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ELEMENTS OF MECHANICS

INCLUDING

KINEMATICS, KINETICS AND STATICS

WITH APPLICATIONS

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NOTE TO THE FIFTH EDITION.

The principal change in this edition is the substitution of another demonstration of the Foucault pendulum (Art. 233). This demonstration is due to Professor W. LeConte Stevens, Washington and Lee University.

Several minor changes have also been made.

T. W. W.

Schenectady, N. Y., November 1904.

CONTENTS.

ELEMENTS OF MECHANICS.

INTRODUCTION.

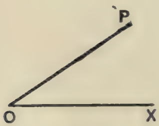
1. MECHANICAL experiences are without doubt of great antiquity. The earliest investigations concerning mechanical principles are ascribed to Archytas of Tarentum (B.C. 400). He is said to have worked out the theory of the pulley. Later, in the writings of Archimedes of Syracuse (B.C. 287–212), are found applications of geometry to various mechanical questions, including a treatise on levers and other machines. From Archimedes to Galileo and Stevinus, a period of nearly two thousand years, no marked advance was made. It is to Galileo and Stevinus that we owe the transition from Mechanics in its original signification as the Science of Machines to Mechanics as the term is now understood—in fact they are to be regarded as the founders of the science of mechanics.

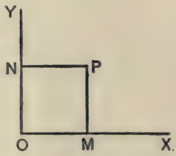
2. The qualities of natural phenomena become known to us through our senses. Certain of these qualities are assumed to be fundamental, in the sense that no one can be expressed in terms of the others. They are incapable of definition and include space, matter, and time. Closely related to these fundamental ideas are the ideas of motion and force.

But although it is not possible to define these qualities, we may consider their mutual relations. In order to investigate these relations it is necessary to compare by measurement the quantities that enter. The science which treats of the relations of matter, motion, and force and of their measurement is called **Mechanics.**

3. The term **body** is applied to a limited portion of matter. Bodies are said to occupy different positions relative to neighboring bodies. We define the position of any point in a body by reference to points in some other body chosen as points of reference.

Thus the position of a point P relative to a chosen point O in the same plane is defined by either (1) the distance OP





and the angle POX made by OP with any known line OX in the plane,—an east and west line, for example,—or (2) the distances PM, PN from two perpendicular lines OX, OY in the plane.

In the first case the distance OP and the angle POX are the polar co-ordinates of P; in the second, PM and PN or OM are the rectangular co-ordinates of P.

4. When a body is changing its position it is said to be in motion. The line drawn through the successive positions occupied forms the **path** of the moving body.

Now as we contemplate the body moving in its path, questions arise as to the influence of the body itself on the motion. We may, however, consider the motion only, apart from the body moving, and study the nature of the path traced out as the body moves from one position to another. Of course no such separation exists. It is a mere abstraction introduced to reduce questions of motion to a purely mathematical form and to serve as an introduction to the more complex problem itself.

This science which investigates motion without considering the nature of the body moved or how the motion is produced is called **Phoronomics** [= law of going] or, more commonly, but less properly, **Kinematics.**

Since in changing position a certain time is taken, the elements of a motion may be said to be distance, direction, and time. Kinematics, therefore, deals with distance, direction, and time, and may be regarded as an extension of geometry by the introduction of the idea of time. Like geometry, it is a purely abstract science resting upon certain ideal assumptions.

The term Kinematics [cinématique] was first proposed by Ampère [1775–1836]. The term Phoronomics (*φορέω*) expresses the idea of mere motion, Kinematics (*κινέω*) involves the idea of the cause of motion.

The tendency at present is rather to restrict the term Kinematics to the geometry of machine parts.

5. When we consider not only the motion but the body moving as well, we pass from kinematics to dynamics (*δύναμις*), the science of force. If a body at rest or in motion has its condition of rest or motion changed, it is usual to say that the change is produced by the action of force. If the form of the body is changed—as in bending a spring—we say that the change of form is due to the operation of force. In the popular sense a force is a *push* or a *pull.* The idea of force seems to be derived from a sense of resistance offered to the use of our muscles, and to muscular effort, or to anything producing like effects we give the name force. The science which treats of the different effects of force on bodies is called **Dynamics.**

6. Each of the two effects of force—change of motion and change of form—furnishes a means of measuring force. Of the two the former mentioned is the more elementary and will be considered first. Forces causing change of size or of shape, or *strain* as it is called, will be discussed in Chapter VIII.

Mechanics is thus divided into Kinematics and Dynamics. For purposes of study each of these divisions has various subdivisions, depending upon the circumstances of the motion, the nature of the body moved or of the forces acting, and which will appear as we proceed.

7. Kinematical Units.—There can be no exact knowledge in physical science without measurement. In order to measure any quantity we must choose some definite quantity of the same kind as standard unit of measurement. Natural standards first suggest themselves, as the palm, foot, span, quadrant of the earth, etc., some variable and others whose values are approached as our methods and instruments are improved. But a standard should as far as possible have a constant value. Hence at present, artificial standards are in general use; their invariability being as certain as that of any natural standard, they can be chosen of dimensions convenient for the purpose in hand and any number of copies can be made with the greatest precision. (See Chapter IX.)

In kinematics the fundamental or independent units are those of *length* or distance and *time*. All other units can be expressed in terms of these two.

(*a*) The standards of length differ in different countries. Two systems of units are in use in Great Britain and the United States, the British and the metric. The British, being the system of every-day life, will be explained first; the metric is explained in Chapter IX. All formulas will be so expressed as to be applicable to any system of units.

The British standard unit of length is the imperial standard **yard**, which is "the straight line or distance between the centres of the transverse lines in the two gold plugs or pins in the bronze bar declared to be the imperial standard" when the bar is at the temperature 62° F.* This bronze bar is deposited in the standards department of the Board of

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* The Weights and Measures Act 1878, 41 & 42 Vict. c. 49.

Trade, London. One third part of the imperial yard is a **foot**, and one twelfth of a foot is an **inch.**

The Troughton 82-inch brass scale obtained by Mr. Hassler for the Coast Survey in 1814 was formerly accepted as a standard of length of the United States, the distance between the twenty-seventh and sixty-third divisions being at 62° F., the standard yard. Direct comparison showed that this distance was equal to the imperial yard at 59.62° F., instead of at 62° F. By reason of its faulty construction and the inferiority of its graduation the Troughton scale "is entirely unsuitable for a standard, and for a long time it has been of historic interest only."

In 1866 the metric system of measures was made lawful throughout the United States, and the yard as known in the Office of Weights and Measures at Washington is defined by the relation

$$1 \text{ yard} = \frac{3600}{3937} \text{ meter},$$

this being the ratio legalized by Congress.

In the absence of any material normal standard of the yard the value of the yard is derived from the standard of the meter, in accordance with the above relation. (Arts. 65, 280.)

(*b*) The standard unit of time throughout the world is the **mean solar day**, which is the average of the intervals between successive transits of the sun's centre across the same meridian. Familiarly it is the time given by two revolutions of the hour-hand of a common clock. The one-twenty-fourth part of a day is an **hour**, the one-sixtieth part of an hour a **minute**, and the one-sixtieth part of a minute a **second.** (See Art. 48.)

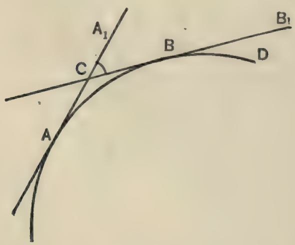
CHAPTER I.

KINEMATICS—MOTION.

8. One body is said to be in motion relative to another body when it changes its position with respect to that other. Change of position implies change of distance or of direction or of both distance and direction. Also in this displacement a certain time is taken, so that the elements of motion are distance, direction, and time. Although we cannot assert that any body in the universe is at rest absolutely, yet it is in most cases sufficient to consider motion referred to some body assumed as fixed. Thus the motion of a railroad train may be referred to the roadbed and depots as fixed, though they all have the motion of the earth.

The more general case will be discussed later on (Art. 42).

9. When all points of a body describe paths equal in magnitude and parallel in direction, the motion is said to be a motion of translation. The path of the body is therefore determined when the path of any point of it is determined.



This conception of dealing with a point greatly simplifies the treatment of problems of translation.

If a point proceeds continually in the same direction, the path is a straight line; if the direction is continually changing, the path is a curve. The direction of motion at any point A of a curvilinear path ABD being the line joining that point to the consecutive point in the path, will be in the direction of the tangent AA_1 to the curve at A. Similarly, at B the direction of motion is along the tangent BB_1, so that in moving from A to B the direction of motion has changed from AA_1 to BB_1, or through the angle A_1CB_1.

10. Velocity.—When a point changes its position, displacement takes place along some continuous path and occupies a certain time.

The rate at which a moving point changes its position is called its **velocity.** Velocity is thus rate of growth of distance.

If the point moves so as to pass over equal distances in equal intervals of time it is said to have a *constant speed.* If the direction also is constant, the point is said to move with *constant velocity.*

The extremity of the minute-hand of a clock moves in a circular path over equal distances in equal intervals, but its direction is continually changing. Its speed is therefore constant, but its velocity is not.

The term speed thus denotes the magnitude of a velocity. However, the term velocity itself is ordinarily used in the sense of speed as well as in the strict sense of speed and direction. In fact in the great majority of cases the direction is assumed to be known, and the magnitude of the velocity is the important thing.

11. Measure of Constant Velocity.—Velocity, if constant, is measured by the number of units of distance described in the unit of time. Thus if a distance s ft were described in t sec, the velocity v would be expressed

$$v = s/t,$$

giving the number of feet described in one second.

To express the measure of a velocity it is necessary to assume a unit of velocity. Since when $s = 1$ and $t = 1$ we have $v = 1$, we must take for *unit of velocity* the velocity of a point which describes unit distance in unit time. The above velocity, which describes v ft in 1 sec, contains v units and may be written

$$v \text{ ft per sec} \quad \text{or} \quad v \text{ ft/sec.}$$

Its numerical measure is v or s/t.

The unit of velocity of 1 ft per sec being expressed in terms of the fundamental kinematical units of length and time, is a *derived unit*. It has no single name generally adopted. The names *vel, velo, fas*, etc., have been proposed.

12. Distance passed over.—If s ft denote the distance passed over in t sec by a point moving with constant velocity, then s/t is the distance passed over in one second and the velocity v is found from

$$v = s/t.$$

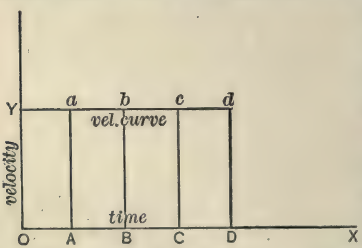
Conversely, if v were known we should determine s from the relation

$$s = vt.$$

This is otherwise evident. For if the point has a constant velocity v ft/sec, it will describe v ft in 1 sec, $2v$ ft in 2 sec, and so on. Hence if s ft is the distance described in t sec,

$$s = vt \text{ ft.}$$

13. Graphical Representation.—A velocity having magnitude and direction as its components may be appropriately represented by a straight line whose length on an assigned scale will show the magnitude and whose direction as indicated by an arrow will show the direction of the velocity.



We may hence give a geometrical representation of uniform motion. Along a straight line OX lay off equal distances OA, AB, . . . to any convenient scale (as 10 sec = 1 in) for as many seconds as the motion has continued. Let the velocities at the points $O, A, B, \ldots$ be represented by $OY, Aa, \ldots$ to any scale (as 10 ft = 1 in). Now since the velocity is constant, the lines $OY, Aa, \ldots$ are equal to one another, and the *curve of velocity* Yd is a straight line parallel to OX, the time line. Also

$$\begin{aligned}\text{Distance passed over} &= OY \times OD\\ &= \text{area } YD.\end{aligned}$$

That is, the number of feet in the distance described would be represented by the number of square feet in the area of the rectangle YD.

Ex. 1. A velocity of 60 miles per hour is 88 feet per second. For

$$v = \frac{60 \text{ m}}{1 \text{ h}} = \frac{60 \times 5280 \text{ ft}}{60 \times 60 \text{ sec}} = 88 \frac{\text{ft}}{\text{sec}} = 88 \text{ ft/sec}.$$

2. In a thunder-storm the clap was heard 6 seconds after the flash was seen. Find the distance of the discharge, the velocity of sound being 1100 ft/sec. *Ans.* 1.25 miles.

3. A passenger sitting in a railroad-car counts 45 telegraph poles (distant 100 ft) passed in one minute: show that the train is running at 50 miles an hour.

4. The minute-hand of a clock is 7 in long: find the linear velocity of its extremity. *Ans.* 11/900 in/sec.

5. Compare the velocities of two points one of which passes over a ft in b sec, and the other b ft in a sec. *Ans.* $a^2 : b^2$.

6. A *knot* being a sea-mile per hour, find how far apart the knots on the log line of a vessel must be placed so that the number of knots which pass over the taffrail in half a minute may give the speed of the vessel in *knots*. *Ans.* 50 ft 8 in.

[The value of the nautical mile adopted by the U. S. Coast and Geodetic Survey is 6080.27 ft; by the English Hydrographic Office, 6080 ft.]

7. A man a ft in height walks along a level street at the rate of c miles an hour in a straight line from an electric light b ft in height: find the velocity of the end of his shadow.

Ans. $bc/(b-a)$ miles/hour.

8. The diameter of the earth being 8000 miles, show that the velocity of a body at the equator due to the earth's rotation is about 17.5 miles/minute.

9. Plot a velocity of 60 miles an hour on a scale of 11 ft/sec = 1 in.

[For plotting it is convenient to use cross-section paper.]

10. AB, AC are any two straight lines. Two bodies P, Q move along these lines with uniform velocities u, v. Find the path of a body R which lies at the middle point of the line PQ.

14. Variable Velocity.—If a point in motion does not pass over equal distances in equal intervals of time the velocity is said to be *variable.*

The expression s/t will then represent the *average velocity* with which s ft are described in t sec. Thus a train in passing over 150 miles in 3 hours would have an average velocity of 50 miles an hour, though its actual velocity at any instant might often differ from this.

The actual velocity at any instant would be determined by finding the limiting value of the average velocity for an indefinitely small distance Δs described during an indefinitely small time Δt, including the instant. Thus

$$v = \text{limit}\ \frac{\Delta s}{\Delta t} = \frac{ds}{dt}.$$

This being the general expression for the velocity at a point includes the case of constant velocity. For

$$ds = v\,dt,$$

and the total distance passed over in the time t is found by summing the distances ds, that is by the integration (= summation) of the expression $v\,dt$ between the proper limits. We have, if v is constant,

$$s = v\int dt$$
$$= vt + c,$$

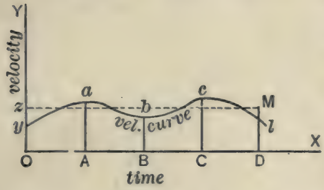
where c is the constant of integration.

If the time is measured from the starting-point of the motion, or $t = 0$ when $s = 0$, then $c = 0$, and

$$s = vt$$

as found above.

15. Variable motion may be illustrated geometrically. For the velocities Oy, Aa, Bb, . . . at the times 0, OA, OB, . . . if plotted and their extremities y, a, b . . . joined would form a line yab . . . not parallel to the time line OX, as in the case of uniform motion. This line would be the velocity curve. The points A, B, . . . may be conceived to be taken so close together that the velocities between may be considered uniform and the figures Oa, Ab, . . . formed to be rectangles. Each rectangle represents a distance passed over, and therefore the total distance would be represented by the sum of these rectangles, that is, by the area Ol.

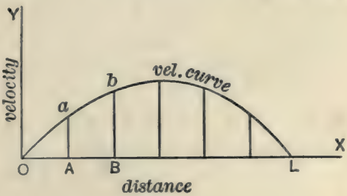


Conceive now a point to start from O, and moving with uniform velocity Oz, to describe the same distance s in the time t as the point moving with the variable velocity. The distance would be represented by the rectangle OM. But with the variable velocities it is represented by the figure Ol. Hence the rectangle OM is equal to the figure Ol, which can happen only when Oz is the *average* (mean) of the values Oy, Aa, Bb, . . . Hence, if we can find the average velocity v, or the velocity of a point which, moving uniformly, passes over the same distance in the same time t as the point moving with variable velocity, we can find the distance s described, from the equation

$$s = vt,$$

and conversely.

16. A velocity curve may also be constructed by laying off OA, OB, . . . along the line OX to represent the distances passed over, and Aa, Bb, . . . at right angles to OX to represent the corresponding velocities. The line through the points a, b, . . . would represent the curve of velocities.



A familiar illustration is afforded by the motion of the piston of a steam-engine. At the beginning and end of its stroke the velocity is zero, at the middle of the stroke the velocity is greatest, and it varies from this value to the end values. The curve of velocity, if plotted, is found to be of a form such as *Oab . . . L* in the figure.

Ex. 1. In September 1895 the Empire State Express made the trip from New York to East Buffalo, 436½ miles, in 407 minutes: find the average speed.

Ans. 64.35 miles/hour, or 94.4 ft/sec.

2. A train runs 29 miles for 2 hours, 30 miles for 3 hours, and 32 miles for 1 hour: find its average velocity.

Ans. 44 ft/sec.

3. An engine makes 100 revolutions a minute. The stroke is 2 ft: find the average piston speed. *Ans.* 400 ft/min.

[Piston speed is usually stated in feet per minute.]

4. A speed of 16.5 knots is equivalent to a speed of 19 miles an hour.

5. In October 1894 the S.S. Lucania made the trip from Queenstown to Sandy Hook Lightship, 2779 nautical miles, in 5 d 7 h 28 min: find the average speed in knots.

Ans. 21.80 knots.

6. Two trains running on parallel tracks pass at a certain place in 6¼ seconds. If each train has the same velocity, and consists of 8 coaches of 52 ft 9 in in length, find the rate per hour. *Ans.* 46 miles/hour.

7. In one of the shafts of the Comstock lode, Nevada, a 36-inch water-wheel is run under a head of 2100 ft at 1150 revolutions per minute: find the peripheral velocity in ft/min. ($\pi = 3\frac{1}{7}$) *Ans.* 10842.9 ft/min.

8. The tunnel of the Cataract Construction Co. at Niagara Falls is 6700 ft long. A chip thrown into the water at the wheel-pit will pass out of the portal in 3.5 minutes. Show that the water has a velocity of about 21 miles an hour.

9. The law of velocity being $v = at + b$, to find the law of distance, distance being reckoned from the initial position.

Ans. $s = at^2/2 + bt$.

10. The law of velocity being $v = c\sqrt{s}$, to find the law of distance. *Ans.* $s = c^2t^2/4$.

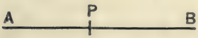
11. A point describes a diameter ($2r$) of a circle with a velocity proportional to the corresponding ordinate, to find the law of distance from the center. *Ans.* $s = r \sin ct$.

r cos ct

17. Composition of Velocities.—A body in motion has one definite path and a definite velocity at each point of its path. But this velocity may be due to several velocities separately communicated to it. Thus a rowboat crosses a stream with a velocity made up of the velocity of rowing and the velocity of the current.

To the single velocity to which the path is due the name of **resultant** velocity is given, and to the separate velocities the name of **component** velocities.

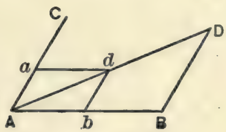
(*a*) *Velocities in the Same Line.*—Suppose a point P in motion along a straight line AB, as a ring sliding along a rod, and that at the same time the rod is moved in the same straight line AB.



The point thus receives two simultaneous velocities and has a single velocity equal to the sum of the separate velocities of ring and rod if in the same direction, and equal to their difference if in the opposite direction.

Ex. A balloon rises with a velocity of 15 ft/sec for one minute, it then descends at the rate of 5 ft/sec for 10 sec and 10 ft/sec for one minute: how far is it now above ground? *Ans.* 250 ft.

(*b*) *Velocities Not in the Same Line.*—Suppose the point P has two simultaneous velocities u, v in fixed directions AC, AB not in the same line. For example, if a ring slide along a rod AC with velocity u, and at the same time AC moves parallel to itself with velocity v along AB.



At the beginning of the motion let the ring be at A. After t sec the rod has moved to BD and the ring has moved a distance BD along the rod and is found at D. Then

$$BD = ut, \quad AB = vt,$$

and therefore

$$BD/AB = u/v.$$

But u and v are constant. Hence the ratio of BD to AB is constant and the ring describes a straight line AD passing through A.

At the end of the first second let BD be in the position bd, then $Ab = u$, $bd = v$, and the ring will be at d on the line AD. Also

$$AD/Ad = BD/bd = ut/u = t/1,$$

or the distances passed over by the ring are proportional to the times. Hence the distance passed over in one second, or the resultant velocity of the ring, is constant.

Again, since Ad represents the resultant velocity, if we draw da parallel to bA, then Aa is equal to bd and $Abda$ is a parallelogram with sides $Aa = u$, $Ab = v$. Hence

If a point possess two simultaneous velocities represented by two straight lines Aa, Ab, their resultant is represented in magnitude and direction by the diagonal Ad of the parallelogram constructed on Aa, Ab as adjacent sides.

This proposition is known as the **parallelogram of velocities.**

18. Motion in a curvilinear path is a result of the parallelogram law on the assumption that each of the elements of the path is rectilinear. (See Art. 9.) The direction at any point of the path is along the tangent at that point.

Ex. 1. Show that the resultant of two velocities OA, OB is represented by $2OD$ when D is the middle point of AB.

2. $ABCD$ is a square and O is the middle point of BC. Find the resultant of the velocities AB, AO, AC.

Ans. $3AO$.

3. Velocities of 8 ft/sec and 10 ft/sec are impressed upon a particle. Find the greatest and least values of their resultant. *Ans.* 18 ft/sec; 2 ft/sec.

4. Show that the resultant velocity diminishes as the angle between the directions of the two component velocities increases.

5. The velocity along AB is 9 ft/sec and along AC 12 ft/sec. If the angle $BAC = 90°$, find the resultant velocity.

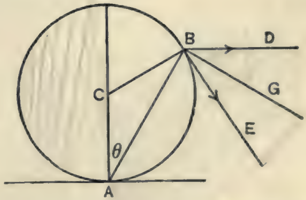
[Draw AB, AC at right angles. Plot on a scale of 12 ft =

1 in. Then $AB = \frac{3}{4}$ in, $AC = 1$ in. Complete the parallelogram $ABCD$. Scale off AD. It measures $1\frac{1}{4}$ in. Hence resultant velocity $(= 1\frac{1}{4} \times 12) = 15$ ft/sec.]

6. A particle has a velocity of 5 ft/sec and an equal velocity at an angle of 120° is communicated to it. Show that the final velocity is 5 ft/sec and that we have an example of change of direction without change of speed.

7. A ball moving with a velocity of 15 ft/sec is struck so as to move off at right angles to its path with a velocity of 20 ft/sec: find the velocity given to it. *Ans.* 25 ft/sec.

8. Show that the direction of motion of any point B on the rim of a wheel running with velocity v on a straight track is perpendicular to AB when A is the point of the wheel in contact with the track at the instant considered.



[For B has two velocities each $= v$, one along the tangent BE and the other along BD parallel to the path of C. The resultant BG bisects the angle EBD. $\therefore ABG = 90°$.

The solution also follows at once from Art. 222.

The path of B is a cycloid. The line BG is a tangent to the path of B at the point B. Hence we have a simple method of drawing a tangent to a cycloid at a given point—a method first suggested by Boscovich and Roberval.

9. Find the magnitude of the resultant velocity in (8) when $\theta = 60°$. *Ans.* v.

10. Draw a tangent to an ellipse at a point P by the method of Roberval.

[The tangent bisects the angle between the focal distances.]

19. It is often convenient to find the resultant by *computing* the diagonal of the parallelogram instead of finding it by a geometrical construction. Thus in Ex. 5 preceding, since $BD = AC$, we find AD by computing the hypotenuse of the right triangle ABD. We have

$$AD^2 = AB^2 + BD^2,$$

or $R^2 = 12^2 + 9^2$

and $R = 15$, as before.

Ex. 1. Solve Ex. 6, 7 Art. 18 by computation.

2. Two velocities of 3 ft/sec, 4 ft/sec are inclined at an angle of 60°. Find their resultant. *Ans.* $\sqrt{37}$ ft/sec.

3. Two velocities u, v are inclined at 120°. Find their resultant. *Ans.* $\sqrt{u^2 - uv + v^2}$.

Show that the resultant makes with v an angle

$$\tan^{-1} u\sqrt{3}/(2v - u).$$

4. Two velocities 15 ft/sec and 36 ft/sec have a resultant of 39 ft/sec. Find the angle between them. *Ans.* 90°.

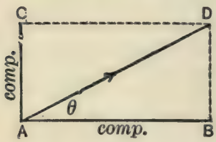
5. If velocities u, v are inclined at 120° and the resultant is inclined to u at 90°, show that $v = 2u$.

6. A point has velocities 3, 3, 5 inclined at 120° to each other. Find their resultant. *Ans.* 2.

What is its direction?

7. Two velocities u, v are inclined at an angle $\theta°$. Show that resultant velocity $= \sqrt{u^2 + 2uv \cos \theta + v^2}$.

20. Resolution of Velocities.—Conversely, a velocity v represented by AD may be broken up or *resolved* into two components AB, AC, being the adjacent sides of the parallelogram constructed on AD as diagonal. This may be done in an indefinitely great number of ways, as an indefinitely great number of parallelograms may be constructed on the same diagonal. Other conditions must be added to render the problem determinate.



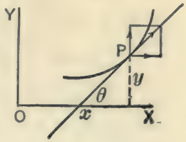
Suppose (1) that the components of AD are to be at right angles and the angle BAD ($= \theta$) is given. Then CAD is known, and the components AB, AC can be plotted. They are thus determined *graphically*.

Since $BD = AC$, it is evident that the magnitudes of the components of AD could be represented by the two sides AB, BD of the triangle ABD. Their values may be found by solving the triangle ABD *trigonometrically*. Thus

$$AB = AD \cos \theta = v \cos \theta,$$
$$AC = AD \sin \theta = v \sin \theta.$$

These are the *rectangular components* of the velocity v.

In general, if a particle is at the point P, whose co-ordinates are x, y at the time t, and if ds/dt is the velocity of the particle in its path and dx/dt, dy/dt the velocities of the particle parallel to the axes of X and Y respectively, and θ the angle which the direction of motion, that is, the tangent at x, y, makes with the axis of X, then, since



$$\cos\theta = \frac{dx}{ds} = \frac{dx}{dt} \bigg/ \frac{ds}{dt},$$

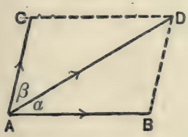
we have
$$\frac{dx}{dt} = \frac{ds}{dt}\cos\theta.$$

Similarly,
$$\frac{dy}{dt} = \frac{ds}{dt}\sin\theta.$$

These are the rectangular components of the velocity ds/dt.

Also,
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2.$$

(2) Let the components AB, AC of AD be not at right angles.



Denote angle BAD by α and CAD by β. Then

$$AB/AD = \sin ADB/\sin ABD$$
$$= \sin\beta/\sin(\alpha+\beta),$$

or
$$AB = v\sin\beta/\sin(\alpha+\beta).$$

Also,
$$AC = v\sin\alpha/\sin(\alpha+\beta).$$

Hence the components of a velocity v in two directions, making given angles with it, are found.

Ex. 1. A ship is sailing N. 30° E. at 8 miles an hour. Find its easterly velocity and its northerly velocity.

Ans. 4 m, $4\sqrt{3}$ m an hour.

2. Find the vertical velocity of a train when moving up a 1% grade at 30 m an hour. *Ans.* 0.3 m an hour.

[A 1% grade is a rise of 1 ft in a horizontal distance of 100 ft. This is the engineering rule.]

4. Show that the components of a velocity v in two directions, making angles of 30°, 60° with it on opposite sides, are $v\sqrt{3}/2$, $v/2$.

21. Acceleration.—In variable motion the velocity changes from point to point of the path. The rate of change of velocity is called **acceleration.**

If a point moves in a straight line with uniform velocity, there is no change of velocity, and therefore no acceleration. If the velocity changes uniformly with the time [grows], the acceleration is *constant;* if it does not change uniformly with the time, it is *variable.*

22. *Measure of Acceleration.*—Acceleration, if constant, is measured by the number of units of velocity added per second.

The unit of acceleration is the acceleration of a point moving with constant velocity which has its velocity changed by the unit of velocity per second. Now unit velocity being unit distance per second, unit acceleration must be unit distance-per-second per second. Thus if an engine 3 minutes after starting has a velocity of 30 ft per sec,

the total change of velocity is 30 ft per sec;
the rate of change of velocity is 30 ft per sec in 3 min
or $\frac{1}{6}$ ft per sec in 1 sec;
and the acceleration is said to be $\frac{1}{6}$ ft per sec per sec.

In general, if a point moves with a velocity that changes uniformly so as to change v ft per sec in t sec, then v/t is the change of velocity in one second, and the acceleration a would be expressed

$$a = v/t \text{ ft per sec per sec.}$$

The *unit acceleration* of 1 ft per sec per sec has no single name in general use; the names *accel, cel, celo, fasp,* etc., have been proposed. We shall use the abbreviation ft/sec² for ft per sec per sec.

Like the unit velocity, the unit acceleration is a *derived* unit.

23. The nature of an acceleration may be indicated by the signs + and −, as the change of velocity is in the same sense or in the opposite sense to that of the original velocity. Hence an acceleration is + if the velocity increases algebraically, and − if it decreases algebraically. To a negative acceleration the name *retardation* is sometimes given.

24. *Motion under Constant Acceleration.*—A point moves in a straight line with a constant acceleration a. If u denotes its initial velocity, it is required to find its velocity v at the end of a time t, and the distance s passed over in this time.

The change of velocity in the time t being $v - u$, the rate of change is $(v - u)/t$. Hence

$$a = (v - u)/t,$$

$$\text{and } \therefore \quad v = u + at, \quad \ldots \ldots \quad (1)$$

giving the velocity at the end of the time t.

Also the distance s passed over is (Art. 15) equal to the product of the average velocity by the time t. But the growth of velocity being constant from beginning to end of the motion, the average velocity is equal to that of the point half-way, that is

$$\text{average vel} = \tfrac{1}{2}\,(\text{initial vel.} + \text{final vel.})$$
$$= \tfrac{1}{2}\,(u + v).$$

Hence $$s = \tfrac{1}{2}\,(u + v) \times t; \quad \ldots \ldots \quad (2)$$

or, putting $v = u + at$,

we have $$s = ut + \tfrac{1}{2}at^2, \quad \ldots \ldots \quad (3)$$

giving the distance passed over in the time t.

The two equations (1) and (3) contain relations between the quantities involved which are independent of one another. Other relations may be deduced from them which are convenient, but which contain no new principle. Thus, eliminating t, we find

$$\begin{aligned} v^2 &= u^2 + 2uat + a^2t^2 && \text{from (1)} \\ &= u^2 + 2a\,(ut + \tfrac{1}{2}at^2) \\ &= u^2 + 2as && \text{from (3)} \end{aligned}$$

$$\text{or } v = \sqrt{u^2 + 2as} \quad . \; . \; . \; . \; . \; . \; . \quad (4)$$

which may be compared with (1)

It is sometimes useful to write (4) in the form

$$v^2/2 - u^2/2 = as \quad . \; . \; . \; . \; . \; . \; . \quad (5)$$

25. If the point had started from rest, then $u = 0$, and the equations become

$$v = at, \qquad \tfrac{1}{2}v^2 = as,$$
$$s = \tfrac{1}{2}at^2, \qquad s = \tfrac{1}{2}vt.$$

These results may be deduced independently. For if the point has a constant acceleration a ft/sec², there will be added a ft/sec to the velocity in each second of the motion. In t sec there will be added at ft/sec. Hence, if v denotes the velocity acquired from rest in t sec, v will be equal to this gain, or

$$v = at. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

$$\begin{aligned} \text{Also} \qquad s &= \text{average velocity} \times \text{time} \\ &= \tfrac{1}{2}v \times t \\ &= \tfrac{1}{2}at \times t \\ &= \tfrac{1}{2}at^2, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2) \end{aligned}$$

giving the distance passed over.

Eliminating t between (1) and (2),

$$\tfrac{1}{2}v^2 = as, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

as before.

26. *Variable Acceleration.*—If the change of velocity is variable, then

$$a = (v - u)/t$$

would give the *average* acceleration with which the velocity changed from u to v in time t.

The actual acceleration at any instant is the limiting value of the average acceleration for an indefinitely small change of velocity Δv during an indefinitely small time Δt, including the instant. We have

$$a = \text{limit}\ \frac{\Delta v}{\Delta t} = \frac{dv}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This may be put in two other forms,* both of which are of frequent application. Thus

$$a = \frac{dv}{dt} = \frac{ds}{dt}\frac{dv}{ds} = v\frac{dv}{ds}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{or} \qquad a = \frac{dv}{dt} = \frac{d\left(\frac{ds}{dt}\right)}{dt} = \frac{d^2s}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

27. The formulas found in Art. 24 follow at once from these equations. Thus (a) let a particle start from a point O with a velocity u: it is required to find its velocity v and distance OP from O at the end of a time t, the acceleration a being constant. Let $OP = s$; then

$$\frac{d^2s}{dt^2} = a.$$

Integrating,

$$\frac{ds}{dt} = at + c,$$

c being the constant of integration. But when $t = 0$, ds/dt or $v = u$; hence $c = u$, and

$$\frac{ds}{dt} = at + u,$$

which gives the velocity at the end of the time t.

Integrating a second time,

$$s = \tfrac{1}{2}at^2 + ut,$$

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* First published by Varignon in 1700.

since when $t = 0$, $s = 0$, and therefore the constant of integration is 0.

(*b*) Let the particle start from O with velocity u, and under a constant acceleration a: required to find its velocity v after passing over a distance s. Here

$$v\frac{dv}{ds} = a.$$

Integrating,

$$v^2/2 = as + c.$$

But when $s = 0$, $v = u$;

$$\therefore \quad v^2/2 = as + u^2/2,$$

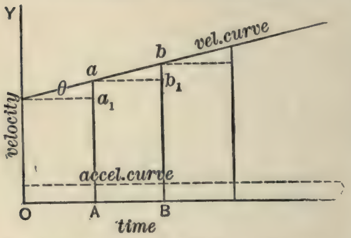
or

$$v^2/2 - u^2/2 = as,$$

as found before.

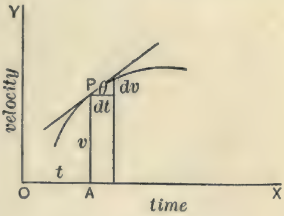
28. *Graphical Representation.*—We have seen how to construct the curve of velocity of a point in motion, either by taking the times as abscissas or the distances as abscissas. We proceed now to show how to construct the curve of acceleration when the curve of velocity has been plotted.

Take the case of a motion in which the times are plotted as abscissas. Let the velocity change uniformly from u to v in the time t. The rate of change of velocity being constant, the velocity curve ab . . . becomes a straight line. The acceleration $a=(v-u)/t$ is represented in the figure by $\tan\theta$, that is, by the tangent of inclination of the line ab . . . to OX.



Hence, if the distances OA, AB, . . . represent one second, the accelerations measured on the velocity scale would be represented from second to second by aa_1, bb_1, . . . all of which are equal to one another. If, therefore, a line be drawn parallel to OX, and at a distance a from it measured on the velocity scale, it will represent the curve of acceleration.

If the rate of velocity is not constant, so that the curve of velocity is curvilinear, then since $a = dv/dt$, the acceleration at a point P (whose co-ordinates are v, t) is represented by $\tan\theta$ when θ is the inclination of the tangent at P to OX. Hence, to plot the acceleration curve, draw tangents to the velocity curve from second to second, and lay off as the ordinates the rise or fall of the tangent measured on the velocity scale. The curve may thus be plotted from point to point.



Ex. 1. In 5 seconds the velocity of a point changes from 200 ft/sec to 100 ft/sec. Find the acceleration.

Ans. $a = -20$ ft/sec².

2. The velocity of a point changes from 20 ft/sec to 10 ft/sec in passing over 75 ft. Find the acceleration and time of motion. *Ans.* $a = -2$ ft/sec², $t = 5$ sec.

Draw a figure illustrating the motion.

3. A point starts from rest. Show that numerically the acceleration, if constant, is equal to twice the distance described in the first second.

4. A point moving with constant acceleration describes 160 ft in the first two seconds of its motion, and 50 ft in the next second. When will it come to rest? When has it a velocity of 20 ft/sec? When of -20 ft/sec?

Ans. 5 sec; 4 sec; 6 sec.

5. A point starts with a velocity u and under a constant acceleration $-a$. Show that it will come to rest in a time u/a, after describing a distance $u^2/2a$.

6. The distances described by a point uniformly accelerated in the first, second, . . . seconds of its motion form an arithmetic progression whose common difference is a.

7. The velocity of a railway train increases uniformly for the first 3 minutes after starting, and during this time the train travels 1 mile. Find the velocity acquired.

Ans. 40 miles/hour.

8. It is observed that a body describes 40, 76, and 112 ft in successive half seconds. Is the motion consistent with uniform acceleration?

9. A train uniformly accelerated takes 5 minutes to run the first mile. How long will it take to run the next mile?

Ans. 2.1 min, nearly.

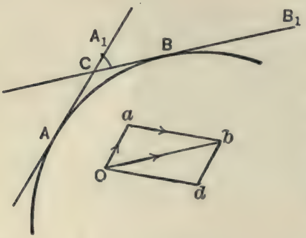
10. In passing between two stations 6 miles apart a train is first constantly accelerated and then equally retarded. The time taken is 12 min. Find the greatest velocity attained.

Ans. 60 miles/hour.

11. In the Westinghouse air-brake trials (1887) on the P. R. R. a train of 50 freight cars running at 36 miles an hour was stopped in 593.5 ft. Find the acceleration of the brake.

Ans. $a = -2.3$ ft/sec^2.

29. *Acceleration in Curvilinear Motion.*—When a point moves in a curve, it must have an acceleration not in the direction of motion. For if the acceleration were in the direction of motion, the path would be a straight line.



Let v_1, v_2 denote the velocities at two points A, B of a curve, and t the time occupied in passing from A to B. The direction of the velocity at A and B is along the tangent at A and B.

From any point O draw Oa parallel to the tangent at A to represent v_1 in magnitude and direction, and Ob parallel to the tangent at B to represent v_2. Complete the parallelogram $Oabd$.

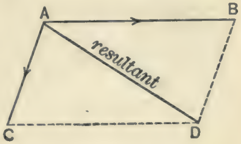
Then the velocities Oa and Od or ab are equivalent to the velocity Ob (Art. 17). Hence ab represents the velocity which must be combined with Oa to produce Ob, and therefore ab is the change of velocity in the time t in both magnitude and direction. The magnitude of the rate of change would be ab/t and its sense is indicated by the direction of ab.

In its most general signification, therefore, acceleration involves change in both magnitude and direction.

If Oa and Ob were in the same straight line, acceleration would be, strictly, rate of change of speed. The term *Quick-*

ening has been suggested for this. But commonly the term acceleration is used in a similar double sense as velocity (Art. 10), and has been so used in the sections preceding.

30. Composition and Resolution of Accelerations.—An acceleration, like a velocity, being a quantity which has magnitude and direction, may be represented by a straight line.



The same method of reasoning as in Art. 17 may be applied to the combination of two accelerations. The statement—called the parallelogram of accelerations—is this:

If a point possess two simultaneous accelerations represented by two straight lines AB, AC, their resultant is represented in magnitude and direction by the diagonal AD of the parallelogram ABDC.

Conversely, an acceleration may be resolved into its components after the manner of Art. 20.

In general an acceleration a at any point x, y of a curvilinear path may be resolved into two components in given directions. The directions usually taken are along the tangent and normal at the point, and in directions parallel to rectangular axes OX, OY. The latter is in general the more convenient.

With the notation of Art. 20 the components parallel to the axes being the rates of increase of dx/dt and of dy/dt would be denoted by d^2x/dt^2 and d^2y/dt^2, respectively. The resultant acceleration a is the sq. root of the sum of the squares of these components.

To find the component acceleration a_t along the tangent at P, we have

$$a_t = \frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta$$

$$= \frac{d^2x}{dt^2}\frac{dx}{ds} + \frac{d^2y}{dt^2}\frac{dy}{ds}$$

$$= \frac{d^2s}{dt^2}\left[\text{by differentiation of } \frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right],$$

or, the tangential component of the acceleration is the same as for rectilinear motion.

The method of resolving parallel to fixed axes the acceleration of a particle and replacing the curvilinear motion by the formulas for rectilinear motion was given by Maclaurin in 1742. This marks a great advance in algebraical mechanics, or, as it is commonly called, analytical mechanics.

31. Notice that although the same law of combination applies to velocities and accelerations, yet a velocity and an acceleration cannot be combined directly. An acceleration continuing for a certain time produces a change of velocity, and this velocity combined with the initial velocity by the parallelogram of velocities gives the final velocity. Or an acceleration continuing for a certain time produces a certain displacement, and this may be combined with the displacement produced by the velocity continued for a certain time. See for illustration Art. 96.

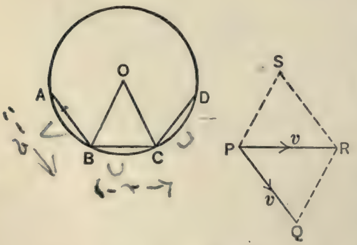
Ex. A particle moves so that the components of its velocity parallel to the co-ordinate axes vary as the corresponding co-ordinates x, y, respectively. Show that the axial accelerations also vary as the co-ordinates.

$$\left[\frac{dx}{dt} = ax; \quad \text{then} \quad \frac{d^2x}{dt^2} = a^2x, \text{ etc.}\right]$$

APPLICATIONS AND ILLUSTRATIONS.

The general principles developed in the preceding pages will now be applied to various important special cases.

32. (1) **Circular Motion.**—When a point describes a circle of radius r with constant velocity v, it experiences a constant acceleration v^2/r directed towards the center of the circle. For suppose a point to move uniformly along a side AB of a regular polygon with velocity v in time t, and that when it arrives at the angle B it receives a velocity u which causes it to move with the same velocity v along BC. Similarly, at C a velocity u which causes it to move along CD, and so on.



From any point P draw PQ parallel to AB and $= v$; PR parallel to BC and $= v$; complete the parallelogram $PQRS$.

Then (Art. 17) PS or QR represents the change of velocity u at B. Also, if O is the center of the polygon, then from the geometry of the figure BO is parallel to QR, or the change of velocity u at B is directed along the radius BO.

Similarly, the change of velocity u at C is directed along the radius CO, and so on.

To find the magnitude of the change of velocity per second: The triangles PQR, OBC are similar.

$$\therefore \quad QR/PQ = BC/OB,$$

or
$$u/v = vt/r,$$

or
$$u/t = v^2/r,$$

giving the rate of change of velocity towards O.

Now when the number of sides is indefinitely increased, the polygon becomes a circle, and the constant velocity v in the polygon becomes constant velocity in the circle.

The changes of velocity which in the polygon took place at the angles, and therefore occurred at intervals, take place from point to point and become continuous in the circle. The rate of change u/t, that is, the acceleration, is equal to the same quantity v^2/r, and is always directed towards the center O, which proves the proposition.

The velocity being constant in magnitude, the acceleration is expended in changing the *direction* of motion. Being always directed towards the center of the circular path, it is said to be *centripetal*.

If T is the time in which the circular path is described, then

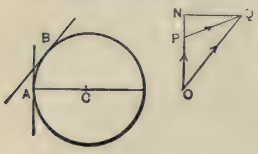
$$vT = 2\pi r.$$

Also,
$$a = v^2/r.$$

Hence
$$a = 4\pi^2 r/T^2.$$

33. *Tangential and Normal Acceleration.*—Consider next the more general case of the acceleration of a point describing a circle in which the velocity in the circumference is not uniform.

Let A, B be two points in the path at an indefinitely small distance Δs apart; Δt the time of moving from A to B; and $\Delta\theta$ the angle between the tangents at A and B.



From any point O draw OP, OQ to represent the velocities v and $v + \Delta v$ at A and B in magnitude and direction. Then PQ, being the velocity which must be compounded with OP to produce OQ, will represent the *change of velocity* between A and B in the time Δt. The limiting value of $PQ/\Delta t$ as B approaches A represents the acceleration at A.

Let fall $QN \perp OP$. Then $PN/\Delta t$, $NQ/\Delta t$ will in the limit represent the accelerations along the tangent AT and the normal AO at A. We have

$$\begin{aligned}
\text{Tangential acceleration} &= \text{lt. } PN/\Delta t \\
&= \text{lt. } \{(v + \Delta v)\cos\Delta\theta - v\}/\Delta t \\
&= dv/dt, \text{ since lt. } \cos\Delta\theta = 1. \\
\text{Normal acceleration} &= \text{lt. } NQ/\Delta t \\
&= \text{lt. } (v + \Delta v)\sin\Delta\theta/\Delta t \\
&= \text{lt. } (v + \Delta v)\frac{\sin\Delta\theta}{\Delta\theta}\cdot\frac{\Delta\theta}{\Delta s}\cdot\frac{\Delta s}{\Delta t} \\
&= v \times 1 \times \frac{1}{r} \times v \\
&= v^2/r.
\end{aligned}$$

See also Arts. 109, 237.

Ex. 1. Find the centripetal acceleration of a point which moves in a circle of 5 ft diameter with a velocity of 5 ft/sec.

Ans. 10 ft/sec^2.

2. Explain how it is that although the particle constantly *gains* velocity along the radius it never *possesses* any such velocity.

["As fast as velocity along the radius is generated, so fast does the direction of the radius change, in the same way that a promise for to-morrow need never be fulfilled because to-morrow never comes."]

3. A man falls overboard from a vessel running at 20 miles an hour. If the masts are 88 ft above the water, find when they will cease to be visible. *Ans.* $20\sqrt{3}$ min.

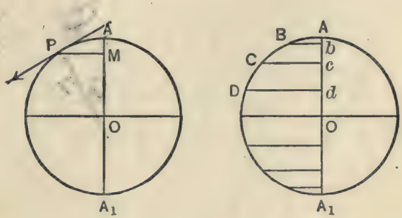
4. The earth if suddenly stopped in its orbit would fall to the sun about one ninth of an inch in the first second.

5. A point describes a circle of radius r so that its projection on a diameter describes the diameter with uniform velocity v: to find the acceleration perpendicular to diameter. *Ans.* $-r^2v^2/y^3$.

6. A point describes a hyperbola $xy = c^2$ with a uniform velocity v: to find the component accelerations parallel to the asymptotes.

Ans. $2c^4v^2/r^4x$, $2c^4v^2/r^4y$ when $r^2 = x^2 + y^2$.

34. (2) **Motion of Oscillation.**—Suppose that while a particle P moves in a circumference of radius r with uniform velocity v, another particle M moves along any diameter AA_1 in such a way that PM is always perpendicular to AA_1; then as P describes the equal arcs AB, BC, . . . M describes the



distances Ab, bc, . . ., until A_1 is reached. As P continues, M returns along the diameter A_1A to A. Thus while P makes a complete revolution, M oscillates back and forth along the diameter AOA_1. The motion of M as it oscillates along the line AOA_1 about O as a centre is called **a simple harmonic motion** (S.H.M.). A motion of oscillation is alternate in direction.

The name harmonic is given to such oscillatory motion because this motion is characteristic of the vibrations of sounding bodies, as tuning-forks, piano-wires, etc.

S.H.M. is the simplest form of harmonic vibration, and is of great importance in mathematical physics.

The time of oscillating from A to A_1 and back to A, being the same as that required for passing once round the circumference in the corresponding circular motion, is $2\pi r/v$, and is called the **period** of the S.H.M. Denote it by T.

The distance OM of the moving particle M at any time from its mean position O is called the **displacement** at that time. Denote it by y.

The distance OA or OB of the extreme position of M from the mean position O is called the **amplitude**. It is the radius of the circle of reference and may be denoted by r.

The fraction of the period of vibration which has elapsed in the passage from A to M is called the **phase** of M at the time considered. It is evidently the ratio of the angle POA to 360° or $\theta/2\pi$ if θ is the circular measure of the angle POA.

The phase of vibration at O is different by 1/4 from that at A, and the phase at A_1 differs by 1/2 from that at A, or A and A_1 are said to be in opposite phases.

35. Let the time be counted from the instant when P is at A. Then if θ denotes the angle POA described in t sec and T the period,

$$t : T = \theta : 2\pi,$$

$$\text{or} \qquad \theta = 2\pi t/T$$

$$= \omega t,$$

where $\omega = 2\pi/T$ is the angle described in one second expressed in circular measure. Hence

$$y = OM = OP \cos\theta = r \cos\omega t,$$

giving the displacement in t seconds.

If from P a perpendicular PN is let fall on OX, then

$$x = ON = r \sin \omega t = r \cos (\omega t - 2\pi/4),$$

or the motion of N is a S.H.M. of the same amplitude and period as M, but differing 1/4 in phase.

36. The acceleration a of M is the component of the acceleration of P in the direction AA_1. But the acceleration of P is along PO, and is equal to v^2/r (Art. 32). Hence

$$\begin{aligned} a_y &= \frac{v^2}{r} \cos POA \\ &= \frac{v^2}{r} \times \frac{y}{r} \\ &= v^2 y / r^2. \end{aligned}$$

Also, since the circular path is described in T seconds with uniform velocity v,

$$vT = 2\pi r.$$

But $$\omega T = 2\pi;$$

$$\therefore \quad v = \omega r,$$

and hence $$a_y = \omega^2 y;$$

or, the circular measure ω of the angle described in one second being constant, the acceleration of M varies as the displacement y from O. (See Art. 101.)

37. Again, since

$$T = 2\pi/\omega,$$

if ω is a constant quantity, it follows that the period of the S.H.M. is independent of the amplitude r. In other words, at whatever distance from O the particle M is started it will return to its starting-point in the same time. This property of oscillating in the same time, whatever the amplitude, is called isochronism, and the vibrations of M are said to be *isochronous*.

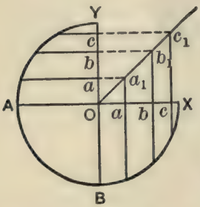
We may write

$$T = 2\pi/\omega$$
$$= 2\pi\sqrt{y/a}$$
$$= 2\pi\sqrt{\frac{\text{displacement at any point}}{\text{acceleration at that point}}},$$

a convenient formula for finding the period.

38. *Composition of S.H.M.'s.*—Simple harmonic motions may be combined into one resultant motion by the parallelogram law as with motions of translation. For example, to find the resultant of two equal S.H.M.'s in directions at right angles to each other and in the same phase.

1. Graphically. Let both motions start from O, one along OX and the other along OY at right angles to OX.



With the equal amplitudes OX, OY as radii describe the semicircles ABX, BAY. The periods being the same, let the semicircles be divided into the same number of equal parts, say eight. The harmonic intervals Oa, ab, . . . along OX, OY are equal.

Since the motions both start from O, the actual position of the particle at the end of the first interval will be at a_1, at the end of the second interval at b_1, and so on. Hence the resultant motion will be along the straight line $Oa_1b_1c_1$ . . . and will be a S.H.M. Since

$$Oa_1 : Ob_1 : \ldots = Oa : Ob : \ldots,$$

the amplitude is evidently $r\sqrt{2}$.

2. Analytically. The equations to the two component S.H.M.'s are

$$x = r \cos \omega t;$$
$$y = r \cos \omega t.$$

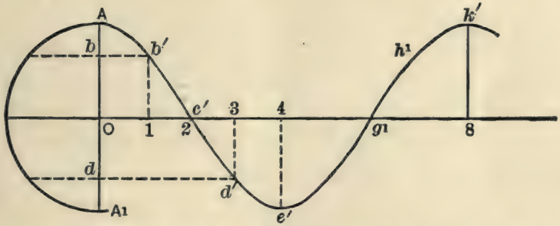
Eliminating t, the equation to the resultant path of the particle is

$$x = y,$$

a straight line inclined at 45° to the axes of X and Y, as before.

39. *Harmonic Curve.*—When a S.H.M. along AA_1 is combined with a uniform motion at right angles to AA_1, the resultant path is called the *harmonic curve.*

To find this curve (1) graphically. Let $O1, 12, \ldots$ represent the distances passed over in equal time-intervals, and $Ab, bo, \ldots$ the distances passed over in the same intervals



by the vibrating particle. At the end of the first interval the particle will be at b', at the end of the second interval at c', and so on. Having reached e' it has made half a vibration, and at k' a complete vibration. The curve $Ab'c'd'e' \ldots k'$ is the harmonic curve.

(2) Analytically. The equation of a S.H.M. along AA_1 is

$$y = r \cos \omega t,$$

and of uniform motion in a straight line

$$s = vt,$$

where v is the velocity of motion and s the displacement in the time t.

To find the locus eliminate t, and

$$\begin{aligned} y &= r \cos \omega s/v \\ &= r \cos 2\pi s/vT \\ &= r \cos 2\pi s/\lambda, \end{aligned}$$

if we place $vT = \lambda$, the distance traveled in the period T in the straight path with velocity v.

Now when

$$
\begin{array}{ll}
s = 0, & y = r, \\
s = \lambda/8, & y = r/\sqrt{2}, \\
s = \lambda/4, & y = 0, \\
s = \lambda/2, & y = -r, \\
s = 3\lambda/4, & y = 0, \\
s = \lambda, & y = r,
\end{array}
$$

and the vibration is completed.

The curve now repeats itself, and the periodicity of the motion of the particle is shown. The time of a period is $2\pi/\omega$, as above.

40. Now instead of the single particle which has the two motions which combined form the harmonic curve conceive particles placed along the line OX extending from O to 8, and made to oscillate along lines perpendicular to OX, with S.H.M.'s differing uniformly in phase. In the figure the S.H.M.'s differ 1/8 in phase. As before, call the period T.

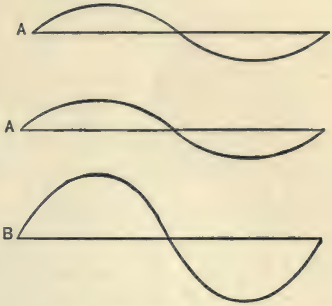
At the end of a certain time the particle 8 is in the highest position k'; the other particles, being times $T/8$, $2T/8$, . . . behind, will be in the positions h', g', . . . shown in the figure. The positions will plainly lie in the harmonic curve.

At the end of the next interval $T/8$, h' will be the highest point, d' the lowest, and at the end of the period, that is, at the end of a complete oscillation, the highest point will have traveled to A.

The form of the curve assumed by the vibrating particles is called a **wave**, and the distance passed over by the wave during a complete oscillation is the **wave-length**, that is, Ak' or its equal $O8$. The wave-length is thus the distance between two points in corresponding positions on the harmonic curve.

If the oscillations had taken place along OX, we should have had *longitudinal* instead of *transverse* waves.

41. Harmonic vibrations in the same line may be combined by first combining each with a perpendicular translation and plotting the harmonic curves. Then, since the motions are in the same straight line, the compound harmonic curve will result from plotting the curve whose ordinates are equal to the algebraic sum of the ordinates of the simple curves.



Thus to compound two equal S.H.M.'s in the same line and having the same amplitude the resultant curve will be a curve of twice the amplitude.

If the curves are of opposite phase, the resultant curve is a straight line, or the motions destroy one another. This illustrates the *interference* of two waves of the same length, but of opposite phase.

Ex. 1. A particle revolving uniformly in a circle will to an eye in the plane of the circle appear to oscillate along a diameter.

[The motion of Jupiter's moons as seen from the earth appears to be approximately a S.H.M.]

2. Show that the upward and downward motion of the connecting-rod of a locomotive is a S.H.M.

3. In the cross-head of an engine a slot is cut perpendicular to the direction of the stroke. If the crank-arm revolving uniformly works with one extremity C in the slot, the motion of C is a S.H.M.

4. If the particle M (Art. 34) be projected on any line at Q, the motion of Q will be a S.H.M. of the same period and phase as M, but of amplitude the projection of the amplitude of M.

[This may be illustrated by viewing the S.H.M. of M in figure on p. 29 obliquely.]

5. Find the average velocity of the point M as it oscillates from A to A_1 and back again to A, if the greatest velocity attained is 1 ft/sec. *Ans.* $2\pi^{-1}$ ft./sec.

6. The velocity of M along the diameter AA_1 is $v \sin \omega t$, the velocity of P in the circle of reference being v.

7. By taking the first and second differentials of the equation $y = r \cos \omega t$ show that

$$\text{vel of } M = -r\omega \sin \omega t = -\omega x;$$
$$\text{accel of } M = -r\omega^2 \cos \omega t = -\omega^2 y;$$

x, y being the co-ordinates of P in the circle of reference.

8. At what point is the velocity of M a maximum, and what is its value? *Ans.* v at the center O.

When is the acceleration greatest, and when least?

9. Combine graphically and analytically two S.H.M.'s in same line and of equal amplitude, period, and phase.

Ans. A S.H.M. of equal period and of twice the amplitude.

10. Compound two S.H.M.'s in the same line of the same amplitude, but differing 1/4 in phase, into a S.H.M. in the same line and of the same period.

$[y = r \cos \omega t + r \cos(\omega t + 2\pi/4) = r\sqrt{2} \cos(\omega t + \pi/4)]$.

11. Find the path of a point which has two equal S.H.M.'s in directions at right angles to each other and differing 1/2 in phase. *Ans.* A straight line.

12. If in (11) the two S.H.M.'s differ 1/4 in phase and also in amplitude, find the path. *Ans.* An ellipse.

13. Uniform circular motion is equivalent to two S.H.M.'s at right angles to each other of equal amplitude and period, but differing 1/4 in phase.

14. A particle has two S.H.M.'s of the same amplitude and phase and in directions at right angles, the periods being as 1 to 2. Find the path. *Ans.* A parabola.

Eliminate t between $x = r \cos \omega t$ and $y = r \cos 2\omega t$. Plot the curve after the manner of Art. 38.

This forms one of a series of curves known as Lissajous' curves. Others may be formed by changing the ratio of the periods.

[For a mechanical method of producing these curves see Art. 118.]

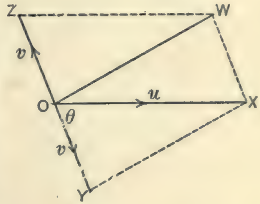
15. If in Ex. 14 the phase differs by 1/4, find the equation of the path. *Ans.* $r^2y^2 = 4x^2(r^2 - x^2)$.

42. Relative Motion.—The motion of a point P has been defined by its change of position with reference to another point O regarded as fixed. This gave the *absolute* motion of P. But if the point O is also in motion, or has an absolute motion with respect to a third point, the motion of P is no longer said to be absolute, but *relative.* This is really the case of bodies in nature, as no point in space is known to be fixed absolutely. Still, for the purpose considered, an assumed point may be regarded as fixed, and in this sense motion is said to be absolute.

The problem of relative motion is really an example of the composition of motions. Suppose, for example, a point A to move with an absolute velocity u, and B with an absolute velocity v, both being referred to the same fixed point O, and in the same straight line; it is required to find the velocity of A relative to B. Conceive both A and B to move in a medium which itself moves with a velocity v, but in the opposite direction to the motion of B. Then B is at rest with reference to the fixed point O. But as the motion of the medium affects A and B alike, their relative motion is unchanged. Hence, as a velocity v has been imparted to A, the velocity of A relative to B will be $u - v$ if both were originally moving in the same direction and $u + v$ if in opposite directions.

As an illustration, take two men A and B walking on a boat's deck from bow to stern, and that the velocity of the boat is equal to the velocity of B. Then B is at rest relative to the shore, and the motion of A relative to B is the same as if the boat were at rest.

43. Consider next when the velocities u, v of the two points A, B are *not* in the same straight line. Suppose the lines OX, OY to represent these velocities in magnitude and direction. Let a velocity v in the direction YO and represented by OZ be imparted to both A and B. The relative motion of A and B is unchanged, and the point B is now at rest. The velocity of A is the resultant of the velocities OZ, OX, that is, is equal to the diagonal OW, which therefore represents the velocity of A relative to B.



The three velocities are represented by the sides of the triangle OXW. Hence, if in a triangle one side OX represent the velocity of A, XW a velocity equal to and opposite that of B, and OX, XW are in the same sense around the triangle, the third side OW taken in the opposite sense around the triangle will represent the velocity of A relative to B.

So, too, if a point X had a velocity XO relative to Z, and at the same time Z had a velocity ZO relative to Y, then evidently XY is the velocity of X relative to Y; that is,

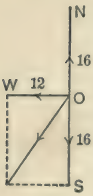
If two sides of a triangle XO, OY taken the same way round represent the velocity of X relative to Z and of Z relative to Y, the third side XY will represent the velocity of X relative to Y.

Corresponding propositions hold for accelerations and for linear displacements.

Any mechanism may be employed to illustrate relative motion by putting a sheet of paper on one of its moving pieces, and a pencil on another moving piece, when the curve traced by the pencil on the paper will represent the relative motion of the two pieces.

Ex. 1. Two trains A and B are running on parallel tracks at 20 and 50 m/h respectively. Find the velocity of A relative to B. *Ans.* -30 m/h or -70 m/h.

2. A steamer is sailing north at 16 miles an hour in an east wind blowing 12 miles an hour. Find the apparent direction of the wind to a passenger on the steamer.



[Let a velocity of 16 miles an hour south be imparted to steamer and wind. The relative motion is unchanged. The steamer is at rest and the velocity of the wind is composed of 12 miles an hour west and 16 miles an hour south. $\therefore$ apparent direction of wind $= \tan^{-1} 12/16$ or $\tan^{-1} 3/4$ east of north.]

3. In Ex. 2 find the direction of the vessel's smoke.

4. Two vessels start at the same time from the same harbor, one sailing east at 12 miles an hour, the other south at 9 miles an hour. Find the velocity of one relative to the other. *Ans.* 15 miles an hour.

5. Two railroad tracks intersect at 60°, and two trains

start at the same instant from the junction at 30 miles an hour each. Find their relative velocity in magnitude and direction.

6. Two bodies A, B move with velocities u, v inclined at an angle θ. Show that the velocity of B with respect to A is $\sqrt{u^2 + v^2 - 2uv \cos \theta}$, and inclined at an angle $\tan^{-1} v \sin \theta/(v \cos \theta - u)$ to the direction of A.

7. Two railroad tracks intersect at 90°. To a passenger in one train traveling at the rate of 32 miles an hour the other seems to have a velocity of 40 miles an hour. Find its absolute velocity. *Ans.* 24 miles an hour.

8. The displacement of A relative to B is s ft south and relative to C it is s ft west. Find the position of C relative to B. *Ans.* $s\sqrt{2}$ S.E.

9. A boat is propelled at 12 miles an hour across a stream flowing at 5 miles an hour in a direction perpendicular to the current. Find the velocity of the boat with reference to the bottom of the channel. *Ans.* 13 m/h.

10. The hour and minute hands of a clock are 6 in and 7 in respectively. Find the relative velocities of their extremities at (1) 6 A.M., (2) 9 A.M., (3) noon, and show that at 3 P.M. the direction of their relative velocity makes an angle $\tan^{-1} 1/14$ with the horizontal.

11. A man traveling eastward at v miles an hour in a wind apparently from the north doubles his speed when the wind appears to blow from the northeast. Show that the wind is really from the northwest and blowing with a velocity of $v\sqrt{2}$ miles an hour.

12. A vessel steams due north at 10 knots in a wind due east. If the vane on the mast is 30° west of south, find the velocity of the wind. *Ans.* 6.65 m/h.

13. Find the linear displacement of the highest point A of a 6-ft locomotive-wheel with reference to the lowest point B while the wheel makes a quarter revolution along a straight track. *Ans.* $6\sqrt{2}$ ft inclined at 45° to the track.

14. Three displacements of a point are parallel to the sides of an equilateral triangle in order and are in magnitude a, $a + 1$, $a + 2$ inches. Show that the magnitude of the resultant displacement is independent of a.

15. Show that the time in which it is possible to cross a road of breadth c ft in a straight line with the least uniform

velocity, between a stream of carriages of breadth b ft following at intervals a ft, moving with velocity v ft/sec, is

$$\frac{c}{v}\left(\frac{a}{b}+\frac{b}{a}\right) \text{ seconds.}$$

16. A point P describes a circle relative to axes through O. Show that relative to parallel axes through P the point O also describes a circle.

Compare the areas of the two circles.

Is a similar proposition true for any curve?

17. Two bicycle racers start with velocities u, u_1, keep up constant accelerations a, a_1, and make a dead heat. Show that the length of the race is

$$\frac{2(u-u_1)(a_1u-au_1)}{(a-a_1)^2}.$$

18. If telegraph-poles are n ft apart, find for how many seconds one must count poles in order that the number counted may equal the number of miles per hour that the train is running, the speed of the train being assumed uniform.

Ans. $15n/22$ seconds.

EXAMINATION.

1. How is the position of a point defined?

2. What ideas are involved in the term displacement?

3. In what sense is a body said to be at rest? in motion?

4. Mention some units of length. What is the British standard of length?

5. Mention some units of time. Mention some natural standards of time besides the day.

6. Define the term velocity, and state how velocity is measured when uniform, and when variable.

7. Define the average velocity of a moving point in any given time.

8. Explain the statement

$$\text{distance} = \text{average velocity} \times \text{time}.$$

9. A knot is, roughly, a velocity of 100 ft/min.

10. Give examples of a body in motion the different parts of which have different velocities.

11. The minute-hand of a clock is twice as long as the hour-hand. Compare the speeds of their extremities.

Ans. 1 : 24.

12. Explain the statement

unit velocity = one foot/one second.

How is velocity represented graphically?

13. Explain the difference between uniform motion and uniformly accelerated motion.

14. Give examples of bodies having accelerations (1) constant in magnitude and direction, (2) constant in magnitude, but not in direction, (3) variable in magnitude and direction.

15. Show that for uniform acceleration from rest

$$s = vt/2 = at^2/2.$$

16. Define acceleration at a given instant of time.

17. Illustrate the meaning of $a = dv/dt$ geometrically.

18. If a body is projected with the velocity u in the direction of a uniform acceleration a, and v be the velocity and s the distance described at the end of the time t, prove

$$(v - u)/a = 2s/(v + u) = t.$$

19. A particle starts with a velocity of 12 ft/sec and the motion is retarded 2 ft/sec². Draw a diagram to find the distance described in 6 seconds.

20. Show how the distance described may be represented graphically when a particle moves with constant acceleration, starting with a velocity u.

21. Show by a diagram that when a particle moves from rest with constant acceleration the distance described is proportional to the square of the time of motion.

22. The distance s described in the nth second of its motion by a particle having an initial velocity u and a uniform acceleration a is given by

$$s = u + (2n - 1)a/2.$$

Compare an acceleration of 1200 yds/min² with an acceleration of 1 ft/sec², the unit of acceleration.

23. Show how two coexistent velocities or accelerations may be combined.

24. Show how to resolve a velocity in two directions at right angles to each other.

25. State the principle of Roberval's method of drawing tangents to a curve.

26. Find the direction and magnitude of the acceleration of a particle which describes a circle of radius r with uniform velocity v.

27. From a point draw lines to represent in magnitude and direction the velocity at the different points of the path of a particle moving uniformly in a circle.

28. Define a simple harmonic motion. Is it a rectilinear motion?

29. A point P moves uniformly in a circle. Show that the velocity and acceleration of the projection M of P on any diameter are proportional to PM and OM respectively, O being the center of the circle.

30. Oscillatory motion is the projection of circular motion upon a diameter.

31. When is motion said to be periodic?

32. The motion of a point along a straight line OX is defined by the equation

$$x = a \cos \omega t,$$

show by finding dx/dt, d^2x/dt^2 that the motion is vibratory and of a period $2\pi/\omega$.

33. Given the velocities of two points A and B to determine the velocity of A relative to B.

34. A person walking rapidly in a vertical rain holds his umbrella towards the front. Explain.

35. Two trains are on parallel tracks. Why is it difficult for a passenger to tell whether his own train or the other is in motion?

36. Explain why the smoke of a steamer is in a parallel plane to the vane on the mast?

37. From a car window a man fires at a buffalo running in

a parallel direction with the train. Show that he must aim in front if his velocity is < that of the buffalo.

38. Find the velocity with which a man must jump backward from a car in motion to fall vertically. ✓

39. Two points move with velocities u and v in opposite directions round a circle. Find their greatest and least relative velocities.

40. Define phoronomics, kinematics.

41. What are the fundamental independent units used in linear kinematics? [Length and time.]

42. Mention some *derived* units.

CHAPTER II.

MATTER IN MOTION. NEWTON'S LAWS OF MOTION.

44. Hitherto we have considered motion in the abstract—how represented and how measured. No reference has been made to the nature of the body moving, and the problem has been as purely ideal as a problem of geometry.

We shall now consider motion with reference to the body moving and the force acting, that is, pass from kinematics to dynamics. This brings us to the region of sense and makes our results capable of verification, or such that we can test computation by observation and measurement. In order to verify results certain fundamental postulates are taken as bases of operation, which will now be stated.

45. The relations of matter, motion, and force, which constitute the science of dynamics, may * be based upon three postulates known as Newton's laws of motion. These laws were known to Galileo and other forerunners of Newton, but were first stated by Newton in concise terms. They are not axiomatic in the sense of a geometrical axiom, because when stated they are not at once assented to. They do not commend themselves to the mind either as true or as false.

In stating Newton's laws certain rude experiments will be indicated which are sufficient to suggest the truth of the laws, but not to establish it. No direct proof is possible. The proof is indirect and is made in this way. Assume the laws true, and certain consequences follow which can be

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* See Art. 217.

tested experimentally. This has been done in so many ways and by so many independent observers, particularly in astronomical work, that we are justified in accepting them as true. For example, the Ephemeris or Nautical Almanac is published several years beforehand, and the predictions made in it and based on these laws are always found to agree with the occurrences when observed. Such, for instance, are the predictions of the times of eclipses of the sun and moon, the positions of the planets, etc.

46. LAW I. *Every body [particle] continues in its state of rest or of uniform motion in a straight line except in so far as it may be compelled by external force to change that state.*

The law lies beyond our experience, as we have no experience of one body not acted upon by another. Our direct experience goes, however, a certain distance in confirmation of the law. Thus, as suggested by Galileo, consider a body placed on a level surface. If at rest it will remain at rest; if in motion it will come to rest after going a distance depending upon the smoothness of the surface. The smoother the surface the farther it goes and the more nearly in a straight line. Conceive a surface perfectly smooth and the air to have no influence on the motion, and we cannot think of any reason why the body should not continue to move uniformly in a straight line.

47. From the law we learn that rest and motion are equally states of a body, the body being wholly without influence on its rest or motion. This property of matter is called **inertia** [the *vis insita* of Newton], and the law itself is often named the law of inertia.

From the law we also learn that by the term forçe is meant a cause of change in motion, not in the sense of moving agent, but in the sense of antecedent. Force is thus not to be regarded as the cause of a state of motion, but of a change of state, from rest to motion, motion to rest, or to an alteration of motion—in a word, of acceleration. Whenever force acts, an acceleration of the motion of the body acted upon is produced.

48. The law guides us in finding a timekeeper. A body in motion and not acted upon by external forces would afford a means of measuring times. For the distances passed over by such a body in equal times are equal.

We know of no permanent motion that is at the same time uniform and rectilinear. The standard motion for the measurement of time is the rotation of the earth on its axis. We *assume* that the earth revolves uniformly or through equal angles in equal times, and find that predictions of astronomical phenomena made on this hypothesis agree closely with subsequent observation.

There is no essential reason why the rotation of some other planet should not be adopted as a standard of uniformity, and the results would not necessarily be the same in the two cases; for there are certain causes (notably the tides) which tend to make our planet rotate more slowly, so that after the lapse of many centuries the period of its rotation may have appreciably increased, and the extent of such retarding influences would be different for different planets.*

49. LAW II. Having learned that a characteristic manifestation of force is acceleration, our next inquiry is as to the relation between force acting, body acted upon, and acceleration produced—in a word, as to how force is measured.

Now it is found that when the same body is exposed to action of the same force it has the same change of motion. Thus the same pull of a spring-balance—equal pulls being measured by equal stretch of spring—gives the same body the same acceleration at all times and places. The same general result is found no matter how the manner of making the experiment is varied.

If two bodies exposed to the action of the same force receive the same acceleration we say that they are of the same **mass**, and if the accelerations are not the same we say that the bodies are of different mass. The term mass is thus applied

* Burton, *Dynamics*, p. 102.

to that physical quality of a body that determines its acceleration. Experiment shows that mass is a definite entity altogether independent of the physical state of the body.

Newton defines "the quantity of any matter as the measure of it by its density and volume conjointly," and states that this quantity is what he shall understand by the term *mass* or *body*.*

50. The quantitative relation between force, mass, and acceleration is given by Newton's second law of motion,

Change of motion is proportional to the impressed force and takes place in the direction of the straight line in which the force is impressed; or, in modern phraseology [see Art. 53 for other statements],

Force [is that which produces acceleration and] *is proportional to the mass m of the body and the acceleration a produced jointly.*

Expressed in symbols, the law gives the relation

$$F = cma,$$

where c is a constant.

In this expression the only unit whose value has been already defined is the unit of acceleration. If one of the two, unit force or unit mass, is assumed, the other is fixed by the equation. It is usual to assume *unit mass* as the mass of a certain piece of metal carefully preserved as standard of reference. (A fuller account of this unit will be given later on.)

Having assumed the unit mass, the unit force may be chosen such that $c = 1$, so that we may write

$$F = ma$$

as the expression of the law.

In this equation when $a = 1$ and $m = 1$ we have $F = 1$, or *unit force* is that force which acting on unit mass produces unit acceleration.

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* Of this definition Mach remarks [*Die Mechanik*, p. 181]: "It is to be observed that the formulation given by Newton which defines mass as the quantity of matter of a body measured by the product of its volume and density, is unfortunate. Since we can only define density as the mass of unit volume, the circle is manifest."

See also Pearson, *Grammar of Science*, p. 358.

51. It follows that with this system of units the equation

$$F = ma$$

means that the force which produces an acceleration a in a body of mass m is expressed by the product ma, called the *mass-acceleration.*

Also, since $a = d^2s/dt^2$ or $a = v\,dv/ds$,

we have $$F = m\frac{d^2s}{dt^2},$$

or $$F = mv\frac{dv}{ds},$$

as the expression of the law.

Any one of these forms is the *general equation of motion* of a particle free to move, the two latter forms being sometimes called the *differential equations of motion.*

The general equation of motion, which is the algebraic statement of the second law of motion, is the connecting link between motion and force. It enables us to pass from the kinematical properties of motion already laid down to questions involving force and mass.

52. Inferences from the law:

(1) If F units of force produce in a mass m an acceleration a, and F_1 units produce in a mass m_1 an acceleration a_1, then

$$F/F_1 = ma/m_1a_1.$$

(a) Now when $m = m_1$, the forces are proportional to the accelerations. Hence

Equal forces are such as produce equal accelerations in equal masses.

(b) When $F = F_1$, the masses are inversely proportional to the accelerations. Hence

Equal masses are such as are equally accelerated by equal forces.

The law thus enables us to express all forces in terms of the unit force, and all masses in terms of the unit mass. (See question 24, p. 68.)

(2) The law implies that when two or more forces act on a particle at the same time, each produces an acceleration in its own direction without reference to the others. In other words, the acceleration produced by a force on a particle is independent of any motion the particle may have and independent of motions produced by other forces acting simultaneously. This is known as the *principle of the independence of forces*. It was first pointed out by Galileo.

That this principle is not axiomatic is evident from the opinion of Descartes. "It is certain that a stone is not equally disposed to receive a new motion or increase of velocity when it is already moving very quickly and when it is moving slowly."

53. The law may be stated in other forms.

(a) Let a force F act on a mass m for a time t, and let v be the velocity acquired. Then if a is the acceleration of motion,

$$F = ma.$$

Also (Art. 25), $$v = at.$$

Eliminating a, we find

$$Ft = mv$$

as the expression of the law.

The product mv of the mass m and its velocity v at any instant is called the **momentum**. The unit of momentum is the momentum of unit mass moving with unit velocity.

The product Ft is called the **impulse** of the force F during the time t. It is expressed in the same unit as mv.

