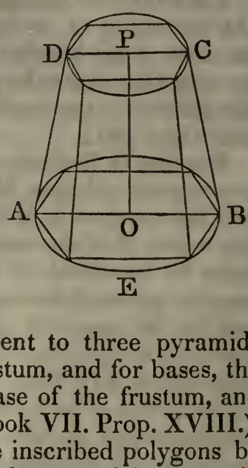
1. That cones of equal altitudes are to each other as their bases;
2. That cones of equal bases are to each other as their altitudes;
3. That similar cones are as the cubes of the diameters of their bases, or as the cubes of their altitudes.

Cor. 2. The solidity of a cone is equivalent to the solidity of a pyramid having an equivalent base and the same altitude (Book VII. Prop. XVII.).

Scholium. Let R be the radius of a cone's base, H its altitude; the solidity of the cone will be $\pi R^2 \times \frac{1}{3}H$, or $\frac{1}{3}\pi R^2 H$.

PROPOSITION VI. THEOREM

The solidity of the frustum of a cone is equal to the sum of the solidities of three cones whose common altitude is the altitude of the frustum, and whose bases are, the upper base of the frustum, the lower base of the frustum, and a mean proportional between them.



Let AEB-CD be the frustum of a cone, and OP its altitude; then will its solidity be equal to

$\frac{1}{3}\pi \times OP \times (AO^2 + DP^2 + AO \times DP)$.

For, inscribe in the lower and upper bases two regular polygons having the same number of sides, and having their homologous sides parallel, each to each. Join the vertices of the homologous angles and there will then be inscribed in the frustum of the cone, the frustum of a regular pyramid. The solidity of the frustum of the pyramid is equivalent to three pyramids having the common altitude of the frustum, and for bases, the lower base of the frustum, the upper base of the frustum, and a mean proportional between them (Book VII. Prop. XVIII.).

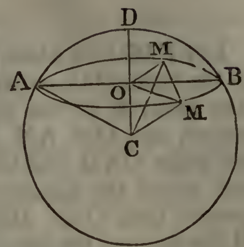
Let now, the number of sides of the inscribed polygons be indefinitely increased: the bases of the frustum of the pyramid will then coincide with the bases of the frustum of the cone, and the two frustums will coincide and become the same solid. Since the area of a circle is equal to $R^2.\pi$ (Book V. Prop. XII. Cor. 2.), the expression for the solidities of the frustum will become

for the first pyramid	$\frac{1}{3}OP \times OA^2\pi$.
for the second	$\frac{1}{3}OP \times PD^2.\pi$
for the third	$\frac{1}{3}OP \times AO \times PD.\pi$; since

$AO \times PD.\pi$ is a mean proportional between $OA^2.\pi$ and $PD^2.\pi$ Hence the solidity of the frustum of the cone is measured by $\frac{1}{3}\pi OP \times (OA^2 + PD^2 + AO \times PD)$.

PROPOSITION VII. THEOREM.

Every section of a sphere, made by a plane, is a circle.



Let AMB be a section, made by a plane, in the sphere whose centre is C. From the point C, draw CO perpendicular to the plane AMB; and different lines CM, CM, to different points of the curve AMB, which terminates the section.

The oblique lines CM, CM, CA, are equal, being radii of the sphere; hence they are equally distant from the perpendicular CO (Book VI. Prop. V. Cor.); therefore all the lines OM, OM, OB, are equal; consequently the section AMB is a circle, whose centre is O.

Cor 1. If the section passes through the centre of the sphere, its radius will be the radius of the sphere; hence all great circles are equal.

Cor. 2. Two great circles always bisect each other; for their common intersection, passing through the centre, is a diameter.

Cor. 3. Every great circle divides the sphere and its surface into two equal parts: for, if the two hemispheres were separated and afterwards placed on the common base, with their convexities turned the same way, the two surfaces would exactly coincide, no point of the one being nearer the centre than any point of the other.

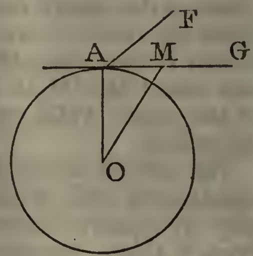
Cor. 4. The centre of a small circle, and that of the sphere, are in the same straight line, perpendicular to the plane of the small circle.

Cor. 5. Small circles are the less the further they lie from the centre of the sphere; for the greater CO is, the less is the chord AB, the diameter of the small circle AMB.

Cor. 6. An arc of a great circle may always be made to pass through any two given points of the surface of the sphere; for the two given points, and the centre of the sphere make three points which determine the position of a plane. But if the two given points were at the extremities of a diameter, these two points and the centre would then lie in one straight line, and an infinite number of great circles might be made to pass through the two given points.

PROPOSITION VIII. THEOREM.

Every plane perpendicular to a radius at its extremity is tangent to the sphere.

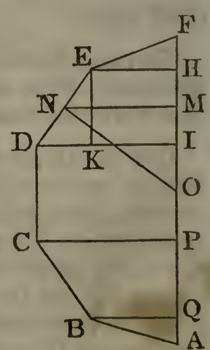


Let FAG be a plane perpendicular to the radius OA, at its extremity A. Any point M in this plane being assumed, and OM, AM, being drawn, the angle OAM will be a right angle, and hence the distance OM will be greater than OA. Hence the point M lies without the sphere; and as the same can be shown for every other point of the plane FAG, this plane can have no point but A common to it and the surface of the sphere; hence it is a tangent plane (Def. 12.)

Scholium. In the same way it may be shown, that two spheres have but one point in common, and therefore touch each other, when the distance between their centres is equal to the sum, or the difference of their radii; in which case, the centres and the point of contact lie in the same straight line.

PROPOSITION IX. LEMMA.

If a regular semi-polygon be revolved about a line passing through the centre and the vertices of two opposite angles, the surface described by its perimeter will be equal to the axis multiplied by the circumference of the inscribed circle.



Let the regular semi-polygon ABCDEF, be revolved about the line AF as an axis: then will the surface described by its perimeter be equal to AF multiplied by the circumference of the inscribed circle.

From E and D, the extremities of one of the equal sides, let fall the perpendiculars EH, DI, on the axis AF, and from the centre O draw ON perpendicular to the side DE: ON will be the radius of the inscribed circle (Book V. Prop. II.). Now, the surface described in the revolution by any one side of the regular polygon, as DE, has been shown to be equal to DE $\times$ *circ.* NM (Prop. IV. Sch.). But since the triangles EDK, ONM, are similar (Book IV. Prop. XXI.), ED : EK or HI : : ON : NM, or as *circ.* ON . *circ.* NM ; hence

$$ED \times circ.\ NM = HI \times circ.\ ON;$$

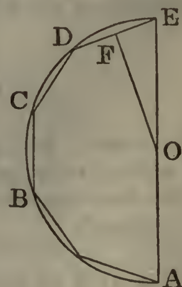
and since the same may be shown for each of the other sides it is plain that the surface described by the entire perimeter i equal to

$$(FH+HI+IP+PQ+QA) \times circ\ ON = AF \times circ.\ ON.$$

Cor. The surface described by any portion of the perimeter, as EDC, is equal to the distance between the two perpendiculars let fall from its extremities on the axis, multiplied by the circumference of the inscribed circle. For, the surface described by DE is equal to HI $\times$ *circ.* ON, and the surface described by DC is equal to IP $\times$ *circ.* ON : hence the surface described by ED+DC, is equal to (HI+IP) $\times$ *circ.* ON, or equal to HP $\times$ *circ.* ON.

PROPOSITION X. THEOREM.

The surface of a sphere is equal to the product of its diameter by the circumference of a great circle.

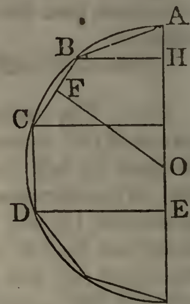


Let ABCDE be a semicircle. Inscribe in it any regular semi-polygon, and from the centre O draw OF perpendicular to one of the sides.

Let the semicircle and the semi-polygon be revolved about the axis AE : the semicircumference ABCDE will describe the surface of a sphere (Def. 8.) ; and the perimeter of the semi-polygon will describe a surface which has for its measure AE $\times$ *circ.* OF (Prop. IX.), and this will be true whatever be the number of sides of the polygon. But if the number of sides of the polygon be indefinitely increased, its perimeter will coincide with the circumference ABCDE, the perpendicular OF will become equal to OE, and the surface described by the perimeter of the semi-polygon will then be the same as that described by the semicircumference ABCDE. Hence the surface of the sphere is equal to AE $\times$ *circ.* OE.

Cor. Since the area of a great circle is equal to the product of its circumference by half the radius, or one fourth of the diameter (Book V. Prop. XII.), it follows that *the surface of a sphere is equal to four of its great circles:* that is, equal to $4\pi.OA^2$ (Book V. Prop. XII. Cor. 2.).

Scholium 1. *The surface of a zone is equal to its altitude multiplied by the circumference of a great circle.*



For, the surface described by any portion of the perimeter of the inscribed polygon, as BC+CD, is equal to EH×*circ.* OF (Prop. IX. Cor.). But when the number of sides of the polygon is indefinitely increased, BC+CD, becomes the arc BCD, OF becomes equal to OA, and the surface described by BC+CD, becomes the surface of the zone described by the arc BCD: hence the surface of the zone is equal to EH×*circ.* OA.

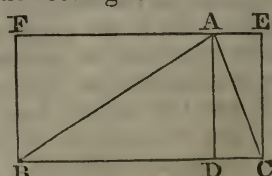
Scholium 2. When the zone has but one base, as the zone described by the arc ABCD, its surface will still be equal to the altitude AE multiplied by the circumference of a great circle.

Scholium 3. Two zones, taken in the same sphere or in equal spheres, are to each other as their altitudes; and any zone is to the surface of the sphere as the altitude of the zone is to the diameter of the sphere.

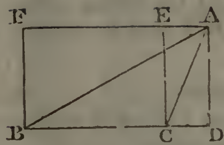
PROPOSITION XI. LEMMA.

If a triangle and a rectangle, having the same base and the same altitude, turn together about the common base, the solid described by the triangle will be a third of the cylinder described by the rectangle.

Let ACB be the triangle, and BE the rectangle.



On the axis, let fall the perpendicular AD: the cone described by the triangle ABD is the third part of the cylinder described by the rectangle AFBD (Prop. V. Cor.); also the cone described by the triangle ADC is the third part of the cylinder described by the rectangle ADCE; hence the sum of the two cones, or the solid described by ABC, is the third part of the two cylinders taken together, or of the cylinder described by the rectangle BCEF.



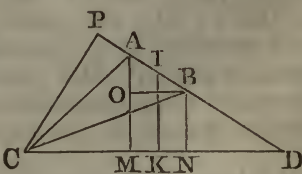
If the perpendicular AD falls without the triangle; the solid described by ABC will, in that case, be the difference of the two cones described by ABD and ACD; but at the same time, the cylinder described by BCEF will be the difference of the two cylinders described by AFBD and AECD. Hence the solid, described by the revolution of the triangle, will still be a third part of the cylinder described by the revolution of the rectangle having the same base and the same altitude.

Scholium. The circle of which AD is radius, has for its measure $\pi \times AD^2$; hence $\pi \times AD^2 \times BC$ measures the cylinder described by BCEF, and $\frac{1}{3}\pi \times AD^2 \times BC$ measures the solid described by the triangle ABC.

PROPOSITION XII. LEMMA.

If a triangle be revolved about a line drawn at pleasure through its vertex, the solid described by the triangle will have for its measure, the area of the triangle multiplied by two thirds of the circumference traced by the middle point of the base.

Let CAB be the triangle, and CD the line about which it revolves.



Produce the side AB till it meets the axis CD in D; from the points A and B, draw AM, BN, perpendicular to the axis, and CP perpendicular to DA produced.

The solid described by the triangle CAD is measured by $\frac{1}{3}\pi \times AM^2 \times CD$ (Prop. XI. Sch.); the solid described by the triangle CBD is measured by $\frac{1}{3}\pi \times BN^2 \times CD$; hence the difference of those solids, or the solid described by ABC, will have for its measure $\frac{1}{3}\pi(AM^2 - BN^2) \times CD$.

To this expression another form may be given. From I, the middle point of AB, draw IK perpendicular to CD; and through B, draw BO parallel to CD: we shall have $AM + BN = 2IK$ (Book IV. Prop. VII.); and $AM - BN = AO$; hence $(AM + BN) \times (AM - NB)$, or $AM^2 - BN^2 = 2IK \times AO$ (Book IV. Prop X.). Hence the measure of the solid in question is expressed by

$$\tfrac{2}{3}\pi \times IK \times AO \times CD.$$

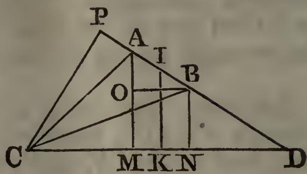
But CP being drawn perpendicular to AB, the triangles ABO DCP will be similar, and give the proportion

$$AO : CP :: AB : CD;$$

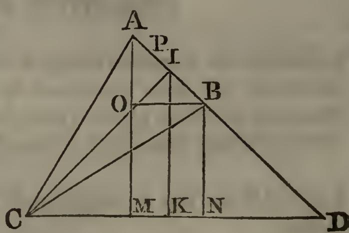
hence $$AO \times CD = CP \times AB;$$

but $CP \times AB$ is double the area of the triangle ABC; hence we have

$$AO \times CD = 2ABC;$$

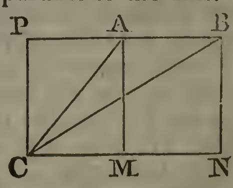


hence the solid described by the triangle ABC is also measured by $\frac{4}{3}\pi \times ABC \times IK$, or which is the same thing, by $ABC \times \frac{2}{3}$*circ.* IK, *circ.* IK being equal to $2\pi \times IK$. Hence *the solid described by the revolution of the triangle* ABC, *has for its measure the area of this triangle multiplied by two thirds of the circumference traced by* I, *the middle point of the base.*



Cor. If the side AC = CB, the line CI will be perpendicular to AB, the area ABC will be equal to $AB \times \frac{1}{2}CI$, and the solidity $\frac{4}{3}\pi \times ABC \times IK$ will become $\frac{2}{3}\pi \times AB \times IK \times CI$. But the triangles ABO, CIK, are similar, and give the proportion AB : BO or MN : : CI : IK; hence $AB \times IK = MN \times CI$; hence the solid described by the isosceles triangle ABC will have for its measure $\frac{2}{3}\pi \times CI^2 \times MN$: that is, *equal to two thirds of* π *into the square of the perpendicular let fall on the base, into the distance between the two perpendiculars let fall on the axis.*

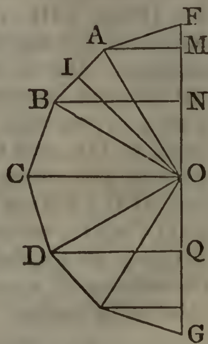
Scholium. The general solution appears to include the supposition that AB produced will meet the axis; but the results would be equally true, though AB were parallel to the axis.



Thus, the cylinder described by AMNB is equal to $\pi.AM^2.MN$; the cone described by ACM is equal to $\frac{1}{3}\pi.AM^2.CM$, and the cone described by BCN to $\frac{1}{3}\pi AM^2\ CN$. Add the first two solids and take away the third; we shall have the solid described by ABC equal to $\pi.AM^2.(MN+\frac{1}{3}CM-\frac{1}{3}CN)$: and since $CN-CM=MN$, this expression is reducible to $\pi.AM^2.\frac{2}{3}MN$, or $\frac{2}{3}\pi.CP^2.MN$; which agrees with the conclusion found above.

PROPOSITION XIII. LEMMA.

If a regular semi-polygon be revolved about a line passing through the centre and the vertices of two opposite angles, the solid described will be equivalent to a cone, having for its base the inscribed circle, and for its altitude twice the axis about which the semi-polygon is revolved.



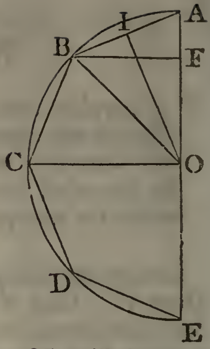
Let the semi-polygon FABG be revolved about FG: then, if OI be the radius of the inscribed circle, the solid described will be measured by $\frac{1}{3}$*area* OI $\times$ 2FG.

For, since the polygon is regular, the triangles OFA, OAB, OBC, &c. are equal and isosceles, and all the perpendiculars let fall from O on the bases FA, AB, &c. will be equal to OI, the radius of the inscribed circle.

Now, the solid described by OAB is measured by $\frac{2}{3}\pi\ \mathrm{OI}^2 \times \mathrm{MN}$ (Prop. XII. Cor.); the solid described by the triangle OFA has for its measure $\frac{2}{3}\pi\mathrm{OI}^2\times\mathrm{FM}$, the solid described by the triangle OBC, has for its measure $\frac{2}{3}\pi\mathrm{OI}^2\times\mathrm{NO}$, and since the same may be shown for the solid described by each of the other triangles, it follows that the entire solid described by the semi-polygon is measured by $\frac{2}{3}\pi\mathrm{OI}^2.(\mathrm{FM}+\mathrm{MN}+\mathrm{NO}+\mathrm{OQ}+\mathrm{QG})$, or $\frac{2}{3}\pi\mathrm{OI}^2\times\mathrm{FG}$; which is also equal to $\frac{1}{3}\pi\mathrm{OI}^2\times 2\mathrm{FG}$. But $\pi.\mathrm{OI}^2$ is the area of the inscribed circle (Book V. Prop. XII. Cor. 2.): hence the solidity is equivalent to a cone whose base is *area* OI, and altitude 2FG.

PROPOSITION XIV. THEOREM.

The solidity of a sphere is equal to its surface multiplied by a third of its radius.



Inscribe in the semicircle ABCDE a regular semi-polygon, having any number of sides, and let OI be the radius of the circle inscribed in the polygon.

If the semicircle and semi-polygon be revolved about EA, the semicircle will describe a sphere, and the semi-polygon a solid which has for its measure $\frac{2}{3}\pi OI^2 \times EA$ (Prop. XIII.); and this will be true whatever be the number of sides of the polygon. But if the number of sides of the polygon be indefinitely increased, the semi-polygon will become the semicircle, OI will become equal to OA, and the solid described by the semi-polygon will become the sphere: hence the solidity of the sphere is equal to $\frac{2}{3}\pi OA^2 \times EA$, or by substituting 2OA for EA, it becomes $\frac{4}{3}\pi.OA^2 \times OA$, which is also equal to $4\pi OA^2 \times \frac{1}{3}OA$. But $4\pi.OA^2$ is equal to the surface of the sphere (Prop. X. Cor.): hence the solidity of a sphere is equal to its surface multiplied by a third of its radius.

Scholium 1. *The solidity of every spherical sector is equal to the zone which forms its base, multiplied by a third of the radius.*

For, the solid described by any portion of the regular polygon, as the isosceles triangle OAB, is measured by $\frac{2}{3}\pi OI^2 \times AF$ (Prop. XII. Cor.); and when the polygon becomes the circle, the portion OAB becomes the sector AOB, OI becomes equal to OA, and the solid described becomes a spherical sector. But its measure then becomes equal to $\frac{2}{3}\pi.AO^2 \times AF$, which is equal to $2\pi.AO \times AF \times \frac{1}{3}AO$. But $2\pi.AO$ is the circumference of a great circle of the sphere (Book V. Prop. XII. Cor. 2.), which being multiplied by AF gives the surface of the zone which forms the base of the sector (Prop. X. Sch. 1.): and the proof is equally applicable to the spherical sector described by the circular sector BOC: hence, the solidity of the spherical sector is equal to the zone which forms its base, multiplied by a third of the radius.

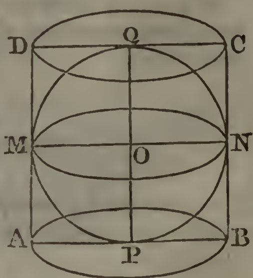
Scholium 2. Since the surface of a sphere whose radius is R, is expressed by $4\pi R^2$ (Prop. X. Cor.), it follows that the surfaces of spheres are to each other as the squares of their radii; and since their solidities are as their surfaces multiplied by their radii, it follows that *the solidities of spheres are to each other as the cubes of their radii, or as the cubes of thei diameters.*

Scholium 3. Let R be the radius of a sphere; its surface will be expressed by $4\pi R^2$, and its solidity by $4\pi R^2 \times \frac{1}{3}R$, or $\frac{4}{3}\pi R^3$. If the diameter is called D, we shall have $R=\frac{1}{2}D$, and $R^3=\frac{1}{8}D^3$: hence the solidity of the sphere may likewise be expressed by

$$\tfrac{4}{3}\pi \times \tfrac{1}{8}D^3=\tfrac{1}{6}\pi D^3.$$

PROPOSITION XV. THEOREM.

The surface of a sphere is to the whole surface of the circumscribed cylinder, including its bases, as 2 is to 3: and the solidities of these two bodies are to each other in the same ratio.



Let MPNQ be a great circle of the sphere; ABCD the circumscribed square: if the semicircle PMQ and the half square PADQ are at the same time made to revolve about the diameter PQ, the semicircle will generate the sphere, while the half square will generate the cylinder circumscribed about that sphere.

The altitude AD of the cylinder is equal to the diameter PQ; the base of the cylinder is equal to the great circle, since its diameter AB is equal to MN; hence, the convex surface of the cylinder is equal to the circumference of the great circle multiplied by its diameter (Prop. 1.). This measure is the same as that of the surface of the sphere (Prop. X.): hence *the surface of the sphere is equal to the convex surface of the circumscribed cylinder.*

But the surface of the sphere is equal to four great circles; hence the convex surface of the cylinder is also equal to four great circles: and adding the two bases, each equal to a great circle, the total surface of the circumscribed cylinder will be equal to six great circles; hence the surface of the sphere is to the total surface of the circumscribed cylinder as 4 is to 6, or as 2 is to 3; which was the first branch of the Proposition.

In the next place, since the base of the circumscribed cylinder is equal to a great circle, and its altitude to the diameter, the solidity of the cylinder will be equal to a great circle multiplied by its diameter (Prop. II.). But the solidity of the sphere is equal to four great circles multiplied by a third of the radius (Prop. XIV.); in other terms, to one great circle multiplied by $\frac{4}{3}$ of the radius, or by $\frac{2}{3}$ of the diameter; hence the sphere is to the circumscribed cylinder as 2 to 3, and consequently the solidities of these two bodies are as their surfaces

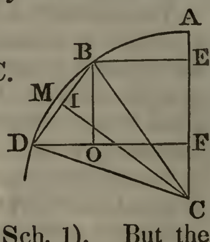
Scholium. Conceive a polyedron, all of whose faces touch the sphere; this polyedron may be considered as formed of pyramids, each having for its vertex the centre of the sphere, and for its base one of the polyedron's faces. Now it is evident that all these pyramids will have the radius of the sphere for their common altitude: so that each pyramid will be equal to one face of the polyedron multiplied by a third of the radius: hence the whole polyedron will be equal to its surface multiplied by a third of the radius of the inscribed sphere.

It is therefore manifest, that the solidities of polyedrons circumscribed about the sphere are to each other as the surfaces of those polyedrons. Thus the property, which we have shown to be true with regard to the circumscribed cylinder, is also true with regard to an infinite number of other bodies.

We might likewise have observed that the surfaces of polygons, circumscribed about the circle, are to each other as their perimeters.

PROPOSITION XVI. PROBLEM.

If a circular segment be supposed to make a revolution about a diameter exterior to it, required the value of the solid which it describes.



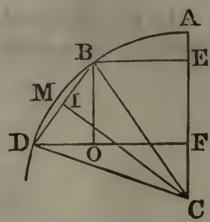
Let the segment BMD revolve about AC. On the axis, let fall the perpendiculars BE, DF; from the centre C, draw CI perpendicular to the chord BD; also draw the radii CB, CD.

The solid described by the sector BCD is measured by $\frac{2}{3}\pi$ $CB^2.EF$ (Prop. XIV. Sch. 1). But the solid described by the isosceles triangle DCB has for its measure $\frac{2}{3}\pi.CI^2.EF$ (Prop. XII. Cor.); hence the solid described by the segment $BMD=\frac{2}{3}\pi.EF.(CB^2-CI^2)$. Now, in the right-angled triangle CBI, we have $CB^2-CI^2=BI^2=\frac{1}{4}BD^2$; hence the solid described by the segment BMD will have for its measure $\frac{2}{3}\pi.EF.\frac{1}{4}BD^2$, or $\frac{1}{6}\pi.BD^2.EF$: that is *one sixth of π into the square of the chord, into the distance between the two perpendiculars let fall from the extremities of the arc on the axis.*

Scholium. The solid described by the segment BMD is to the sphere which has BD for its diameter, as $\frac{1}{6}\pi.BD^2.EF$ is to $\frac{1}{6}\pi.BD^3$, or as EF to BD.

PROPOSITION XVII. THEOREM.

Every segment of a sphere is measured by the half sum of its bases multiplied by its altitude, plus *the solidity of a sphere whose diameter is this same altitude.*



Let BE, DF, be the radii of the two bases of the segment, EF its altitude, the segment being described by the revolution of the circular space BMDFE about the axis FE. The solid described by the segment BMD is equal to $\frac{1}{6}\pi.BD^2.EF$ (Prop. XVI.); and the truncated cone described by the trapezoid BDFE is equal to $\frac{1}{3}\pi.EF.(BE^2+DF^2+BE.DF)$ (Prop. VI.); hence the segment of the sphere, which is the sum of those two solids, must be equal to $\frac{1}{6}\pi.EF.(2BE^2+2DF^2+2BE.DF+BD^2)$ But, drawing BO parallel to EF, we shall have $DO=DF-BE$, hence $DO^2=DF^2-2DF.BE+BE^2$ (Book IV. Prop. IX.); and consequently $BD^2=BO^2+DO^2=EF^2+DF^2-2DF.BE+BE^2$. Put this value in place of BD^2 in the expression for the value of the segment, omitting the parts which destroy each other; we shall obtain for the solidity of the segment,

$$\tfrac{1}{6}\pi EF.(3BE^2+3DF^2+EF^2),$$

an expression which may be decomposed into two parts; the one $\frac{1}{6}\pi.EF.(3BE^2+3DF^2)$, or $EF.\left(\dfrac{\pi.BE^2+\pi.DF^2}{2}\right)$ being the half sum of the bases multiplied by the altitude; while the other $\frac{1}{6}\pi.EF^3$ represents the sphere of which EF is the diameter (Prop. XIV. Sch.): hence every segment of a sphere, &c.

Cor. If either of the bases is nothing, the segment in question becomes a spherical segment with a single base; hence *any spherical segment, with a single base, is equivalent to half the cylinder having the same base and the same altitude*, plus *the sphere of which this altitude is the diameter.*

General Scholium.

Let R be the radius of a cylinder's base, H its altitude: the solidity of the cylinder will be $\pi R^2\times H$, or πR^2H.

Let R be the radius of a cone's base, H its altitude: the solidity of the cone will be $\pi R^2\times\frac{1}{3}H$, or $\frac{1}{3}\pi R^2H$.

Let A and B be the radii of the bases of a truncated cone, H its altitude : the solidity of the truncated cone will be $\frac{1}{3}\pi.H.(A^2+B^2+AB)$.

Let R be the radius of a sphere ; its solidity will be $\frac{4}{3}\pi R^3$.

Let R be the radius of a spherical sector, H the altitude of the zone, which forms its base : the solidity of the sector will be $\frac{2}{3}\pi R^2H$.

Let P and Q be the two bases of a spherical segment, H its altitude : the solidity of the segment will be $\frac{P+Q}{2}.H+\frac{1}{6}\pi.H^3$.

If the spherical segment has but one base, the other being nothing, its solidity will be $\frac{1}{2}PH+\frac{1}{6}\pi H^3$.

BOOK IX.

OF SPHERICAL TRIANGLES AND SPHERICAL POLYGONS

Definitions.

1. A *spherical triangle* is a portion of the surface of a sphere, bounded by three arcs of great circles.

These arcs are named the *sides* of the triangle, and are always supposed to be each less than a semi-circumference. The angles, which their planes form with each other, are the angles of the triangle.

2. A spherical triangle takes the name of *right-angled*, *isosceles*, *equilateral*, in the same cases as a rectilineal triangle.

3. A *spherical polygon* is a portion of the surface of a sphere terminated by several arcs of great circles.

4. A *lune* is that portion of the surface of a sphere, which is included between two great semi-circles meeting in a common diameter.

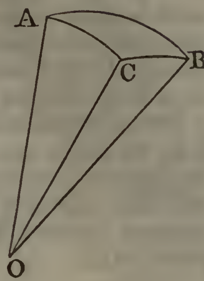
5. A *spherical wedge* or *ungula* is that portion of the solid sphere, which is included between the same great semi-circles, and has the lune for its base.

6. A *spherical pyramid* is a portion of the solid sphere, included between the planes of a solid angle whose vertex is the centre. The *base* of the pyramid is the spherical polygon intercepted by the same planes.

7. The *pole of a circle* of a sphere is a point in the surface equally distant from all the points in the circumference of this circle. It will be shown (Prop. V.) hat every circle, great or small, has always two poles.

PROPOSITION I. THEOREM.

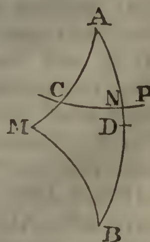
In every spherical triangle, any side is less than the snm of the other two.



Let O be the centre of the sphere, and ACB the triangle; draw the radii OA, OB, OC. Imagine the planes AOB, AOC, COB, to be drawn; these planes will form a solid angle at the centre O; and the angles AOB, AOC, COB, will be measured by AB, AC, BC, the sides of the spherical triangle. But each of the three plane angles forming a solid angle is less than the sum of the other two (Book VI. Prop. XIX.); hence any side of the triangle ABC is less than the sum of the other two.

PROPOSITION II. THEOREM.

The shortest path from one point to another, on the surface of a sphere, is the arc of the great circle which joins the two given points.



Let ANB be the arc of a great circle which joins the points A and B; then will it be the shortest path between them.

1st. If two points N and B, be taken on the arc of a great circle, at unequal distances from the point A, the shortest distance from B to A will be greater than the shortest distance from N to A.

For, about A as a pole describe a circumference CNP. Now, the line of shortest distance from B to A must cross this circumference at some point as P. But the shortest distance from P to A whether it be the arc of a great circle or any other line, is equal to the shortest distance from N to A; for, by passing the arc of a great circle through P and A, and revolving it about the diameter passing through A, the point P may be made to coincide with N, when the shortest distance from P to A will coincide with the shortest distance from N to A: hence, the shortest distance from B to A, will be greater than the shortest distance from N to A, by the shortest distance from B to P.

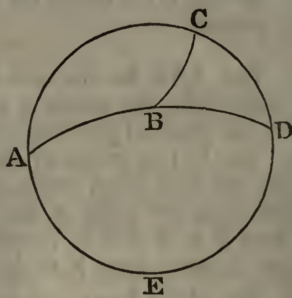
If the point B be taken without the arc AN, still making AB greater than AN, it may be proved in a manner entirely similar to the above, that the shortest distance from B to A will be greater than the shortest distance from N to A.

If now, there be a shorter path between the points B and A, than the arc BDA of a great circle, let M be a point of the short est distance possible; then through M draw MA, MB, arcs of great circles, and take BD equal to BM. By the last theorem, BDA < BM + MA; take BD = BM from each, and there will remain AD < AM. Now, since BM = BD, the shortest path from B to M is equal to the shortest path from B to D: hence if we suppose two paths from B to A, one passing through M and the other through D, they will have an equal part in each; viz. the part from B to M equal to the part from B to D.

But by hypothesis, the path through M is the shortest path from B to A: hence the shortest path from M to A must be less than the shortest path from D to A, whereas it is greater since the arc MA is greater than DA: hence, no point of the shortest distance between B and A can lie out of the arc of the great circle BDA.

PROPOSITION III. THEOREM.

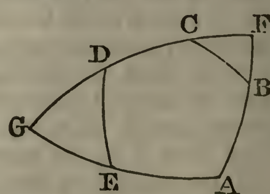
The sum of the three sides of a spherical triangle is less than the circumference of a great circle.



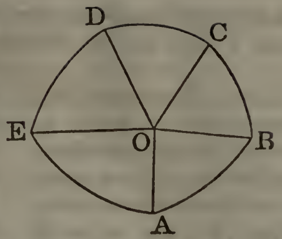
Let ABC be any spherical triangle; produce the sides AB, AC, till they meet again in D. The arcs ABD, ACD, will be semicircumferences, since two great circles always bisect each other (Book VIII. Prop. VII. Cor. 2.). But in the triangle BCD, we have the side BC < BD + CD (Prop I.); add AB + AC to both; we shall have AB + AC + BC < ABD + ACD, that is to say, less than a circumference.

PROPOSITION IV. THEOREM

The sum of all the sides of any spherical polygon is less than the circumference of a great circle.



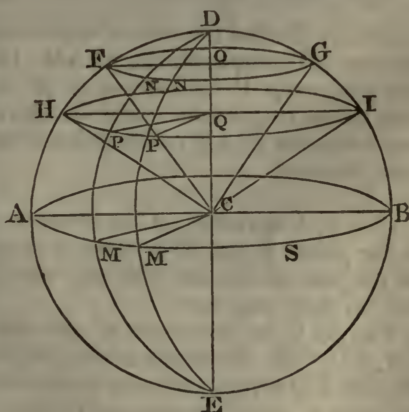
Take the pentagon ABCDE, for example. Produce the sides AB, DC, till they meet in F; then since BC is less than BF + CF, the perimeter of the pentagon ABCDE will be less than that of the quadrilateral AEDF. Again, produce the sides AE, FD, till they meet in G; we shall have ED < EG + DG; hence the perimeter of the quadrilateral AEDF is less than that of the triangle AFG; which last is itself less than the circumference of a great circle; hence, for a still stronger reason, the perimeter of the polygon ABCDE is less than this same circumference.



Scholium. This proposition is fundamentally the same as (Book VI. Prop. XX.); for, O being the centre of the sphere, a solid angle may be conceived as formed at O by the plane angles AOB, BOC, COD, &c., and the sum of these angles must be less than four right angles; which is exactly the proposition here proved. The demonstration here given is different from that of Book VI. Prop. XX.; both, however, suppose that the polygon ABCDE is convex, or that no side produced will cut the figure.

PROPOSITION V. THEOREM.

The poles of a great circle of a sphere, are the extremities of that diameter of the sphere which is perpendicular to the circle; and these extremities are also the poles of all small circles parallel to it.

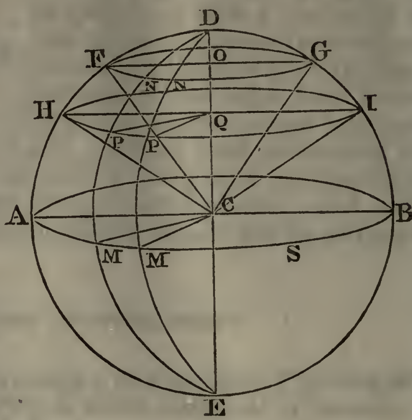


Let ED be perpendicular to the great circle AMB; then will E and D be its poles; as also the poles of the parallel small circles HPI, FNG.

For, DC being perpendicular to the plane AMB, is perpendicular to all the straight lines CA, CM, CB, &c. drawn through its foot in this plane; hence all the arcs DA, DM, DB, &c. are quarters of the circumference. So likewise are all the arcs EA, EM, EB, &c.; hence the points D and E are each equally distant from all the points of the circumference AMB; hence, they are the poles of that circumference (Def. 7.).

Again, the radius DC, perpendicular to the plane AMB, is perpendicular to its parallel FNG; hence, it passes through O the centre of the circle FNG (Book VIII. Prop. VII. Cor. 4.); hence, if the oblique lines DF, DN, DG, be drawn, these oblique lines will diverge equally from the perpendicular DO, and will themselves be equal. But, the chords being equal, the arcs are equal; hence the point D is the pole of the small circle FNG; and for like reasons, the point E is the other pole.

Cor. 1. Every arc DM, drawn from a point in the arc of a great circle AMB to its pole, is a quarter of the circumference, which for the sake of brevity, is usually named a *quadrant*: and this quadrant at the same time makes a right angle with the arc AM. For, the line DC being perpendicular to the plane AMC, every plane DME, passing through the line DC is perpendicular to the plane AMC (Book VI. Prop. XVI.); hence, the angle of these planes, or the angle AMD, is a right angle.



Cor. 2. To find the pole of a given arc AM, draw the indefinite arc MD perpendicular to AM; take MD equal to a quadrant; the point D will be one of the poles of the arc AM: or thus, at the two points A and M, draw the arcs AD and MD perpendicular to AM; their point of intersection D will be the pole required.

Cor. 3. Conversely, if the distance of the point D from each of the points A and M is equal to a quadrant, the point D will be the pole of the arc AM, and also the angles DAM, AMD, will be right angles.

For, let C be the centre of the sphere; and draw the radii CA, CD, CM. Since the angles ACD, MCD, are right angles, the line CD is perpendicular to the two straight lines CA, CM; hence it is perperpendicular to their plane (Book VI. Prop. IV.); hence the point D is the pole of the arc AM; and consequently the angles DAM, AMD, are right angles.

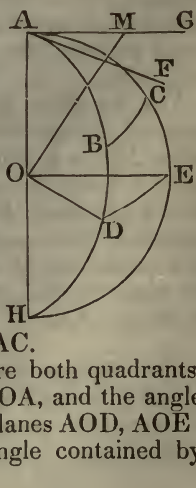
Scholium. The properties of these poles enable us to describe arcs of a circle on the surface of a sphere, with the same facility as on a plane surface. It is evident, for instance, that by turning the arc DF, or any other line extending to the same distance, round the point D, the extremity F will describe the small circle FNG; and by turning the quadrant DFA round the point D, its extremity A will describe the arc of the great circle AMB.

If the arc AM were required to be produced, and nothing were given but the points A and M through which it was to pass, we should first have to determine the pole D, by the intersection of two arcs described from the points A and M as centres, with a distance equal to a quadrant; the pole D being found, we might describe the arc AM and its prolongation, from D as a centre, and with the same distance as before.

In fine, if it be required from a given point P, to let fall a perpendicular on the given arc AM; find a point on the arc AM at a quadrant's distance from the point P, which is done by describing an arc with the point P as a pole, intersecting AM in S: S will be the point required, and is the pole with which a perpendicular to AM may be described passing through the point P.

PROPOSITION VI. THEOREM.

The angle formed by two arcs of great circles, is equal to the angle formed by the tangents of these arcs at their point of intersection, and is measured by the arc described from this point of intersection, as a pole, and limited by the sides, produced if necessary.

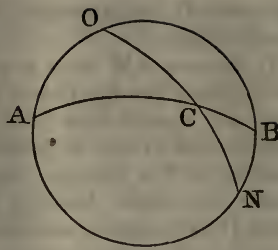


Let the angle BAC be formed by the two arcs AB, AC; then will it be equal to the angle FAG formed by the tangents AF, AG, and be measured by the arc DE, described about A as a pole.

For the tangent AF, drawn in the plane of the arc AB, is perpendicular to the radius AO; and the tangent AG, drawn in the plane of the arc AC, is perpendicular to the same radius AO. Hence the angle FAG is equal to the angle contained by the planes ABO, OAC (Book VI. Def. 4.); which is that of the arcs AB, AC, and is called the angle BAC.

In like manner, if the arcs AD and AE are both quadrants, the lines OD, OE, will be perpendicular to OA, and the angle DOE will still be equal to the angle of the planes AOD, AOE: hence the arc DE is the measure of the angle contained by these planes, or of the angle CAB.

Cor. The angles of spherical triangles may be compared together, by means of the arcs of great circles described from their vertices as poles and included between their sides: hence it is easy to make an angle of this kind equal to a given angle.

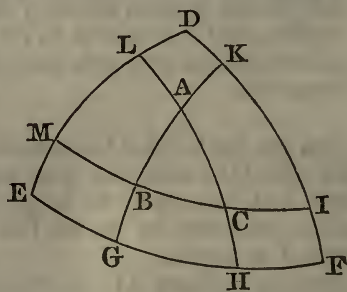


Scholium. Vertical angles, such as ACO and BCN are equal; for either of them is still the angle formed by the two planes ACB, OCN.

It is farther evident, that, in the intersection of two arcs ACB, OCN, the two adjacent angles ACO, OCB, taken together, are equal to two right angles.

PROPOSITION VII. THEOREM.

If from the vertices of the three angles of a spherical triangle, as poles, three arcs be described forming a second triangle, the vertices of the angles of this second triangle, will be respectively poles of the sides of the first.



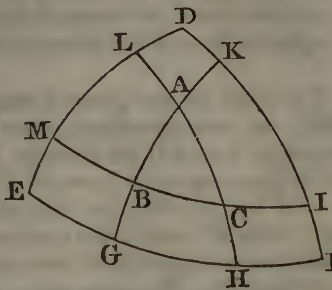
From the vertices A, B, C, as poles, let the arcs EF, FD, ED, be described, forming on the surface of the sphere, the triangle DFE; then will the points D, E, and F, be respectively poles of the sides BC, AC, AB.

For, the point A being the pole of the arc EF, the distance AE is a quadrant; the point C being the pole of the arc DE, the distance CE is likewise a quadrant: hence the point E is removed the length of a quadrant from each of the points A and C; hence, it is the pole of the arc AC (Prop. V. Cor. 3.). It might be shown, by the same method, that D is the pole of the arc BC, and F that of the arc AB.

Cor. Hence the triangle ABC may be described by means of DEF, as DEF is described by means of ABC. Triangles so described are called *polar triangles*, or *supplemental triangles.*

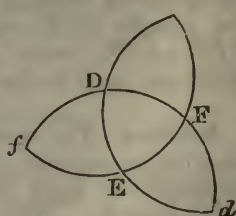
PROPOSITION VIII. THEOREM.

The same supposition continuing as in the last Proposition, each angle in one of the triangles, will be measured by a semicircumference, minus *the side lying opposite to it in the other triangle.*



For, produce the sides AB, AC, if necessary, till they meet EF, in G and H. The point A being the pole of the arc GH, the angle A will be measured by that arc (Prop. VI.). But the arc EH is a quadrant, and likewise GF, E being the pole of AH, and F of AG; hence EH+GF is equal to a semicircumference. Now, EH+GF is the same as EF+GH; hence the arc GH, which measures the angle A, is equal to a semicircumference *minus* the side EF. In like manner, the angle B will be measured by $\frac{1}{2}$*circ.*—DF: the angle C, by $\frac{1}{2}$ *circ.*—DE.

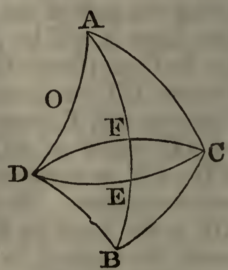
And this property must be reciprocal in the two triangles, since each of them is described in a similar manner by means of the other. Thus we shall find the angles D, E, F, of the triangle DEF to be measured respectively by $\frac{1}{2}$ *circ.*—BC, $\frac{1}{2}$ *circ.*—AC, $\frac{1}{2}$ *circ.*—AB. Thus the angle D, for example, is measured by the arc MI; but MI+BC=MC+BI=$\frac{1}{2}$ *circ.*; hence the arc MI, the measure of D, is equal to $\frac{1}{2}$ *circ.*—BC: and so of all the rest.



Scholium. It must further be observed, that besides the triangle DEF, three others might be formed by the intersection of the three arcs DE, EF, DF. But the proposition immediately before us is applicable only to the central triangle, which is distinguished from the other three by the circumstance (see the last figure) that the two angles A and D lie on the same side of BC, the two B and E on the same side of AC, and the two C and F on the same side of AB.

PROPOSITION IX. THEOREM.

If around the vertices of the two angles of a given spherical triangle, as poles, the circumferences of two circles be described which shall pass through the third angle of the triangle; if then, through the other point in which these circumferences intersect and the two first angles of the triangle, the arcs of great circles be drawn, the triangle thus formed will have all its parts equal to those of the given triangle.



Let ABC be the given triangle, CED, DFC, the arcs described about A and B as poles; then will the triangle ADB have all its parts equal to those of ABC.

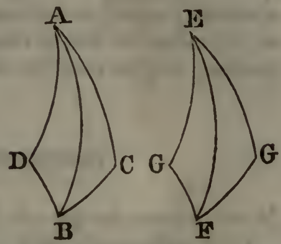
For, by construction, the side AD=AC, DB=BC, and AB is common; hence these two triangles have their sides equal, each to each. We are now to show, that the angles opposite these equal sides are also equal.

If the centre of the sphere is supposed to be at O, a solid angle may be conceived as formed at O by the three plane angles AOB, AOC, BOC; likewise another solid angle may be conceived as formed by the three plane angles AOB, AOD, BOD. And because the sides of the triangle ABC are equal to those of the triangle ADB, the plane angles forming the one of these solid angles, must be equal to the plane angles forming the other, each to each. But in that case we have shown that the planes, in which the equal angles lie, are equally inclined to each other (Book VI. Prop. XXI.); hence all the angles of the spherical triangle DAB are respectively equal to those of the triangle CAB, namely, DAB=BAC, DBA=ABC, and ADB=ACB; hence the sides and the angles of the triangle ADB are equal to the sides and the angles of the triangle ACB.

Scholium. The equality of these triangles is not, however, an absolute equality, or one of superposition; for it would be impossible to apply them to each other exactly, unless they were isosceles. The equality meant here is what we have already named an equality by *symmetry;* therefore we shall call the triangles ACB, ADB, *symmetrical triangles.*

PROPOSITION X. THEOREM.

Two triangles on the same sphere, or on equal spheres, are equal in all their parts, when two sides and the included angle of the one are equal to two sides and the included angle of the other each to each.



Suppose the side AB=EF, the side AC=EG, and the angle BAC=FEG; then will the two triangles be equal in all their parts.

For, the triangle EFG may be placed on the triangle ABC, or on ABD symmetrical with ABC, just as two rectilineal triangles are placed upon each other, when they have an equal angle included between equal sides. Hence all the parts of the triangle EFG will be equal to all the parts of the triangle ABC; that is, besides the three parts equal by hypothesis, we shall have the side BC=FG, the angle ABC=EFG, and the angle ACB=EGF.

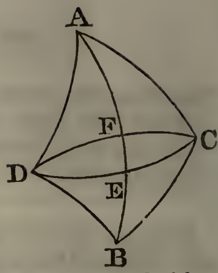
PROPOSITION XI. THEOREM.

Two triangles on the same sphere, or on equal spheres, are equal in all their parts, when two angles and the included side of the one are equal to two angles and the included side of the other, each to each.

For, one of these triangles, or the triangle symmetrical with it, may be placed on the other, as is done in the corresponding case of rectilineal triangles (Book I. Prop. VI.).

PROPOSITION XII. THEOREM.

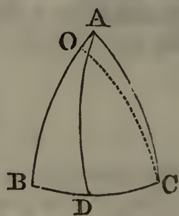
If two triangles on the same sphere, or on equal spheres, have all their sides equal, each to each, their angles will likewise be equal, each to each, the equal angles lying opposite the equal sides.



This truth is evident from Prop. IX, where it was shown, that with three given sides AB, AC, BC, there can only be two triangles ACB, ABD, differing as to the position of their parts, and equal as to the magnitude of those parts. Hence those two triangles, having all their sides respectively equal in both, must either be absolutely equal, or at least *symmetrically* so; in either of which cases, their corresponding angles must be equal, and lie opposite to equal sides.

PROPOSITION XIII. THEOREM.

In every isosceles spherical triangle, the angles opposite the equal sides are equal; and conversely, if two angles of a spherical triangle are equal, the triangle is isosceles.



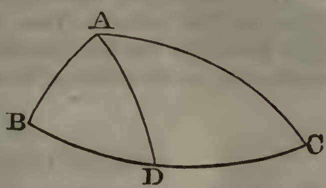
First. Suppose the side AB=AC; we shall have the angle C=B. For, if the arc AD be drawn from the vertex A to the middle point D of the base, the two triangles ABD, ACD, will have all the sides of the one respectively equal to the corresponding sides of the other, namely, AD common, BD=DC, and AB=AC: hence by the last Proposition, their angles will be equal; therefore, B=C.

Secondly. Suppose the angle B=C; we shall have the side AC=AB. For, if not, let AB be the greater of the two; take BO=AC, and draw OC. The two sides BO, BC, are equal to the two AC, BC; the angle OBC, contained by the first two is equal to ACB contained by the second two. Hence the two triangles BOC, ACB, have all their other parts equal (Prop. X.); hence the angle OCB=ABC: but by hypothesis, the angle ABC=ACB; hence we have OCB=ACB, which is absurd; hence it is absurd to suppose AB different from AC; hence the sides AB, AC, opposite to the equal angles B and C, are equal.

Scholium. The same demonstration proves the angle BAD=DAC, and the angle BDA=ADC. Hence the two last are right angles; hence *the arc drawn from the vertex of an isosceles spherical triangle to the middle of the base, is at right angles to that base, and bisects the vertical angle.*

PROPOSITION XIV. THEOREM.

In any spherical triangle, the greater side is opposite the greater angle; and conversely, the greater angle is opposite the greater side.



Let the angle A be greater than the angle B, then will BC be greater than AC; and conversely, if BC is greater than AC, then will the angle A be greater than B.

First. Suppose the angle $A>B$; make the angle $BAD=B$: then we shall have $AD=DB$ (Prop. XIII.): but $AD+DC$ is greater than AC; hence, putting DB in place of AD, we shall have $DB+DC$, or $BC>AC$.

Secondly. If we suppose $BC>AC$, the angle BAC will be greater than ABC. For, if BAC were equal to ABC, we should have $BC=AC$; if BAC were less than ABC, we should then, as has just been shown, find $BC<AC$. Both these conclusions are false: hence the angle BAC is greater than ABC.

PROPOSITION XV. THEOREM.

If two triangles on the same sphere, or on equal spheres, are mutually equiangular, they will also be mutually equilateral.

Let A and B be the two given triangles; P and Q their polar triangles. Since the angles are equal in the triangles A and B, the sides will be equal in their polar triangles P and Q (Prop. VIII.): but since the triangles P and Q are mutually evuilateral, they must also be mutually equiangular (Prop. XII.); and lastly, the angles being equal in the triangles P and Q, it follows that the sides are equal in their polar triangles A and B. Hence the mutually equiangular triangles A and B are at the same time mutually equilateral.

Scholium. This proposition is not applicable to rectilinea. triangles; in which equality among the angles indicates only proportionality among the sides. Nor is it difficult to account for the difference observable, in this respect, between spherical and rectilineal tr·angles. In the Proposition now before us as well as in the preceding ones, which treat of the comparison of triangles, it is expressly required that the arcs be traced on the same sphere, or on equal spheres. Now similar arcs are to each other as their radii; hence, on equal spheres, two triangles cannot be similar without being equal. Therefore it is not strange that equality among the angles should produce equality among the sides.

The case would be different, if the triangles were drawn upon unequal spheres; there, the angles being equal, the triangles would be similar, and the homologous sides would be to each other as the radii of their spheres.

PROPOSITION XVI. THEOREM.

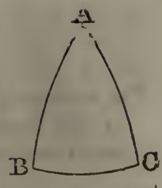
The sum of all the angles in any spherical triangle is less than six right angles, and greater than two.

For, in the first place, every angle of a spherical triangle is less than two right angles: hence the sum of all the three is less than six right angles.

Secondly, the measure of each angle of a spherical triangle is equal to the semicircumference *minus* the corresponding side of the polar triangle (Prop. VIII.); hence the sum of all the three, is measured by the three semicircumferences *minus* the sum of all the sides of the polar triangle. Now this latter sum is less than a circumference (Prop. III.); therefore, taking it away from three semicircumferences, the remainder will be greater than one semicircumference, which is the measure of two right angles; hence, in the second place, the sum of all the angles of a spherical triangle is greater than two right angles.

