ELEMENTS

OF

GEOMETRY,

PLANE AND SPHERICAL TRIGONOMETRY,

AND

CONIC SECTIONS.

BY

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PREFACE

An attempt is made in this volume, to bring the science of geometry, directly to the comprehension of the learner; and to accomplish this end, it is necessary to sweep away some of the rubbish and some of the redundancies which have seemed only to obstruct our progress and becloud our vision.

All attempts to prove what is perfectly obvious to every one without proof, only weakens the mind rather than strengthens it, and hence, we have discarded all such propositions as the following: "All right angles are equal." "Any two sides of a triangle are greater than the third side." "Parallel lines can never meet, however far they may be produced"—and some few others of like character. In almost every treatise on Geometry, the first, or one of the first propositions for demonstration is, "*That all right angles are equal.*" This proposition at once excites in the mind of the intelligent pupil, a mingled sensation of disappointment and indignation,—disappointment, because he expected to learn new truths; indignation, because he feels as if his time and common sense are trifled with.

When he attempts the demonstration, he either has, or has not, a correct idea of a right angle; if he has a correct idea, he cannot demonstrate, or say anything that can be called a demonstration—because the proposition is all embraced in the definition of a right angle.

If he has not the correct idea of the term right angle, he must obtain it before he can commence any demonstration; so, in either case, the proposition is worse than useless.

When he comes to the proposition, that "Any two sides of a triangle, are together, greater than a third side," and is carried through a useless demonstration, he looks about in wonder and perplexity, to discover why it is that he should be dragged through formal techicalities to arrive at the perfectly axiomatic truth, that a straight line is the shortest distance between two points.

Where is the logic of proving that parallel lines will never meet, however far they may be produced, when the very meaning of the term parallel is, that they cannot meet; hence, we say that all attempts to prove what is perfectly obvious, tend more to confuse and weaken, than to strengthen and enlighten.

Notwithstanding we have discarded such like propositions, we have omitted none of the truths therein expressed; for we have put them either in the axioms or definitions, and have made as complete a chain of geometrical truths as are to be found in any other work.

At the same time, no attempt has been made to present all the known propositions in geometry; we have taken such only as, united and combined, will give the pupil complete power over the science, and make his geometrical knowledge *efficient*, *useful*, and *practical*.

In the mathematical sciences, it is necessary to be more or less technical, formal, and exact; but we have made efforts not to be unpleasantly so. We have presumed that the reader will exercise his own judgment in construing our language; and in place of the preciseness of the professor, we have aimed to take the more wholesome and elevated tone of the practical common-sense man of the world. For the sake of perspicuity and brevity, we have freely used the algebraic language; and the whole work supposes that the reader clearly comprehends simple equations, and is able to perform all ordinary operations with them; but this should be no objection to the use of this book—for no treatise on Geometry should be studied prior to Algebra, whatever be the tone and style of the Geometry.

To most persons, Geometry is a very dry and uninteresting study; and from the nature of the human mind it must be so, until the pupil catches the *spirit of the science;* but as a general thing that spirit cannot be infused until some essential advancements have been made; hence, the ill success of many who undertake this study.

It is essential that the teacher should have a clear view of all these particulars; that he should possess the true spirit himself; and then he will be able to animate, encourage, and assist the new beginner, until the daylight of the science breaks in upon his mind.

It is of little use to commence Geometry unless the learner is determined to go through, at least, so far, as to understand Plane Trigonometry. The first propositions are only so many letters in the great alphabet of science, and we must be able to put them together, before we can really perceive their utility and power. These considerations induced us to be very full and practical in the application of Geometry, and if a student can go through this book understandingly, we are sure that his geometrical knowledge will be at once ample and efficient.

With proper encouragement and proper instruction, the learner will begin to discover the beauties of geometrical demonstrations, after passing through the first three books, and when that discovery is made, all serious difficulties will be over. Yet the pupil should not stop there ; for, to receive the benefits of any science, we must have *command over that science.* To receive the benefits of any enterprise, we must carry it through to completion, or be content to lose a part, if not the whole of our labors ; it is emphatically so with this science.

The infinitesimal system has been used in demonstrations to a greater extent in this, than in most other works of like kind, and although the method has been objected to, the objections are neither far-sighted nor philosophical ; a rejection of this method necessarily rejects the differential and integral calculus, and all works based upon them as unscientific and unsound.

In plane and spherical trigonometry, great pains have been taken to show the theoretical beauties of those sciences, as well as their practical application, and for this end, many of the demonstrations have been given both analytically and geometrically. In applying these sciences, more examples are given in this work than any other that I have seen, and such questions and such problems have been chosen, as to show the great power and utility of geometrical science. In confirmation of this, we refer the reader to the various astronomical problems, and in particular to the one, giving general directions for computing the beginning or end of a local solar eclipse.

Those only who pay particular attention to Geometry, will be able to demonstrate the propositions proposed for exercises on pages 100–104 ; they are designed for amateurs in particular ; they are marks of attainment to which all may aspire, but as a general thing they will require more time and attention than can be devoted to them in schools ; therefore, no attempt should be made to solve all of them, before passing on.

In conic sections we have not been as full as some other treatises, especially in respect to the hyperbola, and the reason for our brevity on that curve is, that it is of little or no practical utility ; it is merely a curve of mathematical curiosity. The ellipse and parabola have important relations to astronomy, and projectile motions, and we have taken particular care to demonstrate those properties essential to their application, and further than this would exceed our design ; but we have given this amply and fully ; yet this treatise is not designed to supersede the study of these curves again in Analytical Geometry, and if the student understands the demonstrations here given, he will be able to pursue analysis with great power and facility.

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PLANE TRIGONOMETRY.

SPHERICAL TRIGONOMETRY.

CONIC SECTIONS.

GEOMETRY.

DEFINITIONS.

1. GEOMETRY is the science that estimates and compares distances, positions, and magnitudes.

2. A Point is position, not magnitude, and on paper it is represented by a visible dot, thus •

3. A Line is length, only. The extremities of a line are points.

4. A Right Line has the same direction in every part.

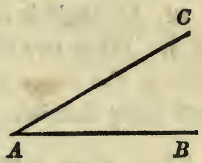
5. A Curved Line is continually changing its direction.

6. A Broken or Crooked Line changes its direction at intervals.

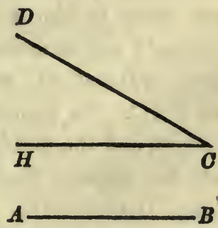
7. An Angle is the difference in the *direction* of two lines.

Two lines drawn from the same point, and in the same direction, are one and the same line.

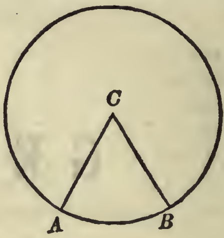
To make an angle apparent, the two lines must meet in a point, as *AB*, and *AC*, which meet at the point *A*.



Two lines, not having the same direction, and not meeting in a point as *AB*, and *CD*, still have an angle existing between them *equal to the difference* in their direction; and to make the angle apparent, take any point in one of the lines, as *C*, and conceive *CH* to lie in the same direction as *AB*. Then the difference in the directions of *CD* and *CH* measures the angle; or *measures the difference* in the directions of *AB* and *CD*.



8. Angles are measured by the number of degrees of a circle included between the two lines which form the angle at the center of the circle. Thus, the portion of the circle between the lines *CA* and *CB* measures the angle at the center of the circle. Every circle is divided into 360°, and the greater the number of degrees between any two lines running from the center, the greater the angle.

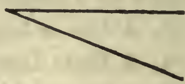


Angles are more indefinitely distinguished by *Acute*, *Obtuse*, and *right angles*.

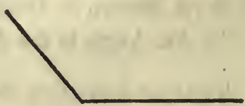
9. A *Right Angle* is formed by one line meeting another so as to make equal angles with the other line.

One line so inclined to another is said to be perpendicular to another.

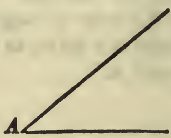
10. An *Acute* Angle is less than a right angle.



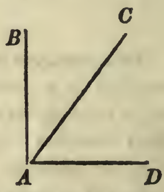
11. An *Obtuse* Angle is greater than a right angle.



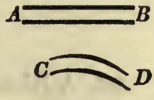
12. An angle is named by a letter at its vertex, as *A*. When two or more angles have their vertices at the same point, this method will not be sufficiently definite.



Thus, when several lines as *AB*, *AC*, *AD*, all meet at the point *A*, several angles are formed; and to define the one formed by the two lines *AB* and *AC*, we must say the angle *CAB*, or *BAC*. To express the angle requires three letters, and the middle one must be at the vertex of the angle. The angle *DAC* is the angle made by the two lines *DA* and *AC*. The angle *DAB* is the angle made by the two lines *DA* and *AB*.



13. Two lines similarly situated and making equal angles with a third line, all being in the same plane, are *parallel.*



Parallel lines may be either right lines, as $A\,B$, or curved lines, as $C\,D$; but at present we are only considering right lines.

Rectilinear parallels have the same absolute direction; and, conversely, lines having the same absolute direction, are parallel.

Two parallel lines cannot be drawn from the same point; for to *fulfill the condition* of parallelism, any attempt to draw them would run them into the same direction, and thus make one line. Conversely, then, two parallel lines cannot meet in a point, however far they may be produced.

14. Superficies are either Plane or Curved.

A Plane Superficies, or a Plane, is that with which a right line may every way coincide. Or, if the line touch the plane in two points, it will touch it in every point; but, if not, it is curved.

15. Plane figures are bounded either by right lines or curves.

16. Plane figures that are bounded by right lines have names according to the number of their sides, or of their angles; for they have as many sides as angles; the least number being three.

17. A figure of three sides and angles is called a triangle; and it receives particular denominations from the relations of its sides and angles.

18. An Equilateral Triangle has three equal sides.

19. An Equiangular Triangle has three equal angles.

Every Equilateral Triangle is also Equiangular.

20. An Isosceles Triangle has two equal sides.

21. A Right Angled Triangle has one right angle.

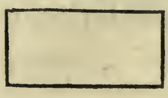
22. An Obtuse Angled Triangle has one obtuse angle.

23. An Acute Angled Triangle has all its three angles acute.

24. A Quadrilateral figure has four sides and four angles.

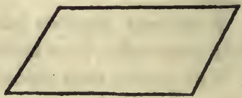
25. A Parallelogram is a quadrilateral which has its opposite sides parallel, and it may take the name of *rectangle, square, rhomboid,* or *rhombus,* according to the relation of its sides and angles.

26. A Rectangle is a parallelogram, having its angles right angles.

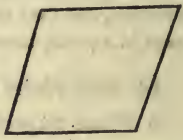


27. A Square has all its sides equal, and all its angles right angles.

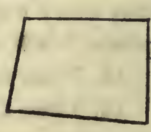
28. A Rhomboid is an oblique angled parallelogram.



29. A Rhombus is an equilateral rhomboid.



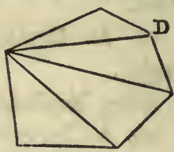
30. A Trapezium is any irregular quadrilateral.



31. A Trapezoid is a quadrilateral which has two opposite sides parallel.

32. A figure of five sides is called a Pentagon; of six, a Hexagon; of eight, an Octagon, &c.; but all these figures are in general called *Polygons.*

33. Diagonals are lines joining any two angles of a polygon not adjacent.

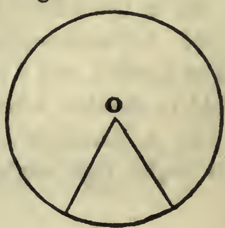


34. Polygons may be similar without being equal; that is, the angles and the number of sides equal, and the length of the sides and the *size* of the figures unequal.

35. A Perimeter of any figure is the sum of all its sides.

36. The Altitude of any figure is the *perpendicular distance* from any side, or any angle, to the opposite side or angle.

37. A Circle is a figure bounded by one uniform curved line, and a certain point within it, from which all straight lines drawn to the curve are equal, and this point is called the center.



EXPLANATION OF TERMS.

1. A Postulate is a position taken; a fact that must be admitted.

2. An Axiom is a self-evident truth; not only too simple to require, *but too simple to admit, of demonstration.*

3. A Proposition is something which is either proposed to be done, or to be demonstrated, and is either a problem or a theorem.

4. A Problem is something proposed to be done.

5. A Theorem is something proposed to be demonstrated.

6. A Lemma is something which is premised, or demonstrated, in order to render what follows more easy.

7. A Corollary is a consequent truth gained immediately from some preceding truth or demonstration.

8. A Scholium is a remark or observation made upon something going before it.

POSTULATES.

1. Let it be granted that a straight line can be drawn from any one point to any other point.

2. That a straight line can be produced to any distance, or terminated at any point.

3. That a circle can be drawn from any center, at any distance from that center.

AXIOMS.

1. *Things which are equal to the same thing are equal to each other.*

2. *When equals are added to equals the wholes are equal.*

3. *When equals are taken from equals the remainders are equal.*

4. *When equals are added to unequals the wholes are unequal.*

5. *When equals are taken from unequals the remainders are unequal.*

6. *Things which are double of the same thing, or equal things, are equal to each other.*

7. *Things which are halves of the same thing are equal.*

8. *Every whole is equal to all its parts taken together.*

9. *Things which coincide, or fill the same space, are identical, or mutually equal in all their parts.*

10. *All right angles are equal to one another.*

11. *Two straight lines cannot inclose a space.*

12. *A straight line is the shortest distance between two points.*

13. *The whole is greater than its part.*

ABBREVIATIONS.

The common algebraical signs will be used in this work, and demonstrations will sometimes be made through the medium of equations; and it is so necessary that the student in Geometry should understand some of the more simple operations of Algebra, that we suppose he is acquainted with the use of the signs. As the words circle, angle, triangle, hypothesis, axiom, are constantly occurring in a course of Geometry, we shall abbreviate them as follows:

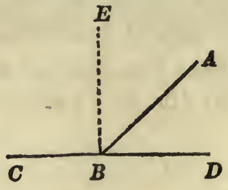
Addition is expressed by		. . . . . .	$+$.
Subtraction	"	"	$-$.
Multiplication	"	"	$\times$.
Equality	"	"	$=$.
Greater than	"	"	$>$.
Less than	"	"	$<$.
Thus: B is greater than A, is written		. .	$B > A$.
B is less than A,	"	" . .	$B < A$.
Let a circle be expressed by		. . . . .	○.
An angle by	"	"	⅃.
A triangle by	"	"	$\triangle$.
The word *hypothesis*		"	(hy.)
Axiom is expressed		"	(ax.)
Theorem	"	"	(th.)
Corollary	"	"	(Cor.)
Perpendicular	"	"	$\perp$.

When the difference of two quantities is expressed, without knowing which is the greater, we use the following symbol, ... $\backsim$.

BOOK I.

THEOREM 1.

When one line meets another, the sum of the two angles which it makes on the same side of the other line, is equal to two right angles.



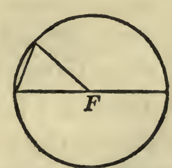
Let AB meet CD; then *we are to demonstrate that the two angles* ABD+ABC= *two right angles.*

If AB does not incline on either side of CD and the angle $ABD=ABC$, then these angles are right angles by definition 9.

But if these angles are unequal, conceive the dotted line, BE, drawn from the point B, so as not to incline on either side; then by the definition, the angles CBE and EBD are right angles; but the angles $CBA+ABD$ make the same sum, or fill the same angular space, as the two angles CBE and EBD; therefore, $CBA+ABD$=two right angles. *Q. E. D.* *

Cor. 1. Hence, all the angles which can be made at any point B, by any number of lines on the same side of the right line CD, are, when taken all together, equal to two right angles.

Cor. 2. And, as all the angles that can be made on the other side of the line CD are also equal to two right angles, therefore all the angles that can be made quite round a point B, by any number of lines, are equal to four right angles.

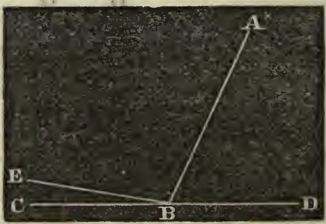


Cor. 3. Hence, also, the whole circumference of a circle, being the sum of the measures of all the angles that can be made about the center F, (def. 8), is the measure of four right angles; consequently, a semicircle, or 180 degrees, is the measure of two right angles; and a quadrant, or 90 degrees, the measure of one right angle.

* The initials of a Latin phrase, meaning "*which was to be demonstrated.*"

THEOREM 2.

If one straight line meets two other straight lines at a common point, forming two angles, which together make two right angles, the two straight lines are one and the same line.



If AB meets the two lines DB and BC at the common point B, and the two angles $DBA+ABC$ =two right angles, *then we are to demonstrate that* DB *and* BC *form one and the same straight line.*

If DB and BC are not in the same line, produce DB to E, making a continued line DE: then by (th. 1) the angles

$$ABD+ABE=2R \quad \text{(2 } R \text{ indicates two right angles.)}$$

But by (hy.) $\quad ABD+ABC=2R$

By subtraction $\quad ABE-ABC=0$

That is, the angle CBE is zero; and DBC is a continued line; or BC falls on BE. *Q. E. D.*

THEOREM 3.

If two straight lines intersect each other, the opposite vertical angles are equal.

If AB and CD intersect each other at E, *we are to demonstrate that the angle* AEC *equals its opposite angle* DEB, *and* AED=CEB.

As AEB is a right line, EA is exactly in the opposite direction from EB; and for the same reason EC is opposite in direction from ED; therefore, the *difference in direction* between EA and EC is equal to the *difference in direction* between EB and ED; or by (def. 7), the angle $AEC=DEB$. In the same manner we can show that the angle $AED=CEB$. *Q. E. D.*

Otherwise: Let $AEC=z$, $AED=y$, and $DEB=x$; *then we are to show that* $x=z$. As AB is a right line, and DE falls upon it, we have, by (th. 1), $\quad x+y=2R$

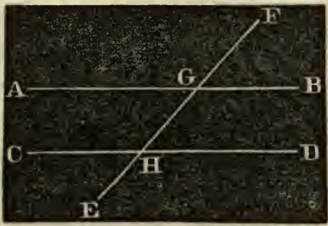
Also, . . . $\quad z+y=2R$

By subtraction, . . $\quad x-z=0$

By transposition, . . $\quad x=z \quad$ *Q. E. D.*

THEOREM 4.

If a straight line falls across two parallel straight lines, the sum of the two interior angles on the same side of the crossing line is equal to two right angles.



Let AB and CD be two parallel lines, and EF running across them; *then we are to demonstrate that the angle* BGH+GHD=2R.

Because GB and HD are parallel, they are equally inclined to the line EF, or have the same difference of direction from that line: Therefore $\angle FGB = \angle GHD$. To each of these equals add the $\angle BGH$.

Then $$FGB+BGH=GHD+BGH.$$

But by (th. 1) the first member of this equation is equal to two right angles: that is, the two interior angles GHD and BGH are together equal to two right angles. *Q. E. D.*

THEOREM 5.

If a straight line falls across two parallel straight lines, the interior alternate angles are equal; and also the opposite exterior angles.

On the supposition that AB and CD are parallel, (see last figure), and EF falls across them, *we are to demonstrate*

1st. That the $\angle AGH$=the alternate $\angle GHD$.

2d. That $AGF=EHD$; or $FGB=CHE$.

By the definition of parallel lines we have

$$FGB=GHD$$

But $FGB=AGH$ (th. 3)

Hence $AGH=GHD$ (ax. 1) *Q. E. D.*

2d. The $\angle FGB=GHD$. But $GHD=CHE$ (th. 3); therefore, $FGB=CHE$. In the same manner we prove that AGF is equal to EHD. *Q. E. D.*

THEOREM 6.

If a straight line falls across two parallel straight lines, the exterior angles are equal to the interior opposite angles on the same side of the crossing line.

If AB and CD are parallel, (see last figure), and EF crosses them, then we are to prove that the exterior $\angle$ $FGB=GHD$

And . . . . $AGF=CHG$

For . . . . $AGH=FGB$ (th. 3)

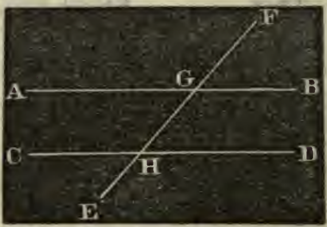
Also . . . . $\underline{AGH=GHD}$ (th. 5)

Hence $FGB=GHD$ (ax. 1)

In the same manner we prove that $AGF=CHG$. *Q. E. D.*

THEOREM 7.

If a straight line falls across two other straight lines, and makes the sum of the two interior angles on the same side equal to two right angles, the two straight lines must be parallel.



Let EF be the line falling across the lines AB and CD, making the two angles $BGH+GHD$=to two right angles; *then we are to demonstrate that* AB *and* CD *must be parallel.*

As EF is a right line, and BA meets it, the two angles (th. 1)

$FGB+BGH=2R$

By (hy.) . $\underline{GHD+BGH=2R}$

By subtraction, $FGB-GHD=0$. That is, there is no difference in the direction of GB and HD from the same line EF; but when there is no difference in the direction of lines (def. 13) the lines are parallel; therefore, AB and CD are parallel. *Q. E. D.*

THEOREM 8.

Parallel lines can never meet, however far they may be produced.

If the lines AB and CD (see last figure) should meet at any distance on either side of EF, they would there form an angle; and if they formed an angle they would not run in the same direction; and not running in the same direction, they would not be parallel; but by (hy.) they are parallel; therefore they cannot meet. *Q. E. D.*

THEOREM 9.

If two straight lines are parallel to a third, they are parallel to each other.

If AB is parallel to EF, and CD also parallel to EF, *then we are to show* that AB is parallel to CD.

Because AB and EF are parallel, they make equal angles with the line HG (def. 13, 2); and because CD and EF are parallel, those two lines make equal angles with the line HG.

Hence AB and CD, making equal angles with another line that falls across them, they are therefore parallel (def. 7). *Q. E. D.*

THEOREM 10.

If two angles have their sides parallel, the two angles will be equal.

Let the two angles be A and DBF; AC parallel to DB, and AH parallel to BF.

On that hypothesis we are to prove that the angle $A=DBF$.

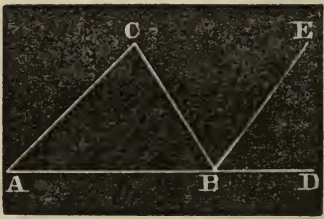
Produce DB, if necessary, to meet AH in G,

Then . $\angle DBF=\angle DGH$ (th. 6)
Also . . $\angle A=\angle DGH$ (th. 6)
Therefore $DBF=A$ (ax. 1) *Q. E. D.*

Scholium. When AH extends in the opposite direction, it is still parallel to BF; but the angle then is the supplemental angle to DBF; that is, equal to FBG.

THEOREM 11.

If any side of a triangle be produced, the exterior angle is equal to the sum of the two interior opposite angles; and the sum of the three angles is equal to two right angles.



Let ABC be any triangle. Produce AB to D. Then *we are to show* that the angle $CBD=\angle A$ $+$the angle C; also, that the angles $A+C+CBA=2R$.

From B conceive BE drawn parallel to AC;

Then $\quad EBD=\angle A \quad$ (th. 6)

By (th. 5) $\quad CBE=\angle C \quad$ (alternate angles).

By addition $\quad \angle CBD=A+C \quad$ *Q. E. D.*

To each of these equals add the angle CBA, and we have

$$CBD+CBA=A+C+CBA$$

But . . $\quad CBD+CBA=2R \quad$ (th. 1)

Therefore $\quad A+C+CBA=2R \quad$ (ax. 1)

That is, the three angles of the triangle are, together, equal to two right angles; and this triangle represents any triangle; therefore, the sum of the three angles of any triangle is equal to two right angles. *Q. E. D.*

Cor. 1. As the exterior angle of any triangle is equal to the sum of the two interior and opposite angles, therefore it is greater than either one of them.

Cor. 2. If two angles in one triangle be equal to two angles in another triangle, the third angles will also be equal, (ax. 3), and the two triangles equiangular.

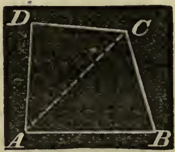
Cor. 3. If one angle in one triangle be equal to one angle in another, the sums of the remaining angles will also be equal (ax. 3).

Cor. 4. If one angle of a triangle be right, the sum of the other two will also be equal to a right angle, and each of them singly will be acute, or less than a right angle.

Cor. 5. The two least angles of every triangle are acute, or each less than a right angle.

THEOREM 12.

In any quadrangle the sum of all the four inward angles is equal to four right angles.



Let $ABCD$ be a quadrangle; then the sum of the four inward angles $A+B+C+D$ is equal to four right angles.

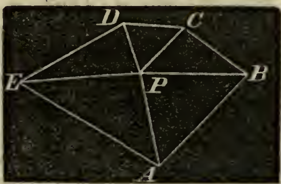
Let the diagonal AC be drawn, dividing the quadrangle into two triangles, ABC, ADC; then, because the sum of the three angles of each of these triangles is equal to two right angles (th. 11), it follows that the sum of all the angles of both triangles which make up the four angles of the quadrangle, must be equal to four right angles (ax. 2). *Q. E. D.*

Cor. 1. Hence if three of the angles be right angles, the fourth will also be a right angle.

Cor. 2. And if the sum of two of the four angles be equal to two right angles, the sum of the remaining two will also be equal to two right angles.

SCHOLIUM.

In any figure bounded by right lines and angles, the sum of all the interior angles is equal to twice as many right angles as the figure has sides, less four right angles.



Let $ABCDE$ be any figure; then the sum of all its inward angles, $A+B+C+D+E$, is equal to twice as many right angles, wanting four, as the figure has sides.

For, from any point P, within it, draw lines PA, PB, PC, &c., to all the angles, dividing the polygon into as many triangles as it has sides. Now the sum of the three angles of each of these triangles, is equal to two right angles (th. 11); therefore the sum of the angles of all the triangles is equal to twice as many right angles as the figure has sides. But the sum of these angles contains the sum of four right angles about the point P: take these away, and the sum of the interior angles of the figure is equal to twice as many right angles as the figure has sides less four right angles. *Q. E. D.*

From this principle we can deduce the following rule to find the sum of the interior angles of any right-lined figure:

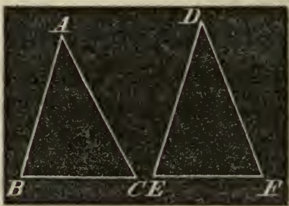
RULE. *Subtract 2 from the number of sides, and multiply the remainder by 2, and the product will be the number of right angles.*

Thus, if the sides be represented by s, then the rule gives $(2s-4)$; nor is the rule varied in *case of a re-entrant angle, as represented at* d *in the figure* a b c d e f. Draw the dotted lines from the angle d to the several opposite angles, making as many triangles as the figure has sides, *less two,* and each triangle has two right angles: hence the rule.

THEOREM 13.

Two triangles which have two sides, and the included angle in the one, equal to the two sides and included angle in the other, are identical, or equal in all respects.

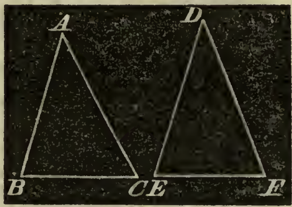
In two △s, ABC and DEF, on the supposition that $AB=DE$, and $AC=DF$, and the ⌋ $A=$ ⌋ D, *we are to prove that* BC *must*=EF, *the* ⌋ B= ⌋E, *and the* ⌋ C= ⌋ F.



Conceive the △ ABC cut out of the the paper, taken up, and placed on the △ DEF in such a manner that the point A shall fall on the point D, and the line AB on the line DE; then the point B will fall on the point E, because the lines are equal. Now, as the ⌋ $A=$ ⌋ D, the line AC must take the same direction as DF, and fall on DF; and as the line $AC=DF$, the point C will fall on F. B being on E and C on F, BC must be exactly on EF, (otherwise, two straight lines would enclose a space ax. 11), and $BC=EF$, and the two magnitudes exactly fill the same space; therefore, the two △s are identical, (ax. 9), and the angle $B=E$, and $C=F$. *Q. E. D.*

THEOREM 14.

When two triangles have a side and two adjacent angles in the one, equal to a side and two adjacent angles in the other, the two triangles are equal in all respects.



In two △s, as ABC and DEF, on the supposition that $BC=EF$, the angle $B=E$, and $C=F$, *we are to prove that* AB=DE, AC=DF, *and the angle* A=D.

Conceive the △ ABC taken up and placed on the △ DEF so that the side BC shall exactly coincide with its equal side EF; then because the angle B is equal to the angle E, the line BA will take the direction of ED, and fall exactly upon it; and because the angle C is equal to the angle F, the line CA will take the direction of FD, and exactly fall upon it; and the two lines BA and CA exactly coinciding with the two lines ED and FD, the point A will fall on D, and the two magnitudes exactly fill the same space; therefore, by (ax. 9) they are identical, and $AB=ED$, $AC=DF$, and the ⌟ $A=$ ⌟D. *Q. E. D.*

THEOREM 15.

If two sides of a triangle are equal, the angles opposite to these sides will be equal.

Let ABC be the triangle; and on the supposition that $AC=CB$, *we are to prove* that the angle $A=B$.

Conceive the angle C divided into two equal angles by the line CD; then we have two △s, ADC and CBD, which have the two sides, AC and CD of the one, equal to the two sides, CB and CD of the other; and the included angle ACD, of the one, equal to BCD of the other: therefore (th. 13), $AD=BD$, and the angle A, opposite to CD of the one triangle, is equal to the angle B, opposite to CD of the other triangle: that is, ⌟ A = ⌟ B. *Q. E. D.*

Cor. 1. As the two triangles ACD and BCD are in all respects equal, the line which bisects the vertical angle of an isosceles △ also bisects the base, and falls perpendicular on the base.

Scholium. Any other point as well as C may be taken in the perpendicular DC, and lines drawn to the extremities A and B; such lines will be equal, as we can prove by theorem 15; hence, we may announce this truth: *That if a perpendicular be drawn from the middle of a line, any point in the perpendicular is at equal distance from the two extremities.*

THEOREM 16.

The greater side of every triangle has the greater angle opposite to it.

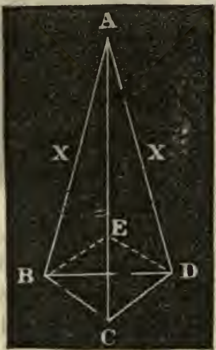
Let ABC be the △; and on the supposition that AC is greater than AB, *we are to prove* that the angle ABC is greater than the ⌋ C. From the greater of the two sides AC, take AD, equal to AB the less, and join BD; thus making two triangles of the original triangle. As $AB = AD$, the ⌋ $ADB =$ the ⌋ ABD (th. 15).

But the ⌋ ADB is the exterior angle of the △ BDC, and therefore greater than C: that is, the ⌋ ABD is greater than the angle C. Much more, then, is the angle ABC greater than C. *Q. E. D.*

THEOREM 17.

If two triangles have two sides of the one equal to two sides of the other, each to each, and an angle opposite one of the equal sides in each triangle equal, then will the two triangles be equal.

Let ABC be one triangle and ADC the other in which $AD = AB$, $BC = DC$, and the angles opposite BC and DC equal, *then will the angle* ABC=ADC, *and* AC *be a converse side.*



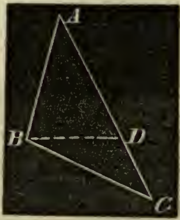
Place the two △'s so that the given angles will come together at A, and lie on the opposite sides of the line AC.

Then because $AB = AD$, ABD is an isosceles △, and the line AC which bisects the angle A is perpendicular to BD and bisects BD (th. 15, cor. 1). Now BC and DC must terminate in the same point C, because $BC = DC$ (th. 15, scholium), therefore, AC is common to the two △'s ABC, ADC; and the △'s are identical. *Q. E. D.*

Scholium. There are, in fact, two cases in this theorem, because $BC = BE$, and $DC = DE$, giving *two pair* of △'s.

THEOREM 18.

The difference of any two sides of a triangle is less than the third side.



Let ABC be the $\triangle$, and let AC be greater than AB; then *we are to prove* that $AC-AB$ is less than BC.

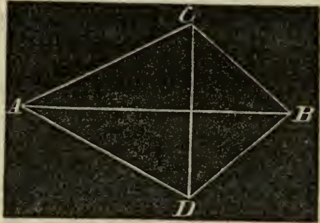
As a straight line is the shortest distance between two points,

Therefore, . $AB+BC > AC$.

From these unequals subtract the equals $AB=AB$, and we have $BC > AC-AB$. (ax. 5). *Q. E. D.*

THEOREM 19.

When two triangles have all three of the sides in one triangle equal to all three in the other, each to each, the two triangles will be identical, and have equal angles opposite equal sides.



In two triangles, as ABC and ABD, on the supposition that the side AB of the one$=AB$ of the other, $AC=AD$, and $BC=BD$, *we are to demonstrate that the angle* ACB=*the angle* ADB, BAC=BAD, *and* ABC=ABD.

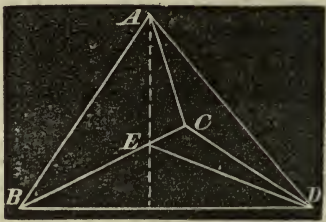
Conceive the two triangles to be joined together by their longest equal sides, and draw the line CD.

Then, in the triangle ACD, because the side AC is equal to AD by (hy.), the angle ACD is equal to the angle ADC (th. 15). In like manner, in the triangle BCD, the angle BCD is equal to the angle BDC, because the side BC is equal to BD. Hence, then, the angle ACD being equal to the angle ADC, and the angle BCD to the angle BDC, by equal additions the sum of the two angles ACD, BCD, is equal to the sum of the two ADC, BDC (ax. 2); that is, the whole angle ACB is equal to the whole angle BDA.

Since then the two sides, AC, CB, are equal to the two sides AD, DB, each to each, by (hy.), and their contained angles ACB, ADB, also equal, the two triangles ABC, ABD, are identical (th. 13), and have their other angles equal, the angle BAC to the angle BAD, and the angle ABC to the angle ABD. *Q. E. D.*

THEOREM A.

If there be two triangles which have the two sides of the one equal to the two sides of the other, each to each, and the included angles unequal, the third sides will be unequal, and the greater side will belong to the triangle which has the greater included angle.



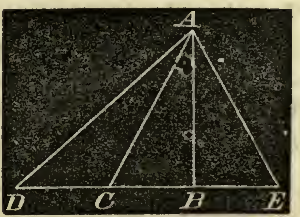
Let ABC be one $\triangle$, and ACD the other $\triangle$. Let AB and AC of the one $\triangle$ be equal to AD and AC of the other $\triangle$. But the angle BAC greater than the angle DAC; *then we are to prove* that the base BC is greater than the base CD.

Conceive the two $\triangle$s joined together so that the shorter sides will be common to them. As $AB=AD$, ABD is an isosceles $\triangle$, from the vertex A draw a line bisecting the angle BAD. This line must meet BC, and will not meet CD, because the $\angle$ BAC is greater than the $\angle$ DAC, and be perpendicular to BD (th. 15). From E, where the perpendicular meets BC, draw ED.

Now . . . . $BE=ED$ (th. 15, scholium).
Add to each EC, then $BC=ED+EC$
But $DE+EC$ is greater than DC;
Therefore . . $BC > DC$. *Q. E. D.*

THEOREM 20.

A perpendicular is the shortest line that can be drawn from any point to a straight line; and if other lines be drawn from the same point to the same straight line, the greater will be at a greater distance from the perpendicular; and lines at equal distances from the perpendicular, on opposite sides, are equal.



Let A be any point without the line DE; and let AB be the perpendicular; AC, AD, and AE oblique lines: then, if BC is less than BD, and $BC=BE$, we are to show,

1st. *That* AB *is less than* AC. 2d. AC *less than* AD. 3d. AC=AE.

In the triangle ABC, as AB is perpendicular by (hy.), the angle ABC is a right angle; then, as it requires the other two angles of the triangle (th. 11) to make another right angle, the angle ACB, is less than a right angle; and as the greater side is always opposite the greater angle, AB is less than AC; and as AC is *any line* differing from AB, therefore AB is the least of any line drawn from A.

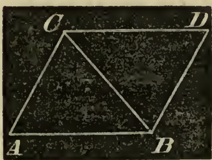
2d. As the two angles ACB and ACD (th. 1) make two right angles, and ACB less than a right angle, therefore ACD is greater than a right angle; consequently, the $\angle$ D is less than a right angle; and, therefore, in the $\triangle$ ACD, AD is greater than AC, or AC is less than AD.

3d. In the $\triangle$s ABC and ABE, AB is common, and $CB=BE$, and the angles at B, right angles; therefore, by (th. 15) $AC=AE$.

Q. E. D.

THEOREM 21.

The opposite sides, and the opposite angles of any parallelogram, are equal to each other.



Let $ABDC$ be a parallelogram. *Then we are to show that* AB=CD, AC=BD, *the angle* A=D, *and the angle* ACD=ABD.

Draw a diagonal, as CB; then, because AB and CD are parallel, the alternate angles ABC and BCD (th. 5) are equal. For the same reason, as AC and BD are parallel, the angles ACB and CBD are equal. Now, in the two triangles ABC and BCD, the side CB is common, and

The $\angle ACB = \angle CBD$ . . (1)

and $\angle BCD = \angle ABC$. (2)

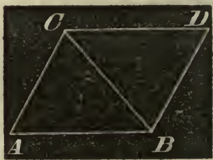
Therefore, the third angle A= the third angle D (th. 11), and by (th. 13) the two $\triangle$s are equal in all respects; that is, the sides opposite the equal angles are equal; or, $AB=CD$, and $AC=BD$. By adding equations (1) and (2), (ax. 2), we have the angle ACD = the angle ABD; therefore, the opposite sides, &c. *Q. E. D.*

Cor. 1. As the sum of all the angles of the quadrilateral is equal to four right angles, and the angle A is always = to the opposite angle D; if, therefore, A is a right angle, D is also a right angle, and all the angles are right angles.

Cor. 2. As the angle ABD, added to the angle A, gives the same sum as the angles of the $\triangle$ ACB; therefore, the two adjacent angles of a parallelogram make two right angles; and this corresponds with the 2d point of theorem 12.

THEOREM 22.

If the opposite sides of a quadrilateral are equal, they are also parallel, and the figure is a parallelogram.



Let $ABDC$ represent any quadrilateral, and on the supposition that $AC=BD$, and $AB=CD$, *we are to prove that* AC *is parallel to* BD, *and* AB *parallel to* CD.

Draw the diagonal CB; then we have two triangles ABC, and CDB, which have the common side CB; and AC of the one$=BD$ of the other, and AB of the one$=CD$ of the other; therefore by (th. 19) the two $\triangle$s are equal, and the angles equal, to which the equal sides are opposite; that is, the angle ACB =the angle CBD, and these are alternate angles; and, therefore, by (th. 5), AC is parallel to BD; and because the angle $ABC=BCD$, AB is parallel to CD, and the figure is a parallelogram *Q. E. D.*

Cor. In this, and also in (th. 21), we proved that the two $\triangle$s which make up the parallelogram are equal; and the same would be true if we drew the diagonal from A to D; and in general we may say, *that the diagonal of any parallelogram bisects the parallelogram.*

THEOREM 23.

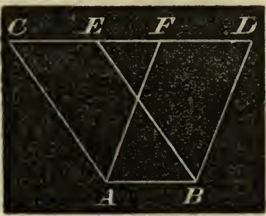
The lines which join the corresponding extremities of two equal and parallel straight lines, are themselves equal and parallel; and the figure thus formed is a parallelogram.

On the supposition that AB is equal and parallel to CD (see last figure), *we are to show that* AC *will be equal and parallel to* BD; and that will make the figure a parallelogram.

Join CB; then because AB and CD are parallel, and CB joins them, the alternate angles ABC and BCD are equal, and the side $AB=CD$, and CB common to the two $\triangle$s ABC and CDB; therefore by (th. 13) the two triangles are equal; that is, $AC=BD$, the angle $A=D$, and $ACB=CBD$; hence, AC is also parallel to BD; and the figure is a parallelogram. *Q. E. D.*

THEOREM 24.

Parallelograms on the same base, and between the same parallels, are equal in surface.

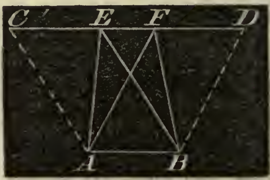


Let $ABEC$ and $ABFD$ be two parallelograms on the same base AB, and between the same parallel lines AB and CD; *then we are to show that these two parallelograms are equal.*

Now CE and FD are equal, because they are each equal to AB (th. 21); and if from the whole line CD we take, in succession, CE and FD, there will remain (ax. 3) $ED=CF$; but $EB=CA$, and $AF=BD$ (th. 21); hence we have two $\triangle$s, CAF and EBD, which have the three sides of the one equal to the three corresponding sides of the other, each to each; and therefore by (th. 19) the two $\triangle$s CAF and EBD are equal. If from the whole figure we take away the $\triangle$ CAF, the parallelogram $ABDF$ remains; and if from the whole figure the other triangle EBD be taken away, the parallelogram $ABEC$ will remain; that is, from the same quantity, if equals are taken (ax. 3), equals will be left; or the parallelogram $ABDF=ABEC$. *Q. E. D.*

THEOREM 25.

Triangles on the same base, and between the same parallels, are equal (in respect to area or surface).

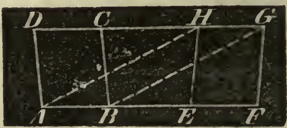


Let the two $\triangle$s ABE and ABF have the same base AB, and between the same parallels AB and CD; *then we are to show that they are equal in surface.*

From B draw a dotted line, BD, parallel to AF; and from A draw a dotted line AC, parallel to BE; and produce EF both ways, if necessary, to C and D; then the parallelogram $ABFD$=the parallelogram $ABCE$ (th. 24). But the $\triangle$ ABE is half the parallelogram $ABCE$, and the $\triangle$ ABF is half the parallelogram $ABDF$; but halves of equals are equal (ax. 7); therefore the $\triangle$ ABE=the $\triangle$ ABF. *Q. E. D.*

THEOREM 26.

Parallelograms on equal bases, and between the same parallels, are equal in area.



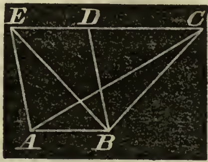
Let $ABCD$, and $EFGH$, be two parallelograms on equal bases, AB and EF, and between the same parallels; *then we are to show that they are equal in area.*

As AB=EF=HG; but lines which join equal and parallel lines, are themselves equal and parallel (th. 23); therefore, if AH and BG be joined, the figure $ABGH$ is a parallelogram=to $ABCD$ (th. 24); and if we turn the whole figure over, the two parallelograms $HEFG$ and $HGBA$, will stand on the same base, HG, and between the same parallels; therefore, $HGEF$=$HGBA$; and consequently (ax. 1) $ABCD$=$EFGH$. *Q. E. D.*

Cor. Triangles on equal bases, and between the same parallels, are equal; for, join BD and EG, the $\triangle$ ABD is half of the parallelogram AC; and the $\triangle$ EFG is half of the equal parallelogram FH; therefore, the $\triangle$ ABD=the $\triangle$ EFG (ax. 7).

THEOREM 27.

If a triangle and a parallelogram be upon the same or equal bases, and between the same parallels, the triangle will be half the parallelogram.

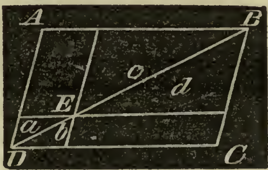


Let *ABC* be a △, and *ABDE* a parallelogram, on the same base *AB*, and between the same parallels; *then we are to show that the* △ ABC *is half of* ABDE.

Draw the diagonal *EB* to the parallelogram; then, because the two △s *ABC* and *ABE* are on the same base, and between the same parallels, they are equal (th. 25); but the △ *ABE* is half the parallelogram *ABDE* (cor. to the 22); therefore the △ *ABC* is half of the same parallelogram (ax. 7). *Q. E. D.*

THEOREM 28.

The complementary parallelograms of any parallelogram which are about its diagonal, are equal to each other.



Let *AC* be a parallelogram, and *BD* its diagonal; take any point, as *E*, in the diagonal, and from it draw lines parallel to its sides; thus forming four parallelograms.

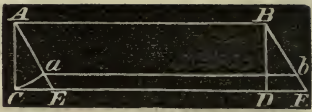
We are now to show that the complementary parallelograms AE *and* EC, *are equal.*

By corollary to theorem 22 we learn that the △ $ADB=$△ DBC. Also by the same (cor.) $a=b$, and $c=d$; therefore by addition . . . $a+c=b+d$.

Now from the whole △ADB take the sum of the two △s $(a+c)$, and from the whole △ DBC take the equal sum $(b+d)$, and the remainders *AE* and *EC* are equal (ax. 3). *Q. E. D.*

THEOREM 29.

The sides of a parallelogram will inclose the greatest space when the angles are right angles.

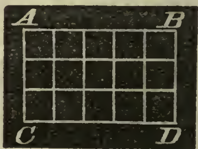


Let *ABDC* be a right angled parallelogram, and *ABba* an oblique angled parallelogram of equal sides to the other; *then we are to show that the right angled parallelogram* ABDC *is greater than the other,* ABba.

We take $Aa=AC$. Then *Aa* is less than *AE*, because the perpendicular *AC*, or its equal *Aa*, is less than any oblique line *AE* (th. 20); therefore the line *ab* is between the two parallels *AB* and *CF*. The parallelogram $ABDC=ABFE$; because they are on the same base *AB*, and between the same parallels (th. 24); but the parallelogram *ABba* is but part of the parallelogram *ABFE*; therefore, *ABFE*, or its equal *ABDC*, is greater than *ABba*; but the parallelogram *ABba* has the same length of sides, respectively, as the parallelogram *ABDC*; therefore the side, &c. *Q. E. D.*

Cor. It is evident, then, that the area of the parallelogram *ABba* will become less and less as its angles become more and more oblique; and greater and greater as its angles become nearer and nearer to right angles.

Scholium. All parallelograms (indeed all figures) are referred to *square units* for their measurement, and the unit may be taken at pleasure; it may be an inch, a foot, a yard, a rod, a mile, &c., according as convenience and propriety may dictate. For example, the parallelogram *ABDC* is measured by the number of *linear units* in *CD*, multiplied into the number of *linear units* in *AC*; the product will be the *square units* in *ABDC*; for conceive *CD* composed of any number of equal parts—say five—and each part some unit of linear measure, and *AC* composed of three such units,

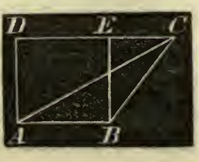


and from each point of division on *CD* draw lines parallel to *AC*; and from each point of division on *AC* draw lines parallel to *CD* or *AB*; then it is as obvious as an axiom that the parallelogram will contain $5\times3=15$ square units; and in general the *areas* of right angled parallelograms are found by multiplying the base by the altitude.

Right angled parallelograms are called rectangles (def. 26), and the altitude of any parallelogram, whether right angled or not, is the *perpendicular distance* between its opposite sides.

THEOREM 30.

The area of any plane triangle is measured by the product of its base into half its altitude; or half the base into the altitude.

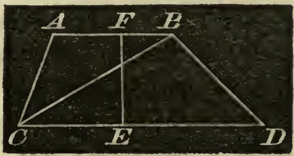


Let ABC represent any triangle, AB its base, and AD at right angles to AB its altitude; *then we are to show that the area of* ABC *is equal to the product of* AB *into one half of* AD; *or the half of* AB *into* AD.

On AB construct the rectangle $ABED$; and the area of this rectangle is measured by AB into AD (scholium to th. 29); but the area of the $\triangle$ ABC is one half this rectangle (th. 27); therefore, &c. *Q. E. D.*

THEOREM 31.

The area of a trapezoid is measured by the half sum of its parallel sides, multiplied into the perpendicular distance between them.



Let $ABDC$ represent any trapezoid, and draw the diagonal BC, which divides it into two triangles, ABC and BCD: CD is the base of one triangle, and AB may be considered as the base of the other; and EF is the common altitude of the two triangles.

Now by the last theorem the area of the triangle CDB is$=\frac{1}{2}CD\times EF$; and the area of the $\triangle\, ABC=\frac{1}{2}AB\times EF$; therefore, by addition, the area of the two $\triangle$s, or of the trapezoid, is equal to $\frac{1}{2}(AB+CD)\times EF$. *Q. E. D.*

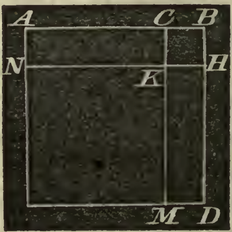
THEOREM 32.

If there be two lines, one of which is divided into any number of parts, the rectangle contained by the two lines is equal to the several rectangles contained by the undivided line, and the several parts of the divided line.

Let AB be one line, and AD the other; and suppose AB divided into any number of parts at the points E, F, G, &c.; then the whole rectangle of the two lines is AH, which is measured by AB into AD; and the rectangle AL is measured by AE into AD; and the rectangle EK is measured by EF into EL, which is equal to EF into AD; and so of all the other partial rectangles; and the truth of the proposition is as obvious as that a whole is equal to the sum of all its parts; and requires no other demonstration than an explanation of exactly what is meant by the words of the text.

THEOREM 33.

If a straight line be divided into any two parts, the square of the whole line is equal to the sum of the squares of the two parts, and twice the rectangle contained by the parts.



Let AB be any line divided into any two parts at the point C; *then we are to show that the square on* AB *is equal to the sum of the squares on* AC *and* CB, *and twice the rectangle of* AC *into* CB.

On AB describe the square (or conceive it described) AD. Through the point C conceive CM drawn parallel to BD; and take $BH=BC$; and through H draw HKN parallel to AB, and CH is the square on CB, by direct construction.

As $AB=BD$, and $CB=BH$, therefore, by subtraction, $AB-CB=BD-BH$; or $AC=HD$. But $NK=AC$, being opposite sides of a parallelogram; and for the same reason $KM=HD$; therefore (ax. 1), $NK=KM$; and the figure NM is a square on NK equal to a square on AC. But the whole square on AB is composed of the two squares CH, NM, and the two complements or rectangles AK and KD; and each of these is AC in length, and BC in width; and each has for its measure AC into CB; therefore the whole square on AB is equal to $AC^2+BC^2+2AC\times CB$. *Q. E. D.*

This may be proved algebraically, thus:

Let w represent any whole right line divided into any two parts a and b; then we shall have the equation

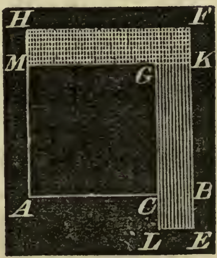
$$w=a+b$$

By squaring $\quad w^2=a^2+b^2+2ab.\quad$ *Q. E. D.*

Scholium. If $a=b$, then $w^2=4a^2$, which shows that the square of any whole line is four times the square of half of it.

THEOREM 34.

The square on the difference of two lines is equal to the sum of the squares of the two lines, diminished by twice the rectangles contained by the lines.



Let AB represent the greater line, BC a lesser line, and AC their difference.

We are now to show that the square on AC *is equal to the sum of the squares on* AB *and* BC, *diminished by twice the rectangle contained by* AB *into* BC.

On AB conceive the square AF to be described; and on CB conceive the square BL described; and on AC describe the square $ACGM$; and produce MG to K.

As $GC=AC$, and $CL=CB$; therefore, by addition, $(GC+CL)$, or GL, is equal $(AC+CB)$, or AB. Therefore the rectangle GE is AB in length, and CB in width; and is measured by AB into BC.

Also $AH=AB$, and $AM=AC$; therefore by subtraction $MH=CB$; and as $MK=AB$, the rectangle HK is AB in length, and CB in width, and it is measured by AB into CB; and the two rectangles GE and HK, are together equal to $2AB\times BC$.

Now the squares on AB and BC make the whole figure $AHFELC$; and from this whole figure, or these two squares, take away the two rectangles HK and GE, and the square on AC only will remain; that is,

$$AC^2=AB^2+BC^2-2AB\times BC.\quad Q.\ E.\ D.$$

This may be proved algebraically, thus:

Let a represent one line, b another and lesser line, and d their difference; then we must have this equation:

$$d=a-b$$

By squaring . . $d^2=a^2+b^2-2ab$.

THEOREM 35.

The difference of the squares of any two lines is equal to the rectangle contained by the sum and difference of the lines.

Let AB be one line, and AC the other, and on them describe the squares AD, AM; then the difference of the squares on AB and on AC is the two rectangles EF and FC. *We are now to show that the measure of these rectangles may be expressed by* (AB+AC) *into* (AB—AC).

The rectangle EF has ED, or its equal AB, for its length; the other has MC, or its equal AC, for its length; therefore, the two together (if we conceive them put between the same parallel lines) will have $(AB+AC)$ for the length; and the common width is CB, which is equal to $(AB-AC)$; therefore, $AB^2-AC^2=(AB+AC)\times(AB-AC)$. *Q. E. D.*

This is proved algebraically thus:

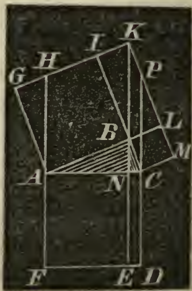
Put a to represent one line, and b another;

Then $a+b$ is their sum, and $a-b$ their difference;

and . . $(a+b)\times(a-b)=a^2-b^2$. *Q. E. D.*

THEOREM 36.

The square described on the hypotenuse of any right angled triangle is equal to the sum of the squares on the other two sides.



Let ABC represent any right angled triangle, the right angle at B.

We are to show that the square on AC *is equal to the sum of two squares; one on* AB, *the other on* BC.

Conceive the three squares, AD, AI, and BM, described on the three sides. Through the point B, draw BNE perpendicular to AC, and produce it to meet the line GI in K.

Produce AF to meet GI in H. If ML be produced, it will meet the point *K*, and *IBLK* will be a right angled parallelogram ; for its opposite sides are parallel, and all its angles right angles.

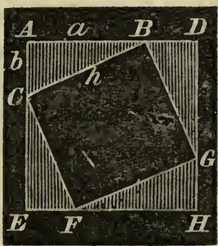
The angle *BAG* is a right angle, and the angle *NAH* is also a right angle ; and from these equals if we subtract the common angle *BAH*, the remaining angle, *BAC*, must be equal to the remaining angle *GAH*. The angle *G* is a right angle, equal to the angle *ABC*; and $AB=AG$; therefore, the two △s *ABC* and *AGH* are equal, and $AH=AC$. But $AC=AF$; therefore $AH=AF$. Now the two parallelograms, *AE* and *AK* are equal, because they are upon equal bases, and between the same parallels, *FH* and *EK* (th. 26).

But the square *AI*, and the parallelogram *AK* are equal, because they are on the same base, *AB*, and between the same parallels, *AB* and *GK*; therefore the square *AI*, and the parallelogram *AE*, being both equal to the same parallelogram *AK*, are equal to each other (ax. 1). In the same manner we may prove the square *BM* equal to the rectangle *ND*; therefore, by addition, the two squares *AI* and *BM*, are equal to the two parallelograms *AE* and *ND*, or to the square *AD*. *Q. E. D.*

Scholium. The two sides *AB* and *BC* may vary, while *AC* remains constant. *AB* may be equal to *BC* ; then the point *N* would be in the middle of *AC*. When *AB* is very near the length of *AC*, and *BC* very small, then the point *N* falls very near to *C*.

Now, as the parallelograms *AE* and *ND* (while *AC* remains unchanged) depend for their relative magnitudes on the position of the point *N*, on the line *AC*, the area *AE* must be to the area *ND* as the line *AN* to *NC* ; that is, *the square on* AB, *must be to the square on* BC, *as the line* AN *to the line* NC.

ANOTHER DEMONSTRATION OF THEOREM 36.



Let *ABC* be a right angled triangle, right angled at *A*. Call *AB*, *a*, *AC*, *b*, and *BC*, *h* : *then we are to show that* $a^2+b^2=h^2$.

Produce *AB* to *D*, making $BD=AC$; and produce *AC* to *E*, making $CE=AB$: then $AD=AE$; and each of these lines is $(a+b)$, and the whole square *AH* is the square of $(a+b)$, and by (th. 33) is a^2+b^2+2ab.

From B draw BG at right angles to CB; and from C draw CF at right angles, the same line CB; then BG and CF must be parallel, and join FG. We must now prove that the four triangles in the square AH are all equal, and that $CGBF$ is the square on CB. As the two angles CBA and CBD make two right angles, (th. 1), and CBG is a right angle by construction, therefore the two angles CBA and GBD make one right angle. But CBA and ACB (cor. 4, th. 11) are also equal to a right angle; and from these equals take the angle CBA, and the angle $GBD =$ the angle ACB. But the angle $A=$ the angle D; both right angles, and BD was made equal to AC; therefore, the two triangles, ABC and GBD, having a side and two angles equal, are in all respects equal, and $CB=BG$. In the same manner we prove $BG=GF$; and therefore CG is a square on CB, and the four triangles are each equal to ABC, and each triangle has for its measure $\frac{1}{2}ab$. The measure of two of these is ab, and the four is $2ab$.

Now . . . $AD^2=a^2+b^2+2ab$

Also . . . $AD^2=h^2+2ab$

By subtraction . $0 \quad =a^2+b^2-h^2$

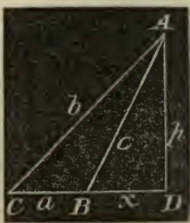
By transposition . $h^2 \quad =a^2+b^2$. *Q. E. D.*

Cor. From this equation we may have

$$h^2-a^2=b^2\text{; or, } (h+a)(h-a)=b^2.$$

THEOREM 37.

In any obtuse angled triangle the square of the side opposite the obtuse angle is greater than the sum of squares on the other two sides, by twice the rectangle of the base, and the distance of the perpendicular from the obtuse angle.



Let ABC be any obtuse angled $\triangle$, obtuse angled at B. Represent the side opposite B by b; opposite A by a; and opposite C by c (and let this be a general form of notation): also represent the perpendicular by p, and DB by x. *Now we are to show that* $b^2=a^2+c^2+2ax$.

By (th. 36) . . . $p^2+(a+x)^2=b^2$ (1)

Also . . . $p^2+ \quad x^2=c^2$ (2)

By expanding equation (1), and subtracting (2), we have

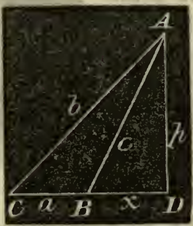
$$a^2+2ax=b^2-c^2$$

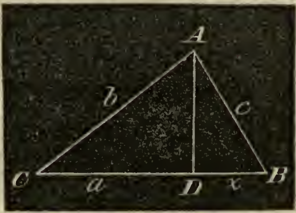
By transposition $\quad b^2=a^2+c^2+2ax. \quad Q. E. D.$

Scholium. This equation is true, whatever be the value of x, and x may be of any value less than CD. When x is very small, B is very near D, and the line c is very near in position and value to p. When $x=0$, c becomes p, and the angle ABC becomes a right angle, and the equation becomes $b^2=a^2+c^2$, corresponding to (th. 36).

THEOREM 38.

In any triangle, the square of a side opposite an acute angle is less than the square of the base, and the other side, by twice the rectangle of the base, and the distance of the perpendicular from the acute angle.





Let ABC, either figure, represent any triangle; C the acute angle, CB the base, and AD the perpendicular, which falls either without or on the base. Then we are to prove that $AB^2 = CB^2+AC^2-2CB\times CD$.

As in (th. 37), put $AB=c$, $AC=b$, $CB=a$, $BD=x$, $AD=p$; and when the perpendicular falls without the base, as in the first figure, $CD=a+x$; when it falls on the base, $CD=a-x$.

Considering the first figure, and by the aid of (th. 36), we have the following equations:

$$p^2+(a+x)^2=b^2 \qquad (1)$$

$$p^2+x^2=c^2 \qquad (2)$$

By expanding (1), and subtracting (2), we have

$$a^2+2ax=b^2-c^2$$

By adding a^2 to both members, and transposing c^2, we have

$$c^2+(2a^2+2ax)=b^2+a^2$$

By transposing the vinculum, and resolving it into factors, we have

$$c^2=a^2+b^2-2a(a+x). \quad Q. E. D.$$

Considering the other figure, we have

$$p^2+a^2-2ax+x^2=b^2 \qquad (1)$$
$$p^2 \qquad +x^2=c^2 \qquad (2)$$

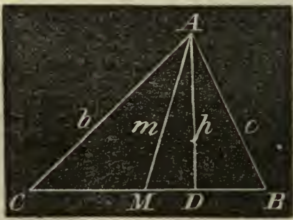
By subtraction $\quad a^2-2ax \quad =b^2-c^2$

By adding a^2 to both members, and transposing c^2, we have

$$c^2+2a^2-2ax=b^2+a^2$$
$$\therefore \quad c^2=b^2+a^2-2a(a-x). \quad Q.E.D.$$

THEOREM 39.

If in any triangle a line be drawn from any angle to the middle of the opposite side, twice the square of this line, together with twice the square of half the side bisected, will be equal to the sum of the squares of the other two sides.



Let ABC be a triangle, its base bisected in M. *Then we are to prove that* $2AM^2+2CM^2=AC^2+AB^2$.

Draw AD perpendicular to the base, and call it p. Put $AC=b$, $AB=c$, $CB=2a$; then $CM=a$, and $MB=a$. Make $MD=x$; then $CD=a+x$, and $DB=a-x$. Put $AM=m$.

Now by (th. 36) we have the two following equations:

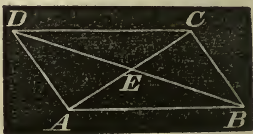
$$p^2+(a-x)^2=c^2 \qquad (1)$$
$$p^2+(a+x)^2=b^2 \qquad (2)$$

By addition $\quad 2p^2+2x^2+2a^2=b^2+c^2$. But $p^2+x^2=m^2$

Therefore $2m^2+2a^2=b^2+c^2$. $\qquad$ *Q. E. D.*

THEOREM 40.

The two diagonals of any parallelogram bisect each other; and the sum of their squares is equal to the sum of the squares of all the four sides of the parallelogram.



Let $ABCD$ be any parallelogram, and draw its diagonals AC and BD.

We are now to show, 1st. *That* $AE=EC$, $DE=EB$. 2d. *That* $AC^2+BD^2=AB^2+BC^2+DC^2+AD^2$.

1. The two triangles ABE and DEC are equal, because $AB = DC$, the angle $ABE =$ the alternate angle EDC, and the vertical angles at E are equal; therefore, AE, the side opposite the angle ABE, is equal to EC, the side opposite the equal angle EDC: also EB, the remaining side of the one $\triangle$ is equal to ED, the remaining side of the other triangle.

2. As ADC is a triangle whose base AC is bisected in E, we have, by (th. 39),

$$2AE^2+2ED^2=AD^2+DC^2 \qquad (1)$$

As ABC is a triangle whose base, AC, is bisected in E, we have

$$2AE^2+2EB^2=AB^2+BC^2 \qquad (2)$$

By adding equations (1) and (2), and observing that

$EB^2=ED^2$, we have

$$4AE^2+4ED^2=AD^2+DC^2+AB^2+BC^2$$

But four times the square of the half of a line is equal to the square of the whole (scholium to th. 33); therefore $4AE^2=AC^2$, and $4ED^2=DB^2$; and by making the substitutions we have

$$AC^2+DB^2=AD^2+DC^2+AB^2+BC^2. \quad Q. E. D.$$

BOOK II.

PROPORTION.

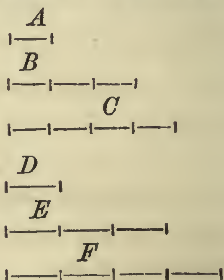
The word Proportion has different shades of meaning, according to the subject to which it is applied: thus, when we say that a person, a building, or a vessel is well *proportioned*, we mean nothing more than that the different parts of the person or thing bear that *general relation* to each other which corresponds to our taste and ideas of beauty or utility, but in a more concise and geometrical sense,

Proportion is the numerical relation which one quantity bears to another of the same kind.

DEFINITIONS AND EXPLANATIONS.

In Geometry, the quantities between which proportion can exist, are of three kinds, only. 1st. *A line to a line.* 2d. *A surface to a surface.* 3d. *A solid to a solid.*

To find the *numerical relation* which one quantity bears to another, we must refer them both to the same standard of measure.



If a quantity, as A, be contained exactly a certain number of times in another quantity, B, the quantity A is said to measure the quantity B; and if the same quantity, A, be contained exactly a certain number of times in another quantity, C, A is also said to be a measure of the quantity C, and it is called a common measure of the quantities B and C; and the quantities B and C will, evidently, bear the same relation to each other that the numbers do which represent the multiple that each quantity is of the common measure A.

Thus, if B contain A three times, and C contain A also three times, B and C being equimultiples of the quantity A, will be equal to each other; and if B contain A three times, and C contain A four times, the proportion between B and C will be the same as the proportion between the numbers 3 and 4.

Again, if a quantity, D, be contained as often in another quantity, E, as A is contained in B, and as often in another quantity, F, as A is contained in C, the ratio of E to F, or the proportion between them, will be the same as the proportion between B and C; and in that case, the quantities B, C, E, and F, are said to be proportional quantities; a relation which is commonly expressed thus, $B : C :: E : F$.

To find the numerical relation that any quantity, as A, has to any other quantity of the same kind as B, we simply divide B by A, and the quotient may appear in the form of a fraction, thus: $\frac{B}{A}$. Now this fraction, or the value of this quotient, is always a *numeral*, whatever quantities may be expressed by A and B.

To find the numerical relation between D and E, we simply divide D by E, or write $\frac{D}{E}$, which denotes the division; and if we find the same quotient as when we divided B by A, then we may write

$$\frac{B}{A}=\frac{D}{E} \qquad (1)$$

If B contains A three times, and D contains E three times, as we have just supposed, equation (1) is nothing more than saying that

$$3=3$$

When we divide one quantity by another to find their *numerical relation*, the quotient thus obtained is called the *ratio*.

When the ratio between two quantities is the same as the ratio between two other quantities, the four quantities constitute a proportion.

N. B. On this single definition rests the whole subject of geometrical proportion.

On this definition, if we suppose that B is any numbei of times A, and D the same number of times E, then

A is to B as E is to D;

Or more concisely:

$A : B = E : D$. *The signs* $: = :$ *meaning equal ratio.*

Now it is manifest, that if E is greater than A, D will be greater than B. If $A=E$, then $B=D$, &c., &c.; and whatever relation or *ratio* A is of E, the same *ratio* B will be of D; and whatever relation B is of A, the same relation D will be of E. This shows that the means may be changed, or made to change places.

Or, . . . $A:E=B:D$, which is the former proportion with the middle terms or *means changed.*

The *first* and *third* of four magnitudes are called the antecedents; the second and fourth, the consequents.

A simple relation or *ratio* exists between any two magnitudes of the same kind; but a proportion, in the full sense of the term, must consist of four quantities.

When the two middle quantities are equal, as,

$$A:B=B:C$$

then the three quantities, A, B, and C, are said to be continued proportionals; and B is said to be the mean proportional between A and C; and C is said to be the third proportional to A and B.

In the proportion $A:B=C:D$, the last D is said to be the fourth proportional to A, B, and C.

By the same rule of expression, A may be called the first proportional, B the second, and C the third; for either one can be found when the other three are given, as we shall subsequently explain.

When quantities have the same constant ratio from one to the other, they are said to be in continued proportion.

Thus: the numbers 1, 2, 4, 8, 16, &c., are in continued proportion; the constant ratio from term to term being 2.

THEOREM 1.

If there be two magnitudes which have a common measure, x, *so that the first magnitude may be expressed by* mx, *the second by* nx; *and two other magnitudes which have a common measure,* y, *so that the first may be expressed by* my, *the second by* ny; *that is, the two common measures* x *and* y *having the same equimultiples,* m *and* n, *to make up the magnitudes; then the four magnitudes will be in geometrical proportion.*

Or . . . $mx:nx=my:ny$

For the *ratio* between mx and nx is $\frac{nx}{mx}=\frac{n}{m}$, and the *ratio* between my and ny is $\frac{ny}{my}=\frac{n}{m}$, the same *ratio;* therefore, by the definition of proportion, these magnitudes are proportional. *Q. E. D.*

Scholium. If we change the means, the magnitudes are still proportional; but the *ratio* between the terms of comparison is different.

Thus: . . $mx : my = nx : ny$.

The ratio between the 1st and 2d, is, $\frac{my}{mx}=\frac{y}{x}$; the ratio between the 3d and 4th is $\frac{ny}{nx}=\frac{y}{x}$, the same ratio as between the other two magnitudes; but as in this latter case we compare different magnitudes, the numeral value of the ratio is different.

But we cannot change the means, unless we then consider the magnitudes existing only in their *numeral relations.* To whatever the magnitudes may refer, whether to lines, surfaces, or solids, the *ratio* is always a mere numeral; therefore, when two ratios stand equal, we may increase or decrease them at pleasure, as will be shown hereafter.

N. B. The first two terms of a proportion are called the *first couplet,* and the last two are called the *second couplet.*

THEOREM 2.

When four magnitudes are in geometrical proportion, the product of the extremes is equal to the product of the means.

Let the four magnitudes be represented by A, B, C, and D.

Then . . . $A : B = C : D$.

Some numeral relation, or ratio, must exist between A and B. Let that ratio be represented by r; that is, B must equal rA.

But, by the definition of proportion, the same relation must exist between C and D as between A and B; or $D = rC$.

Then by substitution we have

$$A : rA = C : rC.$$

The product of the extremes is rCA, and that of the means is ArC; obviously the same. *Q. E. D.*

THEOREM 3.

If three magnitudes be continued proportionals, the product of the extremes is equal to the square of the mean.

Let A, B, and C represent the three magnitudes:

Then . . $A:B=B:C$, by the definition of proportion.

But by theorem 2 (book 2), the product of the extremes is equal to the product of the means; that is, $A\times C=B^2$. *Q. E. D.*

THEOREM 4.

Equimultiples of any two magnitudes have the same ratio as the magnitudes themselves; and the magnitudes and their equimultiples may therefore form a proportion.

Let A and B represent the magnitudes, and mA and mB their equimultiples.

Then . . . $A:B=mA:mB$

For the ratio of A to B is $\frac{B}{A}$, and of mA to mB is $\frac{mB}{mA}=\frac{B}{A}$, the same ratio; therefore, &c. *Q. E. D.*

THEOREM 5.

If four quantities be proportional, they will be proportional when taken inversely.

If $A:B=mA:mB$, then $B:A=mB:mA$;

For in either case, the product of the extremes and means are manifestly equal; or the ratio between the couplets is the same; therefore, &c. *Q. E. D.*

THEOREM 6.

Magnitudes which are proportional to the same proportionals, are proportional to each other.

If . $A:B=P:Q$ }
and . $a:b=P:Q$ } Then we are to prove that $A:B=a:b$.

By the law of proportion $\frac{B}{A}=\frac{Q}{P}$

Also . . . . $\frac{b}{a}=\frac{Q}{P}$

Therefore, by (ax. 1) $\frac{B}{A}=\frac{b}{a}$, or $A:B=a:b$ *Q. E. D.*

Cor. This principle may be extended through any number of proportionals.

THEOREM 7.

If any number of quantities be proportional, then any one of the antecedents will be to its consequent as the sum of all the antecedents is to the sum of all the consequents.

$$\left.\begin{array}{lr} \text{Let} \quad . \quad . \quad . & A:B=C:D \\ \text{And} \quad . \quad . \quad . & C:D=E:F \\ \text{And} \quad . \quad . \quad . & E:F=G:H \\ & \text{\&c.}=\text{\&c.} \end{array}\right\} (1)$$

Then we are to show that

$$A:B=C+E+G \text{ \&c.}:D+F+H, \text{ \&c.}$$

If $A:B$ as $C:D$, then some factor, whole or fractional, multiplied by A, will produce C; and the same factor multiplied by B, will produce D; that is, the proportions (1) become

$$\begin{aligned} A:B&=mA:mB \\ &=nA:nB \\ &=pA:pB \\ &\text{\&c., \&c.} \end{aligned}$$

But, $A:B=mA+nA+pA$, &c $:mB+nB+pB$, &c.

For the ratio . . $\frac{B}{A}=\frac{(m+n+p)B}{(m+n+p)A}$

Now as . . . $mA=C$, $nA=E$, $pA=G$, &c.

Therefore, . $A:B=C+E+G:D+F+H$. *Q. E. D.*

THEOREM 8.

If four magnitudes constitute a proportion, the first will be to the sum of the first and second, as the third is to the sum of the third and fourth.

By hypothesis, $A:B::C:D$; then we are to prove that $A:A+B::C:C+D$.

By the given proportion, $\frac{B}{A}=\frac{D}{C}$.

Add unity to both members, and reducing them to the form of a fraction, we have $\frac{B+A}{A}=\frac{D+C}{C}$. Throwing this equation into its equivalent proportional form, we have

$$A:A+B::C:C+D.$$

N. B. In place of adding unity, subtract it, and we shall find that

$$A:A-B::C:C-D$$

Or . . $$A:B-A::C:D-C.$$

THEOREM 9.

If four magnitudes be proportional, the sum of the first and second is to their difference, as the sum of the third and fourth is to their difference.

Admitting that . $A:B::C:D$, we are to prove that

$$A+B:A-B::C+D:C-D$$

From the same hypothesis, th. 8 gives

$$A:A+B::C:C+D$$

And . . $$A:A-B::C:C-D$$

Changing the means (which will not affect the product of the extremes and means, and of course will not destroy proportionality), and we have

$$A:C::A+B:C+D$$

$$A:C::A-B:C-D$$

Now, by (th. 2), $A+B:C+D::A-B:C-D$

Changing the means, $A+B:A-B::C+D:C-D$

THEOREM 10.

If four magnitudes be proportional, like powers or roots of the same will be proportional.

Admitting . $A:B::C:D$, we are to show that

$$A^n:B^n::C^n:D^n, \text{ and } A^{\frac{1}{n}}:B^{\frac{1}{n}}::C^{\frac{1}{n}}:D^{\frac{1}{n}}$$

By the hypothesis, $\frac{A}{B}=\frac{C}{D}$. Raising both members of this equation to the nth power, and

$$\frac{A^n}{B^n}=\frac{C^n}{D^n}$$

Changing this to the proportion $A^n : B^n :: C^n : D^n$

Resuming again the equation $\frac{A}{B}=\frac{C}{D}$, and taking the nth root of each member, we have $\frac{A^{\frac{1}{n}}}{B^{\frac{1}{n}}}=\frac{C^{\frac{1}{n}}}{D^{\frac{1}{n}}}$. Converting this equation into its equivalent proportion, we have

$$A^{\frac{1}{n}} : B^{\frac{1}{n}} :: C^{\frac{1}{n}} : D^{\frac{1}{n}}$$

Now by the first part of this theorem, we have

$$A^{\frac{m}{n}} : B^{\frac{m}{n}} :: C^{\frac{m}{n}} : D^{\frac{m}{n}}$$ m representing any power whatever, and n representing any root.

THEOREM 11.

If four magnitudes be proportional, also four others, their compound, or product of term by term, will form a proportion.

Admitting that . $A : B :: C : D$
And . . . $X : Y :: M : N$

We are to show that $AX : BY :: MC : ND$

From the first proportion, $\frac{A}{B}=\frac{C}{D}$

From the second, $\frac{X}{Y}=\frac{M}{N}$

Multiply these equations, member by member, and

$$\frac{AX}{BY}=\frac{MC}{ND}$$

Or . . $AX : BY :: MC : ND$

The same would be true in any number of proportions.

THEOREM 12.

Taking the same hypothesis as in (th.11), *we propose to show, that a proportion may be formed by dividing one proportion by the other, term by term.*

By hypothesis, . . $A : B :: C : D$
And . . . $X : Y :: M : N$

Multiply extremes and means, $AD = BC$ (1)

And . . . . . $NX = MY$ (2)

Divide (1) by (2), and . $\frac{A}{X} \times \frac{D}{N} = \frac{C}{M} \times \frac{B}{Y}$

Convert these four terms, which make two equal products, into a proportion, and we shall have

$$\frac{A}{X} : \frac{B}{Y} :: \frac{C}{M} : \frac{D}{N}$$

By comparing this with the given proportions, we find it composed of the quotients of the several terms of the first proportion, divided by the corresponding term of the second.

THEOREM 13.

If four magnitudes be proportional, we may multiply the first couplet or the second couplet, the antecedents or the consequents, or divide them by the same factor, and the results will be proportional in every case.

Suppose . . . . $A : B :: C : D$

Multiply extremes and means, and $AD = BC$ (1)

Multiply this equation by M, and $MAD = MBC$

Now, in this last equation, MA may be considered as a single term or factor, or MD may be so considered. So, in the second member, we may take MB as one factor, or MC. Hence, we may convert this equation into a proportion in four different ways.

Thus, as . . $MA : MB :: C : D$

Or as . . $A : B :: MC : MD$

Or as . . $MA : B :: MC : D$

Or as . . $A : MB :: C : MD$

If we resume the original equation (1), and divide it by any number, M, in place of multiplying it, we can have, by the same course of reasoning,

$$\frac{A}{M} : \frac{B}{M} :: C : D$$

$$A : B :: \frac{C}{M} : \frac{D}{M}$$

$$\frac{A}{M} : B :: \frac{C}{M} : D$$

$$A : \frac{B}{M} :: C : \frac{D}{M}$$

THEOREM 14.

If three magnitudes are in continued proportion, the first is to the third, as the square of the first is to the square of the second.

Let A, B, and C, represent three proportionals.

Then we are to show that $A : C = A^2 : B^2$

By (th. 3) $AC = B^2$

Multiply this equation by the numeral value of A, then we have

$$A^2C = AB^2$$

This equation gives the following proportion:

$$A : C = A^2 : B^2. \qquad Q. E. D.$$

THEOREM 15.

If any one of the four magnitudes which form a proportion, be effaced or unknown, it can be restored by means of the other three.

Let $A : B = C : D$ represent a proportion, and suppose D unknown; then represent it by x

That is . . $A : B = C : x$

The ratio between A and B is the same as between C and x.

Represent the ratio between A and B by r; and as r is always a numeral, whatever quantities are represented by A and B, therefore, $\frac{x}{C} = r$; or $x = rC$; which shows that x or D must be of the same name as C.

When A and B are not commensurable, the *ratio* is expressed by $\frac{B}{A}$ and $x = \frac{CB}{A}$; or, in numbers, the product of the second and third terms divided by the first, will give the fourth, which is the *rule of three* in arithmetic.

In short, as

$$AD = BC, \; A = \frac{BC}{D}, \; B = \frac{AD}{C}, \; C = \frac{AD}{B}, \text{ and } D = \frac{CB}{A}.$$

THEOREM 16.

Parallelograms, and also triangles, having the same or equal altitudes, are to one another as their bases.

Let a represent the number of units, and part of a unit in BC, and b the number of units and part of a unit in BD.

Also let p represent the units and parts of a unit in the perpendicular, AB. Now by (scholium to th. 29 book 1), the parallelogram $ABCE=pa$, and the parallelogram $ABDF=pb$; and as magnitudes must be proportional to themselves,

$$ABCE : ABDF=pa : pb$$

But . . $a : b=pa : pb$ (th. 4 book 2)

Therefore (th. 6 book 2), we have

$$ABCE : ABDF=a : b. \qquad Q. E. D.$$

Cor 1. As triangles on the same base and altitude as parallelograms are halves of parallelograms; and as the halves of quantities are in the same proportion as their wholes; therefore

The . . $\triangle BPC : \triangle BQD=a : b$.

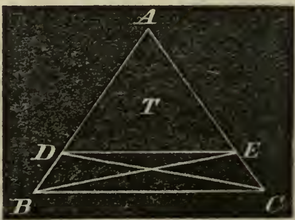
Cor. 2. When parallelograms and triangles have the same or equal basis, they will be to each other as their altitudes; for the proportion $ABCE : ABDF=pa : pb$, as above, is always true; and when a becomes equal to b and p, and p different,

Then . . $ABCE : ABDF=Pa : pa$

Or . . $ABCE : ABDF=P : p$, that is, as their perpendicular altitudes.

THEOREM 17.

Lines drawn parallel to the base of a triangle, cut the sides of the triangle proportionally.



Let ABC be any triangle, and draw DE parallel to the base BC; *then we are to show that*

$$AD : DB=AE : EC.$$

Join DC and BE. The triangle $DEB=$ the $\triangle DEC$, because they are on the same base, DE, and between the same parallels, DE and BC (th. 25 book 1).

Represent the triangle ADE by T, DEB by x, DEC by y; then $x=y$. Now, as the triangles T and x may be considered as having AD and DB for bases, and the perpendicular distance of the point E from AB for altitudes, therefore, by (th. 16, book 2).

$$AD : DB = T : x$$

By reasoning in the same manner in reference to the triangles T and y, they having their common vertex in D, we have the proportion

$$AE : EC = T : y. \quad \text{But } x=y$$

Therefore $\quad AE : EC = T : x$
But $\quad AD : DB = T : x$

Therefore, (th. 6, book 2)

$$AE : EC = AD : DB$$

Or $AD : DB = AE : EC$.

Q. E. D.

Cor. Considering AEB as one triangle, and AED another, having their common vertex in E; and in the same manner, ADC as one, and ADE another, whose vertex is D, then we may have

$$AB : AD = AC : AE$$

For, by taking the proportion

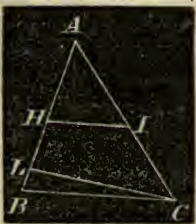
$$AD : DB = AE : EC$$

And by composition, (th. 8 book 2), we have

$$AB : AD = AC : AE.$$

THEOREM 18.

Similar triangles have their sides, about the equal angles, proportional.



Let ABC and DEF be two similar triangles, having the angle $A=D$, $B=E$, and $C=F$; and for the sake of perspicuity, we will suppose AB greater than ED. *Now we are to show that* $AB : AC = DE : DF$; *or* that

$$AB : DE = AC : DF.$$

Conceive the triangle DEF taken up and placed on the triangle ABC, in such a manner that the point D shall fall on A, and the line DE on AB, the point E falling on H. Now, as the angle $E=B$, the line EF, or its representative, HI, will take the direction of BC, and be parallel to BC (def. of parallel lines).

Now the two triangles DEF and AHI are identical; for $AH=DE$, and $A=D$, and $AHI=E$; then $AIH=F$; therefore $AI=DF$, and $HI=EF$. But as HI is parallel to BC, by the last theorem we have

$$AB : AC=AH : AI$$

That is, . . $AB : AC=DE : DF$ *Q. E. D.*

Scholium. If perpendiculars be let fall from like angles in the triangles, to the opposite sides, as CL and FM, such perpendiculars will divide the two triangles into similar partial triangles, and

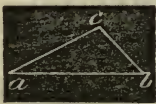
As . . . $AB : DE=AC : DF$

And . . . $CL : MF=AC : DF$

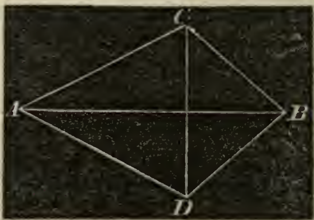
Therefore (th. 6 b. 2) $AB : DE=CL : MF$

THEOREM 19.

If any triangle have its sides respectively proportional to the like sides of another triangle, each to each, then the two triangles will be equiangular.



Let the triangle abc have its sides proportional to the triangle ABC; that is, ac to AC, as cb to CB, and ac to AC, as ab to AB; *then we are to prove* that the $\triangle$ abc is equiangular to the $\triangle$ ABC.



On the other side of the base, AB, and from A, conceive the angle BAD to be drawn $=$ to the angle a; and from the point B, conceive the angle ABD drawn $=$ to the $\angle$ b. Then the third $\angle$ $=$ to the third angle C (th. 11, cor. 2, b. 1); and the $\triangle$ ABD will be equiangular to the $\triangle$ abc by construction.

Therefore, . . $ac : ab=AD : AB$

By hypothesis, . $ac : ab=AC : AB$

Hence, . . $AD : AB=AC : AB$ (th. 6, b. 2).

In this last proportion the consequents are equal; therefore, the antecedents are equal: that is, $AD=AC$

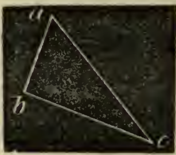
In the same manner we prove that $BD=CB$

But AB is common to the two triangles; therefore, all three of the sides of the $\triangle$ ABD are respectively equal to all three of the sides of the $\triangle$ ABC (th. 19, b. 1).

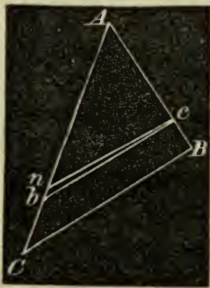
But the $\triangle$ ABD is equiangular to the $\triangle$ abc by construction; therefore, the $\triangle$ ABC is also equiangular to the $\triangle$ abc. *Q. E. D.*

THEOREM 20.

If two triangles have one angle in the one equal to one angle in the other, and the sides about these equal angles, directly, or reciprocally proportional, the two triangles will be equiangular.



Let ABC and abc be two $\triangle$s, and the angle $A=a$, and AC of the one to ac of the other, as AB to ab. *Then we are to show that the angle* $B=b$, *and the angle* $c=C$.



If we take the $\triangle$ abc, turn it over and place the point a on A, ac on AC, and ab on AB, and join cb, then cb will be parallel to CB; for if cb be not parallel to CB, draw cn parallel to CB.

Then $AC : AB :: An : Ac$ (th. 17, b. 2)

Also $AC : AB :: Ab : Ac$ (hy.)

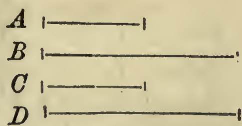
Now as three terms in each of these proportions are the same, the other terms must be equal: that is, $Ab=An$, and cb and cn is the same line. But cn was drawn parallel to CB; that is, cb is parallel to CB; therefore, the angle $C=c$ by the definition of parallel lines. Therefore, &c. *Q. E. D.*

THEOREM 21.

*When four straight lines are in proportion, the product of the extremes is equal to the product of the means.**

Let A, B, C, D, represent the four lines

Then we are to show, geometrically, that $A \cdot D = C \cdot B$.



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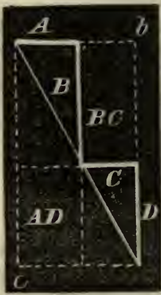
* This proposition has had a symbolical proof, in theorem 2 book 2, but we deem it important to give this geometrical demonstration.



Place A and B at right angles with each other, and draw the hypotenuse. Also place C and D at right angles to each other, and draw its hypotenuse. Then bring the two triangles together, so that C shall be at right angles with B, as represented in the figure.

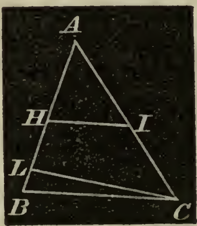
Now, these two $\triangle$s have each a right $\lrcorner$, and the sides about the equal angles, proportional; that is, $A : B = C : D$; therefore, (th. 20, b. 2), the two $\triangle$s are equiangular, and the acute angles which meet at the extremities of B and C, are $=$ to a right angle, and the lines B and C make another right angle, by construction; therefore, the extremities of A, B, C, and D, are in one right line (th. 2 b. 1), and that line is the diagonal of the parallelogram cb. Hence, the complementary parallelograms about this parallelogram are equal (th. 28, b. 1); but one of these is B long, and and C wide, and the other D long, and A wide; therefore,

$$B \times C = A \times D. \quad Q.\ E.\ D.$$

Cor. When $B = C$ then $A \cdot D = B^2$, and B is the mean proportional between A and D.

THEOREM 22.

Similar triangles are to one another as the squares of their like sides.



Let ABC, and DEF, be two similar or equiangular triangles. *Then we are to prove that* $ABC : DEF = AB^2 : DE^2$

By the similarity of the triangles, we have,

$$AB : DE = LC : MF$$

But, . . $AB : DE = AB : DE$

Hence, . . $AB^2 : DE^2 = AB \cdot LC : DE \cdot MF$

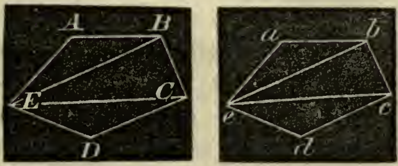
But, by (th. 30, b. 1), $AB \cdot LC$ is double the area of the $\triangle\ ABC$, $DE \cdot MF$ is double of the $\triangle\ DEF$.

Therefore, $\triangle\ ABC : \triangle\ DEF :: AB \cdot LC : DE \cdot MF$

(Th. 6, b. 2). " " $= AB^2 : DE^2$. $Q.\ E.\ D.$

THEOREM 23.