Hence the statement:

The momentum acquired by a body in any time is numerically equal to the impulse of the force which produces it; or, in a word,

change of momentum = impulse.

The term momentum (Lat. *momentum* = *movimentum*, a movement) is the modern equivalent of Newton's phrase "quantity of motion" (*quantitas motus*).

The term impulse was proposed by Belanger (1790–1874) in his *Cours de Mécanique* (1847).

(*b*) Suppose the distance passed over by the mass m under the action of the force F to be s, and that v is the velocity acquired. Then (Art. 25)

$$v^2/2 = as.$$

But $$F = ma.$$

Eliminating a, we have

$$Fs = mv^2/2$$

as the expression of the law.

The product Fs is called the **work done** by the force F acting through the distance s. The unit work is the work done by unit force acting through unit distance.

The expression $mv^2/2$ is the **energy** of the moving body in units of the same name as the unit work.

Hence the statement;

The energy acquired by a moving body is numerically equal to the work which produces it, or

energy change = work done.

The term work was proposed by Coriolis (1792–1843) in his *Traité de Mécanique* (1829); the term energy in this sense by Thomas Young (1773–1829), physicist and Egyptologist.

54. The relations in the preceding article indicate different aspects of the measure of force. The first two given by the equations

$$F = ma,$$
$$Ft = mv,$$

in which force is measured by the mass-acceleration or by the rate of change of momentum with time, are due to Newton. All dynamical questions are by him disposed of with the aid of the ideas of force, mass, and momentum.

The third equation,

$$Fs = mv^2/2,$$

introduces a distinct measure of force, and is the method of Huygens.

The first method is the one most commonly used in elementary mechanics and will be employed at first; the other will be discussed in Chapters VI and VIII.

55. If the acting force F be variable, it may be considered to consist of a succession of constant forces acting for indefinitely small intervals Δt and through indefinitely small distances Δs. The total impulse would be found by the summation of $F\Delta t$ throughout the whole time of motion t, and would be represented by

$$\int_0^t F dt.$$

The total work would be found by the summation of $F\Delta s$ throughout the whole distance of motion s, and would be represented by

$$\int_0^s F ds.$$

An impulse may therefore be defined as the *time-integral* of the force, and work may be defined as the *line-integral* of the force.

56. LAW III. In order to exert force the agent acting must meet a resistance. Thus the hand in motion does not exert force until it meets some object. The object reacts on the hand. Press the table and the table will press the hand. Force is always a *mutual* action: in other words, forces are never single, but act in pairs—one the *action* and the other the *reaction*. This pair of actions between two bodies or two parts of the same body is known as a **stress**. If it is of the nature of a push, preventing approach of the two bodies, it is called *compression* or *pressure;* if of the nature of a pull, preventing separation, it is called *tension;* if of the nature of a shear, preventing sliding, it is called a *shearing stress* or *shear.*

When we speak of a force acting on a body we consider only one of the two bodies between which stress exists. The force is the component of the stress on the body—the action. This was the case in discussing the preceding two laws.

But since a force cannot exist by itself,—forces being *dual,* —the view given in laws I and II is only partial and requires to be supplemented. This is done by the law of stress, or Newton's third law of motion, which is:

When one body acts on another, the reacting force (*reaction*) *is equal in magnitude and opposite in direction to the acting force* (*action*), or, as it may be expressed:

The mutual actions of two bodies are always equal and act in opposite directions.

57. In some cases the relation between the action of the agent and the reaction of the resistance is sufficiently evident. Thus if one body rests upon another it will be granted that the pressure exerted by the upper is equal to the counter-pressure exerted by the lower: if a horse hauls a canal-boat to which he is attached by a rope, the pull of the rope on the horse is equal to its pull on the boat, and so on. But when a stone falls from a height it is not evident whether the action of the earth on the stone is equal to the action of the stone on the earth. Nor is the relation evident between the actions of a magnet and a piece of iron,* nor between bodies widely separated, as the earth and the moon. But the law asserts that in *all* cases the acting force and reacting force are equal.

Newton points out the consequence of denying the truth of the law: "For instance, if the attraction of any part of the earth, say a mountain, upon the remainder of the earth were

* Newton, by placing a magnet in one vessel and the iron in another, and floating both vessels in water so as to touch each other, showed that, as neither vessel was able to propel the other along with itself through the water, the attraction of the iron on the magnet must be equal and opposite to that of the magnet on the iron, both being equal to the pressure between the two vessels. (Maxwell.)

greater or less than that of the remainder of the earth upon the mountain, there would be a residual force acting upon the system of the earth and the mountain as a whole which would cause it to move off with an ever-increasing velocity through infinite space. This is contrary to the first law of motion, which asserts that a body does not change its state of motion unless acted upon by external force."

58. Newton interprets the terms action and reaction in more than one sense. These interpretations are contained in the statements—

When two bodies act upon one another (1) the impressed force on the one is equal and opposite to the impressed force on the other; (2) the gain of momentum by the one is equal to the loss of momentum by the other; (3) the gain of energy by the one is equal to the loss of energy by the other.

Notice that action and reaction take place between *different* bodies. Thus in a tug-of-war the pull along the rope of A on B is equal to the pull of B on A. No matter where the rope is cut, the force on one side of the section is equal to that on the other side.

59. *Summary of the Laws.*—The laws of motion may be summarized in these statements:

(1) Every force is one component of a stress, and the two forces of a stress are equal and opposite.

(2) There is no acceleration without force, and the mass-acceleration produced is directly proportional to the measure of the force acting. In symbols: If a force F acts on a mass m and produces an acceleration a, then

$$F = ma,$$

and if v be the velocity acquired after a time t,

$$Ft = mv;$$

or if v be the velocity acquired in describing a distance s,

$$Fs = \tfrac{1}{2}mv^2.$$

60. Which of the laws of motion do the following statements illustrate?

(*a*) A circus-rider to jump through a hoop and alight on his horse springs vertically upward.

(*b*) "If a stone be dropped from the top of the mast of a ship in motion, the stone will fall at the foot of the mast notwithstanding the motion of the ship."

(*c*) When an omnibus turns a sharp corner there is a tendency to throw the driver from his seat.

(*d*) When a horse tows a canal boat the pull backward on the horse is equal to the pull forward on the boat.

(*e*) When a train stops suddenly the passengers receive a jerk.

(*f*) A cannon-ball has a different effect on a granite wall and on an earth wall.

(*g*) A severe jar is received from a step downward when one expects to step on the level.

(*h*) In suburban-passenger traffic the trains must stop and start quickly. The boiler and machinery are placed over the driving-wheels.

(*i*) Speaking of the third law of motion, Dr. Lodge says: "Action and reaction are equal and opposite. Sometimes an absurd difficulty is felt with regard to this, even by engineers. They say: "If the cart pulls against the horse with precisely the same force as the horse pulls the cart, why does the cart move?" Why on earth not? The cart moves because the horse pulls it, and because nothing else is pulling it back. "Yes," they say, "the cart is pulling back, but what is it pulling back, not itself surely?" "No, the horse." Yes, certainly the cart is pulling the horse; if the cart offered no resistance, what would be the good of the horse? That is what he is for, to overcome the pull-back of the cart; but nothing is pulling the cart back. There is no puzzle at all when once you realize that there are two bodies and two forces acting, and that one force acts on each body."

It has been objected to this that if the cart pulls the horse as much as the horse pulls the cart, there is just as much reason that the cart drag the horse as the horse drag the cart after it.

Does the objection appear to you a valid one?

61. Dynamical Units.—In Art. 7 were given the names of the kinematical units of length and time. Before giving the names of the units of dynamics some explanations are necessary.

The methods of comparing masses and of comparing forces given in Art. 52 are theoretically sufficient. A system of units, known as an absolute system, based upon these considerations and used in theoretical investigations in scientific laboratories, is given in Chapter IX.

But these methods, though capable of being described in succinct terms, are difficult of performance in practice. To compare masses by comparing their accelerations produced by forces assumed equal, or to compare forces by comparing their effects upon masses assumed equal, though useful as a laboratory experiment, is an experiment not capable of much precision with any apparatus at present in existence. Accordingly this dynamical method is in practice replaced by another more easily put in operation and much more precise.*

62. It is a fact of common observation that a body free to move falls towards the earth. It acts as if the earth attracted it. It is assumed as a convenient explanation of the observed phenomenon that there exists a stress between the earth and the body, and to one component of this stress the name *force of gravity* is given.

A body free to move if exposed to the action of the force of gravity is uniformly accelerated. Experiment shows that at the same place this force acts on all bodies in the same way; that is, the acceleration g [initial of gravity] produced by it has no relation to the magnitude or form of the bodies or to the material of which they are composed.

Experiment shows, too, that the value of g is constant so long as we keep to the same place on the earth's surface. It is found to vary with the latitude and the height above sea-level. It is greatest at the poles and least at the equator. It decreases as the height above sea-level increases. (See Arts. 101, 111.)

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* For the comparison of masses by the inertia method see text-books of laboratory physics. For the method by the ballistic balance see Hicks' *Dynamics*, pp. 22–26.

At New York $g = 32.16$, at San Francisco $g = 32.16$, at London $g = 32.19$, at Paris $g = 32.18$, at the equator $g = 32.09$ ft/sec^2. The approximate value 32 ft/sec^2 is very often used for convenience of computation and as being close enough in ordinary cases.

The experiments referred to are carried out with Atwood's Machine (Art. 70); or after Galileo's method, with falling bodies and the inclined plane (Art. 105); or best of all with the pendulum, as first pointed out by Newton (Art. 116).

We add a familiar illustration. If we let drop say a coin and a piece of paper, the coin will reach the floor first; but if we place both in a vessel without a lid and drop the vessel, the bodies inside will reach the floor together. This suggests that when the resistance offered by the air is removed, the time of falling the same distance and consequently the acceleration of gravity on both bodies is the same.

A better method is to let drop the bodies simultaneously from a shelf in a tall glass cylinder from which the air has been pumped out. Both will be seen to strike the bottom at the same instant.

63. Let two bodies m, m' be exposed to the action of the force of gravity. If connected together, as by a thread passing over a smooth pulley, and in equilibrium, the action and reaction through the thread being the same, we have

$$mg = m'g'.$$

But from the preceding article

$$g = g'.$$

Hence

$$m = m',$$

and in this way masses may be compared.

No great accuracy of measurement is to be looked for with this apparatus, simple as it appears (see Ex. 1, Art. 70). It, however, is theoretically the simplest form of balance, and suggests that instrument.

In making measurements of precision the connection between the bodies is made by placing them on the pans of a beam-balance. If the balance remains in equilibrium the bodies are of the same mass. When the comparison is made in this way the term **weight** is by prevailing custom used instead of mass, and it is usual to say that bodies which equipoise in the beam-balance are of the same weight. The process is called weighing; and by assuming a standard unit the weights of all bodies may be expressed in terms of the standard. Instead, therefore, of comparing masses by the dynamical method as required by the fundamental conception of mass, we may make the comparison by the beam-balance—an operation capable of extreme precision. With our present appliances and methods weighings can be carried out within one part in ten millions. "It is claimed that two avoirdupois pounds can be compared with an error not exceeding 0.0002 of a grain."

64. Notice that this method of comparing bodies by the balance is entirely independent of the locality where the experiment is made. The beam-balance does not show what the forces on the two bodies are, but only that these forces are equal at any place. With a spring-balance, however, assuming that it takes the same force to produce the same deflection of the spring, we may compare forces at the same place or at different places.

Thus, at any place, let two bodies be compared by beam-balance and also by spring-balance. If the whole apparatus is carried to another locality of different elevation, the bodies will still equipoise on the beam-balance, but the deflection of the spring-balance will be different from what it was before, though both bodies will produce the same deflection. The weight [or the mass] of a body is therefore constant, but the force of gravity on the body varies from place to place.

65. Gravitation Units—(*a*) *Unit Weight.*—The standard unit used in comparing bodies by weighing is the **pound** (lb). The British standard of weight is the imperial standard pound, which is defined by the Weights and Measures Act 1878, 41 and 42 Vict. c. 49, as "of platinum, the form being that of a cylinder . . . marked P. S. [Parliamentary Standard] 1844, 1 lb." Like the imperial yard it is deposited in the standards department of the Board of Trade, London.

In the United States the weights and measures in common use were introduced from England before the Revolution. In 1828 Congress legalized a troy pound for purposes of coinage. In 1830 the Treasury Department, for custom-house purposes, adopted the Troughton 82-inch brass scale as standard unit of length (Art. 7), the pound avoirdupois derived from the troy pound as unit of weight, the wine gallon of 231 cubic inches as unit of liquid measure, and the Winchester bushel of 2150.42 cubic inches as unit of dry measure.

In 1836 Congress authorized the Secretary of the Treasury to send copies of all weights and measures adopted as standards by the Treasury Department to every State for the use of the State. Some of the States had already legalized standards, but all of them formally adopted these standards, and in this way a practically uniform system of weights and measures was secured throughout the Union. Strictly speaking, however, each State has its own standards, and these standards are entirely independent of the standards of the Office of Weights and Measures at Washington, though copies of them.

In 1866 the metric system of weights and measures was made legal throughout the United States, and the pound was defined in terms of the kilogram by the relation

$$1 \text{ pound avoirdupois} = 1/2.2046 \text{ kilogram.}$$

There being no material normal standard of the pound avoirdupois, its value is derived from the standard of the kilogram in accordance with this relation. (Art. 280.)

The act of 1873 reads: "For the purpose of securing a due conformity in weight of the coins of the United States the brass troy pound weight procured by the Minister of the United States at London in 1827 for the use of the mint and now in the custody of the mint at Philadelphia shall be the standard troy pound of the mint of the United States, conformably to which the coinage thereof shall be regulated."

(*b*) *The unit force* is naturally taken to be equal to the force with which the earth attracts the unit weight, the standard pound. The unit force is named a **pound**.

The pound is called the gravitation unit of force, because its value depends on the force of gravity. This force is not constant over the earth's surface, but varies from place to place. The unit of force defined in Art. 50, being entirely independent of the force of gravity, is known as the absolute unit of force.

With the gravitation system of units the abbreviation lb will be used to indicate mass as found by weighing, and the word pound to indicate force; thus we speak of a body weighing m or M lb, a force of F pounds. It is customary to use the letter w or W instead of m or M with gravitation measures.

66. *Relation of the Units of Force.*—The fundamental units of distance, time, and mass or weight are the same in both the absolute and gravitation systems. The divergence occurs at the derived unit force and related quantities.

A relation between the absolute and gravitation units of force is readily found. For

1 grav. unit force produces in unit weight g units accel.
1 abs. unit force produces in unit mass 1 unit accel.

$\therefore$ 1 gravitation unit of force $= g$ absolute units of force.

We may therefore convert absolute measures of force to gravitation measures by dividing by the value of g at the place in question, and *vice versa*.

Thus a force acting upon a body weighing W and causing an acceleration a is equivalent to

Wa absolute units of force,
Wa/g gravitation units of force.

Similarly, the momentum of a body weighing W and moving with velocity v is

Wv absolute units of momentum,
Wv/g gravitation units of momentum.

The energy of a body weighing W and moving with velocity v is

$Wv^2/2$ absolute units of energy,
$Wv^2/2g$ gravitation units of energy.

In all cases g is *not* to be regarded as a divisor of either one of the factors W or a or v, but as a divisor of the product Wa or Wv or Wv^2, and with a corresponding *change of unit* when the division is performed. Keeping this in mind, there can be no confusion in the use of g in passing from one system of units to another.

Hence rules such as the following are misleading: "The number of units of mass in a body is found by dividing the weight in pounds by the value of g at the place where the weight is determined."

67. *Restatement of Newton's Second Law.*—Comparing Arts. 50, 66, it is evident that, using gravitation measures, the statement of Newton's second law would be

$$F = Wa/g \quad \ldots \ldots \quad (1)$$

when a force F pounds acting on a body weighing W lb produces an acceleration a ft/sec²; or, if v ft/sec is the velocity acquired,

$$Ft = Wv/g, \quad \ldots \ldots \quad (2)$$

where Ft is the *impulse* in second-pounds, and Wv/g is the *momentum* in second-pounds; or, if s ft is the distance passed over,

$$Fs = Wv^2/2g \quad \ldots \ldots \quad (3)$$

where Fs is the *work done* in foot-pounds, and $Wv^2/2g$ is the *energy acquired* in foot-pounds.

Equation (1) may be written in the differential form by putting

$$a = d^2s/dt^2, \quad \text{or} \quad a = v\,dv/ds.$$

68. *Weight.*—With the gravitation system of units, which is the system of every-day life and of the more usual commercial and engineering questions, it is necessary to conform to the terms sanctioned by prevailing custom. Here there is some confusion. Thus the term weight is used in the double sense of quantity and of force*—or rather in a triple sense. (1) In commerce it is used to denote quantity as measured by the beam-balance. For example, when we say the weight of a barrel of flour is 196 lb we indicate quantity. This, too, is the legal sense of the term. The Weights and Measures Act, 1878, in defining the standard pound, states that "the weight *in vacuo* of the platinum weight declared to be the imperial standard shall be the legal standard of weight, and of measures having reference to weight, and shall be called the imperial standard pound, and shall be the only unit or standard measure of weight from which all other weights and all measures having reference to weight shall be ascertained."

(2) Again, we speak of the pressure of a weight, lifting a weight, etc. A pressure may be balanced by the pressure of a weight. The law of action and reaction is the formulation of the equality of pressure and counterpressure in the dynamical sense. Thus the term weight is extended to mean the pull or force of the earth on the body.† A body which weighs W lb is attracted with a force of W pounds.

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* In French, *poids* and *pesanteur*.

† Some writers who define mass as "quantity of matter" introduce into the gravitation system of units the term mass in the sense of quantity, and restrict the term weight to mean the force with which the earth attracts the body. Thus a barrel of flour has a mass of 196 lb, and its weight is expressed as "the weight of 196 lb," or as "196 lb weight." The weight of a mass of 1 cwt is 112 lb weight, etc.

The term mass, however, belongs strictly to an absolute system of units only, and the innovation adds confusion to a system already confused enough. See, for a discussion of this question, *Nature*, vols.

The distinction is stated by Maxwell [*Theory of Heat*, p. 80] as follows: "In fact, the only occasions in common life in which it is required to estimate weight considered as a force is when we have to determine the strength required to lift or carry things, or when we have to make a structure strong enough to support their weight. In all other cases the word weight must be understood to mean *the quantity of the thing as determined by the process of weighing against 'standard weights.'*"

(3) The term is also used to denote the determinate body employed in the beam-balance as "a weight," "a set of weights," etc.

The word pound has a similar variety of meanings. We speak of a pound weight, a pound force, and of a certain body itself as "a pound."

At the close of the last century, in different parts of the world, the word *pound* was applied to 391 units of weight, and the word *foot* to 292 different units of length. Not only were no two of these identical, but in only a few cases were their relative values known with anything like precision. Most of these have since been swept away. (Mendenhall.)

69. It is to be clearly understood that in mechanics we are compelled to recognize two classes of problems, each with its nomenclature and units.

To the problems that ordinarily present themselves in daily life—local questions—the units of the gravitation system are exclusively used. But many questions in electricity and magnetism, in physical science and in astronomy, compel us to drop local considerations. For the purpose of comparing measurements the need of standards universal, and not local, becomes imperative. This was indicated by Newton and clearly pointed out by Gauss, who introduced the absolute system of units (Arts. 50, 65).

There are two series of gravitation units in use in Great Britain and the United States—the British and the metric.

XXXVII–XLVI—particularly a paper by Prof. Greenhill, vol. XLVI, pp. 247-252.

At present the British gravitation system is in most common use. The metric system, however, is legalized by both countries, and is the only system legalized by the United States, the British system having been "inherited from England with the common law."

The absolute system of units is founded on metric measures only—that is, the only absolute system that has obtained universal recognition, the C.G.S. system. (See Chapter IX.)

The student must be familiar with all three sets of measures and with the methods of passing from one to another. For clearness the British system and the metric system are in this book presented separately. So far the British system only has been made use of. But at this point Chapter IX may be read for an account of the metric system. The metric system may then, if thought advisable, be introduced and carried side by side with the British system, or may be used altogether. The tables on pp. 364–367 have been arranged to give a synoptical view of the various nomenclatures and to facilitate conversions from one system to another.

Ex. 1. A force of 5 pounds acts on a weight of 40 lb. Find the acceleration produced. *Ans.* $a = 4$ ft/sec².

2. A 60-lb shot is fired with a velocity of 1200 ft/sec. Find the impulse communicated by the powder.
Ans. 2250 second-pounds.

3. A 16-lb weight falling from a height strikes the ground with a velocity of 16 ft/sec. Find the momentum destroyed.
Ans. 8 second-pounds.

4. In what senses is the term weight used in the following statements?

(*a*) "Every person who sells coal in quantities of less than half a ton in weight shall keep scales and weights of the legal standard, and shall weigh such coal before delivery."—[*The Weights and Measures Act.*]

(*b*) "It will therefore be understood that the expression 'weight of the earth' has no physical meaning. This weight may be anything. It is an indeterminate quantity."—[Nipher's *Electricity,* p. 16.]

5. Explain how to pass from gravitation to absolute measures of force, momentum, and energy; and *vice versa* (Art. 66).

6. A baseball weighing $5\frac{1}{2}$ oz has a velocity of 50 ft/sec. What constant force must the catcher apply to bring it to rest in 1 ft? *Ans.* 13.4 pounds.

7. A baseball has a horizontal velocity of 66 ft/sec. An impulse causes it to travel back on the same line with a velocity of 62 ft/sec. Find the change of momentum.
Ans. 11/8 second-pounds.

8. A railroad train which weighs 150 tons and has a velocity of 60 miles/hour is brought to rest in half a minute by the action of the brakes. Find the average force exerted.
Ans. 27,500 pounds.

9. A 4-lb ball is moving with a velocity of 40 ft/sec and in 10 sec afterwards it is moving with the same speed in the opposite direction. What force has acted during this time?
Ans. 1 pound.

10. A 50-ton train acquires a speed from rest of 10 miles/hour in 5 minutes. How long would it take a 75-ton train drawn by the same engine to acquire a speed of 12 miles/hour? *Ans.* 9 minutes.

11. "In computing the energy of a railroad train I am directed to use the formula $Wv^2/2g$. Now why use $2g$, since that quantity is exclusively an element of falling bodies? Is it not possible to compute the energy by a process entirely independent of gravity considerations?"—[Query in *Sci. Amer.*, 1893.]

12. The 66-ton Canet guns made (1890) for the Japanese navy use projectiles weighing 1034 lb with an initial velocity of 2262 ft/sec. Find the velocity of recoil of the gun.
Ans. 17.7 ft/sec.

13. A 1000-lb shot strikes a target directly with a velocity of 700 ft/sec and rebounds with a velocity of 100 ft/sec. Find the measure of the impulse. *Ans.* 25,000 second-pounds.

14. A ball weighing 1 lb falls on a level floor with a velocity of 100 ft/sec and rebounds with a velocity of 76 ft/sec. Find the average force exerted between ball and floor, supposing the time of impact to be 1/100 sec. *Ans.* 550 pounds.

15. A man who weighs 150 lb moves from the bow to the stern of a boat, a length of 10 ft. If the boat weighs 225 lb find how far it will have moved forward, supposing the resistance of the water not taken into account. *Ans.* 4 ft.

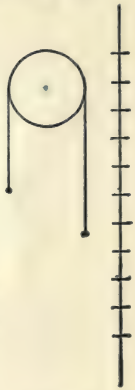
16. A spring-balance is graduated at New York. Find the true weight of a body which weighs 67 lb in this balance at London. *Ans.* 67 lb 1 oz.

[At New York $g = 32.16$, at London $g = 32.19$ ft/sec^2.]

17. A body weighing W lb acted on by a force F pounds, describes the distance S ft in t sec. Show that

velocity acquired $= Fgt/W$ ft/sec,
distance passed over $= Fgt^2/2W$ ft,
momentum acquired $= Ft$ second-pounds.

70. Test of the Laws of Motion.—A laboratory contrivance in common use for illustrating the laws of motion is known as *Atwood's machine*,* the essential features of which are as follows:



Two bodies weighing W_1, W_2 lb are fastened to a light inextensible thread, which passes over a pulley mounted so as to oppose the motion as little as possible. The effect of the pulley is to change the direction, but not to alter the magnitude of the pull.

Let P denote the pull of the thread expressed in pounds. The one body W_1 falls and the other W_2 rises with the same acceleration a along the motion of the thread, and therefore of the same sign.

The force causing an acceleration a ft/sec² on W_1 lb is $W_1 - P$ pounds.

$$\therefore \qquad W_1 - P = W_1 a/g.$$

The force causing an acceleration a ft/sec² on W_2 lb is $P - W_2$ pounds.

$$\therefore \qquad P - W_2 = W_2 a/g.$$

Hence, eliminating P,

$$a = g(W_1 - W_2)/(W_1 + W_2) \text{ ft/sec}^2.$$

By taking W_1 and W_2 of considerable magnitude, but nearly equal, the acceleration a may be reduced to so small a quan-

* Invented by George Atwood, F.R.S. (1746–1807), Cambridge, England.

tity that the distance s fallen through in a given time t is easily observed on a vertical scale attached. The time of motion may be taken from a clock or from a pendulum attached to the apparatus and beating seconds.

The manner of making experiments with the machine will be found described in any book of laboratory physics.

Ex. 1. In an Atwood machine $W_1 = 21$ oz, $W_2 = 20$ oz, and it was observed that in 5 sec the weight W_1 descended 9.5 ft. Find the value of g the acceleration of gravity.

Ans. 31 + ft/sec².

[*Hint.*—Equate the values of a found from $s = at^2/2$ and $a = g(W_1 - W_2)/(W_1 + W_2)$.

The result would seem to show that there is something wrong either with the apparatus or with the observed values of distance or time. The fact is that on account of friction and the resistance of the air the distance and time observed will not be the ideal quantities demanded by the problem, so that no very exact measure of g can be made with this apparatus. The Atwood machine is now regarded as of historic interest mainly.]

2. In an Atwood machine find the pull of the cord if the weights are 3 lb and 2 lb. *Ans.* 2.4 pounds.

3. In an Atwood machine the weights are $P + Q$ and $P - Q$. Show that if a is the acceleration of motion, then $Pa = Qg$.

4. In an Atwood machine the weights are each equal to 31 oz, and a weight of 2 oz is placed on one of them and removed after it has fallen 2 ft. Find the time of fall and the distance through which each weight will move in the next second.

Ans. 2 sec; 2 ft.

5. In an Atwood machine, the sum of the weights being given, show that the pull of the thread will be greatest when the weights are equal.

6. In an Atwood machine the pull of the thread is an harmonic mean between the moving weights.

7. If the thread breaks at any instant, show that when the ascending motion of one of the weights ceases the other will have descended through three times the distance through which the former has ascended in the interval.

EXAMINATION.

1. A body is set in motion and left wholly to itself. What is the nature of the subsequent motion?

2. What is meant by inertia?

[The tendency of a body to continue in its state of rest or of uniform motion in a straight line.]

3. Give illustrations of inertia.

[Ex. The taking up of water by a moving train from a long trough situated between the tracks.]

4. Deduce from the laws of motion a definition of force. Is force necessary to maintain motion?

5. State the law of motion which gives a quantitative statement as to the effect produced on a particle by a force.

6. A constant force produces a uniform acceleration.

7. How may force be measured?

[By deformation caused, as of a spring; by acceleration contributed (Art. 50); by opposing it to the pull of the earth (Art. 65)]

8. Define an impulse. What is the unit?

9. State the parallelogram of momenta.

10. Prove that the average force which, acting for t sec through l ft on a body weighing W lb, is required to produce a velocity of v ft/sec is Wv/gt, or $Wv^2/2gl$ pounds.

If the body is moving with the velocity v ft/sec, what force is required to stop it?

11. Show that force may be defined as:

(1) Rate of change of momentum with time.

(2) Rate of change of energy with distance.

[These questions suggest the answer to a question keenly debated soon after the time of Galileo as to whether force was to be regarded as proportional to the velocity produced or to the square of the velocity. The first view was held by Descartes and his school; the second by Leibnitz. The dispute lasted for nearly sixty years, until D'Alembert dissipated the misunderstanding by showing that with respect to time force is proportional to the velocity, but with respect to distance, to the square of the velocity. Thus $F = mv/t$, or $F = mv^2/2s$.]

12. Compare the momentum of a 20-lb shot moving with a velocity of 500 ft/sec with that of a 2-oz bullet moving at the rate of 500 yds/sec. *Ans.* 160 : 3.

13. What velocity will the force of gravity give to a pound weight in one second? *Ans.* g ft/sec.

14. "One's weight is given, but not—or not in the same way and degree—one's velocity. Weight given, it is only by doubling or trebling his velocity that a man can make his momentum double or treble as needed." (Carlyle, *Friedrich the Second.*) Explain.

15. Show that it necessarily follows from the second law of motion that forces can be represented by straight lines.

16. What are the tests of the equality (1) of two forces, (2) of two weights?

17. A man with a hod on his shoulder falls off a ladder. Find the pressure on his shoulder during the fall.

18. Describe an experiment which involves the principle of the second law of motion, and show how the probability of the law may be inferred from it.

19. State the units of weight, force, momentum, and work in the gravitation system.

20. "A force of 100 pounds is the force which would balance a weight of 100 lb if acted upon by gravity only."

21. Would it be advantageous for a merchant to buy groceries in New York to sell in Cuba if he used the same spring-balance in both places? How would a pair of scales answer?

22. A force of 10 pounds acts upon a weight of 10 lb for 10 sec. Find the momentum acquired in second-pounds.

23. "When you weigh a thing in an ordinary balance, do you find its weight?"

24. It is sometimes said that we reason in a circle in stating the foundations of dynamics, defining force by mass and mass

by force. We say in effect: "Equal forces are those which produce equal accelerations in equal masses; equal masses those in which equal accelerations are produced by equal forces."

["If we assume constancy of mass of a body and of the physical properties say of a spiral spring, there is no difficulty in getting out of this circle of definition. These are assumptions we are entitled to make as the result of experience."]

25. Explain the theory of Atwood's machine for illustrating the laws of motion.

26. Masses may be compared by the impulses required to produce unit velocity.

27. Weights W_1, W_2 lb hang at the ends of a string which passes over a smooth pulley. Find the pull of the string.

Ans. $2W_1W_2/(W_1+W_2)$ pounds.

28. A bucket weighing 25 lb is let down into a well with uniform speed. Find the pull of the rope.

Ans. 25 pounds.

29. "Why when we place an ax upon wood, and on the ax place a heavy weight, is the wood but slightly indented; whereas if we raise the ax without the weight and strike upon the wood the wood is split, although the falling weight is much less than the resting and pressing weight?" (Aristotle.)

[In Aristotle's day the conception that force was measured by mv was unsuspected.—Lewes.]

30. Describe any method of testing whether the acceleration produced by gravity is uniform?

31. Which of Newton's laws implies the principle of the "independence of forces"?

32. What authority have we for saying that the same force will produce the same change of motion in a particle whether the particle is at rest or in motion?

33. Write down the kinematical equation of rectilinear motion.

$$[dv/dt = a, \quad \text{or} \quad d^2s/dt^2 = a.]$$

34. Write down the dynamical equation of rectilinear motion.

$$[mdv/dt = F, \quad \text{or} \quad md^2s/dt^2 = F.]$$

35. Write down the equation of motion of a particle acted upon by (1) a vertical force only, (2) a horizontal force only.

36. When is a train said to be going at full speed?

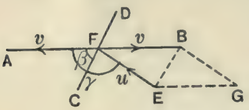
37. If A pulls against B and action and reaction are equal, how is it that A may overcome B?

38. Explain the kick of a gun.

39. Explain how an ice-boat can sail with greater velocity than that of the wind propelling it, assuming that the ice offers no resistance to the boat's motion.

(1) Consider the case of the wind abaft the beam.

As the velocity of the boat increases the resistance of the air on the sail increases, until a velocity is reached when the force of the wind behind the sail is balanced by the reaction of the air in front of the sail. The action and reaction balancing, the boat moves with uniform velocity and at full speed. Call this velocity v.



Let AB represent the keel of the boat, CD the sail, EF the direction of the wind, u its velocity, and β, γ the inclinations of sail and wind to the keel AFB.

Let a velocity $-v$ be imparted to both boat and wind. The boat will be at rest. The velocity $-v$, combined with the velocity u of the wind, will give the velocity direction of the *apparent wind* along EB (Art. 43), which must be parallel to the sail, since at full speed the sail is in a calm. Hence

$$v/u = \sin(\gamma - \beta)/\sin \beta.$$

For a given velocity of wind and a given position of sail u and β are fixed. The value of v is, then, evidently greatest when $\sin(\gamma - \beta)$ is greatest, that is, when $\sin(\gamma - \beta) = 1$, and then $\gamma - \beta = 90°$, or $\gamma = 90° + \beta$. When this is the case,

$$v \sin \beta = u.$$

Now $\sin \beta$ is always less than unity, and therefore

$$v > u,$$

or the velocity of the boat exceeds that of the wind.

Let v_1 denote the component of the boat's velocity in the direction of the wind. Then

$$v_1 = v \cos(180 - \gamma)$$
$$= u \sin(\gamma - \beta) \cos(180 - \gamma)/\sin \beta.$$

This is a maximum when $\gamma = 135° + \beta/2$, and then

$$v_1 = u(1 + \sin \beta)/2 \sin \beta.$$

Now $$v_1 > u$$

if $$1 + \sin \beta > 2 \sin \beta,$$

or if $$\sin \beta < 1, \quad \text{which it is.}$$

Hence the component in the direction of the wind is greater than the velocity of the wind, or the boat can run to leeward faster than by sailing directly before the wind.

If the wind is before the beam, it may be shown in a similar manner that if $\beta < 19.5°$ and $\gamma = 45° + \beta/2$, the component velocity of the boat in the direction of the wind is greater than the velocity of the wind, or it is possible to run to windward faster than the wind.

40. If the wind is nearly abeam, or $\gamma = 85°$ say, and the sail is set so that $\beta = 20°$, show that $v = 2.6u$.

41. An ice-boat runs directly before the wind. What is the greatest velocity it can attain?

42. Which of Newton's laws corrects the opinion held by the ancients that circular motion is *perfect* and *natural*?

CHAPTER III.

DYNAMICS OF A PARTICLE.

71. Having considered the geometrical properties of motion, and also the methods of measuring force, we are ready to study the motion produced in a body by forces of given magnitude.

When a body is acted on by forces, experience shows that various forms of motion may arise. If all of the component particles of the body move through equal distances in the same direction, the motion is a motion of translation. The motion of any particle would in this case give the motion of the body. In a motion of rotation the particles do not move through equal distances in the same direction, those nearest the axis of rotation moving the shortest distance. If a body consisted of a single particle, it would, in its rotation about an axis passing through it, remain in the same position. We shall therefore exclude rotation, and be able to study the translation of a body if we consider the motion of a single particle only.

72. Composition of Forces.—The number of forces acting on a particle may be one or more than one. If we can combine the separate forces into an equivalent single force, we can reduce all cases to that of the action of a single force. The method of combining forces will be our first step, and next we shall consider the motion of a particle under the action of forces.

73. *Representation of Force.*—The elements of a force that completely determine it are:

(1) Its magnitude or the number of units of force in it;

(2) Its position or line of action;

(3) Its direction along the line of action or its sense.

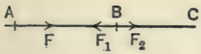
A force acting on a particle, or at a point in a body, may therefore be represented by a straight line AB in the line of action of the force, the length of AB representing the magnitude of the force, the direction from A to B the direction of the force, and the point A the point of application. Each unit of length of AB will represent unit force. But the length of the unit is arbitrary. Hence we may plot forces to any scale we please, as 1 pound = 1 inch, 10 pounds = 1 inch, etc.

In a diagram the direction of a force is conveniently indicated by an arrow-head. When different forces act in the same line, their directions may be indicated by the signs + and −. Thus a force of 2 pounds acting at A along AB

may be written + 2, and an equal force at A along AC would be written − 2. The choice of signs is of course arbitrary.

74. A force acting on a body may be considered to act at any point in the body in the line of action of the force.

Thus let the force F act at A in the direction AC, and let B be a fixed point in the line of action. The action of F at A has an equal and opposite reaction F_1 at B, according to Newton's third law. But F_1 at B would be balanced by F_2 at B, equal and opposite to it, and therefore in the same direction as F. Hence F at A may be replaced by an equal force F_2 at B.



This principle is known as the *transmissibility of force.*

Ex. 1. On a scale of 100 pounds per inch what force would be represented by a line 20 inches long?

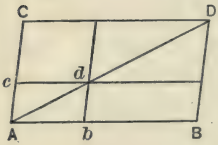
Ans. 2000 pounds.

2. If a force of P pounds be represented by a straight line a inches long, by what straight line will a force of Q pounds be represented? *Ans.* Qa/P inches.

3. Find the force equivalent to the forces 3, 5, − 7 pounds acting downward in the same line.

4. The forces $a + b$, $a - b$, $2a$, in the same line, can be made to balance. How?

75. *Composition of Two Forces (Graphical Method).*—Suppose that F_1, F_2 are two forces which act on a particle w. Let Ab, Ac represent the accelerations a_1, a_2 due to F_1, F_2.



The resultant a of these accelerations is represented by the diagonal Ad of the parallelogram $Abdc$ (Art. 30).

But F_1 being equal to wa_1/g and F_2 to wa_2/g, may be represented by two lines AB, AC in the directions of the accelerations a_1, a_2 produced by them. Complete the parallelogram $ABDC$ and join AD. Then

$$AB : BD = wa_1/g : wa_2/g$$
$$= a_1 : a_2$$
$$= Ab : bd,$$

and therefore Ad, AD coincide in direction.

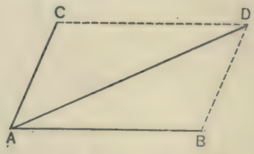
Also, from similar triangles Adb, ADB,

$$AD : DB = Ad : db,$$

or $$AD : wa_2/g = a : a_2,$$

that is, $$AD = wa/g.$$

Hence AD represents the force which produces the acceleration a, and is equivalent to the two forces F_1, F_2 represented by AB, AC. It is therefore called the **resultant force.** Hence



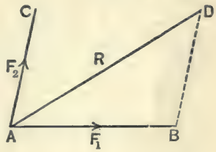
If two forces F_1, F_2 acting on a particle A (or at a point A) be represented by two lines AB, AC, their resultant R is represented by the diagonal AD of the parallelogram $ABDC$ constructed on AB, AC as adjacent sides.

This principle is called the **parallelogram of forces.**

The problem of finding the resultant of two forces is by this proposition reduced to the solution of a geometrical problem.

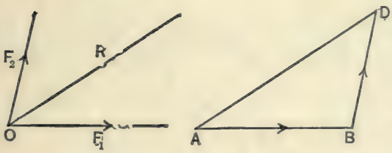
The parallelogram of forces was first clearly formulated by Newton (1642–1727). It is a direct consequence of the second law of motion, and may be employed to test the truth of the law. The principle had been previously employed by Stevinus of Bruges, who derived it from the inclined plane.

76. Consider in the parallelogram figure the manner in which the resultant is formed. The forces F_1, F_2, drawn to scale, are represented by the lines AB, AC. From B the line BD is drawn parallel to AC, and from C the line CD is drawn parallel to AB. The diagonal AD of the parallelogram represents the resultant in magnitude, direction, and position.



This construction is equivalent to the following: Plot the forces F_1, F_2 as before. From B draw BD parallel and equal to AC. Join AD, which represents the resultant.

Still better, by breaking the figure into two parts. Let F_1, F_2 be the forces acting at O. From any point A draw AB to scale equal and parallel to F_1. From B draw BD to scale equal and parallel to F_2. Join AD, which will represent the resultant in *magnitude* and *direction*. To find its *position:* We know that it must pass through O, and hence, if through O we draw a line equal and parallel to AD, we have R in magnitude, direction, and position. We have therefore a *force diagram* and a *construction diagram* purely geometrical. In simple cases there is no confusion when the two overlap and are included in one figure; but in complicated cases it is better to keep them separate, as we shall see later on.



Ex. 1. If forces of 10 pounds and 12 pounds act on a particle, find the greatest and least possible resultants.

Ans. 22 pounds; 2 pounds.

2. Show by a drawing that the value of the resultant decreases as the angle between the forces increases.

3. *ABCDEF* is a regular hexagon. Show that the resultant of the system of forces represented by *AB, AD, AE* is 2*AD*.

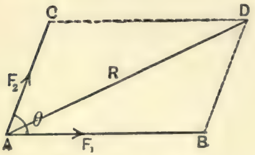
4. Two equal forces *F* act at an angle of 120°. Find *R*.
Ans. $R = F$.

5. Two equal forces act at a point. Through what angle must one of them be turned that their resultant may be turned through a right angle? *Ans.* 180°.

6. Show by a drawing that the resultant of P and $P + Q$ acting at 120° is equal to the resultant of Q and $Q + P$ acting at 120°.

7. If R is the resultant of two forces P and Q, and S the resultant of P and R, show that the resultant of S and Q is $2R$.

77. We may express the resultant R of two forces F_1, F_2 in terms of the forces and their included angle θ. This method of finding R is often more convenient than the graphical method of measuring the diagonal.



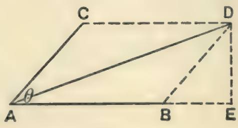
Let $AB = F_1$, $AC = F_2$, $\angle BAC = \theta$. Then from trigonometry we have in the triangle ABD [note $AC = BD$; $\angle BAC + \angle ABD = 180°$]

$$AD^2 = AB^2 - 2AB \,.\, BD \cos ABD + BD^2$$
$$= AB^2 + 2AB \,.\, AC \cos BAC + AC^2,$$

or $$R^2 = F_1^2 + 2F_1F_2 \cos \theta + F_2^2,$$

which gives the *magnitude* of the resultant.

The *line of action* of R may be found by solving the triangle ABD to find the angle BAD. Thus from D let fall DE perpendicular to AB. Then if $\angle DAB = \alpha$, we have



$$\tan \alpha = DE/(AB + BE)$$
$$= F_2 \sin \theta/(F_1 + F_2 \cos \theta).$$

Hence R is completely determined.

The special case of the forces acting in directions at right angles is important. The parallelogram becomes a rectangle, and from the figure

$$R^2 = F_1^2 + F_2^2,$$
$$\tan \alpha = F_2/F_1,$$

whence the resultant is determined.

Ex. 1. If two equal forces F, F are at right angles to one another, then $R = F\sqrt{2}$ and $\alpha = 45°$.

2. Find the resultant of two equal forces, each of 10 pounds, acting at an angle of 30°. *Ans.* 19.3 pounds.

3. If two equal forces F, F are inclined at an angle 2θ, then $R = 2F \cos \theta$ and $\alpha = \theta$.

4. Show from the general formula that the value of R increases as the angle between the forces diminishes, and *vice versa.*

5. When the angle BAC is 0° or 180° the forces are in the same straight line, and the formula for finding R, if correct, should reduce to the sum or difference of the forces. Examine, and see if it does.

6. Find the resultant of P and $P + Q$ acting at 120°, and also the resultant of Q and $Q + P$ acting at 120°.

Ans. $R = \sqrt{P^2 + PQ + Q^2}$ in both cases.

7. The resultant of two forces P and Q at right angles to each other is R. If each force is increased by R, show that the new resultant makes with R an angle

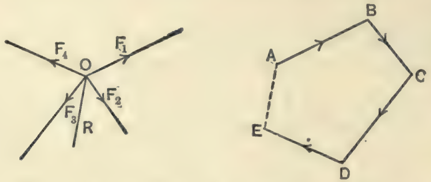
$$\tan^{-1} (P - Q)/(P + Q + R).$$

8. The resultant R of P and Q is equal to P. If P is doubled, show that the resultant R_1 of $2P$ and Q is at right angles to Q, and find its magnitude in terms of P and Q.

78. *Composition of More than Two Forces (Graphical Method).*—Let F_1, F_2, F_3, F_4 represent forces acting on a particle at O; it is required to find their resultant.

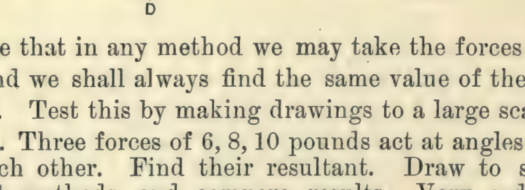
Following the method of Art. 76, from a point A we draw AB equal and parallel to F_1, BC equal and parallel to F_2, CD equal and parallel to F_3, DE equal and parallel to F_4. Join AE, which will represent the resultant in magnitude and direction. For join AC, AD. Then AC represents the resultant of F_1, F_2 in magnitude and direction; AD the resultant of AC and F_3, that is, of F_1, F_2, F_3; and AE the resultant of AD and F_4, that is, of F_1, F_2, F_3, F_4.

AE represents, therefore, the resultant of all the forces at O in magnitude and direction. From O in the force diagram



draw a line R equal and parallel to AE. Then will R represent the resultant of F_1, F_2, F_3, F_4 in magnitude, direction, and position, and be completely determined (see Art. 73).

We might have combined the two diagrams, or we might have derived the resultant directly from the parallelogram of forces, as indicated in the figures below.



Notice that in any method we may take the forces in *any order* and we shall always find the same value of the resultant AE. Test this by making drawings to a large scale.

Ex. 1. Three forces of 6, 8, 10 pounds act at angles of 120° with each other. Find their resultant. Draw to scale by different methods, and compare results. Vary order, and compare.

2. Forces of 1, 2, 3, 4, 5, 6 act at angles of 60°. Find R. Test as in Ex. 1.

3. Forces of 20, 20, 21 pounds act at a point. The angle between the first and second is 120°, and between the second and third 30°. Find R. *Ans.* 29 pounds.

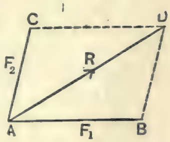
4. Is it necessary that force and construction diagrams be drawn to the same scale?

5. In the construction diagram the lines are drawn parallel to the forces. Would it be allowable to draw them perpendicular to the forces, or inclined at any (the same) angle, θ for example? Test by a drawing.

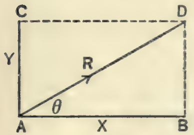
6. If the forces are in the same straight line, what does the force polygon become? what the construction diagram?

7. O is any point in the plane of the triangle ABC, and D, E, F are the middle points of the sides. Show that the system of forces OA, OB, OC is equivalent to the system OD, OE, OF.

79. Resolution of Forces.—By means of the parallelogram of forces, a force R can be found equivalent to two forces F_1, F_2, acting on a particle A. Conversely, the force R acting at A may be resolved into two *component forces* F_1, F_2 acting at A, by constructing on AD ($= R$) as diagonal a parallelogram, and taking the sides AB, AC to represent the components. The problem is similar to that already discussed in Art 20.



When the two components are at right angles their values may readily be found analytically. For example, to resolve a force R represented by the line AD into two components X, Y at right angles along AB, AC.



Complete the rectangle AD.

The components are represented by the sides AB, AC of this rectangle. Then

$$X = AB = AD \cos\theta = R\cos\theta;$$

$$Y = AC = AD\cos(90-\theta) = R\cos(90-\theta) \text{ or } R\sin\theta.$$

Hence *the rectangular component (or resolved part) of a force R in a given direction is equal to the product of the force and the cosine of the angle between the force and the given direction.*

As a check,

$$X^2 + Y^2 = R^2\cos^2\theta + R^2\sin^2\theta$$
$$= R^2,$$

which is also evident from the figure.

Examples to be Solved Graphically.

1. If a force is resolved into two components, prove that the greater component always makes the smaller angle with the force.

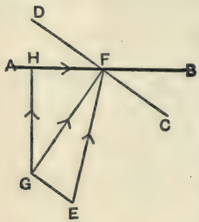
2. Resolve a force of 20 pounds into two components each of which makes an angle of 60° with it.

Ans. Each = 20 pounds.

3. If two forces acting at a point be represented in magnitude and direction by the diagonals of a quadrilateral, their resultant will coincide with that of the forces represented by two opposite sides.

4. Explain the boatmen's saying that there is greater "power" in a horse hauling a canal-boat with a long rope than with a short one. Is the same true of a steam-tug?

5. Discuss the action of the wind in propelling a sailing-vessel.



[Let AB be the keel, CD the sail. Let the force of the wind be represented in magnitude and direction by EF. The component GF of EF, perpendicular to the sail, is the effective component in propelling the ship; the other component, EG, parallel to the sail, is useless. But GF drives the ship forward and sidewise. The component GH of GF, perpendicular to AB, produces side motion or leeway, and the other component, HF, along the keel produces forward motion or headway.]

6. Discuss the action of the rudder of a vessel in counteracting leeway.

7. Show that one effect of the action of the rudder is to diminish the vessel's motion.

Examples to be Solved Analytically.

8. Find the rectangular components of a force 10 when $\theta = 60°, 90°, 120°, 180°, 240°, 270°, 300°$. *Ans.* $5, 5\sqrt{3}$; $0, 10$; $-5, 5\sqrt{3}$; $-10, 0$; $-5, -5\sqrt{3}$; $0, -10$; $5, -5\sqrt{3}$.

[Draw a figure for each case, and explain the sign of the result.]

9. Resolve a force of 10 pounds into two equal components, one of them making an angle of 45° with the force.

Ans. 7 pounds, nearly.

10. The force on a smooth surface from a N. E. wind of strength F is equal to the force from a north wind of strength $F/\sqrt{2}$.

11. Find that rectangular component of a force of 10 pounds which makes an angle of 90° with the force.

12. The pull on the rope of a canal-boat is 100 pounds and the direction of the rope makes an angle of 60° with the parallel banks. Find the force urging the boat forward.

Ans. 50 pounds.

13. Show that the components of a force F in two directions making angles of 30°, 45° with it on opposite sides are $2\,F/(1+\sqrt{3})$ and $\sqrt{2}F/(1+\sqrt{3})$.

If the angles are β, γ, show that the components are $F\sin\beta/\sin(\beta+\gamma)$ and $F\sin\gamma/\sin(\beta+\gamma)$.

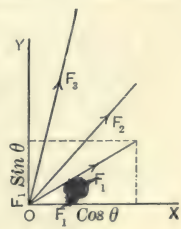
14. In a direct-acting steam-engine the piston-pressure P is equivalent to $P\tan\theta$ perpendicular to its line of action and $P\sec\theta$ along the connecting-rod, θ being the angle of inclination of the connecting-rod to the line of action of the piston.

15. If F be the force of the wind, and β, γ the inclinations of wind and sail to the keel AB of a boat, then (see ex. 5)

$$\text{headway force} = F\sin(\gamma-\beta)\sin\gamma;$$
$$\text{leeway force} = F\sin(\gamma-\beta)\cos\gamma.$$

16. Show that the headway force is greatest when the sail bisects the angle between the boat and the wind.

80. *Composition of Forces (Analytical Method).*—The analytical method of resolving a force into its components leads us to a method of combining forces which is often more convenient than the graphical method given in Art. 78. The two methods may be used to check one another.



Take three forces F_1, F_2, F_3, acting on a particle O. Through O draw *any* two lines OX, OY at right angles to each other, and let θ_1, θ_2, θ_3

denote the angles which the directions of F_1, F_2, F_3 make with OX. The components of

F_1 are $F_1 \cos\theta_1$ along OX, $F_1 \sin\theta_1$ along OY;
F_2 are $F_2 \cos\theta_2$ along OX, $F_2 \sin\theta_2$ along OY;
F_3 are $F_3 \cos\theta_3$ along OX, $F_3 \sin\theta_3$ along OY.

The components along OX being in the same straight line, may be combined by addition into a single force X; that is,

$$F_1 \cos\theta_1 + F_2 \cos\theta_2 + F_3 \cos\theta_3 = X. \quad . \quad . \quad (1)$$

Similarly, the components along OY, being in the same straight line, may be combined into a single force Y, or

$$F_1 \sin\theta_1 + F_2 \sin\theta_2 + F_3 \sin\theta_3 = Y. \quad . \quad . \quad (2)$$

Hence the original forces are equivalent to two forces X, Y acting in directions OX, OY at right angles to each other. The resultant of X, Y must therefore be the resultant of the original forces. Call it R, and let θ be the angle it makes with the axis of X; then

$$R\cos\theta = X, \qquad R\sin\theta = Y. \quad . \quad . \quad . \quad (3)$$

Square and add (remembering that $\cos^2\theta + \sin^2\theta = 1$), and

$$R = \sqrt{X^2 + Y^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which gives the *magnitude* of the resultant.

Divide the second of equations (3) by the first, and

$$\tan\theta = Y/X, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which gives the *direction* of the resultant.

Hence, since the resultant acts at O, it is known in position, magnitude, and direction, and is completely determined.

81. If we equate the values of X, Y in equations (1), (2), (3), we find

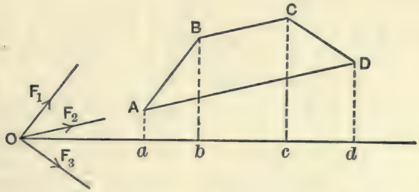
$$R\cos\theta = F_1 \cos\theta_1 + F_2 \cos\theta_2 + F_3 \cos\theta_3;$$
$$R\sin\theta = F_1 \sin\theta_1 + F_2 \sin\theta_2 + F_3 \sin\theta_3.$$

Now OX, OY are *any* two rectangular axes. Hence

The component in any direction of the resultant of a number of forces is equal to the sum of their components in the same direction.

This principle may readily be proved geometrically.

For let F_1, F_2, F_3 be three forces acting at O. Find the resultant $R(= AD)$ by constructing the polygon $ABCD$ (Art. 78).



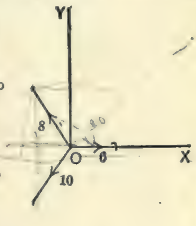
Let OX be the line along which the forces are to be resolved. From A, B, C, D let fall perpendiculars on OX. Then evidently

$$ad = ab + bc + cd.$$

But ad is the component along OX of R, ab of F_1, bc of F_2, and cd of F_3. Hence the proposition is proved.

Ex. 1. Three forces of 6, 8, 10 pounds act on a particle at angles of 120° to each other. Find the resultant in magnitude and direction. (Solved graphically, p. 78.)

Since the direction of OX is arbitrary, we may take it along one of the forces.



(a) Take OX to fall along the force 6. Then

$$\begin{aligned} X &= 6 \cos 0° + 8 \cos 120° + 10 \cos 240° \\ &= 6 - 8 \cos 60° - 10 \cos 60° \\ &= -3; \end{aligned}$$

$$\begin{aligned} Y &= 6 \sin 0° + 8 \sin 120° + 10 \sin 240° \\ &= 6 \sin 0° + 8 \sin 60° - 10 \sin 60° \\ &= -\sqrt{3}. \end{aligned}$$

$$\therefore\ R = \sqrt{9 + 3} = 2\sqrt{3};$$

$$\tan \theta = -\sqrt{3}/-3 = 1/\sqrt{3} \quad \text{and} \quad \theta = 30° \text{ or } 210°,$$

or the resultant is perpendicular to the force 8. Plot it.

(*b*) Take OX along the force 8. Then

$$X = 8\cos 0° - 6\cos 60° - 10\cos 60° \\ = 0;$$

$$Y = 8\sin 0° - 6\sin 60° + 10\sin 60° \\ = 2\sqrt{3}.$$

$$\therefore \quad R = \sqrt{0 + 12} = 2\sqrt{3}, \text{ as before;}$$

$$\tan\theta = 2\sqrt{3}/0 = \infty \quad \text{and} \quad \theta = 90°,$$

or the resultant is perpendicular to the force 8, and in the same position relative to the forces as before.

(*c*) Take OX along the force 10 and solve.

2. The forces of 2, 1, 4 pounds are inclined to the axis of X at angles of 0°, 60°, and 180°. Find the inclination of their resultant to the same axis. *Ans.* 150°.

3. Two forces of 1 and 2 pounds act at an angle of 120°. Show that the direction of the resultant is perpendicular to that of the smaller force.

4. $ABCDEF$ is a regular hexagon. Show that the resultant of the forces represented by AB, AC, AD, AE, AF is $6AB$.

5. A particle placed at the centre of an octagon is acted on by forces in directions tending to each of the angles of the figure and of magnitudes taken in order of 4, 6, 8, 10, 12, 14, 16, 18 pounds. Find the magnitude of the resultant and the angle it makes with the force 8.

Ans. $-8\sqrt{4+2\sqrt{2}}$ pounds; $\tan^{-1}(\sqrt{2}-1)$.

6. Three smooth pegs are driven into a vertical wall and form an equilateral triangle whose base is horizontal. Two equal weights of 10 lb are connected by a thread which is hung over the pegs; find the pressure on each peg.

Ans. $10\sqrt{3}$, $10\sqrt{2-\sqrt{3}}$ pounds.

7. Three forces P, Q, R act at angles α, β, γ. Find their resultant.

Ans. $\sqrt{(P^2+Q^2+R^2+2QR\cos\alpha+2RP\cos\beta+2PQ\cos\gamma)}$.

82. We have now the means of combining the separate forces that act on a particle into a single force producing the same motion as the separate forces. If the forces act so as to neutralize each other, this single force vanishes and the condition of the particle as to rest or motion will remain unchanged. If at rest before the forces begin to act it will remain at rest; if in motion, it will continue to move with uniform velocity in a straight line (Art. 46).

Thus suppose a particle O acted on by a number of forces whose resultant is R. If a force equal and opposite to R be applied, the resultant of the forces now acting on the particle is *nil* and the acceleration produced is *nil*. The forces are said to *equilibrate* one another, and the particle is said to be **in equilibrium.**

A particle is in equilibrium so long as its condition of rest or motion remains unchanged. Equilibrium therefore does not imply rest, but rest implies equilibrium. That branch of dynamics which considers the circumstances for which equilibrium is possible is called **Statics.**

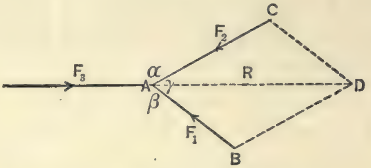
When the forces do not equilibrate an acceleration arises from the resultant force, and the particle has a motion compounded of the motion in its original path and that due to the resultant. That branch of dynamics which considers the circumstances under which change of motion takes place is called **Kinetics.**

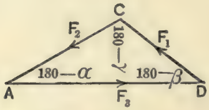
STATICS OF A PARTICLE.

83. The condition of equilibrium of forces acting on a particle is, in general terms, that the resultant of these forces shall be zero, or, in other words, that any one of the forces shall be equal and opposite to the resultant of the others. In most cases, however, more specific rules are necessary. The more important of these we shall now consider.

84. A. *Equilibrium under Three Forces—Lami's Theorem.* —Let three forces F_1, F_2, F_3 not in the same straight line act on the particle A. Let F_1, F_2 be represented by AB, AC, respectively. Find the resultant ($R = DA$) of F_1, F_2 by completing the parallelogram $ABDC$. For equilibrium to exist R and F_3 must be equal and opposite.

Now in the parallelogram the side DC is equal and parallel to BA. Hence the sides DC, CA, AD of the triangle DCA are proportional to the forces.





Notice that the directions of the forces are the same way round the triangle DCA. Thus if the first force is in the direction DC, the second *must* be in the direction CA, and the third in the direction AD. Notice also that the ratios of the sides of the triangle are the same as those of any similar triangle. Hence

If three forces in the same plane acting on a particle keep it in equilibrium they may be represented in magnitude and direction (but not in position) by the three sides of a triangle taken the same way round.

The converse of this proposition is known as the *triangle of forces.*

By means of the triangle DCA we may find a relation between the forces and their included angles. If α, β, γ be the angles between the directions of the forces, then in the triangle the angles are evidently $180° - \alpha, 180° - \beta, 180° - \gamma$; and since the sides of a triangle are as the sines of the opposite angles,

$$DC/\sin(180° - \alpha) = CA/\sin(180° - \beta) = AD/\sin(180° - \gamma),$$

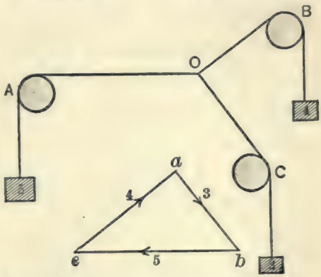
or

$$F_1/\sin\alpha = F_2/\sin\beta = F_3/\sin\gamma;$$

that is, *when three forces acting on a particle keep it in equilibrium, each is proportional to the sine of the angle between the directions of the other two forces.*

This is known as **Lami's Theorem.** It was first published by Father Lami in his *Mécanique* in 1687, the same year that Newton's *Principia* appeared.

Illustration.—Take a piece of board and drive in three smooth pegs, A, B, C, or place three pulleys at A, B, C. Run strings over the pegs, and knot together as at O. Suspend weights from the strings. Draw lines along the strings on the board, and plot the triangle abc with sides parallel to these lines. The sides of this triangle will be found to be proportional to the weights. Thus make the weights 3, 4, 5 oz., and it will be found that one angle of the triangle abc will be 90°.



Ex. 1. If two forces acting at a point are represented in magnitude and direction by two sides of a triangle, show that the third side represents the resultant.

2. Can forces of 5, 6, 13 pounds acting on a particle keep it at rest?

What is the least force that will produce rest?

Ans. 2 pounds.

3. If three forces acting at a point are represented by the sides of a triangle taken the same way round, show that they equilibrate. [Triangle of forces.]

4. If three forces acting at a point equilibrate, show that they may be represented by the three sides of a triangle perpendicular to the directions of the forces and taken the same way round.

5. Three forces acting at a point are determined by the three median lines of a triangle; show that they equilibrate.

6. Show that any force may be resolved into three forces of given magnitude, the direction of one of these forces being assumed.

7. Three forces P, Q, R acting on a particle equilibrate. The angle between P and Q is 90°, between Q and R 120°; show that

$$P : Q : R = \sqrt{3} : 1 : 2.$$

85. B. *Equilibrium under any Forces* (*Graphical Condition*).—If in the force diagram (Art. 78) the direction of R is reversed, the particle O will be in equilibrium under the action of F_1, F_2, F_3, F_4, S, where S is a force equal and opposite to R. Indicate the directions of these forces on the construction diagram and notice that they are the same way round. Notice, too, that $ABCDEA$ is a closed polygon. Hence

If any number of forces in the same plane acting on a particle keep it in equilibrium, they may be represented in magnitude and direction by the sides of a closed polygon taken the same way round.

This is the *graphical condition of equilibrium* of forces acting on a particle.

Ex. 1. $ABCD$ is a quadrilateral, and O the point of intersection of the lines bisecting the opposite sides. Show that forces represented by OA, OB, OC, OD acting at O equilibrate.

2. $ABCDEF$ is a regular hexagon. Show that equal forces acting along AB, CD, EF, AF, ED, CB equilibrate.

3. If forces acting at a point are represented by the sides of a polygon taken the same way round, show that they equilibrate. [Polygon of forces: converse of Art. 85.]

4. If four forces act at a point and are represented in magnitude and direction by the sides of a quadrilateral, either they keep the point at rest or have a resultant which is double a force represented by one of the sides or one of the diagonals, or four times the line joining the middle points of the diagonals.

86. C. *Equilibrium under any Forces* (*Analytical Condition*).—The analytical equivalent of Art. 85 may be deduced from Art. 80. For if the forces acting at O equilibrate, the resultant R must be equal to zero. Hence

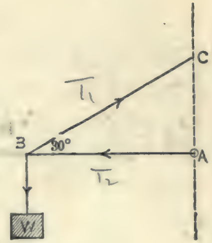
$$X^2 + Y^2 = 0,$$

which, since X^2 and Y^2 are both positive, can only be satisfied by $X = 0$, $Y = 0$, that is, by

$$F_1 \cos\theta_1 + F_2 \cos\theta_2 + \ldots = 0,$$
$$F_1 \sin\theta_1 + F_2 \sin\theta_2 + \ldots = 0.$$

Hence *if any number of forces in the same plane acting on a particle keep it in equilibrium, the sums of the components of the forces along any two straight lines at right angles to each other through the particle are equal to zero.*

These are the *analytical conditions of equilibrium* when any number of forces act on a particle.



Ex. 1. A rod AB whose weight may be neglected is hinged at A, and supports a weight W at B. It is held up by a thread BC fastened to a fixed point C vertically above A. If AB is horizontal and angle $ABC = 30°$, find the pull T_1 of the thread, and the thrust T_2 along the rod AB.

[The point B is in equilibrium under the forces T_1, T_2, W. We have therefore the two relations, resolving along a horizontal line through B as BA,

$$T_2 \cos 0° - T_1 \cos 30° - W \cos 90° = 0,$$

and along a vertical line through B as BW,

$$T_2 \sin 0° - T_1 \sin 30° + W \sin 90° = 0.$$

Solving these equations,

$$T_1 = 2W, \qquad T_2 = \sqrt{3}W.]$$

2. Solve Ex. 1 by Lami's theorem.

3. A thread whose length is $2l$ is fastened at two points A and B in the same horizontal and distant l from each other. The thread carries a smooth ring of weight W. Find the pull of the thread. *Ans.* $W/\sqrt{3}$.

4. In a canal with parallel banks a boat is moored by two ropes attached to posts on the banks. If the ropes are inclined at angles of 30°, 60° to the banks, compare the pulls in them, both ropes being in the same horizontal plane.

Ans. $1 : \sqrt{3}$.

5. A man weighing 160 lb rests in a hammock suspended by ropes which are inclined at 30° and 45° to two vertical posts. Find the pull in each rope.

Ans. 117.1 and 82.8 pounds.

6. A man weighing 160 lb is seated in a loop at the end of a rope 10 ft 3 in long, the other end being fastened to a point above. What horizontal force will pull him 2 ft 3 in from the vertical, and what will be the pull on the rope?

Ans. 36 pounds; 164 pounds.

KINETICS OF A PARTICLE.

87. If a number of forces act on a particle and the resultant be found, a certain motion is due to this resultant. If the particle has this motion, it is said to be *free;* if it has some other motion, the deviation must be owing to the entrance of some cause not accounted for, and the motion is said to be *constrained*. In free motion the particle is isolated from all causes tending to affect its motion except the acting forces, while in constrained motion this is not the case.

We have seen that the position of a particle is defined by its coördinates with reference to certain axes assumed to be fixed. A change in position is represented by changes in these coördinates. Hence the coördinates being either a distance and two angles or three distances, a point is said to have three degrees of freedom to move.

If the point is compelled to remain in an assigned plane (as the plane of the paper), its position is defined by two coordinates, and it is said to have two degrees of freedom and one degree of constraint. Similarly, if compelled to remain at the same distance from a fixed point it would move on the surface of a sphere, and have two degrees of freedom and one of constraint.

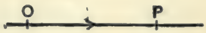
Again, if the point were compelled to remain in two planes, that is, in their line of intersection, it would have one degree of freedom and two of constraint: so also if compelled to remain in one plane and keep at the same distance from a fixed point, that is, to move in a circular path.

If compelled to remain in three planes, it can have only one position, their point of intersection, and is therefore wholly constrained.

88. Free Motion.—By means of the second law of motion (Art. 50) connecting force acting, particle acted upon, and acceleration produced, we are able to extend the geometrical properties of motion to particles acted upon by given forces. We shall first of all consider the particle to be unconstrained and have all degrees of freedom.

Various cases may arise depending on whether the force acting is in the direction of motion of the particle, and whether this force is or is not constant.

89. (A) **Force Constant.**—Let a particle weighing w lb be acted on by a force F pounds in the direction of its motion. The acceleration a produced is found from $F = wa/g$, or



$$a = Fg/w.$$

Let O be the initial position of the particle, P its position at the end of a time t, and let $OP = s$. If u is the initial velocity of the particle, the velocity v at the end of the time t is composed of that due to u and that due to the acceleration a. Hence (Art. 24)

$$v = u + at = u + Fgt/w; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the distance s passed over is given by

$$s = ut + \tfrac{1}{2}at^2 = ut + \tfrac{1}{2}Fgt^2/w. \quad . \quad . \quad . \quad . \quad (2)$$

The same results follow from the differential equation of motion. For let a particle of weight w be acted on by a force F. If s is the distance of the particle from the starting-point at time t, the equation of motion may be written (Art. 67)

$$\frac{w}{g}\frac{d^2s}{dt^2} = F;$$

or, since w, F, g are constant,

$$\frac{d^2s}{dt^2} = a,$$

where the constant quantity Fg/w is denoted by a.

This equation may be developed as in Art. 27, and the values of v, s given above found.

Ex. 1. An ice-boat weighing 1000 lb is driven for 30 sec from rest by a wind force of 100 pounds. Find the velocity acquired and the distance passed over.

Ans. 96 ft/sec; 1440 ft.

2. A train of 100 tons is running at the rate of 45 miles an hour. Find what constant force is required to bring it to rest in one minute. *Ans.* 6875 pounds.

3. The pull of the engine on a train whose weight is 100 tons is 1000 pounds. In what time will the train acquire a velocity of 45 miles an hour? *Ans.* 6 min $52\frac{1}{2}$ sec.

4. What pressure will a man weighing 150 pounds exert on the floor of an elevator descending with an acceleration of 4 ft/sec^2?

[The force causing the man to move is the resultant of the force of gravity on the man (150 pounds) and the upward force N of the platform on the man, that is, to $150 - N$ pounds.

But this force causes an acceleration 4 ft/sec^2 in a weight of 150 lb. Hence, from Newton's second law,

$$150 - N = 150 \times 4/32$$

and $$N = 131.25 \text{ pounds.}]$$

5. Find the pressure when the elevator is ascending with the same acceleration.

6. The scale-pans of a balance each weigh W lb and weights W_1, W_2 are placed in them. Find the pressures on the pans during the motion.

Ans. $2W_1(W + W_2)/(2W + W_1 + W_2)$ pounds and $2W_2(W + W_1)/(2W + W_1 + W_2)$ pounds.

7. A man who is just strong enough to lift 150 lb can lift a barrel of flour of 200 lb from the floor of an elevator while going down with an acceleration of 8 ft/sec^2.

8. A chain 16 ft long is hung over a smooth pin with one end 2 ft higher than the other end and then let go. Show that the chain will run off the pin in about 7/5 second.

90. *Falling Bodies.*—A case of special interest is that of bodies falling freely towards the earth. The acting force is the force of gravity which produces an acceleration g ft/sec^2 vertically downward if the motion takes place in a vacuum, and this no matter what the body may be. For places near the earth's surface the value of g is practically constant. (See Art. 62.)

The doctrine of Aristotle (B.C. 384–322) that the rate at which a body falls depends upon its weight was held until the time of Galileo (A.D. 1564–1642), who in 1590, by letting fall weights from the leaning tower of Pisa, demonstrated its falsity. By giving the true theory of falling bodies Galileo laid the foundation of the modern science of motion. Its development was carried forward by Huygens (1629–1693), who first discussed the dynamics of a system of bodies, Galileo having confined himself to that of a single body. It was, however, reserved for Newton to raise it to the rank of an exact science.

91. In the case of a body falling in the open air it is not true that the motion is uniformly accelerated. The air offers a certain resistance to the motion. The exact law of this resistance is not known. But for simplicity of treatment, and as close enough for most purposes and for moderate heights, it is *assumed* that a body falls in the air with a constant acceleration g ft/sec².

Hence if a body is projected vertically *downward* with a velocity u, its velocity v at the end of a time t would be found by putting $g = a$ in equation (1), Art. 24, or

$$v = u + gt,$$

and the vertical distance y fallen through by putting $g = a$ in equation (3), or

$$y = ut + gt^2/2.$$

Also by eliminating t,

$$v^2/2 = u^2/2 + gy,$$

which gives the velocity at any distance y fallen through.

Ex. Give the corresponding formulas when the body is projected vertically upward. [Write $-g$ for g.]

92. If the body falls from rest these equations reduce to

$$v = gt,$$
$$y = gt^2/2,$$
$$v^2/2 = gy.$$

The last equation gives the value acquired by a body falling through a height h to be

$$v = \sqrt{2gh},$$

and is sometimes said to be the velocity due to the *head* h.

Conversely,

$$h = v^2/2g,$$

which would give the head due to the velocity v.

93. The same results follow from the differential equation of motion. For let a body weighing w lb be projected vertically downward with velocity u. The acting force being w pounds, the equation of motion is

$$\frac{w}{g}\frac{d^2y}{dt^2} = w,$$

or $$\frac{d^2y}{dt^2} = g,$$

the axis of y being vertically downward.

This equation may be developed as in Art 27. The results are

$$v \text{ or } dy/dt = u + gt,$$
$$y = ut + \tfrac{1}{2}gt^2,$$

as already found.

If the body were projected vertically upward the equation of motion would be

$$\frac{d^2y}{dt^2} = -g,$$

and thence $$v = u - gt,$$
$$y = ut - \tfrac{1}{2}gt^2.$$

94. *Special Problems.*—A body is projected vertically upward with a velocity u ft/sec.

(a) To find the time of reaching the highest point of its path. At the highest point the vertical velocity $v = 0$, or

$$u - gt = 0,$$

and $$t = u/g \text{ sec.}$$

(*b*) To find the time of flight.

This would be the time of reaching the starting-point.

At this time $y = 0$, or

$$ut - \tfrac{1}{2}gt^2 = 0,$$

$$\text{and} \qquad t = 0 \text{ or } 2u/g \text{ sec},$$

which shows that the body is twice at the starting-point, once at the beginning of the motion when $t = 0$ and again at the end when $t = 2u/g$. The latter being the time from the beginning to the end of the motion is the time of flight.

The time of rising has been found $= u/g$ sec.

$\therefore$ time of falling $= 2u/g - u/g = u/g$ sec,

or time of rising = time of falling.*

(*c*) To find the greatest height reached.

This will be the value of y at the time u/g.

Substitute this value for t in

$$y = ut - \tfrac{1}{2}gt^2,$$

and we find $\qquad y = u^2/2g$ ft,

the greatest height.

Or it may be found by putting y in the form

$$y = \frac{u^2}{2g} - \left(t - \frac{u}{g}\right)^2 \frac{g}{2},$$

noting that y is greatest when $t - u/g = 0$, and then

$$y = u^2/2g \text{ ft}$$

as before.

The result also follows by putting $v = 0$ in

$$v^2/2 = u^2/2 - gy.$$

(*d*) To find the velocity at any point of the path. Here

$$v = u - gt,$$

---

* Descartes wrote to Mersenne: "I am astonished at what you tell me of having found by experiment that bodies thrown up in the air take neither more nor less time to rise than to fall again."

which gives the velocity at any time t;

$$\text{or} \qquad v^2/2 = u^2/2 - gy,$$

which gives the velocity at any height y.

95. Galileo's experimental demonstration of the formula $y = gt^2/2$ was as follows.

He caused balls to slide down grooves in inclined planes, measured off the distances 1, 4, 9, . . . on the grooves, and observed that the times of descent were represented by the numbers 1, 2, 3, . . . He used inclined planes because by retarding the descent he could observe the motion more accurately than if the balls fell freely. He had to assume that the law of descent was unaltered.

To measure small times he constructed a clock. This consisted of a vessel of water of large cross-section and having a small hole in the bottom. When a ball began to roll the water was allowed to run out and fall on a balance. The orifice was closed when the ball reached the end of its path. The times were as the weights of water discharged.

Much more precise measurements can now be made with Atwood's machine and a modern clock. The leading idea in the Atwood machine—that of diluting the acceleration of gravity—is the same as Galileo's, which he effected by the inclined plane.

Ex. 1. Find the distance passed over by a body falling freely during the sixth second of its fall.

$$\text{Dist. fallen in 6 sec} = 32 \times 6^2/2 = 16 \times 36 \text{ ft};$$
$$\text{Dist. fallen in 5 sec} = 32 \times 5^2/2 = 16 \times 25 \text{ ft}.$$

$$\therefore \text{Dist. fallen in the sixth second } (= 16 \times 11) = 176 \text{ ft}.$$

2. Two bodies are dropped from a height at an interval of 2 sec. Find the distance between them at the end of the next 2 sec. *Ans.* 192 ft.

3. Given that at New York a body falls through 48.24 ft in the second second of its fall, find the value of the acceleration of gravity there. *Ans.* 32.16 ft/sec².