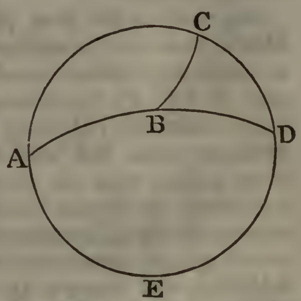
Cor. 1. The sum of all the angles of a spherical triangle is not constant, like that of all the angles of a rectilineal triangle; it varies between two right angles and six, without ever arriving at either of these limits. Two given angles therefore do not serve to determine the third.

Cor. 2. A spherical triangle may have two, or even three of its angles right angles; also two, or even three of its angles obtuse.



Cor. 3. If the triangle ABC is *bi-rectangular,* in other words, has two right angles B and C, the vertex A will be the pole of the base BC; and the sides AB, AC, will be quadrants (Prop. V. Cor. 3.).

If the angle A is also a right angle, the triangle ABC will be *tri-rectangular;* its angles will all be right angles, and its sides quadrants. Two of the tri-rectangular triangles make half a hemisphere, four make a hemisphere, and the tri-rectangular triangle is obviously contained eight times in the surface of a sphere.



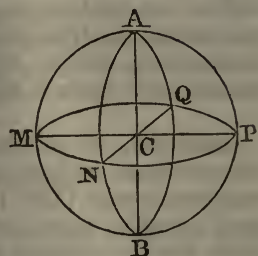
Scholium. In all the preceding observations, we have supposed, in conformity with (Def. 1.) that spherical triangles have always each of their sides less than a semicircumference; from which it follows that any one of their angles is always less than two right angles. For, if the side AB is less than a semicircumference, and AC is so likewise, both those arcs will require to be produced, before they can meet in D. Now the two angles ABC, CBD, taken together, are equal to two right angles; hence the angle ABC itself, is less than two right angles.

We may observe, however, that some spherical triangles do exist, in which certain of the sides are greater than a semicircumference, and certain of the angles greater than two right angles. Thus, if the side AC is produced so as to form a whole circumference ACE, the part which remains, after subtracting the triangle ABC from the hemisphere, is a new triangle also designated by ABC, and having AB, BC, AEDC for its sides. Here, it is plain, the side AEDC is greater than the semicircumference AED; and at the same time, the angle B opposite to it exceeds two right angles, by the quantity CBD.

The triangles whose sides and angles are so large, have been excluded by the Definition; but the only reason was, that the solution of them, or the determination of their parts, is always reducible to the solution of such triangles as are comprehended by the Definition. Indeed, it is evident enough, that if the sides and angles of the triangle ABC are known, it will be easy to discover the angles and sides of the triangle which bears the same name, and is the difference between a hemisphere and the former triangle.

PROPOSITION XVII. THEOREM.

The surface of a lune is to the surface of the sphere, as the angle of this lune, is to four right angles, or as the arc which measures that angle, is to the circumference.



Let AMBN be a lune; then will its surface be to the surface of the sphere as the angle NCM to four right angles, or as the arc NM to the circumference of a great circle.

Suppose, in the first place, the arc MN to be to the circumference MNPQ as some one rational number is to another, as 5 to 48, for example. The circumference MNPQ being divided into 48 equal parts, MN will contain 5 of them; and if the pole A were joined with the several points of division, by as many quadrants, we should in the hemisphere AMNPQ have 48 triangles, all equal, because all their parts are equal. Hence the whole sphere must contain 96 of those partial triangles, the lune AMBNA will contain 10 of them; hence the lune is to the sphere as 10 is to 96, or as 5 to 48, in other words, as the arc MN is to the circumference.

If the arc MN is not commensurable with the circumference, we may still show, by a mode of reasoning frequently exemplified already, that in that case also, the lune is to the sphere as MN is to the circumference.

Cor. 1. Two lunes are to each other as their respective angles.

Cor. 2. It was shown above, that the whole surface of the sphere is equal to eight tri-rectangular triangles (Prop. XVI. Cor. 3.); hence, if the area of one such triangle is represented by T, the surface of the whole sphere will be expressed by 8T This granted, if the right angle be assumed equal to 1, the surface of the lune whose angle is A, will be expressed by $2A \times T$. for,

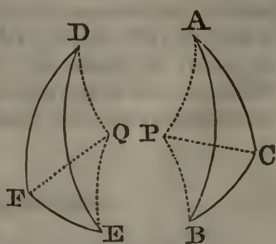
$$4 : A : : 8T : 2A \times T$$

in which expression, A represents such a part of unity, as the angle of the lune is of one right angle

Scholium. The spherical ungula, bounded by the planes AMB, ANB, is to the whole solid sphere, as the angle A is to four right angles. For, the lunes being equal, the spherical ungulas will also be equal ; hence two spherical ungulas are to each other, as the angles formed by the planes which bound them.

PROPOSITION XVIII. THEOREM.

Two symmetrical spherical triangles are equivalent.



Let ABC, DEF, be two symmetrical triangles, that is to say, two triangles having their sides AB=DE, AC=DF, CB=EF, and yet incapable of coinciding with each other: we are to show that the surface ABC is equal to the surface DEF.

Let P be the pole of the small circle passing through the three points A, B, C ;* from this point draw the equal arcs PA, PB, PC (Prop. V.) ; at the point F, make the angle DFQ=ACP, the arc FQ=CP ; and draw DQ, EQ.

The sides DF, FQ, are equal to the sides AC, CP ; the angle DFQ=ACP : hence the two triangles DFQ, ACP are equal in all their parts (Prop. X.) ; hence the side DQ=AP, and the angle DQF=APC.

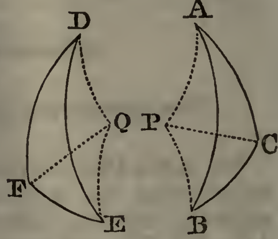
In the proposed triangles DFE, ABC, the angles DFE, ACB, opposite to the equal sides DE, AB, being equal (Prop. XII.). if the angles DFQ, ACP, which are equal by construction, be taken away from them, there will remain the angle QFE, equal to PCB. Also the sides QF, FE, are equal to the sides PC, CB ; hence the two triangles FQE, CPB, are equal in all their parts ; hence the side QE=PB, and the angle FQE=CPB.

Now, the triangles DFQ, ACP, which have their sides respectively equal, are at the same time isosceles, and capable of coinciding, when applied to each other ; for having placed AC on its equal DF, the equal sides will fall on each other, and thus the two triangles will exactly coincide : hence they are equal ; and the surface DQF=APC. For a like reason, the surface FQE=CPB, and the surface DQE=APB ; hence we

* The circle which passes through the three points A, B, C, or which circumscribes the triangle ABC, can only be a small circle of the sphere ; for if it were a great circle, the three sides AB, BC, AC, would lie in one plane, and the triangle ABC would be reduced to one of its sides.

have DQF+FQE—DQE=APC+CPB—APB, or DFE= ABC; hence the two symmetrical triangles ABC, DEF are equal in surface.

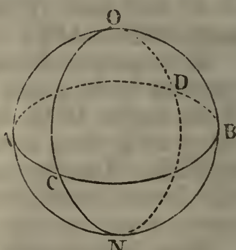
Scholium. The poles P and Q might lie within triangles ABC, DEF: in which case it would be requisite to add the three triangles DQF, FQE, DQE, together, in order to make up the triangle DEF; and in like manner, to add the three triangles APC, CPB, APB, together, in order to make up the triangle ABC: in all other respects, the demonstration and the result would still be the same.



PROPOSITION XIX. THEOREM.

If the circumferences of two great circles intersect each other on the surface of a hemisphere, the sum of the opposite triangles thus formed, is equivalent to the surface of a lune whose angle is equal to the angle formed by the circles.

Let the circumferences AOB, COD, intersect on the hemisphere OACBD; then will the opposite triangles AOC, BOD, be equal to the lune whose angle is BOD.

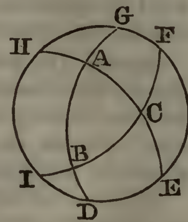


For, producing the arcs OB, OD, on the other hemisphere, till they meet in N, the arc OBN will be a semi-circumference, and AOB one also; and taking OB from each, we shall have BN=AO. For a like reason, we have DN=CO, and BD=AC. Hence, the two triangles AOC, BDN, have their three sides respectively equal; they are therefore symmetrical; hence they are equal in surface (Prop. XVIII.): but the sum of the triangles BDN, BOD, is equivalent to the lune OBNDO, whose angle is BOD: hence, AOC+BOD is equivalent to the lune whose angle is BOD.

Scholium. It is likewise evident that the two spherical pyramids, which have the triangles AOC, BOD, for bases, are together equivalent to the spherical ungula whose angle is BOD.

PROPOSITION XX. THEOREM.

The surface of a spherical triangle is measured by the excess of the sum of its three angles above two right angles, multiplied by the tri-rectangular triangle.



Let ABC be the proposed triangle: produce its sides till they meet the great circle DEFG drawn at pleasure without the triangle. By the last Theorem, the two triangles ADE, AGH, are together equivalent to the lune whose angle is A, and which is measured by 2A.T (Prop. XVII. Cor. 2.). Hence we have ADE+AGH=2A.T; and for a like reason, BGF+BID=2B.T, and CIH+CFE=2C.T But the sum of these six triangles exceeds the hemisphere by twice the triangle ABC, and the hemisphere is represented by 4T; therefore, twice the triangle ABC is equal to 2A.T+2B.T+2C.T—4 T; and consequently, once ABC=(A+B+C—2)T; hence every spherical triangle is measured by the sum of all its angles *minus* two right angles, multiplied by the tri-rectangular triangle.

Cor. 1. However many right angles there may be in the sum of the three angles minus two right angles, just so many tri-rectangular triangles, or eighths of the sphere, will the proposed triangle contain. If the angles, for example, are each equal to $\frac{4}{3}$ of a right angle, the three angles will amount to 4 right angles, and the sum of the angles minus two right angles will be represented by 4—2 or 2; therefore the surface of the triangle will be equal to two tri-rectangular triangles, or to the fourth part of the whole surface of the sphere.

Scholium. While the spherical triangle ABC is compared with the tri-rectangular triangle, the spherical pyramid, which has ABC for its base, is compared with the tri-rectangular pyramid, and a similar proportion is found to subsist between them. The solid angle at the vertex of the pyramid, is in like manner compared with the solid angle at the vertex of the tri-rectangular pyramid. These comparisons are founded on the coincidence of the corresponding parts. If the bases of the pyramids coincide, the pyramids themselves will evidently coincide, and likewise the solid angles at their vertices. From this, some consequences are deduced.

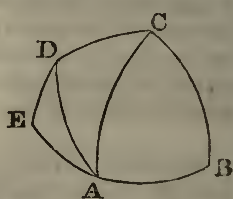
First. Two triangular spherical pyramids are to each other as their bases: and since a polygonal pyramid may always be divided into a certain number of triangular ones, it follows that any two spherical pyramids are to each other, as the polygons which form their bases.

Second. The solid angles at the vertices of these pyramids, are also as their bases; hence, for comparing any two solid angles, we have merely to place their vertices at the centres of two equal spheres, and the solid angles will be to each other as the spherical polygons intercepted between their planes or faces.

The vertical angle of the tri-rectangular pyramid is formed by three planes at right angles to each other: this angle, which may be called a *right solid angle*, will serve as a very natural unit of measure for all other solid angles. If, for example, the the area of the triangle is $\frac{3}{4}$ of the tri-rectangular triangle, then the corresponding solid angle will also be $\frac{3}{4}$ of the right solid angle.

PROPOSITION XXI. THEOREM.

The surface of a spherical polygon is measured by the sum of all its angles, minus *two right angles multiplied by the number of sides in the polygon* less *two, into the tri-rectangular triangle.*



From one of the vertices A, let diagonals AC, AD be drawn to all the other vertices; the polygon ABCDE will be divided into as many triangles *minus* two as it has sides. But the surface of each triangle is measured by the sum of all its angles *minus* two right angles, into the tri-rectangular triangle; and the sum of the angles in all the triangles is evidently the same as that of all the angles of the polygon; hence, the surface of the polygon is equal to the sum of all its angles, diminished by twice as many right angles as it has sides less two, into the tri-rectangular triangle.

Scholium. Let s be the sum of all the angles in a spherical polygon, n the number of its sides, and T the tri-rectangular triangle; the right angle being taken for unity, the surface of the polygon will be measured by

$$\big(s-2\,(n-2,)\big)\ \mathrm{T}, \text{ or } (s-2\,n+4)\ \mathrm{T}$$

APPENDIX.

THE REGULAR POLYEDRONS.

A *regular polyedron* is one whose faces are all equal regular polygons, and whose solid angles are all equal to each other. There are five such polyedrons.

First. If the faces are equilateral triangles, polyedrons may be formed of them, having solid angles contained by three of those triangles, by four, or by five: hence arise three regular bodies, the *tetraedron*, the *octaedron*, the *icosaedron*. No other can be formed with equilateral triangles; for six angles of such a triangle are equal to four right angles, and cannot form a solid angle (Book VI. Prop. XX.).

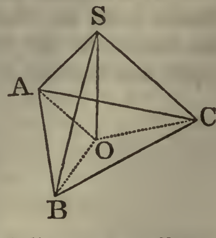
Secondly. If the faces are squares, their angles may be arranged by threes: hence results the *hexaedron* or *cube*. Four angles of a square are equal to four right angles, and cannot form a solid angle.

Thirdly. In fine, if the faces are regular pentagons, their angles likewise may be arranged by threes: the regular *dodecaedron* will result.

We can proceed no farther: three angles of a regular hexagon are equal to four right angles; three of a heptagon are greater.

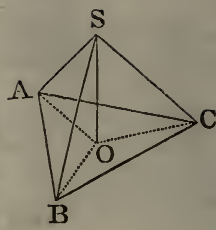
Hence there can only be five regular polyedrons; three formed with equilateral triangles, one with squares, and one with pentagons.

Construction of the Tetraedron.



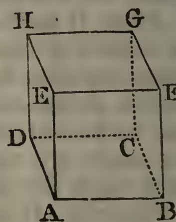
Let ABC be the equilateral triangle which is to form one face of the tetraedron. At the point O, the centre of this triangle, erect OS perpendicular to the plane ABC; terminate this perpendicular in S, so that AS=AB; draw SB, SC: the pyramid S-ABC will be the tetraedron required.

For, by reason of the equal distances OA, OB, OC, the oblique lines SA, SB, SC, are equally removed from the perpendicular SO, and consequently equal (Book VI. Prop. V.). One of them SA=AB; hence the four faces of the pyramid S-ABC, are triangles, equal to the given triangle ABC. And the solid angles of this pyramid are all equal, because each of them is formed by three equal plane angles: hence this pyramid is a regular tetraedron.



Construction of the Hexaedron.

Let ABCD be a given square. On the base ABCD, construct a right prism whose altitude AE shall be equal to the side AB. The faces of this prism will evidently be equal squares; and its solid angles all equal, each being formed with three right angles: hence this prism is a regular hexaedron or cube.



The following propositions can be easily proved.

1. Any regular polyedron may be divided into as many regular pyramids as the polyedron has faces; the common vertex of these pyramids will be the centre of the polyedron; and at the same time, that of the inscribed and of the circumscribed sphere.

2. The solidity of a regular polyedron is equal to its surface multiplied by a third part of the radius of the inscribed sphere.

3. Two regular polyedrons of the same name, are two similar solids, and their homologous dimensions are proportional; hence the radii of the inscribed or the circumscribed spheres are to each other as the sides of the polyedrons.

4. If a regular polyedron is inscribed in a sphere, the planes drawn from the centre, through the different edges, will divide the surface of the sphere into as many spherical polygons, all equal and similar, as the polyedron has faces.

APPLICATION OF ALGEBRA.

TO THE SOLUTION OF

GEOMETRICAL PROBLEMS.

A problem is a question which requires a solution. A geometrical problem is one, in which certain parts of a geometrical figure are given or known, from which it is required to determine certain other parts.

When it is proposed to solve a geometrical problem by means of Algebra, the given parts are represented by the first letters of the alphabet, and the required parts by the final letters, and the relations which subsist between the known and unknown parts furnish the equations of the problem. The solution of these equations, when so formed, gives the solution of the problem.

No general rule can be given for forming the equations. The equations must be independent of each other, and their number equal to that of the unknown quantities introduced (Alg. Art. 103.). Experience, and a careful examination of all the conditions, whether explicit or implicit (Alg. Art. 94,) will serve as guides in stating the questions; to which may be added the following particular directions.

1st. Draw a figure which shall represent all the given parts, and all the required parts. Then draw such other lines as will establish the most simple relations between them. If an angle is given, it is generally best to let fall a perpendicular that shall lie opposite to it; and this perpendicular, if possible, should be drawn from the extremity of a given side.

2d. When two lines or quantities are connected in the same way with other parts of the figure or problem, it is in general, not best to use either of them separately; but to use their sum, their difference, their product, their quotient, or perhaps another line of the figure with which they are alike connected.

3d. When the area, or perimeter of a figure, is given, it is sometimes best to assume another figure similar to the proposed, having one of its sides equal to unity, or some other known quantity. A comparison of the two figures will often give a required part. We will add the following problems.*

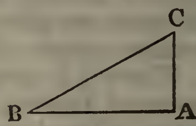
* The following problems are selected from Hutton's Application of Algebra to Geometry, and the examples in Mensuration from his treatise on that subject

APPLICATION OF ALGEBRA

PROBLEM I.

In a right angled triangle BAC, having given the base BA, and the sum of the hypothenuse and perpendicular, it is required to find the hypothenuse and perpendicular.

Put BA$=c=3$, BC$=x$, AC$=y$ and the sum of the hypothenuse and perpendicular equal to $s=9$



Then, $x+y=s=9.$

and $x^2=y^2+c^2$ (Bk. IV. Prop. XI.)

From 1st equ: $x=s-y$

and $x^2=s^2-2sy+y^2$

By subtracting, $0=s^2-2sy-c^2$

or $2sy=s^2-c^2$

hence, $y=\dfrac{s^2-c^2}{2s}=4=$ AC

Therefore $x+4=9$ or $x=5=$ BC.

PROBLEM II.

In a right angled triangle, having given the hypothenuse, and the sum of the base and perpendicular, to find these two sides

Put BC$=a=5$, BA$=x$, AC$=y$ and the sum of the base and perpendicular $=s=7$

Then $x+y=s=7$

and $x^2+y^2=a^2$

From first equation $x=s-y$

or $x^2=s^2-2sy+y^2$

Hence, $y^2=a^2-s^2+2sy-y^2$

or $2y^2-2sy=a^2-s^2$

or $y^2-sy=\dfrac{a^2-s^2}{2}$

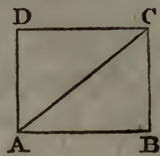
By completing the square $y^2-sy+\frac{1}{4}s^2=\frac{1}{2}a^2-\frac{1}{4}s^2$

or $y=\frac{1}{2}s\pm\sqrt{\frac{1}{2}a^2-\frac{1}{4}s^2}=4$ or 3

Hence $x=\frac{1}{2}s\mp\sqrt{\frac{1}{2}a^2-\frac{1}{4}s^2}=3$ or 4

PROBLEM III.

In a rectangle, having given the diagonal and perimeter, to find the sides.



Let ABCD be the proposed rectangle. Put $AC=d=10$, the perimeter$=2a=28$, or $AB+BC=a=14$: also put $AB=x$ and $BC=y$.

Then, $x^2+y^2=d^2$

and $x+y=a$

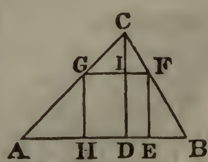
From which equations we obtain,

$$y=\tfrac{1}{2}a\pm\sqrt{\tfrac{1}{2}d^2-\tfrac{1}{4}a^2}=8 \text{ or } 6,$$

and $$x=\tfrac{1}{2}a\mp\sqrt{\tfrac{1}{2}d^2-\tfrac{1}{4}a^2}=6 \text{ or } 8.$$

PROBLEM IV.

Having given the base and perpendicular of a triangle, to find the side of an inscribed square.



Let ABC be the triangle and HEFG the inscribed square. Put $AB=b$, $CD=a$, and HE or $GH=x$: then $CI=a-x$.

We have by similar triangles

$$AB : CD :: GF : CI$$

or $$b : a :: x : a-x$$

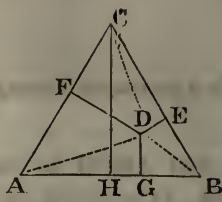
Hence, $$ab-bx=ax$$

or $$x=\frac{ab}{a+b}= \text{ the side of the inscribed square};$$

which, therefore, depends only on the base and altitude of the triangle.

PROBLEM V.

In an equilateral triangle, having given the lengths of the three perpendiculars drawn from a point within, on the three sides: to determine the sides of the triangle.



Let ABC be the equilateral triangle; DG, DE and DF the given perpendiculars let fall from D on the sides. Draw DA, DB, DC to the vertices of the angles, and let fall the perpendicular CH on the base. Let DG$=a$, DE$=b$, and DF$=c$: put one of the equal sides AB $=2x$; hence AH$=x$, and CH$=\sqrt{AC^2-AH^2}=\sqrt{4x^2-x^2}$ $=\sqrt{3x^2}=x\sqrt{3}$.

Now since the area of a triangle is equal to half its base into the altitude, (Bk. IV. Prop. VI.)

$\frac{1}{2}AB\times CH=x\times x\sqrt{3}=x^2\sqrt{3}=$triangle ACB

$\frac{1}{2}AB\times DG=x\times a \quad =ax \quad =$triangle ADB

$\frac{1}{2}BC\times DE=x\times b \quad =bx \quad =$triangle BCD

$\frac{1}{2}AC\times DF=x\times c \quad =cx \quad =$triangle ACD

But the three last triangles make up, and are consequently equal to, the first; hence,

$$x^2\sqrt{3}=ax+bx+cx=x(a+b+c);$$

or $$x\sqrt{3}=a+b+c$$

therefore, $$x=\frac{a+b+c}{\sqrt{3}}$$

REMARK. Since the perpendicular CH is equal to $x\sqrt{3}$, it is consequently equal to $a+b+c$: that is, the perpendicular let fall from either angle of an equilateral triangle on the opposite side, is equal to the sum of the three perpendiculars let fall from any point within the triangle on the sides respectively.

PROBLEM VI.

In a right angled triangle, having given the base and the difference between the hypothenuse and perpendicular, to find the sides.

PROBLEM VII.

In a right angled triangle, having given the hypothenuse and the difference between the base and perpendicular, to determine the triangle.

PROBLEM VIII.

Having given the area of a rectangle inscribed in a given triangle; to determine the sides of the rectangle.

PROBLEM IX.

In a triangle, having given the ratio of the two sides, together with both the segments of the base made by a perpendicular from the vertical angle; to determine the triangle.

PROBLEM X.

In a triangle, having given the base, the sum of the other two sides, and the length of a line drawn from the vertical angle to the middle of the base; to find the sides of the triangle.

PROBLEM XI.

In a triangle, having given the two sides about the vertical angle, together with the line bisecting that angle and terminating in the base; to find the base.

PROBLEM XII.

To determine a right angled triangle, having given the lengths of two lines drawn from the acute angles to the middle of the opposite sides.

PROBLEM XIII.

To determine a right-angled triangle, having given the perimeter and the radius of the inscribed circle.

PROBLEM XIV.

To determine a triangle, having given the base, the perpendicular and the ratio of the two sides.

PROBLEM XV.

To determine a right angled triangle, having given the hypothenuse, and the side of the inscribed square.

PROBLEM XVI.

To determine the radii of three equal circles, described within and tangent to, a given circle, and also tangent to each other.

PROBLEM XVII

In a right angled triangle, having given the perimeter and the perpendicular let fall from the right angle on the hypothenuse, to determine the triangle.

PROBLEM XVIII.

To determine a right angled triangle, having given the hypothenuse and the difference of two lines drawn from the two acute angles to the centre of the inscribed circle.

PROBLEM XIX.

To determine a triangle, having given the base, the perpendicular, and the difference of the two other sides.

PROBLEM XX.

To determine a triangle, having given the base, the perpendicular and the rectangle of the two sides.

PROBLEM XXI.

To determine a triangle, having given the lengths of three lines drawn from the three angles to the middle of the opposite sides.

PROBLEM XXII.

In a triangle, having given the three sides, to find the radius of the inscribed circle.

PROBLEM XXIII.

To determine a right angled triangle, having given the side of the inscribed square, and the radius of the inscribed circle.

PROBLEM XXIV.

To determine a right angled triangle, having given the hypothenuse and radius of the inscribed circle.

PROBLEM XXV.

To determine a triangle, having given the base, the line bisecting the vertical angle, and the diameter of the circumscribing circle.

PLANE TRIGONOMETRY.

In every triangle there are six parts: three sides and three angles. These parts are so related to each other, that if a certain number of them be known or given, the remaining ones can be determined.

Plane Trigonometry explains the methods of finding, by calculation, the unknown parts of a rectilineal triangle, when a sufficient number of the six parts are given.

When three of the six parts are known, and one of them is a side, the remaining parts can always be found. If the three angles were given, it is obvious that the problem would be indeterminate, since all similar triangles would satisfy the conditions.

It has already been shown, in the problems annexed to Book III., how rectilineal triangles are constructed by means of three given parts. But these constructions, which are called *graphic methods*, though perfectly correct in theory, would give only a moderate approximation in practice, on account of the imperfection of the instruments required in constructing them. Trigonometrical methods, on the contrary, being independent of all mechanical operations, give solutions with the utmost accuracy.

These methods are founded upon the properties of lines called trigonometrical lines, which furnish a very simple mode of expressing the relations between the sides and angles of triangles.

We shall first explain the properties of those lines, and the principal formulas derived from them; formulas which are of great use in all the branches of mathematics, and which even furnish means of improvement to algebraical analysis. We shall next apply those results to the solution of rectilineal triangles.

DIVISION OF THE CIRCUMFERENCE.

I. For the purposes of trigonometrical calculation, the circumference of the circle is divided into 360 equal parts, called degrees; each degree into 60 equal parts, called minutes; and each minute into 60 equal parts, called seconds.

The semicircumference, or the measure of two right angles contains 180 degrees; the quarter of the circumference, usually denominated the quadrant, and which measures the right angle, contains 90 degrees.

II. Degrees, minutes, and seconds, are respectively designated by the characters : °, ′, ″ : thus the expression 16° 6′ 15″ represents an arc, or an angle, of 16 degrees, 6 minutes, and 15 seconds.

III. The *complement* of an angle, or of an arc, is what remains after taking that angle or that arc from 90°. Thus the complement of 25° 40′ is equal to 90°—25° 40′=64° 20′; and the complement of 12° 4′ 32″ is equal to 90°—12° 4′ 32″=77° 55′ 28″.

In general, A being any angle or any arc, 90°—A is the complement of that angle or arc. If any arc or angle be added to its complement, the sum will be 90°. Whence it is evident that if the angle or arc is greater than 90°, its complement will be negative. Thus, the complement of 160° 34′ 10″ is —70° 34′ 10″. In this case, the complement, taken positively, would be a quantity, which being subtracted from the given angle or arc, the remainder would be equal to 90°.

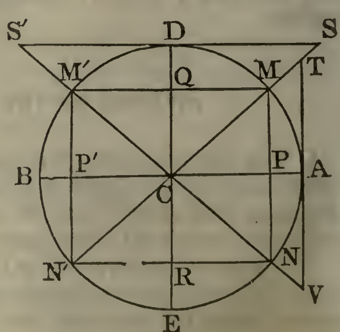
The two acute angles of a right-angled triangle, are together equal to a right angle; they are, therefore, complements of each other.

IV. The *supplement* of an angle, or of an arc, is what remains after taking that angle or arc from 180°. Thus A being any angle or arc, 180°—A is its supplement.

In any triangle, either angle is the supplement of the sum of the two others, since the three together make 180°.

If any arc or angle be added to its supplement, the sum will be 180°. Hence if an arc or angle be greater than 180°, its supplement will be negative. Thus, the supplement of 200° is —20°. The supplement of any angle of a triangle, or indeed of the sum of either two angles, is always positive.

GENERAL IDEAS RELATING TO TRIGONOMETRICAL LINES.



V. The *sine* of an arc is the perpendicular let fall from one extremity of the arc, on the diameter which passes through the other extremity. Thus, MP is the sine of the arc AM, or of the angle ACM.

The *tangent* of an arc is a line touching the arc at one extremity, and limited by the prolongation of the diameter which passes through the other extremity. Thus AT is the tangent of the arc AM, or of the angle ACM.

The *secant* of an arc is the line drawn from the centre of the circle through one extremity of the arc and limited by the tangent drawn through the other extremity. Thus CT is the secant of the arc AM, or of the angle ACM.

The *versed sine* of an arc, is the part of the diameter intercepted between one extremity of the arc and the foot of the sine. Thus, AP is the versed sine of the arc AM, or the angle ACM.

These four lines MP, AT, CT, AP, are dependent upon the arc AM, and are always determined by it and the radius; they are thus designated:

$$\begin{aligned}
MP &= \sin AM, \text{ or } \sin ACM,\\
AT &= \text{tang } AM, \text{ or } \text{tang } ACM,\\
CT &= \sec AM, \text{ or } \sec ACM,\\
AP &= \text{ver-sin } AM, \text{ or } \text{ver-sin } ACM.
\end{aligned}$$

VI. Having taken the arc AD equal to a quadrant, from the points M and D draw the lines MQ, DS, perpendicular to the radius CD, the one terminated by that radius, the other terminated by the radius CM produced; the lines MQ, DS, and CS, will, in like manner, be the sine, tangent, and secant of the arc MD, the complement of AM. For the sake of brevity, they are called the *cosine*, *cotangent*, and *cosecant*, of the arc AM and are thus designated:

$$\begin{aligned}
MQ &= \cos AM, \text{ or } \cos ACM,\\
DS &= \cot AM, \text{ or } \cot ACM,\\
CS &= \text{cosec } AM, \text{ or } \text{cosec } ACM.
\end{aligned}$$

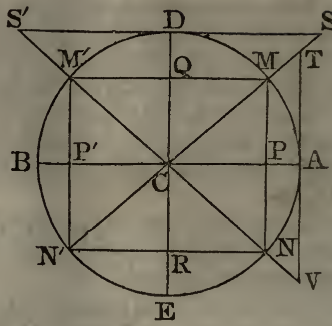
In general, A being any arc or angle, we have

$$\begin{aligned}
\cos A &= \sin (90° - A),\\
\cot A &= \text{tang } (90° - A),\\
\text{cosec } A &= \sec (90° - A).
\end{aligned}$$

The triangle MQC is, by construction, equal to the triangle CPM; consequently CP=MQ: hence in the right-angled triangle CMP, whose hypothenuse is equal to the radius, the two sides MP, CP are the sine and cosine of the arc AM: hence. the cosine of an arc is equal to that part of the radius intercepted between the centre and foot of the sine.

The triangles CAT, CDS, are similar to the equal triangles CPM, CQM; hence they are similar to each other. From these principles, we shall very soon deduce the different relations which exist between the lines now defined: before doing so, however, we must examine the changes which those lines undergo, when the arc to which they relate increases from zero to 180°.

The angle ACD is called the *first quadrant;* the angle DCB, the *second quadrant;* the angle BCE, the *third quadrant;* and the angle ECA. the *fourth quadrant.*



VII. Suppose one extremity of the arc remains fixed in A, while the other extremity, marked M, runs successively throughout the whole extent of the semicircumference, from A to B in the direction ADB.

When the point M is at A, or when the arc AM is zero, the three points T, M, P, are confounded with the point A; whence it appears that the sine and tangent of an arc zero, are zero, and the cosine and secant of this same arc, are each equal to the radius. Hence if R represents the radius of the circle, we have

$$\sin 0=0, \quad \text{tang } 0=0, \quad \cos 0=\mathrm{R}, \quad \sec 0=\mathrm{R}.$$

VIII. As the point M advances towards D, the sine increases, and so likewise does the tangent and the secant; but the cosine, the cotangent, and the cosecant, diminish.

When the point M is at the middle of AD, or when the arc AM is 45°, in which case it is equal to its complement MD, the sine MP is equal to the cosine MQ or CP; and the triangle CMP, having become isosceles, gives the proportion

$$\mathrm{MP} : \mathrm{CM} :: 1 : \sqrt{2},$$

or

$$\sin 45^\circ : \mathrm{R} :: 1 : \sqrt{2}.$$

Hence

$$\sin 45^\circ = \cos 45^\circ = \frac{\mathrm{R}}{\sqrt{2}} = \tfrac{1}{2}\mathrm{R}\sqrt{2}.$$

In this same case, the triangle CAT becomes isosceles and equal to the triangle CDS; whence the tangent of 45° and its cotangent, are each equal to the radius, and consequently we have

$$\text{tang } 45^\circ = \cot 45^\circ = \mathrm{R}.$$

IX. The arc AM continuing to increase, the sine increases till M arrives at D; at which point the sine is equal to the radius, and the cosine is zero. Hence we have

$$\sin 90^\circ = \mathrm{R}, \quad \cos 90^\circ = 0;$$

and it may be observed, that these values are a consequence of the values already found for the sine and cosine of the arc zero; because the complement of 90° being zero, we have

$$\sin 90^\circ = \cos 0^\circ = \mathrm{R}, \text{ and}$$
$$\cos 90^\circ = \sin 0^\circ = 0.$$

As to the tangent, it increases very rapidly as the point M approaches D; and finally when this point reaches D, the tangent properly exists no longer, because the lines AT, CD, being parallel, cannot meet. This is expressed by saying that the tangent of 90° is infinite; and we write tang $90°=\infty$. The complement of 90° being zero, we have

$$\text{tang } 0=\cot 90° \text{ and } \cot 0=\text{tang } 90°.$$

Hence $\cot 90°=0$, and $\cot 0=\infty$.

X. The point M continuing to advance from D towards B, the sines diminish and the cosines increase. Thus M′P′ is the sine of the arc AM′, and M′Q, or CP′ its cosine. But the arc M′B is the supplement of AM′, since AM′+M′B is equal to a semicircumference; besides, if M′M is drawn parallel to AB, the arcs AM, BM′, which are included between parallels, will evidently be equal, and likewise the perpendiculars or sines MP, M′P′. Hence, *the sine of an arc or of an angle is equal to the sine of the supplement of that arc or angle.*

The arc or angle A has for its supplement 180°—A: hence generally, we have

$$\sin A=\sin (180°-A.)$$

The same property might also be expressed by the equation

$$\sin (90°+B)=\sin (90°-B),$$

B being the arc DM or its equal DM′.

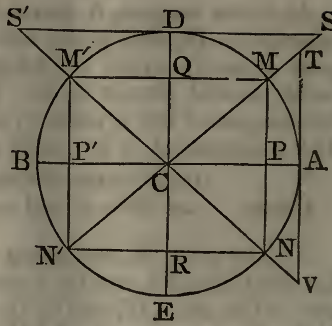
XI. The same arcs AM, AM′, which are supplements of each other, and which have equal sines, have also equal cosines CP, CP′; but it must be observed, that these cosines lie in different directions. The line CP which is the cosine of the arc AM, has the origin of its value at the centre C, and is estimated in the direction from C towards A; while CP′, the cosine of AM′ has also the origin of its value at C, but is estimated in a contrary direction, from C towards B.

Some notation must obviously be adopted to distinguish the one of such equal lines from the other; and that they may both be expressed analytically, *and in the same general formula*, it is necessary to consider all lines which are estimated in one direction as *positive*, and those which are estimated in the contrary direction as *negative*. If, therefore, the cosines which are estimated from C towards A be considered as positive, those estimated from C towards B, must be regarded as negative. Hence, generally, we shall have,

$$\cos A=-\cos (180°-A)$$

that is, *the cosine of an arc or angle is equal to the cosine of its supplement taken negatively.*

The necessity of changing the algebraic sign to correspond



with the change of direction in the trigonometrical line, may be illustrated by the following example. The versed sine AP is equal to the radius CA minus CP the cosine AM: that is,

ver-sin AM$=$R—cos AM.

Now when the arc AM becomes AM′ the versed sine AP, becomes AP′, that is equal to R$+$CP′. But this expression cannot be derived from the formula,

ver-sin AM$=$R—cos AM,

unless we suppose the cosine AM to become negative as soon as the arc AM becomes greater than a quadrant.

At the point B the cosine becomes equal to —R; that is,

$$\cos 180° = -R.$$

For all arcs, such as ADBN′, which terminate in the third quadrant, the cosine is estimated from C towards B, and is consequently negative. At E the cosine becomes zero, and for all arcs which terminate in the fourth quadrant the cosines are estimated from C towards A, and are consequently positive.

The sines of all the arcs which terminate in the first and second quadrants, are estimated above the diameter BA, while the sines of those arcs which terminate in the third and fourth quadrants are estimated below it. Hence, considering the former as positive, we must regard the latter as negative.

XII. Let us now see what sign is to be given to the tangent of an arc. The tangent of the arc AM falls above the line BA, and we have already regarded the lines estimated in the direction AT as positive: therefore the tangents of all arcs which terminate in the first quadrant will be positive. But the tangent of the arc AM′, greater than 90°, is determined by the intersection of the two lines M′C and AT. These lines, however, do not meet in the direction AT; but they meet in the opposite direction AV. But since the tangents estimated in the direction AT are positive, those estimated in the direction AV must be negative: therefore, *the tangents of all arcs which terminate in the second quadrant will be negative.*

When the point M′ reaches the point B the tangent AV will become equal to zero: that is,

$$\text{tang } 180° = 0.$$

When the point M′ passes the point B, and comes into the position N′, the tangent of the arc ADN′ will be the line AT:

hence, *the tangents of all arcs which terminate in the third quadrant are positive.*

At E the tangent becomes infinite: that is.

$$\text{tang } 270° = \infty.$$

When the point has passed along into the fourth quadrant to N, the tangent of the arc ADN′N will be the line AV: hence, *the tangents of all arcs which terminate in the fourth quadrant are negative.*

The cotangents are estimated from the line ED. Those which lie on the side DS are regarded as positive, and those which lie on the side DS′ as negative. Hence, *the cotangents are positive in the first quadrant, negative in the second, positive in the third, and negative in the fourth.* When the point M is at B the cotangent is infinite; when at E it is zero: hence,

$$\cot 180° = -\infty; \cot 270° = 0.$$

Let q stand for a quadrant; then the following table will show the signs of the trigonometrical lines in the different quadrants

	$1q$	$2q$	$3q$	$4q$
Sine	+	+	—	—
Cosine	+	—	—	+
Tangent	+	—	+	—
Cotangent	+	—	+	—

XIII. In trigonometry, the sines, cosines, &c. of arcs or angles greater than 180° do not require to be considered; the angles of triangles, rectilineal as well as spherical, and the sides of the latter, being always comprehended between 0 and 180°. But in various applications of trigonometry, there is frequently occasion to reason about arcs greater than the semicircumference, and even about arcs containing several circumferences. It will therefore be necessary to find the expression of the sines and cosines of those arcs whatever be their magnitude.

We generally consider the arcs as positive which are estimated from A in the direction ADB, and then those arcs must be regarded as negative which are estimated in the contrary direction AEB.

We observe, in the first place, that two equal arcs AM, AN with contrary algebraic signs, have equal sines MP, PN, with contrary algebraic signs; while the cosine CP is the same for both.

The equal tangents AT, AV, as well as the equal cotangents DS, DS′, have also contrary algebraic signs. Hence, calling x the arc, we have in general,

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\text{tang}(-x) = -\text{tang } x$$
$$\cot(-x) = -\cot x$$

By considering the arc AM, and its supplement AM′, and recollecting what has been said, we readily see that,

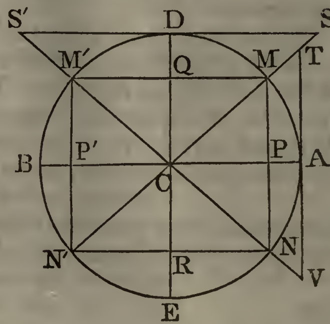
sin (an arc) = sin (its supplement)

cos (an arc) = —cos (its supplement)

tang (an arc) = —tang (its supplement)

cot (an arc) = —cot (its supplement).

It is no less evident, that if one or several circumferences were added to any arc AM, it would still terminate exactly at the point M, and the arc thus increased would have the same sine as the arc AM; hence if C represent a whole circumference or 360°, we shall have $\sin x = \sin (C+x) = \sin x = \sin (2C+x)$, &c. The same observation is applicable to the cosine, tangent, &c.



Hence it appears, that whatever be tne magnitude of x the proposed arc, its sine may always be expressed, with a proper sign, by the sine of an arc less than 180°. For, in the first place, we may subtract 360° from the arc x as often as they are contained in it; and y being the remainder, we shall have $\sin x = \sin y$. Then if y is greater than 180°, make $y = 180° + z$, and we have $\sin y = -\sin z$. Thus all the cases are reduced to that in which the proposed arc is less than 180°; and since we farther have $\sin (90°+x) = \sin (90°-x)$, they are likewise ultimately reducible to the case, in which the proposed arc is between zero and 90°.

XIV. The cosines are always reducible to sines, by means of the formula $\cos A = \sin (90°-A)$; or if we require it, by means of the formula $\cos A = \sin (90°+A)$: and thus, if we can find the value of the sines in all possible cases, we can also find that of the cosines. Besides, as has already been shown, that the negative cosines are separated from the positive cosines by the diameter DE; all the arcs whose extremities fall on the right side of DE, having a positive cosine, while those whose extremities fall on the left have a negative cosine.

Thus from 0° to 90° the cosines are positive; from 90° to 270° they are negative; from 270° to 360° they again become positive; and after a whole revolution they assume the same values as in the preceding revolution, for $\cos (360°+x) = \cos x$.

From these explanations, it will evidently appear, that the sines and cosines of the various arcs which are multiples of the quadrant have the following values:

$\sin 0°=0$	$\sin 90°=R$	$\cos 0°=R$	$\cos 90°=0$
$\sin 180°=0$	$\sin 270°=-R$	$\cos 180°=-R$	$\cos 270°=0$
$\sin 360°=0$	$\sin 450°=R$	$\cos 360°=R$	$\cos 450°=0$
$\sin 540°=0$	$\sin 630°=-R$	$\cos 540°=-R$	$\cos 630°=0$
$\sin 720°=0$	$\sin 810°=R$	$\cos 720°=R$	$\cos 810°=0$
&c.	&c.	&c.	&c.

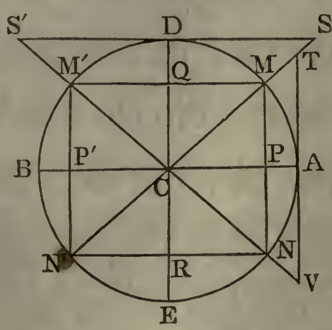
And generally, k designating any whole number we shall have

$$\sin 2k\,.\,90°=0, \qquad \cos (2k+1)\,.\,90°=0,$$
$$\sin (4k+1)\,.\,90°=R, \qquad \cos 4k\,.\,90°=R,$$
$$\sin (4k-1)\,.\,90°=-R, \qquad \cos (4k+2)\,.\,90°=-R.$$

What we have just said concerning the sines and cosines renders it unnecessary for us to enter into any particular detail respecting the tangents, cotangents, &c. of arcs greater than 180°; the value of these quantities are always easily deduced from those of the sines and cosines of the same arcs: as we shall see by the formulas, which we now proceed to explain.

THEOREMS AND FORMULAS RELATING TO SINES, COSINES, TANGENTS, &c.

XV. *The sine of an arc is half the chord which subtends a double arc.*

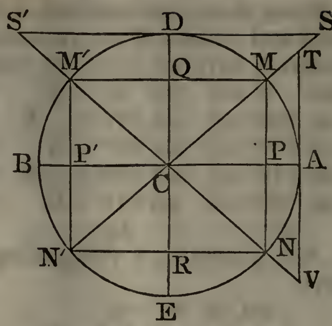


For the radius CA, perpendicular to the chord MN, bisects this chord, and likewise the arc MAN; hence MP, the sine of the arc MA, is half the chord MN which subtends the arc MAN, the double of MA.

The chord which subtends the sixth part of the circumference is equal to the radius; hence

$$\sin \frac{360°}{12} \text{ or } \sin 30°=\tfrac{1}{2}R,$$

in other words, the sine of a third part of the right angle is equal to the half of the radius.



XVI. *The square of the sine of an arc, together with the square of the cosine, is equal to the square of the radius; so that in general terms we have* $\sin^2 A + \cos^2 A = R^2$.

This property results immediately from the right-angled triangle CMP, in which $MP^2 + CP^2 = CM^2$.

It follows that when the sine of an arc is given, its cosine may be found, and reciprocally, by means of the formulas $\cos A = \pm \sqrt{(R^2 - \sin^2 A)}$, and $\sin A = \pm \sqrt{(R^2 - \cos^2 A)}$. The sign of these formulas is $+$, or $-$, because the same sine MP answers to the two arcs AM, AM′, whose cosines CP, CP′, are equal and have contrary signs; and the same cosine CP answers to the two arcs AM, AN, whose sines MP, PN, are also equal, and have contrary signs.

Thus, for example, having found $\sin 30° = \frac{1}{2}R$, we may deduce from it $\cos 30°$, or $\sin 60° = \sqrt{(R^2 - \frac{1}{4}R^2)} = \sqrt{\frac{3}{4}R^2} = \frac{1}{2}R\sqrt{3}$.

XVII. *The sine and cosine of an arc* A *being given, it is required to find the tangent, secant, cotangent, and cosecant of the same arc.*

The triangles CPM, CAT, CDS, being similar, we have the proportions:

$$CP : PM :: CA : AT; \text{ or } \cos A : \sin A :: R : \text{tang } A = \frac{R \sin A}{\cos A}$$

$$CP : CM :: CA : CT; \text{ or } \cos A : R :: R : \sec A = \frac{R^2}{\cos A}$$

$$PM : CP :: CD : DS; \text{ or } \sin A : \cos A :: R : \cot A = \frac{R \cos A}{\sin A}$$

$$PM : CM :: CD : CS; \text{ or } \sin A : R :: R : \text{cosec } A = \frac{R^2}{\sin A}$$

which are the four formulas required. It may also be observed, that the two last formulas might be deduced from the first two, by simply putting $90° - A$ instead of A.

From these formulas, may be deduced the values, with their proper signs, of the tangents, secants, &c. belonging to any arc whose sine and cosine are known; and since the progressive law of the sines and cosines, according to the different arcs to which they relate, has been developed already, it is unnecessary to say more of the law which regulates the tangents and secants.

By means of these formulas, several results, which have already been obtained concerning the trigonometrical lines may be confirmed. If, for example, we make $A=90°$, we shall have $\sin A=R$, $\cos A=0$; and consequently $\tan 90°=\frac{R^2}{0}$, an expression which designates an infinite quantity: for the quotient of radius divided by a very small quantity, is very great, and increases as the divisor diminishes; hence, the quotient of the radius divided by zero is greater than any finite quantity.

The tangent being equal to $R.\frac{\sin}{\cos}$; and cotangent to $R.\frac{\cos}{\sin}$; it follows that tangent and cotangent will both be positive when the sine and cosine have like algebraic signs, and both negative, when the sine and cosine have contrary algebraic signs. Hence, the tangent and cotangent have the same sign in the diagonal quadrants: that is, positive in the 1st and 3d, and negative in the 2d and 4th; results agreeing with those of Art. XII.

The Algebraic signs of the secants and cosecants are readily determined. For, the secant is equal to radius square divided by the cosine, and since radius square is always positive, it follows that the algebraic sign of the secant will depend on that of the cosine: hence, it is positive in the 1st and 4th quadrants and negative in the 2nd and 3rd.

Since the cosecant is equal to radius square divided by the sine, it follows that its sign will depend on the algebraic sign of the sine: hence, it will be positive in the 1st and 2nd quadrants and negative in the 3rd and 4th.

XVIII. The formulas of the preceding Article, combined with each other and with the equation $\sin^2 A+\cos^2 A=R^2$, furnish some others worthy of attention.

First we have $R^2 + \text{tang}^2\ A = R^2 + \frac{R^2 \sin^2 A}{\cos^2 A} = \frac{R^2 (\sin^2 A+\cos^2 A)}{\cos^2 A}=\frac{R^4}{\cos^2 A}$; hence $R^2+\text{tang}^2\ A=\sec^2 A$, a formula which might be immediately deduced from the right-angled triangle CAT. By these formulas, or by the right-angled triangle CDS, we have also $R^2+\cot^2 A=\text{cosec}^2 A$.

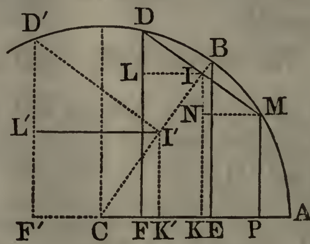
Lastly, by taking the product of the two formulas $\text{tang}\ A=\frac{R \sin A}{\cos A}$, and $\cot A=\frac{R \cos A}{\sin A}$, we have $\text{tang}\ A\times\cot A=R^2$, a formula which gives $\cot A=\frac{R^2}{\text{tang}\ A}$, and $\text{tang}\ A=\frac{R^2}{\cot A}$.

We likewise have $\cot B=\frac{R^2}{\text{tang}\ B}$.

Hence cot A : cot B : : tang B : tang A ; that is, *the cotangents of two arcs are reciprocally proportional to their tangents.*

The formula $\cot A \times \text{tang } A = R^2$ might be deduced immediately, by comparing the similar triangles CAT, CDS, which give AT : CA : : CD : DS, or tang A : R : : R : cot A

XIX. *The sines and cosines of two arcs,* a *and* b. *being given, it is required to find the sine and cosine of the sum or difference of these arcs.*



Let the radius AC=R, the arc AB=a, the arc BD=b, and consequently ABD=$a+b$. From the points B and D, let fall the perpendiculars BE, DF upon AC; from the point D, draw DI perpendicular to BC ; lastly, from the point I draw IK perpendicular, and IL parallel to, AC.

The similar triangles BCE, ICK, give the proportions,

CB : CI : : BE : IK, or R : cos b : : sin a : $\text{IK}=\dfrac{\sin a \cos b}{R}$

CB : CI : : CE : CK, or R : cos b : : cos a : $\text{CK}=\dfrac{\cos a \cos b.}{R}$

The triangles DIL, CBE, having their sides perpendicular, each to each, are similar, and give the proportions,

CB : DI : : CE : DL, or R : sin b : : cos a : $\text{DL}=\dfrac{\cos a \sin b.}{R}$

CB : DI : : BE : IL, or R : sin b : : sin a : $\text{IL}=\dfrac{\sin a \sin b.}{R}$

But we have

IK+DL=DF=sin $(a+b)$, and CK—IL=CF=cos $(a+b)$.

Hence

$$\sin(a+b)=\frac{\sin a \cos b+\sin b \cos a}{R}$$

$$\cos(a+b)=\frac{\cos a \cos b-\sin a \sin b.}{R}$$

The values of sin $(a-b)$ and of cos $(a-b)$ might be easily deduced from these two formulas; but they may be found directly by the same figure. For, produce the sine DI till it meets the circumference at M ; then we have BM=BD=b, and MI=ID=sin b. Through the point M, draw MP perpendicular, and MN parallel to, AC : since MI=DI, we have MN =IL, and IN=DL. But we have IK—IN=MP=sin $(a-b)$, and CK+MN=CP=cos $(a-b)$; hence

$$\sin(a-b)=\frac{\sin a \cos b-\sin b \cos a}{\mathrm{R}}$$

$$\cos(a-b)=\frac{\cos a \cos b+\sin a \sin b}{\mathrm{R}}$$

These are the formulas which it was required to find.