The perimeters of similar figures are to one another as their like sides; and their areas are to one another as the squares of their like sides.



Let $ABCDE$, and $abcde$, be two similar figures; *then we are to show that* EA *is to* ea *as the sum of all the sides* EA+AB, *&c., is to* ea+ab, *&c., and that the area of one is to that of the other,* as EA^2 *to* ea^2, *or* AB^2 *to* ab^2.

As the figures are exactly similar by hypothesis, whatever relation AB is to EA, the same relation ab will be to ea; and if we take

$$\left.\begin{array}{l} AB=mEA \\ BC=nEA \\ CD=pEA \\ DE=qEA \end{array}\right\} \text{ Then we must take } \left\{\begin{array}{l} ab=m(ea) \\ bc=n(ea) \\ cd=p(ea) \\ de=q(ea) \end{array}\right.$$

Now, by (th. 7, b. 2),

$$AE : ea = EA + mEA, \text{ \&c. } : ea + mea, \text{ \&c.}$$

That is,

$EA : ea = P : p$. P and p representing the perimeters of the figures.

As the two figures are exactly similar, whatever part the triangle EAB is of one whole, the same part the triangle eab is of the other whole; therefore,

$$EAB : eab = EABCDE : eabcde.$$

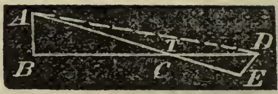
But by (th. 22, b. 2) $\quad EAB : eab = AB^2 : ab^2$

Therefore, by (th. 6, b. 2),

$$EABCDE : eabcde = AB^2 : ab^2. \quad Q.E.D.$$

THEOREM 24.

Two triangles which have an angle in the one, equal to an angle in the other, are to each other as the rectangle of the sides about the equal angles.



Let ABC be one triangle, and CDE the other, and so placed that BC and CD shall be one and the same line. Then if the angle $BCA = ECD$, AC and CE will be in the same line (converse of th. 3, b. 1). Draw the dotted line, AD, and call the triangle $ACD = T$.

We have now to show that the

$$\triangle\ ABC : \triangle\ CDE = BC \cdot CA : CE \cdot CD$$

By (th. 16, b. 2), $\triangle\ ABC : T = BC : CD$

Also, $T : \triangle\ CDE = AC : CE$

By multiplying term by term, and neglecting the common factor in the first couplet, we have,

$$\triangle\ ABC : \triangle\ CDE = AC \cdot BC : CE \cdot CD. \quad Q.\ E.\ D.$$

Scholium. When the sides about the equal angles are proportional, the two $\triangle$s will be similar, and this theorem becomes essentially that of 22; for in that case we shall have,

$$BC : CA = CD : CE.$$

Multiply the first couplet by CA, the last couplet by CE, and changing the means,

$$BC \cdot CA : CE \cdot CD = CA^2 : CE^2$$

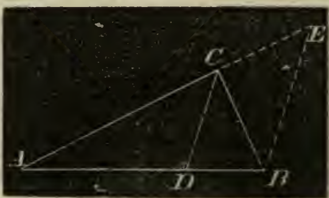
Comparing this proportion with the concluding one, we have,

$$\triangle\ ABC : \triangle\ CDE = CA^2 : CE^2$$

Which is theorem 22 of this book.

THEOREM 25.

If the vertical angle of a triangle be bisected, the bisecting line will cut the base into segments, proportional to the adjacent sides of the triangle.



Let ABC be any triangle, and bisect the vertical angle, C, by the straight line CD. *Then we are to show that*

$$AD : DB = AC : CB.$$

Produce AC to E, making $CE = CB$, and join EB. The exterior angle ACB, of the $\triangle\ CEB$, is equal to the two angles E, and CBE (th. 15, b. 1); but the angle $E = CBE$, because $CB = CE$; therefore the angle ACD, the half of the angle ACB, equals the angle E; hence, DC and BE are parallel (th. 12, b. 1).

Now, as ABE is a triangle, and CD is parallel to BC, therefore, by (th. 17, b. 2), $AD : DB = AC : CE$ or CB. Q. E. D.

THEOREM 26.

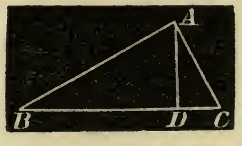
If from the right angle of a right angled triangle, a perpendicular be drawn to the hypotenuse,

1. *The perpendicular divides the triangle into two similar triangles, and each is similar to the whole triangle.*

2. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

3. *The segments of the hypotenuse will be in proportion to the squares of the adjacent sides of the triangle.*

4. *The sum of the squares of the two sides, is equal to the square of the hypotenuse.*



Let BAC be a right angled triangle, right angled at A, and draw AD perpendicular to BC. Put $AB=c$, $AC=b$, and $BC=a$. Put, also, $BD=m$, $DC=n$; then $m+n=a$.

1. The two $\triangle$s, ABC, and ABD, have the common angle, B, and the right angle $BAC=BDA$; therefore, the third angle $C=BAD$, and the two $\triangle$s are equiangular, and therefore similar. In the same manner we prove the $\triangle\ ADC$ similar to the $\triangle\ ABC$, and the two triangles, ABD, ADC, being similar to the same $\triangle$, are similar to each other.

2. As similar triangles have the sides about the equal angles proportional (th. 18, b. 2), therefore,

$$m : AD = AD : n;\ \text{or,}\ m \cdot n = AD^2$$

3. Comparing the triangles ABD, and ABC, the sides about the common angle, B, gives

$$m : c = c : a \qquad (1)$$

Comparing ADC with ABC, we have,

$$n : b = b : a \qquad (2)$$

From proportion (1) we have, $\quad am=c^2 \qquad (3)$

From " (2) " $\quad an=b^2 \qquad (4)$

Divide equation (3) by (4), and $\frac{m}{n}=\frac{c^2}{b^2}$, which shows that the *ratio* between n and m is the same as the *ratio* between b^2 and c^2; or,

$$n : m = b^2 : c^2$$

Or, ... $$m : n = c^2 : b^2$$

4. Add equations (3) and (4), and we have,

$$c^2+b^2=a(n+m)=a^2. \quad Q. E. D.$$

This last equation is theorem 36, book 1.

Scholium. If we take the last equation, $c^2+b^2=a^2$, and transpose b^2, and then separate the second member into factors, we shall have,

$$c^2=a^2-b^2$$
$$=(a+b)(a-b)$$

From this we learn that in any right angled triangle, the hypotenuse, increased by one side, multiplied by the hypotenuse diminished by the same side, is equal to the square of the other side.

BOOK III.

ON THE INVESTIGATION OF THE CIRCLE, THE MEASURE OF ANGLES, AND OTHER THEOREMS IN WHICH THE CIRCLE IS AN IMPORTANT ELEMENT.

DEFINITIONS.

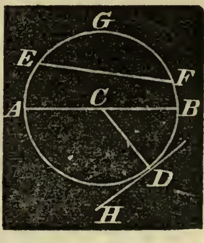
1. A Curve Line is one that is continually changing its direction.

2. A Circle is a figure bounded by one uniform curved line, and all straight lines drawn from a certain point within it to the curve, are equal; and this point is called the center.

3. The entire curve is called the circumference of the circle: any portion of it is called an arch, or arc of the circle.

4. Any single straight line from the center to the circumference, is called the *radius* of the circle.

5. A straight line drawn between any two points on the circum ference, is called a *chord.*



6. The space on either side of a chord, inclosed by the chord and arc, is called a *segment of a circle.*

7. Any chord which passes through the center, is called a *diameter*, and such a chord divides the circle into two equal segments, called *semicircles.*

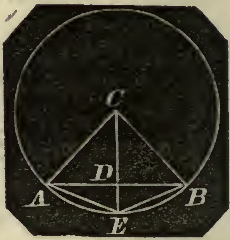
8. A straight line touching the circumference of a circle, at any one point, is called a *tangent to the circle.*

9. The arc, and area between two radii, is called the sector of a circle.

Thus: the marginal figure represents a circle; C is the center, CB, or CD, or CA, or any line from C to the circumference, is a radius. EGF is an arc; EF is a chord; the areas on each side of EF are called *segments.* AB is a diameter; CBD is *a sector;* and HD is a *tangent.*

THEOREM 1.

The radius perpendicular to a chord, bisects the chord, and also the arc of the chord.



Let AB be a chord, C the center of the circle, and CD perpendicular to AB; *then we are to prove that* $AD=BD$, *and* $AE=EB$.

As C is the center of the circle, $AC=CB$, and CD is common to the two $\triangle$s ACD and BCD, and the angles at D being right angles, therefore the two $\triangle$s ADC and BDC are identical, and $AD=DB$, which proves the first part of the theorem.

Now as $AD=DB$, and DE common to the two spaces, ADE and DEB, and the angles at D, right angles, if we conceive the sector CBE turned over and placed on CAE, CE retaining its position, the point B will fall on the point A, because $AD=DB$; then the arc BE will fall on the arc AE; otherwise, there would be points in one or the other arc unequally distant from the center, which is impossible; therefore, the arc AE = the arc EB. *Q. E. D.*

THEOREM 2.

Equal angles, at the center are subtended by equal chords.

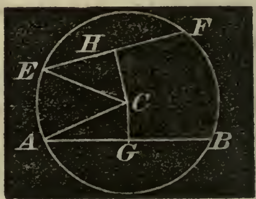
(See figure to last theorem).

Let the angle $ACE=ECB$, then the two isosceles triangles, ACE, and ECB, are equal in all respects, and $AE=EB$.

Q. E. D.

THEOREM 3.

In the same circle, or in equal circles, equal chords are equally distant from the center.



Let AB and EF be equal chords, and C the center of the circle. From C, draw CG and CH perpendicular to the respective chords. These perpendiculars will bisect the chords (th. 1, b. 3), and we shall have $AG=EH$. *We are now to show that* $CG=CH$.

In the two $\triangle$s, ACG and ECH, we have $EC=CA$, $AG=EH$, and the angle $H=$ the angle G, both being right angles ; therefore, the two triangles ACG, and ECH, are identical, and $CG=CH$. *Q. E. D.*

We may demonstrate this theorem analytically, and more generally, as follows :

Let EH represent the half of *any* chord, and put it equal to C. Put $HC=P$, and $CE=R$; R representing the radius of the circle. Then, by (th. 36, b. 1), we have

$$C^2+P^2=R^2 \qquad (1)$$

Also let AG represent the half of *any other* chord, and put it equal to c; and put its distance from the center equal to p; then,

$$c^2+p^2=R^2 \qquad (2)$$

By equating the first members of (1) and (2), we have this general equation : $$C^2+P^2=c^2+p^2 \qquad (3)$$

Now, if $C=c$, that is, the chords equal, then $P^2=p^2$, or $P=p$, the perpendiculars will be equal ; and if $P=p$, then $C=c$; that is, chords equally distant from the center, are equal.

Equation (3) is true, under all circumstances, and if we suppose C greater than c, then P will be less than p; that is, the greater the chord, the nearer it will be to the center.

For if C is greater than c, let d be their difference ;

Then, . . $C=c+d$, and $C^2=c^2+2cd+d^2$

And substitute this value of C^2 in equation (3), and we have,

$$c^2+2cd+d^2+P^2=c^2+p^2$$

By canceling c^2, we have, $2cd+d^2+P^2=p^2$

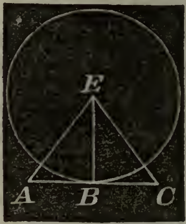
That is P^2 is less than p^2, because it requires $2cd+d^2$ to make equality ; and if P^2 is less than p^2, P is less than p; *that is, the greater chord is at a less distance from the center.*

Cor. If the chord C runs through the center, then P, in equation (3), equals 0, and $C^2=c^2+p^2$. But $R^2=c^2+p^2$, by equation (2), or $C^2=R^2$, or $C=R$, or the semichord becomes the radius, as it manifestly should, in that case.

THEOREM 4.

If any line be drawn tangent to a circle, and from the point of contact a line be drawn to the center of the circle, the tangent and this radius will form a right angle.

A tangent line can meet the circle only at one point, for if the line meets the circles in two points, *and is still a tangent,* it follows that the portion of the circumference between the two points, is a right line; but no part of a circumference is a right line, but a continued curve line; and whenever a right line meets a circle in two points, it must *cut* the circle, and therefore cannot be a tangent.



Now let ABC be a tangent line, touching the circle at the point B, and draw the radius, EB, and the line EC, and EA.

Now we are to show that EB *is perpendicular to* ABC. Because B is the only point in the line ABC which touches the circle, any other line, as EC, or EA, must be greater than EB; therefore, EB is the shortest line that can be drawn from the point E to the line AC; therefore, EB is the perpendicular to AC (th. 20, b. 1). *Q. E. D.*

THEOREM 5.

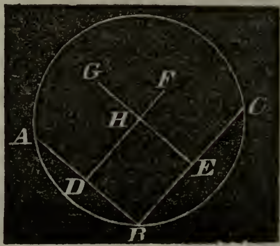
In the same circle, or in equal circles, equal chords subtend or stand on equal portions of the circumference.

Conceive two equal circles, and two equal chords drawn within them. Then conceive one circle taken up and placed upon the other, in such a position that the two equal chords will fall on, and exactly coincide with each other; and then the circles must coincide, because they are equal; and the two segments of the two circles on each side of the equal chords, must also coincide, or the circles could not coincide; and magnitudes which coincide, or exactly fill the same space, are in all respects equal (ax. 9). Therefore

Q. E. D.

THEOREM 6.

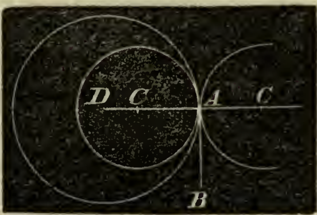
Through three given points, not in the same straight line, one circumference can be made to pass, and but one.



Join AB and BC. If a circle is made to pass through the two points A and B, the line AB will be a chord to such a circle; and if a chord is bisected by a line at right angles, the bisecting line will pass through the center of the circle (th. 1, b. 3); therefore, if we bisect the line AB, and draw DF at right angles from the point of bisection, *any circle that can pass through the points A and B*, must have its center somewhere in the line DF. And, by reasoning in the same way (after we draw EG at right angles from the middle point of BC), any circle that can pass through the points B and C, must have its center somewhere in the line EG. Now, if the two lines, DF, and EG, meet in a common point, that point will be a center, from whence a circle can be drawn to pass through the three points, A, B, and C, and DF and EG will always meet, unless they are parallel, and if they are parallel, it follows that AB and BC must be parallel (definition 13), or be in one and the same straight line; but this can never be the case while the three given points, A, B, and C, are not in the same straight line; therefore the two lines will meet, and from the point H, at which they meet, a circle, and *only one* circle, can be drawn, passing through the three given points. *Q. E. D.*

THEOREM 7.

If two circles touch each other internally, or externally, the two centers and point of contact shall be in one right line.



Let two circles touch each other internally, as represented at A, and through the point A, conceive AB to be a tangent, at the common point. Now, if a line, perpendicular to AB, be drawn from the point A, it must pass through the center of either circle (th. 4, b. 3); and as there can be but one perpendicular from the same point, (th. 20, b. 1), therefore, A, C, and D, the point of contact, and the two centers, must be in one and the same line. *Q. E. D.*

Next, let the circles touch each other externally, and from the point of contact conceive the common tangent, AB, to be drawn.

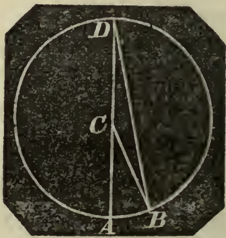
Then a line, AC, perpendicular to AB, will pass through the center of the external circle, (th. 4, b. 3), and a perpendicular, AD, from the same point, A, will pass through the center of the other circle; hence, BAC and BAD are together equal to two right angles; therefore C, A, D, is one continued line (th. 2, b. 1). *Q. E. D.*

Cor. When two circles touch each other internally, the distance between their centers is equal to the difference of their radii; and when they touch each other externally, the distances of their centers are equal to the sum of their radii.

THEOREM 8.

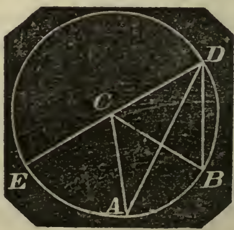
An angle at the circumference of any circle is measured by half the arc on which it stands.

In this work it is taken as an axiom that any angle standing at the center of a circle is measured by the arc on which it stands; and we now proceed to show that the angle at the circumference, is half the angle at the center.



Let ACB be an angle at the center, and D an angle at the circumference, and at first suppose D in a line with AC. *We are now to show that the angle* ACB *is double the angle* D.

Join DB, and the $\triangle$ DCB is an isosceles triangle; for $CD=CB$; and as its exterior angle, ACB, is equal to the two interior angles, D, and CBD, (th. 11, b. 1), and these two angles equal to each other; therefore, ACB is double the angle at D; but ACB is measured by the arc AB; therefore the angle D is measured by half the arc AB.



Now let D be not in a line with AC, but at any point on the circumference (except on AB), and join DC, and produce it to E.

Now by the first part of this theorem,

The angle . $ECB=2EDB$

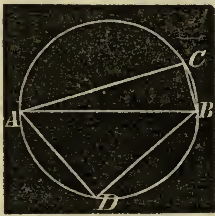
Also, . . $ECA=2EDA$

By subtraction, $ACB=2ADB$

But ACB is measured by the arc AB; therefore ADB or D, is measured by one half of the same arc. *Q. E. D.*

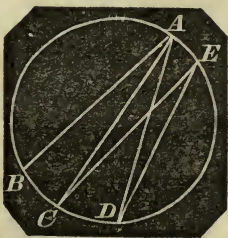
THEOREM 9.

An angle in a semicircle, is a right angle; an angle in a segment, greater than a semicircle, is less than a right angle; and an angle in a segment, less than a semicircle, is greater than a right angle.



If the angle ACB is in a semicircle, the opposite segment, ADB, on which it stands, is also a semicircle, and the angle ACB is measured by half the arc ADB (th. 8, b. 2); that is, half of 180 degrees, or 90 degrees, which is the measure of a right angle.

If the angle ACB is in a segment greater than a semicircle, then the opposite segment is less than a semicircle, and the measure of the angle is less than half of 180 degrees, or less than a right angle. If the angle ACB is in a segment less than a semicircle, then the opposite segment, ADB, on which the angle stands, is greater than a semicircle, and its half, greater than 90 degrees; and, consequently, the angle greater than a right angle. *Q. E. D.*



Scholium. Angles at the circumference, which stand on the same arc of a circle, are equal to one another; for all angles, as CAD, CED, are measured by half the same arc, CD; and having the same measure, they must be equal.

Also, equal angles at the circumference must stand on equal arcs; for the arc, as BC, and CD, being measures of the angles BAC, and CAD, therefore, if the angles are equal, the magnitudes, which measure them, must be equal also.

THEOREM 10.

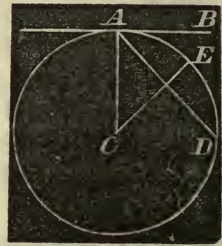
The sum of two opposite angles of any quadrilateral inscribed in a circle, is equal to two right angles.

(See figure to the last theorem.)

Let $ACBD$ represent any quadrilateral inscribed in a circle. The angle ACB has for its measure, half of the arc ADB, and the angle ADB has for its measure, half of the arc ACB; therefore, by addition, the sum of the two opposite angles at C and D, are together measured by half of the whole circumference, or by 180 degrees, or by two right angles. *Q. E. D.*

THEOREM 11.

An angle formed by a tangent and a chord, is measured by one half of the intercepted arc.



Let AB be a tangent, and AD a chord, and A the point of contact; *then we are to show that the angle* BAD *is measured by half the arc* AED.

From A, draw the radius AC; and from the center, C, draw CE perpendicular to AD.

The angle $BAD+DAC=90°$ (th. 4, b. 3)

Also, $C+DAC=90°$ (cor. 4, th. 11, b. 1)

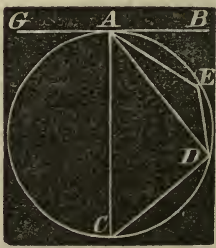
Therefore, by subtraction, $BAD-C=0$

By transposition, the angle $BAD=C$.

But the angle C, at the center of the circle, is measured by the arc AE, the half of AED; therefore, the equal angle, BAD, is also measured by the arc AE, the half of AED. *Q. E. D.*

THEOREM 12.

An angle formed by a tangent and a chord, is equal to an angle in the opposite segment of the circle.



Let AB be a tangent, and AD a chord, and from the point of contact, A, draw any angles, as ACD, and AED, in the segments. *Then we are to show that the angle $BAD=ACD$, and $GAD=AED$.*

By the last theorem, the angle BAD is measured by half the arc AED; and as the angle ACD (th. 8, b. 3) is measured by half of the same arc, therefore the angle $BAD=ACD$.

Again, as $AEDC$ is a quadrilateral, inscribed in a circle, the sum of the opposite angles,

$$ACD + AED = 2 \text{ right angles.} \quad \text{(th. 10, b. 3)}$$

Also, the angles $BAD + DAG = 2$ right angles. (th. 1, b. 1)

By subtraction (and observing that BAD has just been proved equal to ACD), we have,

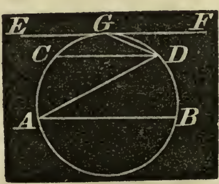
$$AED - DAG = 0$$

Or, ... $AED = DAG$, by transposition.

Q. E. D.

THEOREM 13.

Parallel chords, or a tangent and a parallel chord, intercept equal arcs on the circumference.



Let AB and CD be two parallel chords, and draw the diagonal, AD; and because AB and CD are parallel, the angle DAB = the angle ADC (th. 5, b. 1); but the angle DAB has for its measure, half of the arc BD; and the angle ADC has for its measure, half of the arc AC (th. 8, b. 3); and because the angles are equal, the arcs are equal; that is, the arc BD = the arc AC. *Q. E. D.*

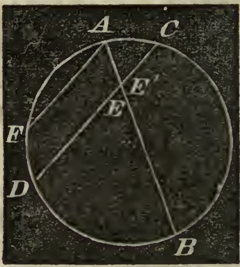
Next, let EF be a tangent, parallel to a chord, CD, and from the point of contact, G, draw GD.

By reason of the parallels, the angle CDG = the angle DGF. But the angle CDG has for its measure, half of the arc CG (th. 9, b. 3); and the angle DGF has for its measure, half of the arc GD (th. 11, b. 3); therefore, these equal measures of equals must be equal; that is, the arc CG = the arc GD. *Q. E. D.*

THEOREM 14.

When two chords intersect each other WITHIN *a circle, the angle thus formed is measured by half the sum of the two intercepted arcs.*

Let AB and CD intersect each other within the circle forming the two angles, E, and E^1, with their opposite vertical and equal angles.



Then we are to show, *that the angle* E *is measured by the half sum of the arcs* AC+BD; *and the angle* E^1 *is measured by the half sum of the arcs* AD+CB.

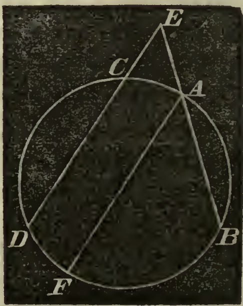
First, draw AF parallel to CD; then, by reason of the parallels, the angle $BAF=E$. But the angle BAF is measured by half of the arc FDB; that is, half of the arc BD, plus half of the arc AC; because $FD=AC$ (th. 13,b. 3).

Now, as the sum of the angles, $E+E^1$, make two right angles, that sum is measured by half the whole circumference.

But the angle E, alone, as we have just determined, is measured by half the sum of the arcs $BD+AC$; therefore, the other angle, E^1, is measured by half of the other parts of the circumference, $AD+CB$. *Q. E. D.*

THEOREM 15.

When two chords intersect, or meet each other WITHOUT *a circle, the angle thus formed is measured by half the difference of the intercepted arcs.*

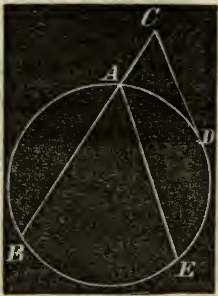


Draw AF parallel to CD; then, by reason of the parallels, the angle E, made by the intersection of the two chords, is equal to the angle BAF. But the angle BAF is measured by half the arc BF; that is, by half the difference between the arcs BD and AC. *Q. E. D.*

N. B. Prolonged chords, to meet without the circle, as ED, and EB, are called secants. They are geometrical, and not trigonometrical secants.

THEOREM 16.

The angle formed by a secant and a tangent, is measured by half the difference of the intercepted arcs.



Let CB be a secant, and CD a tangent. *We are now to show that the angle formed at* C, *is measured by half of the difference of the arcs* BD *and* DA.

From A, draw AE parallel to CD; then the angle $BAE=C$. But the angle BAE is measured by half of the arc BE (th. 8, b. 3); that is, by half of the difference between the arcs BD and AD; for the arc $AD=DE$, and $BD—DE=BE$; therefore the equal angle, C, is measured by half the arc BE. *Q. E. D.*

THEOREM 17.

When two chords intersect each other in a circle, the rectangle of the segments of the one, will be equal to the rectangle of the segments of the other.

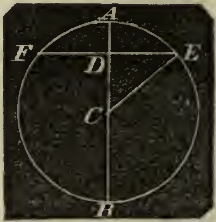
Let AB and CD be two chords intersecting each other in E. *Then we are to show that the rectangle* $AE \times EB = CE \times ED$.

Join AD and CB, forming the two triangles AED and CEB, which are equiangular, and therefore similar; for the angles B and D are equal, because they are both measured by half the arc AC. Also the angles A and C are equal, because each is measured by half the same arc, DB; and the angle $AED=CEB$, because they are vertical angles; hence, the triangles, AED and CEB are equiangular. But equiangular triangles have their sides, about the equal angles, proportional (th. 18, b. 2); therefore, AE and ED, about the angle E, are proportional to CE and EB, about the same angle.

That is, . . $AE : ED :: CE : EB$

Or (th. 21, b. 2), $AE \times EB = ED \times EC$. *Q. E. D.*

Scholium. *When one chord is a diameter, and the other at right angles to it, the rectangle of the segments of the diameter is equal to the square of half the other chord; or half of the bisected chord is a mean proportional between the segments of the diameter.*



For $AD \times DB = FD \times DE$. But if AB passes through the center, C, at right angles to FE, then $FD = DE$ (th. 1, b. 3), and in the place of FD, write its equal, DE, in the last equation, and we have,

$$AD \times DB = DE^2$$

Or, . $AD : DE :: DE : DB$

Put, $DE = x$, $CD = y$, and $CE = R$, the radius of the circle.

Then $AD = R - y$, and $DB = R + y$. With this notation, $AD \times DB$,

Becomes, . . $(R-y)(R+y) = x^2$

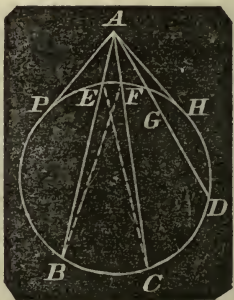
Or, ... $R^2 - y^2 = x^2$

Or, ... $R^2 = x^2 + y^2$

That is, the square of the hypotenuse of the right angled triangle, DCE, *is equal to the sum of the squares of the other two sides.*

THEOREM 18.

If from any point without a circle, any number of secants be drawn, the rectangle formed by any one secant and its external segment, will be equal to the rectangle of any other secant, and its external segment.



Let AB, AC, AD, &c., be secants, and AE, AF, AG, &c., their external segments. *Then we are to show that*

$$AB \times AE = AC \times AF$$

And, $AB \times AE = AD \times AG$, &c.

Join BF and EC; then the two $\triangle$s, AFB and AEC are equiangular; for the angle $B = C$, as each of them is measured by half the same arc, EF; and the angle BAC is common to the two triangles; therefore, the third angles are equal (th. 11, cor. 2, b. 1).

Therefore (th. 18, b. 2), $AB : AF :: AC : AE$

Hence, . . . $AB \times AE = AC \times AF$

In the same manner we may prove that

$AB \times AE = AG \times AD$

And, ... $AC \times AF = AG \times AD$

Q. E. D.

Scholium 1. If we conceive AD to revolve outward, on A, as a fixed point, G and D will come nearer together, and will be exactly together in the tangent AH.

But however far or near G may be to D, we always have,

$$AB \times AE = AD \times AG$$

And, when both AD and AG become AH, we shall have,

$$AB \times AE = \overline{AH}^2$$

Scholium 2. If AH and AP be tangents to the same circle, from the same point on each side of A, they will be equal to each other;

For, . $BA \times AE = AP^2$

Also, . $BA \times AE = AH^2$

Hence (ax. 1), $(AP^2) = (AH^2)$, or $AP = AH$.

This property will enable us to compute the diameter of the earth, whenever we know the visible distance of its regular surface, as seen from any known hight above the surface.

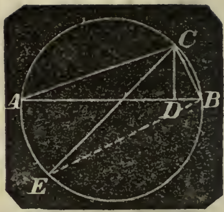
For example, suppose FC to be the diameter of the earth, AF, the hight of a mountain, and AH the distance on sea to the visible horizon. If AF and AH were both known, FC could be computed therefrom. For, let $FC = x$, $AF = h$, and $AH = d$.

Then, . . . $(h+x)h = d^2$, or $x = \frac{d^2}{h} - h$

On this principle, rough estimates of the diameter of the earth have been made; and on this principle the *dip of the horizon* has been computed.

THEOREM 19.

If a circle be described about a triangle, the rectangle of two sides is equal to the rectangle of the perpendicular let fall on to the third side, and the diameter of the circumscribing circle.



Let ABC be the triangle, AC and CB, the sides, CD the perpendicular on the base, and CE the diameter of the circle. *Then we are to show that*

$$AC \times CB = CE \times CD.$$

The two $\triangle$s, ACD and CEB, are equiangular, because $A = E$, both measured by the half of the arc CB. Also, ADC is a right angle, equal to CBE, an angle in a semicircle, and therefore a right angle; hence, the third angle, $ACD = BCE$ (th. 11, cor. 1, b. 1). Therefore (th. 18, b. 2),

$$AC : CD :: EC : CB$$

Hence, . . $AC \times CB = CE \times CD$. *Q. E. D.*

Scholium. *The continued product of three sides of a triangle, is equal to the double area of the triangle into the diameter of its circumscribing circle.*

Multiply both members of the last equation by AB, and we have,

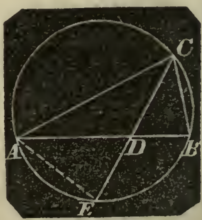
$$AC \times CB \times AB = CE \times (AB \times CD)$$

But CE is the diameter of the circle, and $(AB \times CD)$ = twice the area of the triangle;

Therefore, $AC \times CB \times AB =$ *diameter* $\times 2 \triangle$s.

THEOREM 20.

The square of a line bisecting any angle of a triangle, together with the rectangle of the segments it makes with the opposite side, are equal to the rectangle of the two sides, including the bisected angle.



Let ABC be the triangle, CD the line bisecting the angle C. *Then we are to show that*

$$CD^2 + AD \times DB = AC \times CB.$$

The two $\triangle$s, ACE and CDB, are equiangular, because the angles E and B are equal, both being in the same segment, and the $\angle$ $ACE = BCD$, by hypothesis. Therefore, (th. 18, b. 2),

$$AC : CE :: CD : CB$$

But it is obvious that $CE = CD + DE$, and by substituting this value of CE, in the proportion, we have,

$$AC : (CD + DE) :: CD : CB$$

By multiplying extremes and means,

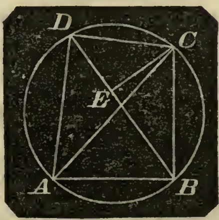
$$CD^2 + DE \times CD = AC \times CB$$

But $DE \times CD = AD \times DB$, by (th. 17, b. 3), which, being substituted, we have,

$$CD^2 + AD \times DB = AC \times CB. \quad Q.\ E.\ D.$$

THEOREM 21.

The rectangle of the two diagonals of any quadrilateral inscribed in a circle, is equal to the sum of the two rectangles of the opposite sides.



Let $ABCD$ be a quadrilateral in a circle; *then we are to show that*

$$AC \times BD = AB \times DC + AD \times BC.$$

From C, let CE be drawn so that the angle DCE shall be equal to angle ACB; and as the angle BAC is equal to the angle CDE, both being in the same segment, therefore, the two triangles, DEC and ABC are equiangular, and we have (th. 18, b. 2),

$$AB : AC :: DE : DC \qquad (1)$$

The two $\triangle$s, ADC and BEC are equiangular; for the angle $DAC = EBC$, both being in the same segment, are measured by half the same arc, DC; and the angle $DCA = ECB$; for $DCE = BCA$; and to each of these add the angle ECA, and $DCA = ECB$; therefore (th. 18, b. 2),

$$AD : AC :: BE : BC \qquad (2)$$

By multiplying the extremes and means in these two proportions, and adding the equations together, we have,

$$(AB \times DC) + (AD \times BC) = (DE + BE) \times AC$$

But, . . . $DE + BE = BD$; therefore,

$$(AB \times DC) + (AD \times BC) = BD \times AC. \quad Q.\ E.\ D.$$

Scholium. When two of the adjacent sides of the quadrilateral are equal, as $AB=BC$, then the resulting equation is,

$$(AB \times DC)+(AB \times AD)=BD \times AC$$

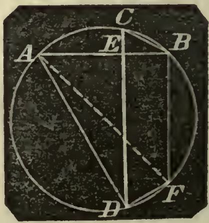
Or, . . $AB \times (DC+AD)=BD \times AC$

Or, . . . $AB : AC :: BD : (CD+AD)$

That is, as one of the equal sides of the quadrilateral, is to the adjoining diagonal, so is the transverse diagonal to the sum of the two unequal sides.

THEOREM 22.

If two chords intersect each other in a circle, at right angles, the sum of the squares of the four segments thus formed, is equal to the square of the diameter of the circle.



Let AB and CD be two chords, intersecting each other at right angles. Draw BF parallel to ED, and join DF and AF. *Now we are to show that*

$$AE^2+EB^2+EC^2+ED^2=AF^2.$$

As BF is parallel to ED, ABF is a right angle, and therefore AF is a diameter (th. 9, b. 3). Also, because BF is parallel to CD, $CB=DF$ (th. 13, b. 3).

Because CEB is a right angle, . $CE^2+EB^2=CB^2=DF^2$

Because AED is a right angle, . $AE^2+ED^2=AD^2$

Adding these two equations, we have,

$$CE^2+EB^2+AE^2+ED^2=DF^2+AD^2$$

But, as AF is a diameter, and ADF a right angle (th. 9, b. 3),

Therefore . $DF^2+AD^2=AF^2$

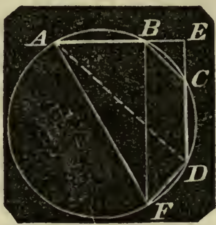
Hence, . . $CE^2+EB^2+AE^2+ED^2=AF^2$. *Q. E. D.*

Scholium. If two chords intersect each other at right angles, in a circle, and their opposite extremities be joined, the two chords thus formed may make two sides of a right angled triangle, of which the diameter of the circle is the hypotenuse.

For AD is one of these chords, and CB is the other; and we have shown that $CB=DF$; and AD and DF are two sides of a right angled triangle, of which AF is the hypotenuse; therefore, AD and CB may be considered the two sides of a right angle, and AF its hypotenuse.

THEOREM 23.

If two secants intersect each other at right angles, the sum of their squares, increased by the sum of the squares of the two parts without the circle, will be equal to the square of the diameter of the circle.



Let AE and ED be two secants intersecting at right angles at the point E. From B, draw BF parallel to CD, and join AF and AD. *Now we are to show that*

$$EA^2+ED^2+EB^2+EC^2=AF^2$$

Because BF is parallel to CD, ABF is a right angle, and consequently AF is a diameter, and $BC=DF$; and because AF is a diameter, ADF is a right angle. As AED is a right angle,

$$AE^2+ED^2=AD^2$$

Also, . . $EB^2+EC^2=BC^2=DF^2$

By addition, $AE^2+ED^2+EB^2+EC^2=AD^2+DF^2=AF^2$.

Q. E. D.

BOOK IV.

PROBLEMS.

In this section, we shall, in most instances, merely show the construction of the problem, and refer to the theorem or theorems that the student may use, to prove that the object is attained by the construction.

In obscure and difficult problems, however, we shall go through the demonstration as though it were a theorem.

PROBLEM 1.

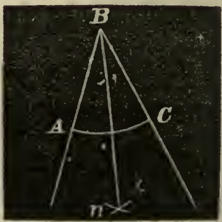
To bisect a given finite straight line.

Let AB be the given line, and from its extremities, A and B, with any radius greater than the half of AB (Post. 3), describe arcs, cutting each other in n and m. Join n and m; and C, where it cuts AB, will be the middle of the line required.

Proof, (th. 15, b, 1, cor. 1).

PROBLEM 2.

To bisect a given angle.

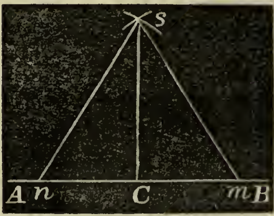


Let ABC be the given angle. With any radius, from the center B, describe the arc AC. From A and C, as centers, with a radius greater than the half of AC, describe arcs, intersecting in n; and join Bn, it will bisect the given angle.

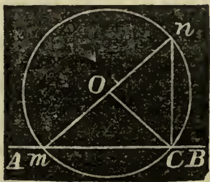
Proof, (th. 19, b. 1).

PROBLEM 3.

From a given point, in a given line, to draw a perpendicular to that line.



Let AB be the given line, and C the given point. Take n and m equal distances on opposite sides of C; and from the points m and n, as centers, with any radius greater than nC or or mC, describe arcs cutting each other in S. Join SC, and it will be the perpendicular required. Proof, (th. 15, b. 1, cor.).

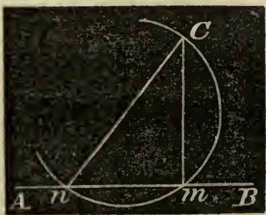


The following is another method, which is preferable, when the given point, C, is at or near the end of the line.

Take any point, O, which is manifestly one side of the perpendicular, and join OC; and with OC, as a radius, describe an arc, cutting AB in m and C. Join mO, and produce it to meet the arc, again, in n; mn is then a diameter to the circle. Join Cn, and it will be the perpendicular required. Proof, (th. 9, b. 3).

PROBLEM 4.

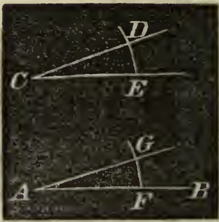
From a given point without a line, to draw a perpendicular to that line.



Let AB be the given line, and C the given point. From C, draw any oblique line, as Cn. Find the middle point of Cn by (problem 1), and from that point, as a center, describe a semicircle, having Cn as a diameter. From the point m, where this semicircle cuts AB, draw Cm, and it will be the perpendicular required. Proof, (th. 9, b. 3).

PROBLEM 5.

At a given point in a line, to make an angle equal to another given angle.



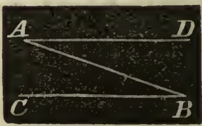
Let A be the given point in the line AB, and DCE the given angle.

From C as a center, with any radius, CE, draw the arc ED.

From A, as a center, with the radius $AF=CE$, describe an indefinite arc; and from F, as a center, with FG as a radius, equal to ED, describe an arc, cutting the other arc in G, and join AG; GAF will be the angle required. Proof, (th. 5, b. 3).

PROBLEM 6.

From a given point, to draw a line parallel to a given line.

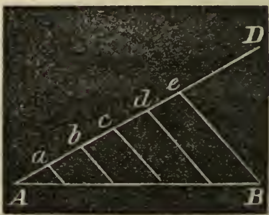


Let A be the given point, and CB the given line. Draw AB, making an angle, ABC; and from the given point, A, in the line AB, draw the angle $BAD=ABC$, by the last problem.

AD and CB make the same angle with AB; they are, therefore, parallel. (Definition of parallel lines).

PROBLEM 7.

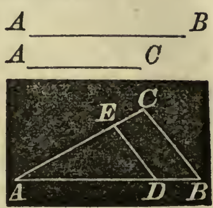
To divide a given line into any number of equal parts.



Let AB represent the given line, and let it be required to divide it into any number of equal parts, say five. From one end of the line A, draw AD, indefinite in both length and position. Take any convenient distance in the dividers, as Aa, and set it off on the line AD; thus making the parts Aa, ab, bc, &c., equal. Through the last point, e, draw EB, and through the points a, b, c, and d, draw parallels to eB (problem 6.); these parallels will divide the line as required Proof (th. 17, b. 2).

PROBLEM 8.

To find a third proportional to two given lines.

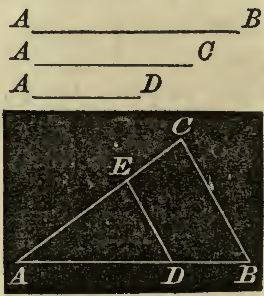


Let *AB* and *AC* be any two lines. Place them at any angle, and join *CB*. On the greater line, *AB*, take *AD*=*AC*, and through *D*, draw *DE* parallel to *BC*; *AE* is the third proportional required.

Proof, (th. 17, b. 2).

PROBLEM 9.

To find a fourth proportional to three given lines.

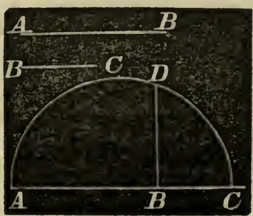


Let *AB*, *AC*, *AD*, represent the three given lines. Place the first two together, at a point forming any angle, as *BAC*, and join *BC*. On *AB* place *AD*, and from the point *D*, draw (problem 6) *DE* parallel to *BC*; *AE* will be the fourth proportional required.

Proof, (th. 17, b. 2).

PROBLEM 10.

To find the middle, or mean proportional, between two given lines.

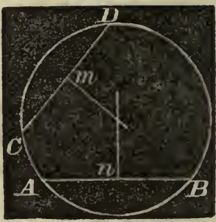


Place *AB* and *BC* in one right line, and, on *AC*, as a diameter, describe a semicircle (postulate 3), and from the point *B*, draw *BD* at right angles to *AC* (problem 3); *BD* is the mean proportional required.

Proof, (scholium to th. 17, b. 3).

PROBLEM 11.

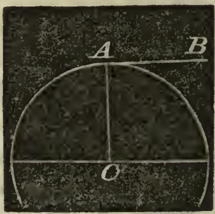
To find the center of a given circle.



Draw any two chords in the given circle, as AB and CD; and from the middle point, n, of AB, draw a perpendicular to AB; and from the middle point, m, draw a perpendicular to CD; and where these two perpendiculars intersect will be the center of the circle. Proof, (th. 1, b. 3).

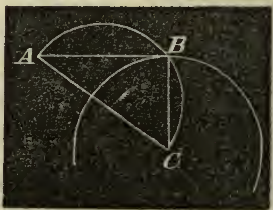
PROBLEM 12.

To draw a tangent to a given circle, from a given point, either in or without the circumference of the circle.



When the given point is in the circumference, as A, draw AC the radius, and from the point A, draw AB perpendicular to AC; AB is the tangent required.

Proof, (th. 4, b. 3).

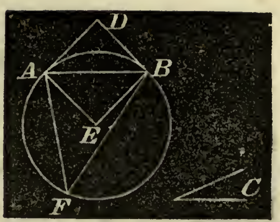


When A is without the circle, draw AC to the center of the circle; and on AC, as a diameter, describe a semicircle; and from the point B, where this semicircle intersects the given circle, draw AB, and it will be tangent to the circle.

Proof, (th. 9, b. 3), and (th. 4, b. 3).

PROBLEM 13.

On a given line, to describe a segment of a circle, that shall contain an angle equal to a given angle.

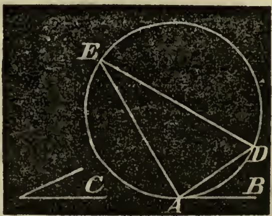


Let AB be the given line, and C the given angle. At the ends of the given line, make angles DAB, DBA, each equal to the given angle, C. Then draw AE, BE, perpendiculars to AD, BD; and with the center, E, and radius, EA or EB, describe a circle; then AFB will be the segment required, as any angle F, made in it, will be equal to the given angle, C.

Proof, (th 11. b. 3), and (th. 8, b. 3).

PROBLEM 14.

To cut a segment from any given circle, that shall contain a given angle.

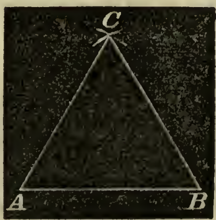


Let C be the given angle. Take any point, as A, in the circumference, and from that point draw the tangent AB; and from the point A, in the line AB, make the angle $BAD=C$ (problem 5), and AED is the segment required.

Proof, (th. 11, b. 3), and (th. 8, b. 3).

PROBLEM 15.

To construct an equilateral triangle on a given finite straight line.

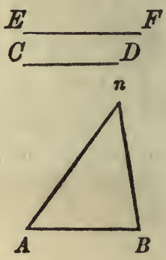


Let AB be the given line, and from one extremity, A, as a center, with a radius equal to AB, describe an arc. At the other extremity, B, with the same radius, describe another arc. From C, where these two arcs intersect, draw CA and CB; ABC will be the triangle required.

The construction is a sufficient demonstration. Or, (ax. 1).

PROBLEM 16.

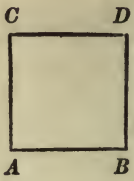
To construct a triangle, having its three sides equal to three given lines, any two of which shall be greater than the third.



Let AB, CD, and EF represent the three lines. Take any one of them, as AB, to be one side of the triangle. From A, as a center, with a radius equal to CD, describe an arc; and from B, as a center, with a radius equal to EF, describe another arc, cutting the former in n. Join An and Bn, and AnB will be the △ required. Proof, (ax. 1).

PROBLEM 17.

To describe a square on a given line.



Let AB be the given line, and from the extremities, A and B, draw AC and BD perpendicular to AB. (Problem 3.)

From A, as a center, with AB as radius, strike an arc across the perpendicular at C; and from C, draw CD parallel to AB; $ACDB$ is the square required. Proof, (th. 21, b. 1.)

PROBLEM 18.

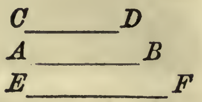
To construct a rectangle, or a parallelogram, whose adjacent sides are equal to two given lines.

Let AB and AC be the two given lines. From the extremities of one line, draw perpendiculars to that line, as in the last problem; and from these perpendiculars, cut off portions equal to the other line; and by a parallel, complete the figure.

When the figure is to be a parallelogram, with oblique angles, describe the angles by problem 5. Proof, (th. 21, b. 1).

PROBLEM 19.

To describe a rectangle that shall be equal to a given square, and have a side equal to a given line.



Let AB be a side of the given square, and CD one side of the required rectangle.

Find the third proportional, EF, to CD and AB (problem 8). Then we shall have,

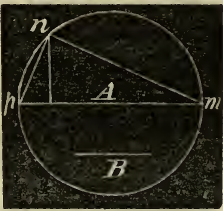
$$CD : AB :: AB : EF$$

Construct a rectangle with the two given lines, CD and EF (problem 18), and it will be equal to the given square, (th. 3, b. 2).

PROBLEM 20.

To construct a square that shall be equal to the difference of two given squares.

Let A represent a side of the greater of two given squares, and B a side of the lesser square.

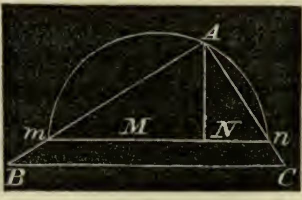


On A, as a diameter, describe a semicircle, and from one extremity, p, as a center, with a radius equal to B, describe an arc, n, and, from the point where it cuts the circumference, draw mn and np; np is the side of a square, which, when constructed, (problem 17), will be equal to the difference of the two given squares. Proof, (th. 9, b. 3, and 36, b. 1.)

PROBLEM 21.

To construct a square, that shall be to a given square, as a line, M, *to a line,* N.

Place M and N in a line, and on the sum describe a semicircle. From the point where they join, draw a perpendicular to meet the circumference in A. Join Am and An, and produce them indefinitely. On Am or An, produced, take $AB=$ to the side of the given square ; and from B, draw BC parallel to mn; AC is a side of the required square.



For, $Am^2 : An^2 :: AB^2 : AC^2$ (th. 17, b. 2.)

Also, $Am^2 : An^2 :: M \; : N$ (scholium to th. 26, b. 2.)

Therefore, $AB^2 : AC^2 :: M \; : N$ (th. 6, b. 2.) *Q. E. D.*

PROBLEM 22.

To cut a line into extreme and mean ratio; that is, so that the whole shall be to the greater part, as that greater is to the less.

Let AB be the line, and from one extremity, B, draw BC at right angles, and equal to half AB.

From C, as a center, and radius CB, describe a circle. Join AC and produce it to F. From A, as a center, and AD radius, describe the arc DE; this arc will divide the line AB, as required.

We are now to show that

$$AB : AE :: AE : EB$$

By (scholium to th. 18, b. 3), we have,

$$AF \times AD = AB^2$$

Or, . . $AF : AB :: AB : AD$

Then, by (th. 8, b. 2), we may have,

$$(AF—AB) : AB :: (AB—AD) : AD$$

As . . $CB = \frac{1}{2}AB = \frac{1}{2}DF$; therefore, $AB = DF$

Hence, . . . $AF—AB = AF—DF = AD = AE$

Therefore, . . $AE : AB :: EB : AE$

By taking the extremes for the means, we have,

$$AB : AE :: AE : EB \qquad Q.\ E.\ D.$$

PROBLEM 23.

To describe an isosceles triangle, having its two equal angles double of the third angle, and the equal sides of any given length.

Let AB be one of the equal sides of the required triangle; and from the point A, with AB radius, strike an arc, BD.

Divide the line AB into extreme and mean ratio by the last problem, and suppose C the point of division, and AC the greater segment.

From the point B, with AC, the greater segment, as radius, strike another arc, cutting the arc BD in D. Join BD, DC, and DA. The triangle ABD is the triangle required.

DEMONSTRATION.

As $AC=BD$, by construction; and as AB is to AC, as AC is to BC, by the division of AB; therefore,

$$AB : BD :: BD : BC$$

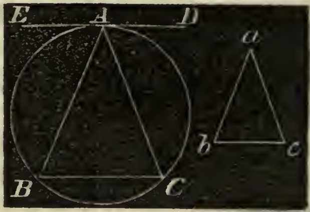
Now, as the terms of this proportion are the sides of the two triangles about the common angle, B, it follows, from (th. 20, b. 2), that the two triangles, ABD and BDC, are equiangular; but the triangle ABD is isosceles; therefore, BDC is isosceles also, and $BD=DC$; but $BD=AC$: hence, $DC=AC$ (ax. 1), and the triangle ACD is isosceles, which gives the angle $CDA=A$. But the exterior angle, $BCD=CDA+A$, (th. 11, b. 1). Therefore, BCD, or its equal $B=CDA+A$; or the angle $B=2A$. Hence, the triangle ABD has each of its angles, at the base, double of the third angle. *Q. E. D.*

Scholium. As the two angles, at the base of the triangle ABD, are equal, and each double of the angle A, it follows that the sum of the three angles is *five times* the angle A. But as the three angles of every triangle always make two right angles, or 180 degrees, therefore, the angle A must be one-fifth of two right angles, or 36 degrees; and BD is a chord of 36 degrees, when AB is a radius to the circle; and ten such chords would extend exactly round the circle.

PROBLEM 24.

Within a given circle to inscribe a triangle, equiangular to a given triangle.

Let ABC be the circle, and abc the given triangle. From any point, as A, draw the tangent EAD to the given circle (problem 12).



From the point A, in the line AD, make the angle DAC= the angle b, (problem 5), and the angle EAB= the angle c, and join BC.

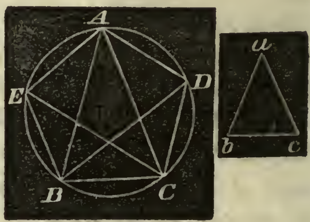
The triangle ABC is inscribed in the circle; it is equiangular to the triangle abc, and is the triangle required.

Proof, (th. 12, b. 3).

PROBLEM 25.

To describe an equilateral and equiangular pentagon in a given circle.

1st. Describe an isosceles triangle, abc, having each of the equal angles, b and c, double of the third angle, a, by problem 23.



2d. Inscribe the triangle ABC, in the given circle, equiangular to the triangle abc, by problem 24; then each of the angles, B and C, is double of the angle A.

3d. Bisect the angles B and C by the lines BD and CE, (problem 3), and join AE, EB, CD, DA, and the figure $AEBCD$ is the pentagon required.

DEMONSTRATION.

By construction, the angles BAC, ABD, DBC, BCE, ECA, are all equal; therefore, by scholium to th. 9, b. 3, the arc BC, AD, DC, AE, and EB, are all equal; and if the arcs are equal the chords AE, EB, &c., are equal. *Q. E. D.*

PROBLEM 26.

To describe an equiangular and equilateral polygon, of six sides, in a circle.

Draw any diameter of the circle, as AB, and from one extremity, B, draw BD equal to BC, the radius of the circle. The arc, BD will be one-sixth part of the whole circumference, and the chord BD will be a side of the regular polygon of six sides.

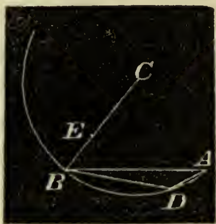
In the △ CBD, as $CB=CD$, and $BD=CB$, by construction the △ is equilateral, and of course equiangular.

But the sum of the three angles of every △, is equal to two right angles, or to 180 degrees; and when the three angles are equal to each other, each one of them must be 60 degrees; but 60 degrees is a sixth parth of 360 degrees, the whole number of degrees in a circle; therefore, the arc whose chord is equal to the radius, is a sixth part of the circumference; and a polygon of six equal sides may be inscribed in a circle, with each side equal to the radius.

Cor. Hence, as BD, is the chord of 60 degrees, and equal to BC or CD, we say generally, *that the chord of* 60 *is equal to radius.*

PROBLEM 27.

To find the side of a regular polygon of fifteen sides, which may be inscribed in any given circle.



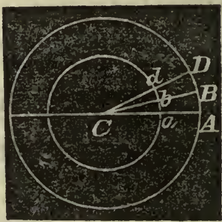
Let CB be the radius of the given circle, and divide it into extreme and mean ratio (problem 22), and make BD equal to CE, the greater part; then BD will be a side of a regular polygon of ten sides (scholium to problem 23). Draw $BA=$ to CB, and it will be a side of a polygon of six sides. Join DA, and that line must be the side of a polygon, which corresponds to the arc of the circle expressed by $\frac{1}{6}$, less $\frac{1}{10}$, of the whole circumference; or $\frac{1}{6}-\frac{1}{10}=\frac{4}{60}=\frac{1}{15}$; that is, one-fifteenth of the whole circumference; or, DA is a side of a regular polygon of 15 sides.

BOOK V.

ON THE PROPORTIONALITIES AND MEASUREMENT OF POLYGONS AND CIRCLES.

THEOREM 1.

The area of any circle is equal to the product of its radius into half of its circumference.



Let CA be the radius of the circle, and AB a very small portion of its circumference, and CAB will be a sector; and we may conceive the whole circle made up of a great number of such sectors; and each sector may be as small as we please; and when *very small*, AB, BD, &c., each one taken separately, may be considered a right line; and the sectors CAB, CBD, &c., will be triangles. The triangle CAB, is measured by the base, CA, multiplied into half the altitude, (th. 30, b. 1) AB; and the triangle CBD is measured by CB, or its equal, CA, into half BD: then the area, or measure of the two triangles, or sectors, is CA, multiplied by the half of AB, plus the half of BD, and so on for all the sectors that compose the circle; therefore, the area of the circle is measured *by the product of the radius into half the circumference.* Q. E. D.

THEOREM 2.

Circumferences of circles are to one another as their radii, and their areas are to one another as the squares of their radii.

Let CA be the radius of a circle (see last figure), and Ca the radius of another circle. Conceive them to be placed upon each other so as to have the same center.

Let AB be a certain definite portion of the circumference of the larger circle, so that m times AB will represent that circumference.

But whatever part AB is of the greater circumference, the same part ab is of the smaller; for the two circles have the same number of degrees, and of course susceptible of division into the same number of sectors. But by proportional triangles we have,

$$CA : Ca :: AB : ab$$

Multiply the last couplet by m (th. 4, b. 2), and we have,

$$CA : Ca :: mAB : mab$$

That is, *as the radius of one circle is to the radius of the other, so is the circumference of the one to the circumference of the other.*

Q. E. D.

To prove the second part of the theorem, represent the larger circle by C, and the smaller by c; and whatever part the sector CAB is of the circle C, the sector Cab is the same part of the circle c.

That is, . $C : c :: CAB : Cab$

But, . . $CAB : Cab :: (CA)^2 : (Ca)^2$ (th. 22, b. 2)

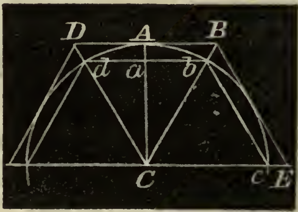
Therefore, . $C : c :: (CA)^2 : (Ca)^2$ (th. 6, b. 2)

Q. E. D.

Scholium. 1. Circles are to one another as the squares of their diameters; for if squares be described about any two circles, such squares will be squares on the diameters of the circles. But each circle is the same proportional part of its circumscribed square; and as like parts of things have the same proportion to each other as the wholes (th. 4, b. 2); therefore, circles are to one another as the squares of their diameters.

Scholium 2. As the circumference of every circle, great or small, is assumed to contain 360 degrees, if we conceive the circumference to be divided into 360 equal parts, and one such part represented by AB, on one circle, or ab on the other, AB and ab will be very near straight lines, and the length of such a line as AB will be greater or less according to the radius of the circle; but its *absolute* length *cannot* be determined until we know the *absolute relation* between the diameter of a circle and its circumference.

To measure the circumference of a circle, or, to discover exactly how many times, and part of a time, it is greater than its diameter, is a problem of some difficulty, and requires patience and care; and it can only be done approximately; for as far as investigations have extended, the circumference of a circle is *incommensurable* with its diameter.



To acquire a very clear and distinct idea of the ratio between the diameter and circumference of a circle, the pupil must commence with first approximations, and proceed with great deliberation.

Conceive a circle described on the radius CA, and in it describe a regular polygon of six sides (problem 26), and each side will be equal to the radius CA; hence the whole *perimeter* of this polygon must be six times the radius, or three times the diameter. Let CA bisect bd in a. Produce Cb and Cd, and through the point A, draw DB parallel to db; DB will then be a side of a regular polygon of six sides, described about the circle, and we can compute the length of this line, DB, as follows: The two triangles, Cbd, and CBD, are equiangular, by construction; therefore,

$$Ca : db :: CA : DB.$$

Now, let us assume CA, or Cd, or the radius of the circle, equal unity; then $db=1$, and the preceding proportion becomes

$$Ca : 1 :: 1 : DB$$

In the right angle triangle Cad, we have,

$$Ca^2+ad^2=Cd^2 \qquad \text{(th. 36, b. 1)}$$

That is, . . $Ca^2+\frac{1}{4}=1$, because $Cd=1$, and $ad=\frac{1}{2}$

By reduction, . . $Ca=\frac{1}{2}\sqrt{3}$, which value of Ca, put in the proportion, we have,

$$\tfrac{1}{2}\sqrt{3} : 1 :: 1 : DB, \text{ or } DB=\frac{2}{\sqrt{3}}$$

But the whole *perimeter* of the circumscribing polygon is six times DB; that is, six times $\dfrac{2}{\sqrt{3}}$, or, $\dfrac{12}{\sqrt{3}}=4\sqrt{3}=6.9282032$.

Thus we have shown, that when the radius of a circle is 1, the perimeter of an inscribed polygon of six sides, is . 6.000000

And of a similar circumscribed polygon, is . . 6.9282032

But, if we call the diameter 1, the perimeter of the inscribed polygon of six equal sides will be, ... 3.0000000

And of the circumscribed, will be 3.4641016

As we would avoid all metaphysical verbiage in science, and come to the point at once, *we lay it down as an axiom*, that when the radius of a circle is 1, and of course the diameter 2, the circumference is *greater* than 6, and less than 6.9282032; and if the diameter is 1, the circumference must be greater than 3, and less than 3.4641016; and this we may call the first approximation to the ratio between the diameter and circumference of a circle.

Scholium 3. As the area of a circle is numerically equal to the radius multiplied by half the circumference (th. 2, b. 5), therefore, if we represent the radius by R, and half the circumference by π, and the area of the circle by a, then we shall have this equation:

$$R\pi = a$$

If we now make $R=1$, this equation gives $\pi=a$; that is, *when the radius of a circle is* 1, *the half circumference is numerically equal to the area.* We will, therefore, seek the *area* of a circle whose radius is unity; and that *area,* if found, will be numerically the half circumference, and by inspecting the last figure, we perceive that it is perfectly axiomatic (the whole is greater than a part), that the *area* of the sector $CbAd$, is greater than the triangle Cbd, and less than the triangle CBD; and the *area* of the whole circle is greater than one polygon, and less than the other. *Finding the* AREA *of a circle, or finding a square which shall be equal to a circle of given diameter, is known as the celebrated problem of squaring the circle.*

THEOREM 3.

Given, the area of a regular inscribed polygon, and the area of a similar circumscribed polygon, to find the areas of a regular inscribed and circumscribed polygon of double the number of sides.

Let C be the center of the circle; AB a side of the given inscribed polygon; EF parallel to AB, a side of the circumscribed polygon.

If AM be joined, and AR and BQ be drawn as tangents, at A and B, AM will be a side of an inscribed polygon of double the number of sides; and $AR = RM$ (scholium 2, th. 18, b. 3), $BQ = QM$, and $AR + RM = RQ =$ the side of the circumscribed polygon of double the number of sides.

The $\triangle$s ARC and RMC, are equal, for $AC = CM$. CR is common to both triangles, and $AR = RM$, tangents from the same point, R; therefore, CR bisects the angle ECM.

Now, as the same construction, and the same reasoning would take place at every one of the equal sectors of the circle, it is sufficient to consider one of them, and whatever is true of that arc, would be true of every one, and true for the whole circle, and its polygons.

To avoid confusion, let p represent the *area* of the given inscribed polygon, and P the *area* of the similar circumscribed polygon. Also let p' represent the area of an inscribed polygon of double the number of sides, and P' the circumscribed polygon of double the number of sides.

As the $\triangle$s ACD and ACM have the common vertex A, they are to each other as their bases, CD to CM; they are also to each other as the polygons of which they form a part.

Hence, . . $p : p' :: CD : CM$ (1)

As AD and EM are parallel, we have,

$$CA : CE :: CD : CM \quad (2)$$

But, because of the common vertex, M, the two $\triangle$s, CAM and CEM, are to each other as CA to CE. But the $\triangle$s are like parts of the polygons p' and P; we have,

Therefore, . $p' : P :: CA : CE$ (3)

That is, . . $p' : P :: CD : CM$ (4) (th. 6, b. 2)

By comparing (1) and (4), we have,

$$p' : P :: p : p', \text{ or } p' = \sqrt{P \times p}$$

That is, the area of p' is a mean proportional between P and p.

The two $\triangle$s, RMC and ERC, having the same vertex, C, are to each other as their bases, MR to RE.

But, because CR bisects the angle ECM, (th. 25, b. 2)

$$MR : RE :: CM : CE$$

But, . . $CM : CE :: CD : CA$ or CM

That is, . $RMC : ERC :: CD : CM$

Or, . $RMC : ERC :: p : p'$

By composition, (th. 8, b. 2),

$$2(RMC) : (RMC+ERC) :: 2p : p+p'$$

But 2 times RMC is P', and $(RMC+ERC)$ is P

Therefore, . . $P' : P :: 2p : p+p'$

Or, ... $P'=\dfrac{2pP}{p+p'}$

Now, P' is known, because $2pP$ is known; and $p+p'$ is also known, as p' has been previously determined. Hence, by means of P and p, we can determine P' and p'. *Q. E. D.*

Scholium. By inspecting the figure in the scholium to theorem 2, we perceive, that if we double the number of sides of the inscribed polygon, we shall more nearly fill up the circle; and if we double the number of sides of the circumscribed polygons, we shall more nearly pare them down to the surface of the circle.

Hence, by continually increasing the sides of the polygons, as indicated by the last theorem, we can find two polygons which shall differ from each other by the smallest conceivable quantity; but the surface of the circle is always between the two polygons; and thus the surface of the circle can be determined to any assignable degree of exactness.

By taking the figure in the scholium to theorem 2, b. 5, we perceive that the area of an inscribed polygon of six sides, to radius unity must be . . $Ca \times da \times 6$

Which is . . $\frac{3}{2}\sqrt{3}$, because $da=\frac{1}{2}$

And, . . . $Ca^2+da^2=Cd^2=1$

Or, . . . $Ca=\frac{1}{2}\sqrt{3}$

Hence, . . . $\frac{1}{2}\sqrt{3}\times\frac{1}{2}\times6=\frac{3}{2}\sqrt{3}=p$, which corresponds with p, in the last theorem.

The area of the circumscribing polygon is measured by

$$CA\times DA\times 6=6DA=3DB.$$

But . . . $Ca : db :: CA : DB$. (th. 17, b. 2.)

That is, . . $\frac{1}{2}\sqrt{3} : 1 :: 1 : DB$, or $BD=\frac{2}{\sqrt{3}}$

Therefore, . . $3DB=\frac{6}{\sqrt{3}}=2\sqrt{3}$, which corresponds with the last theorem.

Having, now, the area of an inscribed and circumscribed polygon of six sides, by applying the last theorem we can readily determine the *area* of an inscribed and a circumscribed polygon of 12 sides.

Thus, . . $p'=\sqrt{pP}=\sqrt{\frac{3}{2}\sqrt{3}\times 2\sqrt{3}}=3$

$$P'=\frac{2pP}{p'+p}=\frac{2\times\frac{3}{2}\sqrt{3}\times 2\sqrt{3}}{3+\frac{3}{2}\sqrt{3}}=\frac{18}{3+\frac{3}{2}\sqrt{3}}=\frac{12}{2+\sqrt{3}}=24-12\sqrt{3}$$

Now let p' and P' be the given polygons, and find others of double the number of sides, and thus continue until the inscribed and circum scribed so nearly coincide, as to determine a very approximate area of the circle.

In this manner we formed the following table:

Number of sides.	Inscribed polygons.	Circumscribed polygons
6	$\frac{3}{2}\sqrt{3}=$ 2.59807621	$2\sqrt{3}=$3.46410161
12	$3=$ 3.0000000	$\frac{12}{2+\sqrt{3}}=$3.2153904
24	$\frac{6}{\sqrt{2+\sqrt{3}}}=$ 3.1058286	3.1596602
48	3.1326287	3.1460863
96	3.1393554	3.1427106
192	3.1410328	3.1418712
384	3.1414519	3.1416616
768	3.1415568	3.1416092
1536	3.1415829	3.1415963
3072	3.1415895	3.1415929
6144	3.1415912	3.1415927

Thus we have found, that when the radius of a circle is 1, the semi-circumference must be more than 3.1415912, and less than 3.1415927; and this is as accurate as can be determined with the small number of decimals here used. To be more accurate we must have more decimal places, and go through a very tedious mechanical operation; but this is not necessary, for the result is well known, and is 3.1415926535897 *plus* other decimal places to the 100th, without termination. This was discovered through the aid of an infinite series in the differential and integral calculus.

The number 3.1416 is the one generally used in practice, as it is much more convenient than a greater number of decimals, and it is sufficiently accurate for all ordinary purposes.

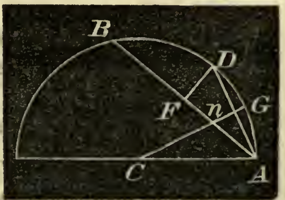
In analytical expressions it has become a general custom with mathematicians to represent this number by the Greek letter π, and, therefore, when any diameter of a circle is represented by D, the circumference of the same circle must be πD. If the radius of a circle is represented by R, the circumference must be represented by $2\pi R$.

As a farther discipline of mind, and for more practical utility, as applicable to trigonometry, we give another method of determining the circumference of a circle, when the diameter is given. It is evident that when we take a small arc, the chord and the arc are nearly of the same length; but the arc is greater than the chord, for the chord is a straight line, and the arc is *curved.* But if we take the half of any small arc, and draw two chords in place of one, such chords taken together, will be much nearer to, and more nearly equal in length to the arc than the one chord of the undivided arc would be.

Now, if we can divide the circumference into *several thousand* equal parts, and can find the length of a chord corresponding to one of these parts, the sum of all these equal chords will be *infinitely near* the circumference of the circle; and the length of such a small chord we can find, *provided* we can first know the chord of any definite arc, and from that deduce the chord of any definite portion of that arc; and this is shown in the following theorem.

THEOREM 4.

Given, the chord of any arc, to determine the chord of half that arc.



Let AB represent a given chord. Bisect the arc AB in D, and join AD. From C, the center of the circle, draw CG perpendicular to AD; and from D, draw DF perpendicular to AB.

From AB *we are to determine* AD. The two $\triangle$s, CAn and AFD, are equiangular; for the angle FAD, at the circumference, is measured by half the arc BD; and nCA, at the center, is measured by half of an equal arc, AD. The right angle, $F=$ the right angle CnA; therefore,

As . . . . $DA : AF :: CA : Cn$.

In the triangle CnA, let $Cn=y$, $nA=x$, and $CA=1$.

Then $AD=2x$; and put $AB=C$; then $AF=\frac{1}{2}C$.

By this notation the preceding proportion becomes

$$2x : \tfrac{1}{2}C :: 1 : y. \quad \text{Hence, } y=\frac{C}{4x}$$

But in the right angled triangle CnA, we have

$$y^2+x^2=1$$

By taking the value of y^2, from the proportion, and reducing, we have the quadratic

$$16x^4-16x^2=-C^2$$

By adding 4 to both members (see Alg. Art. 99), and extracting square root, we have

$$4x^2-2=\pm\sqrt{4-C^2}$$

Therefore, . . . $2x=\sqrt{2-\sqrt{4-C^2}}$

As $2x$ is the value of AD, the expression $(2-\sqrt{4-C^2})^{\frac{1}{2}}$ is the value of the chord of the half of any arc, when C represents the value of the chord of the whole arc. We must take the *minus* sign to the part represented by $\sqrt{4-C^2}$, as the plus sign would give increasing, and not decreasing values.

If we represent the chord of a given arc by C, and the chord of half that arc by C_1, and the chord of half that arc by C_2, and the chord of half that arc again by C_3, &c., &c., we shall have the following series of equations :

$$C= \text{ the first chord}$$

$$(2-\sqrt{4-C^2})^{\frac{1}{2}}=C_1$$

$$(2-\sqrt{4-C_1^2})^{\frac{1}{2}}=C_2$$

$$(2-\sqrt{4-C_2^2})^{\frac{1}{2}}=C_3$$

$$\&c.=\&c.$$

To apply these equations, we observe that in any circle the chord of 60° is equal to the radius (cor. to prob. 26), and if the radius is assumed as unity, we have,

$C=$ chord of 60° $=1.000000000$ sid. ins. pol. of 6 sides.

$(2-\sqrt{4-C^2})^{\frac{1}{2}}=C_1=$ chord of 30° $=.5176380902$ sid. ins. pol. of 12 sides.

$(2-\sqrt{4-C_1^2})^{\frac{1}{2}}=C_2=$ chord of 15° $= .2610523842$ sid.
ins. pol. of 24 sides.

$(2-\sqrt{4-C_2^2})^{\frac{1}{2}}=C_3=$ chord of 7° 30′ $= .1308062583$ sid.
ins. pol. of 48 sides.

$(2-\sqrt{4-C_3^2})^{\frac{1}{2}}=C_4=$ chord of 3° 45 $= .0654381655$ sid.
ins. pol. of 96 sides.

$(2-\sqrt{4-C_4^2})^{\frac{1}{2}}=C_5=$ chord of 1° 52′ 30″ $= .0327234632$ sid.
ins. pol. of 192 sides.

$(2-\sqrt{4-C_5^2})^{\frac{1}{2}}=C_6=$ chord of 56′ 15″ $= .0163622792$ sid.
ins. pol. of 384 sides.

$(2-\sqrt{4-C_6^2})^{\frac{1}{2}}=C_7=$ chord of 28′ 7″ 30‴ $= .0081812080$ sid.
ins. pol. of 768 sides.

$(2-\sqrt{4-C_7^2})^{\frac{1}{2}}=C_8=$ chord of 14′ 3″ 45 ‴ $= .0040906112$ sid.
ins. pol. of 1536 sides.

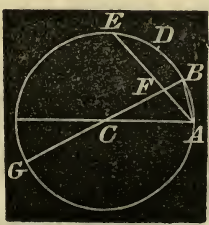
$(2-\sqrt{4-C_8^2})^{\frac{1}{2}}=C_9=$ chord of 7′ &c. $= .0020453068$ sid.
ins. pol. of 3072 sides.

Hence, $.0020453068\times3072=6.2831814896$, is the perimeter of an inscribed polygon of 3072 sides when the radius is 1, or diameter 2. When the diameter is 1, the perimeter is 3.1415907498, which is a a little, and but a little, less than the circumference, as determined by more extended computations.

Although not necessary for practical application, the following beautiful theorem for the analytical tri-section of an arc will not be unacceptable.

THEOREM 5.

Given, the chord of any arc, to determine the chord of one third of such arc.



Let AE be the given chord, and conceive its arc divided into three equal parts, as represented by AB, BD, and DE.

Through the center draw BCG, and join AB. The two △s, CAB and ABF, are equiangular; for the angle FAB, being at the circumference, is measured by half the arc BE, which is equal to AB, and the angle BCA, at the center, is measured by the arc AB; therefore, the angle $FAB = BCA$; but the angle CBA or FBA, is common to both triangles; therefore, the third angle, CAB, of the one triangle, is equal to the third angle, AFB, of the other (th. 11, b. 1, cor. 2), and the two triangles are equiangular and similar.

But the $\triangle\ CBA$ is isosceles; therefore, the $\triangle\ AFB$ is also isosceles, and $AB = AF$, and we have the following proportions:

$$CA : AB :: AB : BF$$

Now let $AE = c$, $AB = x$, $CA = 1$. Then $AF = x$, and $EF = c - x$, and the proportion becomes,

$$1 : x :: x : BF. \quad \text{Hence } BF = x^2$$

Also, ... $FG = 2 - x^2$

As AE and GB are two chords that intersect each other at the point F, we have,

$$GF \times FB = AF \times FE \qquad \text{(th. 17, b. 3)}$$

That is, . . $(2 - x^2)x^2 = x(c - x)$

Or, ... $x^3 - 3x = -c$

If we suppose the arc AF to be 60 degrees, then $c = 1$, and the equation becomes $x^3 - 3x = -1$; a cubic equation, easily resolved by Horner's method (Robinson's Algebra, University Edition, Art. 193), giving $x = .347296+$, the chord of 20°. This again may be taken for the value of c, and a second solution will give the chord of 6° 40′, and so on, trisecting as many times as we please.

If the pupil has carefully studied the foregoing principles, he has the foundation of all geometrical knowledge; but to acquire independence and confidence, it is necessary to receive such discipline of mind as the following exercises furnish.

Some of the examples are mere problems, some are theorems, and some a combination of both. Care has been taken in their selection, that they should be appropriate; not very severe, not such as to try the powers of a professed geometrician, nor such as would be too trifling to engage serious attention.

EXERCISES IN GEOMETRICAL INVESTIGATION.

1. From two given points, to draw two equal straight lines, which shall meet in the same point, in a line given in position.

2. From two given points on the same side of a line, given in position to draw two lines which shall meet in that line, and make equal angles with it.

3. If from a point without a circle, two straight lines be drawn to

the concave part of the circumference, making equal angles with the line joining the same point and the center, the parts of these lines which are intercepted within the circle, are equal.

4. If a circle be described on the radius of another circle, any straight line drawn from the point where they meet, to the outer circumference, is bisected by the interior one.

5. From two given points on the same side of a line given in position, to draw two straight lines which shall contain a given angle, and be terminated in that line.

6. If, from any point without a circle, lines be drawn touching it the angle contained by the tangents is double the angle contained by the line joining the points of contact, and the diameter drawn through one of them.

7. If, from any two points in the circumference of a circle, there be drawn two straight lines to a point, in a tangent, to that circle, they will make the greatest angle when drawn to the point of contact.

8. From a given point within a given circle, to draw a straight line which shall make, with the circumference, an angle, less than any angle made by any other line drawn from that point.

9. If two circles cut each other, the greatest line that can be drawn through the point of intersection, is that which is parallel to the line joining their centers.

10. If, from any point within an equilateral triangle, perpendiculars be drawn to the sides, they are, together, equal to a perpendicular drawn from any of the angles to the opposite side.

11. If the points of bisection of the sides of a given triangle be joined, the triangle, so formed, will be one-fourth of the given triangle.

12. The difference of the angles at the base of any triangle, is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.

13. If, from the three angles of a triangle, lines be drawn to the points of bisection of the opposite sides, these lines intersect each other in the same point.

14. The three straight lines which bisect the three angles of a triangle, meet in the same point.

15. The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of two opposite sides, are, together, half the parallelogram.

16. The figure formed by joining the points of bisection of the sides of a trapezium, is a parallelogram.

17. If squares be described on three sides of a right angled triangle,

and the extremities of the adjacent sides be joined, the triangles so formed, are equal to the given triangle, and to each other.

18. If squares be described on the hypotenuse and sides of a right angled triangle, and the extremities of the sides of the former, and the adjacent sides of the others, be joined, the sum of the squares of the lines joining them, will be equal to five times the square of the hypotenuse.

19. The vertical angle of an oblique-angled triangle, inscribed in a circle, is greater or less than a right angle, by the angle contained between the base, and the diameter drawn from the extremity of the base.

20. If the base of any triangle be bisected by the diameter of its circumscribing circle, and, from the extremity of that diameter, a perpendicular be let fall upon the longer side, it will divide that side into segments, one of which will be equal to half the sum, and the other to half the difference of the sides.

21. A straight line drawn from the vertex of an equilateral triangle, inscribed in a circle, to any point in the opposite circumference, is equal to the two lines together, which are drawn from the extremities of the base to the same point.

22. The straight line bisecting any angle of a triangle inscribed in a given circle, cuts the circumference in a point, which is equidistant from the extremities of the sides opposite to the bisected angle, and from the center of a circle inscribed in the triangle.

23. If, from the center of a circle, a line be drawn to any point in the chord of an arc, the square of that line, together with the rectangle contained by the segments of the chord, will be equal to the square described on the radius.

24. If two points be taken in the diameter of a circle, equidistant from the center, the sum of the squares of the two lines drawn from these points to any point in the circumference, will be always the same.

25. If, on the diameter of a semicircle, two equal circles be described, and in the space included by the three circumferences, a circle be inscribed, its diameter will be $\frac{2}{3}$ the diameter of either of the equal circles.

26. If a perpendicular be drawn from the vertical angle of any triangle to the base, the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.

27. The square described on the side of an equilateral triangle, is equal to three times the square of the radius of the circumscribing circle.

28. The sum of the sides of an isosceles triangle, is less than the sum of any other triangle on the same base and between the same parallels.

29. In any triangle, given one angle, a side adjacent to the given angle, and the difference of the other two sides, to construct the triangle.

30. In any triangle, given the base, the sum of the other two sides, and the angle opposite the base, to construct the triangle.

31. In any triangle, given the base, the angle opposite to the base, and the difference of the other two sides, to construct the triangle.

PROBLEMS REQUIRING THE AID OF ALGEBRA FOR THEIR SOLUTION.

No definite rules can be given for the solution or construction of the following problems; and the pupil can have no other resources than his own natural tact, and the application of his analytical and geometrical knowledge thus far obtained; and if that knowledge is sound and practical, the pupil will have but little difficulty; but if his geometrical acquirements are superficial and fragmentary, the difficulties may be insurmountable: hence, the ease or the difficulty which we experience in resolving such problems, is the test of an efficient or inefficient knowledge of theoretical geometry.

When a problem is proposed requiring the aid of Algebra, draw the figure representing the several parts, both known and unknown. Represent the known parts by the first letters of the alphabet, and the unknown and required parts by the final letters, &c.; and use whatever truths or conditions are available to obtain a sufficient number of equations, and the solution of such equations will give the unknown and required parts the same as in common Algebra.

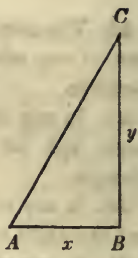
But as we are unable to teach by more general precept, we give the solutions of a few examples, as a guide to the student.

The first two are specimens of the most simple and easy; the last two or three are specimens of the most difficult and complex, or such as might not be readily resolved, in case solutions were not given.

It might be proper to observe that different persons might draw different figures to the more complex problems, and make different equations and give different solutions; but the best solutions are always the most simple.

PROBLEM 1.

Given, the hypotenuse, and the sum of the other two sides of a right angled triangle, to determine the triangle.



Let ABC be the $\triangle$. Put $CB=y$, $AB=x$, $AC=h$, and $CB+AB=s$. Then, by a given condition we we have,

$$x+y=s$$

And, . . $x^2+y^2=h^2$ (th. 36, b. 1)

From these two equations a solution is easily obtained, giving,

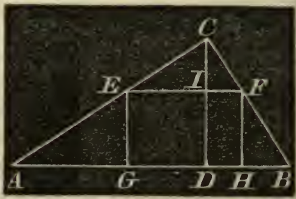
$$x=\tfrac{1}{2}s\pm\tfrac{1}{2}\sqrt{2h^2-s^2} \qquad y=\tfrac{1}{2}s\mp\tfrac{1}{2}\sqrt{2h^2-s^2}$$

If $h=5$, and $s=7$, $x=4$ or 3, and $y=3$ or 4.

N. B. In place of putting x to represent one side, and y the other, we might put $(x+y)$ to represent the greater side, and $(x-y)$ the lesser side; then, . . $x^2+y^2=\frac{h^2}{2}$, and $2x=s$, &c.

PROBLEM 2.

Given, the base and perpendicular of a triangle, to find the side of its inscribed square.



Let ABC be the $\triangle$. $AB=b$, the base, $CD=p$, the perpendicular.

Draw EF parallel to AB, and suppose it equal to EG, a side of the required square; and put $EF=x$.

Then, by proportional $\triangle$s we have,

$$CI : EF :: CD : AB$$

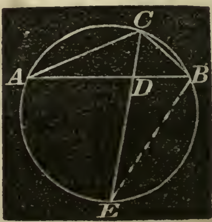
That is, $p-x : x :: p : b$

Hence, . $bp-bx=px$; or, $x=\frac{bp}{b+p}$

That is, the side of the inscribed square is equal to the product of the base and altitude, divided by their sum.

PROBLEM 3.

In a triangle, having given the sides about the vertical angle, and the ine bisecting that angle and terminating in the base, to find the base.



Let ABC be the $\triangle$, and let a circle be circumscribed about it. Divide the arc AEB into two equal parts at the point E, and join EC. This line bisects the vertical angle (th. 9, b. 3, scholium). Join BE.

Put $AD=x$, $DB=y$, $AC=a$, $CB=b$, $CD=c$, and $DE=w$. The two $\triangle$s, ADC and EBC, are equiangular; from which we have,

$$w+c : b :: a : c; \text{ or, } cw+c^2=ab \qquad (1)$$

But, as EC and AB are two chords that intersect each other in a circle, we have, ... $cw=xy$ (th. 17, b. 3)

Therefore, ... $xy+c^2=ab$ (2)

But, as CD bisects the vertical angle, we have,

$$a : b : : x : y \quad \text{(th. 23, b. 2)}$$

Or, . . $x=\frac{ay}{b}$ (3)

Hence, . $\frac{a}{b}y^2+c^2=ab$; or $y=\sqrt{b^2-\frac{c^2b}{a}}$

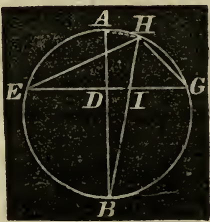
And, ... $x=\frac{a}{b}\sqrt{b^2-\frac{c^2b}{a}}$

Now, as x and y are determined, the base is determined.

N. B. Observe that equation (2) is theorem 20, book 3.

PROBLEM 4.

To determine a triangle, from the base, the line bisecting the vertical angle, and the diameter of the circumscribing circle.



Describe the circle on the given diameter, AB, and divide it in two parts, in the point D, so that $AD\times DB$ shall be equal to the square of one half the given base.

Through D draw EDG at right angles to AB, and EG will be the given base of the triangle.

Put . $AD=n$, $DB=m$, $AB=d$, $DG=b$.

Then, $n+m=d$, and $nm=b^2$; and these two equations will determine n and m; and therefore, n and m we shall consider as known.

Now, suppose EHG to be the required $\triangle$, and join HIB and HA. The two $\triangle$s, AHB, DBI, are equiangular, and therefore, we have,

$$AB : HB : : IB : DB.$$

But HI is a given line, that we will represent by c; and if we put $IB=w$, we shall have $HB=c+w$; then the above proportion becomes,

$$d : c+w : : w : m$$

Now, w can be determined by a quadratic equation; and therefore, IB is a known line.

In the right angled $\triangle$ DBI, the hypotenuse IB, and base DB, are known; therefore, DI is known (th. 36, b. 1); and if DI is known, EI and IG are known.

Lastly, let $EH=x$, $HG=y$, and put $EI=p$, and $IG=q$.

Then, by theorem 20, book 3, $pq+c^2=xy$ (1)

But, ... $x:y::p:q$ (th. 25, b. 2)

Or, ... $x=\frac{py}{q}$ (2)

And, from equations (1) and (2) we can determine x and y, the sides of the $\triangle$; and thus the determination has been attained, carefully and easily, step by step.

PROBLEM 5.

Three equal circles touch each other externally, and thus inclose one acre of ground; what is the diameter in rods of each of these circles?

Draw three equal circles to touch each other externally, and join the three centers, thus forming a triangle. The lines joining the centers will pass through the points of contact (th. 7, b. 3).

Let R represent the radius of these equal circles; then it is obvious that each side of this $\triangle$ is equal to $2R$. The triangle is therefore equilateral, and it incloses the given area, and three equal sectors.

As each sector is a third of two right angles, the three sectors are, together, equal to a semicircle; but the area of a semicircle, whose radius is R, is expressed by $\frac{\pi R^2}{2}$ (th. 3, b. 5, and th. 1, b. 5); and the area of the whole triangle must be $\frac{\pi R^2}{2}+160$; but the area of the $\triangle$ is also equal to R multiplied by the perpendicular altitude, which is $R\sqrt{3}$.

Therefore, . $R^2\sqrt{3}=\frac{\pi R^2}{2}+160$

Or, . $R^2(2\sqrt{3}-\pi)=320$

$$R^2=\frac{320}{2\sqrt{3}-3.1415926}=\frac{3.20}{0.3225}=992.248$$

Hence, $R=31.48+$ rods for the result.

PROBLEM 6.

In a right angled triangle, having given the base and the sum of the perpendicular and hypotenuse, to find these two sides.

PROBLEM 7.

Given, the base and altitude of a triangle, to divide it into three equal parts, by lines parallel to the base.

PROBLEM 8.

In any equilateral $\triangle$, *given the length of the three perpendiculars drawn from any point within, to the three sides, to determine the sides.*

PROBLEM 9.

In a right angled triangle, having given the base (3), *and the difference between the hypotenuse and perpendicular* (1), *to find both these two sides.*

PROBLEM 10.

In a right angled triangle, haviny given the hypotenuse (5), *and the difference between the base and perpendicular* (1), *to determine both these two sides.*

PROBLEM 11.

Having given, the area or measure of the space of a rectangle inscribed in a given triangle, to determine the sides of the rectangle.

PROBLEM 12.

In a triangle, having given the ratio of the two sides, together with both the segments of the base, made by a perpendicular from the vertical angle, to determine the sides of the triangle.

PROBLEM 13.

In a triangle, having given the base, the sum of the other two sides, and the length of a line drawn from the vertical angle to the middle of the base, to find the sides of the triangle.

PROBLEM 14.

To determine a right angled triangle; having given the lengths of two lines drawn from the acute angles to the middle of the opposite sides.

PROBLEM 15.

To determine a right angled triangle; having given the perimeter, and the radius of its inscribed circle.

PROBLEM 16.

To determine a triangle; having given the base, the perpendicular, and the ratio of the two sides.

PROBLEM 17.

To determine a right angled triangle; having given the hypotenuse, and the side of the inscribed square.

PROBLEM 18.

To determine the radii of three equal circles, inscribed in a given circle, to touch each other, and also the circumference of the given circle.

PROBLEM 19.