4. A body is projected vertically upwards with a velocity of 160 ft/sec. Find (1) when it will come to rest, (2) the height to which it will rise. *Ans.* 5 sec; 400 ft.

5. In Ex. 4 find the velocity when the body is at a height of 256 ft. *Ans.* 96 ft/sec.

6. A body is projected vertically upwards with a velocity of

160 ft/sec. Find at what times it is 256 ft above the starting-point and find the total time of flight.

Ans. 8 sec or 2 sec; 10 sec.

7. It is required to project a body vertically to a height of 36 ft. Find the velocity of projection. *Ans.* 48 ft/sec.

8. A stone thrown vertically upwards is observed to be at a height of 96 ft in 2 sec. How much higher will it rise?

Ans. 4 ft.

9. A stone after falling for one second strikes a pane of glass, in breaking through which it loses one half of its velocity. How far will it fall the next second? *Ans.* 32 ft.

10. A body has fallen through 16 ft. With what velocity must another be shot downwards so as to overtake the other in 4 seconds? *Ans.* 36 ft/sec.

11. If s_1, s_2, s_3 are the distances described by a falling body in the pth, qth, rth seconds of its fall, prove

$$s_1(q-r)+s_2(r-p)+s_3(p-q)=0.$$

12. A stone is dropped into a well and after 2 seconds is heard to strike the water. Required the distance x to the surface of the water, the velocity of sound being 1100 ft/sec.

[2 sec = time of fall of stone + time of rise of sound
$= \sqrt{2x/g} + x/1100. \quad \therefore x = 60.5$ ft.]

13. If h is the height fallen through by a body in the nth second of its fall, show that $2h/g$ is an odd integer.

14. A body falls through the same distance at two different places on the earth's surface, and it is observed that the time of falling is t sec less and the velocity acquired n ft/sec greater at one place than at the other. If g_1, g_2 be the accelerations of gravity at the two places, show that

$$g_1 g_2 = n^2/t^2.$$

15. A two-ton hammer falls through 16 ft. Find its velocity. *Ans.* 32 ft/sec.

16. Three bodies are thrown vertically downwards with velocities u_1, u_2, u_3 from heights h_1, h_2, h_3 and reach the ground at the same instant. Show that

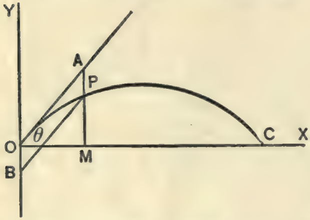
$$(h_1-h_2)/(u_1-u_2)=(h_2-h_3)/(u_2-u_3)=(h_3-h_1)/(u_3-u_1).$$

17. A man in an elevator which is rising with a uniform acceleration a tosses a ball vertically upwards with velocity v and after t seconds overtakes it. Show that

$$a+g=2v/t.$$

96. *Projectiles.*—If a body be projected with a given velocity in a direction not vertical and be acted upon by the force of gravity only, it is called a **projectile.** Neglecting the influence of the air, the body after projection is subject to a vertical acceleration g due to the force of gravity, as in the case of a falling body. The path will result from a combination of the motions due to the velocity of projection and to the vertical acceleration g. The name *trajectory* is often given to it.

Let O be the point of projection, OA the direction of projection, and u the velocity of projection.



The body if subject to the velocity u only would arrive in time t at a point A, where

$$OA = ut;$$

and if subject to the acceleration g only, would arrive at B, where

$$OB = gt^2/2.$$

But since the body receives both displacements simultaneously, its actual position is at P, the opposite vertex of the parallelogram OP to O. The locus of P for different values of t will be the path.

A good illustration is afforded by a jet of water issuing from a hose-pipe, or from an orifice in the side of a vessel.

97. In order to determine the properties of the path, it is convenient to find its equation referred to rectangular axes OX horizontal and OY vertical.

Let the velocity of projection u make an angle θ with OX, and let x, y denote the co-ordinates of P at the time t. Then

$$x = OM$$
$$= ut \cos\theta; \quad \ldots \quad (1)$$
$$y = PM$$
$$= AM - AP \text{ (or } OB)$$
$$= ut \sin\theta - gt^2/2. \quad \ldots \quad (2)$$

Eliminating t between these two equations, we find a relation between the co-ordinates x, y which holds for all values of t and is therefore the equation of the path. This gives

$$y = x \tan \theta - gx^2/2u^2 \cos^2 \theta,$$

which represents a parabola *—the parabola of Apollonius.

Ex. Refer to your Analytical Geometry and show that the co-ordinates of the vertex of this parabola are $u^2 \sin 2\theta/2g$, $u^2 \sin^2 \theta/2g$; and the latus rectum is $2u^2 \cos^2 \theta/g$. What is the distance of the directrix from the vertex?

Ans. $u^2 \cos^2 \theta/2g$.

98. *Special Problems.*—(*a*) To find the resultant velocity v and the direction of motion at any point P of the path.

Resolve the initial velocity u into horizontal and vertical components; that is, into $u \cos \theta$ along OX, and $u \sin \theta$ along OY. The horizontal acceleration is zero and the vertical acceleration is $-g$. Hence at the end of time t

$$\begin{aligned}&\text{horizontal velocity } v_1 = u \cos \theta;\\ &\text{vertical velocity } \quad v_2 = u \sin \theta - gt.\end{aligned}$$

Then, v_1 and v_2 being at right angles,

$$\begin{aligned}v^2 &= v_1^2 + v_2^2\\ &= (u \cos \theta)^2 + (u \sin \theta - gt)^2\\ &= u^2 - 2ut\, g \sin \theta + g^2t^2,\end{aligned}$$

and the *velocity at any time t* is found.

This may be written

$$v^2 = u^2 - 2g(ut \sin \theta - gt^2/2),$$

or

$$v^2/2 = u^2/2 - gy,$$

and the *velocity at any height y* is found.

---

* This was first demonstrated by Galileo. No attempt had been made up to his time to explain curvilinear motion of any kind. (See Art. 99.)

The direction of motion at P makes with the horizontal OX an angle α such that

$$\begin{aligned} \tan \alpha &= v_2/v_1 \\ &= (u \sin \theta - gt)/u \cos \theta \\ &= \tan \theta - gt/u \cos \theta, \end{aligned}$$

giving the *direction at any time* t;

$$\text{or} \qquad \begin{aligned} \tan \alpha &= \sqrt{v_2^2/v_1^2} \\ &= \sqrt{(u^2 \sin^2 \theta - 2gy)/u^2 \cos^2 \theta} \\ &= \sqrt{\tan^2 \theta - 2gy/u^2 \cos^2 \theta}, \end{aligned}$$

giving the *direction at any height* y.

(*b*) To find the time of reaching the highest point of the path.

At this point the vertical velocity $v_2 = 0$.

$$\therefore \qquad u \sin \theta - gt = 0,$$

$$\text{and} \qquad t = u \sin \theta/g,$$

the *time of reaching the highest point.*

(*c*) To find the greatest height reached.

This will be the value of y at the time $u \sin \theta/g$. Substitute this value of t in

$$y = ut \sin \theta - \tfrac{1}{2}gt^2,$$

$$\text{and} \qquad y = u^2 \sin^2 \theta/2g,$$

the *greatest height.*

(*d*) To find the time of flight, that is, the time in which the particle will reach the line OX. At this time $y = 0$, and

$$\therefore \qquad 0 = ut \sin \theta - \tfrac{1}{2}gt^2,$$

$$\text{whence} \qquad t = 0, \quad t = 2u \sin \theta/g;$$

which shows that the particle is twice on the line OX, once at O, the beginning of the motion, when $t = 0$, and again at C, the end of the motion, when $t = 2u \sin \theta/g$. The latter, being the time from the beginning to the end of the motion, is *the time of flight.*

(*e*) To find the range, that is, the distance OC.

The distance OC is described with the constant velocity $u \cos \theta$ in the time $2u \sin \theta/g$. Hence

$$OC = u \cos \theta \times 2u \sin \theta/g = u^2 \sin 2\theta/g,$$

the *range*.

The greatest value $\sin 2\theta$ can have is unity, and this occurs when $2\theta = 90°$ or $\theta = 45°$. Hence, for a given velocity, the *range* is *greatest* when the angle of projection is 45° and its value is u^2/g. This result is not true in practice, as we have not taken into account the resistance of the air. When the resistance of the air is considered, experiment gives an angle of about 34° instead of 45°.

99. It will be instructive to deduce the preceding results from the differential equations of motion.

Take the axes of co-ordinates OX, OY horizontal and vertically upward, respectively, and let x, y be the co-ordinates of the particle in its path at the end of the time t when referred to these axes.

The force acting is wholly vertical. Hence the horizontal component is zero and the equations of motion along OX, OY are, respectively

$$\frac{w}{g}\frac{d^2x}{dt^2} = 0, \qquad \frac{w}{g}\frac{d^2y}{dt^2} = -w,$$

or

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -g,$$

the minus sign entering since the axis of Y is drawn vertically upward.

Integrating with respect to t and noting that $u \cos \theta$ is the initial velocity along the axis of X and $u \sin \theta$ that along the axis of Y, we have

$$\frac{dx}{dt} = u \cos \theta, \qquad \frac{dy}{dt} = u \sin \theta - gt.$$

Integrating a second time,

$$x = tu \cos \theta, \qquad y = tu \sin \theta - \tfrac{1}{2}gt^2,$$

the constants of integration vanishing because when $t = 0$, we have $x = 0$, $y = 0$.

These four equations completely determine the motion of the particle.

Eliminating t between the last two equations, we have

$$y = x \tan\theta - gx^2/2u^2\cos^2\theta,$$

the equation to the parabolic path as already found (p. 99).

Special Problems.—(*a*) To find the velocity v at any time t.

$$\begin{aligned} v^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \\ &= (u\cos\theta)^2 + (u\sin\theta - gt)^2 \\ &= u^2 - 2ut\,g\sin\theta + g^2t^2, \end{aligned}$$

as already found (p. 99).

Also, if α is the angle which the velocity makes with the horizontal,

$$\begin{aligned} \tan\alpha &= dy/dx \\ &= (u\sin\theta - gt)/u\cos\theta \\ &= \tan\theta - gt/u\cos\theta, \end{aligned}$$

as before (p. 100).

(*b*) To find the greatest height and time of reaching it.

Make y a maximum in

$$y = tu\sin\theta - \tfrac{1}{2}gt^2.$$

The first differential coefficient is zero, or

$$0 = dy/dt = u\sin\theta - gt.$$

$$\therefore \quad t = u\sin\theta/g,$$

which gives the time of reaching the greatest height.

This value of t makes

$$y = u^2\sin^2\theta/2g,$$

the value of the greatest height, as before (p. 100).

(*b*) To find the range and its greatest value.

Put $y = 0$ in the equation to the path and

$$x = \tan\theta \times 2u^2 \cos^2\theta/g = u^2 \sin 2\theta/g,$$

as before (p. 101).

This is a maximum when

$$dx/d\theta = 0 \quad \text{or} \quad \cos 2\theta = 0,$$

which gives $\theta = 45°$, as before. Hence

$$\text{greatest range} = u^2/g, \quad \text{as before (p. 101).}$$

100. According to ancient theories of gunnery the trajectory was composed of (i) the *motus violentus* when the velocity was so great that the shot flew in a straight line, the extent of the *motus violentus* being the point-blank range, an error which prevails to this day; (ii) the *motus mixtus*, the curvilinear path of the trajectory; (iii) the *motus naturalis*, in which the body fell vertically downwards, as it tends to in reality in the neighborhood of the vertical asymptote when the resistance of the air is taken into account, so that after all the ancient theory was a fair imitation of the true trajectory.

Galileo combined the *motus violentus OA* and the *motus naturalis AP* into the *motus mixtus OP* [figure, p. 98] throughout the trajectory, and thus obtained a parabola; but when this theory was accepted by artillerists it was necessary to suppose that the velocity of the shot was very much less than its real value, a discrepancy that could not be detected till Robins invented his Ballistic Pendulum, 1740. (See Art. 255.)

When the resistance of the air is taken into account all the simplicity of the preceding theory disappears. (Greenhill.)

Ex. 1. Find the angle of elevation in order to attain a range of 10,000 ft, the velocity of projection being 800 ft/sec. *Ans.* 15°.

2. A man can throw a ball 100 ft vertically upwards. Find the greatest distance he can throw on a level field. *Ans.* 200 ft.

3. Prove that the range for an elevation of 30° is the same as for an elevation of 60°.

4. Compare the greatest heights in the two cases in Ex. 3. *Ans.* 1 : 3.

5. The range is four times the greatest height. Find the angle of projection. *Ans.* 45°.

6. The champion college record (1893) for throwing a base-ball is 349 ft. How high did the ball rise?

Ans. 87 ft 3 in.

Show that the time of flight was about 4.7 sec.

7. Show that numerically

(twice time of flight in sec)^2 = greatest height in ft.

8. A shot is fired horizontally from a roof 16 ft in height. Find the velocity with which it strikes the ground.

Ans. 32 ft/sec.

9. A ball is fired at an angle of 45° so as just to pass over a wall 10 ft high at a distance of 100 yards. How far from the wall will it strike the ground? *Ans.* 10.34 ft.

10. A body is projected horizontally from a given height h with a velocity u. Prove that the equation to the path is

$$2u^2y = gx^2.$$

Show that the range is $u\sqrt{2h/g}$ ft, and find the time of flight.

11. A railroad train is crossing a bridge 64 ft above a river at the rate of 30 miles an hour. At what distance from the abutment must a passenger drop a coin so as just to land at its foot? *Ans.* 88 ft.

12. An arrow shot horizontally from the top of a tree 64 ft high strikes the ground 100 ft from the foot of the tree. Find the time of flight and the initial velocity.

Ans. 2 sec; 50 ft/sec.

13. A stone is thrown with a velocity of 80 ft/sec so as just to pass over two trees 100 ft apart and each 50 ft in height. Show that the time of passing between the trees is 2.5 seconds and that the angle of throw is 60°.

14. When the elevation is α a projectile falls s ft short of the mark aimed at, and when the elevation is β it goes s ft too far. Show that if θ is the elevation for hitting the mark,

then $$2 \sin 2\theta = \sin 2\alpha + \sin 2\beta.$$

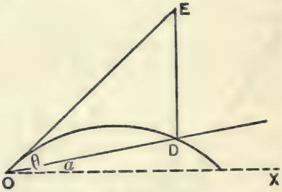
15. Develop a formula for finding the distance to which a stone could be hurled from an l-ft sling swung in a horizontal circle at n revolutions per second and h ft from the ground. *Ans.* $1.57ln\sqrt{h}$.

16. If t_1 be the time from O to any point P of the path of a projectile, and t_2 the time from P to A when OA is the range on a horizontal plane, then the height of P above OA is $\frac{1}{2}gt_1t_2$.

17. A ship sailing uniformly in a straight course with a velocity u is fired at from a battery at the instant when she passes nearest to it, her distance then being h. If v be the velocity of the ball, it is necessary in order to hit that

$$v^2 - u^2 > gh.$$

18. The inclination of a slope OD to the horizontal is α. From the foot O of the slope a gun elevated at an angle θ above the horizontal is fired up the slope with a velocity of u ft/sec. Find the range OD and time of flight t.



[Solved most readily by the method of Art. 96.

Draw DE vertical to meet OE at E. Then

$$OE = ut, \qquad DE = \tfrac{1}{2}gt^2.$$

But the sides of a triangle being proportional to the sines of the opposite angles,

$$DE/OE = \sin(\theta - \alpha)/\sin(90 + \alpha).$$
$$\therefore \tfrac{1}{2}gt^2/ut = \sin(\theta - \alpha)/\cos\alpha$$

and $$t = 2u\sin(\theta - \alpha)/g\cos\alpha.$$

Also $$OD/OE = \sin(90 - \theta)/\sin(90 + \alpha),$$

and $$\therefore OD = ut\cos\theta/\cos\alpha$$
$$= 2u^2\sin(\theta - \alpha)\cos\theta/g\cos^2\alpha.]$$

19. Find the velocity of projection in order that a shot fired at an elevation of 30° should hit an object half a mile distant on a hillside sloping 1 in 40. *Ans.* 319 + ft/sec.

20. If in (18) the shot strikes the plane perpendicularly, show that

$$2\tan(\theta - \alpha)\tan\alpha = 1.$$

21. In (18) show that the maximum range is $u^2/g(1 + \sin\alpha)$.

Show also that the angle of elevation for maximum range bisects the angle between the slope OD and the vertical.

22. Particles projected from a point with velocity u and in the same plane will at the end of any time t lie on a circle whose radius is ut. Plot this circle.

23. A particle is projected with velocity u so as to reach a given point a, b. If θ_1, θ_2 are the possible angles of elevation, show that

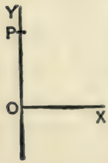
$$\cot\theta_1 + \cot\theta_2 = 2au^2/(2bu^2 + ga^2).$$

24. A shell discharged from a mortar just touched in its flight the top of a steeple, and in 4 seconds after fell at the distance of $3262\frac{2}{3}$ feet from the bottom of the steeple, from whence the report of its fall was heard at the mortar just 12 seconds after the explosion. Find the time of flight and the height of the steeple, taking the velocity of sound 1142 ft/sec. *Ans.* 7 sec ; 192 ft.

25. "Swift of foot was Hiawatha:
He could shoot an arrow from him
And run forward with such fleetness
That the arrow fell behind him!
Strong of arm was Hiawatha:
He could shoot ten arrows upward,
Shoot them with such strength and swiftness
That the tenth had left the bowstring
Ere the first to earth had fallen."

If one second elapsed between the discharge of each of the ten arrows and Hiawatha shot at his greatest range every time, he must have been able to run at least at the rate of 98 miles an hour.

101. (B) Force Variable and in the Direction of Motion.—(*a*) Let the force be directed to a fixed point O as center of force, and let its magnitude be proportional to the distance from O. Let YPO be the line of motion, and let P be the position of the particle w at any time t. If the distance OP is denoted by y, the equation of motion along OY is



$$\frac{w}{g}\frac{d^2y}{dt^2} = -cy,$$

where c is a constant, the minus sign being taken because the acceleration tends to diminish y.

Put $cg/w = \omega^2$, and the equation of motion becomes

$$\frac{d^2y}{dt^2} = -\omega^2 y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Multiply both sides of this equation by $2dy$ and integrate. Then

$$(dy/dt)^2 = C - \omega^2 y^2.$$

Let the motion start at a point distant r from O. Then when $y = r$, dy/dt or $v = 0$. Hence $C = \omega^2 r^2$ and

$$(dy/dt)^2 = \omega^2(r^2 - y^2), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

giving the velocity at any point of the path of a body attracted from rest towards the center of force O from a distance r.

To find the time of describing any distance.

Taking the square root of both members of eq. (2), we find

$$dt = -dy/\omega\sqrt{r^2 - y^2},$$

the minus sign being taken because the motion is towards O, and as t increases y decreases.

Integrating,

$$\omega t = \cos^{-1} y/r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

since $t = 0$ when $y = r$.

This may be written

$$y = r \cos \omega t,$$

the same equation as found in Art. 35. The motion is therefore a S.H.M. of period $2\pi/\omega$, of amplitude r, and whose oscillations are isochronous.

This conclusion is also evident from eq. (2). For $v = 0$ when $y = r$ or $y = -r$, and v has its greatest value when $y = 0$. Hence the particle starting from rest at P increases in velocity until it reaches O, decreases in velocity until Q is reached where $OQ = -r$, when it comes to rest. From Q

it returns under the action of the central force through O to P, when it again comes to rest. Hence it oscillates through the distance $2r$.

The time t of moving from P to O is found by putting $y = 0$ in eq. (3), that is,

$$\omega t = \cos^{-1} 0 = \pi/2,$$

and

$$t = \pi/2\omega.$$

The time of an oscillation being four times this, is $2\pi/\omega$.

Ex. 1. Suppose the earth a sphere 8000 miles in diameter, and that a cannon-ball were dropped at P into a vertical shaft POQ, passing through the center O. The attraction of the sphere on the ball may be taken to vary directly as the distance from the center. Hence, neglecting the resistance of the air, the ball would oscillate between P and Q. Required the time of an oscillation.

The acceleration of gravity at $P = g$ ft/sec².

Hence from equation (1), putting $y = r$, we have

$$\omega^2 r = g,$$

and time $= 2\pi/\omega = 85$ minutes, nearly.

2. Find the velocity of the ball at the earth's center.

Ans. 5 miles/sec nearly.

3. Show that the velocity at any time may be found from $v = -\omega r \sin \omega t$.

4. Compare the time of falling down the first 2000 miles with that of falling down the second 2000 miles. *Ans.* 2 : 1.

The following paragraph shows ignorance of the real conditions:

"Let us now conceive of a huge shaft of 8000 miles in length extending entirely through the earth to our antipodes. If a metal ball be dropped from either end of this shaft it is evident that it will drop 'down,' but we should not expect to see the antipodean metallic sphere come falling 'up' to us, nor would the Celestials in China expect to see the ball dropped from our side come falling 'up' to them."

(*b*) Let the force be inversely proportional to the square of the distance from O.

With the same notation as before, the equation of motion may be written

$$\frac{d^2y}{dt^2} = -\frac{c}{y^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where c is a constant. The minus sign is taken, since y is measured in a direction opposite to that of the motion.

Multiply both sides of the equation by $2dy$ and integrate. Then

$$\left(\frac{dy}{dt}\right)^2 = \frac{2c}{y} + \text{const.}$$

Let the motion start at a point distant r from O. Then when $y = r$, v or $dy/dt = 0$, and

$$v^2 \quad \text{or} \quad \left(\frac{dy}{dt}\right)^2 = 2c\left(\frac{1}{y} - \frac{1}{r}\right), \quad . \quad . \quad . \quad . \quad (2)$$

which gives the velocity at any point of the path of a body falling from rest towards the center of force O from a distance r.

For example, assuming the earth a homogeneous sphere of radius R, and that its attraction on a body outside of its surface is the same as if its weight were concentrated at its center O, we have for the acceleration of a falling body at its surface c/R^2. But this surface acceleration is known to be equal to g, so that

$$c/R^2 = g, \quad \text{or} \quad c = gR^2.$$

Hence the velocity with which a body falling from a height h would reach the earth's surface is found by putting $y = R$, $r = R + h$ in equation (2), and

$$\begin{aligned} v^2 &= 2c\{1/R - 1/(R+h)\} \\ &= 2gR^2\{1/R - 1/(R+h)\} \\ &= 2gh/(1 + h/R). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3) \end{aligned}$$

When the height h is small in comparison with the earth's radius R, the term h/R may be neglected, and we have the ordinary formula for the velocity of falling bodies already given in Art. 92.

To find the time of reaching the center of force.

Take the square root of both members of eq. (2). Then

$$\frac{dt}{dy} = -\sqrt{\frac{r}{2c}}\,\frac{y}{\sqrt{ry - y^2}},$$

the $-$ sign being taken because y decreases as t increases.

Integrating between the limits $y = r$ and $y = 0$, we find

$$t = \pi\sqrt{r^3}/2\sqrt{2c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Ex. 1. If a body fall from an indefinitely great distance it will reach the earth with a velocity of about seven miles a second.

2. Find the velocity of projection so that a body would not return to the earth.

3. Show that the time of reaching a point distant y from the center of force O is

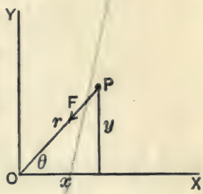
$$\sqrt{r/2c}\left\{\sqrt{ry - y^2} + r \cos^{-1} \sqrt{y/r}\right\}.$$

4. "Men called him Mulciber: and how he fell
From heaven they fabled, thrown by angry Jove
Sheer o'er the crystal battlement: from morn
To noon he fell, from noon to dewy eve,
A summer's day; and with the setting sun
Dropt from the zenith like a falling star,
On Lemnos th' Ægean isle."

Taking the summer's day 15 hours, show that the distance of Lemnos isle from heaven is about one-fourth of the distance to the moon.

5. If the earth were suddenly stopped in its orbit it would fall into the sun in a little over two months, the eccentricity of the orbit being neglected.

102. (C) Force Variable and Not in the Direction of Motion.—Let the particle w be projected in any direction and acted on by the attractive force F. The path will lie in the plane passing through the center of force and the direction of projection.



In this plane let O be the center of force, OX, OY the axes of coördinates, and x, y the coördinates of the particle P at a time t. Let $OP = r$, and the angle $POX = \theta$.

Let the force F vary inversely as the square of the distance r from O or $F = n/r^2$ when n is a constant.

The equations of motion along OX and OY are (Art. 50)

$$\frac{w}{g}\frac{d^2x}{dt^2} = -F\cos\theta = -\frac{n}{r^2}\times\frac{x}{r} = -\frac{nx}{r^3},$$

$$\frac{w}{g}\frac{d^2y}{dt^2} = -F\sin\theta = -\frac{n}{r^2}\times\frac{y}{r} = -\frac{ny}{r^3},$$

or

$$\frac{d^2x}{dt^2} = -\frac{cx}{r^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{d^2y}{dt^2} = -\frac{cy}{r^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

when $c = ng/w$.

The relation between x and y will give the equation to the path of the particle. To find it:

Multiply the first equation by y, the second by x, and subtract.

$$\therefore \quad x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

and by integration

$$x\frac{dy}{dt} - y\frac{dx}{dt} = k, \text{ a constant.}$$

Also, since $x = r\cos\theta$, $y = r\sin\theta$,

$$x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence, eliminating r^2,

$$\frac{d^2x}{dt^2} = -\frac{c}{k}\cos\theta\frac{d\theta}{dt}, \qquad \frac{d^2y}{dt^2} = -\frac{c}{k}\sin\theta\frac{d\theta}{dt};$$

and by integration

$$\frac{dx}{dt} - c_1 = -\frac{c}{k}\sin\theta, \qquad \frac{dy}{dt} - c_2 = \frac{c}{k}\cos\theta$$
$$= -cy/kr, \quad (4) \qquad\qquad = cx/kr, \quad (5)$$

when c_1, c_2 are constants.

Multiply the fourth equation by y, the fifth by x, and subtract, and we have

$$k + c_1y - c_2x = cr/k = c\sqrt{x^2 + y^2}/k,$$

the equation to a conic section with the origin at the focus.

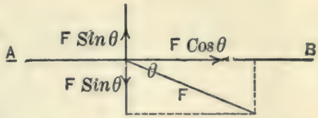
Hence *the path described by a particle under the action of a central force varying inversely as the square of the distance is a conic section whose focus is at the center of force.*

This is the case of planetary motion, the sun being at the center of force.

The further discussion of this problem will be found in works on mathematical astronomy.

103. Constrained Motion.—To a particle acted on by a force F in an assigned direction a certain path results. If the path differs from this, it must be owing to some cause which changes the motion, that is, to the action of another force. Hence if the path is prescribed, we may, by adding forces which with the original force will give a resultant which can produce this path, consider the motion free. The discussion will therefore come under the principles already laid down.

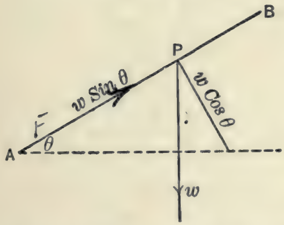
104. Motion on a Horizontal Plane.—Resolve the given force F into its two components $F\cos\theta$ along the plane AB and $F\sin\theta$ at right angles to it, θ being the inclinaton of F to the plane. To each of these forces an acceleration is due. But the particle is constrained so as not to move in the direction of the force $F\sin\theta$. This can be brought about by assuming that the plane exerts an equal force $F\sin\theta$ in the opposite direction, which force is known as the *reaction* of the plane.



As regards the horizontal stress between the particle and the plane, we can say nothing *à priori*. Experiment shows that it depends on the nature of the surfaces in contact. We shall for the present assume that the stress between the particle and the plane is normal only, or, as it is often expressed, that the plane is *smooth*.

If therefore a particle slides on a smooth plane under the action of a force F inclined at an angle θ, the reaction of the plane is $F\sin\theta$ and the force acting along the plane is $F\cos\theta$, which latter, being that to which the motion is due, is the effective force.

105. Motion on an Inclined Plane.—Suppose a particle P to slide down a smooth inclined plane AB under the action of gravity. If the particle weighs w lb, the force acting is w pounds vertically downward.

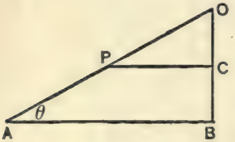


Resolve the vertical force w into two components $w\sin\theta$ along the plane and $w\cos\theta$ at right angles to the plane. The motion down the plane is evidently due to the component $w\sin\theta$ only.

Hence if a denotes the acceleration down the plane, the equation of motion is

$$w \sin \theta = wa/g,$$
$$\text{and } \therefore \qquad a = g \sin \theta.$$

The problem is therefore the same as that of a particle moving freely under an acceleration $g \sin \theta$.



Let the particle start from rest at O to slide down the plane. If the point P is reached in the time t, the velocity v attained will be given by (Art. 25)

$$v = tg \sin \theta. \quad . \quad . \quad . \quad (1)$$

The velocity may also be expressed in terms of the distance OP passed over. Thus (Art. 25)

$$v^2 = 2g \sin \theta \times OP$$
$$= 2g \times OC$$

if PC is let fall $\perp$ OB.

This velocity is the same as that attained at C by the particle falling freely from rest through the distance OC (Art. 92).

Hence the velocity acquired on reaching A, the foot of the plane, is found from

$$v = \sqrt{2gh} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

when $h = OB$, the height of the plane.

If the particle at O had an initial velocity u in the direction of the acceleration down the plane, then, corresponding to eqs. (1) and (2), we should have

$$v = u + tg \sin \theta; \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$
$$v = \sqrt{u^2 + 2gh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If t is the time of sliding down the plane from rest and l represents the length OA, then

$$l = \tfrac{1}{2}g \sin \theta \times t^2.$$

Hence if l be measured and t be observed, we have a method of computing the value of g. On account of the impossibility of finding a smooth plane an accurate value of g is not to be looked for by this method. Galileo, who first employed it, found g to be about 31 ft/sec². See Art. 116.

106. The differential equation of motion of a particle P sliding down an inclined plane is evidently

$$\frac{w}{g}\frac{d^2s}{dt^2} = w \sin \theta,$$

or

$$\frac{d^2s}{dt^2} = g \sin \theta,$$

where s is the distance of P from O at the time t.

This equation may be developed as in Art. 27 and the above results obtained.

Ex. 1. If a body slide down an incline whose length is twice its height, the acceleration of motion is one half that of the same body falling freely.

2. The starting-point of a switchback railroad is 121 ft above the terminus. Neglecting friction, find the terminal velocity of a car which starts from rest. *Ans.* 60 m. an hour.

3. A body starts from rest and slides down a smooth plane of height h and length l ft. Prove that the distance passed over in l sec is $\frac{1}{2}glh$ ft.

4. A car shunted up a 1% grade with a velocity of 60 miles an hour just reaches the top of the incline. Find the time occupied. *Ans.* 4 m 35 sec.

5. A car starts down a 1% grade with a speed of 15 miles an hour. Find the distance passed over in 2 minutes. *Ans.* 4944 ft.

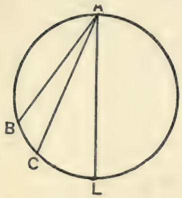
6. Find the time occupied in sliding down an inclined plane of height h and length l ft. *Ans.* $l\sqrt{2/gh}$ sec.

7. A weight of 4 lb is drawn up a smooth plane inclined at 30° by a weight of 4 lb joined to the first by a thread which passes over a smooth peg at the top of the plane and which descends vertically. Show that the acceleration is 8 ft/sec. If the weight on the plane were 8 lb, what would the motion be?

8. A weight of W lb placed on a smooth plane inclined at an angle θ to the horizontal is acted on by a horizontal force of W pounds. Show that the acceleration down the plane is $g\sqrt{2}\sin(\theta - \pi/4)$.

9. A body is projected up a smooth plane with velocity u. Show that it will go a distance $u^2/2g \sin\theta$ in a time $u/g \sin\theta$ and then come to rest.

10. Show that the time of descent from a given point to the center of a circle vertically below it is the same as that to the circumference down a tangent.



11. Prove that the time of descent of a particle starting from the extremity A of a vertical diameter AL is the same along all chords AB, AC, . . . of the circle.

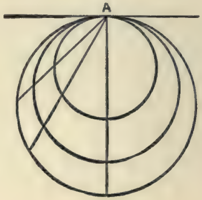
Ans. time $= 2\sqrt{r/g}$.

Would the same be true for chords of the circle ending at the lowest point L?

In this case show that the velocity attained on reaching the lowest point varies as the length of the chord.

12. Ex. 10 was first given by Galileo in this form:

Imagine gutters radiating in a vertical plane from a common point A at different inclinations. Place at A a like number of heavy bodies and let them descend the gutters simultaneously. The bodies will at any instant lie on a circle. The radii of these circles increase as the squares of the times.



[Show that in general $r = gt^2/4$.]

13. Find the line of quickest descent from a given point A to a given straight line BC.

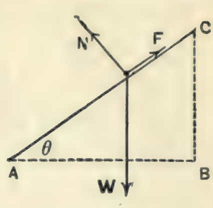
[Through A draw AB horizontal to meet BC at B. Make $BC = BA$. Then AC is the line required. Proof?]

14. The axis of a parabola is vertical. Show that the focal chord down which a particle would slide in the shortest time is inclined to the axis at an angle $\tan^{-1}\sqrt{2}$.

107. Equilibrium on an Inclined Plane.*—When a particle or body weighing W lb, placed on a smooth inclined plane, is acted on by gravity only, the acceleration down the plane is due to

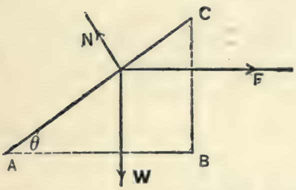
* This is of special interest, as being one of the problems of oblique forces first solved. The solution is due to Simon Stevinus of Bruges, Belgium (1548–1620). It may be found in Whewell's *Mechanics*, p. 44. The modern methods of statics date from Stevinus.

the component along the plane $W \sin \theta$ of the vertical force W pounds. For equilibrium to exist an equal acceleration must be applied in the opposite direction; in other words, the resultant force along the plane must be *nil*. Hence if a force F be applied parallel to the plane it will hold the particle in equilibrium if



$$F = W \sin \theta.$$

Acceleration perpendicular to the plane is prevented by the constraint of the plane, the normal reaction N being equal to the normal component $W \cos \theta$. Hence for equilibrium we have the two conditions,



$$F = W \sin \theta,$$
$$N = W \cos \theta,$$

(2) If a force F be applied parallel to the base of the plane, we shall evidently have equilibrium provided

$$F \cos \theta = W \sin \theta,$$

or $$F = W \tan \theta,$$

and $$N = W \cos \theta + F \sin \theta$$
$$= W \cos \theta + W \tan \theta \sin \theta$$
$$= W \sec \theta.$$

These results may also be obtained

(*a*) Directly by means of Lami's Theorem.

(1) When the force is parallel to the plane:

$$F/\sin (180° - \theta) = N/\sin (90° + \theta) = W/\sin 90°.$$

(2) When the force is parallel to the base:

$$F/\sin(180° - \theta) = N/\sin 90° = W/\sin (90 + \theta),$$

which easily reduce to the same relations as before.

(*b*) By the method of Art. 86.

(1) When the force is parallel to the plane.

Resolve along AC and

$$F - W \cos (90 - \theta) = 0.$$

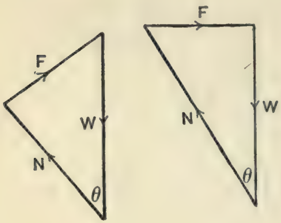
Resolve along a line perpendicular to AC, and

$$N - W \cos \theta = 0.$$

Whence $F = W \sin \theta$, $N = W \cos \theta$, as before.

(2) Similarly, when the force is parallel to the base we find

$$F = W \tan \theta, \quad N = W \sec \theta.$$



(*c*) By the method of Art. 84. For the particle is held in equilibrium by the three forces F, W, N pounds. Construct the triangle of forces by drawing lines parallel to the forces. Solving the right triangles, we have

$$F = W \sin \theta, \quad N = W \cos \theta,$$

or

$$F = W \tan \theta, \quad N = W \sec \theta,$$

as before.

Ex. 1. What force acting horizontally will support a weight of 10 lb on a plane inclined at 45° to the horizon?

Ans. 10 pounds.

2. Find the angle of inclination when a force of 10 pounds along the plane supports a weight of 20 lb. *Ans.* 30°.

3. A man pushes a garden roller weighing 80 lb up a plank 10 ft 3 in long and with one end 2 ft 3 in above the ground. If the handle is horizontal, find the force applied and the pressure of the roller on the plank.

Ans. 18 pounds; 82 pounds.

4. Show that a body of weight W resting on an inclined plane will be held in equilibrium by a force F parallel to the plane if

$$F : W = \text{height} : \text{length},$$

and parallel to the base if

$$F : W = \text{height} : \text{base}.$$

5. A cask weighing 400 lb is lowered into a cellar down a smooth slide inclined at 45° to the vertical. It is lowered by two ropes passing under it, one end of each rope being fixed, while two men pay out the other ends. Find the pull exerted by each man. *Ans.* $100/\sqrt{2}$ pounds.

6. Find the force required to haul a train weighing 100 tons up a 1% grade if the force required on the level is 10 pounds per ton. *Ans.* 3000 pounds.

7. On an inclined plane, if N is the reaction when the force acting is along the plane and N_1 the reaction when an equal force is horizontal, show that the weight $= \sqrt{NN_1}$.

8. On an inclined plane a force P parallel to the plane supports a weight W, and a horizontal force Q will also support W. Show that

$$P^{-2} - Q^{-2} = W^{-2}.$$

9. On an inclined plane a force P acting parallel to the plane can support a weight W_1, and acting horizontally a weight W_2; prove

$$W_1^2 - W_2^2 = P^2.$$

10. A body weighing W lb is kept at rest on a plane whose inclination is θ by a force P acting at an angle α with the plane. Show that

$$P = W \sin \theta / \cos \alpha,$$
$$N = W \cos (\alpha + \theta) / \cos \alpha.$$

11. If two weights W_1, W_2 support each other on a double inclined plane by means of a thread passing over the common vertex of the planes, and θ_1, θ_2 are the inclinations of the planes, then

$$W_1 / W_2 = \sin \theta_2 / \sin \theta_1.$$

108. Motion in a Circle.—Suppose a particle weighing w to move with constant velocity v in the circumference of a circle of radius r. It was shown in Art. 32 that the acceleration of motion is always directed towards the center O of the circular path; in other words, the particle is constrained by a constant force C directed towards O, and therefore having its direction always perpendicular to the direction of motion of the particle. To this force the name **centripetal force** is given.

The motion may be illustrated by supposing the particle attached by a thread to an axis through O and to revolve about this axis with uniform velocity, the motion being due to the initial velocity v and to the pull of the thread, and to these alone.

Suppose that the thread is cut when the particle reaches any point P. Since there is no force now acting, the particle will move in the tangent to the circle at P and with velocity v. This follows from the first law of motion. The pull of the thread acts only in changing the motion from uniform rectilinear to uniform circular motion; it acts towards the center O and constantly changes direction as the particle changes place; it acts constantly, and produces an acceleration a which changes at every moment the direction of the velocity v without changing its magnitude.

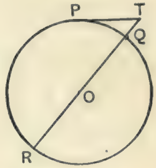
Consider further. The pull in the thread is a stress, the action on the particle towards the center O and the reaction of the particle from O. The two are equal by Newton's third law. The first is the centripetal force, and to the other the name **centrifugal force** is commonly given.

Notice that the centrifugal force does not act on the moving particle, but is *the force with which the moving particle acts upon the constraint when it is constrained to move in a circular path.* In other words, the centrifugal force is the tension of the thread directed outwards from O, the center of the circular path.

The pull of the thread acts as

a centrifugal force on O,
a centripetal force on P.

109. To find the magnitude of the centripetal or centrifugal force.



If the particle were free to move it would proceed in the tangent PT with velocity v, and after an interval t would be at T when

$$PT = vt.$$

But as it is found on the circumference at Q, it must be deflected by the central force C a distance TQ in this interval. Let a be the acceleration due to this force. Then

$$TQ = \tfrac{1}{2}at^2.$$

Prolong TQ through the center O to R. From geometry,

$$TQ \times TR = TP^2.$$

Proceeding to the limit, we have, since limit $TR = 2r$,

$$\tfrac{1}{2}at^2 \times 2r = v^2t^2$$

$$\text{or} \qquad a = v^2/r,$$

the same result as already found in Art. 32.

Now, from Newton's second law,

$$\begin{aligned} C &= wa/g \\ &= wv^2/gr, \end{aligned}$$

the value of the centripetal or centrifugal force.

If the whole circumference is described in t seconds, then

$$tv = 2\pi r$$

$$\text{and} \qquad C = 4\pi^2 wr/gt^2 \text{ pounds,}$$

if w is expressed in lb and r in ft.

If n is the number of revolutions per minute,

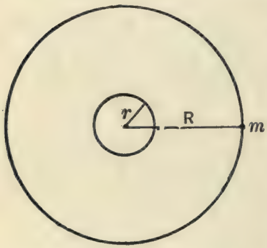
$$v = 2\pi rn/60 \text{ ft/sec}$$

$$\text{and} \qquad C = 0.00034wrn^2 \text{ pounds,}$$

a rule used by machinists.

The term centrifugal force was introduced by Huygens (1629–1695), and is very often misunderstood. See question 60, page 139. It is sometimes defined as the tendency of a body in motion to continue to move in a straight line. This definition may be compared with that given above.

110. A very remarkable application of the idea of centripetal force was made by Newton to test the truth of the law that the acceleration of gravity varies inversely as the square of the distance from the earth's center.* Observation shows that the moon revolves round the earth in an orbit nearly circular and with uniform velocity. If $v =$ velocity of moon, $R =$ radius of orbit, then the acceleration of the moon directed to the center of the earth is v^2/R. If this acceleration is due to gravity, we have



$$g' = v^2/R$$

when g' is the value of g at the distance R from the earth's center. Also, if the acceleration of gravity varies inversely as the square of the distance from the earth's center,

$$g'/g = r^2/R^2$$

when r is the earth's radius.

Hence, eliminating g', we have, as the condition to be satisfied if the hypothesis is true,

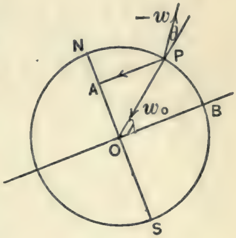
$$v^2R = r^2g.$$

Now from measurement, $R = 240{,}000$ miles, $r = 4000$ miles, $g = 32$ ft/sec², time of revolution of moon $= 27$ days 8 hours

* "In our day the principle is so familiar that we imagine it must have been an easy step to generalize from terrestrial to celestial mechanics. Yet neither Kepler the bold nor Galileo the far-seeing had the courage to make such a generalization. Even Newton was very timid in extending terrestrial to celestial laws."—*Lewes.*

= 2,360,000 seconds nearly, whence $v = 3315$ ft/sec. Substitute these values, and the expression will be found to check.

111. *Effect of the Earth's Rotation.*—Consider a particle at a point P on the earth's surface. Assuming the earth a sphere, the force w_0 on the particle due to gravity alone acts along PO towards the center O. But in consequence of the earth's rotation the particle receives a centripetal acceleration along PA perpendicular to NS, the earth's axis. Hence the force w exerted by the particle on its support is the resultant of w_0 along PO and the centripetal force C reversed (Art. 108).



If the particle were supported by a thread, the direction of the thread would be along w, so that the direction of w is that of a *plumb-line* at P. We may therefore say that the particle P is in equilibrium under

w_0 directed to the center O of the earth;
$-w$ directed upward along the plumb-line;
$-C$ the reversed centripetal force.

Let $\lambda =$ the latitude of $P = \angle POB$;
$r =$ the radius of the earth;
$\theta =$ inclination of plumb-line to radius OP.

Resolving the forces along OP and $\perp$ to OP, we have (Art. 86)

$$-w\cos\theta - C\cos\lambda + w_0 = 0, \quad . \quad . \quad . \quad . \quad (1)$$
$$-w\sin\theta + C\sin\lambda = 0, \quad . \quad . \quad . \quad . \quad (2)$$

from which to find w and θ.

Now $$C = 4\pi^2 w_0 \times PA/gt^2 \qquad \text{(Art. 109)}$$
$$= 4\pi^2 w_0 r\cos\lambda/gt^2.$$

But $t = 24$ hours, $r = 4000$ miles;

$$\therefore \quad C = w_0 \cos\lambda/289,$$

and $$w = w_0(1 - \cos^2\lambda/289), \text{ nearly,}$$

which shows how the attractive force is diminished in consequence of the earth's rotation.

To put it in slightly different form: If g_0 is the acceleration due to gravity alone, and g is the resultant acceleration in the direction of the plumb-line, then

$$g/g_0 = w/w_0,$$

and $$\therefore \quad g = g_0(1 - \cos^2\lambda/289),$$

which shows the variation of g with the latitude.

At the equator, where $\lambda = 0$, experiment gives

$$g = 32.09 \text{ ft/sec}^2.$$

Hence at the equator $g_0 = (32.09 + 0.11)$ ft/sec²,
and at latitude λ

$$\begin{aligned} g &= 32.09 + 0.11 - 0.11\cos^2\lambda \\ &= 32.09 + 0.11\sin^2\lambda \text{ ft/sec}^2. \end{aligned}$$

Also, from eqs. (1) and (2)

$$\tan\theta = \sin\lambda\cos\lambda/289,$$

giving the inclination of the plumb-line at any point to the radius of the earth through that point.

112. Attention was first called by Huygens to the influence of centrifugal force on the acceleration of gravity. A pendulum clock taken from Paris to Cayenne by Richer in 1671–1673 lost time unaccountably, and when adjusted and brought back to Paris it gained an equal amount. When Halley in 1677 went to the island of St. Helena to observe the stars of the southern hemisphere, he found his clock lost so much that the screw at the bottom of the pendulum did not enable him to shorten it sufficiently. Huygens, by demonstrating the greater centrifugal acceleration of the earth near the equator and the consequent diminution of g, cleared up the mystery. (See Art. 115.)

Ex. 1. A stone weighing 4 ounces is whirled 90 times a minute at the end of a thread 3 ft 6 in long. Find the pull of the thread. *Ans.* 2.4 pounds, nearly.

2. A locomotive weighing 60 tons is running at 15 miles an hour on a level track round a curve of 3300 ft radius (about 1° 44′). Show that the lateral pressure on the rails is 550 pounds.

3. Show that the lateral pressure on the rails by a locomotive of weight W lb, and running at the rate of v miles an hour on a curve of r ft radius is $0.067\, Wv^2/r$ pounds.

4. Find the velocity of projection in order that a bullet shot horizontally may travel round the earth continually.

Ans. 5 miles per sec.

5. The center of a steel crank-pin which weighs 16 lb is 12 in from the center of the engine-shaft. The shaft makes 180 revolutions per minute. Find the centrifugal force arising from the pin. *Ans.* 178 pounds nearly.

6. Show that the rubber tire of a bicycle becomes slack when running at more than $\sqrt{\pi g d T / W}$ ft/sec, where W is the weight of the tire in lb, T the tension in pounds, and d the diameter of the wheel in feet.

7. The attractive force of the earth is diminished in consequence of the earth's rotation by 1/289 part.

8. "We know by the mere consideration of centrifugal force that the whole sun attracts each ton of the earth with a force of a little more than a pound, and that the whole earth attracts each ton of the moon with a force of ten ounces."

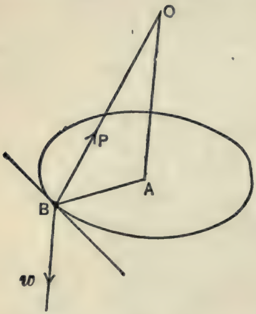
9. Show that the acceleration of a body falling freely at the equator is 1/9 ft/sec² less than it would be if the earth did not revolve on its axis.

10. A spring-balance graduated at New Orleans, in lat. 30°, will be in error 0.09% as shown by the beam-balance on the boundary between the U. S. and Canada in lat. 45°, and an equal amount at the equator, near the mouth of the Amazon.

11. If the earth were to revolve on its axis 17 times as fast as it does, no stress would exist at the equator between the earth and a body resting on its surface.

Per Contra.—The universe is not twice given, with an earth at rest and an earth in motion. It is accordingly not permitted us to say how things would be if the earth did not rotate. (Mach.)

113. *Conical Pendulum.*—Suppose a particle of weight w suspended by a thread from a point O and caused to swing about the vertical axis OA with a uniform velocity v in a circular path. Such an arrangement is called a **conical pendulum.**



Let l be the length of the thread OB, and let T be the time of a complete revolution, or the *period*.

Let B be the position of the particle at any time. Denote the angle between OB and the vertical OA by θ, and the radius AB by r.

The particle B is acted on by two forces—the weight w vertically downwards and the pull P along the thread BO. Since the resultant motion has a uniform velocity v in a circle with center A and radius r, the resultant of the acting forces must be a centripetal force directed to A. The magnitude of this force is wv^2/gr (Art. 109).

Hence the particle B would be in equilibrium under P, w, and a force equal and opposite to the centripetal force wv^2/gr. Resolving vertically and horizontally (Art. 86),

$$-w + P\cos\theta = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$-wv^2/gr + P\sin\theta = 0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also, since T is the period of revolution,

$$vT = 2\pi r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From these equations it readily follows that

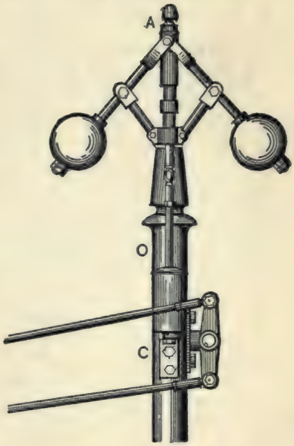
$$\cos\theta = gT^2/4\pi^2 l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$
$$P = w/\cos\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$
$$v^2 = gl\sin\theta\tan\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It is sometimes convenient to write these relations in a different form. Denote the vertical distance OA by h. Then

$$T = 2\pi\sqrt{h/g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$
$$Ph = wl, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$
$$hv^2 = gr^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The relation (7) shows that the period is the same as that of a simple pendulum of length h (Art. 115).

114. The conical pendulum may be used as a regulator of mechanical motion. An apparatus depending upon this principle, known as the *governor*, was applied by James Watt to the steam-engine.*



In the figure, which represents one form of governor, as the speed of the engine increases the spindle AO revolves more quickly, and the balls separate; as it diminishes, the balls come together. The slide O rises and falls accordingly, and by means of a set of levers, C, the steam-valves of the engine are acted on, and the supply of steam admitted to the cylinder regulated.

Ex. 1. Obtain the results (1) and (2) by means of Lami's theorem.

2. By taking moments about A and O in succession, show that

$$Ph = wl, \quad hv^2 = gr^2.$$

3. Show that, roughly,

$$h = (\text{twice period})^2/5.$$

4. Show how to find the value of the acceleration g of gravity by means of a conical pendulum.

[Observe T, measure h, and compute g from $g = 4\pi^2 h/T^2$.]

5. A train is running around a horizontal curve of 5445 ft radius at 30 miles an hour. Show that the surface of a basin

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* "If a pair of common fire-tongs suspended by a cord from the top be made to turn by the twisting or untwisting of the cord the legs will separate from each other with force proportioned to the speed of rotation. Mr. Watt adapted this fact most ingeniously to the regulation of the speed of his steam-engine."

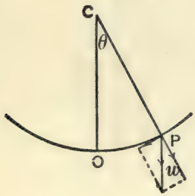
of water on the train will be inclined to the horizontal at an angle $\tan^{-1} 1/90$.

6. Find the length of a Watt governor that will run 60 revolutions per minute. *Ans.* 9.78 in.

7. If n is the number of revolutions per second of a conical pendulum having a thread l ft long and a bob weighing w lb, show that the pull of the thread is $4\pi^2 l n^2 w/g$ pounds.

8. A skater 5 ft 10 in in height in going round a ring 100 ft in diameter leans inward 5 in from the vertical. Find his speed. *Ans.* 10.7 ft/sec.

115. *Simple Pendulum.*—Consider a particle w suspended from a fixed point C by a thread of length l and moving in a vertical arc under the action of the force of gravity. The arrangement is called a **simple pendulum.** (See Art. 119.)



Let P be any position of the particle. Denote the angle between CP and the vertical CO by θ.

The force w on the particle acting vertically downward may be resolved into two rectangular components, $w \sin\theta$ along the tangent at P and $w \cos\theta$ along PC. The component along PC cannot affect the motion in the arc. The motion of the particle therefore depends on the tangential component only.

If a denotes the acceleration along the tangent at the point P, the equation of motion is

$$w \sin\theta = wa/g,$$

and therefore

$$a = g \sin\theta.$$

Now if θ is expressed in circular measure and is a small angle, we may replace $\sin\theta$ by θ. Hence

$$\begin{aligned} a &= g\theta, \text{ approximately, if } \theta \text{ be small;} \\ &= g\,(\text{arc } OP/\text{radius } CO), \\ &= \frac{g}{l} \times OP, \end{aligned}$$

or, since g and l are constant, the acceleration is proportional to the displacement OP, and therefore the motion in the small arc OP is a S. H. M. The time t of an oscillation being one half the period, we have (Art. 37)

$$t = \pi \sqrt{\text{displacement}/\text{acceleration.}}$$
$$= \pi \sqrt{OP/a}$$
$$= \pi \sqrt{l/g},$$

a result independent of the length of the arc. Hence in all small arcs the times of oscillation are the same, and the vibrations of a pendulum are therefore said to be *isochronous.*

Galileo, it is said, first tested this by watching the swinging of a bronze lamp (a masterpiece of Benvenuto Cellini) in the cathedral at Pisa, Italy, and measuring the time by counting the beats of his pulse.

116. A pendulum which makes one oscillation in one second is called a **seconds-pendulum.** If l be its length, we have, since $t = 1$,

$$1 = \pi\sqrt{l/g}$$
or
$$l = g/\pi^2.$$

An approximate value of g is 32.2 ft/sec². Hence

$$l = 39.12 \text{ inches},$$

or a pendulum which beats seconds is about 3 ft 3⅛ in in length. [The length of the meter is 3 ft 3⅜ in.] See Ex. 10, p. 131.

Conversely, if the length l is known, we may find the value of g, the acceleration due to gravity at any place. For then

$$g = \pi^2 l = 10l, \text{ roughly.}$$

To test whether the value of g was the same for all bodies, Newton placed different materials in equal small boxes and suspended them by equal long threads. He noted that the time of oscillation was the same for each used as a pendulum. The resistance of the air being the same in each case, he concluded that the acceleration due to gravity was constant at the same place for all substances, whatever their chemical composition.

117. If a pendulum of length l makes n oscillations in a given time τ at a place where the acceleration of gravity is g, then

$$\tau/n = t = \pi\sqrt{l/g}.$$

Suppose (1) the length of the pendulum changed to $l + \lambda$, when λ is small. By differentiation,

$$-\tau dn/n^2 = \frac{\pi}{2}\sqrt{1/gl}\,dl$$

and $$-dn = ndl/2l = n\lambda/2l,$$

giving the *loss* in the number of oscillations.

Thus, if a seconds-pendulum loses 20 sec a day, its change in length would be found from

$$20 = 86{,}400\lambda/(2 \times 39.12)$$

or $$\lambda = 0.018 \text{ inch}.$$

Suppose (2) the pendulum carried to a place where g has the value $g + \gamma$, the change γ being small and the length l remaining the same. Then

$$\tau dn/n^2 = \frac{\pi}{2}\sqrt{l/g^3}\,dg$$

and $$dn = ndg/2g = n\gamma/2g,$$

giving the *gain* in the number of oscillations.

Suppose (3) the pendulum carried to a height h above the earth's surface. Then, since g varies as $1/r^2$, r being the distance from the earth's center, we have

$$\tau/n = cr\sqrt{l}, \quad \text{where } c \text{ is a constant.}$$

Hence $$-\tau dn/n^2 = c\sqrt{l}\,dr,$$

$$-dn = ndr/r = nh/r,$$

giving the *loss* in the number of oscillations.

(4) If carried to a depth h below the surface, g varies as r, and the *loss* in the number of oscillations would be $nh/2r$.

Ex. 1. If a pendulum of length l vibrates n times in s seconds, prove

$$l\pi^2n^2 = gs^2.$$

2. Find the number of oscillations made by a pendulum a yard long in one minute. *Ans.* 62.57.

3. A plummet attached to a fine wire vibrates 60 times in 3 minutes. Find the length of the wire. *Ans.* 29.36 ft.

4. A seconds-pendulum makes 11 oscillations more in 24 hours at the foot of a mountain than at the top. Find the height h of the mountain. *Ans.* 1/2 mile nearly.

5. A pendulum which beats seconds at the surface when carried to the bottom of a mine loses 5 beats in 24 hours. Find the depth of the mine.

6. A seconds-pendulum is lengthened 1/100 inch. Show that it will lose 11 sec per day.

7. A clock gains 30 min/week. How much should the pendulum be lengthened for correct time?

Ans. 0.006 of its length.

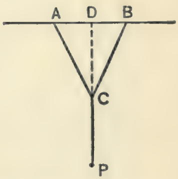
8. A clock keeps correct time at a place where g is 32.24 ft. Show that it will lose 3 min 21 sec/day at a place where g is 32.09 ft.

9. A pendulum clock is in an elevator which is going down with an acceleration of 1 ft/sec^2. Show that the clock loses 15/16 sec/min.

10. To find the length x of a seconds-pendulum. Count the number of vibrations n made in a day by a pendulum of known length l. Then compute x from

$$\sqrt{x/l} = n/86{,}400.$$

118. The paths found analytically in the examples of combining S.H.M.'s (p. 36) and many others may be traced mechanically by an apparatus known as *The Blackburn pendulum.*

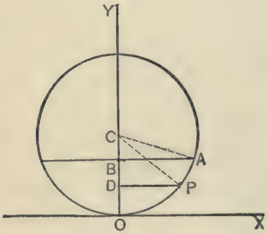


Two threads ACP, BCP fastened at two points A, B in a horizontal line are brought together by a small ring at C sliding over the threads. At P is attached a funnel to carry sand or ink to trace the curve made.

The bob P oscillates in the plane of the paper about C, and also perpendicular to this plane about AB as axis. Hence for small disturbances P has two S.H.M.'s at right angles, of which the period of that in the plane of the paper is $2\pi\sqrt{CP/g}$, and of the other $2\pi\sqrt{PD/g}$. If, for example, $CP = PD/4$, the curve traced out will be a parabola.

119. *Motion on a Vertical Circle.*—We shall now discuss the motion of a particle w constrained to move on a smooth vertical circle under the action of gravity. The problem of the simple pendulum is a special case.

Since a circle may be regarded as the limit of a polygon of an indefinitely great number of sides, the path on the circular arc may be treated as a series of inclined planes. The problem is therefore similar to that of Art. 105.



Take the origin at O, the lowest point of the circle; OX, the axis of X horizontal, and OY the axis of Y vertical, and passing through the center C.

Let A be the initial position of the particle and P its position at the end of a time t; a, b, the coördinates of A; x, y, the coördinates of P; $AP = s$, and angle $OCP = \theta$.

Draw AB and PD parallel to OX.

The direction of motion at P being ultimately along the tangent at P, the equation of motion is (Arts. 67, 20)

$$\frac{w}{g}\frac{d^2s}{dt^2} = -w\frac{dy}{ds},$$

$$\text{or} \qquad \frac{d^2s}{dt^2} = -g\frac{dy}{ds}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the minus sign being taken because y decreases as s increases.

Multiply by $2ds$ and integrate.

$$\therefore \quad v^2 = (ds/dt)^2 = -2gy + C.$$

But when $y = b$, $v = 0$, and $\therefore$ $C = 2gb$.

Hence $\qquad v^2 = 2g(b - y)$

$$= 2g \times BD, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or the velocity at P is the same as would be acquired in falling through the distance BD, the height of A above P (Art. 105).

If we put the angle $OCA = \beta$, equation (2) may be written

$$\begin{aligned} v^2 &= 2g(CD - CB) \\ &= 2gr\,(\cos\theta - \cos\beta), \quad . \quad . \quad . \quad . \quad . \quad (3) \end{aligned}$$

where r is the radius of the circle.

120. To find the time of motion in the small arc AP.

We have, writing ds/dt for v,

$$\left(\frac{ds}{dt}\right)^2 = 2gr\,(\cos\theta - \cos\beta).$$

Also $\qquad ds = rd\theta.$

$$\therefore \quad \sqrt{2g/r}\,dt = -\,d\theta/\sqrt{\cos\theta - \cos\beta}, \quad . \quad . \quad (4)$$

the $-$ sign being taken because θ decreases as t increases. This equation cannot be integrated by the ordinary methods. If, however, β is so small that powers above the second may be neglected, we have

$$\begin{aligned} \cos\theta - \cos\beta &= (1 - \theta^2/2) - (1 - \beta^2/2) \\ &= (\beta^2 - \theta^2)/2. \end{aligned}$$

Hence $\qquad \sqrt{g/r}\,dt = -\,d\theta/\sqrt{\beta^2 - \theta^2}. \quad . \quad . \quad . \quad . \quad . \quad (5)$

Integrating, and noting that when $t = 0$, $\theta = \beta$, we have

$$\sqrt{g/r}\,t = \cos^{-1}(\theta/\beta), \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which gives the value of t, the time sought.

The time of making an oscillation, that is, the time of moving through the arc AOA_1, is found by putting $\theta = -\beta$. This gives

$$t = \pi\sqrt{r/g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the result already deduced in Art. 115.

121. We may also determine the pressure N of the circle on the particle w.

The centripetal force along $PC = wv^2/gr$.

The resultant along PC of the force N and the vertical force w is $N - w \cos \theta$. Hence

$$N - w \cos \theta = wv^2/gr$$

and
$$N = w \cos \theta + wv^2/gr$$
$$= w\{\cos \theta + 2(\cos \theta - \cos \beta)\}$$
$$= w(3 \cos \theta - 2 \cos \beta).$$

Ex. 1. Compare the times of a particle sliding down a small arc AO of a vertical circle to the time of sliding down the chord AO. *Ans.* 11 : 14.

2. A particle starts from the highest point of a smooth vertical circle and slides down the convex side under the action of gravity. Find where it leaves the circle.

Ans. At a depth = radius/3.

3. A weight attached to a thread 2 ft long revolves in a vertical circle. Find the velocity at the highest point of the path that the thread may just remain taut. *Ans.* 8 ft/sec.

[At the highest point the centripetal accel. = the accel. of gravity.]

4. A weight w hanging at the end of a thread of length l is projected with a velocity u so as to describe a vertical circle. Show that the pull P of the thread and the velocity v at any point in the path whose vertical height above the lowest point is h are found from

$$u^2 = v^2 + 2gh;$$
$$Pl/w = u^2/g + l - 3h.$$

5. In (4) show that if $u^2 > 5gl$ the particle will perform complete revolutions.

In this case the pull at the lowest point is not less than $6w$ pounds.

6. A pendulum is let go from a horizontal position. If W is its weight, show that the pull on the thread when the bob is in the lowest position is $3W$.

7. If l_1 is the length of the seconds-pendulum at latitude λ and l the length at the equator, then

$$l_1/l = 1 - \cos^2 \lambda/289.$$

EXAMINATION.

1. State the elements which specify a force.

2. Forces may be represented by straight lines drawn in their direction and of lengths proportional to their magnitudes.

3. Which of Newton's laws implies the principle of the transmissibility of force?

4. Show clearly from Newton's second law that the resultant of two concurrent forces may be found by the same process as the resultant of two concurrent velocities.

5. Show that the parallelogram of forces may be illustrated experimentally.

6. What is meant by a force resolved in a given direction?

7. A force F acts at O in the line OA. Find its component along the line OB.

8. A force may be resolved into two components in any assigned directions.

9. What is meant by the resultant of a number of forces acting at a point?

10. Given two forces and their resultant, show how to find the angle between their directions.

11. Since a force can have no component at right angles to itself, how is it that a ship can be sailed at right angles to the wind?

12. To resolve a force P into two others such that the angle between them is $60°$ and their sum is the greatest possible.

13. Explain the action of the forces by which an arrow is discharged from a bow.

14. A body moves with uniform velocity in a straight line. Find the relation between the acting forces.

15. When are forces said to equilibrate?

16. "Equilibrium is not a balancing of forces, but of the effects of forces." Explain.

17. "Two or more forces can hardly be said to balance each other unless they all act on the same body." Why?

18. State and prove the triangle of forces.

19. State and prove the polygon of forces.

20. State the *graphical* condition of equilibrium when forces act at a point.

21. Three forces equilibrate. Show that no one of them can be greater than the sum of the other two.

22. State and prove Lami's theorem.

23. State the *analytical* conditions of equilibrium when two, three, . . . n forces act at a point.

24. What is meant by saying that the acceleration of a falling body is g ft/sec^2?

["Velocity is poured into the body at that rate."]

25. What is the average velocity of a body during the first second of its fall under the action of gravity?

26. A particle is shot upwards with a velocity u ft/sec. Find the height reached and the time of ascent.

27. The times of falling from rest through two successive equal distances are as $\sqrt{2} - 1 : 1$.

28. Prove that the distances passed over in the first, second, . . . seconds by a body falling freely under gravity are as the numbers 1, 3, 5, . . . respectively.

29. The velocity with which a body must be projected to reach a height h ft is $8\sqrt{h}$ ft/sec.

30. How did Galileo show experimentally that distances fallen through are as the squares of the times of falling?

31. Aristotle asserted that the time of falling of a body is inversely as its weight. Show that this requires the acceleration of gravity to be proportional to the square of the weight.

32. Find the momentum produced when a weight of 20 lb falls through a distance of 25 ft. *Ans.* 25 second-pounds.

33. Discuss the motion of a body acted on by a constant force (1) in the direction of motion; (2) not in the direction of motion.

34. A shot is fired from an elevation in a horizontal direction with a velocity of 1000 ft/sec. Draw a figure representing its position at the end of 1, 1.5, 2, 2.5, 3 seconds.

35. Explain how it is that a body projected at a given angle to the horizon describes a curve.

36. Find the path of a projectile in a vacuum. Show that the problem is one of kinematics.

37. Find the position and velocity of a projectile at the end of a given time t.

38. Find the range of a projectile on (1) a horizontal plane; (2) an inclined plane.

39. A man running with uniform speed along a level road throws a ball upward. In what direction must he throw it that it may return to his hand?

40. Two bodies are projected in any manner under the direction of gravity. Show that their relative velocity is constant throughout the motion.

41. From a balloon sailing horizontally at 60 miles an hour a ball is let drop. Find its direction after $2\frac{3}{4}$ sec.

Ans. 45° to the horizontal.

42. Prove that the elevation required to attain a range R with initial velocity u is given by

$$\sin 2\theta = gR/u^2.$$

43. If R is the range and T the time of flight of a projectile, the angle of elevation θ is given by

$$\tan \theta = gT^2/2R.$$

44. If u_1, u_2 are the velocities at the ends of a focal chord of the path of a projectile and v the horizontal velocity, show that

$$u_1^{-2} + u_2^{-2} = v^{-2}.$$

45. Two projectiles are shot from two points in the same horizontal plane with velocities u, v and at inclinations α, β. Show that if they meet,

$$u \sin \alpha = v \sin \beta.$$

46. Two bodies are projected from the same point at the same time and in the same direction, but with different ve-

locities. Show that the direction of the line joining them at any time is parallel to the line of projection.

[This forms a good illustration of the principle of the independence of forces. Art. 52.]

47. The world's record for putting the shot is 47 ft. Find the time the shot was in the air.

48. Find the velocity acquired by a particle sliding from rest down a length l of a smooth plane inclined at an angle θ to the horizontal.

49. A body slides from rest down a series of smooth inclined planes whose total heights are h ft. Show that the velocity at the bottom is $\sqrt{2gh}$ ft/sec.

50. A series of inclined planes begin at the same point and terminate in the same horizontal. Compare the velocities acquired by bodies sliding down them.

51. The problem of finding the line of quickest descent from a given point to a given line is equivalent to the geometrical problem of describing a circle tangent to the given line and whose highest point shall be the given point.

52. The angle of an incline is 15°. If the pressure on the plane is equal to the acting force, show that the inclination of the force to the plane is 60°.

53. Find the pressure exerted by a barrel of flour (196 lb) on an elevator floor (1) rising with uniform speed, (2) falling at a speed which increases 1 ft in each second.

Ans. 196; $189\frac{7}{8}$ pounds.

54. Give examples of a central stress as a tension and as a pressure.

[Weight at the end of a thread; train passing round a curve.]

55. Define centrifugal force. Illustrate by reference to the preceding question.

56. "When a weight tied to a thread is wheeled about a center the tension upon the thread is measured by the formula $Wv^2/gr =$ pull."

57. A particle moves with uniform velocity in a circle.

Show that the centrifugal force varies as the radius directly and as the square of the time of circuit inversely.

58. A fly-wheel in consequence of too rapid rotation goes to pieces. In what direction do the pieces fly off?

59. The centrifugal force of 1 lb making 1 revolution per minute in a circle of 1 ft radius is 0.000341 pound.

60. "You have heard [said the lecturer] of the wonderful centripetal force by which Divine Wisdom has retained the planets in their orbits round the sun. But it must be clear to you that if there were no other force in action the centripetal force would draw our earth and the other planets into the sun, and universal ruin would ensue. To prevent such a catastrophe the same Divine Wisdom has implanted a centrifugal force of the same amount and directly opposite, etc." Quoted by *De Morgan.*

What was the lecturer's probable meaning?

61. How far will a body fall towards the earth in one minute at the moon's distance? *Ans.* 16 ft.

62. What is the "discount in the attraction of the earth due to its rotation" at the equator?

63. How is the law of gravitation verified by means of the moon's motion?

64. How is it that as an elevator comes to rest in its descent a passenger feels as if he were being lifted up?

65. Prove that a conical pendulum of which the bob describes a horizontal circle at a depth of h inches below the point of support will make $188/\sqrt{h}$ revolutions per minute; and that if the thread is l inches long and the bob weighs W lb the tension of the thread is Wl/h pounds.

66. Find the time of oscillation of a pendulum of length l inches.

67. Describe a method of finding the length of a seconds-pendulum.

68. Show that the period of a conical pendulum is the same as the time of oscillation of a certain simple pendulum.

69. A pendulum-clock is carried in a balloon. Does it gain or lose time as the balloon rises?

70. Show that the lengths of pendulums vibrating at any place are inversely as the squares of the numbers of oscillations.

71. A pendulum is let fall from a height h. Find the height when the pull of the thread is equal to the weight of the plummet. *Ans.* $2h/3$.

72. The vibrations of a simple pendulum are isochronous.

73. Show how the height of a mountain or the depth of a mine may be found by counting the number of oscillations lost by a pendulum which beats seconds on the earth's surface.

74. A pendulum which beats seconds at Paris, where $g = 32.18$, is carried to New York, where $g = 32.16$. Find the number of seconds lost per day. *Ans.* 27 sec.

75. A seconds pendulum if carried to the top of a mountain 1 mile above sea-level would lose between 21 and 22 sec per day.

76. The New York Central R.R. tracks between Albany and Buffalo lie approximately along the 43d parallel of latitude. The weight of the Empire State Express is about 280 tons. A speed of 60 miles an hour is often attained. At that speed find the difference between the vertical pressures on the rails of train east and train west.

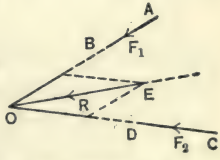
CHAPTER IV.

STATICS OF A BODY.

122. In the preceding chapter the behavior of a particle under the action of forces has been considered. This includes the case of a body under forces acting at the same point, and such that the resulting motion is a motion of translation only.

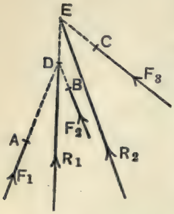
The directions of forces applied to a particle must necessarily all pass through the particle. Applied to a body the directions need not all pass through one point. Besides, forces applied to a body may cause it to change its form, as well as to change its position. To exclude the former we shall assume that the body while under the action of the forces retains an invariable form. It is *not* necessary to assume that the body cannot be made to change its form, or, as it is commonly stated, be rigid, but only that while the forces act the form should remain unchanged.

123. Composition of Concurrent Forces. — Suppose two forces F_1, F_2 to act along the lines AB, CD at the points B, D of a body. It is required to find their resultant.



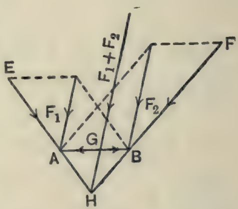
Prolong the lines of action to meet in a point O. The forces F_1, F_2 may be considered to act at this point (Art. 74). Their resultant R is found by completing the parallelogram OE as in Art. 75, and the line of action of

R is along the line EO. The value of R may be computed as in Art. 77.



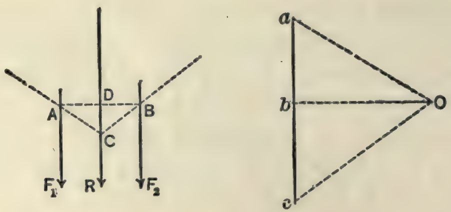
If the number of forces is more than two, combine two of them F_1, F_2 into a resultant R_1; next combine R_1 and F_3 into a resultant R_2; and so on. The final resultant R will be the resultant of all the forces in magnitude and position.

124. Composition of Parallel Forces.—If in Art. 123 the lines of action of F_1, F_2 are parallel, the construction for finding R fails. A special artifice is necessary.



Let F_1 be resolved into any two components EA, BA, and F_2 into the two components FB, AB (Art. 79). Since BA and AB are equal and opposite forces, the two forces F_1, F_2 may be replaced by EA, FB, whose resultant is found as in Art. 123. Hence the resultant of F_1, F_2 is found.

125. With a number of parallel forces the resolutions and compositions required to find R would produce a very



complicated force diagram. To avoid overlapping a construction diagram is introduced.

Let F_1, F_2 be the parallel forces drawn to any convenient scale. From any point a draw ab equal and parallel to F_1, and from b draw bc equal and parallel to F_2.

From any point O draw the lines Oa, Ob, Oc to the points a, b, c.

Now ab, or F_1, may be resolved into the two components aO, Ob, and bc, or F_2, into bO, Oc.

In order to determine the lines of action of these components we transfer the components to the force diagram. Draw any line AC parallel to aO to meet F_1 in A, AB parallel to bO to meet F_2 in B, and BC parallel to Oc. Then F_1 is equivalent to aO along AC, and Ob along BA; F_2 is equivalent to bO along AB, and Oc along BC. But the forces along BA and AB are equal and opposite. Hence F_1 and F_2 are equivalent to aO along AC, and Oc along BC.

Now the resultant of aO and Oc is ac, or $F_1 + F_2$, and the lines of action AC, BC intersect in C, which is therefore a point on the resultant.

Hence, if through C a line equal and parallel to ac is drawn, it will represent the resultant $F_1 + F_2$ of the two parallel forces F_1, F_2 in magnitude, direction, and line of action. The resultant is therefore completely determined (Art. 73).

The rule for plotting R that follows from this is given in Art. 129.

126. We may readily compute the position of the point D in which the resultant cuts AB. From similar triangles Oab, ACD; Ocb, BCD,

$$CD/AD = ab/Ob, \qquad CD/BD = cb/Ob.$$

Eliminate CD and Ob by dividing one equation by the other, and

$$BD/AD = ab/cb = F_1/F_2,$$

which, since the whole distance AB is known, gives the position of the point D.

Notice that the position of the point D does not involve the directions of the forces F_1, F_2. Being independent of their directions, it is a fixed point for forces F_1, F_2 acting at assigned points A, B in any lines of action. (Test this by a drawing.)

The point D is called the **center of parallel forces**—more strictly it is the center of the points of application of the forces.

To sum up: When two parallel forces act,

(*a*) *The magnitude of the resultant = the sum of the forces.*

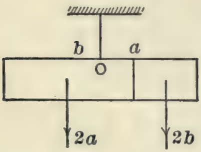
(*b*) *The direction of the resultant = the direction of the forces.*

(*c*) *The line of action of the resultant divides the line joining the points of application of the forces inversely as the forces.*

Ex. Show that $AD = AB \times F_2/(F_1+F_2)$
and $BD = AB \times F_1/(F_1+F_2)$.