The preceding demonstration may seem defective in point of generality, since, in the figure which we have followed, the arcs a and b, and even $a+b$, are supposed to be less than 90°. But first the demonstration is easily extended to the case in which a and b being less than 90°, their sum $a+b$ is greater than 90°. Then the point F would fall on the prolongation of AC, and the only change required in the demonstration would be that of taking $\cos(a+b)=-\mathrm{CF}'$; but as we should, at the same time, have $\mathrm{CF}'=\mathrm{I'L'}-\mathrm{CK'}$, it would still follow that $\cos(a+b)=\mathrm{CK'}-\mathrm{I'L'}$, or $\mathrm{R}\cos(a+b)=\cos a \cos b-\sin a \sin b$. And whatever be the values of the arcs a and b, it is easily shown that the formulas are true: hence we may regard them as established for all arcs. We will repeat and number the formulas for the purpose of more convenient reference.

$$\sin(a+b)=\frac{\sin a \cos b+\sin b \cos a}{\mathrm{R}} \quad (1.).$$

$$\sin(a-b)=\frac{\sin a \cos b-\sin b \cos a}{\mathrm{R}} \quad (2.).$$

$$\cos(a+b)=\frac{\cos a \cos b-\sin a \sin b}{\mathrm{R}} \quad (3.)$$

$$\cos(a-b)=\frac{\cos a \cos b+\sin a \sin b}{\mathrm{R}} \quad (4.)$$

XX. If, in the formulas of the preceding Article, we make $b=a$, the first and the third will give

$$\sin 2a=\frac{2\sin a \cos a}{\mathrm{R}},\ \cos 2a=\frac{\cos^2 a-\sin^2 a}{\mathrm{R}}=\frac{2\cos^2 a-\mathrm{R}^2}{\mathrm{R}}$$

formulas which enable us to find the sine and cosine of the double arc, when we know the sine and cosine of the arc itself.

To express the $\sin a$ and $\cos a$ in terms of $\frac{1}{2}a$, put $\frac{1}{2}a$ for a, and we have

$$\sin a=\frac{2\sin\frac{1}{2}a \cos\frac{1}{2}a}{\mathrm{R}},\quad \cos a=\frac{\cos^2\frac{1}{2}a-\sin^2\frac{1}{2}a}{\mathrm{R}}.$$

To find the sine and cosine of $\frac{1}{2}a$ in terms of a, take th equations

$$\cos^2\tfrac{1}{2}a+\sin^2\tfrac{1}{2}a=\mathrm{R}^2,\text{ and } \cos^2\tfrac{1}{2}a-\sin^2\tfrac{1}{2}a=\mathrm{R}\cos a,$$

there results by adding and subtracting

$$\cos^2\tfrac{1}{2}a=\tfrac{1}{2}\mathrm{R}^2+\tfrac{1}{2}\mathrm{R}\cos a,\text{ and } \sin^2\tfrac{1}{2}a=\tfrac{1}{2}\mathrm{R}^2-\tfrac{1}{2}\mathrm{R}\cos a;$$

whence

$$\sin\tfrac{1}{2}a=\sqrt{(\tfrac{1}{2}\mathrm{R}^2-\tfrac{1}{2}\mathrm{R}\cos a)}=\tfrac{1}{2}\sqrt{2\mathrm{R}^2-2\mathrm{R}\cos a}.$$

If we put $2a$ in the place of a, we shall have,

$$\sin a=\sqrt{(\tfrac{1}{2}R^2-\tfrac{1}{2}R\cos 2a)}=\tfrac{1}{2}\sqrt{2R^2-2R\cos 2a}.$$

$$\cos a=\sqrt{(\tfrac{1}{2}R^2+\tfrac{1}{2}R\cos 2a)}=\tfrac{1}{2}\sqrt{2R^2+2R\cos 2a}.$$

Making, in the two last formulas, $a=45°$, gives $\cos 2a=0$, and $\sin 45°=\sqrt{\tfrac{1}{2}R^2}=R\sqrt{\tfrac{1}{2}}$; and also, $\cos 45°=\sqrt{\tfrac{1}{2}R^2}=R\sqrt{\tfrac{1}{2}}$. Next, make $a=22°\ 30'$, which gives $\cos 2a=R\sqrt{\tfrac{1}{2}}$, and we have $\sin 22°\ 30'=R\sqrt{(\tfrac{1}{2}-\tfrac{1}{2}\sqrt{\tfrac{1}{2}})}$ and $\cos 22°\ 30'=R\sqrt{(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}})}$.

XXI. If we multiply together formulas (1.) and (2.) Art. XIX. and substitute for $\cos^2 a$, $R^2-\sin^2 a$, and for $\cos^2 b$, $R^2-\sin^2 b$; we shall obtain, after reducing and dividing by R^2,

$$\sin(a+b)\sin(a-b)=\sin^2 a-\sin^2 b=(\sin a+\sin b)(\sin a-\sin b).$$

$$\text{or, } \sin(a-b) : \sin a-\sin b :: \sin a+\sin b : \sin(a+b).$$

XXII. The formulas of Art. XIX. furnish a great number of consequences; among which it will be enough to mention those of most frequent use. By adding and subtracting we obtain the four which follow,

$$\sin(a+b)+\sin(a-b)=\frac{2}{R}\sin a\cos b.$$

$$\sin(a+b)-\sin(a-b)=\frac{2}{R}\sin b\cos a.$$

$$\cos(a+b)+\cos(a-b)=\frac{2}{R}\cos a\cos b.$$

$$\cos(a-b)-\cos(a+b)=\frac{2}{R}\sin a\sin b.$$

and which serve to change a product of several sines or cosines into *linear* sines or cosines, that is, into sines and cosines multiplied only by constant quantities.

XXIII. If in these formulas we put $a+b=p$, $a-b=q$, which gives $a=\frac{p+q}{2}$, $b=\frac{p-q}{2}$, we shall find

$$\sin p+\sin q=\frac{2}{R}\sin\tfrac{1}{2}(p+q)\cos\tfrac{1}{2}(p-q)\quad(1.)$$

$$\sin p-\sin q=\frac{2}{R}\sin\tfrac{1}{2}(p-q)\cos\tfrac{1}{2}(p+q)\quad(2.)$$

$$\cos p+\cos q=\frac{2}{R}\cos\tfrac{1}{2}(p+q)\cos\tfrac{1}{2}(p-q)\quad(3.)$$

$$\cos q-\cos p=\frac{2}{R}\sin\tfrac{1}{2}(p+q)\sin\tfrac{1}{2}(p-q)\quad(4.)$$

If we make $q=o$, we shall obtain,

$$\sin p=\frac{2\sin\frac{1}{2}p\cos\frac{1}{2}p}{R}$$

$$R+\cos p=\frac{2\cos^2\frac{1}{2}p}{R}$$

$$R—\cos p=\frac{2\sin^2\frac{1}{2}p}{R}: \text{ hence}$$

$$\frac{\sin p}{R+\cos p}=\frac{\text{tang}\,\frac{1}{2}p}{R}=\frac{R}{\cot\frac{1}{2}p}$$

$$\frac{\sin p}{R—\cos p}=\frac{\cot\frac{1}{2}p}{R}=\frac{R}{\text{tang}\,\frac{1}{2}p}:$$

formulas which are often employed in trigonometrical calculations for reducing two terms to a single one.

XXIV. From the first four formulas of Art XXIII. and the first of Art. XX., dividing, and considering that $\frac{\sin a}{\cos a}=\frac{\text{tang}\,a}{R}=\frac{R}{\cot a}$ we derive the following:

$$\frac{\sin p+\sin q}{\sin p—\sin q}=\frac{\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p—q)}{\cos\frac{1}{2}(p+q)\sin\frac{1}{2}(p—q)}=\frac{\text{tang}\,\frac{1}{2}(p+q)}{\text{tang}\,\frac{1}{2}(p—q)}$$

$$\frac{\sin p+\sin q}{\cos p+\cos q}=\frac{\sin\frac{1}{2}(p+q)}{\cos\frac{1}{2}(p+q)}=\frac{\text{tang}\,\frac{1}{2}(p+q)}{R}$$

$$\frac{\sin p+\sin q}{\cos q—\cos p}=\frac{\cos\frac{1}{2}(p—q)}{\sin\frac{1}{2}(p—q)}=\frac{\cot\frac{1}{2}(p—q)}{R}$$

$$\frac{\sin p—\sin q}{\cos p+\cos q}=\frac{\sin\frac{1}{2}(p—q)}{\cos\frac{1}{2}(p—q)}=\frac{\text{tang}\,\frac{1}{2}(p—q)}{R}$$

$$\frac{\sin p—\sin q}{\cos q—\cos p}=\frac{\cos\frac{1}{2}(p+q)}{\sin\frac{1}{2}(p+q)}=\frac{\cot\frac{1}{2}(p+q)}{R}$$

$$\frac{\cos p+\cos q}{\cos q—\cos p}=\frac{\cos\frac{1}{2}(p+q)\cos\frac{1}{2}(p—q)}{\sin\frac{1}{2}(p+q)\sin\frac{1}{2}(p—q)}=\frac{\cot\frac{1}{2}(p+q)}{\text{tang}\,\frac{1}{2}(p—q)}$$

$$\frac{\sin p+\sin q}{\sin\ (p+q)}=\frac{2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p—q)}{2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p+q)}=\frac{\cos\frac{1}{2}(p—q)}{\cos\frac{1}{2}(p+q)}$$

$$\frac{\sin p—\sin q}{\sin\ (p+q)}=\frac{2\sin\frac{1}{2}(p—q)\cos\frac{1}{2}(p+q)}{2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p+q)}=\frac{\sin\frac{1}{2}(p—q)}{\sin\frac{1}{2}(p+q)}$$

Formulas which are the expression of so many theorems From the first, it follows that the *sum of the sines of two arcs is to the difference of these sines, as the tangent of half the sum of the arcs is to the tangent of half their difference*

XXV. In order likewise to develop some formulas relative to tangents, let us consider the expression

$\text{tang }(a+b)=\frac{\text{R}\sin(a+b)}{\cos(a+b)}$, in which by substituting the values of $\sin(a+b)$ and $\cos(a+b)$, we shall find

$$\text{tang }(a+b)=\frac{\text{R}(\sin a\cos b+\sin b\cos a)}{\cos a\cos b-\sin b\sin a}.$$

Now we have $\sin a=\frac{\cos a\text{ tang }a}{\text{R}}$, and $\sin b=\frac{\cos b\text{ tang }b}{\text{R}}$: substitute these values, dividing all the terms by $\cos a\cos b$; we shall have

$$\text{tang }(a+b)=\frac{\text{R}^2(\text{tang }a+\text{tang }b)}{\text{R}^2-\text{tang }a\text{ tang }b};$$

which is the value of the tangent of the sum of two arcs, expressed by the tangents of each of these arcs. For the tangent of their difference, we should in like manner find

$$\text{tang }(a-b)=\frac{\text{R}^2(\text{tang }a-\text{tang }b)}{\text{R}^2+\text{tang }a\text{ tang }b.}$$

Suppose $b=a$; for the duplication of the arcs, we shall have the formula

$$\text{tang }2a=\frac{2\text{R}^2\text{ tang }a}{\text{R}^2-\text{tang}^2a}:$$

Suppose $b=2a$; for their triplication, we shall have the formula

$$\text{tang }3a=\frac{\text{R}^2(\text{tang }a+\text{tang }2a)}{\text{R}^2-\text{tang }a\text{ tang }2a};$$

in which, substituting the value of tang $2a$, we shall have

$$\text{tang }3a=\frac{3\text{R}^2\text{ tang }a-\text{tang}^3a}{\text{R}^2-3\text{ tang}^2a.}$$

XXVI. *Scholium.* The radius R being entirely arbitrary, is generally taken equal to 1, in which case it does not appear in the trigonometrical formulas. For example the expression for the tangent of twice an arc when $\text{R}=1$, becomes,

$$\text{tang }2a=\frac{2\text{ tang }a}{1-\text{tang}^2a}\cdot$$

If we have an analytical formula calculated to the radius of 1, and wish to apply it to another circle in which the radius is R, we must multiply each term by such a power of R as will make all the terms homogeneous: that is, so that each shall contain the same number of literal factors.

CONSTRUCTION AND DESCRIPTION OF THE TABLES.

XXVII. If the radius of a circle is taken equal to 1, and the lengths of the lines representing the sines, cosines, tangents, cotangents, &c. for every minute of the quadrant be calculated, and written in a table, this would be a table of *natural* sines, cosines, &c.

XXVIII. If such a table were known, it would be easy to calculate a table of sines, &c. to any other radius; since, in different circles, the sines, cosines, &c. of arcs containing the same number of degrees, are to each other as their radii.

XXIX. If the trigonometrical lines themselves were used, it would be necessary, in the calculations, to perform the operations of multiplication and division. To avoid so tedious a method of calculation, we use the logarithms of the sines, cosines, &c.; so that the tables in common use show the values of the logarithms of the sines, cosines, tangents, cotangents, &c. for each degree and minute of the quadrant, calculated to a given radius. This radius is 10,000,000,000, and consequently its logarithm is 10.

XXX. Let us glance for a moment at one of the methods of calculating a table of natural sines.

The radius of a circle being 1, the semi-circumference is known to be 3.14159265358979. This being divided successively, by 180 and 60, or at once by 10800, gives .0002908882086657, for the arc of 1 minute. Of so small an arc the sine, chord, and arc, differ almost imperceptibly from the ratio of equality; so that the first ten of the preceding figures, that is, .0002908882 may be regarded as the sine of $1'$; and in fact the sine given in the tables which run to seven places of figures is .0002909. By Art. XVI. we have for any arc, $\cos = \sqrt{(1-\sin^2)}$. This theorem gives, in the present case, $\cos 1' = .9999999577$. Then by Art. XXII. we shall have

$$2 \cos 1' \times \sin 1' - \sin 0' = \sin 2' = .0005817764$$
$$2 \cos 1' \times \sin 2' - \sin 1' = \sin 3' = .0008726646$$
$$2 \cos 1' \times \sin 3' - \sin 2' = \sin 4' = .0011635526$$
$$2 \cos 1' \times \sin 4' - \sin 3' = \sin 5' = .0014544407$$
$$2 \cos 1' \times \sin 5' - \sin 4' = \sin 6' = .0017453284$$

&c. &c. &c.

Thus may the work be continued to any extent, the whole difficulty consisting in the multiplication of each successive result by the quantity $2 \cos 1' = 1.9999999154$.

Or, the sines of 1′ and 2′ being determined, the work might be continued thus (Art. XXI.):

$$\sin 1' : \sin 2' - \sin 1' :: \sin 2' + \sin 1' : \sin 3$$
$$\sin 2' : \sin 3' - \sin 1' :: \sin 3' + \sin 1' : \sin 4$$
$$\sin 3' : \sin 4' - \sin 1' :: \sin 4' + \sin 1' : \sin 5'$$
$$\sin 4' : \sin 5' - \sin 1' :: \sin 5' + \sin 1' : \sin 6'$$

&c. &c. &c.

In like manner, the computer might proceed for the sines of degrees, &c. thus:

$$\sin 1^\circ : \sin 2^\circ - \sin 1^\circ :: \sin 2^\circ + \sin 1^\circ : \sin 3^\circ$$
$$\sin 2^\circ : \sin 3^\circ - \sin 1^\circ :: \sin 3^\circ + \sin 1^\circ : \sin 4^\circ$$
$$\sin 3^\circ : \sin 4^\circ - \sin 1^\circ :: \sin 4^\circ + \sin 1^\circ : \sin 5^\circ$$

&c. &c. &c.

Above 45° the process may be considerably simplified by the theorem for the tangents of the sums and differences of arcs. For, when the radius is unity, the tangent of 45° is also unity, and tan $(a+b)$ will be denoted thus:

$$\tan (45^\circ + b) = \frac{1 + \tan b}{1 - \tan b}.$$

And this, again, may be still further simplified in practice. The secants and cosecants may be found from the cosines and sines.

TABLE OF LOGARITHMS.

XXXI. If the logarithms of all the numbers between 1 and any given number, be calculated and arranged in a tabular form, such table is called a table of logarithms. The table annexed shows the logarithms of all numbers between 1 and 10,000.

The first column, on the left of each page of the table, is the column of numbers, and is designated by the letter N; the decimal part of the logarithms of these numbers is placed directly opposite them, and on the same horizontal line.

The *characteristic* of the logarithm, or the part which stands to the left of the decimal point, is always known, being 1 less than the places of integer figures in the given number, and therefore it is not written in the table of logarithms. Thus, for all numbers between 1 and 10, the characteristic is 0: for numbers between 10 and 100 it is 1, between 100 and 1000 it is 2, &c.

PROBLEM.

To find from the table the logarithm of any number.

CASE I.

When the number is less than 100.

Look on the first page of the table of logarithms, along the columns of numbers under N, until the number is found; the number directly opposite it, in the column designated Log., is the logarithm sought.

CASE II.

When the number is greater than 100, *and less than* 10,000.

Find, in the column of numbers, the three first figures of the given number. Then, pass across the page, in a horizontal line, into the columns marked 0, 1, 2, 3, 4, &c., until you come to the column which is designated by the fourth figure of the given number: to the four figures so found, two figures taken from the column marked 0, are to be prefixed. If the four figures found, stand opposite to a row of six figures in the column marked 0, the two figures from this column, which are to be prefixed to the four before found, are the first two on the left hand; but, if the four figures stand opposite a line of only four figures, you are then to ascend the column, till you come to the line of six figures: the two figures at the left hand are to be prefixed, and then the decimal part of the logarithm is obtained. To this, the characteristic of the logarithm is to be prefixed, which is always one less than the places of integer figures in the given number. Thus, the logarithm of 1122 is 3.049993.

In several of the columns, designated 0, 1, 2, 3, &c., small dots are found. Where this occurs, a cipher must be written for each of these dots, and the two figures which are to be prefixed, from the first column, are then found in the horizontal line directly below. Thus, the log. of 2188 is 3.340047, the two dots being changed into two ciphers, and the 34 from the column 0, prefixed. The two figures from the colum 0, must also be taken from the line below, if any dots shall have been passed over, in passing along the horizontal line: thus, the logarithm of 3098 is 3.491081, the 49 from the column 0 being taken from the line 310.

CASE III.

When the number exceeds 10,000, *or consists of five or more places of figures.*

Consider all the figures after the fourth from the left hand, as ciphers. Find, from the table, the logarithm of the first four places, and prefix a characteristic which shall be one less than the number of places including the ciphers. Take from the last column on the right of the page, marked D, the number on the same horizontal line with the logarithm, and multiply this number by the numbers that have been considered as ciphers: then, cut off from the right hand as many places for decimals as there are figures in the multiplier, and add the product, so obtained, to the first logarithm: this sum will be the logarithm sought.

Let it be required to find the logarithm of 672887. The log. of 672800 is found, on the 11th page of the table, to be 5.827886, after prefixing the characteristic 5. The corresponding number in the column D is 65, which being multiplied by 87, the figures regarded as ciphers, gives 5655; then, pointing off two places for decimals, the number to be added is 56.55. This number being added to 5.827886, gives 5.827942 for the logarithm of 672887; the decimal part .55, being omitted.

This method of finding the logarithms of numbers, from the table, supposes that the logarithms are proportional to their respective numbers, which is not rigorously true. In the example, the logarithm of 672800 is 5.827886; the logarithm of 672900, a number greater by 100, 5.827951: the difference of the logarithms is 65. Now, as 100, the difference of the numbers, is to 65, the difference of their logarithms, so is 87, the difference between the given number and the least of the numbers used, to the difference of their logarithms, which is 56.55: this difference being added to 5.827886, the logarithm of the less number, gives 5.827942 for the logarithm of 672887. The use of the column of differences is therefore manifest.

When, however, the decimal part which is to be omitted exceeds .5, we come nearer to the true result by increasing the next figure to the left by 1; and this will be done in all the calculations which follow. Thus, the difference to be added. was nearer 57 than 56; hence it would have been more exact to have added the former number.

The logarithm of a vulgar fraction is equal to the logarithm of the numerator minus the logarithm of the denom inator. The logarithm of a decimal fraction is found, *by considering it as a whole number, and then prefixing to the decimal part of its logarithm a negative characteristic, greater by unity than the number of ciphers between the decimal point and the first significant place of figures.* Thus, the logarithm of .0412. is $\bar{2}.614897$.

PROBLEM.

To find from the table, a number answering to a given logarithm.

XXXII Search, in the column of logarithms, for the decimal part of the given logarithm, and if it be exactly found, set down the corresponding number. Then, if the characteristic of the given logarithm be positive, point off, from the left of the number found, one place more for whole numbers than there are units in the characteristic of the given logarithm, and treat the other places as decimals; this will give the number sought.

If the characteristic of the given logarithm be 0, there will be one place of whole numbers; if it be —1, the number will be entirely decimal; if it be —2, there will be one cipher between the decimal point and the first significant figure; if it be —3, there will be two, &c. The number whose logarithm is 1.492481 is found in page 5, and is 31.08.

But if the decimal part of the logarithm cannot be exactly found in the table, take the number answering to the nearest less logarithm; take also from the table the corresponding difference in the column D: then, subtract this less logarithm from the given logarithm; and having annexed a sufficient number of ciphers to the remainder, divide it by the difference taken from the column D, and annex the quotient to the number answering to the less logarithm: this gives the required number, nearly. This rule, like the one for finding the logarithm of a number when the places exceed four, supposes the numbers to be proportional to their corresponding logarithms.

Ex. 1. Find the number answering to the logarithm 1.532708. Here,

The given logarithm, is - - -	1.532708
Next less logarithm of 34,09, is - -	1.532627
Their difference is - - - - -	81

And the tabular difference is 128: hence

128) 81.00 (63

which being annexed to 34,09, gives 34.0963 for the number answering to the logarithm 1.532708.

Ex. 2. Required the number answering to the logarithm 3.233568.

The given logarithm is	3.233568
The next less tabular logarithm of 1712, is	3.233504
Diff.=	64

Tab. Diff.=253) 64.00 (25

Hence the number sought is 1712.25, marking four places of integers for the characteristic 3.

TABLE OF LOGARITHMIC SINES.

XXXIII. In this table are arranged the logarithms of the numerical values of the sines, cosines, tangents, and cotangents. of all the arcs or angles of the quadrant, divided to minutes, and calculated for a radius of 10,000,000,000. The logarithm of this radius is 10. In the first and last horizontal line, of each page, are written the degrees whose logarithmic sines, &c. are expressed on the page. The vertical columns on the left and right, are columns of minutes.

CASE I.

To find, in the table, the logarithmic sine, cosine, tangent, or cotangent of any given arc or angle.

1. If the angle be less than 45°, look in the first horizontal line of the different pages, until the number of degrees be found; then descend along the column of minutes, on the left of the page, till you reach the number showing the minutes; then pass along the horizontal line till you come into the column designated, *sine*, *cosine*, *tangent*, or *cotangent*, as the case may be: the number so indicated, is the logarithm sought. Thus, the sine, cosine, tangent, and cotangent of 19° 55′, are found on page 37, opposite 55, and are, respectively, 9.532312, 9.973215, 9.559097, 10.440903.

2. If the angle be greater than 45°, search along the bottom line of the different pages, till the number of degrees are found; then ascend along the column of minutes, on the right hand side of the page, till you reach the number expressing the minutes; then pass along the horizontal line into the columns designated *tang.*, *cotang.*, *sine*, *cosine*, as the case may be: the number so pointed out is the logarithm required.

It will be seen, that the column designated sine at the top of the page, is designated cosine at the bottom; the one designated tang., by cotang., and the one designated cotang., by tang.

The angle found by taking the degrees at the top of the page, and the minutes from the first vertical column on the left, is the complement of the angle, found by taking the corresponding degrees at the bottom of the page, and the minutes traced up in the right hand column to the same horizontal line. This being apparent, the reason is manifest, why the columns designated sine, cosine, tang., and cotang., when the degrees are pointed out at the top of the page, and the minutes counted downwards, ought to be changed, respectively, into cosine, sine, cotang., and tang., when the degrees are shown at the bottom of the page, and the minutes counted upwards.

If the angle be greater than 90°, we have only to subtract it from 180°, and take the sine, cosine, tangent, or cotangent of the remainder.

The secants and cosecants are omitted in the table, being easily found from the cosines and sines.

For, $\text{sec.}=\frac{R^2}{\text{cos.}}$; or, taking the logarithms, log. sec.$=2$ log. R—log. cos.$=20$—log. cos.; that is, *the logarithmic secant is found by substracting the logarithmic cosine from* 20. And $\text{cosec.}=\frac{R^2}{\text{sine}}$, or log. cosec.$=2$ log. R—log. sine$=20$—log. sine; that is, *the logarithmic cosecant is found by subtracting the logarithmic sine from* 20.

It has been shown that $R^2=\text{tang.}\times\text{cotang.}$; therefore, 2 log. R$=$log. tang.$+$log. cotang.; or $20=$log. tang.$+$log. cotang.

The column of the table, next to the column of sines, and on the right of it, is designated by the letter D. This column is calculated in the following manner. Opening the table at any page, as 42, the sine of 24° is found to be 9.609313; of 24° 1′, 9.609597: their difference is 284; this being divided by 60, the number of seconds in a minute, gives 4.73, which is entered in the column D, omitting the decimal point. Now, supposing the increase of the logarithmic sine to be proportional to the increase of the arc, and it is nearly so for 60″, it follows, that 473 (the last two places being regarded as decimals) is the increase of the sine for 1″. Similarly, if the arc be 24° 20′, the increase of the sine for 1″, is 465, the last two places being decimals. The same remarks are equally applicable in respect of the column D, after the column cosine, and of the column D, between the tangents and cotangents. The column D between the tangents and cotangents, answers to either of these columns; since of the same arc, the log. tang. + log. cotang = 20. Therefore, having two arcs, a and b, log. tang b + log. cotang b = log. tang a + log. cotang a; or, log. tang b — log. tang a = log. cotang a — log. cotang b.

Now, if it were required to find the logarithmic sine of an arc expressed in degrees, minutes, and seconds, we have only to find the degrees and minutes as before; then multiply the corresponding tabular number by the seconds, cut off two places to the right hand for decimals, and then add the product to the number first found, for the sine of the given arc. Thus, if we wish the sine of 40° 26′ 28″.

The sine 40° 26′ - - - -		9.811952 -
Tabular difference = 247		
Number of seconds = 28		
	Product = 69.16, to be added =	69.16
	Gives for the sine of 40° 26′ 28″ =	9.812021.16

The tangent of an arc, in which there are seconds, is found in a manner entirely similar. In regard to the cosine and cotangent, it must be remembered, that they increase while the arcs decrease, and decrease while the arcs are increased, consequently, the proportional numbers found for the seconds must be subtracted, not added.

Ex. To find the cosine 3° 40′ 40″.

Cosine 3° 40′		9.999110
Tabular difference = 13		
Number of seconds = 40		
	Product = 5.20, which being subtracted =	5.20
	Gives for the cosine of 3° 40′ 40′	9.999104.80

CASE II.

To find the degrees, minutes, and seconds answering to any given logarithmic sine, cosine, tangent, or cotangent.

Search in the table, and in the proper column, until the number be found; the degrees are shown either at the top or bottom of the page, and the minutes in the side columns, either at the left or right. But if the number cannot be exactly found in the table, take the degrees and minutes answering to the nearest less logarithm, the logarithm itself, and also the corresponding tabular difference. Subtract the logarithm taken, from the given logarithm, annex two ciphers, and then divide the remainder by the tabular difference : the quotient is seconds, and is to be connected with the degrees and minutes before found ; to be added for the sine and tangent, and subtracted for the cosine and cotangent.

Ex. 1. To find the arc answering to the sine 9.880054
Sine 49° 20′, next less in the table, 9.879963

Tab. Diff. 181)9100(50″

Hence the arc 49° 20′ 50″ corresponds to the given sine 9.880054.

Ex. 2. To find the arc corresponding to cotang. 10.008688.
Cotang 44° 26′, next less in the table 10.008591

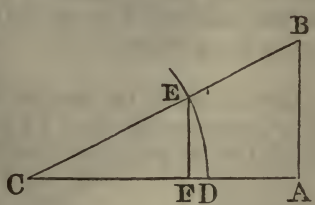
Tab. Diff. 421)9700(23″

Hence, 44° 26′—23″=44° 25′ 37″ is the arc corresponding to the given cotangent 10.008688.

PRINCIPLES FOR THE SOLUTION OF RECTILINEAL TRIANGLES.

THEOREM I.

In every right angled triangle, radius is to the sine of either of the acute angles, as the hypothenuse to the opposite side : and radius is to the cosine of either of the acute angles, as the hypothenuse to the adjacent side.



Let ABC be the proposed triangle, right-angled at A : from the point C as a centre, with a radius CD equal to the radius of the tables, describe the arc DE, which will measure the angle C ; on CD let fall the perpendicular EF, which will be the sine of the angle C, and CF will be its cosine. The triangles CBA, CEF, are similar, and give the proportion,

$$CE : EF :: CB : BA : \text{hence}$$
$$R : \sin C :: BC : BA.$$

But we also have,

$$CE : CF :: CB : CA : \text{hence}$$
$$R : \cos C :: CB : CA.$$

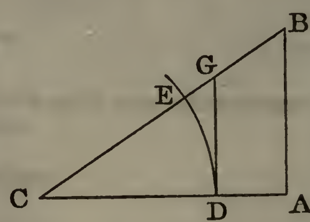
Cor. If the radius $R=1$, we shall have,

$$AB=CB \sin C, \text{ and } CA=CB \cos C.$$

Hence, in every right angled triangle, *the perpendicular is equal to the hypothenuse multiplied by the sine of the angle at the base; and the base is equal to the hypothenuse multiplied by the cosine of the angle at the base; the radius being equal to unity.*

THEOREM II.

In every right angled triangle, radius is to the tangent of either of the acute angles, as the side adjacent to the side opposite.



Let CAB be the proposed triangle.

With any radius, as CD, describe the arc DE, and draw the tangent DG.

From the similar triangles CDG, CAB, we shall have,

$$CD : DG :: CA : AB : \text{hence,}$$
$$R : \text{tang } C :: CA : AB.$$

Cor. 1. If the radius $R=1$,

$$AB=CA \text{ tang } C.$$

Hence, *the perpendicular of a right angled triangle is equal to the base multiplied by the tangent of the angle at the base,* the radius being unity.

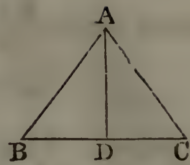
Cor. 2. Since the tangent of an arc is equal to the cotangent of its complement (Art. VI.), the cotangent of B may be substituted in the proportion for tang C, which will give

$$R : \cot B :: CA : AB.$$

THEOREM III.

In every rectilineal triangle, the sines of the angles are to each other as the opposite sides.

Let ABC be the proposed triangle; AD the perpendicular, let fall from the vertex A on the opposite side BC: there may be two cases.



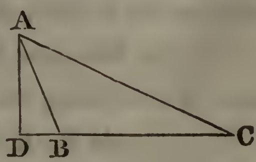
First. If the perpendicular falls within the triangle ABC, the right-angled triangles ABD, ACD, will give,

$$R : \sin B :: AB : AD.$$
$$R : \sin C :: AC : AD.$$

In these two propositions, the extremes are equal; hence,

$$\sin C : \sin B :: AB : AC.$$

Secondly. If the perpendicular falls without the triangle ABC, the right-angled triangles ABD, ACD, will still give the proportions,



$$R : \sin ABD :: AB : AD,$$
$$R : \sin C \quad :: AC : AD;$$

from which we derive

$$\sin C : \sin ABD :: AB : AC.$$

But the angle ABD is the supplement of ABC, or B; hence $\sin ABD = \sin B$; hence we still have

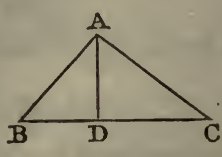
$$\sin C : \sin B :: AB : AC.$$

THEOREM IV.

In every rectilineal triangle, the cosine of either of the angles is equal to radius multiplied by the sum of the squares of the sides adjacent to the angle, minus the square of the side opposite, divided by twice the rectangle of the adjacent sides.

Let ABC be a triangle: then will

$$\cos B = R\frac{AB^2 + BC^2 - AC^2}{2AB \times BC.}$$



First. If the perpendicular falls within the triangle, we shall have $AC^2 = AB^2 + BC^2 - 2BC \times BD$ (Book IV. Prop. XII.); hence $BD = \dfrac{AB^2 + BC^2 - AC^2}{2BC}$. But in the right-angled triangle ABD, we have

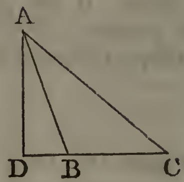
$$R : \cos B :: AB : BD;$$

hence, $\cos B = \frac{R \times BD}{AB}$, or by substituting the value of BD,

$$\cos B = R \times \frac{AB^2 + BC^2 - AC^2}{2AB \times BC}$$

Secondly. If the perpendicular falls without the triangle, we shall have $AC^2 = AB^2 + BC^2 + 2BC \times BD$; hence

$$BD = \frac{AC^2 - AB^2 - BC^2}{2BC}.$$



But in the right-angled triangle BAD, we still have $\cos ABD = \frac{R \times BD}{AB}$; and the angle ABD being supplemental to ABC, or B, we have

$$\cos B = -\cos ABD = -\frac{R \times BD}{AB}.$$

hence by substituting the value of BD, we shall again have

$$\cos B = R \times \frac{AB^2 + BC^2 - AC^2}{2AB \times BC}.$$

Scholium. Let A, B, C, be the three angles of any triangle, *a*, *b*, *c*, the sides respectively opposite them: by the theorem, we shall have $\cos B = R \times \frac{a^2 + c^2 - b^2}{2ac}$. And the same principle, when applied to each of the other two angles, will, in like manner give $\cos A = R \times \frac{b^2 + c^2 - a^2}{2bc}$, and $\cos C = R \times \frac{a^2 + b^2 - c^2}{2ab}$. Either of these formulas may readily be reduced to one in which the computation can be made by logarithms.

Recurring to the formula $R^2 - R \cos A = 2\sin^2 \frac{1}{2}A$ (Art. XXIII.), or $2\sin^2 \frac{1}{2}A = R^2 - R\cos A$, and substituting for $\cos A$, we shall have

$$2\sin^2 \tfrac{1}{2}A = R^2 - R^2 \times \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{R^2 \times 2bc - R^2(b^2 + c^2 - a^2)}{2bc} = R^2 \times \frac{a^2 - b^2 - c^2 + 2bc}{2bc}$$

$$= R^2 \times \frac{a^2 - (b-c)^2}{2bc} = R^2 \times \frac{(a+b-c)\ (a+c-b)}{2bc}.$$ Hence

$$\sin \tfrac{1}{2}A = R\sqrt{\left(\frac{(a+b-c)\ (a+c-b)}{4bc}\right)}.$$

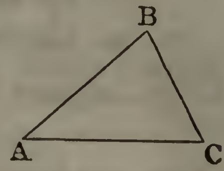
For the sake of brevity, put

$\frac{1}{2}(a+b+c) = p$, or $a+b+c = 2p$; we have $a+b-c = 2p-2c$, $a+c-b = 2p-2b$; hence

$$\sin \tfrac{1}{2}A = R\sqrt{\left(\frac{(p-b)\ (p-c)}{bc}\right)}.$$

THEOREM V.

In every rectilineal triangle, the sum of two sides is to their difference as the tangent of half the sum of the angles opposite those sides, to the tangent of half their difference.



For, AB : BC : : sin C : sin A (Theorem III.). Hence, AB+BC : AB—BC : : sin C+sin A : sin C—sin A. But sinC+sin A : sin C—sin A : : tang$\frac{C+A}{2}$: tang $\frac{C-A}{2}$ (Art. XXIV.) ; hence,

AB+BC : AB—BC : : tang $\frac{C+A}{2}$: tang $\frac{C-A}{2}$, which is the property we had to demonstrate.

With the aid of these five theorems we can solve all the cases of rectilineal trigonometry.

Scholium. The required part should always be found from the given parts ; so that if an error is made in any part of the work, it may not affect the correctness of that which follows.

SOLUTION OF RECTILINEAL TRIANGLES BY MEANS OF LOGARITHMS.

It has already been remarked, that in order to abridge the calculations which are necessary to find the unknown parts of a triangle, we use the logarithms of the parts instead of the parts themselves.

Since the addition of logarithms answers to the multiplication of their corresponding numbers, and their subtraction to the division of their numbers ; it follows, that the logarithm of the fourth term of a proportion will be equal to the sum of the logarithms of the second and third terms, diminished by the logarithm of the first term.

Instead, however, of subtracting the logarithm of the first term from the sum of the logarithms of the second and third terms, it is more convenient to use the *arithmetical complement* of the first term.

The *arithmetical complement* of a logarithm is the number which remains after subtracting the logarithm from 10. Thus 10—9.274687=0.725313 : hence, 0.725313 is the arithmetical complement of 9.274687.

It is now to be shown that, *the difference between two logarithms is truly found, by adding to the first logarithm the arithmetical complement of the logarithm to be subtracted, and diminishing their sum by* 10.

Let $a =$ the first logarithm.
$b =$ the logarithm to be subtracted.
$c = 10-b=$the arithmetical complement of b.

Now, the difference between the two logarithms will be expressed by $a-b$. But from the equation $c=10-b$, we have $c-10=-b$: hence if we substitute for $-b$ its value we shall have

$$a-b=a+c-10,$$

which agrees with the enunciation.

When we wish the arithmetical complement of a logarithm, we may write it directly from the tables, by subtracting the left hand figure from 9, then proceeding to the right, subtract each figure from 9, till we reach the last significant figure, which must be taken from 10 : this will be the same as taking the logarithm from 10.

Ex. From 3.274107 take 2.104729.

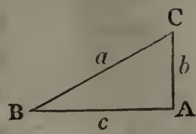
Common method.		*By ar.-comp.*
3.274107		3.274107
2.104729	ar.-comp.	7.895271
Diff. 1.169378	sum	1.169378 after rejecting the 10.

We therefore have, for all the proportions of trigonometry the following

RULE.

Add together the arithmetical complement of the logarithm of the the first term, the logarithm of the second term, and the logarithm of the third term, and their sum after rejecting 10, *will be the logarithm of the fourth term. And if any expression occurs in which the arithmetical complement is twice used,* 20 *must be rejected from the sum.*

SOLUTION OF RIGHT ANGLED TRIANGLES.



Let A be the right angle of the proposed right angled triangle, B and C the other two angles; let a be the hypothenuse, b the side opposite the angle B, c the side opposite the angle C. Here we must consider that the two angles C and B are complements of each other; and that consequently, according to the different cases, we are entitled to assume sin C=cos B, sin B=cos C, and likewise tang B= cot C, tang C=cot B. This being fixed, the unknown parts of a right angled triangle may be found by the first two theorems; or if two of the sides are given, by means of the property, that the square of the hypothenuse is equal to the sum of the squares of the other two sides.

EXAMPLES.

Ex. 1. In the right angled triangle BCA, there are given the hypothenuse a=250, and the side b=240; required the other parts.

$$R : \sin B :: a : b \text{ (Theorem I.).}$$

or, $$a : b :: R : \sin B.$$

When logarithms are used, it is most convenient to write the proportion thus,

As hyp. a	250	ar.-comp. log.	7.602060
To side b	240		2.380211
So is R			10.000000
To sin B	73° 44′ 23″	(after rejecting 10)	9.982271

But the angle C=90°—B=90°—73° 44′ 23″=16° 15′ 37″ or, C might be found by the proportion,

As hyp. a	250	ar.-comp. log.	7.602060
To side b	240		2.380211
So is R			10.000000
To cos C	16° 15′ 37″		9.982271

To find the side c, we say,

As R		ar. comp. log.	0.000000
To tang. C	16° 15′ 37″		9.464889
So is side b	240		2.380211
To side c	70.0003		1.845100

Or the side c might be found from the equation

$$a^2=b^2+c^2.$$

For, $c^2=a^2-b^2=(a+b)\times(a-b)$:

hence, $2\log. c=\log.(a+b)+\log.(a-b)$, or

$\log. c=\frac{1}{2}\log.(a+b)+\frac{1}{2}\log.(a-b)$

	$a+b=250+240=490$	log.	2.690196
	$a-b=250-240=10$	- -	1.000000
			2) 3.690196
Log. c 70	- - - - - -		1.845098

Ex. 2. In the right angled triangle BCA, there are given, side $b=384$ yards, and the angle $B=53°\ 8'$: required the other parts.

To find the third side c.

R : tang B : : c : b (Theorem II.)

or, tang B : R : : b : c. Hence,

As tang	B 53° 8′	ar.-comp.	log.	9.875010
Is to	R	- - -	- -	10.000000
So is side	b 384	- - -	- -	2.584331
To side	c 287.965	- -	- -	2.459341

Note. When the logarithm whose arithmetical complement is to be used, exceeds 10, take the arithmetical complement with reference to 20 and reject 20 from the sum.

To find the hypothenuse a.

R : sin B : : a : b (Theorem I.). Hence,

As sin B 53° 8′	ar. comp.	log.	0.096892
Is to R	- - -	- -	10.000000
So is side b 384	- -	- -	2.584331
To hyp. a 479.98	-	- -	2.681223

Ex. 3. In the right angled triangle BAC, there are given,

side $c=195$, angle $B=47°\ 55'$,

required the other parts.

Ans. Angle $C=42°\ 05'$, $a=290.953$, $b=215.937$.

SOLUTION OF RECTILINEAL TRIANGLES IN GENERAL.

Let A, B, C be the three angles of a proposed rectilineal tri angle ; a, b, c, the sides which are respectively opposite them ; the different problems which may occur in determining three of these quantities by means of the other three, will all be reducible to the four following cases.

CASE I.

Given a side and two angles of a triangle, to find the remaining parts.

First, subtract the sum of the two angles from two right angles, the remainder will be the third angle. The remaining sides can then be found by Theorem III.

I. In the triangle ABC, there are given the angle A=58° 07′, the angle B=22° 37′, and the side c=408 yards: required the remaining angle and the two other sides.

To the angle A - - - - -	=58° 07′
Add the angle B - - - - -	=22° 37′
Their sum - - - - - -	=80° 44′
taken from 180° leaves the angle C -	=99° 16′.

This angle being greater than 90° its sine is found by taking that of its supplement 80° 44′.

To find the side a.

As sine C	99° 16′	ar.-comp. log.	0.005705
Is to sine A	58° 07′	- - - -	9.928972
So is side c	408 -	- - - -	2.610660
So side a	351.024	- - - -	2.545337

To find the side b.

As sine C	99° 16′	ar.-comp. log.	0.005705
Is to sine B	22° 37′	- - - -	9.584968
So is side c	408 -	- - - -	2.610660
To side b	158.976	- - - -	2.201333

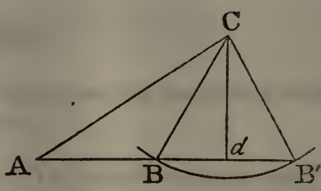
2. In a triangle ABC, there are given the angle A=38° 25′ B=57° 42′, and the side c=400: required the remaining parts.

Ans. Angle C=83° 53′, side a=249.974, side b=340.04.

CASE II.

Given two sides of a triangle, and an angle opposite one of them, to find the third side and the two remaining angles.

1. In the triangle ABC, there are given side AC=216, BC=117, and the angle A=22° 37′, to find the remaining parts.



Describe the triangles ACB, ACB′, as in Prob. XI. Book III.

Then find the angle B by Theorem III.

As side B′C or BC 117	ar.-comp.	log.	7.931814
Is to side AC 216	- - - -	-	2.334454
So is sine A 22° 37′	- - - -	-	9.584968
To sine B′	45° 13′ 55″ or ABC 134° 46′ 05″		9.851236

Add to each A	22° 37′ 00″	22° 37′ 00″
Take their sum	67° 50′ 55″	157° 23′ 05″
From	180° 00′ 00″	180° 00′ 00″
Rem. ACB′	112° 09′ 05″	ACB=22° 36′ 55″

To find the side AB or AB′.

As sine A	22° 37′	ar.-comp.	log.	0.415032
Is to sine ACB′	112° 09′ 05″	- -	-	9.966700
So is side	B′C 117	- -	-	2.068186
To side AB′	281.785	- -	-	2.449918

The ambiguity in this, and similar examples, arises in consequence of the first proportion being true for both the triangles ACB, ACB′. As long as the two triangles exist, the ambiguity will continue. But if the side CB, opposite the given angle, be greater than AC, the arc BB′ will cut the line ABB′ on the same side of the point A, but in one point, and then there will be but one triangle answering the conditions.

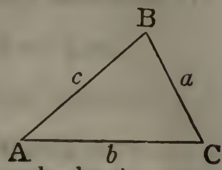
If the side CB be equal to the perpendicular C*d*, the arc BB′ will be tangent to ABB′, and in this case also, there will be but one triangle. When CB is less than the perpendicular C*d*, the arc BB′ will not intersect the base ABB′, and in that case there will be no triangle, or the conditions are impossible.

2. Given two sides of a triangle 50 and 40 respectively, and the angle opposite the latter equal to 32°: required the remaining parts of the triangle.

Ans. If the angle opposite the side 50 be acute, it is equal to 41° 28′ 59″, the third angle is then equal to 106° 31′ 01″, and the third side to 72.368. If the angle opposite the side 50 be obtuse, it is equal to 138° 31′ 01″, the third angle to 9° 28′ 59″, and the remaining side to 12.436.

CASE III.

Given two sides of a triangle, with their included angle, to find the third side and the two remaining angles.



Let ABC be a triangle, B the given angle, and c and a the given sides.

Knowing the angle B, we shall likewise know the sum of the other two angles $C+A=180°—B$, and their half sum $\frac{1}{2}(C+A)=90—\frac{1}{2}B$. We shall next compute the half difference of these two angles by the proportion (Theorem V.),

$$c+a : c—a :: \text{tang } \tfrac{1}{2}(C+A) \text{ or cot } \tfrac{1}{2}B : \text{tang } \tfrac{1}{2}(C—A,)$$

in which we consider $c>a$ and consequently $C>A$. Having found the half difference, by adding it to the half sum $\frac{1}{2}(C+A)$, we shall have the greater angle C; and by subtracting it from the half-sum, we shall have the smaller angle A. For, C and A being any two quantities, we have always,

$$C=\tfrac{1}{2}(C+A)+\tfrac{1}{2}(C—A)$$
$$A=\tfrac{1}{2}(C+A)—\tfrac{1}{2}(C—A).$$

Knowing the angles C and A to find the third side b, we have the proportion.

$$\sin A : \sin B :: a : b$$

Ex. 1. In the triangle ABC, let $a=450$, $c=540$, and the included angle $B=80°$: required the remaining parts.

$$c+a=990,\ c—a=90,\ 180°—B=100°=C+A.$$

As $c+a$	990	ar.-comp.	log.	7.004365
Is to $c—a$	90	- - -	- -	1.954243
So is tang $\frac{1}{2}(C+A)$	50°	- -	-	10.076187
To tang $\frac{1}{2}(C—A)$	6° 11′	- -	-	9.034795

Hence, $50°+6°\ 11'=56°\ 11'=C$; and $50°—6°\ 11'=43°\ 49'=A$.

To find the third side b.

As sine A	43° 49′	ar.-comp.	log.	0.159672
Is to sine B	80°	- - -	- -	9.993351
So is side a	450	-	- - -	2.653213
To side b	640.082	- -	- -	2.806236

Ex. 2. Given two sides of a plane triangle, 1686 and 960, and their included angle 128° 04′: required the other parts.

Ans. Angles, 33° 34′ 39″, 18° 21′ 21″ side 2400

CASE IV

Given the three sides of a triangle, to find the angles.

We have from Theorem IV. the formula,

$$\sin \tfrac{1}{2} A = R\sqrt{\left(\frac{(p-b)\,(p-c)}{bc}\right)} \quad \text{in which}$$

p represents the half sum of the three sides. Hence

$$\sin^2 \tfrac{1}{2} A = R^2\left(\frac{(p-b)\,(p-c)}{bc}\right), \quad \text{or}$$

2 log. sin $\tfrac{1}{2}$A=2 log. R+log. $(p-b)$+log. $(p-c)$—log. c—log. b.

Ex. 1. In a triangle ABC, let $b=40$, $c=34$, and $a=25$: required the angles.

Here $p=\dfrac{40+34+25}{2}=49.5$, $p-b=9.5$, and $p-c=15.5$.

2 Log. R		- - - - - -	20.000000
log. $(p-b)$	9.5	- - - - - -	0.977724
log. $(p-c)$	15.5	- - - - - -	1.190332
—log. c	34	ar.-comp. - -	8.468521
—log. b	40	ar.-comp. - -	8.397940
2 log. sin $\tfrac{1}{2}$ A		- - - - -	19.034517
log. sin $\tfrac{1}{2}$ A	19° 12′ 39″	- - -	9·517258

Angle A=38° 25′ 18″.

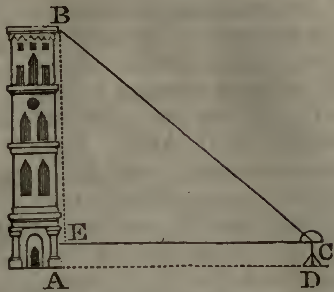
In a similar manner we find the angle B=83° 53′ 18″ and the angle C=57° 41′ 24″.

Ex. 2. What are the angles of a plane triangle whose sides are, $a=60$, $b=50$, and $c=40$?

Ans. 41° 24′ 34″, 55° 46′ 16″ and 82° 49′ 10″.

APPLICATIONS.

Suppose the height of a building AB were required, the foot of it being accessible.



On the ground which we suppose to be horizontal or very nearly so, measure a base AD, neither very great nor very small in comparison with the altitude AB; then at D place the foot of the circle, or whatever be the instrument, with which we are to measure the angle BCE formed by the horizontal line CE parallel to AD, and by the visual ray direct it to the summit of the building. Suppose we find AD or CE=67.84 yards, and the angle BCE=41° 04′: in order to find BE, we shall have to solve the right angled triangle BCE, in which the angle C and the adjacent side CE are known.

To find the side EB.

As R	- - - - - - - - ar.-comp.	-	0.000000
Is to tang. C	41° 04′	- - - - - - - - - -	9.940183
So is EC	67.84	- - - - - - - - - -	1.831486
To EB	59.111	- - - - - - - - - -	1.771669

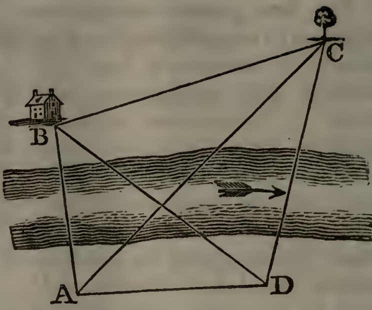
Hence, EB=59.111 yards. To EB add the height of the instrument, which we will suppose to be 1.12 yards, we shall then have the required height AB=60.231 yards.

If, in the same triangle BCE it were required to find the hypothenuse, form the proportion

As cos C 41° 04′	ar.-comp.	- - log.	0.122660
Is to R	- - - - - - - - - - - - - -		10.000000
So is CE	67.84	- - - - - - - - - -	1.831486
To CB	89.98	- - - - - - - - - -	1.954146

Note. If only the summit B of the building or place whose height is required were visible, we should determine the distance CE by the method shown in the following example; this distance and the given angle BCE are sufficient for solving the right angled triangle BCE, whose side, increased by the height of the instrument, will be the height required.

2. To find upon the ground the distance of the point A from an inaccessible object B, we must measure a base AD, and the two adjacent angles BAD, ADB. Suppose we have found AD= 588.45 yards, BAD$=103°$ $55' 55''$, and BDA$=36° 04'$; we shall thence get the third angle ABD$=40° 05''$, and to obtain AB, we shall form the proportion



As sine ABD	$40° 05''$	ar.-comp. - log. -	0.191920
Is to sin BDA	$36° 04'$	- - - - - - - -	9.769913
So is AD	588.45	- - - - - - - -	2.769710
To AB - -	538.943	- - - - - - - -	2.731543

If for another inaccessible object C, we have found the angles CAD$=35° 15'$, ADC$=119° 32'$, we shall in like manner find the distance AC$=1201.744$ yards.

3. To find the distance between two inaccessible objects B and C, we determine AB and AC as in the last example; we shall, at the same time, have the included angle BAC=BAD—DAC. Suppose AB has been found equal to 538.818 yards, AC$=1201.744$ yards, and the angle BAC$=68° 40' 55''$; to get BC, we must resolve the triangle BAC, in which are known two sides and the included angle.