In a right angled triangle, having given the perimeter, or sum of all the sides, and the perpendicular let fall from the right angle on the hypotenuse, to determine the triangle; that is, its sides.

PROBLEM 20.

To determine a right angled triangle; having given the hypotenuse and the difference of two lines, drawn from the two acute angles to the center of the inscribed circle.

PROBLEM 21.

To determine a triangle; having given the base, the perpendicular, and the difference of the two other sides.

PROBLEM 22.

To determine a triangle; having given the base, the perpendicular, and the rectangle, or product of the two sides.

PROBLEM 23.

To determine a triangle; having given the lengths of three lines drawn from the three angles to the middle of the opposite sides.

PROBLEM 24.

In a triangle, having given all the three sides, to find the radius of the inscribed circle.

PROBLEM 25.

To determine a right angled triangle; having given the side of the inscribed square, and the radius of the inscribed circle.

PROBLEM 26.

To determine a triangle, and the radius of the inscribed circle; having given the lengths of three lines drawn from the three angles to the center of that circle.

PROBLEM 27.

To determine a right angled triangle; having given the hypotenuse, and the radius of the inscribed circle.

BOOK VI.

ON THE INTERSECTION OF PLANES.

DEFINITIONS.

The 14th definition of book 1, defines a plane. It is a superfices, having length and breadth, but no thickness.

The surface of still water, the side of a sheet of paper, may give a person some idea of a plane.

A curved surface is not a plane; although we sometimes say, "the plane of the earth's surface."

1. *If any two points be taken in a plane, and a straight line join the points, every point in that line is in the plane.*

2. If any point in such a line should be either above or below the surface, such a surface would not be a plane.

3. A straight line is perpendicular to a plane, when it makes right angles with every straight line which it meets in that plane.

4. Two planes are perpendicular to each other when any straight line drawn in one of the planes, perpendicular to their common section, is perpendicular to the other plane.

5. If two planes cut each other, and from any point in the line of their common section, two straight lines be drawn, at right angles to that line, one in the one plane, and the other in the other plane, the angle contained by these two lines is the angle made by the planes.

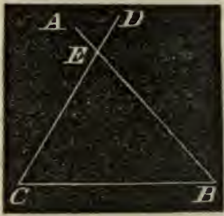
6. A straight line is parallel to a plane when it does not meet the plane, though produced ever so far.

7. Planes are parallel to each other when they do not meet, though produced to any extent.

8. A solid angle is one which is formed by the meeting, in one point, of more than two plane angles, which are not in the same plane with each other.

THEOREM 1.

If any three straight lines meet one another, they are in one plane.

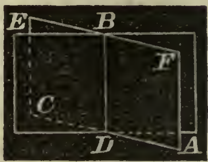


For conceive a plane passing through BC to revolve about that line till it pass through the point E. Then because the points E and C are in that plane, the line EC is in it; and for the same reason, the line EB is in it; and BC is in it, by hypothesis. Hence the lines AB, CD, and BC are all in one plane.

Cor. Any two straight lines which meet each other, are in one plane; and any three points whatever, are in one plane.

THEOREM 2.

If two planes cut one another, the line of their common section is a straight line.



For let B and D, any two points in the line of their common section, be joined by the straight line BD; then because the points B and D are both in the plane AE, the whole line BD is in that plane; and for the same reason BD is in the plane CF. The straight line BD is therefore common to both planes; and it is therefore the line of their common section.

PROPOSITION 3. THEOREM.

If a straight line stand at right angles to each of two other straight lines at their point of intersection, it will be at right angles to the plane of those lines.

Let AB stand at right angles to EF and CD, at their point of intersection A. *Then* AB *will be at right angles to any other line drawn through* A *in the plane, passing through EF,* CD, and, of course, at right angles to the plane itself. (Def. 3.)

Through A, draw any line, AG, in the plane

$EF\ CD$, and from any point G, draw GH parallel to AD. Take $HF=AH$, and join FG and produce it to D. Because HG is parallel to AD, we have

$$FH : HA :: FG : GD$$

But, in this proportion, the first couplet is a ratio of equality; therefore the last couplet is also a ratio of equality,

That is, $FG=GD$, or the line FD is bisected in G.

Join BD, BG, and BF.

Now, in the triangle AFD, as the base FD is bisected in G, we have, . $AF^2+AD^2=2AG^2+2GF^2$ (1) (th. 39 b. 1.)

Also, as DF is the base of the $\triangle\ BDF$, we have by the same theorem, . $BF^2+BD^2=2BG^2+2GF^2$ (2)

By subtracting (1) from (2) and observing that $BF^2-AF^2=AB^2$, because BAF is a right angle; and $BD^2-AD^2=AB^2$, because BAD is a right angle, and we shall then have,

$$AB^2+AB^2=2BG^2-2AG^2$$

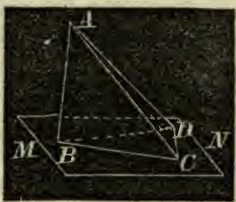
Dividing by 2, and transposing AG^2, and we have,

$$AB^2+AG^2=BG^2$$

This last equation shows that BAG is a right angle. But AG is any line drawn through A, in the plane EF, CD, therefore AB is at right angles to any line in the plane, and, of course, at right angles to the plane itself. *Q. E. D.*

PROPOSITION 4. PROBLEM AND THEOREM.

To draw a straight line perpendicular to a plane, from a given point above it.



Let MN be the plane, and A the point above it. Take, DC, any line on the plane, and draw AC at right angles to it.

From the point C, draw CB on the plane, at right angles to the line DC.

Lastly, from A, draw AB at right angles to the line BC, and join BD. ABC *is a right angle by construction, and now if we can prove that* ABD *is also a right angle, then* AB *is at right angles to the plane, by the last proposition.*

Because ABC is a right angle, we have,

$$AB^2+BC^2=AC^2$$

To both members of this equation, add DC^2 and we have,

$$AB^2+(BC^2+DC^2)=AC^2+DC^2$$

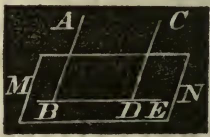
Because BCD is a right angle, $BC^2+DC^2=BD^2$, and because ACD is a right angle, $AC^2+DC^2=AD^2$, and taking these latter values in the last equation, we have,

$AB^2+BD^2=AD^2$; which shows that ABD is a right angle, and proves our proposition. *Q. E. D.*

PROPOSITION 5. THEOREM.

Two straight lines, having the same inclination to a plane, whether perpendicular or oblique, are parallel to one another.

This proposition is axiomatic from our definition of parallel lines; for a stationary plane can have but one position, and the same inclination from any fixed position, must, of course, give parallel lines; but, for the sake of perspicuity, we will give the following as a demonstration.



Let MN be a plane, and AB and CD lines having the same inclination to it.

Then AB *and* CD *are parallel.*

If the lines do not meet the plane, produce them until they do meet it in B and D. Join the points B and D, by the line BD, and produce it to E.

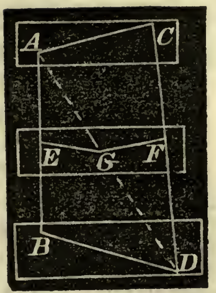
The angle $CDE=ABD$, otherwise the two lines would not have the same inclination to the plane. But when one line, as BE, cuts two others, as AB CD, making the exterior angle, CDE, equal to the interior and opposite angle on the same side, ABE, then the two lines, AB and CD, are parallel. (Converse of th. 6, b. 1).

Q. E. D.

k

PROPOSITION 6. THEOREM.

If two straight lines be drawn in any position through parallel planes, they will be cut proportionally by the planes.



Conceive three planes to be parallel, as represented in the figure, and take any points, A and B, in the first and third planes, and join AB, which passes through the second plane at E.

Also, take any other two points, as C and D, in the first and third planes, and join CD, the line passing through the second plane at F.

Join the two lines by the diagonal AD, which passes through the second plane at G. Join BD, EG, GF, and AC. *We are now to show that,* $AE : EB :: CF : FD$

For the sake of perspicuity, put $AG=X$, and $GD=Y$.

As the planes are parallel, BD is parallel EG; then, in the two triangles ABD and AEG, we have, (th. 17 b. 2).

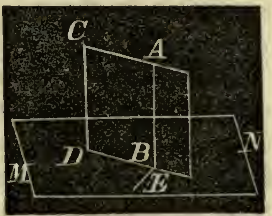
$$AE : EB :: X : Y$$

Also, as the planes are parallel, GF is parallel to AC, and we have, . . $CF : FD :: X : Y$

By comparing the proportions, and applying theorem 6, book 2, we have, . . $AE : EB :: CF : FD$. *Q. E. D.*

PROPOSITION 7. THEOREM.

If a straight line be perpendicular to a plane, all planes passing through that line will be perpendicular to the first-mentioned plane.



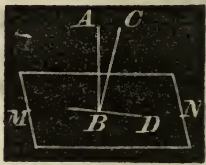
Let MN be a plane, and AB perpendicular to it. Let BC be any other plane, passing through AB; this plane will be perpendicular to MN.

Let BD be the common intersection of the two planes, and from the point B, draw BE at right angles to DB.

Then, as AB is perpendicular to the plane MN, it is perpendicular to every line in that plane, passing through B (def. 3, b. 6); therefore, ABE is a right angle. But the angle ABE (def. 5, b. 6), measures the inclination of the two planes; therefore, the plane CB is perpendicular to the plane MN, and thus we can show that any other plane, passing through AB, will be perpendicular to MN; therefore, &c. *Q. E. D.*

PROPOSITION 8. THEOREM.

From the same point in a plane, but one perpendicular can be erected from the plane.



Let MN be a plane, and B a point in it, and, if possible, let two perpendiculars, BA and BC, be erected.

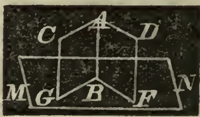
Let BD be drawn on the plane MN, coinciding in direction with the plane passing through these two perpendiculars.

Now, as a perpendicular to a plane is at right angles to every line that can be drawn on the plane, through the foot of the perpendicular, therefore, ABD is a right angle, also CBD is a right angle.

Hence, $ABD = CBD$; the greater equal to the less, which is absurd; therefore, BC must coincide with BA, and be one and the same line; therefore, from the same point, &c. *Q. E. D.*

PROPOSITION 9. THEOREM.

If two planes are perpendicular to a third plane, the common intersection of the two planes will be perpendicular to the third plane.



Let CB and BD be two planes, both perpendicular to the third plane, MN, and let B be the common point to all three of the planes. From B, draw BA at right angles to FB; BA will be in the plane BD. From B, draw also a perpendicular to GB, this will be BA; or, there may be two perpendiculars erected from the same point, which is impossible; therefore, BA is a common section to the two planes BC and BD, and it is at right angles to the two lines BF and BG, on the plane MN. AB is therefore perpendicular to that plane. (Prop. 3, b. 6). *Q. E. D.*

PROPOSITION 10. THEOREM.

If a solid angle be formed by three plane angles, the sum of any two of them is greater than the third.

Let the three angles, BAD, DAC, BAC, form the solid angle A. The sum of any two of these is greater than the third. When these angles are all equal, it is evident that the sum of any two is greater than the third, and the proposition needs demonstration only when one of them, as BAC, is greater than either of the others; we are then to prove that it is less than their sum.

On the line AB, take any point, B, and draw any line, as BD. From the same point, B, make the angle $ABC=ABD$, and join DC. From the point A, and on the plane BAC, draw the angle $BAE=BAD$. Now the two plane triangles BAD and BAE, have a common side, AB, and the angles adjacent equal (th. 14, b. 1); therefore, the two $\triangle$s are, in all respects, equal; and $AD=AE$, and $BD=BE$.

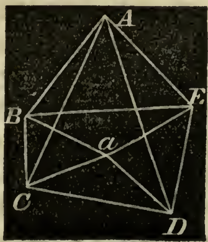
In the triangle BDC, $\quad BC<BD+DC$
But, . . . $\quad BE=BD$
By subtraction, . . $\quad EC<DC$

In the two triangles, DAC and EAC, $DA=AE$, and AC is common, but EC is less than CD; therefore, the angle DAC, opposite DC, is greater than the angle EAC, opposite EC. (Converse of th. A, b. 1).

That is, . . . $\quad DAC>EAC$
But, . . . $\quad DAB=BAE$
By addition, $\quad DAC+DAB>BAC$. (Ax. 2). *Q. E. D.*

PROPOSITION 11. THEOREM.

The sum of any plane angles forming any solid angle, is always less than four right angles.

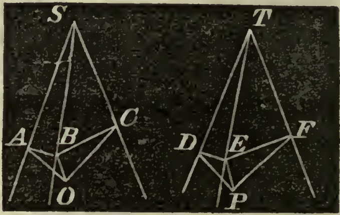


Let the planes which form the solid angle at A, be cut by another plane, which we may call the plane of the base, $BCDE$. Take any point, a, in this plane, and join aB, aC, aD, aE, &c., thus making as many triangles on the plane of the base, as there are triangular planes forming the solid angle A. But as the sum of the angles of every $\triangle$ is two right angles, the sum of all the angles of the $\triangle$s which have their vertex in A, is equal to the sum of all angles of the $\triangle$s which have their vertex in a. But the angles $BCA+ACD$, are, together, greater than the angles $BCa+aCD$, or BCD, by the last proposition. That is, the sum of all the angles at the bases of the $\triangle$s which have their vertex in A, is greater than the sum of all the angles at the bases of the $\triangle$s which have their vertex in a. Therefore, the sum of all the angles at a, is greater than the sum of all the angles at A, but the sum of all the angles at a, is equal to four right angles; therefore, the sum of all the angles at A, is less than four right angles. *Q. E. D.*

PROPOSITION 12. THEOREM.

If two solid angles are formed by three plane angles respectively equal to each other, the planes which contain the equal angles will be equally inclined to each other.

Let the angle $ASC=DTF$, and the angle $ASB=DTE$; also the angle $BSC=ETF$; then will the inclination of the planes, ASC, ASB, be equal to that of the planes DTF, DTE.



Having taken SB at pleasure, draw BO perpendicular to the plane ASC; from the point O, at which that perpendicular meets the plane, draw OA, OC, perpendicular to SA, SC; join AB, BC; next take $TE=SB$; draw EP perpendicular to the plane DTF; from the point P, draw PD, PF, perpendicular to TD, TF; lastly, join DE, EF.

The triangle SAB, is right angled at A, and the triangle TDE, at D; and since the angle $ASB=DTE$, we have $SBA=TED$. Likewise, $SB=TE$; therefore, the triangle SAB is equal to the triangle TDE; hence, $SA=TD$, and $AB=DE$. In like manner it may be shown that, $SC=TF$, and $BC=EF$. That granted, the quadrilateral $SAOC$, is equal to the quadrilateral $TDPF$; for, place the angle ASC, upon its equal DTF; because $SA=TD$, and $SC=TF$, the point A will fall on D, and the point C on F;

and, at the same time, AO, which is perpendicular to SA, will fall on PD, which is perpendicular to TD, and, in like manner, OC on PF; wherefore, the point O will fall on the point P, and AO will be equal to DP. But the triangles AOB, DPE, are right angled at O and P; the hypotenuse $AB=DE$, and the side $AO=DP$; hence, those triangles are equal; hence, the angle $OAB=PDE$. The angle OAB is the inclination of the two planes ASB, ASC; the angle PDE, is that of the two planes DTE, DTF; consequently, those two inclinations are equal to each other. Hence, *If two solid angles are formed, &c.*

Scholium. The angles which form the solid angles at S and T, may be of such relative magnitudes, that the perpendiculars, BO and EP, may not fall within the bases, ASC and DTF; but they will always either fall on the bases or on the planes of the bases produced, and O will have the same relative situation to A, S, and C, as P has to D, T, and F. But, in case that O and P fall on the planes of the bases produced, the angles BCO and EFP, would be obtuse angles; but the demonstration of the problem would not be varied in the least.

BOOK VII.

SOLID GEOMETRY.

The object of Solid Geometry is to estimate and compare the surfaces and magnitudes of solid bodies; and, like Plane Geometry, it must rest on definitions and axioms.

To the definitions already given, we add the following, as being exclusively applicable to Solid Geometry.

Surfaces are measured by *square units;* so solids are measured by *cube units*.

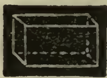
1. A Cube is a solid, bounded by six equal square surfaces, forming eight equal solid angles.

All other solids are referred to a unit of this figure for measurement.

2. A *Prism* is a solid, whose ends are parallel, equal, and form equiangular plane figures; and its sides, connecting these ends, are parallelograms.

3. A prism takes particular names according to the figure of its base or ends, whether triangular, square, rectangular, pentagonal, hexagonal, &c.

4. A right or upright prism, is that which has the planes of the sides perpendicular to the planes of the ends or base.



5. A Parallelopipedon is a prism bounded by six parallelograms, every opposite two of which are equal, alike, and parallel.

6. A rectangular parallelopipedon, is that whose bounding planes are all rectangles, which are perpendicular to each other.

A rectangular parallelopipedon becomes a *cube* when all its planes are equal.

7. A Cylinder is a round prism, having circles for its ends; and is conceived to be formed by the rotation of a right line about the circumferences of two equal and parallel circles, always parallel to the axis.

8. The axis of a cylinder, is the right line joining the

centers of the two parallel circles, about which the figure is described.

9. A Pyramid is a solid, whose base is any right lined plane figure, and its sides triangles, having all their vertices meeting together in a point above the base, called the vertex of the pyramid.

10. A pyramid, like the prism, takes particular names from the figure of the base.

11. A Cone is a convex pyramid, having a circular base, and is conceived to be generated by the rotation of a right line about the circumference of a circle, one end of which is fixed at a point above the plane of that circle.

12. The axis of a cone is the right line joining the vertex, or fixed point, and the center of the circle about which the figure is described.

13. Similar cones and cylinders, are such as have their altitudes and the diameters of their bases proportional.

14. A Sphere is a solid, having but one surface, which is in every part equally convex; and every point on such a surface is equally distant from a certain point within, called the center.

15. A sphere may be conceived as having been generated by the revolution of a semicircle about its axis.

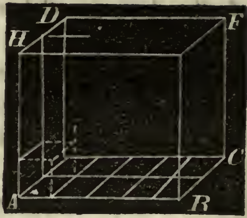
The diameter of such a semicircle is the diameter of the sphere; and the center of the semicircle is the center of the sphere.

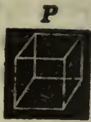
16. The altitude of any solid is the *perpendicular distance* between the parallel planes, one of which is the base of the solid, and the other is a plane, parallel with the plane of the base, passing through the vertex of the solid.

17. The area of the surface is measured by the product of its *length* and *breadth* (as explained by scholium on page 32); and these dimensions are always conceived to be exactly at right angles with each other.

18. In a similar manner, solids are measured by the product of their *length*, *breadth*, and *hight*, when all their dimensions are at right angles with each other.

The product of the length and breadth of a solid, is the measure of the *surface* of its base.





Let P, in the annexed figure, represent the measuring unit, and AF the rectangular solid to be measured.

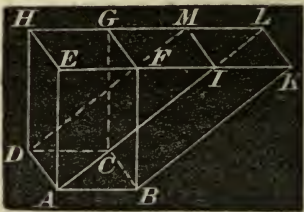
A side of P, is one unit in length, one in breadth, and one in hight; one inch, one foot, one yard, or any other unit that may be taken.

Then, . $1\times1\times1=1$, the *unit cube.*

Now, if the base of the solid, AC, is, as here represented, 5 units in length and 2 in breadth, then it is obvious that $(5\times2=10)$. 10 units, equal to P, can be placed on the base of AC, and no more; and as each of them will occupy a unit of altitude, therefore, 2 *units* of altitude will contain 20 solid units, 3 units of altitude, 30 solid units, and so on; or, in general terms, *the number of square units in the base, multiplied by the linear units in perpendicular altitude, will give the solid units in any rectangular solid.**

THEOREM 1.

Two parallelopipedons on the same base, and of the same altitude, the one rectangular, the other oblique, the opposite sides of which lie in the same planes, will be equal in solidity.



Let AG be the rectangular parallelopipedon on the base AC, and AL the the oblique parallelopipedon, on the same base, AC, and of the same altitude, namely, the perpendicular distance between the parallel planes AC and EL, and the side AF, in the same plane with AK, and the side DG, in the same plane with DL. *Then we are to show, that* the oblique parallelopipedon $ABCDMIKL$, is equivalent to the rectangular parallelopipedon, AG.

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* This is one of those simple and primary truths that admit of no demonstration; for no other truths more simple and elementary than itself can be brought to bear upon it; hence we enunciate it as a definition.

All efforts to prove a proposition which is perfectly obvious, are very unsatisfactory to the mind, and always tend more to confuse than to elucidate.

As the sides of the two solids are in the same plane, *EFK* is one right line; *EF*=*IK*, because each is equal to *AB*. From the whole line *EK*, subtract, successively, *EF* and *IK*; thus showing that *EI*=*FK*. But *BF*=*AE*, and the angle *BFK*= the angle *AEI*; therefore, the △ *BFK*=△ *AEI*. The parallelogram *DE* =*CF*, and the parallelogram *EM*=*FL*; and all the angles at *F* forming the solid angles at that point, are respectively equal to the like angles at *E*.

Hence, the two prisms, *CBFGLK* and *DAEHMI* are equal; for they are bounded by equal planes equally inclined to each other; or, one prism can be conceived to be taken up and placed into the same space occupied by the other; and magnitudes that fill the same space, are equal.

Now, from the whole solid, take the prism *GB—K*, and the upright solid, *AG*, is left; and from the whole solid take the prism *DE—I*, and the oblique solid, *AL*, is left. Hence, by (ax. 3) the rectangular parallelopipedon *AG*, is equivalent to the oblique parallelopipedon *AL*, on the same base and altitude. *Q. E. D.*

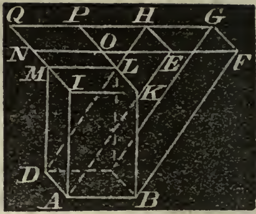
Cor. The measure of the solid *AG*, is the base, *ABCD*, into the perpendicular, *AE* (def. 18, solid ge.); consequently, the measure of the solid, *AL*, is also the same base, multiplied by the same perpendicular.

Scholium. If *EF* and *IK* are in the same line; that is, the sides *AF* and *AK* in the same plane; but the angles *AEH* and *BFG* not right angles, then neither parallelopipedon is rectangular; but they are proved equal in exactly the same manner; that is, by proving the two prisms equal, and subtracting each in succession from the whole solid.

Hence, two oblique parallelopipedons, on the same base, and of the same altitude, whose opposite sides are between the same planes, are equal in solidity.

PROBLEM 2.

Any oblique parallelopipedon is equivalent to a rectangular parallelopipedon on the same base and altitude.



Let AG be any oblique parallelopipedon, and AL a rectangular parallelopipedon, on the same base, DB, and between the same parallel planes, BD and HF. *Then we are to show, that* they are equivalent.

Produce HG and IM; and because they are in the same horizontal plane, and not parallel, they will meet in some point, Q. Also produce FE and KL, and thus form the parallelogram NP. Now conceive another parallelopipedon to stand on the base DB, and its upper base occupying the parallelogram $NP=DB$. Now, by scholium to theorem 1, book 7, the solid, AG, is equal to this *imaginary* solid, AP. But (th. 1, b. 7), the rectangular solid, AL, is also equal to this *imaginary* solid, AP. Therefore, the solid AG is $=$ to the rectangular solid, AL. (Ax). *Q. E. D.*

Cor. *Hence, every parallelopipedon, in whatever direction or degree it is inclined, is measured by the product of its base into its perpendicular altitude.*

THEOREM 3.

Parallelopipedons on the same, or on equal bases, are to one another as their perpendicular altitudes; and parallelopipedons having equal altitudes, are to one another as their bases.

Let P and p represent two parallelopipedons, whose bases are B and b, and altitudes A and a.

Then, by the last theorem, the measure of P is BA, and the measure of p is ba. But, all magnitudes are proportional to their numerical measures; that is, . . . $P : p = BA : ba$

Now, in case $A=a$, we have (th. 4, b. 2), $P : p = B : b$

In case $B=b$, then we have, . . . $P : p = A : a$

Q. E. D.

THEOREM 4.

*Similar parallelopipedons are to one another as the cubes of their like dimensions.**

* This theorem is true for all similar solids.

Let P and p represent two parallelopipedons, as in theorom 3; and let l and n represent the length and breadth of the base of P, and h its altitude.

Also, let l' and n' represent the length and breadth of p, and h' its altitude.

Hence, by cor. to th. 2, b. 7, $P=lnh$, and $p=l'n'h'$.

That is, . . $P : p=lnh : l'n'h'$*

But, by reason of the similarity of the solids,

$$l : l'=n : n'$$

$$n : n'=n : n'$$

And, . . $h : h'=n : n'$

Multiplying these proportions together, term by term, (th. 11 b. 2), we have, . . $lnh : l'n'h'=n^3 : n'^3$

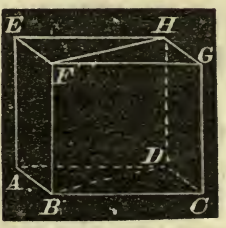
That is, . . $P : p=n^3 : n'^3$ (th. 6, b. 2)

By a little different arrangement of the proportions, we have, $P : p=l^3 : l'^3$

Or, . . . $P : p=h^3 : h'^3$ *Q. E. D.*

THEOREM 5.

Any parallelopipedon may be divided into two equal prisms, by a diagonal plane passing through its opposite edges.



The parallelopipedon may be conceived to be composed of a great multitude of extremely thin parallelograms, all equal to one another; and the diagonal HF divides the parallelogram EG into two equal parts (th. 22, cor. b. 1); and the line HF, passing down through all the parallelograms, from EG to AC, divides each and all of them into two equal parts; that is, the diagonal plane, $HFBD$, divides the parallelopipedon into two equal parts, each of which is a prism. *Q. E. D.*

Otherwise, the two prisms may be proved to be bounded by equal planes and equal angles; therefore, they are magnitudes that exactly fill equal spaces, and are therefore equal. *Q. E. D.*

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* When the three factors are all equal; that is, $l=n=h$, $P : p=l^3 : l'^3$; but in this case, the solids are actual cubes.

Cor. The solidity of a prism is therefore the triangular base, *DBC*, multiplied by its altitude, the perpendicular distance between the planes *AC* and *EG;* or, it may be found by the product of the base, *HGCD*, and half the perpendicular distance between the planes *GD* and *EB*.

THEOREM 6.

All prisms of equal bases and altitudes are equal in solidity, whatever be the figures of the bases.

It is of no consequence what shape a base may be, for it is greater or less, according to the number of square units that may be contained in it; hence, the base of a triangular prism may be considered a square, or rectangular prism, containing the same number of square units as the triangular base; that is, any prism may be considered a rectangular parallelopipedon, whose base is the same in area as the base of the prism; but the solidity of a parallelopipedon is measured by the area of its base by its altitude (def. 18); and therefore, a prism of the same area of base and altitude, has the same measure. *Q. E. D.*

THEOREM 7.

All similar solids are to one another as the cubes of their like dimensions.

By theorem 4, of this book, this proposition is proved true for all similar parallelopipedons; and by theorem 5, all similar parallelopipedons may be divided into two equal parts, thus forming similar prisms. But the halves of things are in the same proportion as their wholes; therefore, all similar prisms are to one another as the cubes of their like dimensions.

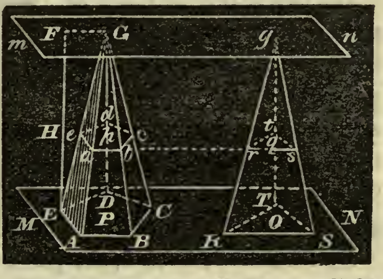
Similar pyramids and similar cones are but the same like parts of similar prisms; and, like parts of wholes, are in the same proportion as the wholes themselves; therefore, our theorem is true for pyramids and cones.

Spheres are like proportional parts of their circumscribing cylinders; and our theorem is true for similar cylinders; it is, therefore, true for spheres.

In short, all similar solids, however irregular the shape, are but like parts of some mathematical figure that may inclose them; and as the theorem is true for the mathematical figures, it is true for any of their like parts; it is, therefore, true for all similar solids whatever. *Q. E. D.*

THEOREM 8.

If a pyramid be cut by a plane which is parallel with its base, the section thus formed will be similar to the base, and its area will be to the area of the base as the square of its perpendicular distance from the vertex, is to the square of the perpendicular altitude of the pyramid.



Let MN and mn be two parallel planes, between which stands any pyramid whose base is P, and vertex G, and perpendicular altitude EF.

On any one of the edges, as GA, take any point a, and draw ab parallel to AB; and from b draw bc parallel to BC. Then, by reason of the parallels (th. 10, b. 1), the angle $abc = ABC$. In this manner we may go round the whole section, whatever be the number of sides: and every angle in the section will be equal to its corresponding angle of the base; that is, the two figures are equiangular, and similar; and as every line of the section is parallel to its corresponding line in the base, therefore, if the base is a plane, the section will be a parallel plane. Produce a line from this plane to the perpendicular at H.

But equiangular plane figures are to one another as the squares of their like sides (th. 23, b. 2); that is,

$$P : p = AB^2 : ab^2$$

But, . $AB^2 : (ab)^2 = GA^2 : Ga^2$ (th's. 17 and 10, b. 2)

And, . $GA^2 : Ga^2 = GE^2 : Ge^2$

And, . $GE^2 : Ge^2 = FE^2 : FH^2$

Multiplying all these proportions together, and at the same time rejecting all the common factors that would otherwise appear in the antecedents and consequents, we have,

$$P : p = FE^2 : FH^2$$

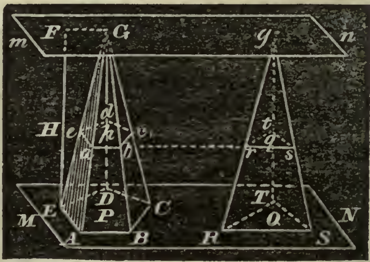
By changing means for extremes, we have,

$$p : P = FH^2 : FE^2 \qquad Q.\ E.\ D.$$

Cor. As the section made by the cutting plane is always similar to the base, it follows that when the base is a polygon of a great number of sides, the section will be a polygon of the same number of sides; and when the base is a circle, the section will be a circle, and so on.

THEOREM 9.

If two pyramids, standing between two parallel planes, be cut by a third parallel plane, the respective sections will be to each other as their bases.



Let two pyramids stand as represented in the figure, and from any point, H, in the perpendicular, pass a plane parallel to the plane MN. By the last theorem, each section of these pyramids is a similar figure to its base.

By theorem 6, book 6, the parallel plane that forms these sections, cuts all lines between the planes MN and mn, proportionally,

Therefore, . $gr : gR = Ge : GE$

And, . . $Ge : GE = FH : FE$

Hence, . . $gr : gR = FH : FE$

By squaring this last proportion, we have,

$$gr^2 : gR^2 = FH^2 : FE^2$$

But, . . $gr^2 : gR^2 = rs^2 : RS^2$

By the application of theorem 6, book 2, to these last two proportions, we have, $FH^2 : FE^2 = rs^2 : RS^2$

But, . . . $p : P = FH^2 : FE^2$ (th. 8, b. 7)

And, . . $rs^2 : RS^2 = q : Q$ (th. 23. b. 2)

Multiplying these three proportions together, term by term, rejecting common factors in antecedents and consequents, we have,

$$p : P = q : Q \qquad Q.\ E.\ D.$$

Cor. *On the supposition that* P=Q, *there results* p=q.

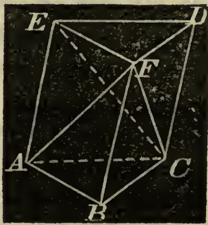
THEOREM 10.

Any two pyramids having equal bases, and situated between the same two parallel planes, or having equal altitudes, are equal.

Take the same figure as for the last theorem, supposing the bases, P and Q, equal, and conceive the perpendicular EF, to be divided by a great multitude of parallel planes, equidistant from each other, and all parallel to the plane MN. By the last theorem, these planes will divide each pyramid into the same number of equal parallel sections, of which the two pyramids may be considered as composed; and, as the sums of equals are equal, therefore, the two pyramids are equal. *Q. E. D.*

THEOREM 11.

Every triangular pyramid is a third part of the triangular prism, having the same base and the same altitude.

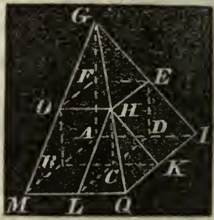


Let $FABC$ be a triangular pyramid; $ABCDEF$ a triangular prism of the same base and the same altitude: the pyramid will be equal to a third of the prism.

Cut off the pyramid $FABC$ from the prism, by a section made along the plane FAC; there will remain the solid $FACDE$, which may be considered as a quadrangular pyramid, whose vertex is F, and whose base is the parallelogram $ACDE$. Draw the diagonal CE; and extend the plane FCE, which will cut the quadrangular pyramid into two triangular ones, $FACE$, $FCDE$. These two triangular pyramids have for their common altitude, the perpendicular let fall from F on the plane $ACDE$. They have equal bases, the triangles ACE, CDE, being halves of the same parallelogram; hence, the two pyramids, $FACE$, $FCDE$, are equivalent (th. 10, b. 7). But the pyramid $FCDE$, and the pyramid $FABC$, have equal bases, ABC, DEF; they have, also, the same altitude, namely, the distance of the parallel planes ABC, DEF; hence these two pyramids are equivalent. Now, the pyramid $FCDE$ has already been proved equivalent to $FACE$; consequently, the three pyramids, $FABC$, $FCDE$, $FACE$, which compose the prism ABD, are all equivalent. Hence, the pyramid, $FABC$ is the third part of the prism ABD, which has the same base, and the same altitude. *Q. E. D.*

Cor. The solidity of a triangular pyramid is equal to a third part of the product of its base by its altitude.

The preceding demonstration is brief, direct, and all that could be desired, provided the learner has a clear conception of the figure as represented on paper; but as we know that this is not generally the case, we give the following method.



Let *ABCDEF* be any rectangular parallelopipedon, and put $AD=a$, $AB=b$, and $AF=h$. Produce *AF* to *G*, making $FG=AF$. Draw *GO* to meet *AB*, produced in *M*. As *FO* is parallel to *AB*, and *AG* double of *AF*, therefore, *AM* is double of *AB*. Join *GE*, and produce it to meet *AD*, in *I*; then, by like reasoning, we shall find *AI* the double of *AD*. Join *GH*, and produce it to meet the plane of *BD*, in *Q*.

The whole figure now comprises two pyramids; one, whose base is *AQ*; the other similar one has *FH* for its base, and the vertex of both, is *G*.

The whole figure also comprises the parallelopipedon *AH*, which is measured by (abh), two prisms, and two equal and similar pyramids. One prism has *DCKI* for its base, and *DE*, for its altitude; the other has *BMLC* for its base, and $BO=DE$, for its altitude.

As each of these bases, *DK* and *BL*, is equal to *AC*, hence, the solidity of these two prisms is expressed by (abh); and the parallelopipedon, and two prisms together, are measured by $2abh$; and, in addition to these, we have two equal pyramids of *unknown* solidity; therefore, let each one be represented by x.

Now, the whole pyramid, whose base is *AQ*, and vertex *G*, is expressed by $(2abh+2x)$.

But the pyramid, whose base is *FH*, and vertex *G*, is expressed by (x).

As these two pyramids are similar, they are to each other as the cubes of their like dimensions; that is, they are to each other as the cube of *GA* to the cube of *GF*. But *GA* is the double of *GF*, by construction. Therefore, $GA^3 : GF^3=8 : 1$

Hence, ... $(2abh+2x) : x=8 : 1$

Product of extremes and means gives, $8x=2abh+2x$

Therefore, ... $x=\frac{1}{3}(abh)$

This last equation shows that the solidity of any pyramid is one-third of any rectangular solid of the same base and altitude.

Cor. This measure of the pyramid is true, whatever be the figure of its base; and when the base is a circle, the pyramid is called a cone; hence, the solidity of a cone is one third of its circumscribing cylinder.

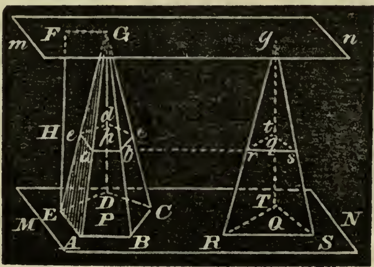
THEOREM 12.

If a pyramid be cut by a plane parallel to its base, the solidity of the frustum that remains after the small pyramid is taken away, is equal to three pyramids of the same altitude as the frustum; one having for its base, the base of the frustum; another, the upper base; and the third, a base which is the mean proportional between the upper and lower bases of the frustum.

(The figure has been previously described in theorem 8.)

Now, by the last theorem, the solidity of the whole pyramid is expressed by $\frac{P(FE)}{3}$, and that of the small pyramid is $\frac{p(FH)}{3}$. The difference of these magnitudes measures the frustum;

That is, . . $\frac{P(FE)-p(FH)}{3}=$the frustum.



To make this expression correspond with the enumeration of this theorem, we must banish FE and FH, and obtain their difference.

By th. 8, book 7, we have,

$FE : FH=\sqrt{P} : \sqrt{p}$ (1)

From this proportion we have, $FE=\frac{(FH)\sqrt{P}}{\sqrt{p}}$, which, substituted in the above expression,

gives, . $\frac{(FH)P\sqrt{p}}{3\sqrt{p}}-\frac{p(FH)}{3}=$ the frustum;

Or, . $(FH)\frac{(P\sqrt{P}-p\sqrt{p})}{3\sqrt{p}}=$ the frustum.

From proportion (1), $FE-FH : FH=\sqrt{P}-\sqrt{p} : \sqrt{p}$ (2)

But $(FE-FH)$ is the altitude of the frustum, which we will designate by a.

Then, from proportion (2), $FH=\frac{a\sqrt{p}}{\sqrt{P}-\sqrt{p}}$

This value of FH, substituted in the last expression for the frustum, gives,

$$\frac{a}{3}\left(\frac{P\sqrt{P}-p\sqrt{p}}{\sqrt{P}-\sqrt{p}}\right)=\text{ the frustum.}$$

By actual division, we have,

$$\frac{a}{3}(P+\sqrt{Pp}+p)=\text{ the frustum ;}$$

Or, . $\frac{1}{3}aP+\frac{1}{3}a\sqrt{Pp}+\frac{1}{3}ap=$ the frustum.

Here we find expressions for three different pyramids, which, together, are equal to the frustum ; one has P for its base, another p, and the third $\sqrt{Pp}$, which is the mean proportional between the two bases, P and p; therefore, a frustum is equal, &c. *Q. E. D.*

Cor. In case $P=p$, the frustum becomes a prism, and the above expression for the three pyramids becomes aP, which is the proper expression for the solidity of a prism.

THEOREM 13.

The convex surface of any regular pyramid is equal to the perimeter of its base, multipled by half its slant hight.

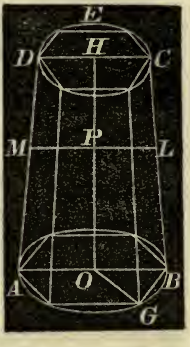
Bisect the side AB in H, and join SH. Since the pyramid is regular, the side SAB is an isosceles triangle ; consequently, SH is perpendicular to AB; hence, SH is the altitude of the triangle, and also the slant hight of the pyramid. For the same reason, each side of the pyramid is an isosceles triangle, whose altitude is the slant hight of the pyramid.

Now, the area of the triangle SAB, is equal to $AB\times\frac{1}{2}SH$; and the area of all the triangles which compose the convex surface of the pyramid, is equal to the sum of their bases. $(AB+BC+CD+DE+EF+AF)\times\frac{1}{2}SH$.

But the sum of these bases, AB, BC, &c., forms the perimeter of the pyramid's base ; and the common altitude, SH, is the slant hight of the pyramid. Therefore, *the convex surface of any regular pyramid, is equal to the perimeter of its base multiplied by half its slant hiyht.*

THEOREM 14.

The convex surface of a frustum of a regular pyramid, is equal to the sum of the perimeter of the two bases multiplied by half the slant hight.



Conceive a regular frustum of a pyramid to exist, as represented in the figure; then each face will be a regular trapezoid, whose surface is measured by the half sum of its parallel sides (th. 31, b. 1), multiplied by the perpendicular distance between them, which is the slant hight of the frustum.

Let S represent a side of the lower base, and s a side of the upper base, and a the slant hight; then the surface of one face is measured by $\frac{1}{2}a\ (S+s)$.

There are just as many of these surfaces as the frustum has sides. Let m represent the number of sides; then the whole surface must be $\frac{1}{2}a(mS+ms)$. But $(mS+ms)$, is the perimeter of the two bases; and $\frac{1}{2}a$ is one-half of the slant hight. Therefore, &c. *Q. E. D.*

Scholium. Let circles be described round the bases of the frustum, as represented in the last figure; and conceive the number of sides to be indefinitely increased; then S and s will be indefinitely small, and m indefinitely great; but however small S and s may be (the corresponding number to m being as much increased), the expression $(mS+ms)$ will still represent the perimeters of the two bases. But, when S and s are indefinitely small, while OA, and DH, that is, the distances from the axis of the frustum from its edges being constant, the perimeter, mS, will become the perimeter of the circle of which OA is the radius; and ms will be the perimeter of the circle of which DH is the radius; that is, $mS=2\pi(AO)$, and $ms=2\pi(DH)$; and by addition,

$$mS+ms=2\pi(AO+DH)$$

But, in this case, $\frac{1}{2}a$ becomes $\frac{1}{2}AD$, one-half the edge of the frustum; and the frustum of the pyramid becomes the frustum of a *cone*, and its surface is measured by

$$\tfrac{1}{2}AD\times 2\pi(AO+DH);\ \text{hence,}$$

The convex surface of a frustum of a cone, is equal to half its sides, multipled by the sum of the circumferences of its two bases.

The above expression is the same as

$$AD\times 2\pi\left(\frac{AO+DH}{2}\right)$$

If we take the middle point, P, between O and H, and draw PM parallel to OA and HD,

Then, . . . $\frac{AO+DH}{2}=PM$, which, substituted, gives . . . . $AD\times 2\pi PM$

That is, the convex surface of the frustum of a cone, is equal to its side, multiplied by the circumference of a circle which is exactly midway between its two bases.

THEOREM 15.

If any regular semi-polygon be revolved about its axis, the surface thus described, will be measured by the product of its axis into the circumference of its inscribed circle.

If the semi-polygon, $DABK$, &c., revolve on its axis, DE, the sides AB, BK, &c., will each describe frustums of cones; and, for investigation, let us take the side AB.

From the middle point, G, draw GI perpendicular to DE. Join GC, and draw AT parallel to DE.

By the scholium to the preceding theorem, the surface described by AB is measured by $AB\times$ cir. GI, which is equal to AT, or HL cir. GC.

That is, . $HL\times 2\pi GC=AB\times 2\pi GI$

The two triangles, ABT and CGI, are similar. As CG is perpendicular to AB, the two angles CGI and IGA, are equal to a right angle. The acute angles of the $\triangle$ ABT are also equal to a right angle.

That is, . $\angle CGI+\angle IGA=\angle BAT+\angle ABT$

But, . . . $\angle IGA= \angle ABT$ (th. 5, b.1)

By subtraction, . $\angle CGI=\angle BAT$

Now, as these two triangles have each a right angle, they are equiangular and similar;

Therefore, . $CG : GI = AB : AT = HL$

Hence, . . $HL \cdot CG = AB \cdot GI$

Multiplying both members of this equation by 2π, we have,

$$HL \cdot 2\pi CG = AB \cdot 2\pi GI$$

Thus we find that the surface described by the side AB, is measured by the product of HL into the circumference of the inscribed circle; and in the same manner we may prove that the surface described by the side AD, is measured by DH into the circumference of the same circle, and so on of every other side; and the surface described by all the sides taken together, is equal to $(DH+HL+LC$, &c.), multiplied into the circumference of the inscribed circle; that is, the surface described by the whole polygon, is equal to DE, the axis of the polygon, into the circumference of its inscribed circle. *Q. E. D.*

THEOREM 16.

The convex surface of a sphere is equal to the product of its diameter into its circumference.

The last theorem is true, whatever be the number of sides of the polygon; and now suppose the number to be indefinitely great; then the sides of the polygon will coincide with the circumference of the circle, and CG becomes CA, and the surface described by the sides of the polygon, is now the surface of the sphere, which is measured by the diameter DE, multiplied into the circumference of the circle $2\pi CA$. *Q. E. D.*

Cor. 1. If we represent the radius of a sphere by R, its circumference is $2\pi R$, and its diameter $2R$; therefore, its convex surface is $4\pi R^2$. The surface of a plane circle, whose radius is R, is πR^2; therefore, the surface of a sphere is 4 times a plane circle of the same diameter.

Cor. 2. The surface of a segment is equal to the circumference of the sphere, multiplied by the thickness of the segment.

Cor. 3. In the same sphere, or in equal spheres, the surfaces of different segments are to each other as their altitudes.

THEOREM 17.

The solidity of a sphere is equal to the product of its surface into a third of its radius.

Let us suppose a sphere to be composed of a great multitude of regular pyramids, whose bases are portions of the surface of the sphere, and their common vertex the center of the sphere; then the altitudes of all such pyramids is the radius of the sphere.

The solidity of one of these pyramids is its base multiplied by $\frac{1}{3}$ of its altitude (th. 11, b. 7); and the solidity of all of these together, will be the sum of all the bases multiplied into $\frac{1}{3}$ of the common altitude. But the sum of all the bases, is the surface of the sphere; and the common altitude is the radius of the sphere; therefore, the solidity of a sphere is equal to its surface multiplied by one third of its radius. *Q. E. D.*

Let R = the radius of the sphere; then (cor. 1, th. 16, b. 7), $4\pi R^2$ is its surface; hence, its solidity must be

$$4\pi R^2 \times \tfrac{1}{3}R = \tfrac{4}{3}\pi R^3.$$

Cor. If r represent the radius of any other sphere, its solidity will be $\frac{4}{3}\pi r^3$; and, by dividing by the constant factors, $\frac{4}{3}\pi$, these two solids are to each other as R^3 to r^3, a result corresponding to theorem 7, book 7.

THEOREM 18.

The solidity of a sphere is two-thirds the solidity of its circumscribing cylinder.

Let R be the radius of the base of an upright cylinder; then, πR^2 will be the area of the base (th. 1, b. 5); but the altitude of a cylinder which will just inclose a sphere, must be $2R$; and the solidity of such a cylinder must be $2\pi R^3$ (def. 18, b. 7). By the last theorem, the solidity of a sphere, whose radius is R, is $\frac{4}{3}\pi R^3$.

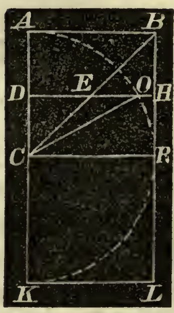
Therefore, the cylinder is to the sphere as $2\pi R^3$ to $\frac{4}{3}\pi R^3$

Or, as . . . . . . 2 to $\frac{4}{3}$

Or, as . . . . . . 1 to $\frac{2}{3}$

Q. E. D.

We give another method of demonstrating this truth, merely for the beauty of the demonstration.



Let AK be the diameter of a semicircle, and also the side of a parallelogram whose width is the radius of the semicircle.

Join the center of the semicircle to either extremity of the parallelogram, as CB, CL. Now conceive the parallelogram to revolve on AK, and it will describe a cylinder; the semicircle will describe a sphere, and the triangle ABC will describe a cone.

In AC, take *any point*, D, and draw DH parallel to AB, and join CO. Then, as $CA=AB$, $CD=DE$. In the right angled triangle CDO, we have,

$$CD^2+DO^2=CO^2 \quad (1)$$

But, . . . $BD^2=DE^2$, and $CO^2=DH^2$

Substituting these values in equation (1), and we have,

$$CE^2+DO^2=DH^2 \quad (2)$$

Multiply every term of this equation by π,

Then, . . $\pi DE^2+\pi DO^2=\pi DH^2$

Now, the first term of this equation, is the measure of the surface of a plane circle, whose radius is DE; the second term is the measure of a plane circle, whose radius is DO; and the second member is the measure of the surface of a plane circle, whose radius is DH. Let each of these surfaces be conceived to be of the same *extremely minute* thickness; then the first term is a section of a cone, the second term is a corresponding section of a sphere, and these two sections are, together, equal to the corresponding section of the cylinder; and this is true for all sections parallel to CR, which compose the cone, the sphere, and the cylinder; therefore, the cone and sphere, together, are equal to the cylinder; but the cone described by the triangle ABC, is $\frac{1}{3}$ of the cylinder described by AR (th. 11, b. 7); therefore, the corresponding section of the sphere, is the remaining *two-thirds*, and the whole sphere is two-thirds of the whole cylinder described by the parallelogram AL.

Q. E. D.

ELEMENTARY PRINCIPLES OF PLANE TRIGONOMETRY.

TRIGONOMETRY in its literal and restricted sense, has for its object, the measure of triangles. When the triangles are on planes, it is plane trigonometry, and when the triangles are on, or conceived to be portions of a sphere, it is spherical trigonometry. In a more enlarged sense, however, this science is the application of the principles of geometry, and numerically connects one part of a magnitude with another, or numerically compares different magnitudes.

As the *sides* and *angles* of triangles are quantities of different kinds, they cannot be *compared* with each other; but the *relation* may be discovered by means of other complete triangles, to which the triangle under investigation can be compared.

Such other triangles are numerically expressed in Table II, and all of them are conceived to have one common point, the center of a circle, and as all possible angles can be formed by two straight lines drawn from the center of a circle, no angle of a triangle can exist whose measure cannot be found in the table of trigonometrical lines.

The measure of an angle is the arc of a circle, intercepted between the two lines which form the angle—the center of the arc always being at the point where the two lines meet.

The arc is measured by *degrees, minutes,* and *seconds,* there being 360 degrees to the whole circle, 60 minutes in one degree, and 60 seconds in one minute. Degrees, minutes, and seconds, are designated by $^\circ$, $'$, $''$. Thus $27^\circ\ 14'\ 21''$, is read 27 degrees, 14 minutes, and 21 seconds.

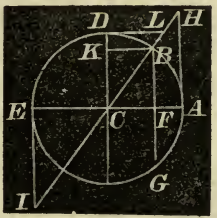
All circles contain the same number of degrees, but the greater the radii the greater is the absolute length of a degree; the circumference of a carriage wheel, the circumference of the earth, or the still greater and indefinite circumference of the heavens, have the same number of degrees; yet the same number of degrees in each and everv circle is precisely the same angle in amount or measure.

As triangles do not contain circles, we can not measure triangles by circular arcs; we must measure them by *other triangles*, that is, by *straight lines*, drawn in and about a circle. from the center.

Such straight lines are called trigonometrical lines, and take particular names, as described by the following

DEFINITIONS.

1. The *sine* of an angle, or an arc, is a line drawn from one end of an arc, perpendicular to a diameter drawn through the other end. Thus, BF is the sine of the arc AB, *and also* of the arc BDE. BK is the sine of the arc BD, it is *also* the cosine of the arc AB, and BF, is the cosine of the arc BD.



N. B. The *complement* of an arc is what it wants of 90°; the *supplement* of an arc is what it what it wants of 180°.

2. The *cosine* of an arc is the perpendicular distance from the center of the circle to the sine of the arc, or it is the same in magnitude as the sine of the complement of the arc. Thus, CF, is the cosine of the arc AB; but $CF=KB$, the sine of BD.

3. The *tangent* of an arc is a line touching the circle in one extremity of the arc, continued from thence, to meet a line drawn through the center and the other extremity.

Thus, AH is the tangent to the arc AB, and DL is the tangent of the arc DB, or the cotangent of the arc AB.

N. B. *The* co, *is but a contraction of the word complement.*

4. The *secant* of an arc, is a line drawn from the center of the circle to the extremity of its tangent. Thus, CH is the secant of the arc AB, or of its supplement BDE.

5. The *cosecant* of an arc, is the secant of the complement. Thus, CL, the secant of BD, is the cosecant of AB.

6. The versed sine of an arc is the difference between the cosine and the radius; that is, AF is the versed sine of the arc AB, and DK is the versed sine of the arc BD.

For the sake of brevity these technical terms are contracted thus: for sine AB, we write *sin.*AB, for cosine AB, we write *cos.*AB, for tangent AB, we write *tan.*AB, &c.

From the preceding definitions we deduce the following obvious consequences:

1st, That when the arc AB, becomes so small as to call it nothing, its sine tangent and versed sine are also nothing, and its secant and cosine are each equal to radius.

2d, The sine and versed sine of a quadrant are each equal to the radius; its cosine is zero, and its secant and tangent are infinite.

3d, The chord of an arc is twice the sine of half the arc. Thus the chord BG, is double of the sine BF.

4th, The sine and cosine of any arc form the two sides of a right angled triangle, which has a radius for its hypotenuse. Thus, CF, and FB, are the two sides of the right angled triangle CFB.

Also, the radius and the tangent always form the two sides of a right angled triangle which has the secant of the arc for its hypotenuse. This we observe from the right angled triangle CAH.

To express these relations analytically, we write

$$\sin.^2+\cos.^2=R^2 \qquad (1)$$

$$R^2+\tan.^2=\sec.^2 \qquad (2)$$

From the two equiangular triangles CFB, CAH, we have

$$CF:FB=CA:AH$$

That is, . $\cos.:\sin.=R:\tan.$ $\quad \tan.=\dfrac{R\sin.}{\cos.}$ (3)

Also, . $CF:CB=CA:CH$

That is, . $\cos:R=R:\sec.$ $\quad \cos.\sec.=R^2$ (4)

The two equiangular triangles CAH, CDL. give

$$CA:AH=DL:DC$$

That is, . $R:\tan.=\cot:R$ $\quad \tan.\cot.=R^2$ (5)

Also, . $CF:FB=DL:DC$

That is, . $\cos.:\sin.=\cot:R$ $\quad \cos.R=\sin.\cot.$ (6)

By observing (4) and (5), we find that

$$\cos.\sec.=\tan.\cot. \qquad (7)$$

Or, . $\cos.:\tan.=\cot.:\sec.$

The *ratios* between the various trigonometrical lines are always the same for the same arc, whatever be the length of the radius; and therefore, we may assume radius of any length to suit our convenience; and the preceding equations will be more concise, and more readily applied, by making radius equal unity. This supposition being made, the preceding becomes

$$\sin.^2+\cos.^2=1 \qquad (1)$$

$$1+\tan.^2=\sec.^2 \qquad (2)$$

$$\tan.=\frac{\sin.}{\cos.} \quad (3) \qquad \cos.=\frac{1}{\sec.} \quad (4)$$

$$\tan.=\frac{1}{\cot.} \quad (5) \qquad \cos.=\sin.\ \cot. \quad (6)$$

The center of the circle is considered the absolute *zero* point, and the different directions from this point are designated by the different signs + and —. On the right of C, toward A, is commonly marked plus (+), then the other direction, toward E, is necessarily minus (—). Above AE is called (+), below that line (—).

If we conceive an arc to commence at A, and increase continuously around the whole circle in the direction of ABD, then the following table will show the mutations of the signs.

	sin.	cos.	tan.	cot.	sec.	cosec.	vers.
1st quadrant.	+	+	+	+	+	+	+
2d "	+	—	—	—	—	+	+
3d "	—	—	+	+	—	—	+
4th "	—	+	—	—	+	—	+

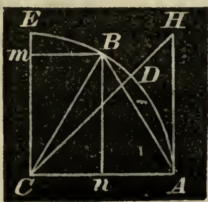
PROPOSITION 1.

The chord of 60° *and the tangent* 45° *are each equal to radius; the sine of* 30° *the versed sine of* 60° *and the cosine of* 60° *are each equal to half the radius.*

(The first truth is proved in problem 15, book 1).

On C=, as radius, describe a quadrant; take AD=45°, AB =60°, and AE=90°, then BE=30°.

Join AB, CB, and draw Bn, perpendicular to CA. Draw Bm, parallel to AC. Make the angle CAH=90°, and draw CDH.



In the △ ABC, the angle ACB=60° by hypothesis; therefore, the sum of the other two angles is (180—60)=120°. But CB=CA, hence the angle CBA= the angle CAB, (th. 15 b. 1), and as the sum of the two is 120°, each one must be 60°; therefore, each of the angles of triangle ABC, is 60°

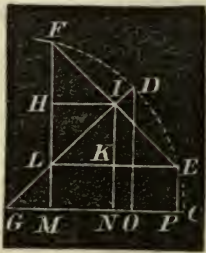
and the sides opposite to equal angles are equal · that is, AB, the chord of 60°, is equal to CA, the radius.

In the △ CAH, the angle CAH is a right angle; and by hypothesis, ACH. is half a right angle; therefore, AHC, is also half a right angle; consequently, $AH=AC$, the tangent of 45°= the radius.

By th. 15, book 1, cor. $Cn=nA$; that is, the cosine and versed sine of 60° are each equal to the half of the radius. As Bn and EC are perpendicular to AC, they are parallel, and Bm is made parallel to Cn; therefore, $Bm=Cn$, or the sine 30°, is the half of radius.

PROPOSITION 2.

Given the sine and cosine of two arcs to find the sine and cosine of the sum, and difference of the same arcs expressed by the sines and cosines of the separate arcs.



Let G be the center of the circle, CD, the greater arc which we shall designate by a, and DF, a less arc, that we designate by b.

Then by the definitions of sines and cosines, $DO=\sin.a$; $GO=\cos.a$; $FI=\sin.b$; $GI=\cos.b$. We are to find FM, which is

$$=\sin.(a+b);\quad GM=\cos.(a+b);$$
$$EP=\sin.(a-b);\quad GP=\cos.(a-b).$$

Because IN is parallel to DO, the two △s GDO, GIN, are equiangular and similar. Also, the △ FHI, is similar to GIN; for the angle FIG, is a right angle; so is HIN; and, from these two equals take away the common angle HIL, leaving the angle $FIH=GIN$. The angles at H and N, are right angles; therefore, the △ FHI, is equiangular, and similar to the △ GIN, and, of course, to the △ GDO; and the side HI, is homologous to IN, and DO.

Again, as $FI=IE$, and IK, parallel to FM,

$$FH=IK, \text{ and } HI=KE.$$

By similar triangles we have

$$GD:DO=GI:IN.$$

That is, . $R:\sin.a=\cos.b:IN$, or $IN=\dfrac{\sin.a\cos.b}{R}$

Also, . $GD:GO=FI:FH$.

That is, . $R:\cos.a=\sin.b:FH$, or $FH=\frac{\cos.a \sin.b}{R}$

Also, . $GD:GO=GI:GN$

That is, . $R:\cos.a=\cos.b:GN$, or $GN=\frac{\cos.a \cos.b}{R}$

Also, . $GD:DO=FI:IH$

That is, . $R:\sin.a=\sin.b:IH$, or $IH=\frac{\sin.a \sin.b}{R}$

By adding the first and second of these equations, we have

$$IN+FH=FM=\sin.(a+b)$$

That is, . $\sin.(a+b)=\frac{\sin.a \cos.b+\cos.a \sin.b}{R}$

By subtracting the second from the first, we have

$$\sin.(a-b)=\frac{\sin.a \cos.b-\cos.a \sin.b}{R}$$

By subtracting the fourth from the third, we have

$GN-IH=GM=\cos.(a+b)$ for the first member.

Hence, . $\cos.(a+b)=\frac{\cos.a \cos.b-\sin.a \sin.b}{R}$

By adding the third and fourth, we have

$$GN+IH=GN+NP=GP=\cos.(a-b)$$

Hence, . $\cos.(a-b)=\frac{\cos.a \cos.b+\sin.a \sin.b}{R}$

Collecting these four expressions, and considering the radius unity, we have

$$(A)\quad\left\{\begin{array}{ll} \sin.(a+b)=\sin.a \cos.b+\cos.a \sin.b & (7) \\ \sin.(a-b)=\sin.a \cos.b-\cos.a \sin.b & (8) \\ \cos.(a+b)=\cos.a \cos.b-\sin.a \sin.b & (9) \\ \cos.(a-b)=\cos.a \cos.b+\sin.a \sin.b & (10) \end{array}\right.$$

Formula (A), accomplishes the objects of the proposition, and from these equations many useful and important deductions can be made. The following, are the most essential:

By adding (7) to (8), we have (11); subtracting (8) from (7), gives (12). Also, (9)+(10) gives (13); (9) taken from (10) gives (14).

$$(B)\quad\left\{\begin{array}{ll} \sin.(a+b)+\sin.(a-b)=2\sin.a \cos b & (11) \\ \sin.(a+b)-\sin.(a-b)=2\cos.a \sin. b & (12) \\ \cos.(a+b)+\cos.(a-b)=2\cos.a \cos.b & (13) \\ \cos.(a-b)-\cos.(a+b)=2\sin. a \sin.b & (14) \end{array}\right.$$

If we put $a+b=A$, and $a-b=B$, then (11) becomes (15), (12) becomes (16), 13 becomes (17), and (14) becomes (18).

$$(C)\left\{\begin{array}{lr} \sin.A+\sin.B=2\sin.\left(\frac{A+B}{2}\right)\cos.\left(\frac{A-B}{2}\right) & (15)\\ \sin.A-\sin.B=2\cos.\left(\frac{A+B}{2}\right)\sin.\left(\frac{A-B}{2}\right) & (16)\\ \cos.A+\cos.B=2\cos.\left(\frac{A+B}{2}\right)\cos.\left(\frac{A-B}{2}\right) & (17)\\ \cos.B-\cos.A=2\sin.\left(\frac{A+B}{2}\right)\sin.\left(\frac{A-B}{2}\right) & (18) \end{array}\right.$$

If we divide (15) by (16), (observing that $\frac{\sin.}{\cos.}=\tan.$ and $\frac{\cos.}{\sin.}=\cot.=\frac{1}{\tan.}$ as we learn by equations (6) and (5) trigonometry), we shall have

$$\frac{\sin.A+\sin.B}{\sin.A-\sin.B}=\frac{\sin.\left(\frac{A+B}{2}\right)}{\cos.\left(\frac{A+B}{2}\right)}\times\frac{\cos.\left(\frac{A-B}{2}\right)}{\sin.\left(\frac{A-B}{2}\right)}=\frac{\tan.\left(\frac{A+B}{2}\right)}{\tan.\left(\frac{A-B}{2}\right)} \quad (19)$$

Whence,

$$\overline{\sin.A+\sin.B}:\overline{\sin.A-\sin.B}=\tan.\left(\frac{A+B}{2}\right):\tan.\left(\frac{A-B}{2}\right)$$

or in words. *The sum of the sines of any two arcs is to the difference of the same sines, as the tangent of the half sum of the same arcs is to the tangent of half their difference.*

By operating in the same way with the different equations in formula (C), we find,

$$(D)\left\{\begin{array}{lr} \frac{\sin.A+\sin.B}{\cos.A+\cos.B}=\tan.\left(\frac{A+B}{2}\right) & (20)\\ \frac{\sin.A+\sin.B}{\cos.B-\cos.A}=\cot.\left(\frac{A-B}{2}\right) & (21)\\ \frac{\sin.A-\sin.B}{\cos.A+\cos.B}=\tan.\left(\frac{A-B}{2}\right) & (22)\\ \frac{\sin.A-\sin.B}{\cos.B-\cos.A}=\cot.\left(\frac{A+B}{2}\right) & (23)\\ \frac{\cos.A+\cos.B}{\cos.B-\cos.A}=\frac{\cot.\left(\frac{A+B}{2}\right)}{\tan.\left(\frac{A-B}{2}\right)} & (24) \end{array}\right.$$

These equations are all true, whatever be the value of the arcs designated by A and B; we may therefore, assign any possible value to either of them, and if in equations (20), (21) and (24), we make $B=O$, we shall have,

$$\frac{\sin.A}{1+\cos.A}=\tan.\frac{A}{2}=\frac{1}{\cot.\frac{1}{2}A} \qquad (25)$$

$$\frac{\sin.A}{1-\cos.A}=\cot.\frac{A}{2}=\frac{1}{\tan.\frac{1}{2}A} \qquad (26)$$

$$\frac{1+\cos.A}{1-\cos.A}=\frac{\cot.\frac{1}{2}A}{\tan.\frac{1}{2}A}=\frac{1}{\tan.^2\frac{1}{2}A} \qquad (27)$$

If we now turn back to formula (A), and divide equation (7) by (9), and (8) by (10), observing at the same time, that $\frac{\sin.}{\cos.}=\tan.$ we shall have,

$$\tan.(a+b)=\frac{\sin a \cos.b+\cos.a \sin.b}{\cos.a \cos.b-\sin.a \sin.b}$$

$$\tan.(a-b)=\frac{\sin.a \cos.b-\cos.a \sin.b}{\cos.a \cos.b+\sin.a \sin.b}$$

By dividing the numerators and denominators of the second members of these equations by ($\cos.a \cos.b$), we find,

$$\tan.(a+b)=\frac{\frac{\sin.a \cos.b}{\cos.a \cos.b}+\frac{\cos.a \sin.b}{\cos.a \cos.b}}{\frac{\cos.a \cos.b}{\cos.a \cos.b}-\frac{\sin.a \sin.b}{\cos.a \cos.b}}=\frac{\tan.a+\tan.b}{1-\tan.a \tan.b} \qquad (28)$$

$$\tan.(a-b)=\frac{\frac{\sin.a \cos.b}{\cos.a \cos.b}-\frac{\cos.a \sin.b}{\cos.a \cos.b}}{\frac{\cos.a \cos.b}{\cos.a \cos.b}+\frac{\sin.a \sin.b}{\cos.a \cos.b}}=\frac{\tan.a-\tan.b}{1+\tan.a \tan.b} \qquad (29)$$

If in equation (11), formula (B), we make $a=b$, we shall have,

$$\sin.2a=2\sin.a \cos.a \qquad (30)$$

Making the same hypothesis in equation (13), gives,

$$\cos.2a+1=2\cos^2.a \qquad (31)$$

The same hypothesis reduces equation (14), to

$$1-\cos.2a=2\sin^2.a \qquad (32)$$

The same hypothesis reduces equation (28), to

$$\tan.2a=\frac{2\tan.a}{1-\tan^2.a} \qquad (33)$$

If we substitute a for $2a$ in (31) and (32), we shall have

$$1+\cos.a=2\cos.^2\tfrac{1}{2}a. \qquad (34)$$

$$\text{and } 1-\cos.a=2\sin.^2\tfrac{1}{2}a. \qquad (35)$$

Recurring again to formula (B), we have, by transposing

$$\sin.(a+b)=2\sin.a\cos.b-\sin.(a-b)$$

$$\sin.(a+b)=2\cos.a\sin.b+\sin.(a-b)$$

If, in the first of these expressions, we make $a=30°$, $2\sin.a$ will equal radius, or unity; and if in the second we make $a=60°$, $2\cos.a$ will also equal unity; these expressions then become,

$$\sin.(30°+b)=\cos.b-\sin.(30°-b) \qquad (36)$$

And . . $$\sin.(60°+b)=\sin.b+\sin.(60-°b) \qquad (37)$$

The sines may be easily continued to 60°, by equation (36), when the sines and cosines of all arcs below 30° have been computed; then, by equation (37), the sines can be readily run up to 90°.

The foregoing equations might have been obtained *geometrically*, but not so easily and concisely.

ON THE CONSTRUCTION OF TABLES OF SINES, TANGENTS, &c.

To explain this, we refer at once to Table II, which contains logarithmic sines, and tangents, and also natural sines and cosines. The natural sines are made to the radius of unity; and, of course, any particular sine is a decimal fraction, expressed by natural numbers. The logarithm of any natural sine, with its index increased by 10, will give the logarithmic sine. Thus, the natural sine of 3° is .052336

The logarithm of this decimal is . . .	—2.718800
To which add	10.
The logarithmic sine of 3° is, therefore, . .	8.718800

In this manner we may find the logarithmic sine of any other arc, when we have the natural sine of the same arc.

If the natural sines and logarithmic sines were on the same radius, the logarithm of the natural sine would be the logarithmic sine, at once. without any increase of the index.

The radius for the logarithmic sines, is arbitrarily taken so large that the index of its logarithm is 10. It might have been more or less; but, by common consent, it is settled at this value; so that the sines of the smallest arcs ever used shall not have a negative index.

In our preceding equations, sin.a, cos.a, &c., referred to *natural sines;* and by such equations we determine their values in natural numbers; and these numbers are put in the table, as seen in table 2, under the heads of *nat. sine*, and *nat. cosine*.

To commence computation, we must know the sine or cosine of some known arc; and we do know the sine and cosine of 30°. The sine of 30° is $\frac{1}{2}$ (prop. 1, trig.), and, hence, $\cos.^2 30°=1-\frac{1}{4}$ (eq. (1) trig.); or, $\cos. 30°=\frac{1}{2}\sqrt{3}$. Now put $2a=30°$, and equation (30) gives

$$2\sin.15° \cos.15°=0.5. \qquad (n)$$

Eq. (1) gives . . $\cos.^2 15°+\sin.^2 15=1. \qquad (n)$

By adding (m) to (n), and extracting square root, we obtain,

$$\cos.15°+\sin.15°=\sqrt{1.5}=1.22474487. \qquad (p)$$

By subtracting (m) from (n), and extracting square root,

$$\cos.15°-\sin.15°=\sqrt{0.5}=0.70710678 \qquad (q)$$

Sub. (q) from (p) gives $2\sin.15°=0.51763709.$

Again, put $2a=15°$, and in like manner apply equations (30) and (1), and we can have the sine and cosine of 7° 30′, and thus we may bisect as many times as we please, but when we get down to any arc under 1′, we can compute the sines by direct proportion.

Also, by theorems 3 and 4, book 5, the semicircumference of a circle whose radius is unity, is 3.14159265; this, divided by 10800, the number of minutes in 180°, will give .0002908882 for the length of the sine or arc of one minute. The logarithm of this number, with its index increased by 10, gives 6.463726, the log. sign of 1′, which is found in the table.

Having the sine and cosine of 1′, we can find the sine and cosine of 2′ by equation (30);

That is, . . $\sin.2a=2\sin.a \cos.a$

Or, . . . $\sin.2'=2\sin.1'\cos.1'$

For the sine of 3′, and every succeeding minute, we apply equation (11), making $a=2'$, and $b=1'$;

That is, . . $\sin.3'=2\sin.2' \cos.1-\sin.1'$

Having the sine of 3′, we obtain the sine of 4′ by the application of the same equation; that is, by making $a=3'$, and $b=1$;

Then, . . . $\sin.4'=2\sin.3' \cos.1-\sin.2'$

$\sin.5'=2\sin.4' \cos.1-\sin.3'$ &c., &c.

When the sine of any arc is known, its cosine is readily determined by the following formula, which is, in substance, equation (1), trigonometry. . . $\cos.=\sqrt{(1+\sin.)(1-\sin.)}$

When the sine and cosine of any arc are known, the sine and cosine of its double, are found from equation (30); and thus, from equations (30), (11), and (1), the sines and cosines of all arcs can be determined.

When the sine and cosine of an arc have been determined through a series of operations, the accuracy of the results should be tested by equation (12) or (14), or by some other equation independent of former operations; and if the two results agree, they may be regarded as accurate.

One independent method will be found by applying theorem 5, book 5. In that theorem we find the chord of 20° is .347296 ; the natural sine, then, of 10°, is .173648. Taken, the chord of 20°, and trisecting the arc by the same problem, we find the chord of 6° 40′ to be .11628; and, of course, the natural sine of 3° 20′ is .05814; and thus, by successive trisections we can obtain the sines, and of course the cosines of certain arcs; and when we arrive at very small arcs, we can compute their increase or decrease by direct proportion.*

Now, if the sine of an arc computed through successive trisections, agrees with the sine of the same arc computed through successive bisections, we must, of course, regard the result as accurate.

When we have the sines and cosines of an arc, the tangent and cotangent are found by (3) $\tan.=\frac{R \sin.}{\cos.}$ (6) $\cot.=\frac{R \cos.}{\sin.}$; and the secant is found by equation (4); that is, $\sec.=\frac{R^2}{\cos.}$

For example, the logarithmic sine of 6°, is 9.019235, and its cosine 9.997614. From these it is required to find the tangent, cotangent, and secant.

R sin.	. .	19.019235
Cos.	. subtract	9.997614
Tan. is	. .	9.021621
R cos.	. .	19.997614
Sin. .	. subtract	9.019235
Cotan. is	. .	10.978379
R^2 is	. .	20.000000
Cos. .	. subtract	9.997674
Secant is	. .	10.002326

* Thus, from theorem 4, book 5, we find the chord of 28′ 7″ 30‴ to be .008181208 ; and wishing to take away 7″ 30‴, we do it by proportion, as follows. The sine of 1′ or 60″ is .0002908882.

Therefore, . 60 : 7½=.0002908882

Or, . . . 8 : 1 =.0002908882 : .000036461

The chord of 28′ 7″ 30‴ is	. .	.008181208
of 7″ 30‴ is	. .	.000036461
of 28′ is	. .	.008144747
The natural sine of 14′ is	. .	.004072373

Now we may halve or double this sine by equation (30).

The secants and cosecants of arcs are not given in our table, because they are very little used in practice; and if any particular secant is required, it can be determined by subtracting the cosine from 20; and the cosecant can be found by subtracting the sine from 20.

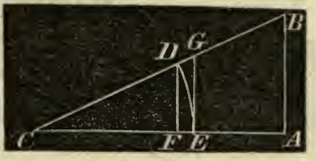
PROPOSITION 3.

In any right angled plane triangle, we may have the following proportions:

1st. *As the hypotenuse is to either side, so is the radius to the sine of the angle opposite to that side.*

2d. *As one side is to the other side, so is the radius to the tangent of the angle adjacent to the first-mentioned side.*

3d. *As one side is to the hypotenuse, so is radius to the secant of the angle adjacent to that side.*



Let CAB represent any right angled triangle, right angled at A. AB and AC are called the sides of the $\triangle$, and CB is called the hypotenuse.

(Here, and in all cases hereafter, we shall represent the angles of a triangle by the large letters A, B, C, and the sides opposite to them, by the small letters a, b, c.)

From either acute angle, as C, take any distance, as CD, *greater* or *less* than CB, and describe the arc DE. This arc measures the angle C. From D, draw DF parallel to BA; and from E, draw EG, also parallel to BA or DF.

By the definitions of sines, tangents, and secants, DF is the sine of the angle C; EG is the tangent, CG the secant, and CF the cosine.

Now, by proportional triangles we have,

$$\left.\begin{array}{lll} CB : BA = CD : DF & \text{or,} & a : c = R : \sin. C \\ CA : AB = CE : EG & \text{or,} & b : c = R : \tan. C \\ CA : CB = CE : CG & \text{or,} & b : a = R : \sec. C \end{array}\right\} Q.\ E.\ D.$$

Scholium. If the hypotenuse of a triangle is made radius, one side is the sine of the angle opposite to it, and the other side is the cosine of the same angle. This is obvious from the triangle CDF.

PROPOSITION 4.

In any triangle, the sines of the angles are to one another as the sides opposite to them.

Let ABC be any triangle. From the points A and B, as centers, with any radius, describe the arcs measuring these angles, and draw pa, CD, and mn, perpendicular to AB.

Then, . . $pa=\sin.A$, $mn=\sin.B$

By the similar $\triangle$s, Apa and ACD, we have,

$$R : \sin.A = b : CD; \text{ or, } R(CD) = b \sin.A \quad (1)$$

By the similar $\triangle$s Bmn and BCD, we have,

$$R : \sin.B = a : CD; \text{ or, } R(CD) = a \sin.B \quad (2)$$

By equating the second members of equations (1) and (2).

$$b \sin.A = a \sin.B.$$

Hence, . $\sin.A : \sin.B = a : b$ } *Q. E. D.*
Or, . . $a : b = \sin A : \sin. B$ }

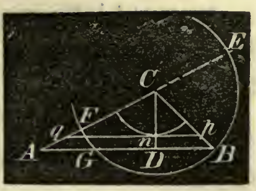
Scholium 1. When either angle is 90°, its sine is radius.

Scholium 2. When CB is less than AC, and the angle B, acute, the triangle is represented by ACB. When the angle B becomes B', it is obtuse, and the triangle is ACB'; but the proportion is equally true with either triangle; for the angle $CB'D=CBA$, and the sine of $CB'D$ is the same as the sine of $AB'C$. In practice we can determine which of these triangles is proposed by the side AB, being greater or less than AC; or, by the angle at the vertex C, being large as ACB, or small as ACB'.

In the solitary case in which AC, CB, and the angle A, are given, and CB less than AC, we can determine both of the $\triangle$s ACB and ACB'; and then we surely have the right one.

PROPOSITION 5.

If from any angle of a triangle, a perpendicular be let fall on the opposite side, or base, the tangents of the segments of the angle are to one another as the segments of the base.



Let ABC be the triangle. Let fall the perpendicular CD, on the side AB.

Take any radius, as Cn, and describe the arc which measures the angle C. From n, draw qnp parallel to AB. Then it is obvious that np is the tangent of the angle DCB, and nq is the tangent of the angle ACD.

Now, by reason of the parallels AB and qp, we have,

$$qn : np = AD : DB$$

That is, $\tan.ACD : \tan.DCB = AD : DB$ Q. E. D.

PROPOSITION 6.

If a perpendicular be let fall from any angle of a triangle to its opposite side or base, this base is to the sum of the other two sides, as the difference of the sides is to the difference of the segments of the base.

(See figure to proposition 5.)

Let AB be the base, and from C, as a center, with the shorter side as radius, describe the circle, cutting AB in G, AC in F, and produce AC to E.

It is obvious that AE is the sum of the sides AC and CB, and AF is their difference.

Also, AD is one segment of the base made by the perpendicular, and $BD=DG$ is the other; therefore, the difference of the segments is AG.

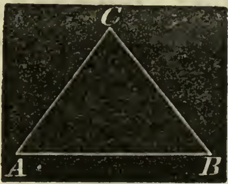
As A is a point without a circle, by theorem 18, book 3, we have,

$$AE \times AF = AB \times AG$$

Hence, . . $AB : AE = AF : AG$ Q. E. D.

PROPOSITION 7.

The sum of any two sides of a triangle, is to their difference, as the tangent of the half sum of the angles opposite to these sides, to the tangent of half their difference.



Let ABC be any plane triangle. Then, by proposition 4, trigonometry, we have,

$$CB : AC = \sin. A : \sin.B$$

Hence,

$$CB + AC : CB - AC = \sin.A + \sin.B : \sin.A - \sin.B \text{ (th. 9 b. 2)}$$

But, tan. $\left(\frac{A+B}{2}\right)$: tan. $\left(\frac{A-B}{2}\right)$=sin.$A$+sin.$B$: sin.A—sin.B (eq. (19), trig.)

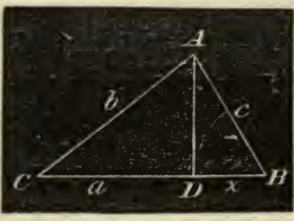
Comparing the two latter proportions (th. 6, b. 2), we have,

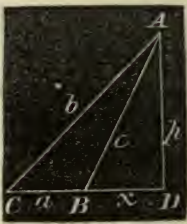
$CB+AC$: $CB-AC$= tan. $\left(\frac{A+B}{2}\right)$: tan. $\left(\frac{A-B}{2}\right)$ *Q. E. D.*

PROPOSITION 8.

Given the three sides of any plane triangle, to find some relation which they must bear to the sines and cosines of the respective angles.

Let ABC be the triangle, and let the perpendicular fall either upon, or without the base, as shown in the figures; and by recurring to theorem 38, book 1, we shall find





$$CD=\frac{a^2+b^2-c^2}{2a} \qquad (1)$$

Now, by proposition 3, trigonometry, we have,

$$R : \cos.C=b : CD$$

Therefore, . $$CD=\frac{b\cos.C}{R} \qquad (2)$$

Equating these two values of CD, and reducing, we have,

$$\cos.C=\frac{R(a^2+b^2-c^2)}{2ab} \qquad (m)$$

In this expression we observe that the part of the numerator which has the minus sign, is the side opposite to the angle; and that the denominator is twice the rectangle of the sides adjacent to the angle. From these observations we at once draw the following expressions for the cosine A, and cosine B.

Thus, . . $$\cos.A=\frac{R(b^2+c^2-a^2)}{2bc} \qquad (n)$$

$$\cos.B=\frac{R(a^2+c^2-b^2)}{2ac} \qquad (p)$$

As these expressions are not convenient for logarithmic computation, we modify them as follows:

If we put $2a=A$, in equation (31), we have,

$$\cos.A+1=2\cos.^2\tfrac{1}{2}A$$

In the preceding expression (n), if we consider radius, unity, and add 1 to both members, we shall have,

$$\cos.A+1=1+\frac{b^2+c^2-a^2}{2bc}$$

Therefore, $$2\cos.^2\tfrac{1}{2}A=\frac{2bc+b^2+c^2-a^2}{2bc}$$

$$=\frac{(b+c)^2-a^2}{2bc}$$

Considering $(b+c)$ as one quantity, and observing that we have the difference of *two squares*, therefore

$(b+c)^2-a^2=(b+c+a)(b+c-a)$; but $(b+c-a)=b+c+a-2a$

Hence, . $$2\cos.^2\tfrac{1}{2}A=\frac{(b+c+a)(b+c+a-2a)}{2bc}$$

Or, . . $$\cos.^2\tfrac{1}{2}A=\frac{\left(\frac{b+c+a}{2}\right)\left(\frac{b+c+a}{2}-a\right)}{bc}$$

By putting $\frac{a+b+c}{2}=s$, and extracting square root, the final result for radius unity, is

$$\cos.\tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}}.$$

For any other radius we must write,

$$\cos.\tfrac{1}{2}A=\sqrt{\frac{R^2s(s-a)}{bc}}$$

By inference, $$\cos.\tfrac{1}{2}B=\sqrt{\frac{R^2s(s-b)}{ac}}$$

Also, . . $$\cos.\tfrac{1}{2}C=\sqrt{\frac{R^2s(s-c)}{ab}}$$

In every triangle, the sum of the three angles must equal 180°; and if one of the angles is small, the other two must be comparatively large; if two of them are small, the third one must be large. The greater angle is always opposite the greater side; hence, by merely inspecting the given sides, any person can decide at once which is the greater angle; and of the three preceding equations, *that one* should be taken which applies to the greater angle, whether that be the particular angle required or not; because the equations bring out the *cosines* to the angles; and the cosines, to very small arcs vary so slowly, that it may be impossible to decide, with sufficient numerical accuracy to what particular arc the cosine belongs. For instance, the cosine 9.999999, carried to the table, applies to several arcs; and, of course, we should not know which one to take; but this difficulty does not exist when the angle is large; therefore, compute the largest angle first, and then compute the other angles by proposition 4.

But we can deduce an expression for the sine of any of the angles, as well as the cosine. It is done as follows:

EQUATIONS FOR THE SINES OF THE ANGLES.

Resuming equation (m), and considering radius, unity, we have,

$$\cos. C = \frac{a^2+b^2-c^2}{2ab}$$

Subtracting each member of this equation from 1, gives

$$1-\cos. C = 1-\left(\frac{a^2+b^2-c^2}{2ab}\right) \qquad (1)$$

Making $2a=C$, in equation (32), then $a=\frac{1}{2}C$,

And . . . $1-\cos. C = 2\sin.^2\frac{1}{2}C$ (2)

Equating the right hand members of (1) and (2),

$$2\sin.^2\tfrac{1}{2}C = \frac{2ab-a^2-b^2+c^2}{2ab}$$

$$= \frac{c^2-(a-b)^2}{2ab}$$

$$= \frac{(c+b-a)(c+a-b)}{2ab}$$

Or, . . . $\sin.^2\frac{1}{2}C = \dfrac{\left(\frac{c+b-a}{2}\right)\left(\frac{c+a-b}{2}\right)}{ab}$

But, . $\dfrac{c+b-a}{2} = \dfrac{c+b+a}{2} - a$ and $\dfrac{c+a-b}{2} = \dfrac{c+a+b}{2} - b$

Put . $\dfrac{a+b+c}{2} = s$, as before; then,

$$\sin.\tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

By taking equation (p), and operating in the same manner, we have . . . $\sin.\frac{1}{2}B = \sqrt{\dfrac{(s-a)(s-c)}{ac}}$

From (n) . . $\sin.\frac{1}{2}A = \sqrt{\dfrac{(s-b)(s-c)}{cb}}$

The preceding results are for radius unity; for any other radius, we must multiply by the number of units in such radius. For the radius of the tables, we write R; and if we put it under the radical sign, we must write R^2; hence, for the sines corresponding with our logarithmic table, we must write the equations thus, . . .

$$\sin.\tfrac{1}{2}A=\sqrt{\frac{R^2(s-b)(s-c)}{bc}}$$

$$\sin.\tfrac{1}{2}B=\sqrt{\frac{R^2(s-a)(s-c)}{ac}}$$

$$\sin.\tfrac{1}{2}C=\sqrt{\frac{R^2(s-a)(s-b)}{ab}}$$

A large angle should not be determined by these equations, for the same reason that a small angle should not be determined from an equation expressing the cosine.

In practice, the equations for cosine are more generally used, because more easily applied.

In the preceding pages we have gone over the whole ground of theoretical plane trigonometry, although several particulars might have been enlarged upon, and more equations in relation to the combinations of the trigonometrical lines, might have been given; but enough has been given to solve every possible case that can arise in the practical application of the science; but to show more clearly the beauty and spirit of this science, and to redeem a promise, we give the following *geometrical demonstrations* of the truths expressed in some of the preceding equations.

From C as the center, with CA as the radius, describe a circle. Take any arc, AB, and call it A; AD a less arc, and call it B; then BD is the difference of the two arcs, and must be designated by $(A-B)$; $AG=AB$; therefore, $DG=A+B$; $EG=\sin.A$; (See fig. p. 154.) $En=\sin.B$; $Gn=\sin.A+\sin.B$;

$$Bn=\sin.A-\sin.B.$$

$$Fm=mD=CH=\cos.B;\ mn=\cos.A;$$

Therefore, $Fm+mn=\cos.A+\cos.B=Fn$;

$$mD-mn=\cos.B-\cos.A=nD;$$

$$DG=2\sin.\left(\frac{A+B}{2}\right)$$

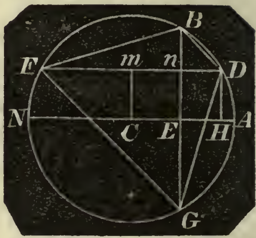
Because . $NF=AD$; $AB+NF=A+B$;

Therefore, . . $180^\circ-(A+B)=\text{arc } FB$;

Or, . . . $90° - \left(\frac{A+B}{2}\right) = \frac{1}{2}\text{arc } FB;$

But the chord FB, is *twice* the sine of $\frac{1}{2}$ arc FB.

That is, $FB = 2\sin.\left(90° - \frac{A+B}{2}\right) = 2\cos.\left(\frac{A+B}{2}\right)$



The angle $nGD = BFD$, because both are measured by one half of the arc BD; that is, by $\left(\frac{A-B}{2}\right)$ and the two triangles GnD, and FnB are similar.

The angle GFn, is measured by

$$\left(\frac{A+B}{2}\right)$$

In the triangle FBG, Fn is drawn from an angle perpendicular to the opposite side; therefore, by Proposition 5, we have,

$$Gn : nB = \tan. GFn : \tan. BFn$$

That is, $\sin.A + \sin.B : \sin.A - \sin.B = \tan.\left(\frac{A+B}{2}\right) : \tan.\left(\frac{A-B}{2}\right)$

This is equation (19).

In the triangle GnD, we have

$$\sin.90° : DG = \sin.nDG : Gn; \quad \sin.nDG = \cos.nGD$$

That is, $1 : 2\sin.\left(\frac{A+B}{2}\right) = \cos.\left(\frac{A-B}{2}\right) : \sin.A + \sin.B$

Or, . $\sin.A + \sin.B = 2\sin.\left(\frac{A+B}{2}\right)\cos.\left(\frac{A-B}{2}\right)$

same as equation (15).

In the triangle FnB, we have,

$$\sin.90 : FB = \sin.BFn : Bn$$

That is, $1 : 2\cos.\left(\frac{A+B}{2}\right) = \sin.\left(\frac{A-B}{2}\right) : \sin.A - \sin.B$

Or, . . $\sin.A - \sin.B = 2\cos.\left(\frac{A+B}{2}\right)\sin.\left(\frac{A-B}{2}\right)$

same as equation (16).