127. Galileo's demonstration of principle (*c*) is very ingenious.

Imagine a uniform beam suspended at its middle point O and in a horizontal position. Let it be divided into two parts by a vertical plane so that the length of one part is $2a$ and of the other $2b$, the whole length of the beam being $2a + 2b$.



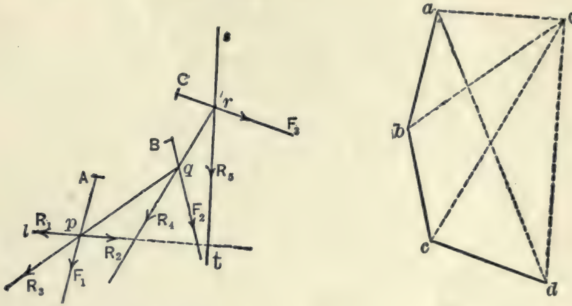
Imagine weights proportional to $2a$, $2b$ suspended at the middle points of the two parts; then the distances of these weights from the point of suspension are evidently b and a respectively.

Hence equilibrium exists if a weight a is suspended at a distance b on one side of the point O, and a weight b suspended at a distance a on the other side of the same point.

128. General Method of Combining Forces (*Graphical*).—We proceed now to show that the method of Art. 125 will in general apply to combining forces in the same plane, whether parallel or not.

Let F_1, F_2, F_3 be the forces plotted to scale. From any point a draw the line ab equal and parallel to F_1, from b draw bc equal and parallel to F_2, and from c draw cd equal and parallel to F_3. The line ad will represent the resultant R of F_1, F_2, F_3 in magnitude and direction (Art. 78).

To find the line of action of R: From any convenient point O draw lines Oa, Ob, Oc, Od to the angular points of the polygon $abcd$. Draw any line lp parallel to aO to meet F_1 in



p, pq parallel to bO to meet F_2 in q, qr parallel to cO to meet F_3 in r, and rs parallel to dO.

Then, remembering that the direction of a force is indicated by the order of the letters on the line representing it,

F_1, or ab, is equivalent to aO along lp and Ob along qp;
F_2, or bc, is equivalent to bO along pq and Oc along rq;
F_3, or cd, is equivalent to cO along qr and Od along sr.

Adding, and noting that the forces along pq and qp are equal and opposite, and that the forces along qr and rq are equal and opposite, we have

F_1, F_2, F_3 equivalent to aO along lp and Od along sr.

But the resultant of aO and Od is ad (or R), and the directions lp, sr of aO, Od intersect in t, which is therefore a point on the resultant R.

Hence, if through t a line is drawn equal and parallel to ad, it will represent the resultant R of F_1, F_2, F_3 in magnitude, direction, and line of action.

129. We hence derive the following rule for finding graphically the resultant of any number of forces in the same plane:

(1) Construct a polygon *abcd* to scale whose sides are equal and parallel to the forces; the closing side *ad* will represent the resultant in magnitude and direction.

When the forces are parallel, this polygon becomes a straight line (Art. 125), sometimes called the *load-line.*

(2) From any point O, called the *pole,* draw lines Oa, Ob, Oc, Od to the angular points of the polygon.

(3) Draw any line lp parallel to aO, pq parallel to bO, qr parallel to cO, and rs parallel to dO.

(4) A line through the intersection of lp and sr equal and parallel to ad will represent the resultant in magnitude, direction, and position.

130. Condition of Equilibrium (*Graphical*).—It is evident that a force equal and opposite to the resultant R would equilibrate the forces F_1, F_2, F_3. Hence *forces in a plane which equilibrate may be represented by the sides of a closed polygon abcd whose sides are parallel and in the same sense as the forces.*

The converse of this, that if forces acting in a plane can be represented by the sides of a closed polygon which are parallel to and in the same sense as the forces they equilibrate, is not true. For the polygon would be the same, no matter what the positions of the forces may be. This condition, in fact, provides against translation only. An additional condition to provide against rotation is necessary.

EXAMPLES. (1) *Forces not Parallel.*

1. Three forces are represented by AD, BC, DB in a parallelogram $ABCD$. Find their resultant. *Ans.* AC.

2. Four forces are represented by the sides AB, BC, CD, AD of a rectangle $ABCD$. Find their resultant.
Ans. $2BC$.

3. Show that if four forces be represented by the four sides of a quadrilateral taken the same way round they cannot equilibrate.

4. Forces 1, 2, 3, 4 pounds act along the sides AB, BC, CD

DA of a square, each side being 12 in long. Find their resultant.

Ans. $2\sqrt{2}$ pounds, parallel to CA and cutting CD in E so that $DE = 18$ in.

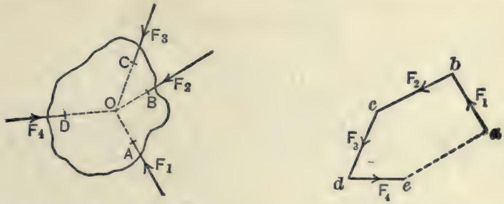
5. Forces 1, 2, 3 pounds act along the sides AB, BC, CA of an equilateral triangle, each side being 12 in long. Find their resultant.

Ans. $\sqrt{3}$ pounds, perpendicular BC and cutting BC in D so that $CD = 6$ in.

6. Three forces P, Q, R are represented in direction by the sides of an equilateral triangle taken the same way round. Show that their resultant is

$$\sqrt{(P^2 + Q^2 + R^2 - PQ - QR - RP)}.$$

7. If the lines of action of forces F_1, F_2, F_3, F_4 meet in a point O, the resultant R is completely determined by drawing



through O a line equal and opposite to ea, the closing side of the polygon whose sides ab, bc, ... are parallel and equal to the forces.

8. In a jib-crane a weight of 20 tons hangs from B. Find the pull P of the tie-rod AB if $AC = 12$ ft, $AB = 6$ ft, $BC = 15$ ft, and the weights of the parts are neglected.

[The jib CB, neglecting its weight, is in equilibrium under P, 20, and the reaction of the hinge at C, which latter must be along CB and equal to the thrust along the jib.

We may consider these three forces acting at B, and that this point is in equilibrium under P, 20, and the thrust along the jib. Hence the three forces are proportional to the sides of the triangle ABC (Art. 84). Hence $P = 10$ tons.]

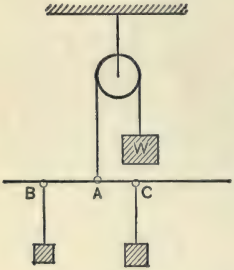
9. The post AC of a jib-crane is 10 ft, the jib CB is inclined at 30°, and the tie AB at 60°, to the vertical. If the weight lifted is 10 tons, find the stresses in AB, BC.

Ans. 10 tons; $10\sqrt{3}$ tons.

(2) *Forces Parallel.*

10. Let BC be a light rod suspended at A. Let weights of 3 oz, 4 oz be attached to it at the points B, C, 7 in apart. Find the upward pull at A and the distances BA, CA.

Ans. 7 oz; 4 in; 3 in.



[Make the apparatus and note if with these weights and measurements it will be in equilibrium.]

11. Draw the figure corresponding to that in Art. 125 when F_1, F_2 act in opposite directions. Show that

$$R = F_1 - F_2.$$

Illustrate by means of the above apparatus.

12. Two like parallel forces of 3 pounds and 5 pounds act at points 2 ft apart. Find their resultant in magnitude and position. *Ans.* 8 pounds; 9 in from 5.

13. Two unlike parallel forces of 3 pounds and 5 pounds act at points 2 ft apart. Find their resultant in magnitude and position. *Ans.* 2 pounds; 3 ft beyond 5.

14. Resolve a force of 100 pounds into two like parallel forces 10 ft apart, one of them being 2 ft from the given force. *Ans.* 80 pounds; 20 pounds.

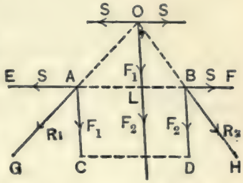
15. O is any point within a triangle ABC. If parallel forces proportional to the areas of the triangles OBC, OCA, OAB act at A, B, C respectively, show that the resultant must pass through O.

16. At the three vertices of a triangle are applied three parallel forces proportional to the opposite sides. Show that the center of parallel forces is at the center of the inscribed circle.

17. The center of three parallel forces at the vertices of a triangle ABC is at the intersection of the perpendiculars to the opposite sides. Show that the forces must be proportional to tan A, tan B, tan C.

18. Show that the resultant R of two parallel forces F_1, F_2 may be found as follows:

At A, B introduce two equal and opposite forces S, $-S$. Find the resultant R_1 of F_1, S, and R_2 of F_2, $-S$.

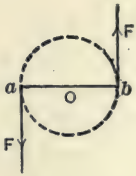


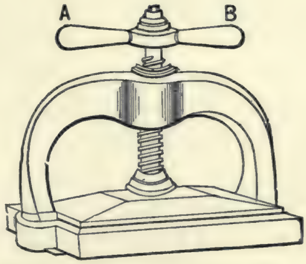
Let R_1, R_2 be transferred to O. Resolve R_1 at O into S and F_1 and R_2 into $-S$ and F_2. Combine S and $-S$, F_1 and F_2, at O.

19. Draw the corresponding figure when F_1, F_2 act in opposite directions.

20. "We have a set of hay-scales, and sometimes we have to weigh wagons that are too long to go on them. Can we get the correct weight by weighing one end at a time and then adding the two weights?"

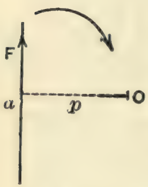
131. Couple—Moment.—If in Art. 125 the two parallel forces F_1, F_2 are equal and act in opposite directions, the points a and c in the construction diagram coincide, and the line ac which represents the resultant becomes zero. Also the lines AC, BC do not intersect, being parallel. Hence the resultant of the forces is zero and its point of application is at an infinite distance. In other words, the two forces cannot be reduced to a single force in a definite position and direction. Another method of measuring the effect of force must therefore be introduced.





To two equal parallel forces acting in opposite directions, but not in the same straight line, the name **couple** is given. A familiar example is seen in the use of a copying-press. The handle is pushed at A and pulled at B, the push and pull if equal forming a couple. In consequence the handle turns about the axis of the screw. So, too, in winding a watch a couple is employed.

132. The tendency of a couple on a body is to cause rotation. Through any point O in the plane of the couple draw aOb perpendicular to the direction of the forces. Consider one of the forces F and let the point O be fixed. The effect of the force in producing rotation about O will evidently depend upon the magnitude of the force and upon the distance aO of its line of action from O. We may say, therefore, that the product $F \times aO$ is a measure of the importance of the force in producing turning. This product is called the **moment** of the force F about the point O, the word moment being used in its old-fashioned sense of importance or influence. Hence the definition:



The moment of a force about a point O is the product of the measure of the force F and the perpendicular p let fall from the point upon the line of action of the force.

The term moment is from Lat. *momentum,* a particle sufficient to turn the scales, a moving cause. The expression "moment of a force" was used by Galileo to denote its effect in setting a machine in motion. Many writers use the term *torque* as the equivalent of "turning moment."

133. *The Unit Moment* is the moment of unit force acting at an arm equal to unit length. It is named a foot-pound, an inch-ton, etc., according to the units of length and force employed. Thus a moment of 10 foot-pounds is ten times the moment of a force of one pound acting at a distance of one foot from the point O.

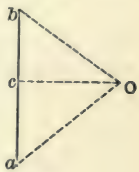
Less frequently the term pound-foot is used for unit of moment.

134. *Sign of the Moment.*—It is evident that the direction of turning about O is as indicated by the arrow in the figure the moment about O being considered $-$ or $+$ according as the tendency of the force F to produce turning is in the direction of motion of the hands of a clock [clockwise] or in the opposite direction * [counter clockwise]. In the figure the moment of F about $O = -Fp$.

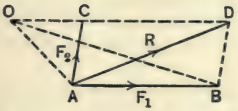
Or, better, since a moment has magnitude and direction, it may be represented by a straight line. Lay off along a line through O normal to the plane of O and F, that is, along the axis of rotation, a length numerically equal to the moment. That end of the line from which the rotation appears counterclockwise indicates the $+$ direction, or

the $+$ direction of the axis of a torque is that in which a right-handed screw would move if driven by the torque.

135. The moment of a force F with reference to a fixed point O may also be represented by an area numerically equal to it. For if ab plotted to scale represent the force F, and Oc the distance p of O from F, then the moment Fp or $ab \times Oc$, is represented numerically by twice the area of the triangle Oab, which has ab for base and Oc for altitude.



136. *Equation of Moments* (*Varignon's Theorem*).—The sum of the moments of two forces F_1, F_2 about any point O in their plane is equal to the moment of their resultant R about the same point.



(a) Let the directions of the forces F_1, F_2 be along the lines AB and AC, and the direction of the resultant R be along AD. From O draw OD parallel to the direction AB of F_1, meeting the directions of F_2 and R in C and D respectively; from D draw DB parallel to CA : then AB, AC, AD represent the forces F_1, F_2, R on the same scale.

Join OA and OB. Then

$$\text{moment of } F_1 \text{ about } O = 2\Delta OAB;$$
$$\text{moment of } F_2 \text{ about } O = 2\Delta OAC;$$
$$\text{moment of } R \text{ about } O = 2\Delta OAD.$$

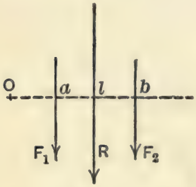
* This is the direction assumed by the whirl of cyclones.

But evidently

$$\Delta OAB + \Delta OAC = \Delta OAD.$$

$\therefore$ mom. of F_1 abt. $O +$ mom. of F_2 abt. $O =$ mom. of R abt. O.

(*b*) If the forces F_1, F_2 are parallel, from O let fall Oab perpendicular to their lines of action and meeting the direction of the resultant R in l. Then (Art. 126)



$$F_1 \times al = F_2 \times bl,$$

or $$F_1(Ol - Oa) = F_2(Ob - Ol),$$

or $$F_1 \times Oa + F_2 \times Ob = (F_1 + F_2)Ol$$
$$= R \times Ol,$$

or mom. of F_1 abt. $O +$ mom. of F_2 abt. $O =$ mom. of R abt. O.

(*c*) Generally if any number of forces in a plane act on a body, the sum of the moments of the forces about any point is equal to the moment of the resultant about the same point.

For, from the preceding, the sum of the moments of any two forces is equal to the moment of their resultant about the same point; of this resultant and a third force, that is, of the three forces, to the moment of their resultant; and so on until the last resultant is reached, which is the resultant of all the forces.

This is known as *Varignon's theorem of moments*.

Ex. 1. Prove Art. 136 (*b*) when the point O lies between F_1 and F_2.

2. A force of 5 pounds acts along one side of an equilateral triangle whose side is 2 ft long. Find the moment about the vertex of the opposite angle. *Ans.* $5\sqrt{3}$ foot-pounds.

3. A force of P pounds acts along the diagonal of a square whose side is $2s$ ft. Find the moments of P about each of the four angular points.

Ans. $0, Ps\sqrt{2}, 0, -Ps\sqrt{2}$ foot-pounds.

4. Find the moment of a force about any point in its line of action.

5. Show that the moments of two forces about any point on their resultant are equal and of opposite sign.

5*a*. Hence find the algebraic sum of the moments of a system of forces about any point on their resultant.

6. Apply the principle of Ex. 4 to find the position of the resultant of forces 3, 4, 5, 6 acting along the sides AB, BC, CD, DA of a square.

[Let the resultant cut AD, CD in E and F. Take moments about E and F and place each sum $= 0$. These two relations determine the positions of E and F. It will be found that the line EF is parallel to the diagonal AC.]

Get the same result by means of the polygon of forces.

7. Forces 1, 2, 3 act along the sides of an equilateral triangle. Show that the resultant is equal $\sqrt{3}$ and cuts the direction of 2 at right angles.

Plot this resultant.

8. Forces P, Q act along the sides AB, AC of an equilateral triangle ABC. Show that their moments about a point D in BC are equal if $BD = sQ/(P + Q)$, where s is a side of the triangle.

9. If P is the thrust along the connecting-rod of an engine, r the crank radius, and the connecting-rod is inclined to the crank axis at 150°, show that the moment of the thrust about the crank-pin is one half the greatest moment possible.

10. At what height from the foot of a tree must one end of a rope of length l ft be fastened so that a given force acting at the other end may have the greatest tendency to overturn the tree? *Ans.* $l/\sqrt{2}$ ft.

11. The post AC of a jib-crane (page 147) is 10 ft, the jib CB is inclined at 30°, and the tie AB at 60° to the vertical. If the weight lifted is 10 tons, find the moment about C tending to upset the crane. *Ans.* $50\sqrt{3}$ foot-tons.

137. *Moment of a Couple.*—In the couple represented by the figure on page 149 the moments of the forces F, F about O are $F \times aO$ and $F \times bO$ units of moment. The total moment about O is $F(aO + bO)$ or $F \times ab$ units of moment.

The distance ab between the forces F, F is called the **arm** of the couple. Hence the definition:

The moment of a couple [*or the torque*] *is the product of one of the forces forming the couple and the arm of the couple.*

The term torque is specially significant when applied to "turning moments" in machine shafting.

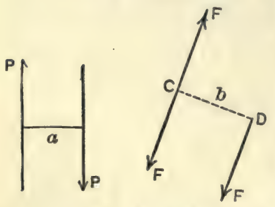
The *unit torque* is the same as the unit moment.

The sign of the torque is negative if the couple tends to cause rotation in the direction of driving a right-handed screw (Art. 134).

138. *Properties of a Couple.*—The moment of a couple depending only on the magnitude of the forces and the distance between them, the effect of a couple is not altered by turning the arm through any angle about one end, nor by moving the arm parallel to its former position in the plane of the couple, nor by changing the couple into another couple having the same moment.

It hence follows that *the resultant of a number of couples in a plane is a couple whose moment is equal to the sum of their moments.*

It also follows that *a single force F and a couple P, P acting in the same plane on a body cannot be in equilibrium.*



For let the moment of the couple be Pa, a being its arm. Replace the couple P, P by a couple F, F of arm b, so that $Fb = Pa$, and place it in the plane so that one of its forces F is opposite to the single force F. The two forces F, F at C are in equilibrium, leaving the single force F at D unbalanced. Hence there cannot be equilibrium.

The theory of couples was first given by Poinsot in his *Statique* (1829).

Ex. 1. A force and a couple acting on a body are equivalent to the single force acting in a direction parallel to its original direction.

2. Two like couples of the same moment are together equal to a single couple of twice the moment.

3. The side of a square $ABCD$ is s inches long. Along the sides AB, CD forces P act, and along AD, CB forces $2P$. Find the moment of the equivalent couple.

Ans. Ps inch-pounds.

4. Three forces are completely represented by the sides of a triangle taken the same way round. Show that they form a couple whose moment is represented by twice the area of the triangle.

5. Show that the sum of the moments of the two forces forming a couple about any point in their plane is equal to the moment of the couple.

139. General Method of Combining Forces (*Analytical*).—It is evident from the preceding articles that—

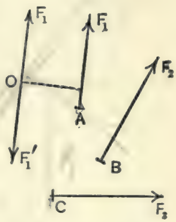
(*a*) *Any system of forces in one plane acting on a body may be reduced to a single force or to a couple.*

For (Art. 123) the forces may be combined two and two until we arrive at a single force or at two forces forming a couple.

This may be put in the slightly different form:

(*b*) *Any system of forces in one plane acting on a body, may be reduced to a single force acting at an assigned point, and a couple.*

For let the forces $F_1, F_2, \ldots$ act at the points $A, B, \ldots$ of the body. At any point O introduce two forces F_1, F_1', each equal to F_1 and of opposite directions. This will not disturb the equilibrium. Hence



$$F_1 \text{ at } A = F_1 \text{ at } A + F_1 \text{ at } O + F_1' \text{ at } O$$
$$= F_1 \text{ at } O + F_1 \text{ at } A + F_1' \text{ at } O.$$

But F_1 at A and F_1' at O form a couple whose moment is $F_1 p_1$, where p_1 is the distance of O from the line of action of F_1. Thus

$$F_1 \text{ at } A = F_1 \text{ at } O \text{ and the couple } F_1 p_1.$$

Similarly,

$$F_2 \text{ at } B = F_2 \text{ at } O \text{ and the couple } F_2 p_2,$$

and so on.

Adding, we have F_1 at A, F_2 at $B, \ldots$, equivalent to equal and parallel forces F_1 at O, F_2 at $O, \ldots$, together with the couples whose moments are $F_1 p_1, F_2 p_2, \ldots$

The forces at O may be combined into a single resultant R by the method of Art. 78 or of Art. 80. The couples may be combined into a single couple G by adding together their moments, so that (Art. 138)

$$G = F_1p_1 + F_2p_2 + \dots$$
$$= \Sigma Fp,$$

where Σ is the symbol of summation.

Hence the proposition is proved.

Cor.—Notice carefully the important principle employed in the preceding demonstration:

A force F acting at a point A is equivalent to (1) *an equal and parallel force F acting at any point O, and* (2) *a couple whose members are $+F$ acting at A and $-F$ at O with an arm p, the perpendicular distance between the members, and a moment Fp.*

Ex. 1. Plot the single force equivalent to a given force of 10 pounds and a given couple consisting of two forces of 4 pounds each at a distance apart of 5 inches.

Ans. 10 pounds distant 2 inches from the given force.

2. A couple and a single force are equivalent to a single force equal and parallel to the given force and at a distance from it found by dividing the moment of the couple by the single force.

3. Compare the above corollary with the second principle of Art. 138.

4. Forces 1, 2, 3, 4 pounds act along the sides AB, BC, CD, DA of a square 12 in long. Find the force passing through the center and couple which are together equal to these forces.

Ans. Force $= 2\sqrt{2}$ pounds along CA; couple $= 60$ inch-pounds.

140. General Conditions of Equilibrium (*Analytical*).—The general conditions of equilibrium may be stated in either of two forms according as we start from one or other of the methods of grouping forces just given.

(a) In general a system of forces acting in one plane on a body may be reduced to a single resultant force or to a couple (Art. 139 (*a*)).

Now a force cannot have zero moment unless the point of moment is in its line of action. The line of action is fixed by two points in it. Hence if the moment of the system about each of three points not in one line is zero, the resultant force must be zero. Also, since forces forming a couple never have zero moment, if the moment of the system is zero no couple can exist. Hence

If forces act in one plane on a body so that the sum of their moments about each of three points not in the same straight line is zero, the body is in equilibrium.

(*b*) In general a system of forces acting in one plane on a body may be reduced to a single force R referred to an arbitrary point O and a single couple G (Art. 139 (*b*)).

Now a single force and a single couple cannot equilibrate (Art. 138). Hence, that the system may be in equilibrium, we must have the two conditions:

1. The resultant force $R \quad = 0$;
2. The resultant couple $G = 0$.

That is, for equilibrium to exist there must be neither translation nor rotation.

These conditions may be put in a form more convenient for computation. For the force R being the resultant of a system of forces parallel to the given forces and acting at a point O, the condition $R = 0$ may be stated in the form given in Art. 86. Also if $G = 0$, then $\Sigma Fp = 0$, or the sum of the moments of the forces about the point O must $= 0$.

Hence the conditions of equilibrium may be stated:

(1) *The sum of the components of the forces in any direction* $OX = 0$;

(2) *The sum of the components of the forces in a* $\perp$ *direction* $OY = 0$;

(3) *The sum of the moments of the forces about any point in their plane* $= 0$.

This method is usually more convenient of application than method (*a*).

Ex. State the conditions of equilibrium if the forces are parallel. *Ans.* $\Sigma F = 0$; $\Sigma Fp = 0$.

Why only two conditions?

141. *Equilibrium under Three Forces.*—The case of equilibrium under the action of three forces admits of special simplification. For it is evident from Art. 124 that three parallel forces acting on a body may equilibrate.

If three forces not parallel equilibrate they must meet in a point. For two of the forces at least intersect. Their resultant acting at the point of intersection must equilibrate the third force; that is, all three must meet in one point. Hence

If three forces in the same plane keep a body in equilibrium they must be parallel or meet in a point.

The great value of this principle in the solution of problems is that the meeting of the lines of action enables us to obtain a geometrical or trigonometrical statement. Also, we can at once apply Lami's Theorem (Art. 84).

The results in any problem might also be found directly,

(1) by resolution along two rectangular axes (Art. 86);

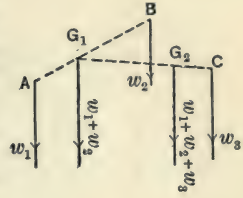
(2) by taking moments (Art. 140);

(3) graphically (Art. 85).

142. In the application of the general conditions of equilibrium of forces acting on a body (Arts. 140, 141) we are met by a difficulty which did not appear when the equilibrium of a particle only was considered (Art. 86). The force of gravity on a particle acts vertically downwards, and its point of application is the particle itself. As every particle of a body is acted on by the force of gravity, we must be able to find the position of the resultant force of gravity on the body before we can apply the general conditions of equilibrium to statical questions. This problem we now proceed to solve.

The subject proper will be resumed in Art. 152.

143. Center of Gravity.—Let $A, B, C, \ldots$ be a system of particles in one plane rigidly connected, and let these particles weigh $w_1, w_2, w_3, \ldots$ lb respectively. The forces due to gravity acting on the particles are $w_1, w_2, w_3, \ldots$ pounds. These forces are all directed towards the earth's center, which is so distant that they may be considered parallel.



Then, as in Art. 126, through whatever angle the system is turned the vertical forces w_1, w_2 at A, B will remain vertical and may without altering their effect be supposed to act as one force $w_1 + w_2$ at G_1, where

$$AG_1/BG_1 = w_2/w_1.$$

Similarly $w_1 + w_2$ at G_1 and w_3 at C, that is, w_1 at A, w_2 at B, w_3 at C, may be supposed to act as one force $w_1 + w_2 + w_3$ at G_2 through whatever angle the system is turned, where

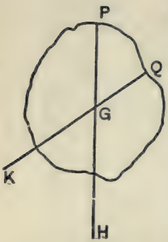
$$G_1G_2/CG_2 = w_3/(w_1 + w_2).$$

Adding thus particle to particle we see that the resultant of the system of forces will pass through a point G in the body such that its position remains the same no matter how the body is turned about. This point G is called the **center of gravity,** or **centroid,** or **mean center** of the body, so that

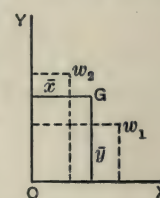
The center of gravity of a body regarded as composed of a system of particles rigidly connected is the point through which the resultant force of gravity on the body passes no matter how the body is turned about.

The idea of the center of gravity is due to Archimedes (B.C. 250). He determined the center of gravity of a triangle, parallelogram, trapezium, parabola, etc.

144. It is evident that if a vertical force equal and opposite to the resultant force of gravity on the body be applied at the center of gravity the body will be in equilibrium in any position. This suggests an experimental method of finding the center of gravity. Thus conceive the body suspended by a thread from a point P. The forces acting are the resultant force of gravity at the center of gravity and the pull of the thread. The lines of action of these forces must lie in the vertical through P. Hence, to find G, suspend the body from P, and strike the vertical PH; next suspend from any other point Q, and strike the vertical QK; the point of intersection of PH and QK will be the center of gravity G required.



145. *Coordinates of the Center of Gravity.*—Let a system of particles $A, B, \ldots$ be referred to horizontal and vertical axes OX, OY in their plane and drawn through any fixed point O. Also let x_1 y_1 be the coordinates of A; x_2, y_2, of B; $\ldots$; $\bar{x}, \bar{y}$, of G, the center of gravity.



The resultant $w_1 + w_2 + \ldots$ of the system of parallel forces $w_1, w_2, \ldots$ acts at G. Also, since the moment of the resultant about a fixed point O is equal to the sum of the moments of the separate forces (Art. 136), we have

$$(w_1 + w_2 + \ldots)\bar{x} = w_1x_1 + w_2x_2 + \ldots,$$

which may be written in the abbreviated form

$$\bar{x} = \Sigma wx/\Sigma w.$$

Similarly,

$$\bar{y} = \Sigma wy/\Sigma w.$$

Ex. 1. Weights 1, 2, 3, 4 oz are placed at the angles O, X, B, Y of a square $OXBY$ whose sides are each 1 ft in length. Find the coordinates of the C. G. referred to OX, OY as axes. *Ans.* 0.5 ft; 0.7 ft.

2. Equal particles are placed at the angles ABC of an equilateral triangle, and at the middle points D, E, F of the sides. Find the C. G. of the whole.

Ans. At G on AD when $GD = AD/3$.

3. Weights of 1, 2, 3 oz are placed at the angles of an equilateral triangle whose sides are 6 inches long. Find the distances of their C. G. from the angles.

Ans. $\sqrt{19}$, $\sqrt{13}$, $\sqrt{7}$ inches.

4*a*. If the velocities $v_1, v_2, \ldots$ of a system of particles $w_1, w_2, \ldots$ parallel to a fixed line OY in their plane are given at any instant, the velocity $\bar{v}$ of their C.G. is found from

$$\bar{v} = \Sigma wv/\Sigma w.$$

[For differentiate $\bar{x} = \Sigma wx/\Sigma w$ with respect to t.]

4*b*. Hence show that the momentum of the system collected at the C.G. is equal to the sum of the momenta of the separate particles.

4*c*. If the particles possess accelerations state the proposition corresponding to (4*a*).

5. Two weights, 1 oz and 2 oz, are joined by a light thread passing over a smooth peg and let go. Prove that during the motion the acceleration of their center of gravity is $g/9$.

6. Two bodies move with uniform speed along two straight lines inclined at an angle θ. Find the locus of their center of gravity. *Ans.* A straight line.

146. It follows from the forms of the expressions Σwx, Σw, in Art. 145, that the finding of the center of gravity is a problem of summation of molecular quantities and therefore one of integration. It is a problem of the integral calculus. In many cases, however, from the shape of the body or by the introduction of some artifice, integration may be avoided.

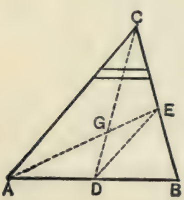
We shall confine ourselves to bodies that are homogeneous or of uniform density; that is, to bodies of such a nature that the particles are equal and the same number of particles make up the same volume in any and every part.

147. If a body of uniform density is symmetrical about a point the C.G. must coincide with this point. Thus a *sphere* being symmetrical about its center, the C.G. will be at the center of the sphere. So the C.G. of a *cube* will be at the intersection of the diagonals.

Again, with a *straight rod* of uniform cross-section, the number of particles on one side of the central section being equal to the number on the other side, the C.G. will be in this section, and if we consider the area of the cross-section to be indefinitely small, so that the rod becomes a straight line, the C.G. will be at the middle point of the rod.

For a *lamina* or plate of uniform thickness the C.G. will lie in a plane midway between the bounding planes. Consider the lamina indefinitely thin, and the C.G. will lie in the lamina itself and at its center of figure, if it has one. Thus the C.G. of a circular lamina is at the centre of the circle, of a rectangular lamina at the intersection of the diagonals, and so on.

148. In a *triangular lamina* not equilateral and therefore without a center of figure we have to introduce a special artifice in order to find the C.G.



Conceive the triangle divided into strips parallel to one side AB and of indefinitely small width. Each strip may be regarded as a uniform rod whose C.G. is at its middle point. These middle points all lie in a line CD, joining C to the middle point of AB. Hence the C.G. of the triangle lies in the line CD.

Similarly the C.G. may be shown to lie in AE, the line joining A to E, the middle point of BC.

Hence it is at G, the intersection of CD and AE.

To find the position of G on CD.

Join DE. Then DE is evidently parallel to AC.

Also, the triangles DGE and CGA are similar.

Hence $$DG/GC = DE/AC = 1/2,$$

and $DG = GC/2$

or $DG = DC/3.$

Similarly, $EG = EA/3.$

Hence *the C. G. of a triangle is on a median line of the triangle at two thirds its length from the vertex.*

Ex. 1. Prove that the C.G. of three equal weights placed at the angular points of a triangle coincides with that of the triangle itself.

[For W at B and W at C are equivalent to $2W$ at D, the middle point of BC. Also, $2W$ at D and W at A are equivalent to $3W$ at G when $DG = AG/2$. But G is the C.G. of the triangle.]

2. Three men support a heavy triangular board at its corners in a horizontal position. Compare the weights supported by each man.

3. The sides of a triangle are 3, 4, 5. Find the distances of the C.G. from the angles. *Ans.* $\sqrt{73}/3, 2\sqrt{13}/3, 5/3$.

4. Equal weights are placed at the angular points of a triangular board and also at the middle points of its sides. Find the C.G. of the system.

5. If G is the C.G. of a triangle ABC, prove

$$3(GA^2 + GB^2 + GC^2) = AB^2 + BC^2 + CA^2.$$

6. A series of triangles of equal area are described on the same base and on the same side of it. Show that their centers of gravity lie in a straight line.

7. A right triangle ABC is suspended from a point P in the hypotenuse AC, and one side AB hangs vertical. Show that $2AP = CP$.

8. G is the C.G. of a triangle ABC, O any point in its plane. The forces represented by OA, OB, OC acting at O have for resultant a force represented by $3OG$.

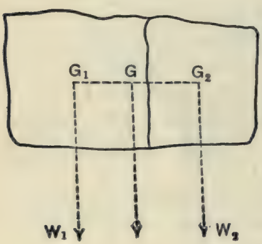
9. A uniform wire is bent into the form of a triangle with sides a, b, c. Show that the distances of the C.G. of the whole from the sides are as

$$(b + c)/a : (c + a)/b : (a + b)/c.$$

10. A triangle ABC, right-angled at C, is suspended successively from A and B. If β, γ be the angles made by AC, BC with the vertical in each position, show that

$$\cot\beta\cot\gamma = 4.$$

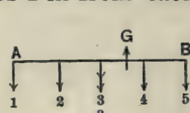
149. *C. G. of a System of Bodies.*—We pass at once to the center of gravity of a system of bodies rigidly connected, by considering that each body may be conceived concentrated into a particle of equal weight acting at the center of gravity of the body. Thus if G_1 is the center of gravity of a body weighing W_1 lb, and G_2 the center of gravity of a body weighing W_2 lb, then the vertical forces at G_1, G_2 being W_1, W_2 pounds, the position of the center of gravity G of the system will be found from



$$W_1 \times G_1G = W_2 \times G_2G.$$

Similarly, if the positions of G, G_1 are given, that of G_2 may be found.

Ex. 1*a*. Weights of 1, 2, 3, 4, 5 lb are strung on a uniform rod AB, whose weight is 3 lb at distances of 4 in from each other. Find the point at which the rod will balance.



[The weight of the rod may be supposed collected at the middle point (Art. 148). We have then to find the C.G. of the weights 1, 2, 6, 4, 5 lb placed 4 in apart. Take moments about A.

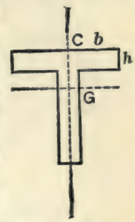
$$(1+2+6+4+5)\times AG = 2\times 4 + 6\times 8 + 4\times 12 + 5\times 16).$$

$$\therefore\ AG = 10\tfrac{2}{9}\text{ in.}$$

Check the result by taking moments about G; also about other points.]

1b. What is the pressure on the point of support?

2. A bicycle weighing w_1 lb has its wheels a in apart. The C.G. of the rider, who weighs w_2 lb is a_2 in behind the front wheel, and the C.G. of the bicycle is a_1 in behind the same wheel. Find the pressure of the rear wheel on the ground. *Ans.* $(w_1a_1 + w_2a_2)/a$ pounds.

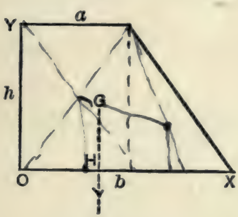


3. A cylindrical jar 6 in deep weighs 3 lb and holds 2 lb of water. Its C.G. is $3\frac{1}{2}$ in from the top when empty. Find its position when the jar is full of water. *Ans.* 3.3 in from the top.

4. Find the C.G. of a T-iron whose depth of flange $= h$, depth of web $= h_1$, breadth of flange $= b$, breadth of web $= b_1$.

Ans. C.G. $= (bh^2 + b_1h_1^2 + 2b_1hh_1)/2(bh + b_1h_1)$.

5. A common form of cross-section of a reservoir wall or embankment wall is a trapezoid whose top and bottom sides are parallel. If top side $= a$, bottom $= b$, and height $= h$, show that



$$OH = \frac{1}{3}\left(a + b - \frac{ab}{a+b}\right);$$

$$GH = \frac{h}{3}\left(\frac{2a+b}{a+b}\right).$$

[Divide the figure into a rectangle and a triangle or into two triangles.]

6. Show that if one fourth part ADE of a triangle ABC is cut off by a line DE parallel to the base BC, the distance of the C.G. of the remainder from the vertex A is 7/9 of the median AF.

7. CA and CB are the arms of a bent lever 2 ft and 4 ft in length, respectively, and inclined at an angle of 60°. Find the distance of their C.G. from C. *Ans.* $\sqrt{21}/3$ ft.

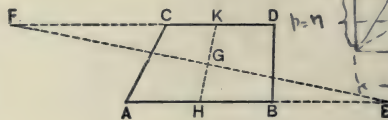
8. A triangular bracket projects 30 ft from a building and weighs 2 tons. A load of 3 tons hangs from the vertex. Find position of the C.G. of the whole.

Ans. 22 ft from the building.

9. In a painter's palette, formed by cutting a small circular disk from a larger one, if the diameters of the disks are 1 to n and the distance between their centers a, show that the dis-

tance of the C.G. of the palette from the center of the larger disk is $a/(n^2 - 1)$.

10. Show that the center of gravity G of a trapezoid lies in the straight line HK joining the middle points of AB and CD, the parallel sides.



11. Show in (10) that

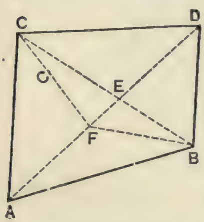
$$HG : KG = AB + 2CD : 2AB + CD.$$

12. Prove the following construction for finding the C.G. of a trapezoid $ABCD$.

Prolong AB, making $BE = CD$; prolong DC, making $CF = AB$. The point of intersection of EF and HK, the line joining the middle points of AB, CD, is the C.G.

[This construction is due to Poinsot.]

13. Prove the following rule for finding the C.G. of a quadrilateral $ABDC$:



Draw the diagonals AD, BC. Make $AF = DE$. The C.G. of the triangle CFB is also that of the quadrilateral.

14. Prove the following rule for finding the C.G. of a quadrilateral.

Join each vertex with the middle point of the opposite side. Join the intersections of the lines from opposite vertices. The intersection of these lines is the C.G.

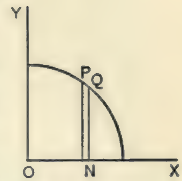
15. If x_1, y_1; x_2, y_2; x_3, y_3; x_4, y_4 are the coordinates of the angular points of a quadrilateral, and x_5, y_5 of the intersection of the diagonals, and $\bar{x}$, $\bar{y}$ of the C.G, then

$$\bar{x} = (x_1 + x_2 + x_3 + x_4 - x_5)/3;$$

$$\bar{y} = (y_1 + y_2 + y_3 + y_4 - y_5)/3.$$

150. *General Method of Finding the C.G.*—We now return to the general formula of Art. 145, which includes all cases, and shall show its application to some of the problems already solved and to others which do not admit of any special artifice in their solution.

Ex. 1. To find the C.G. of a thin lamina in the form of a quadrant of a circle of radius r.



[Take the origin at the center O, and let the lamina h be divided into strips by lines parallel to the axis of Y.

Let PN be any strip; x, y, the coordinates of P; $x + \Delta x$, $y + \Delta y$, the coordinates of Q; and let the lamina weigh δ per unit area. Then

$$\text{area of strip} = y\Delta x;$$
$$\text{force of gravity on strip} = \delta y \Delta x.$$

$+\frac{2}{3}$ KH

The C.G. of the strip being ultimately at its middle point, its coordinates are $x, y/2$. Hence if $\bar{x}, \bar{y}$ denote the coordinates of the C.G. of the quadrant,

$$\bar{x} \times \Sigma \delta y \Delta x = \Sigma \delta x y \Delta x;$$
$$\bar{y} \times \Sigma \delta y \Delta x = \tfrac{1}{2} \Sigma \delta y^2 \Delta x;$$

or in the notation of the calculus, δ being constant,

$$\bar{x} \int_0^r y\,dx = \int_0^r xy\,dx; \qquad \bar{y} \int_0^r y\,dx = \tfrac{1}{2} \int_0^r y^2 dx.$$

Performing the integrations indicated, remembering that $x^2 + y^2 = r^2$, we have

$$\bar{x} = 4r/3\pi = \bar{y},$$

and $$OG = \sqrt{\bar{x}^2 + \bar{y}^2} = 4r\sqrt{2}/3\pi.$$

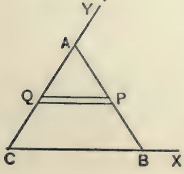
2. To find the C.G. of a thin lamina in the form of a quadrant of an ellipse whose semi-axes are a, b.

Ans. $\bar{x} = 4a/3\pi$; $\bar{y} = 4b/3\pi$.

3. Find the position of the centroid of a semicircle.

Ans. $OG = 4r/3\pi$.

4. To find the C.G. of a triangle ACB whose sides are a, b, c.



[Take C as origin; CB, CA, as axes of X, Y.

Cut into strips parallel to CB. Let PQ be any strip, and x, y the coordinates of P. Then

$$\text{Area of strip } PQ = x\Delta y \sin C.$$

The coordinates of the C.G. of the strip PQ are $x/2, y$, ultimately. Hence

$$\bar{x}\int_0^a xdy = \int_0^a \frac{x^2}{2}dy; \qquad \bar{y}\int_0^b xdy = \int_0^b xydy.$$

But, from the equation of the straight line AB,

$$x/a + y/b = 1.$$

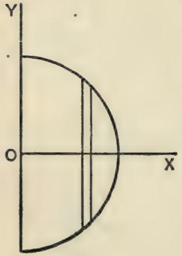
Performing the integrations, we find

$$\bar{x} = a/3; \qquad \bar{y} = b/3;$$

giving the same position of the C.G. as found in Art. 148.]

5. Solve Example 4 using rectangular coordinates.

6. To find the C.G. of a hemisphere of radius r and center O.



[From symmetry the C.G. must lie in OX normal to the base.

Divide the hemisphere into slices by planes normal to OX.

Volume of slice $= \pi y^2 \Delta x$.

Coordinates of C.G. of slice are x, 0.

$$\therefore\ OG\int_0^r \pi y^2 dx = \int_0^r \pi x y^2 dx,$$

and $\qquad OG = 3r/8$.]

Sometimes it is advantageous to introduce polar coordinates.

Ex. 7. To find the centroid of a quadrant of a circle of radius r (see Example 1).

[Take OX as initial line, and r, θ the coordinates of Q. Join OP, OQ, forming with PQ the element area OPQ.

Area $OPQ = \frac{1}{2}OP \times PQ = \frac{1}{2}r^2\Delta\theta$ ultimately.

Coordinates of C.G. of area OPQ are $\frac{2}{3}r\cos\theta$, $\frac{2}{3}r\sin\theta$, ultimately. (Art. 148.)

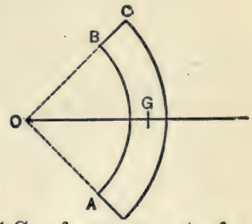
Hence $\quad \bar{x}\int_0^{\frac{\pi}{2}} \frac{1}{2}r^2 d\theta = \int_0^{\frac{\pi}{2}} \frac{2}{3}r\cos\theta \times \frac{1}{2}r^2 d\theta$, etc.]

8. Find after the manner of Ex. 7 the position of the centroid of a semicircle. (Compare Ex. 3.)

9. To find the C.G. of a circular sector AOB if angle $AOB = 2\beta$ and r is the radius.

Ans. $OG = \frac{2}{3}r \sin \beta/\beta = 2$ radius $\times$ chord/3 arc.

10. Find the C.G. of a sector of a circular ring, radius $OA = r_1$, radius $OC = r_2$, angle $AOB = 2\beta$.



Ans.

$$OG = 2(r_2^3 - r_1^3) \sin \beta / 3(r_2^2 - r_1^2)\beta.$$

[A practical illustration is the front surface of a circular arch.]

11. Show that the distance of the C.G. of a segment of a circle from the center is $c^3/12A$ when c is the chord and A the area of the segment.

12. To find the distance of the C.G. of a semi-circumference from the center O. *Ans.* $OG = 2r/\pi$.

13. To find the distance of the C.G. of a quadrantal arc from the center O. *Ans.* $OG = 2\sqrt{2}r/\pi$.

14. To find the distance of the C.G. of a circular arc AB subtending an angle 2β at the center.

Ans. $OG = r \sin \beta/\beta$, or $OG \doteq$ radius $\times$ chord/arc.

15. ABC is a triangle inscribed in a circle center O, and F, G, H are the centers of gravity of the sectors AOB, BOC, COA. Show that

$$AB/OF + BC/OG + CA/OH = 3\pi.$$

151. Having now found the position of the point of application of the resultant force of gravity on a body, it is possible to apply the general conditions of equilibrium to bodies acted on by forces (Art. 140). This application will form the remainder of this chapter.

We shall confine ourselves to forces that lie in the same plane. This will include the great majority of problems that occur in practice; for in structures forces are in general so applied as to be *symmetrical* about a plane, and therefore their resultants lie in this plane, so that the forces may be treated as if all acted in this plane.

152. It is well to call to mind at this place how to find

(1) The resultant of a number of forces acting at a point

(*a*) graphically, Art. 78;
(*b*) analytically, Art. 80.

(2) The resultant of a number of forces acting at different points

(*a*) graphically, Art. 128;
(*b*) analytically, Art. 139.

(3) The conditions of equilibrium when forces act at a point

(*a*) graphically, Art. 85;
(*b*) analytically, Art. 86.

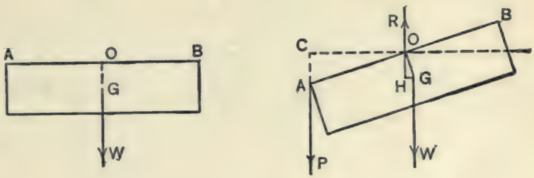
(4) The conditions of equilibrium when forces act on a body

(*a*) graphically (three forces), Art. 141;
(*b*) analytically, Art. 140.

In many applications of Mechanics in Architecture and Bridge-building both the graphical and analytical methods of computation are used. At times one is more convenient than the other, and always one may be used to check the other. This will be illustrated as we proceed.

153. We shall consider first the equilibrium of a single body variously supported, and then that of a system of bodies. The same general principle runs through all cases. When one body is in contact with another at one or more points, the action of one body is equal to the reaction of the other. If, therefore, the support be removed and a force applied to the body in magnitude and direction equal to the reaction of the support on the body, the conditions of equilibrium may be applied to the body as if acted on by the original forces and this reaction. In this way, too, the equilibrium of a body as part of a system may be studied.

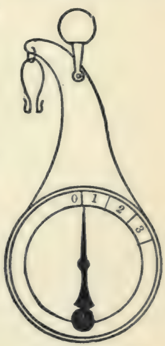
154. Equilibrium of a Body Supported at One Point.—Consider a uniform beam of length $2l$, depth $2h$, and weighing W lb, suspended at its middle point O. The force on the support O is equal to W pounds and acts vertically along OG through the center of gravity G of the beam. A body weigh-



ing P lb, if placed on the beam or suspended from it anywhere except on the line OG, will cause the beam to take an inclined position.

Suppose P to be suspended at the extremity A, and let R denote the reaction of the support at O, and θ the angle of deflection of the beam from the horizontal position.

We may consider the support removed and the beam in equilibrium under the vertical parallel forces P, W, R. Then (Art. 140)



(1) The sum of the forces is zero, or

$$P + W - R = 0.$$

Hence the pressure R on the support is found.

(2) The sum of the moments about any point is zero. Take moments about O and

$$P \times OC - W \times GH + R \times 0 = 0,$$

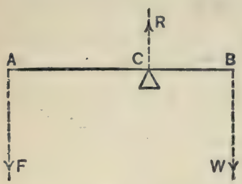
or $$P \times l \cos \theta - W \times h \sin \theta = 0,$$

or $$\tan \theta = Pl/Wh,$$

and the inclination θ is found.

By attaching a pointer to the beam free to move over a graduated arc we have a means of comparing weights. An example is afforded by the common letter-scale.

Ex. 1. Let AB represent a rigid rod (as a crowbar) turning on a fixed support C. Let a force F be applied at A, and let W be the resistance to be balanced at B. Given the lengths of AC, CB, it is required to find the relation between F, W when in equilibrium.



[Neglect for the present the weight of the rod. Let F and W be vertical, and let R denote the reaction at C.

We may consider the support removed and the rod in equilibrium under F, W, R. Then the equations of equilibrium are

$$F + W - R \qquad = 0.$$
$$F \times AC - W \times BC = 0.$$

Hence $$F/BC = W/AC = R/AB,$$

the relation sought.

If W_1 is the weight of the rod acting at its middle point G, the equations of equilibrium become

$$F + W + W_1 - R = 0,$$
$$F \times AC + W_1 \times GC - W \times BC = 0,$$

from which the ratios of F, W, R may be found.

The rod AB is known as a **Lever**, the support C the *fulcrum*, and the distances AC, CB the *arms* of the lever. The relation

$$F/BC = W/AC = R/AB$$

is sometimes called *the principle of the lever*.

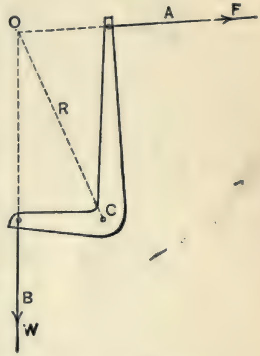
The condition of equilibrium in the case of forces acting at right angles to the arms of a straight lever was given by Archimedes (B.C. 287–212). But it was not until the end of the fifteenth century that the case of forces acting obliquely was solved. This was done by Leonardo da Vinci (1452–1519) of Florence, painter and philosopher.

2. A lever is 2 ft long. Where must the fulcrum be placed that 10 pounds at one end may balance 30 pounds at the other end?

3. From a pole resting on the shoulders of two men a weight W is suspended. It is n times as far from one man as from the other; what does each support?

Ans. $W/(n+1)$, $nW/(n+1)$.

4. Find the relation between F and W in a bell-crank lever. A and B are the bell wires, C is the pivot about which the lever turns.



[The directions of F, W, and the reaction R of the pivot meet in a point O. Hence take moments about C.]

5. A pair of nut-crackers is a inches in length, and a pressure of p pounds will crack a nut placed b inches from the hinge. What weight placed on the nut would crack it?

Ans. pa/b pounds.

6. The handle of a claw-hammer is 1 ft long and the length of the claw is 2 in. A pressure of 25 pounds is applied at the end of the handle. Find what resistance offered by a nail would be overcome.

Ans. 150 pounds.

7. A uniform wire bent into the form of three sides of a square is hung up from one of the angles. Show that the inclination of the first side to the horizontal is $\tan^{-1} 2/3$.

8. ABC is a triangular board having $AB = 5$ in, $BC = 4$ in, $AC = 3$ in. Find the point on BC from which the board if suspended will hang with AB horizontal.

Ans. 7/12 in from C.

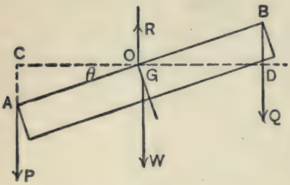
9. A plank 16 ft long and weighing 120 lb projects over a vertical wall in a horizontal plane to a distance of 6 ft A boy weighing 80 lb walks slowly along the plank. When will it begin to topple over?

Ans. When 3 ft from end.

10. Two weights P, Q balance at the ends of a lever whose weight may be neglected. If when the weights are interchanged equilibrium is maintained by adding weights P_1, Q_1 to P, Q, respectively, show that

$$P^2 - Q^2 = QQ_1 - PP_1.$$

155. *Balance.*—Consider a uniform beam hanging in equilibrium in a horizontal position, being suspended from a point O directly above the center of gravity G, and with the extremities A, B equidistant from O and in the same straight line with it. If from A, B are suspended two equal weights P, P, the beam will still remain in a horizontal position, since the moments about O are equal.



Hence the apparatus, which is evidently a lever with equal arms, may be used for comparing equal weights. Attaching pans to A, B, the weights for comparison may be placed in these pans and the operation facilitated. Such an arrangement is the common **Balance.**

If the attached weights are not equal, the beam will not rest in a horizontal position. We proceed to find the position of equilibrium in this case and thence to infer the requisites of a good balance.

Let W be the weight of the beam acting at G, and suppose unequal weights P, Q suspended from A and B, P being the greater. When the beam is again in equilibrium denote the angle of inclination of AB to the horizontal CD by θ.

Let $AO = OB = l$ and $OG = h$.

The beam is in equilibrium under the parallel forces P, Q, W, and the reaction R at O. Hence

$$P + Q + W - R = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Take moments about O, and

$$Pl \cos\theta - Wh \sin\theta - Ql \cos\theta = 0, \quad . \quad . \quad (2)$$

or

$$\tan\theta = (P - Q)l/Wh.$$

Now the balance will indicate small differences $P - Q$ the more clearly the greater the angle θ through which it swings for these differences. But $\tan\theta$ or θ is greatest when l/Wh is greatest, that is, when l is large or the beam has long arms, when W is small or the beam light, when h is small or the center of gravity is just below the point of suspension. Such a balance has great *sensibility,* and is suitable for delicate investigations in Chemistry, Physics, Assaying, etc.

In scales for weighing large weights *stability* rather than sensibility is wanted; that is, for small differences of P and Q the angle of deviation of the beam from the horizontal as shown by tan θ should be small. This requires Wh to be large or the beam to be heavy, with a long distance between the center of gravity G and the point of suspension O. By making the arms long, a balance may be constructed which shall possess in a measure both sensibility and stability. As the two conditions are at variance, the amount of compromise must be decided by the use to which the balance is to be put.

For very accurate work the method called *double weighing* is in use Place the body to be weighed in one pan, and balance with sand placed in the other pan. Remove the body, and balance the sand with standard weights. The weight of the body is then shown by the standard weights, for they produce the same effect as the body itself.

This method, which is due to Borda, is really a method of substitution, and gives a correct result if the balance is not properly adjusted.

156. *To Test a Balance.*—Suppose the beam to rest in a horizontal position when the scale-pans are empty. Weigh a body P first in one pan and then in the other. If the two values obtained are equal to one another, the balance is true.

For let a, b denote the lengths of the arms and W_a, W_b the weights of the scale-pans. Then if W denotes the observed weight of the body P, we have

$$W_a \times a = W_b \times b; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$(P + W_a) \times a = (W + W_b)b; \quad . \quad . \quad . \quad . \quad (2)$$

$$(W + W_a)a = (P + W_b)b. \quad . \quad . \quad . \quad . \quad (3)$$

Hence $a = b$, $W_a = W_b$, and $P = W$, or the arms are equal, the pans are of the same weight, and the observed weight is the weight of the body.

Ex. 1*a*. The beam of an unloaded balance rests in a horizontal position, and the arms are known to be of unequal length. Show that the true weight W of a body which appears to weigh W_1 when placed in one scale and W_2 when placed in the other scale is given by

$$W = \sqrt{W_1 W_2}$$

1*b*. Show that if W_1, W_2 are nearly equal, this may be written closely enough

$$W = (W_1 + W_2)/2.$$

$$\left[\text{For } 2\sqrt{W_1 W_2} = (W_1 + W_2) - (\sqrt{W_1} - \sqrt{W_2})^2.\right]$$

2*a*. If the arms of a balance are of equal length but (from wear) the pans are not of the same weight, show that the true weight W of a body which appears to weigh W_1 when placed in one pan and W_2 when placed in the other pan is found from

$$W = (W_1 + W_2)/2.$$

2*b*. Show that the difference of the weights of the scale-pans is equal to $(W_1 - W_2)/2$.

3. If the C.G. of the beam is not directly under the point of suspension, show that the correct weight may be found as in Ex. 2*a*.

4. Is it fair to infer that in all cases

correct wt. = half-sum of apparent wts. in the two scales?

5. A balance has unequal arms. The apparent weights of a body weighed first in one scale-pan and then in the other are 9 and 11 grains. What error is made by taking 10 grains as the true weight? *Ans.* 0.05 grain.

6. Why must not the C.G. of a beam-balance coincide with the point of suspension?

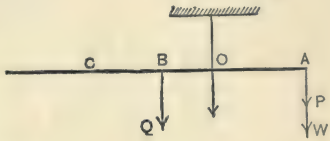
7. In a balance show that if the scale-pans hang freely no error can arise from the weights not being placed in the center of the pans.

8. A body whose weight is 10 lb when placed in one pan of a false balance appears to weigh 9.5 lb. Find its apparent weight when placed in the other pan. *Ans.* 10.526 lb.

9. The arms of a balance are a in and b in. A grocer uses the scale-pans in alternate order in serving customers. Find his gain or loss per lb. *Ans.* $(a-b)^2/2ab$ lb.

10. Discuss the balance if the point of suspension O is at a distance k above the beam.

157. *Steelyard.*—Consider a beam suspended from a point O directly above its center of gravity and hanging in equilibrium in a horizontal position. If from the beam we suspend two bodies of unequal weight P, Q, it will still remain in equilibrium in a horizontal position if



$$P \times AO = Q \times BO.$$

Let the weight P suspended from A be a hook or a scale-pan. If to P we add an unknown weight W, we shall still have equilibrium, provided Q is shifted to a point C such that

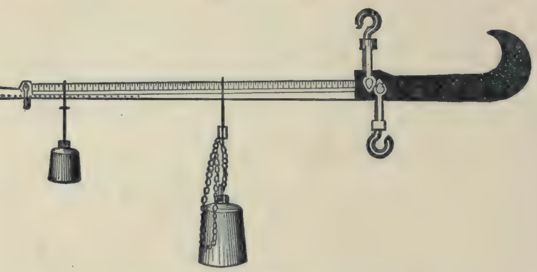
$$(P + W)AO = Q \times CO.$$

Subtract these equations, and

$$W \times AO = Q \times BC,$$

which gives the unknown W as soon as BC is measured.

To save measurements of BC at every weighing of a body, it is convenient to graduate the beam in the first place. Thus suppose $P = 1$ lb, $Q = 2$ lb, and $AO = 4$ in. Then $OB = 2$ in, and a notch can be made at B, which, as the weight Q then balances the pan P only, would be marked 0. Let now $W = 1$ lb; then $BC = AO \times W/Q = 2$ in, and C would be the position at the 1-lb mark. Make $W = 2$ lb and $BD = 4$ in, giving D the 2-lb mark, and so on. Hence in weighing a body it is only necessary to place it in the pan and move the weight Q until the notch is found where the beam will remain horizontal. The number at the notch indicates the weight. This instrument is called a **Steelyard.**



The advantages of a steelyard over the balance are: (1) the exact adjustment of the instrument is made by moving a single weight Q along the rod; (2) when the body to be weighed is heavier than the fixed weight the pressure on the point of support is less than in the balance. The steelyard is therefore better adapted to measure large weights. There is, on the other hand, this advantage in the balance, that by using numerous small weights the reading can be effected with greater precision than by subdividing the arm of the steelyard. (Routh.)

Ex. 1. Graduate a steelyard to weigh half-pounds.

2. If the point of suspension O be *not* over the center of gravity and the movable weight Q placed at a point H holds the steelyard in a horizontal position, show that when the weight P is attached $HB = AO \times P/Q$, and hence show how to graduate the steelyard.

[For AO and Q are constant. $\therefore$ HB varies as P. Hence H is the point from which the graduations must be made. If Q is at B when $P = 1$ lb, then by taking $BD = BH$ and placing Q at D, P will be 2 lb, that is, B is the 2-lb notch, etc.]

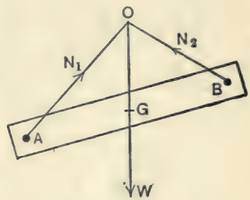
3. A steelyard beam weighs 3 lb, the wt. Q is 4 lb, and the distance of the center of gravity from O is 3 in, and of the point of suspension of the scale A from O 5 in. Show that the 1-lb graduation-marks are at intervals of 5/4 in.

4. A steelyard weighs W lb and is correctly graduated for a movable weight Q. Prove that a weight $2Q$ may be used provided a fixed weight W is suspended at the center of gravity of the steelyard, but the readings must be doubled.

5. A piece is broken off the longer arm of a steelyard. Show that the customer is defrauded.

6. A grocer files the movable weight of his steelyard. Show that he cheats his customers.

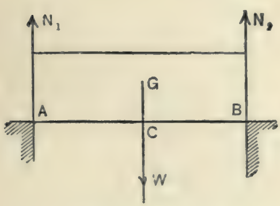
158. Equilibrium of a Body Supported at Two Points.—Consider next a body in which two points are supported, as, for example, a beam supported by two smooth horizontal pins A and B, or a beam resting on two props A and B.



The forces acting are the weight W vertically downwards through G, the center of gravity of the beam, and the reactions N_1, N_2 of the supports A and B. Since the direction of W is known, if from the conditions of the question that of one of the two N_1 or N_2 is given, the direction of the other is found; otherwise the problem is indeterminate.

If the forces are not parallel they must meet in a point, and the solution is given by Art. 141; if they are parallel it is given by Art. 140. As this latter case is of special importance, it is here worked out in full.

Let the two props A, B be in the same horizontal plane. Since the direction of W is vertical, the directions of the reactions N_1, N_2 at A, B must also be vertical. To find these reactions, resolve vertically, and take moments about G; then



$$N_1 + N_2 - W = 0,$$
$$N_1 \times AC - N_2 \times BC = 0,$$

from which N_1, N_2 are found.

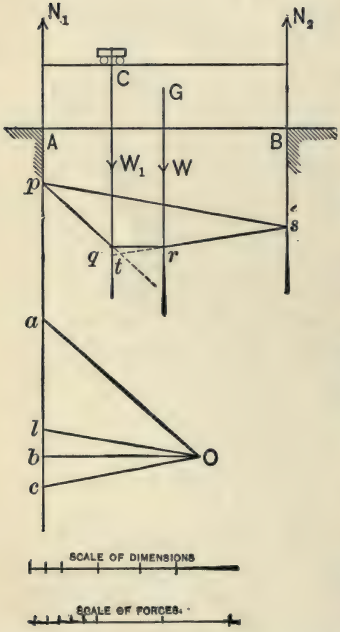
Or take moments about B and A in succession, and

$$-N_1 \times AB + W \times BC = 0,$$
$$N_2 \times AB - W \times AC = 0,$$

from which the same values of N_1, N_2 result.

The pressures N_1, N_2 on the supports may also be determined *graphically*.

Thus suppose the beam AB to carry besides its own weight W a load W_1 at C. Draw ab, bc to scale to represent the vertical forces W_1, W in magnitude and direction. The force represented by the line cba which closes the polygon of forces will hold W_1, W in equilibrium and will be the sum of the reactions N_1, N_2.



Take any pole O and join Oa, Ob, Oc. From any point p in the direction of N_1 draw pq parallel to aO, from q draw qr parallel to bO, and from r draw rs parallel to cO. Join sp, and draw Ol parallel to sp. Then cl, la are the reactions N_2, N_1 sought.

Proof.—The intersection t of pq and sr gives the position of the resultant abc of W_1 and W. (Art. 128.) And cba, the resultant of N_1 and N_2, being equal and opposite to abc, must act upwards through t.

Now cba acting at t is equiv. to cO along ts and Oa along tp.

But cO along ts is equiv. to cl along sB and lO along ps,
and Oa along tp is equiv. to Ol along sp and la along pA.

The forces lO and Ol, being equal and opposite, balance. Hence cba acting at t is equiv. to cl along sB and la along pA; that is, cl, la represent the reactions N_2, N_1 at A, B on the scale of forces.

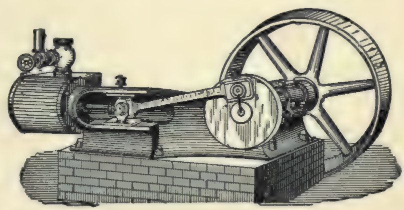
Hence the rule:

(*a*) Form the force polygon or load line *abca* by laying off the forces to scale.

(*b*) Select a convenient pole O and form the equilibrium polygon *pqrs*.

(*c*) Draw Ol parallel to the closing line *sp* of the equilibrium polygon, dividing *ca* into parts *la*, *cl*, which will represent the reactions N_1, N_2 at A and B respectively. (Compare Art. 129.)

159. The graphical method may be employed in finding the stresses in a mechanism. Take, for example, the steam-engine. Let P be the pressure exerted by the piston on the pin A of the cross-head. It is transmitted by the connecting-rod to the crank-pin B, and thence to the crank axis C. If now

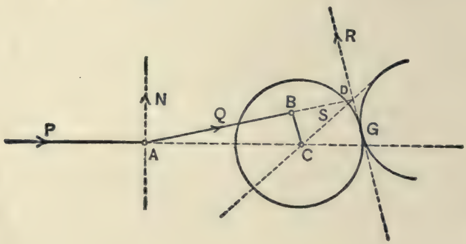


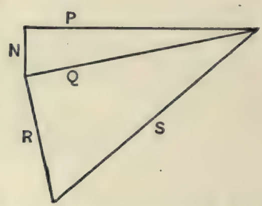
the machinery is driven by a wheel CG on the axis C working in another wheel at G, the resistance R would be tangent to the pitch-circles of these wheels.

Consider the pin A. It is in equilibrium under the pressure P along the axis of the piston-rod, the thrust Q along the connecting-rod, and the reaction N of the guide-bar of the cross-head. Hence plot P to scale, and complete the triangle of forces, from which scale off Q and N.

Again, the wheel CG is in equilibrium under the action of Q, R and the reaction S of the crank axis C. All three must meet in the point D, where Q and R meet. Hence plot the triangle of forces, and scale off S and R. The relation between P, the piston pressure, and R, the force transmitted to the mechanism, is therefore determined.

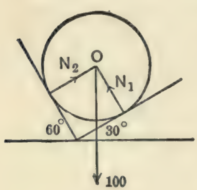
We have neglected the weights of the pieces. The only one important to consider is the weight resting on the crank axis C. Call it W. It acts vertically. Combine Q and W into one resultant R_1. The reaction S will now pass through the intersection of R_1 and R. Hence complete the triangle of forces for R, S, R_1, and scale off S and R. Thus the relation between P and R is found.





Similarly, the weights of all the pieces may be taken into account if desired.

Ex. 1. A spherical shot weighing 100 lb lies between two smooth planes inclined at respectively 30° and 60° to the horizontal. Find the pressure on each plane.



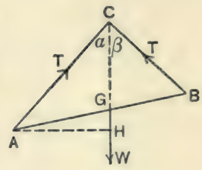
[The pressures of the shot on the planes are perpendicular to the planes. These pressures are balanced by the reactions N_1, N_2 of the planes, which reactions pass through the center of the sphere. The weight of the sphere acts vertically downward at the center.

Hence the center O may be considered in equilibrium under the forces N_1, N_2, 100 pounds, and the planes may be removed. Finish in all the ways indicated in Art. 141.

Ans. $N_1 = 50$ pounds; $N_2 = 50\sqrt{3}$ pounds.]

2. A rod AB whose weight may be neglected and which is 35 in long carries a weight W at G 20 in from the end A.

The rod is carried by a thread 49 in long, tied to the ends A, B and slung over a smooth peg C. Find the pull on the thread and the inclination of the rod to the horizontal when it comes to rest.



[The peg being smooth, the pull T is the same on both sides of the peg. The rod is held in equilibrium by the three forces T, T, W. The directions of T, T pass through C. Hence the direction of W, which is vertical, must pass through C. (Art. 141.)

Let the angles at C be α, β.

By Lami's theorem

$$T/\sin\alpha = T/\sin\beta = W/\sin(\alpha+\beta).$$
$$\therefore\ \alpha=\beta \quad\text{and}\quad T = W/2\cos\alpha.$$

The value of the inclination θ follows from the geometry of the figure.

For $$AC : CB = AG : GB = 20 : 15.$$
$$\therefore\ AC = 28 \quad\text{and}\quad CB = 21.$$

Hence $$\angle ACB = 90^\circ \quad\text{and}\quad \alpha = 45^\circ.$$

Also $$\cos\theta = AH/AG$$
$$= 7/5\sqrt{2}, \quad\text{and } \theta \text{ is found.}$$

Finally, $T = W/2\cos 45 = W/\sqrt{2}$, the pull in the thread.

3. A uniform beam weighing W lb rests on two smooth planes inclined at 30° and 60° to the horizontal. Find the angle which the beam makes with the horizontal in the position of equilibrium, and also the pressures on the planes.

Ans. 30°; $W\sqrt{3}/2$, $W/2$ pounds.

4. If in (3) the angles of inclination of the planes are β, γ, and θ is the inclination of the beam to the horizontal, then

$$2\tan\theta = \cot\beta - \cot\gamma.$$

5. If in (3) the beam is not uniform, but has its C.G. at a distance a from one end and b from the other end, then

$$(a+b)\tan\theta = a\cot\beta - b\cot\gamma.$$

6. A cellar-door AB, hinged at the upper edge A, rests at an angle of 45° with the horizontal. Its weight, W lb may be taken to act at its middle point G. The door is raised by a horizontal pull F applied at the lower edge B. Find F, and also the reaction R at the hinge A.

Ans. $F = W/2$; $R = W\sqrt{5}/2$.

7. A straight rod 4 in long is placed in a smooth hemispherical cup, and when in equilibrium one inch projects over the edge. Find the radius of the cup. *Ans.* $\sqrt{3}$ in.

8. A rod 3 ft long is in equilibrium resting upon a smooth pin and with one end against a smooth vertical wall. If the pin is 1 ft from the wall, show that the inclination θ to the horizontal is given by $3 \cos^3 \theta = 2$.

9. One end of a heavy uniform rod rests against a smooth vertical wall. A smooth ring whose weight may be neglected, attached to a point in the wall by an inextensible thread, slides on the rod. If θ is the angle which the rod makes with the wall when in equilibrium, and if the length of the rod be $2n$ times that of the thread, prove

$$\cot^3 \theta + \cot \theta = n.$$

10. To a point A is fastened one end of a thread of length l, with a smooth ring at the other end B. Through this ring passes another thread with one end fastened at C, distant $2l$ from A and on the same horizontal line with it. A weight W is attached to the other end of this thread. Show that in the position of equilibrium, neglecting weight of thread and ring, the inclination θ of AB to AC is given by

$$\cos \theta = 2 \cos 2\theta,$$

and the inclination ϕ of CB to CA by

$$\sec \phi/2 + \operatorname{cosec} \phi/2 = 4\sqrt{2}.$$

EXAMPLES.—*To be Solved Graphically* (Art. 158).

1. A highway bridge 25 ft long weighs 6 tons. Find the pressures on the abutments when a $2\frac{1}{2}$-ton wagon is one fifth of the distance across.

[Referring to figure of Art. 158, $W = 6$ tons, $W_1 = 2.5$ tons. Draw on a scale of forces of 1 ton = 1 in. Then $ab = 2.5$ in, $bc = 6$ in.

Complete the equilibrium polygon and draw Ol. Then cl

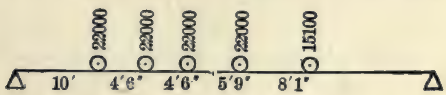
will be found to measure 3.5 in and *la* 5 in, showing that the pressures are 3.5 tons and 5 tons.

Solve also analytically by taking moments about the supports.]

2. A ladder 20 ft long and weighing 75 lb is carried by two men, one at each end. If one man carries 30 lb, how far is the C.G. from the end of the ladder? *Ans.* 12 ft.

3. A beam of 40 ft span weighs 1 ton per running foot. One half of it carries a uniform load (as a train of coal cars) of 2 tons and the other of 3 tons per running foot. Find the pressures on the end supports. *Ans.* 65 tons; 75 tons.

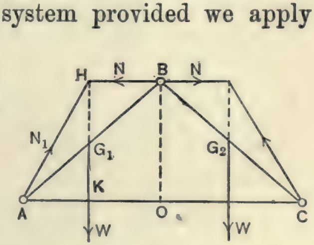
4. A truss of 60 ft span and weighing 100 tons carries an



Erie consolidation engine as in the figure. Find the pressures on the supports. *Ans.* 66.6 tons; 84.9 tons.

160. Equilibrium of a System of Bodies.—If bodies rest against one another or are connected by threads, hinges, etc., so that the system while in equilibrium under external forces can be regarded as remaining unchanged in form, the conditions of Art. 140 are at once applicable. The pulls (tensions) of threads, reactions of hinges, etc., being in pairs and forming stresses, are in equilibrium when the whole system is considered. Hence in writing down the equations of equilibrium these internal stresses may be neglected, the external forces alone being considered.

Also, each body in the system being in equilibrium, we may consider it as severed from the system provided we apply forces equal to the resultant actions of the system on it. The conditions of equilibrium may be applied to each body in succession.



Thus take two equal uniform beams *AB*, *BC* of weight *W* lb each, hinged at *B*, and with the other ends *A*, *C* moving on hinges in the same horizontal plane, the beams being in a vertical plane.

Put the length $AB = l$, the height $BO = h$, and the span $AC = 2a$. The beams being equal, the reactions N, N at B will be horizontal. Also, if the beam BC were removed, AB would be in equilibrium under the forces W, N, and the reaction N_1 of the hinge A, which reaction must pass through H, the intersection of W and N (Art. 141). Hence the direction of N_1 is known. Put its inclination to $AC = \beta$.

The beam AB is in equilibrium under the forces W, N, N_1. Resolve vertically and horizontally, and

$$W - N_1 \sin \beta = 0;$$
$$N - N_1 \cos \beta = 0,$$

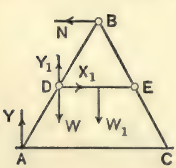
or
$$N_1 = W \operatorname{cosec} \beta;$$
$$= W \sqrt{a^2 + 4h^2}/2h;$$
$$N = W \cot \beta$$
$$= Wa/2h;$$

and the forces N, N_1 are found.

Or, graphically, since the forces on the beam AB are parallel to the sides of the triangle AHK, if we take HK to represent W, the values of N, N_1 will be represented by AK, HK, and may be scaled off or computed.

Ex. 1. Find the values of N, N_1 by taking moments about A, K in succession.

2. Find the value of N_1 by considering the system as a whole in equilibrium under W, W, N_1, N_1 (the internal forces N, N forming a stress) and resolving vertically.



3. Two beams AB, BC, of weight W lb each are hinged at B, and with their middle points D, E joined by a beam DE of weight W_1 lb hinged at D, E. The ends A, C rest upon a smooth horizontal support; to find the reactions at A, D, B, E, C.

[The reactions Y at A and C are vertical. Let the reaction at D due to the weight W_1 be resolved into two components, X_1 horizontal and Y_1 vertical.

The beam DE is in equilibrium,

$$\therefore \quad 2Y_1 = W_1.$$

The beam AB is in equilibrium,

$$\therefore \quad N = X_1,$$
$$Y + Y_1 = W,$$

and taking moments about D,

$$N \sin \theta = Y \cos \theta.$$

Hence Y, Y_1, X_1, and N are found.]

4. What are the values of Y, Y_1, X_1, and N in the above example?

What is the total reaction at D?

5. In a rowboat propelled by two oars the pulls exerted by the rower are equal. Find the resistance of the water to the motion of the boat and the pressure on the rowlock.

[Let P = pull on each oar;
Q = pressure of water on each blade;
R = reaction between each rowlock and oar;
S = resistance of water to motion of boat;
a = distance of hand of rower from rowlock;
b = distance of rowlock from blade of oar.

Now the system as a whole is in equilibrium under the external parallel forces Q, Q, S. Hence

$$Q + Q = S.$$

Also, since each oar is in equilibrium under the parallel forces P, Q, R, by taking moments about the rowlock

$$Pa = Qb.$$
$$\therefore \quad Pa = Sb/2,$$

and the resistance S is found.

The pressure R on the rowlock would be given by taking moments about the blade, or

$$Rb = P(a + b).$$

As a check we have

$$P + Q = R,$$

the forces P, Q, R being parallel.]