As AC+AB	1740.562	ar.-comp. log. -	6.759311
Is to AC—AB	662.926	- - - - - - - -	2.821465
So is tang. $\frac{B+C}{2}$	$55° 39' 32''$	- - - - - -	10.165449
To tang. $\frac{B-C}{2}$	$29° 08' 19''$	- - - - - - -	9.746225

Hence - - - - - - $\frac{B-C}{2}=29° 08' 19''$

But we have - - - - $\frac{B+C}{2}=55° 39' 32''$

Hence - - - - - - - - B $=84° 47' 51''$

and - - - - - - - - C $=26° 31' 13''$

Now, to find the distance BC make the proportion,

As sine B 84° 47′ 51″ ar.-comp.	- log. -	0.001793
Is to sine A 68° 40′ 55″	- - - - - - - - -	9.969218
So is AC 1201.744	- - - - - - - - - -	3.079811
To BC 1124.145	- - - - - - - - - - -	3.050822

4. Wanting to know the distance between two inaccessible objects which lie in a direct line from the bottom of a tower of 120 feet in height, the angles of depression are measured, and found to be, of the nearest, 57°; of the most remote, 25° 30′: required the distance between them.

Ans. 173.656 feet.

5. In order to find the distance between two trees, A and B, which could not be directly measured because of a pool which occupied the intermediate space, the distance of a third point C from each, was measured, viz. CA=588 feet and CB =672 feet, and also the contained angle ACB=55° 40′: required the distance AB.

Ans. 592.967 feet.

6. Being on a horizontal plane, and wanting to ascertain the height of a tower, standing on the top of an inaccessible hill, there were measured, the angle of elevation of the top of the hill 40°, and of the top of the tower 51°: then measuring in a direct line 180 feet farther from the hill, the angle of elevation of the top of the tower was 33° 45′: required the height of the tower.

Ans. 83.9983 feet.

7. Wanting to know the horizontal distance between two inaccessible objects A and B, and not finding any station from which both of them could be seen, two points C and D, were chosen, at a distance from each other equal to 200 yards, from the former of which A could be seen, and from the latter B, and at each of the points C and D a staff was set up. From C a distance CF was measured, not in the direction DC, equal to 200 yards, and from D, a distance DE equal to 200 yards, and the following angles were taken, viz. AFC=83° ACF= 54° 31′, ACD=53° 30′, BDC=156° 25′, BDE=54° 30′, and BED=88° 30′: required the distance AB.

Ans. 345.46 yards.

8. From a station P there can be seen three objects, A, B and C, whose distances from each other are known, viz. AB= 800, AC=600, and BC=400 yards. There are also measured the horizontal angles, APC=33° 45′, BPC=22° 30′. It is required, from these data, to determine the three distances PA, PC and PB.

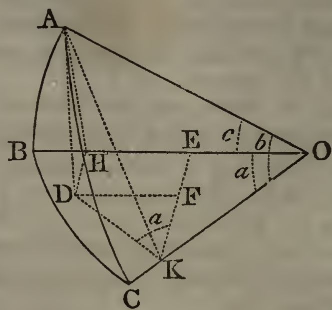
Ans. PA=710.193, PC=1042.522, PB=934.291 yards.

SPHERICAL TRIGONOMETRY.

I. It has already been shown that a spherical triangle is formed by the arcs of three great circles intersecting each other on the surface of a sphere, (Book IX. Def. 1). Hence, every spherical triangle has six parts: the sides and three angles.

Spherical Trigonometry explains the methods of determining, by calculation, the unknown sides and angles of a spherical triangle when any three of the six parts are given.

II. Any two parts of a spherical triangle are said to be of the *same species* when they are both less or both greater than 90°; and they are of different species when one is less and the other greater than 90°.



III. Let ABC be a spherical triangle, and O the centre of the sphere. Let the sides of the triangle be designated by letters corresponding to their opposite angles: that is, the side opposite the angle A by a, the side opposite B by b, and the side opposite C by c. Then the angle COB will be represented by a, the angle COA by b and the angle BOA by c. The angles of the spherical triangle will be equal to the angles included between the planes which determine its sides (Book IX. Prop. VI.).

From any point A, of the edge OA, draw AD perpendicular to the plane COB. From D draw DH perpendicular to OB, and DK perpendicular to OC; and draw AH and AK: the last lines will be respectively perpendicular to OB and OC, (Book VI. Prop. VI.)

The angle DHA will be equal to the angle B of the spherical triangle, and the angle DKA to the angle C.

The two right angled triangles OKA, ADK, will give the proportions

$$R : \sin AOK :: OA : AK, \text{ or, } R \times AK = OA \sin b.$$
$$R : \sin AKD :: AK : AD, \text{ or, } R \times AD = AK \sin C.$$

Hence, $R^2 \times AD = AO \sin b \sin C$, by substituting for AK its value taken from the first equation.

In like manner the triangles AHO, ADH, right angled at H and D, give

$$R : \sin c :: AO : AH, \text{ or } R \times AH = AO \sin c$$
$$R : \sin B :: AH : AD, \text{ or } R \times AD = AH \sin B.$$

Hence, $R^2 \times AD = AO \sin c \sin B$.

Equating this with the value of $R^2 \times AD$, before found, and dividing by AO, we have

$$\sin b \sin C = \sin c \sin B, \text{ or } \frac{\sin C}{\sin B} = \frac{\sin c}{\sin b} \quad (1)$$

or, $\sin B : \sin C :: \sin b : \sin c$ that is,

The sines of the angles of a spherical triangle are to each other as the sines of their opposite sides.

IV. From K draw KE perpendicular to OB, and from D draw DF parallel to OB. Then will the angle $DKF = COB = a$, since each is the complement of the angle EKO.

In the right angled triangle OAH, we have

$$R : \cos c :: OA : OH; \text{ hence}$$
$$AO \cos c = R \times OH = R \times OE + R.DF.$$

In the right-angled triangle OKE

$$R : \cos a :: OK : OE, \text{ or } R \times OE = OK \cos a.$$

But in the right angled triangle OKA

$$R : \cos b :: OA : OK, \text{ or, } R \times OK = OA \cos b.$$

Hence $R \times OE = OA . \dfrac{\cos a \cos b}{R}$

In the right-angled triangle KFD

$$R : \sin a : KD : DF, \text{ or } R \times DF = KD \sin a.$$

But in the right angled triangles OAK, ADK, we have

$$R : \sin b :: OA : AK, \text{ or } R \times AK = OA \sin b$$
$$R : \cos K : AK : KD, \text{ or } R \times KD = AK \cos C$$

hence $KD = \dfrac{OA \sin b \cos C}{R^2}$, and

$R \times DF = \dfrac{OA \sin a \sin b \cos C}{R^2}$: therefore

$$OA \cos c = \frac{OA \cos a \cos b}{R} + \frac{AO \sin a \sin b \cos C}{R^2}, \text{ or}$$

$$R^2 \cos c = R \cos a \cos b + \sin a \sin b \cos C.$$

Similar equations may be deduced for each of the other sides. Hence, generally,

$$\left.\begin{array}{l} R^2 \cos a = R \cos b \cos c + \sin b \sin c \cos A. \\ R^2 \cos b = R \cos a \cos c + \sin a \sin c \cos B. \\ R^2 \cos c = R \cos b \cos a + \sin b \sin a \cos C. \end{array}\right\} \quad (2.)$$

That is, *radius square into the cosine of either side of a spherical triangle is equal to radius into the rectangle of the cosines of the two other sides plus the rectangle of the sines of those sides into the cosine of their included angle.*

V. Each of the formulas designated (2) involves the three sides of the triangle together with one of the angles. These formulas are used to determine the angles when the three sides are known. It is necessary, however, to put them under another form to adapt them to logarithmic computation.

Taking the first equation, we have

$$\cos A = \frac{R^2 \cos a - R \cos b \cos c}{\sin b \sin c}$$

Adding R to each member, we have

$$R + \cos A = \frac{R^2 \cos a + R \sin b \sin c - R \cos b \cos c}{\sin b \sin c}$$

But, $R + \cos A = \dfrac{2 \cos^2 \frac{1}{2} A}{R}$ (Art. XXIII.), and

$R \sin b \sin c - R \cos b \cos c = -R^2 \cos (b+c)$ (Art. XIX.);

$$\text{hence, } \frac{2 \cos^2 \frac{1}{2} A}{R} = \frac{R^2 (\cos a - \cos (b+c))}{\sin b \sin c} =$$

$$2 R \frac{\sin \frac{1}{2}(a+b+c) \sin \frac{1}{2}(b+c-a)}{\sin b \sin c} \quad \text{(Art. XXIII).}$$

Putting $s = a+b+c$, we shall have

$\frac{1}{2}s = \frac{1}{2}(a+b+c)$ and $\frac{1}{2}s - a = \frac{1}{2}(b+c-a)$: hence

$$\left.\begin{array}{l} \cos \frac{1}{2} A = R\sqrt{\dfrac{\sin \frac{1}{2}(s) \sin (\frac{1}{2}s - a)}{\sin b \sin c}} \\ \cos \frac{1}{2} B = R\sqrt{\dfrac{\sin \frac{1}{2}(s) \sin (\frac{1}{2}s - b)}{\sin a \sin c}} \\ \cos \frac{1}{2} C = R\sqrt{\dfrac{\sin \frac{1}{2}(s) \sin (\frac{1}{2}s - c)}{\sin a \sin b}} \end{array}\right\} \quad (3.)$$

Had we subtracted each member of the first equation from R, instead of adding, we should, by making similar reductions, have found

$$\left.\begin{array}{l} \sin\frac{1}{2}A=R\sqrt{\dfrac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a+c-b)}{\sin b\sin c}} \\ \sin\frac{1}{2}B=R\sqrt{\dfrac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(b+c-a)}{\sin a\sin c}} \\ \sin\frac{1}{2}C=R\sqrt{\dfrac{\sin\frac{1}{2}(a+c-b)\sin\frac{1}{2}(b+c-a)}{\sin a\sin b}} \end{array}\right\} (4.)$$

Putting $s=a+b+c$, we shall have

$\frac{1}{2}s-a=\frac{1}{2}(b+c-a)$, $\frac{1}{2}s-b=\frac{1}{2}(a+c-b)$, and $\frac{1}{2}s-c=\frac{1}{2}(a+b-c)$

hence,

$$\left.\begin{array}{l} \sin\frac{1}{2}A=R\sqrt{\dfrac{\sin(\frac{1}{2}s-c)\sin(\frac{1}{2}s-b)}{\sin b\sin c}} \\ \sin\frac{1}{2}B=R\sqrt{\dfrac{\sin(\frac{1}{2}s-c)\sin(\frac{1}{2}s-a)}{\sin a\sin c}} \\ \sin\frac{1}{2}C=R\sqrt{\dfrac{\sin(\frac{1}{2}s-b)\sin(\frac{1}{2}s-a)}{\sin a\sin b}} \end{array}\right\} (5.)$$

VI. We may deduce the value of the side of a triangle in terms of the three angles by applying equations (4.), to the polar triangle. Thus, if a', b', c', A', B', C', represent the sides and angles of the polar triangle, we shall have

$$A=180^\circ-a',\ B=180^\circ-b',\ C=180^\circ-c';$$
$$a=180^\circ-A',\ b=180^\circ-B',\ \text{and } c=180^\circ-C'$$

(Book IX. Prop. VII.): hence, omitting the ′, since the equations are applicable to any triangle, we shall have

$$\left.\begin{array}{l} \cos\frac{1}{2}a=R\sqrt{\dfrac{\cos\frac{1}{2}(A+B-C)\cos\frac{1}{2}(A+C-B)}{\sin B\sin C}} \\ \cos\frac{1}{2}b=R\sqrt{\dfrac{\cos\frac{1}{2}(A+B-C)\cos\frac{1}{2}(B+C-A)}{\sin A\sin C}} \\ \cos\frac{1}{2}c=R\sqrt{\dfrac{\cos\frac{1}{2}(A+C-B)\cos\frac{1}{2}(B+C-A)}{\sin A\sin B.}} \end{array}\right\} (6.)$$

Putting $S=A+B+C$, we shall have

$\frac{1}{2}S-A=\frac{1}{2}(C+B-A)$, $\frac{1}{2}S-B=\frac{1}{2}(A+C-B)$ and $\frac{1}{2}S-C=\frac{1}{2}(A+B-C)$, hence

$$\left.\begin{aligned} \cos\tfrac{1}{2}a&=R\sqrt{\frac{\cos(\frac{1}{2}S-C)\cos(\frac{1}{2}S-B)}{\sin B\sin C}}\\ \cos\tfrac{1}{2}b&=R\sqrt{\frac{\cos(\frac{1}{2}S-C)\cos(\frac{1}{2}S-A)}{\sin A\sin C}}\\ \cos\tfrac{1}{2}c&=R\sqrt{\frac{\cos(\frac{1}{2}S-B)\cos(\frac{1}{2}S-A)}{\sin A\sin B}} \end{aligned}\right\}\ (7.)$$

VII. If we apply equations (2.) to the polar triangle, we shall have

$$-R^2\cos A'=R\cos B'\cos C'-\sin B'\sin C'\cos a'.$$

Or, omitting the ′, since the equation is applicable to any triangle, we have the three symmetrical equations,

$$\left.\begin{aligned} R^2.\cos A&=\sin B\sin C\cos a-R\cos B\cos C\\ R^2.\cos B&=\sin A\sin C\cos b-R\cos A\cos C\\ R^2.\cos C&=\sin A\sin B\cos c-R\cos A\cos B \end{aligned}\right\}\ (8.)$$

That is, *radius square into the cosine of either angle of a spherical triangle, is equal to the rectangle of the sines of the two other angles into the cosine of their included side, minus radius into the rectangle of their cosines.*

VIII. All the formulas necessary for the solution of spherical triangles, may be deduced from equations marked (2.). If we substitute for $\cos b$ in the third equation, its value taken from the second, and substitute for $\cos^2 a$ its value $R^2-\sin^2 a$, and then divide by the common factor $R.\sin a$, we shall have

$$R.\cos c\sin a=\sin c\cos a\cos B+R.\sin b\cos C.$$

But equation (1.) gives $\sin b=\dfrac{\sin B\sin c}{\sin C}$;

hence, by substitution,

$$R\cos c\sin a=\sin c\cos a\cos B+R.\frac{\sin B\cos C\sin c}{\sin C}$$

Dividing by $\sin c$, we have

$$R\frac{\cos c}{\sin c}\sin a=\cos a\cos B+R\frac{\sin B\cos C}{\sin C}.$$

$$\text{But, } \frac{\cos}{\sin}=\frac{\cot}{R} \text{ (Art. XVII.).}$$

Therefore, $\cot c \sin a=\cos a \cos B+\cot C \sin B.$

Hence, we may write the three symmetrical equations,

$$\left.\begin{array}{l}\cot a \sin b=\cos b \cos C+\cot A \sin C \\ \cot b \sin c=\cos c \cos A+\cot B \sin A \\ \cot c \sin a=\cos a \cos B+\cot C \sin B\end{array}\right\} (9.)$$

That is, *in every spherical triangle, the cotangent of one of the sides into the sine of a second side, is equal to the cosine of the second side into the cosine of the included angle, plus the cotangent of the angle opposite the first side into the sine of the included angle.*

IX. We shall terminate these formulas by demonstrating *Napier's Analogies*, which serve to simplify several cases in the solution of spherical triangles.

If from the first and third of equations (2.), $\cos c$ be eliminated, there will result, after a little reduction,

$$R \cos A \sin c=R \cos a \sin b—\cos C \sin a \cos b.$$

By a simple permutation, this gives

$$R \cos B \sin c=R \cos b \sin a—\cos C \sin b \cos a.$$

Hence by adding these two equations, and reducing, we shall have

$$\sin c (\cos A+\cos B)=(R—\cos C) \sin (a+b)$$

$$\text{But since } \frac{\sin c}{\sin C}=\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}, \text{ we shall have}$$

$$\sin c (\sin A+\sin B)=\sin C (\sin a+\sin b), \text{ and}$$

$$\sin c (\sin A—\sin B)=\sin C (\sin a—\sin b).$$

Dividing these two equations successively by the preceding one; we shall have

$$\frac{\sin A+\sin B}{\cos A+\cos B}=\frac{\sin C}{R—\cos C} \cdot \frac{\sin a+\sin b}{\sin (a+b)}.$$

$$\frac{\sin A—\sin B}{\cos A+\cos B}=\frac{\sin C}{R—\cos C} \cdot \frac{\sin a—\sin b}{\sin (a+b)}.$$

And reducing these by the formulas in Articles XXIII. and XXIV., there will result

$$\text{tang } \tfrac{1}{2}(A+B) = \cot \tfrac{1}{2}C . \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}$$

$$\text{tang } \tfrac{1}{2}(A-B) = \cot \tfrac{1}{2}C . \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)}.$$

Hence, two sides a and b with the included angle C being given, the two other angles A and B may be found by the analogies,

$$\cos \tfrac{1}{2}(a+b) : \cos \tfrac{1}{2}(a-b) :: \cot \tfrac{1}{2}C : \text{tang } \tfrac{1}{2}(A+B)$$
$$\sin \tfrac{1}{2}(a+b) : \sin \tfrac{1}{2}(a-b) :: \cot \tfrac{1}{2}C : \text{tang } \tfrac{1}{2}(A-B).$$

If these same analogies are applied to the polar triangle of ABC, we shall have to put $180°-A'$, $180°-B'$, $180°-a'$, $180°-b'$, $180°-c'$, instead of a, b, A, B, C, respectively; and for the result, we shall have after omitting the $'$, these two analogies,

$$\cos \tfrac{1}{2}(A+B) : \cos \tfrac{1}{2}(A-B) :: \text{tang } \tfrac{1}{2}c : \text{tang } \tfrac{1}{2}(a+b)$$
$$\sin \tfrac{1}{2}(A+B) : \sin \tfrac{1}{2}(A-B) :: \text{tang } \tfrac{1}{2}c : \text{tang } \tfrac{1}{2}(a-b)$$

by means of which, when a side c and the two adjacent angles A and B are given, we are enabled to find the two other sides a and b. These four proportions are known by the name of *Napier's Analogies.*

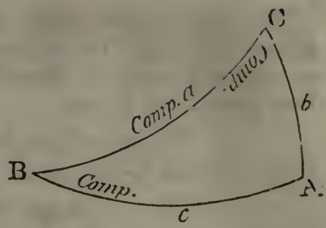
X. In the case in which there are given two sides and an angle opposite one of them, there will in general be two solutions corresponding to the two results in Case II. of rectilineal triangles. It is also plain that this ambiguity will extend itself to the corresponding case of the polar triangle, that is, to the case in which there are given two angles and a side opposite one of them. In every case we shall avoid all false solutions by recollecting,

1st. *That every angle, and every side of a spherical triangle is less than* 180°.

2d. *That the greater angle lies opposite the greater side, and the least angle opposite the least side, and reciprocally.*

NAPIER'S CIRCULAR PARTS.

XI. Besides the analogies of Napier already demonstrated, that Geometer also invented rules for the solution of all the cases of right angled spherical triangles.



In every right angled spherical triangle BAC, there are six parts: three sides and three angles. If we omit the consideration of the right angle, which is always known, there will be five remaining parts, two of which must be given before the others can be determined.

The *circular parts*, as they are called, are the two sides c and b, about the right angle, the complements of the oblique angles B and C, and the complement of the hypothenuse a. Hence there are five circular parts. The right angle A not being a circular part, is supposed not to separate the circular parts c and b, so that these parts are considered as adjacent to each other.

If any two parts of the triangle be given, their corresponding circular parts will also be known, and these together with a required part, will make three parts under consideration. Now, these three parts *will all lie together, or one of them will be separated from both of the others.* For example, if B and c were given, and a required, the three parts considered would lie together. But if B and C were given, and b required, the parts would not lie together; for, B would be separated from C by the part a, and from b by the part c. In either case B is the *middle part.* Hence, when there are three of the circular parts under consideration, *the middle part is that one of them to which both of the others are adjacent, or from which both of them are separated.* In the former case the parts are said to be *adjacent,* and in the latter case the parts are said to be *opposite.*

This being premised, we are now to prove the following rules for the solution of right angled spherical triangles, which it must be remembered apply to the *circular parts*, as already defined.

1st. *Radius into the sine of the middle part is equal to the rectangle of the tangents of the adjacent parts.*

2d. *Radius into the sine of the middle part is equal to the rectangle of the cosines of the opposite parts.*

These rules are proved by assuming each of the five circular parts, in succession, as the middle part, and by taking the extremes first opposite, then adjacent. Having thus fixed the three parts which are to be considered, take that one of the general equations for oblique angled triangles, which shall contain the three corresponding parts of the triangle, together with the right angle: then make $A=90°$, and after making the reductions corresponding to this supposition, the resulting equation will prove the rule for that particular case.

For example, let comp. a be the middle part and the extremes opposite. The equation to be applied in this case must contain a, b, c, and A. The first of equations (2.) contains these four quantities: hence

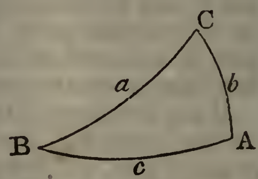
$$R^2 \cos a = R \cos b \cos c + \sin b \sin c \cos A.$$

If $A = 90°$ $\cos A = 0$; hence

$$R \cos a = \cos b \cos c;$$

that is, radius into the sine of the middle part, (which is the complement of a,) is equal to the rectangle of the cosines of the opposite parts.

Suppose now that the complement of a were the middle part and the extremes adjacent. The equation to be applied must contain the four quantities a, B, C, and A. It is the first of equations (8.).



$$R^2 \cos A = \sin B \sin C \cos a - R \cos B \cos C.$$

Making $A = 90°$, we have

$$\sin B \sin C \cos a = R \cos B \cos C, \quad \text{or}$$

$$R \cos a = \cot B \cot C;$$

that is, radius into the sine of the middle part is equal to the rectangle of the tangent of the complement of B into the tangent of the complement of C, that is, to the rectangle of the tangents of the adjacent *circular parts.*

Let us now take the comp. B, for the middle part and the extremes opposite. The two other parts under consideration will then be the perpendicular b and the angle C. The equation to be applied must contain the four parts A, B, C, and b: it is the second of equations (8.),

$$R^2 \cos B = \sin A \sin C \cos b - R \cos A \cos C$$

Making $A = 90°$, we have, after dividing by R,

$$R \cos B = \sin C \cos b.$$

Let comp. B be still the middle part and the extremes adjacent. The equation to be applied must then contain the four parts a, B, c, and A. It is similar to equations (9.).

$$\cot a \sin c = \cos c \cos B + \cot A \sin B$$

But if $A = 90°$, $\cot A = 0$; hence,

$$\cot a \sin c = \cos c \cos B; \quad \text{or}$$

$$R \cos B = \cot a \operatorname{tang} c.$$

And by pursuing the same method of demonstration when each circular part is made the middle part, we obtain the five following equations, which embrace all the cases

$$\left.\begin{array}{l} \text{R} \cos a = \cos b \cos c = \cot \text{B} \cot \text{C} \\ \text{R} \cos \text{B} = \cos b \sin \text{C} = \cot a \ \text{tang}\ c \\ \text{R} \cos \text{C} = \cos c \sin \text{B} = \cot a \ \text{tang}\ b \\ \text{R} \sin b = \sin a \sin \text{B} = \text{tang}\ c \cot \text{C} \\ \text{R} \sin c = \sin a \sin \text{C} = \text{tang}\ b \cot \text{B} \end{array}\right\} \quad (10.)$$

We see from these equations that, *if the middle part is required we must begin the proportion with radius; and when one of the extremes is required we must begin the proportion with the other extreme.*

We also conclude, from the first of the equations, that when the hypothenuse is less than 90°, the sides b and c will be of the same species, and also that the angles B and C will likewise be of the same species. When a is greater than 90°, the sides b and c will be of different species, and the same will be true of the angles B and C. We also see from the two last equations that a side and its opposite angle will always be of the same species.

These properties are proved by considering the algebraic signs which have been attributed to the trigonometrical lines, and by remembering that the two members of an equation must always have the same algebraic sign.

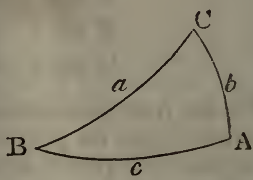
SOLUTION OF RIGHT ANGLED SPHERICAL TRIANGLES BY LOGARITHMS.

It is to be observed, that when any element is discovered in the form of its sine only, there may be two values for this element, and consequently two triangles that will satisfy the question; because, the same sine which corresponds to an angle or an arc, corresponds likewise to its supplement. This will not take place, when the unknown quantity is determined by means of its cosine, its tangent, or cotangent. In all these cases, the sign will enable us to decide whether the element in question is less or greater than 90°; the element will be less than 90°, if its cosine, tangent, or cotangent, has the sign +; it will be greater if one of these quantities has the sign —.

In order to discover the species of the required element of the triangle, we shall annex the minus sign to the logarithms of all the elements whose cosines, tangents, or cotangents, are negative. Then by recollecting that the product of the two extremes has the same sign as that of the means, we can at once determine the sign which is to be given to the required element, and then its species will be known.

EXAMPLES.

1. In the right angled spherical triangle BAC, right angled at A, there are given $a=64° 40'$ and $b=42° 12'$: required the remaining parts.



First, to find the side c.

The hypothenuse a corresponds to the middle part, and the extremes are opposite: hence

$$R \cos a = \cos b \cos c, \quad \text{or}$$

As cos	b	42° 12′	ar.-comp.	log.	0.130296
Is to	R	-	- - -	-	10.000000
So is cos	a	64° 40′	- - -	-	9.631326
To cos	c	54° 43′ 07″	- -	-	9.761622

To find the angle B.

The side b will be the middle part and the extremes opposite: hence

$$R \sin b = \cos(\text{comp. } a) \times \cos(\text{comp. B}) = \sin a \sin B.$$

As sin	a	64° 40′	ar.-comp.	log.	0.043911
Is to sin	b	42° 12′	- - -	-	9.827189
So is	R	-	- - -	-	10.000000
To sin	B	48° 00′ 14″	- - -	-	9.871100

To find the angle C.

The angle C is the middle part and the extremes adjacent; hence

$$R \cos C = \cot a \tan b.$$

As	R	-	ar.-comp.	log.	0.000000
Is to cot	a	64° 40′	- - -	-	9.675237
So is tang	b	42° 12′	- - -	-	9.957485
To cos	C	64° 34′ 46″	- -	-	9.632722

2. In a right angled triangle BAC, there are given the hypothenuse $a=105° 34'$, and the angle $B=80° 40'$: required the remaining parts.

To find the angle C.

The hypothenuse will be the middle part and the extremes adjacent: hence,

$$R \cos a = \cot B \cot C.$$

As cot	B 80° 40′	ar.-comp.	log.		0.784220+
Is to cos	a 105° 34′	- -	-	-	9.428717—
So is	R	- -	- -	-	10.000000+
To cot	C 148° 30′ 54″	-	-	-	10.212937—

Since the cotangent of C is negative the angle C is greater than 90°, and is the supplement of the arc which would correspond to the cotangent, if it were positive.

To find the side c.

The angle B will correspond to the middle part, and the extremes will be adjacent: hence,

$$R \cos B = \cot a \operatorname{tang} c.$$

As cot	a 105° 34′	ar.-comp.	log.		0.555053—
Is to	R	- -	- -	-	10.000000+
So is cos	B 80° 40′	- -	-	-	9.209992+
To tang	c 149° 47′ 36″	-	-	-	9.765045—

To find the side b.

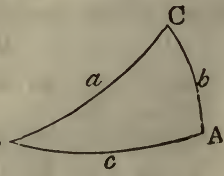
The side b will be the middle part and the extremes opposite: hence,

$$R \sin b = \sin a \sin B.$$

As	R -	ar. comp.	log.	-	0.000000
To sin	a 105° 34′	- -	- -	-	9.983770
So is sin	B 80° 40′	- -	- -	-	9.994212
To sin	b 71°54′ 33″	- -	- -	-	9.977982

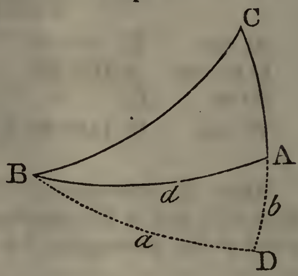
OF QUADRANTAL TRIANGLES.

A *quadrantal* spherical triangle is one which has one of its sides equal to 90°.



Let BAC be a quadrantal triangle in which the side $a=90°$. If we pass to the corresponding polar triangle, we shall have $A'=180°-a=90°$, $B'=180°-b$, $C'=180°-c$, $a'=180°-A$, $b'=180°-B$, $c'=180°-C$; from which we see, that the polar triangle will be right angled at A′, and hence every case may be referred to a right angled triangle.

But we can solve the quadrantal triangle by means of the right angled triangle in a manner still more simple.



In the quadrantal triangle BAC, in which BC=90°, produce the side CA till CD is equal to 90°, and conceive the arc of a great circle to be drawn through B and D. Then C will be the pole of the arc BD, and the angle C will be measured by BD (Book IX. Prop. VI.), and the angles CBD and D will be right angles. Now before the remaining parts of the quadrantal triangle can be found, at least two parts must be given in addition to the side BC=90°; in which case two parts of the right angled triangle BDA, together with the right angle, become known. Hence the conditions which enable us to determine one of these triangles, will enable us also to determine the other.

3. In the quadrantal triangle BCA, there are given CB=90°, the angle C=42° 12′, and the angle A=115° 20′: required the remaining parts.

Having produced CA to D, making CD=90° and drawn the arc BD, there will then be given in the right angled triangle BAD, the side a=C=42° 12′, and the angle BAD=180°—BAC=180°—115° 20′=64° 40′, to find the remaining parts.

To find the side *d*.

The side *a* will be the middle part, and the extremes opposite: hence,

$$R \sin a = \sin A \sin d.$$

As sin	A	64° 40′	ar.-comp.	log.	0.043911
Is to	R	- - - -	-	-	10.000000
So is sin	*a*	42° 12′	- - -	-	9.827189
To sin	*d*	48° 00′ 14″	- -	-	9.871100

To find the angle B.

The angle A will correspond to the middle part, and the extremes will be opposite: hence

$$R \cos A = \sin B \cos a.$$

As cos	*a*	42° 12′	ar.-comp.	log.	0.130296
Is to	R	- - - -	-	-	10.000000
So is cos	A	64° 40′	- - -	-	9.631326
To sin	B	35° 16′ 53″	- -	-	9.761622

To find the side b.

The side b will be the middle part, and the extremes adjacent: hence,

$$R \sin b = \cot A \tan g\, a.$$

As	R	-	ar.-comp. log.	0.000000
Is to cot	A	64° 40′	- - - -	9.675237
So is tang	a	42° 12′	- - - -	9.957485
To sin	b	25° 25′ 14″	- - -	9.632722

Hence, CA$=90°-b=90°-25° 25' 14''$ $=64° 34' 46''$
CBA$=90°-$ABD$=90°-35° 16' 53''=54° 43' 07''$
BA$=d$ - - - - $=48° 00' 15''$.

4. In the right angled triangle BAC, right angled at A, there are given $a=115° 25'$, and $c=60° 59'$: required the remaining parts.

Ans. $\begin{cases} B=148° 56' 45'' \\ C=\ 75° 30' 33'' \\ b=152° 13' 50''. \end{cases}$

5. In the right angled spherical triangle BAC, right angled at A, there are given $c=116° 30' 43''$, and $b=29° 41' 32''$: required the remaining parts.

Ans. $\begin{cases} C=103° 52' 46'' \\ B=\ 32° 30' 22'' \\ a=112° 48' 58''. \end{cases}$

6. In a quadrantal triangle, there are given the quadrantal side $=90°$, an adjacent side $=115° 09'$, and the included angle $=115° 55'$: required the remaining parts.

Ans. $\begin{cases} \text{side,} & 113° 18' 19'' \\ \text{angles,} & \begin{cases} 117° 33' 52'' \\ 101° 40' 07''. \end{cases} \end{cases}$

SOLUTION OF OBLIQUE ANGLED TRIANGLES BY LOGARITHMS.

There are six cases which occur in the solution of oblique angled spherical triangles.

1. Having given two sides, and an angle opposite one of them.

2. Having given two angles, and a side opposite one of them.

3. Having given the three sides of a triangle, to find the angles.

4. Having given the three angles of a triangle, to find the sides.

5. Having given two sides and the included angle.

6. Having given two angles and the included side.

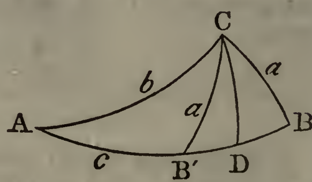
CASE I.

Given two sides, and an angle opposite one of them, to find the remaining parts.

For this case we employ equation (1.);

$$\text{As } \sin a : \sin b :: \sin A : \sin B.$$

Ex. 1. Given the side $a=44°\ 13'\ 45''$, $b=84°\ 14'\ 29''$ and the angle $A=32°\ 26'\ 07''$: required the remaining parts.



To find the angle B.

As sin	a	$44°\ 13'\ 45''$	ar.-comp.	log.	0.156437
Is to sin	b	$84°\ 14'\ 29''$	- -	-	9.997803
So is sin	A	$32°\ 26'\ 07''$	- -	-	9.729445
To sin	B	$49°\ 54'\ 38''$ or sin B′ $130°\ 5'\ 22''$			9.883685

Since the sine of an arc is the same as the sine of its supplement, there will be two angles corresponding to the logarithmic sine 9.883685 and these angles will be supplements of each other. It does not follow however that both of them will satisfy all the other conditions of the question. If they do, there will be two triangles ACB′, ACB; if not, there will be but one.

To determine the circumstances under which this ambiguity arises, we will consider the 2d of equations (2.).

$$R^2 \cos b = R \cos a \cos c + \sin a \sin c \cos B.$$

from which we obtain

$$\cos B = \frac{R^2 \cos b - R \cos a \cos c}{\sin a \sin c}.$$

Now if cos b be greater than cos a, we shall have

$$R^2 \cos b > R \cos a \cos c,$$

or the sign of the second member of the equation will depend on that of cos b. Hence cos B and cos b will have the same sign, or B and b will be of the same species, and there will be but one triangle.

But when $\cos b > \cos a$, $\sin b < \sin a$: hence,

If the sine of the side opposite the required angle be less than the sine of the other given side, there will be but one triangle.

If however, $\sin b > \sin a$, the $\cos b$ will be less than $\cos a$, and it is plain that such a value may then be given to c as to render

$$R^2 \cos b < R \cos a \cos c,$$

or the sign of the second member may be made to depend on $\cos c$.

We can therefore give such values to c as to satisfy the two equations

$$+\cos B = \frac{R^2 \cos b - R \cos a \cos c}{\sin a \sin c}$$

$$-\cos B = \frac{R^2 \cos b - R \cos a \cos c}{\sin a \sin c}.$$

Hence, *if the sine of the side opposite the required angle be greater than the sine of the other given side, there will be two triangles which will fulfil the given conditions.*

Let us, however, consider the triangle ACB, in which we are yet to find the base AB and the angle C. We can find these parts most readily by dividing the triangle into two right angled triangles. Draw the arc CD perpendicular to the base AB: then in each of the triangles there will be given the hypothenuse and the angle at the base. And generally, when it is proposed to solve an oblique angled triangle by means of the right angled triangle, we must so draw the perpendicular that *it shall pass through the extremity of a given side, and lie opposite to a given angle.*

To find the angle C, in the triangle ACD.

As cot	A	32° 26′ 07″	ar.-comp.	log.	9.803105
Is to	R	- -	- -	-	10.000000
So is cos	b	84° 14′ 29″	- -	-	9.001465
To cot	ACD	86° 21′ 06″	- -	-	8.804570

To find the angle C in the triangle DCB.

As cot	B	49° 54′ 38″	ar.-comp.	log.	0.074810
Is to	R	- -	- -	-	10.000000
So is cos	a	44° 13′ 45″	- -	-	9.855250
To cot	DCB	49° 35′ 38″	- -	-	9.930060

Hence $\quad$ ACB=135° 56′ 47″.

To find the side AB.

As sin	A $32° 26' 07''$	ar.-comp.	log.	0.270555
Is to sin	C $135° 56' 47''$	- -	-	9.842191
So is sin	a $44° 13' 45''$	- -	-	9.843563
To sin	c $115° 16' 29''$	- -	-	9.956309

The arc $64° 43' 31''$, which corresponds to sin c is not the value of the side AB: for the side AB must be greater than b, since it lies opposite to a greater angle. But $b=84° 14' 29''$: hence the side AB must be the supplement of $64° 43' 31''$, or $115° 16' 29''$.

Ex. 2. Given $b=91° 03' 25''$, $a=40° 36' 37''$, and $A=35° 57' 15''$: required the remaining parts, when the obtuse angle B is taken.

$$Ans. \begin{cases} B=115° 35' 41'' \\ C=\ 58° 30' 57'' \\ c=\ 70° 58' 52'' \end{cases}$$

CASE II.

Having given two angles and a side opposite one of them, to find the remaining parts.

For this case, we employ the equation (1.)

$$\sin A : \sin B :: \sin a : \sin b.$$

Ex. 1. In a spherical triangle ABC, there are given the angle $A=50° 12'$, $B=58° 8'$, and the side $a=62° 42'$; to find the remaining parts.

To find the side b.

As sin	A $50° 12'$	ar.-comp.	log.	0.114478
Is to sin	B $58° 08'$	- - -	-	9.929050
So is sin	a $62° 42'$	- - -	-	9.948715
To sin	b $79° 12' 10''$, or $100° 47' 50''$			9.992243

We see here, as in the last example, that there are two arcs corresponding to the 4th term of the proportion, and these arcs are supplements of each other, since they have the same sine. It does not follow, however, that both of them will satisfy all the conditions of the question. If they do, there will be two triangles; if not, there will be but one.

To determine when there are two triangles, and also when there is but one, let us consider the second of equations (8.)

$R^2 \cos B = \sin A \sin C \cos b - R \cos A \cos C$, which gives

$$\cos b = \frac{R^2 \cos B + R \cos A \cos C}{\sin A \sin C}.$$

Now, if cos B be greater than cos A we shall have

$$R^2 \cos B > R \cos A \cos C,$$

and hence the sign of the second member of the equation will depend on that of cos B, and consequently cos b and cos B will have the same algebraic sign, or b and B will be of the same species. But when cos B > cos A the sin B < sin A : hence

If the sine of the angle opposite the required side be less than the sine of the other given angle, there will be but one solution.

If, however, sin B > sin A, the cos B will be less than cos A, and it is plain that such a value may then be given to cos C, as to render

$$R^2 \cos B < R \cos A \cos C,$$

or the sign of the second member of the equation may be made to depend on cos C. We can therefore give such values to C as to satisfy the two equations

$$+\cos b = \frac{R^2 \cos B + R \cos A \cos C}{\sin A \sin C}, \quad \text{and}$$

$$-\cos b = \frac{R^2 \cos B + R \cos A \cos C}{\sin A \sin C}.$$

Hence, *if the sine of the angle opposite the required side be greater than the sine of the other given angle there will be two solutions.*

Let us first suppose the side b to be less than 90°, or equal to 79° 12′ 10″.

If now, we let fall from the angle C a perpendicular on the base BA, the triangle will be divided into two right angled triangles, in each of which there will be two parts known besides the right angle.

Calculating the parts by Napier's rules we find,

$C = 130° 54′ 28$

$c = 119° 03′ 26″$.

If we take the side $b = 100° 47′ 50″$, we shall find

$C = 156° 15′ 06″$

$c = 152° 14′ 18″$.

Ex. 2. In a spherical triangle ABC there are given A=103° 59′ 57″, B=46° 18′ 7″, and a=42° 8′ 48″; required the remaining parts.

There will but one triangle, since sin B$<$sin A.

$$Ans. \begin{cases} b = 30° \\ C = 36° \ 7' \ 54'' \\ c = 24° \ 3' \ 56''. \end{cases}$$

CASE III.

Having given the three sides of a spherical triangle to find the angles.

For this case we use equations (3.).

$$\cos \tfrac{1}{2} A = R \sqrt{\frac{\sin \frac{1}{2} s \sin (\frac{1}{2}s - a)}{\sin b \sin c}}$$

Ex. 1. In an oblique angled spherical triangle there are given a=56° 40′, b=83° 13′ and c=114° 30′; required the angles.

$$\tfrac{1}{2}(a+b+c) = \tfrac{1}{2}s \quad = 127° \ 11' \ 30''$$
$$\tfrac{1}{2}(b+c-a) = (\tfrac{1}{2}s - a) = 70° \ 31' \ 30''.$$

Log sin $\frac{1}{2}s$ 127° 11′ 30″	- - -	9.901250
log sin $(\frac{1}{2}s-a)$ 70° 31′ 30″	- - -	9.974413
—log sin b 83° 13′	ar.-comp.	0.003051
—log sin c 114° 30′	ar.-comp.	0.040977
Sum	- - - - - - -	19.919691
Half sum =log cos $\frac{1}{2}$A 24° 15′, 39″	- -	9.959845

Hence, angle A=48° 31′ 18″.

The addition of twice the logarithm of radius, or 20, to the numerator of the quantity under the radical just cancels the 20 which is to be subtracted on account of the arithmetical complements, to that the 20, in both cases, may be omitted.

Applying the same formulas to the angles B and C, we find,

B= 62° 55′ 46″

C=125° 19′ 02″.

Ex. 2. In a spherical triangle there are given a=40° 18′ 29″. b=67° 14′ 28″, and c=89° 47′ 6″ : required the three angles.

$$Ans. \begin{cases} A = 34° \ 22' \ 16'' \\ B = 53° \ 35' \ 16'' \\ C = 119° \ 13' \ 32' \end{cases}$$

CASE IV.

Having given the three angles of a spherical triangle, to find the three sides.

For this case we employ equations (7.)

$$\cos \tfrac{1}{2}a = R\sqrt{\frac{\cos(\frac{1}{2}S-B)\cos(\frac{1}{2}S-C)}{\sin B \sin C}}.$$

Ex. 1. In a spherical triangle ABC there are given $A=48°$ $30'$, $B=125°\ 20'$, and $C=62°\ 54'$; required the sides.

$\frac{1}{2}(A+B+C)=\frac{1}{2}S=\ 118°\ 22'$
$(\frac{1}{2}S-A)\ \ -\ \ =\ \ 69°\ 52'$
$(\frac{1}{2}S-B)\ \ -\ \ =-\ 6°\ 58'$
$(\frac{1}{2}S-C)\ \ -\ \ =\ \ 55°\ 28'$

Log cos $(\frac{1}{2}S-B)$	$-6°\ 58'$	- - -	9.996782
log cos $(\frac{1}{2}S-C)$	$55°\ 28'$	- - -	9.753495
—log sin B	$125°\ 20'$	ar.-comp.	0.088415
—log sin C	$62°\ 54'$	ar.-comp.	0.050506
Sum	- - - - -	- -	19.889198
Half sum=log cos $\frac{1}{2}A=28°\ 19'\ 48''$		-	9.944599

Hence, side $a=56°\ 39'\ 36''$.

In a similar manner we find,

$b=114°\ 29'\ 58''$
$c=\ 83°\ 12'\ 06''$.

Ex. 2. In a spherical triangle ABC, there are given $A=109°$ $55'\ 42''$, $B=116°\ 38'\ 33''$, and $C=120°\ 43'\ 37''$; required the three sides.

$$Ans. \begin{cases} a=\ 98°\ 21'\ 40'' \\ b=109°\ 50'\ 22'' \\ c=115°\ 13'\ 26'' \end{cases}$$

CASE V.

Having given in a spherical triangle, two sides and their included angle, to find the remaining parts.

For this case we employ the two first of Napier's Analogies.

$\cos \frac{1}{2}(a+b) : \cos \frac{1}{2}(a-b) :: \cot \frac{1}{2}C : \text{tang } \frac{1}{2}(A+B)$

$\sin \frac{1}{2}(a+b) : \sin \frac{1}{2}(a-b) :: \cot \frac{1}{2}C : \text{tang } \frac{1}{2}(A-B)$.

Having found the half sum and the half difference of the angles A and B, the angles themselves become known; for, the greater angle is equal to the half sum plus the half difference, and the lesser is equal to the half sum minus the half difference.

The greater angle is then to be placed opposite the greater side. The remaining side of the triangle can then be found by Case II.

Ex. 1. In a spherical triangle ABC, there are given $a=68°$ $46'$ $2''$, $b=37°$ $10'$, and $C=39°$ $23'$; to find the remaining parts

$\frac{1}{2}(a+b)=52°\ 58'\ 1''$, $\frac{1}{2}(a-b)=15°\ 48'\ 1''$, $\frac{1}{2}C=19°\ 41'\ 30''$.

As cos	$\frac{1}{2}(a+b)$	52° 58′ 1″	log. ar.-comp.	0.220210
Is to cos	$\frac{1}{2}(a-b)$	15° 48′ 1″	- - -	9.983271
So is cot	$\frac{1}{2}C$	19° 41′ 30″	- - -	10.446254
To tang	$\frac{1}{2}(A+B)$	77° 22′ 25″	- - -	10.649735

As sin	$\frac{1}{2}(a+b)$	52° 58′ 1″	log. ar.-comp.	0.097840
Is to sin	$\frac{1}{2}(a-b)$	15° 48′ 1″	- - -	9.435016
So is cot	$\frac{1}{2}C$	19° 41′ 30″	- - -	10.446254
To tang	$\frac{1}{2}(A-B)$	43° 37′ 21″	- - -	9.979110

Hence, $A=77°\ 22'\ 25''+43°\ 37'\ 21''=120°\ 59'\ 46''$

$B=77°\ 22'\ 25''-43°\ 37'\ 21''=\ 33°\ 45'\ 04''$

side c - - - - $=\ 43°\ 37'\ 37''$.

Ex. 2. In a spherical triangle ABC, there are given $b=83°$ $19'$ $42''$, $c=23°$ $27'$ $46''$, the contained angle $A=20°$ $39'$ $48''$; to find the remaining parts.

$$Ans. \begin{cases} B=156°\ 30'\ 16'' \\ C=\ \ 9°\ 11'\ 48'' \\ a=\ 61°\ 32'\ 12''. \end{cases}$$

CASE VI.

In a spherical triangle, having given two angles and the included side to find the remaining parts.

For this case we employ the second of Napier's Analogies.

$$\cos \tfrac{1}{2}(A+B) : \cos \tfrac{1}{2}(A-B) :: \text{tang } \tfrac{1}{2}c : \text{tang } \tfrac{1}{2}(a+b)$$

$$\sin \tfrac{1}{2}(A+B) : \sin \tfrac{1}{2}(A-B) :: \text{tang } \tfrac{1}{2}c : \text{tang } \tfrac{1}{2}(a-b).$$

From which a and b are found as in the last case. The remaining angle can then be found by Case I.

Ex. 1. In a spherical triangle ABC, there are given A=81 38′ 20″, B=70° 9′ 38″, c=59° 16′ 23″; to find the remaining parts.

$\tfrac{1}{2}(A+B)$=75° 53′ 59″, $\tfrac{1}{2}(A-B)$=5° 44′ 21″, $\tfrac{1}{2}c$=29° 38′ 11″.

As cos	$\tfrac{1}{2}(A+B)$	75° 53′ 59″	log. ar.-comp.	0.613287
To cos	$\tfrac{1}{2}(A-B)$	5° 44′ 21″	- -	9.997818
So is tang	$\tfrac{1}{2}c$	29° 38′ 11″	- -	9.755051
To tang	$\tfrac{1}{2}(a+b)$	66° 42′ 52″	- -	10.366156

As sin	$\tfrac{1}{2}(A+B)$	75° 53′ 59″	log. ar.-comp.	0.013286
To sin	$\tfrac{1}{2}(A-B)$	5° 14′ 21″	- -	9.000000
So is tang	$\tfrac{1}{2}c$	29° 38′ 11″	- -	9.755051
To tang	$\tfrac{1}{2}(a-b)$	3° 21′ 25″	- -	8.768337

Hence a=66° 42′ 52″+3° 21′ 25″=70° 04′ 17″
b=66° 42′ 52″—3° 21′ 25″=63° 21′ 27″
angle C - - - =64° 46′ 33″.

Ex. 2. In a spherical triangle ABC, there are given A=34° 15′ 3″, B=42° 15′ 13″, and c=76° 35′ 36″; to find the remaining parts.

Ans. $\begin{cases} a = 40° \ 0′ \ 10″ \\ b = 50° \ 10′ \ 30″ \\ C = 121° \ 36′ \ 19″ \end{cases}$

MENSURATION OF SURFACES.

The area, or content of a surface, is determined by finding how many times it contains some other surface which is assumed as the unit of measure. Thus, when we say that a square yard contains 9 square feet, we should understand that one square foot is taken for the unit of measure, and that this unit is contained 9 times in the square yard.

The most convenient unit of measure for a surface, is a square whose side is the linear unit in which the linear dimensions of the figure are estimated. Thus, if the linear dimensions are feet, it will be most convenient to express the area in square feet; if the linear dimensions are yards, it will be most convenient to express the area in square yards, &c.

We have already seen (Book IV. Prop. IV. Sch.), that the term, rectangle or product of two lines, designates the rectangle constructed on the lines as sides; and that the numerical value of this product expresses the number of times which the rectangle contains its unit of measure.

PROBLEM I.

To find the area of a square, a rectangle, or a parallelogram.

Rule.—*Multiply the base by the altitude, and the product will be the area* (Book IV. Prop. V.).

1. To find the area of a parallelogram, the base being 12.25 and the altitude 8.5. *Ans.* 104.125.

2. What is the area of a square whose side is 204.3 feet? *Ans.* 41738.49 *sq. ft.*

3. What is the content, in square yards, of a rectangle whose base is 66.3 feet, and altitude 33.3 feet? *Ans.* 245.31.

4. To find the area of a rectangular board, whose length is $12\frac{1}{2}$ feet, and breadth 9 inches. *Ans.* $9\frac{3}{8}$ *sq. ft.*

5. To find the number of square yards of painting in a parallelogram, whose base is 37 feet, and altitude 5 feet 3 inches. *Ans.* $21\frac{7}{12}$.

PROBLEM II.

To find the area of a triangle.

CASE I.

When the base and altitude are given.

Rule.—*Multiply the base by the altitude, and take half the product. Or, multiply one of these dimensions by half the other* (Book IV. Prop. VI.).

1. To find the area of a triangle, whose base is 625 and altitude 520 feet. *Ans.* 162500 *sq. ft.*

2. To find the number of square yards in a triangle, whose base is 40 and altitude 30 feet. *Ans.* $66\frac{2}{3}$.

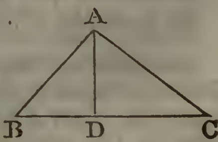
3. To find the number of square yards in a triangle, whose base is 49 and altitude $25\frac{1}{4}$ feet. *Ans.* 68.7361.

CASE II.

When two sides and their included angle are given.

RULE.—*Add together the logarithms of the two sides and the logarithmic sine of their included angle; from this sum subtract the logarithm of the radius, which is* 10, *and the remainder will be the logarithm of double the area of the triangle. Find, from the table, the number answering to this logarithm, and divide it by* 2; *the quotient will be the required area.*

Let BAC be a triangle, in which there are given BA, BC, and the included angle B.



From the vertex A draw AD, perpendicular to the base BC, and represent the area of the triangle by Q. Then,

$$R : \sin B :: BA : AD \text{ (Trig. Th. I.)}:$$

hence,
$$AD = \frac{BA \times \sin B}{R}.$$

$$\text{But, } Q = \frac{BC \times AD}{2} \text{ (Book IV. Prop. VI.)};$$

hence, by substituting for AD its value, we have

$$Q = \frac{BC \times BA \times \sin B}{2R}, \text{ or } 2Q = \frac{BC \times BA \times \sin B}{R}.$$

Taking the logarithms of both numbers, we have

$$\log. 2Q = \log. BC + \log. BA + \log. \sin B - \log. R;$$

which proves the rule as enunciated.