In the triangle FBn, we have,

$$\sin.90 : FB = \cos.BFn : Fn$$

That is, $1 : 2\cos.\left(\frac{A+B}{2}\right) = \cos.\left(\frac{A-B}{2}\right) : \cos.A + \cos.B$

Or, $\cos.A+\cos.B=2\cos.\left(\frac{A+B}{2}\right)\cos.\left(\frac{A-B}{2}\right)$ same as equation (17).

In the triangle GnD, we have,

$$\sin.90° : GD=\sin.nGD : nD$$

That is, . $1 : 2\sin.\left(\frac{A+B}{2}\right)=\sin.\left(\frac{A-B}{2}\right) : \cos.B-\cos.A$,

same as equation (18).

In the triangle FGn, we have,

$$\sin.GFn : Gn=\cos.GFn : Fn$$

That is, $\sin.\frac{A+B}{2} : \sin.A+\sin.B=\cos.\frac{A+B}{2} : \cos.A+\cos.B$

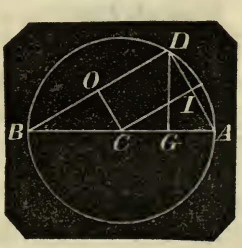
Or, $(\sin.A+\sin.B)\cos.\left(\frac{A+B}{2}\right)=(\cos.A+\cos.B)\sin.\left(\frac{A+B}{2}\right)$

Or, . . $\frac{\sin.A+\sin.B}{\cos.A+\cos.B}=\frac{\sin.\frac{A+B}{2}}{\cos.\frac{A+B}{2}}=\tan.\left(\frac{A+B}{2}\right)$

same as equation (20).

We give a few more geometrical demonstrations from the following figure:

Let the arc $AD=A$; then $DG=\sin.A$; $CG=\cos.A$;
$DI=\sin.\frac{1}{2}A$; $AD=2\sin.\frac{1}{2}A$; $CI=\cos.\frac{1}{2}A$;
$CI=DO$; $DB=2DO=2\cos.\frac{1}{2}A$.



The angle DBA, is measured by half AD; that is, by $\frac{1}{2}A$.

Also, . $ADG=DBA=\frac{1}{2}A$.

Now in the triangle BDG, we have,

$$\sin.DBG : DG=\sin.90° : BD$$

That is, $\sin.\frac{1}{2}A : \sin.A=1 : 2\cos.\frac{1}{2}A$

Or, . $\sin.A=2\sin.\frac{1}{2}A\cos.\frac{1}{2}A$

same as equation (30).

In the same triangle

$$\sin.90° : BD=\sin.BDG : BG;\ \sin.BDG=\cos.DBG;$$

That is, . $1 : 2\cos.\frac{1}{2}A=\cos.\frac{1}{2}A : 1+\cos.A$

Or, . $2\cos^2.\frac{1}{2}A=1+\cos.A$, same as equation (34).

In the triangle DGA, we have,

$$\sin.90^\circ : AD = \sin.GDA : GA$$

That is, . $1 : 2\sin.\tfrac{1}{2}A = \sin.\tfrac{1}{2}A : 1 - \cos.A$

Or, . $2\sin.^2\tfrac{1}{2}A = 1 - \cos.A$, same as equation (35).

By similar triangles, we have,

$$BA : AD = AD : AG$$

That is, . $2 : 2\sin.\tfrac{1}{2}A = 2\sin.\tfrac{1}{2}A :$ versed $\sin.A$

Or, . versed $\sin.A = 2\sin.^2\tfrac{1}{2}A$.

APPLICATION OF THE PRINCIPLES OF TRIGONOMETRY.

Every triangle consists of six parts; three sides, and three angles; and to determine all the parts, three of them must be given, and at least *one of these parts must be a side,* because two triangles may have equal angles, and their sides be very different in respect to magnitude

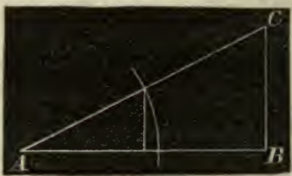
In right angled plane triangles, the right angle is always given; and if two other parts, and *one a side,* be given, it will be sufficient for the complete determination of all the other parts.

Before the invention of logarithms, the numerical computations for the parts of a triangle were all made by arithmetical proportion, as in the rule of three, through the help of natural sines and cosines; but the operations, in many cases, were extremely laborious. For mere curiosity, we will use natural sines to solve the following triangle.

Given, the hypotenuse of a right angled triangle, 840.4 *feet, and one of the oblique angles,* 38° 16′, *to find the other parts.*

The two oblique angles, together, make 90° (th. 11, b. 1, cor. 4); therefore, the other angle is 51° 44′.

sin. 38° 16′ As $1 : 38^\circ\ 16' = AC : CB$



But the natural sine of 38°, 16′ is .61932 and $AC = 840.4$.

Therefore, $1 : .61932 = 840.4 : CB$

$$\begin{array}{r} 840.4 \\ \hline 247728 \\ 247728 \\ 495456 \\ \hline CB = 520.476528 \end{array}$$

For the side AB, we have the following proportion:

$$1 : \cos.38° \; 16' = AC : AB$$

That is, . . $1 : .78513 = 840.4 : AB$

$$\begin{array}{r} 8404 \\ \hline 314052 \\ 314052 \\ 628104 \\ \hline AB = 659.823252 \end{array}$$

Before we go into logarithmic computation, it is important to say a word or two in relation to the nature of logarithms.

Logarithms are *exponential* numbers; and Algebra teaches us, that the addition of the exponents of like quantities multiplies the quantities, and the subtraction of the exponents divides the quantities.

Hence, by logarithms, we perform multiplication by addition, and division by subtraction.

EXPLANATION OF THE TABLES.

For the computation of logarithms, we refer at once to Algebra; here we shall point out the manner of finding them in the tables, and some of their uses. The logarithm of 1, is 0; of 10, is 1.00000; of 100, is 2.00000, &c. Hence, the logarithm of any number between 1 and 10, must be a *decimal;* between 10 and 100, must be 1 *and a decimal;* between 100 and 1000, must be 2 *and a decimal.* The whole number belonging to a logarithm, is called its *index.* The index is never put in the tables (except from 1 to 100, and need not be put there), because we always know what it is. It is always one less than the number of digits in the whole number. Thus, the number 3754 has 3 for the index to its logarithm, because the number consists of 4 digits; that is, the logarithm is 3, *and some decimal.*

The number 347.921 has 2 for the index of its logarithm, because the number is between 347 and 348, and 2 is the index for the logarithms of all numbers over 100, and less than 1000.

All numbers consisting of the same figures, whether integral, fractional, or mixed, have logarithms consisting of the same *decimal* part. The logarithms would differ only in their indices.

Thus, .	the number	7956.	has	3.900695	for its log.
	the number	795.6	has	2.900695	"
	the number	79.56	has	1.900695	"
	the number	7.956	has	0.900695	"
	the number	.7956	has	—1.900695	"
	the number	.07956	has	—2.900695	"

From this we perceive that we must take the logarithm out of the table for a mixed number or a decimal, the same as if the figures expressed an entire number; and then, to *prefix* the index, we must consider the *value* of the number.

The decimal part of a logarithm is always positive; but the index becomes negative when the number is a decimal; and the smaller the decimal, the greater the negative index.

To prefix the index to a decimal, count the decimal point as 1, and every cipher as 1, up to the first significant figure, and this is the negative index.

For example, find the logarithm of the decimal .0000831.

Num. 0000831 log. —5.919601

The point is counted one, and each of the ciphers is counted one; therefore the index is *minus five.*

The smaller the decimal, the greater the negative index; and when the decimal becomes 0, the logarithm is *negatively infinite.*

Hence, the logarithmic sine of 0° is *negatively infinite,* however great the radius.

The logarithm of any number consisting of four figures, or less, is taken out of the table directly, and without the least difficulty.

Thus, to find the logarithm of the number 3725, we find 372, at the side of the table, and run down the column marked 5 at the top, and we find opposite the former, and under the latter, .571126, for the decimal part of the logarithm.

Hence,	the logarithm of 3725	is	3.571126
	the logarithm of 37250	is	4.571126
	the logarithm of 37.25	is	1.571126, &c.

Find the logarithm of the number 834785.

This number is so large that we cannot find it in the table, but we can find the numbers 8347 and 8348. The logarithms of these numbers are the same as the logarithms of the numbers 834700 and 834800, except the indices.

	834700	log.	5.921530
	834800	log.	5.921582
Difference, . .	100		52

Now, our proposed number, 834785, is between the two preceding numbers; and, of course, its logarithm lies between the two preceding logarithms; and, without further comment, we may proportion to it thus, . . . 100 : 85=52 : 44.2

Or, . . . 1. : .85=52 : 44.2

To the logarithm . . .			5.921530
Add			44
Hence, the logarithm of	834785	is	5.921574
the logarithm of	8.34785	is	0.921574

From this we draw the following rule to find the log. of any number consisting of more than four places of figures.

RULE.—*Take out the logarithm of the four superior places, directly from the table, and take the difference between this logarithm and the next greater logarithm in the table. Multiply this difference by the inferior places of figures in the number, as a decimal.*

Example. Find the logarithm of 357.32514.

" the logarithm of 3573. decimal part is .553033

The difference between this and the next greater in the table, is 122.

The figures not included in the above logarithm, are

```
                      .2514
Multiply by  . . .     122
                     -----
                      5028
                     5028
                    2514
                    -------
                    30.6708
```

This result shows that 31 should be added to the decimal part of the logarithm already found; that is, the logarithm of the proposed number,

357.32514 is 2.553064

The logarithm of 357325.14 is 5.553064

We will now give the *converse* of this problem; that is, we give the decimal part of a logarithm, .553064, to find the figures corresponding.

The next less logarithm in the table, is .553033, corresponding to the figure 3573. The difference between our given logarithm and the one next less in the table, is 31; and the difference between two consecutive logarithms in this part of the table, is 122. Now divide 31 by 122, and write the quotient after the number 3573.

```
That is,  . . .  122)31. (254
                     244
                     ---
                      660
                      610
                      ---
                       500
                       488
                       ---
```

The figures, then, are 3573254, which corresponds to the decima logarithm .553064; and the value of these figures will, of course, depend on the index to the logarithm.

From this, we draw the following rule to find the number corresponding to a given logarithm.

RULE.—*If the given logarithm is not in the table, find the one next less, and take out the four figures corresponding; and if more than four figures are required, take the difference between the given logarithm and the next less in the table, and divide that difference by the difference of the two consecutive logarithms in the table, the one less, the other greater than the given logarithm; and the figures arising in the quotient, as many as may be required, must be annexed to the former figures taken from the table.*

EXAMPLES.

1. Given, the logarithm 3.743210, to find its corresponding number true to *three* places of decimals. *Ans.* 5536.182

2. Given, the logarithm 2.633356, to find its corresponding number true to two places of decimals. *Ans.* 429.89

3. Given, the logarithm —3.291746, to find its corresponding number. *Ans.* .0019577

TABLE II.

This table contains logarithmic sines and tangents, and natural sines and cosines. We shall confine our explanations to the logarithmic sines and cosines.

The sine of every degree and minute of the quadrant is given, directly, in the table, commencing at 0°, and extending to 45°, at the head of the table; and from 45° to 90°, at the foot of the table, increasing backward.

The same column that is marked sine, at the top, is marked cosine at the bottom; and the reason for this is apparent to any one who has examined the definitions of sines.

The difference of two consecutive logarithms is given, corresponding to *ten* seconds. Removing the decimal point one figure, will give the difference for *one* second; and if we multiply this difference by any proposed number of seconds, we shall have a difference corresponding to that number of seconds, above the logarithm, corresponding to the preceding degree and minute.

For example, find the sine of 19° 17′ 22″.

The sine of 19° 17′, taken directly from the table, is	9.518829
The difference for 10″ is 60.2; for 1″, is 6.02×22 . .	133
Hence, 19° 17′ 22″ sine is	9.518952

From this it will be perceived that there is no difficulty in obtaining the sine or tangent, cosine or cotangent, of any angle greater than 30′.

Conversely. Given the logarithmic sine 9.982412, to find its corresponding arc. The sine next less in the table, is 9.982404, and gives the arc 73° 48′. The difference between this and the given sine, is 8, and the difference for 1″, is .61; therefore, the number of seconds corresponding to 8, must be discovered by dividing 8 by the decimal .61, which gives 13. Hence, the arc sought is 73° 48′ 13″.

These operations are too obvious to require a rule. When the arc is very small, such arcs as are sometimes required in astronomy, it is necessary to be very accurate; and for that reason we omitted the difference for seconds for all arcs under 30′. Assuming that the sines and tangents of arcs under 30′ vary in the same proportion as the arcs themselves, we can find the sine or tangent of any very small arc to great accuracy, as follows:

The sine of 1′, as expressed in the table, is . .	6.463726
Divide this by 60; that is, subtract logarithm . .	1.778151
The logarithmic sine of 1″, therefore, is . . .	4.685575
Now, for the sine of 17″, add the logarithm of 17 .	1.230449
Logarithmic sine of 17″, is	5.916024

In the same manner we may find the sine of any other small arc.

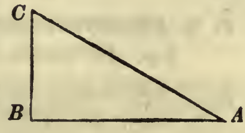
For example, find the sine of 14′ 21½″; that is, 861″5

To logarithmic sine of 1″, is, ...	4.685575
Add logarithm of 861.5	2.935255
Logarithmic sine of 14′ 21½″	7.620830

Without further preliminaries, we may now preceed to practical

EXAMPLES.

2. In a right angled triangle, ABC, given the base, AB, 1214, and the angle A, 51° 40′ 30″, to find the other parts.



To find BC.

As radius	. .	10.000000
: tan.A	51° 40′ 30″	10.102119
:: AB	1214 .	3.084219
: BC	1535.8 .	3.186338

N. B. When the first term of a logarithmic proportion is radius, the resulting logarithm is found by adding the second and third logarithms, rejecting 10 in the index, which is dividing by the first term.

In all cases we add the second and third logarithms together; which, in logarithms, is multiplying these terms together; and from that sum we subtract the first logarithm, whatever it may be, which is dividing by the first term.

To find AC.

As sin. C, or cos. A 51° 40′ 30″	.	9.792477
: AB 1214	.	3.084219
:: Radius .	.	10.000000
: AC 1957.7	.	3.291742

To find this resulting logarithm, we subtracted the first logarithm from the second, conceiving its index to be 13.

Let ABC represent any plane triangle, right angled at B.

1. Given AC 73.26, and the angle A 49° 12′ 20″; required the other parts? *Ans.* The angle C 40° 47′ 40″, BC 55.46, and AB 47.87.

2. Given AB 469.34, and the angle A 51° 26 17″, to find the other parts? *Ans.* The angle C 38° 33′ 43″, BC 588.7, and AC 752.9.

3. Given AB 493, and the angle C 20° 14′; required the remaining parts? *Ans.* The angle A 69° 46′, BC 1338, and AC 1425.

4. Let AB=331, the angle A=49° 14′; what are the other parts? *Ans.* AC 506.9, BC 383.9, and the angle C 40° 46′.

5. If AC=45, and the angle C=37° 22′, what are the remaining parts? *Ans.* AB 27.31, BC 35.76, and the angle A 52° 38′

6. Given AC 4264.3, and the angle A 56° 29′ 13″, to find the remaining parts. *Ans.* AB 2354.4, BC 3555.4, and the angle C 33° 30′ 47″.

7. If AB=44.2, and the angle A=31° 12′ 49″, what are the other parts? *Ans.* AC 49.35, BC 25.57, and the angle C 58° 47′ 11″.

8. If AB=8372.1, and BC=694.73, what are the other parts? *Ans.* AC 8400.9, the angle C 85° 15′, and the angle A 4° 45′.

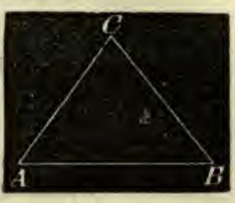
9. If AB be 63.4, and AC be 85.72, what are the other parts? *Ans.* BC 57.7, the angle C 47° 42′, and the angle A 42° 18′

10. Given AC 7269, and AB 3162, to find the other parts. *Ans.* BC 6546, the angle C 25° 47′ 7″, and the angle A 64° 12′ 53″.

11. Given AC 4824, and BC 2412, to find the other parts. *Ans.* The angle A 30° 00′, the angle C 60° 00′, and AB 4178

OBLIQUE ANGLED TRIGONOMETRY.

EXAMPLE 1.



In the triangle ABC, given $AB=376$, the angle $A=48° 3'$, and the angle $B=40° 14'$, to find the other parts.

As the sum of the three angles of every triangle is always 180°, the third angle, C, must be $180°—88° 17'=91° 43'$.

To find AC.

As sin.91° 43′	.	9.999805
: AB 376 .	.	2.575188
:: sin. B 40° 14′ .		9.810167
		12.385355
: AC 243 .	.	2.385550

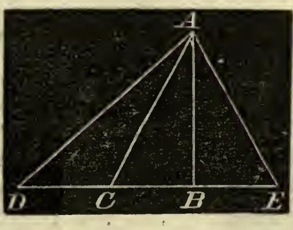
Observe, that the sine of 91° 43′ is the same as the cosine of 1° 43′.

To find BC.

As sin.91° 43′	.	9.999805
: AB 376	.	2.575188
:: sin.A48° 3′	.	9.871414
		12.446602
: BC 279.8 .	.	2.446797

EXAMPLE 2.

In a plane triangle, given two sides, and an angle opposite one of them, to determine the other parts.



Let $AD=1751$. feet, one of the given sides. The angle $D=31° 17\ 19''$, and the side opposite, 1257.5. From these data, we are required to find the other side, and the other two angles.

In this case we do not know whether AC or AE represents 1257.5, because $AC=AE$. If we take AC for the other given side, then DC is the other required side, and DAC is the vertical angle. If we take AE for the other given side, then DE is the required side, and DAE is the vertical angle ; but in such cases we determine both triangles.

To find the angle $E=C$.

(Prop. 4.) As	$AC=AE=1257.5$	log.	3.099508
:	D 31° 17′ 19″	sin.	9.715460
::	AD 1751 .	log.	3.243286
			12.958746
$E=C$:	46° 18′ . .	sin.	9.859238

From 180° take 46° 18′, and the remainder is the angle DCA $=133°$ 42′.

The angle $DAC=ACE-D$ (th. 11, b. 1); that is,

$DAC=46°\ 18'-31°\ 17'\ 19''=15°\ 0'\ 41''$

The angles D and E, taken from 180°, give $DAE=102°$ 24′ 41″.

To find DC.

As	sin.D 31° 17′ 19′	log.	9.715460
:	AC 1257.5 .	log.	3.099508
::	sin.DAC 15° 0′ 41″	log.	9.413317
			12.512825
:	DC 626.86 . .		2.797165

To find DE.

As	sin.D 31° 17′ 17″ .	9.715460
:	AE 1257.5 . .	3.099508
::	sin.102° 24′ 41′ . .	9.989730
		13.089238
:	DE 2364.7 . .	3.373778

N. B. To make the triangle possible, AC must not be less than AB, the sine of the angle D, when DA is made radius.

EXAMPLE 3.

In any plane triangle, given two sides and the included angle, to find the other parts.

Let $AD=1751$ (see last figure), $DE=2364.5$, and the included angle $D=31°$ 17′ 19″. We are required to find DE, the angle DAE, and angle E. Observe that the angle E must be less than the angle DAE, because it is opposite a less side.

From	180°
Take D	31° 17′ 19″
Sum of the other two angles	$=148°$ 42′ 41″ (th. 11, b. 1)
$\frac{1}{2}$ sum	$=$ 74° 21′ 20″

By proposition 7,

$$DE+DA : DE-DA = \tan.74^\circ\ 21'\ 20'' : \tan.\tfrac{1}{2}(DAE-E)$$

That is,

$$4115.5 : 613.5 = \tan.74^\circ\ 21'\ 20'' : \tfrac{1}{2}(DAE-E)$$

Tan.74° 21′ 20″ .	10.552778
613.5 . .	2.787815
	13.340593
4115.5 log. (sub.)	3.614423
$\frac{1}{2}(DAE-E)$ tan.28° 1′ 36″	9.726170

But the half sum and half difference of any two quantities are equal to the greater of the two ; and the half sum, less the half difference, is equal the less.

Therefore, to	74°	21′	20″
Add .	28	1	36
$DAE=$	102°	22′	56″
$E=$	46	19	44

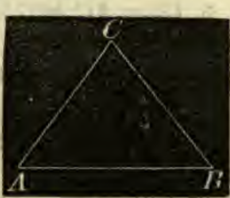
To find AE.

As sin.E 46° 19′ 44″ .	9.859323
: DA 1751 . . .	3.243286
:: sin.D 31° 17′ 19″ .	9.715460
	12.958746
: AE 1257.2 . . .	3.099423

EXAMPLE 4.

Given the three sides of a plane triangle to find the angles.

Given $AC=1751$, $CB=1257.5$, $AB=2364.5$



If we take the formula for cosines, we will compute the greatest angle, which is C. To correspond with the formula,

$$\cos.\tfrac{1}{2}C=\sqrt{\frac{R^2s(s-c)}{ab}}$$ we must take $a=1257.5$ $b=1751$, and $c=2364.5$

The half sum of these is, $s=2686.5 \cdot s-c=322$

R^2 . .	20.000000
$s=2686.5$.	3.429187
$s-c=322$.	2.507856
Numerator, log.	25.937043

R^2	.	20.000000
s=2686.5	.	3.429187
$s-c$=322	.	2.507856
Numerator, log.		25.937043
a 1257.5	3.099508	
b 1751.	3.243286	
Denominator, log.	6.342794	6.342794
		2)19.594249
$\frac{1}{2}C$= 51° 11′ 10″	cos.	9.797124
C=102 22 20		

The remaining angles may now be found by problem 4.

We give the following examples for practical exercises:

Let ABC represent any oblique angled triangle.

1. Given AB 697, the angle A 81° 30′ 10″, and the angle B 40° 30′ 44″, to find the other parts.

Ans. AC 534, BC 813, and the angle C 57° 59′ 6″.

2. If AC=720.8, the angle A=70° 5′ 22″, and the angle B=59° 35′ 36″, required the other parts.

Ans. AB 643.2, BC 785.8, and the angle C 50° 19′ 2″.

3. Given BC 980.1, the angle A 7° 6′ 26″, and the angle B 106° 2′ 23″, to find the other parts.

Ans. AB 7284, AC 7613.3, and the angle C 66° 51′ 11″.

4. Given AB 896.2, BC 328.4, and the angle C 113° 45′ 20″, to find the other parts.

Ans. AC 712, the angle A 19° 35′ 48″, and the angle B 46° 38′ 52″.

5. Given AC 4627, BC 5169, and the angle A 70° 25′ 12″, to find the other parts.

Ans. AB 4328, the angle B 57° 29′ 56″, and the angle C 52° 4′ 52″.

6. Given AB 793.8, BC 481.6, and AC 500.0, to find the angles.

Ans. The angle A 35° 15′ 32″, the angle B 36° 49′ 18″, and the angle C 107° 55′ 10″.

7. Given AB 100.3, BC 100.3, and AC 100.3, to find the angles.

Ans. The angle A 60°, the angle B 60°, and the angle C 60°.

8. Given AB 92.6, BC 46.3, and AC 71.2, to find the angles.

Ans. The angle A 29° 17′ 22″, the angle B 48° 47′ 31″, and the angle C 101° 55′ 8″.

9. Given AB 4963, BC 5124, and AC 5621, to find the angles.

Ans. The angle A 57° 30′ 28″, the angle B 67° 42′ 36″, and the angle C 54° 46′ 56″.

10. Given AB 728.1, BC 614.7, and AC 583.8, to find the angles.

Ans. The angle A 54° 32′ 52″, the angle B 50° 40′ 58″, and the angle C 74° 46′ 10″.

11. Given AB 96.74, BC 83.29, and AC 111.42, to find the angles.

Ans. The angle A 46° 30′ 45″, the angle B 76° 3′ 45″, and the angle C 57° 25′ 30′.

12. Given AB 363.4, BC 148.4, and the angle B 102° 18′ 27″, to find the other parts.

Ans. The angle A 20° 9′ 17″, the side $AC. = 420.8$, and the angle C 57° 32′ 16″.

13. Given AB 632, BC 494, and the angle A 20° 16′, to find the other parts, C being acute.

Ans. The angle C 26° 18′ 19″, the angle B 133° 25′ 41″, and AC 1035.86.

14. Given AB 53.9, AC 46 21, and the angle B 58°16′, to find the other parts.

Ans. The angle A 38° 58′, the angle C 82° 46, and BC 34,16.

15. Given AB 2163, BC 1672, and the angle C 112° 18′ 22″, to find the other parts.

Ans. AC 877.2, the angle B 22° 2′ 16″, and the angle A 45° 39′ 22″.

16. Given AB 496, BC 496, and the angle B 38° 16′, to find the other parts.

Ans. AC 325.1, the angle A 70° 52′ and the angle C 70° 52′.

17. Given AB 428, the angle C 49° 16′, and $(AC+BC)$ 918, to find the other parts, the angle B being obtuse.

Ans. The angle A 38° 44′ 48″, the angle B 91° 59′ 12″, AC 564.49, and BC 353.5.

18. Given AC 126, the angle B 29° 46′, and $(AB—BC)$ 43, to find the other parts.

Ans. The angle A 55° 51′ 32″, the angle C 94° 22′ 28″, AB 253.05, and BC 210°.54.

19. Given AB 1269, AC 1837, and the angle A 53° 16′ 20″, to find the other parts.

Ans. The angle B 83° 23′ 47″, the angle C 43° 19′ 53″, and BC 1482.16.

APPLICATION OF TRIGONOMETRY TO MEASURING THE HIGHT AND DISTANCES OF VISIBLE OBJECTS.

In this useful application of trigonometry, a base line is always supposed to be measured, or given in length; and by means of a quadrant, sextant, circle, theodolite, or some other instrument for measuring angles, such angles are measured as connected with the base line, and the objects whose hights or distances it is proposed to determine, enable us to compute, from the principles of trigonometry, what those hights or distances are.

Sometimes, particularly in marine surveying, horizontal angles are determined by the compass; but the varying effect of surrounding bodies on the needle, even in situations little removed from each other, and the general construction of the instrument itself, render it unfit to be applied in the determination of angles where anything like precision is required.

The following examples present sufficient variety to guide the student in determining what will be the most eligible mode of proceeding in any case that is likely to occur in practice.

EXAMPLE 1.

Being desirous of finding the distance between two distant objects, C and D, I measured a base AB, of 384 yards, on the same horizontal plane with the objects C and D. At A, I found the angle DAB=48° 12′, and CAB=89° 18′; at B the angle ABC was 46° 14′, and ABD 87° 4′. It is required from these data to compute the distance between C and D.

From the angle CAB, take the angle DAB; the remainder, 41° 6′, is the angle CAD. To the angle DBA, add the angle DAB, and 44° 44′, the supplement of the sum, is the angle ADB. In the same way the angle ACB, which is the supplement of the sum of CAB and CBA, is found to be 44° 28′. Hence, in the triangles ABC and ABD, we have

As sin. ACB 44° 28	.	9.845405
: AB 384 yards .	.	2.584331
:: sin. ABC 46° 14′	.	9.858635
		12.442996
. AC 395.9 yards .	.	2.597561

As sin. ADB 44° 44′	.	9.847454
: AB 384 yards	. .	2.584331
:: sin. ABD 87° 4′ .	.	9.999431
		12.583762
: AD 544.9 yards .	.	2.736308

Then, in the triangle CAD, we have given the sides CA and AD, and the included angle CAD, to find CD; to compute which we proceed thus:

The supplement of the angle CAD is the sum of the angles ACD, and ADC;

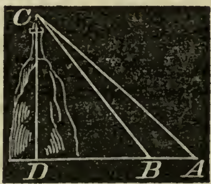
Hence, . $\frac{ACD+ADC}{2}=69°\ 27'$, and, by proportion we have,

As $AD+AC$	.	940.8	2.937497
: $AD-AC$	. .	149	2.173186
:: tan. $\frac{ACD+ADC}{2}$		69° 27′	10.426108
			12.599294
: tan. $\frac{ACD-ADC}{2}$		22 54	9.625797
the angle ACD	sum	92 21	
the angle ADC .	diff.	46 33	

As sin. ADC 46° 33′	.	9.860922
: AC 395.9 yards .	.	2.597585
:: sin. CAD 41° 6′	.	9.817813
		12.415398
: CD 358.5 yards .	.	2.554476

EXAMPLE 2.

To determine the altitude of a lighthouse, I observed the elevation of its top above the level sand on the seashore, to be 15° 32′ 18″, and measuring directly from it, along the sand 638 yards, I then found its elevation to be 9° 56′ 26″; required the hight of the lighthouse.



Let CD represent the hight of the lighthouse above the level of the sand, and let B be the first station, and A the second; then the angle CBD is 15° 32′ 18″, and the angle CAB is 9° 56′ 26″; therefore, the angle ACB, which is the difference of the angles CBD and CAB, is 5° 35′ 52″.

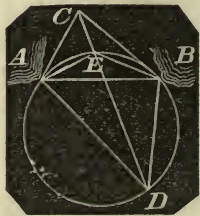
Hence,	As sin. ACB 5° 35′ 52″ .	8.989201
	: AB 638 . . .	2.804821
	:: sin. angle A 9° 56′ 26″	9.237107
		12.041928
	: BC 1129.06 yards .	3.052727

As radius	10.000000
: BC 1129.06 . .	3.052727
:: sin. CBD 15° 32′ 18″ .	9.427945
	12.480672
: DC 302.46 yards .	2.480672

EXAMPLE 3.

Coming from sea, at the point D, I observed two headlands, A and B, and inland, at C, a steeple, which appeared between the headlands. I found, from a map, that the headlands were 5.35 from each other; that the distance from A to the steeple was 2.8 miles, and from B to the steeple 3.47 miles; and I found with a sextant, that the angle ADC was 12° 15′, and the angle BDC 15° 30′. Required my distance from each of the headlands, and from the steeple.

CONSTRUCTION.



The angle between the two headlands is the sum of 15° 30′ and 12° 15′, or 27° 45′. Take the double, 55° 30′. Conceive AB to be the chord of a circle, and the segment on one side of it to be 55° 30; and, of course, the other will be 304° 30′. The point D will be somewhere in the circumference of this circle. Consider that point as determined, and join CD.

In the triangle ABC we have all the sides, and, of course, we can find all the angles; and if the angle ACB is less than (180°—(27° 45′))=152° 15′, then the circle cuts the line CD, in a point E, and C is without the circle.

Join AE, BE, AD, and DB. $AEBD$ is a quadrilateral in a circle, and $AEB+ADB=180°$.

The angle $ADE=$ the angle ABE, because both are measured by half the arc AE. Also, $EDB=EAB$, for a similar reason.

Now, in the triangle AEB, its side AB, and all its angles, are known; and from thence AE can be computed. Then, having the two sides AC and AE of the triangle AEC, and the included angle CAE, we can find the angle AEC, and, of course, its supplement, AED. Then, in the triangle AED we have the side AE, and the two angles AED and ADE, from which we can find AD.

The computation, at length, is as follows:

To find AE.

angle EAB	15° 30′	As sin.AEB	152° 15′	9.668027
angle EBA	12 15	: AB	5.35	.728354
	27 45	:: sin.ABE	12° 15′	9.326700
	180 0			10.855054
angle AEB	152 15	: AE	2.438	.387027

To find the angle BAC.

BC	3.47		
AB	5.35	log.	.728354
AC	2.80	log.	.447158
	2)11.62		1.175512
	5.81	log.	.764176
BC	2.34	log.	.369216
			20
			21.133392
			2)19.957880
	17° 41′ 58″	cos.	9.978940
	2		
angle BAC	35 23 56		
angle EAB	15 30		
angle EAC	19 53 56		
	180		
	2)160 6 4		
	80 3 2	$\frac{AEC+ACE}{2}$	

To find the angles AEC and ACE.

As $AC+AE$	5.238	.719165
: $AC-AE$	.362	—1.558709
.: tan.$\frac{AEC+ACE}{2}$	80° 3′ 2″	10.755928
		10.314637
tan.$\frac{AEC-ACE}{2}$	21 30 12	9.595472

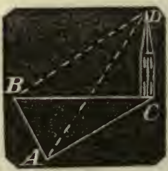
angle AEC	101° 33′ 14″ sum	
angle ACE or ACD	58 32 50 diff.	
angle CDA	12 15	
	70 47 50 supplement	109° 12′ 10″ angle CAD
		35 23 56 angle CAB
		73 48 14 angle BAD

To find AD.

As sin.ADC 12° 15′ .	9.326700
: AC 2.8 . . .	.447158
:: sin.ACD 58° 32′ 50″	9.930985
	10.378143
: AD 11.26 miles .	1.051443

EXAMPLE 4.

The elevation of a spire at one station was 23° 50′ 17″, and the horizontal angle at this station, between the spire and another station, was 93° 4′ 20″. The horizontal angle at the latter station, between the spire and the first station, was 54° 28′ 36″, and the distance between the two stations, 416 feet. Required the hight of the spire.



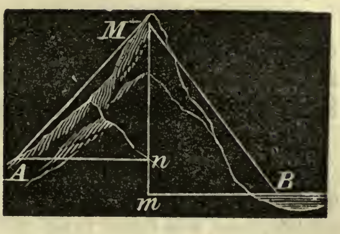
Let CD be the spire, A the first station, and B the second; then the vertical angle CAD is 23° 50′ 17″; and as the horizontal angles CAB and CBA are 93° 4′ 20″, and 54° 28′ 36″ respectively, the angle ACB, the supplement of their sum, is 32° 27′ 4″.

To find AC.

As sin.BCA 32° 27′ 3″ .	9.729634
: side AB 416 . .	2.619093
:: sin.ABC 54° 28′ 36″ .	9.910560
	12.529653
: side AC 631 . .	2.800019

To find DC.

As radius	10.000000
: side AC 631 . .	2.800019
:: tan.DAC 23° 50′ 17″ .	9.645270
: DC 278.8 . .	2.445289



By the application of the fourth example we can compute the different elevations of different planes, provided the same object is visible from them.

For example, let M be a prominent tree or rock near the top of a mountain, and by observations taken at A, we can determine the perpendicular Mn. By like observations we can determine the perpendicular Mm. The difference between these two perpendiculars, is nm, or the difference in the elevation between the two points A and B. But if the distances between A and n, or B and m, are considerable, or more than two or three miles, corrections must be made for the convexity of the earth ; but for less distances such corrections are not necessary.

EXAMPLES FOR EXERCISE.

1. Required the hight of a wall whose angle of elevation is observed, at the distance of 463 feet, to be 16° 21′ ? *Ans.* 135.8 feet.

2. The angle of elevation of a hill is, near its bottom, 31° 18′, and 214 yards further off, 26° 18′. Required the perpendicular hight of the hill, and the distance of the perpendicular from the first station.

Ans. The hight of the hill is 565.2, and the distance of the perpendicular from the first station, is 929.6.

3. The wall of a tower which is 149.5 feet in hight, makes, with a line drawn from the top of it to a distant object on the horizontal plane, an angle of 57° 21′. What is the distance of the object from the bottom of the tower ? *Ans.* 233.3 feet.

4. From the top of a tower, whose hight was 138 feet, I took the angles of depression of two objects which stood in a direct line from the bottom of the tower, and upon the same horizontal plane with it. The depression of the nearer object was found to be 48° 10′, and that of the further, 18° 52′. What was the distance of each from the bottom of the tower ?

Ans. Distance of the nearer 123.5, and of the farther 403.8 feet.

5. Being on the side of a river, and wishing to know the distance of a house on the other side, I measured 312 yards in a right line by the side of the river, and then found that the two angles, one at each end of this line, subtended by the other end and the house, were 31° 15′ and 86° 27′. What was the distance between each end of the line and the house ? *Ans.* 351.7, and 182.8 yards.

6. Having measured a base of 260 yards in a straight line, close by one side of a river, I found that the two angles, one at each end of the line, subtended by the other end and a tree close to the opposite bank, were 40° and 80°. What was the breadth of the river?

Ans. 190.1 yards.

7. From an eminence of 268 feet in perpendicular hight, the angle of depression of the top of a steeple which stood on the same horizontal plane, was found to be 40° 3′, and of the bottom 56° 18′. What was the hight of the steeple? *Ans.* 117.8 feet.

8. Wanting to know the distance between two objects which were separated by a morass, I measured the distance from each to a point where I could see them both; the distances were 1840 and 1428 yards, and the angle which, at that point, the objects subtended, was 36° 18′ 24″. Required their distance. *Ans.* 1090.85 yards.

9. From the top of a mountain, three miles in hight, the visible horizon appeared depressed 2° 13′ 27″. Required the diameter of the earth, and the distance of the boundary of the visible horizon.

Ans. Diameter of the earth 7958 miles, distance of the horizon 154.54 miles.

10. From a ship a headland, was seen bearing north, 39° 23′ east. After sailing 20 miles north, 47° 49′ west, the same headland was observed to bear north, 87° 11′ east. Required the distance of the headland from the ship at each station?

Ans. The distance at the first station was 19.09, and at the second 26.96 miles.

11. The top of a tower, 100 feet above the level of the sea, was seen as on the surface of the sea, from the masthead of a ship, 90 feet above the water. The diameter of the earth being 7960 miles, what was the distance between the observer and the object?

Ans. 23.9 plus $\frac{1}{13}$ for refraction = 25.7 miles.

12. From the top of a tower, by the seaside, of 143 feet high, it was observed that the angle of depression of a ship's bottom, then at anchor, measured 35°; what, then, was the ship's distance from the bottom of the wall? *Ans.* 204.22 feet.

13. Wanting to know the breadth of a river, I measured a base of 500 yards in a straight line close by one side of it; and at each end of this line I found the angles subtended by the other end and a tree close to the bank on the other side of the river, to be 53° and 79° 12′. What then, was the perpendicular breadth of the river? *Ans.* 529.48 yards.

14. What is the perpendicular hight of a hill, its angle of elevation taken at the bottom of it, being 46°, and 200 yards further off, on a level with the bottom, the angle was 31°? *Ans.* 286.28 yards.

15. Wanting to know the hight of an inaccessible tower; at the least distance from it, on the same horizontal plane, I took its angle of elevation equal to 58°; then going 300 feet directly from it, found the angle there to be only 32°; required its hight, and my distance from it at the first station.

Ans. $\begin{cases} \text{Hight} & 307.53. \\ \text{Distance} & 192.15. \end{cases}$

16. Two ships of war, intending to cannonade a fort, are, by the shallowness of the water, kept so far from it, that they suspect their guns cannot reach it with effect. In order, therefore, to measure the distance, they separate from each other a quarter of a mile, or 440 yards, then each ship observes and measures the angle which the other ship and fort subtends, which angles are 83° 45′ and 85° 15′. What, then, is the distance between each ship and the fort?

Ans. $\begin{cases} 2292.26 \\ 2298.05 \end{cases}$ yards.

17. A point of land was observed by a ship, at sea, to bear east-by-south;* and after sailing north-east 12 miles, it was found to bear south-east-by-east. It is required to determine the place of that headland, and the ship's distance from it at the last observation?

Ans. 26.0728 miles.

18. Wanting to know my distance from an inaccessible object, 0, on the other side of a river; and having no instrument for taking angles, but only a chain or chord for measuring distances; from each of two stations, A and B, which were taken at 500 yards asunder, I measured in a direct line from the object 0, 100 yards, viz., AC and BD, each equal to 100 yards; also the diagonal AD measured 550 yards, and the diagonal BC 560. What, then, was the distance of the object 0 from each station A and B?

Ans. $\begin{cases} A0 & 536.25. \\ B0 & 500.09. \end{cases}$

19. A navigator found, by observation, that the vertex of a certain mountain, which he supposed to be 45 minutes of a degree distant, had an altitude above the sea horison of 31′ 20″. Now, on the supposition that the earth's radius is 3956 miles, and the observer's *dip* was 4′ 15″, what was the hight of the mountain? *Ans.* 3960 feet.

N. B. This should be diminished by about its one-eleventh part for the influence of horizontal refraction.

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* That is, one point south of east. A point of the compass is 11° 15.

SPHERICAL TRIGONOMETRY.

Spherical Geometry is nothing more than the general principles of geometry applied to the various sections of a sphere; and spherical trigonometry, is but the general principles of plane trigonometry applied to triangles resting on a surface of a sphere, and the planes of the sides of the triangles passing through the center of the sphere.

DEFINITIONS.

1. A sphere is a solid whose surface is equally convex in every part, and every point of the surface is equally distant from one point within, and this point is called the center. A sphere may be conceived to be generated by the revolution of a semicircle about its diameter.

If the center of the semicircle rests at the same point, the position of the diameter may be in any direction or position, and the revolution of the semicircle will describe the same sphere.

2. Any plane that passes through the center of the sphere, divides the solid and the surface into two equal parts.

3. Any two planes that pass through the center of a sphere, intersect each other on the opposite points of the sphere, because the section of any two planes is a right line (th. 2, b. 6).

4. A great circle on a sphere, is one whose plane passes through the center of the sphere.

5. Every great circle has poles, two points on the sphere directly opposite to each other and equally distant from every point on the great circle.

The distance from any pole to its equator in *any direction*, is one fourth of the whole distance round the sphere.

6. Any point on a sphere may be a pole to *some great circle*.

7. A spherical triangle is formed by the intersection of three great circles on a sphere. Conceive three radii drawn from the three angular points to the center of the sphere, thence forming a solid angle. The angles of the three planes which form this solid angle at the center, are the three angles which measure the sides of the triangle, and the inclination of these planes to each other form the angles of the triangle.

8. The complete measure of a spherical triangle, is but the complete measure of a solid angle at the center of a sphere; and this solid angle is the same, whatever be the radius of the sphere.

9. Every great circle, or portion of a great circle on the surface of a sphere, has its poles; conversely, every pole, or the point where two circles intersect, has *its equator* 90° distant, and the portion of this equator between the two sides, or the two sides produced, measures the spherical angle at the pole.

The inclination of two tangents of two arcs formed at their point of intersection, also measures the spherical angle. (Def. 5, to b. 6).

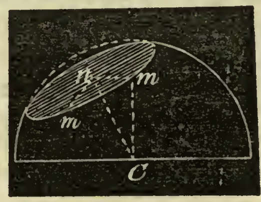
10. We can always draw one, and only one great circle through any two points on the surface of a sphere; for the two given points and the center of the sphere, give three points, and through three points only one plane can be made to pass (cor. th. 1, b. 6).

PROPOSITION 1.

Every section of a sphere by a plane is a circle.

If the plane passes through the center of the sphere, the section is evidently a circle, for every point on the surface of the sphere is equally distant from the center. These sections are great circles, and all great circles on the same sphere are equal to each other.

Now let the cutting plane not pass through the center. From the center C, let fall Cn perpendicular to the plane; and when a line is perpendicular to a plane, it is perpendicular to all lines that can be drawn in that plane (th. 3, b. 6); therefore, any line as nm in the plane, is at right angles to Cn. Hence $nm=\sqrt{Cm^2-Cn^2}$.



But nm is any line in the plane, from the point n to the surface of the sphere, and this value for nm is invariable, and it is the radius of a circle whose center is n.

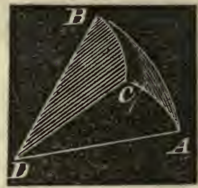
N. B. These circles are called small circles, and are greater or less, as they are nearer or more remote from the center C.

Small circles on a sphere, are never considered as sides of spherical triangles. We again repeat, that sides of spherical triangles must be portions of *great* circles, and each side must be less than 180°.

PROPOSITION 2.

Any two sides of a spherical triangle are together greater than the third.

Let AB, AC, and BC, be the three sides of the triangle, and D the center of the sphere.

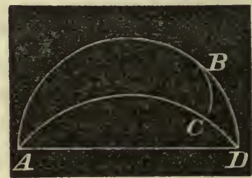


The arcs AB, AC, and BC, are measured by the angles of the planes that form the solid angle at D. But any two of these angles are together greater than the third (th. 10, b. 6). Therefore, any two sides of the triangle are together, greater than the third. *Q. E. D.*

PROPOSITION 3.

The sum of the three sides of any spherical triangle is less than the circumference of a great circle.

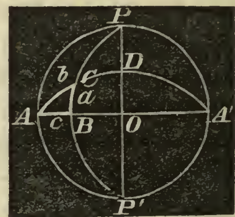
Let ABC be a triangle; the two sides AB, AC, produced, will meet at the point on the sphere which is directly opposite to A; and the arcs ABD, and ACD, are together equal to a great circle. But by the last proposition, BC is less than the two arcs BD and DC. Therefore, AB, BC, and AC, are together less than $ABD+ACD$; that is, less than a great circle. *Q. E. D.*



PROPOSITION 4.

Every right angled spherical triangle must have a complemental, supplemental, and four quadrantal triangles in the same hemisphere.

Let ABC, be a right angled spherical triangle, right angled at B.



Produce the sides AB and AC, and they will meet at A', the opposite point on the sphere. Produce BC, both ways, 90° from the point B, to P and P', which are therefore, poles to the arc AB (def. 9, spherics). Through A, P, and the center of the sphere, pass a plane cutting the sphere into two equal parts, forming a great circle on the sphere, which great circle will be represented by the plane circle $PAP'A$ on the paper. At right angles to this plane, pass another plane, cutting the sphere into two equal parts; this great circle is represented on the paper, by the straight line POP'. A and A', are the poles to the great circle POP'. P and P', are the poles to the great circle ABA'.

As PC, PD and CD, are portions of great circles on a sphere, CPD is a spherical triangle, and it is *complemental* to the given triangle ABC; because CD is the complement of AC, CP the complement of BC, and PD is the complement of DO, or of the angle A. Again, the triangle $A'BC$, is *supplemental* to ABC, because $A'=A$; $A'C$ is the supplement of AC, and $A'B$ is the supplement of AB. ACP is a spherical triangle, and one of its sides, AP, is a quadrant, and it is therefore called a quadrantal triangle. So also, are the triangles $A'CP$, ACP', and $P'CA'$, quadrantal triangles.

Cor. In every triangle there are *six* elements; three sides and three angles, which are sometimes called parts.

Now, if all the parts of the triangle ABC are known, the parts of the complemental triangle PCD, are also known, and the supplemental triangle $A'BC$, must be as completely known.

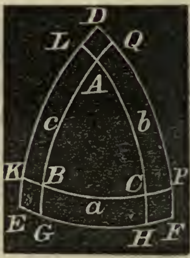
When the triangle PCD is known, the triangles ACP and $A'PC$ are also known, for the side PD, measures the angles PAC and $PA'C$, and the angle CPD, added to the right angle $A'PD$, gives the angle $A'PC$, and CPA, is supplemental to this. Hence a solution of any right angled spherical triangle, is a solution to its complemental, supplemental, and all its quadrantal triangles.

Definition. Every triangle, together with its supplemental triangle, form what is called a *Lune*. Thus, the triangles ABC, and $A'BC$, form a lune; PCD and $P'CD$, form a lune; PAC and $P'AC$, also form a lune.

It is obvious, that the surface of the lune $PAP'B$, is to the surface of the sphere, as the arc AB, is to the whole circumference.

PROPOSITION 5.

If there be three arcs of great circles whose poles are the angular points of a spherical triangle, such arcs, if produced, will form another triangle, whose sides will be supplemental to the angles of the first triangle, and the sides of the first triangle will be supplemental to the angles of the second



Let the arcs of the three great circles be GH, PQ, KL, whose poles are respectively A, B, and C. Produce the three arcs until they meet in E, D, and F. *We are now to show,* that E is the pole to the great circle AC; D the pole of the great circle BC; F the pole to the great circle AB. Also, that the side EF, is supplemental to the angle A; ED to the angle C; and DF to the angle B; and also, that, the side AC, is supplemental to the angle E, &c.

Any pole is 90° from any point on its great circle, and therefore, as A is the pole to the great circle GH, the point A, is 90° from the point E. As C is the pole of the great circle LK, C is 90° from any point in that great circle; therefore, C is 90° from the point E, and E, being 90° from both A and C, it is the pole of the arc AC. In the same manner, we may prove that D is the pole of BC, and F the pole of AB.

Because A is the pole of the arc GH, the arc GH measures the angle A (def. 9 spherics); for the same reason, PQ measures the angle B, and LK measures the angle C.

Because E is the pole of the arc AC, $EH=90°$

Or, ... $EG+GH=90°$

For a like reason, . . $FH+GH=90°$

Adding these two equations, and observing that $GH=A$, and afterward transposing one A, we have,

$$EG+GH+FH=180°-A.$$

Or, ... $EF=180°-A$

In like manner, ... $FD=180°-B$ (a)

And, ... $ED=180°-C$

But the arc $(180°-A)$, is a supplemental arc to A, by the definition of arcs; therefore, the three sides of the triangle EDF, are supplements of the angles A, B, C, of the triangle ABC.

Again, as E, is the pole of the arc AC, the whole angle E, is measured by the whole arc LH.

But, ... $AC+CH=90°$

Also, ... $AC+AL=90°$

By addition, . . $AC+AC+CH+AL=180°$

By transposition, . $AC+CH+AL=180°-AC$

$$\left.\begin{array}{ll}\text{That is, ... } & LH, \text{ or } E=180°-AC \\ \text{In the same manner, . . } & F=180°-AB \\ \text{And, ... } & D=180°-BC\end{array}\right\} (b)$$

That is, the sides of the first triangle, are supplemental to the angles of the second triangle. *Q. E. D.*

PROPOSITION 6.

The sum of the three angles of any spherical triangle, is greater than two right angles, and less than six right angles.

Turn to equations (a), of the last proposition, and add them together. The first member of the equation so formed will be the sum of three sides of a spherical triangle, which sum we may designate by S. The other member will be 6 right angles (there being 2 right angles in each 180°) less the three angles A, B, and C.

That is, . . $S=6$ right angles$-(A+B+C)$

By proposition 3, the sum S, is less than 4 right angles; therefore, to it add s, a sufficient quantity to make 4 right angles.

Then, 4 right angles$=6$ right angles$-(A+B+C)+s$

Drop 4 right angles from both members, and transpose $(A+B+C)$

Then, . $A+B+C=2$ right angles$+s$

That is, the three angles of a spherical triangle, make a greater sum than two right angles by the indefinite quantity s, which quantity is called the *spherical excess*, and is greater or less according to the size of the triangle.

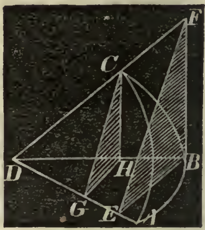
Again the sum of the angles is less than 6 right angles. There are but *three* angles to any triangle, and no one of them can come up to 180°, or 2 right angles. For an angle is the inclination of two lines or two planes; and when two planes incline by 180°, the planes are parallel, or are in one and the same plane; therefore, as neither angle can equal 2 right angles, the three can never equal 6 right angles. *Q. E. D.*

Scholium. By merely inspecting the figure to proposition 4, we perceive that the triangle PAB, has two right angles; one at A, the other at B, besides the third angle APB.

The triangle $P'A'O$, has 3 right angles. The triangle $A'P'C$, has two of its angles, each greater than a right angle.

PROPOSITION. 7.

With the sines of the sides, and the tangent of ONE SIDE *of any right angled spherical triangle, two plane triangles can be formed that will be similar, and similarly situated.*



Let ABC, be a spherical triangle, right angled at B; and let D be the center of the sphere. Because the angle CBA, is a right angle, the plane CDB, is perpendicular to the plane DBA. From C, let fall CH, perpendicular to the plane DBA, and as the plane CBD is perpendicular to the plane DBA, CH will lie in the plane CBD, and be perpendicular to the line DB, and perpendicular to all lines that can be drawn in the plane DBA, from the point H (th. 3, b. 6).

Draw HG perpendicular to DA, and join GC; GC will lie wholly in the plane CDA (def. of planes), and CHG is a right angled triangle, right angled at H.

We will now demonstrate that the angle DGC, *is a right angle.*

The right angled $\triangle$ CHG, gives $CH^2+HG^2=CG^2$ (1)

The right angled $\triangle$ DGH, gives $DG^2+HG^2=DH^2$ (2)

By subtraction, . . . $CH^2-DG^2=CG^2-DH^2$ (3)

By transposition, . . . $CH^2+DH^2=CG^2+DG^2$ (4)

But the first member of the equation (4), is equal to CD^2; because CDH, is a right angled triangle;

Therefore, ... $CD^2=GC^2+DG^2$

Hence, CD, is the hypotenuse to the right angled triangle DGC (th. 36, b. 1).

From the point B, draw BE at right angles to DA, and BF at right angles to DB, in the plane CDB extended; the point F being in the line DC. Join EF, and as F is in the plane CDA, and E is in the same plane, the line EF, is in the plane CDA. *Now we are to show, that the triangle* CHG *is similar, and similarly situated to the triangle* BEF.

As HG and BE are both at right angles to DA, they are parallel; and as CH and BF are both at right angles to DB, they are parallel; and by reason of the parallels, the angles GHC and EBF, are equal; but GHC is a right angle; therefore, EBF is also a right angle.

Now as GH and BE are parallel, and CH and BF parallel, we have,

$$DH : DB = HG : BE$$

And, . . . $DH : DB = HC : BF$

Therefore, . . $HG : BE = HC : BF$ (th. 6, b. 2)

Or, ... $HG : HC = BE : BF$

Here, then, are two triangles, having an angle in the one equal to an angle in the other, and the sides about the equal angles proportional; the two triangles are therefore equiangular (th. 20, b. 2); and they are similarly situated, for their sides make equal angles at H and B with the same line, DB. *Q. E. D.*

Scholium. By the definition of sines, cosines, and tangents, we perceive, that CH is the sine of the arc BC, DH is its cosine, and BF its tangent; CG is the sine of the arc AC, and DG its cosine. Also, BE is the sine of the arc AB, and DE is the cosine of the same arc. With this figure we are prepared to demonstrate the following theorems.

PROPOSITION 8. THEOREM 1.

In any right angled spherical triangle, the sine of one side is to the tangent of the other side, as radius is to the tangent of the angle adjacent to the first-mentioned side.

Or, as the sine of one side is to the tangent of the other side, so is the cotangent of the angle, adjacent to the first-mentioned side, to the radius.

In the right angled plane triangle EBF, we have,

$$EB : BF = R : \tan.BEF$$

That is, . $\sin. c : \tan.a = R : \tan.A$ *Q. E. D.*

A modification of this proposition demonstrates the latter part of the theorem. By reference to equation (5), plane trigonometry, we shall find that, $\tan.A.\ \cot.A = R^2$; therefore, $\tan.A = \dfrac{R^2}{\cot.A}$

Substituting this value for tangent A, in the preceding proposition, and dividing the last couplet by R, we shall have.

$$\sin. c : \tan.a = 1 : \frac{R}{\cot.A}$$

Or, . . . $\sin. c : \tan.a = \cot.A : R$ *Q. E. D.*

Or, ... $R \sin. c = \tan.a \cot.A$ (1)

Cor. By changing the construction, drawing the tangent to AB, in place of the tangent to BC, and proceeding in a similar manner, we have, $R\ \sin.a=\tan.c\ \cot.C$ (2)

PROPOSITION 9. THEOREM. 2.

In any right angled spherical triangle, the sine of the right angle is to the sine of the hypotenuse, as the sine of either of the other angles to the sine of the side opposite to that angle.

N. B. For the sake of perspicuity, if not of brevity, we will represent the angles of the triangle, by A, B, C, and of the sides or arcs opposite to these angles by a, b, c; that is, a opposite A, &c.

The sine of 90°, or radius, is designated by R.

In the plane triangle CHG, we have,

$$\sin.CHG : CG=\sin.CGH : CH$$

That is, . . $R : \sin.b=\sin.A : \sin.a$ *Q. E. D.*

Or, . . . $R\sin.a=\sin.b\ \sin.A$ (3)

Cor. By a change in the construction of the figure, drawing a tangent to AB, &c., we shall have,

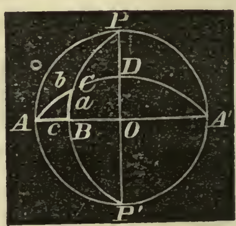
$R : \sin.\ b=\sin.\ C : \sin.c$ *Q. E. D.*

Or, . . . $R\sin.c=\sin.b\ \sin.C$ (4)

Scholium. Collecting the four preceding equations drawn from theorems 1 and 2, we have,

$$\left.\begin{array}{ll}(1) & R\ \sin.c=\tan.a\ \cot.A\\ (2) & R\sin.a=\tan.c\ \cot.C\\ (3) & R\sin.a=\sin.b\ \sin.A\\ (4) & R\ \sin.c=\sin.b\ \sin.C\end{array}\right\}$$

These equations refer to the right angled triangle ABC; but the principles are true for any right angled spherical triangle. Let us apply them to the right angled triangle PDC, the complemental triangle to ABC.



Making this application, equation (1) becomes,

$R\sin.CD=\tan.PD\cot.C$ (n)

(2) becomes $R\sin.PD=\tan.CD\cot.P$ (m)

(3) becomes $R\sin.PD=\sin.PC\sin.C$ (o)

(4) becomes $R\sin.CD=\sin.PC\sin.P$ (p)

By observing that $\sin.CD=\cos.AC=\cos.b$,

And that . . $\tan.PD=\cot.DO=\cot.A$, &c; and by running equations (n), (m), (o), and (p), back into the triangle ABC, and we shall have,

$$\left.\begin{array}{ll}(5) & R\cos.b=\cot.A\ \cot.C \\ (6) & R\cos.A=\cot.b\ \tan.c \\ (7) & R\cos.A=\cos.a\ \sin.C \\ (8) & R\cos.b=\cos.a\ \cos.c\end{array}\right\}$$

By observing equation (6), we find that the second member refers to sides adjacent to the angle A. The same relation holds in respect to the angle C, and gives,

$$(9)\quad R\cos.C=\cot.b\ \tan.a$$

Making the same observations on (7), we infer,

$$(10)\quad R\cos.C=\cos.c\ \sin.A$$

Observation 1. Several of these equations can be deduced geometrically without the least difficulty. For example, take the figure to proposition 7. Observe the parallels in the plane DBA, which give, $DB : DH=DE : DG$

That is, . . $R : \cos.a=\cos.c : \cos.b$

A result identical with equation (8), and in words is expressed thus: *As radius is to cosine of one side, so is the cosine of the other side, to the cosine of the hypotenuse.*

Observation 2. Equations numbered from (1) to (10), cover every possible case that can occur in right angled spherical trigonometry, but the combinations are too various to be remembered, and readily applied to practical use.

We can remedy this inconvenience, by taking the *complement* of the hypotenuse, and the *complements* of the two oblique angles, in place of the arcs themselves.

Thus b is the hypotenuse, and let b' be its complement.

Then, $b+b'=90°$; or, $b=90°-b'$; and, $\sin.b=\cos.b'$, $\cos.b=\sin.b'$; $\tan.b=\cot.b'$. In the same manner if A' is the complement to A,

Then, . $\sin.A=\cos.A'$; $\cos.A=\sin.A'$; and, $\tan.A=\cot.A'$; and similarly, $\sin.C=\cos.C'$; $\cos.C=\sin.C'$, and $\tan.C=\cot.C'$.

Substituting these values for b, A, and C, in the foregoing *ten* equations (a and c remaining the same), we have,

NAPIER'S CIRCULAR PARTS.

(11) $R\ \sin.c = \tan.a\ \tan.A'$
(12) $R\ \sin.a = \tan.c\ \tan.C'$
(13) $R\ \sin.a = \cos.b'\ \cos.A'$
(14) $R\ \sin.c = \cos.b'\ \cos.C'$
(15) $R\ \sin.b' = \tan.A'\ \tan.C'$
(16) $R\ \sin.A' = \tan.b'\ \tan.c$
(17) $R\ \sin.A' = \cos.a\ \cos.C'$
(18) $R\ \sin.b' = \cos.a\ \cos.c$
(19) $R\ \sin.C' = \tan.b'\ \tan.a$
(20) $R\ \sin.C' = \cos.c\ \cos.A'$

Omitting the consideration of the right angle there are five parts.—Each part taken as a middle part, is connected to its adjacent parts by one equation, and to its extreme parts by another equation; and therefore, ten equations are required for the combinations of all the parts.

These equations are very remarkable, because the first members are all composed of radius into *some sine*, and the second members are all composed of the product of *two tangents*, or *two cosines*.

To condense these equations into words, for the purpose of assisting the memory, we will refer them, any one of them, directly to the right angled triangle ABC, in the last figure.

When the right angle is left out of the question, a right angled triangle consists of *five* parts—*three* sides, and *two* angles. Let any one of these parts be called a *middle part*, then two other parts will lie adjacent to this part, and two *opposite to it*, that is, separated from it by two other parts.

For instance, take equation (11), and call c the *middle* part, then A' and a will be adjacent parts, and C' and b' opposite parts. Again, take a as a *middle part*, then c and C' will be adjacent parts, and A' and b' will be opposite parts; and thus we may go round the triangle.

Take any equation from (11) to (20), and consider the middle part in the first member of the equation, and we shall find that they correspond to these two *invariable and comprehensive rules*.

1. *The radius into the sine of the middle part equals the product of the tangents of the adjacent parts.*

2. *The radius into the sine of the middle part equals the product of the cosines of the opposite parts.*

These rules are known as Napier's Rules, because they were first brought forth by that distinguished mathematician, who was also the inventor of logarithms.

We caution the pupil to be very particular to take the *complements* of the hypotenuse, and the complements of the oblique angles.

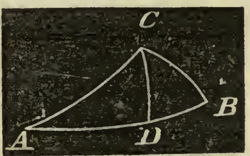
OBLIQUE ANGLED SPHERICAL TRIGONOMETRY.

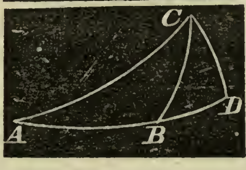
The preceding investigations have had reference to right angled spherical trigonometry only; but the application of these principles cover oblique angled trigonometry also, for every oblique angled spherical triangle may be considered as made up of the sum or difference of two right angled spherical triangles. With this explanatory remark, we give,

PROPOSITION 9. THEOREM. 3.

In all spherical triangles, the sines of the sides are to each other, as the sines of the angles opposite to them.

This was proved in relation to right angled triangles in theorem 2, and we now apply the principle to oblique angled triangles.





Let ABC, be the triangle, and let CD be perpendicular to AB, or to AB produced as represented in the margin.

Then by theorem 2, we have,

$$R : \sin.AC = \sin.A : \sin.CD$$

Also, $\quad \sin.CB : R = \sin.CD : \sin.B.$

By multiplying these two proportions term by term, and leaving out the common factor R, in the first couplet, and the common factor sin. CD, in the second, we have, $\quad \sin.CB : \sin.AC = \sin.A : \sin.B.$ *Q. E. D.*

Cor. From the truth of this theorum, it follows, that the angles at the base of an isosceles triangle are equal, and that in every spherical triangle the greater angle is opposite the greater side.

PROPOSITION 10. THEOREM 4.

In any spherical triangle, if an arc be let fall from any angle to the opposite side as a base, or to the base produced, the cosines of the other two sides will be to each other as the cosines of the segments of the base.

By the application of equation (8) to the last figure, we have,

$$R \cos.AC = \cos.AD \cos.DC$$

Similarly, . $R \cos.BC = \cos.DC \cos.BD$

Dividing one of these equations by the other, omitting common factors in numerators and denominators, we have,

$$\frac{\cos.AC}{\cos.BC} = \frac{\cos.AD}{\cos.BD}$$

Or, . $\cos.AC : \cos.BC = \cos.AD : \cos.BD$. *Q. E. D.*

PROPOSITION 11. THEOREM 5.

If from any angle of a spherical triangle, a perpendicular be let fall on the base, or on the base produced, the tangents of the segments of the base will be to each other reciprocally proportional to the cotangents of the segments of the angle.

By the application of equation (2) to the last figure, we have,

$$R \sin.CD = \tan.AD \cot.ACD$$

Similarly, . $R \sin.CD = \tan.BD \cot.BCD$

Therefore, by equality,

$$\tan.AD \cot.ACD = \tan.BD \cot.BCD$$

Or, . $\tan.AD : \tan.BD = \cot.BCD : \cot.ACD$. *Q. E. D.*

PROPOSITION 12. THEOREM 6.

The same construction remaining, the cosines of the angles at the extremities of the segments of the base, are to each other as the sines of the segments of the opposite angle.

Equation (7) applied to the triangle ACD, gives

$$R \cos.A = \cos.CD \sin.ACD \quad (s)$$

Also, . . $R \cos.B = \cos.CD \sin.BCD \quad (t)$

Dividing equation (s) by (t), gives

$$\frac{\cos.A}{\cos.B}=\frac{\sin.ACD}{\sin.BCD}$$

Or, . . $\cos.B : \cos.A = \sin.BCD : \sin.ACD$. *Q. E. D.*

PROPOSITION 13. THEOREM 7.

The same construction remaining, the sines of the segments of the base, are to each other as the cotangents of the adjacent angles.

Equation (1), applied to the triangle ACD, gives

$$R \sin.AD = \tan.CD \cot.A \qquad (s)$$

Similarly, . $$R \sin.BD = \tan.CD \cot.B \qquad (t)$$

Dividing (s) by (t), gives

$$\frac{\sin.AD}{\sin.BD}=\frac{\cot.A}{\cot.B}$$

Or, . $\sin.BD : \sin.AD = \cot.B : \cot.A$. *Q. E. D.*

PROPOSITION 14. THEOREM 8.

The same construction remaining, the cotangents of the two sides are to each other as the cosines of the segments of the angle.

Equation (9), applied to the triangle ACD, gives

$$R \cos.ACD = \cot.AC \tan.CD \qquad (s)$$

Similarly, . $$R \cos.BCD = \cot.BC \tan.CD \qquad (t)$$

Dividing (s) by (t), gives

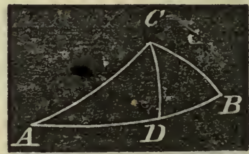
$$\frac{\cos.ACD}{\cos.BCD}=\frac{\cot.AC}{\cot.BC}$$

Or, . $\cot.AC : \cot.BC = \cos.ACD : \cos.BCD$. *Q. E. D.*

Remark. The preceding theorems enable us to solve any spherical triangle, right angled or oblique, when any three of the *six* parts are given. But oblique angled spherical triangles we have thus far considered as composed of two right angled triangles; and it is sometimes a little troublesome to select the theorems or equations which apply to the case in question. To remedy this inconvenience, we will at once seek a relation between the cosines and sines of an angle of any spherical triangle, and the sines and cosines of its sides. Therefore, we investigate the following propositions.

PROPOSITION 15. PROBLEM.

Investigate, and show the relation between the cosine of an angle of a spherical triangle, and the sines and cosines of its sides.



Let ABC be a spherical triangle, and CD a perpendicular from the angle C on to the side AB, or on to the side AB produced. Then, by proposition 10, th. 4,

$\cos.AC : \cos.CB = \cos.AD : \cos.BD$ (1)

When CD falls within the triangle,

$$BD=(AB-AD)$$

When CD falls without the triangle,

$$BD=(AD-AB)$$

Hence, . $\cos.BD=\cos.(AD-AB)$

Now, $\cos.(AB-AD)=\cos.(AD-AB)$, because each of them is equal to $\cos.AB \cos.AD+\sin.AB \sin.AD$. (Plane trig. eq. 10.)

This value of $\cos.BD$, put in proportion (1), gives

$$\cos.AC : \cos.CB=\cos.AD : \cos.AB \cos.AD+\sin.AB \sin.AD \quad (2)$$

Dividing the last couplet of proportion (2) by $\cos.AD$, observing that . . . $\frac{\sin.AD}{\cos.AD}=\tan.AD$, and we have

$$\cos.AC : \cos.CB=1 : \cos.AB+\sin.AB \tan.AD \quad (3)$$

By applying equation (6) to the triangle ACD, taking the radius as unity, we have $\cos.A=\cot.AC \tan.AD$ (k)

But, . $\tan.AC \cot.AC=1$ (eq. 5, plane trig.) (l)

Multiply equation (k) by $\tan.AC$, observing equation (l), and we have . $\tan.AC \cos.A=\tan.AD$

Substituting this value of $\tan.AD$, in proportion (3), we have

$$\cos.AC : \cos.CB=1 : \cos.AB+\sin.AB \tan.AC \cos.A \quad (4)$$

Multiplying extremes and means, gives

$$\cos. CB = \cos. AC \cos. AB + \sin. AB (\cos. AC \tan. AC) \cos. A$$

But, . . $\tan. AC = \frac{\sin. AC}{\cos. AC}$, or, $\cos. AC \tan. AC = \sin. AC$

Therefore, . $\cos. CB = \cos. AC \cos. AB + \sin. AB \sin. AC \cos. A$

Hence, . $\cos. A = \frac{\cos. CB - \cos. AC \cos. AB}{\sin. AB \sin. AC}$ (F) final result.*

By processes perfectly similar, like theorems may be deduced for the angles B and C.

If the sides opposite the angles A, B, and C, be respectively represented by a, b, and c, the formula will be expressed thus :

$$\left.\begin{aligned} \cos. A &= \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c} \\ \cos. B &= \frac{\cos. b - \cos. a \cos. c}{\sin. a \sin. c} \\ \cos. C &= \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b} \end{aligned}\right\} (S)$$

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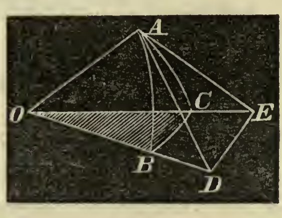
* As this equation has been denominated "*The fundamental formula of Spherical Trigonometry*," and as it is susceptible of a more geometrical demonstration, we give the following, which we believe will be very acceptable to every lover of mathematical science.



Let ABC be a spherical triangle, and O the center of the sphere.

From the angle A, draw AD tangent to the arc AB, and AE tangent to the arc AC. OD and OE, drawn from the center of the sphere to the extremities of the tangents, are, of course, secants. OD is the secant of AB, and OE the secant of the arc AC.

Because AD is a tangent, it is perpendicular to the radius OA. For the same reason AE is perpendicular to the same radius OA. But OA is the common intersection of the two planes AOB and AOC, and hence, by definition 5, book 6, the angle DAE is the inclination of the two planes AOB and AOC, and is, therefore, equal to the spherical angle A. As is customary, let the side opposite to A be designated by a, opposite B by b, opposite C by c.

These formulas are not adapted to the use of logarithms; and the use of *natural sines and cosines* would lead to tedious operations; we must, therefore, make some advantageous mutations, or the equations will be useless; hence the following investigations:

In equation (35), plane trigonometry, we find

$$1+\cos.A=2\cos^2\tfrac{1}{2}A$$

Therefore, $$2\cos.^2\tfrac{1}{2}A=1+\frac{\cos.a-\cos.b\cos.c}{\sin.b\sin.c}$$

$$=\frac{(\sin.b\sin.c-\cos.b\cos.c)+\cos.a}{\sin.b\sin.c}\quad(m)$$

But, . $\cos.(b+c)=\cos.b\cos.c-\sin.c\sin.b$ (9), plane trigonometry. By comparing this last equation with the second member of equation (m), we perceive that equation (m) is readily reduced to

$$2\cos.^2\tfrac{1}{2}A=\frac{\cos.a-\cos(b+c)}{\sin.b\sin.c}$$

Then, $AD=\tan.c$, $AE=\tan.b$, $OD=\sec.c$, $OE=\sec.b$.

Designate DE by x, and observe that the angle BOC is measured by the arc $BC=a$.

Now, to the two plane triangles ODE and ADE, if we apply equation (m), proposition 8, plane trigonometry, we shall have

$$\cos.a=\frac{\sec.^2c+\sec.^2b-x^2}{2\sec.c\sec.b}$$

$$\cos.A=\frac{\tan.^2c+\tan.^2b-x^2}{2\tan.c\tan.b}$$

Clearing these two equations of fractions, and subtracting the latter from the former, and observing, that for any arc, $\sec.^2-\tan.^2=R^2$; and if R is unity, as it is in this case, we shall have,

$$2\sec.c\sec.b\cos.a-2\tan.c\tan.b\cos.A=2$$

Dividing by 2, and substituting the values of the secants and tangents from equations (4) and (5), plane trigonometry,

Namely, . $\sec.=\dfrac{1}{\cos.}$, $\tan.=\dfrac{\sin.}{\cos.}$, we shall then have,

$$\frac{\cos.a}{\cos.c\cos.b}-\frac{\sin.c\sin.b\cos.A}{\cos.c\cos.b}=1$$

Considering $(b+c)$ as one arc, and then making application of equation (18), plane trigonometry, we have,

$$2\cos^2\tfrac{1}{2}A=\frac{2\sin.\left(\frac{a+b+c}{2}\right)\sin.\left(\frac{b+c-a}{2}\right)}{\sin.b\,\sin.c}$$

But, . $\frac{b+c-a}{2}=\frac{b+c+a}{2}-a$; and if we put S to represent $\frac{b+c+a}{2}$, we shall have

$$\cos^2\frac{A}{2}=\frac{\sin.S\,\sin.(S-a)}{\sin.b\,\sin.c}$$

Or, . . $\cos.\frac{A}{2}=\sqrt{\frac{\sin.S\,\sin.(S-a)}{\sin.b\,\sin.c}}$

The right hand member of this equation gives the value of the

---

Clearing of fractions, transposing, and changing signs, will give

$$\sin.c\,\sin.b\,\cos.A=\cos.a-\cos.c\,\cos.b$$

Therefore, . . . $\cos.A=\frac{\cos.a-\cos.c\,\cos.b}{\sin.c\,\sin.b}$

For the sake of the mathematical exercise, I will suppose we have the three sides of a spherical triangle, as follows:

$a=70°\ 4'\ 18''$, $b=59°\ 16'\ 23''$, and $c=63°\ 21'\ 27''$, from which we require the angle A, and we have no other formula except the above equation, *and logarithms are not yet invented.*

From the table of natural sines and cosines, we find

$\cos.a=0.34090$

$\cos.b=0.51191$ $\sin.b=0.8791$

$\cos.c=0.44840$ $\sin.c=0.8938$

By the multiplication of decimals, retaining *only five* places, we find,

$\cos.b\,\cos.c=0.22953$, and $\sin.b\,\sin.c=0.76786$

From $\cos.a$	. .	0.34890
Take $\cos.b\,\cos.c$	.	0.22953

0.76786)0.11137(0.14505=$\cos.A$

By comparing this decimal with the table, we find it very nearly corresponds to 81° 40′. The true value of A is 81° 38′ 20″

cosine when the radius is unity. To a greater radius, the cosine would be greater; and in just the same proportion as the radius increases, all the trigonometrical lines increase; therefore, to adapt the above equation to our tables where the radius is R, we must write R in the second member, as a factor; and if we put it under the radical sign, we must write R^2.

For the other angles we shall have precisely similar equations;

That is . .
$$\left.\begin{aligned}\cos.\frac{A}{2}&=\sqrt{\frac{R^2\sin.S\sin.(S-a)}{\sin.b\sin.c}}\\ \cos.\frac{B}{2}&=\sqrt{\frac{R^2\sin.S\sin.(S-b)}{\sin.a\sin.c}}\\ \cos.\frac{C}{2}&=\sqrt{\frac{R^2\sin.S\sin.(S-c)}{\sin.a\sin.b}}\end{aligned}\right\}\ (T)$$

Formulas, for the sines of the angles, are obtained as follows: From equation (32), plane trigonometry, we obtain

$$2\sin.^2\tfrac{1}{2}A=1-\cos.A.$$

Substituting the value of cos.A, taken from equation (S), and

we have .
$$2\sin.^2\tfrac{1}{2}A=1-\frac{\cos.a-\cos.b\cos.c}{\sin.b\sin.c}$$
$$=\frac{(\sin.b\sin.c+\cos.b\cos.c)-\cos.a}{\sin.b\sin.c}$$

But, $\cos.(b\backsim c)=\sin.b.\sin.c+\cos.b\cos.c$ ((10) plane trig.)

This equation reduces the preceding one to

$$2\sin.^2\tfrac{1}{2}A=\frac{\cos.(b\backsim c)-\cos.a}{\sin.b\sin.c}$$

Considering $(b\backsim c)$ as a single arc, and applying equation (18), plane trigonometry, we have

$$2\sin.^2\tfrac{1}{2}A=\frac{2\sin.\left(\frac{a+b-c}{2}\right)\sin.\left(\frac{a+c-b}{2}\right)}{\sin.b\sin.c}$$

But, $\frac{a+b-c}{2}=\frac{a+b+c}{2}-c=S-c$, if we put $S=\frac{a+b+c}{2}$

Also, $\frac{a+c-b}{2}=\frac{a+b+c}{2}-b=S-b$

Dividing the preceding equation by 2, and making these substitutions, we have,

$$\sin.\tfrac{1}{2}A=\frac{\sin.(S-c)\sin.(S-b)}{\sin.b\ \sin.c},\ \text{when radius is unity.}$$

When radius is R, we have

$$\left.\begin{array}{ll} & \sin.\tfrac{1}{2}A=\sqrt{\dfrac{R^2\sin.(S-c)\sin.(S-b)}{\sin.b\ \sin.c}} \\ \text{Similarly,} & \sin.\tfrac{1}{2}B=\sqrt{\dfrac{R^2\sin.(S-a)\sin.(S-c)}{\sin.a\ \sin.c}} \\ \text{And,}\quad . & \sin.\tfrac{1}{2}C=\sqrt{\dfrac{R^2\sin.(S-a)\sin.(S-b)}{\sin.a\ \sin.b}} \end{array}\right\}\quad (U)$$

To apply to our tables, R^2 must be put under the radical sign. We shall show the application of these formulas, and those in equations (S), hereafter.

From (30), plane trigonometry, we have

$$\sin.A=2\ \sin.\tfrac{1}{2}A\ \cos.\tfrac{1}{2}A$$

Squaring, . $\sin.^2A=4\ \sin.^2\tfrac{1}{2}A\ \cos.^2\tfrac{1}{2}A \qquad (t)$

Square the first equation in (T), and multiply it by the square of the first equation in (U), and four times their product is

$$4\ \sin.^2\tfrac{1}{2}A\ \cos.^2\tfrac{1}{2}A=\frac{4\ R^4\sin.S\sin.(S-a)\sin.(S-b)\sin.(S-c)}{\sin.^2 b\ \sin.^2 c}$$

Comparing the first member with equation (t), we have

$$\sin.^2 A=\frac{4\ R^4\sin.S\sin.(S-a)\sin.(S-b)\sin.(S-c)}{\sin.^2 b\ \sin.^2 c}\qquad (u)$$

By operating in the same manner with the several equations in (T) and (U), we have

$$\sin.^2 B=\frac{4\ R^4\sin.S\ \sin.(S-a)\sin.(S-b)\sin.(S-c)}{\sin.^2 a\ \sin.^2 c}\qquad (v)$$

The numerators of the second members of (u) and (v), are the same; and if we divide (u) by (v), and extract the square root, we shall have

$$\frac{\sin.A}{\sin.B}=\frac{\sin.a}{\sin.b}$$

Or, . . $\sin.B : \sin.A=\sin.b : \sin.a$, a truth that was demonstrated in proposition 9, spherical trigonometry.

We have again demonstrated it in this manner, to show that equation (F), from which all the preceding equations arose, is really the fundamental equation of spherical trigonometry.

A spherical triangle consists of six parts; three sides, and three angles; and there are certain relations existing between them; but the combinations of these relations have their limits; and when we have gone through these relations, if we continue to combine equations, we shall only fall on truths previously demonstrated, and this is exemplified by our last operations.

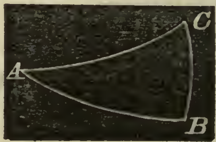
APPLICATION.

SOLUTION OF RIGHT ANGLED SPHERICAL TRIANGLES.

1. At a certain time the sun's longitude was 40° 29′ 30″, and the obliquity of the ecliptic 23° 27′ 32″. What was the declination?

Ans. 14° 58′ 52″.

This example presents a right angled spherical triangle, ABC. The hypotenuse, AC=40° 29′ 30″, and the angle A=23° 27′ 32″, and the side, CB, is required. By our system of notation, $AC=b$, $BC=a$.



This can be solved by equation (3) or (13), which are essentially the same; that is.

$$R\ \sin.a=\sin.b\ \sin.A$$

$\sin.b=\sin.40°\ 29'\ 30''$	.	9.812470
$\sin.A=\sin.23°\ 27'\ 32''$	.	9.599985
Ans. $\sin.a=\sin.14°\ 58'\ 52''$	.	9.412455

Rejecting 10 in the index, is the same as dividing by the radius, as the equation requires.

2. At a certain time, the *difference* between the longitude of the *sun* and *moon*, was 76° 10′ 20″, and the moon's latitude, at the same time, was 5° 9′ 12″ north. What was the true angular distance between the centers of the sun and moon? *Ans.* 76° 13′ 45″.

This problem presents a right angled spherical triangle, whose base AB=76° 10′ 20″, and perpendicular BC=5° 9′ 12″. The hypotenuse AC, is required. Equation (8) or (18) solves it.

$c=76°\ 10'\ 20''$	cos.	.	9.378406
$a=\ \ 5°\ \ 9'\ 12''$	cos.	.	9.998241
$b=76°\ 13'\ 45''$	cos.	.	9.376647

3. An astronomer observed the sun to pass his meridian on a certain day when his astronomical clock gave 2 h. 9 min. 33 sec. for the siderial time, and the altitude was such as to give the declination of 13° 5′ 6″ north. What was the sun's longitude, and what was the obliquity of the ecliptic ? *Ans.* Lon. 34° 39′ 46″. Obliq. eclip. 23° 27′ 26″.

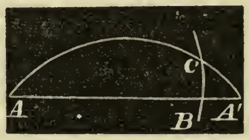
This problem presents a right angled spherical triangle, giving its base and perpendicular, and demanding the hypotenuse, and the angle at the base.

2 h. 9 m. 33 s.$=c=$32° 23 15	cos.	.	9.926571
$a=$13 5 6	cos.	.	9.988575
$b=$34 39 46	cos.	.	9.915146

To find A, we apply equation (3) or (13), as they are one and the same.

R sin.a	. . .	19.354869
sin.b (subtract)	.	9.754918
$A=$23° 27′ 26″	. .	9.599951

At a certain time the sun's longitude will be 150° 33′ 20″, and the obliquity of the ecliptic 23° 27′ 29″. Required its right ascension and declination. *Ans.* R. A. 152° 37′ 28″; Dec. 11° 17′ 7″N.



Observation. This problem presents a right angled spherical triangle, whose base and hypotenuse are each greater than 90°; and in cases of this kind, let the pupil observe, *that the base is greater than the hypotenuse*, and the oblique angle opposite the base, is greater than a right angle. In all cases, a triangle and its supplemental triangle, make a *lune*. It is 180° from one pole to its opposite, whatever great circle be traversed. It is 180° along the equator ABA', and also 180° along the ecliptic ACA'; and the lune always gives two triangles; and when the sides of one of them are greater than 90°, we take its supplemental triangle, as in this case we operate on the triangle $A'CB$.

But $A'C$ is greater than $A'B$; therefore, AB is greater than AC. The angle $A'CB$ is less than 90°; therefore, ACB is greater than 90°, because the two angles together make two right angles.

These facts are technically expressed, by saying, that the sides and opposite angles are of the *same affection**; and if the two sides of a right angled spherical triangle are of the *same affection*, the hypotenuse

* *Same affection:* that is, both greater, or both less than 90°. *Different affection:* the one greater, the other less than 90°.

will be less than 90°; and of *different affection*, the hypotenuse will be greater than 90°.

If, in every instance, we make a natural construction of the figure and use common judgment, it will be impossible to doubt whether an arc must be taken greater or less than 90°.

We now solve the triangle $A'CB$, $A'C$=29° 26′ 40″.

To find BC.	Eq. (3) or (13).	b sin. 29° 26′ 40″	.	9.691594
		A sin. 23° 27′ 29″	.	9.599984
		a sin. 11° 17′ 7″	.	9.291578

To find $A'B$, we use equation (1) or (11), thus :

tan.	11° 17′ 7″	.	9.300016
cot.	23° 27′ 29″	.	10.362674
c sin.	27° 22′ 32″	.	9.662590
	180		
AB=152° 37′ 28″			

We select the following examples to exercise the pupils in right angled spherical trigonometry:

1. In the right angled spherical triangle ABC, given AB 118° 21′ 4″, and the angle A 23° 40′ 12″, to find the other parts.

Ans. AC 116° 17′ 55″, the angle C 100° 59 26″, and BC 21° 5′ 42″.

2. In the right angled spherical triangle ABC, given AB 53° 14′ 20″, and the angle A 91° 25′ 53″, to find the other parts.

Ans. AC 91° 4′ 9″, the angle C 53° 15′ 8″, and BC 91° 47′ 11″.

3. In the right angled spherical triangle ABC, given AB 102° 50′ 25″, and the angle A 113° 14′ 37″, to find the other parts.

Ans. AC 84° 51′ 36″, the angle C 101° 46′ 57″, and BC 113° 46′ 27″.

4. In the right angled shpherical triangle ABC, given AB 48° 24′ 16″, and BC 59° 38′ 27″, to find the other parts.

Ans. AC 70° 23′ 42″, the angle A 66° 20′ 40″, and the angle C 52° 32′ 55″.

5. In the right angled spherical triangle ABC, given AB 151° 23′ 9″, and BC 16° 35′ 14″ to find the other parts.

Ans. AC 147° 16′ 51″, the angle C 117° 37′ 21″, and the angle A 31° 52′ 50″.

6. In the right angled spherical triangle *ABC*, given *AB* 73° 4′ 31″, and *AC* 86° 12′ 15″, to find the other parts.

Ans. *BC* 76° 51′ 20″, the angle *A* 77° 24′ 23″, and the angle *C* 73° 29′ 40″.

7. In the right angled spherical triangle *ABC*, given *AC* 118° 32′ 12″, and *AB* 47° 26′ 35″, to find the other parts.

Ans. *BC* 134° 56′ 20″, the angle *A* 126° 19′ 2″, and the angle *C* 56° 58′ 44″.

8. In the right angled spherical triangle *ABC*, given *AB* 40° 18′ 23″, and *AC* 100° 3′ 7″, to find the other parts.

Ans. The angle *A* 98° 38′ 53″, the angle *C* 41° 4′ 6″, and *BC* 103° 13′ 52″.

9. In the right angled spherical triangle *ABC*, given *AC* 61° 3′ 22″, and the angle *A* 49° 28′ 12″, to find the other parts.

Ans. *AB* 49° 36′ 6″, the angle *C* 60° 29′ 19″, and *BC* 41° 41′ 32″.

10 In the right angled spherical triangle *ABC*, given *AB* 29° 12′ 50″, and the angle *C* 37° 26′ 21″, to find the other parts ?

Ans. Ambiguous; the angle *A* 65° 27′ 58″ or its supplement, *AC* 53° 24′ 13″ or its supplement, *BC* 46° 55′ 2″ or its supplement.

11. In the right angled spherical triangle *ABC*, given *AB* 100° 10′ 3″, and the angle *C* 90° 14′ 20″, to find the other parts.

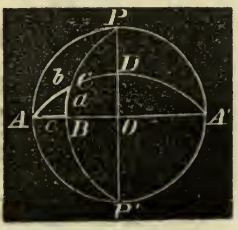
Ans. Ambiguous; *AC* 100° 9′ 55″ or its supplement, *BC* 1° 19′ 53″ or its supplement, and the angle *A* 1° 21′ 8″ or its supplement.

12. In the right angled spherical triangle *ABC*, given *AB* 54° 21′ 35″, and the angle *C* 61° 2′ 15″, to find the other parts.

Ans. Ambiguous; *BC* 129° 28′ 28″ or its supplement, *AC* 111° 44′ 34″ or its supplement, and the angle *A* 123° 47′ 44″ or its supplement.

13. In the right angled spherical triangle *ABC*, given *AB* 121° 26′ 25″, and the angle *C* 111° 14′ 37″, to find the other parts.

Ans. Ambiguous; the angle *A* 136° 0′ 3′ or its supplement, *AC* 66°′ 15′ 38″ or its supplement, and *BC* 140° 30′ 56″ or its supplement.



The solution of right angled spherical triangles includes, also, the solution of *quadrantal* triangles, as may be seen by inspecting the adjoining figure. *When we have one quadrantal triangle, we have four, which fill up the whole hemisphere.*

To effect the solution of either of the four quadrantal triangles *APC*, *AP′C*, *A′PC*, or

$A'P'C$, it is sufficient to solve the small right angled spherical triangle ABC.