6. In an 8-oar boat the oars are 10 ft long, and the distance of the hand of each rower from the rowlock is 2.5 ft. If the pull of each rower is 75 pounds, find the force exerted on the boat. *Ans.* 400 pounds.

7*a*. Two weights P, Q rest on the outside of a smooth vertical hoop and are connected by a thread which subtends a right angle at the center of the hoop. Find the inclination θ of the thread to the horizontal when in the position of equilibrium. *Ans.* $\theta = \tan^{-1}(P - Q)/(P + Q)$.

7*b*. Show that if T denotes the pull in the thread, then

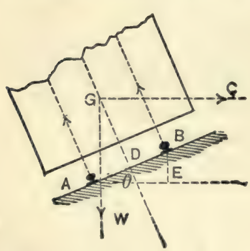
$$T^2/P^2 + T^2/Q^2 = 2.$$

8. Two smooth spheres, each 1 ft in diameter and weighing 10 lb, are placed inside a hollow cylinder of 20 in diameter, open at both ends and resting on a horizontal table. Find the weight of the cylinder when just on the point of overturning. *Ans.* 8 lb.

9. Two equal cylindrical sawlogs, of weight W lb each, are in contact with each other and with the bottom and sides of a truck in which they are placed. A third equal log is placed on the other two. Find the pressure on the sides of the truck.

Ans. $W/2\sqrt{3}$ pounds.

10*a*. Find the proper elevation BE of the outer rail on a railroad track for a given velocity v of engine weighing W lb, and on a curve of radius r ft, in order that there may be no flange or lateral pressure on the rails.



[The forces acting are W pounds vertically downward through G, the center of gravity of the engine, and the reactions N_1, N_2 of the rails. Since there is no flange pressure, the reactions must be perpendicular to the track.

The resultant motion is due to a force C directed to the center of the curve (Art. 108). Hence the engine may be said to be in equilibrium under W, N_1, N_2, and $-C$.

Let θ be the inclination of AB to the horizontal.

Resolve the forces along AB and

$$-C\cos\theta + W\sin\theta = 0.$$

But (Art. 109)

$$C = Wv^2/gr \text{ pounds.}$$
$$\therefore \quad \tan\theta = C/W = v^2/gr.$$

If θ is small, $\tan\theta = BE/AB =$ elevation/gauge of track.

$$\therefore \quad \text{elevation of outer rail} = \text{gauge} \times v^2/gr.$$

For standard gauge of 4 ft $8\frac{1}{2}$ in this gives

$$\text{elevation of outer rail} = 7v^2/4r \text{ inches, nearly,}$$

v being expressed in feet per second, and r in feet.]

10*b*. If the velocity V is expressed in miles per hour, show that

$$\text{elevation of outer rail} = 15V^2/4r \text{ inches, nearly,}$$

the radius r being, as before, expressed in feet.

10*c*. If the velocity V is expressed in miles per hour, and D is the degree of curve, then

$$\text{elevation of outer rail} = V^2D/1530 \text{ inches, nearly.}$$

11. A train is running round a curve of radius r with velocity v. Show that the weight of a carriage is divided between the outer and inner rails in the ratio of

$$gra + v^2h \quad \text{to} \quad gra - v^2h,$$

where h is the height of the C.G. of the carriage above the rails, and $2a$ is the distance between the rails.

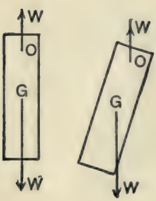
12. A train is running round a level curve of 3025 ft radius at 15 miles an hour. Show that the deflection of a plumb-line suspended in a car from the vertical is $\tan^{-1} 1/200$.

13. Find the greatest velocity v a locomotive can have to be just on the point of overturning on a curved level track

of radius r ft, the center of gravity of the locomotive being 6 ft above the rails, and the gauge of the track 4 ft $8\frac{1}{2}$ inches.

Ans. $3.55\sqrt{r}$ ft/sec, nearly.

161. Stability.—A body in equilibrium under the action of forces may be at rest or move with uniform velocity in a straight line. Consider it at rest in some one position. If displaced from this position it may remain at rest under the forces acting, or it may begin to move. Thus let a rod



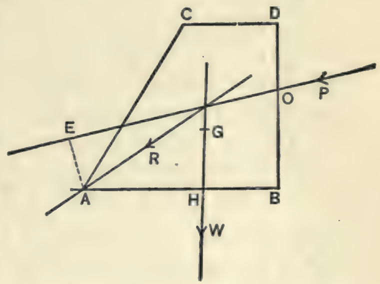
suspended from a fixed point O be in equilibrium. The force acting, the weight W vertically downward, is balanced by the reaction of the support. If the rod is displaced, the forces at O and G are no longer in the same line but form a couple tending to turn the rod. If O is above G, the tendency of the rod is to return to its original position, and the rod in its original position is said to be in *stable equilibrium*. The stability is measured by the torque, that is, by the moment of W about the point of turning O.

If O is below G, the tendency of the rod when disturbed is to move farther from its original position, and the equilibrium is *unstable*. If O is at G, the rod will remain in any position and the equilibrium is *neutral*.

162. An important case is that in which a body rests on a horizontal plane. If the body is symmetrical and is under the action of gravity only, the resultant force of gravity acts vertically through the center of gravity and must be balanced by the resultant vertical reaction of the plane. The problem may thus be treated as if all the forces acted in one plane.

If the body is not symmetrical and the resultant vertical force falls within the base of the body resting on the plane, equilibrium will exist; if it does not so fall, the weight downward and equal reaction upward will form a couple and the body will rotate and topple over.

Suppose, for example, $ABCD$ to be the cross-section of a wall built to withstand the pressure of earth on one side, as the wall of a railroad embankment. Such a wall is called a retaining wall. We assume that the wall cannot give way except in one piece and by being overturned about the edge A. Let P be the resultant of the forces acting on the side BD of the wall, and let O be the point of application of P. The weight W of the wall acts vertically downward through the center of gravity G.



The tendency to overturn about the edge A is measured by the moment of P about A, that is, by $P \times AE$. The tendency to withstand overturning is measured by $W \times AH$. The stability against rotation depends upon the difference of these two moments.

Hence in the position of limiting equilibrium when the wall is just on the point of turning about A we have the relation

$$P \times AE = W \times AH.$$

For the condition of security against the shearing force causing sliding see Ex. 21, p. 222.

The problem of stability is further discussed in Art. 219.

Ex. 1. A cubical block is placed on an inclined plane. If kept from slipping, find the inclination of the plane when just on the point of rolling over. *Ans.* 45°.

2. The silver dollar is 1.5 in diameter and 0.1 in thick. Show that a cylindrical pile of 100 dollars, but no more, will stand if placed on a desk sloping 3 in 20.

3. A triangular board whose sides are 2, 5, $\sqrt{13}$ inches will just stand if placed with the side 2 on a horizontal table.

4. A triangular lamina whose sides are a, b, c can just stand on the side c when placed on a smooth table. Show that

$$a^2 - b^2 = 3c^2.$$

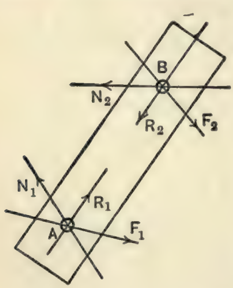
5*a*. A circular table weighing W lb has three equal legs at equidistant points on its circumference. The table is placed on a level floor. Neglecting the weight of the legs, find the smallest weight which, hung upon the edge, will be upon the point of upsetting the table. *Ans.* W lb.

5*b*. If the table has four legs at equidistant points, find the least weight that will upset it. *Ans.* $W(\sqrt{2}+1)$ lb.

5*c*. If the table is square and the legs are at the middle points of the sides, show that any weight greater than W placed at one of the corners will upset the table.

163. Graphical Statics.—If in Art. 160 the beams, instead of being connected at B by a hinge rigidly attached to the beams, are connected by a pin distinct from both beams, and also rest upon pins at A, C distinct from the beams, the problem is much simplified.

Consider the beam AB. Suppose the forces acting on the beam, including its weight, to be combined into a single force F, and let F be resolved into two components F_1 at A and F_2 at B.



Let N_1, N_2 be the reactions of the pins at A, B on the beam. The beam is in equilibrium under the forces F_1, N_1 at A, and F_2, N_2 at B. Combine F_1, N_1 into a single force R_1, and F_2, N_2 into a single force R_2. Then the beam being in equilibrium, the forces R_1, R_2 must be equal and act in opposite directions. Hence they form a *stress*.

Since R_1 is the resultant of F_1 and N_1, a force R_3, equal and opposite R_1, would keep F_1 and N_1 in equilibrium.

Now since each pin in a framework connects two or more beams, the resultant action of these beams on the pin A, say, is equal to N_1. Hence F_1, the action of the component of the external force on the pin, R_3, the action of the beam AB on the pin, and N_1, the action of the other beams on the pin A, are in equilibrium.

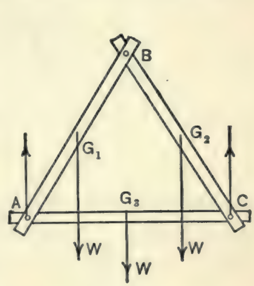
In general, each pin is in equilibrium under the action of the external force at the pin and the forces in the beams that meet at the pin, the directions of these forces being along the beams.

The external force F being known in magnitude and direction, and the directions of the forces along the beams being known from the form of the structure, the computation of the magnitude of these forces (commonly called stresses) forms a simple problem of forces meeting at a point, and may be solved graphically or analytically.

A careful study of the examples will show that in some cases the graphical solution is to be preferred, and in others the analytical. The graphical solution is in general the more convenient if the structure is at all complex.

164. The determination of stresses in beams pinned together and subjected to external forces generally in the form of loads of some kind is very important in architecture and engineering. The subject belongs to a special branch of mechanics, known as *graphical statics.* We add a short sketch of its application to simple framed structures.

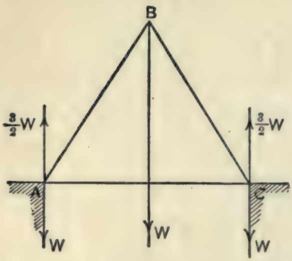
165. Jointed Frames.—As the simplest possible example of a jointed frame, let us consider three beams hinged by pins at A, B, C, and resting on supports at A, C in the same horizontal plane. This is known as a *triangular truss.* Suppose the beams all alike and weighing W lb each. The reactions of the supports A, C balance the weights of the beams and act vertically upwards. Hence the external forces acting and keeping the truss in equilibrium are as in the figure.



Now transfer the weights to the pins. Thus,

$$W \text{ at } G_1 = \tfrac{1}{2} W \text{ at } A + \tfrac{1}{2} W \text{ at } B,$$
$$W \text{ at } G_2 = \tfrac{1}{2} W \text{ at } B + \tfrac{1}{2} W \text{ at } C,$$
$$W \text{ at } G_3 = \tfrac{1}{2} W \text{ at } A + \tfrac{1}{2} W \text{ at } C;$$

$\therefore$ W at $G_1 + W$ at $G_2 + W$ at $G_3 = W$ at $A + W$ at $B + W$ at C,

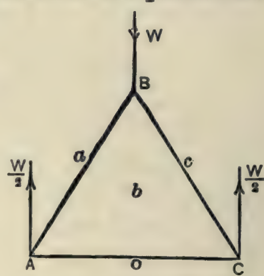


or the weights at G_1, G_2, G_3 are replaced by weights W at A, W at B, and W at C.

The total reactions at A, C being equal, each is one half the total weight, or is equal to $3\,W/2$.

Combining the upward and downward forces, we have, finally the force W at B vertically downward, and the resultant forces (or reactions) $W/2$ at A and $W/2$ at C vertically upward, keeping the truss in equilibrium.

We have thus transferred the weights of the beams to the joints, and can now consider the beams as without weight, and indicating direction only. The resulting stresses in the pieces we next find.

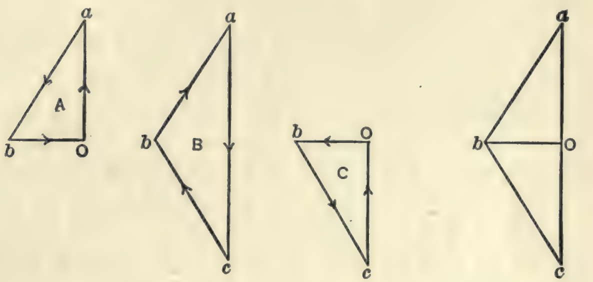


Since the weights on *each* pin are in equilibrium with the stresses produced in the pieces meeting at the pin, the condition of Art. 85 is at once applicable Consider the forces at each pin in order.

(a) Pin A. The forces acting are $W/2$ vertically upwards, and the unknown stresses in BA, CA, and these three forces keep the pin in equilibrium. Draw Oa to scale (fig. A) to represent $W/2$. From a draw ab parallel to BA, and bO parallel to AC, closing the triangle. Then ab, bO represent on the same scale the stresses in AB, AC. The direction of $W/2$ or Oa is known to be vertically upwards. And since for equilibrum Oa, ab, bO must be taken the same way round, their directions are as in the figure.

Transfer these directions to the truss diagram. The stress ab in AB is towards A, showing that the piece AB is in *compression*, or is a **Strut**; the stress bO in AC from A, showing that the piece AC is in *tension*, or is a **Tie**.

(*b*) Pin *B*. The forces equilibrating are the stress in *AB* towards *B* (being equal and opposite that towards *A*) and the force *W* vertically downwards, which are known, and the



stress in *CB*, which is unknown. Draw (fig. B) *ba* to represent the stress in *AB*, *ac* to represent *W*; then *cb* will represent the stress in *CB*. Transferring the directions to the stress diagram, we see that *CB* is a strut.

(*c*) Pin *C*. The forces equilibrating are the stress in *BC*, *W*/2, and the stress in *AC*, all of which are known. For check the diagram may be drawn as in Fig. C.

166. It is evident that we should have saved labor by adding the second figure to the first and the third to the sum as in the fourth figure, which is the complete *stress diagram*.

In practice it is convenient to consider the stress diagram as in two parts. Thus the line *ac* is the polygon of external forces, *ac* being the downward force at *B* balanced by the upward forces *cO* at *C* and *Oa* at *A*, and is complete in itself. The closing of this line shows that the external forces have been properly estimated. On this force polygon as base the stress diagram is added step by step by passing from pin to pin as indicated.

In case the truss is symmetrical, as in our example, it is only *necessary* to consider the first half of the pins. But it is safer to consider all of them, as the symmetry of the drawing will furnish a test of its accuracy.

Notice carefully that a study of the stress diagram shows not only the *amount* of stress but the *kind* of stress in any piece, and therefore whether a strut or a tie should be employed.

167. For tracing the connection between the pieces themselves and the stresses in them as shown by the stress diagram, an exceedingly convenient system of notation, due to Prof. Henrici, London, but usually known as Bow's notation, is in common use.

A beam or a force is named by letters placed on either side of it. Thus, in the second figure, p. 194, *Ob* is the tie *AC*, *Oa* the reaction $W/2$ at the left support, *ab* the strut *AB*, *ac* the force *W* at *B*, and so on. These letters carried into the stress diagram *aOcb* give us *ab* the stress in the piece *ab*, *cb* the stress in *cb*, *Ob* the stress in the rod *Ob*. The letters *A*, *B*, *C* at the pins do not enter the stress diagram and are not necessary.

168. The following examples should be solved both graphically and analytically. The first will be worked out in detail by both methods.

The general outline of the graphical solution is this:

(1) Draw the truss to scale from the dimensions given.

(2) Compute the pin loads.

(3) Compute the resultant reactions of the supports.

(4) Draw the force polygon to scale.

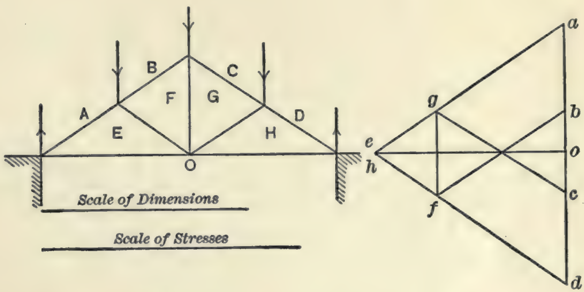
(5) Draw the stress diagram on the force polygon as base.

(6) Scale off the stresses and tabulate, indicating forces in compression by the sign $-$, and pieces in tension by the sign $+$.

The analytical solution is carried out by equating the moments of the lateral forces and of the stresses developed in the pieces. See page 198.

Ex. 1. In a roof of 32 ft span and height 12 ft the trusses are 10 ft apart, and the pieces *EF*, *GH* come to the middle points of the rafters. If the weight of the roof-covering is 25 lb/ft^2, draw the stress diagram and scale off the stresses.

(*a*) *Graphical Solution.*—(1) Draw the truss to scale from the given dimensions. Take the scale of dimensions 5 ft = 1 in.



(2) The length of each rafter $= \sqrt{12^2 + 16^2} = 20$ ft.

The load on each rafter $= 20 \times 25 \times 10 = 5000$ pounds.

The pin loads *AB, BC, CD* are each 2500 pounds and act vertically downward as indicated in the figure.

(3) The resultant reaction at each pier $= 7500/2 = 3750$ pounds vertically upward.

(4) Take the scale of stresses 1000 pounds = 1 in.

Form the force polygon by laying off $ab = 2500$ pounds, $bc = 2500$ pounds, $cd = 2500$ pounds, $do = 3750$ pounds, $oa = 3750$ pounds, thus closing the polygon or *load line ad.*

(5) Next the stress diagram.

Begin at the left-hand pin. The forces acting are $OA =$ 3750 pounds and the unknown stresses in *AE, EO.* From *a* draw a line *ae* parallel to the piece *AE,* and from *o* a line *oe* parallel to the piece *OE* intersecting *ae* in *e,* thus forming the stress triangle *oae.* Scale off *ae, oe,* and we find the stresses in the pieces *AE, OE* to be 6250 and 5000 pounds respectively.

To find the character of the stresses note that *OA* acts upward. Carrying the direction the same way round the triangle *oae* and transferring these directions to the pieces themselves, we find the piece *AE* in compression and the piece *OE* in tension.

Proceed to the next pin to the right. The stresses are represented by the sides of the quadrilateral *abfe,* of which *ab, ae* are known. The unknown stresses are *bf, fe,* which may be scaled off and their directions determined as before.

Next take the pin at the vertex. The stresses in the pieces meeting here are represented by the sides of the crossed figure *bcfgb*, of which *bc*, *bf* are known.

(6) Finally, the results may be tabulated as follows:

Name of Piece.	*AE* or *DH*	*EO* or *HO*	*BF* or *CG*	*EF* or *GH*	*FG*
Stress	− 6250	+ 5000	− 4170	− 2080	+ 2500

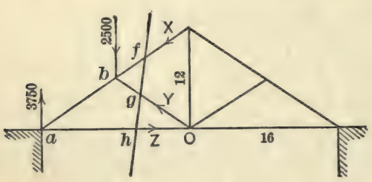
NOTE.—Instead of scaling off *AE*, *EO*, etc., we may compute their values. For the triangle *oae* is similar to the triangle formed by the left half of the truss diagram, whose dimensions are known. Then

$$3750 \text{ (or } oa) : ae = 12 : 20;$$
$$3750 \text{ (or } oa) : oe = 12 : 16;$$

and hence $ae = 6250$; $oe = 5000$; as before.

The computed and measured stresses *may* differ a few pounds, depending on the scale used. The allowable difference depends upon the character of the work.

(*b*) *Analytical Solution.*—Suppose the truss divided into two parts by the plane *fgh*. Consider the equilibrium of the part to the left of this plane.



The part to the right may be removed if we conceive forces x, y, z applied equal and opposite to the stresses in the pieces. Equilibrium will exist between the external forces 3750 pounds at a and 2500 pounds at b and these forces. The directions in which the forces 3750, 2500 tend to turn the truss is indicated by the arrows. To counteract this tendency the forces X, Y, Z must act as shown in the figure.

In forming the equation of moments (Art. 136) labor is saved if we take the point of moment for any unknown force at the intersection of the directions of the other two unknown forces. Thus, to find x, take moments about O and, noting that the length of the perpendicular from O on the direction of x is 9.6, we have

$$9.6X + 2500 \times 8 - 3750 \times 16 = 0,$$

or $$X = 4167 \text{ pounds.}$$

To find Y take moments about a,

then $$9.6Y - 2500 \times 8 = 0,$$
or $$Y = 2083 \text{ pounds.}$$

To find Z take moments about b,

then $$6Z - 3750 \times 8 = 0,$$
or $$Z = 5000 \text{ pounds,}$$

the values already found.

The directions of the arrows show that X and Y are forces of compression and Z of tension.

Similarly, by taking sections in other places we may find the stresses in all of the pieces.

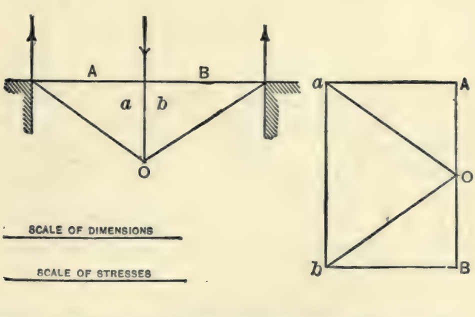
(α) Why are a, b, O the best points for taking moments in finding X, Y, Z? Would other points do? Try and see.

(β) Explain clearly how it is that the taking of moments about a, b, c gives the same results as the application of the three conditions of equilibrium of Art. 140*b*.

2. In a triangular roof-truss the rafters are $2\frac{1}{2}$ ft apart and the roofing material weighs 20 lb/ft². The span is 24 ft and height 5 ft. Find the stresses in the rafters.

Ans. 845 pounds.

3. Show that the stress diagram for the truss represented



in the figure, loaded at the center over the vertical piece *ab*, known as the king-post, is as in the margin.

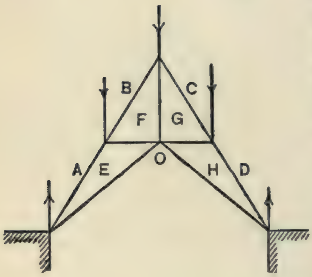
4. A foot-bridge 18 ft span and 6 ft breadth has a crowd

of people on it equal to 100 lb/ft² floor surface. The king-posts of the two trusses are 3 ft in depth. Find the stresses.

Ans. Stress on post $ab = 2700$ pounds.

5. In (3) the span is $2l$, depth d. Show that the compression in Aa is $Wl/2d$, and find the tension in Oa and the stress in the vertical ab. *Ans.* $W\sqrt{d^2 + l^2}/2d$; W.

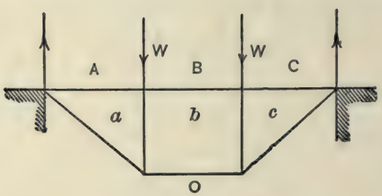
6. Find the stresses in the roof-truss represented in the figure. The span is 24 ft, rise of ridge 16 ft, rise of hip 8 ft; the trusses are 5 ft apart, and the weight of the roof-covering is 20 lb/ft².



Ans.

AE or *DH*	*BF* or *CG*	*EF* or *GH*	*EO* or *HO*	*FG*
− 3750	− 2500	− 750	+ 2700	+ 3000

7. Draw a stress diagram for a queen-post truss. The queen-posts ab, bc divide the span into three equal parts, and



the truss is loaded at the joints with weights W.

8. A foot-bridge (queen-post) of span 24 ft, breadth 7 ft, length of queen-posts 3 ft, carries a load of 100 lb/ft² of floor. Find the stresses developed, the queen-posts dividing the span into three equal parts.

Ans. Aa, Bb, or $Cc = -7467$; Oa or $Oc = +7970$;
ab or $bc = -2800$; $Ob = +7467$ pounds.

EXAMINATION.

1. Show how to find graphically the resultant of any number of coplanar forces acting on a body.

2. Is a system of coplanar forces equivalent to a single force? If not, state the exceptions.

3. Show how to find the resultant of two like parallel forces and its line of action. When the forces are unlike and parallel, how then?

4. Define the moment of a force with reference to an assigned point.

5. Why is it difficult to hold a heavy weight at arm's length?

[President Lincoln "could take a heavy ax and, grasping it with his thumb and forefinger at the extreme end of the handle, hold it out in a horizontal line from his body."]

6. Distinguish between the terms moment and momentum.

7. Three men are to carry a stick 18 ft long and weighing 200 lb, each to sustain one third of the weight. One man is to lift from the end and the other two by means of a cross-bar. Where must the cross-bar be placed?

8. An inch is taken as the unit of length. What is the geometrical representation of the unit of moment?

9. State and prove Varignon's theorem of moments (1) for forces not parallel, (2) for parallel forces.

10. Is there any reason why a man should put his shoulder to the spoke rather than to the body of a wagon in helping it uphill?

11. A couple can never be balanced by a single force.

12. The moment of a couple can never be zero.

13. State the changes which a couple may undergo without altering its statical effect.

14. Show how to find a couple equivalent to a number of couples having the same plane.

15. Forces acting along the sides of a polygon in order and proportional to the sides in magnitude may be reduced to a single couple.

16 Show that couples may be combined according to the parallelogram law.

17. The sum of the moments of the two forces of a couple about any point in their plane is constant.

18. The moment of a couple may be represented by the area of the parallelogram formed by the two forces of the couple as opposite sides.

19. Show how to combine analytically any number of coplanar forces acting on a body.

20. If any number of forces acting at a point equilibrate, the algebraic sum of the components of the forces in *any* two directions must each be equal to zero.

21. Three forces in a plane equilibrate. Show that one of two conditions must be satisfied.

22. State the conditions of equilibrium of any number of coplanar forces acting on a body.

23. The conditions to be satisfied for parallel forces in equilibrium are

$$\Sigma F = 0, \qquad \Sigma Fp = 0.$$

24. A body free to move in a plane has one point fixed. State the condition of equilibrium.

25. Two torques (P, P), (Q, Q) with arms a, b are in equilibrium if $Pa - Qb = 0$.

26. "In problems of equilibrium it is necessary to consider bodies to be perfectly rigid." Is this true? Consider a bridge, for example.

27. Find the pressures on the rails due to a given pressure P on the locomotive crank-pin.

28. Three forces represented by the three median lines AD, BE, CF of a triangle ABC equilibrate (Art. 140a).

29. Define the C.G. of a system of heavy particles, and show that in every case there exists only *one* such point.

[Suppose there are two such points G_1, G_2.

The direction of the resultant force of gravity is towards the earth's center no matter how the body is turned.

Turn the body so that the line G_1, G_2 is perpendicular to the direction of this force.

The force cannot therefore pass through both G_1 and G_2.]

30. A flat disk is not perfectly circular. How would you find its C.G.?

31. A body supported at its C.G. will remain at rest in any position.

32. A person going uphill appears to lean forwards and going down to lean backwards. Explain.

33. State a rule for finding the centroid of a uniform triangular (1) lamina, (2) wire.

34. Find the centroid of a trapezoid, the parallel sides being a and b respectively and their distance apart d.

35. Given the weights and C.G. of a body and of one part of it to find the C.G. of the remainder.

36. In an inclined plane AB a weight P suspended by a thread passing over the highest point B balances a weight Q on the plane. Show that the C.G. of P and Q remains at the same height above the base whatever the positions of the weights. (Torricelli's principle.)

37. Find the C.G. of a quadrilateral.

38. P is a point without a triangle ABC. Show that the resultant of the forces represented by PA, PB, PC passes through the C.G. of the triangle.

39. The three sections of a fishing-rod are 4 ft, 2 ft, and 1 ft long and weigh 8 oz, 6 oz, and 4 oz respectively. Find the C.G. of the rod when drawn out to its full length.

40*a*. How does the property that every body has but one C.G. help us to solve geometrical theorems?

[Place weights at certain points. Combine in different order to find the C.G. of the system. The various portions of the C.G. found must coincide.]

40*b*. The median lines of a triangle pass through one point.

40*c*. The lines joining the middle points of the opposite sides of a quadrilateral bisect each other.

40*d*. The line joining the middle point of the two diagonals of a quadrilateral and the lines joining the middle points of the opposite sides all intersect in one point.

41. Explain the positions of force, resistance, and fulcrum in the following levers: Wheelbarrow, spade, pair of scissors, the forearm.

42. A man carries a bundle at the end of a stick which passes over his shoulder. Show that the pressure on his shoulder varies inversely as the distance of his hand from his shoulder.

43. Find the position in which a beam-balance will rest when loaded with unequal weights.

44. In a beam-balance if the C.G. coincides with the point of suspension the balance remains in equilibrium in all positions.

45. Show that Borda's method of double weighing gives the correct weight of a body no matter how false the balance used.

46. In a beam-balance great sensitiveness and quick weighing are to a certain extent incompatible.

47. In a common steelyard the length of the graduations is inversely proportional to the movable weight.

48. Two smooth spheres, each weighing W lb, rest in contact between two smooth planes inclined at 30° and 60° to the horizontal. Find the position of equilibrium.

Ans. Inclination of line joining centers to horizontal $= \tan^{-1} 3/4$.

49. A rod a yard long rests over the edge of a vertical hollow cylinder $2\frac{1}{4}$ inches diameter with one end against the inner surface. Show that in the position of equilibrium the rod is inclined at 60° to the horizontal.

50. Forces P, Q, R act at a point O and keep it in equilibrium. If any line cut their directions in A, B, C, prove

$$P/OA + Q/OB + R/OC = 0.$$

51. If forces P, Q, R acting at the center O of a circular lamina along the radii OA, OB, OC equilibrate forces P', Q', R' acting along the sides BC, CA, AB of the inscribed triangle ABC, show that

$$PP'/BC + QQ'/CA + RR'/AB = 0.$$

52. If D is the degree of curve, show that the elevation of outer rail required for an engine speed of 30 miles an hour in passing round a circular curve is

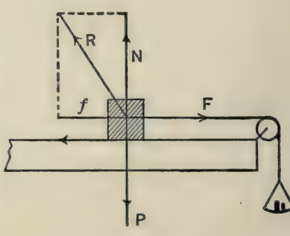
$3D/5$ inches, nearly.

CHAPTER V.

FRICTION.

169. As already stated, greater simplicity and clearness are secured by considering the properties of bodies one at a time, and thus leading up to the actual state of the case, which is quite complicated, since bodies in nature possess many properties. Thus far the surface of a body has been assumed to be perfectly smooth, that is (Art. 104), to offer no resistance to the motion of a body in contact with it. But in reality we know that if one body be moved along another (as a book along a table) a certain resistance will be offered to the motion. The resistance arises from irregularities in the surfaces in contact, from elevations and depressions which fit more or less closely into one another. To it the name **Friction** is given.

Suppose a body weighing W lb to rest on a horizontal plane and to be acted on by a vertical force of Q pounds. The total vertical force $(P = Q + W)$ pounds is equilibrated by the vertical reaction N of the plane. If now a small force is applied parallel to the plane, the motion of the body is prevented by the equal force of friction called into play. If the force be increased, the friction is equally increased until a certain limit is reached when a further increase of force causes motion. The friction between two bodies thus accommodates itself to the acting force up to a certain limit when



motion takes place, and the amount called into play when the body is just on the point of moving is called *limiting friction.* Or, since the body is just on the point of moving and the acting forces may be regarded as equilibrating one another, the friction is named *static friction.*

If the surfaces in contact were smooth, the reaction N would be normal to the table. But now that the plane is not smooth the reaction R is necessarily no longer normal. The component of R normal to the plane would be the reaction N as between smooth surfaces, and the component along the plane would be the friction f between the two surfaces.

170. Laws of Static Friction.—From experiment it is found that between surfaces with little or no lubrication, under moderate loads, and just beginning to slide on one another, or when the velocity of motion is small, the amount of static friction is—

(1) *Proportional to the normal stress between the surfaces in contact.*

(2) *Independent of the areas in contact, the normal stress remaining the same.*

(3) *Dependent on the material of which the bodies are composed.*

Hence from law (1) the friction f corresponding to the normal stress N is given by

$$f = \mu N, \quad \text{or} \quad f/N = \mu,$$

where μ is a constant depending on the material and character of the surfaces in contact. It is called the **coefficient of friction,** and its value must be determined by experiment (Arts. 175, 179).

171. The above "laws" of friction were deduced from experiments made with surfaces having little or no lubrication, and moving with low velocities; and for such conditions only are they to be depended on. In machines, however, surfaces without lubrication and moving with low velocities are the exception, and we there have an entirely different set of conditions. The friction is now *kinetic.** Recent experiments show that even with surfaces of the same material, the character of the lubrication, the thickness of the lubricating film, the load, the velocity, the form of the surfaces whether flat or curved, the areas in contact, and the temperature of the surfaces have each great influence on the friction produced. The films that coat the surfaces slide over one another when the pressure is not excessive. No general relation between the friction (f) and the pressure (N) producing it has yet been deduced depending on these conditions. Special experiments are necessary in all cases where the conditions differ in any of the points mentioned from those entering into experiments already made.

We may, however, in general write

$$f = \mu N,$$

where the value of μ is determined by the conditions of the problem. These conditions will show whether we may assume the so-called laws of statical friction or have recourse to special experiment. Roughly, the coefficient of friction μ for well-lubricated surfaces is from $\frac{1}{4}$ to $\frac{1}{10}$ that for dry surfaces.

172. When one surface rolls on another, the resistance encountered is termed *rolling friction*. Consider a heavy wheel rolling over a horizontal surface. The wheel from its weight compresses the surface and is itself compressed at the place of support. Thus the area in contact is greater than if both were rigid, and the wheel is forced to climb an elevation in order to move forward. The elevation occurs at every new position of the wheel, and the resistance is continuous. The amount of this frictional resistance depends on the nature of the materials in contact.

Hence the importance of smooth hard roads for carriages

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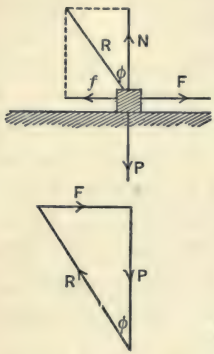
* The distinction between static and kinetic friction was first pointed out by Coulomb (1736–1806). The laws of static friction were enunciated by Coulomb, and confirmed by the later experiments of Morin at Metz in 1837–1838.

and bicycles. Hence, too, the importance of tracks well ballasted, and of heavy rails whether for locomotive or street-car traffic. In railway work when the road is of heavy steel rails the rolling friction is found to be small in comparison with the bearing friction.



173. Angle of Friction—In the experiment of Art. 169 the three forces F, P, R hold the body in equilibrium and may be represented by the sides of a right triangle. The reaction R is equivalent to the two forces f along the plane and N normal to it.

The angle ϕ between the directions of R and N is called the **angle of friction.**

Now, since the body is in equilibrium under F, N, f, P,

$$F - f = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$N - P = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also, since the body is in equilibrium under F, R, P,

$$F - R \sin \phi = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$P - R \cos \phi = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But by definition $\qquad f = \mu N. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$

Hence $\qquad \mu = f/N = F/P = \tan \phi,$

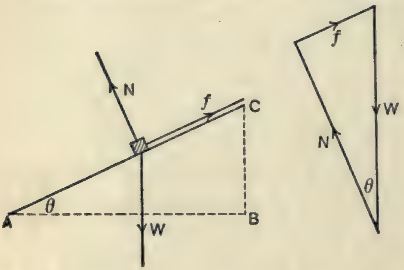
or *the coefficient of friction is equal to the tangent of the angle of friction.*

Ex. 1. Find the coefficient of friction when the angle of repose is 30°; 45°. *Ans.* $1/\sqrt{3} = .577$; 1.

2. The coefficient of friction is 0.4. Find the angle of friction. *Ans.* 21° 48′.

174. The fact that the reaction R, which holds the forces F and P in equilibrium, makes with the normal to the surfaces in contact an angle equal to the angle of friction ϕ, gives a key to the application of the graphical method of the polygon of forces to problems involving friction. The method is especially valuable in cases where the forces are interlaced as in mechanisms.

175. Measurement of the Coefficient of Friction.—Suppose a body of weight W to rest on a table, and that the table is tilted about the edge A. We have now the body resting on an inclined plane AC. Continue tilting until an inclination θ is reached when the body is just on the point of moving down the plane. At this point the forces holding it in equilibrium are W vertically downward, N normal to the plane, and f along the plane. Draw the triangle of forces, and



$$f/N = \tan\,\theta.$$

But since f is the limiting friction,

$$f/N = \tan\,\phi.$$

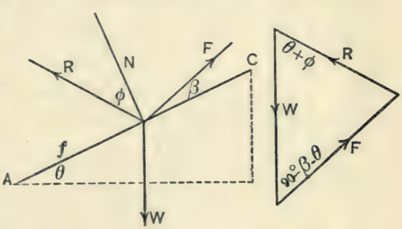
Hence $\theta = \phi$, or the angle of inclination of the plane, is equal to ϕ, the angle of friction. For this reason the angle of friction is called the **Angle of Repose.**

This gives an experimental method of measuring the angle of friction, and thence the coefficient of friction between two bodies P, Q. For form an inclined plane of one of the bodies, P, and on this place the other, Q. Tilt the plane until Q just begins to slide: the tangent of the angle of inclination is the coefficient of friction for the two bodies.

176. The following coefficients of friction may be regarded as average values, to be used when no special experiments covering the cases under consideration are possible. The circumstances may be such that the tabular values are very far from the truth. Indeed, at present we may be said to be acquainted with no quantitative laws of friction of much value.

Wood on wood or metal, surface		dry........	0.4 to 0.6
" " "	"	lubricated .	0.1 to 0.2
Metal on metal,	"	dry.......	0.2
" "	"	lubricated..	0.075
Steel on ice................................			0.02

177. Equilibrium on a Rough Incline.—To find the limits between which a force F must lie to hold in equilibrium a body of weight W on the rough incline AC, the inclination θ of the plane to the horizon, the inclination β of the force to the plane, and the angle of friction ϕ being given.



(*a*) Let the body be on the point of moving *up* the plane.

The friction f acts *down* the plane.

The forces acting are F, W, N, and f. Or, since the resultant R of N and f makes an angle ϕ with the normal to the plane, the forces acting may be taken to be F, W, R. Then by Lami's theorem

$$F/\sin(\theta + \phi) = W/\sin(90 + \beta - \phi),$$
$$\text{or} \qquad F = W \sin(\theta + \phi)/\cos(\beta - \phi),$$

(*b*) Let the body be on the point of moving *down* the plane· Then similarly

$$F = W \sin(\theta - \phi)/\cos(\beta + \phi).$$

Hence for any force lying between these values of F the body remains in equilibrium and is not on the point of moving either up or down the plane. These values of F are the *limiting values* required.

178. These results may also be obtained by the method of Art. 86. For when the body is on the point of moving *up* the plane, we have, resolving along and perpendicular to the plane,

$$F \cos \beta - f - W \sin \theta = 0,$$
$$F \sin \beta + N - W \cos \theta = 0.$$

Also, $\qquad f = \mu N, \quad \text{and} \quad \mu = \tan \phi.$

The resulting value of F will be found to agree with that just given in (*a*).

The value of the resultant reaction of the plane will be found to be $W \cos (\beta + \theta)/\cos (\beta - \phi)$.

Ex. 1. A weight of 60 lb rests on a rough level floor. Find the least horizontal force that will move it, the coefficient of friction being 0.5. *Ans.* 30 pounds.

Find the resultant reaction of the floor.

Ans. $30\sqrt{5}$ pounds.

2. In Ex. 1 find the least force inclined at 45° to the floor that will move the weight. *Ans.* $20\sqrt{2}$ pounds.

Find the resultant reaction of the floor.

Ans. $20\sqrt{5}$ pounds.

What is its direction?

3. The weight on the driving-wheels of a locomotive is twenty tons, and the coefficient of friction is 0.2. Find the greatest pull the engine is capable of. *Ans.* 4 tons.

4. Find the least force that will drag a body weighing 100 lb along a rough horizontal plane, the coefficient of friction being $1/\sqrt{3}$. *Ans.* 50 pounds.

Find the resultant reaction of the plane.

Ans. $50\sqrt{3}$ pounds.

5*a*. Find the angle β which a given force F must make with a horizontal plane that a weight W may just be on the point of sliding on the plane, the angle of friction being ϕ.

Ans. β is found from $F \cos (\beta - \phi) = W \sin \phi$.

5*b*. For what value of β is the force F the least possible, and what is the value of the least force?

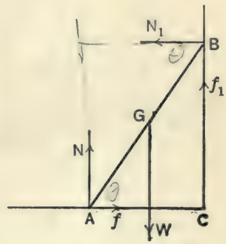
Ans. $\beta = \phi$; $F = W \sin \phi$.

5*c*. What is the resultant reaction of the plane?

Ans. $W \cos \phi$.

6. Find the least angle of inclination of a wooden incline that stone blocks may slide down it under gravity only ($\mu = 1/\sqrt{3}$). *Ans.* 30°.

7. The foot of a ladder of weight W and length l rests on the ground at A, and the top at B against a rough vertical wall. Find its inclination θ when on the point of sliding, the coefficient of friction in each case being 0.5.



[The forces acting are W at G the middle point of AB, the reaction N of the ground at A normal to AC, and the friction f along AC; the reaction N_1 at B normal to CB, and the friction f_1 along CB.

The forces equilibrating, the conditions of Art. 86 must be satisfied. Hence, resolving vertically, horizontally, and taking moments about G,

$$N + f_1 - W = 0, \quad \ldots \ldots \quad (1)$$
$$f - N_1 = 0, \quad \ldots \ldots \quad (2)$$
$$-Nl\cos\theta + fl\sin\theta + N_1 l\sin\theta + f_1 l\cos\theta = 0. \quad (3)$$

Also, since the ladder is just on the point of sliding, we have from the first of Morin's laws (Art. 170)

$$f = N/2, \qquad f_1 = N_1/2.$$

Hence, by substitution in (3),

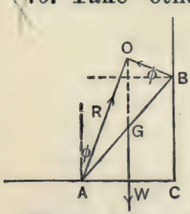
$$\tan\theta = 3/4.$$

From equations (1) and (2) we have the reactions

$$N = 4W/5, \qquad N_1 = 2W/5.]$$

7a. Is there any advantage in taking moments about G in the preceding solution?

7b. Take other points and find from which the most simple equation results. Try A and B for example.



7c. Deduce the relation $\tan\theta = 3/4$ from geometrical considerations.

[The resultant R of N and f makes an angle ϕ with the normal at A.

The resultant R_1 of N_1 and f_1 makes an angle ϕ with the normal at B.

There being three forces R, R_1, W, they must meet in a point O (Art. 141), which, since the direction and position of W are fixed, must lie on the vertical passing through G.

Evidently $\angle AOB = 90°$. But $AG = GB$. $\therefore GO = GB$ and $90 - \phi = \theta + \phi$. Hence

$$\tan\theta = \cot 2\phi = 3/4, \qquad \text{since } \cot\phi = 2.$$

The reactions N and N_1 may now be found by Lami's theorem. Find them.]

7d. If the coefficients of friction at A and B are μ and μ_1, show that

$$\tan\theta = (1 - \mu\mu_1)/2\mu.$$

Obtain this in two ways as above.

7e. If the wall is smooth and the coefficient of friction between ground and ladder is μ, the inclination of the ladder to the horizontal is $\cot^{-1} 2\mu$.

8. Find the horizontal force necessary to push a body weighing W lb up a rough incline, the angle of inclination of the plane to the horizon being θ and the angle of friction ϕ.
Ans. $W \tan(\theta + \phi)$.

What is the force necessary to keep the body from sliding down? *Ans.* $W \tan(\theta - \phi)$.

9. A weight W is just supported by friction on a plane inclined at an angle θ to the horizon. Show that it cannot be moved up the plane by any horizontal force less than $W \tan 2\theta$.

Examine the special case of $\theta = 45°$.

10. A ladder 20 ft long rests at 45° against a rough vertical wall. A man whose weight is twice that of the ladder mounts it. When will the ladder begin to slip, the coefficient of friction being 0.5 at either end? *Ans.* After going up 13 ft.

11*a*. A body weighing W lb is placed on a rough plane inclined at an angle θ to the horizon. Find the limits between which a force acting parallel to the plane must lie in order to keep the body from moving.

Ans. $W(\sin\theta \pm \mu\cos\theta)$ or $W\sin(\theta \pm \phi)/\cos\phi$ if $\mu = \tan\phi$.

11*b*. If the force necessary to pull the body up the plane be n times that required to keep it from sliding down, then

$$(n-1)\tan\theta = (n+1)\tan\phi.$$

12. A weight can just be sustained on a rough plane in-

clined at 45° by a force acting horizontally or by an equal force acting along the plane. Show that

$$\mu = \sqrt{2} - 1.$$

13. On a hill sloping 1 in 50 a loaded sled weighing 1 ton is kept from sliding down. Show that the pull of the horses may vary from 360 to 440 pounds, the coefficient of friction between sled and snow being 0.2.

14. A rod resting within a rough sphere subtends a right angle at the center of the sphere. If the rod is on the point of moving, show that its inclination to the horizontal is twice the angle of friction.

15. If in (14) the rod subtends an angle 2β at the center, then

$$2 \tan \theta = \tan (\beta + \phi) - \tan (\beta - \phi).$$

16. The angle of a wooden incline is 68°. Show that it is impossible to drag a wooden block up the plane by a horizontal force, the coefficient of friction being 0.4.

17. The force necessary to haul a train at uniform speed on a 1% grade is 3.5 times that on the level. Show that the coefficient of friction is 1/250.

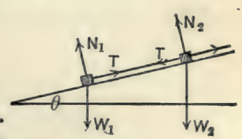
18. A uniform beam rests on two rough inclined planes. Show that in the position of equilibrium the inclination θ of the beam to the horizontal is given by

$$2 \tan \theta = \cot (\alpha + \phi) - \cot (\beta - \phi),$$

when α, β are the inclinations of the planes and ϕ is the angle of friction.

19*a*. Two weights W_1, W_2, joined by a thread, are placed on a rough narrow inclined plane. If the coefficients of friction between weights and plane are μ_1, μ_2, show that the weights will just begin to slide down when the plane is tilted to an angle

$$\tan^{-1} (\mu_1 W_1 + \mu_2 W_2)/(W_1 + W_2).$$



[This is a good illustration of Art. 160. Consider the system as a whole. Then the tensions T, T are internal forces. For equilibrium, resolving along the plane and perpendicular to it,

$$W_1 \sin\theta + W_2 \sin\theta = \mu_1 N_1 + \mu_2 N_2,$$
$$W_1 \cos\theta + W_2 \cos\theta = N_1 + N_2.$$

But since the first weight is in equilibrium under W_1, N, T,

$$N_1 = W_1 \cos\theta;$$

and since the second weight is in equilibrium under W_2, N, T,

$$N_2 = W_2 \cos\theta.$$

Hence after a simple reduction

$$\tan\theta = (\mu_1 W_1 + \mu_2 W_2)/(W_1 + W_2).]$$

19*b*. Solve the problem by considering the weights in equilibrium under W_1, N, T and W_2, N, T respectively, and equating the two values of T.

19*c*. Show that the tension of the thread is

$$W \sin\theta - \mu W \cos\theta.$$

20. Three equal weights are joined as in Ex. 19. If the coefficients of friction are 0.4, 0.5, 0.6, show that the slope of the plane is as 1 to 2.

179. Motion on a Rough Plane.—Suppose a body weighing W lb to slide on a rough horizontal plane and let μ be the coefficient of friction, to determine the motion.

The normal force being W pounds, the friction is μW pounds. Hence the motion may be said to be due to a force of $-\mu W$ pounds.

Let a be the acceleration produced. Then (Art. 67)

$$-\mu W = Wa/g,$$

and ∴ $$a = -\mu g.$$

If u is the initial velocity, we have (Art. 24)

$$v = u - \mu gt,$$
$$s = ut - \tfrac{1}{2}\mu gt^2,$$

which will give the velocity acquired and distance passed over at the end of a time t.

Application.—Coulomb and Morin used an apparatus similar to that shown in Art. 169 to determine the coefficient of friction. The coefficient of static friction was given by the force F necessary to bring the body P from rest to motion. The coefficient of kinetic friction was computed by noting the time t taken by P to slide over a distance s. Here

$$\text{moving force} = F - \mu P.$$

If a is the acceleration produced,

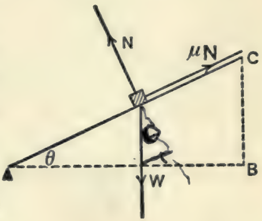
$$(F + P)a/g = F - \mu P.$$

But $$s = \tfrac{1}{2}at^2;$$

$$\therefore \quad \mu = F/P - 2(F + P)s/Pgt^2,$$

and μ is found.

180. Motion on a Rough Incline.—Let the plane be inclined to the horizontal at an angle θ and let μ be the coefficient of friction.



If the body is moving *down* the plane, the forces acting are W vertically downwards, the reaction N normal to the plane, and the friction μN *up* the plane.

The force causing motion down the plane is the sum of the components along the plane AC, or $W \sin \theta - \mu N$. Hence, if a is the acceleration of motion (Art. 67),

$$W \sin \theta - \mu N = Wa/g. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Resolving the forces perpendicular to AC, we have

$$N = W \cos \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Eliminating N between (1) and (2),

$$a = g(\sin \theta - \mu \cos \theta),$$

or, putting $\mu = \tan \phi$,

$$a = g \sin(\theta - \phi)/\cos\phi, \quad . \quad . \quad . \quad . \quad (3)$$

which gives the acceleration down the plane.

Similarly, if the body is projected up the plane, the accleration (found by changing the sign of μ) is

$$g \sin(\theta + \phi)/\cos\phi.$$

The acceleration being found, the velocity at any time and the distance passed over are known from Art. 24.

Ex. 1. A curling-stone is projected along ice with a velocity of 16 ft/sec. If the coefficient of friction is 0.1, find in what time it will come to rest and how far it will travel.

Ans. 5 sec; 40 ft.

2. A weight of 8 lb rests on a rough floor. Find the least horizontal force that will give it an acceleration of 4 ft/sec^2, the coefficient of friction being 0.5. *Ans.* 5 pounds.

3. A body with initial velocity u slides along a rough horizontal plane on which the coefficient of friction is μ. Show that it will come to rest in a time $u/\mu g$ after passing over a distance $u^2/2\mu g$.

4. A train moving at 40 miles an hour on a level track is brought to rest by friction in half a minute. Prove that the coefficient of friction is 11/180.

5. A weight W_1 lb is moved along a rough table by a weight W_2 lb attached to a thread passing over a smooth pin at the edge of the table. Find the pull of the thread.

Ans. $W_1 W_2(1 + \mu)/(W_1 + W_2)$ pounds.

6. The inclination of a plane is 45° and the coefficient of friction 0.75. Show that the time taken by a body to slide down this plane is twice what it would be if the plane were smooth.

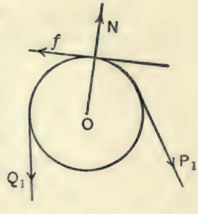
7. A body is projected up a rough plane inclined at an angle θ to the horizon. If t_1 is the time of ascending and t_2 the time of descending, show that

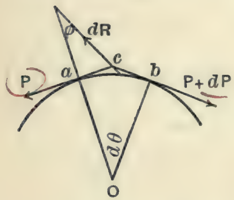
$$(t_1^2 + t_2^2)\cot\theta = (t_2^2 - t_1^2)\cot\phi,$$

where ϕ is the angle of friction.

8. A train of 100 tons (excluding engine) runs up a 1% grade with an acceleration of 1 ft/sec^2. If the friction is 10 pounds per ton, find the pull on the drawbar between engine and train. *Ans.* $4\frac{5}{8}$ tons.

181. Belt Friction.—This is an interesting and important application of the preceding principles. Consider a belt or rope passing over (or around) a cylindrical block securely fastened, and let P_1, Q_1 denote the forces at the ends of the belt. Suppose P_1 to be just on the point of overcoming Q_1, or that the belt is moving with uniform velocity. The forces acting on the belt, its weight being neglected, are P_1, Q_1, the reaction of the cylinder, and the friction of the cylinder. In order to find the relation between them conceive the arc of contact of the belt cut into elements of indefinitely small length, find the relation for one element, and then sum up for the whole arc.





The pull of the belt increases from Q_1 to P_1. Consider an element ab of the belt and let P denote the pull at a and $P + dP$ the pull at b. The third force acting is the reaction dR of the pulley, which makes an angle ϕ equal to the angle of friction with the radius (Art. 174).

The forces which hold ab in equilibrium P, $P + dP$, and dR, being three in number, must intersect in one point c (Art. 141). Call the angle $aOb = d\theta$.

Then by Lami's theorem

$$P/\sin(90 + \phi + d\theta) = (P + dP)/\sin(90 - \phi),$$

or $$P/(\cos\phi - d\theta\sin\phi) = (P + dP)/\cos\phi,$$

since $d\theta$ is a small angle, and therefore $\cos d\theta = 1$, $\sin d\theta = d\theta$.

Hence, clearing of fractions, we have ultimately

$$dP = P\tan\phi\, d\theta$$
$$= \mu P d\theta,$$

putting $\tan\phi = \mu$ the coefficient of friction.

Summing up all the elements ab of the rope in contact, noting that the limits of P are P_1 and Q_1, and calling the angle subtended by the arc of contact θ, we have, if θ is expressed in circular measure,

$$\int_{Q_1}^{P_1} dP/P = \int_0^\theta \mu\, d\theta,$$

or $$\log_e P_1/Q_1 = \mu\theta,$$

or $$P_1 = Q_1 e^{\mu\theta},$$

when $e(=2.718)$ is the base of the natural system of logarithms.

Transferring to the common system of logarithms, as more convenient for computation,

$$\log_{10} P_1/Q_1 = 0.434\mu\theta.$$

182. If the weight Q_1 is on the point of slipping down instead of being drawn up, the action is the same as if P_1 and Q_1 changed places, and therefore $Q_1 = P_1 e^{\mu\theta}$.

The conditions are the same and the relation between P_1 and Q_1 the same if the cylinder instead of being fixed is capable of turning about O, and no slipping occurs, axle friction being neglected.

Ex. 1. Obtain the relation

$$dP = \mu P d\theta$$

by resolving the forces along and perpendicular to the tangent at a.

2. A rope passing over a wooden cylinder supports a barrel of flour weighing 196 lb. Find the force which will just raise the barrel, the coefficient of friction being 0.4.
Ans. 689 pounds.

3. In Ex. 2 find the force that will just keep the barrel from slipping down. *Ans.* 56 pounds.

4. Find the number n of turns of rope round a snubbing-post that a man pulling P pounds may just be able to hold a canal-boat pulling Q pounds. *Ans.* $2\pi\mu n = \log_e Q/P$.

5. A chain is wrapped twice round an iron drum. Find the coefficient of friction if a pull of 100 pounds just supports 50 tons. *Ans.* 0.55.

6. If $\mu = 0.25$, and a rope passes twice round a post, prove that any force will balance another more than twenty times as great.

7. If the coefficient of friction between belt and pulley is 0.3, and 0.4 of the pulley is embraced by the belt, show that $P = 2Q$, nearly.

EXAMINATION.

1. When is a body said to be smooth? When rough?

[As it exerts force on another body normal or not to the surface in contact.]

2. "Friction is a self-adjusting force." Explain.

3. The slopes of railroad embankments vary. Explain.

4. What is the usual rule for slopes in a railroad cut through sand and gravel? [$1\frac{1}{2}$ to 1.]

5. State the laws of limiting friction and define the coefficient of friction.

6. Show that coefficient of friction = tan (angle of friction).

7. Give an experimental method of finding the coefficient of friction between two surfaces.

8. What horizontal force will cause a state bordering on motion in a weight of 10 lb lying on a level floor if the coefficient of friction is 0.5?

9. The total resistance to motion of a body on a rough incline and just on the point of motion is $N\sqrt{1+\mu^2}$, where N is the normal force.

10. A body is held in equilibrium on a rough inclined plane by a force along the slope. Find the limits of this force.

11. The force required to haul a sled up hill is least when the inclination of the tongue is equal to the angle of friction.

12. Simon Stevinus of Bruges (1548–1620) remarked that the question, "What force will support a wagon on an inclined plane? is a statical question, but that the question, What force

will move the wagon? requires additional considerations to be introduced." Discuss this statement.

13. A die rests on a rough board and the board is tilted. Will the die slide or topple over?

14. A cylindrical jar 3 in in diameter and 1 ft in height rests on a rough table ($\mu = 1/3$) and the table is tilted. Show that the jar will topple over before it slides.

15. A sphere cannot rest upon an inclined plane however rough.

16. A cubical box is half filled with water and placed upon a rough incline. Show that it will slide or topple over as $\mu < > 1$.

17. What is meant by the "friction of adhesion" of a locomotive?

[Force at circum. of driver/weight on driver.]

18. The weight on the driving-wheels of the St. Clair tunnel decapod locomotive is 195,000 lb. Taking the coefficient of traction to be 600 pounds per ton, find the hauling force on the drawbar. *Ans.* 58,500 pounds.

19. The tractive force required to propel a bicycle over a smooth level surface is 0.01 of the load. Calling the load 150 lb, a force of one and a half pounds would be required to move the wheel forward.

20. Explain the contrivance known as *friction-wheels.* What is the advantage of ball-bearings for bicycles? Sketch in section such a bearing.

21. In the figure of Art. 162 the condition of security of the wall against sliding on the base AB is that the horizontal component of the force P is less than μW.

22. Account for girder-rails taking the place of thin rails for economy in railroad traffic.

[The blows from the wheels would distort the rails so that rolling friction would be enormous.]

23. A tug-boat running at 15 miles/hour begins to turn on a curve of 150 ft radius. Show that objects on the deck will

slide unless the coefficient of friction exceeds 0.1, the deck being supposed to remain horizontal.

24. Prove that starting from scratch the shortest time over 100 yards is about 15 seconds, taking the adhesion of the feet as 1/12 the weight.

25. A drawer of length l can be pulled out by one handle unless the distance of the handle from the center exceeds $l/2\mu$, where μ is the coefficient of friction.

26. A rod l ft long and weighing 1 lb per foot lies on a rough table, the coefficient of friction being μ. The rod is pulled at right angles to its length by a force at one end. Show that the point P about which it begins to turn is at a distance $l/\sqrt{2}$ ft from that end.

CHAPTER VI.

WORK AND ENERGY.

183. In Art. 53 it was shown that if a force F acts upon a particle w through a distance s, and v is the velocity acquired, we have the relation

$$Fs = wv^2/2$$

F being expressed in absolute units, or

$$Fs = wv^2/2g$$

F being expressed in gravitation units.

This equation, which was first stated by Huygens, being an algebraic statement of the second law of motion, may consequently be used as the starting-point of the science of dynamics. It was so used first by Lagrange, and often since his time by other writers. (Art. 217.)

We proceed to discuss the separate terms of this equation and then the equation itself.

184. Work.—When a particle acted on by a force is displaced in the direction of the force, the force is said to *do work* in moving the particle, and the work done, depending both on the force and the displacement, is expressed by their product. Thus, if a weight of W lb falls through a height of h ft, the acting force is W pounds, and the work done *by* the force of gravity is expressed by the product Wh. If the weight is raised through h ft, the resisting force of gravity W pounds is overcome through h ft, and the work done *against* the force of gravity is expressed by $-Wh$.

In general, when work is being done by a force F, constant in magnitude and line of action, the point of application of the force is being displaced through a distance s in the direction of the force, and *the work done* W *by the force is measured by the product of the force F and the displacement s*, or

$$\mathsf{W} = Fs.$$

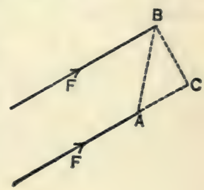
If the motion be uniform and R denote the resisting force overcome, we may write

$$\mathsf{W} = -Rs$$

as the expression for the work done *on* the resistance.

Work may therefore be defined as *the overcoming of resistance.*

185. The definition of work done may be stated in an equivalent form, which is often convenient. Let the particle on which the force F acts at A be displaced to B so that AB is the total displacement. Let fall BC perpendicular to CA, and denote the angle BAC by θ.



Then by definition

$$\mathsf{W} = F \times AC.$$

$$\text{But} \quad AC = AB \cos\theta.$$

$$\therefore \quad \mathsf{W} = F \times AB \cos\theta,$$

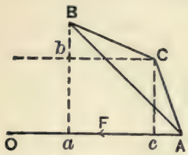
$$\text{or} \quad \mathsf{W} = F \cos\theta \times AB,$$

or *the work done by a force acting obliquely to the path of a particle is measured by the product of the force and the projection on its direction of the total displacement, or by the product of the component of the force along the total displacement by that displacement.*

When $\theta = 90°$, then $\cos\theta = 0$ and $\mathsf{W} = 0$. Hence, when the displacement is at right angles to the direction of the force, the work done by the force is *nil.*

Ex. In a pendulum find the work done by the pull of the rod on the bob as it swings to and fro.

186. In the definition nothing is said about the nature of the path of the point of application of the force. We shall now show that the work done does not depend on the form of the path.



Let the particle on which the force F acts at A be moved to B either in the straight path AB or in the broken path ACB. Let fall Bba, Cc perpendicular to OA, the direction of F. Then by definition

$$\begin{aligned}\text{work done in straight path } AB &= F \times Aa;\\ \text{work done in broken path } ACB &= F \times Ac + F \times ca\\ &= F(Ac + ca)\\ &= F \times Aa,\end{aligned}$$

or the work done is the same in the two paths.

The same result evidently follows if the broken path consists of any number of parts instead of two. And this no matter what the magnitude of these parts. But a curve is the limit of an indefinitely great number of indefinitely short paths. Hence the result is true for a curvilinear path. So that

The work done by a force on a particle in passing from one position A to another position B is entirely independent of the path of the particle between A and B.

187. Work under Variable Force.—If the acting force F is not constant in magnitude, we may conceive the path s described by its point of application between two points A and B divided into an indefinitely great number of parts, each part so small that the force may be considered constant throughout it. Let $PQ (= \Delta s)$ denote one of these small portions of the path. Then, the line of action of F at P being along PQ, the work done through PQ is $F \times \Delta s$, and the total work done in moving through the distance s is found by summing the elementary works $F \times \Delta s$, and would be expressed by

$$\int_B^A F ds.$$

More generally, if the line of action of F at P is not in the direction of PQ, let Δp denote the projection of the displacement of the point of application of F along the direction of F; then the elementary work done is $F \times \Delta p$, and the total work done between two positions A and B would be expressed by

$$\int_B^A F\,dp = \int_B^A F \cos\theta\,ds.$$

188. *Unit of Work.*—In the general formula for work

$$\mathsf{W} = Fs;$$

taking $F = 1$, $s = 1$, we have $\mathsf{W} = 1$; and therefore the *unit of work* is taken to be the work done by unit force acting through unit distance.

The engineering unit of work or unit of everyday life is based on the gravitation measure of force, and is the work done by a force of one pound acting through a distance of one foot. This is usually called a **foot-pound.** Thus the work done in raising a body weighing 100 lb vertically through 6 feet is the work done in overcoming a force of 100 pounds through 6 feet, and is 600 foot-pounds.

Ex. 1. The tractive force of a locomotive is 10 tons. Find the work done in hauling a train one mile.

Ans. 105,600,000 ft-pounds.

2. The steam-pressure on the piston of an engine is 50 pounds/in^2, the area of the piston 300 in^2, and the stroke is 4 ft. Find the work done in one stroke.

Ans. 60,000 ft-pounds.

3. A hole is punched through a wrought-iron plate 1/2 in thick, the punch exerting a uniform pressure of 42 tons. Find the work done. *Ans.* 3500 ft-pounds.

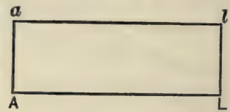
4. A horse hauling a wagon exerts a constant pull of 100 pounds and travels at the rate of 2 miles an hour. How much work will be done in 5 min? *Ans.* 88,000 ft-pounds.

5. Find the work done in lifting a chain which hangs vertically, its length being l ft and its weight W lb.

Ans. $Wl/2$ ft-pounds.

189. *Graphical Representation of Work.*—Work done may be represented graphically.

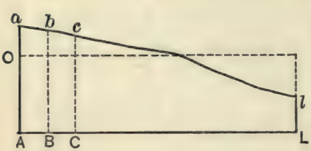
(1) Let the acting force F be constant in magnitude and direction. Let AL plotted to scale represent the displacement of the point of application of the force, and let Aa plotted to scale and perpendicular to AL represent the force at A. Then, since the force is uniform throughout, the work done $AL \times Aa$ is represented by the area of the rectangle Al.



Ex. Plot on a scale of 5 pounds = 1 in and 10 ft = 1 in the work done in overcoming a resistance of 20 pounds through a distance of 20 ft.

(2) If the force F is variable, the work done is expressed as already explained in Art. 187.

Let AL represent the displacement, and let Aa, Ll perpendicular to AL represent the initial and final values of the force F in the direction of motion. Let AL be divided into a large number n of equal parts, and let Bb, Cc, . . . represent the corresponding forces. A line through the extremities $a, b, c, \ldots l$ forms the curve of resistance.



The element work $F\Delta p$ is represented by the area $BCcb$ ultimately, and the total work $\int Fdp$ by the sum of these element areas, that is, by the area $ALla$.

The value of the work done may be computed approximately from the diagram.

For the work done from A to B may be taken to be that due to the force $\frac{1}{2}(Aa + Bb)$ acting through the distance AB or AL/n, or

$$\text{the work done through } AB = \tfrac{1}{2}(Aa + Bb)AL/n.$$

Similarly, the work done through $BC = \frac{1}{2}(Bb + Cc)AL/n$.

. . . .

Hence by addition

total work done $= \frac{1}{2}(Aa + 2Bb + 2Cc + \ldots + Ll)AL/n$.

By means of certain contrivances the curve $abc \ldots l$ may be plotted mechanically by the force itself, as, for example, in the steam-engine by means of the indicator. Having the curve, the mean force AO may be found by stretching a thread so as to have equal areas above and below it. Or the area may be read off at once by an Amsler polar planimeter, and the work done found directly. Or the indicator drawing or "card" may be divided up by drawing equidistant ordinates, the lengths of these ordinates scaled off, and the formula above applied. All of these methods are at times useful.

Ex. 1. Draw a diagram for the work done in raising the chain in Ex. 5, p. 227.

Ans. A right triangle, the force decreasing from W to 0.

2. Draw a diagram for the work done in lifting a chain vertically by one end from the ground.

Ans. A rt. triangle. (Plot it.)

190. *Work of Raising a System of Particles from One Position to Another.*—Let the particles weigh $w_1, w_2, \ldots$ and let $x_1, x_2, \ldots$ denote their respective heights above a fixed horizontal plane. Let the system as a whole weigh W, and let $\bar{x}$ denote the height of its C.G. above the plane. Then

$$W\bar{x} = w_1x_1 + w_2x_2 + \ldots \quad . \quad . \quad . \quad . \quad (1)$$

Suppose the system displaced so that w_1 is raised through a height h_1, w_2 through a height $h_2, \ldots$ and the C.G. of the whole through a height $\bar{h}$. Then

$$W(\bar{h} + \bar{x}) = w_1(h_1 + x_1) + w_2(h_2 + x_2) + \ldots \quad . \quad (2)$$

Subtracting (1) from (2),

$$W\bar{h} = w_1h_1 + w_2h_2 + \ldots \quad . \quad . \quad . \quad . \quad (3)$$

But by definition $w_1h_1 + w_2h_2 + \ldots$ is the work done in raising the particles from one position to the other. And this is equal to $W\bar{h}$, the work done in raising the total weight W through the height $\bar{h}$ through which the C.G. of the system has been raised.

The same method of treatment may readily be applied to the case of raising a body in parts.

It is evident that in general it requires less work to compute the value of $W\bar{h}$ than that of Σwh. Hence the utility of the proposition.

Ex. 1. A cylindrical shaft 14 ft in diameter has to be sunk to a depth of 10 fathoms through chalk whose weight is 144 lb/ft³. Find the work done.

Ans. 39,916,800 foot-pounds.

2. How many foot-pounds must be expended in raising from the ground the materials for building a column 66 ft 8 in high and 21 ft square, the material weighing 112 lb/ft³.

Ans. 109,760,000 ft-pounds.

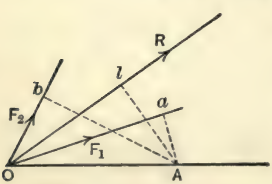
3. Find the work done in raising a Venetian blind having n slats, the weight of each slat being w lb and their distance apart a ft. *Ans.* $awn(n + 1)/2$ ft-pounds.

4. In pumping 1000 gallons from a water-tank with vertical sides the surface of the water is lowered 5 ft. Find the work done, the discharge being 10 ft above the original surface. (A gallon of water weighs $8\frac{1}{3}$ lb.)

Ans. 104,167 foot-pounds.

Draw the work diagram.

191. Principle of Work.—Let forces $F_1, F_2, \ldots$ act at a point causing a displacement OA. Let R be the resultant of the forces.



From A let fall perpendiculars on the directions of the forces, and let $\theta_1, \theta_2, \ldots$ be the inclinations of these directions to OA, and θ the inclination of the resultant R. Then, (Art. 81) the sum of the components of the forces along OA

being equal to the component of the resultant along OA, we have

$$F_1 \cos\theta_1 + F_2 \cos\theta_2 + \ldots = R \cos\theta,$$

or $$F_1 \times Oa/OA + F_2 \times Ob/OA + \ldots = R \times Ol/OA,$$

or $$F_1 \times Oa + F_2 \times Ob + \ldots = R \times Ol;$$

that is, *the work done by a system of forces acting at a point is equal to that done by their resultant.*

The equation may be written

$$F_1 \times Oa + F_2 \times Ob + \ldots - R \times Ol = 0;$$

which shows that *if a system of forces acting at a point equilibrate and the system receives a displacement the algebraic sum of the works done by the forces is equal to zero.*

Conversely, *if any number of forces act at a point, the condition of equilibrium is that the sum of the works done by the forces for every displacement shall be equal to zero.*

For let F denote the resultant and p the resolved part of the displacement parallel to the resultant. Then the work done by the forces is equal to Fp. But this by hypothesis is zero. As p is *not* zero, F must be zero, or the forces must equilibrate.

This is called the **principle of work** as applied to forces acting at a point or to forces acting on a particle.

It is to be noticed that the displacement need not actually be made. It may be conceived to be made, the geometrical relations of the parts remaining the same. The displacement is then said to be *virtual* [hypothetical], and the work done by the forces virtual work.

192. *Equilibrium of a System.*—Let a system of particles forming a body be acted on by forces whose points of application remain at invariable distances from each other. The forces acting on the particles consist of the external forces and of the forces arising from the connection of the particles among themselves. The condition of equilibrium of any particle is (Art. 191) that the sum of the works done by the forces external and internal is for every displacement equal to zero. Hence, summing from particle to particle, the condition of equilibrium for the body is that the sum of the works done by the forces external and internal is for every displacement equal to zero. But the work done by the internal forces, being that due to the mutual actions and reactions of the particles, necessarily vanishes for the whole system. Hence it is necessary to consider the external forces only, and *the condition of equilibrium is that the sum of the works done by the external forces is equal to zero for every displacement of the body.*

The same principle will apply to a *system* of bodies rigidly connected, or so connected that the geometrical relations existing among the parts are not disturbed by a given displacement, and hence to what is called a **machine**.

Ex. 1. If two particles of a system are joined by an inelastic thread, the work of the pull of the thread is *nil.*

[For let the pull T at A be transferred to B. The pulls at B being equal and opposite, the sum of the works done is *nil.*]

2. If a body turn about a fixed point, the work of the reaction is *nil.*

[For there is no displacement.]

3. If a body slide on a smooth plane, the work of the reaction is *nil.*

[Displacement is at right angles to force.]

193. Machines.—In a machine the separate pieces are so constrained that when one piece moves every other receives a determinate motion relative to it and to the others. The motion of the pieces being thus constrained, the direction of motion is altogether independent of that of the acting force. Hence the paths of the parts and their relative velocities may be treated as a kinematical problem when we consider the mechanism only.

A machine is a mechanism to which a driving force is applied. The object of a machine is the overcoming of force at one place by means of a force applied at another place. The acting force doing work on the machine is called the *driving force*,* and the force overcome is called the *resistance.* The pieces in constraint are termed *elements,* and can occur only in pairs. No single body can form a machine. Thus a tow-line, though it transmits force, is not a machine. Neither is a bridge, because the parts do not move relatively to one another.

The condition of equilibrium for forces acting on a machine is *that the algebraic sum of the works done by the external forces is equal to zero for any displacement which does not disturb the geometrical relations of the parts.*

The displacement must be so chosen as to be in conformity with the motion of the parts when the machine is in operation. No work is then done by or against the internal forces.

If, therefore, F is the driving force, R the resistance, x the displacement of F, and y the displacement of R, the condition of equilibrium may be written

$$Fx - Ry = 0.$$

This principle was first stated by Simon Stevinus of Bruges in investigating the equilibrium of systems of pulleys. It was also noted by Galileo in the case of the inclined plane, and was given by him as the condition to be satisfied for the equilibrium of a machine.

In the generalized form, for any system of forces in equilibrium, the principle was named the principle of **Virtual Velocities** by John Bernoulli [1717] and by this name it is still very generally known.

194. Machines (*Friction Neglected*). — In general in a machine the force of friction enters as a resistance; but for simplicity we shall, first of all, neglect friction and consider the parts to be perfectly smooth. This is an ideal case, and may be looked upon as a first approximation to the actual.

We proceed to find the relation between driving force and

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* Often called "the power." It is best to reserve "power" to denote the rate of doing work. (Art. 209.)

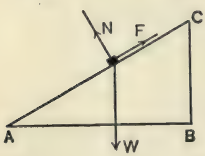
resistance in some simple forms of machines, the motion being uniform, that is, the acting forces are in equilibrium.

It is usually stated that "all machines are either certain simple machines known as the mechanical powers,—the lever, inclined plane, pulley, wedge, screw,—or combinations of two or more of these mechanical powers." But as Prof. Kennedy very justly remarks in his *Mechanics of Machinery*, "It is not worth while to discuss a theory so hopelessly inconsistent with facts as this."

See, for a full discussion, Reuleaux, *Theoretische Kinematik*.

195. *Inclined Plane.*—A body weighing W rests on a plane inclined to the horizon, and is constrained so as to be capable of sliding parallel to the greatest slope of the plane and of no other motion—in a groove, for example. The mechanism forms an example of what is called a *sliding pair*.

(a) Let the body be held from sliding down the plane by a force F acting parallel to the greatest slope.



The force F may be regarded as the acting force and the vertical force W as the resistance.

In bringing the body from A to C

work done by $F = F \times AC$;
work done by $W = -W \times CB$.

Hence, since there is equilibrium,

$$F \times AC - W \times CB = 0,$$

or $$W/F = AC/BC,$$

giving the *force ratio*, or *mechanical advantage*, or *theoretical advantage*.

The same result could be found by considering the constraint of the plane replaced by the reaction N and the body in equilibrium under F, W, N. Then, if the displacement is from A to C,

$$\text{work done by } F = F \times AC;$$
$$\text{work done by } N = 0;$$
$$\text{work done by } W = -W \times BC;$$

and for equilibrium (uniform motion)

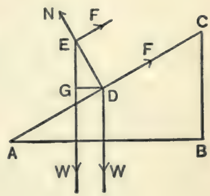
$$F \times AC + 0 - W \times BC = 0,$$

or $$F \times AC = W \times BC, \text{ as before.}$$

This equation also shows that *the work done in moving a body up a smooth plane is equal to that done in raising it through the vertical height of the plane.*

To find the reaction N.

Let the body be displaced through a distance DE perpendicular to the plane. Then in the displacement



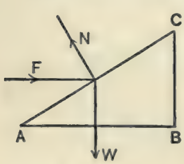
$$\text{work done by } N = N \times DE;$$
$$\text{work done by } F = 0;$$
$$\text{work done by } W = -W \times EG.$$

Hence for equilibrium

$$N \times DE - W \times EG = 0,$$

and $$N/W = EG/DE = AB/AC.$$

(*b*) If the force F which holds W in equilibrium is parallel to the base, then in moving from A to C W acts through BC, and F acts through AB, each in its line of action. Hence



$$F \times AB - W \times BC = 0,$$

or $$W/F = AB/BC,$$

giving the mechanical advantage.