1. What is the area of a triangle whose sides are, BC= 125.81, BA=57.65, and the included angle B=57° 25′?

$$\text{Then, } \log. 2Q = \begin{cases} +\log. BC & 125.81 \ldots\ldots & 2.099715 \\ +\log. BA & 57.65 \ldots\ldots & 1.760799 \\ +\log. \sin B & 57^\circ 25' \ldots\ldots & 9.925626 \\ -\log. R \ldots\ldots\ldots\ldots & & -10. \end{cases}$$

log. 2Q 3.786140

and 2Q=6111.4, or Q=3055.7, the required area.

2. What is the area of a triangle whose sides are 30 and 40 and their included angle 28° 57′ ? *Ans.* 290.427.

3. What is the number of square yards in a triangle of which the sides are 25 feet and 21.25 feet, and their included angle 45° ? *Ans.* 20.8694.

CASE III.

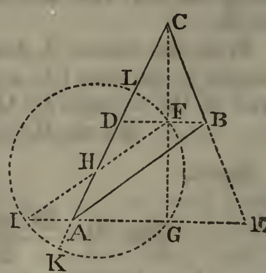
When the three sides are known.

RULE.—1. *Add the three sides together, and take half their sum*

2. *From this half-sum subtract each side separately.*

3. *Multiply together the half-sum and each of the three remainders, and the product will be the square of the area of the triangle. Then, extract the square root of this product, for the required area.*

Or, *After having obtained the three remainders, add together the logarithm of the half-sum and the logarithms of the respective remainders, and divide their sum by 2: the quotient will be the logarithm of the area.*



Let ABC be the given triangle. Take CD equal to the side CB, and draw DB; draw AE parallel to DB, meeting CB produced, in E: then CE will be equal to CA. Draw CFG perpendicular to AE and DB, and it will bisect them at the points G and F. Draw FHI parallel to AB, meeting CA in H, and EA produced, in I. Lastly, with the centre H and radius HF, describe the circumference of a circle, meeting CA produced in K: this circumference will pass through I, because AI=FB=FD, therefore, HF=HI ; and it will also pass through the point G, because FGI is a right angle.

Now, since HA=HD, CH is equal to half the sum of the sides CA, CB ; that is, $CH=\frac{1}{2}CA+\frac{1}{2}CB$; and since HK is equal to $\frac{1}{2}IF=\frac{1}{2}AB$, it follows that

$$CK=\tfrac{1}{2}AC+\tfrac{1}{2}CB+\tfrac{1}{2}AB=\tfrac{1}{2}S,$$

by representing the sum of the sides by S.

Again, $HK=HI=\frac{1}{2}IF=\frac{1}{2}AB$, or $KL=AB$.

Hence, $CL=CK-KL=\frac{1}{2}S-AB$,

and $AK=CK-CA=\frac{1}{2}S-CA$,

and $AL=DK=CK-CD=\frac{1}{2}S-CB$.

Now, $AG\times CG=$ the area of the triangle ACE,

and $AG\times FG=$ the area of the triangle ABE ;

therefore, $AG\times CF=$ the area of the triangle ACB

Also, by similar triangles,

$$AG : CG :: DF : CF, \text{ or } AI : CF;$$

therefore, $AG \times CF =$ triangle $ACB = CG \times DF = CG \times AI$;
consequently, $AG \times CF \times CG \times AI =$ square of the area ACB.

But $\quad CG \times CF = CK \times CL = \frac{1}{2}S(\frac{1}{2}S - AB)$,

and $\quad AG \times AI = AK \times AL = (\frac{1}{2}S - CA) \times (\frac{1}{2}S - CB)$;

therefore, $AG \times CF \times CG \times AI = \frac{1}{2}S(\frac{1}{2}S - AB) \times (\frac{1}{2}S - CA) \times (\frac{1}{2}S - CB)$, which is equal to the square of the area of the triangle ACB.

1. To find the area of a triangle whose three sides are 20, 30, and 40.

20	45	45	45 half-sum.
30	20	30	40
40	—	—	—
—	25 1st rem.	15 2d rem.	5 3d rem
2)90			
—			
45 half-sum.			

Then, $45 \times 25 \times 15 \times 5 = 84375$.

The square root of which is 290.4737, the required area.

2. How many square yards of plastering are there in a triangle whose sides are 30, 40, and 50 feet? *Ans.* $66\frac{2}{3}$.

PROBLEM III.

To find the area of a trapezoid.

Rule.—*Add together the two parallel sides: then multiply their sum by the altitude of the trapezoid, and half the product will be the required area* (Book IV. Prop. VII.).

1. In a trapezoid the parallel sides are 750 and 1225, and the perpendicular distance between them is 1540; what is the area? *Ans.* 152075.

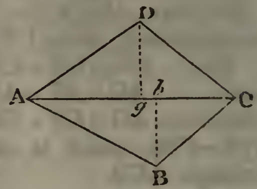
2. How many square feet are contained in a plank, whose length is 12 feet 6 inches, the breadth at the greater end 15 inches, and at the less end 11 inches? *Ans.* $13\frac{1}{2}\frac{3}{4}$ *sq. ft.*

3. How many square yards are there in a trapezoid, whose parallel sides are 240 feet, 320 feet, and altitude 66 feet? *Ans.* $2053\frac{1}{3}$.

PROBLEM IV.

To find the area of a quadrilateral.

Rule.—*Join two of the angles by a diagonal, dividing the quadrilateral into two triangles. Then, from each of the other angles let fall a perpendicular on the diagonal: then multiply* *the diagonal by half the sum of the two perpendiculars, and the product will be the area.*



1. What is the area of the quadrilateral ABCD, the diagonal AC being 42, and the perpendiculars Dg, Bb, equal to 18 and 16 feet?

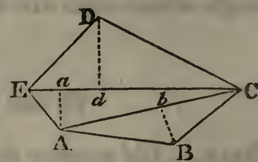
Ans. 714.

2. How many square yards of paving are there in the quadrilateral whose diagonal is 65 feet, and the two perpendiculars let fall on it 28 and $33\frac{1}{2}$ feet? *Ans.* $222\frac{1}{12}$.

PROBLEM V.

To find the area of an irregular polygon.

RULE.—*Draw diagonals dividing the proposed polygon into trapezoids and triangles. Then find the areas of these figures separately, and add them together for the content of the whole polygon.*



1. Let it be required to determine the content of the polygon ABCDE, having five sides.

Let us suppose that we have measured the diagonals and perpendiculars, and found AC=36.21, EC=39.11, Bb=4, Dd=7.26, Aa=4.18, required the area.

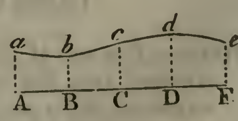
Ans. 296.1292.

PROBLEM VI.

To find the area of a long and irregular figure, bounded on one side by a right line.

RULE.—1. *At the extremities of the right line measure the perpendicular breadths of the figure, and do the same at several intermediate points, at equal distances from each other.*

2. *Add together the intermediate breadths and half the sum of the extreme ones: then multiply this sum by one of the equal parts of the base line: the product will be the required area, very nearly.*



Let AEea be an irregular figure, having for its base the right line AE. At the points A, B, C, D, and E, equally distant from each other, erect the perpendiculars Aa, Bb, Cc, Dd, Ee, to the base line AE, and designate them respectively by the letters a, b, c, d, and e.

Then, the area of the trapezoid $ABba = \frac{a+b}{2} \times AB$,

the area of the trapezoid $BCcb = \frac{b+c}{2} \times BC$,

the area of the trapezoid $CDdc = \frac{c+d}{2} \times CD$,

and the area of the trapezoid $DEed = \frac{d+e}{2} \times DE$;

hence, their sum, or the area of the whole figure, is equal to

$$\left(\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+d}{2}+\frac{d+e}{2}\right) \times AB,$$

since AB, BC, &c. are equal to each other. But this sum is also equal to

$$\left(\frac{a}{2}+b+c+d+\frac{e}{2}\right) \times AB,$$

which corresponds with the enunciation of the rule.

1. The breadths of an irregular figure at five equidistant places being 8.2, 7.4, 9.2, 10.2, and 8.6, and the length of the base 40, required the area.

8.2	4)40
8.6	——
——	10 one of the equal parts.
2(16.8	——
——	
8.4 mean of the extremes.	
7.4	35.2 sum.
9.2	10
10.2	——
——	352 = area.
35.2 sum.	——
——	

2. The length of an irregular figure being 84, and the breadths at six equidistant places 17.4, 20.6, 14.2, 16.5, 20.1, and 24.4; what is the area? *Ans.* 1550.64.

PROBLEM VII.

To find the area of a regular polygon.

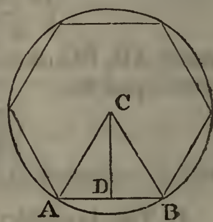
Rule I.—*Multiply half the perimeter of the polygon by the apothem, or perpendicular let fall from the centre on one of the sides, and the product will be the area required* (Book V Prop. IX.).

Remark I.—The following is the manner of determining the perpendicular when only one side and the number of sides of the regular polygon are known:—

First, divide 360 degrees by the number of sides of the polygon, and the quotient will be the angle at the centre; that is, the angle subtended by one of the equal sides. Divide this angle by 2, and half the angle at the centre will then be known.

Now, the line drawn from the centre to an angle of the polygon, the perpendicular let fall on one of the equal sides, and half this side, form a right-angled triangle, in which there are known, the base, which is half the equal side of the polygon, and the angle at the vertex. Hence, the perpendicular can be determined.

1. To find the area of a regular hexagon, whose sides are 20 feet each.



6)360°

60°=ACB, the angle at the centre.

30°=ACD, half the angle at the centre

Also, CAD=90°—ACD=60°; and AD=10.

Then, as sin ACD	30°, ar. comp.	. . . . . .	0 301030
: sin CAD	60°	. . . . . .	9.937531
: AD	10	. . . . . .	1.000000
: CD	17.3205	. . . . . .	1.238561

Perimeter =120, and half the perimeter =60.

Then, 60 × 17.3205=1039.23, the area.

2. What is the area of an octagon whose side is 20?

Ans. 1931.36886.

Remark II.—The area of a regular polygon of any number of sides is easily calculated by the above rule. Let the areas of the regular polygons whose sides are unity or 1, be calculated and arranged in the following

TABLE.

Names.	Sides.	Areas.
Triangle	3 . . .	0.4330127
Square	4 . . .	1.0000000
Pentagon	5 . . .	1.7204774
Hexagon	6 . . .	2.5980762
Heptagon	7 . . .	3.6339124
Octagon	8 . . .	4.8284271
Nonagon	9 . . .	6.1818242
Decagon	10 . . .	7.6942088
Undecagon . . .	11 . . .	9.3656399
Dodecagon . . .	12 . . .	11.1961524

Now, since the areas of similar polygons are to each other as the squares of their homologous sides (Book IV. Prop. XXVII.), we shall have

$$1^2 : \text{tabular area} :: \text{any side squared} : \text{area}.$$

Or, to find the area of any regular polygon, we have

RULE II.—1. *Square the side of the polygon.*

2. *Then multiply that square by the tabular area set opposite the polygon of the same number of sides, and the product will be the required area.*

1. What is the area of a regular hexagon whose side is 20?

$$20^2=400, \quad \text{tabular area} =2.5980762.$$

Hence, $2.5980762 \times 400=1039.2304800$, as before.

2. To find the area of a pentagon whose side is 25.

Ans. 1075.298375.

3. To find the area of a decagon whose side is 20.

Ans. 3077.68352.

PROBLEM VIII.

To find the circumference of a circle when the diameter is given, or the diameter when the circumference is given.

RULE.—*Multiply the diameter by* 3.1416, *and the product will be the circumference; or, divide the circumference by* 3.1416, *and the quotient will be the diameter.*

It is shown (Book V. Prop. XIV.), that the circumference of a circle whose diameter is 1, is 3.1415926, or 3.1416. But since the circumferences of circles are to each other as their radii or diameters, we have, by calling the diameter of the second circle d,

$$1 : d :: 3.1416 : \text{circumference},$$

or, $$d \times 3.1416= \text{circumference}.$$

Hence, also. $$d=\frac{\text{circumference}}{3.1416}$$

1. What is the circumference of a circle whose diameter is 25? *Ans.* 78.54.

2. If the diameter of the earth is 7921 miles, what is the circumference? *Ans.* 24884.6136.

3. What is the diameter of a circle whose circumference is 11652.1904? *Ans.* 37.09.

4. What is the diameter of a circle whose circumference is 6850? *Ans.* 2180.41.

PROBLEM IX

To find the length of an arc of a circle containing any number of degrees.

RULE.—*Multiply the number of degrees in the given arc by* 0.0087266, *and the product by the diameter of the circle.*

Since the circumference of a circle whose diameter is 1, is 3.1416, it follows, that if 3.1416 be divided by 360 degrees, the quotient will be the length of an arc of 1 degree: that is, $\frac{3.1416}{360}=0.0087266=$ arc of one degree to the diameter 1. This being multiplied by the number of degrees in an arc, the product will be the length of that arc in the circle whose diameter is 1; and this product being then multiplied by the diameter, will give the length of the arc for any diameter whatever.

REMARK.—When the arc contains degrees and minutes, reduce the minutes to the decimal of a degree, which is done by dividing them by 60.

1. To find the length of an arc of 30 degrees, the diameter being 18 feet. *Ans.* 4.712364.

2. To find the length of an arc of 12° 10′, or $12\frac{1}{6}$°, the diameter being 20 feet. *Ans.* 2.123472.

3. What is the length of an arc of 10° 15′, or $10\frac{1}{4}$°, in a circle whose diameter is 68? *Ans.* 6.082396.

PROBLEM X.

To find the area of a circle.

RULE I.—*Multiply the circumference by half the radius* (Book V. Prop. XII.).

RULE II.—*Multiply the square of the radius by* 3.1416 (Book V. Prop. XII. Cor. 2).

1. To find the area of a circle whose diameter is 10 and circumference 31.416. *Ans.* 78.54.

2. Find the area of a circle whose diameter is 7 and circumference 21.9912. *Ans.* 38.4846.

3. How many square yards in a circle whose diameter is $3\frac{1}{2}$ feet? *Ans.* 1.069016.

4. What is the area of a circle whose circumference is 12 feet? *Ans.* 11.4595.

PROBLEM XI.

To find the area of the sector of a circle.

RULE I.—*Multiply the arc of the sector by half the radius* (Book V. Prop. XII. Cor. 1).

RULE II.—*Compute the area of the whole circle: then say, as* 360 *degrees is to the degrees in the arc of the sector, so is the area of the whole circle to the area of the sector.*

1. To find the area of a circular sector whose arc contains 18 degrees, the diameter of the circle being 3 feet. *Ans.* 0.35343.

2. To find the area of a sector whose arc is 20 feet, the radius being 10. *Ans.* 100.

3. Required the area of a sector whose arc is 147° 29′, and radius 25 feet. *Ans.* 804.3986

PROBLEM XII.

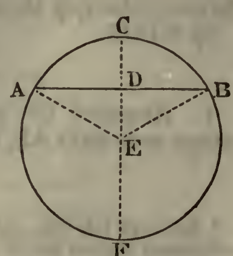
To find the area of a segment of a circle.

RULE.—1. *Find the area of the sector having the same arc, by the last problem.*

2. *Find the area of the triangle formed by the chord of the segment and the two radii of the sector.*

3. *Then add these two together for the answer when the segment is greater than a semicircle, and subtract them when it is less.*

1. To find the area of the segment ACB, its chord AB being 12, and the radius EA, 10 feet.



As EA	10 ar. comp.	. .	9.000000
: AD	6	. . . .	0.778151
:: sin D	90°	. . . . . .	10.000000
: sin AED	36° 52′ = 36.87		9.778151
	2		

73.74 = the degrees in the arc ACB

Then, $0.0087266 \times 73.74 \times 20 = 12.87 = \text{arc ACB, nearly}$

$$\begin{array}{r} 5 \\ \hline 64.35 = \text{area EACB.} \\ \hline \end{array}$$

Again, $\sqrt{EA^2 - AD^2} = \sqrt{100 - 36} = \sqrt{64} = 8 = ED$;
and $6 \times 8 = 48 =$ the area of the triangle EAB.

Hence, sect. $EACB - EAB = 64.35 - 48 = 16.35 = ACB$.

2. Find the area of the segment whose height is 18, the diameter of the circle being 50. *Ans.* 636.4834.

3. Required the area of the segment whose chord is 16, the diameter being 20. *Ans.* 44.764.

PROBLEM XIII.

To find the area of a circular ring: that is, the area included between the circumferences of two circles which have a common centre.

Rule.—*Take the difference between the areas of the two circles.* Or, *subtract the square of the less radius from the square of the greater, and multiply the remainder by* 3.1416.

For the area of the larger is . . . . $R^2\pi$
and of the smaller . . . . $r^2\pi$

Their difference, or the area of the ring, is $(R^2 - r^2)\pi$.

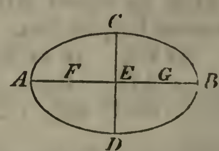
1. The diameters of two concentric circles being 10 and 6, required the area of the ring contained between their circumferences. *Ans.* 50.2656.

2. What is the area of the ring when the diameters of the circles are 10 and 20? *Ans.* 235.62.

PROBLEM XIV.

To find the area of an ellipse, or oval.*

Rule.—*Multiply the two semi-axes together, and their product by* 3.1416.



1. Required the area of an ellipse whose semi-axes AE, EC, are 35 and 25. *Ans.* 2748.9.

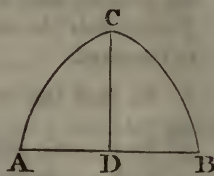
* Although this rule, and the one for the following problem, cannot be demonstrated without the aid of principles not yet considered, still it was thought best to insert them, as they complete the rules necessary for the mensuration of planes.

2. Required the area of an ellipse whose axes are 24 and 18.
Ans. 339.2928.

PROBLEM XV.

To find the area of any portion of a parabola.

Rule.—*Multiply the base by the perpendicular height, and tak two-thirds of the product for the required area.*



1. To find the area of the parabola ACB, the base AB being 20 and the altitude CD, 18.
Ans. 240.

2 Required the area of a parabola, the base being 20 and the altitude 30. *Ans.* 400.

MENSURATION OF SOLIDS.

The mensuration of solids is divided into two parts.

1st. The mensuration of their surfaces; and,

2dly. The mensuration of their solidities.

We have already seen, that the unit of measure for plane surfaces is a square whose side is the unit of length.

A curved line which is expressed by numbers is also referred to a unit of length, and its numerical value is the number of times which the line contains its unit. If, then, we suppose the linear unit to be reduced to a right line, and a square constructed on this line, this square will be the unit of measure for curved surfaces.

The unit of solidity is a cube, the face of which is equal to the superficial unit in which the surface of the solid is estimated, and the edge is equal to the linear unit in which the linear dimensions of the solid are expressed (Book VII. Prop. XIII. Sch.).

The following is a table of solid measures:—

1728	cubic inches	=	1 cubic foot.
27	cubic feet	=	1 cubic yard.
$4492\frac{1}{8}$	cubic feet	=	1 cubic rod.
282	cubic inches	=	1 ale gallon.
231	cubic inches	=	1 wine gallon
2150.42	cubic inches	=	1 bushel.

OF POLYEDRONS, OR SURFACES BOUNDED BY PLANES.

PROBLEM I.

To find the surface of a right prism.

Rule.—*Multiply the perimeter of the base by the altitude, and the product will be the convex surface* (Book VII. Prop. I.). *To this add the area of the two bases, when the entire surface is required.*

1. To find the surface of a cube, the length of each side being 20 feet. *Ans.* 2400 *sq. ft.*

2. To find the whole surface of a triangular prism, whose base is an equilateral triangle, having each of its sides equal to 18 inches, and altitude 20 feet. *Ans.* 91.949.

3. What must be paid for lining a rectangular cistern with lead at 2*d.* a pound, the thickness of the lead being such as to require 7*lbs.* for each square foot of surface; the inner dimensions of the cistern being as follows, viz. the length 3 feet 2 inches, the breadth 2 feet 8 inches, and the depth 2 feet 6 inches? *Ans.* 2*l.* 3*s.* $10\frac{5}{9}$*d.*

PROBLEM II.

To find the surface of a regular pyramid.

Rule.—*Multiply the perimeter of the base by half the slant height, and the product will be the convex surface* (Book VII. Prop. IV.): *to this add the area of the base, when the entire surface is required.*

1. To find the convex surface of a regular triangular pyramid, the slant height being 20 feet, and each side of the base 3 feet. *Ans.* 90 *sq. ft.*

2. What is the entire surface of a regular pyramid, whose slant height is 15 feet, and the base a pentagon, of which each side is 25 feet? *Ans.* 2012.798.

PROBLEM III.

To find the convex surface of the frustum of a regular pyramid.

Rule.—*Multiply the half-sum of the perimeters of the two bases by the slant height of the frustum, and the product will be the convex surface* (Book VII. Prop. IV. Cor.).

1. How many square feet are there in the convex surface of the frustum of a square pyramid, whose slant height is 10 feet, each side of the lower base 3 feet 4 inches, and each side of the upper base 2 feet 2 inches? *Ans.* 110 *sq. ft.*

2. What is the convex surface of the frustum of an heptagonal pyramid whose slant height is 55 feet, each side of the lower base 8 feet, and each side of the upper base 4 feet? *Ans.* 2310 *sq. ft.*

PROBLEM IV

To find the solidity of a prism.

RULE.—1. *Find the area of the base.*

2. *Multiply the area of the base by the altitude, and the product will be the solidity of the prism* (Book VII. Prop. XIV.).

1. What is the solid content of a cube whose side is 24 inches? *Ans.* 13824.

2. How many cubic feet in a block of marble, of which the length is 3 feet 2 inches, breadth 2 feet 8 inches, and height or thickness 2 feet 6 inches? *Ans.* $21\frac{1}{9}$.

3. How many gallons of water, ale measure, will a cistern contain, whose dimensions are the same as in the last example? *Ans.* $129\frac{1}{4}\frac{7}{7}$.

4. Required the solidity of a triangular prism, whose height is 10 feet, and the three sides of its triangular base 3, 4, and 5 feet. *Ans.* 60.

PROBLEM V.

To find the solidity of a pyramid.

RULE.—*Multiply the area of the base by one-third of the altitude, and the product will be the solidity* (Book VII. Prop. XVII.).

1. Required the solidity of a square pyramid, each side of its base being 30, and the altitude 25. *Ans.* 7500.

2. To find the solidity of a triangular pyramid, whose altitude is 30, and each side of the base 3 feet. *Ans.* 38.9711.

3. To find the solidity of a triangular pyramid, its altitude being 14 feet 6 inches, and the three sides of its base 5, 6, and 7 feet. *Ans.* 71.0352.

4. What is the solidity of a pentagonal pyramid, its altitude being 12 feet, and each side of its base 2 feet? *Ans.* 27.5276.

5. What is the solidity of an hexagonal pyramid, whose altitude is 6.4 feet, and each side of its base 6 inches? *Ans.* 1.38564.

PROBLEM VI.

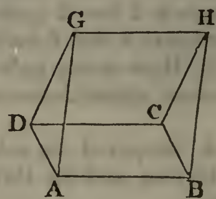
To find the solidity of the frustum of a pyramid.

Rule.—*Add together the areas of the two bases of the frustum and a mean proportional between them, and then multiply the sum by one-third of the altitude* (Book VII. Prop. XVIII.).

1. To find the number of solid feet in a piece of timber, whose bases are squares, each side of the lower base being 15 inches, and each side of the upper base 6 inches, the altitude being 24 feet. *Ans.* 19.5.

2. Required the solidity of a pentagonal frustum, whose altitude is 5 feet, each side of the lower base 18 inches, and each side of the upper base 6 inches. *Ans.* 9.31925.

Definitions.



1. *A wedge* is a solid bounded by five planes: viz. a rectangle ABCD, called the base of the wedge; two trapezoids ABHG, DCHG, which are called the sides of the wedge, and which intersect each other in the edge GH; and the two triangles GDA, HCB, which are called the ends of the wedge.

When AB, the length of the base, is equal to GH, the trapezoids ABHG, DCHG, become parallelograms, and the wedge is then one-half the parallelopipedon described on the base ABCD, and having the same altitude with the wedge.

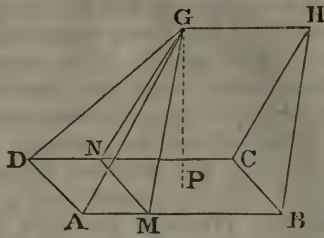
The altitude of the wedge is the perpendicular let fall from any point of the line GH, on the base ABCD.

2. *A rectangular prismoid* is a solid resembling the frustum of a quadrangular pyramid. The upper and lower bases are rectangles, having their corresponding sides parallel, and the convex surface is made up of four trapezoids. The altitude of the prismoid is the perpendicular distance between its bases.

PROBLEM VII.

To find the solidity of a wedge.

Rule.—*To twice the length of the base add the length of the edge. Multiply this sum by the breadth of the base, and then by the altitude of the wedge, and take one-sixth of the product for the solidity.*



Let $L=AB$, the length of the base.

$l=GH$, the length of the edge.

$b=BC$, the breadth of the base.

$h=PG$, the altitude of the wedge.

Then, $L-l=AB-GH=AM$.

Suppose AB, the length of the base, to be equal to GH, the length of the edge, the solidity will then be equal to half the parallelopipedon having the same base and the same altitude (Book VII. Prop. VII.). Hence, the solidity will be equal to $\frac{1}{2}blh$ (Book VII. Prop. XIV.).

If the length of the base is greater than that of the edge, let a section MNG be made parallel to the end BCH. The wedge will then be divided into the triangular prism BCH–M, and the quadrangular pyramid G–AMND.

The solidity of the prism $=\frac{1}{2}bhl$, the solidity of the pyramid $=\frac{1}{3}bh(L-l)$; and their sum, $\frac{1}{2}bhl+\frac{1}{3}bh(L-l)=\frac{1}{6}bh3l+\frac{1}{6}bh2L-\frac{1}{6}bh2l=\frac{1}{6}bh(2L+l)$.

If the length of the base is less than the length of the edge, the solidity of the wedge will be equal to the difference between the prism and pyramid, and we shall have for the solidity of the wedge,

$$\tfrac{1}{2}bhl-\tfrac{1}{3}bh(l-L)=\tfrac{1}{6}bh3l-\tfrac{1}{6}bh2l+\tfrac{1}{6}bh2L=\tfrac{1}{6}bh(2L+l).$$

1. If the base of a wedge is 40 by 20 feet, the edge 35 feet, and the altitude 10 feet, what is the solidity?

Ans. 3833.33.

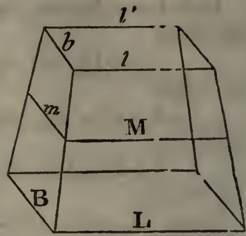
2. The base of a wedge being 18 feet by 9, the edge 20 feet, and the altitude 6 feet, what is the solidity?

Ans. 504.

PROBLEM VIII.

To find the solidity of a rectangular prismoid.

Rule.—*Add together the areas of the two bases and four times the area of a parallel section at equal distances from the bases: then multiply the sum by one-sixth of the altitude.*



Let L and B be the length and breadth of the lower base, l and b the length and breadth of the upper base, M and m the length and breadth of the section equidistant from the bases, and h the altitude of the prismoid.

Through the diagonal edges L and l let a plane be passed, and it will divide the prismoid into two wedges, having for bases, the bases of the prismoid, and for edges the lines L and $l'=l$.

The solidity of these wedges, and consequently of the prismoid, is

$$\tfrac{1}{6}Bh(2L+l)+\tfrac{1}{6}bh(2l+L)=\tfrac{1}{6}h(2BL+Bl+2bl+bL).$$

But since M is equally distant from L and l, we have

$$2M=L+l, \quad \text{and } 2m=B+b;$$

hence, $\quad 4Mm=(L+l)\times(B+b)=BL+Bl+bL+bl.$

Substituting 4Mm for its value in the preceding equation, and we have for the solidity

$$\tfrac{1}{6}h(BL+bl+4Mm).$$

Remark.—This rule may be applied to any prismoid whatever. For, whatever be the form of the bases, there may be inscribed in each the same number of rectangles, and the number of these rectangles may be made so great that their sum in each base will differ from that base, by less than any assignable quantity. Now, if on these rectangles, rectangular prismoids be constructed, their sum will differ from the given prismoid by less than any assignable quantity. Hence the rule is general.

1. One of the bases of a rectangular prismoid is 25 feet by 20, the other 15 feet by 10, and the altitude 12 feet; required the solidity. *Ans.* 3700.

2. What is the solidity of a stick of hewn timber whose ends are 30 inches by 27, and 24 inches by 18, its length being 24 feet? *Ans.* 102 feet

OF THE MEASURES OF THE THREE ROUND BODIES.

PROBLEM IX.

To find the surface of a cylinder.

Rule.—*Multiply the circumference of the base by the altitude, and the product will be the convex surface* (Book VIII. Prop I.). *To this add the areas of the two bases, when the entire surface is required.*

1 What is the convex surface of a cylinder, the diameter of whose base is 20, and whose altitude is 50?

Ans. 3141.6.

2. Required the entire surface of a cylinder, whose altitude is 20 feet, and the diameter of its base 2 feet.

Ans. 131.9472.

PROBLEM X.

To find the convex surface of a cone.

Rule.—*Multiply the circumference of the base by half the side* (Book VIII. Prop. III.) : *to which add the area of the base, when the entire surface is required.*

1. Required the convex surface of a cone, whose side is 50 feet, and the diameter of its base $8\frac{1}{2}$ feet. *Ans.* 667.59.

2. Required the entire surface of a cone, whose side is 36 and the diameter of its base 18 feet. *Ans.* 1272.348.

PROBLEM XI.

To find the surface of the frustum of a cone.

Rule.—*Multiply the side of the frustum by half the sum of the circumferences of the two bases, for the convex surface* (Book VIII. Prop. IV.) : *to which add the areas of the two bases, when the entire surface is required.*

1. To find the convex surface of the frustum of a cone, the side of the frustum being $12\frac{1}{2}$ feet, and the circumferences of the bases 8.4 feet and 6 feet. *Ans.* 90.

2. To find the entire surface of the frustum of a cone, the side being 16 feet, and the radii of the bases 3 feet and 2 feet.

Ans. 292.1688.

PROBLEM XII.

To find the solidity of a cylinder.

Rule.—*Multiply the area of the base by the altitude* (Book VIII. Prop. II.).

1. Required the solidity of a cylinder whose altitude is 12 feet, and the diameter of its base 15 feet. *Ans.* 2120.58.

2. Required the solidity of a cylinder whose altitude is 20 feet, and the circumference of whose base is 5 feet 6 inches.

Ans. 48.144.

PROBLEM XIII.

To find the solidity of a cone.

RULE.—*Multiply the area of the base by the altitude, and take one-third of the product* (Book VIII. Prop. V.).

1. Required the solidity of a cone whose altitude is 27 feet, and the diameter of the base 10 feet. *Ans.* 706.86.

2. Required the solidity of a cone whose altitude is $10\frac{1}{2}$ feet, and the circumference of its base 9 feet. *Ans.* 22.56.

PROBLEM XIV.

To find the solidity of the frustum of a cone.

RULE.—*Add together the areas of the two bases and a mean proportional between them, and then multiply the sum by one-third of the altitude* (Book VIII. Prop. VI.).

1. To find the solidity of the frustum of a cone, the altitude being 18, the diameter of the lower base 8, and that of the upper base 4. *Ans.* 527.7888.

2. What is the solidity of the frustum of a cone, the altitude being 25, the circumference of the lower base 20, and that of the upper base 10? *Ans.* 464.216.

3. If a cask, which is composed of two equal conic frustums joined together at their larger bases, have its bung diameter 28 inches, the head diameter 20 inches, and the length 40 inches how many gallons of wine will it contain, there being 231 cubic inches in a gallon? *Ans.* 79.0613.

PROBLEM XV.

To find the surface of a sphere.

RULE I.—*Multiply the circumference of a great circle by the diameter* (Book VIII. Prop. X.).

RULE II.—*Multiply the square of the diameter, or four times the square of the radius, by* 3.1416 (Book VIII. Prop. X. Cor.).

1. Required the surface of a sphere whose diameter is 7. *Ans.* 153.9384.

2. Required the surface of a sphere whose diameter is 24 inches. *Ans.* 1809.5616 *in.*

3. Required the area of the surface of the earth, its diameter being 7921 miles. *Ans.* 197111024 *sq. miles.*

4. What is the surface of a sphere, the circumference of its great circle being 78.54? *Ans.* 1963.5.

PROBLEM XVI.

To find the surface of a spherical zone.

RULE.—*Multiply the altitude of the zone by the circumference of a great circle of the sphere, and the product will be the surface* (Book VIII. Prop. X. Sch. 1).

1. The diameter of a sphere being 42 inches, what is the convex surface of a zone whose altitude is 9 inches?

Ans. 1187.5248 *sq. in.*

2. If the diameter of a sphere is $12\frac{1}{2}$ feet, what will be the surface of a zone whose altitude is 2 feet?

Ans. 78.54 *sq. ft.*

PROBLEM XVII.

To find the solidity of a sphere.

RULE I.—*Multiply the surface by one-third of the radius* (Book VIII. Prop. XIV.).

RULE II.—*Cube the diameter, and multiply the number thus found by* $\frac{1}{6}\pi$: *that is, by* 0.5236 (Book VIII. Prop. XIV. Sch. 3).

1. What is the solidity of a sphere whose diameter is 12?

Ans. 904.7808.

2. What is the solidity of the earth, if the mean diameter be taken equal to 7918.7 miles? *Ans.* 259992792083.

PROBLEM XVIII.

To find the solidity of a spherical segment.

RULE.—*Find the areas of the two bases, and multiply their sum by half the height of the segment; to this product add the solidity of a sphere whose diameter is equal to the height of the segment* (Book VIII. Prop. XVII.).

REMARK.—When the segment has but one base, the other is to be considered equal to 0 (Book VIII. Def. 14).

1. What is the solidity of a spherical segment, the diameter of the sphere being 40, and the distances from the centre to the bases, 16 and 10. *Ans.* 4297.7088.

2. What is the solidity of a spherical segment with one base the diameter of the sphere being 8, and the altitude of the segment 2 feet? *Ans.* 41.888.

3. What is the solidity of a spherical segment with one base, the diameter of the sphere being 20, and the altitude of the segment 9 feet? *Ans.* 1781.2872.

PROBLEM XIX.

To find the surface of a spherical triangle.

Rule.—1. *Compute the surface of the sphere on which the triangle is formed, and divide it by* 8; *the quotient will be the surface of the tri-rectangular triangle.*

2. *Add the three angles together; from their sum subtract* 180°, *and divide the remainder by* 90°: *then multiply the tri-rectangular triangle by this quotient, and the product will be the surface of the triangle* (Book IX. Prop. XX.).

1. Required the surface of a triangle described on a sphere whose diameter is 30 feet, the angles being 140°, 92°, and 68°. *Ans.* 471.24 *sq. ft.*

2. Required the surface of a triangle described on a sphere of 20 feet diameter, the angles being 120° each. *Ans.* 314.16 *sq. ft.*

PROBLEM XX.

To find the surface of a spherical polygon.

Rule.—1. *Find the tri-rectangular triangle, as before.*

2. *From the sum of all the angles take the product of two right angles by the number of sides less two. Divide the remainder by* 90°, *and multiply the tri-rectangular triangle by the quotient: the product will be the surface of the polygon* (Book IX. Prop. XXI.).

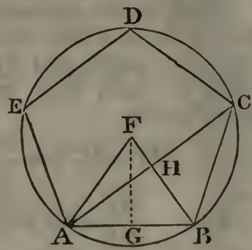
1. What is the surface of a polygon of seven sides, described on a sphere whose diameter is 17 feet, the sum of the angles being 1080°? *Ans.* 226.98.

2. What is the surface of a regular polygon of eight sides, described on a sphere whose diameter is 30, each angle of the polygon being 140°? *Ans.* 157.08

OF THE REGULAR POLYEDRONS.

In determining the solidities of the regular polyedrons, it becomes necessary to know, for each of them, the angle contained between any two of the adjacent faces. The determination of this angle involves the following property of a regular polygon, viz.—

Half the diagonal which joins the extremities of two adjacent sides of a regular polygon, is equal to the side of the polygon multiplied by the cosine of the angle which is obtained by dividing 360° *by twice the number of sides: the radius being equal to unity.*



Let ABCDE be any regular polygon. Draw the diagonal AC, and from the centre F draw FG, perpendicular to AB. Draw also AF, FB; the latter will be perpendicular to the diagonal AC, and will bisect it at H (Book III. Prop. VI. Sch.).

Let the number of sides of the polygon be designated by n: then,

$$\mathrm{AFB}=\frac{360°}{n}, \qquad \text{and } \mathrm{AFG}=\mathrm{CAB}=\frac{360°}{2n}.$$

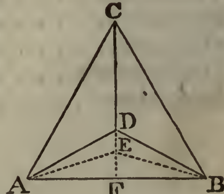
But in the right-angled triangle ABH, we have

$$\mathrm{AH}=\mathrm{AB}\cos \mathrm{A}=\mathrm{AB}\cos\frac{360°}{2n} \quad \text{(Trig. Th. I. Cor.)}$$

Remark 1.—When the polygon in question is the equilateral triangle, the diagonal becomes a side, and consequently half the diagonal becomes half a side of the triangle.

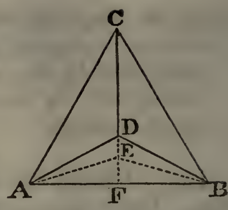
Remark 2.—The perpendicular $\mathrm{BH}=\mathrm{AB}\sin\frac{360°}{2n}$ (Trig. Th. I. Cor.).

To determine the angle included between the two adjacent faces of either of the regular polyedrons, let us suppose a plane to be passed perpendicular to the axis of a solid angle, and through the vertices of the solid angles which lie adjacent. This plane will intersect the convex surface of the polyedron in a regular polygon; the number of sides of this polygon will be equal to the number of planes which meet at the vertex of either of the solid angles, and each side will be a diagonal of one of the equal faces of the polyedron.



Let D be the vertex of a solid angle, CD the intersection of two adjacent faces, and ABC the section made in the convex surface of the polyedron by a plane perpendicular to the axis through D.

Through AB let a plane be drawn perpendicular to CD, produced if necessary, and suppose AE, BE, to be the lines in which this plane intersects the adjacent faces. Then will AEB be the angle included between the adjacent faces, and FEB will be half that angle, which we will represent by $\frac{1}{2}$A.



Then, if we represent by n the number of faces which meet at the vertex of the solid angle, and by m the number of sides of each face, we shall have, from what has already been shown,

$$BF=BC\cos\frac{360°}{2n}, \quad \text{and } EB=BC\sin\frac{360°}{2m}.$$

But $\dfrac{BF}{EB}=\sin FEB=\sin\frac{1}{2}A$, to the radius of unity:

hence,
$$\sin\tfrac{1}{2}A=\frac{\cos\dfrac{360°}{2n}}{\sin\dfrac{360°}{2m}}.$$

This formula gives, for the plane angle formed by every two adjacent faces of the

Tetraedron	70° 31′ 42″
Hexaedron	90°
Octaedron	109° 28′ 18″
Dodecaedron	116° 33′ 54″
Icosaedron	138° 11′ 23″

Having thus found the angle included between the adjacent faces, we can easily calculate the perpendicular let fall from the centre of the polyedron on one of its faces, when the faces themselves are known.

The following table shows the solidities and surfaces of the regular polyedrons, when the edges are equal to 1.

A TABLE OF THE REGULAR POLYEDRONS WHOSE EDGES ARE 1.

Names.	No. of Faces.	Surface.	Solidity.
Tetraedron	4	1.7320508	0.1178513
Hexaedron	6	6.0000000	1.0000000
Octaedron.	8	3.4641016 . . .	0.4714045
Dodecaedron	12	20.6457288	7.6631189
Icosaedron	20	8.6602540	2.1816950

PROBLEM XXI.

To find the solidity of a regular polyedron.

Rule I.—*Multiply the surface by one-third of the perpendicular let fall from the centre on one of the faces, and the product will be the solidity.*

Rule II.—*Multiply the cube of one of the edges by the solidity of a similar polyedron, whose edge is* 1.

The first rule results from the division of the polyedron into as many equal pyramids as it has faces. The second is proved by considering that two regular polyedrons having the same number of faces may be divided into an equal number of similar pyramids, and that the sum of the pyramids which make up one of the polyedrons will be to the sum of the pyramids which make up the other polyedron, as a pyramid of the first sum to a pyramid of the second (Book II. Prop. X.); that is, as the cubes of their homologous edges (Book VII. Prop. XX.); that is, as the cubes of the edges of the polyedron.

1. What is the solidity of a tetraedron whose edge is 15?
Ans. 397.75.
2. What is the solidity of a hexaedron whose edge is 12?
Ans. 1728.
3. What is the solidity of a octaedron whose edge is 20?
Ans. 3771.236.
4. What is the solidity of a dodecaedron whose edge is 25?
Ans. 119736.2328.
5. What is the solidity of an icosaedron whose side is 20?
Ans. 17453.56

A TABLE

OF

LOGARITHMS OF NUMBERS

FROM 1 TO 10,000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	26	1.414973	51	1.707570	76	1.880814
2	0.301030	27	1.431364	52	1.716003	77	1.886491
3	0.477121	28	1.447158	53	1.724276	78	1.892095
4	0.602060	29	1.462398	54	1.732394	79	1.897627
5	0.698970	30	1.477121	55	1.740363	80	1.903090
6	0.778151	31	1.491362	56	1.748188	81	1.908485
7	0.845098	32	1.505150	57	1.755875	82	1.913814
8	0.903090	33	1.518514	58	1.763428	83	1.919078
9	0.954243	34	1.531479	59	1.770852	84	1.924279
10	1.000000	35	1.544068	60	1.778151	85	1.929419
11	1.041393	36	1.556303	61	1.785330	86	1.934498
12	1.079181	37	1.568202	62	1.792392	87	1.939519
13	1.113943	38	1.579784	63	1.799341	88	1.944483
14	1.146128	39	1.591065	64	1.806180	89	1.949390
15	1.176091	40	1.602060	65	1.812913	90	1.954243
16	1.204120	41	1.612784	66	1.819544	91	1.959041
17	1.230449	42	1.623249	67	1.826075	92	1.963788
18	1.255273	43	1.633468	68	1.832509	93	1.968483
19	1.278754	44	1.643453	69	1.838849	94	1.973128
20	1.301030	45	1.653213	70	1.845098	95	1.977724
21	1.322219	46	1.662758	71	1.851258	96	1.982271
22	1.342423	47	1.672098	72	1.857333	97	1.986772
23	1.361728	48	1.681241	73	1.863323	98	1.991226
24	1.380211	49	1.690196	74	1.869232	99	1.995635
25	1.397940	50	1.698970	75	1.875061	100	2.000000

N. B. In the following table, in the last nine columns of each page, where the first or leading figures change from 9's to 0's, points or dots are introduced instead of the 0's through the rest of the line, to catch the eye, and to indicate that from thence the annexed first two figures of the Logarithm in th second column stand in the next lower line.

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	.300	.724	1147	1570	1993	2415	424
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	419
104	7033	7451	7868	8284	8700	9116	9532	9947	.361	.775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	.195	.600	1004	1408	1812	2216	2619	3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	.207	.602	.998	396
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	389
112	9218	9606	9993	.380	.766	1153	1538	1924	2309	2694	386
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	.320	379
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	372
117	8186	8557	8928	9298	9668	..38	.407	.776	1145	1514	369
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	9543	9904	.266	.626	.987	1347	1707	2067	2426	360
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	.258	.611	.963	1315	1667	2018	2370	2721	3071	351
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	..26	34[illegible]
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	340
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	.253	338
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	113943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	.245	330
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	..12	323
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	315
138	9879	.194	.508	.822	1136	1450	1763	2076	2389	2702	314
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	146128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
141	9219	9527	9835	.142	.449	.756	1063	1370	1676	1982	307
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	.168	.469	.769	1068	301
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
151	8977	9264	9552	9839	.126	.413	.699	.985	1272	1558	287
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	..51	281
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	8657	8932	9206	9481	9755	..29	.303	.577	.850	1124	274
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
160	204120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
162	9515	9783	..51	.319	.586	.853	1121	1388	1654	1921	267
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	.193	256
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742	254
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
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176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	.176	245
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
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180	255273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
186	9513	9746	9980	.213	.446	.679	.912	1144	1377	1609	233
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	278754	8982	9211	9439	9667	9895	.123	.351	.578	.806	228
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
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193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	22[illegible]
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197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
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199	8853	9071	9289	9507	9725	9943	.161	.378	.595	.813	218
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202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
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207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	322219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
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214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	8456	8656	8855	9054	9253	9451	9650	9849	..47	.246	199
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222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
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225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
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229	9835	..25	.215	.404	.593	.783	.972	1161	1350	1539	189
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231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	9216	9401	9587	9772	9958	.143	.328	.513	.698	.883	185
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	..30	181
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241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
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245	9166	9343	9520	9698	9875	..51	.228	.405	.582	.759	177
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
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251	9674	9847	..20	.192	.365	.538	.711	.883	1056	1228	173
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	55.7	5688	5858	6029	6199	6370	171
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259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	414973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
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265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
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272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
273	6163	6322	6481	6640	6798	6957	7116	7275	7433	7592	159
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	9333	9491	9648	9806	9964	.122	.279	.437	.594	.752	158
276	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323	157
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
78	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
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283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
288	9392	9543	9694	9845	9995	.146	.296	.447	.597	.748	151
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292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
295	9822	9969	.116	.263	.410	.557	.704	.851	.998	1145	147
296	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610	146
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
300	477121	7266	7411	7555	7700	7844	7989	8133	8278	8422	145
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302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299	144
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
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305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
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307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
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309	9958	..99	.239	.380	.520	.661	.801	.941	1081	1222	140
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
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312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
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317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291	137
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319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	505150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
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324	510545	0679	0813	0947	1081	1215	1349	1482	1616	1750	134
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326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4414	133
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	518514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
331	9828	9959	..90	.221	.353	.484	.615	745	.876	1007	131
332	521138	1269	1400	1530	1661	1792	1922	2053	2183	2314	131
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334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
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349	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	124
350	544068	4192	4316	4440	4564	4688	4812	4936	5060	5183	124
351	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	124
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353	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
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355	550228	0351	0473	0595	0717	0840	0962	1084	1206	1328	122
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357	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
359	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
360	556303	6423	6544	6664	6785	6905	7026	7146	7267	7387	120
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367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
370	568202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
371	9374	9491	9608	9725	9842	9959	..76	.193	.309	.426	117
372	570543	0660	0776	0893	1010	1126	1243	1359	1476	1592	117
373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
374	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
375	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
376	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
377	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	115
378	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	115
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380	579784	9898	..12	.126	.241	.355	.469	.583	.697	.811	114
381	580925	1039	1153	1267	1381	1495	1608	1722	1836	1950	114
382	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	114
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	113
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	113
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	113
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	112
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389	9950	..61	.173	.284	.396	.507	.619	.730	.842	.953	112
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392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	111
393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	110
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	110
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	110
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398	9883	9992	.101	.210	.319	.428	.537	.646	755	.864	109
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410	612784	2890	2996	3102	3207	3313	3419	3525	3630	3736	106
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	106
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413	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	105
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	105
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
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449	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97
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454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	96
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462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
463	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	94
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470	672098	2190	2283	2375	2467	2560	2652	2744	2836	2929	92
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	92
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474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	92
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	91
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492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
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506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
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527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
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533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
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535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
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540	732394	2474	2555	2635	2715	2796	2876	2956	3037	3117	80
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545	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
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550	740363	0442	0521	0600	0678	0757	0836	0915	0994	1073	79
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552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2646	79
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
560	748188	8266	8343	8421	8498	8576	8653	8731	8808	8885	77
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562	9736	9814	9891	9968	..45	.123	.200	.277	.354	.431	77
563	750508	0586	0663	0740	0817	0894	0971	1048	1125	1202	77
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567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
570	755875	5951	6027	6103	6180	6256	6332	6408	6484	6560	76
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
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591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
599	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
600	778151	8224	8296	8368	8441	8513	8585	8658	8730	8802	72
601	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	72
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603	780317	0389	0461	0533	0605	0677	0749	0821	0893	0965	72
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606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
610	785330	5401	5472	5543	5615	5686	5757	5828	5899	5970	71
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
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614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
616	9581	9651	9722	9792	9863	9933	...4	..74	.144	.215	70
617	790285	0356	0426	0496	0567	0637	0707	0778	0848	0918	70
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
620	792392	2462	2532	2602	2672	2742	2812	2882	2952	3022	70
621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	70
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
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624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
630	799341	9409	9478	9547	9616	9685	9754	9823	9892	9961	69
631	800029	0098	0167	0236	0305	0373	0442	0511	0580	0648	69
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	69
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
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648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
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653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
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658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
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665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
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667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
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690	838849	8912	8975	9038	9101	9164	9227	9289	9352	9415	63
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694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	63
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703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
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706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
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708	850033	0095	0156	0217	0279	0340	0401	0462	0524	0585	61
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715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
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717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
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767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
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775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
776	9862	9918	9974	..30	..86	.141	.197	.253	.309	.365	56
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779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
780	892095	2150	2206	2262	2317	2373	2429	2484	2540	2595	56
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	56
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
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791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
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797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
800	903090	3144	3199	3253	3307	3361	3416	3470	3524	3578	54
801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
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803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
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808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
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812	9556	9610	9663	9716	9770	9823	9877	9930	9984	..37	53
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818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
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825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
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835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
840	924279	4331	4383	4434	4486	4538	4589	4641	4693	4744	52
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
850	929419	9470	9521	9572	9623	9674	9725	9776	9827	9879	51
851	9930	9981	..32	..83	.134	.185	.236	.287	.338	.389	51
852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898	51
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
860	934498	4549	4599	4650	4700	4751	4801	4852	4902	4953	50
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	50
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
868	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
870	939519	9569	9619	9669	9719	9769	9819	9869	9918	9968	50
871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467	50
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	50
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
880	944483	4532	4581	4631	4680	4729	4779	4828	4877	4927	49
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	949390	9439	9488	9536	9585	9634	9683	9731	9780	9829	49
891	9878	9926	9975	..24	..73	.121	.170	.219	.267	.316	49
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803	49
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
900	954243	4291	4339	4387	4435	4484	4532	4580	4628	4677	48
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	48
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	48
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	959041	9089	9137	9185	9232	9280	9328	9375	9423	9471	48
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
912	9995	..42	..90	.138	.185	.233	.280	.328	.376	.423	48
913	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	47
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	963788	3835	3882	3929	3977	4024	4071	4118	4165	4212	47
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6786	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	968483	8530	8576	8623	8670	8716	8763	8810	8856	8903	47
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
933	9882	9928	9975	..21	..68	.114	.161	.207	.254	.300	47
934	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765	46
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
940	973128	3174	3220	3266	3313	3359	3405	3451	3497	3543	46
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	977724	7769	7815	7861	7906	7952	7998	8043	8089	8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	982271	2316	2362	2407	2452	2497	2543	2588	2633	2678	45
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	986772	6817	6861	6906	6951	6996	7040	7085	7130	7175	45
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	9895	9939	9983	..28	..72	.117	.161	.206	.250	.294	44
978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738	44
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	991226	1270	1315	1359	1403	1448	1492	1536	1580	1625	44
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	995635	5679	5723	5767	5811	5854	5898	5942	5986	6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6	7	8	9	D.

A TABLE

OF

LOGARITHMIC

SINES AND TANGENTS

FOR EVERY

DEGREE AND MINUTE

OF THE QUADRANT.