To the half lune $AP'B$, we add the triangle ABC, and we have the quadrantal triangle $AP'C$; and by subtracting the same from the equal half lune APB, we have the quadrantal triangle PAC.

When we have the side, AC, of the same triangle, we have its supplement, $A'C$, which is a side of the triangle $A'PC$, and of $A'P'C$. When we have the side, CB, of the small triangle, by adding it to 90°, we have $P'C$, a side of the triangle $A'P'C$; and subtracting it from 90°, we have PC, a side of the triangle APC, and $A'PC$.

EXAMPLES.

1. *In a quadrantal triangle, there are given the quadrantal side,* 90°, *a side adjacent,* 42° 21′, *and the angle opposite this last side, equal to* 36° 31′. *Required the other parts.*

By this enumeration we cannot decide whether the triangle APC or $AP'C$, is the one required, for $AC=42° 21'$ belongs equally to both triangles. The angle $APC=AP'C=36° 31'=AB$.

We operate wholly on the triangle ABC.

To find the angle A, call it the *middle part*.

Then, $R\cos.(CAB)=R\sin.PAC=\cot.AC.\tan.AB$

cot. $AC=$	42° 21′	10.040231
tan. $AB=$	36 31	9.869473
cos. $CAB=$	35 40 51	9.909704
	90	
$PAC=$	54 19 9	
$P'AC=$	125 40 51	

To find the angle C, call it the *middle part*.

$R\cos.ACB=\sin.CAB\cos.AB$

sin. $CAB=$	35° 40 51″	9.765869
cos. $AB=$	36 31	9.905085
cos. $ACB=$	62 2 45	9.670954
	180	
$ACP=A'CP'=$	117 57 15	

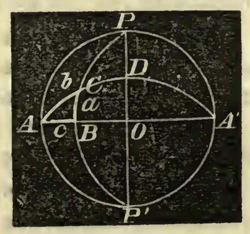
To find the side BC, call it the *middle part.*

$$R\ \sin.BC = \tan.AB\ \cot.ACB.$$

$\tan.AB=$	36° 31′ 0″	9.869473
$\cot.ACB=$	62 2′ 45″	9.724835
$\sin.BC=$	23 8′ 11″	9.594308
	90	
$PC=$	66 51′ 49″	
$P'C=$	113 8′ 11″	

We now have all the sides, and all the angles of the *four* triangles in question.

2. *In a quadrantal spherical triangle, having given the quadrantal side,* 90°, *an adjacent side,* 115°, 09′, *and the included angle,* 115° 55′, *to find the other parts.*



This enunciation clearly points out the particular triangle $A'P'C$. $A'P'=90°$; and conceive $A'C=115°\ 09'$. Then the angle $P'A'C=115°\ 55'=P'D$.

From the angle $P'A'C$ take 90° or $P'A'B$, and the remainder is the angle $OA'D=BAC$ $=25°\ 55'$.

We here again operate on the triangle ABC. $A'C$, taken from 180°, gives . . 64° 51′$=AC$

To find BC, we call it the *middle part.*

$$R\ \sin.BC = \sin.AC\ \sin.BAC.$$

$\sin.AC=$	64° 51′	.	9.956744
$\sin.BAC=$	25 55′	.	9.640544
$\sin.BC=$	23 18′ 19″		8.597288
	90		
$P'C=$	113 18′ 19″		

To find AB we call it the *middle part.*

$$R\ \sin.AB = \tan.BC\ \cot.BAC.$$

$\tan.BC=$	23° 18′ 19″	.	9.634251
$\cot.BAC=$	25 55′ .	.	9.313423
$\sin.AB=$	62 26′ 8″	.	9.947674
	180		
$A'B=$	117 33′ 52″=the angle $A'P'C$		

To find the angle C, we call it the *middle part.*

$$R\ \cos. C = \cot. AC\ \tan. BC$$

$\cot. AC=$	64° 51′ .	. 9.671634
$\tan. BC=$	23 18′ 19″	. 9.634251
$\cos. C=$	78	9.305885
	180 19′ 53″	
$P'CA'=$	101 40′ 7″	

Thus we have found the side $P'C$=113° 18′ 19″
The angle $A'P'C$=117° 33′ 52″ *Ans.*
" $P'CA'$=101° 40′ 7″

3. In a quadrantal triangle, given the quadrantal side, 90°, a side adjacent, 67° 3′, and the included angle, 49° 18′, to find the other parts.

Ans. The remaining side is 53° 5′ 46″, the angle opposite the quadrantal side, 108° 32′ 27″, and the remaining angle, 60° 48′ 54″.

4. In a quadrantal triangle, given the quadrantal side, 90°, one angle adjacent, 118° 40′ 36″, and the side opposite this last mentioned angle, 113° 2′ 28″, to find the other parts.

Ans. The remaining side is 54° 38′ 57″, the angle opposite, 51° 2′ 35″, and the angle opposite the quadrantal side is 72° 26′ 21″.

5. In a quadrantal triangle, given the quadrantal side, 90, and the two adjacent angles, one 69° 13′ 46″, the other 72° 12′ 4″, to find the other parts.

Ans. One of the remaining sides is 70° 8′ 39″, the other is 73° 17′ 29″, and the angle opposite the quadrantal side is 96° 13′ 23″.

6. In a quadrantal triangle, given the quadrantal side, 90°, one adjacent side, 86° 14′ 40″, and the angle opposite to that side, 37° 12′ 20″, to find the other parts.

Ans. The remaining side is 4° 43′ 2″, the angle opposite, 2° 51′ 23″, and the angle opposite the quadrantal side, 142° 42′ 2″.

7. In a quadrantal triangle, given the quadrantal side, 90°, and the other two sides, one 118° 32′ 16″, the other 67° 48′ 40″, to find the other parts—the three angles.

Ans. The angles are 64° 32′ 21″, 121° 3′ 40″, and 77° 11′ 6″; the greater angle opposite the greater side, of course.

8. In a quadrantal triangle, given the quadrantal side, 90°, the angle opposite, 104° 41′ 17″, and one adjacent side, 73° 21′ 6″, to find the other parts.

Ans. The remaining side is 49° 42′ 18″, and the remaining angles are 47° 32′ 39″, and 67° 56′ 13″.

OBLIQUE ANGLED SPHERICAL TRIGONOMETRY.

ALL cases of oblique angled spherical trigonometry may be solved by right angled trigonometry, except two ; because every oblique angled spherical triangle is composed of the sum or difference of two right angled spherical triangles.

When a side and two of the angles, or *an angle and two of the sides are given*, to find the other parts, conform to the following directions :

Let a perpendicular be drawn from an extremity of a given side, and opposite a given angle or its supplement ; this will form two right angled spherical triangles ; and one of them will have its hypotenuse and one of its adjacent angles given, from which all its other parts can be computed ; and some of these parts will become as known parts to the other triangle, from which all its parts can be computed.

To facilitate these computations, we here give a summary of the practical truths demonstrated in the foregoing propositions.

1. *The sines of the sides of spherical triangles are proportional to the sines of their opposite angles.*

2. *The sines of the segments of the base, made by a perpendicular from the opposite angle, are proportional to the cotangents of their adjacent angles.*

3. *The cosines of the segments of the base are proportional to the cosines of the adjacent sides of the triangle.*

4. *The tangents of the segments of the base are proportional to the the tangents of the opposite segments of the vertical angles.*

5. *The cosines of the angles at the base, are proportional to the sines of the corresponding segments of the vertical angles.*

6. *The cosines of the segments of the vertical angles are proportional to the cotangents of the adjoining sides of the triangle.*

The two cases in which right angled triangles are not used, are,
1st. When the three sides are given to find the angles ; and,
2d. When the three angles are given to find the sides.

The first of these cases is the most important of all, and for that reason great attention has been given to it, and two series of equations, (T) and (U), have been deduced to facilitate its solution.

We now apply the following equation to find the angle A, of the triangle ABC, whose sides are a, b, c. $a=70° \ 4' \ 18''$. $b=63° \ 21' \ 27''$. $c=59° \ 16' \ 23''$. a is opposite A, b is opposite B. and c is opposite C.

$$\cos.\tfrac{1}{2}A=\sqrt{\frac{R^2\sin.S\sin.(S-a)}{\sin.b\sin.c}}$$

We write the second member of this equation thus:

$$\sqrt{\left(\frac{R}{\sin.b}\right)\left(\frac{R}{\sin.c}\right)\sin.S\sin.(S-a)}$$

showing four distinct logarithms.

The logarithm corresponding to $\frac{R}{\sin.b}$ is the sin.b subtracted from 10; and $\frac{R}{\sin.c}$ is the sin.c subtracted from 10, which we call *sin.complement.*

$BC=a=$	70°	4′	18″		
$AB=c=$	59°	16′	23″	sin.com.	0.065697
$AC=b=$	63°	21′	27″	sin.com.	0.048749
	2)192	42	8		
$S=$	96	21	4″	sin.	9.997326
$S-a=$	26	16	46	sin.	9.646158
					2)19.757930
$\frac{1}{2}A=$	40	49	10	cos.	9.878965
			2		
$A=$	81	38	20		

When we apply the equation to find the angle A, we write a first, at the top of the column; when we apply the equation to find the angle B, we write b at the top of the column. Thus,

To find the angle B

$$\cos.\tfrac{1}{2}B=\sqrt{\frac{R^2\sin.S\sin.(S-b)}{\sin.a\sin.c}}$$

$$=\sqrt{\left(\frac{R}{\sin.a}\right)\left(\frac{R}{\sin.c}\right)(\sin.S)\sin.(S-b)}$$

$b=$	63°	21′	27″		
$c=$	59	16	23	sin.com.	.065697
$a=$	70	4	18	sin.com.	.026857
	2)192	42	8		
$S=$	96	21	4	sin.	9.997326
$S-b=$	32	59	37	sin.	9.736034
					2)19.825874
$\frac{1}{2}B=$	35	4	49	cos.	9.912937
			2		
$B=$	70	9	38		

By the other equation in formula (T), we can find the angle C; but, for the sake of variety, we will find the angle C by the application of the third equation in formula (U).

$$\sin.\tfrac{1}{2}C=\sqrt{\frac{R^2\sin.(S-b)\sin.(S-a)}{\sin.b\sin.a}}$$

$$=\sqrt{\left(\frac{R}{\sin.b}\right)\left(\frac{R}{\sin.a}\right)\sin.(S-b)\sin.(S-a)}$$

$c=$	59°	16′	23″	.	.	
$a=$	70	4	18	sin.com.		.026817
$b=$	63	21	27	sin.com.		.048479
2)192		42	8			
$S=$	96	21	4			
$S-a=$	26	16	46	sin.	.	9.646158
$S-b=$	32	59	37	sin.	.	9.736034
						2)19.457758
$\frac{1}{2}C=$	32°	23′	17″	sin.	.	9.778879
			2			
$C=$	64	46	34			

To show the harmony and practical utility of these two sets of equations, we will find the angle A, from the equation

$$\sin.\tfrac{1}{2}A=\sqrt{\left(\frac{R}{\sin.b}\right)\left(\frac{R}{\sin.c}\right)\sin.(S-b)\sin.(S-c)}$$

$a=$	70	4′	18″			
$b=$	63	21	27	sin.com.		.048749
$c=$	59	16	23	sin.com.		.065697
2)192		42	8			
$S=$	96	21	4			
$S-b=$	32	59	37	sin. .		. 9.736034
$S-c=$	37	4	41	sin. .		. 9.780247
						2)19.630727
$\frac{1}{2}A=$	40°	49′	10″	sin. .		. 9.815363
			2			
$A=$	81	38	20			

2. In a spherical triangle ABC, given the angle A, 38° 19′ 18″, the angle B, 48° 0′ 10″, and the angle C, 121° 8′ 6″, to find the sides a, b, c.

Apply proposition 5, spherics.

$A=38°\ 19'\ 18''$	supplement	$141°\ 40'\ 42''$
$B=48\quad 0\quad 10$	supplement	$131\quad 59\quad 50$
$C=121\quad 8\quad 6$	supplement	$58\quad 51\quad 54$

We now find the angles to the spherical triangle, whose sides are these supplements.

Thus, .	$141°\ 40'\ 42''$		
	$131\quad 59\quad 50$	sin.com.*	.128909
	$58\quad 51\quad 54$	sin.com.	.067551
	2)332 32 26		
	166 16 13	sin. .	9.375375
	24 35 31	sin. .	9.619253
			2)19.191088
	$66°\ 47'\ 37\frac{1}{2}''$	cos. .	9.595543
	2		
angle	$=133\quad 35\quad 15$		
supp.	$=46\quad 24\quad 45=a$ of the original triangle.		

In the same manner we find $b=60°\ 14'\ 25''$ $c=89°\ 1'\ 14''$

EXAMPLES FOR EXERCISE.

1. In any triangle, ABC, whose sides are a, b, c, given $b=118°2'\ 14''$, $c=120°\ 18'\ 33''$, and the included angle $A=27°\ 22'\ 34''$, to find the other parts.

Ans. $a=23°\ 57'\ 13''$, angle $B=91°\ 26'\ 44''$, and $C=102°\ 5'\ 54''$.

2. Given $A=81°\ 38'\ 17''$, $B=70°\ 9'\ 38''$, and $C=64°\ 46'\ 32''$, to find the sides a, b, and c.

Ans. $a=70°\ 4'\ 18''$, $b=63°\ 21'\ 27''$, and $c=59°\ 16'\ 23''$.

3. Given the three sides $a=93°\ 27'\ 34''$, $b=100°\ 4'\ 26''$, and $c=96°\ 14'\ 50''$, to find the angles A, B, and C.

Ans. $A=94°\ 39'\ 4''$, $B=100°\ 32'\ 19''$, and $C=96°\ 58'\ 36''$.

4. Given two sides, $b=84°\ 16'$, $c=81°\ 12'$, and the angle $C=80°\ 28'$, to find the other parts.

Ans. The result is ambiguous, for we may consider the angle B as acute or obtuse. If the angle B is acute, then $A=97°\ 13'\ 45''$, $B=83°\ 11'\ 24''$, and $a=96°\ 13'\ 33''$.

If B is obtuse, then $A=21°\ 16'\ 44''$, $B=96°\ 48'\ 36''$, and $a=21°\ 19'\ 29''$

---

* The sine complement of $131°\ 59'\ 50''$, is the same as the sine complement of $48°\ 0'\ 10''$.

5. Given one side, $c=64°\ 26'$, and the angles adjacent, $A=49°$, and $B=52°$, to find the other parts.

Ans. $b=45°\ 56'\ 46''$, $a=43°\ 29'\ 49''$, and $C=98°\ 28'\ 5''$.

6 Given the three sides, $a=90°$, $b=90°$, $c=90°$, to find the angles A, B, and C. *Ans.* $A=90°$, $B=90°$, and $C=90°$.

7. Given the two sides, $a=77°\ 25'\ 11''$, and $c=128°\ 13'\ 47''$, and the angle C, $131°\ 11'\ 12''$ to find the other parts.

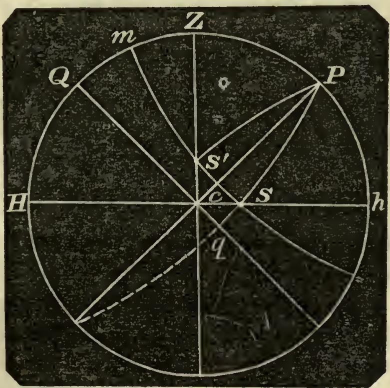
Ans. $b=84°\ 29'\ 24''$, $A=69°\ 14'$, and $B=72°\ 28'\ 46'$.

8. Given the three sides, a, b, c, $a=68°\ 34'\ 13''$, $b=59°\ 21'\ 18$, and $c=112°\ 16'\ 32''$, to find the angles A, B, and C.

Ans. $A=45°\ 26'\ 12''$, $B=41°\ 11'\ 6''$, $C=134°\ 54'\ 27''$

APPLICATION.

Spherical trigononometry becomes a science of incalculable importance in its connection with geography, navigation, and astronomy; for neither of these subjects can be understood without it; and to stimulate the student to a study of the science, we here attempt to give him a glimpse at some of its points of application.



Let the lines in the annexed figure represent circles in the heavens above and around us.

Let Z be the zenith, or the point just overhead, Hch the horizon, PZH the meridian in the heavens, P the pole of the earth's equator; then Ph is the latitude of the observer, and PZ is the co.latitude. Qcq is a portion of the equator, and the dotted, curved line, $mS'S$, parallel to the equator, is the parallel of the sun's declination at some particular time; and in this figure the sun's declination is supposed to be north. By the revolution of the earth on its axis, the sun is apparently brought from the horizon, at S, to the meridian, at m; and from thence it is carried down on the same curve, on the other side of the meridian; and this apparent motion of the sun (or any other celestial body) makes angles at the pole P, which are in direct proportion to their times of description.

The apparent straight line, Zc, is what is denominated, in astronomy, the *prime vertical;* that is, the east and west line through the zenith, passing through the *east* and *west* points in the horizon.

When the latitude of the place is north, and the declination is also north, as is represented in this figure, the sun rises and sets on the horizon to the north of the east and west points, and the distance is measured by the arc cS, on the horizon.

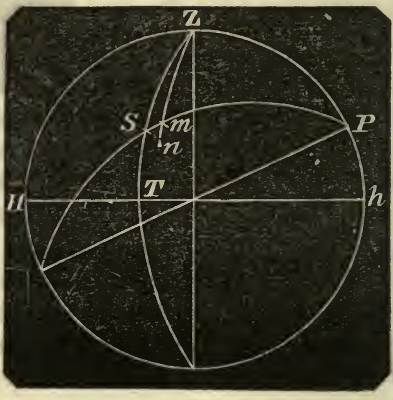
This arc can be found by means of the right angled spherical triangle cqS, right angled at q. Sq is the sun's declination, and the angle Scq is equal to the *co.latitude* of the place; for the angle Pch is the latitude, and the angle Scq is its complement.

The side cq, a portion of the equator, measures the angle cPq, the time of the sun's rising or setting before or after *six*, apparent time. Thus we perceive that this little triangle cSq, is a very important one.

When the sun is exactly *east* or *west*, it can be determined by the triangle ZPS'; the side PZ is known, being the co.latitude; the angle PZS' is a right angle, and the side PS' is the sun's polar distance. Here, then, is the hypotenuse and side of a right angled spherical triangle given, from which the other parts can be computed. The angle ZPS' is the time from noon, and the side ZS' is the sun's zenith distance at that time.

FORMULA FOR TIME.

The most important problem in navigation is that of finding the time from the altitude of the sun, when the sun's declination and the latitude of the observer are given.



This problem will be understood by the triangle PZS. When the sun is on the meridian, it is then apparent noon. When not on the meridian, we can determine the interval from noon by means of the triangle $PZS;$ for we can know all its sides; and the angle at P, changed into time at the rate of 15° to one hour, will give the time from apparent noon, when any particular altitude, as TS, may have been observed. PS is known by the sun's declination at about the time; and PZ is known, if the observer knows his latitude.

Having these three sides, we can always find the sought angle at the pole, by the equations already given in formulas (T), or (U); but these formulas require the use of the *co.latitude* and the *co.altitude*, and the practical navigator is very averse to taking the trouble of finding the complements of arcs, when he is quite certain that formulas can be made, which comprise but the arcs themselves.

The practical man, also, *very properly* demands the most concise practical results. No matter how much labor is spent in theorizing, provided we arrive at practical brevity; and for the especial accommodation of seamen, the following formula for finding time has been deduced.

From the fundamental equation of spherical trigonometry, taken from page 191 we have,

$$\cos.P = \frac{\cos.ZS - \cos.PZ \cos.PS}{\sin.PZ \sin.PS}$$

Now, in place of $\cos.ZS$, we take $\sin.ST$, which is, in fact, the same thing, and in place of $\cos.PZ$, we take sin.lat., which is also the same.

In short, let $A=$ the altitude of the sun, $L=$ the latitude of the observer, and $D=$ the sun's polar distance.

Then, . . $\cos.P = \dfrac{\sin.A - \sin.L \cos.D}{\cos.L \sin.D}$

But, . $2 \sin.^2 \frac{1}{2}P = 1 - \cos.P$ (See eq. 32, page 143.)

Therefore, $2 \sin.^2 \frac{1}{2}P = 1 - \dfrac{\sin.A - \sin.L \cos.D}{\cos.L \sin.D}$

$$= \frac{(\cos.L \sin.D + \sin.L \cos.D) - \sin.A}{\cos.L \sin.D}$$

$$= \frac{\sin.(L+D) - \sin.A}{\cos.L \sin.D}$$

Considering $(L+D)$ as a single arc, and applying equation (16), plane trigonometry, we have, after dividing by 2,

$$\sin.^2 \tfrac{1}{2}P = \frac{\cos.\left(\frac{L+D+A}{2}\right)\sin.\left(\frac{L+D-A}{2}\right)}{\cos.L \sin.D}$$

But, $\frac{L+D-A}{2} = \frac{L+D+A}{2} - A$ and if we assume

$S = \frac{L+D+A}{2}$, we shall have,

$$\sin.^2 \tfrac{1}{2}P = \frac{\cos.S \sin.(S-A)}{\cos.L \sin.D}$$

Or, $\sin. \tfrac{1}{2}P = \sqrt{\frac{\cos.S \sin\ (S-A)}{\cos.L \sin.D}}$

This is the final result, when the radius is unity, and when the radius is greater by R, then the sin. $\tfrac{1}{2}P$, will be greater by R; and, therefore, the value of this sine, corresponding to our tables is,

$$\sin. \tfrac{1}{2}P = \sqrt{\left(\frac{R}{\cos.L}\right)\left(\frac{R}{\sin D}\right)\cos.S \sin.(S-A)}$$

This equation is known as the sailor's formula for time, and a very concise and beautiful formula it is; it is used by thousands who have little knowledge of how it is obtained, or who know little of the amount of science there is wrapt up in it.

When the observer has logarithmic tables that contain *secants* and *cosecants*, the above equation can be modified.

Because, $\sec.L = \frac{R^2}{\cos.L}$ and $\operatorname{cosec}.D = \frac{R^2}{\sin.D}$

(See equations, plane trigonometry, page 138.)

Therefore, $\sin.\tfrac{1}{2}P = \sqrt{\left(\frac{\sec.L}{R}\right)\left(\frac{\operatorname{cosec}.D}{R}\right)\cos.S \sin.(S-A)}$

Here, then, we have *four* distinct logarithms to be added together and divided by 2, which is extracting square root.

The first logarithm is the secant of the latitude, diminished by the index 10; the second is the cosecant of the polar distance, diminished by the index 10; the third is the cosine of the half sum of altitude, latitude, and polar distance; and the fourth is the sine of an arc, found by diminishing this half sum by the altitude.

Navigators retain this formula in memory by the following words:

Altitude—latitude—polar distance—half sum—remainder; secant—cosecant—cosine—sine.

EXAMPLE.

In latitude 39° 6′ 20″ north, when the sun's declination was 12° 3′ 10″, north, the true altitude* of the sun's center was observed to be 30° 10′ 40″, *rising*. What was the apparent time?

Alt.	30°	10′	30″	.	
Lat.	39	6	20	cos.com.	.110146
P.D.	77	56	50	sin.com.	.009680
	2)147	13	40		
S=	73	36	50	cos. .	9.450416
(*S*—*A*)=	43	26	20	sin. .	9.837299
					2)19.407541
	30	22	5	sin.	9.703770
			2		
P=	60	44	10		

This angle, converted into time, at the rate of 15° to one hour, or 4 minutes to 1°, gives 4h. 2m. 56s. from apparent noon; and as the sun was rising, it was before noon, or

7h. 57m. 4s. A. M

If to this the equation of time were given and applied, we should have the mean time; and if such time were compared to a clock or watch, we could determine its error. A good observer, with a good instrument, can, in this manner, determine the local time within 4 or 5 seconds.

* The instrument used, the manner of taking the altitude, its correction for refraction, semidiameter, and other practical or circumstantial details, do not belong to a work of this kind, but to a work on practical astronomy or navigation.

The great importance of determining the exact time, at sea, is to determine the longitude, which is but the difference of the local time between the observer's meridian and the assumed prime meridian.

A timepiece, of nice and delicate construction, called a chronometer, by its rate of motion and adjustment, will show the time at Greenwich, or at any other known meridian to which it refers; and this time, compared with an observation on the sun, will determine the amount of difference in local times, which is, in substance, longitude.

The same triangle, PZS, gives the bearing of the sun, which is is called its azimuth; that is, the angle PZS is the azimuth from the north, and its supplement, HZS, is its azimuth from the south. This is the true bearing; and if the bearing per compass is the same, then the compass has no variation; if different, the amount of difference gives the amount of the variation of the compass.

HOW TO MANAGE A LOCAL SOLAR ECLIPSE.

We shall touch this subject only so far as to show the application and utility of spherical trigonometry.

The angular semidiameter of the sun is about 15′, and that of the moon, about the same; and, of course, when an eclipse of the sun commences or ends, the apparent distance between the sun and moon cannot be greater than about 32′, or a little more than half a degree.

The nautical almanac, or the astronomical tables, will give us the time when the sun and moon fall into line on the same meridian of *right ascension,* and give us, also, their difference in declinations, at the same time, together with all the other necessary elements, such as semidiameters, horizontal parallax, hourly motions, &c.

Now let us take the time when the moon is in conjunction with the sun in *right ascension,* and demand the apparent distance between the centers of the sun and moon, as seen from any particular locality.

By the time as given in the nautical almanac, we know the sun's distance from the *local* meridian, either east or west.

Look at the last figure. Let S represent the position of the sun's center, P the pole, and Z the zenith of the observer.

Then, in the triangle ZPS, we know the two sides, ZP and PS; and from the apparent time, we know their included angle, ZPS.

The declination of both sun and moon is also given in the nautical almanac, corresponding to this time; and their difference gives the space which we represent by Sm, on our figure. From the triangle PZm (two sides and angle included), compute Zm and the angle ZmP.

The effect of parallax is to depress the body in a vertical direction; and if m is its true place, as seen from the center of the earth, n may represent its apparent place, as seen by the observer, whose zenith is Z.

The arc mn is computed from the horizontal parallax, by the following proportion, p representing the lunar horizontal parallax.

$$\text{Rad.} : \cos. ☽ \text{ app.altitude} = p : mn.$$

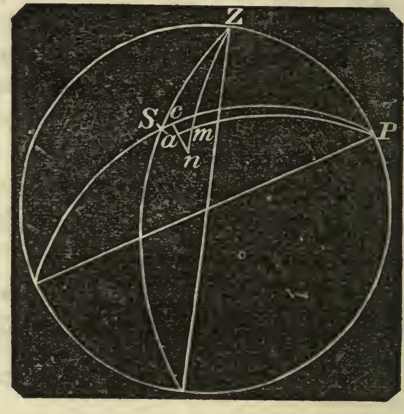
The angle $Smn = ZmP$, and the angle ZmP is computed from the triangle PZm. Now, the triangle Smn is always very small; the sides are never more than a degree in length, and are generally much less; and it therefore may be regarded as a plane triangle, with two sides, Sm and mn, and the angle Smn, between them, given. From these data we can compute the distance between S and n; and if that distance is less than the sum of the semidiameters of the sun and moon, the sun must then be in an eclipse—otherwise it is not.

But whether the distance between S and n is less, equal, or greater than the semidiameters of the sun and moon, by it the computer can assume an approximate time for the beginning or end of the eclipse, as the case may be.

In case the computer wishes to compute the apparent distance between *sun* and *moon*, corresponding to any other time than that of conjunction in *right ascension*, he may assume any interval before or after that period; and by the moon's motion from the sun during that interval, he can put the moon in its true place, at m.

Now, by the help of the spherical triangle PZm, and the moon's horizontal parallax, the distance mn can be computed as before;

and by means of the little triangle *mna*, we compute the distances *na* and *am*. The distance *na* is parallax in right ascension, and *ma* is parallax in declination. Parallax increases the moon's right ascension when the moon is east of the meridian, and diminishes it when west of the meridian.



Now, the difference between *PS* and *Pa*, is the apparent difference of declination of the sun and moon; and *nc* is the apparent difference of right ascension of the same bodies; *ca* is the real difference in right ascension. The distances *Sc* and *cn*,* expressed in *seconds* of *arc* as linear units, form two sides of a right angled plane triangle; and the distance *Sn*, the hypotenuse, is the apparent distance between the center of the sun and the center of the moon; and just at the commencement or end of an eclipse, that distance will be equal to the semidiameter of the sun, added to the semidiameter of the moon.

But it would be only accident if an operator should assume the exact time of the beginning or end of an eclipse; but the distance *Sn*, computed, would indicate whether the eclipse had already commenced or ended, or would commence or end within some very short interval of time.

Astronomers, however, are in the habit of taking two intervals of time, about 10 or 15 minutes asunder, between which they know the eclipse will commence, and compute the apparent distance, *Sn*, for these two periods; one of them will be less, and the other greater than the sum of the two semidiameters; and thus they find data to proportion to the commencement or end in question.

By the same principles astronomers compute the beginning and end of occultations.

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* The number of seconds in *cn* must be multiplied by the cosine of the declination, because *cn* is an arc of a small circle.

MISCELLANEOUS ASTRONOMICAL EXAMPLES.

1. In latitude 40° 48′ north, the sun bore south 78° 16′ west, at 3h. 37m. 59s. P. M., apparent time. Required his altitude and declination.

Ans. The altitude 36° 46′, and declination 15° 32′ north.

2. In north latitude, when the sun's declination was 14° 20′ north, his altitudes, at two different times on the same forenoon, were 43° 7′+,* and 67° 10′+; and the change of his azimuth, in the interval, 45° 2 . Required the latitude. *Ans.* 34° 20′ north.

3. In latitude 16° 4′ north, when the sun's declination is 23° 2′ north. Required the time in the afternoon, and the sun's altitude and bearing when his azimuth neither increases nor decreases.

Ans. Time 3h. 9m. 26s. P. M., altitude 45° 1′, and bearing north 73° 16′ west.

4. The sun set south west ½ south, when his declination was 16° 4′ south. Required the latitude. *Ans.* 69° 1′ north.

5. The altitude of the sun, when on the equator, was 14° 28′+, bearing east 22° 30′ south. Required the latitude and time.

Ans. Latitude 56° 1′, and time 7h. 46m. 12s. A. M.

6. The altitude of the sun was 20° 41′ at 2h. 20m. P. M, when his declination was 10° 28′ south. Required his azimuth and the latitude. *Ans.* Azimuth south 37° 5′ west, latitude 51° 58′ north.

7. If, on August 11, 1840, Spica set 2h. 26m. 14s. before Arcturus, hight of the eye 15 feet, required the north latitude.

Ans. 38° 46′ north.

8. If, on November 14, 1829, Menkar rise 48m. 3s. before Aldebaran, hight of the eye 17 feet, required the north latitude.

Ans. 39° 33′ north.

9. In latitude 16° 40′ north, when the sun's declination was 23° 18′ north, I observed him twice, in the same forenoon, bearing north 68° 30′ east. Required the times of observation, and his altitude at each time.

Ans. Times 6h. 15m. 40s. A. M., and 10h. 32m. 48s. A. M., altitudes 9° 59′ 36″, and 68° 29′ 42″.

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* Plus means rising ; and, of course, forenoon.

LUNAR OBSERVATIONS.

The moon revolves through a great circle of the celestial sphere in about 27 days and 8 hours; and astronomers are able to designate its exact position in respect to the stars, corresponding to any definite time.

But the observer is supposed to be at the center of the earth. The moon is never seen by an observer in *exactly its true plane,* unless the observer is in a line between the center of the earth and the center of the moon; that is, unless the moon is in the zenith of the observer; in all other positions the moon is depressed by parallax, and appears nearer to those stars which are below her, and further from those that are above her, than would appear from the center of the earth.

The true distance between the sun and moon, or between a star and the moon, can be deduced from the apparent distance, by the application of spherical trigonometry.

The apparent altitudes of the two objects must be taken, and corrected for parallax and refraction.

Let Z be the zenith of the observer, S' the apparent place of the sun or star, and S its true place; also, let m' be the apparent place of the moon, and m its true place, as seen from the center of the earth.

With the observed sides of the spherical triangle $ZS'm'$, we compute the angle at Z; then, in the triangle ZSm we have the two sides ZS and Zm, and the included angle at Z, from which we compute the side Sm, which is the *true distance.*

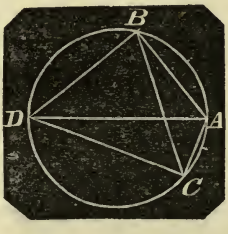
To the definite, true distance, there is a corresponding definite *Greenwich* time, which the practical navigator can find with the utmost facility. This time at the *first meridian,* compared with the local time deduced from the altitude of the sun, will of course give the longitude.

To deduce the true distance from the apparent, is called *working a lunar,* and is a subject of considerable perplexity to the young navigator; but, by means of auxiliary tables, and rules for delicate approximations, science and art have nearly overcome all difficulties, and a good operator can now work a lunar in about *five minutes.*

We here only give a view of the scientific principles involved. For complete practical knowledge we must consult books on navigation.

APPENDIX TO TRIGONOMETRY.

For the benefit of those who may desire to cultivate a taste for mathematical science, we give the following exercises, which are designed to strengthen the powers for geometrical investigations.



To demonstrate equations (7), (8), (9), and (10), geometrically, the pupil must be fully impressed with the following principles:

1. *An angle in a semicircle is a right angle.*

2. *If one side of a right angled triangle is made the sine of its opposite angle, the other side will be the cosine of the same angle.*

(See proposition 3, page 147.)

3. Any chord is double the sine of half the arc. (See observation 3. page 138.)

4. Observe theorem 21, book 3.

Now from A, any point on a circle, take AB, the double of any arc designated by a, and AC, double of any arc designated by b. Draw AD, the diameter, and consider its value equal 2, twice the radius of unity. Join BD and DC.

Then, by reason of the quadrilateral in a circle, we have,

$$AD \cdot BC = AB \cdot DC + AC \cdot BD \qquad (1)$$

But, $\left.\begin{array}{l} AB=2 \sin.a \\ BD=2 \cos.a \end{array}\right\}$ Also, $\left.\begin{array}{l} AC=2 \sin.b \\ DC=2 \cos.b \end{array}\right\}$

$$BC=2 \sin.(a+b), \text{ and } AD=2$$

Substituting these values in (1), we have

$$4 \sin.(a+b)=2 \sin.a\ 2 \cos.b+2 \cos.a\ 2 \sin.b$$

Dividing by 4, and

$$\sin.(a+b)=\sin a \cos.b+\cos.a \sin.b$$

Now let the arc $CAB=2a$, and $AB=2b$; then $AC=2a-2b$

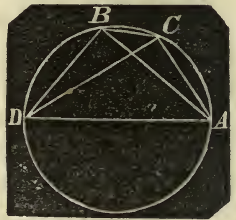
And, . $CB=2\sin.a$, $AC=2\sin.(a-b)$, $BD=2\cos.b$

$$AB=2\sin.b, \quad DC=2\cos.(a-b)$$

Substituting these values in equation (1), we have

$$4\sin.a=2\sin.b\ 2\cos.(a-b)+2\sin.(a-b)2\cos.b$$

Dividing by 4, $\sin.a=\sin.b\cos.(a-b)+\sin.(a-b)\cos.b$



To demonstrate equation (8.) Let the arc $AB=2a$, $AC=2\,b$;

Then, . $BC=2(a-b)$

And, by reason of the quadrilateral,

$$AB{\cdot}DC=BC{\cdot}AD+AC{\cdot}BD \quad (2)$$

But, . $\left.\begin{array}{l}AB=2\sin.a\\BD=2\cos.a\end{array}\right\}$ Also, $\left.\begin{array}{l}AC=2\sin.b\\DC=2\cos.b\end{array}\right\}$

$$AD=2, \text{ and } BC=2\sin.(a-b)$$

These values substituted above, and we have

$$2\sin.a\ 2\cos.b=4\sin.(a-b)+2\sin.b\ 2\cos.a$$

Dividing by 4, transposing, &c.,

And $\sin.(a-b)=\sin.a\cos.b-\sin.b\cos.a$

Again, let the *arc* $AC=2a$, the *arc* $CB=2b$; then the *arc*

$$ACB=2(a+b),$$

And the chord $\left.\begin{array}{l}AB=2\sin.(a+b)\\BD=2\cos.(a+b)\end{array}\right\}$ $\left.\begin{array}{l}AC=2\sin a\\DC=2\cos.a\end{array}\right\}$

$$AD=2, \text{ and } BC=2\sin.b$$

Substituting these values in equation (2), we have,

$$2\cos.a\ 2\sin.(a+b)=4\sin.b+2\sin.a\ 2\cos.(a+b)$$

Dividing by 4,

$$\cos.a\sin.(a+b)=\sin.b+\sin.a\cos.(a+b)$$

To demonstrate the truth of equation (10), we use the last figure, conceiving the arc AC to be $2a$, the arc BD to be $2b$.

Then the arc BC will be measured by $(180°—2(a+b))$; its half will therefore be measured by $90°—(a+b)$.

But, . $2\sin.(90°—a+b)=2\cos.(a+b)=BC$

On this hypothesis,

$$\text{The chord } \left.\begin{array}{l} AC=2\sin.a \\ CD=2\cos.a \end{array}\right\} \text{ Also, } \left.\begin{array}{l} DB=2\sin.b \\ AB=2\cos.b \end{array}\right\}$$

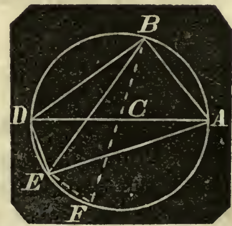
$$AD=2, \text{ and } BC=2\cos.(a+b)$$

Substituting these values in equation (2), we have

$$2\cos.b\ 2\cos.a=4\cos.(a+b)+2\sin.a\ 2\sin.b$$

Dividing and transposing,

$$\cos.(a+b)=\cos.a\cos.b—\sin.a\sin.b$$



To demonstrate equation (10). Draw the diameter AD, and on one side of it take the arc $AB=2a$, and on the other side take the arc $DE=2b$. Join BD, AE, and BE. From B, draw BCF through the center of the circle; then the arc DEF = the arc AB, and EF is the difference of the arcs AB and DE; it is therefore measured by $2(a—b)$.

Now, in the quadrilateral $ABDE$, we have

$$AD\cdot BE=AB\cdot DE+DB\cdot AE$$

$$\left.\begin{array}{l} AB=2\sin.a \\ BD=2\cos.a \end{array}\right\} \text{ Also, } \left.\begin{array}{l} DE=2\sin.b \\ AE=2\cos.b \end{array}\right\}$$

$$AD=2, \text{ and } BE=2\cos.(a—b)$$

These values, substituted in the last equation, will give

$$4\cos.(a—b)=2\sin.a\ 2\sin.b+2\cos.a\ 2\cos.b$$

$$\cos.(a—b)=\sin.a\sin.b+\cos.a\cos.b$$

PROBLEMS FOR EXERCISE.

1. Show, *geometrically*, that $\text{rad.}\cdot(\text{rad.}+\cos.A)=2\cos^2\frac{A}{2}$; that $\text{rad.}\cdot(\text{rad}—\cos.A)=2\sin^2\frac{A}{2}$; that $\text{rad.}\cdot\sin.2A=2\sin.A\cdot\cos.A$;

2. Prove that $\tan.A+\tan.B=\frac{\sin.(A+B)}{\cos.A\cdot\cos.B}$, radius being unity.

3. Demonstrate, *geometrically*, that $\text{rad.}\cdot\sec.2A=\tan.A\ \tan.2A+\text{rad}^2$.

4. Show that in any plane triangle, the base is to the sum of the other two sides, as the sine of half the vertical angle is to the cosine of half the difference of the angles at the base.

5. Show that the base of a plane triangle is to the difference of the other two sides, as the cosine of half the vertical angle is to the sine of half the difference of the angles at the base.

6. The difference of two sides of a triangle, is to the difference of the segments of a third side, made by a perpendicular from the opposite angle, as the sine of half the vertical angle is to the cosine of half the difference of the angles at the base; required the proof.

NOTE.

When we give our attention to the relations existing between the arc of a circle and its sine, cosine, and tangent, it becomes very desirable to find some law which will invariably and unconditionally *numerically connect* the arc with its trigonometrical lines; and the object has been accomplished, though not in as elementary a manner as is desirable for a work like this.

In the calculus the process is clear and simple; but simple as it may be, the reader must first understand the calculus before it can be even comprehensible to him.

We give the following investigation, independent of the calculus, taken from the French works of Legendre, with our own modifications and illustrations. By a little careful study, any one can thoroughly comprehend it, who is familiar with algebraic equations, and understands the *binomial theorem.*

LEMMA.

If there be an algebraic equation in which the members consist of quantities, part real and part imaginary, then the real quantities in the two members are equal, and the imaginary quantities are equal.

N. B. Imaginary quantities contain the factor $\sqrt{-1}$, and such quantities are, emphatically, *imaginary;* they have no real existence.

Suppose we have an equation in which the sum of the real quantities in the first member is represented by A; and the sum of the like quantities in the second member by B. Also, the sum of the imaginary quantities in the first member, suppose represented by $S\sqrt{-1}$, and the sum of the like quantities in the second member by $T\sqrt{-1}$; that is, suppose the following equation to exist.

$$A+S\sqrt{-1}=B+T\sqrt{-1}$$

Then, $\quad A=B$, and $S\sqrt{-1}=T\sqrt{-1}$

If A is not equal to B, one must be greater than the other; and as they are supposed to be real and definite quantities, their difference must be real and definite; and, therefore, we can represent it by the definite quantity D.

That is, suppose A greater than B by D; then the equation becomes

$$B+D+S\sqrt{-1}=B+T\sqrt{-1}$$

Strike out B from both members, and transpose $S\sqrt{-1}$

Then, $\quad D=T\sqrt{-1}-S\sqrt{-1}=(T-S)\sqrt{-1}$

That is, a real quantity equal to an imaginary one—a perfect *absurdity;* and this absurdity is in consequence of supposing A not equal to B; therefore, we must admit that $A=B$.

It necessarily follows that

$$S\sqrt{-1}=T\sqrt{-1}$$

Let a represent any arc, the radius unity; then,

$$\cos.^2a+\sin.^2a=1$$

Conceive the first member as composed of the two factors,

$$\cos.a+h\sin.a, \text{ and } \cos.a-h\sin.a$$

The product of these two factors, is

$\cos.^2a-h^2\sin.^2a$; and, by hypothesis, this product must equal the first member of the equation; that is,

$$\cos.^2a-h^2\sin.^2a=\cos.^2a+\sin.^2a$$

Dropping $\cos.^2a$ from both members, there remains

$$-h^2\sin.^2a=\sin.^2a$$

Dividing by $\sin.^2 a$, and changing signs, we have

$h^2=-1$, or $h=+\sqrt{-1}$, which shows that the coefficient, h, is imaginary.*

The different powers of h are

$$h=+1\sqrt{-1},\ h^2=-1,\ h^3=-1\sqrt{-1},\ h^4=+1,\ h^5=+\sqrt{-1},\ h^6=-1,$$

and so on. Observe that all the even powers of h are rational quantities; in short, units, with the signs *plus* and *minus* alternating.

Thus, . $h^2=-1$, $h^4=+1$, $h^6=-1$, $h^8=+1$, and so on.

All the odd powers are *imaginary*, and the signs alternating.

If we multiply the two similar factors,

$$\cos.a+h\sin.a$$

And, . . . $\cos.b+h\sin.b$

Product will be, $\cos.a\cos.b+(\sin.a\cos.b+\cos.a\sin.b)h+h^2\sin.a\sin.b$

Now let $h=\sqrt{-1}$, and $h^2=-1$; then this product is

$$(\cos.a\cos.b-\sin.a\sin.b)+(\sin.a\cos.b+\cos.a\sin.b)\sqrt{-1}$$

Comparing this expression with equations (9) and (7), page 141, we perceive that it is the same as

$$\cos.(a+b)+\sin.(a+b)\sqrt{-1};$$

Hence, $(\cos.a+h\sin.a)(\cos.b+h\sin.a)=\cos.(a+b)+h\sin.(a+b)$

In case we give to h its particular imaginary value, $\sqrt{-1}$

It is very remarkable that the product of these factors can be found by simply adding the arcs, which is a property analagous to logarithms.

If we make $a=b$ in the preceding equation, we have

$$(\cos.a+h\sin.a)(\cos.a+h\sin.a)=\cos.2a+h\sin.2a \qquad (1)$$

$$(\cos.a+h\sin.a)(\cos.2a+h\sin.2a)=\cos.3a+h\sin.3a \qquad (2)$$

$$(\cos.a+h\sin.a)(\cos.3a+h\sin.3a)=\cos.4a+h\sin.4a \qquad (3)$$

and so on.

The first member of equation (1), is

$$(\cos.a+h\sin.a)^2$$

---

* This investigation shows, also, that the sum of any two squares may be regarded as the product of two binomial factors.

Thus, . . . $x^2+y^2=(x+y\sqrt{-1})(x-y\sqrt{-1})$

The first member of equation (2), is

$(\cos.a+h\sin.a)^3$, and so on. Therefore, in general, if n is taken to represent any entire number whatever, we shall have,

$$\cos.na+h\sin.na=(\cos.a+h\sin.a)^n$$

But, $\quad(\cos.a+h\sin.a)^n=\cos.^na(1+h\tan.a)^n$

Because, $\quad\dfrac{\sin.a}{\cos.a}=\tan.a$

Hence, $\quad\cos.na+h\sin.na=\cos.^na(1+h\tan.a)^n \qquad (4)$

Expanding the binomial in the second member, we have

$$(1+h\tan.a)^n=1+nh\tan.a+n\frac{n-1}{2}h^2\tan^2a+n\frac{n-1}{2}\ \frac{n-2}{3}h^3\tan^3a,\ \ \&c.$$

Substituting the expanded binomial in equation (4), it becomes

$$\cos.na+h\sin.na=$$

$$\cos.^na(1+nh\tan.a+n\frac{n-1}{2}h^2\tan.^2a+n\frac{n-1}{2}\ \frac{n-2}{3}h^3\tan.^3a,\ \ \&c.)$$

Calling to mind the principles explained in the preceding lemma, and recollecting that all the terms containing the odd powers of h must be imaginary, and all the other terms real, therefore, we may put $\cos.na$ equal to all the real quantities in the series, multiplied by the factor $\cos.^na$; and the *imaginary* quantity $h\sin.na$, must be put equal to all the terms in the series containing the odd powers of h, and the whole multiplied by the factor $\cos.^na$.

But as every term of this equation will contain h, we can divide by h, and thus convert every odd power into an even power, and change the equation from imaginary terms to real terms.

Thus, by equating the parts of the preceding equation, we have

$$\cos.na=$$

$$\cos^na(1+n\ \frac{n-1}{2}\ h^2\tan.^2a+n\ \frac{n-1}{2}\ \frac{n-2}{3}\ \frac{n-3}{4}\ h^4\tan.^4a+\ \&c.)$$

$$\sin.na=\cos.^na(n\tan.a+n\ \frac{n-1}{2}\ \frac{n-2}{3}\ h^2\tan.^3a+n\ \frac{n-1}{2}\ \frac{n-2}{3}\ \frac{n-3}{4}\ \frac{n-4}{5}\ h^4\tan.^5a+\ \&c.)$$

Put $x=na$. Then $n=\frac{x}{a}$. Also observe that $h^2=-1$, and $h^4=1$, and so on, alternately. Making these substitutions, the preceding equations become

$$\cos.x=\cos.^n a\left(1-\frac{x\cdot x-a}{1\cdot 2}\,\frac{\tan^2 a}{a^2}+\frac{x(x-a)(x-2a)(x-3a)}{1\cdot 2\cdot 3\cdot 4}\,\frac{\tan.^4 a}{a4}\ \&c.\right)$$

$$\sin.x=\cos.^n a\left(\frac{x}{1}\,\frac{\tan.a}{a}-\frac{x(x-a)(x-2a)}{1\cdot 2\cdot 3}\,\frac{\tan.^3 a}{a^3}\right.$$
$$\left.\frac{x(x-a)(x-2a)(x-3a)(x-4a)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\,\frac{\tan.^5 a}{a^5}\ \&c.\right)$$

In these equations the arc a may be taken of any value whatever, and when a represents a very small arc, $\frac{\tan.a}{a}$ is very near unity, and is exactly unity when $a=0$.

Also, when $a=0$, $\cos.a=1$, and any power of 1 is 1; therefore, $\cos.^n a=1$. Making these substitutions, the final results will be,

$$\cos.x=1-\frac{x^2}{1\cdot 2}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}-\frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\&c.$$

$$\sin.x=x-\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{x^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}+\&c.$$

To apply these equations, and show their practical utility in the primary computions for the natural sines and cosines, we require the natural sine and cosine of 3°.

When radius is unity, the arc of 180° is 3.14159265.

Therefore, the arc of 3° is .052359877.

Hence, . . . $-\frac{x^2}{2}=-0.001370733$

And, ... $\frac{x^4}{24}=+0.000000313$

Therefore, from . . . 1.000000313

Take 0.001370733

$\cos.x=$ 0.998629580 the cos. of 3°.

$x=$ 0.052359877

$-\frac{x^3}{6}=$ 6 000023923

$\frac{x^5}{120}=$ 0.000000003

$\sin.x=$ 0.052335957 the sin. of 3°.

In like manner we may compute the sine and cosine of any other arc. But the greater the arc, the slower the series will converge; and, in case of large arcs, a greater number of terms must be taken to obtain a result of equal exactness; the series, however, is never used for large arcs, but the combinations of other formulas are then used. These formulas are more practical than any other hitherto given for the same object; but their theoretical investigation is supposed to require more power than a learner can at first possess.

CONIC SECTIONS.

DEFINITIONS.

1. Conic Sections are the figures made by a plane, cutting a cone.

2. There are *five* different figures that can be made by a plane cutting a cone, namely: a *triangle*, a *circle*, an *ellipse*, a *parabola*, and an *hyperbola*.

Remark. The three last mentioned are commonly regarded as embracing the whole of conic sections; but with equal propriety the triangle and the circle might be admitted into the same family. On the other hand we may examine the properties of the ellipse, the parabola, and the hyperbola, in like manner as we do a triangle or a circle, without any reference to a cone, whatever.

It is important to study these curves on account of their extensive application to astronomy and other sciences.

3. If a plane cut a cone through its vertex, and terminate in any part of its base, the section will evidently be a triangle.

4. If a plane cut an upright cone parallel to its base, the section will be a circle.

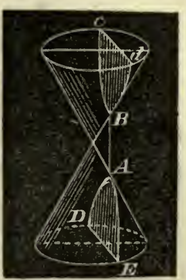
5. If a plane cut a cone obliquely through both sides of the cone, the section will represent a curve, called an ellipse.

6. If a plane cut a cone *parallel* to one side of the cone, or what is the same thing, if the cutting plane and the side of the cone make equal angles with the base, then the section will represent a parabola.

7. If a plane cut a cone, making a greater angle with the base than the side of the cone makes, then the section is an hyperbola.

8. And if all the sides of a cone be continued through the vertex forming an opposite equal cone, and the plane be also continued to cut the opposite cone, this latter section will be the opposite hyperbola to the former

9. The vertices of any section are the points where the cutting plane meets the opposite sides of the cone, or the sides of the vertical triangular section, as A and B.



Hence the ellipse, and the opposite hyperbolas, have each two vertices; but the parabola only one; unless we consider the other as at an infinite distance.

10. The axis, or transverse diameter of a conic section, is the line or distance AB between the vertices.

Hence, the axis of a parabola is infinite in length, AB being only a part of it.

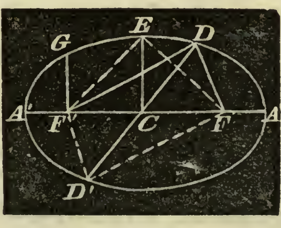
THE ELLIPSE.

When we know how to describe a circle, we can give a definition of it; and without conceiving it to be a conic section, we can go on and investigate its properties. So with the ellipse. When we know how to describe it, we can give a definition of it, and go on and investigate its properties; and we shall do so without conceiving it to be a conic section.

PROBLEM.

To describe an Ellipse.

Take any two points, as F and F'. Take a thread, longer than the distance between F and F', and fasten one extremity at the point F, the other at F'. Then take a pencil and put it in the *loop*, and move the pencil entirely round the fixed points, keeping the thread at equal tension in every part. The pencil thus passing round the points F and F', describes a curve, as is represented in the adjoining figure, and it is called an ellipse; hence an ellipse may be defined as on the following page:



DEFINITIONS.

1. An ellipse is a plane curve, confined by two fixed points; and the sum of the distances from any point in the curve to the fixed points, is constantly the same.

2. The two fixed points are called the *foci.*

3. The center is the point C, the middle point between the foci.

4. A *diameter* is a straight line through the center, and terminated both ways by the curve.

5. The extremities of a diameter are called its *vertices.*

Thus, DD' is a diameter, and D and D' are its *vertices.*

6. The *major axis* is the diameter which passes through the *foci.*

Thus, AA' is the *major axis.*

7. The *minor axis* is the diameter at right angles to the major axis. Thus CE is the *semi* minor axis.

8. The distance between the center and either focus is *called* the *excentricity* when the *semi major* axis is unity.

That is, the *excentricity* is the ratio between CA and CF; or it is $\frac{CF}{CA}$; and, of course, always less than unity. The less the excentricity, the nearer the ellipse approaches the circle.

9. A tangent is a straight line which meets the curve in one point, only; and, being produced, does not cut it.

10. An ordinate to a diameter is a straight line drawn from any point of the curve, *parallel to a tangent,* passing through one of the vertices of *that* diameter.

N. B. A diameter and its ordinate are not at right angles, unless the diameter be either the *major* or *minor* axis.

11. The points into which a diameter is divided by an ordinate, are called *abscissas.*

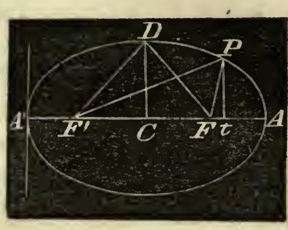
12. The *parameter* of a diameter is the double ordinate which passes through one of the foci.

13. The parameter of the major axis is called the principal parameter, or *latus-rectum.* Thus, $F'G$ is one half of the principal parameter.

14. A subtangent is that part of the axis produced, which is included between a tangent and the ordinate drawn from the point of contact.

PROPOSITION 1. THEOREM.

The major axis is always equal to the sum of the two lines drawn from any point in the curve to the foci.



Suppose the pencil at D to revolve along in the loop, holding the threads $F'D$ and FD at equal tension; and when D arrives at A, there will be *two lines* of threads between F and A Hence, the entire length of the threads will be measured by $F'F+2FA$. Also, when D arrives at A', the length of the threads is measured by $FF'+2F'A'$.

Therefore, . $FF'+2FA=FF'+2F'A'$

Hence, ... $FA=F'A'$

From the expression $FF'+2FA$, take away FA, and add $F'A'$, and the sum will not be changed, and we have

$$FF'+2FA=A'F'+FF'+FA=A'A$$

Hence, . . . $F'D+FD=A'A$ *Q. E. D.*

PROPOSITION 2. THEOREM.

The distance from either focus to the extremity of the minor axis, is equal to half the major axis.

As $F'C=CF$ (see last figure), and CD is at right angles to $F'F$, therefore, . . . $F'D=FD$.

But, . . . $F'D+FD=A'A$

Or, ... $2FD=A'A$

Or, ... $FD=$ half $A'A$, or CA. *Q. E. D.*

Scholium. *Half the minor axis is a mean proportional between the distance from either focus to the principal vertices.*

In the right angled triangled FCD we have

$$CD^2=FD^2-FC^2$$

But, ... $FD=AC$

Therefore, . . $CD^2=AC^2-FC^2$

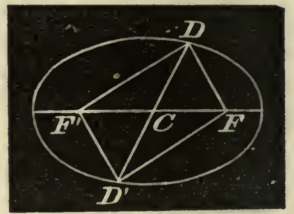
$$=(AC+FC)(AC-FC)$$

$$=AF'\times AF$$

Or, . . . $AF:CD=CD:FA'$

PROPOSITION 3. THEOREM.

Every diameter is bisected in the center.

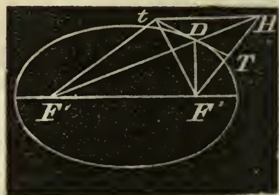


Let D be any point in the curve, and C the center. Join DC, and produce it. From F' draw $F'D'$ parallel to FD; and from F draw FD' parallel to $F'D$. The figure $DFD'F'$ is a parallelogram by construction; and therefore its opposite sides are equal.

Hence, the sum of the two sides $F'D'$ and $D'F$ is equal to $F'D$ and DF; therefore, by definition 1, the point D' is in the ellipse. But the two diagonals of a parallelogram bisect each other; therefore, $DC=CD'$, and the diameter DD' is bisected at the center, C, and DD' represents any diameter. Therefore, &c. *Q. E. D.*

PROPOSITION 4. THEOREM.

A tangent to the ellipse makes equal angles with the two straight lines drawn from the point of contact to the foci.



Let F and F' be the foci, and D any point in the curve. Join $F'D$ and FD, and produce $F'D$ to H, making $DH=DF$, and join FH. Bisect FH in T. Join TD and produce it to t.

Now by theorem 15, book 1, the angle $FDT=$ the angle HDT, and $HDT=$ its opposite vertical angle, $F'Dt$.

Therefore, . . $FDT=F'Dt$

It now remains to be shown that Tt is a tangent, and only meets the curve at the point D.

If possible, let it meet the curve in some other point, as t, and join Ft, tH, and $F't$.

By theorem 15, book 1, $Ft=tH$

To each of these add $F't$;

Then, . . $F't+tH=F't+Ft$

But $F't+tH$ are, together, greater than $F'H$, because a straight line is the shortest distance between two points; that is, $F't+Ft$, the two lines from the foci, are, together, greater than FH, or greater than $F'D+FD$; therefore, the point t is without the ellipse, and t is any point in the line Tt, except D; therefore, Tt is a tangent, touching the ellipse at D, and it makes equal angles with the lines drawn from the point of contact to the foci.

Q. E. D.

Cor. The tangents at the vertices of either axis are perpendicular to that axis; and as the ordinates are parallel to the tangents, it follows that all ordinates to the major or minor axis must cut one axis at right angles, and be parallel to the other axis.

Scholium. Any point in the curve may be considered as a point in a tangent to the curve at that point.

It is found by experiment that *light, heat,* and *sound,* when they approach to, are reflected off, from any surface at equal angles; that is, any and every single ray makes the angle of reflection equal to the angle of incidence.

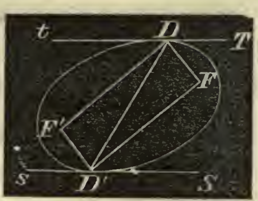
Therefore, if a light is placed at one focus of an ellipse, and the sides a reflecting surface, the reflections will concentrate at the other focus. If the sides of a room be elliptical, and a stove is placed at one focus, it will concentrate heat at the other.

Whispering galleries are made on this principle, and all theaters and large assembly rooms should more or less approximate to this figure. The concentration of the rays of heat from one of these points to the other, is the reason why they are called the *foci,* or burning points.

PROPOSITION 5. THEOREM.

Tangents to the ellipse, at the vertices of the diameter, are parallel to one another.

Let DD' be the diameter, and F' and F the foci. Join $F'D$, $F'D'$, FD, and FD'.



Draw the tangents, Tt and Ss, one through the point D, the other through the point D'. These tangents will be parallel.

By proposition 3, $F'D'FD$ is a parallelogram, and the angle $F'D'F$ is equal to its opposite angle, $F'DF$.

But the sum of all the angles that can be made on one side of a line, is equal to two right angles.

Therefore, by leaving out the equal angles which form the opposite angles of the parallelogram, we have

$$sD'F'+SD'F'=tDF'+TDF.$$

But, by proposition 4, $sD'F'=SD'F$; therefore, their sum is double of either one of them, and the above equation may be changed to . . $2SD'F=2tDF'$

Or, . . . $SD'F=tDF'$

But DF' and $D'F$ are parallel; therefore, $SD'F$ and tDF' are, in effect, alternate angles, showing that Tt and Ss are parallel.

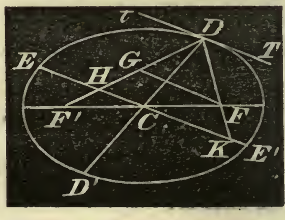
Q. E. D.

Cor. If tangents be drawn through the vertices of any two conjugate diameters, they will form a parallelogram circumscribing the ellipse.

PROPOSITION 6. THEOREM.

If, from the vertex of any diameter, straight lines are drawn through the foci, meeting the conjugate diameter, the part intercepted by the conjugate, is equal to half the major axis.

Let DD' be the diameter, and Tt the tangent. Draw EE' parallel to Tt. Join $F'D$ and DF, and produce DF to K; and from F draw FG parallel to EE' or Tt.



Now, by reason of the parallels,

we have the following equations among the angles.

$$\left.\begin{matrix} tDG=DGF \\ TDF=DFG \end{matrix}\right\} \text{ Also, } \left.\begin{matrix} tDG=DHK \\ TDF=DKH \end{matrix}\right\}$$

But, by proposition 4, $tDG=TDF$

Therefore, by equality, $DGF=DFG$

And, . . . $DHK=DKH$

Hence, the triangle DGF is *isosceles;* also, the triangle DHK is *isosceles.* Whence, . $DG=DF$, and $DH=DK$.

Because HC is parallel to FG, and $F'C=CF$,

Therefore, . . $F'H=HG$

Add . . . $DF=DG$

$$F'H+DF=DH$$

But the sum of the lines in both members of this equation is $F'D+DF$, which is equal to the major axis of the ellipse; therefore, either member is half the major axis; that is, DH, or its equal, DK, is each equal to half the major axis. *Q. E. D.*

PROPOSITION 7. THEOREM.

Perpendiculars from the foci of an ellipse upon a tangent, meet the tangent in the circumference of a circle, whose diameter is the major axis.

Let $F'F$ be the foci, C the center, and D a point in the ellipse, through which passes the tangent Tt. Join $F'D$ and FD, and produce $F'D$ to H, making $DH=FD$, and produce FD to G, making $DG=F'D$. Then $F'H$ and FG are each equal to the major axis, $A'A$.

Join FH, meeting the tangent in T, and join $F'G$, meeting it in t. Draw the dotted lines, CT and Ct.

By proposition 4, the angle $FDT=$ the angle $F'Dt$; and observing that opposite vertical angles are equal, therefore, the four angles formed by lines crossing at D, are all equal.

The triangles $DF'G$ and DHF are *isosceles* by construction, and as their vertical angles at D are bisected by the line Tt, therefore, $F't=tG$, and $FT=TH$.

Comparing the triangles $F'GF$ and $F'Ct$, we find FC equals the half of $F'F$, and $F't$ the half of FG; therefore, Ct is the half of FG. But $A'A=FG$; hence, $Ct=\frac{1}{2}A'A=CA$.

Comparing the triangles $FF'H$ and FCT, we find the sides FH and FF' cut proportionally in T and C; therefore, they are equiangular and similar, and CT is parallel to $F'H$, and equal to half of it. That is, CT is equal to CA; and CA, CT, and Ct, are all equal; and hence a circle described from the center, C, at the distance of CA, will pass through the points T and t. Therefore, perpendiculars, &c.

Q. E. D.

PROPOSITION 8. THEOREM.

The product of the perpendiculars from the foci upon a tangent, is equal to the square of half the minor axis.

Produce TC and GF' (see figure to the last proposition), and they will meet in the circle, at S; for FT and $F't$ are both perpendicular to the same line, Tt; they are, therefore, parallel; and the two triangles CFT and $CF'S$, having a side, FC, of the one, equal to CF', of the other, and their respective angles equal, therefore $CS=CT$, and S is in the circle, and $SF'=FT$.

Now, as $A'A$ and St are two lines that intersect each other in a circle, therefore, (th.17, b. 3)

$$SF'\times F't=A'F'\times F'A$$

$$FT\times F't=A'F'\times F'A$$

But, by the scholium to proposition 2, it is shown that

$A'F'\times F'A=$ the square of half the minor axis.

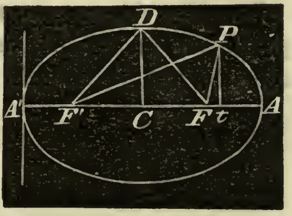
Hence, . . $FT\times F't=$ the square of half the minor axis.

Therefore, the product, &c. *Q. E. D.*

Cor. The two triangles, FTD and $F'tD$, are similar, and from them we have . $TF : F't=FD : DF'$; that is, *perpendiculars let fall from the foci upon a tangent, are to each other as the distances of the point of contact from the foci.*

PROPOSITION 9. PROBLEM.

Given the major axis and the distance between the foci of any ellipse, to find the relation between an abscissa of the major axis and its corresponding ordinate.



Let F' and F be the foci, C the center, and put CF', or $CF=c$, and $CA=A$. Then $F'D=A$, and in the triangle $F'DC$ or FDC, if the hypotenuse FD and FC are both known, then DC is known; therefore, we may put $CD=B$, and consider A, B, and c, known quantities.

Take any point on the major axis, as t, and draw tP at right angles to $A'A$.

Measuring from the point A', $A't$ is the *abscissa*, and tP is the corresponding *ordinate*.

The problem requires us to find the mathematical relation between these two lines. We can find it by the aid of the two right angled triangles $F'tP$ and FtP.

Put . . $A't=x$, and $tP=y$

Then . . $F't=A't-A'F'=x-(A-c)=x+c-A$

And . . $Ft=A't-A'F=x-(A+c)=x-c-A$

Put . . $F'P=r$, and $F'P=r'$

Then, . $F'P+FP=r'+r=2A \quad (1)$

In the triangle $F'Pt$ we have

$$(x+c-A)^2+y^2=r'^2 \quad (2)$$

In the triangle FPt we have

$$(x-c-A)^2+y^2=r^2 \quad (3)$$

By subtracting (3) from (2), expanding and reducing, we obtain

$$4cx-4cA=r'^2-r^2 \quad (4)$$

Or, . . . $4c(x-A)=(r'+r)(r'-r) \quad (5)$

But the first factor in the second member of equation (5) is equal to $2A$; hence we have

$$r'-r=\frac{2c}{A}(x-A) \qquad (6)$$

But, . . . $r'+r=2A$ (7)

By adding (6) and (7), then dividing by 2, and then subtracting (6) from (7), and dividing by 2, we have the two following equations:

$$r'=A+\frac{c}{A}(x-A) \qquad (8)$$

$$r=A-\frac{c}{A}(x-A) \qquad (9)$$

It should be observed that equations (8) and (9) are expressions for lines, one of which is called radius rector in astronomy.

By squaring equation (9), and comparing it with equation (3), equating the two values of r^2, we shall then have

$$x^2+c^2+A^2-2cx-2Ax+2cA+y^2=$$

$$A^2-2c(x-A)+\frac{c^2}{A^2}(x-A)^2$$

Or, . $x^2+c^2-2Ax+y^2=\frac{c^2}{A^2}(x^2-2xA+A^2)$

Or, $A^2x^2+c^2A^2-2A^3x+A^2y^2=c^2x^2-2c^2xA+c^2A^2$

Or, . $A^2y^2+(A^2-c^2)x^2=(A^2-c^2)2Ax$

Observing that $A^2-c^2=B^2$, the square of the semi minor axis, and substituting this value, the preceding equation becomes

$$A^2y^2+B^2x^2=2AB^2x$$

Hence, ... $y^2=\frac{B^2}{A^2}(2Ax-x^2)$ (10)

Or . . . . $y=\pm\frac{B}{A}\sqrt{2Ax-x^2}$ (11)

We cannot reduce this equation to lower terms, or condense it to a more simple form; and, therefore, it must rest as the final result; and, in the language of *analytical* geometry, it is called *the equation of the ellipse.*

Any definite value may be assigned to x, not greater than $2A$, and when any particular value is assigned, the equation will give the corresponding value of the *ordinate*, y, and as y has the double sign, it shows that y may be drawn both above and below $A'A$, or shows that the curve is symmetrical on both sides of $A'A$.

Now let us examine the result when particular values are given to x. At the point A' $x=0$; and this value of x put in the equation, gives $y=0$; obviously the proper result. Again, suppose $x=2A$, and this value of x put in the equation, gives

$$y=\pm\frac{B}{A}\sqrt{4A^2-4A^2}=\pm\frac{B}{A}\times 0$$

That is, $y=0$, for that point, also.

If we suppose $x=3A$, y will come out *imaginary;* showing that there is no *real* value to y beyond the point A; and in this way imaginary equations have real practical utility.

If we suppose $x=A$, then y will become $CD=B$.

If we make $A'F'=x$, then $x=A-c$; and this value put in the equation, gives . $y=\pm\frac{B}{A}\sqrt{(2A-x)(A-c)}$

$$=\pm\frac{B}{A}\sqrt{(A+c)(A-c)}=\pm\frac{B^2}{A}$$

By the definition, the double ordinate from either focus, is called the *parameter;* and we perceive by this equation that the semi parameter is the third proportional to the *major* and *minor* axes;

For, . . $A:B=B:y$; a proportion that gives the preceding equation.

It is sometimes most convenient to take C, the center of the ellipse, for the *zero* point, in place of the point A', one extremity of the major axis.

If we make this change, it will cause no changes in the ordinate y, but x, in the equation for the ellipse, must be diminished by A; and x, a measure from that point, can never be greater than A, but it can have the double sign plus or minus. At the point A', x will be equal to *minus* A, and at the other extremity of the major axis, x will be equal to *plus* A.

To change the equation $y^2=\frac{B^2}{A^2}(2Ax-x^2)$ into its equivalent expression, when the origin of x is changed from A' to C, we must put $x-A=x'$. Hence, x and x' designate the *same point* on the axis; and if x is less than A, then x' is negative.

If . $x-A=x'$, then $x=A+x'$

$$(2Ax-x^2)=(2A-x)x=(A-x')(A+x')=A^2-x'^2$$

Hence, $y^2=\frac{B^2}{A^2}(A^2-x'^2)=B^2-\frac{B^2x'^2}{A^2}$

Or, . . $A^2y^2+B^2x'^2=A^2B^2$

We may omit the accent of x, for x, or x', is only a different symbol for *any point* on the major axis corresponding to the ordinate y. The accent was only taken to avoid confusion while changing the *zero* point; therefore, the following equation is the equation for the ellipse, the zero point being the center.

$$A^2y^2+B^2x^2=A^2B^2$$

In case $A=B$, the ellipse becomes a circle, and the equation becomes . . $A^2y^2+A^2x^2=A^4$

Or, ... $y^2+x^2=A^2$

This last equation is obviously the equation of the circle, y being the sine of any arc, x its cosine, and A the radius.

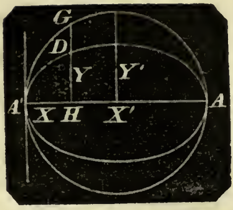
The change in the zero point from the vertex of the major axis to the center, changes equations (8) and (9) into

$$\left.\begin{aligned} r'&=A+\frac{cx'}{A} \\ r&=A-\frac{cx'}{A} \end{aligned}\right\} \quad (m)$$

Or, without the accent, $r'=A+\frac{cx}{A}$, and $r=A-\frac{cx}{A}$

PROPOSITION 10. THEOREM.

The squares of the ordinates of the major axis are to each other as the rectangles of their corresponding abscissas.



Let y be any ordinate, and x its corresponding abscissa. Then, by the last proposition, we shall have

$$y^2=\frac{B^2}{A^2}(2A-x)x$$

Let y' be any other ordinate, and x' its corresponding abscissa, and by the same proposition we must have

$$y'^2=\frac{B^2}{A^2}(2A-x')x'$$

Dividing one of these equations by the other, omitting common factors in the numerator and denominator of the second member of the new equation, we have

$$\frac{y^2}{y'^2}=\frac{(2A-x)x}{(2A-x')x'}$$

Hence, $\quad y^2 : y'^2=(2A-x)x : (2A-x')x'$

By simply inspecting the figure, we cannot fail to perceive that $(2A-x)$, and x, are the abscissas corresponding to the ordinate y, and $(2A-x')$ and x', are the two corresponding to y'. Therefore, the squares of the ordinates, &c. *Q. E. D.*

PROPOSITION 11. THEOREM.

If a circle be described on the major axis of an ellipse, and any ordinate be drawn common to both the circle and the ellipse, the ordinate corresponding to the circle is to the part corresponding to the ellipse as the major axis of the ellipse is to its minor axis.

On $A'A$ (see figure to last proposition), as a diameter, describe a circle. Draw any ordinate, as GH. The part DH is y, of the last proposition.

The proportion in the last proposition is true, and y and y' may be any two ordinates, whatever. And now suppose y' represents the semi minor axis; then x' will equal A, and $2A-x'=A$. Taking this hypothesis, the proportion referred to becomes

$$y^2 : B^2=(2A-x)x : A^2$$

Changing the means, and observing that

$$(2A-x)x=GH^2 \qquad \text{(th. 17, b. 3, scholium.)}$$

We have, . $y^2 : GH^2 = B^2 : A^2$

Taking extremes for means, and extracting the square root of every term, we have

$$GH : y = A : B \qquad Q. E. D.$$

PROPOSITION 12. THEOREM.

The area of an ellipse is a mean proportional between two circles—the one described on the minor, and the other on the major axis.

On the major axis describe a circle, as in the figure, and draw GH, any ordinate, and conceive it to be a *broad line*, covering portions of both the circle and the ellipse.

By the last proposition we have

$$A : B = GH : y$$
$$= GH' : y'$$
$$= GH'' : y''$$

That is, GH', y'; GH'', y'', &c., are other ordinates, all in the same proportion of A to B; and thus we can conceive the whole areas of both circle and ellipse, made up of ordinates, each and all of which are in the proportion of A to B. Now, by applying theorem 7, book 2, we have

$$A : B = GH + GH', \text{ \&c.} : y + y', \text{ \&c.}$$

That is, . $A : B =$ area circle : area ellipse

But the area of the circle on the major axis, is πA^2 (th. 1, b. 5.)

Substituting this, and the proportion becomes

$$A : B = \pi A^2 : \text{area ellipse.}$$

Or, . . area ellipse $= \pi AB$

Which is the mean proportional between (πA^2) and (πB^2), the expressions for the areas of the two circles, one on the major diameter, and the other on the minor diameter. *Q. E. D.*

Scholium. Hence the rule in mensuration to find the area of an ellipse.

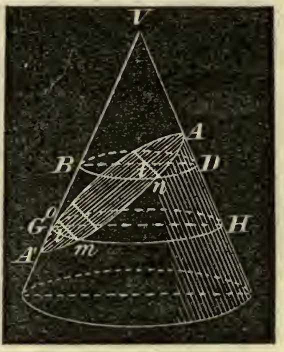
RULE. Multiply together the semi major and semi minor axes, and multiply that product by 3.1416.

PROPOSITION 13. THEOREM.

If a cone be cut by a plane, making an angle with the base less than that made by the side of the cone, the section is an ellipse.

Let VGH, be a plane passing through the axis of a cone, $Anmo$, another plane perpendicular to the former, cutting both sides of the cone but not parallel with the base of the cone, then the figure $AnmA'o$, will be an ellipse, AA' being its major axis.

Take any point, t, and in the plane AnA' draw tn, at right angles to AA'. and as the plane AnA' is perpendicular to the plane VGH, tn is at right angles to all lines that can be drawn in the plane VGH, from the point t; therefore. tn is at right angles to BD. Through the point t, conceive BD drawn parallel to the base of the cone, and it will be a diameter to a circular section of the cone passing through the point n.



In the same manner take any other point in AA' as l, and draw lm at right angles to $A'A$, &c; and GmH will be a circular section passing through the point m.

Now by the similar triangles AtD, AlH, $A'lG$, $A'tB$, we have

$$At : Al = Dt : Hl$$

$$A't : A'l = Bt : Gl$$

By multiplying these proportions together (th. 11, b. 2), term, by term, we have

$$At \cdot A't : Al \cdot A'l = Dt \cdot Bt : Hl \cdot Gl$$

But by reason of the circle BnD, $\left.\begin{array}{l} Bt\cdot Dt=tn^2 \\ Hl\cdot Gl=lm^2 \end{array}\right\}$ (th. 17, b. 2).
" circle GmH,

Hence, . . $At\cdot A't : Al\cdot A'l = tn^2 : lm^2$

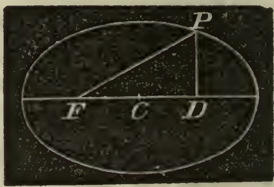
This last proportion shows the same property as demonstrated in Proposition 10; therefore, this section of the cone is an ellipse.

Q. E. D

Scholium. Hence the propriety of calling an ellipse a *conic section.*

PROPOSITION 14. PROBLEM.

Given the major axis, the distance between the center and either focus of an ellipse, and the angle made between the major axis and a radius drawn from either focus to any point in the ellipse to find an expression for that radius.



Let F be a focus, and FP any radius, and put the angle $PFD=v$.

From proposition 9, equation (m) we find that

$$FP=r=A+\frac{cx}{A}$$

an equation in which A represents the semi major axis, c the distance FC, and x the distance CD.

Now by trigonometry we have

$$1 : \cos.v = r : c+x$$

Whence, . . . $x=r\cos.v-c$

Substituting this value of x in the equation for the radius, we have

$$r=A+\frac{cr\cos.v-c^2}{A}$$

$$Ar=A^2+cr\cos.v-c^2$$

Hence, . $(A-c\cos.v)r=A^2-c^2$

Or, ... $r=\frac{A^2-c^2}{A-c\cos.v}$

This equation shows the value of r in known quantities, and of course it is the expression required.

Scholium. The excentricity of an ellipse is the distance from the center to either focus, when the semi major axis is taken as unity. Designate the excentricity by e, then $1:e=A:c$

Hence, ... $c=eA$

Substituting this value of c in the preceding equation, we have

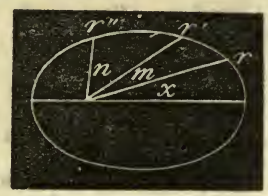
$$r=\frac{A^2-e^2A^2}{A-eA\cos.v}=\frac{A(1-e^2)}{1-e\cos.v}$$

This equation gives an expression for FP, when the angle PFD is less than 90°; when greater than 90°, the expression is

$$\frac{A(1-e^2)}{1+e\cos.v}$$

PROPOSITION 15. PROBLEM.

Given the relative values of three different radii, drawn from the focus of an ellipse, together with the angles between them, to find the relative major axis of the ellipse, the excentricity, and the position of the major axis, or its angle from one of the given radii.



Let r, r', and r'', represent the three given radii, the angle between r and r' equal m, and between r and r'' equal n. The angle between the radius r and the major axis is supposed to be unknown, and we therefore, call it x.

From the last proposition, we have

$$r=\frac{A(1-e^2)}{1-e\cos.x} \qquad (1)$$

$$r'=\frac{A(1-e^2)}{1-e\cos(x+m)} \qquad (2)$$

$$r''=\frac{A(1-e^2)}{1-e\cos.(x+n)} \qquad (3)$$

Equating $A(1-e^2)$ obtained from (1) and (2), and we have

$$r-re\cos.x=r'-r'e\cos.(x+m)$$

Or, $$e=\frac{r-r'}{r\cos.x-r'\cos.(x+m)} \qquad (4)$$

In like manner from (1) and (3),

$$r-re\cos.x=r''-r''e\cos.(x+n)$$

$$e=\frac{r-r''}{r\cos.x-r''\cos.(x+n)} \qquad (5)$$

Equating (4) and (5), we have

$$\frac{r-r'}{r\cos.x-r'\cos.(x+m)}=\frac{r-r''}{r\cos.x-r''\cos.(x+n)}$$

$$\frac{r-r'}{r-r''}=\frac{r\cos.x-r'\cos.(x+m)}{r\cos.x-r''\cos.(x+n)}$$

$$=\frac{r\cos.x-r'\cos.x\cos.m+r'\sin.x\sin.m}{r\cos.x-r''\cos.x\cos.n+r''\sin.x\sin.n}$$

$$=\frac{r-r'\cos.m+r'\sin.m\tan.x}{r-r''\cos.n+r''\sin.n\tan.x}$$

For the sake of perspicuity and brevity, put $r-r'=d$, And $r-r''=d'$. The known quantity $r-r'\cos.m=a$. And $r-r''\cos.n=b$. Then the preceding equation becomes,

$$\frac{d}{d'}=\frac{a+r'\sin.m\tan.x}{b+r''\sin.n\tan.x}$$

$$db+dr''\sin.n\tan.x=ad'+d'r'\sin.m\tan.x$$

$$(dr''\sin.n-d'r'\sin.m)\tan.x=ad'-db$$

$$\tan.x=\frac{ad'-db}{dr''\sin.n-d'r'\sin.m}$$

The value of x found by this last equation, determines the position of the major axis.

Having x, equation (4) or (5), will give the excentricity e. Equations (1), (2), and (3), contain A, the semi major axis as a *common factor*, it does not therefore affect the relative values of r, r', and r'', and as A disappears in the subsequent part of the investigation, it shows that the angle x and the eccentricity e, are entirely independent of the magnitude of the ellipse; they only determine its figure. To apply the preceding formulas, we propose the following

EXAMPLE.

On the first day of August 1846, *an astronomer observed the sun's longitude to be* 128° 47′ 31″, *and by comparing this observation with observations made on the previous and subsequent days, he found its motion in longitude was then at the rate of* 57′ 24″ 9 *per day. By like observations, made on the first of September, he determined the sun's longitude to be* 158° 37′ 46″, *and its mean daily motion for that time* 58′ 6″ 6; *and at a third time, on the* 10*th of October, the observed longitude was* 196° 48′ 4″, *and mean daily motion* 59′ 22″ 9. *From these data is required the longitude of the solar apogee, and the excentricity of the apparent solar orbit.*

It is demonstrated in astronomy, that the relative distances to the sun, when the earth is in different parts of its orbit, must be to each other inversely as the *square root* of the sun's apparent angular motion at the several points; therefore, $(r)^2$, $(r')^2$, and $(r'')^2$, must be in proportion to

$$\frac{1}{57'\,24''\,9},\quad \frac{1}{58'\,6''\,6},\ \text{and}\ \frac{1}{59'\,22''\,9}$$

Or as the numbers,

$$\frac{1}{3444.9},\quad \frac{1}{3486.6},\ \text{and}\ \frac{1}{3562.9}.$$

Multiply by 3562.9 and the proportion will not be changed, and we may put

$$r=\left(\frac{3562.9}{3444.9}\right)^{\frac{1}{2}},\qquad r'=\left(\frac{3562.9}{3486.6}\right)^{\frac{1}{2}},\ \text{and}\ r''=1.$$

By the aid of logarithms, we soon find

$$r=1.016982 \qquad r'=1.010857 \text{ and } r''=1.$$

Hence, $r-r'=d=0.006125$, $\quad r-r''=d'=0.016982$

$$\begin{array}{rrrr} & 158° & 37' & 46'' \\ & 128 & 47 & 31 \\ \hline m= & 29 & 50 & 15 \end{array} \qquad \begin{array}{rrrr} & 196° & 48' & 4'' \\ & 128 & 47 & 31 \\ \hline n= & 68 & 0 & 33 \end{array}$$

To correspond with the formulas, we must take the *natural* sine and cosine of m and n,

$$m=29^\circ\ 50'\ 15''\ \text{sin.}\ .497542\ .\ \text{cosine}\ .867440$$

$$n=68\quad 0\ 33\quad \text{sin.}\ .927238\ .\ \text{cosine}\ .374472$$

$$r—r'\cos.m=a=0.140172$$

$$r—r''\cos.n=b=0.642510$$

$$ad'=(0.140172)(0.016982)=0.0023796$$

$$bd=(0.64251)(0.006125)=0.0039358$$

$$d'r'\sin.m=0.0085405$$

$$dr''\sin.n=0.0056793$$

$$\tan.x=\frac{ad'—bd}{dr''\sin.n—d'r'\sin.m}=\frac{db—ad'}{d'r'\sin.m—dr''\sin.n}$$

$$=\frac{.0015562}{.0028612}=\frac{155.62}{286.12}$$

This numerical result corresponds to radius unity; to compare it with our tables and take out the arc, we must take out the logarithm of the numerator, increase its index by 10, and subtract the logarithm of the denominator,

Thus, .	155.62 log. .	12.192080
	286.12 log. .	2.456548
	$x=$ 30° 23′ 40″ tan.	9.735532
From, ...		128° 47′ 31″
Take, x		28° 32′ 24″
Longitude of the apogee, .		100 14 57

The true longitude at that time was 99° 40′.

The result of any one set of observations, are but first approximations, of course; but we did not adduce this example to teach astronomy, but to teach the properties of the ellipse.

To find the excentricity, we apply equation (5), observing that $r''\cos.(x+n)$ must be subtracted, but when $(x+n)$ is greater than

90° (as it is in this case) it becomes negative, and substracting a negative quantity gives an increase,

Thus, $e=\frac{r-r''}{r\cos.x-r''\cos.(x+n)}=\frac{.016982}{.887+.114}=\frac{.016982}{1.001}$

This gives $e=0.01696$; its true value is, 0.01678.

Our value of x is a little too small which is the principal cause of the difference.

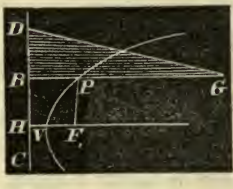
THE PARABOLA.

DEFINITIONS.

1. A *parabola* is a plane curve, every point of which is equally distant from a fixed point and a given straight line.

2. The given point is called the *focus*, and the given line is called the *directrix*.

To describe a parabola.



Let CD be the given line, and F a given point. Take a square, as DBG, and to one side of it, GB, attach a thread, and let the thread be of the *same length as the side* GB *of the square.* Fasten one end of the thread at the point G, the other end at F.

Put the other side of the square against the given line, CD, and with a pencil, P, in the thread, bring the thread up to the side of the square. Slide one side of the square along the line CD, and at the same time keep the thread close against the other side, permitting the thread to slide round the pencil P. As the side of the square, BD, is moved along the line CD, the pencil will describe the curve represented as passing through the points V and P.

$$GP+PF=\text{ the thread}$$

$$GP+PB=\text{ the thread}$$

By subtraction $PF-PB=0$ or $PF=PB$

This result is true at any and every position of the point P; that is, it is true for every point on the curve corresponding to definition 1.

Hence, . . $FV=VH$

If the square be turned over and moved in the opposite direction, the other part of the parabola, the other side of the line FH, may be described.

3. A *diameter* to a parabola is a straight line drawn through any point of the curve *perpendicular to the directrix*. Thus, the line HF is a diameter; also, BG is a diameter; and all diameters are parallel to one another.

4. The point in which the diameter cuts the curve, is called the *vertex* of that diameter.

5. The diameter which passes through the focus, is called the principal diameter, and sometimes it is called the *axis* of the parabola.

A *tangent* is a line touching the curve at a point, and if produced, does not cut the curve. Thus, AC is a tangent, at the point B.

7. An *ordinate* to a diameter is a straight line drawn from any point in the curve to meet the diameter, and is parallel to a tangent passing through the vertex of that diameter. Thus, BD is a diameter, and ED an ordinate from the point E. ED is parallel to the tangent AB, drawn through the vertex B.

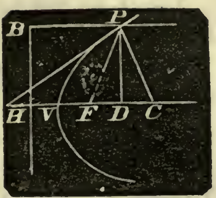
It will be proved in proposition 15, that $ED=DG$; and hence, EG is called a *double ordinate.*

8. An *abscissa* is the part of a diameter between the vertex and an ordinate. Thus, BD is an abscissa, and DE is its corresponding ordinate.

9. The *parameter* of any diameter is the double ordinate which passes through the focus. Thus, IH, which is parallel to AB, and passes through the focus F, is the *parameter* of the particular diameter BD.

10. The parameter to the principal diameter is called the principal parameter, or *latus-rectum.*

In a general sense, the *parameter*, or *latus-rectum*, means the constant quantity that enters into the equation of a curve. In a parabola it is a *third proportional* to any abscissa, and its ordinate.



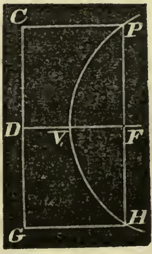
11. A *normal* is a line drawn perpendicular to a tangent from its point of contact, and is terminated by the axis.

12. A *subnormal* is the part of the axis intercepted between the normal and the corresponding ordinate.

Thus, PC is a normal, and DC is the corresponding *subnormal*, or line *under* the normal. Similarly, HD is a line under the tangent, and is called a *subtangent*.