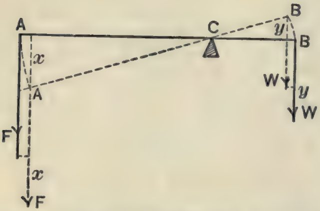
Ex. 1. Show in (*b*) that $N/W = AC/AB$.

2. A weight of 100 lb is hauled up an incline of 1 in 100 and 1000 ft long. Find the work done.

Ans. 1000 ft-pounds.

3. Draw a diagram of the work done in moving a weight of 10 lb up a smooth incline 5 ft high.

196. *The Straight Lever.*—If a bar is constrained to move about an axis or fulcrum, plane motion only being possible, it is a *lever.* A lever—that is, bar and fulcrum—forms an example of a *turning-pair.*



Let the driving force F and the resistance W be both vertical.

While the point of application of F descends a distance x the resistance W will ascend a certain distance y. Then for equilibrium

$$Fx - Wy = 0.$$

But from similar triangles

$$x : y = AC : BC.$$

Hence $$F \times AC = W \times BC,$$

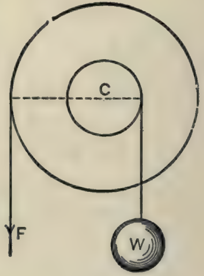
which is the "principle of the lever." (Art. 154.)

Ex. 1. What is the mechanical advantage?

2. If the force F and resistance R are not vertical, show that

$$F \times AC = R \times BC.$$

197. *The Wheel and Axle.*—Two cylinders fastened together move freely on a common axis C which is horizontal and works in fixed bearings. A force F acts by a cord coiled round the larger cylinder (or *wheel*), and balances a weight W hanging from a cord coiled round the smaller cylinder (or *axle*).



The mechanism is equivalent to a lever with *unequal* arms, the axis corresponding to the fulcrum of the lever, and the radii to the arms. It is called the *wheel and axle,* and forms a turning-pair.

While the driving force F descends a distance equal to the circumference of the wheel the weight W is raised a distance equal to the circumference of the axle. Hence for equilibrium

$$F \times 2\pi r_1 - W \times 2\pi r_2 = 0,$$

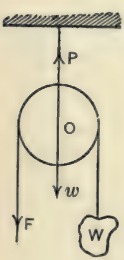
$$\text{or} \qquad W/F = r_1/r_2,$$

giving the mechanical advantage.

Ex. 1. Obtain the relation between F and W by the "principle of moments."
2. Find the vertical force on the axis.

198. *The Pulley and Cord.*—A pulley is a wheel with a groove round its outer edge [sheave], and capable of revolving freely about an axis through its center O. This axis is fixed in a frame or *block* to which a hook is attached.

This mechanism consists of three links, the cord, sheave, and frame, the term link being applied to bodies arranged so as to give the required motion.



(*a*) The single fixed pulley.

The driving force F and the weight W to be raised are both vertical, the constraint preventing cross-motion.

If F descend any distance x, W will ascend an equal distance x. Hence for equilibrium

$$Fx - Wx = 0,$$

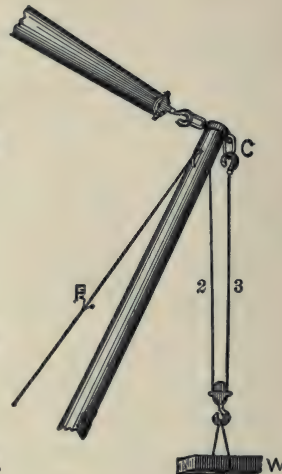
$$\text{and} \qquad W/F = 1.$$

If P is the force on the support and w the weight of the pulley, then, the four parallel forces F, w, W, P being in equilibrium, we have

$$\begin{aligned} P &= F + W + w \\ &= 2W + w. \end{aligned}$$

(*b*) The single movable pulley.

In the figure we have a single fixed and a single movable pulley.



If F descends any vertical distance x, W will ascend a distance $x/2$. Hence for equilibrium

$$Fx - Wx/2 = 0,$$

$$\text{or} \qquad W/F = 2.$$

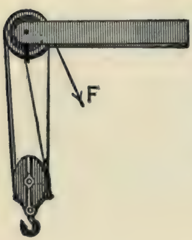
If the weight w of the pulley is taken into account, then evidently

$$W + w = 2F.$$

Ex. Find the pull on the hook C.

(*c*) Pulley tackle.

In the figure is represented a tackle, the upper and lower blocks each containing two sheaves, and the same rope passing round all.



The motion is not strictly in one plane, but we may assume it to be so as a first approximation.

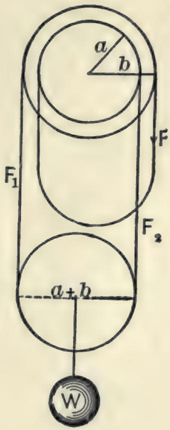
While F descends a distance x, W ascends a distance $x/4$.

$$\therefore \quad Fx - Wx/4 = 0,$$

$$\text{and} \qquad W/F = 4.$$

(*d*) The Weston differential pulley consists in the upper block of two sheaves of slightly different radii a, b fastened together, and in the lower block of a single sheave of diameter $a + b$. Each block is a turning-pair. An endless chain passes round the sheaves as in the figure. Notches are cut or teeth set in the upper-block sheaves which fit the links of the chain.

While the upper block makes one revolution the lower is raised an amount equal to one half the difference of the cir-



cumferences of the two pulleys in the upper block. Hence for equilibrium

$$F \times 2\pi b - W \times (\pi b - \pi a) = 0,$$

$$\text{or} \qquad W/F = 2b/(b - a).$$

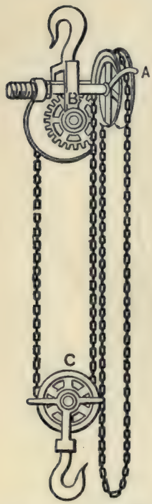
Ex. 1. In a Weston pulley the diameters of the pulleys in the upper block are 7 and 8 inches. Find the theoretical advantage. *Ans.* 16.

2. Obtain the relation between F and W by the method of moments.

3. The diameters of the pulleys are 6 and 7 in and the weight is to be raised at the rate of an inch a second. Find the rate at which the chain must be pulled. *Ans.* 70 ft/min.

Show that to raise the weight 1 ft between seven and eight revolutions must be made.

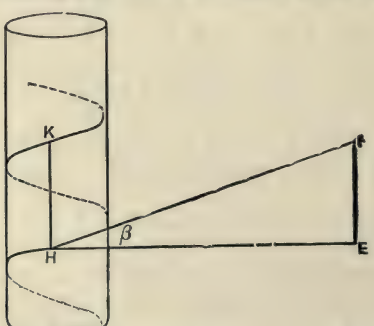
199. A common form of hoisting apparatus is the worm-wheel gear, shown in the figure. A light endless chain passes over a pulley *A*, which has teeth in the groove to fit the links. This is the driving-chain. On the axle of the pulley is a worm which works in the toothed wheel fixed to the pulley *B*. Over the pulley *B* passes the movable end of the heavy chain that runs under the block *C*. This is the lifting-chain.



To *C* the weight to be raised is attached.

Ex. The driving and driven pulleys *A* and *B* are each 1 ft in diameter and the number of teeth in the toothed wheel is 24. What weight could be raised by a pull of 10 pounds applied to the endless chain? *Ans.* 480 lb.

200. *The Screw.*—In turning-pairs the motion is simply a motion of rotation. We may, however, conceive the rotating body not only to revolve about the axis of rotation, but to advance along the axis, the motions being in perpendicular planes. The combination of the two motions gives rise to screw motion, and the two bodies form a *twisting-pair.*



Thus suppose *H* to be a point on the surface of a cylinder which is revolving in a bearing or nut and at the same time advancing uniformly along its axis.

By the motion of rotation alone, while the cylinder makes a revolution, H would describe a path equal to the circumference. Develop this in the line HE. But the motion of translation carries it a distance equal to EF. Hence it is found at K, and the path HK would be traced by wrapping the triangle HEF about the cylinder. The path of the point H is called the *thread* of the screw, and the distance HK or EF between consecutive threads the *pitch*.

The inclination of the thread to the axis is given by the angle $EHF\ (=\beta)$. Now

$$\tan\beta = EF/HE$$
$$= \text{pitch/circum. of cylinder.}$$

201. Take the bell-bottom jack-screw (p. 249) with force F applied at the end of a lever-arm l to raise a weight W with uniform speed.

While the lever-arm makes one revolution, that is, while the force moves through a distance $2\pi l$, the weight is raised a distance equal to the pitch p of the screw. Hence for equilibrium

$$F \times 2\pi l - W \times p = 0,$$
$$\text{or} \qquad F/W = p/2\pi l.$$

202. In machine tools sliding motion is commonly produced by means of screws. The relation between the pitch, number of revolutions, and rate of sliding is very simple. Thus if p is the pitch in inches, n the number of revolutions per minute, and v the velocity in ft/min, then evidently

$$np = 12v.$$

For example, if a screw moves a slide at 16 ft/min, and has a pitch of 1.5 in, the number of revolutions required would be 128 per minute.

Ex. 1. In a pulley tackle the driving force descends 1 ft while the weight to be raised ascends 1 in. What force will raise 1 ton? *Ans.* $166\frac{2}{3}$ pounds.

2. Find the relation between F and W in the copying-press (p. 149), $2l$ being the length of the handle.

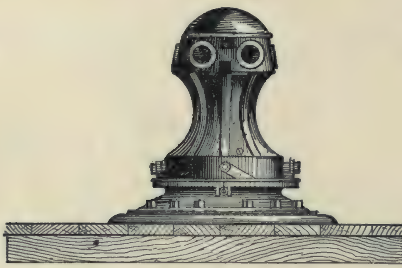
Ans. $F = W \times$ pitch of screw$/4\pi l$.

W. R. M. Christie

3. A weight of 400 lb is being raised by a pair of double pulley-blocks. The rope is fastened to the upper block, and the parts of the rope (whose weight may be disregarded) are considered vertical. Each block weighs 10 lb. Find the pressure on the axle of the upper block.

Ans. 522.5 pounds.

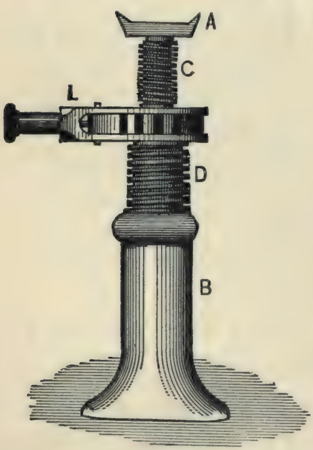
4. The axle of a capstan is 2 ft in diameter. If four sail-



ors push with a force of 40 pounds each at the ends of spikes 4 ft long, find the weight of the anchor that is lifted.

5. A man has to raise a weight and has only one pulley at his disposal. How must he apply it to gain the greatest advantage?

6. In a differential pulley the diameters of the pulleys of the compound sheave are a, b in. Find how many revolutions are required to raise the weight c in. *Ans.* $2c/\pi(a - b)$.

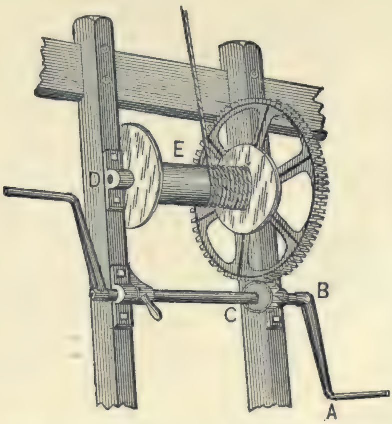


7. In a telescopic jack-screw a smaller screw C works in a companion nut cut in the larger screw D, which latter works in a nut in the fixed block B. The block A being fixed, the upper screw does not rotate. If l is the length of the lever-arm, find the relation between F and W.

Ans. $F \times 2\pi l = W \times$ diff. of pitches of screws.

8. A screw moves the table of a planing-machine 12.5 ft a minute and makes 100 revolutions per minute. Find the pitch. *Ans.* 1.5 inch.

9. In a derrick-winch the crank AB is l in leverage, the

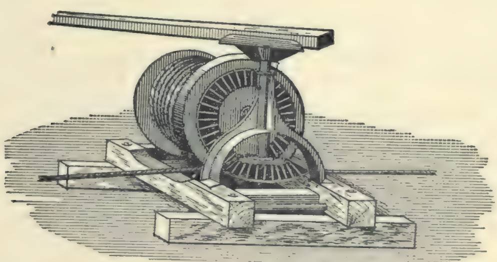


gears n to 1, and the drum E is d in diameter. Find the two-man-power capacity, each man exerting a force of p pounds. *Ans.* $4pln/d$ pounds.

In general show that for one man exerting a force F a weight W will be raised, given by

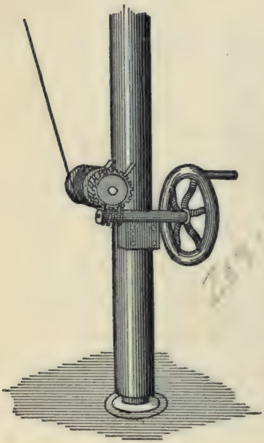
$$\frac{F}{W}=\frac{\text{rad. drum}}{\text{length arm}}\times\frac{\text{no. of teeth in pinion}}{\text{no. of teeth in wheel}}.$$

10. In a winch, given that the cranks have 18 in leverage, the gears are 4 to 1, the drum is 6 in in diameter, and the



capacity with two-man power is 3 tons. Find the force exerted by each man. *Ans.* 125 pounds.

11. In a hoisting-machine the gears are 36 to 36 teeth, the drum 21 in in diameter, and the load for one horse $1\frac{1}{2}$ tons. Find the pull exerted by the horse at the end of a 7-ft horizontal lever. *Ans.* 375 pounds.



12. In a combination of single-threaded worm and wheel used in hoists the worm-wheel has n teeth, the radius of the driving-wheel is l in in length, and the radius of the drum around which the lifting-rope winds is r in. Find the relation between F and W. *Ans.* $Fln = Wr$.

203. Machines (*Friction Considered*).—To overcome frictional resistance in a machine work must be done. Part of the driving force is taken up in doing this work, and is, as it were, absorbed. Although it serves no useful purpose, it is not lost. (Art. 216.)

The work absorbed in overcoming friction is measured as the work done on any other resistance. We have

$$\text{work against friction} = \text{friction} \times \text{distance described}$$
$$= \mu N \times s,$$

where N is the normal pressure, μ the coefficient of friction. and s the distance described in the direction of the resistance.

Thus the work done in overcoming the frictional resistance in one revolution of a journal of diameter d ft and carrying a load of W lb is $\mu W \times \pi d$ ft-pounds. In case the journal revolves n times per minute the work absorbed is

$$n \times \mu W \times \pi d \text{ ft-pounds per minute.}$$

In a machine, owing to friction between the pieces, part of the work done by the driving force is wasted, so that the resulting useful work is less than the total work done by the driving force in the first place. We have, in fact,

$$\text{total work} = \text{useful work} + \text{useless work.}$$

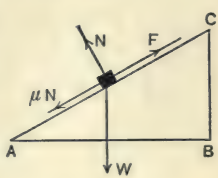
The ratio of the useful work to the total work is known as the *efficiency* of the machine, so that

$$\text{efficiency} = \text{useful work}/\text{total work};$$

or, as sometimes expressed,

$$\text{efficiency} = \text{work got out}/\text{work put in}.$$

204. *The Inclined Plane.*—To find the force F necessary to haul a weight W *up* a rough incline AC, the coefficient of friction being μ and the force F parallel to the plane.



Let N denote the normal reaction. The friction μN acts *down* the plane, the forces being as in the figure.

Then, if the displacement is from A to C, we have for equilibrium

$$F \times AC - W \times BC - \mu N \times AC = 0. \quad . \quad . \quad (1)$$

Resolving the forces *perpendicular* to the plane,

$$N - W \cos \theta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Eliminating N between (1) and (2),

$$\begin{aligned} F \times AC &= W \times BC + \mu W \cos \theta \times AC \\ &= W \times BC + \mu W \times AB, \quad . \quad . \quad . \quad . \quad (3) \end{aligned}$$

or *the work done in hauling the body up the plane AC is equal to that done in raising it through the height BC and in hauling it along the base AB, the coefficient of friction being the same on AC and AB.*

The work wasted is (Art. 196)

$$\mu W \times AB.$$

205. In case the force F acts *down* the plane, it may be shown similarly that

$$F \times AC = \mu W \times AB - W \times BC,$$

and the work wasted is the same as before.

If $\mu W \times AB = W \times BC$, then $F = 0$, and the body would slide down the plane with uniform velocity.

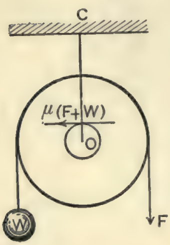
Ex. 1. Find the work done in hauling a sled weighing 500 lb half a mile, the coefficient of friction being 0.2.
Ans. 264,000 ft-pounds.

2. Find the work done in hauling a train of 100 tons one mile up a 1% grade, the resistance being 8 pounds per ton.
Ans. 7392 foot-tons.

3. "On a grade of 1 in 10 a bicycle-rider, in addition to the tractive force, actually lifts one tenth of his weight and that of the machine."

206. *The Pulley.*—(*a*) The single fixed pulley.

Let F be the driving force and W be the weight to be raised. The normal reaction is $F + W$, and the friction $\mu(F + W)$, if μ is the coefficient of friction.



Let r = radius of axle, a = radius of sheave.

Let a complete revolution of the pulley be made; then, summing the work done, we have for equilibrium

$$F \times 2\pi a - W \times 2\pi a - \mu(F + W) \times 2\pi r = 0,$$

or $$F(a - \mu r) = W(a + \mu r),$$

and $$W/F = (a - \mu r)/(a + \mu r)$$
$$= 1 - 2\mu r/a, \text{ nearly},$$
$$= 1/\lambda, \text{ suppose},$$

the symbol λ being introduced for convenience of writing.

In pulley tackle so much depends on the stiffness of the ropes that it is of little use to determine the effect of axle-friction only. The formulas in all practical cases are empirical and the investigation outside our province.

Ex. 1. If the axle were smooth, then $F = W$.

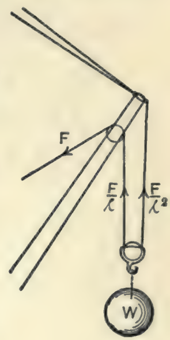
2. The effect of the friction is the same as the raising of an additional weight $(\lambda - 1) W$ if the axle were smooth.

3. Find the tension of the cord supporting the pulley.

4. Show that the efficiency $= 1/\lambda$.

(b) The single movable pulley.

In a derrick where the rope passes under the movable pulley and over the fixed pulley let F be the force applied and W the weight to be raised.



From the preceding it follows that the pulls in the three parts of the cord are F, F/λ, and F/λ^2, respectively.

Hence, considering the cords which hold the lower pulley to be parallel,

$$W = F/\lambda + F/\lambda^2,$$

and $$W/F = (\lambda + 1)/\lambda^2,$$

Ex. 1. If friction is neglected, show that $W = 2F$.

2. With two double-sheave blocks

$$W = F(\lambda^4 - 1)/(\lambda^5 - \lambda^4).$$

3. With two double-sheave blocks, neglecting friction, $W = 4F$. Hence find the loss due to friction.

4. In a double-sheave tackle the under block is fixed and the upper movable. Find the pull on the support A and the loss due to friction.

(c) The differential pulley.

Let F be the force applied and W the weight which is on the point of being raised (figure, page 239).

Consider the lower pulley. Let F_1 denote the pull of the driving chain and F_2 that of the driven. Then from (a)

$$F_1 = \lambda F_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But $$F_1 + F_2 = W. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the upper block F and F_2 drive and F_1 is driven; F and F_1 act on the larger sheaf, and F_2 reduced to this sheaf is $F_2 a/b$. Then

$$F + F_2 a/b = \lambda F_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eliminating F_1, F_2 from these equations,

$$W/F = b(1 + \lambda)/(\lambda^2 b - a),$$

which gives the mechanical advantage.

When the weight is on the point of slipping down, that is, when W acts as driving force and F as resistance, then, writing $1/\lambda$ for λ, we have

$$F/W = (b - \lambda^2 a)/b\lambda(1 + \lambda)$$

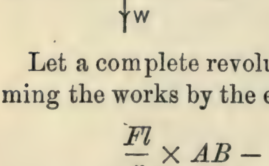
as the condition to be fulfilled.

If the driving force F is removed, then whatever be the weight we must have, in order that no back action may take place,

$$b - \lambda^2 a = 0,$$

which gives the relation between the radii, that this may be possible. If $\lambda = 1.1$ then $b/a = 1.2$, and the diameters of the pulleys are roughly as 5 to 6. For these dimensions the chain will not slip, whatever weight is being raised. The practical value of the pulley lies largely in this circumstance.

207. *The Screw.*—Consider a screw-jack with the driving force F acting at the end of a lever l and a weight W on the point of being raised, the coefficient of friction being μ.



The screw unwrapped is an inclined plane with the force parallel to the base (Art. 200).

Let a complete revolution of the screw be made; then, summing the works by the external forces, we have for equilibrium

$$\frac{Fl}{r} \times AB - W \times BC - \mu N \times AC = 0. \quad . \quad . \quad (1)$$

Resolving the forces perpendicular to AC,

$$N - W \cos \beta - \frac{Fl}{r} \sin \beta = 0; \quad . \quad . \quad . \quad . \quad (2)$$

∴ by substituting for N its value from (2) in (1),

$$\frac{Fl}{r} = W\frac{BC}{AB} + \mu\frac{AC}{AB}\left(W\cos\beta + \frac{Fl}{r}\sin\beta\right),$$

$$= W\tan\beta + \mu\sec\beta\left(W\cos\beta + \frac{Fl}{r}\sin\beta\right),$$

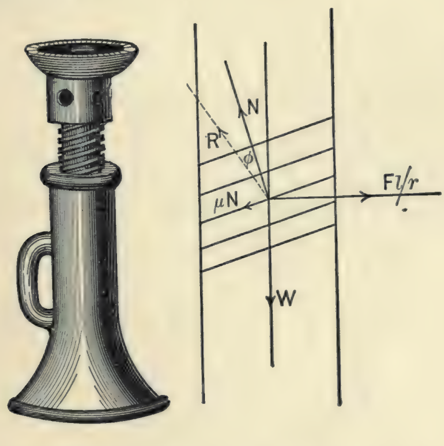
or $Fl = Wr\tan(\beta + \phi)$, ... (3)

the result required.

The useful work is $W \times AB \tan\beta$, and the total work is $W \times AB \tan(\beta + \phi)$.

$$\therefore \quad \text{efficiency} = \tan\beta/\tan(\beta + \phi).$$

208. The relation between F and W may also be found graphically.



The resultant R of N and μN makes an angle ϕ with the normal. Hence lay off W to scale, complete the triangle of forces, and scale off Fl/r and R.

From the triangle we have at once

$$Fl/r = W\tan(\beta + \phi).$$

If the weight W is on the point of moving *down* and the acting force is denoted by F_1, then it may be shown in a similar manner that

$$F_1 l/r = W \tan(\beta - \phi).$$

Hence if the force applied has any value between F and F_1 the screw will be in equilibrium when supporting a weight W.

Ex. 1. If $\beta = \phi$ we have $F_1 = 0$. What does this mean?

2. The circumference of a screw is 4 in, there are 2 threads to the inch, and the coefficient of friction is 0.2. Find the force applied at the end of a lever 14 in long that will raise a weight of 100 lb. *Ans.* 1.5 pounds.

What force would just support the weight? *Ans.* 1/3 pound.

3. If $\beta = 0$, the efficiency is null. Explain.

4. In what other case is the efficiency null?

5. Show that the efficiency of a screw is greatest when the pitch angle is 45°, nearly.

6. In a screw the pitch angle is 45° and the coefficient of friction 0.16. Find the efficiency. *Ans.* 0.72.

7. If $\tan\beta = 0.1$, $\tan\phi = 0.01$, prove efficiency $= 0.9$;
$\tan\beta = 0.1$, $\tan\phi = 0.1$, " " $= 0.5$;
$\tan\beta = 0.1$, $\tan\phi = 0.2$, " " $= 0.3$.

Hence show the importance of lubrication.

8. The efficiency of a rectangular screw is a maximum when

$$\beta = (\pi - 2\phi)/4.$$

209. Power. Activity.—It is plain from the definition of work that any small force can do work of any magnitude provided sufficient time is given. Hence, in order to compare *agents* which do work, it is necessary to take the time employed into consideration. To indicate the rate of doing work or the amount of work performed in a given interval the term **power** or **activity** is used. Thus to raise one ton of coal through 300 ft in 10 min would require an expenditure of 2000 × 300 ft-pounds of work in 10 min, or of 60,000 ft-pounds per min, or of 1,000 ft-pounds per sec, either one of which would be an expression for the power of the agent.

In general, if a resistance of R pounds is overcome at a velocity of v ft/sec, then

$$Rv \text{ ft-pounds/sec}$$

will measure the power of the agent.

By the *unit power* or unit rate of working is meant the power of an agent which can do unit work in unit time. The gravitation unit is the foot-pound per second or the foot-pound per minute.

In engineering work the foot-pound per minute being too small a unit for most purposes, the multiple unit of the **horse-power** has been introduced. A horse-power is defined as the power of an agent which can do a work of 33,000 foot-pounds per minute or of 550 foot-pounds per second. It does not give the average power of a horse; but as a unit of this size is convenient, the name introduced by Savery and defined by James Watt has been retained. Carefully notice that the term horse-power does *not* express an amount of work but the rate of doing work.

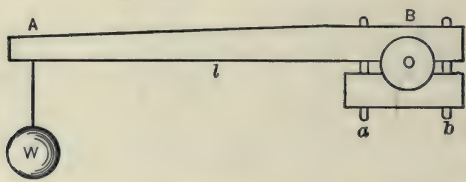
Conversely, $33{,}000 \times 60$ ($= 1{,}980{,}000$) ft-pounds is called a **horse-power-hour**, a term frequently employed in engineering. The term horse-power-hour is thus a unit of work and not a unit of power.

Watt derived his horse-power from actual experiments carried out at Barclay and Perkins' brewery, London, with certain of the very heavy horses belonging to the firm. These horses were caused to raise a weight from the bottom of a deep well by pulling horizontally on a rope passing over a pulley. He found that a horse could walk 2.5 miles an hour and at the same time raise a weight of 100 lb. This is equivalent to $2.5 \times 5280 \times 100/60 = 22{,}000$ foot-pounds per minute. By adding 50 per cent to this he obtained 33,000. The margin, 11,000 foot-pounds, he allowed for loss in engine-friction, etc.

Gen. Morin's estimate of the power of a horse is 26,150 ft-pounds per minute.

210. For determining the power developed by a steam-engine or other machine, some form of friction-brake is used. The idea is to balance the work done by the machine by a frictional resistance, compute this resistance, and thence find the power of the machine. The brake *absorbs* the work to be measured.

Let O be a shaft of radius r, to which the brake AB is fastened. By means of the screws a, b the friction of the



brake on the shaft may be regulated. Suppose it adjusted so that the engine develops a friction f, just sufficient to balance a body of weight W placed at the end A of the beam. Then the moments of f and W about O must be equal, or

$$fr = Wl.$$

Suppose the shaft to revolve uniformly n times per minute. Then, assuming that the friction for uniform motion of the shaft is the same as at the point of just beginning to move, we have

$$\begin{aligned}\text{Work done in one min} &= \text{friction developed in } n \text{ revs.}\\ &= f \times 2\pi r \times n\\ &= 2\pi n Wl.\end{aligned}$$

If W is expressed in pounds and l in feet, then the

$$\begin{aligned}\text{H.P.} &= 2\pi n Wl/33{,}000\\ &= 0.00019 n Wl.\end{aligned}$$

Since the total work done by a machine is given by the indicated horse-power (Art. 189) and the useful work by the braked horse-power, we may define the *efficiency* of the machine to be the ratio of the B.H.P. to the I.H.P., or

$$\textbf{efficiency} = \textbf{B.H.P./I.H.P.}$$

Ex. 1. Show that one H.P. = 396,000 inch-pounds/minute
= 198 inch-tons/minute.

2. 22 tons of coal are to be hoisted through 50 yards in 10 min. Find the H.P. of engine necessary. *Ans.* 20 H.P.

3. A traction-engine weighing 5 tons hauls a load of 10 tons at 8 miles an hour, the resistance being 20 pounds per ton. Find the H.P. exerted. *Ans.* 6.4 H.P.

4. A belt passing round two pulleys moves with a velocity of 10 ft/sec. Find the H.P. transmitted if the difference of tension of the belt above and below the pulleys is 1100 pounds. *Ans.* 20 H.P.

5. Find the H.P. required to propel a train weighing W tons at V miles an hour on a level track, the resistance being p pounds per ton. *Ans.* $WpV/375$.

6. Find the H.P. required by a vessel to overcome a resistance of R pounds at a speed of V knots.
Ans. $RV/325.66$ horse-power.

Hence show that one horse-power, or 550 ft-pounds per second, is about 326 knot-pounds.

7. A shaft 14 ft in diameter is sunk in gravel to a depth of 10 fathoms in 10 days of 10 hours each. Taking the weight of the gravel at 100 lb/ft^3, find the H.P. expended in lifting out the gravel. *Ans.* 0.14 H.P.

8. "At 80 miles an hour a pull of 4 pounds 11 ounces represents one horse-power."

9. Find the horse-power necessary to pump out the St. Mary's Falls Canal Lock, Sault Ste. Marie, in 24 hours, the length of the lock being 500 ft, width 80 ft, and depth of water 18 ft, the water being delivered at a height of 42 ft above the bottom of the lock. *Ans.* 31.2 H.P.

10. A belt can stand a pull of 100 pounds only. Find the least speed at which it can be driven to transmit 20 H.P.
Ans. 110 ft/sec.

11. How many gallons of water would be raised per

minute from a mine 600 ft deep by an engine of 175 H.P., supposing a gallon of water to weigh $8\frac{1}{3}$ lb.?

Ans. 1155 gals.

12. The average flow over Niagara Falls is 270,000 cubic feet per second. The height of fall is 161 feet. Show that the H.P. developed is in round numbers 5 million.

13. The U. S. war-ship Columbia has a speed of 23 knots with an indicated horse-power of 22,000. Find the resistance offered by the water to her passage. *Ans.* 156 tons.

14. In testing a Corliss engine running at 100 revolutions per minute, the lever-arm of the brake employed was $10\frac{1}{2}$ ft and the weight attached 2000 lb. Find the H.P. developed.

Ans. 400 H.P.

15. "A C. and C. electric motor shows on a Prony brake a pull of 5 ounces on a 1-ft lever, that is, 2 ft-pounds per revolution, or about $\frac{1}{10}$ H.P. at 1500 revolutions per minute." Check the conclusions in this statement.

16. In a Corliss engine running at 100 pounds pressure per sq in and 100 revolutions per minute, the diameter of the cylinder is 18 in and length of stroke 42 in. If the brake was used on a pulley 6 ft in diameter and keyed to the engine-shaft, find the friction on the face of the pulley.

Ans. $f = 9450$ pounds.

17. A 6-ton fly-wheel on a 14-in axle makes 90 revolutions per minute. Find the H.P. absorbed in friction, the coefficient of friction being 0.1. *Ans.* 12 H.P.

18. A steam-hoist of 3 H.P. is found to raise a weight of 10 tons to a height of 50 ft in 20 min. How many ft-pounds of work are wasted by friction in a day of 10 hours?

Ans. 29,400,000 foot-pounds.

19. A pumping-engine of piston area 100 in^2, steam-pressure 60 pounds/in^2, length of stroke 3 ft, and number of revolutions per min 25, raises 500 gallons of water per minute a height of 50 ft. Find the efficiency. *Ans.* 0.23, nearly.

20. A train weighing 100 tons runs at 42 miles an hour on a level track, the resistance being 8 pounds per ton. Find its speed up a 1% grade (1 ft rise in 100 ft) if the engine-power is unchanged. *Ans.* 12 miles/hour.

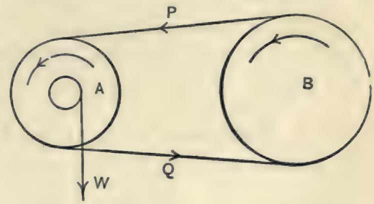
21. A traction-engine weighing 5 tons can haul 15 tons on a level, the coefficient of friction being 0.02. Find the net load it can haul up a 1% grade. *Ans.* $8\frac{1}{3}$ tons.

22. A train weighs W tons and the resistance is p pounds

per ton. Find the horse-power necessary to propel it at a speed of V miles an hour up a grade of $n\%$.

Ans. $WV(p + 20n)/375$ horse-power.

23. Find the speed of a driving-pulley 3.5 ft in diameter to transmit 6 H.P., the driving force of the belt being 150 pounds. *Ans.* 120 revolutions/minute.



24*a*. The horse-power transmitted by an endless belt passing over two pulleys is

$$\pi dn(P - Q)/33000$$

when d = diameter of driving-pulley in ft,
n = no. of revolutions per minute of driving-pulley,
P, Q = the pulls in the belt on the taut and slack sides expressed in pounds.

24*b*. If T is the torque of the driver, then

$$\text{H.P.} = Tn/5250.$$

25. A train of 100 tons is hauled by an engine of 150 H.P. The resistance is 14 pounds per ton. Find the greatest velocity that the engine can attain. *Ans.* 40 miles/hour.

26. Check this statement: "55 pounds mean effective pressure at 600 ft piston speed gives one H.P. for each sq in of piston area."

27. Prove H.P. of an engine = $SNAP/33000$ where

S = length of stroke in ft;
N = no. of strokes per minute;
A = piston area in square inches;
P = mean steam-pressure in pounds per square inch of piston area.

28. Find the work done per hour at the crank-pin of an engine revolving 40 times a minute and acting against a resistance of 7000 pounds, the radius of the crank being 18 inches. *Ans.* 79,200 ft-tons.

29. Show that

$$\text{tractive pull of a locomotive (in pounds)} = d^2ps/D$$

where d = diameter of cylinders in inches;
p = mean effective pressure in pounds per sq. inch;
s = stroke in inches;
D = diameter of driving-wheels in inches.

30. Show that the cylinder diameter of an engine that will produce n horse-power at a piston velocity of s ft per minute under a mean effective pressure of p pounds per sq in is $205 \sqrt{n/ps}$ inches nearly.

211. Energy.—We have seen that when the forces acting on a body are in equilibrium or the body moves with a uniform velocity, the sum of the works done is zero; that is, the work done by the resultant driving force is equal to that done on the resistance. Now the action of a force is to cause acceleration. If the motion is uniform, the acceleration caused by the driving force is balanced by the equal and opposite acceleration caused by the resistance. But if the acceleration caused by the driving force exceeds that caused by the resistance, velocity is gained, and the motion is not uniform.

Thus let a force F pounds act on a body weighing W lb through a distance s ft and let v ft/sec be the velocity acquired. Then (Art. 183)

$$Fs = Wv^2/2g.$$

Now Fs is the work done by F in passing over a distance s in its line of action, and therefore a body W in acquiring a velocity v from rest must have $Wv^2/2g$ units of work done upon it.

Conversely, the force F which will generate a velocity v in acting through a distance s, will destroy the same velocity if acting through the same distance in the opposite direction, or the body by virtue of its velocity v can do Fs or $Wv^2/2g$ units of work in giving up that velocity and coming to rest. This capacity which a body possesses of doing work in consequence of its velocity is known as **energy**. Hence the measure of the energy of a body which weighs W and has a velocity v is $Wv^2/2g$ units of work. We may therefore state the equation

$$Fs = Wv^2/2g.$$

If a body is in motion under the action of force, the work done in passing from one position to another is equal to the energy produced.

212. More generally, if a force F acting on a body which weighs W is opposed by a uniform resistance R in the same line of action, then the net driving force is $F - R$. Let a be the acceleration produced; then, from Newton's second law,

$$F - R = Wa/g.$$

If s is the distance passed over and the velocity is changed from u to v in passing over this distance, then

$$as = v^2/2 - u^2/2.$$

Eliminating a,

$$Fs = Rs + Wv^2/2g - Wu^2/2g;$$

which may be stated:

The work done by a force on a body (or system of bodies with configuration remaining the same) is equal to the work done against the resistance together with the change of energy generated in the body (or system of bodies).

213. The possession of energy as defined implies motion. But the motion may not appeal directly to our senses. A body may possess energy by virtue of its position. For example, let a body be thrown upward. The body possesses energy as shown by its capacity to overcome obstacles. But its visible energy is continually diminishing, and becomes zero when the highest point is reached. In its fall it acquires visible energy until at the starting-point this energy is equal to the initial energy.

In rising the energy was expended in overcoming the resistance offered by gravity. The force of gravity is prepared to restore in the return to the initial position the energy abstracted. The body and the earth thus form a system of give and take—a connected system in which the sum total of the energy remains the same. But, instead of saying that as one member of a system loses energy some other member of that system gains energy, it is convenient to confine our attention to one member of the system only, and say that as it loses energy of motion (*kinetic* energy) it stores or gains energy by virtue of its position (*potential* energy); or, as it is sometimes expressed, by virtue of the configuration of the system.

The kinetic energy of a body or the energy which it possesses in virtue of its motion is measured by the work it can do against resistance in parting with this motion. Thus the kinetic energy of a body weighing W lb and moving with a velocity v ft/sec is $Wv^2/2g$ foot-pounds of work.

The potential energy of a body, or the energy it possesses by virtue of its configuration, is measured by the work it can do in passing from its present configuration to some standard configuration. Thus a weight W lb, if raised to a height h ft, would possess a potential energy of Wh foot-pounds of work, because it could do that amount of work in falling to its original position—in driving a clock, for example.

The term potential energy was introduced by Prof. Macquorn Rankine, and the term kinetic energy by Lord Kelvin.

214. Unit of Energy.—The energy expended being equal to the work done, the unit of energy must be the same as the unit of work. In the British system the unit is therefore the *foot-pound.*

The expression Wv^2/g, introduced by Leibnitz (1646–1716) under the name *vis viva,* or living force, is still made use of by some writers. Its value is equal to twice the kinetic energy. In view of the prominence of the doctrine of energy in modern physics, the term kinetic energy is to be preferred, and *vis viva,* being quite superfluous, should be dropped. The Leibnitz term *vis mortua,* as the name for pressure, is long dead.

Galileo used the terms "momentum," "impulse," and "energy" indiscriminately.

215. For illustration, consider the relation between the kinetic and potential energies in the following simple examples:

(1) A body weighing W lb falls from A to B through a height h ft.

$$\text{The p.e. at } A = \text{work in falling from } A \text{ to } B$$
$$= Wh \text{ foot-pounds.}$$

$$\text{The k.e. at } A = 0.$$

$$\therefore \text{ total energy at } A = Wh \text{ foot-pounds.}$$

Let C be any point in AB and let $AC = s$ ft. When the weight W has fallen a distance s it has acquired a velocity v, so that $v^2 = 2gs$ (Art. 91). Then

$$\text{k.e. at } C = Wv^2/2g = Ws;$$

$$\text{p.e. at } C = W(h - s).$$

$$\therefore \text{ total energy at } C = Ws + W(h - s)$$
$$= Wh \text{ foot-pounds}$$
$$= \text{total energy at } A.$$

But C is *any* point, and therefore the sum of the kinetic and potential energies of the body is constant throughout the fall.

(2) Consider a body sliding down a smooth inclined plane of height h and length l under the action of gravity.

This is very similar to the preceding, and its development is left to the student.

(3) A body sliding down a rough inclined plane AB under the action of gravity.

With the same notation as before, μ being the coefficient of friction,

$$\text{k.e. at } A = 0;$$

$$\text{p.e. at } A = Wh$$
$$= Wl \sin\theta.$$

$$\therefore \text{ total energy at } A = Wl \sin\theta \text{ ft-pounds.}$$

The acceleration down the plane $= g(\sin\theta - \mu\cos\theta)$;

$$\text{velocity at } B = \sqrt{2g(\sin\theta - \mu\cos\theta)l};$$

$$\text{k.e. at } B = \tfrac{1}{2} W \times 2g(\sin\theta - \mu\cos\theta)l/g$$
$$= Wl(\sin\theta - \mu\cos\theta)$$

$$\text{p.e. at } B = 0;$$

$$\therefore \text{ total energy at } B = Wl(\sin\theta - \mu\cos\theta) \text{ foot-pounds.}$$

216. In the last case, instead of the energies at A and B being equal as in (1) and (2), there is a loss $\mu Wl \cos\theta$ in passing from A to B. It would thus appear that when energy passes from one form to another it is not always capable of being changed into its original form—at least with the same degree of readiness. The loss $\mu Wl \cos\theta$ is explained by saying that this energy has been converted into other forms of energy, principally the molecular energy called heat. Experiment shows that heat may be expressed as a definite number of foot-pounds of work. Thus Joule found that the amount of heat necessary to raise one pound of water from 0° to 1° F. is capable of doing 778 foot-pounds of work.

217. These examples are simple illustrations of a general principle, known as the **conservation of energy**, which is stated by Maxwell as follows:

The energy of a system is a quantity which can neither be increased nor diminished by any actions between the parts of the system, though it may be transformed into any of the forms of which energy is susceptible; in a word,

The energy of a closed system is a constant quantity, or

Energy is indestructible.

218. The principle of the conservation of energy is the greatest of all physical laws. It is founded upon experimental evidence most extensive, so that no doubt of its applicability to all forces in nature is now entertained. It includes Newton's laws of motion and the principle of work, and hence upon it the science of dynamics may be based. It cannot, however, be deduced from Newton's laws of motion, as it is more comprehensive than they.

It is perhaps more in accordance with the spirit of modern science to consider space, time, matter, and energy as the fundamental concepts of Mechanics rather than space, time, matter, and force (Art. 2). The further principle necessary on which to base the science would in the former case be that of the conservation of energy, while with the latter Newton's laws of motion are commonly employed. The latter method, which is that followed in this book, is given as being more readily appreciated by the beginner. But as it involves difficulties from which the first is free, it seems likely that the notion of energy will in time be adopted universally as the fundamental one rather than that of force.

Ex. 1*a*. Find the greatest velocity v attained by a car weighing 20 tons if hauled along a level, straight track by a pull of 1000 pounds through a distance of half a mile and against a constant resistance of 15 pounds per ton.

Ans. 54.4 ft/sec.

1*b*. Show that the car will come to rest after traveling $1\frac{1}{6}$ miles and in 3.77 min if the pull ceases at the end of half a mile.

2. A weight of 25 lb has fallen through 25 ft. Find the rate at which work is being done. *Ans.* 0.9 horse-power.

3. A cannon when fired recoils with a velocity of 10 ft/sec and runs up a platform having an incline of 1 in 4. Find the distance it goes before coming to rest. *Ans.* 6 ft 3 in.

4. A body weighing 64 lb and moving east with a velocity of 3 ft/sec receives a blow such that the velocity due to it is 4 ft/sec north. Find the resultant kinetic energy.

Ans. 25 foot-pounds.

[This resultant = the *sum* of the separate kinetic energies.]

5. Find the work done in stopping a 100-lb shot moving with a velocity of 1000 ft/sec.

Ans. 1,562,500 foot-pounds.

6. Find the force exerted in stopping a train of 250 tons in 1000 ft from a velocity of 30 miles an hour.

Ans. 15,125 pounds.

7. A shot pierces a target of a certain thickness. Show that to pierce one of 4 times the thickness twice the velocity is necessary.

8. What is the H.P. of an engine that will deliver 10,000 gallons of water per minute with a velocity of 10 ft/sec, if 10% is wasted by leakage? *Ans.* 4.4 H.P.

9. A blacksmith's helper using a 16-lb sledge strikes 20 times a minute and with a velocity of 30 ft/sec. Find his rate of work. *Ans.* 3/22 H.P.

10. A stone is thrown with a horizontal velocity of 50 ft/sec. Find the velocity with which it strikes the ground, which is horizontal and 6 ft below the point of projection.

Ans. 53.7 ft/sec.

11. Show that to give a train a velocity of 20 miles an hour requires the same energy as to lift it vertically through a height of 13.4 feet.

12. A hoisting-engine lifts an elevator weighing 1 ton through 50 ft when it attains a velocity of 4 ft/sec. If the steam is shut off, how much higher will it rise? *Ans.* 3 in.

13. In (12) find the time of rising 50 ft, supposing the motion uniformly accelerated, and also find the H.P. of the engine. *Ans.* 25 sec; 7.3 H.P.

14. Show that the energy stored in a train of weight W lb and moving with a velocity of V miles per hour is $WV^2/30$ foot-pounds.

15. A train of 100 tons is running at 30 miles an hour up a 2% grade. Find the H.P. required, the resistance on a level being 10 pounds per ton, due to axle-friction chiefly.

16*a*. In a locomotive running on the level at 30 miles an hour the tractive force is 8 tons. Taking the resistance of friction as 10 pounds per ton, find the number of 20-ton cars that can be hauled if engine and tender weigh 100 tons.

Ans. 75 cars.

16*b*. Find the number that would be hauled up a 2% grade.

Ans. 11 cars.

16*c*. Find the H.P. exerted in the former case.

Ans. 1280 H.P.

17*a*. An engine exerts on a car weighing 20,000 lb a net pull of 2 pounds per ton. Find the energy stored in the car after going $2\frac{1}{2}$ miles. *Ans.* 264,000 foot-pounds.

17*b*. If shunted to a level side track when the frictional resistance is 10 pounds per ton, find how far it will run before coming to rest? *Ans.* 1/2 mile.

17*c*. If shunted on a side track with a 1% grade, how far will it run before coming to rest? *Ans.* 1/6 mile.

17*d*. If there are brakes on half the wheels, and these are applied with a pressure of half a wheel-load, how far will the car run up a 1% grade, the coefficient of friction between wheel and brake-shoe being 0.2? *Ans.* 203 ft, nearly.

18. In the Westinghouse brake tests (Jan. 1887) at Weehawken a passenger-train moving 22 miles an hour on a down grade of 1% was stopped in 91 ft. There was 94% of the train braked. Taking the frictional resistance as 8 pounds per ton, find the net brake resistance per ton and the grade to which this is equivalent. *Ans.* 393 pounds; 20.9 per cent.

19. The tractive force of an engine is P tons. If the weight of engine and train is W tons and the frictional resistance n pounds per ton, show that in going up an a% grade the velocity acquired in t seconds from rest will be Qgt ft/sec, and the energy 0.5 WQ^2gt^2 ft-tons, where

$$Q = P/W - a/100 - n/2000.$$

219. Stability of Equilibrium.—Conceive a body in equilibrium to be slightly displaced: the forces acting will tend either to restore the body to its original position or to move it farther from that position, or will themselves equilibrate. In the first case the original position is said to be one of *stable*, in the second of *unstable*, and in the third of *neutral*, equilibrium so far as this displacement is concerned. In the special case when the forces continue to act in parallel directions, act at the same points, and remain of the same magnitude, the condition of neutral equilibrium is called *astatic*. Thus a body supported at its center of gravity is in astatic equilibrium.

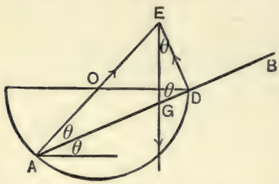
We shall consider the force of gravity to be the only external force acting. When displacement occurs and the body tends to return, the force of gravity does work in this return. Hence the potential energy is greater than in the position of equilibrium, or the position of equilibrium is one of minimum potential energy. But gravity alone acting, the potential energy depends on the height x of the center of gravity above a fixed horizontal plane. Hence in a position of stable equilibrium the center of gravity is in its lowest position.

Similarly, in a position of unstable equilibrium, the center of gravity is in its highest position, and in neutral equilibrium it remains in the same position after displacement.

The problem thus reduces itself to a determination of whether the *height* of the C.G. above a fixed horizontal plane is a minimum or a maximum, or is stationary.

The "Moving Stone" at Lexington is one of the most remarkable freaks of nature in the State of Kentucky. In the rear of the grounds attached to the home of the late Gov. Gilmer is a huge boulder, standing alone on the edge of a stream. Resting directly upon this boulder is another weighing at least twenty tons. This upper boulder rests upon a stone pinnacle not more than two feet square, and so evenly balanced that (although the slightest touch will cause it to rock to and fro) a hundred horses could not pull it from the socket. (*St. Louis Republic.*)

Ex. 1. A spoon rests in a hemispherical cup: to determine the nature of the equilibrium.



Let r = radius of cup, $2l$ = length of spoon.

Let the C.G. of the spoon be at its middle point G. The three forces acting, the weight and the reactions at A and D cut in a point E.

Let x denote the *depth* of G below OD. Then

$$x = r \sin 2\theta - l \sin \theta.$$

$$\therefore\ dx/d\theta = 2r \cos 2\theta - l \cos \theta.$$

Put $dx/d\theta = 0$, and we have for the position of equilibrium

$$\cos \theta = (l + \sqrt{l^2 + 32r^2})/8r, \quad . \quad . \quad . \quad . \quad (1)$$

the + sign of the radical being taken because θ is less than a right angle necessarily.

Also, in this position

$$d^2x/d\theta^2 = -4r \sin 2\theta + l \sin \theta,$$

which is + or − according as

$$4r \sin 2\theta <> l \sin \theta,$$

$$\text{or} \qquad \cos \theta <> l/8r.$$

But $\cos\theta > l/8r$ from eq. (1). Hence $d^2x/d\theta^2$ is negative, or x is a maximum and the equilibrium is stable.

2. A yard-stick rests over the edge of a cylindrical jar $2\frac{1}{4}$ in diameter with one end against the inner surface. Show that in the position of equilibrium the stick is inclined at 60° to the horizontal and that the equilibrium is unstable.

Show also that in order that the jar may not tumble over, its weight must be at least six times that of the stick.

3. A hemisphere rests on a horizontal table. Show that the equilibrium is stable.

4. A hemisphere, radius r, rests with its flat surface on the top of a sphere, radius R. Show that the equilibrium is stable or unstable as $R > < 3r/8$.

5. A cylindrical block of radius r with a hemispherical end rests on a horizontal table. Find the height of the cylinder for neutral equilibrium. *Ans.* $r/\sqrt{2}$.

6. A hemisphere, radius r, rests in neutral equilibrium with its curved surface on the top of a fixed sphere, radius R. Show that

$$5r = 3R.$$

EXAMINATION.

1. "An unbalanced force always does work."

2. How is the work done by the force of gravity on a falling body measured?

3. How is the work done against friction measured?

4. Give examples of work done against resistances.

5. Define the terms foot-pound, foot-ton, and inch-pound.

6. Find the work done against gravity in going upstairs.

7. Equal forces act for the same time upon unequal weights w_1, w_2. Find the relation between

(1) the momenta generated by the forces;
(2) the works done by the forces.

8. Define a lever, and find the conditions of equilibrium on a straight lever.

9. "The reason why a force acting at a greater distance from the fulcrum moves a weight more easily is because it describes a larger circle." (Aristotle.)

10. Put a 5-lb weight into one of the pans of an ordinary balance and drop a 4-lb weight into the other pan. The beam will be tilted, and if the 4-lb weight is quickly removed it may lead an onlooker to think it really weighs more than the other.

11. In catching a flying ball the player lets his hand be carried in the direction of motion of the ball. Why?

12. Find the relation between the work done in dragging a body up an inclined plane or in lifting it through the height of the plane when (*a*) smooth, (*b*) rough.

Also find the condition that a body may slide down a rough incline with uniform speed.

13. Find the condition of equilibrium in a system of pulleys in which the same cord goes round all the sheaves.

14. The work of raising a body in sections is

$$Wh \text{ foot-pounds,}$$

where W is the total weight in lb and h is the height in ft through which the C.G. has been raised.

15. In a screw-jack the pitch of the screw is 1 inch, the lever is 2 ft long, and the force applied at the end of the lever 25 pounds. Find the weight that can be lifted, friction being neglected.

16. What is meant by backlash in screw gears?

[Slack between screw and nut.]

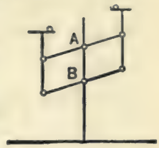
17. Find the forces that will (1) just support, (2) just move a rough screw when it carries a given weight.

18. The mechanical advantage of a smooth screw = (circumference described by acting force)/(pitch of screw).

19. Define the efficiency of a machine. In all cases the efficiency must be a proper fraction.

20. In a certain machine the work that should be obtained theoretically is 1600 ft-tons, but the amount actually obtained is 1200 ft-tons. Find the efficiency. *Ans.* 75%.

21. Explain why in a letter-balance [Roberval balance] it does not matter at what place on the scale-pans the weights are placed.



[The Roberval balance consists of a jointed parallelogram in which two opposite sides turn freely about their middle points A and B. On the other sides, which are always vertical, are fixed the scale-pans.

A druggist's scales are an example.]

22. An ocean steamer is running at n knots when the engines indicate N horse-power. Show that the resistance offered by the water is, roughly, $N/6n$ tons.

23. The weight on the drivers of an engine is W lb, the adhesion is $1/n$ part, the diameter of the cylinders d in, the diameter of the driver D in, the length of stroke s in, the mean steam-pressure per sq in p pounds; then

$$WD = d^2nps.$$

24. Give the units of activity.

25. What is meant by the term horse-power? By horse-power-hour?

26. What is the ratio of man-power to horse-power? [From 1/5 to 1/10.]

26*a*. May a man work at a higher rate than this?

[Yes, for a short time. He may work at the rate of a horse-power for a minute or two.

It does not follow that a one-horse-power engine could replace a horse in all cases, because a horse could make a spurt and for a short time work at the rate of 3- or 4-horse power.]

27. Compute the horse-power necessary to propel a train weighing W tons at a speed of V miles an hour (1) on the level, (2) up a grade of $n\%$, if the resistance is p pounds per ton. (See Examples, p. 254.)

Ans. H.P. on level $= .0027\,WVp$;
on grade $= .0027\,WV(p + 20n)$.

28. The tractive power of the engine [870, Empire State

Express] is a fraction over 11 lb per pound of average effective cylinder-pressure. This represents at 52.1 miles/hour very nearly 16 H.P. (*Engineer*, London.)

Check the conclusion.

29. If P pounds represents the driving force and v ft/sec the velocity, show that the power is represented by Pv ft-pounds/sec.

30. A turbine can utilize 75% of the energy of falling water. Show that the effective horse-power of a waterfall is

$$Qh/700, \text{ nearly,}$$

where $Q =$ the number of cubic feet of water per minute,
$h =$ the height of the fall in feet.

31. Give examples of bodies possessing kinetic energy.

32. A particle free to move is acted on by a force. Show that the work done by the force is equal to the gain of kinetic energy by the particle.

33. Show that force may be defined as rate of change of kinetic energy with the distance.

34. "If we multiply one half the momentum of every particle of a body by its velocity and add the results, we shall get the kinetic energy of the body."

35. How much kinetic energy does a body weighing W lb lose when its velocity changes from u to v ft/sec?

36. An athlete can make a longer running jump than standing jump.

37. The weight of a train is 95.5 tons, and the drawbar pull is 6 pounds per ton. Find the H.P. required to keep the train running at 25 miles an hour. *Ans.* 38.2 H.P.

37*a*. Find the energy required to bring the train to full speed from rest. *Ans.* 2000 foot-tons, nearly.

38. A train weighing W lb resting on a straight level track is acted on by a horizontal pull of F pounds for t seconds. Show that the energy acquired is

$$F^2gt^2/2W \text{ foot-pounds.}$$

39. The work done by an impulse of I second-pounds in changing the velocity of a weight W lb from u to v ft/sec is

$$I(u + v)/2 \text{ foot-pounds.}$$

40. The kinetic energy of a moving body is independent of the direction of the motion.

41. Show that there is no parallelogram of kinetic energies.

42. Find the H.P. of an engine that will discharge a gallons of water per minute from a depth of b ft and with a velocity of v ft/sec. (A gallon of water weighs $8\frac{1}{3}$ lb.)

Ans. $a(b + v^2/2g)/3960$ H.P.

43. State what is meant by the conservation of energy.

44. Show from the principle of the conservation of energy that if a weight slide from rest down a smooth inclined plane of height h,

$$v^2 = 2gh.$$

45. A particle is shot up a smooth tube with velocity v. Show that it will come to rest after reaching a point whose height above the point of projection is $v^2/2g$.

46. How is the potential energy of a body related to its stability?

47. A hemisphere rests on a rough inclined plane. Is the equilibrium stable?

48. Point out the errors in the following:

(1) "A car of weight W and velocity V miles per hour in passing from the tangent to a circular curve of radius R has its direction suddenly changed by the impulse

$$WV^2/15R,$$

causing a corresponding shock."

(2) "An equally accurate formula is the weight multiplied by the fall in feet = the momentum of foot-pounds of work."

(3) "A collision between two bicycles, each with a 150-lb rider, spinning at the moderate speed of seven miles an hour, would result in a smash-up with a force of 3000 pounds."

(4) "How much energy is radiated by the sun is very well known now, and is reckoned to be about 10,000 horse-power per square foot of its surface."

49. In 1895 a train on the Lake Shore R.R. made a run of 86 miles at the rate of 73 miles an hour. The weight of the train was 250 tons. Taking the train resistance on the straight level track to be 15 pounds per ton at this speed, show that the engine must have developed about 730 horse-power.

50. The engine in (49) was a 10-wheeler, having drivers 5 ft 8 in in diameter, and cylinders 17 × 24 in. Show that to develop 730 horse-power the average effective cylinder-pressure must have been about 37 pounds per square inch.

[A pressure of over 60 lb/in^2 has been kept up at this speed. See *R.R. Gazette*, Dec. 6, 1895.]

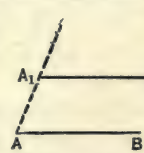
CHAPTER VII.

DYNAMICS OF ROTATION.

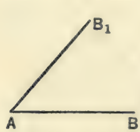
220. The cases of the motion of a single particle and of a motion of translation of a body, together with some of the more simple problems connected with rotation about a fixed axis, have been discussed in the preceding chapters. We now proceed to treat with some degree of fullness the motion of a body into which translation and rotation both enter.

The most important case and the only one we shall consider is that in which the particles of the body move in parallel planes. In this case the position of the body is determined when the positions of two points in it in a plane parallel to the plane of motion are known. For one point being fixed, rotation only is possible about this point. But a second point fixed fixes the body itself. Changes of position of the body will be determined by considering the changes of position of these two points fixed in the body.

221. Consider a body displaced from one position to another. Let A, B be the initial position of two fixed points in the body and A_1, B_1 the new positions after displacement, the displacement taking place in the plane of the paper.



(1) If A_1B_1 is parallel to AB, the body may be moved from its first position to its second position by a motion parallel to AA_1 or BB_1. This a motion of translation only.

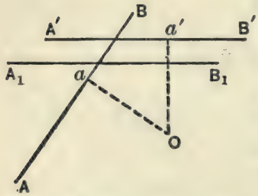


(2) If A_1 coincides with A, the body may be moved from its first position to its second by a rotation about A. This a motion of rotation only.

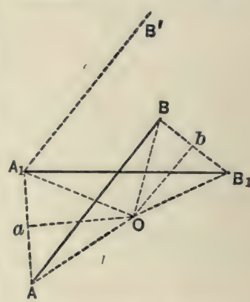
(3) Let A_1B_1 assume any position relative to AB. This

displacement may be considered to occur in either of two ways:

a. From any point O in the plane of the points let fall Oa perpendicular to the line AB joining A and B. Let the body be rotated about O until AaB assumes the position $A'a'B'$ parallel to A_1B_1. A motion parallel to $A'A_1$ will bring the points A', B' into coincidence with A_1, B_1; that is, the body may be moved from the first position to the second by a rotation about any assumed point and a translation.



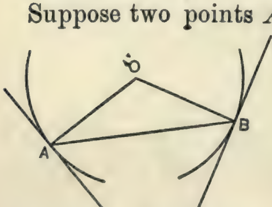
b. Join AA_1 and BB_1. Bisect these lines at a, b and let the perpendiculars aO, bO intersect in O. Then evidently $OA = OA_1$, $OB = OB_1$, and $\angle AOA_1 = \angle BOB_1$. Hence, if the body be rotated about O through the angle AOA_1, the point A will fall on A_1 and B on B_1; that is, the body will be moved from the first position to the second by a rotation about O.



It is evident from the construction that the position of the point O changes from instant to instant with the change of position of AB. In any position it is only the center of rotation for an instant, so that O is called the **instantaneous center of rotation.**

222. This may be still further illustrated.

Suppose two points A, B of a body to have any motion in the plane of the paper. The points A, B will each trace out a path. Consider A. The line joining two consecutive positions of A will give the direction of motion in the path. This line is the direction of the tangent to the path at A. Since an indefinitely great



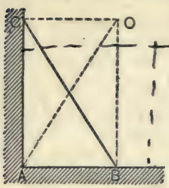
number of curves may have a common tangent at a point, it follows that this tangent is quite independent of the form of the path. Hence *for the instant* we may consider the path to be a circular arc. The perpendicular AO to the tangent will pass through the center of the circle, and, conversely, the direction of motion at A for the instant will be perpendicular to the radius of the circle. Hence the instantaneous motion of A is the same as if it took place in a circle with center somewhere on AO. Similarly, the motion of B is the same as if in a circle with center somewhere on BO. But O is common to AO and BO. Hence the instantaneous motion of A and B, and therefore of the line AB, is a motion of rotation about a point O as instantaneous center.

The points A, B are any two points in the body. Hence, *whatever the plane motion of the body, it is always possible to find a point O such that for the instant the motion about it shall represent the actual motion;* in other words, at any instant one point O is at rest, and the other points are moving in directions perpendicular to the lines joining them to this point.

If the radii AO, BO do not intersect, the tangents to the paths at A, B are parallel, and the motion is a motion of translation. The radii being parallel may be said to intersect at infinity, and hence a motion of translation may be regarded as a rotation about a center at an infinite distance.

223. In general, the instantaneous center O will vary in position from instant to instant. The locus or path described by it is called a *centrode.* But in case the radii AO, BO continue to intersect in the same point O, as the motion progresses the instantaneous center becomes a permanent or fixed center. For example, a wagon-wheel revolves about the axle as a permanent center, but with reference to the ground it revolves about the point of contact as an instantaneous center. The path traced by the wheel on the ground is the centrode.

Ex. 1. A ladder BC slides between a vertical wall and the ground, which is horizontal. Find the instantaneous center and the centrode.

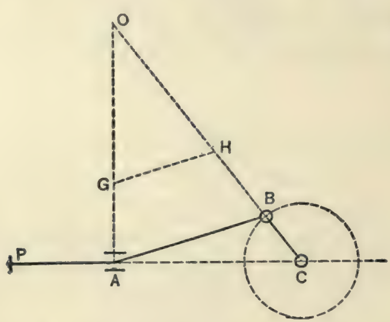


[The paths are along AB, AC. Hence the instantaneous center is at the intersection O of the perpendiculars BO, CO. It is evident that $AO = BC$, the length of the ladder, and therefore O is at a constant distance from A. Hence the centrode is a circle, with A as center.]

2. What is the direction of motion of any point G in BC at any instant? *Ans.* $\perp$ to GO.

3. The Empire State Express engine 999 has driving-wheels 7 ft 2 in diameter. Find the height above the track of a point on the circumference that has half the velocity of the highest point of the wheel. *Ans.* 1 ft 9.5 in.

224. In the case of moving bodies rigidly connected together the determination of the velocity of one with respect



to another may be based on the preceding. For illustration take the ordinary steam-engine. The mechanism itself has been already shown in Art. 159.

Suppose we wish to find the velocity v_1 of the piston P relative to the velocity v_2 of the crank-pin B. The velocity of the piston is the same as that of the extremity A of the connecting-rod AB. The velocity of the crank-pin B is the same as that of the extremity B of the connecting-rod.

Hence the relation sought is the same as that between the velocities of the extremities A and B of the connecting-rod.

The bed-plate is fixed. The extremity A of the connecting-rod moves in a straight line PC, and, the direction of motion being along PC, the instantaneous center is in a line AO at right angles to PC. The extremity B moves in a circle of center C, and therefore the instantaneous center is in the line CB. Hence the connecting-rod is *for the instant* in the condition of a body turning about an axis through O, the intersection of AO and CB. Consequently

$$v_1 : v_2 = OA : OB,$$

or the velocities of piston and crank-pin at any instant are as the distances of the cross-head A and crank-pin B from the instantaneous center O.

If, therefore, the velocity of one of the two, piston or crank-pin, is given, that of the other follows at once. Thus, suppose the crank-pin to have a velocity of 10 ft/sec. Lay off to scale a distance $BH = 10$ ft, and draw HG parallel to BA. Then since

$$HB : GA = OB : OA,$$

we scale off GA as the velocity of the piston.

By drawing the crank in different positions, and finding the corresponding positions of G, a curve will result, the ordinates of which will give the velocity of the piston through its stroke.

Ex. 1. If ω is the angular velocity of the crank B and v the speed of the piston P, then

$$v = \omega \times CR$$

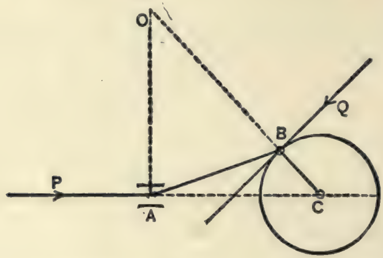
where R is the intersection of AB and the perpendicular CR to CA at C.

2. If v denote the piston velocity, u the velocity of the

crank-pin B, l the length of AB, r the length of CB, and θ the angle ACB, show that

$$v/u = \sin \theta(1 + r \cos \theta/l) \text{ nearly.}$$

225. The relation between the piston-pressure P and the crank-pin resistance Q, when the connecting-rod is inclined



at any angle, has already been found in Art. 159, but may be solved more simply by aid of the instantaneous center of rotation.

For if v_1 is the velocity of the piston and v_2 that of the crank-pin at any instant, then, O being the instantaneous center, we have (Art. 224)

$$v_1 : v_2 = OA : OB.$$

But from the principle of work

$$P \times v_1 - Q \times v_2 = 0.$$

Hence $$P \times OA = Q \times OB,$$

the relation sought.

The value of Q for a given piston-pressure will thus vary according to the position of the connecting-rod. It may be represented graphically, as in the case of the indicator-diagram (Art. 189).

The *average* of the values of Q for a complete revolution of the crank corresponding to a given piston-pressure P will be found by equating the work done by each of the two forces. We have, if r is the radius of the crank-arm and S the length of the stroke,

$$Q \times 2\pi r = P \times 2S.$$

$$\text{But} \qquad S = 2r.$$

$$\therefore \quad Q = 2P/\pi.$$

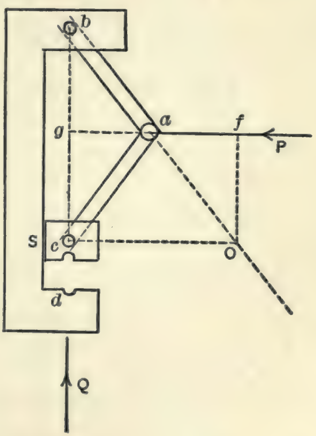
the relation required.

Ex. 1. In an engine the diameter of the cylinder is 14 in, and the steam-pressure 75 pounds/in². Find the average value of the force acting on the crank-pin.

Ans. 7350 pounds.

2. In (1) find the force acting when the crank stands at 60°, and the ratio of the connecting-rod to the crank is $5\frac{1}{2}$.

3. In a steam-riveting machine the piston-pressure P is applied at the joint a, and the rivet squeezed between the jaws c, d. Find the relation between P and the force Q exerted on the rivet.

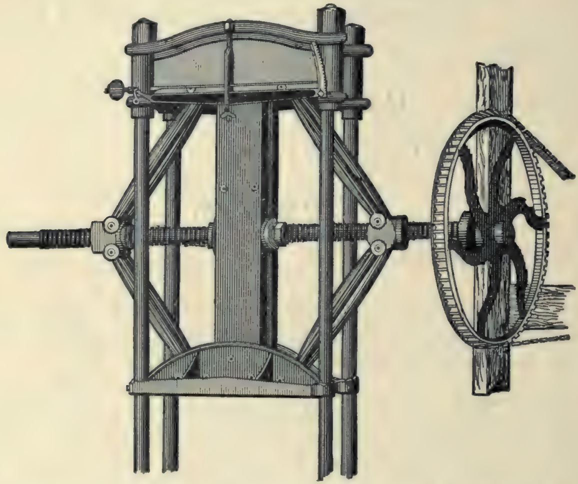


[The instantaneous center is at O, where ba and the perpendicular through c to the sliding surface S intersect. Then

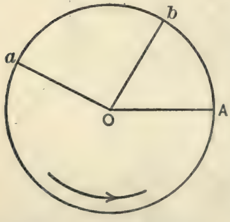
$$P \times fO = Q \times cO.$$

As a approaches g, cO diminishes; and when a reaches g, Q becomes indefinitely great. Hence the advantage of the apparatus in that an enormous pressure may be produced by a moderate force acting through a small distance.

This is an example of the toggle-joint, a mechanism of very considerable importance. It is applied, for example, in the railroad air-brake, in cider, oil, and other presses, etc. In the figure below is shown part of a power-screw oil-press.]



226. Angular Velocity.—If the motion of a body is a motion of rotation about a fixed axis, each particle describes a circumference whose center is in the axis. Since each circumference is described in the same time, the linear velocities of the particles in the circular paths must be proportional to the distances of the particles from the axis—the greater the distance the greater the velocity. The linear velocity of the body, however, cannot be said to be equal to that of any particle, as in a motion of translation.



Suppose O to be an axis perpendicular to the plane of the paper, and A any particle of the body. When the particle has reached the position b, it has described the angle AOb. It is evident that every particle of the body would describe an angle equal to this. The velocity of rotation may therefore be defined in terms of the angle of rotation. The rate at which the angle is described is the **angular velocity** of the particle, and being the same for every particle, is also the angular velocity of the body.

Thus, if the motion is uniform and the angle AOa is described in t sec, the angular velocity ω of the body is measured by the angle described in 1 sec, or

$$\omega = \angle AOa/t.$$

227. The *unit of angular velocity* is naturally taken to be unit angle described in one second. The unit angle employed is the unit of circular measure, being the angle AOb, which subtends an arc Ab equal in length to the radius AO. This angle is called a radian,* so that the unit of angular velocity is *one radian per second.*

Thus, if a body makes n revolutions per second, the number of radians described per second, or the angular velocity, is $2\pi n$; that is,

$$\omega = 2\pi n \text{ radians/sec.}$$

Notice that

$$1 \text{ revolution/sec} = 2\pi \text{ radians/sec.}$$

228. *Relation of Angular and Linear Velocity.*—The angular velocity of every point of the body has the same value. The relation between this and the linear velocity v of any particle A situated at a distance r from the axis of rotation follows at once. For the time of motion being t sec, we have

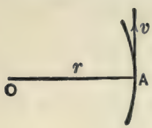
$$\begin{aligned} \omega &= \angle AOa/t \\ &= \text{arc } Aa/rt \\ &= v/r, \end{aligned}$$

the relation sought.

* It is shown in treatises on trigonometry that

$$\text{one radian} = 57.3°.$$

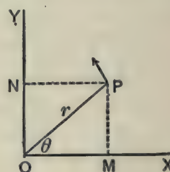
229. The direction of motion of a particle A in a circular path, or of a particle A of a body revolving about an axis O, is for an indefinitely small arc perpendicular to AO, that is, along the tangent to the path at A. We may therefore place



$$\text{tangential velocity } v = r\omega$$

if ω is the angular velocity of the particle about O.

Let the particle be referred to fixed rectangular axes OX, OY, and let x, y be its coordinates when in any position P. Denote the angle POX by θ and OP by r.



The components v_x, v_y of the velocity v of the particle parallel to OX, OY are

$$v_x = -v \sin\theta = -\omega r \sin\theta = -\omega y,$$
$$v_y = v \cos\theta \quad = \omega r \cos\theta \quad = \omega x,$$

where ω is the angular velocity about O.

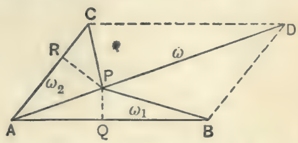
230. If the angular velocity is not constant, the actual angular velocity at any instant is determined by finding the limiting value of the average angular velocity for an indefinitely small angle $\Delta\theta$ described during an indefinitely small time Δt, including the instant. We have

$$\omega = \text{limit } \Delta\theta/\Delta t$$
$$= d\theta/dt \text{ radians/sec.}$$

231. *Graphical Representation.*—An angular velocity about an axis having magnitude and direction may be represented by a straight line. This line is taken along the axis of rotation, and therefore perpendicular to the plane of rotation, and of length proportional to the magnitude of the angular velocity. The positive direction of the axis is taken to be that in which a right-handed screw would move if placed along the axis and turned with the body.

232. *Composition of Angular Velocities.*—In the case of a rigid body having a number of angular velocities one point must be fixed, or may be considered fixed for the instant, and the angular velocities may be combined in a similar way to that in which linear velocities are combined by means of the parallelogram law.

Thus let two concurrent angular velocities ω_1, ω_2 about two axes AB, AC be represented by the lines AB, AC, and let P be any point lying in the diagonal AD of the parallelogram $ABDC$.



Let fall the perpendiculars PQ, PR on AB, AC.

The linear velocity of P due to the angular velocity about the axis AB is represented by $AB \times PQ$ or $2\Delta APB$. Also the linear velocity of P due to the angular velocity about the axis AC is represented by $AC \times PR$ or $2\Delta APC$. Hence

$$\begin{aligned}\text{resultant velocity of } P &= 2\Delta APB - 2\Delta APC \\ &= 0,\end{aligned}$$

or P is at rest. But P is any point in AD. Hence every point in AD has a resultant velocity due to the two rotations equal to zero; that is, AD is the direction of the resultant axis of rotation.

To find the magnitude of the resultant angular velocity ω about AD.

Consider a point Q of the body situated on AB. It has an angular velocity ω about AD, ω_2 about AC, and 0 about AB.

$$\begin{aligned}\text{The linear vel. of } Q \text{ about } AD &= \omega \times \perp \text{ from } Q \text{ on } AD \\ &= \omega \times AQ \sin QAD;\end{aligned}$$

$$\begin{aligned}\text{The linear vel. of } Q \text{ about } AC &= \omega_2 \times \perp \text{ from } Q \text{ on } AC \\ &= AC \times AQ \sin QAC;\end{aligned}$$

$$\text{The linear vel. of } Q \text{ about } AB = 0.$$

Hence, since AD is the resultant axis,

$$\omega \times AQ \sin QAD = AC \times AQ \sin QAC,$$

$$\text{and} \qquad \omega = AC \times AD/AC$$
$$= AD,$$

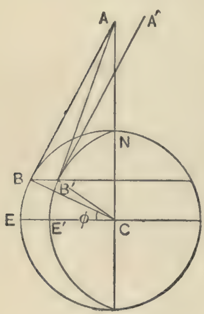
or ω is represented in magnitude by the diagonal AD of the parallelogram.

If, then, θ is the angle between the two axes AB and AC, we have, as in Art. 19,

$$\omega^2 = \omega_1^2 + 2\omega_1\omega_2 \cos\theta + \omega_2^2.$$

233. *Resolution of Angular Velocities.*—Any angular velocity may be resolved into components after the manner of linear velocities.

An interesting application is afforded by Foucault's pendulum. The apparatus consists of a long pendulum freely suspended and set oscillating in a vertical plane. A horizontal table placed below the pendulum appears to revolve in a direction opposite that of the hands of a watch. The problem is to find the angular velocity of the plane of oscillation of the pendulum relative to the table.



Let B be position of point at latitude ϕ at a given moment, and B' the position at end of short time Δt, during which the earth has rotated through a small angle, $\Delta\theta$, whose equatorial measure is the arc EE'. Draw tangents BA, $B'A$, cutting the earth's axis prolonged at A. Draw $B'A'$ parallel to BA.