N. B The minutes in the left-hand column of each page, increasing downwards, belong to the degrees at the top; and those increasing upwards, in the right-hand column, belong to the degrees below.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	0.000000		10.000000		0.000000		Infinite.	60
1	6.463726	501717	000000	00	6.463726	501717	13.536274	59
2	764756	293485	000000	00	764756	293483	235244	58
3	940847	208231	000000	00	940847	208231	059153	57
4	7.065786	161517	000000	00	7.065786	161517	12.934214	56
5	162696	131968	000000	00	162696	131969	837304	55
6	241877	111575	9.999999	01	241878	111578	758122	54
7	308824	96653	999999	01	308825	99653	691175	53
8	366816	85254	999999	01	366817	85254	633183	52
9	417968	76263	999999	01	417970	76263	582030	51
10	463725	68988	999998	01	463727	68988	536273	50
11	7.505118	62981	9.999998	01	7.505120	62981	12.494880	49
12	542906	57936	999997	01	542909	57933	457091	48
13	577668	53641	999997	01	577672	53642	422328	47
14	609853	49938	999996	01	609857	49939	390143	46
15	639816	46714	999996	01	639820	46715	360180	45
16	667845	43881	999995	01	667849	43882	332151	44
17	694173	41372	999995	01	694179	41373	305821	43
18	718997	39135	999994	01	719003	39136	280997	42
19	742477	37127	999993	01	742484	37128	257516	41
20	764754	35315	999993	01	764761	35136	235239	40
21	7.785943	33672	9.999992	01	7.785951	33673	12.214049	39
22	806146	32175	999991	01	806155	32176	193845	38
23	825451	30805	999990	01	825460	30806	174540	37
24	843934	29547	999989	02	843944	29549	156056	36
25	861662	28388	999988	02	861674	28390	138326	35
26	878695	27317	999988	02	878708	27318	121292	34
27	895085	26323	999987	02	895099	26325	104901	33
28	910879	25399	999986	02	910894	25401	089106	32
29	926119	24538	999985	02	926134	24540	073866	31
30	940842	23733	999983	02	940858	23735	059142	30
31	7.955082	22980	9.999982	02	7.955100	22981	12.044900	29
32	968870	22273	999981	02	968889	22275	031111	28
33	982233	21608	999980	02	982253	21610	017747	27
34	995198	20981	999979	02	995219	20983	004781	26
35	8.007787	20390	999977	02	8.007809	20392	11.992191	25
36	020021	19831	999976	02	020045	19833	979955	24
37	031919	19302	999975	02	031945	19305	968055	23
38	043501	18801	999973	02	043527	18803	956473	22
39	054781	18325	999972	02	054809	18327	945191	21
40	065776	17872	999971	02	065806	17874	934194	20
41	8.076500	17441	9.999969	02	8.076531	17444	11.923469	19
42	086965	17031	999968	02	086997	17034	913003	18
43	097183	16639	999966	02	097217	16642	902783	17
44	107167	16265	999964	03	107202	16268	892797	16
45	116926	15908	999963	03	116963	15910	883037	15
46	126471	15566	999961	03	126510	15568	873490	14
47	135810	15238	999959	03	135851	15241	864149	13
48	144953	14924	999958	03	144996	14927	855004	12
49	153907	14622	999956	03	153952	14627	846048	11
50	162681	14333	999954	03	162727	14336	837273	10
51	8.171280	14054	9.999952	03	8.171328	14057	11 828672	9
52	179713	13786	999950	03	179763	13790	820237	8
53	187985	13529	999948	03	188036	13532	811964	
54	196102	13280	999946	03	196156	13284	803844	6
55	204070	13041	999944	03	204126	13044	795874	5
56	211895	12810	999942	04	211953	12814	788047	4
57	219581	12587	999940	04	219641	12590	780359	3
58	227134	12372	999938	04	227195	12376	772805	2
59	234557	12164	999936	04	234621	12168	765379	1
60	241855	11963	999934	04	241921	11967	758079	0
	Cosine		Sine		Cotang.		Tang.	M.

89 Degrees.

M.	Sine	D.	Cosine	D.	Tang.	D	Cotang.	
0	8.241855	11963	9.999934	04	8.241921	11967	11.758079	60
1	249033	11768	999932	04	249102	11772	750898	59
2	256094	11580	999929	04	256165	11584	743835	58
3	263042	11398	999927	04	263115	11402	736885	57
4	269881	11221	999925	04	269956	11225	730044	56
5	276614	11050	999922	04	276691	11054	723309	55
6	283243	10883	999920	04	283323	10887	716677	54
7	289773	10721	999918	04	289856	10726	710144	53
8	296207	10565	999915	04	296292	10570	703708	52
9	302546	10413	999913	04	302634	10418	697366	51
10	308794	10266	999910	04	308884	10270	691116	50
11	8.314954	10122	9.999907	04	8.315046	10126	11.684954	49
12	321027	9982	999905	04	321122	9987	678878	48
13	327016	9847	999902	04	327114	9851	672886	47
14	332924	9714	999899	05	333025	9719	666975	46
15	338753	9586	999897	05	338856	9590	661144	45
16	344504	9460	999894	05	344610	9465	655390	44
17	350181	9338	999891	05	350289	9343	649711	43
18	355783	9219	999888	05	355895	9224	644105	42
19	361315	9103	999885	05	361430	9108	638570	41
20	366777	8990	999882	05	366895	8995	633105	40
21	8.372171	8880	9.999879	05	8.372292	8885	11.627708	39
22	377499	8772	999876	05	377622	8777	622378	38
23	382762	8667	999873	05	382889	8672	617111	37
24	387962	8564	999870	05	388092	8570	611908	36
25	393101	8464	999867	05	393234	8470	606766	35
26	398179	8366	999864	05	398315	8371	601685	34
27	403199	8271	999861	05	403338	8276	596662	33
28	408161	8177	999858	05	408304	8182	591696	32
29	413068	8086	999854	05	413213	8091	586787	31
30	417919	7996	999851	06	418068	8002	581932	30
31	8.422717	7909	9.999848	06	8.422869	7914	11.577131	29
32	427462	7823	999844	06	427618	7830	572382	28
33	432156	7740	999841	06	432315	7745	567685	27
34	436800	7657	999838	06	436962	7663	563038	26
35	441394	7577	999834	06	441560	7583	558440	25
36	445941	7499	999831	06	446110	7505	553890	24
37	450440	7422	999827	06	450613	7428	549387	23
38	454893	7346	999823	06	455070	7352	544930	22
39	459301	7273	999820	06	459481	7279	540519	21
40	463665	7200	999816	06	463849	7206	536151	20
41	8.467985	7129	9.999812	06	8.468172	7135	11.531828	19
42	472263	7060	999809	06	472454	7066	527546	18
43	476498	6991	999805	06	476693	6998	523307	17
44	480693	6924	999801	06	480892	6931	519108	16
45	484848	6859	999797	07	485050	6865	514950	15
46	488963	6794	999793	07	489170	6801	510830	14
47	493040	6731	999790	07	493250	6738	506750	13
48	497078	6669	999786	07	497293	6676	502707	12
49	501080	6608	999782	07	501298	6615	498702	11
50	505045	6548	999778	07	505267	6555	494733	10
51	8.508974	6489	9.999774	07	509200	6496	11.490800	9
52	512867	6431	999769	07	513098	6439	486902	8
53	516726	6375	999765	07	516961	6382	483039	7
54	520551	6319	999761	07	520790	6326	479210	6
55	524343	6264	999757	07	524586	6272	475414	5
56	528102	6211	999753	07	528349	6218	471651	4
57	531828	6158	999748	07	532080	6165	467920	3
58	635523	6106	999744	07	535779	6113	464221	2
59	539186	6055	999740	07	539447	6062	460553	1
60	542819	6004	999735	07	543084	6012	456916	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8.542819	6004	9.999735	07	8.543084	6012	11.456916	60
1	546422	5955	999731	07	546691	5962	453309	59
2	549995	5906	999726	07	550268	5914	449732	58
3	553539	5858	999722	08	553817	5866	446183	57
4	557054	5811	999717	08	557336	5819	442664	56
5	560540	5765	999713	08	560828	5773	439172	55
6	563999	5719	999708	08	564291	5727	435709	54
7	567431	5674	999704	08	567727	5682	432273	53
8	570836	5630	999699	08	571137	5638	428863	52
9	574214	5587	999694	08	574520	5595	425480	51
10	577566	5544	999689	08	577877	5552	422123	50
11	8.580892	5502	9.999685	08	8.581208	5510	11.418792	49
12	584193	5460	999680	08	584514	5468	415486	48
13	587469	5419	999675	08	587795	5427	412205	47
14	590721	5379	999670	08	591051	5387	408949	46
15	593948	5339	999665	08	594283	5347	405717	45
16	597152	5300	999660	08	597492	5308	402508	44
17	600332	5261	999655	08	600677	5270	399323	43
18	603489	5223	999650	08	603839	5232	396161	42
19	606623	5186	999645	09	606978	5194	393022	41
20	609734	5149	999640	09	610094	5158	389906	40
21	8.612823	5112	9.999635	09	8.613189	5121	11.386811	39
22	615891	5076	999629	09	616262	5085	383738	38
23	618937	5041	999624	09	619313	5050	380687	37
24	621962	5006	999619	09	622343	5015	377657	36
25	624965	4972	999614	09	625352	4981	374648	35
26	627948	4938	999608	09	628340	4947	371660	34
27	630911	4904	999603	09	631308	4913	368692	33
28	633854	4871	999597	09	634256	4880	365744	32
29	636776	4839	999592	09	637184	4848	362816	31
30	639680	4806	999586	09	640093	4816	359907	30
31	8.642563	4775	9.999581	09	8.642982	4784	11.357018	29
32	645428	4743	999575	09	645853	4753	354147	28
33	648274	4712	999570	09	648704	4722	351296	27
34	651102	4682	999564	09	651537	4691	348463	26
35	653911	4652	999558	10	654352	4661	345648	25
36	656702	4622	999553	10	657149	4631	342851	24
37	659475	4592	999547	10	659928	4602	340072	23
38	662230	4563	999541	10	662689	4573	337311	22
39	664968	4535	999535	10	665433	4544	334567	21
40	667689	4506	999529	10	668160	4526	331840	20
41	8.670393	4479	9.999524	10	8.670870	4488	11.329130	19
42	673080	4451	999518	10	673563	4461	326437	18
43	675751	4424	999512	10	676239	4434	323761	17
44	678405	4397	999506	10	678900	4417	321100	16
45	681043	4370	999500	10	681544	4380	318456	15
46	683665	4344	999493	10	684172	4354	315828	14
47	686272	4318	999487	10	686784	4328	313216	13
48	688863	4292	999481	10	689381	4303	310619	12
49	691438	4267	999475	10	691963	4277	308037	11
50	693998	4242	999469	10	694529	4252	305471	10
51	8.696543	4217	9.999463	11	8.697081	4228	11.302919	9
52	699073	4192	999456	11	699617	4203	300383	8
53	701589	4168	999450	11	702139	4179	297861	7
54	704090	4144	999443	11	704646	4155	295354	6
55	706577	4121	999437	11	707140	4132	292860	5
56	709049	4097	999431	11	709618	4108	290382	4
57	711507	4074	999424	11	712083	4085	287917	3
58	713952	4051	999418	11	714534	4062	285465	2
59	716383	4029	999411	11	716972	4040	283028	1
60	718800	4006	999404	11	719396	4017	280604	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8.718800	4006	9.999404	11	8.719396	4017	11.280604	60
1	721204	3984	999398	11	721806	3995	278194	59
2	723595	3962	999391	11	724204	3974	275796	58
3	725972	3941	999384	11	726588	3952	273412	57
4	728337	3919	999378	11	728959	3930	271041	56
5	730688	3898	999371	11	731317	3909	268683	55
6	733027	3877	999364	12	733663	3889	266337	54
7	735354	3857	999357	12	735996	3868	264004	53
8	737667	3836	999350	12	738317	3848	261683	52
9	739969	3816	999343	12	740626	3827	259374	51
10	742259	3796	999336	12	742922	3807	257078	50
11	8.744536	3776	9.999329	12	8.745207	3787	11.254793	49
12	746802	3756	999322	12	747479	3768	252521	48
13	749055	3737	999315	12	749740	3749	250260	47
14	751297	3717	999308	12	751989	3729	248011	46
15	753528	3698	999301	12	754227	3710	245773	45
16	755747	3679	999294	12	756453	3692	243547	44
17	757955	3661	999286	12	758668	3673	241332	43
18	760151	3642	999279	12	760872	3655	239128	42
19	762337	3624	999272	12	763065	3636	236935	41
20	764511	3606	999265	12	765246	3618	234754	40
21	8.766675	3588	9.999257	12	8.767417	3600	11.232583	39
22	768828	3570	999250	13	769578	3583	230422	38
23	770970	3553	999242	13	771727	3565	228273	37
24	773101	3535	999235	13	773866	3548	226134	36
25	775223	3518	999227	13	775995	3531	224005	35
26	777333	3501	999220	13	778114	3514	221886	34
27	779434	3484	999212	13	780222	3497	219778	33
28	781524	3467	999205	13	782320	3480	217680	32
29	783605	3451	999197	13	784408	3464	215592	31
30	785675	3431	999189	13	786486	3447	213514	30
31	8.787736	3418	9.999181	13	8.788554	3431	11.211446	29
32	789787	3402	999174	13	790613	3414	209387	28
33	791828	3386	999166	13	792662	3399	207338	27
34	793859	3370	999158	13	794701	3383	205299	26
35	795881	3354	999150	13	796731	3368	203269	25
36	797894	3339	999142	13	798752	3352	201248	24
37	799897	3323	999134	13	800763	3337	199237	23
38	801892	3308	999126	13	802765	3322	197235	22
39	803876	3293	999118	13	804758	3307	195242	21
40	805852	3278	999110	13	806742	3292	193258	20
41	8.807819	3263	9.999102	13	8.808717	3278	11.191283	19
42	809777	3249	999094	14	810683	3262	189317	18
43	811726	3234	999086	14	812641	3248	187359	17
44	813667	3219	999077	14	814589	3233	185411	16
45	815599	3205	999069	14	816529	3219	183471	15
46	817522	3191	999061	14	818461	3205	181539	14
47	819436	3177	999053	14	820384	3191	179616	13
48	821343	3163	999044	14	822298	3177	177702	12
49	823240	3149	999036	14	824205	3163	175795	11
50	825130	3135	999027	14	826103	3150	173897	10
51	8.827011	3122	9.999019	14	8.827992	3136	11.172008	9
52	828884	3108	999010	14	829874	3123	170126	8
53	830749	3095	999002	14	831748	3110	168252	7
54	832607	3082	998993	14	833613	3096	166387	6
55	834456	3069	998984	14	835471	3083	164529	5
56	836297	3056	998976	14	837321	3070	162679	4
57	838130	3043	998967	15	839163	3057	160837	3
58	839956	3030	998958	15	840998	3045	159002	2
59	841774	3017	998950	15	842825	3032	157175	1
60	843585	3000	998941	15	844644	3019	155356	0
	Cosine		Sine		Cotang.		Tang.	M.

86 Degrees

M.	Sine	D.	Cosine	D.	Tang.	D	Cotang.	
0	8.843585	3005	9.998941	15	8.844644	3019	11.155356	60
1	845387	2992	998932	15	846455	3007	153545	59
2	847183	2980	998923	15	848260	2995	151740	58
3	848971	2967	998914	15	850057	2982	149943	57
4	850751	2955	998905	15	851846	2970	148154	56
5	852525	2943	998896	15	853628	2958	146372	55
6	854291	2931	998887	15	855403	2946	144597	54
7	856049	2919	998878	15	857171	2935	142829	53
8	857801	2907	998869	15	858932	2923	141068	52
9	859546	2896	998860	15	860686	2911	139314	51
10	861283	2884	998851	15	862433	2900	137567	50
11	8.863014	2873	9.998841	15	8.864173	2888	11.135827	49
12	864738	2861	998832	15	865906	2877	134094	48
13	866455	2850	998823	16	867632	2866	132368	47
14	868165	2839	998813	16	869351	2854	130649	46
15	869868	2828	998804	16	871064	2843	128936	45
16	871565	2817	998795	16	872770	2832	127230	44
17	873255	2806	998785	16	874469	2821	125531	43
18	874938	2795	998776	16	876162	2811	123838	42
19	876615	2786	998766	16	877849	2800	122151	41
20	878285	2773	998757	16	879529	2789	120471	40
21	8.879949	2763	9.998747	16	8.881202	2779	11 118798	39
22	881607	2752	998738	16	882869	2768	117131	38
23	883258	2742	998728	16	884530	2758	115470	37
24	884903	2731	998718	16	886185	2747	113815	36
25	886542	2721	998708	16	887833	2737	112167	35
26	888174	2711	998699	16	889476	2727	110524	34
27	889801	2700	998689	16	891112	2717	108888	33
28	891421	2690	998679	16	892742	2707	107258	32
29	893035	2680	998669	17	894366	2697	105634	31
30	894643	2670	998659	17	895984	2687	104016	30
31	8.896246	2660	9.998649	17	8.897596	2677	11.102404	29
32	897842	2651	998639	17	899203	2667	100797	28
33	899432	2641	998629	17	900803	2658	099197	27
34	901017	2631	998619	17	902398	2648	097602	26
35	902596	2622	998609	17	903987	2638	096013	25
36	904169	2612	998599	17	905570	2629	094430	24
37	905736	2603	998589	17	907147	2620	092853	23
38	907297	2593	998578	17	908719	2610	091281	22
39	908853	2584	998568	17	910285	2601	089715	21
40	910404	2575	998558	17	911846	2592	088154	20
41	8.911949	2566	9.998548	17	8.913401	2583	11.086599	19
42	913488	2556	998537	17	914951	2574	085049	18
43	915022	2547	998527	17	916495	2565	083505	17
44	916550	2538	998516	18	918034	2556	081966	16
45	918073	2529	998506	18	919568	2547	080432	15
46	919591	2520	998495	18	921096	2538	078904	14
47	921103	2512	998485	18	922619	2530	077381	13
48	922610	2503	998474	18	924136	2521	075864	12
49	924112	2494	998464	18	925649	2512	074351	11
50	925609	2486	998453	18	927156	2503	072844	10
51	8.927100	2477	9.998442	18	8.928658	2495	11.071342	9
52	928587	2469	998431	18	930155	2486	069845	8
53	930068	2460	998421	18	931647	2478	068353	7
54	931544	2452	998410	18	933134	2470	066866	6
55	933015	2443	998399	18	934616	2461	065384	5
56	934481	2435	998388	18	936093	2453	063907	4
57	935942	2427	998377	18	937565	2445	062435	3
58	937398	2419	998366	18	939032	2437	060968	2
59	938850	2411	998355	18	940494	2430	059506	1
60	940296	2403	998344	18	941952	2421	058048	0
	Cosine		Sine		Cotang.		Tang.	M.

85 Degrees.

M	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8.940296	2403	9.998344	19	8.941952	2421	11.058048	60
1	941738	2394	998333	19	943404	2413	056596	59
2	943174	2387	998322	19	944852	2405	055148	58
3	944606	2379	998311	19	946295	2397	053705	57
4	946034	2371	998300	19	947734	2390	052266	56
5	947456	2363	998289	19	949168	2382	050832	55
6	948874	2355	998277	19	950597	2374	049403	54
7	950287	2348	998266	19	952021	2366	047979	53
8	951696	2340	998255	19	953441	2360	046559	52
9	953100	2332	998243	19	954856	2351	045144	51
10	954499	2325	998232	19	956267	2344	043733	50
11	8.955894	2317	9.998220	19	8.957674	2337	11.042326	49
12	957284	2310	998209	19	959075	2329	040925	48
13	958670	2302	998197	19	960473	2323	039527	47
14	960052	2295	998186	19	961866	2314	038134	46
15	961429	2288	998174	19	963255	2307	036745	45
16	962801	2280	998163	19	964639	2300	035361	44
17	964170	2273	998151	19	966019	2293	033981	43
18	965534	2266	998139	20	967394	2286	032606	42
19	966893	2259	998128	20	968766	2279	031234	41
20	968249	2252	998116	20	970133	2271	029867	40
21	8.969600	2244	9.998104	20	8.971496	2265	11.028504	39
22	970947	2238	998092	20	972855	2257	027145	38
23	972289	2231	998080	20	974209	2251	025791	37
24	973628	2224	998068	20	975560	2244	024440	36
25	974962	2217	998056	20	976906	2237	023094	35
26	976293	2210	998044	20	978248	2230	021752	34
27	977619	2203	998032	20	979586	2223	020414	33
28	978941	2197	998020	20	980921	2217	019079	32
29	980259	2190	998008	20	982251	2210	017749	31
30	981573	2183	997996	20	983577	2204	016423	30
31	8.982883	2177	9.997984	20	8.984899	2197	11.015101	29
32	984189	2170	997972	20	986217	2191	013783	28
33	985491	2163	997959	20	987532	2184	012468	27
34	986789	2157	997947	20	988842	2178	011158	26
35	988083	2150	997935	21	990149	2171	009851	25
36	989374	2144	997922	21	991451	2165	008549	24
37	990660	2138	997910	21	992750	2158	007250	23
38	991943	2131	997897	21	994045	2152	005955	22
39	993222	2125	997885	21	995337	2146	004663	21
40	994497	2119	997872	21	996624	2140	003376	20
41	8.995768	2112	9.997860	21	8.997908	2134	11.002092	19
42	997036	2106	997847	21	999188	2127	000812	18
43	998299	2100	997835	21	9.000465	2121	10.999535	17
44	999560	2094	997822	21	001738	2115	998262	16
45	9.000816	2087	997809	21	003007	2109	996993	15
46	002069	2082	997797	21	004272	2103	995728	14
47	003318	2076	997784	21	005534	2097	994466	13
48	004563	2070	997771	21	006792	2091	993208	12
49	005805	2064	997758	21	008047	2085	991953	11
50	007044	2058	997745	21	009298	2080	990702	10
51	9.008278	2052	9.997732	21	9.010546	2074	10.989454	9
52	009510	2046	997719	21	011790	2068	988210	8
53	010737	2040	997706	21	013031	2062	986969	7
54	011962	2034	997693	22	014268	2056	985732	6
55	013182	2029	997680	22	015502	2051	984498	5
56	014400	2023	997667	22	016732	2045	983268	4
57	015613	2017	997654	22	017959	2040	982041	3
58	016824	2012	997641	22	019183	2033	980817	2
59	018031	2006	997628	22	020403	2028	979597	1
60	019235	2000	997614	22	021620	2023	978380	0
	Cosine		Sine		Cotang.		Tang.	M

84 Degrees.

M	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.019235	2000	9.997614	22	9.021620	2023	10.978380	60
1	020435	1995	997601	22	022834	2017	977166	59
2	021632	1989	997588	22	024044	2011	975956	58
3	022825	1984	997574	22	025251	2006	974749	57
4	024016	1978	997561	22	026455	2000	973545	56
5	025203	1973	997547	22	027655	1995	972345	55
6	026386	1967	997534	23	028852	1990	971148	54
7	027567	1962	997520	23	030046	1985	969954	53
8	028744	1957	997507	23	031237	1979	968763	52
9	029918	1951	997493	23	032425	1974	967575	51
10	031089	1947	997480	23	033609	1969	966391	50
11	9.032257	1941	9.997466	23	9.034791	1964	10.965209	49
12	033421	1936	997452	23	035969	1958	964031	48
13	034582	1930	997439	23	037144	1953	962856	47
14	035741	1925	997425	23	038316	1948	961684	46
15	036896	1920	997411	23	039485	1943	960515	45
16	038048	1915	997397	23	040651	1938	959349	44
17	039197	1910	997383	23	041813	1933	958187	43
18	040342	1905	997369	23	042973	1928	957027	42
19	041485	1899	997355	23	044130	1923	955870	41
20	042625	1894	997341	23	045284	1918	954716	40
21	9.043762	1889	9.997327	24	9.046434	1913	10.953566	39
22	044895	1884	997313	24	047582	1908	952418	38
23	046026	1879	997299	24	048727	1903	951273	37
24	047154	1875	997285	24	049869	1898	950131	36
25	048279	1870	997271	24	051008	1893	948992	35
26	049400	1865	997257	24	052144	1889	947856	34
27	050519	1860	997242	24	053277	1884	946723	33
28	051635	1855	997228	24	054407	1879	945593	32
29	052749	1850	997214	24	055535	1874	944465	31
30	053859	1845	997199	24	056659	1870	943341	30
31	054966	1841	9.997185	24	9.057781	1865	10.942219	29
32	056071	1836	997170	24	058900	1869	941100	28
33	057172	1831	997156	24	060016	1855	939984	27
34	058271	1827	997141	24	061130	1851	938870	26
35	059367	1822	997127	24	062240	1846	937760	25
36	060460	1817	997112	24	063348	1842	936652	24
37	061551	1813	997098	24	064453	1837	935547	23
38	062639	1808	997083	25	065556	1833	934444	22
39	063724	1804	997068	25	066655	1828	933345	21
40	064806	1799	997053	25	067752	1824	932248	20
41	9.065885	1794	9.997039	25	9.068846	1819	10.931154	19
42	066962	1790	997024	25	069938	1815	930062	18
43	068036	1786	997009	25	071027	1810	928973	17
44	069107	1781	996994	25	072113	1806	927887	16
45	070176	1777	996979	25	073197	1802	926803	15
46	071242	1772	996964	25	074278	1797	925722	14
47	072306	1768	996949	25	075356	1793	924644	13
48	073366	1763	996934	25	076432	1789	923568	12
49	074424	1759	996919	25	077505	1784	922495	11
50	075480	1755	996904	25	078576	1780	921424	10
51	9.076533	1750	9.996889	25	9.079644	1776	10.920356	9
52	077583	1746	996874	25	080710	1772	919290	8
53	078631	1742	996858	25	081773	1767	918227	7
54	079676	1738	996843	25	082833	1763	917167	6
55	080719	1733	996828	25	083891	1759	916109	5
56	081759	1729	996812	26	084947	1755	915053	4
57	082797	1725	996797	26	086000	1751	914000	3
58	083832	1721	996782	26	087050	1747	912950	2
59	084864	1717	996766	26	088098	1743	911902	1
60	085894	1713	996751	26	089144	1738	910856	0
	Cosine		Sine		Cotang.		Tang	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.085894	1713	9.996751	26	9.089144	1738	10.910856	60
1	086922	1709	996735	26	090187	1734	909813	59
2	087947	1704	996720	26	091228	1730	908772	58
3	088970	1700	996704	26	092266	1727	907734	57
4	089990	1696	996688	26	093302	1722	906698	56
5	091008	1692	996673	26	094336	1719	905664	55
6	092024	1688	996657	26	095367	1715	904633	54
7	093037	1684	996641	26	096395	1711	903605	53
8	094047	1680	996625	26	097422	1707	902578	52
9	095056	1676	996610	26	098446	1703	901554	51
10	096062	1673	996594	26	099468	1699	900532	50
11	9.097065	1668	9.996578	27	9.100487	1695	10.899513	49
12	098066	1665	996562	27	101504	1691	898496	48
13	099065	1661	996546	27	102519	1687	897481	47
14	100062	1657	996530	27	103532	1684	896468	46
15	101056	1653	996514	27	104542	1680	895458	45
16	102048	1649	996498	27	105550	1676	894450	44
17	103037	1645	996482	27	106556	1672	893444	43
18	104025	1641	996465	27	107559	1669	892441	42
19	105010	1638	996449	27	108560	1665	891440	41
20	105992	1634	996433	27	109559	1661	890441	40
21	9.106973	1630	9.996417	27	9.110556	1658	10.889444	39
22	107951	1627	996400	27	111551	1654	888449	38
23	108927	1623	996384	27	112543	1650	887457	37
24	109901	1619	996368	27	113533	1646	886467	36
25	110873	1616	996351	27	114521	1643	885479	35
26	111842	1612	996335	27	115507	1639	884493	34
27	112809	1608	996318	27	116491	1636	883509	33
28	113774	1605	996302	28	117472	1632	882528	32
29	114737	1601	996285	28	118452	1629	881548	31
30	115698	1597	996269	28	119429	1625	880571	30
31	9.116656	1594	9.996252	28	9.120404	1622	10.879596	29
32	117613	1590	996235	28	121377	1618	878623	28
33	118567	1587	996219	28	122348	1615	877652	27
34	119519	1583	996202	28	123317	1611	876683	26
35	120469	1580	996185	28	124284	1607	875716	25
36	121417	1576	996168	28	125249	1604	874751	24
37	122362	1573	996151	28	126211	1601	873789	23
38	123306	1569	996134	28	127172	1597	872828	22
39	124248	1566	996117	28	128130	1594	871870	21
40	125187	1562	996100	28	129087	1591	870913	20
41	9.126125	1559	9.996083	29	9.130041	1587	10.869959	19
42	127060	1556	996066	29	130994	1584	869006	18
43	127993	1552	996049	29	131944	1581	868056	17
44	128925	1549	996032	29	132893	1577	867107	16
45	129854	1545	996015	29	133839	1574	866161	15
46	130781	1542	995998	29	134784	1571	865216	14
47	131706	1539	995980	29	135726	1567	864274	13
48	132630	1535	995963	29	136667	1564	863333	12
49	133551	1532	995946	29	137605	1561	862395	11
50	134470	1529	995928	29	138542	1558	861458	10
51	9.135387	1525	9.995911	29	9.139476	1555	10.860524	9
52	136303	1522	995894	29	140409	1551	859591	8
53	137216	1519	995876	29	141340	1548	858660	7
54	138128	1516	995859	29	142269	1545	857731	6
55	139037	1512	995841	29	143196	1542	856804	5
56	139944	1509	995823	29	144121	1539	855879	4
57	140850	1506	995806	29	145044	1535	854956	3
58	141754	1503	995788	29	145966	1532	854034	2
59	142655	1500	995771	29	146885	1529	853115	1
60	143555	1496	995753	29	147803	1526	852197	0
	Cosine		Sine		Cotang.		Tang.	M

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.143555	1496	9.995753	30	9.147803	1526	10.852197	60
1	144453	1493	995735	30	148718	1523	851282	59
2	145349	1490	995717	30	149632	1520	850368	58
3	146243	1487	995699	30	150544	1517	849456	57
4	147136	1484	995681	30	151454	1514	848546	56
5	148026	1481	995664	30	152363	1511	847637	55
6	148915	1478	995646	30	153269	1508	846731	54
7	149802	1475	995628	30	154174	1505	845826	53
8	150686	1472	995610	30	155077	1502	844923	52
9	151569	1469	995591	30	155978	1499	844022	21
10	152451	1466	995573	30	156877	1496	843123	50
11	9 153330	1463	9.995555	30	9.157775	1493	10.842225	49
12	154208	1460	995537	30	158671	1490	841329	48
13	155083	1457	995519	30	159565	1487	840435	47
14	155957	1454	995501	31	160457	1484	839543	46
15	156830	1451	995482	31	161347	1481	838653	45
16	157700	1448	995464	31	162236	1479	837764	44
17	158569	1445	995446	31	163123	1476	836877	43
18	159435	1442	995427	31	164008	1473	835992	42
19	160301	1439	995409	31	164892	1470	835108	41
20	161164	1436	995390	31	165774	1467	834226	40
21	9.162025	1433	9.995372	31	9.166654	1464	10.833346	39
22	162885	1430	995353	31	167532	1461	832468	38
23	163743	1427	995334	31	168409	1458	831591	37
24	164600	1424	995316	31	169284	1455	830716	36
25	165454	1422	995297	31	170157	1453	829843	35
26	166307	1419	995278	31	171029	1450	828971	34
27	167159	1416	995260	31	171899	1447	828101	33
28	168008	1413	995241	32	172767	1444	827233	32
29	168856	1410	995222	32	173634	1442	826366	31
30	169702	1407	995203	32	174499	1439	825501	30
31	9.170547	1405	9.995184	32	9.175362	1436	10.824638	29
32	171389	1402	995165	32	176224	1433	823776	28
33	172230	1399	995146	32	177084	1431	822916	27
34	173070	1396	995127	32	177942	1428	822058	26
35	173908	1394	995108	32	178799	1425	821201	25
36	174744	1391	995089	32	179655	1423	820345	24
37	175578	1388	995070	32	180508	1420	819492	23
38	176411	1386	995051	32	181360	1417	818640	22
39	177242	1383	995032	32	182211	1415	817789	21
40	178072	1380	995013	32	183059	1412	816941	20
41	9.178900	1377	9.994993	32	9.183907	1409	10.816093	19
42	179726	1374	994974	32	184752	1407	815248	18
43	180551	1372	994955	32	185597	1404	814403	17
44	181374	1369	994935	32	186439	1402	813561	16
45	182196	1366	994916	33	187280	1399	812720	15
46	183016	1364	994896	33	188120	1396	811880	14
47	183834	1361	994877	33	188958	1393	811042	13
48	184651	1359	994857	33	189794	1391	810206	12
49	185466	1356	994838	33	190629	1389	809371	11
50	186280	1353	994818	33	191462	1386	808538	10
51	9.187092	1351	9.994798	33	9.192294	1384	10.807706	9
52	187903	1348	994779	33	193124	1381	806876	8
53	188712	1346	994759	33	193953	1379	806047	7
54	189519	1343	994739	33	194780	1376	805220	6
55	190325	1341	994719	33	195606	1374	804394	5
56	191130	1338	994700	33	196430	1371	803570	4
57	191933	1336	994680	33	197253	1369	802747	3
58	192734	1333	994660	33	198074	1366	801926	2
59	193534	1330	994640	33	198894	1364	801106	1
60	194332	1328	994620	33	199713	1361	800287	0
	Cosine		Sine		Cotang.		Tang.	M.

81 Degrees.

M.	Sine	D.	Cosine	D	Tang.	D.	Cotang.	
0	9.194332	1328	9.994620	33	9.199713	1361	10.800287	60
1	195129	1326	994600	33	200529	1359	799471	59
2	195925	1323	994580	33	201345	1356	798655	58
3	196719	1321	994560	34	202159	1354	797841	57
4	197511	1318	994540	34	202971	1352	797029	56
5	198302	1316	994519	34	203782	1349	796218	55
6	199091	1313	994499	34	204592	1347	795408	54
7	199879	1311	994479	34	205400	1345	794600	53
8	200666	1308	994459	34	206207	1342	793793	52
9	201451	1306	994438	34	207013	1340	792987	51
10	202234	1304	994418	34	207817	1338	792183	50
11	9.203017	1301	9.994397	34	9.208619	1335	10.791381	49
12	203797	1299	994377	34	209420	1333	790580	48
13	204577	1296	994357	34	210220	1331	789780	47
14	205354	1294	994336	34	211018	1328	788982	46
15	206131	1292	994316	34	211815	1326	788185	45
16	206906	1289	994295	34	212611	1324	787389	44
17	207679	1287	994274	35	213405	1321	786595	43
18	208452	1285	994254	35	214198	1319	785802	42
19	209222	1282	994233	35	214989	1317	785011	41
20	209992	1280	994212	35	215780	1315	784220	40
21	9.210760	1278	9.994191	35	9.216568	1312	10.783432	39
22	211526	1275	994171	35	217356	1310	782644	38
23	212291	1273	994150	35	218142	1308	781858	37
24	213055	1271	994129	35	218926	1305	781074	36
25	213818	1268	994108	35	219710	1303	780290	35
26	214579	1266	994087	35	220492	1301	779508	34
27	215338	1264	994066	35	221272	1299	778728	33
28	216097	1261	994045	35	222052	1297	777948	32
29	216854	1259	994024	35	222830	1294	777170	31
30	217609	1257	994003	35	223606	1292	776394	30
31	9.218363	1255	9.993981	35	9.224382	1290	10.775618	29
32	219116	1253	993960	35	225156	1288	774844	28
33	219868	1250	993939	35	225929	1286	774071	27
34	220618	1248	993918	35	226700	1284	773300	26
35	221367	1246	993896	36	227471	1281	772529	25
36	222115	1244	993875	36	228239	1279	771761	24
37	222861	1242	993854	36	229007	1277	770993	23
38	223606	1239	993832	36	229773	1275	770227	22
39	224349	1237	993811	36	230539	1273	769461	21
40	225092	1235	993789	36	231302	1271	768698	20
41	9.225833	1233	9.993768	36	9.232065	1269	10.767935	19
42	226573	1231	993746	36	232826	1267	767174	18
43	227311	1228	993725	36	233586	1265	766414	17
44	228048	1226	993703	36	234345	1262	765655	16
45	228784	1224	993681	36	235103	1260	764897	15
46	229518	1222	993660	36	235859	1258	764141	14
47	230252	1220	993638	36	236614	1256	763386	13
48	230984	1218	993616	36	237368	1254	762632	12
49	231714	1216	993594	37	238120	1252	761880	11
50	232444	1214	993572	37	238872	1250	761128	10
51	9.233172	1212	9.993550	37	9.239622	1248	10.760378	9
52	233899	1209	993528	37	240371	1246	759629	8
53	234625	1207	993506	37	241118	1244	758882	7
54	235349	1205	993484	37	241865	1242	758135	6
55	236073	1203	993462	37	242610	1240	757390	5
56	236795	1201	993440	37	243354	1238	756646	4
57	237515	1199	993418	37	244097	1236	755903	3
58	238235	1197	993396	37	244839	1234	755161	2
59	238953	1195	993374	37	245579	1232	754421	1
60	239670	1193	993351	37	246319	1230	753681	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.239670	1193	9.993351	37	9.246319	1230	10.753681	60
1	240386	1191	993329	37	247057	1228	752943	59
2	241101	1189	993307	37	247794	1226	752206	58
3	241814	1187	993285	37	248530	1224	751470	57
4	242526	1185	993262	37	249264	1222	750736	56
5	243237	1183	993240	37	249998	1220	750002	55
6	243947	1181	993217	38	250730	1218	749270	54
7	244656	1179	993195	38	251461	1217	748539	53
8	245363	1177	993172	38	252191	1215	747809	52
9	246069	1175	993149	38	252920	1213	747080	51
10	246775	1173	993127	38	253648	1211	746352	50
11	9.247478	1171	9.993104	38	9.254374	1209	10.745626	49
12	248181	1169	993081	38	255100	1207	744900	48
13	248883	1167	993059	38	255824	1205	744176	47
14	249583	1165	993036	38	256547	1203	743453	46
15	250282	1163	993013	38	257269	1201	742731	45
16	250980	1161	992990	38	257990	1200	742010	44
17	251677	1159	992967	38	258710	1198	741290	43
18	252373	1158	992944	38	259429	1196	740571	42
19	253067	1156	992921	38	260146	1194	739854	41
20	253761	1154	992898	38	260863	1192	739137	40
21	9.254453	1152	9.992875	38	9.261578	1190	10.738422	39
22	255144	1150	992852	38	262292	1189	737708	38
23	255834	1148	992829	39	263005	1187	736995	37
24	256523	1146	992806	39	263717	1185	736283	36
25	257211	1144	992783	39	264428	1183	735572	35
26	257898	1142	992759	39	265138	1181	734862	34
27	258583	1141	992736	39	265847	1179	734153	33
28	259268	1139	992713	39	266555	1178	733445	32
29	259951	1137	992690	39	267261	1176	732739	31
30	260633	1135	992666	39	267967	1174	732033	30
31	9.261314	1133	9.992643	39	9.268671	1172	10.731329	29
32	261994	1131	992619	39	269375	1170	730625	28
33	262673	1130	992596	39	270077	1169	729923	27
34	263351	1128	992572	39	270779	1167	729221	26
35	264027	1126	992549	39	271479	1165	728521	25
36	264703	1124	992525	39	272178	1164	727822	24
37	265377	1122	992501	39	272876	1162	727124	23
38	266051	1120	992478	40	273573	1160	726427	22
39	266723	1119	992454	40	274269	1158	725731	21
40	267395	1117	992430	40	274964	1157	725036	20
41	9.268065	1115	9.992406	40	9.275658	1155	10.724342	19
42	268734	1113	992382	40	276351	1153	723649	18
43	269402	1111	992359	40	277043	1151	722957	17
44	270069	1110	992335	40	277734	1150	722266	16
45	270735	1108	992311	40	278424	1148	721576	15
46	271400	1106	992287	40	279113	1147	720887	14
47	272064	1105	992263	40	279801	1145	720199	13
48	272726	1103	992239	40	280488	1143	719512	12
49	273388	1101	992214	40	281174	1141	718826	11
50	274049	1099	992190	40	281858	1140	718142	10
51	9.274708	1098	9.992166	40	9.282542	1138	10.717458	9
52	275367	1096	992142	40	283225	1136	716775	8
53	276024	1094	992117	41	283907	1135	716093	7
54	276681	1092	992093	41	284588	1133	715412	6
55	277337	1091	992069	41	285268	1131	714732	5
56	277991	1089	992044	41	285947	1130	714053	4
57	278644	1087	992020	41	286624	1128	713376	3
58	279297	1086	991996	41	287301	1126	712699	2
59	279948	1084	991971	41	287977	1125	712023	1
60	280599	1082	991947	41	288652	1123	711348	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.280599	1082	9.991947	41	9.288652	1123	10.711348	60
1	281248	1081	991922	41	289326	1122	710674	59
2	281897	1079	991897	41	289999	1120	710001	58
3	282544	1077	991873	41	290671	1118	709329	57
4	283190	1076	991848	41	291342	1117	708658	56
5	283836	1074	991823	41	292013	1115	707987	55
6	284480	1072	991799	41	292682	1114	707318	54
7	285124	1071	991774	42	293350	1112	706650	53
8	285766	1069	991749	42	294017	1111	705983	52
9	286408	1067	991724	42	294684	1109	705316	51
10	287048	1066	991699	42	295349	1107	704651	50
11	9.287687	1064	9.991674	42	9.296013	1106	10.703987	49
12	288326	1063	991649	42	296677	1104	703323	48
13	288964	1061	991624	42	297339	1103	702661	47
14	289600	1059	991599	42	298001	1101	701999	46
15	290236	1058	991574	42	298662	1100	701338	45
16	290870	1056	991549	42	299322	1098	700678	44
17	291504	1054	991524	42	299980	1096	700020	43
18	292137	1053	991498	42	300638	1095	699362	42
19	292768	1051	991473	42	301295	1093	698705	41
20	293399	1050	991448	42	301951	1092	698049	40
21	9.294029	1048	9.991422	42	9.302607	1090	10.697393	39
22	294658	1046	991397	42	303261	1089	696739	38
23	295286	1045	991372	43	303914	1087	696086	37
24	295913	1043	991346	43	304567	1086	695433	36
25	296539	1042	991321	43	305218	1084	694782	35
26	297164	1040	991295	43	305869	1083	694131	34
27	297788	1039	991270	43	306519	1081	693481	33
28	298412	1037	991244	43	307168	1080	692832	32
29	299034	1036	991218	43	307815	1078	692185	31
30	299655	1034	991193	43	308463	1077	691537	30
31	9.300276	1032	9.991167	43	9.309109	1075	10.690891	29
32	300895	1031	991141	43	309754	1074	690246	28
33	301514	1029	991115	43	310398	1073	689602	27
34	302132	1028	991090	43	311042	1071	688958	26
35	302748	1026	991064	43	311685	1070	688315	25
36	303364	1025	991038	43	312327	1068	687673	24
37	303979	1023	991012	43	312967	1067	687033	23
38	304593	1022	990986	43	313608	1065	686392	22
39	305207	1020	990960	43	314247	1064	685753	21
40	305819	1019	990934	44	314885	1062	685115	20
41	9.306430	1017	9.990908	44	9.315523	1061	10.684477	19
42	307041	1016	990882	44	316159	1060	683841	18
43	307650	1014	990855	44	316795	1058	683205	17
44	308259	1013	990829	44	317430	1057	682570	16
45	308867	1011	990803	44	318064	1055	681936	15
46	309474	1010	990777	44	318697	1054	681303	14
47	310080	1008	990750	44	319329	1053	680671	13
48	310685	1007	990724	44	319961	1051	680039	12
49	311289	1005	990697	44	320592	1050	679408	11
50	311893	1004	990671	44	321222	1048	678778	10
51	9.312495	1003	9.990644	44	9.321851	1047	10.678149	9
52	313097	1001	990618	44	322479	1045	677521	8
53	313698	1000	990591	44	323106	1044	676894	7
54	314297	998	990565	44	323733	1043	676267	6
55	314897	997	990538	44	324358	1041	675642	5
56	315495	996	990511	45	324983	1040	675017	4
57	316092	994	990485	45	325607	1039	674393	3
58	316689	993	990458	45	326231	1037	673769	2
59	317284	991	990431	45	326853	1036	673147	1
60	317879	990	990404	45	327475	1035	672525	0
	Cosine		Sine		Cotang.		Tang.	

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.317879	990	9.990404	45	9.327474	1035	10 672526	60
1	318473	988	990378	45	328095	1033	671905	59
2	319066	987	990351	45	328715	1032	671285	58
3	319658	986	990324	45	329334	1030	670666	57
4	320249	984	990297	45	329953	1029	670047	56
5	320840	983	990270	45	330570	1028	669430	55
6	321430	982	990243	45	331187	1026	668813	54
7	322019	980	990215	45	331803	1025	668197	53
8	322607	979	990188	45	332418	1024	667582	52
9	323194	977	990161	45	333033	1023	666967	51
10	323780	976	990134	45	333646	1021	666354	50
11	9.324366	975	9.990107	46	9.334259	1020	10.665741	49
12	324950	973	990079	46	334871	1019	665129	48
13	325534	972	990052	46	335482	1017	664518	47
14	326117	970	990025	46	336093	1016	663907	46
15	326700	969	989997	46	336702	1015	663298	45
16	327281	968	989970	46	337311	1013	662689	44
17	327862	966	989942	46	337919	1012	662081	43
18	328442	965	989915	46	338527	1011	661473	42
19	329021	964	989887	46	339133	1010	660867	41
20	329599	962	989860	46	339739	1008	660261	40
21	9.330176	961	9.989832	46	9.340344	1007	10.659656	39
22	330753	960	989804	46	340948	1006	659052	38
23	331329	958	989777	46	341552	1004	658448	37
24	331903	957	989749	47	342155	1003	657845	36
25	332478	956	989721	47	342757	1002	657243	35
26	333051	954	989693	47	343358	1000	656642	34
27	333624	953	989665	47	343958	999	656042	33
28	334195	952	989637	47	344558	998	655442	32
29	334766	950	989609	47	345157	997	654843	31
30	335337	949	989582	47	345755	996	654245	30
31	9.335906	948	9.989553	47	9.346353	994	10.653647	29
32	336475	946	989525	47	346949	993	653051	28
33	337043	945	989497	47	347545	992	652455	27
34	337610	944	989469	47	348141	991	651859	26
35	338176	943	989441	47	348735	990	651265	25
36	338742	941	989413	47	349329	988	650671	24
37	339306	940	989384	47	349922	987	650078	23
38	339871	939	989356	47	350514	986	649486	22
39	340434	937	989328	47	351106	985	648894	21
40	340996	936	989300	47	351697	983	648303	20
41	9.341558	935	9.989271	47	9.352287	982	10.647713	19
42	342119	934	989243	47	352876	981	647124	18
43	342679	932	989214	47	353465	980	646535	17
44	343239	931	989186	47	354053	979	645947	16
45	343797	930	989157	47	354640	977	645360	15
46	344355	929	989128	48	355227	976	644773	14
47	344912	927	989100	48	355813	975	644187	13
48	345469	926	989071	48	356398	974	643602	12
49	346024	925	989042	48	356982	973	643018	11
50	346579	924	989014	48	357566	971	642434	10
51	9.347134	922	9.988985	48	9.358149	970	10.641851	9
52	347687	921	988956	48	358731	969	641269	8
53	348240	920	988927	48	359313	968	640687	7
54	348792	919	988898	48	359893	967	640107	6
55	349343	917	988869	48	360474	966	639526	5
56	349893	916	988840	48	361053	965	638947	4
57	350443	915	988811	49	361632	963	638368	3
58	350992	914	988782	49	362210	962	637790	2
59	351540	913	988753	49	362787	961	637213	1
60	352088	911	988724	49	363364	960	636636	0
	Cosine		Sine		Cotang.		Tang	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.352088	911	9.988724	49	9.363364	960	10.636636	60
1	352635	910	988695	49	363940	959	636060	59
2	353181	909	988666	49	364515	958	635485	58
3	353726	908	988636	49	365090	957	634910	57
4	354271	907	988607	49	365664	955	634336	56
5	354815	905	988578	49	366237	954	633763	55
6	355358	904	988548	49	366810	953	633190	54
7	355901	903	988519	49	367382	952	632618	53
8	356443	902	988489	49	367953	951	632047	52
9	356984	901	988460	49	368524	950	631476	51
10	357524	899	988430	49	369094	949	630906	50
11	9.358064	898	9.988401	49	9.369663	948	10.630337	49
12	358603	897	988371	49	370232	946	629768	48
13	359141	896	988342	49	370799	945	629201	47
14	359678	895	988312	50	371367	944	628633	46
15	360215	893	988282	50	371933	943	628067	45
16	360752	892	988252	50	372499	942	627501	44
17	361287	891	988223	50	373064	941	626936	43
18	361822	890	988193	50	373629	940	626371	42
19	362356	889	988163	50	374193	939	625807	41
20	362889	888	988133	50	374756	938	625244	40
21	9.363422	887	9.988103	50	9.375319	937	10.624681	39
22	363954	885	988073	50	375881	935	624119	38
23	364485	884	988043	50	376442	934	623558	37
24	365016	883	988013	50	377003	933	622997	36
25	365546	882	987983	50	377563	932	622437	35
26	366075	881	987953	50	378122	931	621878	34
27	366604	880	987922	50	378681	930	621319	33
28	367131	879	987892	50	379239	929	620761	32
29	367659	877	987862	50	379797	928	620203	31
30	368185	876	987832	51	380354	927	619646	30
31	9.368711	875	9.987801	51	9.380910	926	10.619090	29
32	369236	874	987771	51	381466	925	618534	28
33	369761	873	987740	51	382020	924	617980	27
34	370285	872	987710	51	382575	923	617425	26
35	370808	871	987679	51	383129	922	616871	25
36	371330	870	987649	51	383682	921	616318	24
37	371852	869	987618	51	384234	920	615766	23
38	372373	867	987588	51	384786	919	615214	22
39	372894	866	987557	51	385337	918	614663	21
40	373414	865	987526	51	385888	917	614112	20
41	9.373933	864	9.987496	51	9.386438	915	10.613562	19
42	374452	863	987465	51	386987	914	613013	18
43	374970	862	987434	51	387536	913	612464	17
44	375487	861	987403	52	388084	912	611916	16
45	376003	860	987372	52	388631	911	611369	15
46	376519	859	987341	52	389178	910	610822	14
47	377035	858	997310	52	389724	909	610276	13
48	377549	857	987279	52	390270	908	609730	12
49	378063	856	987248	52	390815	907	609185	11
50	378577	854	987217	52	391360	906	608640	10
51	9.379089	853	9.987186	52	9.391903	905	10.608097	9
52	379601	852	987155	52	392447	904	607553	8
53	380113	851	987124	52	392989	903	607011	7
54	380624	850	987092	52	393531	902	606469	6
55	381134	849	987061	52	394073	901	605927	5
56	381643	848	987030	52	394614	900	605386	4
57	382152	847	986998	52	395154	899	604846	3
58	382661	846	986967	52	395694	898	604306	2
59	383168	845	986936	52	396233	897	603767	1
60	383675	844	986904	52	396771	896	603229	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.383675	844	9.986904	52	9.396771	896	10.603229	60
1	384182	843	986873	53	397309	896	602691	59
2	384687	842	986841	53	397846	895	602154	58
3	385192	841	986809	53	398383	894	601617	57
4	385697	840	986778	53	398919	893	601081	56
5	386201	839	986746	53	399455	892	600545	55
6	386704	838	986714	53	399990	891	600010	54
7	387207	837	986683	53	400524	890	599476	53
8	387709	836	986651	53	401058	889	598942	52
9	388210	835	986619	53	401591	888	598409	51
10	388711	834	986587	53	402124	887	597876	50
11	9.389211	833	9.986555	53	9.402656	886	10.597344	49
12	389711	832	986523	53	403187	885	596813	48
13	390210	831	986491	53	403718	884	596282	47
14	390708	830	986459	53	404249	883	595751	46
15	391206	828	986427	53	404778	882	595222	45
16	391703	827	986395	53	405308	881	594692	44
17	392199	826	986363	54	405836	880	594164	43
18	392695	825	986331	54	406364	879	593636	42
19	393191	824	986299	54	406892	878	593108	41
20	393685	823	986266	54	407419	877	592581	40
21	9.394179	822	9.986234	54	9.407945	876	10.592055	39
22	394673	821	986202	54	408471	875	591529	38
23	395166	820	986169	54	408997	874	591003	37
24	395658	819	986137	54	409521	874	590479	36
25	396150	818	986104	54	410045	873	589955	35
26	396641	817	986072	54	410569	872	589431	34
27	397132	817	986039	54	411092	871	588908	33
28	397621	816	986007	54	411615	870	588385	32
29	398111	815	985974	54	412137	869	587863	31
30	398600	814	985942	54	412658	868	587342	30
31	9.399088	813	9.985909	55	9.413179	867	10.586821	29
32	399575	812	985876	55	413699	866	586301	28
33	400062	811	985843	55	414219	865	585781	27
34	400549	810	985811	55	414738	864	585262	26
35	401035	809	985778	55	415257	864	584743	25
36	401520	808	985745	55	415775	863	584225	24
37	402005	807	985712	55	416293	862	583707	23
38	402489	806	985679	55	416810	861	583190	22
39	402972	805	985646	55	417326	860	582674	21
40	403455	804	985613	55	417842	859	582158	20
41	9.403938	803	9.985580	55	9.418358	858	10.581642	19
42	404420	802	985547	55	418873	857	581127	18
43	404901	801	985514	55	419387	856	580613	17
44	405382	800	985480	55	419901	855	580099	16
45	405862	799	985447	55	420415	855	579585	15
46	406341	798	985414	56	420927	854	579073	14
47	406820	797	985380	56	421440	853	578560	13
48	407299	796	985347	56	421952	852	578048	12
49	407777	795	985314	56	422463	851	577537	11
50	408254	794	985280	56	422974	850	577026	10
51	9.408731	794	9.985247	56	9.423484	849	10.576516	9
52	409207	793	985213	56	423993	848	576007	8
53	409682	792	985180	56	424503	848	575497	7
54	410157	791	985146	56	425011	847	574989	6
55	410632	790	985113	56	425519	846	574481	5
56	411106	789	985079	56	426027	845	573973	4
57	411579	788	985045	56	426534	844	573466	3
58	412052	787	985011	56	427041	843	572959	2
59	412524	786	984978	56	427547	843	572453	1
60	412996	785	984944	56	428052	842	571948	0
	Cosine		Sine		Cotang.		Tang	M

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.412996	785	9.984944	57	9.428052	842	10.571948	60
1	413467	784	984910	57	428557	841	571443	59
2	413938	783	984876	57	429062	840	570938	58
3	414408	783	984842	57	429566	839	570434	57
4	414878	782	984808	57	430070	838	569930	56
5	415347	781	984774	57	430573	838	569427	55
6	415815	780	984740	57	431075	837	568925	54
7	416283	779	984706	57	431577	836	568423	53
8	416751	778	984672	57	432079	835	567921	52
9	417217	777	984637	57	432580	834	567420	51
10	417684	776	984603	57	433080	833	566920	50
11	9.418150	775	9.984569	57	9.433580	832	10.566420	49
12	418615	774	984535	57	434080	832	565920	48
13	419079	773	984500	57	434579	831	565421	47
14	419544	773	984466	57	435078	830	564922	46
15	420007	772	984432	58	435576	829	564424	45
16	420470	771	984397	58	436073	828	563927	44
17	420933	770	984363	58	436570	828	563430	43
18	421395	769	984328	58	437067	827	562933	42
19	421857	768	984294	58	437563	826	562437	41
20	422318	767	984259	58	438059	825	561941	40
21	9.422778	767	9.984224	58	9.438554	824	10.561446	39
22	423238	766	984190	58	439048	823	560952	38
23	423697	765	984155	58	439543	823	560457	37
24	424156	764	984120	58	440036	822	559964	36
25	424615	763	984085	58	440529	821	559471	35
26	425073	762	984050	58	441022	820	558978	34
27	425530	761	984015	58	441514	819	558486	33
28	425987	760	983981	58	442006	819	557994	32
29	426443	760	983946	58	442497	818	557503	31
30	426899	759	983911	58	442988	817	557012	30
31	9.427354	758	9.983875	58	9.443479	816	10.556521	29
32	427809	757	983840	59	443968	816	556032	28
33	428263	756	983805	59	444458	815	555542	27
34	428717	755	983770	59	444947	814	555053	26
35	429170	754	983735	59	445435	813	554565	25
36	429623	753	983700	59	445923	812	554077	24
37	430075	752	983664	59	446411	812	553589	23
38	430527	752	983629	59	446898	811	553102	22
39	430978	751	983594	59	447384	810	552616	21
40	431429	750	983558	59	447870	809	552130	20
41	9.431879	749	9.983523	59	9.448356	809	10.551644	19
42	432329	749	983487	59	448841	808	551159	18
43	432778	748	983452	59	449326	807	550674	17
44	433226	747	983416	59	449810	806	550190	16
45	433675	746	983381	59	450294	806	549706	15
46	434122	745	983345	59	450777	805	549223	14
47	434569	744	983309	59	451260	804	548740	13
48	435016	744	983273	60	451743	803	548257	12
49	435462	743	983238	60	452225	802	547775	11
50	435908	742	983202	60	452706	802	547294	10
51	9.436353	741	9.983166	60	9.453187	801	10.546813	9
52	436798	740	983130	60	453668	800	546332	8
53	437242	740	983094	60	454148	799	545852	7
54	437686	739	983058	60	454628	799	545372	6
55	438129	738	983022	60	455107	798	544893	5
56	438572	737	982986	60	455586	797	544414	4
57	439014	736	982950	60	456064	796	543936	3
58	439456	736	982914	60	456542	796	543458	2
59	439897	735	982878	60	457019	795	542981	1
60	440338	734	982842	60	457496	794	542504	0
	Cosine		Sine		Cotang.		Tang.	M.