PROPOSITION 1. THEOREM.

The latus-rectum is four times the distance from the focus to the vertex.



Let PVH be a parabola, F the focus, and V the principal vertex. PH, at right angles to DF, through the point F, is the *latus-rectum*.

We are to prove that $PH=4FV$.

Because PH is parallel to CG, and CP, GH, parallel to DF, the two figures, CF and FG, are parallelograms.

Therefore, . $CP=DF$, and $GH=DF$

Or, . $CP+GH=2DF$ (1)

But by the definition of the curve,

$$DF=2VF,\quad CP=PF, \text{ and } GH=HF$$

Substitute these values in equation (1), and we have

$$PF+FH=PH=4FV. \qquad Q.\ E.\ D.$$

Cor. As $CP=PF$, and the angles at F, D, and C, right angles, $PFDC$ is a square.

PROPOSITION 2. THEOREM.

Any point within a parabola is nearer to the focus than to the directrix; and any point without a parabola is at a greater distance from the focus than from the directrix.

Let A be any point within the curve, and from it draw AB perpendicular to the directrix.

As A is within the curve, AB must necessarily cut the curve in some point. Let P be that point, and join PF and AF.

By the definition of the curve, $PB=PF$. To each of these add PA, and $AB=AP+PF$. But $AP+PF$ are, together, greater than AF, because a straight line is the shortest distance between two points; therefore, AB is greater than AF.

Again, let A' be a point without the curve—it is nearer to the directrix than to the focus.

Draw $A'F$; and as A' is without the curve, this line must necessarily meet the curve in some point, as P. Draw PB and $A'B'$ perpendicular to the directrix, and join $A'B$.

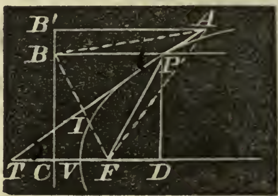
$$A'P+PB=A'F$$

But, $\quad A'P+PB>A'B$; that is, $A'F>A'B$

But $A'B$, being the hypotenuse of the right angled triangle $A'B'B$, it is greater than $A'B'$. But $A'F$ is greater than $A'B$; much more then is $A'F$ greater than $A'B'$; therefore, any point, &c. *Q. E. D.*

PROPOSITION 3. THEOREM.

The line which bisects the angle which is formed by the two lines drawn from any point in the curve, one to the focus, the other perpendicular to the directrix, is a tangent to the curve at that point.



Let P be any point in the curve. Draw PF to the focus, and PB perpendicular to the directrix. Let PT be so drawn as to bisect the angle BPF. Then PT will touch the parabola at the point P, and be tangent to the curve.

Join BF, and PBF is an isosceles triangle; therefore, the angle $PBI=$ the angle PFI. The angle $BPI=$ the angle FPI, by hypothesis; hence, the two triangles BPI and PIF, being equiangular, and having PI common, are in all respects equal, and PI is perpendicular to BF, and $BI=FI$.

It now remains to be shown that any other point than P, in the line APT, is without the curve.

Take any other point in the line TP, as A, and draw the dotted lines AF and AB. They are equal. (Th. 15, b. 1, scholium.)

But AB being the hypotenuse of the right angled triangle $AB'B$ it is greater than AB'; that is, AF is greater than AB'; consequently A is without the curve, as proved by the last proposition.

In the same manner it may be proved that any other point in the line AT is without the curve, except the point P. AT is, therefore, a tangent to the curve at the point P. *Q. E. D.*

Cor. 1. A line of light, parallel to the axis, striking the point of the parabola at P, will be reflected to F; because the angle of incidence is equal to the angle of reflection; and the same will be true at every point of the curve; hence, if a reflecting mirror have a parabolic surface, all the rays of light that meet it parallel with the axis, will be reflected to the focus; and for this reason many attempts have been made to form perfect parabolic mirrors for reflecting telescopes.

If a light be placed at the focus of such a mirror, it will reflect all its rays in one direction; hence, in certain situations, parabolic mirrors have been made for lighthouses, for the purpose of throwing all the light seaward.

Cor. 2. The angle BPF continually increases, as the pencil P moves toward V, and at V it becomes equal to two right angles; and the tangent at V is perpendicular to the axis, which is called the vertical tangent.

Cor. 3. Since an ordinate to any diameter is parallel to the tangent at the vertex, *an ordinate* to the axis is perpendicular to the axis.

PROPOSITION 4. THEOREM.

If a tangent be drawn from any point in the curve to the axis produced, the extremities of the tangent are equally distant from the focus.

Let PT (see figure to the last proposition) be a tangent, meeting the curve at P, and the axis at T. Then we are to prove that

$$PF=FT$$

PB is parallel to FT; therefore, the angle $BPT=$ the angle PTF. But $BPT=TPF$. (Prop. 3.)

Hence, the angle $PTF=$ the angle TPF; consequently, the triangle TFP is isosceles, and $PF=TF$. *Q. E. D.*

PROPOSITION 5. THEOREM.

The subtangent to the axis is bisected by the vertex.

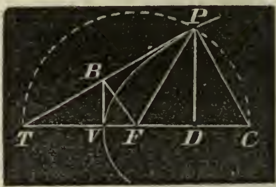
From the point P (see last figure) draw PD, an ordinate to the axis. DT is a subtangent, and it is bisected at V. As PD is parallel to BC, and PB parallel to CD, $PBCD$ is a parallelogram.

Therefore, . .	$PB=CD$	
But, . . .	$PB=PF$, by the definition of the curve.	
And, . . .	$PF=FT$. (Prop. 5.)	
Therefore, . .	$CD=FT$	
That is, . .	$DV+VC=TV+VF$	
But, . .	$VC=VF$	
By subtraction,	$DV=TV$	*Q. E. D.*

Cor. Hence, to draw a tangent to any point P, draw the ordinate PD, and take $VT=VD$, and join TP; it will be a tangent at P.

PROPOSITION 6. THEOREM.

If, from any point in a parabola, a tangent and a normal be drawn, both terminated in the axis, these two lines will be chords of a circle, of which the focus is the center, and the distance to the point P, the radius.



Let P be the point, F the focus, and TVC the axis. Draw PD perpendicular to the axis, and take $TV=VD$ (cor. to last prop.) and join TP, which is the tangent from P. From P draw PC, at right angles to TP; then PC, is the normal. (Def. 11.)

Draw PF. By proposition 4, $PF=FT$. Now, if FP be made radius, and a semicircle described, the points T, P, and C, will be in the circumference, and TC will be the diameter.

Hence TPC is a right angle, and $FP=FC$, and TP and PC, are chords to this circle; therefore, if from any point &c.

Q. E. D.

PROPOSITION 7. THEOREM.

The subnormal is equal to half the latus rectum.

Take the figure to the last proposition. By the definition of the curve. $FP=DV+VF=FD+2VF$

Or, . $2VF=FP-FD$ (1)

$\underline{CD=FC-FD}$ (2)

By subtracting (2) from (1), and observing that $FP=FC$, we have, $2VF-CD=0$

Or, . . $CD=2VF$

But CD is the subnormal, and $2VF$ is half the *latus rectum;* therefore, the subnormal &c. *Q. E. D.*

PROPOSITION 8. THEOREM.

If a perpendicular be drawn from the focus to any tangent, the point of intersection will be in the vertical tangent.

From the focus F (see last figure), draw FB perpendicular to PT, and as the triangle PFT is isosceles (Prop. 4), and PF and FT the equal sides; the line from the vertex F, perpendicular to the base, bisects the base; therefore, $TB=BP$.

As VB and PD are both perpendicular to the axis, they are therefore parallel.

Hence, . . $TV : VD=TB : BP$ (th. 17, b. 2).

But, . . . $TV=VD$

Therefore, . . $TB=BP$

That is, a line from F perpendicular, to PT, and a line from V perpendicular to the axis, both cut the tangent PT into two equal parts, and therefore, meet in the same point, B.

Hence: If a perpendicular, &c. *Q. E. D.*

Cor. 1. The two triangles VBF and PBF, are similar, for they are both right angled triangles, and the angle PFB=the angle VFB.

Hence, . . $VF : FB = FB : PF$

That is, *the perpendicular from the focus to any tangent, is a mean proportional between the distances of the focus from the vertex, and from the point of contact.*

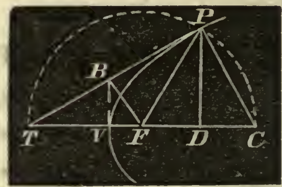
Scholium. From the preceding proportion, we have

$$VF \cdot PF = FB^2$$

But VF, remains constant for the same parabola; therefore, *the distance from the focus to the point of contact varies, as the square of the perpendicular drawn from the focus upon the tangent.*

PROPOSITION 9. PROBLEM.

Find the equation of the curve, or the mathematical relation between any abscissa on the axis, and its corresponding ordinate.



Let V be taken as the zero point. Put $VD=x$, $PD=y$, and let $2p$ represent the *parameter*. As TPC, is a right angled triangle, right angled at P, PD is a mean proportional between TD and DC. (Scho. to th. 17, b. 3).

But, ... $TD=2x$ (Prop. 5).

And, ... $DC=p$ (Prop. 7).

Therefore by multiplication, $TD \cdot DC = y^2 = 2px$

By taking the square root, $y = \pm\sqrt{2px}$, the double sign shows two equal values to y, the one above, the other below the axis; hence, the curve is symmetrical in respect to its focus and axis.

PROPOSITION 10. THEOREM.

The squares of ordinates to the axis are to one another, as their corresponding abscissas.

By the last proposition, any ordinate represented by y, and its corresponding abscissa represented by x, are connected together by the following equation.

$$y^2=2px \qquad (1)$$

Any other ordinate represented by y', and its corresponding abscissa represented by x', have a like connection.

That is, . . $y'^2=2px' \qquad (2)$

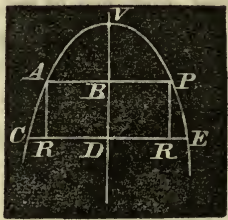
Dividing (2) by (1), omitting the common factor $2p$, and we have

$$\frac{y'^2}{y^2}=\frac{x'}{x}$$

Or, . . $y'^2 : y^2=x' : x$ *Q. E. D.*

PROPOSITION 11. THEOREM.

As the parameter of the axis is to the sum of any two ordinates, so is the difference of those ordinates to the difference of their abscissas.



Let CVE be a portion of a parabola, V the vertex, VD the axis, VB and VD abscissas, and PB and ED their corresponding ordinates.

Put $VB=x$, $VD=x'$, $PB=y$,

And $ED=y'$

Then, $AR=x'-x$, $RE=y'+y$, and $CR=y'-y$

From Proposition 10.

$$y'^2=2px'$$

$$y^2=2px$$

By subtraction, $y'^2-y^2=2p(x'-x)$

Or, . . $(y'+y)(y'-y)=2p(x'-x)$

$$\left.\begin{array}{ll}\text{Or, . . .} & 2p : y'+y=y'-y : x'-x \\ \text{Or, . . .} & 2p : RE=CR : AR\end{array}\right\} \quad Q.\ E.\ D.$$

Cor. Take the product of the extremes and means of this last proportion and we have

$$(2p)AR = CR \cdot RE$$

But, . . . $(2p)x' = y^2$ (Prop. 10).

By division, . . . $\dfrac{AR}{x'} = \dfrac{CR \cdot RE}{y'^2}$

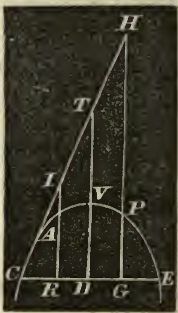
Or, ... $\dfrac{AR}{VD} = \dfrac{CR \cdot RE}{DE'^2}$

Or, . . . $VD : AR = DE^2 : CR \cdot RE$

That is, any abscissa of the axis, is to any other diamater, so is the square of the ordinate to the rectangle of the segments of the double ordinate.

PROPOSITION 12. THEOREM.

If a tangent be drawn from any point of a parabola, and from any point in the tangent a line be drawn parallel to the axis, and terminated in the double ordinate, this line will be cut by the curve in the same proportion as the line cuts the double ordinate.



Let CT be a tangent for the point C, V the vertex, VD the axis, and CE the double ordinate $CD=y$ $VD=x$

Take any point I, in the tangent, and draw IR parallel to VD, cutting the curve at A. Then we are to show

That . . $IA : AR = CR : RE$

Produce DV to T, and observe, that

$$DV = VT,$$

Or, ... $DT = 2DV$ (Prop. 5).

By similar $\triangle$s, . . $CR : RI = CD : DT$

$$= y : 2x$$

By eq. of the curve . $2p : 2y = y : 2x$

By equality, . . . $CR : RI = 2p : (2y)CE$

Proposition 11, . . $2p : RE = CR : AR$

Prod. term, by term, $2p \cdot CR : RI \cdot RE = 2p \cdot CR : CE \cdot AR$

In this last proportion the antecedents are equal; therefore, the consequents are equal.

Hence, . $RI \cdot RE = CE \cdot AR$

Or, . . $RI : AR = CE : RE$

By division, $(RI - AR) : AR = (CE - RE) : RE$

That is, . $IA : AR = CR : RE$ *Q. E. D.*

Cor. The same is true, if a line be drawn from any other point of the tangent.

Therefore, . $HP : PG = CG : GE$

PROPOSITION 13. THEOREM.

If any points be taken on a tangent, and from thence lines be drawn parallel to the axis to meet the curve, the length of such lines will be to each other as the squares of the distances of the points from the point of contact measured on the tangent.

Let CH be a tangent to a parabola, and I and H any points taken upon it. Let DV be the axis produced to T. Draw IR parallel to VD, meeting the curve at A; and also, draw HG parallel to VD, meeting the curve at P.

We are now to prove, that

$$IA : HP = CI^2 : CH^2$$

By the last proposition, we have

$$IA : AR = CR : RE$$

Multiplying the last couplet by CR, and substituting the value of $CR \cdot RE$ taken from corollary to Proposition 11, and

$$IA : AR = CR^2 : \frac{AR \cdot CD^2}{VD}$$

Dividing the second and fourth terms by AR, and afterward multiplying the same terms by VD, observing that $VD = VT$, then we have

$$IA : VT = CR^2 : CD^2$$

But by similar triangles,

$$CI^2 : CT^2 = CR^2 : CD^2$$

Therefore, by equality,

$$IA : TV = CI^2 : CT^2$$

In the same manner, we may prove that

$$HP : TV = CH^2 : CT^2$$

Dividing one of these proportions by the other, term by term,

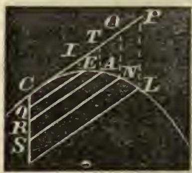
And, . . $$\frac{IA}{HP} : 1 = \frac{CI^2}{CH^2} : 1$$

Or, . . . $$IA : HP = CI^2 : CH^2$$ *Q. E. D.*

Application. Conceive CH to be the direction of a projectile, and undisturbed by the resistance of the air, or the force of gravity, it would move along the line CH, passing over equal distances in equal times. Now let gravity act in the direction of IR, and as bodies fall in proportion to the squares of the times of descent, therefore, IA, TV, HP, &c., must be to each other, as the squares CI^2, CT^2, CH^2, &c; that is the real path of a projectile undisturbed by atmospheric resistance must have the same property, as just demonstrated in this proposition. In other words, the path of a projectile is *some parabola*, more or less curved according to the direction and intensity of the projectile force.

PROPOSITION 14. THEOREM.

The abscissas of any diameter are to each other as the squares of their corresponding ordinates.



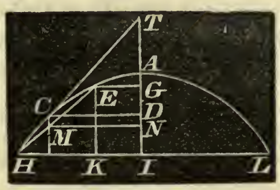
By the definition of a diameter, it must be the axis, or parallel to the axis; and ordinates to any diameter must be parallel to the tangent drawn through the vertex of that diameter. Hence, if CS is a diameter, and CP a tangent, and I, T, and O, any points on the tangent, and from thence lines drawn parallel to the axis to meet the curve, and from thence lines parallel to the tangent to meet the diameter, the figures so formed will be parallelograms, and their opposite sides equal.

By the last proposition, IE, TA, &c., are to each other as CI^2, CT^2, &c.; that is, CQ, CR, &c., are to each other as QE^2, RA^2, &c.; or the abscissas are as the squares of their corresponding ordinates. *Q. E. D.*

REMARK. This is the same property as was proved in relation to the axis and its ordinates in proposition 10.

PROPOSITION 15. THEOREM.

If a line be drawn parallel to any tangent, and cut the curve in two points, and from these points ordinates be drawn to the axis, and another from the point of contact of the tangent, then the three ordinates will be in arithmetical progression.



Let CT be a tangent, and HE parallel to it. Draw the ordinates EG, CD, and HI.

Then, . $EG+HI=2CD$

From the similar triangles, HKE, CDT, we have

$$HK : KE=CD : DT=2AD$$

By prop. 11, . $2p : KL=HK : KE$

Therefore, by (th. 6, b2.) $2p : KL=CD : 2AD$

By eq. of the curve, $2p : 2CD=CD : 2AD$

By comparing the two preceding proportions, we find that KL must equal $2CD$. But by inspecting the figure, we perceive that

$$KL=LI+IK=HI+EG$$

That is, . . $HI+EG=2CD$ *Q. E. D.*

Scholium. As CD is the arithmetical mean between GE and HI, if we draw CM parallel to AI, and draw MN parallel to CD, it will equal CD; hence, MN being midway in value between EG and HI, and parallel to them, it must meet the lines HE and GI in their midway points. *That is, the diameter* CM *cuts its ordinate* HE *in two equal parts; and as* HE *is any ordinate, therefore, the diameter cuts all its ordinates into two equal parts.*

PROPOSITION 16. THEOREM.

A parabola is a conic section, the cone being cut by a plane parallel to its side.

Let the cone be cut, or conceived to be cut, by the plane VMN passing through its axis, and then conceive this plane cut by the plane DAI, perpendicular to the first plane, and so inclined that AH shall be parallel to VM.

Draw MN and KL perpendicular to the axis of the cone, and make them diameters of parallel circles, whose planes are at right angles to the plane VMN.

From the points F and H, where AH meets KL and MN, draw FG and HI at right angles to AH; and because the plane DAI is at right angles to the plane VMN, FG is at right angles to KL, and HI is at right angles to MN.

Now, from the similar triangles, AFL, AHN, we have

$$AF : AH = FL : HN$$

By reason of the parallels, $KF=MH$; therefore, by multiplying the last couplet we have

$$AF : AH = FL \cdot KF : HN \cdot MH$$

But, by reason of the semicircles MIN, KGL,

$$KF \cdot FL = FG^2, \text{ and } MH \cdot HN = HI^2 \text{ (th. 17, b. 3.)}$$

Consequently, . $AF : AH = FG^2 : HI^2$

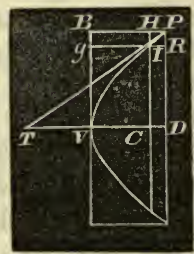
This is the same property as was demonstrated in proposition 10: therefore, the nature of the curve is the same. *Q. E. D.*

Cor. Hence, $\dfrac{FG^2}{AF} = \dfrac{HI^2}{AH}$ and $\dfrac{FG^2}{AF}$, or $\dfrac{HI}{AH}$ is a third proportional, and a constant quantity, which we have called $2p$, the parameter by definition 10.

Remark. We might have commenced the subject of the parabola by assuming it a conic section of this kind, and then sought out its other properties.

PROPOSITION 17. THEOREM.

Every segment of a parabola at right angles with its axis, is two-thirds of its circumscribing rectangle.



Let P be any point in the curve, and PT a tangent. Draw PD and DT. Take any very small portion of the tangent, as PI—so small as to consider it as coinciding with the curve, without sensible errors. Draw IG, Ig, making the two rectangles BR, HD.

Let us now investigate the relation between these two rectangles.

As customary, put $PD=y$, $VD=x$; then, $PB=x$, and $DT=2x$. (Prop. 5.)

The rectangle . $BR=x(PR)$, and $HD=y(RI)$

By similar triangles

$$PR : RI=y : 2x$$

Multiply the *first* and *third* terms of this proportion by x, and the *second* and *fourth* by y. We then have

$$x(PR) : y(RI)=xy : 2xy$$
$$=1 : 2$$

The whole rectangle $BVDP$ is divided into two spaces by the curve—the one within the curve, the other external to it. And we perceive by the above proportion that the small rectangle, BR, external to the curve, is to its corresponding rectangle, HD, within the curve, as 1 to 2.

By taking any other small portion of the curve, as well as PI, and drawing its external and internal rectangle, we can prove in the same manner that they will be to each other as 1 to 2; and thus we can fill up the whole external and internal spaces, and they will be to each other as 1 to 2. Hence, the space within the curve is *two-thirds* of the whole rectangle BD, and the same is true of the spaces on the other side of the axis. Therefore, every segment, &c. *Q. E. D.*

PROPOSITION 18. THEOREM.

If a parabola revolve on its axis, the solid generated is equal to one half of its circumscribing cylinder.

Take the figure to the last proposition, and conceive the parabola to revolve on the axis VD, and find the relation between the two solids generated by the two parallelograms BR and HD. The parallelogram HD will generate a cylinder, whose diameter is $2y$, and length RI.

The parallelogram BR will generate a circular band, whose length is x, and thickness PR.

The solidity of the cylinder $=\pi y^2(RI)$

The solidity of the band $=(\pi y^2-\pi(y-PR)^2)x$

These two quantities are in the proportion of

$$y^2(RI)$$

$$(2y(PR)-PR^2)x$$

By rejecting the very small quantity $(PR)^2$ as being very inconsiderable in connection with the other term, we have

Sol. of cylinder : sol. of band $=y^2(RI) : 2xy(PR)$

But, as in the preceding proposition,

$$PR : RI=y : 2x$$

Or, . . . $2x(PR)=y(RI)$

Or, . . . $2xy(PR)=y^2(RI)$

This equation shows that the last terms in the preceding proportion are equal; therefore,

sol. of cylinder : sol. of band $=1 : 1$

Or the solidities of the cylinder and band are equal; and the same is true of every pair of corresponding solids; and the sum of the *parabaloid* is all the *minute* cylinders which make up the solid generated by the revolution of the parabola, (called a parabaloid); and the sum of all the *minute* bands makes up the solid exterior to the parabaloid. Hence, the parabaloid is equal to half its circumscribing cylinder. *Q. E. D.*

THE HYPERBOLA.

DEFINITIONS.

1. An *hyperbola* is a plane curve, confined by two fixed points called the *foci*, and the difference of the distances of each and every point in the curve from the two fixed points, is constantly equal to a *given line*.

REMARK 1. The distance between the foci, is also supposed to be known; and the *given line* must be less than the distance between the fixed points; that is, less than the distance between the *foci*.

REMARK 2. The ellipse is a curve, confined by two fixed points called the *foci*, and the *sum* of two lines drawn from any point in the curve, is constantly equal to a given line. In the hyperbola, the *difference* of two lines drawn from any point in the curve, to the fixed points, is equal to the given line. The ellipse is but a single curve, and the *foci* are within it; but it will be shown in the course of our investigation, that the hyperbola *consists of two equal and opposite branches*, and the least distance between them is the given line.

2. The line joining the *foci*, and produced, if necessary, is called the axis of the hyperbola.

3. The middle point of the straight line which joins the *foci*, is called the *center* of the hyperbola.

4. The *excentricity*, is the distance from the center to either focus.

5. A diameter is any straight line passing through the center and terminated by two opposite hyperbolas.

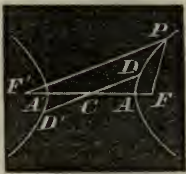
6. The extremities of a diameter are called its *vertices*.

7. A *tangent* is a straight line which meets the curve only in one point, and being produced, does not *cut* the curve.

8. An *ordinate* to a diameter, is a straight line drawn from any point of the curve to meet the diameter produced, and is parallel to the tangent at the vertex of the diameter.

9. An *abscissa*, is the distance between the tangent point and its corresponding ordinate, measured on the diameter produced.

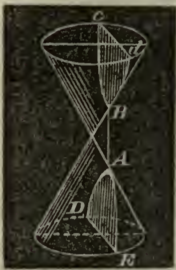
10. The *parameter* is a double ordinate, passing through the focus. The *principal parameter* passes through the focus at right angles to the axis.



REMARK. Thus, let $F'F$ be two fixed points. Draw a line between them, and bisect it in C. Take CA, CA', each equal to half the given line, and CA may be any distance *less* than CF; $A'A$ is the given line, and is called the *major** *axis* of the hyperbola. Now let us suppose the curve already found and represented by ADP. Take any point, as P, and join PF and PF'; then by Definition 1, the difference between PF' and PF must be equal to the given line $A'A$, and conversely if $PF'—PF=A'A$, then P is a point in the curve.

By taking any point, P, in the curve, and joining PF and PF', a triangle PFF' is always formed, having $F'F$ for its base and $A'A$ for the difference of the sides; and these are all the *conditions* necessary to define the curve.

As a triangle can be formed *directly opposite* to $PF'F$, which shall be in all respects exactly equal to it, the two triangles having $F'F$ for a common side; the difference of the other two sides of this opposite triangle will be equal to $A'A$, and correspond with the condition of the curve; hence, a curve can be formed about the focus F' exactly similar and equal to the curve about the focus F.



In short, F' and A' have the same situation in respect to C, as F and A have to C, and the line FF' is common to all the points; therefore if a curve can pass about the focus F, a like curve can pass about the focus F', and this is illustrated by the adjoining figure, representing a plane cutting vertical cones.

Any line drawn through C, and terminated by the opposite curves, is called a diameter; thus, DD' is a diameter, and by a very simple demonstration we can prove that it is bisected in C.

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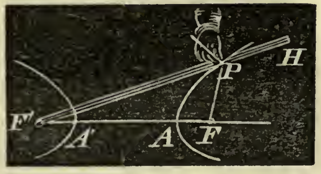
*The term *major axis* implies that there is a *minor axis*, but where it is, we cannot at present determine; when we find such a line, we will give it its proper name.

PROPOSITION 1. PROBLEM.

To describe an hyperbola.



Take a ruler $F'H$, and fasten one end at the point F', on which the ruler may turn as a hinge. At the other end of the ruler attach a thread, and let it be less than the ruler by the given line $A'A$. Fasten the other end of the thread at F.

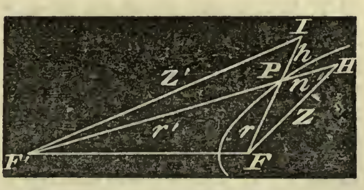
With a pencil, P, press the thread against the ruler and keep it at equal tension between the points H and F. Let the ruler turn on the point F', keeping the pencil close to the ruler and letting the thread slide round the pencil; the pencil will thus describe a curve on the paper.

If the ruler be changed and made to revolve about the other focus as a fixed point, the opposite branch of the curve can be described.

In all positions of P, except when at A or A', PF' and PF will be two sides of a triangle, and the difference of these two sides is constantly equal to the difference between the ruler and the thread; but that difference was made equal to the given line $A'A$; hence, by Definition 1, the curve thus described, must be an hyperbola.

PROPOSITION 2. THEOREM.

If two straight lines be drawn from a point without an hyperbola to the foci, the excess of the one above the other will be less than the major axis; but if the two straight lines be drawn from a point within an hyperbola to the foci, the excess of one above the other will be greater than the major axis.



Explanatory Note. In this and all subsequent propositions, we shall consider but one branch of the curve; that about the focus F.

The distance between any point, P, on the curve, and the focus F, will be represented by r, and between P and the focus F' by r'.

Let I be a point without the curve; join IF, IF', and as F is within the curve, the line IF necessarily cuts the curve at some point P. Let the line without the curve be represented by h.

Put $F'I=z'$, and corresponding to the nature of the curve, put $r'-r=a$, or $r'=r+a$.

Add h to both members of this last equation, and

$$r'+h=r+h+a$$

But the first member of this equation is the sum of two sides of a triangle, and of course greater than its third side z'; therefore, increase z' by t to make it equal to $r'+h$.

Then, . . . $z'+t=(r+h)+a$

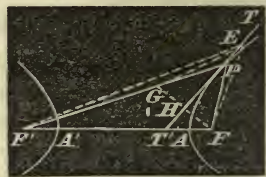
Or, . . $z'-(r+h)=a-t$

That is, the difference between IF' and IF, is less than a, the major axis. In a similar manner, we may demonstrate that $HF'-HF$ is greater than a. *Q. E. D.*

PROPOSITION 3. THEOREM.

A tangent to the hyperbola bisects the angle contained by lines drawn from the point of contact to the foci.

Let F', F be the foci and P any point on the curve, draw PF' PF and bisect the angle $F'PF$ by the line TT'; this line will be a tangent at P.



If TT' be a tangent at P, every other point on this line will be without the curve.

Take $PG=PF$ and join GF, TT' bisects GF, and any point in the line TT' is at equal distances from F and G (th. 15 b. 1). By the definition of the curve $F'G=A'A$ the given line. Now take any other point than P in TT' as E, and join EF', EF and EG, $EF=EG$.

Therefore, . $EF'-EF=EF'-EG$. But $EF'-EG$, is less than $F'G$, because the difference of any two sides of a triangle is less than the third side (th. 18 b. 1). That is, $EF'-EF$ is less than $A'A$; consequently the point E is without the curve (Prop. 2), and as E is any point on the line TT' except P; therefore, the line, TT', which bisects the angle at P, is a tangent to the curve at that point. *Q. E. D.*

Scholium. It should be observed, that the *variable* point in the curve, as P joined to the two *invariable* points F' and F form a triangle, and that the tangent of the curve at the point P, bisects the angle of that triangle at P.

But when any angle of a triangle is bisected, the bisecting line cuts the base into segments proportional to the other sides (th. 23 b. 2).

Therefore, . . $F'P : PF = F'T' : T'F$

Or, ... $r' : r = F'T' : T'F$

But as r' must be greater than r by a given quantity a.

Therefore, . . $r+a : r = F'T' : T'F$

Or, . . . $1+\frac{a}{r} : 1 = F'T' : T'F$

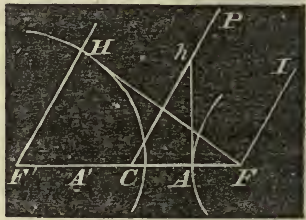
Let it be observed, that a is a constant quantity, and r a variable one, which can increase without limit, and when r is *immensely* great in respect to a, the fraction $\frac{a}{r}$ is *extremely minute*, and the first term of the above proportion, does not in any *practical* sense differ from the second; therefore, in that case, the *third* term does not essentially differ from the *fourth;* that is, $F'T'$ does not *essentially* differ from FT' when r, or the distance of P from F is *immensely* great. *Hence, the tangent at any point P, of the hyperbola, can never cross the line FF′ at its middle point,* but it may approach within the *least imaginable* distance to that point.

THE ASYMPTOTES.

The direction of a line passing through the center of opposite hyperbolas to which a tangent may approach within the *least imaginable* distance is called an asymptote.

PROPOSITION 4. PROBLEM.

To draw an asymptote to an hyperbola and find its angle with the axis.



Let FF' be the foci of an hyperbola and $A'A$ the major axis, and C the center. From F' as a center with a radius equal $A'A$, describe a circle. From the other focus F, draw FH a tangent to this circle, and from the center F' and through the point of contact H, draw the line $F'H$, and let it be indefinitely produced. From C, draw CP parallel to FH, and from F, draw FI also parallel to $F'H$; then the three lines $F'H$, CP and FI, are all perpendicular to FH, and therefore, will never meet, however far they may be produced.

Now suppose $F'H$ and FI to make the *slightest possible* inclination toward CP, and if they equally incline, it is evident that they would meet in the same point P, and the less the inclination from right angles, *the greater the distance to* P, and PHF would form an *isosceles* triangle, having FH for its base, and PH, PF for its equal sides, and if PH and PF are anything less than *infinity*, the point P will be in the hyperbola; for, by our supposition the *infinitely* slight inclination at H, does not prevent us from taking $PF'F$ as a triangle, and the difference of the sides PF', PF, is $F'H=A'A$.

Hence CP is a line to which the curve *can constantly approach, but never meet*, or can meet it only at an infinite distance, and this line is called an *asymptote.*

To obtain an expression for its angle with FF' we observe that the triangle $F'HF$ is right angled at H, and FF' and $A'A$ are always considered as known lines, but $A'A=F'H$.

Hence, . $F'F : A'A = \sin. 90° : \sin. HFF'$, or $\cos. PCF$

In analytical geometry $A'A=a$, and $AF=c$;

Therefore, . . . $FF'=a+2c$, $F'H=a$

And, ... $FH=\sqrt{4ac+4c^2}=2\sqrt{ac+c^2}$

If from the point A, we draw Ah at right angles to FC, the two triangles $F'HF$, CAh, will be similar, and give the proportion

$$F'H : HF = CA : Ah$$

That is, . $a : 2\sqrt{ac+c^2} = \frac{1}{2}a : Ah = \sqrt{(a+c)c}$

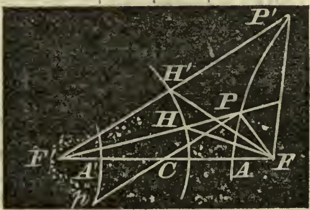
From the preceding equation, we perceive that Ah is a mean proportional between FA and AF'.

The double of the line Ah, drawn at right angles to FF' through the point C, is what mathematicians have arbitrarily termed the *minor axis*. Hence, they give this rule for drawing an *asymptote*.

RULE.—*From either vertex of the major axis draw a line at right angles to that axis equal to half the minor axis, connect the center* C *to the other extremity, and the connecting line produced is the asymptote.*

PROPOSITION 5. PROBLEM.

To describe an hyperbola by points.



Let F, F' be the foci and $A'A$ the major axis, and C the center.

From F' as a center with $A'A$ radius, describe a portion of a circle as represented in the figure. From F', draw any line as $F'P$, cutting the circle in H and join FH. From F, draw the line FP, making the angle

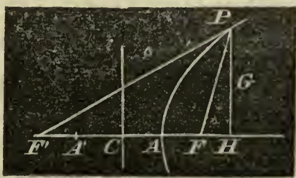
$$HFP = PHF$$

It is obvious, then, that P must be in the curve. In the same manner we find P', or any other point. By joining the points P and C, and producing it so that $PC = Cp$, we shall have p, a point in the opposite branch of the hyperbola, and in the same manner we can find other points in the opposite branch.

PROPOSITION 6. PROBLEM.

Find the equation of the curve in relation to the center and major axis.

Let F' F, be the foci, C the center, and $A'A$ the major axis. Take any point, P, on the curve, and draw the perpendicular PH, join PF PF'.



Put $CA = a$, AF', $AF = c$, $CF = d$, $CH = x$, $PH = y$. $PF = r$, $PF' = r'$.

Then $FH = x - d$, or if H falls between A and F, then $FH = d - x$, but in either case the result will be the same, because $(x-d)^2 = (d-x)^2$.

By the definition of the curve, we have

$$r'-r=2a \qquad (1)$$

The $\triangle$ *PHF'* gives . $r'^2=(d+x)^2+y^2$ (2)

The $\triangle$ *PHF* gives . $r^2=(x-d)^2+y^2$ (3)

By subtraction, . $r'^2-r^2=4dx$ (4)

Divide (4) by (1) and $r'+r=\frac{2dx}{a}$ (5)

Subtract (1) from (5) and $2r=\frac{2dx}{a}-2a$ (6)

Or, ... $r=\frac{dx}{a}-a$ (7)

Combining (7) and (3) $\frac{d^2x^2}{a^2}-2dx+a^2=x^2-2dx+d^2+y^2$

Or, . . . $(d^2-a^2)x^2=(d^2-a^2)a^2+a^2y^2$ (8)

But the quantity (d^2-a^2) is called the square of half the minor axis by common consent, and it is designated by b^2; a is half the major axis; therefore,

$$b^2x^2=a^2b^2+a^2y^2 \qquad (9)$$

Or, . . . $a^2y^2-b^2x^2=-a^2b^2$ *the equation of the curve.*

By giving different values to x, the corresponding values of y may be found. If we make $x=a$, y becomes o, which shows that the curve commences at the point A. If we make $x=a$, y again becomes o, showing the opposite point in the other branch of the curve. If we make x less than a, y becomes imaginary, showing that there is no curve in a perpendicular direction between A' and A.

If in equation (8) we make $x=d$, PH or y will be half the *parameter* by the definition of parameter. The equation then becomes

$$d^4-a^2d^2=a^2d^2-a^4+a^2y^2$$

Or, . . $d^4-2a^2d^2+a^4=a^2y^2$

Or, . . . $d^2-a^2=ay$

Or, ... $\frac{b^2}{a}=y$

Hence, ... $a:b=b:y$

That is, the parameter is a third proportional to the major and minor axes.

There are many other properties of the hyperbola not here demonstrated, but being of little or no practical importance, we omit them.

LOGARITHMIC TABLES;

ALSO A TABLE OF

NATURAL AND LOGARITHMIC

SINES, COSINES, AND TANGENTS,

TO EVERY MINUTE OF THE QUADRANT.

LOGARITHMS OF NUMBERS

FROM

1 TO 10000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0 000000	26	1 414973	51	1 707570	76	1 880814
2	0 301030	27	1 431364	52	1 716003	77	1 886491
3	0 477121	28	1 447158	53	1 724276	78	1 892095
4	0 602060	29	1 462398	54	1 732394	79	1 897627
5	0 698970	30	1 477121	55	1 740363	80	1 903090
6	0 778151	31	1 491362	56	1 748188	81	1 908485
7	0 845098	32	1 505150	57	1 755875	82	1 913814
8	0 903090	33	1 518514	58	1 763428	83	1 919078
9	0 954243	34	1 531479	59	1 770852	84	1 924279
10	1 000000	35	1 544068	60	1 778151	85	1 929419
11	1 041393	36	1 556303	61	1 785330	86	1 934498
12	1 079181	37	1 568202	62	1 792392	87	1 939519
13	1 113943	38	1 579784	63	1 799341	88	1 944483
14	1 146128	39	1 591065	64	1 806180	89	1 949390
15	1 176091	40	1 602060	65	1 812913	90	1 954243
16	1 204120	41	1 612784	66	1 819544	91	1 959041
17	1 230449	42	1 623249	67	1 826075	92	1 963788
18	1 255273	43	1 633468	68	1 832509	93	1 968483
19	1 278754	44	1 643453	69	1 838849	94	1 973128
20	1 301030	45	1 653213	70	1 845098	95	1 977724
21	1 322219	46	1 662578	71	1 851258	96	1 982271
22	1 342423	47	1 672098	72	1 857333	97	1 986772
23	1 361728	48	1 681241	73	1 863323	98	1 991226
24	1 380211	49	1 690196	74	1 869232	99	1 995635
25	1 397940	50	1 698970	75	1 875061	100	2 000000

N. B. In the following table, in the last nine columns of each page, where the first or leading figures change from 9's to 0's, points or dots are now introduced instead of the 0's through the rest of the line, to catch he eye, and to indicate that from thence the corresponding natural numbers in the first column stands in the *next lower line*, and its annexed first two figures of the Logarithms in the second column.

N.	0	1	2	3	4	5	6	7	8	9
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891
101	4321	4750	5181	5609	6038	6466	6894	7321	7748	8174
102	8600	9026	9451	9876	.300	.724	1147	1570	1993	2415
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616
104	7033	7451	7868	8284	8700	9116	9532	9947	.361	.775
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978
107	9384	9789	.195	.600	1004	1408	1812	2216	2619	3021
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028
109	7426	7825	8223	8620	9017	9414	9811	.207	.602	.998
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830
112	9218	9606	9993	.380	.766	1153	1538	1924	2309	2694
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	.320
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815
117	8186	8557	8928	9298	9668	..38	.407	.776	1145	1514
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819
120	9181	9543	9904	.266	.626	.987	1347	1707	2067	2426
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552
123	9905	.258	.611	.963	1315	1667	2018	2370	2721	3071
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	1026
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	.253
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	0245
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	..12
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564
138	9879	.194	.508	.822	1136	1450	1763	2076	2389	2702
139	143015	3327	3630	3951	4263	4574	4885	5196	5507	5818
140	6128	6438	6748	7058	7367	7676	7985	8294	8603	8911
141	9219	9527	9835	.142	.449	.756	1063	1370	1676	1982
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061
144	8362	8664	8965	9266	9567	9868	.168	.469	.769	1068
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802

N.	0	1	2	3	4	5	6	7	8	9
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689
151	8977	9264	9552	9839	.126	.413	.699	.985	1272	1558
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	..51
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623
157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382
158	8657	8932	9206	9481	9755	..29	.303	.577	.850	1124
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247
162	9515	9783	..51	.319	.586	.853	1121	1388	1654	1921
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579
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165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	.193
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276
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181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279
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187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927
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193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446
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208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012
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211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131
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237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394
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239	8398	8580	8761	8943	9124	9306	9487	9668	9849	..30
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248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025
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254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370
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278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449
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284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242
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296	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610
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298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526
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329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382
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335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501
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352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652
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358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973
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361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589
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366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909
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377	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377
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383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720
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393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681
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398	9883	9992	.101	.210	.319	.428	.537	.646	.755	.864
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402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419
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407	9594	9701	9808	9914	..21	.128	.234	.341	.447	.554
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409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845
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416	9293	9198	9302	9406	9511	9615	9719	9824	9928	..32
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419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146
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424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287
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431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388
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440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324
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443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262
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452	5138	5235	5331	5427	5526	5619	5715	5810	5906	6002
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960
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455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870
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472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516
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484	4854	4935	5025	5114	5204	5294	5383	5473	5563	5652
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486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331
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497	6356	5444	6531	6618	6706	6793	6880	6968	7055	7142
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014
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545	6397	6476	6556	6636	6715	6795	6874	6954	7034	7113
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701
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552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2646
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431
554	3510	3558	3667	3745	3823	3902	3980	4058	4136	4215
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556
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592	2322	2395	2468	3542	2615	2688	2762	2835	2908	2981
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354
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605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832
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609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259
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621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418
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624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811
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627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890
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632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071
637	4139	4208	4276	4354	4412	4480	4548	4616	4685	4753
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433
639	5501	5669	5637	5705	5773	5841	5908	5976	6044	6112
640	6180	6248	6316	6384	6451	6519	6587	6655	6723	6790
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648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178
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671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305
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674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239
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715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064
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734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819
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792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218
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797	1458	1513	1567	1622	1676	1736	1785	1840	1894	1948
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803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281
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843	5828	5874	5931	5982	6034	6085	6137	6188	6240	6291
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854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915
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866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969
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871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467
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883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8365
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341
890	9390	9439	9488	9536	9585	9634	9683	9731	9780	9829
891	9878	9926	9975	..24	..73	.121	.170	.219	.267	.316
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260
896	2308	2356	2405	2453	2502	2550	2599	2647	5696	2744
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194

N.	0	1	2	3	4	5	6	7	8	9
900	954243	4291	4339	4387	4435	4484	4532	4580	4628	4677
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947
912	9995	..42	..90	.138	.185	.233	.280	.328	.376	.423
913	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835
933	9882	9928	9975	..21	..68	.114	.161	.207	.254	.300
934	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082
940	3128	3174	3220	3266	3313	3359	3405	3451	3497	3543
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304
947	6350	6396	6442	6488	6533	6579	6925	6671	6717	6763
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678

N.	0	1	2	3	4	5	6	7	8	9
950	977724	7769	7815	7861	7906	7952	7998	8043	8089	8135
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382
967	5426	5471	5516	5561	5606	5651	5699	5741	5786	5830
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960
975	9005	9049	9093	9138	9183	9227	9272	9316	9361	9405
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850
977	9895	9939	9983	..28	..72	.117	.161	.206	.250	.294
978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182
980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713
988	4757	4801	4845	4886	4933	4977	5021	5065	5108	5152
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591
990	5635	5679	5723	5767	5811	5854	5898	5942	5986	6030
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652
997	8695	8739	8792	8826	8869	8913	8956	9000	9043	9087
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957

′	Sine.	D.10″	Cosine.	D.10″	Tang.	D.10″	Cotang.	N.sine.	N. cos.	
0	0.000000		10.000000		0.000000		Infinite.	00000	100000	60
1	6.463726		000000		6.463726		13.536274	00029	100000	59
2	764756		000000		764756		235244	00058	100000	58
3	940847		000000		940847		059153	00087	100000	57
4	7.065786		000000		7.065786		12.934214	00116	100000	56
5	162696		000000		162696		837304	00145	100000	55
6	241877		9.999999		241878		758122	00175	100000	54
7	308824		999999		308825		691175	00204	100000	53
8	366816		999999		366817		633183	00233	100000	52
9	417968		999999		417970		582030	00262	100000	51
10	463725		999998		463727		536273	00291	100000	50
11	7.505118		9.999998		7.505120		12.494880	00320	99999	49
12	542906		999997		542909		457091	00349	99999	48
13	577668		999997		577672		422328	00378	99999	47
14	609853		999996		609857		390143	00407	99999	46
15	639816		999996		639820		360180	00436	99999	45
16	667845		999995		667849		332151	00465	99999	44
17	694173		999995		694179		305821	00495	99999	43
18	718997		999994		719003		280997	00524	99999	42
19	742477		999993		742484		257516	00553	99998	41
20	764754		999993		764761		235239	00582	99998	40
21	7.785943		9.999992		7.785951		12.214049	00611	99998	39
22	806146		999991		806155		193845	00640	99998	38
23	825451		999990		825460		174540	00669	99998	37
24	843934		999989		843944		156056	00698	99998	36
25	861663		999988		861674		138326	00727	99997	35
26	878695		999988		878708		121292	00756	99997	34
27	895085		999987		895099		104901	00785	99997	33
28	910879		999986		910894		089106	00814	99997	32
29	926119		999985		926134		073866	00844	99996	31
30	940842		999983		940858		059142	00873	99996	30
31	7.955082	2298	9.999982	0.2	7.955100	2298	12.044900	00902	99996	29
32	968870	2227	999981	0.2	968889	2227	031111	00931	99996	28
33	982233	2161	999980	0.2	982253	2161	017747	00960	99995	27
34	995198	2098	999979	0.2	995219	2098	004781	00989	99995	26
35	8.007787	2039	999977	0.2	8.007809	2039	11.992191	01018	99995	25
36	020021	1983	999976	0.2	020045	1983	979955	01047	99995	24
37	031919	1930	999975	0.2	031945	1930	968055	01076	99994	23
38	043501	1880	999973	0.2	043527	1880	956473	01105	99994	22
39	054781	1832	999972	0.2	054809	1833	945191	01134	99994	21
40	065776	1787	999971	0.2	065806	1787	934194	01164	99993	20
41	8.076500	1744	9.999969	0.2	8.076531	1744	11.923469	01193	99993	19
42	086965	1703	999968	0.2	086997	1703	913003	01222	99993	18
43	097183	1664	999966	0.2	097217	1664	902783	01251	99992	17
44	107167	1626	999964	0.3	107202	1627	892797	01280	99992	16
45	116926	1591	999963	0.3	116963	1591	883037	01309	99991	15
46	126471	1557	999961	0.3	126510	1557	873490	01338	99991	14
47	135810	1524	999959	0.3	135851	1524	864149	01367	99991	13
48	144953	1492	999958	0.3	144996	1493	855004	01396	99990	12
49	153907	1462	999956	0.3	153952	1463	846048	01425	99990	11
50	162681	1433	999954	0.3	162727	1434	837273	01454	99989	10
51	8.171280	1405	9.999952	0.3	8.171328	1406	11.828672	01483	99989	9
52	179713	1379	999950	0.3	179763	1379	820237	01513	99989	8
53	187985	1353	999948	0.3	188036	1353	811964	01542	99988	7
54	196102	1328	999946	0.3	196156	1328	803844	01571	99988	6
55	204070	1304	999944	0.3	204126	1304	795874	01600	99987	5
56	211895	1281	999942	0.4	211953	1281	788047	01629	99987	4
57	219581	1259	999940	0.4	219641	1259	780359	01658	99986	3
58	227134	1237	999938	0.4	227195	1238	772805	01687	99986	2
59	234557	1216	999936	0.4	234621	1217	765379	01716	99985	1
60	241855		999934		241921		758079	01745	99985	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

89 Degrees.

′	Sine.	D.10″	Cosine.	D.10″	Tang.	D.10″	Cotang.	N. sine.	N. cos.	
0	8.241855	1196	9.999934	0.4	8.241921	1197	11.758079	01742	99985	60
1	249033	1177	999932	0.4	249102	1177	750898	01774	99984	59
2	256094	1158	999929	0.4	256165	1158	743835	01803	99984	58
3	263042	1140	999927	0.4	263115	1140	736885	01832	99983	57
4	269881	1122	999925	0.4	269956	1122	730044	01862	99983	56
5	276614	1105	999922	0.4	276691	1105	723309	01891	99982	55
6	283243	1088	999920	0.4	283323	1089	716677	01920	99982	54
7	289773	1072	999918	0.4	289856	1073	710144	01949	99981	53
8	296207	1056	999915	0.4	296292	1057	703708	01978	99980	52
9	302546	1041	999913	0.4	302634	1042	697366	02007	99980	51
10	308794	1027	999910	0.4	308884	1027	691116	02036	99979	50
11	8.314954	1012	9.999907	0.4	8.315046	1013	11.684954	02065	99979	49
12	321027	998	999905	0.4	321122	999	678878	02094	99978	48
13	327016	985	999902	0.4	327114	985	672886	02123	99977	47
14	332924	971	999899	0.5	333025	972	666975	02152	99977	46
15	338753	959	999897	0.5	338856	959	661144	02181	99976	45
16	344504	946	999894	0.5	344610	946	655390	02211	99976	44
17	350181	934	999891	0.5	350289	934	649711	02240	99975	43
18	355783	922	999888	0.5	355895	922	644105	02269	99974	42
19	361315	910	999885	0.5	361430	911	638570	02298	99974	41
20	366777	899	999882	0.5	366895	899	633105	02327	99973	40
21	8.372171	888	9.999879	0.5	8.372292	888	11.627708	02356	99972	39
22	377499	877	999876	0.5	377622	879	622378	02385	99972	38
23	382762	867	999873	0.5	382889	867	617111	02414	99971	37
24	387962	856	999870	0.5	388092	857	611908	02443	99970	36
25	393101	846	999867	0.5	393234	847	606766	02472	99969	35
26	398179	837	999864	0.5	398315	837	601685	02501	99969	34
27	403199	827	999861	0.5	403338	828	596662	02530	99968	33
28	408161	818	999858	0.5	408304	818	591696	02560	99967	32
29	413068	809	999854	0.5	413213	809	586787	02589	99966	31
30	417919	800	999851	0.6	418068	800	581932	02618	99966	30
31	8.422717	791	9.999848	0.6	8.422869	791	11.577131	02647	99965	29
32	427462	782	999844	0.6	427618	783	572382	02676	99964	28
33	432156	774	999841	0.6	432315	774	567685	02705	99963	27
34	436800	766	999838	0.6	436962	766	563038	02734	99963	26
35	441394	758	999834	0.6	441560	758	558440	02763	99962	25
36	445941	750	999831	0.6	446110	750	553890	02792	99961	24
37	450440	742	999827	0.6	450613	743	549387	02821	99960	23
38	454893	735	999823	0.6	455070	735	544930	02850	99959	22
39	459301	727	999820	0.6	459481	728	540519	02879	99959	21
40	463665	720	999816	0.6	463849	720	536151	02908	99958	20
41	8.467985	712	9.999812	0.6	8.468172	713	11.531828	02938	99957	19
42	472263	706	999809	0.6	472454	707	527546	02967	99956	18
43	476498	699	999805	0.6	476693	700	523307	02996	99955	17
44	480693	692	999801	0.6	480892	693	519108	03025	99954	16
45	484848	686	999797	0.7	485050	686	514950	03054	99953	15
46	488963	679	999793	0.7	489170	680	510830	03083	99952	14
47	493040	673	999790	0.7	493250	674	506750	03112	99952	13
48	497078	667	999786	0.7	497293	668	502707	03141	99951	12
49	501080	661	999782	0.7	501298	661	498702	03170	99950	11
50	505045	655	999778	0.7	505267	655	494733	03199	99949	10
51	8.508974	649	9.999774	0.7	8.509200	650	11.490800	03228	99948	9
52	512867	643	999769	0.7	513098	644	486902	03257	99947	8
53	516726	637	999765	0.7	516961	638	483039	03286	99946	7
54	520551	632	999761	0.7	520790	633	479210	03316	99945	6
55	524343	626	999757	0.7	524586	627	475414	03345	99944	5
56	528102	621	999753	0.7	528349	622	471651	03374	99943	4
57	531828	616	999748	0.7	532080	616	467920	03403	99942	3
58	535523	611	999744	0.7	535779	611	464221	03432	99941	2
59	539186	605	999740	0.7	539447	606	460553	03461	99940	1
60	542819		999735		543084		456916	03490	99939	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

88 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	8.542819	600	9.999735	0.7	8.543084	602	11.456916	03490	99939	60
1	546422	595	999731	0.7	546691	596	453309	03519	99938	59
2	549995	591	999726	0.7	550268	591	449732	03548	99937	58
3	553539	586	999722	0.8	553817	587	446183	03577	99936	57
4	557054	581	999717	0.8	557336	582	442664	03606	99935	56
5	560540	576	999713	0.8	560828	577	439172	03635	99934	55
6	563999	572	999708	0.8	564291	573	435709	03664	99933	54
7	567431	567	999704	0.8	567727	568	432273	03693	99932	53
8	570836	563	999699	0.8	571137	564	428863	03723	99931	52
9	574214	559	999694	0.8	574520	559	425480	03752	99930	51
10	577566	554	999689	0.8	577877	555	422123	03781	99929	50
11	8.580892	550	9.999685	0.8	8.581208	551	11.418792	03810	99927	49
12	584193	546	999680	0.8	584514	547	415486	03839	99926	48
13	587469	542	999675	0.8	587795	543	412205	03868	99925	47
14	590721	538	999670	0.8	591051	539	408949	03897	99924	46
15	593948	534	999665	0.8	594283	535	405717	03926	99923	45
16	597152	530	999660	0.8	597492	531	402508	03955	99922	44
17	600332	526	999655	0.8	600677	527	399323	03984	99921	43
18	603489	522	999650	0.8	603839	523	396161	04013	99919	42
19	606623	519	999645	0.8	606978	519	393022	04042	99918	41
20	609734	515	999640	0.9	610094	516	389906	04071	99917	40
21	8.612823	511	9.999635	0.9	8.613189	512	11.386811	04100	99916	39
22	615891	508	999629	0.9	616262	508	383738	03129	99915	38
23	618937	504	999324	0.9	619313	505	380687	04159	99913	37
24	621962	501	999619	0.9	622343	501	377657	04188	99912	36
25	624965	497	999614	0.9	625352	498	374648	04217	99911	35
26	627948	494	999608	0.9	628340	495	371660	04246	99910	34
27	630911	490	999603	0.9	631308	491	368692	04275	99909	33
28	633854	487	999597	0.9	634256	488	365744	04304	99907	32
29	636776	484	999592	0.9	637184	485	362816	04333	99906	31
30	639680	481	999586	0.9	640093	482	359907	04362	99905	30
31	8.642563	477	9.999581	0.9	8.642982	478	11.357018	04391	99904	29
32	645428	474	999575	0.9	645853	475	354147	04420	99902	28
33	648274	471	999570	0.9	648704	472	351296	04449	99901	27
34	651102	468	999564	0.9	651537	469	348463	04478	99900	26
35	653911	465	999558	1.0	654352	466	345648	04507	99898	25
36	656702	462	999553	1.0	657149	463	342851	04536	99897	24
37	659475	459	999547	1.0	659928	460	340072	04565	99896	23
38	662230	456	999541	1.0	662689	457	337311	04594	99894	22
39	664968	453	999535	1.0	665433	454	334567	04623	99893	21
40	667689	451	999529	1.0	668160	453	331840	04653	99892	20
41	8.670393	448	9.999524	1.0	8.670870	449	11.329130	04682	99890	19
42	673080	445	999518	1.0	673563	446	326437	04711	99889	18
43	675751	442	999512	1.0	676239	443	323761	04740	99888	17
44	678405	440	999506	1.0	678900	442	321100	04769	99886	16
45	681043	437	999500	1.0	681544	438	318456	04798	99885	15
46	683665	434	999493	1.0	684172	435	315828	04827	99883	14
47	686272	432	999487	1.0	6[illegible]6784	433	313216	04856	99882	13
48	688863	429	999481	1.0	689381	430	310619	04885	99881	12
49	691438	427	999475	1.0	691963	428	308037	04914	99879	11
50	693998	424	999469	1.0	694529	425	305471	04943	99878	10
51	8.696543	422	9.999463	1.1	8.697081	423	11.302919	04972	99876	9
52	699073	419	999456	1.1	699617	420	300383	05001	99875	8
53	701589	417	999450	1.1	702139	418	297861	05030	99873	7
54	704090	414	999443	1.1	704246	415	295354	05059	99872	6
55	706577	412	999437	1.1	707140	413	292860	05088	99870	5
56	709049	410	999431	1.1	709618	411	290382	05117	99869	4
57	711507	407	999424	1.1	702083	408	287917	05146	99867	3
58	713952	405	999418	1.1	714534	406	285465	05175	99866	2
59	716383	403	999411	1.1	716972	404	283028	05205	99864	1
60	718800		999404		719396		280604	05234	99863	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

87 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	8.718800	401	9.999404	1.1	8.719396	402	11.280604	05234	99863	60
1	721204	398	999398	1.1	721806	399	278194	05263	99861	59
2	723595	396	999391	1.1	724204	397	275796	05292	99860	58
3	725972	394	999384	1.1	726588	395	273412	05321	99858	57
4	728337	392	999378	1.1	728959	393	271041	05350	99857	56
5	730688	390	999371	1.1	731317	391	268683	05379	99855	55
6	733027	388	999364	1.2	733663	389	266337	05408	99854	54
7	735354	386	999357	1.2	735996	387	264004	05437	99852	53
8	737667	384	999350	1.2	738317	385	261683	05466	99851	52
9	739969	382	999343	1.2	740626	383	259374	05495	99849	51
10	742259	380	999336	1.2	742922	381	257078	05524	99847	50
11	8.744536	378	9.999329	1.2	8.745207	379	11.254793	05553	99846	49
12	746802	376	999322	1.2	747479	377	252521	05582	99844	48
13	749055	374	999315	1.2	749740	375	250260	05611	99842	47
14	751297	372	999308	1.2	751989	373	248011	05640	99841	46
15	753528	370	999301	1.2	754227	371	245773	05669	99839	45
16	755747	368	999294	1.2	756453	369	243547	05698	99838	44
17	757955	366	999286	1.2	758668	367	241332	05727	99836	43
18	760151	364	999279	1.2	760872	365	239128	05756	99834	42
19	762337	362	999272	1.2	763065	364	236935	05785	99833	41
20	764511	361	999265	1.2	765246	362	234754	05814	99831	40
21	8.766675	359	9.999257	1.2	8.767417	360	11.232583	05844	99829	39
22	768828	357	999250	1.3	769578	358	230422	05873	99827	38
23	770970	355	999242	1.3	771727	356	228273	05902	99826	37
24	773101	353	999235	1.3	773866	355	226134	05931	99824	36
25	775223	352	999227	1.3	775995	353	224005	05960	99822	35
26	777333	350	999220	1.3	778114	351	221886	05989	99821	34
27	779434	348	999212	1.3	780222	350	219778	06018	99819	33
28	781524	347	999205	1.3	782320	348	217680	06047	99817	32
29	783605	345	999197	1.3	784408	346	215592	06076	99815	31
30	785675	343	999189	1.3	786486	345	213514	06105	99813	30
31	8.787736	342	9.999181	1.3	8.788554	343	11.211446	06134	99812	29
32	789787	340	999174	1.3	790613	341	209387	06163	99810	28
33	791828	339	999166	1.3	792662	340	207338	06192	99808	27
34	793859	337	999158	1.3	794701	338	205299	06221	99806	26
35	795881	335	999150	1.3	796731	337	203269	06250	99804	25
36	797894	334	999142	1.3	798752	335	201248	06279	99803	24
37	799897	332	999134	1.3	800763	334	199237	06308	99801	23
38	801892	331	999126	1.3	802765	332	197235	06337	99799	22
39	803876	329	999118	1.3	804858	331	195242	06366	99797	21
40	805852	328	999110	1.3	806742	329	193258	06395	99795	20
41	8.807819	326	9.999102	1.3	8.808717	328	11.191283	06424	99793	19
42	809777	325	999094	1.4	810683	326	189317	06453	99792	18
43	811726	323	999086	1.4	812641	325	187359	06482	99790	17
44	813667	322	999077	1.4	814589	323	185411	06511	99788	16
45	815599	320	999069	1.4	816529	322	183471	06540	99786	15
46	817522	319	999061	1.4	818461	320	181539	06569	99784	14
47	819436	318	999053	1.4	820384	319	179616	06598	99782	13
48	821343	316	999044	1.4	822298	318	177702	06627	99780	12
49	823240	315	999036	1.4	824205	316	175795	06656	99778	11
50	825130	313	999027	1.4	826103	315	173897	06685	99776	10
51	8.827011	312	9.999019	1.4	8.827992	314	11.172008	06714	99774	9
52	828884	311	999010	1.4	829874	312	170126	06743	99772	8
53	830749	309	999002	1.4	831748	311	168252	06773	99770	7
54	832607	308	998993	1.4	833613	310	166387	06802	99768	6
55	834456	307	998984	1.4	835471	308	164529	06831	99766	5
56	836297	306	998976	1.4	837321	307	162679	06860	99764	4
57	838130	304	998967	1.5	839163	306	160837	06889	99762	3
58	839956	303	998958	1.5	840998	304	159002	06918	99760	2
59	841774	302	998950	1.5	842825	303	157175	06947	99758	1
60	843585		998941		844644		155356	06976	99756	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

'	Sine.	D. 10"	Cosine.	D. 10"	Tang.	D. 10"	Cotang.	N. sine.	N. cos.	
0	8.843585	300	9.998941	1.5	8.844644	302	11.155356	06976	99756	60
1	845387	299	998932	1.5	846455	301	153545	07005	99754	59
2	847183	298	998923	1.5	848260	299	151740	07034	99752	58
3	848971	297	998914	1.5	850057	298	149943	07063	99750	57
4	850751	295	998905	1.5	851846	297	148154	07092	99748	56
5	852525	294	998896	1.5	853628	296	146372	07121	99746	55
6	854291	293	998887	1.5	855403	295	144597	07150	99744	54
7	856049	292	998878	1.5	857171	293	142829	07179	99742	53
8	857801	291	998869	1.5	858932	292	141068	07208	99740	52
9	859546	290	998860	1.5	860686	291	139314	07237	99738	51
10	861283	288	998851	1.5	862433	290	137567	07266	99736	50
11	8.863014	287	9.998841	1.5	8.864173	289	11.135827	07295	99734	49
12	864738	286	998832	1.5	865906	288	134094	07324	99731	48
13	866455	285	998823	1.6	867632	287	132368	07353	99729	47
14	868165	284	998813	1.6	869351	285	130649	07382	99727	46
15	869868	283	998804	1.6	871064	284	128936	07411	99725	45
16	871565	282	998795	1.6	872770	283	127230	07440	99723	44
17	873255	281	998785	1.6	874469	282	125531	07469	99721	43
18	874938	279	998776	1.6	876162	281	123838	07498	99719	42
19	876615	279	998766	1.6	877849	280	122151	07527	99716	41
20	878285	277	998757	1.6	879529	279	120471	07556	99714	40
21	8.879949	276	9.998747	1.6	8.881202	278	11.118798	07585	99712	39
22	881607	275	998738	1.6	882869	277	117131	07614	99710	38
23	883258	274	998728	1.6	884530	276	115470	07643	99708	37
24	884903	273	998718	1.6	886185	275	113815	07672	99705	36
25	886542	272	998708	1.6	887833	274	112167	07701	99703	35
26	888174	271	998699	1.6	889476	273	110524	07730	99701	34
27	889801	270	998689	1.6	891112	272	108888	07759	99699	33
28	891421	269	998679	1.6	892742	271	107258	07788	99696	32
29	893035	268	998669	1.7	894366	270	105634	07817	99694	31
30	894643	267	998659	1.7	895984	269	104016	07846	99692	30
31	8.896246	266	9.998649	1.7	8.897596	268	11.102404	07875	99689	29
32	897842	265	998639	1.7	899203	267	100797	07904	99687	28
33	899432	264	998629	1.7	900803	266	099197	07933	99685	27
34	901017	263	998619	1.7	902398	265	097602	07962	99683	26
35	902596	262	998609	1.7	903987	264	096013	07991	99680	25
36	904169	261	998599	1.7	905570	263	094430	08020	99678	24
37	905736	260	998589	1.7	907147	262	092853	08049	99676	23
38	907297	259	998578	1.7	908719	261	091281	08078	99673	22
39	908853	258	998568	1.7	910285	260	089715	08107	99671	21
40	910404	257	998558	1.7	911846	259	088154	08136	99668	20
41	8.911949	257	9.998548	1.7	8.913401	258	11.086599	08165	99666	19
42	913488	256	998537	1.7	914951	257	085049	08194	99664	18
43	915022	255	998527	1.7	916495	256	083505	08223	99661	17
44	916550	254	998516	1.8	918034	256	081966	08252	99659	16
45	918073	253	998506	1.8	919568	255	080432	08281	99657	15
46	919591	252	998495	1.8	921096	254	078904	08310	99654	14
47	921103	251	998485	1.8	922619	253	077381	08339	99652	13
48	922610	250	998474	1.8	924136	252	075864	08368	99649	12
49	924112	249	998464	1.8	925649	251	074351	08397	99647	11
50	925609	249	998453	1.8	927156	250	072844	08426	99644	10
51	8.927100	248	9.998442	1.8	8.928658	249	11.071342	08455	99642	9
52	928587	247	998431	1.8	930155	249	069845	08484	99639	8
53	930068	246	998421	1.8	931647	248	068353	08513	99637	7
54	931544	245	998410	1.8	933134	247	066866	08542	99635	6
55	933015	244	998399	1.8	934616	246	065384	08571	99632	5
56	934481	243	998388	1.8	936093	245	063907	08600	99630	4
57	935942	243	998377	1.8	937565	244	062435	08629	99627	3
58	937398	242	998366	1.8	939032	244	060968	08658	99625	2
59	938850	241	998355	1.8	940494	243	059506	08687	99622	1
60	940296		998344		941952		058048	08716	99619	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	'

85 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	8.940296	240	9.998344	1.9	8.941952	242	11.058048	08716	99619	60
1	941738	239	998333	1.9	943404	241	056596	08745	99617	59
2	943174	239	998322	1.9	944852	240	055148	08774	99614	58
3	944606	238	998311	1.9	946295	240	053705	08803	99612	57
4	946034	237	998300	1.9	947734	239	052266	08831	99609	56
5	947456	236	998289	1.9	949168	238	050832	08860	99607	55
6	948874	235	998277	1.9	950597	237	049403	08889	99604	54
7	950287	235	998266	1.9	952021	237	047979	08918	99602	53
8	951696	234	998255	1.9	953441	236	046559	08947	99599	52
9	953100	233	998243	1.9	954856	235	045144	08976	99596	51
10	954499	232	998232	1.9	956267	234	043733	09005	99594	50
11	8.955894	232	9.998220	1.9	8.957674	234	11.042326	09034	99591	49
12	957284	231	998209	1.9	959075	233	040925	09063	99588	48
13	958670	230	998197	1.9	960473	232	039527	09092	99586	47
14	960052	229	998186	1.9	961866	231	038134	09121	99583	46
15	961429	229	998174	1.9	963255	231	036745	09150	99580	45
16	962801	228	998163	1.9	964639	230	035361	09179	99578	44
17	964170	227	998151	1.9	966019	229	033981	09208	99575	43
18	965534	227	998139	2.0	967394	229	032606	09237	99572	42
19	966893	226	998128	2.0	968766	228	031234	09266	99570	41
20	968249	225	998116	2.0	970133	227	029867	09295	99567	40
21	8.969600	224	9.998104	2.0	8.971496	226	11.028504	09324	99564	39
22	970947	224	998092	2.0	972855	226	027145	09353	99562	38
23	972289	223	998080	2.0	974209	225	025791	09382	99559	37
24	973628	222	998068	2.0	975560	224	024440	09411	99556	36
25	974962	222	998056	2.0	976906	224	023094	09440	99553	35
26	976293	221	998044	2.0	978248	223	021752	09469	99551	34
27	977619	220	998032	2.0	979586	222	020414	09498	99548	33
28	978941	220	998020	2.0	980921	222	019079	09527	99545	32
29	980259	219	998008	2.0	982251	221	017749	09556	99542	31
30	981573	218	997996	2.0	983577	220	016423	09585	99540	30
31	8.982883	218	9.997984	2.0	8.984899	220	11.015101	09614	99537	29
32	984189	217	997972	2.0	986217	219	013783	09642	99534	28
33	985491	216	997959	2.0	987532	218	012468	09671	99531	27
34	986789	216	997947	2.0	988842	218	011158	09700	99528	26
35	988083	215	997935	2.1	990149	217	009851	09729	99526	25
36	989374	214	997922	2.1	991451	216	008549	06758	99523	24
37	990660	214	997910	2.1	992750	216	007250	09787	99520	23
38	991943	213	997897	2.1	994045	215	005955	09816	99517	22
39	993222	212	997885	2.1	995337	215	004663	09845	99514	21
40	994497	212	997872	2.1	996624	214	003376	09874	99511	20
41	8.995768	211	9.997860	2.1	8.997908	213	11.002092	09903	99508	19
42	997036	211	997847	2.1	999188	213	000812	09932	99505	18
43	998299	210	997835	2.1	9.000465	212	10.999535	09961	99503	17
44	999560	209	997822	2.1	001738	211	998262	09990	99500	16
45	9.000816	209	997809	2.1	003007	211	996993	10019	99497	15
46	002069	208	997797	2.1	004272	210	995728	10048	99494	14
47	003318	208	997784	2.1	005534	210	994466	10077	99491	13
48	004563	207	997771	2.1	006792	209	993208	10106	99488	12
49	005805	206	997758	2.1	008047	208	991953	10135	99485	11
50	007044	206	997745	2.1	009298	208	990702	10164	99482	10
51	9.008278	205	9.997732	2.1	9.010546	207	10.989454	10192	99479	9
52	009510	205	997719	2.1	011790	207	988210	10221	99476	8
53	010737	204	997706	2.1	013031	206	686969	10250	99473	7
54	011962	203	997693	2.2	014268	206	985732	10279	99470	6
55	013182	203	997680	2.2	015502	205	984498	10308	99467	5
56	014400	202	997667	2.2	016732	204	983268	10337	99464	4
57	015613	202	997654	2.2	017959	204	983041	10366	99461	3
58	016824	201	997641	2.2	019183	203	980817	10395	99458	2
59	018031	201	997628	2.2	020403	203	979597	10424	99455	1
60	019235		997614		021620		978380	10453	99452	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.019235	200	9.997614	2.2	9.021620	202	10.978380	10453	99452	60
1	020435	199	997601	2.2	022834	202	977166	10482	99449	59
2	021632	199	997588	2.2	024044	201	975956	10511	99446	58
3	022825	198	997574	2.2	025251	201	974749	10540	99443	57
4	024016	198	997561	2.2	026455	200	973545	10569	99440	56
5	025203	197	997547	2.2	027655	199	972345	10597	99437	55
6	026386	197	997534	2.3	028852	199	971148	10626	99434	54
7	027567	196	997520	2.3	030046	198	969954	10655	99431	53
8	028744	196	997507	2.3	031237	198	968763	10684	99428	52
9	029918	195	997493	2.3	032425	197	967575	10713	99424	51
10	031089	195	997480	2.3	033609	197	966391	10742	99421	50
11	9.032257	194	9.997466	2.3	9.034791	196	10.965209	10771	99418	49
12	033421	194	997452	2.3	035969	196	964031	10800	99415	48
13	034582	193	997439	2.3	037144	195	962856	10829	99412	47
14	035741	192	997425	2.3	038316	195	961684	10858	99409	46
15	036896	192	997411	2.3	039485	194	960515	10887	99406	45
16	038048	191	997397	2.3	040651	194	959349	10916	99402	44
17	039197	191	997383	2.3	041813	193	958187	10945	99399	43
18	040342	190	997369	2.3	042973	193	957027	10973	99396	42
19	041485	190	997355	2.3	044130	192	955870	11002	99393	41
20	042625	189	997341	2.3	045284	192	954716	11031	99390	40
21	9.043762	189	9.997327	2.4	9.046434	191	10.953566	11060	99386	39
22	044895	180	997313	2.4	047582	191	952418	11089	99383	38
23	046026	188	997299	2.4	048727	190	951273	11118	99380	37
24	047154	187	997285	2.4	049869	190	950131	11147	99377	36
25	048279	187	997271	2.4	051008	189	948992	11176	99374	35
26	049400	186	997257	2.4	052144	189	947856	11205	99370	34
27	050519	186	997242	2.4	053277	188	946723	11234	99367	33
28	051635	185	997228	2.4	054407	188	945593	11263	99364	32
29	052749	185	997214	2.4	055535	187	944465	11291	99360	31
30	053859	184	997199	2.4	056659	187	943341	11320	99357	30
31	9.054966	184	9.997185	2.4	9.057781	186	10.942219	11349	99354	29
32	056071	184	997170	2.4	058900	186	941100	11378	99351	28
33	057172	183	997156	2.4	060016	185	939984	11407	99347	27
34	058271	183	997141	2.4	061130	185	938870	11436	99344	26
35	059367	182	997127	2.4	062240	185	937760	11465	99341	25
36	060460	182	997112	2.4	063348	184	936652	11494	99337	24
37	061551	181	997098	2.4	064453	184	935547	11523	99334	23
38	062639	181	997083	2.5	065556	183	934444	11552	99331	22
39	063724	180	997068	2.5	066655	183	933345	11580	99327	21
40	064806	180	997053	2.5	067752	182	932248	11609	99324	20
41	9.065885	179	9.997039	2.5	9.068846	182	10.931154	11638	99320	19
42	066962	179	997024	2.5	069038	181	930062	11667	99317	18
43	068036	179	997009	2.5	071027	181	928973	11696	99314	17
44	069107	178	996994	2.5	072113	181	927887	11725	99310	16
45	070176	178	996979	2.5	073197	180	926803	11754	99307	15
46	071242	177	996964	2.5	074278	180	925722	11783	99303	14
47	072306	177	996949	2.5	075356	179	924644	11812	99300	13
48	073366	176	996934	2.5	076432	179	923568	11840	99297	12
49	074424	176	996919	2.5	077505	178	922495	11869	99293	11
50	075480	175	996904	2.5	078576	178	921424	11898	99290	10
51	9.076533	175	9.996889	2.5	9.079644	178	10.920356	11927	99286	9
52	077583	175	996874	2.5	080710	177	919290	11956	99283	8
53	078631	174	996858	2.5	081773	177	918227	11985	99279	7
54	079676	174	996843	2.5	082833	176	917167	12014	99276	6
55	080719	173	996828	2.5	083891	176	916109	12043	99272	5
56	081759	173	996812	2.6	084947	175	915053	12071	99269	4
57	082797	172	996797	2.6	086000	175	914000	12100	99265	3
58	083832	172	996782	2.6	087050	175	912950	12129	99262	2
59	084864	172	996766	2.6	088098	174	911902	12158	99258	1
60	085894		996751		089144		910856	12187	99255	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

83 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.085894	171	9.996751	2.6	9.089144	174	10.910856	12187	99255	60
1	086922	171	996735	2.6	090187	173	909813	12216	99251	59
2	087947	170	996720	2.6	091228	173	908772	12245	99248	58
3	088970	170	996704	2.6	092266	173	907734	12274	99244	57
4	089990	170	996688	2.6	093302	172	906698	12302	99240	56
5	091008	169	996673	2.6	094336	172	905664	12331	99237	55
6	092024	169	996657	2.6	095367	171	904633	12360	99233	54
7	093037	168	996641	2.6	096395	171	903605	12389	99230	53
8	094047	168	996625	2.6	097422	171	902578	12418	99226	52
9	095056	168	996610	2.6	098446	170	901554	12447	99222	51
10	096062	167	996594	2.6	099468	170	900532	12476	99219	50
11	9.097065	167	9.996578	2.7	9.100487	169	10.899513	12504	99215	49
12	098066	166	996562	2.7	101504	169	898496	12533	99211	48
13	099065	166	996546	2.7	102519	169	897481	12562	99208	47
14	100062	166	996530	2.7	103532	168	896468	12591	99204	46
15	101056	165	996514	2.7	104542	168	895458	12620	99200	45
16	102048	165	996498	2.7	105550	168	894450	12649	99197	44
17	103037	164	996482	2.7	106556	167	893444	12678	99193	43
18	104025	164	996465	2.7	107559	167	892441	12706	99189	42
19	105010	164	996449	2.7	108560	166	891440	12735	99186	41
20	105992	163	996433	2.7	109559	166	890441	12764	99182	40
21	9.106973	163	9.996417	2.7	9.110556	166	10.889444	12793	99178	39
22	107951	163	996400	2.7	111551	165	888449	12822	99175	38
23	108927	162	996384	2.7	112543	165	887457	12851	99171	37
24	109901	162	996368	2.7	113533	165	886467	12880	99167	36
25	110873	162	996351	2.7	114521	164	885479	12908	99163	35
26	111842	161	996335	2.7	115507	164	884493	12937	99160	34
27	112809	161	996318	2.7	116491	164	883509	12966	99156	33
28	113774	160	996302	2.8	117472	163	882528	12995	99152	32
29	114737	160	996285	2.8	118452	163	881548	13024	99148	31
30	115698	160	996269	2.8	119429	162	880571	13053	99144	30
31	9.116656	159	9.996252	2.8	9.120404	162	10.879596	13081	99141	29
32	117613	159	996235	2.8	121377	162	878623	13110	99137	28
33	118567	159	996219	2.8	122348	161	877652	13139	99133	27
34	119519	158	996202	2.8	123317	161	876683	13168	99129	26
35	120469	158	996185	2.8	124284	161	875716	13197	99125	25
36	121417	158	996168	2.8	125249	160	874751	13226	99122	24
37	122362	157	996151	2.8	126211	160	873789	13254	99118	23
38	123306	157	996134	2.8	127172	160	872828	13283	99114	22
39	124248	157	996117	2.8	128130	159	871870	13312	99110	21
40	125187	156	996100	2.8	129087	159	870913	13341	99106	20
41	9.126125	156	9.996083	2.9	9.130041	159	10.869959	13370	99102	19
42	127060	156	996066	2.9	130994	158	869006	13399	99098	18
43	127993	155	996049	2.9	131944	158	868056	13427	99094	17
44	128925	155	996032	2.9	132893	158	867107	13456	99091	16
45	129854	154	996015	2.9	133839	157	866161	13485	99087	15
46	130781	154	995998	2.9	134784	157	865216	13514	99083	14
47	131706	154	995980	2.9	135726	157	864274	13543	99079	13
48	132630	153	995963	2.9	136667	156	863333	13572	99075	12
49	133551	153	995946	2.9	137605	156	862395	13600	99071	11
50	134470	153	995928	2.9	138542	156	861458	13629	99067	10
51	9.135387	152	9.995911	2.9	9.139476	155	10.860524	13658	99063	9
52	136303	152	995894	2.9	140409	155	859591	13687	99059	8
53	137216	152	995876	2.9	141340	155	858660	13716	99055	7
54	138128	152	995859	2.9	142269	154	857731	13744	99051	6
55	139037	151	995841	2.9	143196	154	856804	13773	99047	5
56	139944	151	995823	2.9	144121	154	855879	13802	99043	4
57	140850	151	995806	2.9	145044	153	854956	13831	99039	3
58	141754	150	995788	2.9	145966	153	854034	13860	99035	2
59	142655	150	995771	2.9	146885	153	853115	13889	99031	1
60	143555		995753		147803		852197	13917	99027	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

82 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.143555	150	9.995753	3.0	9.147803	153	10.852197	13917	99027	60
1	144453	149	995735	3.0	148718	152	851282	13946	99023	59
2	145349	149	995717	3.0	149632	152	850368	13975	99019	58
3	146243	149	995699	3.0	150544	152	849456	14004	99015	57
4	147136	148	995681	3.0	151454	151	848546	14033	99011	56
5	148026	148	995664	3.0	152363	151	847637	14061	99006	55
6	148915	148	995646	3.0	153269	151	846731	14090	99002	54
7	149802	147	995628	3.0	154174	150	845826	14119	98998	53
8	150686	147	995610	3.0	155077	150	844923	14148	98994	52
9	151569	147	995591	3.0	155978	150	844022	14177	98990	51
10	152451	147	995573	3.0	156877	150	843123	14205	98986	50
11	9.153330	146	9.995555	3.0	9.157775	149	10.842225	14234	98982	49
12	154208	146	995537	3.0	158671	149	841329	14263	98978	48
13	155083	146	995519	3.0	159565	149	840435	14292	98973	47
14	155957	145	995501	3.1	160457	148	839543	14320	98969	46
15	156830	145	995482	3.1	161347	148	838653	14349	98965	45
16	157700	145	995464	3.1	162236	148	837764	14378	98961	44
17	158569	144	995446	3.1	163123	148	836877	14407	98957	43
18	159435	144	995427	3.1	164008	147	835992	14436	98953	42
19	160301	144	995409	3.1	164892	147	835108	14464	98948	41
20	161164	144	995390	3.1	165774	147	834226	14493	98944	40
21	9.162025	143	9.995372	3.1	9.166654	146	10.833346	14522	98940	39
22	162885	143	995353	3.1	167532	146	832468	14551	98936	38
23	163743	143	995334	3.1	168409	146	831591	14580	98931	37
24	164600	142	995316	3.1	169284	145	830716	14608	98927	36
25	165454	142	995297	3.1	170157	145	829843	14637	98923	35
26	166307	142	995278	3.1	171029	145	828971	14666	98919	34
27	167159	142	995260	3.1	171899	145	828101	14695	98914	33
28	168008	141	995241	3.2	172767	144	827233	14723	98910	32
29	168856	141	995222	3.2	173634	144	826366	14752	98906	31
30	169702	141	995203	3.2	174499	144	825501	14781	98902	30
31	9.170547	140	9.995184	3.2	9.175362	144	10.824638	14810	98897	29
32	171389	140	995165	3.2	176224	143	823776	14838	98893	28
33	172230	140	995146	3.2	177084	143	822916	14867	98889	27
34	173070	140	995127	3.2	177942	143	822058	14896	98884	26
35	173908	139	995108	3.2	178799	142	821201	14925	98880	25
36	174744	139	995089	3.2	179655	142	820345	14954	98876	24
37	175578	139	995070	3.2	180508	142	819492	14982	98871	23
38	176411	139	995051	3.2	181360	142	818640	15011	98867	22
39	177242	138	995032	3.2	182211	141	817789	15040	98863	21
40	178072	138	995013	3.2	183059	141	816941	15069	98858	20
41	9.178900	138	9.994993	3.2	9.183907	141	10.816093	15097	98854	19
42	179726	137	994974	3.2	184752	141	815248	15126	98849	18
43	180551	137	994955	3.2	185597	140	814403	15155	98845	17
44	181374	137	994935	3.2	186439	140	813561	15184	98841	16
45	182196	137	994916	3.3	187280	140	812720	15212	98836	15
46	183016	136	994896	3.3	188120	140	811880	15241	98832	14
47	183834	136	994877	3.3	188958	139	811042	15270	98827	13
48	184651	136	994857	3.3	189794	139	810206	15299	98823	12
49	185466	136	994838	3.3	190629	139	809371	15327	98818	11
50	186280	135	994818	3.3	191462	139	808538	15356	98814	10
51	9.187092	135	9.994798	3.3	9.192294	138	10.807706	15385	98809	9
52	187903	135	994779	3.3	193124	138	806876	15414	98805	8
53	188712	135	994759	3.3	193953	138	806047	15442	98800	7
54	189519	134	994739	3.3	194780	138	805220	15471	98796	6
55	190325	134	994719	3.3	195606	137	804394	15500	98791	5
56	191130	134	994700	3.3	196430	137	803570	15529	98787	4
57	191933	134	994680	3.3	197253	137	802747	15557	98782	3
58	192734	133	994660	3.3	198074	137	801926	15586	98778	2
59	193534	133	994640	3.3	198894	136	801106	15615	98773	1
60	194332		994620		199713		800287	15643	98769	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

81 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.194332	133	9.994620	3.3	9.199713	136	10.800287	15643	98769	60
1	195129	133	994600	3.3	200529	136	799471	15672	98764	59
2	195925	132	994580	3.3	201345	136	798655	15701	98760	58
3	196719	132	994560	3.4	202159	135	797841	15730	98755	57
4	197511	132	994540	3.4	202971	135	797029	15758	98751	56
5	198302	132	994519	3.4	203782	135	796218	15787	98746	55
6	199091	131	994499	3.4	204592	135	795408	15816	98741	54
7	199879	131	994479	3.4	205400	134	794600	15845	98737	53
8	200666	131	994459	3.4	206207	134	793793	15873	98732	52
9	201451	131	994438	3.4	207013	134	792987	15902	98728	51
10	202234	130	994418	3.4	207817	134	792183	15931	98723	50
11	9.203017	130	9.994397	3.4	9.208619	133	10.791381	15959	98718	49
12	203797	130	994377	3.4	209420	133	790580	15988	98714	48
13	204577	130	994357	3.4	210220	133	789780	16017	98709	47
14	205354	129	994336	3.4	211018	133	788982	16046	98704	46
15	206131	129	994316	3.4	211815	133	788185	16074	98700	45
16	206906	129	994295	3.4	212611	132	787389	16103	98695	44
17	207679	129	994274	3.5	213405	132	786595	16132	98690	43
18	208452	128	994254	3.5	214198	132	785802	16160	98686	42
19	209222	128	994233	3.5	214989	132	735011	16189	98681	41
20	209992	128	994212	3.5	215780	131	784220	16218	98676	40
21	9.210760	128	9.994191	3.5	9.216568	131	10.783432	16246	98671	39
22	211526	127	994171	3.5	217356	131	782644	16275	98667	38
23	212291	127	994150	3.5	218142	131	781858	16304	98662	37
24	213055	127	994129	3.5	218926	130	781074	16333	98657	36
25	213818	127	994108	3.5	219710	130	780290	16361	98652	35
26	214579	127	994087	3.5	220492	130	779508	16390	98648	34
27	215338	126	994066	3.5	221272	130	778728	16419	98643	33
28	216097	126	994045	3.5	222052	130	777948	16447	98638	32
29	216854	126	994024	3.5	222830	129	777170	16476	98633	31
30	217609	126	994003	3.5	223606	129	776394	16505	98629	30
31	9.218363	125	9.993981	3.5	9.224382	129	10.775618	16533	98624	29
32	219116	125	993960	3.5	225156	129	774844	16562	98619	28
33	219868	125	993939	3.5	225929	129	774071	16591	98614	27
34	220618	125	993918	3.5	226700	128	773300	16620	98609	26
35	221367	125	993896	3.6	227471	128	772529	16648	98604	25
36	222115	124	993875	3.6	228239	128	771761	16677	98600	24
37	222861	124	993854	3.6	229007	128	770993	16706	98595	23
38	223606	124	993832	3.6	229773	127	770227	16734	98590	22
39	224349	124	993811	3.6	230539	127	769461	16763	98585	21
40	225092	123	993789	3.6	231302	127	768698	16792	98580	20
41	9.225833	123	9.993768	3.6	9.232065	127	10.767935	16820	98575	19
42	226573	123	993746	3.6	232826	127	767174	16849	98570	18
43	227311	123	993725	3.6	233586	126	766414	16878	98565	17
44	228048	123	993703	3.6	234345	126	765655	16906	98561	16
45	228784	122	993681	3.6	235103	126	764897	16935	98556	15
46	229518	122	993660	3.6	235859	126	764141	16964	98551	14
47	230252	122	993638	3.6	236614	126	763386	16992	98546	13
48	230984	122	993616	3.6	237368	125	762632	17021	98541	12
49	231714	122	993594	3.7	238120	125	761880	17050	98536	11
50	232444	121	993572	3.7	238872	125	761128	17078	98531	10
51	9.233172	121	9.993550	3.7	9.239622	125	10.760378	17107	98526	9
52	233899	121	994528	3.7	240371	125	759629	17136	98521	8
53	234625	121	993506	3.7	241118	124	758882	17164	98516	7
54	235349	120	993484	3.7	241865	124	758135	17193	98511	6
55	236073	120	993462	3.7	242610	124	757390	17222	98506	5
56	236795	120	993440	3.7	243354	124	756646	17250	98501	4
57	237515	120	993418	3.7	244097	124	755903	17279	98496	3
58	238235	120	993396	3.7	244839	123	755161	17308	98491	2
59	238953	119	993374	3.7	245579	123	754421	17336	98486	1
60	239670		993351		246319		753681	17365	98481	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