The initial plane of the pendulum is ABC and the new plane $A'B'C$; hence the deviation is $\Delta\theta = A'B'A = B'AB$, or, in radial measure,

$$\Delta\theta = \frac{BB'}{AB}.$$

Let T = periodic time. Then $EE' = 2\pi R\frac{\Delta t}{T}$. Also $BB' = EE' \cos\phi$ and $AB = R \cot\phi$. Hence

$$\Delta\theta = \frac{2\pi R \cos\phi}{TR \cot\phi}\Delta t. \quad \therefore \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T}\sin\phi.$$

In the limit this becomes the rate of angular change,

$$\frac{d\theta}{dt} = \omega = \frac{2\pi}{T}\sin\phi.$$

Integrating, $\theta = \frac{2\pi}{T}t\sin\phi, = 360° \times \frac{t}{T}\sin\phi.$

If the hour be taken as unit of time, $T = 24$, and

$$\theta = 15t\sin\phi.$$

If ω denote the earth's angular velocity, $= \frac{2\pi}{T}$, the expression for the total deviation of the plane of the Foucault pendulum in time t becomes

$$\theta = \omega t\sin\phi.$$

Ex. A Foucault pendulum is set vibrating at New Orleans in lat. 30°. After what interval will it oscillate in the initial plane of oscillation? *Ans.* 1 day.

What is the time of a complete revolution of the pendulum?

234. Angular Acceleration.—The rate of change of angular velocity about an axis is called the **angular acceleration.**

The unit of angular velocity being one radian per second, the *unit of angular acceleration* is taken one radian per second per second, or, as it may be written, 1 radian/sec².

235. If a body start from rest with a uniform angular acceleration α, then after a time t the angular velocity ω would be given by

$$\omega = \alpha t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also the angle θ passed through is given by (Art. 25)

$$\theta = \alpha t^2/2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Eliminating t between (1) and (2),

$$\omega^2/2 = \alpha\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

These results may be compared with those of Art. 25.

Ex. If the body instead of starting from rest had an initial angular velocity ω_0, show that (compare Art. 24)

$$\omega = \omega_0 + \alpha t,$$
$$\theta = \omega_0 t + \alpha t^2/2,$$
$$\omega^2/2 - \omega_0^2/2 = \alpha\theta.$$

236. If the angular acceleration is not constant, the actual angular acceleration at any instant would be determined by finding the limiting value of the average angular acceleration for an indefinitely small angular velocity $\Delta\omega$ acquired in an indefinitely small time Δt, including the instant. We have

$$\begin{aligned} \alpha &= \text{limit } \Delta\omega/\Delta t \\ &= d\omega/dt \\ &= d^2\theta/dt^2 \text{ radians/sec}^2, \end{aligned}$$

or, as it may be written,

$$\alpha = \omega d\omega/d\theta \text{ radians/sec}^2.$$

These results may be compared with those of Art. 26.

237. *Tangential and Normal Acceleration.*—The acceleration of a particle w of a body revolving about an axis may be resolved into two components at right angles to each other. Let v be the linear velocity at any instant along the tangent to the path, ω the angular velocity about the axis, and r the radius of the path. Then (Art. 33)

$$\begin{aligned} \text{tangential acceleration} &= dv/dt \\ &= d(\omega r)/dt \\ &= r d\omega/dt \\ &= r\alpha; \\ \text{normal acceleration} &= v^2/r \\ &= \omega^2 r^2/r \\ &= \omega^2 r. \end{aligned}$$

Hence we may write

$$\begin{aligned} \text{tangential force} &= wr\alpha/g, \\ \text{normal force} &= w\omega^2 r/g, \end{aligned}$$

where ω and α may be expressed in any of the differential forms given above.

238. Note the analogy between the kinematical formulas for translation and the corresponding formulas for rotation. We have

Translation.	Rotation.
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = ut + at^2/2$	$\theta = \omega_0 t + \alpha t^2/2$
$as = v^2/2 - u^2/2$	$\alpha\theta = \omega^2/2 - \omega_0^2/2.$

Corresponding differential equations are

$$v = \frac{ds}{dt} \qquad\qquad \omega = \frac{d\theta}{dt}$$

$$a = \frac{d^2s}{dt^2} \qquad\qquad \alpha = \frac{d^2\theta}{dt^2}$$

$$a = v\frac{dv}{ds} \qquad\qquad \alpha = \omega\frac{d\omega}{d\theta}.$$

239. Example.—A wheel is revolving 50 times per second. It is brought to rest with a uniform angular retardation in 10 seconds. Find the number of turns it makes before coming to rest.

Here $\omega_0 = 2\pi \times 50 = 100\pi$ radians/sec.

$\omega = 0, \qquad t = 10$ sec.

Now
$$\omega = \omega_0 + \alpha t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or $\quad 0 = 100\pi + 10\alpha;$

$\therefore \quad \alpha = -10\pi$ radians/sec.

Also
$$\alpha\theta = \omega^2/2 - \omega_0^2/2,$$

or $\quad -10\pi\theta = 0 - (100\pi)^2/2;$

$\therefore \quad \theta = 500\pi$ radians,

and number of revolutions $= 500\pi/2\pi$

$= 250.$

Ex. 1. If ω is expressed in degrees, show that

$$v = 2\pi\omega r/360°.$$

2. A body makes n revolutions per second. Show that the angular velocity is $2\pi n$ radians/sec.

3. A belt passes over a pulley d ft in diameter and making n revolutions per min. Find its linear velocity.

Ans. πdn ft/min.

4. A wheel 4 ft in diameter revolves 420 times per minute. Find the angular velocity and the linear velocity of a point 1.5 ft from the center. *Ans.* 14π radians/sec; 21π ft/sec.

5. The crank of an engine makes n revolutions per min. Its radius is r ft. Find the linear velocity of the crank-pin in ft/sec. *Ans.* $\pi rn/30$.

6. A locomotive is running at 45 miles an hour. The driving-wheels are 6 ft in diameter and the stroke is 2 ft. Find the average piston velocity in ft/sec.

Ans. $44/\pi$ ft/sec.

7. A wheel making 20 revolutions a second is brought gradually to rest in 10 seconds. How many revolutions has it made after the brake was applied?

Ans. 100 revolutions.

8. One of the 12 spokes of a carriage-wheel is vertical. Find the velocity of the extremity of the first spoke in advance if the velocity of the carriage is 7.5 miles an hour.

Ans. $11\sqrt{2 + \sqrt{3}}$ ft/sec.

9. A particle describes a circle of radius r with uniform velocity v. Find its angular velocity about any point on the circumference. *Ans.* $v/2r$.

10. A coin, radius r, is rolled along a table. If v denotes the linear velocity of its center and ω its angular velocity about the point of contact with the table, then

$$v = \omega r.$$

11*a*. A system has two component rotations of 2 and 3 radians/sec about axes inclined at 60°. Show that the resultant rotation is $\sqrt{19}$ radians/sec.

11*b*. What is the position of the resultant axis of rotation?

12*a*. If a body has two angular velocities ω_1, ω_2 about parallel axes through A, B, show that the resultant ω is found from

$$\omega = \omega_1 \pm \omega_2,$$

and is about an axis through a point C such that

$$\omega_1 \times AC = \omega_2 \times BC.$$

12*b*. Give the corresponding proposition for parallel translations.

13. If u, v denote the component velocities of translation of a point P parallel to rectangular axes OX, OY, and ω the angular velocity of P about O, then

$$\text{total velocity of } P \text{ parallel to } OX = u - \omega y;$$

$$\text{total velocity of } P \text{ parallel to } OY = v + \omega x.$$

240. Dynamical Equations of Motion.—We have seen (Art. 221) that the plane motion of a rigid body may be indicated in two ways:

1. As a translation of the body and a rotation about an axis through any point and perpendicular to the plane of motion; or

2. As a rotation about an axis which changes from instant to instant—the instantaneous axis of rotation.

The cases in which the motion is wholly a translation or wholly a rotation about a fixed axis are evidently included.

There are two cases to be considered: (A) the motion due to continuous forces, (B) the initial motion due to impulses.

241. Motion under Continuous Forces.—When a single particle w_1 at rest and free to move is acted on by a force F_1, the acceleration a' produced is in the direction of F_1, and its value is found from (Art. 67)

$$F_1 = w_1 a'/g.$$

But if the particle, instead of being free to move, forms one of a rigid system, then, besides the external force F_1, other forces act on the particle arising from the mutual actions of the particles. Let R_1 be the resultant of these internal reactions on w_1.

The acceleration of w_1 is now due not to F_1, but to the resultant of F_1 and R_1. Denote this acceleration by a_1.

Then $w_1 a/g$, called the **effective force** on the particle, is the resultant of F_1 and R_1, or, in other words, F_1, R_1, and $w_1 a_1/g$ reversed,* are in equilibrium.

Similarly, for a particle w_2 the forces F_2, R_2, and $w_2 a_2/g$ reversed, are in equilibrium.

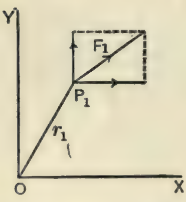
Summing up for the whole system, and assuming as a consequence of the third law of motion that the internal actions and reactions $R_1, R_2, \ldots$ of the system are in equilibrium among themselves, we conclude that

The external forces $F_1, F_2, \ldots$ and the reversed effective forces for all the particles form a system of forces in equilibrium.

Hence the solution of the problem is reduced to the statical problem of Art. 140.

[The above principle is known as *D'Alembert's principle*, having been enunciated by D'Alembert in 1742. It is not an independent principle, but an immediate consequence of Newton's laws of motion. Its great use is that it enables us to state dynamical propositions in a statical form.]

For the first two conditions of equilibrium (Art. 140), that the sum of the forces parallel to two rectangular axes should each equal zero, let $X_1, Y_1; X_2, Y_2; \ldots$ denote the components of the external forces $F_1, F_2, \ldots$ parallel to the rectangular axes OX, OY drawn through a fixed point O. Also let $w_1 a_x'/g, w_1 a_y'/g; w_2 a_x''/g, w_2 a_y''/g; \ldots$ denote the components of the effective forces parallel to these axes, $a_x', a_y'; a_x'', a_y''; \ldots$ being the component accelerations of the particles $w_1, w_2, \ldots$



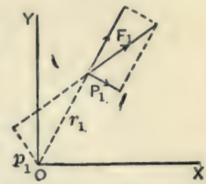
Then, resolving parallel to the axes, we have

$$\Sigma X = \Sigma w a_x/g, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\Sigma Y = \Sigma w a_y/g. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* The reversed effective force—wa/g is sometimes called the *force of inertia* or the *inertia resistance* of the particle.

For the third condition of equilibrium, that the sum of the moments of the forces acting—that is, of the external forces and the effective forces reversed—should be equal to zero, let $p_1, p_2, \ldots$ denote the distances of the directions of the external forces $F_1, F_2, \ldots$ from O, and $r_1, r_2, \ldots$ the distances of the particles $P_1, P_2, \ldots$ on which these forces act from O.



Now, since each particle moves in a circle about the axis O, it is convenient in taking moments to resolve the effective forces into components along and perpendicular to the radii $OP_1, OP_2, \ldots$

The component forces at P_1 are (Art. 237)

$$w_1 r_1 \omega^2/g \text{ along } OP_1,$$

$$w_1 r_1 \alpha/g \perp OP_1,$$

where ω is the angular velocity and α the angular acceleration of the particle.

Hence, taking moments about O for all the forces, we have

$$F_1 p_1 + F_2 p_2 + \ldots = w_1 r_1^2 \alpha/g + w_2 r_2^2 \alpha/g + \ldots,$$

the angular acceleration α being the same for all of the particles.

This may be written

$$\Sigma Fp = \frac{\alpha}{g} \Sigma wr^2,$$

since α/g is a constant factor in each term.

The left-hand member of this equation is the ordinary expression for the statical moment or torque. The factor Σwr^2 of the right-hand member, which is independent of the angular acceleration α, is called the second moment or **moment of inertia** of the body relative to the axis. It is usually denoted by the letter I, so that

$$\Sigma Fp = I\alpha/g. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence the angular acceleration α of a body about the axis of rotation is found whether fixed or not.

The three equations (1), (2), (3), above, are the equations of motion of a rigid body when acted on by continuous forces in the same plane.

[The term moment of inertia was introduced by Euler in order to express in analogous terms the formulas for linear and angular acceleration. Thus for translation, using absolute units (Art. 50),

$$F = ma,$$

or force = inertia × linear acceleration, . . . (1)

if we use the term inertia instead of the term mass as Euler did.

For rotation

$$\Sigma Fp = \Sigma mr^2 \times \alpha.$$

Here ΣFp is moment of force and α is angular acceleration. Hence, if for Σmr^2 is written "moment of inertia," we have, analogous to (1),

moment of force = moment of inertia × angular acceleration.]

242. The point O selected as fixed point is usually the C.G. of the body, and the motion is reduced to a translation of this point and a rotation about it.

Thus if $\bar{a}_x$, $\bar{a}_y$ are the accelerations of the C.G. of the body parallel to the axes, then (Art. 146)

$$W\bar{a}_x = \Sigma w\bar{a}_x, \qquad W\bar{a}_y = \Sigma w\bar{a}_y.$$

Hence $$\Sigma X = W\bar{a}_x/g, \qquad \Sigma Y = W\bar{a}_y/g,$$

or *the C. G. of a body moves as if the whole weight were concentrated at that point and the external forces were to act on it parallel to their original directions.*

The motion of rotation about the C.G. is given by

$$\Sigma Fp = I\alpha/g.$$

Being given the initial velocities of translation and rotation, the displacement and resulting velocities after any term may be found from Art. 24 for translation and from Art. 235 for rotation. Hence the motion is completely determined.

Ex. 1. A fly-wheel weighing W lb is rotating at n revolutions per minute about its axle, whose radius is r ft. If the driving-belt is slipped, in what time t will the wheel come to rest, μ being the coefficient of friction?

Let ω = the angular velocity to be destroyed
$= 2\pi n/60$ radians/sec;
α = the angular acceleration required to destroy ω in t sec
$= \omega/t$ radians/sec^2.

Now, since the wheel rotates about a fixed axis,

$$\text{torque} = I\alpha/g,$$
or
$$\mu W \times r = I\pi n/30gt,$$
and
$$t = I\pi n/30\mu Wrg \text{ seconds.}$$

2. Show that the number of revolutions made by the wheel before it comes to rest is $I\pi n^2/3600\mu Wrg$.

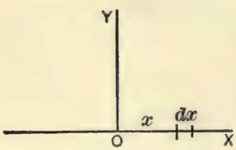
243. It is evident that before we can find numerical results in any case we must be able to compute the value of I, the moment of inertia of the rotating body. The method of doing this we proceed to explain in the following sections.

The sections treating of moments of inertia, like the sections on center of gravity (Arts. 143–151), form a sort of interlude, and might be placed in a treatise on the integral calculus—perhaps better placed there. The subject of mechanics proper is resumed in Art. 251.

244. Moment of Inertia.—The form of the expression for the moment of inertia about an axis, Σwr^2, shows that we may define it as the sum of the products of the particles $w_1, w_2, \ldots$ of a rigid body into the squares of their distances $r_1, r_2, \ldots$ from the axis of rotation. The finding of moments of inertia is therefore a problem of summation of indefinitely small quantities, and not a mechanical question at all. This summation may in some cases be made without the calculus, as in the first example following, which for illustration is solved in both ways.

In the case of bodies of irregular shape it is in general most convenient to determine the moment of inertia experimentally. A method of doing this is indicated in example 4, page 314.

Ex. 1. To find the moment of inertia I of a thin uniform rod, weight W, length l, about an axis OY through its center O, and at right angles to the rod.



[Conceive the rod cut into elements of indefinitely small length Δx, and let x be the distance of any one of these elements from O.

Let each unit of length weigh δ, then the length Δx will weigh $\delta \Delta x$, and the moment of inertia of this element about the axis is $\delta \Delta x \times x^2$. Hence for the whole rod

$$I = \int_{-\frac{l}{2}}^{+\frac{l}{2}} \delta x^2 dx = \delta l^3/12 = Wl^2/12.$$

Or thus: Suppose the rod divided into a large number $2n$ of equal parts. The length of each part is $l/2n$ and its weight is $\delta l/2n$. The distances of these parts from O may be taken to be the distances of their centers of gravity from O, that is, $l/4n$, $3l/4n$, . . . Hence, taking half the rod,

$$\tfrac{1}{2}I = \frac{\delta l}{2n}\left(\frac{l}{4n}\right)^2 + \frac{\delta l}{2n}\left(\frac{3l}{4n}\right)^2 + \dots \text{ to } n \text{ terms}$$

$$= \frac{\delta l^3}{32n^3}(1^2 + 3^2 + \dots \text{ to } n \text{ terms})$$

$$= \delta l^3/24 \text{ when } n \text{ is indefinitely great,}$$

and $\qquad I = Wl^2/12$, as before.]

1a. Show that the moment of inertia about a perpendicular axis through one extremity is $Wl^2/3$.

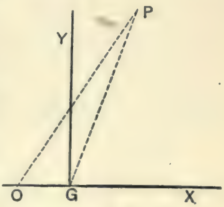
245. *Unit Moment of Inertia.*—The unit moment of inertia is the moment of inertia of a particle of unit weight situated at unit distance from the axis of rotation. No special name is in use, but it is necessary in giving moments of inertia to state the units employed in computing them. The name *newton* has been suggested as an appropriate name for the unit moment in the British gravitation system.

If the pound weight were made into a thin cylinder of 1 ft radius and caused to revolve about the central axis it would have unit moment of inertia.

246. The computation of moments of inertia may be facilitated by the aid of the following two propositions:

(1) *The moment of inertia of a body about any axis is equal to the moment of inertia about a parallel axis through the center of gravity, together with the product of the weight of the body into the square of the distance between the two axes.*

For suppose the two parallel axes through a point O and the center of gravity G to lie in a plane perpendicular to the plane of the paper. Take G as origin, the plane of the paper the plane of XY, and OGX the axis of X. Let x, y denote the coordinates of any particle P weighing w. Call the distance $GO = h$. Then if I denote the moment of inertia about an axis through O, we have



$$I = \Sigma w\{y^2 + (x + h)^2\}$$
$$= \Sigma w(y^2 + x^2) + 2h\Sigma wx + h^2\Sigma w,$$

since the distance h is constant.

Of the three expressions in the right-hand member, the first is equal to $\bar{I}$ the moment of inertia about G; the second is equal to zero, since G is the center of gravity (Art. 145); and the third is equal to Wh^2, where W is the total weight of the body. Hence

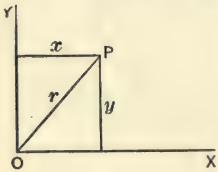
$$I = \bar{I} + Wh^2,$$

which proves the proposition.

This proposition is due to Lagrange.

(2) *The moment of inertia of a plane lamina about an axis through any point O and perpendicular to its plane* [polar moment] *is equal to the sum of the moments of inertia about two rectangular axes through O and in the plane* [rectangular moment].

Let OX, OY be rectangular axes through any point O in the plane of the lamina, x, y the coordinates of P any particle weighing w, and r the distance OP. Then



$$r^2 = x^2 + y^2.$$

$$\therefore \quad wr^2 = wx^2 + wy^2;$$

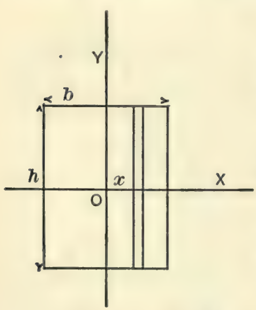
and summing up for the particles in the whole lamina,

$$\Sigma wr^2 = \Sigma wx^2 + \Sigma wy^2,$$

or $$I \text{ about } O = I \text{ about } OY + I \text{ about } OX,$$

or $$I = I_y + I_x,$$

which proves the proposition.



Ex. 2. Find the I of a thin rectangular lamina or plate of breadth b and depth h, about an axis through its center of gravity O and parallel to h.

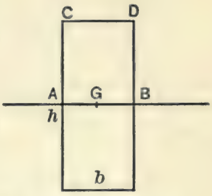
[Conceive the lamina cut into strips parallel to the axis, and of breadth Δx. Let x denote the distance of one of these strips from the axis, and let it weigh δ per unit area. The strip will weigh $\delta h \Delta x$. Hence

$$I = \int_{-\frac{b}{2}}^{+\frac{b}{2}} \delta h \times dx \times x^2 = \tfrac{1}{12}\delta h b^3 = \tfrac{1}{12} W b^2,$$

where the whole lamina weighs W.]

3. If in (2) the axis is parallel to the side b show that

$$I = Wh^2/12.$$



4. Show that the I of a rectangular lamina breadth b, depth h about the side b is $Wh^2/3$. (For example, a door about its hinges.)

[Deduce from Ex. (3) by aid of Art. 246, and also solve independently.]

5. Find the I of a rectangular lamina, breadth b, depth h, about an axis through its center of gravity and perpendicular to its plane. *Ans.* $I = W(b^2 + h^2)/12$.

6. The I of a square plate of side a about a diagonal is $Wa^2/12$.

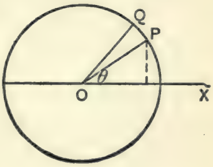
7. Show that the I of a square lamina about *any* axis in its plane and through its center is the same and equal to $Wa^2/12$.

8. Find the I of a rectangular plate about a diagonal, the sides of the rectangle being a, b.

Ans. $I = Wa^2b^2/6(a^2 + b^2)$.

9. Find the I of a thin circular ring of radius r and weight W about an axis through its center and perpendicular to its plane. *Ans.* $I = Wr^2$.

10. Find the I of a circular ring, radius r and weight W about a diameter.



[Conceive the ring cut into elements PQ subtending an angle $d\theta$ at the center O. Then weight of $PQ = \delta \times rd\theta$. Hence

$$I = \int_0^{2\pi} \delta r d\theta \times r^2 \sin^2 \theta$$

$$= \delta r^3 \int_0^{2\pi} \frac{1 - \cos 2\theta}{2} d\theta$$

$$= \pi \delta r^3$$

$$= Wr^2/2.]$$

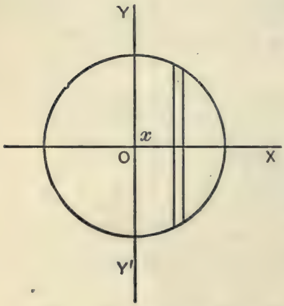
11. Find the I of a circular ring or wire about a tangent.

Ans. $I = 3\,Wr^2/2$.

[Deduce from Ex. 10 by aid of Art. 246, and also solve independently.]

12. Find the I of a quadrantal wire about an axis through one extremity and perpendicular to its plane.

Ans. $I = 2Wr^2(1 - 2/\pi)$.



13. Find the I of a thin circular plate, radius r and weight W, about a diameter.

[Conceive the plate cut into strips parallel to the axis, and of breadth Δx. Then the equation to the circle being $x^2 + y^2 = r^2$, the length of a strip at a distance x from O is $2\sqrt{r^2 - x^2}$, and area of strip $= 2\sqrt{r^2 - x^2}\Delta x$. Hence

$$I = \int_{-r}^{+r} 2\delta x^2 \sqrt{r^2 - x^2}dx = \delta\pi r^4/4 = Wr^2/4.$$

The simplest way of finding the value of this integral is to put $x = r \sin\theta$. The limits of integration then become $+\pi/2, -\pi/2$.]

Solve the problem, using polar coordinates.

14. Find the I of a thin circular plate about a tangent.

Ans. $5Wr^2/4$.

15. Find the I of a circular plate of radius r and weight W about a perpendicular axis through its center.

Ans. $I = Wr^2/2$.

[May be deduced from Ex. 13 and Art. 246 (2), or independently, as follows:

Conceive the plate composed of concentric rings. Let x denote the distance of any ring from the center and Δx its width. Then the ring weighs $\delta \times 2\pi x\,\Delta x$. Hence

$$I = \int_0^r 2\pi\delta x dx \times x^2$$

$$= \pi\delta r^4/2$$

$$= Wr^2/2.]$$

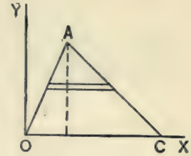
Conversely, Ex. 13 may now be deduced from Ex. 15.

16. Find the moment of inertia of a grindstone 2 ft in diameter and 6 inches thick about its axis, if the stone weighs 125 lb/ft³.

Ans. 98.2.

17. Find the I of a ring of radii r_1, r_2 about a $\perp$ axis through its center. *Ans.* $I = W(r_1^2 + r_2^2)/2$.

18. Find the I of a triangular lamina of base b and height h about the base.



[Divide the lamina into strips parallel to the base.

Let y = distance of any strip from the base;

Δy = width of strip.

Then length of strip : $b = h - y : h$,

or length of strip $= b(h - y)/h$,

area of strip $= b(h-y)\Delta y/h$.

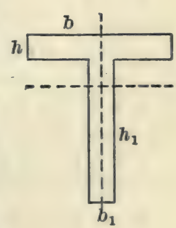
$$\therefore \quad I = \int_0^h \delta b y^2 (h - y) dy/h$$
$$= \delta b h^3/12$$
$$= Wh^2/6.]$$

19. Find the I of a triangle of base b, height h, about an axis through its center of gravity and parallel to the base. *Ans.* $I = W(h^2/6 - h^2/9) = Wh^2/18$.

20. Find the I of a triangle of base b, height h, about an axis through its vertex and parallel to the base b. *Ans.* $I = Wh^2/2$.

21. Find the I of a hexagon of side a about a diagonal. *Ans.* $I = 5Wa^2/4$.

22. Find the I of a T-iron, breadth of flange $= b$, breadth of web $= b_1$, depth of flange $= h$, depth of web $= h_1$, about an axis through the center of gravity. (See Ex. 4, p. 165.) *Ans.* $bh^3/12 + b_1h_1^3/12 + (h+h_1)^2/4(b^{-1}h^{-1} + b_1^{-1}h_1^{-1})$.



23. Find the I of a sphere of radius r about a diameter as axis.

[Conceive the sphere divided into slices of width dx by planes perpendicular to the axis OX. Let the distance of any slice from the center O be x. Then radius of slice $= \sqrt{r^2 - x^2}$, and volume of slice $= \pi(r^2 - x^2)\Delta x$.

The moment of inertia of the slice about OX is (Ex. 15)

$$\tfrac{1}{2}\delta\pi(r^2 - x^2)^2 \Delta x.$$

Summing up for the whole sphere, that is, integrating between the limits $x = r$ and $x = -r$, we have

$$\begin{aligned} I &= 8\delta\pi r^5/15 \\ &= 2Wr^2/5, \end{aligned}$$

since weight of sphere $= 4\delta\pi r^3/3$.]

24. Find the I of a sphere, radius r, about a tangent line.
Ans. $I = W(2r^2/5 + r^2) = 7Wr^2/5$.

25. Show, by differentiating the result of Ex. 15, that the moment of inertia of an indefinitely thin ring of radius r about an axis through its center and perpendicular to its plane is Wr^2.

26. Show, by differentiating the result of Ex. 23, that the moment of inertia of an indefinitely thin spherical shell about a diameter is $2Wr^2/3$.

27. Find the I of an elliptical plate about (1) its major axis, (2) its minor axis, and (3) about an axis through its center and perpendicular to its plane.
Ans. $Wb^2/4$; $Wa^2/4$; $W(a^2 + b^2)/4$.

28. Find the I of a right cone, height h and radius of base r, about an axis through the vertex parallel to the base (solution similar to that of Ex. 18).
Ans. $I = 3W(r^2 + 4h^2)/20$.

29. Find the I of a right cone, height h and radius of base r, about an axis through its center of gravity and parallel to the base. *Ans.* $I = 3W(r^2 + h^2/4)/20$.

30. Find the I of a right cone, height h and radius of base r, about its own axis. *Ans.* $I = 3Wr^2/10$.

31. Prove that I is the same for all parallel axes situated at equal distances from the center of gravity.

32. Of all parallel axes the I about that which passes through the center of gravity is the least.

247. Radius of Gyration.—The general expression for the moment of inertia of a series of particles rigidly connected and weighing $w_1, w_2, \ldots$ about an axis situated $r_1, r_2, \ldots$ from the particles is Σwr^2. The whole series of particles forms a body weighing Σw or W. We may conceive the body concentrated into a single particle weighing W, and a distance k from the axis may always be found, so that the moment of inertia of the particle W about the axis is equal to the sum of the moments of inertia of the separate particles of the body, or

$$Wk^2 = \Sigma wr^2.$$

To this distance k the name **radius of gyration** is commonly given.

248. If the axis passes through the C.G. of the body, we may write

$$W\bar{k}^2 = \Sigma wr^2$$

when $\bar{k}$ is called the *principal radius of gyration.*

The relation between k and $\bar{k}$ follows at once. For (Art. 246)

$$\Sigma wr^2 = \Sigma w\bar{r}^2 + Wh^2,$$

$$\text{or} \qquad Wk^2 = W\bar{k}^2 + Wh^2,$$

$$\text{or} \qquad k^2 = \bar{k}^2 + h^2,$$

the relation sought.

249. *Reduced Weight.*—We may write

$$I = W_1k_1^2,$$

where k_1 is *any assumed distance* from the axis of rotation and $W_1 = I/k_1^2$; that is, we can replace the rotating body by an equivalent particle of weight W_1 at any distance k_1 from the axis of rotation by dividing the moment of inertia of the body about the axis of rotation by the square of the assumed distance. The weight of this particle is called the *reduced weight.*

The reduction is of frequent application in mechanisms where many pieces have to be considered. For a simple illustration take the case of a fly-wheel not properly centered, Art. 238; also the compound pendulum, Art. 251. A more complicated example will be found in Weisbach's *Hoisting Machinery*, p. 143.

250. The results in the following table for the square of the radius of gyration about an axis through the center of gravity should be carefully checked.

1. Straight rod, length l,
   axis ⊥ rod — $\bar{k}^2 = l^2/12$

2. Rectangular lamina, breadth b, depth h,
   axis ∥ side h — $\bar{k}^2 = b^2/12$
   axis ⊥ plane — $\bar{k}^2 = (b^2+h^2)/12$

3. Square, side a,
   axis a diagonal — $\bar{k}^2 = a^2/12$
   axis ⊥ plane — $\bar{k}^2 = a^2/6$

4. Circular disk, diameter d,
   axis a diameter — $\bar{k}^2 = d^2/16$
   axis ⊥ plane — $\bar{k}^2 = d^2/8$

5. Circular ring, diameter d,
   axis a diameter — $\bar{k}^2 = d^2/8$
   axis ⊥ plane — $\bar{k}^2 = d^2/4$

6. Triangular lamina, base b, height h,
   axis parallel base — $\bar{k}^2 = h^2/18$

7. A sphere, diameter d,
   axis a diameter — $\bar{k}^2 = d^2/10$

8. A spherical shell, diameter d,
   axis a diameter — $\bar{k}^2 = d^2/6$

9. Rectangular prism, sides a, b, c,
   axis ⊥ face bc — $\bar{k}^2 = (b^2+c^2)/12$

[Since the prism may be conceived to consist of an infinite number of plates, each of which has the same radius of gyration with respect to an axis through the centre and perpendicular to their planes, the radius of gyration of the prism is the same as that of any plate.]

10. Circular cylinder, length l, diameter d,
axis the axis of the cylinder $\bar{k}^2 = d^2/8$
axis ⊥ axis of cylinder $\bar{k}^2 = d^2/16 + l^2/12$

11. Right cone, altitude h, diameter of base d,
axis ⊥ to axis of cone $\bar{k}^2 = 3(d^2+h^2)/80$
axis the axis of the cone $\bar{k}^2 = 3d^2/40$.

12. Hollow cylinder, inner diameter d_1, outer diameter d_2,
axis the axis of the cylinder $\bar{k}^2 = (d_1^2+d_2^2)/8$

251. The subject proper is now resumed. It was interrupted at Arts. 245, 246 by the discussion of moments of inertia.

Compound Pendulum.—A most important case on account of its applications is that of a body oscillating about a horizontal axis under the action of gravity.

Let G be the C.G. of the rotating body, W the weight of the body, C the axis of rotation, and ω the angular velocity.

Let the distance $CG = h$, and the angle of swing $= \theta$ at the instant considered.

The external forces acting are the weight W at G, and the reaction of the axis which may be resolved into horizontal and vertical components X and Y.

The effective forces on a particle w_1 distant r_1 from the axis are $w_1 r_1 \omega^2/g$ along r_1, and $w_1 r_1 \alpha/g$ perpendicular to r_1 (Art. 237). The resultant of these effective forces acting at G is evidently

$$Wh\omega^2/g \text{ along } CG \quad \text{and} \quad Wh\alpha/g \perp CG.$$

Hence resolving horizontally, vertically, and taking moments about C,

$$Wh\alpha \cos \theta/g - Wh\omega^2 \sin \theta/g = X, \quad . \quad . \quad . \quad . \quad (1)$$

$$Wh\alpha \sin \theta/g + Wh\omega^2 \cos \theta/g = Y - W, \quad . \quad . \quad (2)$$

$$I\alpha/g - Wh \sin \theta = 0, \quad . \quad . \quad . \quad . \quad (3)$$

the equations of motion.

The third equation gives the angular acceleration

$$\begin{aligned} \alpha &= Wgh \sin \theta/I \\ &= g \sin \theta/l, \end{aligned}$$

if I/Wh is denoted by l.

But the linear acceleration a at a distance l from C is $l\alpha$. Hence

$$\begin{aligned} a &= l\alpha \\ &= g \sin \theta. \end{aligned}$$

Now (Art. 115) this is the equation of motion of a simple pendulum oscillating under the action of gravity.

Hence the angular motion of the body about the axis is the same as that of a simple pendulum under the same initial circumstances and whose length is l. This length l or I/Wh is called the length of the simple equivalent pendulum, and the oscillating body is called a **compound pendulum.**

(*a*) *Time of Swing.*—The time t of an oscillation (single swing) will be given by (Art. 115)

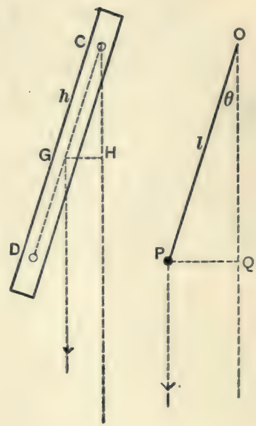
$$\begin{aligned} t &= \pi \sqrt{l/g} \\ &= \pi \sqrt{I/Whg}, \end{aligned}$$

or, as it may be written,

$$t = \pi \sqrt{(\bar{k}^2 + h^2)/hg},$$

since $I = W(\bar{k}^2 + h^2)$, the first term being the moment of inertia about C, and h the distance CG.

(*b*) *Principle of Reversion.*—A point D at a distance l from C, the point of suspension, is called the *centre of oscillation,* for the reason that the time of oscillation of the whole pendulum is the same as that of a simple pendulum of length l and swinging about C. Denote the distance DG by h_1, so that $h + h_1 = l$.



Suppose now the pendulum inverted, and suspended from D instead of from C. Now

$$CD = OP$$
$$= (\bar{k}^2 + h^2)/h$$
$$= h + \bar{k}^2/h$$

and $$CG = h.$$

$$\therefore \quad GD = \bar{k}^2/h.$$

Hence $CG \, . \, GD = \bar{k}^2$.

This may be written

$$DG \, . \, GC = \bar{k}^2;$$

which shows that if D is made the center of suspension, C will become the centre of oscillation, as was first pointed out by Huygens.

Hence the points of suspension and of oscillation can be interchanged without changing the time of oscillation, and appropriately therefore a pendulum with points of suspension situated as C, D is known as a *reversion pendulum.*

(*c*) *Determination of g.*—The pendulum furnishes one of the most accurate methods of determining g, the acceleration due to gravity at the earth's surface. We have for the time of an oscillation

$$t = \pi \sqrt{l/g};$$

whence $$g = \pi^2 l/t^2.$$

If now t be observed, and l the length of the simple equivalent pendulum, can be found, we may at once compute g.

Borda in 1792 constructed a pendulum as like a simple pendulum as possible, of a platinum sphere suspended by a very fine wire. Here, if r is the radius of the sphere

$$\bar{k}^2 = 2r^2/5,$$

and hence $$g = \pi^2(h + 2r^2/5h)/t^2.$$

The time of swing t was observed, and the distance h from the point of suspension to the center of the sphere measured. Hence g is found.

The first pendulum constructed on the Huygens' principle of reversion was Kater's in 1818. Other forms by Repsold, Mendenhall, etc., with methods of finding l and t, are described in books of laboratory physics.

(*d*) *Pressure on the Axis of Support.*—Returning to the equation of motion, we note that there are three equations but four unknowns, X, Y, ω, α. A fourth relation is afforded by the pendulum motion (Art. 255),

$$I\omega^2/2g = Wh(\cos\theta - \cos\beta).$$

Hence X, Y may be computed.

For example, the pressure of a large bell when swinging.

Ex. 1. A rod of length l is suspended at one end and caused to oscillate. Find the length of the equivalent simple pendulum. *Ans.* $2l/3$.

2. A thin circular ring, diameter d, and a small ball, suspended by a fine thread of length d, are caused to oscillate about a horizontal axis perpendicular to the plane of the ring. Show that the two will oscillate together.

3. A circular disk 16 in diameter makes small oscillations about a horizontal tangent. Find the length of the equivalent simple pendulum. *Ans.* 10 in.

4. A sphere 20 in diameter makes small oscillations about a horizontal tangent. Find the depth of the center of oscillation below the axis. *Ans.* 14 in.

5. A rod 1 ft long is suspended from a point 3 in from one end. Find the time of a small oscillation.

Ans. $\pi\sqrt{7/12g}$ sec.

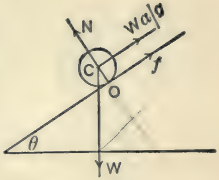
6. In a compound pendulum the time of oscillation is a minimum when $h = \bar{k}$.

7. A sphere of radius r oscillates about a horizontal tangent. Find the length of the equivalent simple pendulum.
Ans. $7r/5$.

8. A cube of edge a oscillates about an edge. Find the length of the equivalent simple pendulum. *Ans.* $2\sqrt{2}a/3$.

9. A st. rod of length r slips into a hemispherical bowl of radius r. Show that the rod will oscillate in the same time as a pendulum of length $5r/3\sqrt{3}$.

10. A cylindrical disc of radius r and weighing W lb rolls without sliding down an inclined plane from rest under the action of gravity. Determine the motion.



The external forces are W pounds at C vertically downwards, the reaction N normal to the plane, and the friction f up the plane. Unless there were friction the disc would slide.

The point of contact O is the instantaneous center of rotation. Then, a being the linear acceleration of the center of gravity C, the resultant effective force along the plane is Wa/g.

Resolving along the plane, normal to the plane and taking moments about C, the three equations of motion are

$$Wa/g = W \sin \theta - f; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$N = W \cos \theta; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$fr = I\alpha/g. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also
$$I = Wr^2/2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and since the disc rolls without sliding,

$$a = r\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence we find

$$a = 2g \sin \theta/3; \qquad \alpha = 2g \sin \theta/3r.$$

The velocity v at any distance s from the starting-point is given by

$$\begin{aligned} v^2 &= 2as \\ &= 2s \times 2g \sin \theta/3 \\ &= 4gh/3, \end{aligned}$$

if h is the height of the starting-point above the point in question.

$$\text{The force of friction } f = I\alpha/gr$$
$$= W \sin \theta/3.$$
$$\text{The reaction } N \qquad = W \cos \theta.$$

The angular acceleration α may be found at once by considering the disc as rotating for the instant about the instantaneous center of rotation O.

Taking moments about O,

$$Wr \sin \theta = I\alpha/g = 3\,Wr^2\alpha/2g,$$

$$\text{and} \qquad \alpha = 2g \sin \theta/3r, \quad \text{as before.}$$

If both rolling and sliding occur we cannot write $a = r\alpha$, but have the relation $f = \mu N$ instead, μ being the coefficient of friction.

If the body slide without friction, $a = g \sin \theta$.

11. A spherical shot rolls down a plane 70 ft long and inclined at 30° to the horizon. Find its velocity at the bottom. *Ans.* 40 ft/sec.

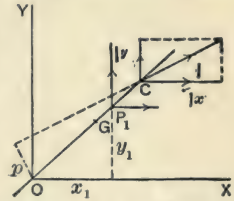
12. A sphere will roll and not slide down an inclined plane if the coefficient of friction is greater than 2 tan $\alpha/7$ where α is the inclination of the plane.

13. A ball is rolled up a 1% incline with an initial velocity of 4 ft/sec. How far will it run? *Ans.* 37.5 ft.

14. Show that an empty keg will roll down an incline slower than the keg filled solid—as with nails or sand.

252. Initial Motion Due to Impulse.—The impulse given to a particle at rest is measured by the momentum of the particle. Reasoning as in Art. 241, if an impulse be given to a system the internal actions and reactions of the particles will on the whole balance, and the external impulse will therefore be measured by the momentum of the system. Considering, then, the system in equilibrium under the external impulse and the momentum reversed, the conditions of Art. 140 may be applied.

In general, instantaneous motion of a body may be represented by a rotation about the instantaneous axis or by a translation and a rotation about a fixed axis O of the body. Let the impulse I communicated to the rod OGC be resolved into components I_x, I_y parallel to rectangular axes OX, OY.



Let P_1 be a particle of the rod weighing w_1; x_1, y_1 the coordinates of P_1; and r_1 the distance of P_1 from O. Let ω denote the angular velocity of the system of particles composing the body.

The initial velocity of P_1 is $\omega r_1 \perp$ to OP_1 (Art. 229). Its components along OX, OY are $= -\omega y_1$ and $= \omega x_1$ respectively. The momenta of the particle in these directions are $-w_1\omega y_1/g$ and $w_1\omega x_1/g$.

The first and second conditions of equilibrium give

$$I_x = -w_1\omega y_1/g - w_2\omega y_2/g - \dots$$
$$= -W\omega\bar{y}/g. \qquad (1)$$
$$I_y = W\omega\bar{x}/g \qquad (2)$$

where $\bar{x}$, $\bar{y}$ are the coordinates of the C.G. of the body.

The resultant impulse is therefore $W\omega\bar{r}/g$ where $\bar{r} = OG$ and is $\perp$ to OG.

Hence the motion is the same as if the whole weight were collected at the C.G. and to rotate about the point O

For the third condition of equilibrium take moments about the axis O; then, if p is the perpendicular from O in the direction of I,

$$Ip = w_1\omega y_1^2/g + w_1\omega x_1^2/g + \dots$$
$$= w_1\omega r_1^2/g + w_2\omega r_2^2/g + \dots$$
$$= \omega\Sigma(wr^2)/g$$
$$= I\omega/g \qquad (3)$$

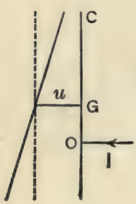
when I is the moment of inertia about O.

Hence

angular velocity $\omega =$ *moment of impulse/moment of inertia.*

As in the case of continuous forces, it is, in general, most convenient to take the C.G. as origin.

Ex. 1. If a rod CGO at rest and free to move receives an impulse ɪ at O and perpendicular to OG, the line joining O to the C.G. of the rod, it is required to determine the motion.



Let $W =$ the weight of the rod, $u =$ the linear velocity of G, $\omega =$ the angular velocity about G, and $h =$ the distance GO.

The equations of motion are:

$$\text{ɪ} = Wu/g; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{ɪ}h = I\omega/g$$
$$= W\bar{k}^2\omega/g. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence u and ω are found.

The velocity of a point C on $GO = u +$ velocity about G
$$= u - \omega \times CG.$$

If the rod begins to move about C,

$$0 = u - \omega \times CG,$$

and $$CG = u/\omega$$
$$= \bar{k}^2/GO;$$

or $$CG.GO = \bar{k}^2.$$

The point O is named the *center of percussion*, and the corresponding axis C the *axis of spontaneous rotation*.

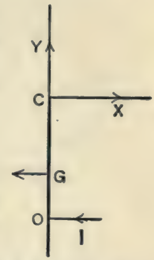
If l is the length of the rod, then $\bar{k}^2 = l^2/12$ and the center of percussion O is at a distance $2l/3$ from C.

"A knowledge of the center of percussion, gained instinctively or otherwise, enables the workman to wield his tools with increased power, and gives greater force to the cut of the swordsman, so that with some physical strength he may perform the feat of cutting a sheep in half or severing, *à la* Richard the Lion-hearted, an iron bar."

2. A rod of length l and free to rotate about a fixed axis C receives an impulse I perpendicular to the rod at O, distant c from C. Find the impulse on the axis.

Ans. $\mathrm{I}\{1 - ch/(h^2 + \bar{k}^2)\}$ where $h = CG$.

Hence deduce example 1.



3. A rod is suspended by two vertical threads at its ends. One thread is suddenly cut. Show that the initial pull in the other is halved.

4. A rod AB is swinging about the end A, and when horizontal the end B is fixed. Show that the jerks at the ends are as 1 to 2.

5. At what point must a rod AB a yard long be struck perpendicularly that one end A may be initially at rest?

Ans. 2 ft from A.

253. Energy of Motion.—The kinetic energy of a body in motion may be found *for any instant* by considering the motion as given by an angular velocity about the axis of motion, fixed or instantaneous.

Let ω radians/sec be the angular velocity about the axis and r ft the distance of any particle w lb from the axis. Then, if v ft/sec denote the linear velocity of the particle at the given instant,

$$\begin{aligned}\text{energy of the particle} &= wv^2/2g \\ &= w\omega^2r^2/2g \text{ foot-pounds.}\end{aligned}$$

Hence, by summing up the energies of all the particles and noting that ω is the same for each particle, we have

$$\begin{aligned}\text{energy of the body} &= \Sigma w\omega^2r^2/2g \\ &= \omega^2\Sigma wr^2/2g \\ &= I\omega^2/2g \text{ foot-pounds}\end{aligned}$$

where I is the moment of inertia about the axis of motion.

Take, for example, a railroad-car while in motion. The wheel may be considered a disk of radius r ft. If its weight is

W lb, the moment of inertia about the point of contact with the rail [the instantaneous center] is $3\,Wr^2/2$.

Also, if the speed of the car, and therefore of the center of the wheel, is u ft/sec, and ω radians/sec the angular velocity of the wheel, then $u = \omega r$.

Hence

$$\begin{aligned}\text{energy of wheel} &= I\omega^2/2g\\ &= 3\,Wr^2\omega^2/4g\\ &= 3\,Wu^2/4g \text{ foot-pounds.}\end{aligned}$$

254. In finding the energy of motion it is in many cases more convenient to refer to the center of gravity than to the center of rotation.

Let p denote the distance between the center of gravity and the axis of rotation, and $\bar{I}$ the moment of inertia about the parallel axis through the center of gravity. Then (Art. 246)

$$I = \bar{I} + Wp^2,$$

and therefore

$$I\omega^2/2g = \bar{I}\omega^2/2g + Wp^2\omega^2/2g.$$

But $p\omega =$ the linear velocity of the center of gravity. Call it u. Then

$$\text{energy of motion} = \bar{I}\omega^2/2g + Wu^2/2g.$$

Hence *the energy of motion is equal to the energy of rotation about the center of gravity and the energy of translation of the body moving with the linear velocity of the center of gravity.*

Thus in the preceding example $\bar{I} = Wr^2/2$, and

$$\begin{aligned}\text{energy of motion} &= \tfrac{1}{2}\,Wr^2\omega^2/2g + Wu^2/2g\\ &= 3\,Wu^2/4g,\end{aligned}$$

since $u = \omega r$, as found before.

Ex. 1. A circular disk (as a coin) rolls on its edge in a vertical plane. Compare the rotation energy with the total energy of motion. *Ans.* 1/3.

2. A hoop rolls in a vertical plane. Show that the energy of rotation is 1/2 the total kinetic energy.

3. A loaded car weighs 40,000 lb, the eight wheels 4000 lb, and the speed is 30 miles an hour. Find the kinetic energy stored. *Ans.* 1,391,500 foot-pounds.

4. If I is the moment of inertia of a fly-wheel and n the number of revolutions per second, then

$$\text{energy of rotation} = 5n^2I/8.$$

5. If the weight of the wheel in (4) is W lb and the greater part is contained in a ring whose mean diameter is d ft, then

$$\text{energy of rotation} = 5d^2n^2W/32 \text{ foot-pounds},$$

a working rule.

6. A fly-wheel weighs 15 tons, and its diameter is 20 ft; the wheel makes 60 revolutions per minute. Find the energy stored. *Ans.* 1,875,000 ft-pounds.

7. The axle of the wheel is 14 in in diameter. If the wheel is disconnected, how many revolutions will it make before coming to rest, the coefficient of friction being 0.8? *Ans.* 21.31.

8. In order to control an engine against its own variations and for external work it is necessary to call upon the fly-wheel for 60,000 foot-pounds, and at the same time a change of velocity from 160 to 140 revolutions per minute is allowable. The wheel is to be 10 ft in diameter. Compute its weight. *Ans.* 1.2 tons.

9. Find the energy of rotation of the earth, considering it a uniform sphere of density 5.6 and of diameter 8000 miles. *Ans.* $10^{30}/5$ foot-pounds.

10. The energy of a sphere rolling without sliding along a plane is to that of an equal sphere sliding without rolling and with the same velocity of the center of gravity as 7 to 5.

$$[\text{Energy of rolling} = I\omega^2/2g$$
$$= 7Wr^2\omega^2/10g,$$

the instantaneous axis being tangent to sphere and plane.

$$\text{Energy of sliding} = Wv^2/2g.$$

Also, $v = \omega r$.

Hence the result.]

11. The kinetic energy acquired by a sphere in moving from rest down a smooth plane is to that acquired by an equal sphere rolling without sliding down a rough plane of the same inclination and length as 7 to 5.

12. The weight of a fly-wheel is W lb, and its diameter d_1 inches. If it is making n revolutions per minute, find in how many revolutions it will be stopped by the friction of the axle if its diameter is d_2 inches and the coefficient of friction μ.

Ans. $\pi d_1^2 n^2/86400 \mu d_2 g$.

13. Examine the following statement: "Every engineer knows that a thing so balanced as to stand in any position is not necessarily balanced for running; that a 4-lb weight at 3 in from the axis of rotation though balanced statically by a 1-lb weight at 12 in from the axis is not balanced by it dynamically. On the contrary, a 4-lb weight at 5 in is balanced by a 1-lb weight at 10 in from the axis."

255. Equation of Energy.—When forces act on a system of particles, the change of kinetic energy is equal to the work done by the forces. But in the case where the particles are rigidly connected the internal forces, being equal and opposite, do no work on the system as a whole. Hence the change in kinetic energy is due to the external forces only, and is equal to the work done by them.

If, then, F is the resultant external force and s the displacement of its point of application, the equation of energy is

$$I\omega^2/2g = Fs. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the force is not constant, then at any instant we have

$$d(I\omega^2/2g) = Fds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

as the expression of the equation of energy.

Expanding (2),

$$I\omega d\omega/g = Fds = Fpd\theta$$

when $d\theta$ is the angle of rotation and p is the distance of F from the axis. That is,

$$I\alpha/g = Fp, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

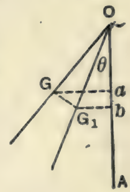
the same equation as found in Art. 241 by D'Alembert's principle.

Compare with the various forms of statement of Newton's second law in Art. 67.

Ex. 1. To find the time of oscillation of a compound pendulum (Art. 251).

Let O be the point of suspension, OA the vertical through O, G the position of the center of gravity when the pendulum is at the end of its swing, G_1 the position of G when $\angle G_1OA = \theta$, and ω the angular velocity there.

Denote $\angle GOA$ by β and let $OG = h$.



The work done by gravity as G moves to G_1

$$= W \times ab$$
$$= Wh(\cos\theta - \cos\beta).$$

The kinetic energy acquired $= I\omega^2/2g$.

Hence $$I\omega^2/2g = Wh(\cos\theta - \cos\beta),$$

and $$\omega^2 = 2Wgh(\cos\theta - \cos\beta)/I, \quad . \; . \quad (1)$$

which gives the velocity in any position.

For a simple pendulum of length l and weight of bob W we have, since $I = Wl^2$,

$$Wl^2\omega^2/2g = Wl(\cos\theta - \cos\beta),$$

or $$\omega^2 = 2g(\cos\theta - \cos\beta)/l. \quad . \; . \; . \; . \quad (2)$$

Comparing the two equations (1) and (2), we see that if a simple pendulum of length I/Wh be set oscillating simultaneously with the compound pendulum, the two will have the same angular velocity for the same angle θ, and therefore the same time of oscillation. Hence the time of oscillation, being that of the simple pendulum, is given by (Art. 115)

$$t = \pi\sqrt{l/g} = \pi\sqrt{I/Wgh},$$

as found before (Art. 251).

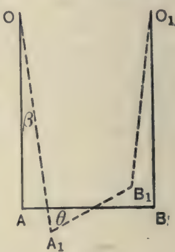
2. Find the initial angular velocity of a simple pendulum of length l that it may swing through a given angle β.

Ans. $\omega = 2\sqrt{g/l}\,\sin \frac{1}{2}\beta$.

3. Find the initial angular velocity of a pendulum 39 in long so that it shall swing through 30°.

Ans. $\omega = 1.62$ radians/sec.

4. A horizontal rod of length $2a$ hangs by two parallel strings, each of length h, attached to its ends. If it be twisted horizontally through a small angle and allowed to oscillate, it is required to find the time of an oscillation.



This is the problem of *bifilar suspension*, and is of great importance in physical and electrical investigations.

Let the angle of displacement of $AB = \theta$, and let the angle of displacement of the thread from the vertical due to moving $AB = \beta$. Then

$$a\theta = \text{arc } AA_1 = h\beta. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The height to which AB is raised in twisting through angle θ is $h - h\cos\beta$.

The work done in raising $AB = Wh(1 - \cos\beta)$.

Hence
$$\begin{aligned} I\omega^2/2g &= Wh(1-\cos\beta) \\ &= Wh\beta^2/2 \quad (\text{if } \beta \text{ is small}) \\ &= Wha^2\theta^2/2h^2, \quad \text{from (1)}, \end{aligned}$$

or
$$\omega^2 = Wga^2\theta^2/Ih.$$

For a simple pendulum of length l and weight of bob W we have, if the angle of displacement is θ (Ex. 1, p. 313),

$$\begin{aligned} \omega^2 &= 2g(1-\cos\theta)/l \\ &= g\theta^2/l. \end{aligned}$$

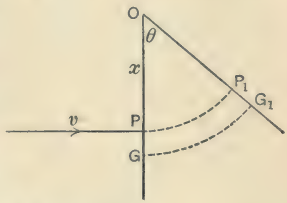
Hence

length of simple equivalent pendulum $l = Ih/Wa^2$,

and time of oscillation
$$\begin{aligned} &= \pi\sqrt{l/g} \\ &= \pi\sqrt{Ih/Wa^2g}. \end{aligned}$$

Conversely, if the time of an oscillation be noted, the moment of inertia of the bar may be computed from this relation. This indeed is the main problem of bifilar suspension.

5. The bob of a pendulum consists of a box filled with sand. The pendulum is deflected from rest through an observed angle θ by a shot fired horizontally and striking the box at a point P when $OP = x$. If the pendulum weighs W lb, the shot w lb, and the distance $OG = h$ where G is the center of gravity, it is required to find the velocity v with which the shot strikes in terms of these observed quantities.



Ans. $v = (W + w)h\sqrt{2gh(1 - \cos\theta)}/wx$ ft/sec.

This instrument, called the *ballistic pendulum*, introduced by Benjamin Robins in 1742, and at one time largely used for finding the velocities of cannon-balls and rifle-bullets, is now almost entirely discarded for more accurate methods depending upon electric contacts.

5*a*. If there is no impulse on the axis, the shot must strike at a point P such that $OP = I/Wh$.

[P is the center of percussion. (See Ex. 1, p. 308.)]

6. A disk of radius r and weight W rolls down an inclined plane of height h under the action of gravity. To find the velocity at the bottom.

Let u = velocity of center of gravity down the plane;
ω = the angular velocity.

Then

kinetic energy acquired = work done by gravity,

or $$\bar{I}\omega^2/2g + Wu^2/2g = Wh.$$

But $$\bar{I} = Wr^2/2, \qquad u = r\omega.$$

$$\therefore \quad u^2 = 4gh/3, \quad \text{as found before (p. 306)}$$

This may be solved at once by referring to the instantaneous center of rotation instead of to the C.G. of the disk. In this case

$$I\omega^2/2g = Wh,$$

and $$I = 3Wr^2/2.$$

$$\therefore \quad u^2 = 4gh/3, \quad \text{as above.}$$

7. A solid sphere and a solid cylinder of equal radii roll from rest down the same inclined plane. Compare their times of descent. *Ans.* $\sqrt{14}/\sqrt{15}$.

8. A circular disk of radius r rolls down an equal fixed disk under the action of gravity. Show that the velocity acquired when the line joining their centers makes an angle θ with the vertical is found from

$$3v^2 = 8gr(1 - \cos\theta).$$

9. Where should a stop be placed behind a thin vertical door, 6 feet high and 2.5 feet wide?

Ans. 10 in from outer edge and 3 ft from top.

If the stop is placed on the floor (as with a railroad-car door), account for the "twist."

10. If the diameter of Sisyphus' spherical stone be 2 ft, which he continually rolls up the surface of a semi-globular mountain half a mile high, what vertical distance will the stone have rolled down under the force of gravity when it leaves the mountain? *Ans.* 1088 ft.

How far from the foot of the mountain does the stone fall?

EXAMINATION.

1. When has a moving body a motion of translation?

[When all points describe equal and similar paths.]

2. When has a moving body a motion of rotation?

[When it has two points fixed.]

3. The motion of a particle in a plane may be represented by a radial velocity dr/dt and a transverse velocity $rd\theta/dt$.

4. It is always possible to represent the plane motion of a body at any instant by a motion of rotation about a certain point.

5. Define the instantaneous center of rotation.

6. Define angular velocity and explain how it is measured.

7. Find the angular velocity of the extremity of the minute-hand of a clock. *Ans.* $\pi/1800$ rad/sec.

8. A body makes 30 turns per minute. Show that its average angular velocity is π.

9. The angular velocity of a body in latitude $\lambda = \pi/12$ radians/hour.

10. Show that 1 rev/min = 0.10472 radians/sec.

11. A particle starts from rest and moves in a circle with a uniform acceleration of 8 radians/sec^2. Find the time of completing the first revolution. *Ans.* $\sqrt{\pi/2}$ sec.

12. If a particle weighs w lb, the center-seeking force required to keep it moving with uniform angular velocity ω in a circle of radius r is $wr\omega^2/g$ pounds.

13. Show how to combine two angular velocities about parallel axes.

14. Give the unit of angular acceleration.

15. A Foucault pendulum is set up at the north pole. Find the time of a complete revolution. *Ans.* 1 day.

If set up at the equator, how then?

16. If a disk rolls on a plane, the velocity of translation of its center is equal to the product of its angular velocity about its center and its radius.

17. Show that the activity or power of a rotating armature is equal to torque $\times$ angular velocity.

18. Define the terms moment of inertia and radius of gyration.

19. State two propositions which abbreviate computations of moments of inertia.

20. The moment of inertia of a lamina about a central axis perpendicular to its plane is

$$\text{weight} \times \text{sum of sqs of} \perp \text{semi-axes}/3, \text{ or } 4,$$

according as the lamina is a rectangle or circle.

State the corresponding proposition for the sphere.

21. Show that the moment of inertia of a thin hollow cylinder, 2 ft diameter and weight 1 lb, about the axis of the cylinder is the unit moment of inertia.

22. The kinetical behavior of a body cannot be determined until we know the value of the radius of gyration of the body.

23. State D'Alembert's principle and give an example of its application.

24. Show how to find the angular acceleration of a body about a fixed axis when the body is acted upon by an external force.

25. Prove moment of impulse = moment of momentum.

26*a*. A circular cylinder with its axis horizontal rolls down an inclined plane. Show that one third of the acceleration of gravity is used in turning the cylinder.

26*b*. If the inclination of the plane is 30°, the distance passed over by the cylinder in 6 seconds is $6g$.

27. A sphere rolls down a plane of inclination θ. Show that the motion of the center parallel to the plane is that of a particle moving with a uniform acceleration $5g \sin \theta/7$.

28. Show that change of momentum about a fixed axis is equal to the moment of the impressed forces about the axis.

29. When is a machine said to be *balanced?*

[When the relative movements of its parts do not tend to make it vibrate as a whole.]

30. Apply the principle of the conservation of energy to find the time of swing of a compound pendulum.

31. What is meant by the center of oscillation? center of percussion?

32. Find experimentally the center of percussion in a baseball-bat.

33. Where is the centre of percussion in a hammer?

34. A foot rule is held lightly at one end between finger and thumb so as to hang vertically. If struck 4 inches from the lower end, there is no pull on the fingers. If struck at 3 or 6 inches, what happens?

35. Why is the lighter end of a baseball-bat held in the hand?

36. A semicircular wire, length l, is bent at its middle point into two quadrants having a common tangent. It oscillates about an axis through this middle point. Find the length of the simple equivalent pendulum.

Ans. $l(\pi - 2)/\pi$.

37. In ringing a bell the swing is through an angle θ. Compute the pressure on the supports.

38. Explain how to determine the moment of inertia of a bar by using bifilar suspension.

39. An armor-plate suspended by chains is struck normally by a shot at a point P vertically below the C.G. of the plate. Find the axis about which the plate revolves.

40. A fly-wheel of a tons weight and b ft diameter makes c revolutions per minute. Find the energy accumulated.

Ans. $0.087ab^2c^2$ ft-pounds, nearly.

41. A fly-wheel is not perfectly centered. How would you compute the centrifugal force?

42. "The outside diameter of an engine fly-wheel is 80 in, width of face 26 in, average thickness of rim 5 in, revolutions per minute 175 Show that the centrifugal force of the rim is 260,136.35 pounds." (Exam. paper.)

[If the wheel is properly balanced, is not the centrifugal force *nil?*]

43. The work stored in a fly-wheel is quadrupled if the angular velocity is doubled.

44. "There is more energy stored in a ton of car-wheel than in a ton of car-body." (*R. R. Gazette,* 1892.)

How much more when the speed of the train is 30 miles/hour?

CHAPTER VIII.

ELASTIC SOLIDS.

256. Observation and experiment show that forces acting on a body may change its motion, its form, or its size. As most simple, we have considered first of all changes of motion only. The action of the external forces was conceived to be resisted by the internal reactions of the particles on one another in such a way that the particles retained their original distances from one another, so that changes of form or of size did not take place. The conditions of equilibrium and of change of motion on this hypothesis have been developed in the preceding chapters.

Experience shows that no perfectly rigid body exists in nature; the body yields to the external forces, and the internal reactions do not prevent changes of form or of size. If the body returns towards its original configuration on the removal of the forces it is said to be *elastic*.

257. Conceive an elastic body under the action of forces to be cut by a plane. The particles of the body act and react on one another across the plane. The actions distributed over one side of the plane may be regarded as combined into one force whose place of application is the plane itself. The action and reaction form a **stress**, though the term stress is frequently used for either component.

The *unit stress* is unit force per unit area; as, for example, 1 pound per square inch, written 1 pound/in^2. The average stress over a surface (intensity of stress) is found by dividing the total stress on the surface by the area of the surface.

Ex. A wire of 0.1 in diameter is stretched by a 5 lb weight. Find the stress across any section, neglecting the weight of the wire. *Ans.* $2000/\pi$ pounds/in².

258. When an elastic solid is subjected to the action of a stress it suffers distortion. Any change of form or of size is known as a **strain.*** Remove the stress and the solid returns to its original dimensions. This is observed to be true for stresses up to a certain amount. When that amount is exceeded the solid will not return to its original form on removing the stress, but will assume another form between the two, or a *permanent set,* as it is called. Increase the stress, and the solid will finally be ruptured. The limit of unit stress up to which a body of unit cross-section may be subjected without producing permanent sets is called the **elastic limit** of the material in question.

Until this limit is reached experiment shows that *strain is proportional to the stress producing it.* This is known as **Hooke's law,** having been stated by Hooke in 1678 under the form *Ut tensio sic vis.* It is analogous to Newton's second law of motion in that it connects stress and strain as the second law connects force and motion.

259. The ratio of a stress to the strain which is produced in any body is called a **modulus of elasticity** for that body.

(1) If the body is acted on equally in all directions by stresses normal to its surface, the size is changed but not the form, and the strain is a *compression* or an *extension,* the stress being compressive or tensile. Thus if V denotes the volume when in the natural state, v the change of volume [dilatation], and p the unit stress, then v/V is the unit strain, and the ratio unit stress/unit strain is the *modulus of elasticity of volume.*

* Rankine in 1850 introduced the term strain to denote the definite change in the size or shape of a body produced by a stress. The term includes distortion, deflection, elongation, etc.

But the innovation has not been universally adopted. Many writers continue to use strain in the sense of force and as synonymous with stress. For example, architects and engineers use the term "strain-sheet" rather than "stress-sheet" for a diagram of stresses in structures.

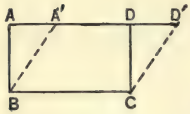
It is usually denoted by the letter k, so that

$$k = p \div v/V = pV/v.$$

It is evident that k is of the same name as unit stress; for unit strain being the ratio of change of volume to original volume is an abstract number. In the British system of units k is usually expressed in pounds/in². For steel,

$$k = 28 \times 10^6 \text{ pounds/in}^2.$$

(2) If the external stress produces change of form but not of size, it is called a shearing stress or **shear** and the distortion a shearing strain.



(*a*) Suppose the stress tangential.

Let $ABCD$ be a rectangular block on the base BC. Conceive it built of infinitely thin sheets placed on BC. If BC is fixed and the sheets are moved in their own planes parallel to BC, the block is displaced into the position $A'BCD'$ say, and has undergone a shear.

The unit strain is the difference of displacement of any two planes divided by the distance between them. Thus

$$\text{unit strain} = AA'/AB.$$

The shearing stress is parallel to BC. If p represent the stress on unit area, then the ratio

$$\begin{aligned}\text{unit stress/unit strain} &= p \div AA'/AB.\\ &= p \times AB/AA'\end{aligned}$$

is the *modulus of elasticity of form*, or the *rigidity*.

It is denoted by the letter n, so that n, like k, is expressed in pounds/in². For steel, $n = 12 \times 10^6$ pounds/in².

(*b*) Suppose the stress to twist the body about an axis. A common example occurs in machine shafting, in which the bodies twisted are circular cylinders. A similar case occurs in many delicate physical instruments in which wires undergo twist.

Consider a cylinder fastened at one end and acted on by external forces. If the resultant of these forces is a couple in a plane perpendicular to the axis, it will cause a twist about the axis. This is balanced by the molecular reaction of the particles of the body, giving rise to an equal and opposite couple in a parallel plane. We therefore say that a cylinder subject to two equal and opposite couples in planes perpendicular to the axis is in a state of **torsion.**

If we suppose the cylinder built up of circular plates, we may conceive them to slide upon one another in the twisting, just as the leaves of a thick book when one cover is held firmly and the other is pulled sideways.

Let the cylinder when deflected from its position of rest by a torque T applied perpendicular to its length come to rest when the angle of torsion is θ. Then it was shown by Coulomb experimentally that

$$T = b\theta$$

where b is a constant, called the *constant of torsion.*

260. Consider a uniform wire, radius r, clamped at one end and carrying a weight attached to the free end. Let the wire be deflected through an angle θ from its position of rest by the torque T. Then (Arts. 241, 259)

$$I\alpha = T = b\theta$$

where α is the angular acceleration and I the moment of inertia of the wire about the axis.

Hence α varies as θ and αr varies as θr, or the acceleration of any point varies as the displacement of the point from the position of no torsion. The motion of each point is therefore a S.H.M., and the wire will oscillate about the position of no torsion. The apparatus forms a *torsion pendulum.*

Hence the time of an oscillation is given by (Art. 115)

$$t = \pi \sqrt{\theta r / \alpha r}$$
$$= \pi \sqrt{I/b}.$$

261. The torsion pendulum may be used to determine the constant of torsion. For let a weight of known moment of inertia be suspended and the time of oscillation observed. Then b may be computed.

Conversely, when b is known, the moment of inertia of any suspended body may be computed. (See Ex. 7, p. 336.)

In many physical investigations the instruments employed involve the use of apparatus suspended by a fiber and caused to oscillate. Thus the Coulomb torsion balance is used for the measurement of small forces and the Cavendish apparatus for measuring the density of the earth.

In his classical researches (1894) on the constant of gravitation, Prof. C. V. Boys found fibers of quartz to possess the properties demanded in a suspension fiber to a greater degree than any material hitherto used—metallic wires or silk fibers, for example.

262. Let an elastic body be subjected to longitudinal compression or extension. Let l denote the original length, λ the change of length, P the stress producing the change, and A the cross-section. Then

$$\text{longitudinal stress} = P/A,$$
$$\text{longitudinal strain} = \lambda/l,$$

and the ratio

$$\text{long. stress/long. strain} = P/A \div \lambda/l$$

is called **Young's modulus of elasticity.**

It is denoted by the letter E, so that

$$E = Pl/A\lambda,$$

and is expressed in pounds/in². For steel $E = 30 \times 10^6$ pounds/in².

The work done by the unit stress in causing the strain λ is equal to the unit stress multiplied by the average strain $\lambda/2$, or

$$W = P\lambda/2.$$

The work which a body can do in returning to its original dimensions after it has been strained up to the elastic limit is the *work of resilience.*

Ex. 1. A rod 0.1 in² cross-section and 10 ft long is suspended from one end. A weight of 1 ton is hung from the lower end and the wire increases in length 0.1 in. Find the unit stress. *Ans.* 20,000 pounds/in².

Also show that the modulus of elasticity = 24,000,000 pounds/in².

2. Can a steel rod 1/2 in × 1/2 in safely carry a load of 4 tons? the elastic limit of steel being 50,000 pounds/in²?

Ans. Yes; the stress is within the elastic limit.

3. Show that Young's modulus may be defined as a stress which would double the length of a bar of unit cross-section, the bar remaining within the elastic limit.

4. A steel rod 50 ft long and 2 in² cross-section is stretched 1/25 in by a weight of 2 tons. Compute Young's modulus for steel. *Ans.* 30×10^6 pounds/in².

5. A steel bar 3 ft long and 2 in × 1 in section was subjected to a tensile stress of 60 tons. The elongation was 0.05 in. Required the work of resilience.

Ans. 3000 inch-pounds.

6. A wrought-iron rod 50 ft long and 2 in² cross-section is subjected to a pull of 25 tons. Taking the modulus of elasticity = 30×10^6 pounds/in², find the elongation.

Ans. 0.5 inch.

The further development of this subject will be found in treatises on the strength and elasticity of materials.

IMPACT.

263. In Arts. 53, 67 it was shown that the second law of motion might be algebraically expressed

$$Ft = Wv,$$

F being in absolute units; or

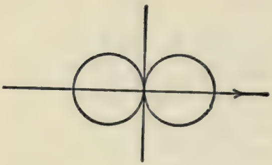
$$Ft = Wv/g,$$

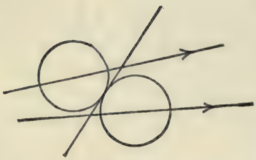
F being in gravitation units.

In words,

the impulse imparted = the momentum acquired.

264. Impact of Two Bodies.—If two bodies come into contact, a collision or *impact* is said to take place. This impact is *direct* if the bodies are moving in the direction of the common normal at the point of contact; *indirect* if they are not moving in this direction.





If the bodies are spheres, the common normal is the line joining their centers.

265. When two bodies impinge, the impulse [action] received by the one must be equal to the impulse [reaction] received by the other. This follows from the law of stress. But the impulses are measured by the momenta acquired. Hence, if by the impact the velocity u ft/sec of the body weighing W lb is changed to v ft/sec say, and the velocity u_1 ft/sec of the body weighing W_1 lb is changed to v_1 ft/sec, then

momentum lost by first $= Wv/g - Wu/g$ second-pounds;

momentum acquired by second $= W_1u_1/g - W_1v_1/g$ second-pounds.

Hence $$Wv/g - Wu/g = W_1u_1/g - W_1v_1/g,$$

or $$Wv/g + W_1v_1/g = Wu/g + W_1u_1/g; \quad . \quad . \quad (1)$$

that is, *the sum of the momenta after impact is equal to the sum of the momenta before impact.*

In order to determine the unknowns v and v_1 we must have another relation between them. Now it is found by experiment that when two bodies impinge directly the difference of velocities after impact bears a constant ratio to the difference of their velocities before impact so long as the materials of the bodies are the same, but these differences are in opposite directions. If this constant ratio, which is called the *coefficient of restitution* of the two bodies, is denoted by the letter e, we have

$$v - v_1 = - e(u - u_1) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

as the second relation between v and v_1. Solving (1) and (2), we find

$$v = \frac{Wu + W_1u_1 - eW_1(u - u_1)}{W + W_1},$$

$$v_1 = \frac{Wu + W_1u_1 + eW_1(u - u_1)}{W + W_1},$$

giving the velocities after impact.

Also, the impulse I on the body W is measured by the change of momentum produced. Hence

$$\begin{aligned} \text{I} &= W(u-v)/g \\ &= WW_1(1 + e)(u - u_1)/(W + W_1)g. \end{aligned}$$

The impulse on the other body is of course equal to this and opposite in sign.

266. The value of the coefficient of restitution e depends on the material composing the bodies. From its definition it follows that the extreme values of e are 0 and 1. If $e = 0$, or the bodies are *inelastic*, then

$$v = v_1,$$

or the bodies move together with a common velocity after impact. If $e = 1$, then

$$v - v_1 = - (u - u_1),$$

or the difference of their velocities after impact is the same as it was before impact, but in the opposite direction. In this case the bodies are *perfectly elastic.*

No examples of either perfectly inelastic or of perfectly elastic bodies occur in nature. But some bodies with very little elasticity, as clay, for example, may be regarded as belonging to the first class, and others, as glass, to the second class. For two glass balls $e = 0.94$; two ivory, $e = 0.81$; two cast iron, $e = 0.66$; two lead, $e = 0.2$.

267. The special case of *direct impact of a sphere on a smooth fixed plane* may be noticed.

The plane being fixed, we have

$$u_1 = 0, \qquad v_1 = 0.$$

But from the experimental law

$$v - v_1 = -e(u - u_1).$$

Hence there results

$$v = -eu,$$

or the velocity of recoil is reversed in direction.

The impulse on the sphere is measured by the change of momentum, and is $= W(u - v)/g = Wu(1 + e)/g$.

The impulse on the plane is equal and opposite to this.

Ex. 1. A perfectly inelastic ball impinges on a plane perpendicularly. Show that there is no recoil.

2. Two inelastic balls are brought to rest by the impact. Prove that they must have been moving in opposite directions with velocities inversely proportional to their weights.

3. Two balls of equal weight are perfectly elastic. Prove that after impact they will exchange velocities.

4. A series of equal elastic balls are placed in contact in a straight line. An equal ball impinges directly on them. Show that all will remain at rest but the last, which will fly off.

5. Find the elasticity of two balls of weights w and W in order that if W impinges on w at rest it will itself be brought to rest. *Ans.* W/w.

6. A ball falls from a height of 16 ft above a level floor.

Find the velocity of rebound and the height to which the ball will rebound if the coefficient of restitution is 0.75.

Ans. 24 ft/sec; 9 ft.

7. A ball falls from a height h above a level floor and rebounds to a height h_1. Show that

$$h_1 = he^2$$

where e is the coefficient of restitution.

For example, if the height is 64 ft and the ball hops four times, show that the height of the fourth hop is 3 in, the value of e being 1/2.

[Conversely, ball and floor being of the same material, we can by this method find the value of e experimentally.]

8. A ball falls from a height h above a level floor. Show that the whole distance described before the body ceases to rebound is $h(1 + e^2)/(1 - e^2)$ and the time taken is

$$\sqrt{2h(1 + e)^2/g(1 - e)^2}.$$

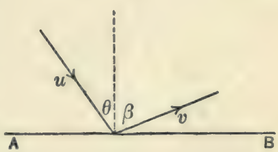
[For distance $= h + 2he^2 + 2he^4 + \ldots$ ad inf.]

9. A ball weighing 4 lb falls from a height of 9 ft on a level floor and rebounds to a height of 4 ft. Find the impulse.

Ans. 5 second-pounds.