74 Degrees.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang	
0	9.440338	734	9.982842	60	9.457496	794	10.542504	60
1	440778	733	982805	60	457973	793	542027	59
2	441218	732	982769	61	458449	793	541551	58
3	441658	731	982733	61	458925	792	541075	57
4	442096	731	982696	61	459400	791	540600	56
5	442535	730	982660	61	459875	790	540125	55
6	442973	729	982624	61	460349	790	539651	54
7	443410	728	982587	61	460823	789	539177	53
8	443847	727	982551	61	461297	788	538703	52
9	444284	727	982514	61	461770	788	538230	51
10	444720	726	982477	61	462242	787	537758	50
11	9.445155	725	9.982441	61	9.462714	786	10.537286	49
12	445590	724	982404	61	463186	785	536814	48
13	446025	723	982367	61	463658	785	536342	47
14	446459	723	982331	61	464129	784	535871	46
15	446893	722	982294	61	464599	783	535401	45
16	447326	721	982257	61	465069	783	534931	44
17	447759	720	982220	62	465539	782	534461	43
18	448191	720	982183	62	466008	781	533992	42
19	448623	719	982146	62	466476	780	533524	41
20	449054	718	982109	62	466945	780	533055	40
21	9.449485	717	9.982072	62	9.467413	779	10.532587	39
22	449915	716	982035	62	467880	778	532120	38
23	450345	716	981998	62	468347	778	531653	37
24	450775	715	981961	62	468814	777	531186	36
25	451204	714	981924	62	469280	776	530720	35
26	451632	713	981886	62	469746	775	530254	34
27	452060	713	981849	62	470211	775	529789	33
28	452488	712	981812	62	470676	774	529324	32
29	452915	711	981774	62	471141	773	528859	31
30	453342	710	981737	62	471605	773	528395	30
31	9.453768	710	9.981699	63	9.472068	772	10.527932	29
32	454194	709	981662	63	472532	771	527468	28
33	454619	708	981625	63	472995	771	527005	27
34	455044	707	981587	63	473457	770	526543	26
35	455469	707	981549	63	473919	769	526081	25
36	455893	706	981512	63	474381	769	525619	24
37	456316	705	981474	63	474842	768	525158	23
38	456739	704	981436	63	475303	767	524697	22
39	457162	704	981399	63	475763	767	524237	21
40	457584	703	981361	63	476223	766	523777	20
41	9.458006	702	9.981323	63	9.476683	765	10.523317	19
42	458427	701	981285	63	477142	765	522858	18
43	458848	701	981247	63	477601	764	522399	17
44	459268	700	981209	63	478059	763	521941	16
45	459688	699	981171	63	478517	763	521483	15
46	460108	698	981133	64	478975	762	521025	14
47	460527	698	981095	64	479432	761	520568	13
48	460946	697	981057	64	479889	761	520111	12
49	461364	696	981019	64	480345	760	519655	11
50	461782	695	980981	64	480801	759	519199	10
51	9.462199	695	9.980942	64	9.481257	759	10.518743	9
52	462616	694	980904	64	481712	758	518288	8
53	463032	693	980866	64	482167	757	517833	7
54	463448	693	980827	64	482621	757	517379	6
55	463864	692	980789	64	483075	756	516925	5
56	464279	691	980750	64	483529	755	516471	4
57	464694	690	980712	64	483982	755	516018	3
58	465108	690	980673	64	484435	754	515565	2
59	465522	689	980635	64	484887	753	515113	1
60	465935	688	980596	64	485339	753	514661	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D	Cosine	D.	Tang.	D.	Cotang.	
0	9.465935	688	9.980596	64	9.485339	755	10.514661	60
1	466348	688	980558	64	485791	752	514209	59
2	466761	687	980519	65	486242	751	513758	58
3	467173	686	980480	65	486693	751	513307	57
4	467585	685	980442	65	487143	750	512857	56
5	467996	685	980403	65	487593	749	512407	55
6	468407	684	980364	65	488043	749	511957	54
7	468817	683	980325	65	488492	748	511508	53
8	469227	683	980286	65	488941	747	511059	52
9	469637	682	980247	65	489390	747	510610	51
10	470046	681	980208	65	489838	746	510162	50
11	9.470455	680	9.980169	65	9.490286	746	10 509714	49
12	470863	680	980130	65	490733	745	509267	48
13	471271	679	980091	65	491180	744	508820	47
14	471679	678	980052	65	491627	744	508373	46
15	472086	678	980012	65	492073	743	507927	45
16	472492	677	979973	65	492519	743	507481	44
17	472898	676	979934	66	492965	742	507035	43
18	473304	676	979895	66	493410	741	506590	42
19	473710	675	979855	66	493854	740	506146	41
20	474115	674	979816	66	494299	740	505701	40
21	9.474519	674	9.979776	66	9.494743	740	10.505257	39
22	474923	673	979737	66	495186	739	504814	38
23	475327	672	979697	66	495630	738	504370	37
24	475730	672	979658	66	496073	737	503927	36
25	476133	671	979618	66	496515	737	503485	35
26	476536	670	979579	66	496957	736	503043	34
27	476938	669	979539	66	497399	736	502601	33
28	477340	669	979499	66	497841	735	502159	32
29	477741	668	979459	66	498282	734	501718	31
30	478142	667	979420	66	498722	734	501278	30
31	9.478542	667	9.979380	66	9.499163	733	10.500837	29
32	478942	666	979340	66	499603	733	500397	28
33	479342	665	979300	67	500042	732	499958	27
34	479741	665	979260	67	500481	731	499519	26
35	480140	664	979220	67	500920	731	499080	25
36	480539	663	979180	67	501359	730	498641	24
37	480937	663	979140	67	501797	730	498203	23
38	481334	662	979100	67	502235	729	497765	22
39	481731	661	979059	67	502672	728	497328	21
40	482128	661	979019	67	503109	728	496891	20
41	9.482525	660	9.978979	67	9.503546	727	10.496454	19
42	482921	659	978939	67	503982	727	496018	18
43	483316	659	978898	67	504418	726	495582	17
44	483712	658	978858	67	504854	725	495146	16
45	484107	657	978817	67	505289	725	494711	15
46	484501	657	978777	67	505724	724	494276	14
47	484895	656	978736	67	506159	724	493841	13
48	485289	655	978696	68	506593	723	493407	12
49	485682	655	978655	68	507027	722	492973	11
50	486075	654	978615	68	507460	722	492540	10
51	9.486467	653	9.978574	68	9.507893	721	10.492107	9
52	486860	653	978533	68	508326	721	491674	8
53	487251	652	978493	68	508759	720	491241	7
54	487643	651	978452	68	509191	719	490809	6
55	488034	651	978411	68	509622	719	490378	5
56	488424	650	978370	68	510054	718	489946	4
57	488814	650	978329	68	510485	718	489515	3
58	489204	649	978288	68	510916	717	489084	2
59	489593	648	978247	68	511346	716	488654	1
60	489982	648	978206	68	511776	716	488224	0
	Cosine		Sine		Cotang.		Tang.	M.

71 Degrees.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.489982	648	9.978206	68	9.511776	716	10.488224	60
1	490371	648	978165	68	512206	716	487794	59
2	490759	647	978124	68	512635	715	487365	58
3	491147	646	978083	69	513064	714	486936	57
4	491535	646	978042	69	513493	714	486507	56
5	491922	645	978001	69	513921	713	486079	55
6	492308	644	977959	69	514349	713	485651	54
7	492695	644	977918	69	514777	712	485223	53
8	493081	643	977877	69	515204	712	484796	52
9	493466	642	977835	69	515631	711	484369	51
10	493851	642	977794	69	516057	710	483943	50
11	9.494236	641	9.977752	69	9.516484	710	10.483516	49
12	494621	641	977711	69	516910	709	483090	48
13	495005	640	977669	69	517335	709	482665	47
14	495388	639	977628	69	517761	708	482239	46
15	495772	639	977586	69	518185	708	481815	45
16	496154	638	977544	70	518610	707	481390	44
17	496537	637	977503	70	519034	706	480966	43
18	496919	637	977461	70	519458	706	480542	42
19	497301	636	977419	70	519882	705	480118	41
20	497682	636	977377	70	520305	705	479695	40
21	9.498064	635	9.977335	70	9.520728	704	10.479272	39
22	498444	634	977293	70	521151	703	478849	38
23	498825	634	977251	70	521573	703	478427	37
24	499204	633	977209	70	521995	703	478005	36
25	499584	632	977167	70	522417	702	477583	35
26	499963	632	977125	70	522838	702	477162	34
27	500342	631	977083	70	523259	701	476741	33
28	500721	631	977041	70	523680	701	476320	32
29	501099	630	976999	70	524100	700	475900	31
30	501476	629	976957	70	524520	699	475480	30
31	9.501854	629	9.976914	70	9.524939	699	10.475061	29
32	502231	628	976872	71	525359	698	474641	28
33	502607	628	976830	71	525778	698	474222	27
34	502984	627	976787	71	526197	697	473803	26
35	503360	626	976745	71	526615	697	473385	25
36	503735	626	976702	71	527033	696	472967	24
37	504110	625	976660	71	527451	696	472549	23
38	504485	625	976617	71	527868	695	472132	22
39	504860	624	976574	71	528285	695	471715	21
40	505234	623	976532	71	528702	694	471298	20
41	9.505608	623	9.976489	71	9.529119	693	0.470881	19
42	505981	622	976446	71	529535	693	470465	18
43	506354	622	976404	71	529950	693	470050	17
44	506727	621	976361	71	530366	692	469634	16
45	507099	620	976318	71	530781	691	469219	15
46	507471	620	976275	71	531196	691	468804	14
47	507843	619	976232	72	531611	690	468389	13
48	508214	619	976189	72	532025	690	467975	12
49	508585	618	976146	72	532439	689	467561	11
50	508956	618	976103	72	532853	689	467147	10
51	9 509326	617	9.976060	72	9.533266	688	10.466734	9
52	509696	616	976017	72	533679	688	466321	8
53	510065	616	975974	72	534092	687	465908	7
54	510434	615	975930	72	534504	687	465496	6
55	510803	615	975887	72	534916	686	465084	5
56	511172	614	975844	72	535328	686	464672	4
57	511540	613	975800	72	535739	685	464261	3
58	511907	613	975757	72	536150	685	463850	2
59	512275	612	975714	72	536561	684	463439	1
60	512642	612	975670	72	536972	684	463028	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.512642	612	9.975670	73	9.536972	684	10.463028	60
1	513009	611	975627	73	537382	683	462618	59
2	513375	611	975583	73	537792	683	462208	58
3	513741	610	975539	73	538202	682	461798	57
4	514107	609	975496	73	538611	682	461389	56
5	514472	609	975452	73	539020	681	460980	55
6	514837	608	975408	73	539429	681	460571	54
7	515202	608	975365	73	539837	680	460163	53
8	515566	607	975321	73	540245	680	459755	52
9	515930	607	975277	73	540653	679	459347	51
10	516294	606	975233	73	541061	679	458939	50
11	9.516657	605	9.975189	73	9.541468	678	10.458532	49
12	517020	605	975145	73	541875	678	458125	48
13	517382	604	975101	73	542281	677	457719	47
14	517745	604	975057	73	542688	677	457312	46
15	518107	603	975013	73	543094	676	456906	45
16	518468	603	974969	74	543499	676	456501	44
17	518829	602	974925	74	543905	675	456095	43
18	519190	601	974880	74	544310	675	455690	42
19	519551	601	974836	74	544715	674	455285	41
20	519911	600	974792	74	545119	674	454881	40
21	9.520271	600	9.974748	74	9.545524	673	10.454476	39
22	520631	599	974703	74	545928	673	454072	38
23	520990	599	974659	74	546331	672	453669	37
24	521349	598	974614	74	546735	672	453265	36
25	521707	598	974570	74	547138	671	452862	35
26	522066	597	974525	74	547540	671	452460	34
27	522424	596	974481	74	547943	670	452057	33
28	522781	596	974436	74	548345	670	451655	32
29	523138	595	974391	74	548747	669	451253	31
30	523495	595	974347	75	549149	669	450851	30
31	9.523852	594	9.974302	75	9.549550	668	10.450450	29
32	524208	594	974257	75	549951	668	450049	28
33	524564	593	974212	75	550352	667	449648	27
34	524920	593	974167	75	550752	667	449248	26
35	525275	592	974122	75	551152	666	448848	25
36	525630	591	974077	75	551552	666	448448	24
37	525984	591	974032	75	551952	665	448048	23
38	526339	590	973987	75	552351	665	447649	22
39	526693	590	973942	75	552750	665	447250	21
40	527046	589	973897	75	553149	664	446851	20
41	9.527400	589	9.973852	75	9.553548	664	10.446452	19
42	527753	588	973807	75	553946	663	446054	18
43	528105	588	973761	75	554344	663	445656	17
44	528458	587	973716	76	554741	662	445259	16
45	528810	587	973671	76	555139	662	444861	15
46	529161	586	973625	76	555536	661	444464	14
47	529513	586	973580	76	555933	661	444067	13
48	529864	585	973535	76	556329	660	443671	12
49	530215	585	973489	76	556725	660	443275	11
50	530565	584	973444	76	557121	659	442879	10
51	9.530915	584	9.973398	76	9.557517	659	10.442483	9
52	531265	583	973352	76	557913	659	442087	8
53	531614	582	973307	76	558308	658	441692	7
54	531963	582	973261	76	558702	658	441298	6
55	532312	581	973215	76	559097	657	440903	5
56	532661	581	973169	76	559491	657	440509	4
57	533009	580	973124	76	559885	656	440115	3
58	533357	580	973078	76	560279	656	439721	2
59	533704	579	973032	77	560673	655	439327	1
60	534052	578	972986	77	561066	655	438934	0
	Cosine		Sine		Cotang.		Tang.	M.

70 Degrees

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.534052	578	9.972986	77	9.561066	655	10.438934	60
1	534399	577	972940	77	561459	654	438541	59
2	534745	577	972894	77	561851	654	438149	58
3	535092	577	972848	77	562244	653	437756	57
4	535438	576	972802	77	562636	653	437364	56
5	535783	576	972755	77	563028	653	436972	55
6	536129	575	972709	77	563419	652	436581	54
7	536474	574	972663	77	563811	652	436189	53
8	536818	574	972617	77	564202	651	435798	52
9	537163	573	972570	77	564592	651	435408	51
10	537507	573	972524	77	564983	650	435017	50
11	9.537851	572	9.972478	77	9.565373	650	10.434627	49
12	538194	572	972431	78	565763	649	434237	48
13	538538	571	972385	78	566153	649	433847	47
14	538880	571	972338	78	566542	649	433458	46
15	539223	570	972291	78	566932	648	433068	45
16	539565	570	972245	78	567320	648	432680	44
17	539907	569	972198	78	567709	647	432291	43
18	540249	569	972151	78	568098	647	431902	42
19	540590	568	972105	79	568486	646	431514	41
20	540931	568	972058	78	568873	646	431127	40
21	9.541272	567	9.972011	78	9.569261	645	10.430739	39
22	541613	567	971964	78	569648	645	430352	38
23	541953	566	971917	78	570035	645	429965	37
24	542293	566	971870	78	570422	644	429578	36
25	542632	565	971823	78	570809	644	429191	35
26	542971	565	971776	78	571195	643	428805	34
27	543310	564	971729	79	571581	643	428419	33
28	543649	564	971682	79	571967	642	428033	32
29	543987	563	971635	79	572352	642	427648	31
30	544325	563	971588	79	572738	642	427262	30
31	9.544663	562	9.971540	79	9.573123	641	10.426877	29
32	545000	562	971493	79	573507	641	426493	28
33	545338	561	971446	79	573892	640	426108	27
34	545674	561	971398	79	574276	640	425724	26
35	546011	560	971351	79	574660	639	425340	25
36	546347	560	971303	79	575044	639	424956	24
37	546683	559	971256	79	575427	639	424573	23
38	547019	559	971208	79	575810	638	424190	22
39	547354	558	971161	79	576193	638	423807	21
40	547689	558	971113	79	576576	637	423424	20
41	9.548024	557	9.971066	80	9.576958	637	10.423041	19
42	548359	557	971018	80	577341	636	422659	18
43	548693	556	970970	80	577723	636	422277	17
44	549027	556	970922	80	578104	636	421896	16
45	549360	555	970874	80	578486	635	421514	15
46	549693	555	970827	80	578867	635	421133	14
47	550026	554	970779	80	579248	634	420752	13
48	550359	554	970731	80	579629	634	420371	12
49	550692	553	970683	80	580009	634	419991	11
50	551024	553	970635	80	580389	633	419611	10
51	9.551356	552	9.970586	80	9.580769	633	10.419231	9
52	551687	552	970538	80	581149	632	418851	8
53	552018	552	970490	80	581528	632	418472	7
54	552349	551	970442	80	581907	632	418093	6
55	552680	551	970394	80	582286	631	417714	5
56	553010	550	970345	81	582665	631	417335	4
57	553341	550	970297	81	583043	630	416957	3
58	553670	549	970249	81	583422	630	416578	2
59	554000	549	970200	81	583800	629	416200	1
60	554329	548	970152	81	584177	629	415823	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.554329	548	9.970152	81	9.584177	629	10.415823	60
1	554658	548	970103	81	584555	629	415445	59
2	554987	547	970055	81	584932	628	415068	58
3	555315	547	970006	81	585309	628	414691	57
4	555643	546	969957	81	585686	627	414314	56
5	555971	546	969909	81	586062	627	413938	55
6	556299	545	969860	81	586439	627	413561	54
7	556626	545	969811	81	586815	626	413185	53
8	556953	544	969762	81	587190	626	412810	52
9	557280	544	969714	81	587566	625	412434	51
10	557606	543	969665	81	587941	625	412059	50
11	9.557932	543	9.969616	82	9.588316	625	10.411684	49
12	558258	543	969567	82	588691	624	411309	48
13	558583	542	969518	82	589066	624	410934	47
14	558909	542	969469	82	589440	623	410560	46
15	559234	541	969420	82	589814	623	410186	45
16	559558	541	969370	82	590188	623	409812	44
17	559883	540	969321	82	590562	622	409438	43
18	560207	540	969272	82	590935	622	409065	42
19	560531	539	969223	82	591308	622	408692	41
20	560855	539	969173	82	591681	621	408319	40
21	9.561178	538	9.969124	82	9.592054	621	10.407946	39
22	561501	538	969075	82	592426	620	407574	38
23	561824	537	969025	82	592798	620	407202	37
24	562146	537	968976	82	593170	619	406829	36
25	562468	536	968926	83	593542	619	406458	35
26	562790	536	968877	83	593914	618	406086	34
27	563112	536	968827	83	594285	618	405715	33
28	563433	535	968777	83	594656	618	405344	32
29	563755	535	968728	83	595027	617	404973	31
30	564075	534	968678	83	595398	617	404602	30
31	9.564396	534	9.968628	83	9.595768	617	10.404232	29
32	564716	533	968578	83	596138	616	403862	28
33	565036	533	968528	83	596508	616	403492	27
34	565356	532	968479	83	596878	616	403122	26
35	565676	532	968429	83	597247	615	402753	25
36	565995	531	968379	83	597616	615	402384	24
37	566314	531	968329	83	597985	615	402015	23
38	566632	531	968278	83	598354	614	401646	22
39	566951	530	968228	84	598722	614	401278	21
40	567269	530	968178	84	599091	613	400909	20
41	9.567587	529	9.968128	84	9.599459	613	10.400541	19
42	567904	529	968078	84	599827	613	400173	18
43	568222	528	968027	84	600194	612	399806	17
44	568539	528	967977	84	600562	612	399438	16
45	568856	528	967927	84	600929	611	399071	15
46	569172	527	967876	84	601296	611	398704	14
47	569488	527	967826	84	601662	611	398338	13
48	569804	526	967775	84	602029	610	397971	12
49	570120	526	967725	84	602395	610	397605	11
50	570435	525	967674	84	602761	610	397239	10
51	9.570751	525	9.967624	84	9.603127	609	10.396873	9
52	571066	524	967573	84	603493	609	396507	8
53	571380	524	967522	85	603858	609	396142	7
54	571695	523	967471	85	604223	608	395777	6
55	572009	523	967421	85	604588	608	395412	5
56	572323	523	967370	85	604953	607	395047	4
57	572636	522	967319	85	605317	607	394683	3
58	572950	522	967268	85	605682	607	394318	2
59	573263	521	967217	85	606046	606	393954	1
60	573575	521	967166	85	606410	606	393590	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.573575	521	9.967166	85	9.606410	606	10.393590	60
1	573888	520	967115	85	606773	606	393227	59
2	574200	520	967064	85	607137	605	392863	58
3	574512	519	967013	85	607500	605	392500	57
4	574824	519	966961	85	607863	604	392137	56
5	575136	519	966910	85	608225	604	391775	55
6	575447	518	966859	85	608588	604	391412	54
7	575758	518	966808	85	608950	603	391050	53
8	576069	517	966756	86	609312	603	390688	52
9	576379	517	966705	86	609674	603	390326	51
10	576689	516	966653	86	610036	602	389964	50
11	9.576999	516	9.966602	86	9.610397	602	10.389603	49
12	577309	516	966550	86	610759	602	389241	48
13	577618	515	966499	86	611120	601	388880	47
14	577927	515	966447	86	611480	601	388520	46
15	578236	514	966395	86	611841	601	388159	45
16	578545	514	966344	86	612201	600	387799	44
17	578853	513	966292	86	612561	600	387439	43
18	579162	513	966240	86	612921	600	387079	42
19	579470	513	966188	86	613281	599	386719	41
20	579777	512	966136	86	613641	599	386359	40
21	9.580085	512	9 966085	87	9.614000	598	10.386000	39
22	580392	511	966033	87	614359	598	385641	38
23	580699	511	965981	87	614718	598	385282	37
24	581005	511	965928	87	615077	597	384923	36
25	581312	510	965876	87	615435	597	384565	35
26	581618	510	965824	87	615793	597	384207	34
27	581924	509	965772	87	616151	596	383849	33
28	582229	509	965720	87	616509	596	383491	32
29	582535	509	965668	87	616867	596	383133	31
30	582840	508	965615	87	617224	595	382776	30
31	9.583145	508	9.965563	87	9 617582	595	10.382418	29
32	583449	507	965511	87	617939	595	382061	28
33	583754	507	965458	87	618295	594	381705	27
34	584058	506	965406	87	618652	594	381348	26
35	584361	506	965353	88	619008	594	380992	25
36	584665	506	965301	88	619364	593	380636	24
37	584968	505	965248	88	619721	593	380279	23
38	585272	505	965195	88	620076	593	379924	22
39	585574	504	965143	88	620432	592	379568	21
40	585877	504	965090	88	620787	592	379213	20
41	9.586179	503	9.965037	88	9.621142	592	10.378858	19
42	586482	503	964984	88	621497	591	378503	18
43	586783	503	964931	88	621852	591	378148	17
44	587085	502	964879	88	622207	590	377793	16
45	587386	502	964826	88	622561	590	377439	15
46	587688	501	964773	88	622915	590	377085	14
47	587989	501	964719	88	623269	589	376731	13
48	588289	501	964666	89	623623	589	376377	12
49	588590	500	964613	89	623976	589	376024	11
50	588890	500	964560	89	624330	588	375670	10
51	9.589190	499	9.964507	89	9.624683	588	10.375317	9
52	589489	499	964454	89	625036	588	374964	8
53	589789	499	964400	89	625388	587	374612	7
54	590088	498	964347	89	625741	587	374259	6
55	590387	498	964294	89	626093	587	373907	5
56	590686	497	964240	89	626445	586	373555	4
57	590984	497	964187	89	626797	586	373203	3
58	591282	497	964133	89	627149	586	372851	2
59	591580	496	964080	89	627501	585	372499	1
60	591878	496	964026	89	627852	585	372148	0
′	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang	D.	Cotang.	
0	9.591878	496	9.964026	89	9.627852	585	10.372148	60
1	592176	495	963972	89	628203	585	371797	59
2	592473	495	963919	89	628554	585	371446	58
3	592770	495	963865	90	628905	584	371095	57
4	593067	494	963811	90	629255	584	370745	56
5	593363	494	963757	90	629606	583	370394	55
6	593659	493	963704	90	629956	583	370044	54
7	593955	493	963650	90	630306	583	369694	53
8	594251	493	963596	90	630656	583	369344	52
9	594547	492	963542	90	631005	582	368995	51
10	594842	492	963488	90	631355	582	368645	50
11	9.595137	491	9.963434	90	9.631704	582	10.368296	49
12	595432	491	963379	90	632053	581	367947	48
13	595727	491	963325	90	632401	581	367599	47
14	596021	490	963271	90	632750	581	367250	46
15	596315	490	963217	90	633098	580	366902	45
16	596609	489	963163	90	633447	580	366553	44
17	596903	489	963108	91	633795	580	366205	43
18	597196	489	963054	91	634143	579	365857	42
19	597490	488	962999	91	634490	579	365510	41
20	597783	488	962945	91	634838	579	365162	40
21	9.598075	487	9.962890	91	9.635185	578	10.364815	39
22	598368	487	962836	91	635532	578	364468	38
23	598660	487	962781	91	635879	578	364121	37
24	598952	486	962727	91	636226	577	363774	36
25	599244	486	962672	91	636572	577	363428	35
26	599536	485	962617	91	636919	577	363081	34
27	599827	485	962562	91	637265	577	362735	33
28	600118	485	962508	91	637611	576	362389	32
29	600409	484	962453	91	637956	576	362044	31
30	600700	484	962398	92	638302	576	361698	30
31	9.600990	484	9.962343	92	9.638647	575	10.361353	29
32	601280	483	962288	92	638992	575	361008	28
33	601570	483	962233	92	639337	575	360663	27
34	601860	482	962178	92	639682	574	360318	26
35	602150	482	962123	92	640027	574	359973	25
36	602439	482	962067	92	640371	574	359629	24
37	602728	481	962012	92	640716	573	359284	23
38	603017	481	961957	92	641060	573	358940	22
39	603305	481	961902	92	641404	573	358596	21
40	603594	480	961846	92	641747	572	358253	20
41	9.603882	480	9.961791	92	9.642091	572	10.357909	19
42	604170	479	961735	92	642434	572	357566	18
43	604457	479	961680	92	642777	572	357223	17
44	604745	479	961624	93	643120	571	356880	16
45	605032	478	961569	93	643463	571	356537	15
46	605319	478	961513	93	643806	571	356194	14
47	605606	478	961458	93	644148	570	355852	13
48	605892	477	961402	93	644490	570	355510	12
49	606179	477	961346	93	644832	570	355168	11
50	606465	476	961290	93	645174	569	354826	10
51	9 606751	476	9.961235	93	9.645516	569	10.354484	9
52	607036	476	961179	93	645857	569	354143	8
53	607322	475	961123	93	646199	569	353801	7
54	607607	475	961067	93	646540	568	353460	6
55	607892	474	961011	93	646881	568	353119	5
56	608177	474	960955	93	647222	568	352778	4
57	608461	474	960899	93	647562	567	352438	3
58	608745	473	960843	94	647903	567	352097	2
59	609029	473	960786	94	648243	567	351757	1
60	609313	473	960730	94	648583	566	351417	0
	Cosine		Sine		Cotang.		Tang.	M.

66 Degrees.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.609313	473	9.960730	94	9.648583	566	10.351417	60
1	609597	472	960674	94	648923	566	351077	59
2	609880	472	960618	94	649263	566	350737	58
3	610164	472	960561	94	649602	566	350398	57
4	610447	471	960505	94	649942	565	350058	56
5	610729	471	960448	94	650281	565	349719	55
6	611012	470	960392	94	650620	565	349380	54
7	611294	470	960335	94	650959	564	349041	53
8	611576	470	960279	94	651297	564	348703	52
9	611858	469	960222	94	651636	564	348364	51
10	612140	469	960165	94	651974	563	348026	50
11	9.612421	469	9.960109	95	9.652312	563	10.347688	49
12	612702	468	960052	95	652650	563	347350	48
13	612983	468	959995	95	652988	563	347012	47
14	613264	467	959938	95	653326	562	346674	46
15	613545	467	959882	95	653663	562	346337	45
16	613825	467	959825	95	654000	562	346000	44
17	614105	466	959768	95	654337	561	345663	43
18	614385	466	959711	95	654674	561	345326	42
19	614665	466	959654	95	655011	561	344989	41
20	614944	465	959596	95	655348	561	344652	40
21	9.615223	465	9.959539	95	9.655684	560	10.344316	39
22	615502	465	959482	95	656020	560	343980	38
23	615781	464	959425	95	656356	560	343644	37
24	616060	464	959368	95	656692	559	343308	36
25	616338	464	959310	96	657028	559	342972	35
26	616616	463	959253	96	657364	559	342636	34
27	616894	463	959195	96	657699	559	342301	33
28	617172	462	959138	96	658034	558	341966	32
29	617450	462	959081	96	658369	558	341631	31
30	617727	462	959023	96	658704	558	341296	30
31	9.618004	461	9.958965	96	9.659039	558	10.340961	29
32	618281	461	958908	96	659373	557	340627	28
33	618558	461	958850	96	659708	557	340292	27
34	618834	460	958792	96	660042	557	339958	26
35	619110	460	958734	96	660376	557	339624	25
36	619386	460	958677	96	660710	556	339290	24
37	619662	459	958619	96	661043	556	338957	23
38	619938	459	958561	96	661377	556	338623	22
39	620213	459	958503	97	661710	555	338290	21
40	620488	458	958445	97	662043	555	337957	20
41	9.620763	458	9.958387	97	9 662376	555	10.337624	19
42	621038	457	958329	97	662709	554	337291	18
43	621313	457	958271	97	663042	554	336958	17
44	621587	457	958213	97	663375	554	336625	16
45	621861	456	958154	97	663707	554	336293	15
46	622135	456	958096	97	664039	553	335961	14
47	622409	456	958038	97	664371	553	335629	13
48	622682	455	957979	97	664703	553	335297	12
49	622956	455	957921	97	665035	553	334965	11
50	623229	455	957863	97	665366	552	334634	10
51	9.623502	454	9.957804	97	9.665697	552	10.334303	9
52	623774	454	957746	98	666029	552	333971	8
53	624047	454	957687	98	666360	551	333640	7
54	624319	453	957628	98	666691	551	333309	6
55	624591	453	957570	98	667021	551	332979	5
56	624863	453	957511	98	667352	551	332648	4
57	625135	452	957452	98	667682	550	332318	3
58	625406	452	957393	98	668013	550	331987	2
59	625677	452	957335	98	668343	550	331657	1
60	625948	451	957276	98	668672	550	331328	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.625948	451	9.957276	98	9.668673	550	10.331327	60
1	626219	451	957217	98	669002	549	330998	59
2	626490	451	957158	98	669332	549	330668	58
3	626760	450	957099	98	669661	549	330339	57
4	627030	450	957040	98	669991	548	330009	56
5	627300	450	956981	98	670320	548	329680	55
6	627570	449	956921	99	670649	548	329351	54
7	627840	449	956862	99	670977	548	329023	53
8	628109	449	956803	99	671306	548	328694	52
9	628378	448	956744	99	671634	547	328366	51
10	628647	448	956684	99	671963	547	328037	50
11	9.628916	447	9.956625	99	9.672291	547	10.327709	49
12	629185	447	956566	99	672619	546	327381	48
13	629453	447	956506	99	672947	546	327053	47
14	629721	446	956447	99	673274	546	326726	46
15	629989	446	956387	99	673602	546	326398	45
16	630257	446	956327	99	673929	545	326071	44
17	630524	446	956268	99	674257	545	325743	43
18	630792	445	956208	100	674584	545	325416	42
19	631059	445	956148	100	674910	544	325090	41
20	631326	445	956089	100	675237	544	324763	40
21	9.631593	444	9.956029	100	9.675564	544	10.324436	39
22	631859	444	955969	100	675890	544	324110	38
23	632125	444	955909	100	676216	543	323784	37
24	632392	443	955849	100	676543	543	323457	36
25	632658	443	955789	100	676869	543	323131	35
26	632923	443	955729	100	677194	543	322806	34
27	633189	442	955669	100	677520	542	322480	33
28	633454	442	955609	100	677846	542	322154	32
29	633719	442	955548	100	678171	542	321829	31
30	633984	441	955488	100	678496	542	321504	30
31	9.634249	441	9.955428	101	9.678821	541	10.321179	29
32	634514	440	955368	101	679146	541	320854	28
33	634778	440	955307	101	679471	541	320529	27
34	635042	440	955247	101	679795	541	320205	26
35	635306	439	955186	101	680120	540	319880	25
36	635570	439	955126	101	680444	540	319556	24
37	635834	439	955065	101	680768	540	319232	23
38	636097	438	955005	101	681092	540	318908	22
39	636360	438	954944	101	681416	539	318584	21
40	636623	438	954883	101	681740	539	318260	20
41	9.636886	437	9.954823	101	9.682063	539	10.317937	19
42	637148	437	954762	101	682387	539	317613	18
43	637411	437	954701	101	682710	538	317290	17
44	637673	437	954640	101	683033	538	316967	16
45	637935	436	954579	101	683356	538	316644	15
46	638197	436	954518	102	683679	538	316321	14
47	638458	436	954457	102	684001	537	315999	13
48	638720	435	954396	102	684324	537	315676	12
49	638981	435	954335	102	684646	537	315354	11
50	639242	435	954274	102	684968	537	315032	10
51	9.639503	434	9.954213	102	9.685290	536	10.314710	9
52	639764	434	954152	102	685612	536	314388	8
53	640024	434	954090	102	685934	536	314066	7
54	640284	433	954029	102	686255	536	313745	6
55	640544	433	953968	102	686577	535	313423	5
56	640804	433	953906	102	686898	535	313102	4
57	641064	432	953845	102	687219	535	312781	3
58	641324	432	953783	102	687540	535	312460	2
59	641584	432	953722	103	687861	534	312139	1
60	641842	431	953660	103	688182	534	311818	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.641842	431	9.953660	103	9.688182	534	10.311818	60
1	642101	431	953599	103	688502	534	311498	59
2	642360	431	953537	103	688823	534	311177	58
3	642618	430	953475	103	689143	533	310857	57
4	642877	430	953413	103	689463	533	310537	56
5	643135	430	953352	103	689783	533	310217	55
6	643393	430	953290	103	690103	533	309897	54
7	643650	429	953228	103	690423	533	309577	53
8	643908	429	953166	103	690742	532	309258	52
9	644165	429	953104	103	691062	532	308938	51
10	644423	428	953042	103	691381	532	308619	50
11	9.644680	428	9.952980	104	9.691700	531	10.308300	49
12	644936	428	952918	104	692019	531	307981	48
13	645193	427	952855	104	692338	531	307662	47
14	645450	427	952793	104	692656	531	307344	46
15	645706	427	952731	104	692975	531	307025	45
16	645962	426	952669	104	693293	530	306707	44
17	646218	426	952606	104	693612	530	306388	43
18	646474	426	952544	104	693930	530	306070	42
19	646729	425	952481	104	694248	530	305752	41
20	646984	425	952419	104	694566	529	305434	40
21	9.647240	425	9.952356	104	9.694883	529	10.305117	39
22	647494	424	952294	104	695201	529	304799	38
23	647749	424	952231	104	695518	529	304482	37
24	648004	424	952168	105	695836	529	304164	36
25	648258	424	952106	105	696153	528	303847	35
26	648512	423	952043	105	696470	528	303530	34
27	648766	423	951980	105	696787	528	303213	33
28	649020	423	951917	105	697103	528	302897	32
29	649274	422	951854	105	697420	527	302580	31
30	649527	422	951791	105	697736	527	302264	30
31	9.649781	422	9.951728	105	9.698053	527	10.301947	29
32	650034	422	951665	105	698369	527	301631	28
33	650287	421	951602	105	698685	526	301315	27
34	650539	421	951539	105	699001	526	300999	26
35	650792	421	951476	105	699316	526	300684	25
36	651044	420	951412	105	699632	526	300368	24
37	651297	420	951349	106	699947	526	300053	23
38	651549	420	951286	106	700263	525	299737	22
39	651800	419	951222	106	700578	525	299422	21
40	652052	419	951159	106	700893	525	299107	20
41	9.652304	419	9.951096	106	9.701208	524	10.298792	19
42	652555	418	951032	106	701523	524	298477	18
43	652806	418	950968	106	701837	524	298163	17
44	653057	418	950905	106	702152	524	297848	16
45	653308	418	950841	106	702466	524	297534	15
46	653558	417	950778	106	702780	523	297220	14
47	653808	417	950714	106	703095	523	296905	13
48	654059	417	950650	106	703409	523	296591	12
49	654309	416	950586	106	703723	523	296277	11
50	654558	416	950522	107	704036	522	295964	10
51	9.654808	416	9.950458	107	9.704350	522	10.295650	9
52	655058	416	950394	107	704663	522	295337	8
53	655307	415	950330	107	704977	522	295023	7
54	655556	415	950266	107	705290	522	294710	6
55	655805	415	950202	107	705603	521	294397	5
56	656054	414	950138	107	705916	521	294084	4
57	656302	414	950074	107	706228	521	293772	3
58	656551	414	950010	107	706541	521	293459	2
59	656799	413	949945	107	706854	521	293146	1
60	657047	413	949881	107	707166	520	292834	0
	Cosine		Sine		Cotang.		Tang.	M.

M	Sine	D	Cosine	D.	Tang.	D.	Cotang.	
0	9.657047	413	9.949881	107	9.707166	520	10.292834	60
1	657295	413	949816	107	707478	520	292522	59
2	657542	412	949752	107	707790	520	292210	58
3	657790	412	949688	108	708102	520	291898	57
4	658037	412	949623	108	708414	519	291586	56
5	658284	412	949558	108	708726	519	291274	55
6	658531	411	949494	108	709037	519	290963	54
7	658778	411	949429	108	709349	519	290651	53
8	659025	411	949364	108	709660	519	290340	52
9	659271	410	949300	108	709971	518	290029	51
10	659517	410	949235	108	710282	518	289718	50
11	9.659763	410	9.949170	108	9.710593	518	10.289407	49
12	660009	409	949105	108	710904	518	289096	48
13	660255	409	949040	108	711215	518	288785	47
14	660501	409	948975	108	711525	517	288475	46
15	660746	409	948910	108	711836	517	288164	45
16	660991	408	948845	108	712146	517	287854	44
17	661236	408	948780	109	712456	517	287544	43
18	661481	408	948715	109	712766	516	287234	42
19	661726	407	948650	109	713076	516	286924	41
20	661970	407	948584	109	713386	516	286614	40
21	9.662214	407	9.948519	109	9.713696	516	10.286304	39
22	662459	407	948454	109	714005	516	285995	38
23	662703	406	948388	109	714314	515	285686	37
24	662946	406	948323	109	714624	515	285376	36
25	663190	406	948257	109	714933	515	285067	35
26	663433	405	948192	109	715242	515	284758	34
27	663677	405	948126	109	715551	514	284449	33
28	663920	405	948060	109	715860	514	284140	32
29	664163	405	947995	110	716168	514	283832	31
30	664406	404	947929	110	716477	514	283523	30
31	9.664648	404	9.947863	110	9.716785	514	10.283215	29
32	664891	404	947797	110	717093	513	282907	28
33	665133	403	947731	110	717401	513	282599	27
34	665375	403	947665	110	717709	513	282291	26
35	665617	403	947600	110	718017	513	281983	25
36	665859	402	947533	110	718325	513	281675	24
37	666100	402	947467	110	718633	512	281367	23
38	666342	402	947401	110	718940	512	281060	22
39	666583	402	947335	110	719248	512	280752	21
40	666824	401	947269	110	719555	512	280445	20
41	9.667065	401	9.947203	110	9.719862	512	10.280138	19
42	667305	401	947136	111	720169	511	279831	18
43	667546	401	947070	111	720476	511	279524	17
44	667786	400	947004	111	720783	511	279217	16
45	668027	400	946937	111	721089	511	278911	15
46	668267	400	946871	111	721396	511	278604	14
47	668506	399	946804	111	721702	510	278298	13
48	668746	399	946738	111	722009	510	277991	12
49	668986	399	946671	111	722315	510	277685	11
50	669225	399	946604	111	722621	510	277379	10
51	9.669464	398	9.946538	111	9.722927	510	10.277073	9
52	669703	398	946471	111	723232	509	276768	8
53	669942	398	946404	111	723538	509	276462	7
54	670181	397	946337	111	723844	509	276156	6
55	670419	397	946270	112	724149	509	275851	5
56	670658	397	946203	112	724454	509	275546	4
57	670896	397	946136	112	724759	508	275241	3
58	671134	396	946069	112	725065	508	274935	2
59	671372	396	946002	112	725369	508	274631	1
60	671609	396	945935	112	725674	508	274326	0
	Cosine		Sine		Cotang.		Tang.	M.