80 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N.sine.	N. cos.	
0	9.239670	119	9.993351	3.7	9.246319	123	10.753681	17365	98481	60
1	240386	119	993329	3.7	247057	123	752943	17393	98476	59
2	241101	119	993307	3.7	247794	123	752206	17422	98471	58
3	241814	119	993285	3.7	248530	122	751470	17451	98466	57
4	242526	118	993262	3.7	249264	122	750736	17479	98461	56
5	243237	118	993240	3.7	249998	122	750002	17508	98455	55
6	243947	118	993217	3.8	250730	122	749270	17537	98450	54
7	244656	118	993195	3.8	251461	122	748539	17565	98445	53
8	245363	118	993172	3.8	252191	121	747809	17594	98440	52
9	246069	117	993149	3.8	252920	121	747080	17623	98435	51
10	246775	117	993127	3.8	253648	121	746352	17651	98430	50
11	9.247478	117	9.993104	3.8	9.254374	121	10.745626	17680	98425	49
12	248181	117	993081	3.8	255100	121	744900	17708	98420	48
13	248883	117	993059	3.8	255824	120	744176	17737	98414	47
14	249583	116	993036	3.8	256547	120	743453	17766	98409	46
15	250282	116	993013	3.8	257269	120	742731	17794	98404	45
16	250980	116	992990	3.8	257990	120	742010	17823	98399	44
17	251677	116	992967	3.8	258710	120	741290	17852	98394	43
18	252373	116	992944	3.8	259429	120	740571	17880	98389	42
19	253067	116	992921	3.8	260146	119	739854	17909	98383	41
20	253761	115	992898	3.8	260863	119	739137	17937	98378	40
21	9.254453	115	9.992875	3.8	9.261578	119	10.738422	17966	98373	39
22	255144	115	992852	3.8	262292	119	737708	17995	98368	38
23	255834	115	992829	3.9	263005	119	736995	18023	98362	37
24	256523	115	992806	3.9	263717	118	736283	18052	98357	36
25	257211	114	992783	3.9	264428	118	735572	18081	98352	35
26	257898	114	992759	3.9	265138	118	734862	18109	98347	34
27	258583	114	992736	3.9	265847	118	734153	18138	98341	33
28	259268	114	992713	3.9	266555	118	733445	18166	98336	32
29	259951	114	992690	3.9	267261	118	732739	18195	98331	31
30	260633	113	992666	3.9	267967	117	732033	18224	98325	30
31	9.261314	113	9.992643	3.9	9.268671	117	10.731329	18252	98320	29
32	261994	113	992619	3.9	269375	117	730625	18281	98315	28
33	262673	113	992596	3.9	270077	117	729923	18309	98310	27
34	263351	113	992572	3.9	270779	117	729221	18338	98304	26
35	264027	113	992549	3.9	271479	116	728521	18367	98299	25
36	264703	112	992525	3.9	272178	116	727822	18395	98294	24
37	265377	112	992501	3.9	272876	116	727124	18424	98288	23
38	266051	112	992478	4.0	273573	116	726427	18452	98283	22
39	266723	112	992454	4.0	274269	116	725731	18481	98277	21
40	267395	112	992430	4.0	274964	116	725036	18509	98272	20
41	9.268065	111	9.992406	4.0	9.275658	115	10.724342	18538	98267	19
42	268734	111	992382	4.0	276351	115	723649	18567	98261	18
43	269402	111	992359	4.0	277043	115	722957	18595	98256	17
44	270069	111	992335	4.0	277734	115	722266	18624	98250	16
45	270735	111	992311	4.0	278424	115	721576	18652	98245	15
46	271400	111	992287	4.0	279113	115	720887	18681	98240	14
47	272064	110	992263	4.0	279801	114	720199	18710	98234	13
48	272726	110	992239	4.0	280488	114	719512	18738	98229	12
49	273388	110	992214	4.0	281174	114	718826	18767	98223	11
50	274049	110	992190	4.0	281858	114	718142	18795	98218	10
51	9.274708	110	9.992166	4.0	9.282542	114	10.717458	18824	98212	9
52	275367	110	992142	4.0	283225	114	716775	18852	98207	8
53	276024	109	992117	4.1	283907	113	716093	18881	98201	7
54	276681	109	992093	4.1	284588	113	715412	18910	98196	6
55	277337	109	992069	4.1	285268	113	714732	18938	98190	5
56	277991	109	992044	4.1	285947	113	714053	18967	98185	4
57	278644	109	992020	4.1	286624	113	713376	18995	98179	3
58	279297	109	991996	4.1	287301	113	712699	19024	98174	2
59	279948	108	991971	4.1	287977	112	712023	19052	98168	1
60	280599		991947		288652		711348	19081	98163	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

79 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.280599	108	9.991947	4.1	9.288652	112	10.711348	19081	98163	60
1	281248	108	991922	4.1	289326	112	710674	19109	98157	59
2	281897	108	991897	4.1	289999	112	710001	19138	98152	58
3	282544	108	991873	4.1	290671	112	709329	19167	98146	57
4	283190	108	991848	4.1	291342	112	708658	19195	98140	56
5	283836	107	991823	4.1	292013	111	707987	19224	98135	55
6	284480	107	991799	4.1	292682	111	707318	19252	98129	54
7	285124	107	991774	4.2	293350	111	706650	19281	98124	53
8	285766	107	991749	4.2	294017	111	705983	19309	98118	52
9	286408	107	991724	4.2	294684	111	705316	19338	98112	51
10	287048	107	991699	4.2	295349	111	704651	19366	98107	50
11	9.287687	106	9.991674	4.2	9.296013	111	10.703987	19395	98101	49
12	288326	106	991649	4.2	296677	110	703323	19423	98096	48
13	288964	106	991624	4.2	297339	110	702661	19452	98090	47
14	289600	106	991599	4.2	298001	110	701999	19481	98084	46
15	290236	106	991574	4.2	298662	110	701338	19509	98079	45
16	290870	106	991549	4.2	299322	110	700678	19538	98073	44
17	291504	105	991524	4.2	299980	110	700020	19566	98067	43
18	292137	105	991498	4.2	300638	109	699362	19595	98061	42
19	292768	105	991473	4.2	301295	109	698705	19623	98056	41
20	293399	105	991448	4.2	301951	109	698049	19652	98050	40
21	9.294029	105	9.991422	4.2	9.302607	109	10.697393	19680	98044	39
22	294658	105	991397	4.2	303261	109	696739	19709	98039	38
23	295286	104	991372	4.3	303914	109	696086	19737	98033	37
24	295913	104	991346	4.3	304567	109	695433	19766	98027	36
25	296539	104	991321	4.3	305218	108	694782	19794	98021	35
26	297164	104	991295	4.3	305869	108	694131	19823	98016	34
27	297788	104	991270	4.3	306519	108	693481	19851	98010	33
28	298412	104	991244	4.3	307168	108	692832	19880	98004	32
29	299034	104	991218	4.3	307815	108	692185	19908	97998	31
30	299655	103	991193	4.3	308463	108	691537	19937	97992	30
31	9.300276	103	9.991167	4.3	9.309109	107	10.690891	19965	97987	29
32	300895	103	991141	4.3	309754	107	690246	19994	97981	28
33	301514	103	991115	4.3	310398	107	689602	20022	97975	27
34	302132	103	991090	4.3	311042	107	688958	20051	97969	26
35	302748	103	991064	4.3	311685	107	688315	20079	97963	25
36	303364	102	991038	4.3	312327	107	687673	20108	97958	24
37	303979	102	991012	4.3	312967	106	687033	20136	97952	23
38	304593	102	990986	4.3	313608	106	686392	20165	97946	22
39	305207	102	990960	4.3	314247	106	685753	20193	97940	21
40	305819	102	990934	4.4	314885	106	685115	20222	97934	20
41	9.306430	102	9 990908	4.4	9.315523	106	10.684477	20250	97928	19
42	307041	102	990882	4.4	316159	106	683841	20279	97922	18
43	307650	101	990855	4.4	316795	106	683205	20307	97916	17
44	308259	101	990829	4.4	317430	105	682570	20336	97910	16
45	308867	101	990803	4.4	318064	105	681936	20364	97905	15
46	309474	101	990777	4.4	318697	105	681303	20393	97899	14
47	310080	101	990750	4.4	319329	105	680671	20421	97893	13
48	310685	101	990724	4.4	319961	105	680039	20450	97887	12
49	311289	100	990697	4.4	320592	105	679408	20478	97881	11
50	311893	100	990671	4.4	321222	105	678778	20507	97875	10
51	9.312495	100	9.990644	4.4	9.321851	104	10.678149	20535	97869	9
52	313097	100	990618	4.4	322479	104	677521	20563	97863	8
53	313698	100	990591	4.4	323106	104	676894	20592	97857	7
54	314297	100	990565	4.4	323733	104	676267	20620	97851	6
55	314897	100	990538	4.4	324358	104	675642	20649	97845	5
56	315495	100	990511	4.5	324983	104	675017	20677	97839	4
57	316092	99	990485	4.5	325607	104	674393	20706	97833	3
58	316689	99	990458	4.5	326231	104	673769	20734	97827	2
59	317284	99	990431	4.5	326853	104	673147	20763	97821	1
60	317879		990404		327475		672525	20791	97815	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.317879	99.0	9.990404	4.5	9.327474	103	10.672526	20791	97815	60
1	318473	98.8	990378	4.5	328095	103	671905	20820	97809	59
2	319066	98.7	990351	4.5	328715	103	671285	20848	97803	58
3	319658	98.6	990324	4.5	329334	103	670666	20877	97797	57
4	320249	98.4	990297	4.5	329953	103	670047	20905	97791	56
5	320840	98.3	990270	4.5	330570	103	669430	20933	97784	55
6	321430	98.2	990243	4.5	331187	103	668813	20962	97778	54
7	322019	98.0	990215	4.5	331803	102	668197	20990	97772	53
8	322607	97.9	990188	4.5	332418	102	667582	21019	97766	52
9	323194	97.7	990161	4.5	333033	102	666967	21047	97760	51
10	323780	97.6	990134	4.5	333646	102	666354	21076	97754	50
11	9.324366	97.5	9.990107	4.6	9.334259	102	10.665741	21104	97748	49
12	324950	97.3	990079	4.6	334871	102	665129	21132	97742	48
13	325534	97.2	990052	4.6	335482	102	664518	21161	97735	47
14	326117	97.0	990025	4.6	336093	102	663907	21189	97729	46
15	326700	96.9	989997	4.6	336702	101	663298	21218	97723	45
16	327281	96.8	989970	4.6	337311	101	662689	21246	97717	44
17	327862	96.6	989942	4.6	337919	101	662081	21275	97711	43
18	328442	96.5	989915	4.6	338527	101	661473	21303	97705	42
19	329021	96.4	989887	4.6	339133	101	660867	21331	97698	41
20	329599	96.2	989860	4.6	339739	101	660261	21360	97692	40
21	9.330176	96.1	9.989832	4.6	9.340344	101	10.659656	21388	97686	39
22	330753	96.0	989804	4.6	340948	101	659052	21417	97680	38
23	331329	95.8	989777	4.6	341552	100	658448	21445	97673	37
24	331903	95.7	989749	4.7	342155	100	657845	21474	97667	36
25	332478	95.6	989721	4.7	342757	100	657243	21502	97661	35
26	333051	95.4	989693	4.7	343358	100	656642	21530	97655	34
27	333624	95.3	989665	4.7	343958	100	656042	21559	97648	33
28	334195	95.2	989637	4.7	344558	100	655442	21587	97642	32
29	334766	95.0	989609	4.7	345157	100	654843	21616	97636	31
30	335337	94.9	989582	4.7	345755	100	654245	21644	97630	30
31	9.335906	94.8	9.989553	4.7	9.346353	99.4	10.653647	21672	97623	29
32	336475	94.6	989525	4.7	346949	99.3	653051	21701	97617	28
33	337043	94.5	989497	4.7	347545	99.2	652455	21729	97611	27
34	337610	94.4	989469	4.7	348141	99.1	651859	21758	97604	26
35	338176	94.3	989441	4.7	348735	99.0	651265	21786	97598	25
36	338742	94.1	989413	4.7	349329	98.8	650671	21814	97592	24
37	339306	94.0	989384	4.7	349922	98.7	650078	21843	97585	23
38	339871	93.9	989356	4.7	350514	98.6	649486	21871	97579	22
39	340434	93.7	989328	4.7	351106	98.5	648894	21899	97573	21
40	340996	93.6	989300	4.7	351697	98.3	648303	21928	97566	20
41	9.341558	93.5	9.989271	4.7	9.352287	98.2	10.647713	21956	97560	19
42	342119	93.4	989243	4.7	352876	98.1	647124	21985	97553	18
43	342679	93.2	989214	4.7	353465	98.0	646535	22013	97547	17
44	343239	93.1	989186	4.7	354053	97.9	645947	22041	97541	16
45	343797	93.0	989157	4.7	354640	97.7	645360	22070	97534	15
46	344355	92.9	989128	4.8	355227	97.6	644773	22098	97528	14
47	344912	92.7	989100	4.8	355813	97.5	644187	22126	97521	13
48	345469	92.6	989071	4.8	356398	97.4	643602	22155	97515	12
49	346024	92.5	989042	4.8	356982	97.3	643018	22183	97508	11
50	346579	92.4	989014	4.8	357566	97.1	642434	22212	97502	10
51	9.347134	92.2	9.988985	4.8	9.358149	97.0	10.641851	22240	97496	9
52	347687	92.1	988956	4.8	358731	96.9	641269	22268	97489	8
53	348240	92.0	988927	4.8	359313	96.8	640687	22297	97483	7
54	348792	91.9	988898	4.8	359893	96.7	640107	22325	97476	6
55	349343	91.7	988869	4.8	360474	96.6	639526	22353	97470	5
56	349893	91.6	988840	4.8	361053	96.5	638947	22382	97463	4
57	350443	91.5	988811	4.9	361632	96.3	638368	22410	97457	3
58	350992	91.4	988782	4.9	362210	96.2	637790	22438	97450	2
59	351540	91.3	988753	4.9	362787	96.1	637213	22467	97444	1
60	352088		988724		363364		636636	22495	97437	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N.sine	N. cos.	
0	9.352088	91.1	9.988724	4.9	9.363364	96.0	10.636636	22495	97437	60
1	352635	91.0	988695	4.9	363940	95.9	636060	22523	97430	59
2	353181	90.9	988666	4.9	364515	95.8	635485	22552	97424	58
3	353726	90.8	988636	4.9	365090	95.7	634910	22580	97417	57
4	354271	90.7	988607	4.9	365664	95.5	634336	22608	97411	56
5	354815	90.5	988578	4.9	366237	95.4	633763	22637	97404	55
6	355358	90.4	988548	4.9	366810	95.3	633190	22665	97398	54
7	355901	90.3	988519	4.9	367382	95.2	632618	22693	97391	53
8	356443	90.2	988489	4.9	367953	95.1	632047	22722	97384	52
9	356984	90.1	988460	4.9	368524	95.0	631476	22750	97378	51
10	357524	89.9	988430	4.9	369094	94.9	630906	22778	97371	50
11	9.358064	89.8	9.988401	4.9	9.369663	94.8	10.630337	22807	97365	49
12	358603	89.7	988371	4.9	370232	94.6	629768	22835	97358	48
13	359141	89.6	988342	4.9	370799	94.5	629201	22863	97351	47
14	359678	89.5	988312	5.0	371367	94.4	628633	22892	97345	46
15	360215	89.3	988282	5.0	371933	94.3	628067	22920	97338	45
16	360752	89.2	988252	5.0	372499	94.2	627501	22948	97331	44
17	361287	89.1	988223	5.0	373064	94.1	626936	22977	97325	43
18	361822	89.0	988193	5.0	373629	94.0	626371	23005	97318	42
19	362356	88.9	988163	5.0	374193	93.9	625807	23033	97311	41
20	362889	88.8	988133	5.0	374756	93.8	625244	23062	97304	40
21	9.363422	88.7	9.988103	5.0	9.375319	93.7	10.624681	23090	97298	39
22	363954	88.5	988073	5.0	375881	93.5	624119	23118	97291	38
23	364485	88.4	988043	5.0	376442	93.4	623558	23146	97284	37
24	365016	88.3	988013	5.0	377003	93.3	622997	23175	97278	36
25	365546	88.2	987983	5.0	377563	93.2	622437	23203	97271	35
26	366075	88.1	987953	5.0	378122	93.1	621878	23231	97264	34
27	366604	88.0	987922	5.0	378681	93.0	621319	23260	97257	33
28	367131	87.9	987892	5.0	379239	92.9	620761	23288	97251	32
29	367659	87.7	987862	5.0	379797	92.8	620203	23316	97244	31
30	368185	87.6	987832	5.1	380354	92.7	619646	23345	97237	30
31	9.368711	87.5	9.987801	5.1	9.380910	92.6	10.619090	23373	97230	29
32	369236	87.4	987771	5.1	381466	92 5	618534	23401	97223	28
33	369761	87.3	987740	5.1	382020	92.4	617980	23429	97217	27
34	370285	87.2	987710	5.1	382575	92.3	617425	23458	97210	26
35	370808	87.1	987679	5.1	383129	92.2	616871	23486	97203	25
36	371330	87.0	987649	5.1	383682	93.1	616318	23514	97·96	24
37	371852	86.9	987618	5.1	384234	92.0	615766	23542	97189	23
38	372373	86.7	987588	5.1	384786	91.9	615214	23571	97182	22
39	372894	86.6	987557	5.1	385337	91.8	614663	23599	97176	21
40	373414	86.5	987526	5.1	385888	91.7	614112	23627	97169	20
41	9.373933	86.4	9.987496	5.1	9.386438	91.5	10.613562	23656	97162	19
42	374452	86.3	987465	5.1	386987	91.4	613013	23684	97155	18
43	374970	86.2	987434	5.1	387536	91.3	612464	23712	97148	17
44	375487	86.1	987403	5.2	388084	91.2	611916	23740	97141	16
45	376003	86.0	987372	5.2	388631	91.1	611369	23769	97134	15
46	376519	85.9	987341	5.2	389178	91.0	610822	23797	97127	14
47	377035	85.8	987310	5.2	389724	90.9	610276	23825	97120	13
48	377549	85.7	987279	5.2	390270	90.8	609730	23853	97113	12
49	378063	85.6	987248	5.2	390815	90.7	609185	23882	97106	11
50	378577	85.4	987217	5.2	391360	90.6	608640	23910	97100	10
51	9.379089	85.3	9.987186	5.2	9.391903	90.5	10.608097	23938	97093	9
52	379601	85.2	987155	5.2	392447	90.4	607553	23966	97086	8
53	380113	85.1	987124	5.2	392989	90.3	607011	23995	97079	7
54	380624	85.0	987092	5.2	393531	90.2	606469	24023	97072	6
55	381134	84.9	987061	5.2	394073	90.1	605927	24051	97065	5
56	381643	84.8	987030	5.2	394614	90.0	605386	24079	97058	4
57	382152	84.7	986998	5.2	395154	89.9	604846	24108	97051	3
58	382661	84.6	986967	5.2	395694	89.8	604306	24136	97044	2
59	383168	84.5	986936	5.2	396233	89.7	603767	24164	97037	1
60	383675		986904		396771		603229	24192	97030	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.383675	84.4	9.986904	5.2	9.396771	89.6	10.603229	24192	97030	60
1	384182	84.3	986873	5.3	397309	89.6	602691	24220	97023	59
2	384687	84.2	986841	5.3	397846	89.5	602154	24249	97015	58
3	385192	84.1	986809	5.3	398383	89.4	601617	24277	97008	57
4	385697	84.0	986778	5.3	398919	89.3	601081	24305	97001	56
5	386201	83.9	986746	5.3	399455	89.2	600545	24333	96994	55
6	386704	83.8	986714	5.3	399990	89.1	600010	24362	96987	54
7	387207	83.7	986683	5.3	400524	89.0	599476	24390	96980	53
8	387709	83.6	986651	5.3	401058	88.9	598942	24418	96973	52
9	388210	83.5	986619	5.3	401591	88.8	598409	24446	96966	51
10	388711	83.4	986587	5.3	402124	88.7	597876	24474	96959	50
11	9.389211	83.3	9.986555	5.3	9.402656	88.6	10.597344	24503	96952	49
12	389711	83.2	986523	5.3	403187	88.5	596813	24531	96945	48
13	390210	83.1	986491	5.3	403718	88.4	596282	24559	96937	47
14	390708	83.0	986459	5.3	404249	88.3	595751	24587	96930	46
15	391206	82.8	986427	5.3	404778	88.2	595222	24615	96923	45
16	391703	82.7	986395	5.3	405308	88.1	594692	24644	96916	44
17	392199	82.6	986363	5.4	405836	88.0	594164	24672	96909	43
18	392695	82.5	986331	5.4	406364	87.9	593636	24700	96902	42
19	393191	82.4	986299	5.4	406892	87.8	593108	24728	96894	41
20	393685	82.3	986266	5.4	407419	87.7	592581	24756	96887	40
21	9.394179	82.2	9.986234	5.4	9.407945	87.6	10.592055	24784	96880	39
22	394673	82.1	986202	5.4	408471	87.5	591529	24813	96873	38
23	395166	82.0	986169	5.4	408997	87.4	591003	24841	96866	37
24	395658	81.9	986137	5.4	409521	87.4	590479	24869	96858	36
25	396150	81.8	986104	5.4	410045	87.3	589955	24897	96851	35
26	396641	81.7	986072	5.4	410569	87.2	589431	24925	96844	34
27	397132	81.7	986039	5.4	411092	87.1	588908	24954	96837	33
28	397621	81.6	986007	5.4	411615	87.0	588385	24982	96829	32
29	398111	81.5	985974	5.4	412137	86.9	587863	25010	96822	31
30	398600	81.4	985942	5.4	412658	86.8	587342	25038	96815	30
31	9.399088	81.3	9.985909	5.5	9.413179	86.7	10.586821	25066	96807	29
32	399575	81.2	985876	5.5	413699	86.6	586301	25094	96800	28
33	400062	81.1	985843	5.5	414219	86.5	585781	25122	96793	27
34	400549	81.0	985811	5.5	414738	86.4	585262	25151	96786	26
35	401035	80.9	985778	5.5	415257	86.4	584743	25179	96778	25
36	401520	80.8	985745	5.5	415775	86.3	584225	25207	96771	24
37	402005	80.7	985712	5.5	416293	86.2	583707	25235	96764	23
38	402489	80.6	985679	5.5	416810	86.1	583190	25263	96756	22
39	402972	80.5	985646	5.5	417326	86.0	582674	25291	96749	21
40	403455	80.4	985613	5.5	417842	85.9	582158	25320	96742	20
41	9.403938	80.3	9.985580	5.5	9.418358	85.8	10.581642	25348	96734	19
42	404420	80.2	985547	5.5	418873	85.7	581127	25376	96727	18
43	404901	80.1	985514	5.5	419387	85.6	580613	25404	96719	17
44	405382	80.0	985480	5.5	419901	85.5	580099	25432	96712	16
45	405862	79.9	985447	5.5	420415	85.5	579585	25460	96705	15
46	406341	79.8	985414	5.6	420927	85.4	579073	25488	96697	14
47	406820	79.7	985380	5.6	421440	85.3	578560	25516	96690	13
48	407299	79.6	985347	5.6	421952	85.2	578048	25545	96682	12
49	407777	79.5	985314	5.6	422463	85.1	577537	25573	96675	11
50	408254	79.4	985280	5.6	422974	85.0	577026	25601	96667	10
51	9.408731	79.4	9.985247	5.6	9.423484	84.9	10.576516	25629	96660	9
52	409207	79.3	985213	5.6	423993	84.8	576007	25657	96653	8
53	409682	79.2	985180	5.6	424503	84.8	575497	25685	96645	7
54	410157	79.1	985146	5.6	425011	84.7	574989	25713	96638	6
55	410632	79.0	985113	5.6	425519	84.6	574481	25741	96630	5
56	411106	78.9	985079	5.6	426027	84.5	573973	25766	96623	4
57	411579	78.8	985045	5.6	426534	84.4	573466	25798	96615	3
58	412052	78.7	985011	5.6	427041	84.3	572959	25826	96608	2
59	412524	78.6	984978	5.6	427547	84.3	572453	25854	96600	1
60	412996		984944		428052		571948	25882	96593	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

75 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.412996	78.5	9.984944	5.7	9.428052	84.2	10.571948	25882	96593	60
1	413467	78.4	984910	5.7	428557	84.1	571443	25910	96585	59
2	413938	78.3	984876	5.7	429062	84.0	570938	25938	96578	58
3	414408	78.3	984842	5.7	429566	83.9	570434	25966	96570	57
4	414878	78.2	984808	5.7	430070	83.8	569930	25994	96562	56
5	415347	78.1	984774	5.7	430573	83.8	569427	26022	96555	55
6	415815	78.0	984740	5.7	431075	83.7	568925	26050	96547	54
7	416283	77.9	984706	5.7	431577	83.6	568423	26079	96540	53
8	416751	77.8	984672	5.7	432079	83.5	567921	26107	96532	52
9	417217	77.7	984637	5.7	432580	83.4	567420	26135	96524	51
10	417684	77.6	984603	5.7	433080	83.3	566920	26163	96517	50
11	9.418150	77.5	9.984569	5.7	9.433580	83.2	10.566420	26191	96509	49
12	418615	77.4	984535	5.7	434080	83.2	565920	26219	96502	48
13	419079	77.3	984500	5.7	434579	83.1	565421	26247	96494	47
14	419544	77.3	984466	5.7	435078	83.0	564922	26275	96486	46
15	420007	77.2	984432	5.8	435576	82.9	564424	26303	96479	45
16	420470	77.1	984397	5.8	436073	82.8	563927	26331	96471	44
17	420933	77.0	984363	5.8	436570	82.8	563430	26359	96463	43
18	421395	76.9	984328	5.8	437067	82.7	562933	26387	96456	42
19	421857	76.8	984294	5.8	437563	82.6	562437	26415	96448	41
20	422318	76.7	984259	5.8	438059	82.5	561941	26443	96440	40
21	9.422778	76.7	9.984224	5.8	9.438554	82.4	10.561446	26471	96433	39
22	423238	76.6	984190	5.8	439048	82.3	560952	26500	96425	38
23	423697	76.5	984155	5.8	439543	82.3	560457	26528	96417	37
24	424156	76.4	984120	5.8	440036	82.2	559964	26556	96410	36
25	424615	76.3	984085	5.8	440529	82.1	559471	26584	96402	35
26	425073	76.2	984050	5.8	441022	82.0	558978	26612	96394	34
27	425530	76.1	984015	5.8	441514	81.9	558486	26640	96386	33
28	425987	76.0	983981	5.8	442006	81.9	557994	26668	96379	32
29	426443	76.0	983946	5.8	442497	81.8	557503	26696	96371	31
30	426899	75.9	983911	5.8	442988	81.7	557012	26724	96363	30
31	9.427354	75.8	9.983875	5.8	9.443479	81.6	10.556521	26752	96355	29
32	427809	75.7	983840	5.9	443968	81.6	556032	26780	96347	28
33	428263	75.6	983805	5.9	444458	81.5	555542	26808	96340	27
34	428717	75.5	983770	5.9	444947	81.4	555053	26836	96332	26
35	429170	75.4	983735	5.9	445435	81.3	554565	26864	96324	25
36	429623	75.3	983700	5.9	445923	81.2	554077	26892	96316	24
37	430075	75.2	983664	5.9	446411	81.2	553589	26920	96308	23
38	430527	75.2	983629	5.9	446898	81.1	553102	26948	96301	22
39	430978	75.1	983594	5.9	447384	81.0	552616	26976	96293	21
40	431429	75.0	983558	5.9	447870	80.9	552130	27004	96285	20
41	9.431879	74.9	9.983523	5.9	9.448356	80.9	10.551644	27032	96277	19
42	432329	74.9	983487	5.9	448841	80.8	551159	27060	96269	18
43	432778	74.8	983452	5.9	449326	80.7	550674	27088	96261	17
44	433226	74.7	983416	5.9	449810	80.6	550190	27116	96253	16
45	433675	74 6	983381	5.9	450294	80.6	549706	27144	96246	15
46	434122	74.5	983345	5.9	450777	80.5	549223	27172	96238	14
47	434569	74.4	983309	5.9	451260	80.4	548740	27200	96230	13
48	435016	74.4	983273	6.0	451743	80.3	548257	27228	96222	12
49	435462	74.3	983238	6.0	452225	80.2	547775	27256	96214	11
50	435908	74.2	983202	6.0	452706	80.2	547294	27284	96206	10
51	9.436353	74.1	9.983166	6.0	9.453187	80.1	10.546813	27312	96198	9
52	436798	74.0	983130	6.0	453668	80.0	546332	27340	96190	8
53	437242	74.0	983094	6.0	454148	79.9	545852	27368	96182	7
54	437686	73.9	983058	6.0	454628	79.9	545372	27396	96174	6
55	438129	73.8	983022	6.0	455107	79.8	544893	27424	96166	5
56	438572	73.7	982986	6.0	455586	79.7	544414	27452	96158	4
57	439014	73.6	982950	6.0	456064	79.6	543936	27480	96150	3
58	439456	73.6	982914	6.0	456542	79.6	543458	27508	96142	2
59	439897	73.5	982878	6.0	457019	79.5	542981	27536	96134	1
60	440338		982842		457496		542504	27564	96126	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.440338	73.4	9.982842	6.0	9.457496	79.4	10.542504	27564	96126	60
1	440778	73.3	982805	6.0	457973	79.3	542027	27592	96118	59
2	441218	73.2	982769	6.1	458449	79.3	541551	27620	96110	58
3	441658	73·1	982733	6.1	458925	79.2	541075	27648	96102	57
4	442096	73.1	982696	6.1	459400	79.1	540600	27676	96094	56
5	442535	73.0	982660	6.1	459875	79.0	540125	27704	96086	55
6	442973	72.9	982624	6.1	460349	79.0	539651	27731	96078	54
7	443410	72.8	982587	6.1	460823	78.9	539177	27759	96070	53
8	443847	72.7	982551	6.1	461297	78.8	538703	27787	96062	52
9	444284	72.7	982514	6.1	461770	78.9	538230	27815	96054	51
10	444720	72.6	982477	6.1	462242	78.7	537758	27843	96046	50
11	9.445155	72.5	9.982441	6.1	9.462714	78.6	10.537286	27871	96037	49
12	445590	72.4	982404	6.1	463186	78.5	536814	27899	96029	48
13	446025	72.3	982367	6.1	463658	78.5	536342	27927	96021	47
14	446459	72.3	982331	6.1	464129	78.4	535871	27955	96013	46
15	446893	72.2	982294	6.1	464599	78.3	535401	27983	96005	45
16	447326	72.1	982257	6.1	465069	78.3	534931	28011	95997	44
17	447759	72.0	982220	6.2	465539	78.2	534461	28039	95989	43
18	448191	72.0	982183	6.2	466008	78.1	533992	28067	95981	42
19	448623	71.9	982146	6.2	466476	78.0	533524	28095	95972	41
20	449054	71.8	982109	6.2	466945	78.0	533055	28123	95964	40
21	9.449485	71.7	9.982072	6.2	9.467413	77.9	10.532587	28150	95956	39
22	449915	71.6	982035	6.2	467880	77.8	532120	28178	95948	38
23	450345	71.6	981998	6.2	468347	77.8	531653	28206	95940	37
24	450775	71.5	981961	6.2	468814	77.7	531186	28234	95931	36
25	451204	71.4	981924	6.2	469280	77.6	530720	28262	95923	35
26	451632	71.3	981886	6.2	469746	77.5	530254	28290	95915	34
27	452060	71.3	981849	6.2	470211	77.5	529789	28318	95907	33
28	452488	71.2	981812	6.2	470676	77.4	529324	28346	95898	32
29	452915	71.1	981774	6.2	471141	77.3	528859	28374	95890	31
30	453342	71.0	981737	6.2	471605	77.3	528395	28402	95882	30
31	9.453768	71.0	9.981699	6.3	9.472068	77.2	10.527932	28429	95874	29
32	454194	70.9	981662	6.3	472532	77.1	527468	28457	95865	28
33	454619	70.8	981625	6.3	472995	77.1	527005	28485	95857	27
34	455044	70.7	981587	6.3	473457	77.0	526543	28513	95849	26
35	455469	70.7	981549	6.3	473919	76.9	526081	28541	95841	25
36	455893	70.6	981512	6.3	474381	76.9	525619	28569	95832	24
37	456316	70.5	981474	6.3	474842	76.8	525158	28597	95824	23
38	456739	70.4	981436	6.3	475303	76.7	524697	28625	95816	22
39	457162	70.4	981399	6.3	475763	76.7	524237	28652	95807	21
40	457584	70.3	981361	6.3	476223	76.6	523777	28680	95799	20
41	9.458006	70.2	9.981323	6.3	9.476683	76.5	10.523317	28708	95791	19
42	458427	70.1	981285	6.3	477142	76.5	522858	28736	95782	18
43	458848	70.1	981247	6.3	477601	76.4	522399	28764	95774	17
44	459268	70.0	981209	6.3	478059	76.3	521941	28792	95766	16
45	459688	69.9	981171	6.3	478517	76.3	521483	28820	95757	15
46	460108	69.8	981133	6.4	478975	76.2	521025	28847	95749	14
47	460527	69.8	981095	6.4	479432	76.1	520568	28875	95740	13
48	460946	69.7	981057	6.4	479889	76.1	520111	28903	95732	12
49	461364	69.6	981019	6.4	480345	76.0	519655	28931	95724	11
50	461782	69.5	980981	6.4	480801	75.9	519199	28959	95715	10
51	9.462199	69.5	9.980942	6.4	9.481257	75.9	10.518743	28987	95707	9
52	462616	69.4	980904	6.4	481712	75.8	518288	29015	95698	8
53	463032	69.3	980866	6.4	482167	75.7	517833	29042	95690	7
54	463448	69.3	980827	6.4	482621	75.7	517379	29070	95681	6
55	463864	69.2	980789	6.4	483075	75.6	516925	29098	95673	5
56	464279	69.1	980750	6.4	483529	75.5	516471	29126	95664	4
57	464694	69.0	980712	6.4	483982	75.5	516018	29154	95656	3
58	465108	69.0	980673	6.4	484435	75.4	515565	29182	95647	2
59	465522	68.9	980635	6.4	484887	75.3	515113	29209	95639	1
60	465935		980596		485339		514661	29247	95630	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.465935	68.8	9.980596	6.4	9.485339	75.3	10.514661	29237	95630	60
1	466348	68.8	980558	6.4	485791	75.2	514209	29265	95622	59
2	466761	68.7	980519	6.5	486242	75.1	513758	29293	95613	58
3	467173	68.6	980480	6.5	486693	75.1	513307	29321	95605	57
4	467585	68.5	980442	6.5	487143	75.0	512857	29348	95596	56
5	467996	68.5	980403	6.5	487593	74.9	512407	29376	95588	55
6	468407	68.4	980364	6.5	488043	74.9	511957	29404	95579	54
7	468817	68.3	980325	6.5	488492	74.8	511508	29432	95571	53
8	469227	68.3	980286	6.5	488941	74.7	511059	29460	95562	52
9	469637	68.2	980247	6.5	489390	74.7	510610	29487	95554	51
10	470046	68.1	980208	6.5	489838	74.6	510162	29515	95545	50
11	9.470455	68.0	9.980169	6.5	9.490286	74.6	10.509714	29543	95536	49
12	470863	68.0	980130	6.5	490733	74.5	509267	29571	95528	48
13	471271	67.9	980091	6.5	491180	74.4	508820	29599	95519	47
14	471679	67.8	980052	6.5	491627	74.4	508373	29626	95511	46
15	472086	67.8	980012	6.5	492073	74.3	507927	29654	95502	45
16	472492	67.7	979973	6.5	492519	74.3	507481	29682	95493	44
17	472898	67.6	979934	6.6	492965	74.2	507035	29710	95485	43
18	473304	67.6	979895	6.6	493410	74.1	506590	29737	95476	42
19	473710	67.5	979855	6.6	493854	74.0	506146	29765	95467	41
20	474115	67.4	979816	6.6	494299	74.0	505701	29793	95459	40
21	9.474519	67.4	9.979776	6.6	9.494743	74.0	10.505257	29821	95450	39
22	474923	67.3	979737	6.6	495186	73.9	504814	29849	95441	38
23	475327	67.2	979697	6.6	495630	73.8	504370	29876	95433	37
24	475730	67.2	979658	6.6	496073	73.7	503927	29904	95424	36
25	476133	67.1	979618	6.6	496515	73.7	503485	29932	95415	35
26	476536	67.0	979579	6.6	496957	73.6	503043	29960	95407	34
27	476938	66.9	979539	6.6	497399	73.6	502601	29987	95398	33
28	477340	66.9	979499	6.6	497841	73.5	502159	30015	95389	32
29	477741	66.8	979459	6.6	468282	73.4	501718	30043	95380	31
30	478142	66.7	979420	6.6	498722	73.4	501278	30071	95372	30
31	9.478542	66.7	9.979380	6.6	9.499163	73.3	10.500837	30098	95363	29
32	478942	66.6	979340	6.6	499603	73.3	500397	30126	95354	28
33	479342	66.5	979300	6.7	500042	73.2	499958	30154	95345	27
34	479741	66.5	979260	6.7	500481	73.1	499519	30182	95337	26
35	480140	66.4	979220	6.7	500920	73.1	499080	30209	95328	25
36	480539	66.3	979180	6.7	501359	73.0	498641	30237	95319	24
37	480937	66.3	979140	6.7	501797	73.0	498203	30265	95310	23
38	481334	66.2	979100	6.7	502235	72.9	497765	30292	95301	22
39	481731	66.1	979059	6.7	502672	72.8	497328	30320	95293	21
40	482128	66.1	979019	6.7	503109	72.8	496891	30348	95284	20
41	9.482525	66.0	9.978979	6.7	9.503546	72.7	10.496454	30376	95275	19
42	482921	65.9	978939	6.7	503982	72.7	496018	30403	95266	18
43	483316	65.9	978898	6.7	504418	72.6	495582	30431	95257	17
44	483712	65.8	978858	6.7	504854	72.5	495146	30459	95248	16
45	484107	65.7	978817	6.7	505289	72.5	494711	30486	95240	15
46	484501	65.7	978777	6.7	505724	72.4	494276	30514	95231	14
47	484895	65.6	978736	6.7	506159	72.4	493841	30542	95222	13
48	485289	65.5	978696	6.8	506593	72.3	493407	30570	95213	12
49	485682	65.5	978655	6.8	507027	72.2	492973	30597	95204	11
50	486075	65.4	978615	6.8	507460	72.2	492540	30625	95195	10
51	9.486467	66.3	9.978574	6.8	9.507893	72.1	10.492107	30653	95186	9
52	486860	65.3	978533	6.8	508326	72.1	491674	30680	95177	8
53	487251	65.2	978493	6.8	508759	72.0	491241	30708	95168	7
54	487643	65.1	978452	6.8	509191	71.9	490809	30736	95159	6
55	488034	65.1	978411	6.8	509622	71.9	490378	30763	95150	5
56	488424	65.0	978370	6.8	510054	71.8	489946	30791	95142	4
57	488814	65.0	978329	6.8	510485	71.8	489515	30819	95133	3
58	489204	64.9	978288	6.8	510916	71.7	489084	30846	95124	2
59	489593	64.8	978247	6.8	511346	71.6	488654	30874	95115	1
60	489982		978206		511776		488224	30902	95106	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.489982	64.8	9.978206	6.8	9.511776	71.6	10.488224	30902	95106	60
1	490371	64.8	978165	6.8	512206	71.6	487794	30929	95097	59
2	490759	64.7	978124	6.8	512635	71.5	487365	30957	95088	58
3	491147	64.6	978083	6.9	513064	71.4	486936	30985	95079	57
4	491535	64.6	978042	6.9	513493	71.4	486507	31012	95070	56
5	491922	64.5	978001	6.9	513921	71.3	486079	31040	95061	55
6	492308	64.4	977959	6.9	514349	71.3	485651	31068	95052	54
7	492695	64.4	977918	6.9	514777	71.2	485223	31095	95043	53
8	493081	64.3	977877	6.9	515204	71.2	484796	31123	95033	52
9	493466	64.2	977835	6.9	515631	71.1	484369	31151	95024	51
10	493851	64.2	977794	6.9	516057	71.0	483943	31178	95015	50
11	9.494236	64.1	9.977752	6.9	9.516484	71.0	10.483516	31206	95006	49
12	494621	64.1	977711	6.9	516910	70.9	483090	31233	94997	48
13	495005	64.0	977669	6.9	517335	70.9	482665	31261	94988	47
14	495388	63.9	977628	6.9	517761	70.8	482239	31289	94979	46
15	495772	63.9	977586	6.9	518185	70.8	481815	31316	94970	45
16	496154	63.8	977544	7.0	518610	70.7	481390	31344	94961	44
17	496537	63.7	977503	7.0	519034	70.6	480966	31372	94952	43
18	496919	63.7	977461	7.0	519458	70.6	480542	31399	94943	42
19	497301	63.6	977419	7.0	519882	70.5	480118	31427	94933	41
20	497682	63.6	977377	7.0	520305	70.5	479695	31454	94924	40
21	9.498064	63.5	9.977335	7.0	9.520728	70.4	10.479272	31482	94915	39
22	498444	63.4	977293	7.0	521151	70.3	478849	31510	94906	38
23	498825	63.4	977251	7.0	521573	70.3	478427	31537	94897	37
24	499204	63.3	977209	7.0	521995	70.3	478005	31565	94888	36
25	499584	63.2	977167	7.0	522417	70.2	477583	31593	94878	35
26	499963	63.2	977125	7.0	522838	70.2	477162	31620	94869	34
27	500342	63.1	977083	7.0	523259	70.1	476741	31648	94860	33
28	500721	63.1	977041	7.0	523680	70.1	476320	31675	94851	32
29	501099	63.0	976999	7.0	524100	70.0	475900	31703	94842	31
30	501476	62.9	976957	7.0	524520	69.9	475480	31730	94832	30
31	9.501854	62.9	9.976914	7.0	9.524939	69.9	10.475061	31758	94823	29
32	502231	62.8	976872	7.1	525359	69.8	474641	31786	94814	28
33	502607	62.8	976830	7.1	525778	69.8	474222	31813	94805	27
34	502984	62.7	976787	7.1	526197	69.7	473803	31841	94795	26
35	503360	62.6	976745	7.1	526615	69.7	473385	31868	94786	25
36	503735	62.6	976702	7.1	527033	69.6	472967	31896	94777	24
37	504110	62.5	976660	7.1	527451	69.6	472549	31923	94768	23
38	504485	62.5	976617	7.1	527868	69.5	472132	31951	94758	22
39	504860	62.4	976574	7.1	528285	69.5	471715	31979	94749	21
40	505234	62.3	976532	7.1	528702	69.4	471298	32006	94740	20
41	9.505608	62.3	9.976489	7.1	9.529119	69.3	10.470881	32034	94730	19
42	505981	62.2	976446	7.1	529535	69.3	470465	32061	94721	18
43	506354	62.2	976404	7.1	529950	69.3	470050	32089	94712	17
44	506727	62.1	976361	7.1	530366	69.2	469634	32116	94702	16
45	507099	62.0	976318	7.1	530781	69.1	469219	32144	94693	15
46	507471	62.0	976275	7.1	531196	69.1	468804	32171	94684	14
47	507843	61.9	976232	7.2	531611	69.0	468389	32199	94674	13
48	508214	61.9	976189	7.2	532025	69.0	467975	32227	94665	12
49	508585	61.8	976146	7.2	532439	68.9	467561	32250	94656	11
50	508956	61.8	976103	7.2	532853	68.9	467147	32282	94646	10
51	9.509326	61.7	9.976060	7.2	9.533266	68.8	10.466734	32309	94637	9
52	509696	61.6	976017	7.2	533679	68.8	466321	32337	94627	8
53	510065	61.6	975974	7.2	534092	68.7	465908	32364	94618	7
54	510434	61.5	975930	7.2	534504	68.7	465496	32392	94609	6
55	510803	61.5	975887	7.2	534916	68.6	465084	32419	94599	5
56	511172	61.4	975844	7.2	535328	68.6	464672	32447	94590	4
57	511540	61.3	975800	7.2	535739	68.5	464261	32474	94580	3
58	511907	61.3	975757	7.2	536150	68.5	463850	32502	94571	2
59	512275	61.2	975714	7.2	536561	68.4	463439	32529	94561	1
60	512642		975670		536972		463028	32557	94552	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

71 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.512642	61.2	9.975670	7.3	9.536972	68.4	10.463028	32557	94552	60
1	513009	61.1	975627	7.3	537382	68.3	462618	32584	94542	59
2	513375	61.1	975583	7.3	537792	68.3	462208	32612	94533	58
3	513741	61.0	975539	7.3	538202	68.2	461798	32639	94523	57
4	514107	60.9	975496	7.3	538611	68.2	461389	32667	94514	56
5	514472	60.9	975452	7.3	539020	68.1	460980	32694	94504	55
6	514837	60.8	975408	7.3	539429	68.1	460571	32722	94495	54
7	515202	60.8	975365	7.3	539837	68.0	460163	32749	94485	53
8	515566	60.7	975321	7.3	540245	68.0	459755	32777	94476	52
9	515930	60.7	975277	7.3	540653	67.9	459347	32804	94466	51
10	516294	60.6	975233	7.3	541061	67.9	458939	32832	94457	50
11	9.516657	60.5	9.975189	7.3	9.541468	67.8	10.458532	32859	94447	49
12	517020	60.5	975145	7.3	541875	67.8	458125	32887	94438	48
13	517382	60.4	975101	7.3	542281	67.7	457719	32914	94428	47
14	517745	60.4	975057	7.3	542688	67.7	457312	32942	94418	46
15	518107	60.3	975013	7.3	543094	67.6	456906	32969	94409	45
16	518468	60.3	974969	7.4	543499	67.6	456501	32997	94399	44
17	518829	60.2	974925	7.4	543905	67.5	456095	33024	94390	43
18	519190	60.1	974880	7.4	544310	67.5	455690	33051	94380	42
19	519551	60.1	974836	7.4	544715	67.4	455285	33079	94370	41
20	519911	60.0	974792	7.4	545119	67.4	454881	33106	94361	40
21	9.520271	60.0	9.974748	7.4	9.545524	67.3	10.454476	33134	94351	39
22	520631	59.9	974703	7.4	545928	67.3	454072	33161	94342	38
23	520990	59.9	974659	7.4	546331	67.2	453669	33189	94332	37
24	521349	59.8	974614	7.4	546735	67.2	453265	33216	94322	36
25	521707	59.8	974570	7.4	547138	67.1	452862	33244	94313	35
26	522066	59.7	974525	7.4	547540	67.1	452460	33271	94303	34
27	522424	59.6	974481	7.4	547943	67.0	452057	33298	94293	33
28	522781	59.6	974436	7.4	548345	67.0	451655	33326	94284	32
29	523138	59.5	974391	7.4	548747	66.9	451253	33353	94274	31
30	523495	59.5	974347	7.5	549149	66.9	450851	33381	94264	30
31	9.523852	59.4	9.974302	7.5	9.549550	66.8	10.450450	33408	94254	29
32	524208	59.4	974257	7.5	549951	66.8	450049	33436	94245	28
33	524564	59.3	974212	7.5	550352	66.7	449648	33463	94235	27
34	524920	59.3	974167	7.5	550752	66.7	449248	33490	94225	26
35	525275	59.2	974122	7.5	551152	66.6	448848	33518	94215	25
36	525630	59.1	974077	7.5	551552	66.6	448448	33545	94206	24
37	525984	59.1	974032	7.5	551952	66.5	448048	33573	94196	23
38	526339	59.0	973987	7.5	552351	66.5	447649	33600	94186	22
39	526693	59.0	973942	7.5	552750	66.5	447250	33627	94176	21
40	527046	58.9	973897	7.5	553149	66.4	446851	33655	94167	20
41	9.527400	58.9	9.973852	7.5	9.553548	66.4	10.446452	33682	94157	19
42	527753	58.8	973807	7.5	553946	66.3	446054	33710	94147	18
43	528105	58.8	973761	7.5	554344	66.3	445656	33737	94137	17
44	528458	58.7	973716	7.6	554741	66.2	445259	33764	94127	16
45	528810	58.7	973671	7.6	555139	66.2	444861	33792	94118	15
46	529161	58.6	973625	7.6	555536	66.1	444464	33819	94108	14
47	529513	58.6	973580	7.6	555933	66.1	444067	33846	94098	13
48	529864	58.5	973535	7.6	556329	66.0	443671	33874	94088	12
49	530215	58.5	973489	7.6	556725	66.0	443275	33901	94078	11
50	530565	58.4	973444	7.6	557121	65.9	442879	33929	94068	10
51	9.530915	58.4	9.973398	7.6	9.557517	65.9	10.442483	33956	94058	9
52	531265	58.3	973352	7.6	557913	65.9	442087	33983	94049	8
53	531614	58.2	973307	7.6	558308	65.8	441692	34011	94039	7
54	531963	58.2	973261	7.6	558702	65.8	441298	34038	94029	6
55	532312	58.1	973215	7.6	559097	65.7	440903	34065	94019	5
56	532661	58.1	973169	7.6	559491	65.7	440509	34093	94009	4
57	533009	58.0	973124	7.6	559885	65.6	440115	34120	93999	3
58	533357	58.0	973078	7.6	560279	65.6	439721	34147	93989	2
59	533704	57.9	973032	7.7	560673	65.5	439327	34175	93979	1
60	534052		972986		561066		438934	34202	93969	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.534052	57.8	9.972986	7.7	9.561066	65.5	10.438934	34202	93969	60
1	534399	57.7	972940	7.7	561459	65.4	438541	34229	93959	59
2	534745	57.7	972894	7.7	561851	65.4	438149	34257	93949	58
3	535092	57.7	972848	7.7	562244	65.3	437756	34284	93939	57
4	535438	57.6	972802	7.7	562636	65.3	437364	34311	93929	56
5	535783	57.6	972755	7.7	563028	65.3	436972	34339	93919	55
6	536129	57.5	972709	7.7	563419	65.2	436581	34366	93909	54
7	536474	57.4	972663	7.7	563811	65.2	436189	34393	93899	53
8	536818	57.4	972617	7.7	564202	65.1	435798	34421	93889	52
9	537163	57.3	972570	7.7	564592	65.1	435408	34448	93879	51
10	537507	57.3	972524	7.7	564983	65.0	435017	34475	93869	50
11	9.537851	57.2	9.972478	7.7	9.565373	65.0	10.434627	34503	93859	49
12	538194	57.2	972431	7.8	565763	64.9	434237	34530	93849	48
13	538538	57.1	972385	7.8	566153	64.9	433847	34557	93839	47
14	538880	57.1	972338	7.8	566542	64.9	433458	34584	93829	46
15	539223	57.0	972291	7.8	566932	64.8	433068	34612	93819	45
16	539565	57.0	972245	7.8	567320	64.8	432680	34639	93809	44
17	539907	56.9	972198	7.8	567709	64.7	432291	34666	93799	43
18	540249	56.9	972151	7.8	568098	64.7	431902	34694	93789	42
19	540590	56.8	972105	7.8	568486	64.6	431514	34721	93779	41
20	540931	56.8	972058	7.8	568873	64.6	431127	34748	93769	40
21	9.541272	56.7	9.972011	7.8	9.569261	64.5	10.430739	34775	93759	39
22	541613	56.7	971964	7.8	569648	64.5	430352	34803	93748	38
23	541953	56.6	971917	7.8	570035	64.5	429965	34830	93738	37
24	542293	56.6	971870	7.8	570422	64.4	429578	34857	93728	36
25	542632	56.5	971823	7.8	570809	64.4	429191	34884	93718	35
26	542971	56.5	971776	7.8	571195	64.3	428805	34912	93708	34
27	543310	56.4	971729	7.9	571581	64.3	428419	34939	93698	33
28	543649	56.4	971682	7.9	571967	64.2	428033	34966	93688	32
29	543987	56.3	971635	7.9	572352	64.2	427648	34993	93677	31
30	544325	56.3	971588	7.9	572738	64.2	427262	35021	93667	30
31	9.544663	56.2	9.971540	7.9	9.573123	64.1	10.426877	35048	93657	29
32	545000	56.2	971493	7.9	573507	64.1	426493	35075	93647	28
33	545338	56.1	971446	7.9	573892	64.0	426108	35102	93637	27
34	545674	56.1	971398	7.9	574276	64.0	425724	35130	93626	26
35	546011	56.0	971351	7.9	574660	63.9	425340	35157	93616	25
36	546347	56.0	971303	7.9	575044	63.9	424956	35184	93606	24
37	546683	55.9	971256	7.9	575427	63.9	424573	35211	93596	23
38	547019	55.9	971208	7.9	575810	63.8	424190	35239	93585	22
39	547354	55.8	971161	7.9	576193	63.8	423807	35266	93575	21
40	547689	55.8	971113	7.9	576576	63.7	423424	35293	93565	20
41	9.548024	55.7	9.971066	8.0	9.576958	63.7	10.423041	35320	93555	19
42	548359	55.7	971018	8.0	577341	63.6	422659	35347	93544	18
43	548693	55.6	970970	8.0	577723	63.6	422277	35375	93534	17
44	549027	55.6	970922	8.0	578104	63.6	421896	35402	93524	16
45	549360	55.5	970874	8.0	578486	63.5	421514	35429	93514	15
46	549693	55.5	970827	8.0	578867	63.5	421133	35456	93503	14
47	550026	55.4	970779	8.0	579248	63.4	420752	35484	93493	13
48	550359	55.4	970731	8.0	579629	63.4	420371	35511	93483	12
49	550692	55.3	970683	8.0	580009	63.4	419991	35538	93472	11
50	551024	55.3	970635	8.0	580389	63.3	419611	35565	93462	10
51	9.551356	55.2	9.970586	8.0	9.580769	63.3	10.419231	35592	93452	9
52	551687	55.2	970538	8.0	581149	63.2	418851	35619	93441	8
53	552018	55.2	970490	8.0	581528	63.2	418472	35647	93431	7
54	552349	55.1	970442	8.0	581907	63.2	418093	35674	93420	6
55	552680	55.1	970394	8.0	582286	63.1	417714	35701	93410	5
56	553010	55.0	970345	8.1	582665	63.1	417335	35728	93400	4
57	553341	55.0	970297	8.1	583043	63.0	416957	35755	93389	3
58	553670	54.9	970249	8.1	583422	63.0	416578	35782	93379	2
59	554000	54.9	970200	8.1	583800	62.9	416200	35810	93368	1
60	554329		970152		584177		415823	35837	93358	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N.sine.	N. cos.	
0	9.554329	54.8	9.970152	8.1	9.584177	62.9	10.415823	35837	93358	60
1	554658	54.8	970103	8.1	584555	62.9	415445	35864	93348	59
2	554987	54.7	970055	8.1	584932	62.8	415068	35891	93337	58
3	555315	54.7	970006	8.1	585309	62.8	414691	35918	93327	57
4	555643	54.6	969957	8.1	585686	62.7	414314	35945	93316	56
5	555971	54.6	969909	8.1	586062	62.7	413938	35973	93306	55
6	556299	54.5	969860	8.1	586439	62.7	413561	36000	93295	54
7	556626	54.5	969811	8.1	586815	62.6	413185	36027	93285	53
8	556953	54.4	969762	8.1	587190	62.6	412810	36054	93274	52
9	557280	54.4	969714	8.1	587566	62.5	412434	36081	93264	51
10	557606	54.3	969665	8.1	587941	62.5	412059	36108	93253	50
11	9.557932	54.3	9.969616	8.2	9.588316	62.5	10.411684	36135	93243	49
12	558258	54.3	969567	8.2	588691	62.4	411309	36162	93232	48
13	558583	54.2	969518	8.2	589066	62.4	410934	36190	93222	47
14	558909	54.2	969469	8.2	589440	62.3	410560	36217	93211	46
15	559234	54.1	969420	8.2	589814	62.3	410186	36244	93201	45
16	559558	54.1	969370	8.2	590188	62.3	409812	36271	93190	44
17	559883	54.0	969321	8.2	590562	62.2	409438	36298	93180	43
18	560207	54.0	969272	8.2	590935	62.2	409065	36325	93169	42
19	560531	53.9	969223	8.2	591308	62.2	408692	36352	93159	41
20	560855	53.9	969173	8.2	591681	62.1	408319	36379	93148	40
21	9.561178	53.8	9.969124	8.2	9.592054	62.1	10.407946	36406	93137	39
22	561501	53.8	969075	8.2	592426	62.0	407574	36434	93127	38
23	561824	53.7	969025	8.2	592798	62.0	407202	36461	93116	37
24	562146	53.7	968976	8.2	593170	61.9	406829	36488	93106	36
25	562468	53.6	968926	8.3	593542	61.9	406458	36515	93095	35
26	562790	53.6	968877	8.3	593914	61.8	406086	36542	93084	34
27	563112	53.6	968827	8.3	594285	61.8	405715	36569	93074	33
28	563433	53.5	968777	8.3	594656	61.8	405344	36596	93063	32
29	563755	53.5	968728	8.3	595027	61.7	404973	36623	93052	31
30	564075	53.4	968678	8.3	595398	61.7	404602	36650	93042	30
31	9.564396	53.4	9.968628	8.3	9.595768	61.7	10.404232	36677	93031	29
32	564716	53.3	968578	8.3	596138	61.6	403862	36704	93020	28
33	565036	53.3	968528	8.3	596508	61.6	403492	36731	93010	27
34	565356	53.2	968479	8.3	596878	61.6	403122	36758	92999	26
35	565676	53.2	968429	8.3	597247	61.5	402753	36785	92988	25
36	565995	53.1	968379	8.3	597616	61.5	402384	36812	92978	24
37	566314	53.1	968329	8.3	597985	61.5	402015	36839	92967	23
38	566632	53.1	968278	8.3	598354	61.4	401646	36867	92956	22
39	566951	53.0	968228	8.4	598722	61.4	401278	36894	92945	21
40	567269	53.0	968178	8.4	599091	61.3	400909	36921	92935	20
41	9.567587	52.9	9.968128	8.4	9.599459	61.3	10.400541	36948	92926	19
42	567904	52.9	968078	8.4	599827	61.3	400173	36975	92913	18
43	568222	52.8	968027	8.4	600194	61.2	399806	37002	92902	17
44	568539	52.8	967977	8.4	600562	61.2	399438	37029	92892	16
45	568856	52.8	967927	8.4	600929	61.1	399071	37056	92881	15
46	569172	52.7	967876	8.4	601296	61.1	398704	37083	92870	14
47	569488	52.7	967826	8.4	601662	61.1	398338	37110	92859	13
48	569804	52.6	967775	8.4	602029	61.0	397971	37137	92849	12
49	570120	52.6	967725	8.4	602395	61.0	397605	37164	92838	11
50	570435	52.5	967674	8.4	602761	61.0	397239	37191	92827	10
51	9.570751	52.5	9.967624	8.4	9.603127	60.9	10.396873	37218	92816	9
52	571066	52.4	967573	8.4	603493	60.9	396507	37245	92805	8
53	571380	52.4	967522	8.5	603858	60.9	396142	37272	92794	7
54	571695	52.3	967471	8.5	604223	60.8	395777	37299	92784	6
55	572009	52.3	967421	8.5	604588	60.8	395412	37326	92773	5
56	572323	52.3	967370	8.5	604953	60.7	395047	37353	92762	4
57	572636	52.2	967319	8.5	605317	60.7	394683	37380	92751	3
58	572950	52.2	967268	8.5	605682	60.7	394318	37407	92740	2
59	573263	52.1	967217	8.5	606046	60.6	393954	37434	92729	1
60	573575		967166		606410		393590	37461	92718	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.573575	52.1	9.967166	8.5	9.606410	60.6	10.393590	37461	92718	60
1	573888	52.0	967115	8.5	606773	60.6	393227	37488	92707	59
2	574200	52.0	967064	8.5	607137	60.5	392863	37515	92697	58
3	574512	51.9	967013	8.5	607500	60.5	392500	37542	92686	57
4	574824	51.9	966961	8.5	607863	60.4	392137	37569	92675	56
5	575136	51.9	966910	8.5	608225	60.4	391775	37595	92664	55
6	575447	51.8	966859	8.5	608588	60.4	391412	37622	92653	54
7	575758	51.8	966808	8.5	608950	60.3	391050	37649	92642	53
8	576069	51.7	966756	8.6	609312	60.3	390688	37676	92631	52
9	576379	51.7	966705	8.6	609674	60.3	390326	37703	92620	51
10	576689	51.6	966653	8.6	610036	60.2	389964	37730	92609	50
11	9.576999	51.6	9.966602	8.6	9.610397	60.2	10.389603	37757	92598	49
12	577309	51.6	966550	8.6	610759	60.2	389241	37784	92587	48
13	577618	51.5	966499	8.6	611120	60.1	388880	37811	92576	47
14	577927	51.5	966447	8.6	611480	60.1	388520	37838	92565	46
15	578236	51.4	966395	8.6	611841	60.1	388159	37865	92554	45
16	578545	51.4	966344	8.6	612201	60.0	387799	37892	92543	44
17	578853	51.3	966292	8.6	612561	60.0	387439	37919	92532	43
18	579162	51.3	966240	8.6	612921	60.0	387079	37946	92521	42
19	579470	51.3	966188	8.6	613281	59.9	386719	37973	92510	41
20	579777	51.2	966136	8.6	613641	59.9	386359	37999	92499	40
21	9.580085	51.2	9.966085	8.7	9.614000	59.8	10.386000	38026	92488	39
22	580392	51.1	966033	8.7	614359	59.8	385641	38053	92477	38
23	580699	51.1	965981	8.7	614718	59.8	385282	38080	92466	37
24	581005	51.1	965928	8.7	615077	59.7	384923	38107	92455	36
25	581312	51.0	965876	8.7	615435	59.7	384565	38134	92444	35
26	581618	51.0	965824	8.7	615793	59.7	384207	38161	92432	34
27	581924	50.9	965772	8.7	616151	59.6	383849	38188	92421	33
28	582229	50.9	965720	8.7	616509	59.6	383491	38215	92410	32
29	582535	50.9	965668	8.7	616867	59.6	383133	38241	92399	31
30	582840	50.8	965615	8.7	617224	59.5	382776	38268	92388	30
31	9.583145	50.8	9.965563	8.7	9.617582	59.5	10.382418	38295	92377	29
32	583449	50.7	965511	8.7	617939	59.5	382061	38322	92366	28
33	583754	50.7	965458	8.7	618295	59.4	381705	38349	92355	27
34	584058	50.6	965406	8.7	618652	59.4	381348	38376	92343	26
35	584361	50.6	965353	8.8	619008	59.4	380992	38403	92332	25
36	584665	50.6	965301	8.8	619364	59.3	380636	38430	92321	24
37	584968	50.5	965248	8.8	619721	59.3	380279	38456	92310	23
38	585272	50.5	965195	8.8	620076	59.3	379924	38483	92299	22
39	585574	50.4	965143	8.8	620432	59.2	379568	38510	92287	21
40	585877	50.4	965090	8.8	620787	59.2	379213	38537	92276	20
41	9.586179	50.3	9.965037	8.8	9.621142	59.2	10.378858	38564	92265	19
42	586482	50.3	964984	8.8	621497	59.1	378503	38591	92254	18
43	586783	50.3	964931	8.8	621852	59.1	378148	38617	92243	17
44	587085	50.2	964879	8.8	622207	59.0	377793	38644	92231	16
45	587386	50.2	964826	8.8	622561	59.0	377439	38671	92220	15
46	587688	50.1	964773	8.8	622915	59.0	377085	38698	92209	14
47	587989	50.1	964719	8.8	623269	58.9	376731	38725	92198	13
48	588289	50.1	964666	8.9	623623	58.9	376377	38752	92186	12
49	588590	50.0	964613	8.9	623976	58.9	376024	38778	92175	11
50	588890	50.0	964560	8.9	624330	58.8	375670	38805	92164	10
51	9.589190	49.9	9.964507	8.9	9.624683	58.8	10.375317	38832	92152	9
52	589489	49.9	964454	8.9	625036	58.8	374964	38859	92141	8
53	589789	49.9	964400	8.9	625388	58.7	374612	38886	92130	7
54	590088	49.8	964347	8.9	625741	58.7	374259	38912	92119	6
55	590387	49.8	964294	8.9	626093	58.7	373907	38939	92107	5
56	590686	49.7	964240	8.9	626445	58.6	373555	38966	92096	4
57	590984	49.7	964187	8.9	626797	58.6	373203	38993	92085	3
58	591282	49.7	964133	8.9	627149	58.6	372851	39020	92073	2
59	591580	49.6	964080	8.9	627501	58.5	372499	39046	92062	1
60	591878		964026		627852		372148	39073	92050	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

67 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10′	Cotang.	N. sine.	N. cos.	
0	9.591878	49.6	9.964026	8.9	9.627852	58.5	10.372148	39073	92050	60
1	592176	49.5	963972	8.9	628203	58.5	371797	39100	92039	59
2	592473	49.5	963919	8.9	628554	58.5	371446	39127	92028	58
3	592770	49.5	963865	9.0	628905	58.4	371095	39153	92016	57
4	593067	49.4	963811	9.0	629255	58.4	370745	39180	92005	56
5	593363	49.4	963757	9.0	629606	58.3	370394	39207	91994	55
6	593659	49.3	963704	9.0	629956	58.3	370044	39234	91982	54
7	593955	49.3	963650	9.0	630306	58.3	369694	39260	91971	53
8	594251	49.3	963596	9.0	630656	58.3	369344	39287	91959	52
9	594547	49.2	963542	9.0	631005	58.2	368995	39314	91948	51
10	594842	49.2	963488	9.0	631355	58.2	368645	39341	91936	50
11	9.595137	49.1	9.963434	9.0	9.631704	58.2	10.368296	39367	91925	49
12	595432	49.1	963379	9.0	632053	58.1	367947	39394	91914	48
13	595727	49.1	963325	9.0	632401	58.1	367599	39421	91902	47
14	596021	49.0	963271	9.0	632750	58.1	367250	39448	91891	46
15	596315	49.0	963217	9.0	633098	58.0	366902	39474	91879	45
16	596609	48.9	963163	9.0	633447	58.0	366553	39501	91868	44
17	596903	48.9	963108	9.1	633795	58.0	366205	39528	91856	43
18	597196	48.9	963054	9.1	634143	57.9	365857	39555	91845	42
19	597490	48.8	962999	9.1	634490	57.9	365510	39581	91833	41
20	597783	48.8	962945	9.1	634838	57.9	365162	39608	91822	40
21	9.598075	48.7	9.962890	9.1	9.635185	57.8	10.364815	39635	91810	39
22	598368	48.7	962836	9.1	635532	57.8	364468	39661	91799	38
23	598660	48.7	962781	9.1	635879	57.8	364121	39688	91787	37
24	598952	48.6	962727	9.1	636226	57.7	363774	39715	91775	36
25	599244	48.6	962672	9.1	636572	57.7	363428	39741	91764	35
26	599536	48.5	962617	9.1	636919	57.7	363081	39768	91752	34
27	599827	48.5	962562	9.1	637265	57.7	362735	39795	91741	33
28	600118	48.5	962508	9.1	637611	57.6	362389	39822	91729	32
29	600409	48.4	962453	9.1	637956	57.6	362044	39848	91718	31
30	600700	48.4	962398	9 2	638302	57.6	361698	39875	91706	30
31	9.600990	48.4	9.962343	9.2	9.638647	57.5	10.361353	39902	91694	29
32	601280	48.3	962288	9.2	638992	57.5	361008	39928	91683	28
33	601570	48.3	962233	9.2	639337	57.5	360663	39955	91671	27
34	601860	48.2	962178	9.2	639682	57.4	360318	39982	91660	26
35	602150	48.2	962123	9.2	640027	57.4	359973	40008	91648	25
36	602439	48.2	962067	9.2	640371	57.4	359629	40035	91636	24
37	602728	48.1	962012	9.2	640716	57.3	359284	40062	91625	23
38	603017	48.1	961957	9.2	641060	57.3	358940	40088	91613	22
39	603305	48.1	961902	9.2	641404	57.3	358596	40115	91602	21
40	603594	48.0	961846	9.2	641747	57.2	358253	40141	91590	20
41	9.603882	48.0	9.961791	9.2	9.642091	57.2	10.357909	40168	91578	19
42	604170	47.9	961735	9.2	642434	57.2	357566	40195	91566	18
43	604457	47.9	961680	9.2	642777	57.2	357223	40221	91555	17
44	604745	47.9	961624	9.3	643120	57.1	356880	40248	91543	16
45	605032	47 8	961569	9.3	643463	57.1	356537	40275	91531	15
46	605319	47.8	961513	9.3	643806	57.1	356194	40301	91519	14
47	605606	47.8	961458	9.3	644148	57.0	355852	40328	91508	13
48	605892	47.7	961402	9.3	644490	57.0	355510	40355	91496	12
49	606179	47.7	961346	9.3	644832	57.0	355168	40381	91484	11
50	606465	47.6	961290	9.3	645174	56.9	354826	40408	91472	10
51	9.606751	47.6	9.961235	9.3	9.645516	56.9	10.354484	40434	91461	9
52	607036	47.6	961179	9.3	645857	56.9	354143	40461	91449	8
53	607322	47.5	961123	9.3	646199	56.9	353801	40488	91437	7
54	607607	47.5	961067	9.3	646540	56.8	353460	40514	91425	6
55	607892	47.4	961011	9.3	646881	56.8	353119	40541	91414	5
56	608177	47.4	960955	9.3	647222	56.8	352778	40567	91402	4
57	608461	47.4	960899	9.3	647562	56.7	352438	40594	91390	3
58	608745	47.3	960843	9.4	647903	56.7	352097	40621	91378	2
59	609029	47.3	960786	9.4	648243	56.7	351757	40647	91366	1
60	609313		960730		648583		351417	40674	91355	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.609313	47.3	9.960730	9.4	9.648583	56.6	10.351417	40674	91355	60
1	609597	47.2	960674	9.4	648923	56.6	351077	40700	91343	59
2	609880	47.2	960618	9.4	649263	56.6	350737	40727	91331	58
3	610164	47.2	960561	9.4	649602	56.6	350398	40753	91319	57
4	610447	47.1	960505	9.4	649942	56.5	350058	40780	91307	56
5	610729	47.1	960448	9.4	650281	56.5	349719	40806	91295	55
6	611012	47.0	960392	9.4	650620	59.5	349380	40833	91283	54
7	611294	47.0	960335	9.4	650959	56.4	349041	40860	91272	53
8	611576	47.0	960279	9.4	651297	56.4	348703	40886	91260	52
9	611858	46.9	960222	9.4	651636	56.4	348364	40913	91248	51
10	612140	46.9	960165	9.4	651974	56.3	348026	40939	91236	50
11	9.612421	46.9	9.960109	9.5	9.652312	56.3	10.347688	40966	91224	49
12	612702	46.8	960052	9.5	652650	56.3	347350	40992	91212	48
13	612983	46.8	959995	9.5	652988	56.3	347012	41019	91200	47
14	613264	46.7	959938	9.5	653326	56.2	346674	41045	91188	46
15	613545	46.7	959882	9.5	653663	56.2	346337	41072	91176	45
16	613825	46.7	959825	9.5	654000	56.2	346000	41098	91164	44
17	614105	46.6	959768	9.5	654337	56.1	345663	41125	91152	43
18	614385	46.6	959711	9.5	654174	56.1	345326	41151	91140	42
19	614665	46.6	959654	9.5	655011	56.1	344989	41178	91128	41
20	614944	46.5	959596	9.5	655348	56.1	344652	41204	91116	40
21	9.615223	46.5	9.959539	9.5	9.655684	56.0	10.344316	41231	91104	39
22	615502	46.5	959482	9.5	656020	56.0	343980	41257	91092	38
23	615781	46.4	959425	9.5	656356	56.0	343644	41284	91080	37
24	616060	46.4	959368	9.5	656692	55.9	343308	41310	91068	36
25	616338	46.4	959310	9.6	657028	55.9	342972	41337	91056	35
26	616616	46.3	959253	9.6	657364	55.9	342636	41363	91044	34
27	616894	46.3	959195	9.6	657699	55.9	342301	41390	91032	33
28	617172	46.2	959138	9.6	658034	55.8	341966	41416	91020	32
29	617450	46.2	959081	9.6	658369	55.8	341631	41443	91008	31
30	617727	46.2	959023	9.6	658704	55.8	341296	41469	90996	30
31	9.618004	46.1	9.958965	9.6	9.659039	55.8	10.340961	41496	90984	29
32	618281	46.1	958908	9.6	659373	55.7	340627	41522	90972	28
33	618558	46.1	958850	9.6	659708	55.7	340292	41549	90960	27
34	618834	46.0	958792	9.6	660042	55.7	339958	41575	90948	26
35	619110	46.0	958734	9.6	660376	55.7	339624	41602	90936	25
36	619386	46.0	958677	9.6	660710	55.6	339290	41628	90924	24
37	619662	45.9	958619	9.6	661043	55.6	338957	41655	90911	23
38	619938	45.9	958561	9.6	661377	55.6	338623	41681	90899	22
39	620213	45.9	958503	9.7	661710	55.5	338290	41707	90887	21
40	620488	45.8	958445	9.7	662043	55.5	337957	41734	90875	20
41	9.620763	45.8	9.958387	9.7	9.662376	55.5	10.337624	41760	90863	19
42	621038	45.7	958329	9.7	662709	55.4	337291	41787	90851	18
43	621313	45.7	958271	9.7	663042	55.4	336958	41813	90839	17
44	621587	45.7	958213	9.7	663375	55.4	336625	41840	90826	16
45	621861	45.6	958154	9.7	663707	55.4	336293	41866	90814	15
46	622135	45.6	958096	9.7	664039	55.3	335961	41892	90802	14
47	622409	45.6	958038	9.7	664371	55.3	335629	41919	90790	13
48	622682	45.5	957979	9.7	664703	55.3	335297	41945	90778	12
49	622956	45.5	957921	9.7	665035	55.3	334965	41972	90766	11
50	623229	45.5	957863	9.7	665366	55.2	334634	41998	90753	10
51	9.623512	45.4	9.957804	9.7	9.665697	55.2	10.334303	42024	90741	9
52	623774	45.4	957746	9.8	666029	55.2	333971	42051	90729	8
53	624047	45.4	957687	9.8	666360	55.1	333620	42077	90717	7
54	624319	45.3	957628	9.8	666691	55.1	333309	42104	90704	6
55	624591	45.3	957570	9.8	667021	55.1	332979	42130	90692	5
56	624863	46.3	957511	9.8	667352	55.1	332648	42156	90680	4
57	625135	45.2	957452	9.8	667682	55.0	332318	42183	90668	3
58	625406	45.2	957393	9.8	668013	55.0	331987	42209	90655	2
59	625677	45.2	957335	9.8	668343	55.0	331657	42235	90643	1
60	625948		957276		668672		331328	42262	90631	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

65 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.625948	45.1	9.957276	9.8	9.668673	55.0	10.331327	42262	90631	60
1	626219	45.1	957217	9.8	669002	54.9	330998	42288	90613	59
2	626490	45.1	957158	9.8	669332	54.9	330668	42315	90606	58
3	626760	45.0	957099	9.8	669661	54.9	330339	42341	90594	57
4	627030	45.0	957040	9.8	669991	54.8	330009	42367	90582	56
5	627300	45.0	956981	9.8	670320	54.8	329680	42394	90569	55
6	627570	44.9	956921	9.9	670649	54.8	329351	42420	90557	54
7	627840	44.9	956862	9.9	670977	54.8	329023	42446	90545	53
8	628109	44.9	956803	9.9	671306	54.7	328694	42473	90532	52
9	628378	44.8	956744	9.9	671634	54.7	328366	42499	90520	51
10	628647	44.8	956684	9.9	671963	54.7	328037	42525	90507	50
11	9.628916	44.7	9.956625	9.9	9.672291	54.7	10.327709	42552	90495	49
12	629185	44.7	956566	9.9	672619	54.6	327381	42578	90483	48
13	629453	44.7	956506	9.9	672947	54.6	327053	42604	90470	47
14	629721	44.6	956447	9.9	673274	54.6	326726	42631	90458	46
15	629989	44.6	956387	9.9	673602	54.6	326398	42657	90446	45
16	630257	44.6	956327	9.9	673929	54.5	326071	42683	90433	44
17	630524	44.6	956268	9.9	674257	54.5	325743	42709	90421	43
18	630792	44.5	956208	10.0	674584	54.5	325416	42736	90408	42
19	631059	44.5	956148	10.0	674910	54.4	325090	42762	90396	41
20	631326	44.5	956089	10.0	675237	54.4	324763	42788	90383	40
21	9.631593	44.4	9.956029	10.0	9.675564	54.4	10.324436	42815	90371	39
22	631859	44.4	955969	10.0	675890	54.4	324110	42841	90358	38
23	632125	44.4	955909	10.0	676216	54.3	323784	42867	90346	37
24	632392	44.3	955849	10.0	676543	54.3	323457	42894	90334	36
25	632658	44.3	955789	10.0	676869	54.3	323131	42920	90321	35
26	632923	44.3	955729	10.0	677194	54.3	322806	42946	90309	34
27	633189	44.2	955669	10.0	677520	54.2	322480	42972	90296	33
28	633454	44.2	955609	10.0	677846	54.2	322154	42999	90284	32
29	633719	44.2	955548	10.0	678171	54.2	321829	43025	90271	31
30	633984	44.1	955488	10.0	678496	54.2	321504	43051	90259	30
31	9.634249	44.1	9.955428	10.1	9.678821	54.1	10.321179	43077	90246	29
32	634514	44.0	955368	10.1	679146	54.1	320854	43104	90233	28
33	634778	44.0	955307	10.1	679471	54.1	320529	43130	90221	27
34	635042	44.0	955247	10.1	679795	54.1	320205	43156	90208	26
35	635306	43.9	955186	10.1	680120	54.0	319880	43182	90196	25
36	635570	43.9	955126	10.1	680444	54.0	319556	43209	90183	24
37	635834	43.9	955065	10.1	680768	54.0	319232	43235	90171	23
38	636097	43.8	955005	10.1	681092	54.0	318908	43261	90158	22
39	636360	43.8	954944	10.1	681416	53.9	318584	43287	90146	21
40	636623	43.8	954883	10.1	681740	53.9	318260	43313	90133	20
41	9.636886	43.7	9.954823	10.1	9.682063	53.9	10.317937	43340	90120	19
42	637148	43.7	954762	10.1	682387	53.9	317613	43366	90108	18
43	637411	43.7	954701	10.1	682710	53.8	317290	43392	90095	17
44	637673	43.7	954640	10.1	683033	53.8	316967	43418	90082	16
45	637935	43.6	954579	10.1	683356	53.8	316644	43445	90070	15
46	638197	43.6	954518	10.2	683679	53.8	316321	43471	90057	14
47	638458	43.6	954457	10.2	684001	53.7	315999	43497	90045	13
48	638720	43.5	954396	10.2	684324	53.7	315676	43523	90032	12
49	638981	43.5	954335	10.2	684646	53.7	315354	43549	90019	11
50	639242	43.5	954274	10.2	684968	53.7	315032	43575	90007	10
51	9.639503	43.4	9.954213	10.2	9.685290	53.6	10.314710	43602	89994	9
52	639764	43.4	954152	10.2	685612	53.6	314388	43628	89981	8
53	640024	43.4	954090	10.2	685934	53.6	314066	43654	89968	7
54	640284	43.3	954029	10.2	686255	53.6	313745	43680	89956	6
55	640544	43.3	953968	10.2	686577	53.5	313423	43706	89943	5
56	640804	43.3	953906	10.2	686898	53.5	313102	43733	89930	4
57	641064	43.2	953845	10.2	687219	53.5	312781	43759	89918	3
58	641324	43.2	953783	10.2	687540	53.5	312460	43785	89905	2
59	641584	43.2	953722	10.3	687861	53.4	312139	43811	89892	1
60	641842		953660		688182		311818	43837	89879	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