10. A sphere impinges directly on an equal sphere at rest. Show that their velocities after impact are as 1 to 3, the coefficient of restitution being 0.5.

268. *Oblique Impact of a Sphere on a Fixed Smooth Plane AB.*—Let u be the velocity before impact, and v the velocity after impact; θ the inclination of u to the normal, and β the inclination of v.



Resolve the velocities along and normal to the plane. The plane being smooth, it exerts a normal pressure only. Hence the impact may be considered direct, with velocity $u \cos \theta$ before and $v \cos \beta$ after impact, and

$$\therefore\ v \cos \beta = -e \times -u \cos \theta = eu \cos \theta.$$

Also, since the pressure is normal, the action along the plane is unchanged by the impact, or

$$v \sin \beta = u \sin \theta.$$

Hence v and β are found.

Ex. 1. What are the values of v and β above?

Ans. $v = u\sqrt{\sin^2\theta + e^2\cos^2\theta}$; $\tan\beta = \tan\theta/e$.

2. If the elasticity be perfect, show that the angle of incidence θ is equal to the angle of reflection β and the velocity is unchanged.

3. If the impact is direct, show that $v = eu$, as already found in Art. 267.

4. Show that the impulse of the sphere on the plane is

$$W(1 + e)u \cos\theta/g \text{ sec-pounds}$$

if the sphere weighs W lb.

5. To hit a ball Q by a ball P after reflection from the edge CA of a billiard-table. "Aim at a point B as far behind the edge CA as Q is in front of it." Prove this.

6. A ball impinges on an equal ball at rest at an angle of 45° to the line of impact. Prove that if both are perfectly elastic their velocities will be equal after impact.

7. A perfectly elastic ball falls from rest for one second and strikes a smooth plane inclined at 45° to the vertical. After what interval will it again strike the plane? *Ans.* 2 sec.

8. Find the angle at which a ball must strike a plane so that its direction after impact may be at right angles to its former direction, the coefficient of restitution being 1/3.

Ans. 30° to the normal.

Show that a perfectly inelastic ball will after impact run along the plane.

9. Two balls weighing 4 lb and 8 lb are moving with equal velocities of 8 ft/sec in opposite parallel directions, and impinge at an angle of 30° with the line joining their centers. If $e = 0.5$, show that the impulse on the first ball $= \sqrt{3}$ second-pounds.

What is the impulse on the other ball?

10. A shell is fired from a mortar on the ground, and after once ricocheting rises so that at its greatest altitude it just passes over a wall. If θ be the angle subtended by the wall at the mortar, show that the angle of elevation α is given by

$$\tan\alpha = 20 \tan\theta,$$

the coefficient of restitution between ground and shell being 0.5.

269. Change of Energy by Impact.—Consider the direct impact of two bodies weighing W and W_1 lb respectively.

With our usual notation

$\mathbf{W} =$ energy before impact $= Wu^2/2g + W_1u_1^2/2g$ ft-pounds.
$\mathbf{W}_1 =$ energy after impact $= Wv^2/2g + W_1v_1^2/2g$ ft-pounds.

The change of energy produced by the impact is the difference of these two expressions. Substitute for v, v_1 their values from Art. 265, and we find after reduction

$$\mathbf{W} - \mathbf{W}_1 = (1 - e^2)\frac{WW_1}{W + W_1}(u - u_1)^2/2g.$$

When $e = 1$, or the bodies are perfectly elastic, then $\mathbf{W} = \mathbf{W}_1$, and there is no change of energy. When $e < 1$, or $e = 0$, then $\mathbf{W} > \mathbf{W}_1$, and the expression indicates a loss of energy produced by the impact.

But from the principle of the conservation of energy (Art. 217) there can be no loss of energy in the system. When energy disappears in one form it reappears in another form. In the present case the energy of impact is broken into two parts, one in producing motion of the impinging bodies, and the other, the so-called loss, in producing sound, heat, etc., and it may be in deforming the bodies.

270. Whether the change of energy produced by impact is to be regarded as a loss or not depends upon the end to be attained. If that is the propulsion of a missile or the driving of a pile, then change of form, heat, etc., are prejudicial, and the energy used in producing them is lost. If, on the other hand, change of form is the main thing, as in molding metal under a hammer or in riveting, this so-called loss becomes the useful energy, and the energy of motion useful in the former case becomes prejudicial in this.

Ex. 1. Two trains weighing 60 tons and 80 tons come into collision with velocities of 60 miles/hour and 45 miles/hour respectively. Find the energy expended in the destruction of the cars, supposing them inelastic.

Ans. 25,410,000 foot-pounds.

2. Two trains of equal weight, moving with velocities of 30 miles an hour each and in opposite directions, collide.

Show that the loss of energy produced by the impact is the same as in the case of a train moving at 60 miles an hour striking another at rest.

In the latter case find the velocity with which the débris will be moved along the track.

Also, show that before impact the total energy in the one case is double that in the other.

3. Find the loss of kinetic energy if a ball weighing 10 lb falls from a height of 16 ft and rebounds after striking the ground to a height of 4 ft. *Ans.* 120 ft-pounds.

4. A ball weighing w lb falls from a height h ft on a fixed plane. Show that the loss of energy from the impact is $(1 - e^2)wh$ ft-pounds, the coefficient of restitution being e.

271. *Applications.*—The case of impact that occurs most frequently in practice is when the bodies are inelastic and one is at rest before impact. Here $u_1 = 0$ and $e = 0$. Hence from Art. 265

$$v = v_1 = Wu/(W + W_1).$$

Consider, for example, the pile-driver.

It is assumed that the pile is inelastic, that there is no deformation of the head of the pile, the blow being instantaneous, and that the resistance of the ground is uniform.

Suppose a pile weighing W_1 lb is driven vertically s ft into the ground by a ram weighing W lb falling through a height of h ft. Let v ft/sec be the velocity of ram and pile after the blow is given. The velocity of the ram before striking being denoted by u ft/sec, we have

$$u^2 = 2gh,$$
$$v = Wu/(W + W_1).$$

The kinetic energy of ram and pile causing penetration is

$$\tfrac{1}{2}(W+W_1)v^2/g = \tfrac{1}{2}W^2u^2/(W+W_1)g$$
$$= W^2h/(W+W_1) \text{ ft-pounds.}$$

The work done by the force of gravity on ram and pile is

$$(W+W_1)s \text{ ft-pounds.}$$

If F pounds denotes the average resisting force offered by the ground, we have by the principle of work

$$Fs = W^2h/(W+W_1) + (W+W_1)s,$$

and F is found.

At the last blow, the value of s being small, the second term may be disregarded in comparison with the first, and we have

$$Fs = W^2h/(W+W_1).$$

Still more approximately, by neglecting the weight of the pile in comparison with that of the ram,

$$Fs = Wh.$$

For example, to find the ultimate load a pile weighing 500 lb could carry if the last blow from a height of 25 ft of a one-ton ram sinks the pile one inch. The three formulas give 482,500, 480,000, 600,000 pounds, respectively.

272. The kinetic energy dissipated by the blow is evidently

$$Wh - W^2h/(W+W_1) = WW_1h/(W+W_1) \text{ ft-pounds,}$$

and appears principally as sound and heat.

It is useful to notice that, the energy of the ram before impact being Wh, the loss of energy will be less the more nearly $W^2/(W+W_1)$ is equal to W, that the more nearly $W/(W+W_1)$ is equal to unity, that is, the greater W is in comparison with W_1. Hence the ram should be large in weight compared with the pile.

With the riveting-hammer, steam-hammer, etc., in which change of form is the end to be attained, the useful work done by the hammer depends on $WW_1h/(W + W_1)$, which is the more nearly equal to Wh the greater W_1 is in comparison with W; that is, the heavier the anvil is in comparison with the hammer.

This was first put in practice by Nasmyth, the inventor of the steam-hammer. "I may mention," he says, "that pile-driving had before been conducted on the cannon-ball principle. A small mass of iron was drawn slowly up and suddenly let down on the head of the pile at a high velocity. This was *destructive*, not *impulsive*, action. Sometimes the pile was shivered into splinters without driving it into the soil; in many cases the head of the pile was shattered into matches, and this in spite of the hoop of iron about it. On the contrary, I employed great mass and moderate velocity. The fall of the steam-hammer block was only 3 or 4 ft, but it went on at 80 blows the minute, and the soil into which the pile was driven never had time to grip or thrust it up."

273. In building, the *safe load* to be carried by the pile is some fractional part of the load P required to drive the pile, say one nth part.

Then $nP = F$, and

$$nPs = W^2h/(W + W_1),$$

or
$$P = W^2h/(W + W_1)ns.$$

The fraction $1/n$ is to be determined by experience. An average value is 1/4.

Ex. 1. A steam-hammer weighing 500 lb has a stroke of 3 ft. If the piston-pressure is 1000 pounds and the blow is vertical, find the work delivered in 6 blows.

Ans. 27,000 foot-pounds.

2. A pile weighing half a ton is driven 12 ft into the ground by 30 blows of a hammer weighing 2 tons falling 30 ft. Prove that it would require 120 tons in addition to the hammer to be superposed on the pile to drive it down slowly, supposing the resistance of the ground uniform.

3. Find the safe load for a pile weighing 500 lb to carry if the pile sinks 0.1 ft at the last blow under the 5-ft fall of a 500-lb ram. *Ans.* 3125 lbs.

4. A pile is driven s ft vertically into the ground by n blows of a steam-hammer fastened to the head of the pile. Given p the mean pressure of the steam in pounds/in^2, d the diameter of the piston in inches, l the length of the stroke in ft, W the weight in lb of the moving parts of the hammer, and W_1 the weight in lb of the pile and fixed parts of the hammer attached to it, and R the mean resistance of the ground in pounds, prove

$$nW(W + \pi pd^2/4)l = Rs(W + W_1).$$

5. In firing from a rifle of weight W lb a bullet of weight W_1 lb with velocity v ft/sec, show that the energy of recoil is $W_1^2v^2/2Wg$.

6*a*. Prove that the mean resistance of the wood is 204 pounds to a nail weighing 1 oz, supposing a hammer weighing 1 lb striking it with a velocity of 34 ft/sec drives the nail 1 in into a fixed block of wood.

6*b*. If the block is free to move and weighs 68 lb, prove that the hammer will drive the nail only 64/65 in.

6*c*. Prove that the nail is 0.0052 sec and 0.005128 sec in penetrating the wood in the two cases, during which the block if free will move 0.015 in.

7. A hammer weighing W lb strikes a nail weighing W_1 lb with a velocity of u ft/sec and drives it s ft into a piece of wood which is fixed in position. If the resistance of the wood is R pounds, find the time of penetration.

Ans. Wu/Rg sec.

8. If water flowing in a pipe 50 ft long, with velocity 24 ft/sec, is shut off in 0.1 sec by a stop-valve, show that the water-pressure in the pipe near the valve is increased by 162.5 pounds/in^2.

EXAMINATION.

1. Distinguish stress and strain. In what senses is the term strain used?

2. What is meant by the elastic limit? How is the working stress related to it?

3. State Hooke's law.

4. Define the modulus of elasticity of volume. Of form.

5. What is meant by Young's modulus of elasticity?

6. State Coulomb's law of torsion.

7. Explain how to determine moments of inertia by means of the torsion pendulum.

[Suspend a body whose moment of inertia (I_1) is required and note the time of oscillation (t_1). Replace it by a body whose moment of inertia (I) is known by computation or otherwise, and note the time of oscillation (t). Then

$$I_1 = It_1^2/t^2.]$$

8. A nut is placed on a table, the forefinger of the left hand placed upon a suture of the nut, and a blow given to the finger by the right fist. The finger will be uninjured, but the nut will almost certainly be cracked.

Explain the general principle involved.

9. State the experimental law on which the determination of the motion of two elastic balls after impact depends.

10. Two inelastic spheres are moving in one straight line. Find the result of a collision upon their velocities.

11. An inelastic sphere impinges on another of twice its weight at rest. Show that the impinging body loses 2/3 of its velocity by the impact.

12. A ball weighing 2 lb impinges obliquely upon an 8-lb ball at rest. The coefficient of restitution is 0.25. Show that after impact the balls move off at right angles.

13. Show that an inelastic ball will, after impact with a smooth plane, slide along the plane.

14. Two perfectly elastic spheres collide directly. Show that their kinetic energy is unaltered by the impact.

15. A body collides with an equal body at rest. Show that the energy before impact cannot exceed twice the energy after impact.

16. A block weighing 100 lb falls through 25 ft and is brought to rest in 1/10 sec. Find the average pressure exerted on the block. *Ans.* 1250 pounds.

17. A nail is driven 1/10 in into wood by a blow from a 2-lb hammer having a velocity of 16 ft/sec. Find the average pressure offered by the wood. *Ans.* 960 pounds.

18. An 80-ton gun discharges an 800-lb shot with a velocity of 1500 ft/sec. If the recoil is resisted by a constant pressure of 15 tons, how far will the gun recoil? *Ans.* 4.7 ft.

19. In driving a nail of weight w_1 lb with a hammer of weight w lb and velocity u ft/sec the energy spent in producing deformation of the head of the nail, sound, etc., is

$$\frac{ww_1}{w + w_1} u^2/2g \text{ foot-pounds,}$$

and the energy spent in driving the nail is

$$\frac{w^2}{w + w_1} u^2/2g \text{ foot-pounds.}$$

20. A shot weighing w lb is fired from a gun whose weight is W lb with velocity v ft/sec relative to the gun. Show that the actual velocities of shot and gun are $Wv(W + w)$ and $wv/(W + w)$ ft/sec respectively.

21. Prove that if a link of a chain which is coiled up loosely near the edge of a table is allowed to hang over the edge the chain will slip over with a uniform acceleration $g/3$.

22. What is the effect of a charge between light and heavy cavalry, the light cavalry having the greater energy and the heavy the greater momentum?

CHAPTER IX.

METRIC UNITS.

274. In Chapter II it was explained that there are two systems of units in use in Mechanics, the absolute system and the gravitation system. The British gravitation system, being that in most common use in this country, has been employed in this book so far. There is no system of absolute units depending upon British measures that is generally acknowledged or is likely to come into use.*

The metric system of measures, on the other hand, provides a nomenclature for absolute and gravitation units. The metric gravitation system, like the British gravitation system, is local in character; the absolute system, known as the C.G.S. system, is cosmopolitan, having been devised for international purposes.

For clear understanding of the units at present in use it will be necessary to go into considerable detail and to present the subject in historical order.

The Standards of the Archives.

275. (*a*) *Unit of Length.*—In 1790 the French government, at the suggestion of Talleyrand, asked the Academy of Sciences to propose a scheme for the reformation of the confusion existing among the weights and measures of the country. The report of the committee of the Academy—which included Laplace, Lagrange, and Borda—was adopted in 1791. The committee resolved that the $1/10^7$ part of an arc

* The system involving the "poundal" as unit of force is quite superfluous.

of the meridian extending from the equator to the pole should be the basis of the standard of length and be called a **meter.**

Accordingly an arc of the meridian was measured and the length of the meter determined. The standard of the meter, which consists of a platinum bar about 1 in wide and 1/7 in thick, was constructed by Borda, presented to the National Assembly by Laplace, and on the same day deposited among the Archives of France. It is hence called the *mètre des archives.*

The meter thus fixed upon as the standard of length was defined by a law of the French Republic as the distance at 0° C., the temperature of melting ice, between the *ends* of the Borda platinum bar just described.

Later determinations have shown that the length of the quadrant is not exactly 10^7 meters. According to Clarke it has a length of 10,002,015 meters, being 2015 meters [5/4 mile] in excess. Other determinations give different values. Hence if the meter were lost or destroyed it would be impossible to replace it with any great degree of accuracy by reference to the quadrant of the meridian as natural unit.

276. It is probable that some other natural unit, as the wave-length of homogeneous light, may be chosen as the ultimate standard of length. Michelson found (1893) that the meter and length of a light-wave of red cadmium light might be compared with the same degree of accuracy as is now possible in the comparison of two meter bars.* He says: "If, therefore, the meter and all its copies were lost or destroyed, they could be replaced by new ones which would not differ from the originals more than do these among themselves."

"The science of the eighteenth century sought to render itself immortal by basing its standard units upon the solid earth; but the science of the nineteenth century soars far

* The meter = 1,553,163.5 wave-lengths (Michelson); = 1,553,163.6 wave-lengths (Benoit).

beyond the solar system and connects its units with the ultimate atoms which constitute the universe itself."

277. (*b*) *Unit of Weight.*—The standard of weight is the **kilogram,** which is a certain cylinder of platinum constructed under the direction of the French Academy at the same time as the meter, and with it deposited among the archives of France. It is hence called the *kilogramme des archives.*

The kilogram was intended to equipoise in a vacuum a cubic decimeter or *liter* of water at 3.9° C., the temperature of greatest density of water; that is, it should be of the same weight. But as weight and length have no natural relation, it was impossible that this relation could be exactly realized. The numerical determination changes as methods of comparison become more perfect, though within small limits.

Without doubt two weights can be compared at least a thousand times more accurately than either of them can be reproduced by weighing a specified volume of water, and for that reason the kilogram, like the English pound, can now be regarded only as an arbitrary standard of which copies must be taken by direct comparison. (Harkness.)

The International Standards.

278. In 1875, by the concurrent action of the principal governments of the world—seventeen in number—an international bureau of weights and measures was established at the Pavillon de Breteuil, Sèvres, near Paris, France. Under the direction of the international committee two ingots were cast of platinum-iridium in the proportion of nine parts of the former to one of the latter metal. From one of these a number of meter-bars were prepared, and from the other a number of kilogram-weights. These standards of length and of weight were intercompared. After the comparison had been made, one of the bars, the length of which at 0° C. was found equal to that of the Mètre des Archives, and one of the kilograms, with which the others were compared, were selected as *international prototype standards.* The others were in 1889 distributed by lot to the different governments and are called *national prototype standards.* Those apportioned to the United States—meter number 27, kilogram number 20—are in the keeping of the National Bureau of Standards, Washington. The standards allotted to Great Britain are meter number 16 and kilogram number 18, deposited with the Standards Department of the Board of Trade, London. The ultimate standards are the international prototype standards at Breteuil, which are defined as follows:

The **International Standard Meter** is derived from the Mètre des Archives, and its length is defined by the distance between two lines at 0° Centigrade on a platinum-iridium bar deposited at the International Bureau of Weights and Measures. This bar is in transverse section nearly of the form of the letter X, known as the Tresca form.

The **International Standard Kilogram** is a lump of platinum-iridium deposited at the same place, and its weight in vacuo is the same as that of the Kilogramme des Archives.

Like the Kilogramme des Archives it is in shape a cylinder, having the same diameter as altitude (39 mm).

279. Of the two definitions of the liter given in Art. 277 the international committee decided that one should be chosen which is most convenient for purposes of measurement. They accordingly in 1880 resolved that the *liter* (l) should denote the volume of a kilogram of water at the temperature of maximum density. The 1/1000 part of the liter is called the *milliliter* (ml).

The liter is therefore not precisely the same as the cubic decimeter, nor the milliliter the same as the cubic centimeter.

280. The international prototype meter and kilogram are now regarded by the National Bureau of Standards, Washington, as the standards of length and weight in the United States. In the absence of any material normal standards the units in ordinary use, the yard and the pound, are derived from the international units in accordance with the act of July 28, 1866. This act gives the relations:

$$1 \text{ yard} = 3600/3937 \text{ meter};$$
$$1 \text{ lb avoir.} = 1/2.2046 \text{ kilogram.}$$

The British imperial yard is equal to 36/39.370113 meter, and the imperial pound is equal to 1/2.2046212 kilogram.

The following are the principal determinations of the value of the meter in terms of the inch:

Date.	Authority.	Inches.
1818	Kater......	39.37079
1835	Baily.................	39.369678
1866	Clarke................	39.370432
1885	Comstock............	39.36985
1893	Rogers...............	39.370155
	International Bureau...	39.370113

No value of the inch in terms of the meter nor of the pound in terms of the kilogram has been adopted by international agreement.

281. The subdivisions and multiples of the meter with their equivalents in British measures are as follows. The abbreviations are those given by the international committee and should be used.

Micromillimeter........	$\mu\mu$	$1/10^9$ m	
Micron.................	μ	$1/10^6$ m	
Millimeter..............	mm	1/1000 m	0.03937 in
Centimeter..............	cm	1/100 m	
Decimeter...............	dm	1/10 m	
Meter...................	m	1 m	39.37 in
Decameter..............		10 m	
Hectometer............		100 m	
Kilometer.......... ...	km	1000 m	0.6214 mile
Myriameter............		10000 m	

The subdivisions and multiples of the gram with their abbreviations and equivalents in British measures are as follows :

Microgram..............	γ	$1/10^6$ g	
Milligram................	mg	1/1000 g	0.01543 grain.
Centigram..............	cg	1/100 g	
Decigram................	dg	1/10 g	
Gram..................	g	1 g	15.43235 grains
Decagram..............		10 g	
Hectogram..............		100 g	
Kilogram................	kg	1,000 g	2.2046 lb.
Myriagram..............		10,000 g	
Quintal..................	q	100,000 g	
Tonneau (Millier)........	t	1,000,000 g	

The following approximate values are for many purposes close enough:

LENGTH.

1 millimeter	= 1/25 inch	1 inch	= 2.5 centimeters
1 centimeter	= 2/5 inch	1 foot	= 30.5 centimeters
1 meter	= 1.1 yard	1 yard	= 0.9 meter
1 kilometer	= 5/8 mile	1 mile	= 1609.3 meters

WEIGHT.

1 gram	= 15.4 grains	1 ounce	= 28.3 grams
1 kilogram	= 2.2 lb	1 lb	= 453.6 grams
1 tonne	= 2205 lb	1 ton	= 10/11 tonne

282. The metric system is in use among nearly all the nations of Europe, of America, North and South, also in Japan. The principal exceptions among civilized nations are the United States, Great Britain, and Russia. In 1864 its use was made legal but not compulsory in Great Britain, and in 1866 in the United States. The act of 1866 reads:

"It shall be lawful throughout the United States of America to employ the weights and measures of the metric system; and no contract or dealing or pleading in any court shall be deemed invalid or liable to objection because the weights or measures expressed or referred to therein are weights or measures of the metric system.

"The Secretary of the Treasury is hereby authorized and directed to furnish each State, to be delivered to the Governor thereof, one set of the standard weights and measures of the Metric System for the use of the States respectively."

This was the first general legislation upon the subject in the United States, and the metric system is thus the first and thus far the only system made legal throughout the country. (See Art. 65.) It is now used in the U. S. Coast and Geodetic Survey, the U. S. Pharmacopœia, and to some extent in the Mint, the Geological Survey, the Post Office, and the Weather Bureau.

Our fractional silver coins represent metric weights. Thus the 50-cent piece weighs 125 decigrams, the 25-cent piece 625 centigrams, etc.

283. The standards being defined, we proceed to explain the two systems of metric units in use in Mechanics. (See Arts. 65–69.)

(1) The gravitation system, used in commerce and the arts.

(2) The absolute system, used in theoretical investigations and in laboratory work.

Gravitation Units.—The fundamental units of the metric gravitation system are the units of length, weight, and time.

The unit *length* is the **meter** [m] or **centimeter** [cm]. (Art. 278.)

The unit *weight* is the **kilogram** [kg] or **gram** [g]. (Art. 278.)

The unit *time* is the **second** [sec]. (Art. 7.)

The principal derived units are:

The unit *velocity*, which describes unit distance in unit time, is denoted by the symbols 1 cm/sec; 1 m/sec; 1 m/min; etc., according to the units of length and time employed.

The unit *acceleration,* which changes the velocity one unit in unit time, is denoted by the symbols 1 cm/sec^2; 1 m/sec^2; 1 m/min^2; etc.

The unit *force,* or the force of gravity on the unit of weight [the kilogram or the gram], is called the **kilogram** or the **gram.**

The term kilogram is used in a double sense, corresponding to the use of the term pound in the British system. (Art. 68.)

The unit *impulse* is the impulse of unit force [1 kilogram] in unit time [1 second], and is called a **second-kilogram.**

The unit *momentum* is the momentum of 1 kg moving with unit velocity, and is the same as the unit of impulse.

Thus the momentum of W kg moving with a velocity of v m/sec is Wv/g second-kilograms, g being equal to 9.81 m/sec^2.

The unit *work,* or the work done by unit force acting through unit distance, is called the **kilogrammeter** [kgm] or the **gram-centimeter** [gcm].

It may also be defined as the work done in lifting unit weight through unit height.

Thus a force of F kilograms acting through s meters does a work of Fs kgm. Or the work done in lifting a weight of W kg through a height of h meters is Wh kgm.

The unit *energy* is the same as the unit of work. (Art. 214.) Thus the energy of W kg moving with a velocity of v m/sec is $Wv^2/2g$ kgm.

The energy stored in a weight of W kg at a height of h meters above the earth's surface is Wh kgm.

The unit *activity* [or power] is the **kilogrammeter per second** [kgm/sec]. The enlarged unit is the *cheval vapeur,* which is 75 kgm/sec.

I.

Ex. 1. A horse trots 12 km in 1 h 40 min. Find his average speed. *Ans.* 2 m/sec.

2. Show that an acceleration of 500 $cm/sec^2 = 18\ km/min^2$.

3. Show that the velocity of the earth round the sun is 3×10^6 cm/sec, the mean distance of the earth and the sun being 1.487×10^{13} cm.

4. A point has velocities of 4, 4, 8 cm/sec inclined at 120° to each other. Find the resultant velocity.

Ans. 4 cm/sec.

5. Find the centripetal acceleration of a point which moves in a circular path of 1 m diameter with a velocity of 10 cm/sec. *Ans.* 2 cm/sec^2.

6. A railroad train timed past posts 1 km apart took 2 min and 1 min to pass over two consecutive distances. Find the velocity at the middle post, assuming the acceleration constant. *Ans.* 50 km/hour.

If t_1, t_2 were the times, show that the acceleration is

$$2(t_2^{-1} - t_1^{-1})/(t_2 + t_1)\ km/sec^2.$$

II.

7. A force of 5 kilograms acts on a weight of 10 kg. Find the distance described in 10 sec. *Ans.* 245.25 m.

8. Show that a brake resistance of 102 kilograms per tonne will bring to rest a train running at 72 km/hour in about 20 sec.

9. If a carriage weighing W kg is placed on a smooth level road and acted on by a pull of P kilograms for t sec, find the velocity acquired. *Ans.* $v = Pgt/W$.

Find the momentum acquired.

10. Explain why a waterfall h meters high can support a column of water $2h$ meters high.

III.

11. Two forces of 4 kilograms and 7 kilograms act at an angle of 55°. Show that their resultant is 9 kilograms 855 grams.

12. The wind is blowing from the southwest. Show that a vessel may be sailed due west.

13. Two cords making an angle of 60° support a chandelier weighing 50 kg. Find the pull in each cord.

Ans. $50/\sqrt{3}$ kilograms.

14. Three forces of 3, 4, 5 kilograms act at 120°. Find their resultant.

15. A locomotive weighing 98.1 tonnes runs round a curve of 800 m radius with a velocity of 72 km/hour. Find the centrifugal force. *Ans.* 5000 kilograms.

16. A ball is thrown upwards with a velocity of 20 m/sec. Find its velocity when half way up.

17. A marline spike falling from a mast down a hatchway took t sec to fall to the bottom of the hold, a depth of h meters. Find the velocity with which it struck.

Ans. $h/t + gt/2$ m/sec.

18. The wheels of a train running at 36 km/hour on coming to a drop of 5 mm in the rails will go about 32 cm before touching the rails.

IV.

19. Weights of 1, 2, 3 g are placed at the angles of an equilateral triangle whose sides are 12 cm in length. Find the distance of the C.G. from the angle 1. *Ans.* $2\sqrt{19}$ cm.

20. A body appears to weigh 240 g when placed in one pan of a balance, and 250 g when placed in the other pan. Find its real weight. *Ans.* $100\sqrt{6}$ g.

21. The length of the beam of a false balance is 1 m. A body placed in one scale weighs 8 kg, and in the other 8.1 kg. Find the lengths of the arms of the balance.

22. A steelyard is 1 m long and weighs 2 kg. It is suspended at a point 10 cm from one end. The movable weight is 1 kg. Find the greatest weight that can be weighed.

Ans. 17 kg.

V.

23. A 10-kg weight slides with constant speed down a rough plane which rises 1 in 10. Find the resistance.

Ans. 1 kilogram.

24. A 10-kg weight rests on a plane whose inclination is 30°. Find the force of friction called into action.

Ans. 5 kilograms.

25. What force will haul a 10-kg weight along a rough

table with a uniform acceleration of 1 m/sec², the coefficient of friction being 0.5? *Ans.* 6.02 kilograms.

VI.

26. Find the number of kgm necessary to raise 200 cm³ of water to a height of 1 km. *Ans.* 200 kgm.

27. The pitch of a screw is 30 mm, and the force exerted by each of two men with levers 2 m long is 15 kilograms. Find the greatest weight that can be raised. *Ans.* 12.57 tonnes.

28. In a differential pulley the radii of the upper sheaves are 190 mm and 200 mm. If the pull on the chain is 20 kilograms, find the greatest weight that can be raised.
Ans. 800 kg.

29. Find the distance in which a boat weighing 50 tonnes and moving at the rate of $4\frac{1}{2}$ km/hour can be brought up by a rope round a post if the greatest pull the rope is capable of sustaining is 1 tonne. *Ans.* 4 m nearly.

30. Show that a train running at 36 km/hour can be brought to rest in about 34 m by the brakes, supposing them to press on the wheels with 3/4 of the weight of the train and that the coefficient of friction is 0.2.

31. A train starts from a station at the foot of an incline of 30° and of length l meters. If the train weighs W kg and it reaches the top of the incline with a velocity v m/sec, show that the work done is $W(l/2 + v^2/2g)$ kgm.

284. Absolute Units.—As already explained in Chapter II, a system of units is absolute when it does not involve the acceleration of gravity at any place.

The fundamental units are those of length, time, and mass; the term mass being used instead of weight because in an absolute system comparisons are made fundamentally by kinetical methods and not by weighing. Comparisons of masses may, however, be made by the beam balance (Art. 63).

285. *C.G.S. Units.*—Gauss and Weber, who first developed the absolute system of units, employed the millimeter as unit of length, the milligram as unit of mass, and the second as unit of time.

In 1873 a committee of the British Association recommended as dynamical and electrical units the centimeter, gram, and second as the fundamental units, and also recommended a system of nomenclature. Additions were made to the nomenclature in 1888.

This system, involving as fundamental units the centimeter as unit of length, the gram as unit of mass, and the mean solar second as unit of time, is called the C.G.S.—centimeter-gram-second—system. It is growing more and more international in character and is used in theoretical investigations in Astronomy, Physics, Electricity, and Mechanics to the exclusion of every other system.

The principal derived units are as follows:

The unit *velocity* which describes unit distance in unit time [1 cm in 1 sec] is called a **kine** (κινέω).

The unit *acceleration* which changes the velocity 1 unit in time [1 kine in 1 sec] is called a **spoud** (σπουδή).

The unit *force* which produces unit acceleration in unit mass [1 spoud in 1 gram] is called a **dyne** (δύναμις). Thus a force of F dynes acting on m grams produces an acceleration of a cm/sec², given by

$$F = ma.$$

The unit *momentum* is the momentum of unit mass moving with unit velocity [1 gram with velocity 1 kine], and is called a **bole** (βόλος). Thus the momentum of m grams moving with a velocity of v cm/sec is mv boles.

The unit *impulse* which generates unit momentum is also called a **bole**. Thus a force of F dynes acting for an interval of t sec on a mass of m grams generates an impulse of Ft boles, and is equivalent to mv boles if v is the velocity change.

The unit *work*, which is the work done by unit force acting through unit distance [1 dyne through 1 cm], is called an **erg** (ἔργον). Thus a force of F dynes acting through s cm does a work of Fs ergs.

The unit *energy* is the same as the unit work, the **erg**. Thus the energy of m grams moving with a velocity of v cm/sec is $mv^2/2$ ergs.

286. *M.K.S. Units.*—In practical electricity and in mechanical pursuits many of the C.G.S. units are found to be inconveniently small. Accordingly what is known as the M.K.S. system, in which the meter, kilogram, and second are the fundamental units, is being adopted as a *working system.* It has as yet received only a partial nomenclature.

In this system

The unit *distance* is 1 meter [$= 10^2$ centimeters].

The unit *mass* is 1 kilogram [$= 10^3$ grams].

The unit *time* is 1 second.

The unit *force* which produces unit acceleration [10^2 cm/sec^2] in unit mass [10^3 grams] is consequently 10^5 dynes. The name *gauss* * has been suggested for this unit.

Another unit of force is the **megadyne**, being one million dynes.

The unit *work* is the work done by unit force acting through unit distance [10^5 dynes through 10^2 cm], that is, 10^7 ergs, and is called a **joule**,† so that

$$1 \text{ joule} = 10^7 \text{ ergs}.$$

Another unit of work is the **megalerg**, or a million ergs, being the work done by a megadyne acting through a cm.

The unit *energy* is the same as the unit of work, the **joule**. Thus the energy of m kilograms moving with a velocity of v meters/sec is $mv^2/2$ joules.

The unit *power* is the power of an agent which can do a work of one joule per second, and is called a **watt**,‡ so that

$$1 \text{ watt} = 10^7 \text{ ergs/sec}.$$

The **kilowatt** [1000 watts] is the unit of power in electric

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* After Gauss of Göttingen (1777–1855), who first published an absolute system of units of dynamics.

† After Joule of Manchester (1818–1889), who first measured accurately the mechanical equivalent of heat.

‡ After James Watt of Glasgow (1736–1819), who made the steam-engine a commercial success.

lighting. Roughly, 3 kilowatts are equivalent to 4 horse-powers.

The **kilowatt-hour** is analogous to the horse-power-hour (Art. 209), and is a unit of work or energy.

So also the **watt-second** is a unit of energy, and is equivalent to a joule.

I.

Ex. 1. A point moves with a uniform speed of 100 kines. In what time will it pass over 1 km? *Ans.* 16 min 40 sec.

2. Show that the velocity of a point on the equator arising from the earth's rotation is about 46,300 kines, the mean radius of the earth being $2\times 10^9/\pi$ cm.

3. A particle starts with a velocity of 100 kines and moves under an acceleration of -2 spouds. In what time will it come to rest? *Ans.* 50 sec.

4. A ball is dropped in an elevator which is rising with an acceleration of 1 spoud. Find its acceleration relative to the floor of the elevator.

5. A ball is dropped from the top of an elevator 2.45 m high. Find the time in which it will reach the platform when the elevator is descending with an acceleration of 490 spouds. *Ans.* 1 sec.

6. The maximum piston-speed of an engine cross-head is 200 kines. Find the average speed, neglecting the obliquity of the connecting-rod. *Ans.* $400/\pi$ kines.

7. Find the distance passed over in the fifth second from rest of a body falling at a place where $g = 980$ spouds. *Ans.* 44.1 m.

II.

8. A body of mass 10 g is acted on by a force of 10 dynes. Find the resultant acceleration. *Ans.* 1 spoud.

9. Find the momentum of a body of mass 10 g (1) moving with a velocity of 10 kines, (2) acted on by a force of 10 dynes for 10 sec. *Ans.* 100 boles.

What is the energy in each case?

10. A body of mass 10 g moving with a velocity of 10 kines is acted on by a force of 10 dynes for 10 sec. The inclination of the force to the original direction of motion is 60°. Find the final velocity. *Ans.* $10\sqrt{3}$ kines.

11. A body of mass 10 g produces a certain deflection in a spring-balance at a place where the acceleration of gravity is

981 spouds, and at another place it is noted that it requires 10.01 g to produce the same deflection. Find the value of the acceleration of gravity at this place.

Ans. 980.02 spouds.

12. A body of mass m grams is moving with a velocity of v cm/sec. In what time will a force of F dynes reduce it to rest?

13. Find the force which acting on 1 kg for 10 sec will produce a velocity of 1 kine. *Ans.* 100 dynes.

14. If 1 gram of air occupies 773 cm^3 and behaves like a cloud of inelastic particles, prove that the pressure of the wind against a plane perpendicular to its direction is $0.00129v^2$ barads, if the velocity of the wind is v kines.

III.

15. A body slides down a smooth plane inclined at 30° to the horizon. Through how many cm will it fall in the second second? *Ans.* $735\frac{3}{4}$ cm.

16. A body of mass m g revolves uniformly in a circle of radius r cm and makes n rev/min. Show that

$$\text{the centrifugal force} = mr(\pi n/30)^2 \text{ dynes.}$$

If t sec is the time of revolution,

$$\text{the centrifugal force} = mr(2\pi/t)^2 \text{ dynes.}$$

17. Find the length of a seconds-pendulum at a place where $g = 981$ spouds. *Ans.* 99.3 cm.

18. If the speed of a bicycle is 18 km/hour, show that a piece of mud thrown from the highest point of the front wheel 70 cm in diameter will strike the ground $10/\sqrt{7}$ meter in advance of the point of contact of the wheel with the ground.

19. Forces whose magnitudes are as 3 to 4 act at a point and in directions at right angles. If their resultant is 2 dynes, find the forces. *Ans.* 1.2, 1.6 dynes.

20. A mass of 6 grams is held up by two threads, one horizontal and the other inclined at 30° to the vertical. Find the pulls in the threads.

Ans. $3924\sqrt{3}$, $1962\sqrt{3}$ dynes.

21. Resolve a force of 12 dynes into two equal forces each inclined at 30° to it. *Ans.* $4\sqrt{3}$ dynes.

22. A cylinder 10 cm in diameter stands on a rough plane and is just on the point of tumbling over when the inclina-

tion of the plane to the horizontal is 30°. Find the height of the cylinder. *Ans.* $10\sqrt{3}$ cm.

23. A kg weight is thrown vertically upward with a velocity of 981 kines. Find its energy at the instant of propulsion and also after 1 sec. *Ans.* 481,180,500 ergs; 0.

24. How much work is done against gravity by a man whose mass is 80 kg in walking a distance of 1 km up a mountain path inclined at 30° to the level?

Ans. 392,400 joules.

25. A 5-kg cannon-ball is discharged with a velocity of 500 m/sec. Show that the energy = 625,000 joules.

26. A kg weight is lifted through 1 m. Find the increase of potential energy. *Ans.* 9.81 joules.

27. A gun recoils with a velocity of 327 cm/sec and runs up an incline of 1 in 4 to a distance of 218 cm. Find the value of the acceleration of gravity at that place.

Ans. 981 cm/sec^2.

28. A particle of mass 1 gram moving with a S.H.M. vibrates 100 times a second. If the range of vibration is 1 cm. at its mean position, find the energy of the particle.

29. The energy of a particle of mass m which executes a simple harmonic motion of amplitude a cm and period T sec is

$$\tfrac{1}{2}ma^2\omega^2 \sin^2 \omega t$$

where $\omega = 2\pi/T$.

30. The kinetic energy of translation of the earth relative to the sun is $10^{33} \times 2.69$ joules, the radius being $2 \times 10^9/\pi$ cm, and the density 5.6 g/cm.^{3}

31. The electrical power unit adopted at Niagara Falls is 5000 electrical horse-power. Show that this is equivalent to 3700 kilowatts.

287. Change of Units—Dimensions.—The numerical value or measure of a quantity is the number of units that must be added together to produce the given quantity; in other words, it is the ratio of the quantity to the unit. If the unit is changed, the numerical value is changed in inverse ratio. Thus a line 100 cm long may be stated as being 1000 mm long or 1 m long, the unit in the first change being 10 times as small and in the second 100 times as large as the original unit. Hence the numerical measure varies inversely as the unit. Of course the quantity itself is quite independent of the unit employed to measure it—just as the matter of this book is in no way affected by the size of type used by the printer.

288. In mechanics certain units are assumed as fundamental units. They are called fundamental because a change in any one of them does not imply a change in any of the others. They are arbitrary, being chosen as the result of circumstances or as seems most convenient for the purpose on hand. Those usually but not necessarily taken are the units of length, time, and mass.* They are denoted by the symbols **L**, **T**, and **M** respectively.

But besides the fundamental units there are other units which are systematically derived from them by definition, and which therefore depend on them. A system of this kind in which a magnitude can be expressed in terms of the fundamental units is called an absolute system. The C.G.S. system is an absolute system, the derived units being systematically defined in terms of the units of length, time, and mass.

289. The relation between a derived unit and the fundamental units from which it is derived may be stated as an algebraic formula. The indices of the fundamental units in the formula of the derived unit are called the **dimensions** of this unit. The formula itself is a dimensional formula. Knowing the dimensions, we can at once write down the change in the value of a derived unit due to any changes in the fundamental units of that system, or we can convert units of one absolute system into those of another.

290. The doctrine of dimensions was first given by Fourier in 1821 in his *Analytical Theory of Heat*, but brought into use by Maxwell.

The view of dimensional formulas here given is the primary one that the formulas indicate numerical relations between

* Maxwell's statement that "the whole system of civilized life is symbolized by a foot rule, a clock, and a set of weights" is for the present too sweeping. All electrical and magnetic units have not yet been expressed in terms of these units.

units; in a word, that they are change-ratios. As the tendency of modern science is to express everything dynamically, it is probable that dimensional formulas may be so expressed as to indicate the *nature* of the quantity—the way in which it is built up. At present they do not do this. For example, the table on p. 367 will show that torque and work, though by no means identical, are of the same dimensions.

For a proposed extension of their meaning along quaternion lines see *Phil. Mag.*, Sept. 1892.

291. The principal derived or secondary units are as follows:

1. *Velocity.*—From the definition

$$\text{unit velocity} = \text{unit length}/\text{unit time},$$

or

$$\mathsf{V} = \mathsf{L}/\mathsf{T} = \mathsf{LT}^{-1},$$

and the unit of velocity is said to be of one dimension with respect to length and minus one dimension with respect to time.

If we change to units L_1, T_1 of length and time when

$$\mathsf{L}_1 = a\mathsf{L}, \qquad \mathsf{T}_1 = b\mathsf{T},$$

then

$$\text{unit vel.} = \mathsf{L}_1/\mathsf{T}_1 = \frac{a}{b}\mathsf{L}/\mathsf{T}.$$

Ex. Find the unit of velocity if the units of length and time are the mile and minute.

$$\text{Unit vel.} = \mathsf{L}_1/\mathsf{T}_1 = 5280\mathsf{L}/60\mathsf{T} = 88\mathsf{L}/\mathsf{T};$$

that is, the unit of velocity is 88 ft/sec.

2. *Acceleration.*—Unit acceleration is rate of change of unit velocity. Hence

$$\mathsf{A} = \mathsf{V}/\mathsf{T} = \mathsf{L}/\mathsf{T}^2 = \mathsf{LT}^{-2},$$

and is said to be of dimensions 1 in length and -2 in time.

3. *Force.*—Unit force gives unit acceleration to unit mass. Hence

$$\mathsf{F} = \mathsf{MA} = \mathsf{ML}/\mathsf{T}^2 = \mathsf{LMT}^{-2}.$$

4. *Impulse.*—Unit impulse is given by unit force acting for unit time. Its notation therefore is

$$FT = LMT^{-2} \times T = LMT^{-1}.$$

5. *Momentum.*—Unit momentum is given by unit mass moving with unit velocity. Its notation therefore is

$$MV = LMT^{-1},$$

or an impulse is of the same dimensions as a momentum.

This also follows from unit impulse being that which generates or destroys unit momentum.

6. *Work.*—Unit work is given by unit force acting through unit distance. Hence

$$F \times L = LMT^{-2} \times L = L^2MT^{-2}.$$

7. *Kinetic energy* (translation) is of the same dimensions as work. For

$$\tfrac{1}{2}MV^2 = \tfrac{1}{2}ML^2T^{-2}.$$

8. *Power* or *Activity.*—Unit power is unit work per unit time. Hence

$$\begin{aligned}\text{unit power} &= W/T\\ &= L^2MT^{-2}/T\\ &= L^2MT^{-3}.\end{aligned}$$

9. *Density.*—Unit density is unit mass per unit volume. Hence

$$\text{unit density} = M/L^3.$$

10. *Pressure.*—Unit pressure is unit force per unit area. Hence

$$\begin{aligned}\text{unit pressure} &= F/L^2\\ &= LMT^{-2}/L^2\\ &= L^{-1}MT^{-2}.\end{aligned}$$

11. *Torque.*—Unit torque is unit force into unit length. Hence

$$\begin{aligned}\text{unit torque} &= F \times L\\ &= LMT^{-2} \times L\\ &= L^2MT^{-2}.\end{aligned}$$

12. *Heat.*—In the dynamical theory of heat it is necessary to consider the dimensions of the unit of heat.

The unit of heat is the *calorie*, or the quantity of heat required to raise unit mass through unit temperature, or 1°. Hence

$$\text{dimensions of unit heat} = \mathsf{DM}$$

when D denotes a degree of temperature on any scale.

Now to produce a unit of heat requires J units of work when J is known as Joule's equivalent, or

$$1 \text{ unit heat} = \mathsf{J} \text{ units work.}$$

Let, as the result of experiment,

$$\mathsf{H} \text{ units heat} = \mathsf{W} \text{ units work.}$$

$$\therefore \quad 1/\mathsf{H} = \mathsf{J}/\mathsf{W},$$

or

$$\mathsf{J} = \mathsf{W}/\mathsf{H}.$$

Hence the dimensions of J are

$$\mathsf{L^2MT^{-2}/DM = D^{-1}L^2T^{-2}},$$

which enables us to find the value of J when the units of work and of temperature are changed from one system to another.

292. By considering the dimensions of the units involved in a mechanical formula we may check at least its *possibility* and thus save the labor of going over the demonstration. For example, in Art. 28 it was found that

$$v^2 = ar.$$

Considering the dimensions only, we have for the separate terms, r being a length,

$$\mathsf{L^2T^{-2}}, \quad \mathsf{LT^{-2} \times L}, \quad \text{or} \quad \mathsf{L^2T^{-2}},$$

and the expression is therefore homogeneous, each term being of the same dimension. Hence $v^2 = ar$ is possibly true.

A similar test would show that such an expression as

$$m^2t + 2Ft^2 = 3m^2s$$

is absurd.

If, then, in the course of an investigation we had arrived at an equation of this form, we should at once conclude that there was an error somewhere. By applying the principle we may check our work at intervals and save carrying along mistakes.

293. It is often required to pass from an absolute to a gravitation system of units and *vice versa*. In these cases the rule given in Art. 66 must be observed.

For example, to find the number of watts in a horse-power:

Let $x =$ the number of ergs per sec in 1 H.P.

Now 1 H.P. = 550 foot-pounds per sec.

Then

$$x \times \mathbf{L}^2\mathbf{M}\mathbf{T}^{-3} = 32.2 \times 550 \times \mathbf{L}_1{}^2\mathbf{M}_1\mathbf{T}^{-3},$$

the second being the unit of time in both systems.

But

$$100 \text{ cm} = 39.37/12 \text{ ft.} \qquad 1000 \text{ g} = 2.2046 \text{ lb.}$$
$$\therefore \quad \mathbf{L}_1 = 1200/39.37\mathbf{L}. \qquad \therefore \quad \mathbf{M}_1 = 1000/2.2046\mathbf{M}.$$

Hence
$$x = 32.2 \times 550 \times \left(\frac{1200}{39.37}\right)^2 \times \frac{1000}{2.2046}$$
$$= 746 \times 10^7 \text{ ergs/sec}$$
$$= 746 \text{ watts.}$$

1. Show that a velocity of 1 kine = 0.0328 ft/sec.
2. " " " " " 1 ft/sec = 30.48 cm/sec.
3. " " " " " 1 mile/hour = 44.7 cm/sec.
4. " " " " " 1 km/hour = 27.78 cm/sec.
5. " " " " " 1 km/hour = 0.62137 miles/hour.
6. " " " " " 1 knot = 1853.25 m/hour.
7. " " an acceleration of 1 spoud = 0.0328 ft/sec².
8. " " " " " 1 ft/sec² = 30.48 cm/sec².
9. Express in miles per hour the speed of a vessel which steams at the rate of 16.5 knots. *Ans.* 19 miles/hour.

10. Find the numerical value of a velocity of 10 ft/sec if the fundamental units of length and time are the yard and minute. *Ans.* 200.

11. Find the unit of velocity if one meter is the unit of length and one spoud the unit of acceleration. *Ans.* 10 kines.

12. Show that the foot per second and mile per hour units of velocity are as 15 : 22.

13. Find the numerical value of g (981 spouds) if one meter is the unit of length and one minute the unit of time. *Ans.* 36 g.

14. The unit of velocity is 1 mile per minute and the unit of time 1 minute. Find the unit of acceleration. *Ans.* 22/15 ft/sec^2.

15. If the acceleration of gravity is taken as the unit of acceleration, and a velocity of $43\frac{7}{11}$ miles an hour the unit of velocity, find the units of time and distance. *Ans.* 2 sec; 128 ft.

16. Find the unit of time if g is taken as unity, one ft being the unit of length. *Ans.* $1/\sqrt{g}$ sec.

17. The unit of length is a meter, the unit of mass a kilogram, and the unit of time a minute. Find the unit of force. *Ans.* 250/9 dynes.

18. If the unit of length is 1 meter, the unit of velocity 100 kines, and the unit of energy 100 ergs, find the unit of power. *Ans.* 10^{-5} watt.

19. Taking **L**, **T**, **W** as units of length, time, and energy, find the units of mass, momentum, and force. *Ans.* $\mathsf{WL^{-2}T^2}$; $\mathsf{WL^{-1}T}$; $\mathsf{WL^{-1}}$.

20. Show that

1 gram [force]	= 981 dynes.
1 grain [force]	= 63.57 dynes.
1 gram-centimeter	= 981 ergs.
1 kilogrammeter	= 10^8 ergs, approx.,
	= 9.81 joules.
1 pound [force]	= 445,000 dynes
	= 2/5 megadyne, roughly.
1 foot-pound	= 1.356 joules
	= 0.138 kilogrammeter.
1 joule	= 3/4 foot-pound, approx.
1 watt	= 3/4 foot-pound/sec, approx.
1 horse-power	= 745.9 watts.
1 kilowatt	= $1\frac{1}{3}$ horse-power, approx.
1 poncelet	= 981 watts.

1 watt-second = 10 megalergs.
3 kilowatt-hours = 4 horse-power-hours.
1 pound/ft² = 479 barads.

21. Show by considering its dimensions that

$$mvt + 2Ft^2 = 3ms$$

is a possible expression.

22. Show by a consideration of dimensions that the distance described from rest by a body acted on by a force in the direction of its motion varies as the square of the time.

23. Show that the equation

$$Fs = mv^2/2$$

is homogeneous.

24. The mean value of Joule's equivalent J is 4.2×10^7 ergs when the unit of work is the erg and the unit of temperature 1° C. Express this in foot-pounds when the unit of work is the foot-pound and the unit of temperature 1° F.

Let $x =$ the number of foot-pounds required.

Then $x \times 32.2 \times \mathsf{L}^2\mathsf{D}^{-1}\mathsf{T}^{-2} = 4.2 \times 10^7 \times \mathsf{L}_1{}^2\mathsf{D}_1{}^{-1}\mathsf{T}^{-2}$.

But $100 \text{ cm} = 39.37/12 \text{ ft}$. $5° \text{ C.} = 9° \text{ F.}$

$\therefore \mathsf{L}_1 = 39.37/1200\mathsf{L}$. $\therefore \mathsf{D}_1 = 9\mathsf{D}/5$,

and
$$x = \frac{4.2 \times 10^7}{32.2} \times \left(\frac{39.37}{1200}\right)^2 \times \frac{5}{9}$$
$$= 780.1 \text{ foot-pounds.}$$

[Griffith's value (1893) is 779.97. The value in common use is 772, which is certainly too small.]

25. Find the value of J when the unit of work is the kilogrammeter and the unit of temperature 1° C.

Ans. 0.428 kgm.

EXAMINATION.

1. Describe the international standard units of length and weight.

2. What is the international unit of time?

3. How are the inch and lb in the United States defined by the act of 1866?

[In terms of the meter and kilogram by the relations 1 m = 39.37 in, 1 kg = 2.2046 lb.]

4. What are the legal equivalents in Great Britain according to the act of 1878?

Ans. 1 m = 39.37079 in; 1 kg = 2.2046212 lb.

5. Explain the prefixes micro-, milli-, centi-, deci-, deca-, hecto-, kilo-, myria-.

6. How many mm in an inch? How many cm in a foot?

7. Show that 8 kilometers is equal to 5 miles, nearly.

8. Show that 10 kilograms is equal to 22 lb, nearly.

9. Should the abbreviation c.c. be used for cubic centimeter?

[No, use cm^3.]

10. What is a metrical tonne?

11. How many cm^3 in a fluid ounce?

12. How many ounces in a liter of water?

13. Is this true: 1 tonne = 1 m^3 water; 1 milligram = 1 mm^3 water?

14. How many kilograms in a quintal?

15. What is a fundamental unit in mechanics? a derived unit?

16. State the fundamental C.G.S. units, the fundamental British units, commonly employed. What other units might be chosen?

17. Describe the C.G.S. system of units.

18. What fundamental units does the absolute unit of force involve?

19. Define the terms erg, joule, watt, and watt-second.

20. Express an erg in British units.

Ans. 7.37×10^{-8} foot-pounds.

21. A *force de cheval* is 75 kgm/sec. Express it in British units. *Ans.* 542.48 foot-pounds/sec.

22. Show that 1 *force de cheval* = 736 watts.

23. What is a micron? a megadyne? a megalerg?

24. A megadyne is roughly equivalent to a kilogram.

25. Give the adopted abbreviations for micron, meter, cubic centimeter, square millimeter, gram, kilogram, kilogrammeter, quintal, tonne.

26. "The pressure of the atmosphere upon every cm^2 of the earth's surface is 1033.3 grams, *a little more than a megadyne.*" Check the statement in italics.

27. A watt represents 10 megalergs per second.

28. Point out the errors in the following:

(*a*) "The service of express trains from London to Edinburgh has received an acceleration of one hour."

(*b*) "The couple exerted is evidently nc dynes."

(*c*) "The momentum of the needle when moving with the angular velocity ω is $I\omega^2/2$."

(*d*) "The momentum is all expended in balancing the couple."

(*e*) "The C.G.S. unit of work is the gram-centimeter."

29. How does the length of the meter compare with that of the seconds-pendulum?

30. How many watts in a horse-power?

31. "A kilowatt is a unit of power and involves the idea of time. A kilowatt-hour is a unit of energy or work, precisely like the term foot-pound, or horse-power-hour, or joule, into which it is directly convertible." Examine this statement.

32. Is the following rule of thumb true?—

"A motor yielding a horse-power for every kilowatt consumed has an efficiency of 75 per cent."

33. Show that 1 joule = 0.00000028 kilowatt-hour.

34. Show that 1 kilowatt-hour = 1.34 horse-power-hours.

35. Show that 1 watt-second = 1 joule.

36. What two functions do dimensional formulas perform in physical investigations?

What other function has been suggested?

37. Show that the equation

$$Ft = mv$$

is homogeneous.

38. Is the equation

$$astv + sv^2 = g^2t^4$$

homogeneous?

39. What is meant by a barad? [A pressure of 1 dyne/cm^2.]

40. Resolve a force so that its component in a given direction shall have a given value.

41. Show how a stick may be balanced on the finger at a constant inclination to the vertical by making the finger describe a horizontal circle with given velocity.

42. A projectile is fired with velocity u kines and at an elevation θ. What are the horizontal and vertical components of its velocity after 2 seconds?

43. Determine the height to which the mud-splashes from the hind wheel will reach on the back of a bicyclist.

44. A mass m grams is moving with velocity u kines. A constant force acts on it in the direction of the motion until the velocity is v kines. Prove that the work done by the force is

$$mv^2/2 - mu^2/2 \text{ ergs.}$$

45. A mass of 1 decigram executes 250 simple harmonic vibrations in 1 sec. If the amplitude of the vibration is 2.5 mm, find the maximum force exerted.

Ans. $6250\pi^2$ dynes.

46. A simple pendulum is pulled aside until the bob is raised h ft above the horizontal through the lowest point and then let go. Find its velocity as it swings through the lowest point.

47. A vessel steams directly away from the rock of Gibraltar at a speed of 20 knots. Show that the rock appears to sink with a constant acceleration of $5.5/10^5$ ft/sec^2.

Hence show how a passenger on board could compute the height of the rock.

48. The actual relation of prototype meter No. 16 is

$$\text{prototype } 16 = 1 \text{ m} - 0.6\mu \pm 0.1\mu \text{ at } 0° \text{ C.},$$

so that it may be said that the meter has been verified with a probable accuracy of 0.1 part in a million.

Explain the meaning of this last statement.

MECHANICAL UNITS—BRITISH TO METRIC.

Length.	1 inch = 25.4000 * millimeters (mm)	log 1.40483
	1 foot = 0.304801 meter (m)	" 9.48402
	1 yard = 0.914402 meter	" 9.96114
	1 mile = 1.60935 * kilometers (km)	" 0.20665
Area.	1 sq inch = 645.16 sq millimeters (mm^2)	log 2.80966
	1 sq foot = 0.09290 sq meter (m^2)	" 8.96802
	1 sq yard = 0.83613 sq meter	" 9.92227
	1 sq mile = 2.59000 sq kilometers (km^2)	" 0.41330
Capacity.	1 U. S. gallon (231 in^3) = 3.78543 * liters (l)	log 0.57812
	1 imperial gallon (277.463 in^3) = 4.54683 liters	" 0.65771
Weight.	1 grain = 64.7989 * milligrams (mg)	log 1.81157
	1 ounce (avoir.) = 28.3495 * grams (g)	" 1.45255
	1 lb (avoir.) = 453.5924 * grams	" 2.65667
	1 ton (2000 lb) = 907.1849 kilograms (kg)	" 2.95770
Force.	1 grain = 63.57 dynes.	
	1 pound = 4.45×10^5 dynes.	
Stress.	1 pound/sq inch = 70.307 grams/sq centimeter	log 1.84700
	1 pound/sq foot = 4.8824 kilograms/sq meter	" 0.68863
Energy.	1 foot-pound = 1.3563×10^7 ergs	
	= 0.13825 kilogrammeters (kgm)	log 9.14067
	1 horse-power-hour = 1,980,000 foot-pounds.	
Activity (Power).	1 foot-pound/minute = 0.0226 watt	log 8.35411
	1 horse-power = 33,000 foot-pounds/minute	
	= 746 watts	
	= 76 kilogrammeters/second.	
Velocity.	1 foot/second = 0.3048 meter/second	log 9.48402
	1 mile/hour = 0.4470 m/sec	" 9.65031
	= 1.6093 km/hr	" 0.20665

* These are the values given by the Office of Weights and Measures, Washington, based on the act of July 28, 1866.

MECHANICAL UNITS—METRIC TO BRITISH.

Length.	1 millimeter (mm) = 0.03937 inch (in).........	log	8.59517
	1 meter (m) = 39.37 * inches............	"	1.59517
	= 3.28083 * feet (ft)........	"	0.51598
	1 kilometer (km) = 0.62137 * mile...........	"	9.79335
Area.	1 sq centimeter (cm^2) = 0.1550 * sq inches (in^2).	log	9.19033
	1 sq meter (m^2) = 10.764 * sq feet (ft^2)....	"	1.03197
	= 1.196 * sq yards... ...	"	0.07773
	1 sq kilometer (km^2) = 0.3861 sq mile........	"	9.58670
Capacity.	1 liter (l) = 61.023 * cubic inches (in^3)..........	log	1.78549
	= 1.0567 * U. S. quarts..............	"	0.02395
	= 0.8797 imperial quart.............	"	9.94433
Weight.	1 gram (g) = 15.432 * grains..............	log	1.18842
	1 kilogram (kg) = 2.2046 * lb avoir............	"	0.34333
	1 tonne (t) = 1.1023 * tons (2000 lb)......	"	0.04230
	= 0.9842 ton (2240 lb)	"	9.99308
Force.	1 dyne = 0.01573 grain....................	log	8.19673
	= 0.00102 gram...................	"	7.00860
	1 megadyne = 10^6 dynes.		
	1 gram = 981 dynes....	log	2.99167
	1 kilogram = 9.81×10^5 dynes.		
Stress.	1 gram/sq centimeter = 0.01422 pound/sq inch.	log	8.15290
	1 kilogram/sq meter = 0.20482 pound/sq foot..	"	9.31137
	1 barad = 0.00102 gram/sq cm....	"	7.00860
Energy.	1 erg = 7.37×10^{-8} foot-pounds.		
	1 megalerg = 10^6 ergs.		
	1 joule = 10^7 ergs = 1 watt-second.		
	1 kilowatt-hour = 1.34 horse-power-hours.		
	1 gram-centimeter = 981 ergs................	log	2.99167
	1 kilogrammeter = 9.81 joules		
	= 7.2330 foot-pounds.......	"	0.85932
Activity.	1 watt = 1 joule/second		
	= 0.10194 kilogrammeter/second.......	log	9.00834
	= 44.2385 foot-pounds/minute..........	"	1.64580
	1 kilowatt = 1.34 horse-power.		
	1 force de cheval = 75 kilogrammeters/second.		
	1 poncelet = 100 kilogrammeters/second		
	= 981 watts.................	log	2.99167

GRAVITATION UNITS.

Dynamical Quantities.	Symbols.	Defining Equations.	British Units. Names.	British Units. Abbreviations.	Metric Units. Names.	Metric Units. Abbreviations.
Length	l, s	....	foot	ft	meter	m
Time......	t	....	second	sec	second	sec
Weight	W, w	....	pound	lb	kilogram	kg
Velocity.. .	v	$v=s/t$	foot per sec	ft/sec	meter per second	m/sec
Acceleration......	a	$a=v/t$	foot per sec per sec	ft/sec²	meter per sec per sec	m/sec²
Force	F	$F=wa/g$	pound	....	kilogram	
Impulse....	$\mathbf{I}$	$\mathbf{I}=Ft$	sec'd-pound	....	second-kilogram	
Momentum	$\mathbf{I}$	$\mathbf{I}=wv/g$	sec'd-pound	....	second-kilogram	
Work......	$\mathbf{W}$	$\mathbf{W}=Fs$	foot-pound	....	kilogrammeter	kgm
Energy	$\mathbf{W}$	$\mathbf{W}=\frac{1}{2}wv^2/g$	foot-pound	....	kilogrammeter	kgm
Power (activity) ...	P	$P=\mathbf{W}/t$	ft-pound per sec; horsepower	ft-pound/sec	kgm per sec	kgm/sec
Moment (torque) .	T	$T=Fl$				
Moment of inertia...	I	$I=\Sigma wr^2$				

ABSOLUTE UNITS.

Dynamical Quantities.	Symbols.	Defining Equations.	Dimensions.	C.G.S. Units.		Practical Units.	
				Names.	Abbreviations.	Names.	Abbreviations.
Length......	l, s	....	L	centimeter	cm	meter	m
Time	t	....	T	second	sec	second	sec
Mass...	M, m	....	M	gram	g	kilogram	kg
Velocity, linear.. ..	v	$v=s/t$	LT^{-1}	kine			
Acceleration, lin'ar	a	$a=v/t$	LT^{-2}	spoud			
Velocity, angular...	ω	$\omega=v/r$	T^{-1}	radian/sec			
Accelerat'n, angular...	a	$a=\omega/t$	T^{-2}	radian/sec²			
Force	F	$F=ma$	LMT^{-2}	dyne	....	megadyne	
Impulse	I	$I=Ft$	LMT^{-1}	bole			
Momentum.	I	$I=mv$	LMT^{-1}	bole			
Work	W	$W=Fs$	L^2MT^{-2}	erg	....	{ joule mega-lerg	
Energy.... .	W	$W=\frac{1}{2}mv^2$	L^2MT^{-2}	erg	....	joule	J
Power... ...	P	$P=W/t$	L^2MT^{-3}	....	....	watt; kilowatt	kw
Moment (torque) ..	T	$T=Fl$	L^2MT^{-2}				
Moment of inertia..	I	$I=\Sigma mr^2$	L^2M				
Density. ..	δ	$\delta=m/l^3$	LM^{-3}				
Pressure....	p	$p=F/l^2$	$L^{-1}MT^{-2}$	barad			
Modulus of elasticity .	E	....	$L^{-1}MT^{-2}$				

SYNOPSIS FOR READY REFERENCE.

NOTATION.

α = acceleration (angular).
a = acceleration (linear).
a_x, a_y = acceleration along axes of X and Y.
C = centripetal force.
C. G. = center of gravity.
e = logarithmic base: coefficient of restitution.
E = energy.
f = friction.
F = force.
g = acceleration due to gravity.
G = resultant couple or torque.
h = height.
H.P. = horse-power.
I = impulse.
I = moment of inertia.
k = radius of gyration.
m, M = mass (absolute units).
N = normal reaction.
p = arm of couple.
P = power.
r = radius of circle.
R = resultant.
s, y = distance passed over.
S.H.M. = simple harmonic motion.
t = time.
T = period of a S.H.M.
ω = velocity (angular).
u, v = velocity (linear).
v_x, v_y = velocity along axes of X, Y.
w, W = weight (gravitation units).
x, y = coordinates of a point.
$\bar{x}, \bar{y}$ = coordinates of C.G.
β, γ, θ = angles of inclination.
ϕ = angle of friction.
μ = coefficient of friction.

MECHANICS.

Two main divisions, Kinematics and Dynamics (p. 3).

A. Kinematics.

Motion of translation.	Motion of rotation.
Velocity uniform (p. 8)	uniform (p. 274)
$v = s/t$	$\omega = \theta/t$
$s = vt$	
variable (p. 10)	variable (p. 280)
$v = ds/dt$	$\omega = d\theta/dt$

Composition of velocities (p. 16), (p. 17); (p. 281)

$$R = \sqrt{u^2 + 2uv\cos\theta + v^2} \qquad \omega = \sqrt{\omega_1^2 + 2\omega_1\omega_2\cos\theta + \omega_2^2}$$

$$R = \sqrt{v_x^2 + v_y^2} \qquad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad \omega = \sqrt{\omega_x^2 + \omega_y^2}$$

Rectangular components (p. 17)

$$v_x = v\cos\theta \qquad \frac{dx}{dt} = \frac{ds}{dt}\cos\theta$$

$$v_y = v\sin\theta \qquad \frac{dy}{dt} = \frac{ds}{dt}\sin\theta$$

Non-rectangular components (p. 17)

Relation of angular and linear velocity (p. 279)

$$v = \omega r$$

Acceleration

uniform (p. 19)	variable (p. 20)	p. 284	p. 284
$a = (v - u)/t$	$a = \dfrac{dv}{dt}$		$\alpha = \dfrac{d\omega}{dt}$
$v = u + at$	$= \dfrac{d^2s}{dt^2}$	$\omega = \omega_0 + \alpha t$	$= \dfrac{d^2\theta}{dt^2}$

Acceleration

uniform (p. 19)	variable (p. 20)	p. 284	p. 284
$s = \frac{1}{2}(u + v)$	$a = v\dfrac{dv}{ds}$		$\alpha = \omega\dfrac{d\omega}{dt}$
$s = ut + at^2/2$		$\theta = \omega_0 t + \alpha t^2/2$	
$as = v^2/2 - u^2/2$		$\alpha\theta = \omega^2/2 - \omega_0{}^2/2.$	

Projectiles—acted on by gravity only.

Body falling freely from rest (p. 92). [Equation of motion.]

$$v = gt; \quad y = gt^2/2; \quad v^2/2 = gs \qquad \frac{d^2y}{dt^2} = g \ \text{(p. 94)}$$

Body projected vertically with velocity u (p. 92).

$$v = u \pm gt; \quad y = ut \pm \tfrac{1}{2}gt^2; \quad v^2/2 - u^2/2 = gy. \quad \frac{d^2y}{dt^2} = \pm\, g$$

Body projected vertically up with velocity u (p. 94).

Time of flight $= 2u/g$ $\qquad \dfrac{d^2y}{dt^2} = -\, g$

Greatest height $= u^2/2g$

Velocity at any time $= u - gt$

Velocity at any height $y = \sqrt{u^2 - 2gy}$

Body projected with velocity u at angle θ to horizontal (p.98).

Path a parabola.

$v = \sqrt{u^2 - 2utg \sin\theta + g^2t^2}$

$$\left\{\begin{aligned} \frac{d^2x}{dt^2} &= 0 \ \text{(p. 101)} \\ \frac{d^2y}{dt^2} &= -\, g \end{aligned}\right.$$

$v = \sqrt{u^2 - 2gy}$

Time of flight $= 2u \sin\theta/g$

Greatest height $= u^2 \sin^2\theta/2g$

Range $= u^2 \sin 2\theta/g$

Direction $= \tan^{-1}(\tan\theta - gt/u\cos\theta)$

$= \tan^{-1}(\tan^2\theta - 2gy/u^2\cos^2\theta)^{\frac{1}{2}}$

Motion in a circle:

Uniform (p. 27)	Variable (p. 28)
$a = v^2/r$	tang. accel. $= \dfrac{dv}{dt}$
$vT = 2\pi r$	normal accel. $= v^2/r$
$a = 4\pi^2 r/T^2$	

Simple harmonic motion (p. 29; p. 107). [Equation of motion.]

$y = r \cos \omega t, \quad x = r \cos (\omega t - \pi/2)$

$a_y = v^2 y/r^2 = \omega^2 y$ $\qquad \dfrac{d^2y}{dt^2} = -\omega^2 y$ (p. 107)

$T = 2\pi/\omega = 2\pi \sqrt{\text{displac./accel.}}$

Simple pendulum (p. 128; p. 133)

time of an oscillation $= \pi \sqrt{l/g}$ $\qquad \dfrac{d^2s}{dt^2} = -g\dfrac{dy}{ds}$

Seconds pendulum (p. 129).

length $= g/\pi^2$

Blackburn pendulum (p. 131).

Motion under gravity down a smooth incline (p. 113)

velocity at bottom $= \sqrt{2gh}$ $\qquad \dfrac{d^2s}{dt} = g \sin \theta$

velocity at time t $= tg \sin \theta$

distance described in time $t = t^2 g \sin \theta/2$

time of sliding down plane $= l\sqrt{2/gh}$

Motion down a rough incline (p. 217). [Equation of motion.]

$a = g(\sin \theta - \mu \cos \theta)$ $\qquad \dfrac{d^2s}{dt^2} = g(\sin \theta - \mu \cos \theta)$

B. Dynamics.

Newton's Laws:

Law of inertia (p. 45).
Law of mass-acceleration (p. 47).
Law of action and reaction (p. 51).
Summary of laws (p. 53).
Algebraic statement of law 2:

Absolute units (pp. 49, 348)	Gravitation units (pp. 59, 344)
$F = ma$	$F = wa/g$
$Ft = mv$	$Ft = wv/g$
$Fs = mv^2/2$	$Fs = wv^2/2g$

Illustrative apparatus. Atwood's machine (p. 65).

Composition of two forces acting at a point (p. 76).

$$R^2 = F_1^2 + 2F_1F_2 \cos\theta + F_2^2$$

$$\beta = \tan^{-1}(F_2 \sin\theta)/(F_1 + F_2 \cos\theta).$$

Resolution of a force R into rectangular components (p. 79)

$$X = R\cos\theta; \quad Y = R\sin\theta.$$

Composition of any forces at a point [analytical method] (p. 81).

$$X = F_1 \cos\theta_1 + F_2 \cos\theta_2 + \ldots; \quad Y = F_1 \sin\theta_1 + F_2 \sin\theta_2 + \ldots$$

$$R = \sqrt{X^2 + Y^2} \qquad \beta = \tan^{-1}(Y/X)$$

Composition of any forces at a point O [graphical method] (p. 77).

From any point draw lines equal and parallel to the forces. The closing side of the polygon is the resultant in magnitude and direction.

Composition of any forces acting on a body [graphical method] (pp. 145, 146).

Statement in Art. 129.

Composition of any forces on a body [analytical method] (p. 155).

Any coplanar system may be reduced—

(*a*) to a single force or to a torque;

(*b*) to a single force at an assigned point and a torque.

Motion under various laws of force.

Force constant and in the direction of motion (p. 91)

$$F = \frac{w}{g}\frac{d^2y}{dt^2} = c.$$

Force variable and in the direction of motion.

(*a*) Force proportional to distance from a fixed point O (p. 106).

$$F = \frac{w}{g}\frac{d^2y}{dt^2} = -cy.$$

(*b*) Force inversely proportional to square of distance from O (p. 109).

$$F = \frac{w}{g}\frac{d^2y}{dt^2} = -\frac{c}{y^2}.$$

Force variable and not in direction of motion (p. 111).

Force varies inversely as square of distance from O. Planetary motion.

Motion on a vertical circle under gravity (p. 132).

$$F = \frac{w}{g}\frac{d^2s}{dt^2} = -w\frac{dy}{ds}$$

pressure of circle on particle $= w(3\cos\theta - 2\cos\beta)$.

Centripetal force (p. 121).

A particle of weight w moves with constant velocity v in the circumference of a circle of radius r in time t, and making n revolutions per minute.

$$C = wv^2/gr = 4\pi^2 wr/gt^2 = 0.00034wrn^2.$$

Conical pendulum or governor (p. 125).

$$T = 2\pi\sqrt{h/g};\quad v = r\sqrt{g/h};\quad P = wl/h.$$

Elevation β of outer rail on a curve of radius r (pp. 127, 188).

$$\beta = \tan^{-1}(v^2/gr).$$

Value of acceleration g of gravity (p. 127).

$$g = 4\pi^2 h/T^2;\ T \text{ is observed and } h \text{ is measured.}$$

Variation of g with the latitude λ of the place of observation (p. 123).

$$g = g_0(1 - \cos^2\lambda/289).$$

Inclination θ of plumb-line at latitude λ to earth's radius.

$$\theta = \tan^{-1}(\sin\lambda\cos\lambda/289).$$

Equilibrium (p. 85).

(1) Forces acting at a point O.

Graphical condition. The forces may be represented in magnitude and direction by the sides of a closed polygon taken the same way round (p. 88).

Analytical condition. The sums of the component forces in any directions OX and OY, at right angles to each other, are each equal to zero (p. 89).

Case of three forces. Each force must be proportional to the sine of the angle between the other two forces (p. 87).

Special problem.

Inclined plane: Smooth (p. 116) Rough (p. 211)

(a) Force parallel plane.

$F = W\sin\theta,\ N = W\cos\theta$ $\qquad F = W\sin(\theta \pm \phi)/\cos\phi$

(*b*) Force parallel base,

$F = W \tan \theta,\ N = W \sec \theta \qquad F = W \tan (\theta \pm \phi)$

(2) Forces acting at different points.

Graphical condition. No general statement (p. 146).

Case of three forces. The forces must meet in a point or be parallel (p. 158).

Analytical condition (p. 157).

- The sum of the component forces in any direction $OX = 0$.
- The sum of the component forces in a perpendicular direction $OY = 0$.
- The sum of the moments about any point in the plane of the forces $= 0$.

[Problem of Calculus.

To find centroid of a surface.

$$\bar{x} = \Sigma wx / \Sigma w \qquad \bar{y} = \Sigma wy / \Sigma w \qquad \text{(p. 160)}$$

$$\bar{x} = \int xydx \Big/ \int ydx \qquad \bar{y} = \tfrac{1}{2}\int y^2dx \Big/ \int ydx \ \text{(p. 167)}$$

$$\bar{x} = \int\int xdxdy \Big/ \int\int dxdy$$

$$\bar{y} = \int\int ydxdy \Big/ \int\int dxdy$$

$$\bar{x} = \int\int r^2 \cos \theta d\theta dr \Big/ \int\int rd\theta dr$$

$$\bar{y} = \int\int r^2 \sin \theta d\theta dr \Big/ \int\int rd\theta dr.]$$

Special problems.

Body supported at one point (p. 171).

The lever (p. 172); the balance (p. 174); the steel-yard (p. 177).

Translation	Rotation (p. 309)
$Wv^2/2g$	$I\omega^2/2g$

Total energy = energy of rotation about C. G. + energy of translation

$$= I\omega^2/2g + Wu^2/2g \text{ (p. 310).}$$

INDEX.

4 28

175