62 Degrees.

M.	Sine	D.	Cosine	D	Tang.	D.	Cotang.	
0	9.671609	396	9.945935	112	9.725674	508	0.274326	60
1	671847	395	945868	112	725979	508	274021	59
2	672084	395	945800	112	726284	507	273716	58
3	672321	395	945733	112	726588	507	273412	57
4	672558	395	945666	112	726892	507	273108	56
5	672795	394	945598	112	727197	507	272803	55
6	673032	394	945531	112	727501	507	272499	54
7	673268	304	945464	113	727805	506	272195	53
8	673505	394	945396	113	728109	506	271891	52
9	673741	393	945328	113	728412	506	271588	51
10	673977	393	945261	113	728716	506	271284	50
11	9.674213	393	9.945193	113	9.729020	506	0.270980	49
12	674448	392	945125	113	729323	505	270677	48
13	674684	392	945058	113	729626	505	270374	47
14	674919	392	944990	113	729929	505	270071	46
15	675155	392	944922	113	730233	505	269767	45
16	675390	391	944854	113	730535	505	269465	44
17	675624	391	944786	113	730838	504	269162	43
18	675859	391	944718	113	731141	504	268859	42
19	676094	391	944650	113	731444	504	268556	41
20	676328	390	944582	114	731746	504	268254	40
21	9.676562	390	9.944514	114	9.732048	504	10.267952	39
22	676796	390	944446	114	732351	503	267649	38
23	677030	390	944377	114	732653	503	267347	37
24	677264	389	944309	114	732955	503	267045	36
25	677498	389	944241	114	733257	503	266743	35
26	677731	389	944172	114	733558	503	266442	34
27	677964	388	944104	114	733860	502	266140	33
28	678197	388	944036	114	734162	502	265838	32
29	678430	388	943967	114	734463	502	265537	31
30	678663	388	943899	114	734764	502	265236	30
31	9 678895	387	9.943830	114	9.735066	502	10.264931	29
32	679128	387	943761	114	735367	502	264633	28
33	679360	387	943693	115	735668	501	264332	27
34	679592	387	943624	115	735969	501	264031	26
35	679824	386	943555	115	736269	501	263731	25
36	680056	386	943486	115	736570	501	263430	24
37	680288	386	943417	115	736871	501	263129	23
38	680519	385	943348	115	737171	500	262829	22
39	680750	385	943279	115	737471	500	262529	21
40	680982	385	943210	115	737771	500	262229	20
41	9.681213	385	9.943141	115	9.738071	500	10.261929	19
42	681443	384	943072	115	738371	500	261629	18
43	681674	384	943003	115	738671	499	261329	17
44	681905	384	942934	115	738971	499	261029	16
45	682135	384	942864	115	739271	499	260729	15
46	682365	383	942795	116	739570	499	260430	14
47	682595	383	942726	116	739870	499	260130	13
48	682825	383	942656	116	740169	499	259831	12
49	683055	383	942587	116	740468	498	259532	11
50	683284	382	942517	116	740767	498	259233	10
51	9.683514	382	9.942448	116	9.741066	498	10.258934	9
52	683743	382	942378	116	741365	498	258635	8
53	683972	382	942308	116	741664	498	258336	7
54	684201	381	942239	116	741962	497	258038	6
55	684430	381	942169	116	742261	497	257739	5
56	684658	381	942099	116	742559	497	257441	4
57	684887	380	942029	116	742858	497	257142	3
58	685115	380	941959	116	743156	497	256844	2
59	685343	380	941889	117	743454	497	256546	1
60	685571	380	941819	117	743752	496	256248	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.685571	380	9.941819	117	9.743752	496	10.256248	60
1	685799	379	941749	117	744050	496	255950	59
2	686027	379	941679	117	744348	496	255652	58
3	686254	379	941609	117	744645	496	255355	57
4	686482	379	941539	117	744943	496	255057	56
5	686709	378	941469	117	745240	496	254760	55
6	686936	378	941398	117	745538	495	254462	54
7	687163	378	941328	117	745835	495	254165	53
8	687389	378	941258	117	746132	495	253868	52
9	687616	377	941187	117	746429	495	253571	51
10	687843	377	941117	117	746726	495	253274	50
11	9.688069	377	9.941046	118	9.747023	494	10.252977	49
12	688295	377	940975	118	747319	494	252681	48
13	688521	376	940905	118	747616	494	252384	47
14	688747	376	940834	118	747913	494	252087	46
15	688972	376	940763	118	748209	494	251791	45
16	689198	376	940693	118	748505	493	251495	44
17	689423	375	940622	118	748801	493	251199	43
18	689648	375	940551	118	749097	493	250903	42
19	689873	375	940480	118	749393	493	250607	41
20	690098	375	940409	118	749689	493	250311	40
21	9.690323	374	9.940338	118	9.749985	493	10.250015	39
22	690548	374	940267	118	750281	492	249719	38
23	690772	374	940196	118	750576	492	249424	37
24	690996	374	940125	119	750872	492	249128	36
25	691220	373	940054	119	751167	492	248833	35
26	691444	373	939982	119	751462	492	248538	34
27	691668	373	939911	119	751757	492	248243	33
28	691892	373	939840	119	752052	491	247948	32
29	692115	372	939768	119	752347	491	247653	31
30	692339	372	939697	119	752642	491	247358	30
31	9.692562	372	9.939625	119	9.752937	491	10.247063	29
32	692785	371	939554	119	753231	491	246769	28
33	693008	371	939482	119	753526	491	246474	27
34	693231	371	939410	119	753820	490	246180	26
35	693453	371	939339	119	754115	490	245885	25
36	693676	370	939267	120	754409	490	245591	24
37	693898	370	939195	120	754703	490	245297	23
38	694120	370	939123	120	754997	490	245003	22
39	694342	370	939052	120	755291	490	244709	21
40	694564	369	938980	120	755585	489	244415	20
41	9.694786	369	9.938008	120	9.755878	489	10.244122	19
42	695007	369	938836	120	756172	489	243828	18
43	695229	369	938763	120	756465	489	243535	17
44	695450	368	938691	120	756759	489	243241	16
45	695671	368	938619	120	757052	489	242948	15
46	695892	368	938547	120	757345	488	242655	14
47	696113	368	938475	120	757638	488	242362	13
48	696334	367	938402	121	757931	488	242069	12
49	696554	367	938330	121	758224	488	241776	11
50	696775	367	938258	121	758517	488	241483	10
51	9.696995	367	9.938185	121	9.758810	488	10.241190	9
52	697215	366	938113	121	759102	487	240898	8
53	697435	366	938040	121	759395	487	240605	7
54	697654	366	937967	121	759687	487	240313	6
55	697874	366	937895	121	759979	487	240021	5
56	698094	365	937822	121	760272	487	239728	4
57	698313	365	937749	121	760564	487	239436	3
58	698532	365	937676	121	760856	486	239144	2
59	698751	365	937604	121	761148	486	238852	1
60	698970	364	937531	121	761439	486	238561	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.698970	364	9.937531	121	9.761439	486	10.238561	60
1	699189	364	937458	122	761731	486	238269	59
2	699407	364	937385	122	762023	486	237977	58
3	699626	364	937312	122	762314	486	237686	57
4	699844	363	937238	122	762606	485	237394	56
5	700062	363	937165	122	762897	485	237103	55
6	700280	363	937092	122	763188	485	236812	54
7	700498	363	937019	122	763479	485	236521	53
8	700716	363	936946	122	763770	485	236230	52
9	700933	362	936872	122	764061	485	235939	51
10	701151	362	936799	122	764352	484	235648	50
11	9.701368	362	9.936725	122	9.764643	484	10.235357	49
12	701585	362	936652	123	764933	484	235067	48
13	701802	361	936578	123	765224	484	234776	47
14	702019	361	936505	123	765514	484	234486	46
15	702236	361	936431	123	765805	484	234195	45
16	702452	361	936357	123	766095	484	233905	44
17	702669	360	936284	123	766385	483	233615	43
18	702885	360	936210	123	766675	483	233325	42
19	703101	360	936136	123	766965	483	233035	41
20	703317	360	936062	123	767255	483	232745	40
21	9.703533	359	9.935988	123	9.767545	483	10.232455	39
22	703749	359	935914	123	767834	483	232166	38
23	703964	359	935840	123	768124	482	231876	37
24	704179	359	935766	124	768413	482	231587	36
25	704395	359	935692	124	768703	482	231297	35
26	704610	358	935618	124	768992	482	231008	34
27	704825	358	935543	124	769281	482	230719	33
28	705040	358	935469	124	769570	482	230430	32
29	705254	358	935395	124	769860	481	230140	31
30	705469	357	935320	124	770148	481	229852	30
31	9.705683	357	9.935246	124	9.770437	481	10.229563	29
32	705898	357	935171	124	770726	481	229274	28
33	706112	357	935097	124	771015	481	228985	27
34	706326	356	935022	124	771303	481	228697	26
35	706539	356	934948	124	771592	481	228408	25
36	706753	356	934873	124	771880	480	228120	24
37	706967	356	934798	125	772168	480	227832	23
38	707180	355	934723	125	772457	480	227543	22
39	707393	355	934649	125	772745	480	227255	21
40	707606	355	934574	125	773033	480	226967	20
41	9.707819	355	9.934499	125	9.773321	480	10.226679	19
42	708032	354	934424	125	773608	479	226392	18
43	708245	354	934349	125	773896	479	226104	17
44	708458	354	934274	125	774184	479	225816	16
45	708670	354	934199	125	774471	479	225529	15
46	708882	353	934123	125	774759	479	225241	14
47	709094	353	934048	125	775046	479	224954	13
48	709306	353	933973	125	775333	479	224667	12
49	709518	353	933898	126	775621	478	224379	11
50	709730	353	933822	126	775908	478	224092	10
51	9.709941	352	9.933747	126	9.776195	478	10.223805	9
52	710153	352	933671	126	776482	478	223518	8
53	710364	352	933596	126	776769	478	223231	7
54	710575	352	933520	126	777055	478	222945	6
55	710786	351	933445	126	777342	478	222658	5
56	710997	351	933369	126	777628	477	222372	4
57	711208	351	933293	126	777915	477	222085	3
58	711419	351	933217	126	778201	477	221799	2
59	711629	350	933141	126	778487	477	221512	1
60	711839	350	933066	126	778774	477	221226	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.711839	350	9.933066	126	9.778774	477	10.221226	60
1	712050	350	932990	127	779060	477	220940	59
2	712260	350	932914	127	779346	476	220654	58
3	712469	349	932838	127	779632	476	220368	57
4	712679	349	932762	127	779918	476	220082	56
5	712889	349	932685	127	780203	476	219797	55
6	713098	349	932609	127	780489	476	219511	54
7	713308	349	932533	127	780775	476	219225	53
8	713517	348	932457	127	781060	476	218940	52
9	713726	348	932380	127	781346	475	218654	51
10	713935	348	932304	127	781631	475	218369	50
11	9.714144	348	9.932228	127	9.781916	475	10.218084	49
12	714352	347	932151	127	782201	475	217799	48
13	714561	347	932075	128	782486	475	217514	47
14	714769	347	931998	128	782771	475	217229	46
15	714978	347	931921	128	783056	475	216944	45
16	715186	347	931845	128	783341	475	216659	44
17	715394	346	931768	128	783626	474	216374	43
18	715602	346	931691	128	783910	474	216090	42
19	715809	346	931614	128	784195	474	215805	41
20	716017	346	931537	128	784479	474	215521	40
21	9.716224	345	9.931460	128	9.784764	474	10.215236	39
22	716432	345	931383	128	785048	474	214952	38
23	716639	345	931306	128	785332	473	214668	37
24	716846	345	931229	129	785616	473	214384	36
25	717053	345	931152	129	785900	473	214100	35
26	717259	344	931075	129	786184	473	213816	34
27	717466	344	930998	129	786468	473	213532	33
28	717673	344	930921	129	786752	473	213248	32
29	717879	344	930843	129	787036	473	212964	31
30	718085	343	930766	129	787319	472	212681	30
31	9.718291	343	9.930688	129	9.787603	472	10.212397	29
32	718497	343	930611	129	787886	472	212114	28
33	718703	343	930533	129	788170	472	211830	27
34	718909	343	930456	129	788453	472	211547	26
35	719114	342	930378	129	788736	472	211264	25
36	719320	342	930300	130	789019	472	210981	24
37	719525	342	930223	130	789302	471	210698	23
38	719730	342	930145	130	789585	471	210415	22
39	719935	341	930067	130	789868	471	210132	21
40	720140	341	929989	130	790151	471	209849	20
41	9.720345	341	9.929911	130	9.790433	471	10.209567	19
42	720549	341	929833	130	790716	471	209284	18
43	720754	340	929755	130	790999	471	209001	17
44	720958	340	929677	130	791281	471	208719	16
45	721162	340	929599	130	791563	470	208437	15
46	721366	340	929521	130	791846	470	208154	14
47	721570	340	929442	130	792128	470	207872	13
48	721774	339	929364	131	792410	470	207590	12
49	721978	339	929286	131	792692	470	207308	11
50	722181	339	929207	131	792974	470	207026	10
51	9.722385	339	9.929129	131	9.793256	470	10.206744	9
52	722588	339	929050	131	793538	469	206462	8
53	722791	338	928972	131	793819	469	206181	7
54	722994	338	928893	131	794101	469	205899	6
55	723197	338	928815	131	794383	469	205617	5
56	723400	338	928736	131	794664	469	205336	4
57	723603	337	928657	131	794945	469	205055	3
58	723805	337	928578	131	795227	469	204773	2
59	724007	337	928499	131	795508	468	204492	1
60	724210	337	928420	131	795789	468	204211	0
	Cosine		Sine		Cotang.		Tang.	M.

58 Degrees

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.724210	337	9.928420	132	9.795789	468	10.204211	60
1	724412	337	928342	132	796070	468	203930	59
2	724614	336	928263	132	796351	468	203649	58
3	724816	336	928183	132	796632	468	203368	57
4	725017	336	928104	132	796913	468	203087	56
5	725219	336	928025	132	797194	468	202806	55
6	725420	335	927946	132	797475	468	202525	54
7	725622	335	927867	132	797755	468	202245	53
8	725823	335	927787	132	798036	467	201964	52
9	726024	335	927708	132	798316	467	201684	51
10	726225	335	927629	132	798596	467	201404	50
11	9.726426	334	9.927549	132	9.798877	467	10.201123	49
12	726626	334	927470	133	799157	467	200843	48
13	726827	334	927390	133	799437	467	200563	47
14	727027	334	927310	133	799717	467	200283	46
15	727228	334	927231	133	799997	466	200003	45
16	727428	333	927151	133	800277	466	199723	44
17	727628	333	927071	133	800557	466	199443	43
18	727828	333	926991	133	800836	466	199164	42
19	728027	333	926911	133	801116	466	198884	41
20	728227	333	926831	133	801396	466	198604	40
21	9.728427	332	9.926751	133	9.801675	466	10.198325	39
22	728626	332	926671	133	801955	466	198045	38
23	728825	332	926591	133	802234	465	197766	37
24	729024	332	926511	134	802513	465	197487	36
25	729223	331	926431	134	802792	465	197208	35
26	729422	331	926351	134	803072	465	196928	34
27	729621	331	926270	134	803351	465	196649	33
28	729820	331	926190	134	803630	465	196370	32
29	730018	330	926110	134	803908	465	196092	31
30	730216	330	926029	134	804187	465	195813	30
31	9.730415	330	9.925949	134	9.804466	464	10.195534	29
32	730613	330	925868	134	804745	464	195255	28
33	730811	330	925788	134	805023	464	194977	27
34	731009	329	925707	134	805302	464	194698	26
35	731206	329	925626	134	805580	464	194420	25
36	731404	329	925545	135	805859	464	194141	24
37	731602	329	925465	135	806137	464	193863	23
38	731799	329	925384	135	806415	463	193585	22
39	731996	328	925303	135	806693	463	193307	21
40	732193	328	925222	135	806971	463	193029	20
41	9.732390	328	9.925141	135	9.807249	463	10.192751	19
42	732587	328	925060	135	807527	463	192473	18
43	732784	328	924979	135	807805	463	192195	17
44	732980	327	924897	135	808083	463	191917	16
45	733177	327	924816	135	808361	463	191639	15
46	733373	327	924735	136	808638	462	191362	14
47	733569	327	924654	136	808916	462	191084	13
48	733765	327	924572	136	809193	462	190807	12
49	733961	326	924491	136	809471	462	190529	11
50	734157	326	924409	136	809748	462	190252	10
51	9.734353	326	9.924328	136	9.810025	462	10.189975	9
52	734549	326	924246	136	810302	462	189698	8
53	734744	325	924164	136	810580	462	189420	7
54	734939	325	924083	136	810857	462	189143	6
55	735135	325	924001	136	811134	461	188866	5
56	735330	325	923919	136	811410	461	188590	4
57	735525	325	923837	136	811687	461	188313	3
58	735719	324	923755	137	811964	461	188036	2
59	735914	324	923673	137	812241	461	187759	1
60	736109	324	923591	137	812517	461	187483	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang	D.	Cotang.	
0	9.736109	324	9.923591	137	9.812517	461	10.187482	60
1	736303	324	923509	137	812794	461	187206	59
2	736498	324	923427	137	813070	461	186930	58
3	736692	323	923345	137	813347	460	186653	57
4	736886	323	923263	137	813623	460	186377	56
5	737080	323	923181	137	813899	460	186101	55
6	737274	323	923098	137	814175	460	185825	54
7	737467	323	923016	137	814452	460	185548	53
8	737661	322	922933	137	814728	460	185272	52
9	737855	322	922851	137	815004	460	184996	51
10	738048	322	922768	138	815279	460	184721	50
11	9.738241	322	9.922686	138	9.815555	459	10.184445	49
12	738434	322	922603	138	815831	459	184169	48
13	738627	321	922520	138	816107	459	183893	47
14	738820	321	922438	138	816382	459	183618	46
15	739013	321	922355	138	816658	459	183342	45
16	739206	321	922272	138	816933	459	183067	44
17	739398	321	922189	138	817209	459	182791	43
18	739590	320	922106	138	817484	459	182516	42
19	739783	320	922023	138	817759	459	182241	41
20	739975	320	921940	138	818035	458	181965	40
21	9.740167	320	9.921857	139	9.818310	458	10.181690	39
22	740359	320	921774	139	818585	458	181415	38
23	740550	319	921691	139	818860	458	181140	37
24	740742	319	921607	139	819135	458	180865	36
25	740934	319	921524	139	819410	458	180590	35
26	741125	319	921441	139	819684	458	180316	34
27	741316	319	921357	139	819959	458	180041	33
28	741508	318	921274	139	820234	458	179766	32
29	741699	318	921190	139	820508	457	179492	31
30	741889	318	921107	139	820783	457	179217	30
31	9.742080	318	9.921023	139	9.821057	457	10.178943	29
32	742271	318	920939	140	821332	457	178668	28
33	742462	317	920856	140	821606	457	178394	27
34	742652	317	920772	140	821880	457	178120	26
35	742842	317	920688	140	822154	457	177846	25
36	743033	317	920604	140	822429	457	177571	24
37	743223	317	920520	140	822703	457	177297	23
38	743413	316	920436	140	822977	456	177023	22
39	743602	316	920352	140	823250	456	176750	21
40	743792	316	920268	140	823524	456	176476	20
41	9.743982	316	9.920184	140	9.823798	456	10.176202	19
42	744171	316	920099	140	824072	456	175928	18
43	744361	315	920015	140	824345	456	175655	17
44	744550	315	919931	141	824619	456	175381	16
45	744739	315	919846	141	824893	456	175107	15
46	744928	315	919762	141	825166	456	174834	14
47	745117	315	919677	141	825439	455	174561	13
48	745306	314	919593	141	825713	455	174287	12
49	745494	314	919508	141	825986	455	174014	11
50	745683	314	919424	141	826259	455	173741	10
51	9.745871	314	9.919339	141	9.826532	455	10.173468	9
52	746059	314	919254	141	826805	455	173195	8
53	746248	313	919169	141	827078	455	172922	7
54	746436	313	919085	141	827351	455	172649	6
55	746624	313	919000	141	827624	455	172376	5
56	746812	313	918915	142	827897	454	172103	4
57	746999	313	918830	142	828170	454	171830	3
58	747187	312	918745	142	828442	454	171558	2
59	747374	312	918659	142	828715	454	171285	1
60	747562	312	918574	142	828987	454	171013	0
	Cosine		Sine		Cotang.		Tang.	M.

56 Degrees.

M.	Sine	D.	Cosine	D.	Tang.	D	Cotang.	
0	9.747562	312	9.918574	142	9.828987	454	10.171013	60
1	747749	312	918489	142	829260	454	170740	59
2	747936	312	918404	142	829532	454	170468	58
3	748123	311	918318	142	829805	454	170195	57
4	748310	311	918233	142	830077	454	169923	56
5	748497	311	918147	142	830349	453	169651	55
6	748683	311	918062	142	830621	453	169379	54
7	748870	311	917976	143	830893	453	169107	53
8	749056	310	917891	143	831165	453	168835	52
9	749243	310	917805	143	831437	453	168563	51
10	749429	310	917719	143	831709	453	168291	50
11	9.749615	310	9.917634	143	9.831981	453	10.168019	49
12	749801	310	917548	143	832253	453	167747	48
13	749987	309	917462	143	832525	453	167475	47
14	750172	309	917376	143	832796	453	167204	46
15	750358	309	917290	143	833068	452	166932	45
16	750543	309	917204	143	833339	452	166661	44
17	750729	309	917118	144	833611	452	166389	43
18	750914	308	917032	144	833882	452	166118	42
19	751099	308	916946	144	834154	452	165846	41
20	751284	308	916859	144	834425	452	165575	40
21	9.751469	308	9.916773	144	9.834696	452	10.165304	39
22	751654	308	916687	144	834967	452	165033	38
23	751839	308	916600	144	835238	452	164762	37
24	752023	307	916514	144	835509	452	164491	36
25	752208	307	916427	144	835780	451	164220	35
26	752392	307	916341	144	836051	451	163949	34
27	752576	307	916254	144	836322	451	163678	33
28	752760	307	916167	145	836593	451	163407	32
29	752944	306	916081	145	836864	451	163136	31
30	753128	306	915994	145	837134	451	162866	30
31	9 753312	306	9.915907	145	9.837405	451	10.162595	29
32	753495	306	915820	145	837675	451	162325	28
33	753679	306	915733	145	837946	451	162054	27
34	753862	305	915646	145	838216	451	161784	26
35	754046	305	915559	145	838487	450	161513	25
36	754229	305	915472	145	838757	450	161243	24
37	754412	305	915385	145	839027	450	160973	23
38	754595	305	915297	145	839297	450	160703	22
39	754778	304	915210	145	839568	450	160432	21
40	754960	304	915123	146	839838	450	160162	20
41	9.755143	304	9.915035	146	9.840108	450	10.159892	19
42	755326	304	914948	146	840378	450	159622	18
43	755508	304	914860	146	840647	450	159353	17
44	755690	304	914773	146	840917	449	159083	16
45	755872	303	914685	146	841187	449	158813	15
46	756054	303	914598	146	841457	449	158543	14
47	756236	303	914510	146	841726	449	158274	13
48	756418	303	914422	146	841996	449	158004	12
49	756600	303	914334	146	842266	449	157734	11
50	756782	302	914246	147	842535	449	157465	10
51	9.756963	302	9.914158	147	9.842805	449	10.157195	9
52	757144	302	914070	147	843074	449	156926	8
53	757326	302	913982	147	843343	449	156657	7
54	757507	302	913894	147	843612	449	156388	6
55	757688	301	913806	147	843882	448	156118	5
56	757869	301	913718	147	844151	448	155849	4
57	758050	301	913630	147	844420	448	155580	3
58	758230	301	913541	147	844689	448	155311	2
59	758411	301	913453	147	844958	448	155042	1
60	758591	301	913365	147	845227	448	154773	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.758591	301	9.913365	147	9.845227	448	10.154773	60
1	758772	300	913276	147	845496	448	154504	59
2	758952	300	913187	148	845764	448	154236	58
3	759132	300	913099	148	846033	448	153967	57
4	759312	300	913010	148	846302	448	153698	56
5	759492	300	912922	148	846570	447	153430	55
6	759672	299	912833	148	846839	447	153161	54
7	759852	299	912744	148	847107	447	152893	53
8	760031	299	912655	148	847376	447	152624	52
9	760211	299	912566	148	847644	447	152356	51
10	760390	299	912477	148	847913	447	152087	50
11	9.760569	298	9.912388	148	9.848181	447	10.151819	49
12	760748	298	912299	149	848449	447	151551	48
13	760927	298	912210	149	848717	447	151283	47
14	761106	298	912121	149	848986	447	151014	46
15	761285	298	912031	149	849254	447	150746	45
16	761464	298	911942	149	849522	447	150478	44
17	761642	297	911853	149	849790	446	150210	43
18	761821	297	911763	149	850058	446	149942	42
19	761999	297	911674	149	850325	446	149675	41
20	762177	297	911584	149	850593	446	149407	40
21	9.762356	297	9.911495	149	9.850861	446	10.149139	39
22	762534	296	911405	149	851129	446	148871	38
23	762712	296	911315	150	851396	446	148604	37
24	762889	296	911226	150	851664	446	148336	36
25	763067	296	911136	150	851931	446	148069	35
26	763245	296	911046	150	852199	446	147801	34
27	763422	296	910956	150	852466	446	147534	33
28	763600	295	910866	150	852733	445	147267	32
29	763777	295	910776	150	853001	445	146999	31
30	763954	295	910686	150	853268	445	146732	30
31	9.764131	295	9.910596	150	9.853535	445	10.146465	29
32	764308	295	910506	150	853802	445	146198	28
33	764485	294	910415	150	854069	445	145931	27
34	764662	294	910325	151	854336	445	145664	26
35	764838	294	910235	151	854603	445	145397	25
36	765015	294	910144	151	854870	445	145130	24
37	765191	294	910054	151	855137	445	144863	23
38	765367	294	909963	151	855404	445	144596	22
39	765544	293	909873	151	855671	444	144329	21
40	765720	293	909782	151	855938	444	144062	20
41	9.765896	293	9.909691	151	9.856204	444	10.143796	19
42	766072	293	909601	151	856471	444	143529	18
43	766247	293	909510	151	856737	444	143263	17
44	766423	293	909419	151	857004	444	142996	16
45	766598	292	909328	152	857270	444	142730	15
46	766774	292	909237	152	857537	444	142463	14
47	766949	292	909146	152	857803	444	142197	13
48	767124	292	909055	152	858069	444	141931	12
49	767300	292	908964	152	858336	444	141664	11
50	767475	291	908873	152	858602	443	141398	10
51	9.767649	291	9.908781	152	9.858868	443	10.141132	9
52	767824	291	908690	152	859134	443	140866	8
53	767999	291	908599	152	859400	443	140600	7
54	768173	291	908507	152	859666	443	140334	6
55	768348	290	908416	153	859932	443	140068	5
56	768522	290	908324	153	860198	443	139802	4
57	768697	290	908233	153	860464	443	139536	3
58	768871	290	908141	153	860730	443	139270	2
59	769045	290	908049	153	860995	443	139005	1
60	769219	290	907958	153	861261	443	138739	0
	Cosine		Sine		Cotang.		Tang.	M.

54 Degrees.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.769219	290	9.907958	153	9.861261	443	10.138739	60
1	769393	289	907866	153	861527	443	138473	59
2	769566	289	907774	153	861792	442	138208	58
3	769740	289	907682	153	862058	442	137942	57
4	769913	289	907590	153	862323	442	137677	56
5	770087	289	907498	153	862589	442	137411	55
6	770260	288	907406	153	862854	442	137146	54
7	770433	288	907314	154	863119	442	136881	53
8	770606	288	907222	154	863385	442	136615	52
9	770779	288	907129	154	863650	442	136350	51
10	770952	288	907037	154	863915	442	136085	50
11	9.771125	288	9 906945	154	9.864180	442	10.135820	49
12	771298	287	906852	154	864445	442	135555	48
13	771470	287	906760	154	864710	442	135290	47
14	771643	287	906667	154	864975	441	135025	46
15	771815	287	906575	154	865240	441	134760	45
16	771987	287	906482	154	865505	441	134495	44
17	772159	287	906389	155	865770	441	134230	43
18	772331	286	906296	155	866035	441	133965	42
19	772503	286	906204	155	866300	441	133700	41
20	772675	286	906111	155	866564	441	133436	40
21	9.772847	286	9.906018	155	9.866829	441	10.133171	39
22	773018	286	905925	155	867094	441	132906	38
23	773190	286	905832	155	867358	441	132642	37
24	773361	285	905739	155	867623	441	132377	36
25	773533	285	905645	155	867887	441	132113	35
26	773704	285	905552	155	868152	440	131848	34
27	773875	285	905459	155	868416	440	131584	33
28	774046	285	905366	156	868680	440	131320	32
29	774217	285	905272	156	868945	440	131055	31
30	774388	284	905179	156	869209	440	130791	30
31	9.774558	284	9.905085	156	9.869473	440	10.130527	29
32	774729	284	904992	156	869737	440	130263	28
33	774899	284	904898	156	870001	440	129999	27
34	775070	284	904804	156	870265	440	129735	26
35	775240	284	904711	156	870529	440	129471	25
36	775410	283	904617	156	870793	440	129207	24
37	775580	283	904523	156	871057	440	128943	23
38	775750	283	904429	157	871321	440	128679	22
39	775920	283	904335	157	871585	440	128415	21
40	776090	283	904241	157	871849	439	128151	20
41	9.776259	283	9.904147	157	9.872112	439	10.127888	19
42	776429	282	904053	157	872376	439	127624	18
43	776598	282	903959	157	872640	439	127360	17
44	776768	282	903864	157	872903	439	127097	16
45	776937	282	903770	157	873167	439	126833	15
46	777106	282	903676	157	873430	439	126570	14
47	777275	281	903581	157	873694	439	126306	13
48	777444	281	903487	157	873957	439	126043	12
49	777613	281	903392	158	874220	439	125780	11
50	777781	281	903298	158	874484	439	125516	10
51	9 777950	281	9.903203	158	9.874747	439	10.125253	9
52	778119	281	903108	158	875010	439	124990	8
53	778287	280	903014	158	875273	438	124727	7
54	778455	280	902919	158	875536	438	124464	6
55	778624	280	902824	158	875800	438	124200	5
56	778792	280	902729	158	876063	438	123937	4
57	778960	280	902634	158	876326	438	123674	3
58	779128	280	902539	159	876589	438	123411	2
59	779295	279	902444	159	876851	438	123149	1
60	779463	279	902349	159	877114	438	122886	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.779463	279	9.902349	159	9.877114	438	10.122886	60
1	779631	279	902253	159	877377	438	122623	59
2	779798	279	902158	159	877640	438	122360	58
3	779966	279	902063	159	877903	438	122097	57
4	780133	279	901967	159	878165	438	121835	56
5	780300	278	901872	159	878428	438	121572	55
6	780467	278	901776	159	878691	438	121309	54
7	780634	278	901681	159	878953	437	121047	53
8	780801	278	901585	159	879216	437	120784	52
9	780968	278	901490	159	879478	437	120522	51
10	781134	278	901394	160	879741	437	120259	50
11	9.781301	277	9.901298	160	9.880003	437	10.119997	49
12	781468	277	901202	160	880265	437	119735	48
13	781634	277	901106	160	880528	437	119472	47
14	781800	277	901010	160	880790	437	119210	46
15	781966	277	900914	160	881052	437	118948	45
16	782132	277	900818	160	881314	437	118686	44
17	782298	276	900722	160	881576	437	118424	43
18	782464	276	900626	160	881839	437	118161	42
19	782630	276	900529	160	882101	437	117899	41
20	782796	276	900433	161	882363	436	117637	40
21	9.782961	276	9.900337	161	9.882625	436	10.117375	39
22	783127	276	900240	161	882887	436	117113	38
23	783292	275	900144	161	883148	436	116852	37
24	783458	275	900047	161	883410	436	116590	36
25	783623	275	899951	161	883672	436	116328	35
26	783788	275	899854	161	883934	436	116066	34
27	783953	275	899757	161	884196	436	115804	33
28	784118	275	899660	161	884457	436	115543	32
29	784282	274	899564	161	884719	436	115281	31
30	784447	274	899467	162	884980	436	115020	30
31	9.784612	274	9.899370	162	9.885242	436	10.114758	29
32	784776	274	899273	162	885503	436	114497	28
33	784941	274	899176	162	885765	436	114235	27
34	785105	274	899078	162	886026	436	113974	26
35	785269	273	898981	162	886288	436	113712	25
36	785433	273	898884	162	886549	435	113451	24
37	785597	273	898787	162	886810	435	113190	23
38	785761	273	898689	162	887072	435	112928	22
39	785925	273	898592	162	887333	435	112667	21
40	786089	273	898494	163	887594	435	112406	20
41	9.786252	272	9.898397	163	9.887855	435	10.112145	19
42	786416	272	898299	163	888116	435	111884	18
43	786579	272	898202	163	888377	435	111623	17
44	786742	272	898104	163	888639	435	111361	16
45	786906	272	898006	163	888900	435	111100	15
46	787069	272	897908	163	889160	435	110840	14
47	787232	271	897810	163	889421	435	110579	13
48	787395	271	897712	163	889682	435	110318	12
49	787557	271	897614	163	889943	435	110057	11
50	787720	271	897516	163	890204	434	109796	10
51	9.787883	271	9.897418	164	9.890465	434	10.109535	9
52	788045	271	897320	164	890725	434	109275	8
53	788208	271	897222	164	890986	434	109014	7
54	788370	270	897123	164	891247	434	108753	6
55	788532	270	897025	164	891507	434	108493	5
56	788694	270	896926	164	891768	434	108232	4
57	788856	270	896828	164	892028	434	107972	3
58	789018	270	896729	164	892289	434	107711	2
59	789180	270	896631	164	892549	434	107451	1
60	789342	269	896532	164	892810	434	107190	0
	Cosine		Sine		Cotang.		Tang.	M.

52 Degrees

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.789342	269	9.896532	164	9.892810	434	10.107190	60
1	789504	269	896433	165	893070	434	106930	59
2	789665	269	896335	165	893331	434	106669	58
3	789827	269	896236	165	893591	434	106409	57
4	789988	269	896137	165	893851	434	106149	56
5	790149	269	896038	165	894111	434	105889	55
6	790310	268	895939	165	894371	434	105629	54
7	790471	268	895840	165	894632	433	105368	53
8	790632	268	895741	165	894892	433	105108	52
9	790793	268	895641	165	895152	433	104848	51
10	790954	268	895542	165	895412	433	104588	50
11	9.791115	268	9.895443	166	9.895672	433	10 104328	49
12	791275	267	895343	166	895932	433	104068	48
13	791436	267	895244	166	896192	433	103808	47
14	791596	267	895145	166	896452	433	103548	46
15	791757	267	895045	166	896712	433	103288	45
16	791917	267	894945	166	896971	433	103029	44
17	792077	267	894846	166	897231	432	102769	43
18	792237	266	894746	166	897491	433	102509	42
19	792397	266	894646	166	897751	433	102249	41
20	792557	266	894546	166	898010	433	101990	40
21	9.792716	266	9.894446	167	9.898270	433	10.101730	39
22	792876	266	894346	167	898530	433	101470	38
23	793035	266	894246	167	898789	433	101211	37
24	793195	265	894146	167	899049	432	100951	36
25	793354	265	894046	167	899308	432	100692	35
26	793514	265	893946	167	899568	432	100432	34
27	793673	265	893846	167	899827	432	100173	33
28	793832	265	893745	167	900086	432	099914	32
29	793991	265	893645	167	900346	432	099654	31
30	794150	264	893544	167	900605	432	099395	30
31	9.794308	264	9.893444	168	9.900864	432	10.099136	29
32	794467	264	893343	168	901124	432	098876	28
33	794626	264	893243	168	901383	432	098617	27
34	794784	264	893142	168	901642	432	098358	26
35	794942	264	893041	168	901901	432	098099	25
36	795101	264	892940	168	902160	432	097840	24
37	795259	263	892839	168	902419	432	097581	23
38	795417	263	892739	168	902679	432	097321	22
39	795575	263	892638	168	902938	432	097062	21
40	795733	263	892536	168	903197	431	096803	20
41	9.795891	263	9.892435	169	9.903455	431	10.096545	19
42	796049	263	892334	169	903714	431	096286	18
43	796206	263	892233	169	903973	431	096027	17
44	796364	262	892132	169	904232	431	095768	16
45	796521	262	892030	169	904491	431	095509	15
46	796679	262	891929	169	904750	431	095250	14
47	796836	262	891827	169	905008	431	094992	13
48	796993	262	891726	169	905267	431	094733	12
49	797150	261	891624	169	905526	431	094474	11
50	797307	261	891523	170	905784	431	094216	10
51	9.797464	261	9.891421	170	9.906043	431	10 093957	9
52	797621	261	891319	170	906302	431	093698	8
53	797777	261	891217	170	906560	431	093440	7
54	797934	261	891115	170	906819	431	093181	6
55	798091	261	891013	170	907077	431	092923	5
56	798247	261	890911	170	907336	431	092664	4
57	798403	260	890809	170	907594	431	092406	3
58	798560	260	890707	170	907852	431	092148	2
59	798716	260	890605	170	908111	430	091889	1
60	798872	260	890503	170	908369	430	091631	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine.	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.798872	260	9.890503	170	9.908369	430	10.091631	60
1	799028	260	890400	171	908628	430	091372	59
2	799184	260	890298	171	908886	430	091114	58
3	799339	259	890195	171	909144	430	090856	57
4	799495	259	890093	171	909402	430	090598	56
5	799651	259	889990	171	909660	430	090340	55
6	799806	259	889888	171	909918	430	090082	54
7	799962	259	889785	171	910177	430	089823	53
8	800117	259	889682	171	910435	430	089565	52
9	800272	258	889579	171	910693	430	089307	51
10	800427	258	889477	171	910951	430	089049	50
11	9.800582	258	9.889374	172	9.911209	430	10.088791	49
12	800737	258	889271	172	911467	430	088533	48
13	800892	258	889168	172	911724	430	088276	47
14	801047	258	889064	172	911982	430	088018	46
15	801201	258	888961	172	912240	430	087760	45
16	801356	257	888858	172	912498	430	087502	44
17	801511	257	888755	172	912756	430	087244	43
18	801665	257	888651	172	913014	429	086986	42
19	801819	257	888548	172	913271	429	086729	41
20	801973	257	888444	173	913529	429	086471	40
21	9.802128	257	9.888341	173	9.913787	429	10.086213	39
22	802282	256	888237	173	914044	429	085956	38
23	802436	256	888134	173	914302	429	085698	37
24	802589	256	888030	173	914560	429	085440	36
25	802743	256	887926	173	914817	429	085183	35
26	802897	256	887822	173	915075	429	084925	34
27	803050	256	887718	173	915332	429	084668	33
28	803204	256	887614	173	915590	429	084410	32
29	803357	255	887510	173	915847	429	084153	31
30	803511	255	887406	174	916104	429	083896	30
31	9.803664	255	9.887302	174	9.916362	429	10.083638	29
32	803817	255	887198	174	916619	429	083381	28
33	803970	255	887093	174	916877	429	083123	27
34	804123	255	886989	174	917134	429	082866	26
35	804276	254	886885	174	917391	429	082609	25
36	804428	254	886780	174	917648	429	082352	24
37	804581	254	886676	174	917905	429	082095	23
38	804734	254	886571	174	918163	428	081837	22
39	804886	254	886466	174	918420	428	081580	21
40	805039	254	886362	175	918677	428	081323	20
41	9.805191	254	9.886257	175	9.918934	428	10.081066	19
42	805343	253	886152	175	919191	428	080809	18
43	805495	253	886047	175	919448	428	080552	17
44	805647	253	885942	175	919705	428	080295	16
45	805799	253	885837	175	919962	428	080038	15
46	805951	253	885732	175	920219	428	079781	14
47	806103	253	885627	175	920476	428	079524	13
48	806254	253	885522	175	920733	428	079267	12
49	806406	252	885416	175	920990	428	079010	11
50	806557	252	885311	176	921247	428	078753	10
51	9.806709	252	9.885205	176	9.921503	428	10.078497	9
52	806860	252	885100	176	921760	428	078240	8
53	807011	252	884994	176	922017	428	077983	7
54	807163	252	884889	176	922274	428	077726	6
55	807314	252	884783	176	922530	428	077470	5
56	807465	251	884677	176	922787	428	077213	4
57	807615	251	884572	176	923044	428	076956	3
58	807766	251	884466	176	923300	428	076700	2
59	807917	251	884360	176	923557	427	076443	1
60	808067	251	884254	177	923813	427	076187	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D	Tang.	D.	Cotang.	
0	9.808067	251	9.884254	177	9.923813	427	10.076187	60
1	808218	251	884148	177	924070	427	075930	59
2	808368	251	884042	177	924327	427	075673	58
3	808519	250	883936	177	924583	427	075417	57
4	808669	250	883829	177	924840	427	075160	56
5	808819	250	883723	177	925096	427	074904	55
6	808969	250	883617	177	925352	427	074648	54
7	809119	250	883510	177	925609	427	074391	53
8	809269	250	883404	177	925865	427	074135	52
9	809419	249	883297	178	926122	427	073878	51
10	809569	249	883191	178	926378	427	073622	50
11	9.809718	249	9.883084	178	9.926634	427	10.073366	49
12	809868	249	882977	178	926890	427	073110	48
13	810017	249	882871	178	927147	427	072853	47
14	810167	249	882764	178	927403	427	072597	46
15	810316	248	882657	178	927659	427	072341	45
16	810465	248	882550	178	927915	427	072085	44
17	810614	248	882443	178	928171	427	071829	43
18	810763	248	882336	179	928427	427	071573	42
19	810912	248	882229	179	928683	427	071317	41
20	811061	248	882121	179	928940	427	071060	40
21	9.811210	248	9.882014	179	9.929196	427	10.070804	39
22	811358	247	881907	179	929452	427	070548	38
23	811507	247	881799	179	929708	427	070292	37
24	811655	247	881692	179	929964	426	070036	36
25	811804	247	881584	179	930220	426	069780	35
26	811952	247	881477	179	930475	426	069525	34
27	812100	247	881369	179	930731	426	069269	33
28	812248	247	881261	180	930987	426	069013	32
29	812396	246	881153	180	931243	426	068757	31
30	812544	246	881046	180	931499	426	068501	30
31	9.812692	246	9.880938	180	9.931755	426	10.068245	29
32	812840	246	880830	180	932010	426	067990	28
33	812988	246	880722	180	932266	426	067734	27
34	813135	246	880613	180	932522	426	067478	26
35	813283	246	880505	180	932778	426	067222	25
36	813430	245	880397	180	933033	426	066967	24
37	813578	245	880289	181	933289	426	066711	23
38	813725	245	880180	181	933545	426	066455	22
39	813872	245	880072	181	933800	426	066200	21
40	814019	245	879963	181	934056	426	065944	20
41	9.814166	245	9.879855	181	9.934311	426	10.065689	19
42	814313	245	879746	181	934567	426	065433	18
43	814460	244	879637	181	934823	426	065177	17
44	814607	244	879529	181	935078	426	064922	16
45	814753	244	879420	181	935333	426	064667	15
46	814900	244	879311	181	935589	426	064411	14
47	815046	244	879202	182	935844	426	064156	13
48	815193	244	879093	182	936100	426	063900	12
49	815339	244	878984	182	936355	426	063645	11
50	815485	243	878875	182	936610	426	063390	10
51	9.815631	243	9.878766	182	9.936866	425	10.063134	9
52	815778	243	878656	182	937121	425	062879	8
53	815924	243	878547	182	937376	425	062624	7
54	816069	243	878438	182	937632	425	062368	6
55	816215	243	878328	182	937887	425	062113	5
56	816361	243	878219	183	938142	425	061858	4
57	816507	242	878109	183	938398	425	061602	3
58	816652	242	877999	183	938653	425	061347	2
59	816798	242	877890	183	938908	425	061092	1
60	816943	242	877780	183	939163	425	060837	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.816943	242	9.877780	183	9.939163	425	10.060837	60
1	817088	242	877670	183	939418	425	060582	59
2	817233	242	877560	183	939673	425	060327	58
3	817379	242	877450	183	939928	425	060072	57
4	817524	241	877340	183	940183	425	059817	56
5	817668	241	877230	184	940438	425	059562	55
6	817813	241	877120	184	940694	425	059306	54
7	817958	241	877010	184	940949	425	059051	53
8	818103	241	876899	184	941204	425	058796	52
9	818247	241	876789	184	941458	425	058542	51
10	818392	241	876678	184	941714	425	058286	50
11	9.818536	240	9.876568	184	9.941968	425	10.058032	49
12	818681	240	876457	184	942223	425	057777	48
13	818825	240	876347	184	942478	425	057522	47
14	818969	240	876236	185	942733	425	057267	46
15	819113	240	876125	185	942988	425	057012	45
16	819257	240	876014	185	943243	425	056757	44
17	819401	240	875904	185	943498	425	056502	43
18	819545	239	875793	185	943752	425	056248	42
19	819689	239	875682	185	944007	425	055993	41
20	819832	239	875571	185	944262	425	055738	40
21	9.819976	239	9.875459	185	9.944517	425	10.055483	39
22	820120	239	875348	185	944771	424	055229	38
23	820263	239	875237	185	945026	424	054974	37
24	820406	239	875126	186	945281	424	054719	36
25	820550	238	875014	186	945535	424	054465	35
26	820693	238	874903	186	945790	424	054210	34
27	820836	238	874791	186	946045	424	053955	33
28	820979	238	874680	186	946299	424	053701	32
29	821122	238	874568	186	946554	424	053446	31
30	821265	238	874456	186	946808	424	053192	30
31	9.821407	238	9.874344	186	9.947063	424	10.052937	29
32	821550	238	874232	187	947318	424	052682	28
33	821693	237	874121	187	947572	424	052428	27
34	821835	237	874009	187	947826	424	052174	26
35	821977	237	873896	187	948081	424	051919	25
36	822120	237	873784	187	948336	424	051664	24
37	822262	237	873672	187	948590	424	051410	23
38	822404	237	873560	187	948844	424	051156	22
39	822546	237	873448	187	949099	424	050901	21
40	822688	236	873335	187	949353	424	050647	20
41	9.822830	236	9.873223	187	9.949607	424	10.050393	19
42	822972	236	873110	188	949862	424	050138	18
43	823114	236	872998	188	950116	424	049884	17
44	823255	236	872885	188	950370	424	049630	16
45	823397	236	872772	188	950625	424	049375	15
46	823539	236	872659	188	950879	424	049121	14
47	823680	235	872547	188	951133	424	048867	13
48	823821	235	872434	188	951388	424	048612	12
49	823963	235	872321	188	951642	424	048358	11
50	824104	235	872208	188	951896	424	048104	10
51	9.824245	235	9.872095	189	9.952150	424	10.047850	9
52	824386	235	871981	189	952405	424	047595	8
53	824527	235	871868	189	952659	424	047341	7
54	824668	234	871755	189	952913	424	047087	6
55	824808	234	871641	189	953167	423	046833	5
56	824949	234	871528	189	953421	423	046579	4
57	825090	234	871414	189	953675	423	046325	3
58	825230	234	871301	189	953929	423	046071	2
59	825371	234	871187	189	954183	423	045817	1
60	825511	234	871073	190	954437	423	045563	0
	Cosine		Sine		Cotang.		Tang.	M.

M	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.825511	234	9.871073	190	9.954437	423	10.045563	60
1	825651	233	870960	190	954691	423	045309	59
2	825791	233	870846	190	954945	423	045055	58
3	825931	233	870732	190	955200	423	044800	57
4	826071	233	870618	190	955454	423	044546	56
5	826211	233	870504	190	955707	423	044293	55
6	826351	233	870390	190	955961	423	044039	54
7	826491	233	870276	190	956215	423	043785	53
8	826631	233	870161	190	956469	423	043531	52
9	826770	232	870047	191	956723	423	043277	51
10	826910	232	869933	191	956977	423	043023	50
11	9.827049	232	9.869818	191	9.957231	423	10.042769	49
12	827189	232	869704	191	957485	423	042515	48
13	827328	232	869589	191	957739	423	042261	47
14	827467	232	869474	191	957993	423	042007	46
15	827606	232	869360	191	958246	423	041754	45
16	827745	232	869245	191	958500	423	041500	44
17	827884	231	869130	191	958754	423	041246	43
18	828023	231	869015	192	959008	423	040992	42
19	828162	231	868900	192	959262	423	040738	41
20	828301	231	868785	192	959516	423	040484	40
21	9.828439	231	9.868670	192	9.959769	423	10.040231	39
22	828578	231	868555	192	960023	423	039977	38
23	828716	231	868440	192	960277	423	039723	37
24	828855	230	868324	192	960531	423	039469	36
25	828993	230	868209	192	960784	423	039216	35
26	829131	230	868093	192	961038	423	038962	34
27	829269	230	867978	193	961291	423	038709	33
28	829407	230	867862	193	961545	423	038455	32
29	829545	230	867747	193	961799	423	038201	31
30	829683	230	867631	193	962052	423	037948	30
31	9.829821	229	9.867515	193	9.962306	423	10.037694	29
32	829959	229	867399	193	962560	423	037440	28
33	830097	229	867283	193	962813	423	037187	27
34	830234	229	867167	193	963067	423	036933	26
35	830372	229	867051	193	963320	423	036680	25
36	830509	229	866935	194	963574	423	036426	24
37	830646	229	866819	194	963827	423	036173	23
38	830784	229	866703	194	964081	423	035919	22
39	830921	228	866586	194	964335	423	035665	21
40	831058	228	866470	194	964588	422	035412	20
41	9.831195	228	9.866353	194	9.964842	422	10.035158	19
42	831332	228	866237	194	965095	422	034905	18
43	831469	228	866120	194	965349	422	034651	17
44	831606	228	866004	195	965602	422	034398	16
45	831742	228	865887	195	965855	422	034145	15
46	831879	228	865770	195	966109	422	033891	14
47	832015	227	865653	195	966362	422	033638	13
48	832152	227	865536	195	966616	422	033384	12
49	832288	227	865419	195	966869	422	033131	11
50	832425	227	865302	195	967123	422	032877	10
51	9.832561	227	9.865185	195	9.967376	422	10.032624	9
52	832697	227	865068	195	967629	422	032371	8
53	832833	227	864950	195	967883	422	032117	7
54	832969	226	864833	196	968136	422	031864	6
55	833105	226	864716	196	968389	422	031611	5
56	833241	226	864598	196	968643	422	031357	4
57	833377	226	864481	196	968896	422	031104	3
58	833512	226	864363	196	969149	422	030851	2
59	833648	226	864245	196	969403	422	030597	1
60	833783	226	864127	196	969656	422	030344	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.833783	226	9.864127	196	9.969656	422	10.030344	60
1	833919	225	864010	196	969909	422	030091	59
2	834054	225	863892	197	970162	422	029838	58
3	834189	225	863774	197	970416	422	029584	57
4	834325	225	863656	197	970669	422	029331	56
5	834460	225	863538	197	970922	422	029078	55
6	834595	225	863419	197	971175	422	028825	54
7	834730	225	863301	197	971429	422	028571	53
8	834865	225	863183	197	971682	422	028318	52
9	834999	224	863064	197	971935	422	028065	51
10	835134	224	862946	198	972188	422	027812	50
11	9.835269	224	9.862827	198	9.972441	422	10.027559	49
12	835403	224	862709	198	972694	422	027306	48
13	835538	224	862590	198	972948	422	027052	47
14	835672	224	862471	198	973201	422	026799	46
15	835807	224	862353	198	973454	422	026546	45
16	835941	224	862234	198	973707	422	026293	44
17	836075	223	862115	198	973960	422	026040	43
18	836209	223	861996	198	974213	422	025787	42
19	836343	223	861877	198	974466	422	025534	41
20	836477	223	861758	199	974719	422	025281	40
21	9.836611	223	9.861638	199	9.974973	422	10.025027	39
22	836745	223	861519	199	975226	422	024774	38
23	836878	223	861400	199	975479	422	024521	37
24	837012	222	861280	199	975732	422	024268	36
25	837146	222	861161	199	975985	422	024015	35
26	837279	222	861041	199	976238	422	023762	34
27	837412	222	860922	199	976491	422	023509	33
28	837546	222	860802	199	976744	422	023256	32
29	837679	222	860682	200	976997	422	023003	31
30	837812	222	860562	200	977250	422	022750	30
31	9.837945	222	9.860442	200	9.977503	422	10.022497	29
32	838078	221	860322	200	977756	422	022244	28
33	838211	221	860202	200	978009	422	021991	27
34	838344	221	860082	200	978262	422	021738	26
35	838477	221	859962	200	978515	422	021485	25
36	838610	221	859842	200	978768	422	021232	24
37	838742	221	859721	201	979021	422	020979	23
38	838875	221	859601	201	979274	422	020726	22
39	839007	221	859480	201	979527	422	020473	21
40	839140	220	859360	201	979780	422	020220	20
41	9.839272	220	9.859239	201	9.980033	422	10.019967	19
42	839404	220	859119	201	980286	422	019714	18
43	839536	220	858998	201	980538	422	019462	17
44	839668	220	858877	201	980791	421	019209	16
45	839800	220	858756	202	981044	421	018956	15
46	839932	220	858635	202	981297	421	018703	14
47	840064	219	858514	202	981550	421	018450	13
48	840196	219	858393	202	981803	421	018197	12
49	840328	219	858272	202	982056	421	017944	11
50	840459	219	858151	202	982309	421	017691	10
51	9.840591	219	9.858029	202	9.982562	421	10.017438	9
52	840722	219	857908	202	982814	421	017186	8
53	840854	219	857786	202	983067	421	016933	7
54	840985	219	857665	203	983320	421	016680	6
55	841116	218	857543	203	983573	421	016427	5
56	841247	218	857422	203	983826	421	016174	4
57	841378	218	857300	203	984079	421	015921	3
58	841509	218	857178	203	984331	421	015669	2
59	841640	218	857056	203	984584	421	015416	1
60	841771	218	856934	203	984837	421	015163	0
	Cosine		Sine		Cotang.		Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.841771	218	9.856934	203	9.984837	421	10.015163	60
1	841902	218	856812	203	985090	421	014910	59
2	842033	218	856690	204	985343	421	014657	58
3	842163	217	856568	204	985596	421	014404	57
4	842294	217	856446	204	985848	421	014152	56
5	842424	217	856323	204	986101	421	013899	55
6	842555	217	856201	204	986354	421	013646	54
7	842685	217	856078	204	986607	421	013393	53
8	842815	217	855956	204	986860	421	013140	52
9	842946	217	855833	204	987112	421	012888	51
10	843076	217	855711	205	987365	421	012635	50
11	9.843206	216	9.855588	205	9.987618	421	0.012382	49
12	843336	216	855465	205	987871	421	012129	48
13	843466	216	855342	205	988123	421	011877	47
14	843595	216	855219	205	988376	421	011624	46
15	843725	216	855096	205	988629	421	011371	45
16	843855	216	854973	205	988882	421	011118	44
17	843984	216	854850	205	989134	421	010866	43
18	844114	215	854727	206	989387	421	010613	42
19	844243	215	854603	206	989640	421	010360	41
20	844372	215	854480	206	989893	421	010107	40
21	9.844502	215	9.854356	206	9.990145	421	10.009855	39
22	844631	215	854233	206	990398	421	009602	38
23	844760	215	854109	206	990651	421	009349	37
24	844889	215	853986	206	990903	421	009097	36
25	845018	215	853862	206	991156	421	008844	35
26	845147	215	853738	206	991409	421	008591	34
27	845276	214	853614	207	991662	421	008338	33
28	845405	214	853490	207	991914	421	008086	32
29	845533	214	853366	207	992167	421	007833	31
30	845662	214	853242	207	992420	421	007580	30
31	9.845790	214	9.853118	207	9.992672	421	10 007328	29
32	845919	214	852994	207	992925	421	007075	28
33	846047	214	852869	207	993178	421	006822	27
34	846175	214	852745	207	993430	421	006570	26
35	846304	214	852620	207	993683	421	006317	25
36	846432	213	852496	208	993936	421	006064	24
37	846560	213	852371	208	994189	421	005811	23
38	846688	213	852247	208	994441	421	005559	22
39	846816	213	852122	208	994694	421	005306	21
40	846944	213	851997	208	994947	421	005053	20
41	9.847071	213	9.851872	208	9.995199	421	10.004801	19
42	847199	213	851747	208	995452	421	004548	18
43	847327	213	851622	208	995705	421	004295	17
44	847454	212	851497	209	995957	421	004043	16
45	847582	212	851372	209	996210	421	003790	15
46	847709	212	851246	209	996463	421	003537	14
47	847836	212	851121	209	996715	421	003285	13
48	847964	212	850996	209	996968	421	003032	12
49	848091	212	850870	209	997221	421	002779	11
50	848218	212	850745	209	997473	421	002527	10
51	9.848345	212	9.850619	209	9.997726	421	10.002274	9
52	848472	211	850493	210	997979	421	002021	8
53	848599	211	850368	210	998231	421	001769	7
54	848726	211	850242	210	998484	421	001516	6
55	848852	211	850116	210	998737	421	001263	5
56	848979	211	849990	210	998989	421	001011	4
57	849106	211	849864	210	999242	421	000758	3
58	849232	211	849738	210	999495	421	000505	2
59	849359	211	849611	210	999748	421	000253	1
60	849485	211	849485	210	10.000000	421	000000	0
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