64 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.641842	43.1	9.953660	10.3	9.688182	53.4	10.311818	43837	89879	60
1	642101	43.1	953599	10.3	688502	53.4	311498	43863	89867	59
2	642360	43.1	953537	10.3	688823	53.4	311177	43889	89854	58
3	642618	43.0	953475	10.3	689143	53.3	310857	43916	89841	57
4	642877	43.0	953413	10.3	689463	53.3	310537	43942	89828	56
5	643135	43.0	953352	10.3	689783	53.3	310217	43968	89816	55
6	643393	43.0	953290	10.3	690103	53.3	309897	43994	89803	54
7	643650	42.9	953228	10.3	690423	53.3	309577	44020	89790	53
8	643908	42.9	953166	10.3	690742	53.2	309258	44046	89777	52
9	644165	42.9	953104	10.3	691062	53.2	308938	44072	89764	51
10	644423	42.8	953042	10.3	691381	53.2	308619	44098	89752	50
11	9.644680	42.8	9.952980	10.4	9.691700	53.1	10.308300	44124	89739	49
12	644936	42.8	952918	10.4	692019	53.1	307981	44151	89726	48
13	645193	42.7	952855	10.4	692338	53.1	307662	44177	89713	47
14	645450	42.7	952793	10.4	692656	53.1	307344	44203	89700	46
15	645706	42.7	952731	10.4	692975	53.1	307025	44229	89687	45
16	645962	42.6	952669	10.4	693293	53.0	306707	44255	89674	44
17	646218	42.6	952606	10.4	693612	53.0	306388	44281	89662	43
18	646474	42.6	952544	10.4	693930	53.0	306070	44307	89649	42
19	646729	42.5	952481	10.4	694248	53.0	305752	44333	89636	41
20	646984	42.5	952419	10.4	694566	52.9	305434	44359	89623	40
21	9.647240	42.5	9.952356	10.4	9.694883	52.9	10.305117	44385	89610	39
22	647494	42.4	952294	10.4	695201	52.9	304799	44411	89597	38
23	647749	42.4	952231	10.4	695518	52.9	304482	44437	89584	37
24	648004	42.4	952168	10.5	695836	52.9	304164	44464	89571	36
25	648258	42.4	952106	10.5	696153	52.8	303847	44490	89558	35
26	648512	42.3	952043	10.5	696470	52.8	303530	44516	89545	34
27	648766	42.3	951980	10.5	696787	52.8	303213	44542	89532	33
28	649020	42.3	951917	10.5	697103	52.8	302897	44568	89519	32
29	649274	42.2	951854	10.5	697420	52.7	302580	44594	89506	31
30	649527	42.2	951791	10.5	697736	52.7	302264	44620	89493	30
31	9.649781	42.2	9.951728	10.5	9.698053	52.7	10.301947	44646	89480	29
32	650034	42.2	951665	10.5	698369	52.7	301631	44672	89467	28
33	650287	42.1	951602	10.5	698685	52.6	301315	44698	89454	27
34	650539	42.1	951539	10.5	699001	52.6	300999	44724	89441	26
35	650792	42.1	951476	10.5	699316	52.6	300684	44750	89428	25
36	651044	42.0	951412	10.5	699632	52.6	300368	44776	89415	24
37	651297	42.0	951349	10.6	699947	52.6	300053	44802	89402	23
38	651549	42.0	951286	10.6	700263	52.5	299737	44828	89389	22
39	651800	41.9	951222	10.6	700578	52.5	299422	44854	89376	21
40	652052	41.9	951159	10.6	700893	52.5	299107	44880	89363	20
41	9.652304	41.9	9.951096	10.6	9.701208	52.4	10.298792	44906	89350	19
42	652555	41.8	951032	10.6	701523	52.4	298477	44932	89337	18
43	652806	41.8	950968	10.6	701837	52.4	298163	44958	89324	17
44	653057	41.8	950905	10.6	702152	52.4	297848	44984	89311	16
45	653308	41.8	950841	10.6	702466	52.4	297534	45010	89298	15
46	653558	41.7	950778	10.6	702780	52.3	297220	45036	89285	14
47	653808	41.7	950714	10.6	703095	52.3	296905	45062	89272	13
48	654059	41.7	950650	10.6	703409	52.3	296591	45088	89259	12
49	654309	41.6	950586	10.6	703723	52.3	296277	45114	89245	11
50	654558	41.6	950522	10.7	704036	52.2	295964	45140	89232	10
51	9.654808	41.6	9.950458	10.7	9.704350	52.2	10.295650	45166	89219	9
52	655058	41.6	950394	10.7	704663	52.2	295337	45192	89206	8
53	655307	41.5	950330	10.7	704977	52.2	295023	45218	89193	7
54	655556	41.5	950366	10.7	705290	52.2	294710	45243	89180	6
55	655805	41.5	950202	10.7	705603	52.1	294397	45269	89167	5
56	656054	41.4	950138	10.7	705916	52.1	294084	45295	89153	4
57	656302	41.4	950074	10.7	706228	52.1	293772	45321	89140	3
58	656551	41.4	950010	10.7	706541	52.1	293459	45347	89127	2
59	656799	41.3	949945	10.7	706854	52.1	293146	45373	89114	1
60	657047		949881		707166		292834	45399	89101	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.657047	41.3	9.949881	10.7	9.707166	52.0	10.292834	45399	89101	60
1	657295	41.3	949816	10.7	707478	52.0	292522	45425	89087	59
2	657542	41.2	949752	10.7	707790	52.0	292210	45451	89074	58
3	657790	41.2	949688	10.8	708102	52.0	291898	45477	89061	57
4	658037	41.2	949623	10.8	708414	51.9	291586	45503	89048	56
5	658284	41.2	949558	10.8	708726	51.9	291274	45529	89035	55
6	658531	41.1	949494	10.8	709037	51.9	290963	45554	89021	54
7	658778	41.1	949429	10.8	709349	51.9	290651	45580	89008	53
8	659025	41.1	949364	10.8	709660	51.9	290340	45606	88995	52
9	659271	41.0	949300	10.8	709971	51.8	290029	45632	88981	51
10	659517	41.0	949235	10.8	710282	51.8	289718	45658	88968	50
11	9.659763	41.0	9.949170	10.8	9.710593	51.8	10.289407	45684	88955	49
12	660009	40.9	949105	10.8	710904	51.8	289096	45710	88942	48
13	660255	40.9	949040	10.8	711215	51.8	288785	45736	88928	47
14	660501	40.9	948975	10.8	711525	51.7	288475	45762	88915	46
15	660746	40.9	948910	10.8	711836	51.7	288164	45787	88902	45
16	660991	40.8	948845	10.8	712146	51.7	287854	45813	88888	44
17	661236	40.8	948780	10.9	712456	51.7	287544	45839	88875	43
18	661481	40.8	948715	10.9	712766	51.6	287234	45865	88862	42
19	661726	40.7	948650	10.9	713076	51.6	286924	45891	88848	41
20	661970	40.7	948584	10.9	713386	51.6	286614	45917	88835	40
21	9.662214	40.7	9.948519	10.9	9.713696	51.6	10.286304	45942	88822	39
22	662459	40.7	948454	10.9	714005	51.6	285995	45968	88808	38
23	662703	40.6	948388	10.9	714314	51.5	285686	45994	88795	37
24	662946	40.6	948323	10.9	714624	51.5	285376	46020	88782	36
25	663190	40.6	948257	10.9	714933	51.5	285067	46046	88768	35
26	663433	40.5	948192	10.9	715242	51.5	284758	46072	88755	34
27	663677	40.5	948126	10.9	715551	51.4	284449	46097	88741	33
28	663920	40.5	948060	10.9	715860	51.4	284140	46123	88728	32
29	664163	40.5	947995	11.0	716168	51.4	283832	46149	88715	31
30	664406	40.4	947929	11.0	716477	51.4	283523	46175	88701	30
31	9.664648	40.4	9.947863	11.0	9.716785	51.4	10.283215	46201	88688	29
32	664891	40.4	947797	11.0	717093	51.3	282907	46226	88674	28
33	665133	40.3	947731	11.0	717401	51.3	282599	46252	88661	27
34	665375	40.3	947665	11.0	717709	51.3	282291	46278	88647	26
35	665617	40.3	947600	11.0	718017	51.3	281983	46304	88634	25
36	665859	40.2	947533	11.0	718325	51.3	281675	46330	88620	24
37	666100	40.2	947467	11.0	718633	51.2	281367	46355	88607	23
38	666342	40.2	947401	11.0	718940	51.2	281060	46381	88593	22
39	666583	40.2	947335	11.0	719248	51.2	280752	46407	88580	21
40	666824	40.1	947269	11.0	719555	51.2	280445	46433	88566	20
41	9.667065	40.1	9.947203	11.0	9.719862	51.2	10.280138	46458	88553	19
42	667305	40.1	947136	11.1	720169	51.1	279831	46484	88539	18
43	667546	40.1	947070	11.1	720476	51.1	279524	46510	88526	17
44	667786	40.0	947004	11.1	720783	51.1	279217	46536	88512	16
45	668027	40 0	946937	11.1	721089	51.1	278911	46561	88499	15
46	668267	40.0	946871	11.1	721396	51.1	278604	46587	88485	14
47	668506	39.9	946804	11.1	721702	51.0	278298	46613	88472	13
48	668746	39.9	946738	11.1	722009	51.0	277991	46639	88458	12
49	668986	39.9	946671	11.1	722315	51.0	277685	46664	88445	11
50	669225	39.9	946604	11.1	722621	51.0	277379	46690	88431	10
51	9.669464	39.8	9.946538	11.1	9.722927	51.0	10.277073	46716	88417	9
52	669703	39.8	946471	11.1	723232	50.9	276768	46742	88404	8
53	669942	39.8	946404	11.1	723538	50.9	276462	46767	88390	7
54	670181	39.7	946337	11.1	723844	50.9	276156	46793	88377	6
55	670419	39.7	946270	11.2	724149	50.9	275851	46819	88363	5
56	670658	39.7	946203	11.2	724454	50.9	275546	46844	88349	4
57	670896	39.7	946136	11.2	724759	50.8	275241	46870	88336	3
58	671134	39.6	946069	11.2	725065	50.8	274935	46896	88322	2
59	671372	39.6	946002	11.2	725369	50.8	274631	46921	88308	1
60	671609		945935		725674		274326	46947	88295	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.671609	39.6	9.945935	11.2	9.725674	50.8	10.274326	46947	88295	60
1	671847	39.5	945868	11.2	725979	50.8	274021	46973	88281	59
2	672084	39.5	945800	11.2	726284	50 7	273716	46999	88267	58
3	672321	39.5	945733	11.2	726588	50.7	273412	47024	88254	57
4	672558	39.5	945666	11.2	726892	50.7	273108	47050	88240	56
5	672795	39.4	945598	11.2	727197	50.7	272803	47076	88226	55
6	673032	39.4	945531	11.2	727501	50.7	272499	47101	88213	54
7	673268	39.4	945464	11.3	727805	50.6	272195	47127	88199	53
8	673505	39.4	945396	11.3	728109	50.6	271891	47153	88185	52
9	673741	39.3	945328	11.3	728412	50.6	271588	47178	88172	51
10	673977	39.3	945261	11.3	728716	50.6	271284	47204	88158	50
11	9.674213	39.3	9.945193	11.3	9.729020	50.6	10.270980	47229	88144	49
12	674448	39.2	945125	11.3	729323	50.5	270677	47255	88130	48
13	674684	39.2	945058	11.3	729626	50.5	270374	47281	88117	47
14	674919	39.2	944990	11.3	729929	50.5	270071	47306	88103	46
15	675155	39.2	944922	11.3	730233	50.5	269767	47332	88089	45
16	675390	39.1	944854	11.3	730535	50.5	269465	47358	88075	44
17	675624	39.1	944786	11.3	730838	50.4	269162	47383	88062	43
18	675859	39.1	944718	11.3	731141	50.4	268859	47409	88048	42
19	676094	39.1	944650	11.3	731444	50.4	268556	47434	88034	41
20	676328	39.0	944582	11.4	731746	50.4	268254	47460	88020	40
21	9.676562	39.0	9.944514	11.4	9.732048	50.4	10.267952	47486	88006	39
22	676796	39.0	944446	11.4	732351	50.3	267649	47511	87993	38
23	677030	39.0	944377	11.4	732653	50.3	267347	47537	87979	37
24	677264	38.9	944309	11.4	732955	50.3	267045	47562	87965	36
25	677498	38.9	944241	11.4	733257	50.3	266743	47588	87951	35
26	677731	38.9	944172	11.4	733558	50.3	266442	47614	87937	34
27	677964	38.8	944104	11.4	733860	50.2	266140	47639	87923	33
28	678197	38.8	944036	11.4	734162	50.2	265838	47665	87909	32
29	678430	38.8	943967	11.4	734463	50.2	265537	47690	87896	31
30	678663	38.8	943899	11.4	734764	50.2	265236	47716	87882	30
31	9.678895	38.7	9.943830	11.4	9.735066	50.2	10.264934	47741	87868	29
32	679128	38.7	943761	11.4	735367	50.2	264633	47767	87854	28
33	679360	38.7	943693	11.5	735668	50.1	264332	47793	87840	27
34	679592	38.7	943624	11.5	735969	50.1	264031	47818	87826	26
35	679824	38.6	943555	11.5	736269	50.1	263731	47844	87812	25
36	680056	38.6	943486	11.5	736570	50.1	263430	47869	87798	24
37	680288	38.6	943417	11.5	736871	50.1	263129	47895	87784	23
38	680519	38.5	943348	11.5	737171	50.0	262829	47920	87770	22
39	680750	38.5	943279	11.5	737471	50.0	262529	47946	87756	21
40	680982	38.5	943210	11.5	737771	50.0	262229	47971	87743	20
41	9.681213	38.5	9.943141	11.5	9.738071	50.0	10.261929	47997	87729	19
42	681443	38.4	943072	11.5	738371	50.0	261629	48022	87715	18
43	681674	38.4	943003	11.5	738671	49.9	261329	48048	87701	17
44	681905	38.4	942934	11.5	738971	49.9	261029	48073	87687	16
45	682135	38.4	942864	11.5	739271	49.9	260729	48099	87673	15
46	682365	38.3	942795	11.6	739570	49.9	260430	48124	87659	14
47	682595	38.3	942726	11.6	739870	49.9	260130	48150	87645	13
48	682825	38.3	942656	11.6	740169	49.9	259831	48175	87631	12
49	683055	38.3	942587	11.6	740468	49.8	259532	48201	87617	11
50	683284	38.2	942517	11.6	740767	49.8	259233	48226	87603	10
51	9.683514	38.2	9.942448	11.6	9.741066	49.8	10.258934	48252	87589	9
52	683743	38.2	942378	11.6	741365	49.8	258635	48277	87575	8
53	683972	38.2	942308	11.6	741664	49.8	258336	48303	87561	7
54	684201	38.1	942239	11.6	741962	49.7	258038	48328	87546	6
55	684430	38.1	942169	11.6	742261	49.7	257739	48354	87532	5
56	684658	38.1	942099	11.6	742559	49.7	257441	48379	87518	4
57	684887	38.0	942029	11.6	742858	49.7	257142	48405	87504	3
58	685115	38.0	941959	11.6	743156	49.7	256844	48430	87490	2
59	685343	38.0	941889	11.7	743454	49.7	256546	48456	87476	1
60	685571		941819		743752		256248	48481	87462	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.685571	38.0	9.941819	11.7	9.743752	49.6	10.256248	48481	87462	60
1	685799	37.9	941749	11.7	744050	49.6	255950	48506	87448	59
2	686027	37.9	941679	11.7	744348	49.6	255652	48532	87434	58
3	686254	37.9	941609	11.7	744645	49.6	255355	48557	87420	57
4	686482	37.9	941539	11.7	744943	49.6	255057	48583	87406	56
5	686709	37.8	941469	11.7	745240	49.6	254760	48608	87391	55
6	686936	37.8	941398	11.7	745538	49.5	254462	48634	87377	54
7	687163	37.8	941328	11.7	745835	49.5	254165	48659	87363	53
8	687389	37.8	941258	11.7	746132	49.5	253868	48684	87349	52
9	687616	37.7	941187	11.7	746429	49.5	253571	48710	87335	51
10	687843	37.7	941117	11.7	746726	49.5	253274	48735	87321	50
11	9.688069	37.7	9.941046	11.8	9.747023	49.4	10.252977	48761	87306	49
12	688295	37.7	940975	11.8	747319	49.4	252681	48786	87292	48
13	688521	37.6	940905	11.8	747616	49.4	252384	48811	87278	47
14	688747	37.6	940834	11.8	747913	49.4	252087	48837	87264	46
15	688972	37.6	940763	11.8	748209	49.4	251791	48862	87250	45
16	689198	37.6	940693	11.8	748505	49.3	251495	48888	87235	44
17	689423	37.5	940622	11.8	748801	49.3	251199	48913	87221	43
18	689648	37.5	940551	11.8	749097	49.3	250903	48938	87207	42
19	689873	37.5	940480	11.8	749393	49.3	250607	48964	87193	41
20	690098	37.5	940409	11.8	749689	49.3	250311	48989	87178	40
21	9.690323	37.4	9.940338	11.8	9.749985	49.3	10.250015	49014	87164	39
22	690548	37.4	940267	11.8	750281	49.2	249719	49040	87150	38
23	690772	37.4	940196	11.8	750576	49.2	249424	49065	87136	37
24	690996	37.4	940125	11.9	750872	49.2	249128	49090	87121	36
25	691220	37.3	940054	11.9	751167	49.2	248833	49116	87107	35
26	691444	37.3	939982	11.9	751462	49.2	248538	49141	87093	34
27	691668	37.3	939911	11.9	751757	49.2	248243	49166	87079	33
28	691892	37.3	939840	11.9	752052	49.1	247948	49192	87064	32
29	692115	37.2	939768	11.9	752347	49.1	247653	49217	87050	31
30	692339	37.2	939697	11.9	752642	49.1	247358	49242	87036	30
31	9.692562	37.2	9.939625	11.9	9.752937	49.1	10.247063	49268	87021	29
32	692785	37.1	939554	11.9	753231	49.1	246769	49293	87007	28
33	693008	37.1	939482	11.9	753526	49.1	246474	49318	86993	27
34	693231	37.1	939410	11.9	753820	49.0	246180	49344	86978	26
35	693453	37.1	939339	11.9	754115	49.0	245885	49369	86964	25
36	693676	37.0	939267	12.0	754409	49.0	245591	49394	86949	24
37	693898	37.0	939195	12.0	754703	49.0	245297	49419	86935	23
38	694120	37.0	939123	12.0	754997	49.0	245003	49445	86921	22
39	694342	37.0	939052	12.0	755291	49.0	244709	49470	86906	21
40	694564	36.9	938980	12.0	755585	48.9	244415	49495	86892	20
41	9.694786	36.9	9.938908	12.0	9.755878	48.9	10.244122	49521	86878	19
42	695007	36.9	938836	12.0	756172	48.9	243828	49546	86863	18
43	695229	36.9	938763	12.0	756465	48.9	243535	49571	86849	17
44	695450	36.8	938691	12.0	756759	48.9	243241	49596	86834	16
45	695671	36.8	938619	12.0	757052	48.9	242948	49622	86820	15
46	695892	36.8	938547	12.0	757345	48.8	242655	49647	86805	14
47	696113	36.8	938475	12.0	757638	48.8	242362	49672	86791	13
48	696334	36.7	938402	12.1	757931	48.8	242069	49697	86777	12
49	696554	36.7	938330	12.1	758224	48.8	241776	49723	86762	11
50	696775	36.7	938258	12.1	758517	48.8	241483	49748	86748	10
51	9.696995	36.7	9.938185	12.1	9.758810	48.8	10.241190	49773	86733	9
52	697215	36.6	938113	12.1	759102	48.7	240898	49798	86719	8
53	697435	36.6	938040	12.1	759395	48.7	240605	49824	86704	7
54	697654	36.6	937967	12.1	759687	48.7	240313	49849	86690	6
55	697874	36.6	937895	12.1	759979	48.7	240021	49874	86675	5
56	698094	36.5	937822	12.1	760272	48.7	239728	49899	86661	4
57	698313	36.5	937749	12.1	760564	48.7	239436	49924	86646	3
58	698532	36.5	937676	12.1	760856	48.6	239144	49950	86632	2
59	698751	36.5	937604	12.1	761148	48.6	238852	49975	86617	1
60	698970		937531		761439		238561	50000	86603	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.698970	36.4	9.937531	12.1	9.761439	48.6	10.238561	50000	86603	60
1	699189	36.4	937458	12.2	761731	48.6	238269	50025	86588	59
2	699407	36.4	937385	12.2	762023	48.6	237977	50050	86573	58
3	699626	36.4	937312	12.2	762314	48.6	237686	50076	86559	57
4	699844	36.3	937238	12.2	762606	48.5	237394	50101	86544	56
5	700062	36.3	937165	12.2	762897	48.5	237103	50126	86530	55
6	700280	36.3	937092	12.2	763188	48.5	236812	50151	86515	54
7	700498	36.3	937019	12.2	763479	48.5	236521	50176	86501	53
8	700716	36.3	936946	12.2	763770	48.5	236230	50201	86486	52
9	700933	36.2	936872	12.2	764061	48.5	235939	50227	86471	51
10	701151	36.2	936799	12.2	764352	48.4	235648	50252	86457	50
11	9.701368	36.2	9.936725	12.2	9.764643	48.4	10.235357	50277	86442	49
12	701585	36.2	936652	12.3	764933	48.4	235067	50302	86427	48
13	701802	36.1	936578	12.3	765224	48.4	234776	50327	86413	47
14	702019	36.1	936505	12.3	765514	48.4	234486	50352	86398	46
15	702236	36.1	936431	12.3	765805	48.4	234195	50377	86384	45
16	702452	36.1	936357	12.3	766095	48.4	233905	50403	86369	44
17	702669	36.0	936284	12.3	766385	48.3	233615	50428	86354	43
18	702885	36.0	936210	12.3	766675	48.3	233325	50453	86340	42
19	703101	36.0	936136	12.3	766965	48.3	233035	50478	86325	41
20	703317	36.0	936062	12.3	767255	48.3	232745	50503	86310	40
21	9.703533	35.9	9.935988	12.3	9.767545	48.3	10.232455	50528	86295	39
22	703749	35.9	935914	12.3	767834	48.3	232166	50553	86281	38
23	703964	35.9	935840	12.3	768124	48.2	231876	50578	86266	37
24	704179	35.9	935766	12.4	768413	48.2	231587	50603	86251	36
25	704395	35.9	935692	12.4	768703	48.2	231297	50628	86237	35
26	704610	35.8	935618	12.4	768992	48.2	231008	50654	86222	34
27	704825	35.8	935543	12.4	769281	48.2	230719	50679	86207	33
28	705040	35.8	935469	12.4	769570	48.2	230430	50704	86192	32
29	705254	35.8	935395	12.4	769860	48.1	230140	50729	86178	31
30	705469	35.7	935320	12.4	770148	48.1	229852	50754	86163	30
31	9.705683	35.7	9.935246	12.4	9.770437	48.1	10.229563	50779	86148	29
32	705898	35.7	935171	12.4	770726	48.1	229274	50804	86133	28
33	706112	35.7	935097	12.4	771015	48.1	228985	50829	86119	27
34	706326	35.6	935022	12.4	771303	48.1	228697	50854	86104	26
35	706539	35.6	934948	12.4	771592	48.1	228408	50879	86089	25
36	706753	35.6	934873	12.4	771880	48.0	228120	50904	86074	24
37	706967	35.6	934798	12.5	772168	48.0	227832	50929	86059	23
38	707180	35.5	934723	12.5	772457	48.0	227543	50954	86045	22
39	707393	35.5	934649	12.5	772745	48.0	227255	50979	86030	21
40	707606	35.5	934574	12.5	773033	48.0	226967	51004	86015	20
41	9.707819	35.5	9.934499	12.5	9.773321	48.0	10.226679	51029	86000	19
42	708032	35.4	934424	12.5	773608	47.9	226392	51054	85985	18
43	708245	35.4	934349	12.5	773896	47.9	226104	51079	85970	17
44	708458	35.4	934274	12.5	774184	47.9	225816	51104	85956	16
45	708670	35.4	934199	12.5	774471	47.9	225529	51129	85941	15
46	708882	35.3	934123	12.5	774759	47.9	225241	51154	85926	14
47	709094	35.3	934048	12.5	775046	47.9	224954	51179	85911	13
48	709306	35.3	933973	12.5	775333	47.9	224667	51204	85896	12
49	709518	35.3	933898	12.6	775621	47.8	224379	51229	85881	11
50	709730	35.3	933822	12.6	775908	47.8	224092	51254	85866	10
51	9.709941	35.2	9.933747	12.6	9.776195	47.8	10.223805	51279	85851	9
52	710153	35.2	933671	12.6	776482	47.8	223518	51304	85836	8
53	710364	35.2	933596	12.6	776769	47.8	223231	51329	85821	7
54	710575	35.2	933520	12.6	777055	47.8	222945	51354	85806	6
55	710786	35.1	933445	12.6	777342	47.8	222658	51379	85792	5
56	710967	35.1	933369	12.6	777628	47.7	222372	51404	85777	4
57	711208	35.1	933293	12.6	777915	47.7	222085	51429	85762	3
58	711419	35.1	933217	12.6	778201	47.7	221799	51454	85747	2
59	711629	35.0	933141	12.6	778487	47.7	221512	51479	85732	1
60	711839		933066		778774		221226	51504	85717	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

59 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.711839	35.0	9.933066	12.6	9.778774	47.7	10.221226	51504	85717	60
1	712050	35.0	932990	12.7	779060	47.7	220940	51529	85702	59
2	712260	35.0	932914	12.7	779346	47.6	220654	51554	85687	58
3	712469	34.9	932838	12.7	779632	47.6	220368	51579	85672	57
4	712679	34.9	932762	12.7	779918	47.6	220082	51604	85657	56
5	712889	34.9	932685	12.7	780203	47.6	219797	51628	85642	55
6	713098	34.9	932609	12.7	780489	47.6	219511	51653	85627	54
7	713308	34.9	932533	12.7	780775	47.6	219225	51678	85612	53
8	713517	34.8	932457	12.7	781060	47.6	218940	51703	85597	52
9	713726	34.8	932380	12.7	781346	47.5	218654	51728	85582	51
10	713935	34.8	932304	12.7	781631	47.5	218369	51753	85567	50
11	9.714144	34.8	9.932228	12.7	9.781916	47.5	10.218084	51778	85551	49
12	714352	34.7	932151	12.7	782201	47.5	217799	51803	85536	48
13	714561	34.7	932075	12.8	782486	47.5	217514	51828	85521	47
14	714769	34.7	931998	12.8	782771	47.5	217229	51852	85506	46
15	714978	34.7	931921	12.8	783056	47.5	216944	51877	85491	45
16	715186	34.7	931845	12.8	783341	47.5	216659	51902	85476	44
17	715394	34.6	931768	12.8	783626	47.4	216374	51927	85461	43
18	715602	34.6	931691	12.8	783910	47.4	216090	51952	85446	42
19	715809	34.6	931614	12.8	784195	47.4	215805	51977	85431	41
20	716017	34.6	931537	12.8	784479	47.4	215521	52002	85416	40
21	9.716224	34.5	9.931460	12.8	9.784764	47.4	10.215236	52026	85401	39
22	716432	34.5	931383	12.8	785048	47.4	214952	52051	85385	38
23	716639	34.5	931306	12.8	785332	47.3	214668	52076	85370	37
24	716846	34.5	931229	12.9	785616	47.3	214384	52101	85355	36
25	717053	34.5	931152	12.9	785900	47.3	214100	52126	85340	35
26	717259	34.4	931075	12.9	786184	47.3	213816	52151	85325	34
27	717466	34.4	930998	12.9	786468	47.3	213532	52175	85310	33
28	717673	34.4	930921	12.9	786752	47.3	213248	52200	85294	32
29	717879	34.4	930843	12.9	787036	47.3	212964	52225	85279	31
30	718085	34.3	930766	12.9	787319	47.2	212681	52250	85264	30
31	9.718291	34.3	9.930688	12.9	9.787603	47.2	10.212397	52275	85249	29
32	718497	34.3	930611	12.9	787886	47.2	212114	52299	85234	28
33	718703	34.3	930533	12.9	788170	47.2	211830	52324	85218	27
34	718909	34.3	930456	12.9	788453	47.2	211547	52349	85203	26
35	719114	34.2	930378	12.9	788736	47.2	211264	52374	85188	25
36	719320	34.2	930300	13.0	789019	47.2	210981	52399	85173	24
37	719525	34.2	930223	13.0	789302	47.1	210698	52423	85157	23
38	719730	34.2	930145	13.0	789585	47.1	210415	52448	85142	22
39	719935	34.1	930067	13.0	789868	47.1	210132	52473	85127	21
40	720140	34.1	929989	13.0	790151	47.1	209849	52498	85112	20
41	9.720345	34.1	9.929911	13.0	9.790433	47.1	10.209567	52522	85096	19
42	720549	34.1	929833	13.0	790716	47.1	209284	52547	85081	18
43	720754	34.0	929755	13.0	790999	47.1	209001	52572	85066	17
44	720958	34.0	929677	13.0	791281	47.1	208719	52597	85051	16
45	721162	34 0	929599	13.0	791563	47.0	208437	52621	85035	15
46	721366	34.0	929521	13.0	791846	47.0	208154	52646	85020	14
47	721570	34.0	929442	13.0	792128	47.0	207872	52671	85005	13
48	721774	33.9	929364	13.1	792410	47.0	207590	52696	84989	12
49	721978	33.9	929286	13.1	792692	47.0	207308	52720	84974	11
50	722181	33.9	929207	13.1	792974	47.0	207026	52745	84959	10
51	9.722385	33.9	9.929129	13.1	9.793256	47.0	10.206744	52770	84943	9
52	722588	33.9	929050	13.1	793538	46.9	206462	52794	84928	8
53	722791	33.8	928972	13.1	793819	46.9	206181	52819	84913	7
54	722994	33.8	928893	13.1	794101	46.9	205899	52844	84897	6
55	723197	33.8	928815	13.1	794383	46.9	205617	52869	84882	5
56	723400	33.8	928736	13.1	794664	46.9	205336	52893	84866	4
57	723603	33.7	928657	13.1	794945	46.9	205055	52918	84851	3
58	723805	33.7	928578	13.1	795227	46.9	204773	52943	84836	2
59	724007	33.7	928499	13.1	795508	46.8	204492	52967	84820	1
60	724210		928420		795789		204211	52992	84805	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

58 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.724210	33.7	9.928420	13.2	9.795789	46.8	10.204211	52992	84805	60
1	724412	33.7	928342	13.2	796070	46.8	203930	53017	84789	59
2	724614	33.6	928263	13.2	796351	46.8	203649	53041	84774	58
3	724816	33.6	928183	13.2	796632	46.8	203368	53066	84759	57
4	725017	33.6	928104	13.2	796913	46.8	203087	53091	84743	56
5	725219	33.6	928025	13.2	797194	46.8	202806	53115	84728	55
6	725420	33.5	927946	13.2	797475	46.8	202525	53140	84712	54
7	725622	33.5	927867	13.2	797755	46.8	202245	53164	84697	53
8	725823	33.5	927787	13.2	798036	46.7	201964	53189	84681	52
9	726024	33.5	927708	13.2	798316	46.7	201684	53214	84666	51
10	726225	33.5	927629	13.2	798596	46.7	201404	53238	84650	50
11	9.726426	33.4	9.927549	13.2	9.798877	46.7	10.201123	53263	84635	49
12	726626	33.4	927470	13.3	799157	46.7	200843	53288	84619	48
13	726827	33.4	927390	13.3	799437	46.7	200563	53312	84604	47
14	727027	33.4	927310	13.3	799717	46.7	200283	53337	84588	46
15	727228	33.4	927231	13.3	799997	46.6	200003	53361	84573	45
16	727428	33.3	927151	13.3	800277	46.6	199723	53386	84557	44
17	727628	33.3	927071	13.3	800557	46.6	199443	53411	84542	43
18	727828	33.3	926991	13.3	800836	46.6	199164	53435	84526	42
19	728027	33.3	926911	13.3	801116	46.6	198884	53460	84511	41
20	728227	33.3	926831	13.3	801396	46.6	198604	53484	84495	40
21	9.728427	33.2	9.926751	13.3	9.801675	46.6	10.198325	53509	84480	39
22	728626	33.2	926671	13.3	801955	46.6	198045	53534	84464	38
23	728825	33.2	926591	13.3	802234	46.5	197766	53558	84448	37
24	729024	33.2	926511	13.4	802513	46.5	197487	53583	84433	36
25	729223	33.1	926431	13.4	802792	46.5	197208	53607	84417	35
26	729422	33.1	926351	13.4	803072	46.5	196928	53632	84402	34
27	729621	33.1	926270	13.4	803351	46.5	196649	53656	84386	33
28	729820	33.1	926190	13.4	803630	46.5	196370	53681	84370	32
29	730018	33.0	926110	13.4	803908	46.5	196092	53705	84355	31
30	730216	33.0	926029	13.4	804187	46.5	195813	53730	84339	30
31	9.730415	33.0	9.925949	13.4	9.804466	46.4	10.195534	53754	84324	29
32	730613	33.0	925868	13.4	804745	46.4	195255	53779	84308	28
33	730811	33.0	925788	13.4	805023	46.4	194977	53804	84292	27
34	731009	32.9	925707	13.4	805302	46.4	194698	53828	84277	26
35	731206	32.9	925626	13.4	805580	46.4	194420	53853	84261	25
36	731404	32.9	925545	13.5	805859	46.4	194141	53877	84245	24
37	731602	32.9	925465	13.5	806137	46.4	193863	53902	84230	23
38	731799	32.9	925384	13.5	806415	46.3	193585	53926	84214	22
39	731996	32.8	925303	13.5	806693	46.3	193307	53951	84198	21
40	732193	32.8	925222	13.5	806971	46.3	193029	53975	84182	20
41	9.732390	32.8	9.925141	13.5	9.807249	46.3	10.192751	54000	84167	19
42	732587	32.8	925060	13.5	807527	46.3	192473	54024	84151	18
43	732784	32.8	924979	13.5	807805	46.3	192195	54049	84135	17
44	732980	32.7	924897	13.5	808083	46.3	191917	54073	84120	16
45	733177	32.7	924816	13.5	808361	46.3	191639	54097	84104	15
46	733373	32.7	924735	13.6	808638	46.2	191362	54122	84088	14
47	733569	32.7	924654	13.6	808916	46.2	191084	54146	84072	13
48	733765	32.7	924572	13.6	809193	46.2	190807	54171	84057	12
49	733961	32.6	924491	13.6	809471	46.2	190529	54195	84041	11
50	734157	32.6	924409	13.6	809748	46.2	190252	54220	84025	10
51	9.734353	32.6	9.924328	13.6	9.810025	46.2	10.189975	54244	84009	9
52	734549	32.6	924246	13.6	810302	46.2	189698	54269	83994	8
53	734744	32.5	924164	13.6	810580	46.2	189420	54293	83978	7
54	734939	32.5	924083	13.6	810857	46.2	189143	54317	83962	6
55	735135	32.5	924001	13.6	811134	46.1	188866	54342	83946	5
56	735330	32.5	923919	13.6	811410	46.1	188590	54366	83930	4
57	735525	32.5	923837	13.6	811687	46.1	188313	54391	83915	3
58	735719	32.4	923755	13.7	811964	46.1	188036	54415	83899	2
59	735914	32.4	923673	13.7	812241	46.1	187759	54440	83883	1
60	736109		923591		812517		187483	54464	83867	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.736109	32.4	9.923591	13.7	9.812517	46.1	10.187482	54464	83867	60
1	736303	32.4	923509	13.7	812794	46.1	187206	54488	83851	59
2	736498	32.4	923427	13.7	813070	46.1	186930	54513	83835	58
3	736692	32.3	923345	13.7	813347	46.0	186653	54537	83819	57
4	736886	32.3	923263	13.7	813623	46.0	186377	54561	83804	56
5	737080	32.3	923181	13.7	813899	46.0	186101	54586	83788	55
6	737274	32.3	923098	13.7	814175	46.0	185825	54610	83772	54
7	737467	32.3	923016	13.7	814452	46.0	185548	54635	83756	53
8	737661	32.2	922933	13.7	814728	46.0	185272	54659	83740	52
9	737855	32.2	922851	13.7	815004	46.0	184996	54683	83724	51
10	738048	32.2	922768	13.8	815279	46.0	184721	54708	83708	50
11	9.738241	32.2	9.922686	13.8	9.815555	45.9	10.184445	54732	83692	49
12	738434	32.2	922603	13.8	815831	45.9	184169	54756	83676	48
13	738627	32.1	922520	13.8	816107	45.9	183893	54781	83660	47
14	738820	32.1	922438	13.8	816382	45.9	183618	54805	83645	46
15	739013	32.1	922355	13.8	816658	45.9	183342	54829	83629	45
16	739206	32.1	922272	13.8	816933	45.9	183067	54854	83613	44
17	739398	32.1	922189	13.8	817209	45.9	182791	54878	83597	43
18	739590	32.0	922106	13.8	817484	45.9	182516	54902	83581	42
19	739783	32.0	922023	13.8	817759	45.9	182241	54927	83565	41
20	739975	32.0	921940	13.8	818035	45.8	181965	54951	83549	40
21	9.740167	32.0	9.921857	13.9	9.818310	45.8	10.181690	54975	83533	39
22	740359	32.0	921774	13.9	818585	45.8	181415	54999	83517	38
23	740550	31.9	921691	13.9	818860	45.8	181140	55024	83501	37
24	740742	31.9	921607	13.9	819135	45.8	180865	55048	83485	36
25	740934	31.9	921524	13.9	819410	45.8	180590	55072	83469	35
26	741125	31.9	921441	13.9	819684	45.8	180316	55097	83453	34
27	741316	31.9	921357	13.9	819959	45.8	180041	55121	83437	33
28	741508	31.8	921274	13.9	820234	45.8	179766	55145	83421	32
29	741699	31.8	921190	13.9	820508	45.7	179492	55169	83405	31
30	741889	31.8	921107	13.9	820783	45.7	179217	55194	83389	30
31	9.742080	31.8	9.921023	13.9	9.821057	45.7	10.178943	55218	83373	29
32	742271	31.8	920939	14.0	821332	45.7	178668	55242	83356	28
33	742462	31.7	920856	14.0	821606	45.7	178394	55266	83340	27
34	742652	31.7	920772	14.0	821880	45.7	178120	55291	83324	26
35	742842	31.7	920688	14.0	822154	45.7	177846	55315	83308	25
36	743033	31.7	920604	14.0	822429	45.7	177571	55339	83292	24
37	743223	31.7	920520	14.0	822703	45.7	177297	55363	83276	23
38	743413	31.6	920436	14.0	822977	45.6	177023	55388	83260	22
39	743602	31.6	920352	14.0	823250	45.6	176750	55412	83244	21
40	743792	31.6	920268	14.0	823524	45.6	176476	55436	83228	20
41	9.743982	31.6	9.920184	14.0	9.823798	45.6	10.176202	55460	83212	19
42	744171	31.6	920099	14.0	824072	45.6	175928	55484	83195	18
43	744361	31.5	920015	14.0	824345	45.6	175655	55509	83179	17
44	744550	31.5	919931	14.1	824619	45.6	175381	55533	83163	16
45	744739	31.5	919846	14.1	824893	45.6	175107	55557	83147	15
46	744928	31.5	919762	14.1	825166	45.6	174834	55581	83131	14
47	745117	31.5	919677	14.1	825439	45.5	174561	55605	83115	13
48	745306	31.4	919593	14.1	825713	45.5	174287	55630	83098	12
49	745494	31.4	919508	14.1	825986	45.5	174014	55654	83082	11
50	745683	31.4	919424	14.1	826259	45.5	173741	55678	83066	10
51	9.745871	31.4	9.919339	14.1	9.826532	45.5	10.173468	55702	83050	9
52	746059	31.4	919254	14.1	826805	45.5	173195	55726	83034	8
53	746248	31.3	919169	14.1	827078	45.5	172922	55750	83017	7
54	746436	31.3	919085	14.1	827351	45.5	172649	55775	83001	6
55	746624	31.3	919000	14.1	827624	45.5	172376	55799	82985	5
56	746812	31.3	918915	14.2	827897	45.4	172103	55823	82969	4
57	746999	31.3	918830	14.2	828170	45.4	171830	55847	82953	3
58	747187	31.2	918745	14.2	828442	45.4	171558	55871	82936	2
59	747374	31.2	918659	14.2	828715	45.4	171285	55895	82920	1
60	747562		918574		828987		171013	55919	82904	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.747562	31.2	9.918574	14.2	9.828987	45.4	10.171013	55919	82904	60
1	747749	31.2	918489	14.2	829260	45.4	170740	55943	82887	59
2	747936	31.2	918404	14.2	829532	45.4	170468	55968	82871	58
3	748123	31.1	918318	14.2	829805	45.4	170195	55992	82855	57
4	748310	31.1	918233	14.2	830077	45.4	169923	56016	82839	56
5	748497	31.1	918147	14.2	830349	45.3	169651	56040	82822	55
6	748683	31.1	918062	14.2	830621	45.3	169379	56064	82806	54
7	748870	31.1	917976	14.3	830893	45.3	169107	56088	82790	53
8	749056	31.0	917891	14.3	831165	45.3	168835	56112	82773	52
9	749243	31.0	917805	14.3	831437	45.3	168563	56136	82757	51
10	749426	31.0	917719	14.3	831709	45.3	168291	56160	82741	50
11	9.749615	31.0	9.917634	14.3	9.831981	45.3	10.168019	56184	82724	49
12	749801	31.0	917548	14.3	832253	45.3	167747	56208	82708	48
13	749987	30.9	917462	14.3	832525	45.3	167475	56232	82692	47
14	750172	30.9	917376	14.3	832796	45.3	167204	56256	82675	46
15	750358	30.9	917290	14.3	833068	45.2	166932	56280	82659	45
16	750543	30.9	917204	14.3	833339	45.2	166661	56305	82643	44
17	750729	30.9	917118	14.4	833611	45.2	166389	56329	82626	43
18	750914	30.8	917032	14.4	833882	45.2	166118	56353	82610	42
19	751099	30.8	916946	14.4	834154	45.2	165846	56377	82593	41
20	751284	30.8	916859	14.4	834425	45.2	165575	56401	82577	40
21	9.751469	30.8	9.916773	14.4	9.834696	45.2	10.165304	56425	82561	39
22	751654	30.8	916687	14.4	834967	45.2	165033	56449	82544	38
23	751839	30.8	916600	14.4	835238	45.2	164762	56473	82528	37
24	752023	30.7	916514	14.4	835509	45.2	164491	56497	82511	36
25	752208	30.7	916427	14.4	835780	45.1	164220	56521	82495	35
26	752392	30.7	916341	14.4	836051	45.1	163949	56545	82478	34
27	752576	30.7	916254	14.4	836322	45.1	163678	56569	82462	33
28	752760	30.7	916167	14.5	836593	45.1	163407	56593	82446	32
29	752944	30.6	916081	14.5	836864	45.1	163136	56617	82429	31
30	753128	30.6	915994	14.5	837134	45.1	162866	56641	82413	30
31	9.753312	30.6	9.915907	14.5	9.837405	45.1	10.162595	56665	82396	29
32	753495	30.6	915820	14.5	837675	45.1	162325	56689	82380	28
33	753679	30.6	915733	14.5	837946	45.1	162054	56713	82363	27
34	753862	30.5	915646	14.5	838216	45.1	161784	56736	82347	26
35	754046	30.5	915559	14.5	838487	45.0	161513	56760	82330	25
36	754229	30.5	915472	14.5	838757	45.0	161243	56784	82314	24
37	754412	30 5	915385	14.5	839027	45.0	160973	56808	82297	23
38	754595	30.5	915297	14.5	839297	45.0	160703	56832	82281	22
39	754778	30.4	915210	14.5	839568	45.0	160432	56856	82264	21
40	754960	30.4	915123	14.6	839838	45.0	160162	56880	82248	20
41	9.755143	30.4	9.915035	14.6	9.840108	45.0	10.159892	56904	82231	19
42	755326	30.4	914948	14.6	840378	45.0	159622	56928	82214	18
43	755508	30.4	914860	14.6	840647	45.0	159353	56952	82198	17
44	755690	30.4	914773	14.6	840917	44.9	159083	56976	82181	16
45	755872	30.3	914685	14.6	841187	44.9	158813	57000	82165	15
46	756054	30.3	914598	14.6	841457	44.9	158543	57024	82148	14
47	756236	30.3	914510	14.6	841726	44.9	158274	57047	82132	13
48	756418	30.3	914422	14.6	841996	44.9	158004	57071	82115	12
49	756600	30.3	914334	14.6	842266	44.9	157734	57095	82098	11
50	756782	30.2	914246	14.7	842535	44.9	157465	57119	82082	10
51	9.756963	30.2	9.914158	14.7	9.842805	44.9	10.157195	57143	82065	9
52	757144	30.2	914070	14.7	843074	44.9	156926	57167	82048	8
53	757326	30.2	913982	14.7	843343	44.9	156657	57191	82032	7
54	757507	30.2	913894	14.7	843612	44.9	156388	57215	82015	6
55	757688	30.1	913806	14.7	843882	44.8	156118	57238	81999	5
56	757869	30.1	913718	14.7	844151	44.8	155849	57262	81982	4
57	758050	30.1	913630	14.7	844420	44.8	155580	57286	81965	3
58	758230	30.1	913541	14.7	844689	44.8	155311	57310	81949	2
59	758411	30.1	913453	14.7	844958	44.8	155042	57334	81932	1
60	758591		913365		845227		154773	57358	81915	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.758591	30.1	9.913365	14.7	9.845227	44.8	10.154773	57358	81915	60
1	758772	30.0	913276	14.7	845496	44.8	154504	57381	81899	59
2	758952	30.0	913187	14.8	845764	44.8	154236	57405	81882	58
3	759132	30.0	913099	14.8	846033	44.8	153967	57429	81865	57
4	759312	30.0	913010	14.8	846302	44.8	153698	57453	81848	56
5	759492	30.0	912922	14.8	846570	44.7	153430	57477	81832	55
6	759672	29.9	912833	14.8	846839	44.7	153161	57501	81815	54
7	759852	29.9	912744	14.8	847107	44.7	152893	57524	81798	53
8	760031	29.9	912655	14.8	847376	44.7	152624	57548	81782	52
9	760211	29.9	912566	14.8	847644	44.7	152356	57572	81765	51
10	760390	29.9	912477	14.8	847913	44.7	152087	57596	81748	50
11	9.760569	29.8	9.912388	14.8	9.848181	44.7	10.151819	57619	81731	49
12	760748	29.8	912299	14.9	848449	44.7	151551	57643	81714	48
13	760927	29.8	912210	14.9	848717	44.7	151283	57667	81698	47
14	761106	29.8	912121	14.9	848986	44.7	151014	57691	81681	46
15	761285	29.8	912031	14.9	849254	44.7	150746	57715	81664	45
16	761464	29.8	911942	14.9	849522	44.7	150478	57738	81647	44
17	761642	29.7	911853	14.9	849790	44.6	150210	57762	81631	43
18	761821	29.7	911763	14.9	850058	44.6	149942	57786	81614	42
19	761999	29.7	911674	14.9	850325	44.6	149675	57810	81597	41
20	762177	29.7	911584	14.9	850593	44.6	149407	57833	81580	40
21	9.762356	29.7	9.911495	14.9	9.850861	44.6	10.149139	57857	81563	39
22	762534	29.6	911405	14.9	851129	44.6	148871	57881	81546	38
23	762712	29.6	911315	15.0	851396	44.6	148604	57904	81530	37
24	762889	29.6	911226	15.0	851664	44.6	148336	57928	81513	36
25	763067	29.6	911136	15.0	851931	44.6	148069	57952	81496	35
26	763245	29.6	911046	15.0	852199	44.6	147801	57976	81479	34
27	763422	29.6	910956	15.0	852466	44.6	147534	57999	81462	33
28	763600	29.5	910866	15.0	852733	44.5	147267	58023	81445	32
29	763777	29.5	910776	15.0	853001	44.5	146999	58047	81428	31
30	763954	29.5	910686	15.0	853268	44.5	146732	58070	81412	30
31	9.764131	29.5	9.910596	15.0	9.853535	44.5	10.146465	58094	81395	29
32	764308	29.5	910506	15.0	853802	44.5	146198	58118	81378	28
33	764485	29.4	910415	15.0	854069	44.5	145931	58141	81361	27
34	764662	29.4	910325	15.1	854336	44.5	145664	58165	81344	26
35	764838	29.4	910235	15.1	854603	44.5	145397	58189	81327	25
36	765015	29.4	910144	15.1	854870	44.5	145130	58212	81310	24
37	765191	29.4	910054	15.1	855137	44.5	144863	58236	81293	23
38	765367	29.4	909963	15.1	855404	44.5	144596	58260	81276	22
39	765544	29.3	909873	15.1	855671	44.4	144329	58283	81259	21
40	765720	29.3	909782	15.1	855938	44.4	144062	58307	81242	20
41	9.765896	29.3	9.909691	15.1	9.856204	44.4	10.143796	58330	81225	19
42	766072	29.3	909601	15.1	856471	44.4	143529	58354	81208	18
43	766247	29.3	909510	15.1	856737	44.4	143263	58378	81191	17
44	766423	29.3	909419	15.1	857004	44.4	142996	58401	81174	16
45	766598	29.2	909328	15.2	857270	44.4	142730	58425	81157	15
46	766774	29.2	909237	15.2	857537	44.4	142463	58449	81140	14
47	766949	29.2	909146	15.2	857803	44.4	142197	58472	81123	13
48	767124	29.2	909055	15.2	858069	44.4	141931	58496	81106	12
49	767300	29.2	908964	15.2	858336	44.4	141664	58519	81089	11
50	767475	29.1	908873	15.2	858602	44.3	141398	58543	81072	10
51	9.767649	29.1	9.908781	15.2	9.858868	44.3	10.141132	58567	81055	9
52	767824	29.1	908690	15.2	859134	44.3	140866	58590	81038	8
53	767999	29.1	908599	15.2	859400	44.3	140600	58614	81021	7
54	768173	29.1	908507	15.2	859666	44.3	140334	58637	81004	6
55	768348	29.0	908416	15.3	859932	44.3	140068	58661	80987	5
56	768522	29.0	908324	15.3	860198	44.3	139802	58684	80970	4
57	768697	29.0	908233	15.3	860464	44.3	139536	58708	80953	3
58	768871	29.0	908141	15.3	860730	44.3	139270	58731	80936	2
59	769045	29.0	908049	15.3	860995	44.3	139005	58755	80919	1
60	769219		907958		861261		138739	58779	80902	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.769219	29.0	9.907958	15.3	9.861261	44.3	10.138739	58779	80902	60
1	769393	28.9	907866	15.3	861527	44.3	138473	58802	80885	59
2	769566	28.9	907774	15.3	861792	44.2	138208	58826	80867	58
3	769740	28.9	907682	15.3	862058	44.2	137942	58849	80850	57
4	769913	28.9	907590	15.3	862323	44.2	137677	58873	80833	56
5	770087	28.9	907498	15.3	862589	44.2	137411	58896	80816	55
6	770260	28.8	907406	15.3	862854	44.2	137146	58920	80799	54
7	770433	28.8	907314	15.4	863119	44.2	136881	58943	80782	53
8	770606	28.8	907222	15.4	863385	44.2	136615	58967	80765	52
9	770779	28.8	907129	15.4	863650	44.2	136350	58990	80748	51
10	770952	28.8	907037	15.4	863915	44.2	136085	59014	80730	50
11	9.771125	28.8	9.906945	15.4	9.864180	44.2	10.135820	59037	80713	49
12	771298	28.7	906852	15.4	864445	44.2	135555	59061	80696	48
13	771470	28.7	906760	15.4	864710	44.2	135290	59084	80679	47
14	771643	28.7	906667	15.4	864975	44.1	135025	59108	80662	46
15	771815	28.7	906575	15.4	865240	44.1	134760	59131	80644	45
16	771987	28.7	906482	15.4	865505	44.1	134495	59154	80627	44
17	772159	28.7	906389	15.5	865770	44.1	134230	59178	80610	43
18	772331	28.6	906296	15.5	866035	44.1	133965	59201	80593	42
19	772503	28.6	906204	15.5	866300	44.1	133700	59225	80576	41
20	772675	28.6	906111	15.5	866564	44.1	133436	59248	80558	40
21	9.772847	28.6	9.906018	15.5	9.866829	44.1	10.133171	59272	80541	39
22	773018	28.6	905925	15.5	867094	44.1	132906	59295	80524	38
23	773190	28.6	905832	15.5	867358	44.1	132642	59318	80507	37
24	773361	28.5	905739	15.5	867623	44.1	132377	59342	80489	36
25	773533	28.5	905645	15.5	867887	44.1	132113	59365	80472	35
26	773704	28.5	905552	15.5	868152	44.0	131848	59389	80455	34
27	773875	28.5	905459	15.5	868416	44.0	131584	59412	80438	33
28	774046	28.5	905366	15.6	868680	44.0	131320	59436	80422	32
29	774217	28.5	905272	15.6	868945	44.0	131055	59459	80403	31
30	774388	28.4	905179	15.6	869209	44.0	130791	59482	80386	30
31	9.774558	28.4	9.905085	15.6	9.869473	44.0	10.130527	59506	80368	29
32	774729	28.4	904992	15.6	869737	44.0	130263	59529	80351	28
33	774899	28.4	904898	15.6	870001	44.0	129999	59552	80334	27
34	775070	28.4	904804	15.6	870265	44.0	129735	59576	80316	26
35	775240	28.4	904711	15.6	870529	44.0	129471	59599	80299	25
36	775410	28.3	904617	15.6	870793	44.0	129207	59622	80282	24
37	775580	28.3	904523	15.6	871057	44.0	128943	59646	80264	23
38	775750	28.3	904429	15.7	871321	44.0	128679	59669	80247	22
39	775920	28.3	904335	15.7	871585	44.0	128415	59693	80230	21
40	776090	28.3	904241	15.7	871849	43.9	128151	59716	80212	20
41	9.776259	28.3	9.904147	15.7	9.872112	43.9	10.127888	59739	80195	19
42	776429	28.2	904053	15.7	872376	43.9	127624	59763	80178	18
43	776598	28.2	903959	15.7	872640	43.9	127360	59786	80160	17
44	776768	28.2	903864	15.7	872903	43.9	127097	59809	80143	16
45	776937	28.2	903770	15.7	873167	43.9	126833	59832	80125	15
46	777106	28.2	903676	15.7	873430	43.9	126570	59856	80108	14
47	777275	28.1	903581	15.7	873694	43.9	126306	59879	80091	13
48	777444	28.1	903487	15.7	873957	43.9	126043	59902	80073	12
49	777613	28.1	903392	15.8	874220	43.9	125780	59926	80056	11
50	777781	28.1	903298	15.8	874484	43.9	125516	59949	80038	10
51	9.777950	28.1	9.903202	15.8	9.874747	43.9	10.125253	59972	80021	9
52	778119	28.1	903108	15.8	875010	43.9	124990	59995	80003	8
53	778287	28.0	903014	15.8	875273	43.8	124727	60019	79986	7
54	778455	28.0	902919	15.8	875536	43.8	124464	60042	79968	6
55	778624	28.0	902824	15.8	875800	43.8	124200	60065	79951	5
56	778792	28.0	902729	15.8	876063	43.8	123937	60089	79934	4
57	778960	28.0	902634	15.8	876326	43.8	123674	60112	79916	3
58	779128	28.0	902539	15.9	876589	43.8	123411	60135	79899	2
59	779295	27.9	902444	15.9	876851	43.8	123149	60158	79881	1
60	779463		902349		877114		122886	60182	79864	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.779463	27.9	9.902349	15.9	9.877114	43.8	10.122886	60182	79864	60
1	779631	27.9	902253	15.9	877377	43.8	122623	60205	79846	59
2	779798	27.9	902158	15.9	877640	43.8	122360	60228	79829	58
3	779966	27.9	902063	15.9	877903	43.8	122097	60251	79811	57
4	780133	27.9	901967	15.9	878165	43.8	121835	60274	79793	56
5	780300	27.8	901872	15.9	878428	43.8	121572	60298	79776	55
6	780467	27.8	901776	15.9	878691	43.8	121309	60321	79758	54
7	780634	27.8	901681	15.9	878953	43.7	121047	60344	79741	53
8	780801	27.8	901585	15.9	879216	43.7	120784	60367	79723	52
9	780968	27.8	901490	15.9	879478	43.7	120522	60390	79706	51
10	781134	27.8	901394	16.0	879741	43.7	120259	60414	79688	50
11	9.781301	27.7	9.901298	16.0	9.880003	43.7	10.119997	60437	79671	49
12	781468	27.7	901202	16.0	880265	43.7	119735	60460	79653	48
13	781634	27.7	901106	16.0	880528	43.7	119472	60483	79635	47
14	781800	27.7	901010	16.0	880790	43.7	119210	60506	79618	46
15	781966	27.7	900914	16.0	881052	43.7	118948	60529	79600	45
16	782132	27.7	900818	16.0	881314	43.7	118686	60553	79583	44
17	782298	27.6	900722	16.0	881576	43.7	118424	60576	79565	43
18	782464	27.6	900626	16.0	881839	43.7	118161	60599	79547	42
19	782630	27.6	900529	16.0	882101	43.7	117899	60622	79530	41
20	782796	27.6	900433	16.1	882363	43.6	117637	60645	79512	40
21	9.782961	27.6	9.900337	16.1	9.882625	43.6	10.117375	60668	79494	39
22	783127	27.6	900242	16.1	882887	43.6	117113	60691	79477	38
23	783282	27.5	900144	16.1	883148	43.6	116852	60714	79459	37
24	783458	27.5	900047	16.1	883410	43.6	116590	60738	79441	36
25	783623	27.5	899951	16.1	883672	43.6	116328	60761	79424	35
26	783788	27.5	899854	16.1	883934	43.6	116066	60784	79406	34
27	783953	27.5	899757	16.1	884196	43.6	115804	60807	79388	33
28	784118	27.5	899660	16.1	884457	43.6	115543	60830	79371	32
20	784282	27.4	899564	16.1	884719	43.6	115281	60853	79353	31
39	784447	27.4	899467	16.2	884980	43.6	115020	60876	79335	30
31	9.784612	27.4	9.899370	16.2	9.885242	43.6	10.114758	60899	79318	29
32	784776	27.4	899273	16.2	885503	43.6	114497	60922	79300	28
33	784941	27.4	899176	16.2	885765	43.6	114235	60945	79282	27
34	785105	27.4	899078	16.2	886026	43.6	113974	60968	79264	26
35	785269	27.3	898981	16.2	886288	43.6	113712	60991	79247	25
36	785433	27.3	898884	16.2	886549	43.5	113451	61015	79229	24
37	785597	27.3	898787	16.2	886810	43.5	113190	61038	79211	23
38	785761	27.3	898689	16.2	887072	43.5	112928	61061	79193	22
39	785925	27.3	898592	16.2	887333	43.5	112667	61084	79176	21
40	786089	27.3	898494	16.3	887594	43.5	112406	61107	79158	20
41	9.786252	27.2	9.898397	16.3	9.887855	43.5	10.112145	61130	79140	19
42	786416	27.2	898299	16.3	888116	43.5	111884	61153	79122	18
43	786579	27.2	898202	16.3	888377	43.5	111623	61176	79105	17
44	786742	27.2	898104	16.3	888639	43.5	111361	61199	79087	16
45	786906	27.2	898006	16.3	888900	43.5	111100	61222	79069	15
46	787069	27.2	897908	16.3	889160	43.5	110840	61245	79051	14
47	787232	27.1	897810	16.3	889421	43.5	110579	61268	79033	13
48	787395	27.1	897712	16.3	889682	43.5	110318	61291	79016	12
49	787557	27.1	897614	16.3	889943	43.5	110057	61314	78998	11
50	787720	27.1	897516	16.4	890204	43.4	109796	61337	78980	10
51	9.787883	27.1	9.897418	16.4	9.890465	43.4	10.109535	61360	78962	9
52	788045	27.1	897320	16.4	890725	43.4	109275	61383	78944	8
53	788208	27.1	897222	16.4	890986	43.4	109014	61406	78926	7
54	788370	27.0	897123	16.4	891247	43.4	108753	61429	78908	6
55	788532	27.0	897025	16.4	891507	43.4	108493	61451	78891	5
56	788694	27.0	896926	16.4	891768	43.4	108232	61474	78873	4
57	788856	27.0	896828	16.4	892028	43.4	107972	61497	78855	3
58	789018	27.0	896729	16.4	892289	43.4	107711	61520	78837	2
59	789180	27.0	896631	16.4	892549	43.4	107451	61543	78819	1
60	789342		896532		892810		107190	61566	78801	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.789342	26.9	9.896532	16.4	9.892810	43.4	10.107190	61566	78801	60
1	789504	26.9	896433	16.5	893070	43.4	106930	61589	78783	59
2	789665	26.9	896335	16.5	893331	43.4	106669	61612	78765	58
3	789827	26.9	896236	16.5	893591	43.4	106409	61635	78747	57
4	789988	26.9	896137	16.5	893851	43.4	106149	61658	78729	56
5	790149	26.9	896038	16.5	894111	43.4	105889	61681	78711	55
6	790310	26.8	895939	16.5	894371	43.4	105629	61704	78694	54
7	790471	26.8	895840	16.5	894632	43.3	105368	61726	78676	53
8	790632	26.8	895741	16.5	894892	43.3	105108	61749	78658	52
9	790793	26.8	895641	16.5	895152	43.3	104848	61772	78640	51
10	790954	26.8	895542	16.5	895412	43.3	104588	61795	78622	50
11	9.791115	26.8	9.895443	16.6	9.895672	43.3	10.104328	61818	78604	49
12	791275	26.7	895343	16.6	895932	43.3	104068	61841	78586	48
13	791436	26.7	895244	16.6	896192	43.3	103808	61864	78568	47
14	791596	26.7	895145	16.6	896452	43.3	103548	61887	78550	46
15	791757	26.7	895045	16.6	896712	43.3	103288	61909	78532	45
16	791917	26.7	894945	16.6	896971	43.3	103029	61932	78514	44
17	792077	26.7	894846	16.6	897231	43.3	102769	61955	78496	43
18	792237	26.6	894746	16.6	897491	43.3	102509	61978	78478	42
19	792397	26.6	894646	16.6	897751	43.3	102249	62001	78460	41
20	792557	26.6	894546	16.6	898010	43.3	101990	62024	78442	40
21	9.792716	26.6	9.894446	16.7	9.898270	43.3	10.101730	62046	78424	39
22	792876	26.6	894346	16.7	898530	43.3	101470	62069	78405	38
23	793035	26.6	894246	16.7	898789	43.3	101211	62092	78387	37
24	793195	26.5	894146	16.7	899049	43.2	100951	62115	78369	36
25	793354	26.5	894046	16.7	899308	43.2	100692	62138	78351	35
26	793514	26.5	893946	16.7	899568	43.2	100432	62160	78333	34
27	793673	26.5	893846	16.7	899827	43.2	100173	62183	78315	33
28	793832	26.5	893745	16.7	900086	43.2	099914	62206	78297	32
29	793991	26.5	893645	16.7	900346	43.2	099654	62229	78279	31
30	794150	26.4	893544	16.7	900605	43.2	099395	62251	78261	30
31	9.794308	26.4	9.893444	16.8	9.900864	43.2	10.099136	62274	78243	29
32	794467	26.4	893343	16.8	901124	43.2	098876	62297	78225	28
33	794626	26.4	893243	16.8	901383	43.2	098617	62320	78206	27
34	794784	26.4	893142	16.8	901642	43.2	098358	62342	78188	26
35	794942	26.4	893041	16.8	901901	43.2	098099	62365	78170	25
36	795101	26.4	892940	16.8	902160	43.2	097840	62388	78152	24
37	795259	26.3	892839	16.8	902419	43.2	097581	62411	78134	23
38	795417	26.3	892739	16.8	902679	43.2	097321	62433	78116	22
39	795575	26.3	892638	16.8	902938	43.2	097062	62456	78098	21
40	795733	26.3	892536	16.8	903197	43.1	096803	62479	78079	20
41	9.795891	26.3	9.892435	16.9	9.903455	43.1	10.096545	62502	78061	19
42	796049	26.3	892334	16.9	903714	43.1	096286	62524	78043	18
43	796206	26.3	892233	16.9	903973	43.1	096027	62547	78025	17
44	796364	26.2	892132	16.9	904232	43.1	095768	62570	78007	16
45	796521	26.2	892030	16.9	904491	43.1	095509	62592	77988	15
46	796679	26.2	891929	16.9	904750	43.1	095250	62615	77970	14
47	796836	26.2	891827	16.9	905008	43.1	094992	62638	77952	13
48	796993	26.2	891726	16.9	905267	43.1	094733	62660	77934	12
49	797150	26.1	891624	16.9	905526	43.1	094474	62683	77916	11
50	797307	26.1	891523	17.0	905784	43.1	094216	62706	77897	10
51	9.797464	26.1	9.891421	17.0	9.906043	43.1	10.093957	62728	77879	9
52	797621	26.1	891319	17.0	906302	43.1	093698	62751	77861	8
53	797777	26.1	891217	17.0	906560	43.1	093440	62774	77843	7
54	797934	26.1	891115	17.0	906819	43.1	093181	62796	77824	6
55	798091	26.1	891013	17.0	907077	43.1	092923	62819	77806	5
56	798247	26.1	890911	17.0	907336	43.1	092664	62842	77788	4
57	798403	26.0	890809	17.0	907594	43.1	092406	62864	77769	3
58	798560	26.0	890707	17.0	907852	43.1	092148	62887	77751	2
59	798716	26.0	890605	17.0	908111	43.0	091889	62909	77733	1
60	798872		890503		908369		091631	62932	77715	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

51 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.798772	26.0	9.890503	17.0	9.908369	43.0	10.091631	62932	77715	60
1	799028	26.0	890400	17.1	908628	43.0	091372	62955	77696	59
2	799184	26.0	890298	17.1	908886	43.0	091114	62977	77678	58
3	799339	25.9	890195	17.1	909144	43.0	090856	63000	77660	57
4	799495	25.9	890093	17.1	909402	43.0	090598	63022	77641	56
5	799651	25.9	889990	17.1	909660	43.0	090340	63045	77623	55
6	799806	25.9	889888	17.1	909918	43.0	090082	63068	77605	54
7	799962	25.9	889785	17.1	910177	43.0	089823	63090	77586	53
8	800117	25.9	889682	17.1	910435	43.0	089565	63113	77568	52
9	800272	25.8	889579	17.1	910693	43.0	089307	63135	77550	51
10	800427	25.8	889477	17.1	910951	43.0	089049	63158	77531	50
11	9.800582	25.8	9.889374	17.2	9.911209	43.0	10.088791	93180	77513	49
12	800737	25.8	889271	17.2	911467	43.0	088533	63203	77494	48
13	800892	25.8	889168	17.2	911724	43.0	088276	63225	77476	47
14	801047	25.8	889064	17.2	911982	43.0	088018	63248	77458	46
15	801201	25.8	888961	17.2	912240	43.0	087760	63271	77439	45
16	801356	25.7	888858	17.2	912498	43.0	087502	63293	77421	44
17	801511	25.7	888755	17.2	912756	43.0	087244	63316	77402	43
18	801665	25.7	888651	17.2	913014	42.9	086986	63338	77384	42
19	801819	25.7	888548	17.2	913271	42.9	086729	63361	77366	41
20	801973	25.7	888444	17.3	913529	42.9	086471	63383	77347	40
21	9.802128	25.7	9.888341	17.3	9.913787	42.9	10.086213	63406	77329	39
22	802282	25.6	888237	17.3	914044	42.9	085956	63428	77310	38
23	802436	25.6	888134	17.3	914302	42.9	085698	63451	77292	37
24	802589	25.6	888030	17.3	914560	42.9	085440	63473	77273	36
25	802743	25.6	887926	17.3	914817	42.9	085183	63496	77255	35
26	802897	25.6	887822	17.3	915075	42.9	084925	63518	77236	34
27	803050	25.6	887718	17.3	915332	42.9	084668	63540	77218	33
28	803204	25.6	887614	17.3	915590	42.9	084410	63563	77199	32
29	803357	25.5	887510	17.3	915847	42.9	084153	63585	77181	31
30	803511	25.5	887406	17.4	916104	42.9	083896	63608	77162	30
31	9.803664	25.5	9.887302	17.4	9.916362	42.9	10.083638	63630	77144	29
32	803817	25.5	887198	17.4	916619	42.9	083381	63653	77125	28
33	803970	25.5	887093	17.4	916877	42.9	083123	63675	77107	27
34	804123	25.5	886989	17.4	917134	42.9	082866	63698	77088	26
35	804276	25.4	886885	17.4	917391	42.9	082609	63720	77070	25
36	804428	25.4	886780	17.4	917648	42.9	082352	63742	77051	24
37	804581	25.4	886676	17.4	917905	42.9	082095	63765	77033	23
38	804734	25.4	886571	17.4	918163	42.8	081837	63787	77014	22
39	804886	25.4	886466	17.4	918420	42.8	081580	63810	76996	21
40	805039	25.4	886362	17.5	918677	42.8	081323	63832	76977	20
41	9.805191	25.4	9.886257	17.5	9.918934	42.8	10.081066	63854	76959	19
42	805343	25.3	886152	17.5	919191	42.8	080809	63877	76940	18
43	805495	25.3	886047	17.5	919448	42.8	080552	63899	76921	17
44	805647	25.3	885942	17.5	919705	42.8	080295	63922	76903	16
45	805799	25.3	885837	17.5	919962	42.8	080038	63944	76884	15
46	805951	25.3	885732	17.5	920219	42.8	079781	63966	76866	14
47	806103	25.3	885627	17.5	920476	42.8	079524	63989	76847	13
48	806254	25.3	885522	17.5	920733	42.8	079267	64011	76828	12
49	806406	25.2	885416	17.5	920990	42.8	079010	64033	76810	11
50	806557	25.2	885311	17.6	921247	42.8	078753	64056	76791	10
51	9.806709	25.2	9.885205	17.6	9.921503	42.8	10.078497	64078	76772	9
52	806860	25.2	885100	17.6	921760	42.8	078240	64100	76754	8
53	807011	25.2	884994	17.6	922017	42.8	077983	64123	76735	7
54	807163	25.2	884889	17.6	922274	42.8	077726	64145	76717	6
55	807314	25.2	884783	17.6	922530	42.8	077470	64167	76698	5
56	807465	25.1	884677	17.6	922787	42.8	077213	64190	76679	4
57	807615	25.1	884572	17.6	923044	42.8	076956	64212	76661	3
58	807766	25.1	884466	17.6	923300	42.8	076700	64234	76642	2
59	807917	25.1	884360	17.6	923557	42.7	076443	64256	76623	1
60	808067		884254		923813		076187	64279	76604	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.808067	25.1	9.884254	17.7	9.923813	42.7	10.076187	64279	76604	60
1	808218	25.1	884148	17.7	924070	42.7	075930	64301	76586	59
2	808368	25.1	884042	17.7	924327	42.7	075673	64323	76567	58
3	808519	25.0	883936	17.7	924583	42.7	075417	64346	76548	57
4	808669	25.0	883829	17.7	924840	42.7	075160	64368	76530	56
5	808819	25.0	883723	17.7	925096	42.7	074904	64390	76511	55
6	808969	25.0	883617	17.7	925352	42.7	074648	64412	76492	54
7	809119	25.0	883510	17.7	925609	42.7	074391	64435	76473	53
8	809269	25.0	883404	17.7	925865	42.7	074135	64457	76455	52
9	809419	24.9	883297	17.8	926122	42.7	073878	64479	76436	51
10	809569	24.9	883191	17.8	926378	42.7	073622	64501	76417	50
11	9.809718	24.9	9.883084	17.8	9.926634	42.7	10.073366	64524	76398	49
12	809868	24.9	882977	17.8	926890	42.7	073110	64546	76380	48
13	810017	24.9	882871	17.8	927147	42.7	072853	64568	76361	47
14	810167	24.9	882764	17.8	927403	42.7	072597	64590	76342	46
15	810316	24.8	882657	17.8	927659	42.7	072341	64612	76323	45
16	810465	24.8	882550	17.8	927915	42.7	072085	64635	76304	44
17	810614	24.8	882443	17.8	928171	42.7	071829	64657	76286	43
18	810763	24.8	882336	17.9	928427	42.7	071573	64679	76267	42
19	810912	24.8	882229	17.9	928683	42.7	071317	64701	76248	41
20	811061	24.8	882121	17.9	928940	42.7	071060	64723	76229	40
21	9.811210	24.8	9.882014	17.9	9.929196	42.7	10.070804	64746	76210	39
22	811358	24.7	881907	17.9	929452	42.7	070548	64768	76192	38
23	811507	24.7	881799	17.9	929708	42.7	070292	64790	76173	37
24	811655	24.7	881692	17.9	929964	42.6	070036	64812	76154	36
25	811804	24.7	881584	17.9	930220	42.6	069780	64834	76135	35
26	811952	24.7	881477	17.9	930475	42.6	069525	64856	76116	34
27	812100	24.7	881369	17.9	930731	42.6	069269	64878	76097	33
28	812248	24.7	881261	18.0	930987	42.6	069013	64901	76078	32
29	812396	24.6	881153	18.0	931243	42.6	068757	64923	76059	31
30	812544	24.6	881046	18.0	931499	42.6	068501	64945	76041	30
31	9.812692	24.6	9.880938	18.0	9.931755	42.6	10.068245	64967	76022	29
32	812840	24.6	880830	18.0	932010	42.6	067990	64989	76003	28
33	812988	24.6	880722	18.0	932266	42.6	067734	65011	75984	27
34	813135	24.6	880613	18.0	932522	42.6	067478	65033	75965	26
35	813283	24.6	880505	18.0	932778	42.6	067222	65055	75946	25
36	813430	24.5	880397	18.0	933033	42.6	066967	65077	75927	24
37	813578	24.5	880289	18.1	933289	42.6	066711	65100	75908	23
38	813725	24.5	880180	18.1	933545	42.6	066455	65122	75889	22
39	813872	24.5	880072	18.1	933800	42.6	066200	65144	75870	21
40	814019	24.5	879963	18.1	934056	42.6	065944	65166	75851	20
41	9.814166	24.5	9.879855	18.1	9.934311	42.6	10.065689	65188	75832	19
42	814313	24.5	879746	18.1	934567	42.6	065433	65210	75813	18
43	814460	24.4	879637	18.1	934823	42.6	065177	65232	75794	17
44	814607	24.4	879529	18.1	935078	42.6	064922	65254	75775	16
45	814753	24.4	879420	18.1	935333	42.6	064667	65276	75756	15
46	814900	24.4	879311	18.1	935589	42.6	064411	65298	75738	14
47	815046	24.4	879202	18.2	935844	42.6	064156	65320	75719	13
48	815193	24.4	879093	18.2	936100	42.6	063900	65342	75700	12
49	815339	24.4	878984	18.2	936355	42.6	063645	65364	75680	11
50	815485	24.3	878875	18.2	936610	42.6	063390	65386	75661	10
51	9.815631	24.3	9.878766	18.2	9.936866	42.5	10.063134	65408	75642	9
52	815778	24.3	878656	18.2	937121	42.5	062879	65430	75623	8
53	815924	24.3	878547	18.2	937376	42.5	062624	65452	75604	7
54	816069	24.3	878438	18.2	937632	42.5	062368	65474	75585	6
55	816215	24.3	878328	18.2	937887	42.5	062113	65496	75566	5
56	816361	24.3	878219	18.3	938142	42.5	061858	65518	75547	4
57	816507	24.2	878109	18.3	938398	42.5	061602	65540	75528	3
58	816652	24.2	877999	18.3	938653	42.5	061347	65562	75509	2
59	816798	24.2	877890	18.3	938908	42.5	061092	65584	75490	1
60	816943		877780		939163		060837	65606	75471	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.816943	24.2	9.877780	18.3	9.939163	42.5	10.060837	65606	75471	60
1	817088	24.2	877670	18.3	939418	42.5	060582	65628	75452	59
2	817233	24.2	877560	18.3	939673	42.5	060327	65650	75433	58
3	817379	24.2	877450	18.3	939928	42.5	060072	65672	75414	57
4	817524	24.1	877340	18.3	940183	42.5	059817	65694	75395	56
5	817668	24.1	877230	18.4	940438	42.5	059562	65716	75375	55
6	817813	24.1	877120	18.4	940694	42.5	059306	65738	75356	54
7	817958	24.1	877010	18.4	940949	42.5	059051	65759	75337	53
8	818103	24.1	876899	18.4	941204	42.5	058796	65781	75318	52
9	818247	24.1	876789	18.4	941458	42.5	058542	65803	75299	51
10	818392	24.1	876678	18.4	941714	42.5	058286	65825	75280	50
11	9.818536	24.0	9.876568	18.4	9.941968	42.5	10.058032	65847	75261	49
12	818681	24.0	876457	18.4	942223	42.5	057777	65869	75241	48
13	818825	24.0	876347	18.4	942478	42.5	057522	65891	75222	47
14	818969	24.0	876236	18.5	942733	42.5	057267	65913	75203	46
15	819113	24.0	876125	18.5	942988	42.5	057012	65935	75184	45
16	819257	24.0	876014	18.5	943243	42.5	056757	65956	75165	44
17	819401	24.0	875904	18.5	943498	42.5	056502	65978	75146	43
18	819545	23.9	875793	18.5	943752	42.5	056248	66000	75126	42
19	819689	23.9	875682	18.5	944007	42.5	055993	66022	75107	41
20	819832	23.9	875571	18.5	944262	42.5	055738	66044	75088	40
21	9.819976	23.9	9.875459	18.5	9.944517	42.5	10.055483	66066	75069	39
22	820120	23.9	875348	18.5	944771	42.4	055229	66088	75050	38
23	820263	23.9	875237	18.5	945026	42.4	054974	66109	75030	37
24	820406	23.9	875126	18.6	945281	42.4	054719	66131	75011	36
25	820550	23.8	875014	18.6	945535	42.4	054465	66153	74992	35
26	820693	23.8	874903	18.6	945790	42.4	054210	66175	74973	34
27	820836	23.8	874791	18.6	946045	42.4	053955	66197	74953	33
28	820979	23.8	874680	18.6	946299	42.4	053701	66218	74934	32
29	821122	23.8	874568	18.6	946554	42.4	053446	66240	74915	31
30	821265	23.8	874456	18.6	946808	42.4	053192	66262	74896	30
31	9.821407	23.8	9.874344	18.6	9.947063	42.4	10.052937	66284	74876	29
32	821550	23.8	874232	18.7	947318	42.4	052682	66306	74857	28
33	821693	23.7	874121	18.7	947572	42.4	052428	66327	74838	27
34	821835	23.7	874009	18.7	947826	42.4	052174	66349	74818	26
35	821977	23.7	873896	18.7	948081	42.4	051919	66371	74799	25
36	822120	23.7	873784	18.7	948336	42.4	051664	66393	74780	24
37	822262	23.7	873672	18.7	948590	42.4	051410	66414	74760	23
38	822404	23.7	873560	18.7	948844	42.4	051156	66436	74741	22
39	822546	23.7	873448	18.7	949099	42.4	050901	66458	74722	21
40	822688	23.6	873335	18.7	949353	42.4	050647	66480	74703	20
41	9.822830	23.6	9.873223	18.7	9.949607	42.4	10.050393	66501	74683	19
42	822972	23.6	873110	18.8	949862	42.4	050138	66523	74663	18
43	823114	23.6	872998	18.8	950116	42.4	049884	66545	74644	17
44	823255	23.6	872885	18.8	950370	42.4	049630	66566	74625	16
45	823397	23.6	872772	18.8	950625	42.4	049375	66588	74606	15
46	823539	23.6	872659	18.8	950879	42.4	049121	66610	74586	14
47	823680	23.5	872547	18.8	951133	42.4	048867	66632	74567	13
48	823821	23.5	872434	18.8	951388	42.4	048612	66653	74548	12
49	823963	23.5	872321	18.8	951642	42.4	048358	66675	74522	11
50	824104	23.5	872208	18.8	951896	42.4	048104	66697	74509	10
51	9.824245	23.5	9.872095	18.9	9.952150	42.4	10.047850	66718	74489	9
52	824386	23.5	871981	18.9	952405	42.4	047595	66740	74470	8
53	824527	23.5	871868	18.9	952659	42.4	047341	66762	74451	7
54	824668	23.4	871755	18.9	952913	42.4	047087	66783	74431	6
55	824808	23.4	871641	18.9	953167	42.3	046833	66805	74412	5
56	824949	23.4	871528	18.9	953421	42.3	046579	66827	74392	4
57	825090	23.4	871414	18.9	953675	42.3	046325	66848	74373	3
58	825230	23.4	871301	18.9	953929	42.3	046071	66870	74353	2
59	825371	23.4	871187	18.9	954183	42.3	045817	66891	74334	1
60	825511		871073		954437		045563	66913	74314	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.825511	23.4	9.871073	19.0	9.954437	42.3	10.045563	66913	74314	60
1	825651	23.3	870960	19.0	954691	42.3	045309	66935	74295	59
2	825791	23.3	870846	19.0	954945	42.3	045055	66956	74276	58
3	825931	23.3	870732	19.0	955200	42.3	044800	66978	74256	57
4	826071	23.3	870618	19.0	955454	42.3	044546	66999	74237	56
5	826211	23.3	870504	19.0	955707	42.3	044293	67021	74217	55
6	826351	23.3	870390	19.0	955961	42.3	044039	67043	74198	54
7	826491	23.3	870276	19.0	956215	42.3	043785	67064	74178	53
8	826631	23.3	870161	19.0	956469	42.3	043531	67086	74159	52
9	826770	23.2	870047	19.1	956723	42.3	043277	67107	74139	51
10	826910	23.2	869933	19.1	956977	42.3	043023	67129	74120	50
11	9.827049	23.2	9.869818	19.1	9.957231	42.3	10.042769	67151	74100	49
12	827189	23.2	869704	19.1	957485	42.3	042515	67172	74080	48
13	827328	23.2	869589	19.1	957739	42.3	042261	67194	74061	47
14	827467	23.2	869474	19.1	957993	42.3	042007	67215	74041	46
15	827606	23.2	869360	19.1	958246	42.3	041754	67237	74022	45
16	827745	23.2	869245	19.1	958500	42.3	041500	67258	74002	44
17	827884	23.1	869130	19.1	958754	42.3	041246	67280	73983	43
18	828023	23.1	869015	19.2	959008	42.3	040992	67301	73963	42
19	828162	23.1	868900	19.2	959262	42.3	040738	67323	73944	41
20	828301	23.1	868785	19.2	959516	42.3	040484	67344	73924	40
21	9.828439	23.1	9.868670	19.2	9.959769	42.3	10.040231	67366	73904	39
22	828578	23.1	868555	19.2	960023	42.3	039977	67387	73885	38
23	828716	23.1	868440	19.2	960277	42.3	039723	67409	73865	37
24	828855	23.0	868324	19.2	960531	42.3	039469	67430	73846	36
25	828993	23.0	868209	19.2	960784	42.3	039216	67452	73826	35
26	829131	23.0	868093	19.2	961038	42.3	038962	67473	73806	34
27	829269	23.0	867978	19.3	961291	42.3	038709	67495	73787	33
28	829407	23.0	867862	19.3	961545	42.3	038455	67516	73767	32
29	829545	23.0	867747	19.3	961799	42.3	038201	67538	73747	31
30	829683	23.0	867631	19.3	962052	42.3	037948	67559	73728	30
31	9.829821	22.9	9.867515	19.3	9.962306	42.3	10.037694	67580	73708	29
32	829959	22.9	867399	19.3	962560	42.3	037440	67602	73688	28
33	830097	22.9	867283	19.3	962813	42.3	037187	67623	73669	27
34	830234	22.9	867167	19.3	963067	42.3	036933	67645	73649	26
35	830372	22.9	867051	19.3	963320	42.3	036680	67666	73629	25
36	830509	22.9	866935	19.4	963574	42.3	036426	67688	73610	24
37	830646	22.9	866819	19.4	963827	42.3	036173	67709	73590	23
38	830784	22.9	866703	19.4	964081	42.3	035919	67730	73570	22
39	830921	22.8	866586	19.4	964335	42.3	035665	67752	73551	21
40	831058	22.8	866470	19.4	964588	42.2	035412	67773	73531	20
41	9.831195	22.8	9.866353	19.4	9.964842	42.2	10.035158	67795	73511	19
42	831332	22.8	866237	19.4	965095	42.2	034905	67816	73491	18
43	831469	22.8	866120	19.4	965349	42.2	034651	67837	73472	17
44	831606	22.8	866004	19.5	965602	42.2	034398	67859	73452	16
45	831742	22.8	865887	19.5	965855	42.2	034145	67880	73432	15
46	831879	22.8	865770	19.5	966109	42.2	033891	67901	73413	14
47	832015	22.7	865653	19.5	966362	42.2	033638	67923	73393	13
48	832152	22.7	865536	19.5	966616	42.2	033384	67944	73373	12
49	832288	22.7	865419	19.5	966869	42.2	033131	67965	73353	11
50	832425	22.7	865302	19.5	967123	42.2	032877	67987	73333	10
51	9.832561	22.7	9.865185	19.5	9.967376	42.2	10.032624	68008	73314	9
52	832697	22.7	865068	19.5	967629	42.2	032371	68029	73294	8
53	832833	22.7	864950	19.5	967883	42.2	032117	68051	73274	7
54	832969	22.6	864833	19.6	968136	42.2	031864	68072	73254	6
55	833105	22.6	864716	19.6	968389	42.2	031611	68093	73234	5
56	833241	22.6	864598	19.6	968643	42.2	031357	68115	73215	4
57	833377	22.6	864481	19.6	968896	42.2	031104	68136	73195	3
58	833512	22.6	864363	19.6	969149	42.2	030851	68157	73175	2
59	833648	22.6	864245	19.6	969403	42.2	030597	68179	73155	1
60	833783		864127		969656		030344	68200	73135	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

47 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N.sine.	N. cos.	
0	9.833783	22.6	9.864127	19.6	9.969656	42.2	10.030344	68200	73135	60
1	833919	22.5	864010	19.6	969909	42.2	030091	68221	73116	59
2	834054	22.5	863892	19.7	970162	42.2	029838	68242	73096	58
3	834189	22.5	863774	19.7	970416	42.2	029584	68264	73076	57
4	834325	22.5	863656	19.7	970669	42.2	029331	68285	73056	56
5	834460	22.5	863538	19.7	970922	42.2	029078	68306	73036	55
6	834595	22.5	863419	19.7	971175	42.2	028825	68327	73016	54
7	834730	22.5	863301	19.7	971429	42.2	028571	68349	72996	53
8	834865	22.5	863183	19.7	971682	42.2	028318	68370	72976	52
9	834999	22.4	863064	19.7	971935	42.2	028065	68391	72957	51
10	835134	22.4	862946	19.8	972188	42.2	027812	68412	72937	50
11	9.835269	22.4	9.862827	19.8	9.972441	42.2	10.027559	68434	72917	49
12	835403	22.4	862709	19.8	972694	42.2	027306	68455	72897	48
13	835538	22.4	862590	19.8	972948	42.2	027052	68476	72877	47
14	835672	22.4	862471	19.8	973201	42.2	026799	68497	72857	46
15	835807	22.4	862353	19.8	973454	42.2	026546	68518	72837	45
16	835941	22.4	862234	19.8	973707	42.2	026293	68539	72817	44
17	836075	22.3	862115	19.8	973960	42.2	026040	68561	72797	43
18	836209	22.3	861996	19.8	974213	42.2	025787	68582	72777	42
19	836343	22.3	861877	19.8	974466	42.2	025534	68603	72757	41
20	836477	22.3	861758	19.9	974719	42.2	025281	68624	72737	40
21	9.836611	22.3	9.861638	19.9	9.974973	42.2	10.025027	68645	72717	39
22	836745	22.3	861519	19.9	975226	42.2	024774	68666	72697	38
23	836878	22.3	861400	19.9	975479	42.2	024521	68688	72677	37
24	837012	22.2	861280	19.9	975732	42.2	024268	68709	72657	36
25	837146	22.2	861161	19.9	975985	42.2	024015	68730	72637	35
26	837279	22.2	861041	19.9	976238	42.2	023762	68751	72617	34
27	837412	22.2	860922	19.9	976491	42.2	023509	68772	72597	33
28	837546	22.2	860802	19.9	976744	42.2	023256	68793	72577	32
29	837679	22.2	860682	20.0	976997	42.2	023003	68814	72557	31
30	837812	22.2	860562	20.0	977250	42.2	022750	68835	72537	30
31	9.837945	22.2	9.860442	20.0	9.977503	42.2	10.022497	68857	72517	29
32	838078	22.1	860322	20.0	977756	42.2	022244	68878	72497	28
33	838211	22.1	860202	20.0	978009	42.2	021991	68899	72477	27
34	838344	22.1	860082	20.0	978262	42.2	021738	68920	72457	26
35	838477	22.1	859962	20.0	978515	42.2	021485	68941	72437	25
36	838610	22.1	859842	20.0	978768	42.2	021232	68962	72417	24
37	838742	22.1	859721	20.1	979021	42.2	020979	68983	72397	23
38	838875	22.1	859601	20.1	979274	42.2	020726	69004	72377	22
39	839007	22.1	859480	20.1	979527	42.2	020473	69025	72357	21
40	839140	22.0	859360	20.1	979780	42.2	020220	69046	72337	20
41	9.839272	22.0	9.859239	20.1	9.980033	42.2	10.019967	69067	72317	19
42	839404	22.0	859119	20.1	980286	42.2	019714	69088	72297	18
43	839536	22.0	858998	20.1	980538	42.2	019462	69109	72277	17
44	839668	22.0	858877	20.1	980791	42.1	019209	69130	72257	16
45	839800	22.0	858756	20.2	981044	42.1	018956	69151	72236	15
46	839932	22.0	858635	20.2	981297	42.1	018703	69172	72216	14
47	840064	21.9	858514	20.2	981550	42.1	018450	69193	72196	13
48	840196	21.9	858393	20.2	981803	42.1	018197	69214	72176	12
49	840328	21.9	858272	20.2	982056	42.1	017944	69235	72156	11
50	840459	21.9	858151	20.2	982309	42.1	017691	69256	72136	10
51	9.840591	21.9	9.858029	20.2	9.982562	42.1	10.017438	69277	72116	9
52	840722	21.9	857908	20.2	982814	42.1	017186	69298	72095	8
53	840854	21.9	857786	20.2	983067	42.1	016933	69319	72075	7
54	840985	21.9	857665	20.3	983320	42.1	016680	69340	72055	6
55	841116	21.8	857543	20.3	983573	42.1	016427	69361	72035	5
56	841247	21.8	857422	20.3	983826	42.1	016174	69382	72015	4
57	841378	21.8	857300	20.3	984079	42.1	015921	69403	71995	3
58	841509	21.8	857178	20.3	984331	42.1	015669	69424	71974	2
59	841640	21.8	857056	20.3	984584	42.1	015416	69445	71954	1
60	841771		856934		984837		015163	69466	71934	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

46 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.841771	21.8	9.856934	20.3	9.984837	42.1	10.015163	69466	71934	60
1	841902	21.8	856812	20.3	985090	42.1	014910	69487	71914	59
2	842033	21.8	856690	20.4	985343	42.1	014657	69508	71894	58
3	842163	21.7	856568	20.4	985596	42.1	014404	69529	71873	57
4	842294	21.7	856446	20.4	985848	42.1	014152	69549	71853	56
5	842424	21.7	856323	20.4	986101	42.1	013899	69570	71833	55
6	842555	21.7	856201	20.4	986354	42.1	013646	69591	71813	54
7	842685	21.7	856078	20.4	986607	42.1	013393	69612	71792	53
8	842815	21.7	855956	20.4	986860	42.1	013140	69633	71772	52
9	842946	21.7	855833	20.4	987112	42.1	012888	69654	71752	51
10	843076	21.7	855711	20.5	987365	42.1	012635	69675	71732	50
11	9.843206	21.6	9.855588	20.5	9.987618	42.1	10.012382	69696	71711	49
12	843336	21.6	855465	20.5	987871	42.1	012129	69717	71691	48
13	843466	21.6	855342	20.5	988123	42.1	011877	69737	71671	47
14	843595	21.6	855219	20.5	988376	42.1	011624	69758	71650	46
15	843725	21.6	855096	20.5	988629	42.1	011371	69779	71630	45
16	843855	21.6	854973	20.5	988882	42.1	011118	69800	71610	44
17	843984	21.6	854850	20.5	989134	42.1	010866	69821	71590	43
18	844114	21.5	854727	20.6	989387	42.1	010613	69842	71569	42
19	844243	21.5	854603	20.6	989640	42.1	010360	69862	71549	41
20	844372	21.5	854480	20.6	989893	42.1	010107	69883	71529	40
21	9.844502	21.5	9.854356	20.6	9.990145	42.1	10.009855	69904	71508	39
22	844631	21.5	854233	20.6	990398	42.1	009602	69925	71488	38
23	844760	21.5	854109	20.6	990651	42.1	009349	69946	71468	37
24	844889	21.5	853986	20.6	990903	42.1	009097	69966	71447	36
25	845018	21.5	853862	20.6	991156	42.1	008844	69987	71427	35
26	845147	21.5	853738	20.6	991409	42.1	008591	70008	71407	34
27	845276	21.4	853614	20.7	991662	42.1	008338	70029	71386	33
28	845405	21.4	853490	20.7	991914	42.1	008086	70049	71366	32
29	845533	21.4	853366	20.7	992167	42.1	007833	70070	71345	31
30	845662	21.4	853242	20.7	992420	42.1	007580	70091	71325	30
31	9.845790	21.4	9.853118	20.7	9.992672	42.1	10.007328	70112	71305	29
32	845919	21.4	852994	20.7	992925	42.1	007075	70132	71284	28
33	846047	21.4	852869	20.7	993178	42.1	006822	70153	71264	27
34	846175	21.4	852745	20.7	993430	42.1	006570	70174	71243	26
35	846304	21.4	852620	20.7	993683	42.1	006317	70195	71223	25
36	846432	21.3	852496	20.8	993936	42.1	006064	70215	71203	24
37	846560	21.3	852371	20.8	994189	42.1	005811	70236	71182	23
38	846688	21.3	852247	20.8	994441	42.1	005559	70257	71162	22
39	846816	21.3	852122	20.8	994694	42.1	005306	70277	71141	21
40	846944	21.3	851997	20.8	994947	42.1	005053	70298	71121	20
41	9.847071	21.3	9.851872	20.8	9.995199	42.1	10.004801	70319	71100	19
42	847199	21.3	851747	20.8	995452	42.1	004548	70339	71080	18
43	847327	21.3	851622	20.8	995705	42.1	004295	70360	71059	17
44	847454	21.2	851497	20.9	995957	42.1	004043	70381	71039	16
45	847582	21.2	851372	20.9	996210	42.1	003790	70401	71019	15
46	847709	21.2	851246	20.9	996463	42.1	003537	70422	70998	14
47	847836	21.2	851121	20.9	996715	42.1	003285	70443	70978	13
48	847964	21.2	850996	20.9	996968	42.1	003032	70463	70957	12
49	848091	21.2	850870	20.9	997221	42.1	002779	70484	70937	11
50	848218	21.2	850745	20.9	997473	42.1	002527	70505	70916	10
51	9.848345	21.2	9.850619	20.9	9.997726	42.1	10.002274	70525	70896	9
52	848472	21.1	850493	21.0	997979	42.1	002021	70546	70875	8
53	848599	21.1	850368	21.0	998231	42.1	001769	70567	70855	7
54	848726	21.1	850242	21.0	998484	42.1	001516	70587	70834	6
55	848852	21.1	850116	21.0	998737	42.1	001263	70608	70813	5
56	848979	21.1	849990	21.0	998989	42.1	001011	70628	70793	4
57	849106	21.1	849864	21.0	999242	42.1	000758	70649	70772	3
58	849232	21.1	849738	21.0	999495	42.1	000505	70670	70752	2
59	849359	21.1	849611	21.0	999748	42.1	000253	70690	70731	1
60	849485		849485		10.000000		000000	70711	70711	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

45 Degrees.